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On the almost-palindromic width of free groups

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A R T I C L E I N F O

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A B S T R A C T

We answer a question of Bardakov (Kourovka Notebook, Problem 19.8) which asks for the existence of a pair of natural numbers (c, m) with the property that every element in the free group on the two-element set $\{a, b\}$ can be represented as a concatenation of c, or fewer, m-almost-palindromes in letters $a^{\pm 1}, b^{\pm 1}$. Here, an m-almost-palindrome is a word which can be obtained from a palindrome by changing at most m letters. We show that no such pair (c, m) exists. In fact, we show that the analogous result holds for all non-abelian free groups.

1. Introduction and statement of the main result

For a group G, an element $g \in G$, and a generating set $X \subseteq G$, let $\text{length}(g, X)$ be the minimal $n \geq 0$ with the property that there exist $x_1, \ldots, x_n \in X$ for which $g = x_1^{\pm 1} \cdots x_n^{\pm 1}$, that is, the minimal number of elements from the generating set X necessary to generate g. Let us define the width of G with respect to the generating set X as

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$$\text{width}(G, X) = \max_{g \in G} \text{length}(g, X)\,,$$

or $\text{width}(G, X) = \infty$, if the maximum does not exist.

If B is a set, we denote by F_B the free group on B whose elements are all reduced words with letters in $B^{\pm 1} = B \cup \{b^{-1} \mid b \in B\}$, with the operation given by concatenation and subsequent reduction. We denote by W_B the free monoid on the set $B^{\pm 1}$ whose elements are now all words with letters in $B^{\pm 1}$, with concatenation as operation. There is the homomorphism of monoids $r_B \colon W_B \to F_B$ which sends a word to its corresponding reduced word, and which is left-inverse to the inclusion of F_B into W_B.

For a word $w \in W_B$, let $\text{rev}(w)$ be its reverse word, that is, the word given by the letters of w in reverse order. A palindrome is a word $p \in W_B$ with the property that $p = \text{rev}(p)$. An m-almost-palindrome is a word which differs from a palindrome by a change of at most m letters. In other words, $\tilde{p} \in W_B$ is an m-almost-palindrome if there exists a palindrome p with the property that $d(p, \tilde{p}) \leq m$, where d is the Hamming distance on the set of all words in W_B whose number of letters equals the number of letters of $\tilde{p}$, or, equivalently, if $d(\tilde{p}, \text{rev}(\tilde{p})) \leq 2m$. Thus, the palindromes are exactly the 0-almost-palindromes. Let us denote by $P_{B,m}$ the set of all m-almost-palindromes and observe that the image $r_B(P_{B,m})$ is a generating set for the free group F_B, for every $m \geq 0$.

If F is a free group, together with a basis B of F, then there is the canonical isomorphism $\phi_B \colon F_B \to F$. Furthermore, note that for all m the set $X_{B,m} = \phi_B(r_B(P_{B,m}))$ is a generating set for F. In fact, we have an increasing sequence $X_{B,0} \subseteq X_{B,1} \subseteq \ldots$ of generating sets of F. The value of the expression $\text{width}(F, X_{B,m})$ does not depend on the choice of the basis B of F, since for two bases B, B', the automorphism of F induced by a bijection $B \to B'$ sends $X_{B,m}$ to $X_{B',m}$. We call this value the m-almost-palindromic width of the free group F and write $\text{AP-width}(F, m) = \text{width}(F, X_{B,m})$.

If F is trivial, or a free group of rank one, then it is readily seen that for all $m \geq 0$ the m-almost-palindromic width of F is zero, or one. However, Bardakov, Shpilrain, and Tolstykh [1] show that, for a non-abelian free group F (i.e., a free group which is neither trivial nor infinite cyclic), the 0-almost-palindromic width (that is, the palindromic width) of F is infinite. Extending on this, Bardakov [2, Problem 19.8] asks whether there exists a pair of natural numbers (c, m) with the property that every element of $F_{\{a,b\}}$ can be represented by a concatenation of c, or fewer, m-almost-palindromes in letters $a^{\pm 1}, b^{\pm 1}$. In other words, he asks whether there is an $m \in \mathbb{N}$ for which the m-almost-palindromic width of F is finite, assuming that F is a free group of rank two.

Of course, the same question can be asked without the restriction on the rank of F. In this note, we answer this question negatively for all non-abelian free groups, building on the argument by Bardakov, Shpilrain, and Tolstykh for the case $m = 0$.

Theorem 1. *If F is a non-abelian free group, then, for all $m \geq 0$, the m-almost-palindromic width of F is infinite.*

Throughout this note, whenever $u, w \in W_B$, in writing uw we always mean the concatenation of the words u and w, that is, the result under the operation in the monoid W_B, in order to avoid ambiguity which may arise from the fact that $F_B \subseteq W_B$. For instance, we explicitly write $r_B(uw)$ to denote the result under the operation in the group F_B, if u and w are reduced words. Moreover, if $u, w \in W_B$ are words, we mean by $u = w$ that the two are equal as elements of W_B, i.e., in a letter-by-letter way. We denote the empty word by ε. If $t \in B^{\pm 1}$ and $k \in \mathbb{N}$, then we denote by t^k the word which consists of k-times the letter t. In particular, $t^0 = \varepsilon$, while t^1 denotes the word which consists of the single letter t. If $t \in B$ and $k < 0$, then we write $t^k = (t^{-1})^{-k}$, that is, $(-k)$-times the letter $t^{-1} \in B^{\pm 1}$.

2. A map with a useful property

If B is a set, and $w \in F_B$, then, for $n \geq 0$, we may write $w = t_1^{k_1} \cdots t_n^{k_n}$, where $t_i \in B$ and $k_i \in \mathbb{Z} \setminus \{0\}$ for all $i \in \{1, \ldots, n\}$, and where $t_i \neq t_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. In this way, the number n, as well as the k_i and t_i are uniquely determined. If $n = 0$, this means that $w = \varepsilon$. We call the words ${t_i}^{k_i}$, for $i \in \{1, \ldots, n\}$, the syllables of the reduced word w.

Let the map $\Delta_B \colon W_B \to \mathbb{Z}$ be defined in two steps as follows: First, if $w \in F_B$ is of the form $w = t^k$ for $t \in B, k \in \mathbb{Z}$, then we define

$$\Delta_B(w) = \Delta_B(t^k) = 0\,.$$

In particular, $\Delta_B(\varepsilon) = 0$. Otherwise, $w \in F_B$ has at least two syllables, that is, it is of the form $w = t_1^{k_1} \cdots t_n^{k_n}$, where the $t_i^{k_i}$ are the syllables of w, with $t_i \in B$ for all $i \in \{1, \ldots, n\}$, for $n \geq 2$. In this case, we set

$$\Delta_B(w) = \Delta_B(t_1^{k_1} \cdots t_n^{k_n}) = \sum_{i=1}^{n-1} \operatorname{sign}(|k_{i+1}| - |k_i|)\,.$$

Here, $\operatorname{sign} \colon \mathbb{Z} \to \mathbb{Z}$ is the map which takes the value -1 on all negative integers, the value 1 on all positive integers, and for which $\operatorname{sign}(0) = 0$. This concludes the definition of Δ_B on F_B. In order to extend this definition to all of W_B, we define

$$\Delta_B(w) = \Delta_B(r_B(w))\,,$$

for a not necessarily reduced word $w \in W_B$.

The map Δ_B has the following useful property which resembles [1, Lemma 1.3].

Lemma 1. *Let $n \geq 0$ be an integer and $w_1, \ldots, w_n \in W_B$. Then*

$$\left| \Delta_B(w_1 \cdots w_n) - \sum_{i=1}^{n} \Delta_B(w_i) \right| \leq 6n\,.$$

Proof. If $n = 0$, then $\Delta_B(w_1 \cdots w_n) = \Delta_B(\varepsilon) = 0$, as well as $\sum_{i=1}^{n} \Delta_B(w_i) = 0$. Thus, for $n = 0$ the inequality holds. If $n = 1$, the inequality is obviously true.

For $n = 2$, we consider two cases: First, if $r_B(w_2) = r_B(w_1)^{-1}$, then $\Delta_B(w_1 w_2) = \Delta_B(r_B(w_1 w_2)) = \Delta_B(r_B(w_1) r_B(w_2)) = \Delta_B(\varepsilon) = 0$. Also, it is readily seen that $\Delta_B(w_1) + \Delta_B(w_2) = \Delta_B(r_B(w_1)) + \Delta_B(r_B(w_1)^{-1}) = 0$, so, in this case,

$$|\Delta_B(w_1 w_2) - (\Delta_B(w_1) + \Delta_B(w_2))| = 0 \leq 12 = 6n\,.$$

Otherwise, let $r_B(w_1) = t_n^{k_n} \cdots t_1^{k_1}$ and $r_B(w_2) = s_1^{l_1} \cdots s_m^{l_m}$ be the decompositions of $r_B(w_1)$ and $r_B(w_2)$ into syllables, where $t_i \in B, k_i \in \mathbb{Z} \setminus \{0\}$ for all $i \in \{1, \ldots, n\}$, and $s_i \in B, l_i \in \mathbb{Z} \setminus \{0\}$ for all $i \in \{1, \ldots, m\}$. Let j be the largest index with the property that $0 \leq j \leq \min\{n, m\}$, and that $t_1^{-k_1} \cdots t_j^{-k_j} = {s_1}^{l_1} \cdots {s_j}^{l_j}$. We set $z = {s_1}^{l_1} \cdots {s_j}^{l_j}$ (i.e., $z = \varepsilon$, if $j = 0$) and, since $r_B(w_2) \neq r_B(w_1)^{-1}$, we may write

$$r_B(w_1) = u_1 t^k z^{-1}\,,$$
$$r_B(w_2) = z t^l u_2\,,$$

where $t \in B$ is a letter, $k, l \in \mathbb{Z}$ with $k + l \neq 0$, where neither u_1 ends with the letter t or t^{-1}, nor u_2 begins with t or t^{-1}, and where u_1 and z^{-1} each consist of whole syllables of $r_B(w_1)$ (i.e., the sum of the numbers of syllables of the reduced words u_1, t^k, and z^{-1} is equal to the number of syllables of $r_B(w_1)$), and where z and u_2 each consist of whole syllables of $r_B(w_2)$. Here, it may happen that u_1, u_2, or z, of course, are empty. Also, one of the numbers k and l may be zero.

Hence, we observe that

$$\Delta_B(w_1) = \Delta_B(u_1) + \epsilon_1 + \Delta_B(z^{-1})\,,$$
$$\Delta_B(w_2) = \Delta_B(z) + \epsilon_2 + \Delta_B(u_2)\,,$$

where $\epsilon_1, \epsilon_2 \in \{-2, -1, 0, 1, 2\}$. Furthermore,

$$\Delta_B(w_1 w_2) = \Delta_B(u_1 t^k z^{-1} z t^l u_2) = \Delta_B(u_1 t^{k+l} u_2) = \Delta_B(u_1) + \epsilon_3 + \Delta_B(u_2)\,,$$

where, similarly, $\epsilon_3 \in \{-2, -1, 0, 1, 2\}$. Now,

$$|\Delta_B(w_1 w_2) - (\Delta_B(w_1) + \Delta_B(w_2))| = |-\epsilon_1 - \epsilon_2 + \epsilon_3| \leq 6 \leq 12 = 6n\,.$$

Finally, for $n \geq 3$, we proceed by induction: Assuming that the assertion holds for $n - 1$, and applying the considerations for the case $n = 2$, we see that

$$\begin{aligned}\left|\Delta_B(w_1\cdots w_n)-\sum_{i=1}^{n}\Delta_B(w_i)\right| &= \Bigg|\Delta_B(w_1\cdots w_n)-(\Delta_B(w_1\cdots w_{n-1})+\Delta_B(w_n))\\ &\qquad +(\Delta_B(w_1\cdots w_{n-1})+\Delta_B(w_n))-\sum_{i=1}^{n}\Delta_B(w_i)\Bigg|\\ &\le \left|\Delta_B(w_1\cdots w_n)-(\Delta_B(w_1\cdots w_{n-1})+\Delta_B(w_n))\right|\\ &\qquad +\left|\Delta_B(w_1\cdots w_{n-1})-\sum_{i=1}^{n-1}\Delta_B(w_i)\right|\\ &\le 6+6(n-1)\\ &= 6n\,,\end{aligned}$$

completing the proof. □

3. Bounds for some values of the map

Throughout this section, let B be a fixed set. We simplify notation and put $F = F_B$, $W = W_B$, $P_m = P_{B,m}$, $r = r_B$, and $\Delta = \Delta_B$. In this context, the following propositions hold.

Proposition 1. *If $\tilde{p} \in P_m$ is an m-almost-palindrome, then*

$$\Delta(\tilde{p}) \le 24m + 12\,.$$

Proof. Let p be a palindrome from which $\tilde{p}$ is obtained by changing m or fewer letters. Let $t_1, \ldots, t_\alpha \in B^{\pm 1}$ be all the letters of p which are subject to the change, in the order in which they appear within p, so that

$$p = w_1 t_1^1 w_2 t_2^1 \cdots t_\alpha^1 w_{\alpha+1}\,,$$

where the w_i are (potentially empty) words. Here, $\alpha \le m$.

By Lemma 1, we have

$$\Delta(p) - \sum_{i=1}^{\alpha+1}\Delta(w_i) - \sum_{i=1}^{\alpha}\Delta(t_i^1) \ge -6\cdot(2\alpha+1)\,.$$

Here, p is a palindrome, and the t_i^1 are words with exactly one letter, so $\Delta(p) = 0$, as well as $\Delta(t_i^1) = 0$ for all $i \in \{1, \ldots, \alpha\}$. Hence, the latter inequality can be simplified to yield

$$\sum_{i=1}^{\alpha+1}\Delta(w_i) \le 12\alpha + 6\,.$$

Let $\tilde{t}_i$ be the letter t_i after the relevant change, so that

$$\tilde{p} = w_1\tilde{t}_1^1 w_2\tilde{t}_2^1 \cdots \tilde{t}_\alpha^1 w_{\alpha+1}\,.$$

Still, we have $\Delta(\tilde{t}_i^1) = 0$, for all $i \in \{1, \ldots, \alpha\}$.

Now, using Lemma 1, together with the above observations, we arrive at the desired inequality

$$\begin{aligned}\Delta(\tilde{p}) &= \Delta(w_1\tilde{t}_1^1 w_2\tilde{t}_2^1 \cdots \tilde{t}_\alpha^1 w_{\alpha+1}) \\ &\le \left(\sum_{i=1}^{\alpha+1} \Delta(w_i)\right) + \left(\sum_{i=1}^{\alpha} \Delta(\tilde{t}_i^1)\right) + 6\cdot(2\alpha+1) \\ &\le (12\alpha+6) + 0 + (12\alpha+6) \\ &= 24\alpha + 12 \\ &\le 24m + 12\,. \quad \square\end{aligned}$$

Proposition 2. *Let $c \ge 0$ be an integer, and let $\tilde{p}_1, \ldots, \tilde{p}_c \in P_m$ be m-almost-palindromes. Then*

$$\Delta(\tilde{p}_1 \cdots \tilde{p}_c) \le 24mc + 18c\,.$$

Proof. We use Lemma 1 and then apply Proposition 1 to see that

$$\begin{aligned}\Delta(\tilde{p}_1 \cdots \tilde{p}_c) &\le \left(\sum_{i=1}^{c} \Delta(\tilde{p}_i)\right) + 6c \\ &\le c\cdot(24m+12) + 6c \\ &= 24mc + 18c\,. \quad \square\end{aligned}$$

4. Proof of the main result

Proof of Theorem 1. Let F be a non-abelian free group, and $m \ge 0$. Since F is non-abelian, a basis B has at least two distinct elements $a, b \in B$. Consider the sequence of words $(w_n)_{n\in\mathbb{N}}$ defined by

$$w_n = a^1b^1a^2b^2 \cdots a^nb^n \in F_B\,.$$

It is readily seen that $\Delta_B(w_n) = n - 1$. In particular, $\Delta_B(w_n) \to \infty$ for $n \to \infty$.

On the other hand, for $c \ge 0$, Proposition 2 shows that Δ_B is bounded from above on the set $r_B({P_{B,m}}^c) \subseteq F_B$, where ${P_{B,m}}^c = \{p_1 \cdots p_c \,|\, p_i \in P_{B,m} \text{ for all } i \in \{1, \ldots, c\}\}$. Hence, $r_B({P_{B,m}}^c) \neq F_B$ for all $c \ge 0$. This implies that for all $c \ge 0$ it holds that also

$$\phi_B(r_B({P_{B,m}}^c)) = \{x_1 \cdots x_c \,|\, x_i \in X_{B,m} \text{ for all } i \in \{1, \ldots, c\}\} \neq F\,.$$

Thus, AP-width$(F, m) = \infty$. □

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