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Exploring prospective teachers' stances in making sense of students' mathematical ideas

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Abstract
In this paper, we explore the critical practice of making sense of students' mathematical ideas. We extend previous research by studying stances prospective teachers adopt, the extent or depth to which they do so, and the types of prospective teachers making sense of students' mathematical ideas. Analyzing the responses of 123 prospective teachers to students' different ideas on an ambiguous mathematical task, our study identifies various stances—descriptive, evaluative, comparative, interpretive, inquiry-based, connective, and projective—and explores the complexity of attributing value, meaning, and significance to student ideas. Our findings offer insights into various types of making sense of students' ideas and suggest that different kinds of attributions are at play for the purposes of observation, assessment, understanding and projection/prediction.

Keywords Stances of interpretation · Student mathematical thinking · Ambiguous mathematical problem · Teacher noticing · Prospective primary teachers

Introduction

The work of mathematics teaching is complex and involves a variety of practices, such as preparing mathematics for teaching (Scheiner et al., 2022), analyzing students' mathematical work (Baldinger, 2020) and responding to students' mathematical thinking in the moment (Jacobs & Empson, 2016). In this regard, making sense of students' mathematical ideas is considered fundamental to these practices of mathematics teaching and is therefore the subject of this paper. Teachers make sense of students' ideas when they engage with students' mathematical thinking—for example, by looking at and listening

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to students' mathematical reasoning and work (Davis, 1997; Empson & Jacobs, 2008; Wallach & Even, 2005). Making sense of students' ideas is crucial to basing instruction on students' thinking, which has been shown to enhance mathematical learning for all students (Fennema et al., 1996; Franke et al., 2001).

Making sense of students' ideas is also a critical part of noticing students' thinking, which has attracted much attention in teacher education research over the past two decades (for overviews, see Dindyal et al., 2021; Kaiser et al., 2023; König et al., 2022; Scheiner, 2016; Stahnke et al., 2016; Weyers et al., 2024). Although teacher noticing is arguably embodied, cultural and positional in important ways (Scheiner, 2021), much of the research on teacher noticing has understood noticing as being composed of two psychological processes: paying attention to and making sense of students' thinking (Sherin et al., 2011). Teachers' attention and ability to make sense of students' thinking, in turn, form the basis for responding to students' thinking (Jacobs et al., 2010; Kaiser et al., 2017; Sánchez-Matamoros et al., 2019), which can vary widely in its potential to support student learning (Stockero et al., 2022). Teacher noticing, as such, is consequential in important ways (Schoenfeld, 2011).

A key feature of classroom instruction is that teachers are confronted with a variety of ways in which students approach and solve mathematical problems. Making sense of students' different approaches and solutions to mathematical problems and recognizing their vividness can be challenging, especially for prospective teachers. In this paper, we therefore identify and explore how prospective teachers make sense of different *valid* student solutions to ambiguous mathematical problems. In doing so, we draw on previous research that has identified different stances—i.e., modes or manners of interpretation—that teachers adopt when making sense of students' ideas.[1]

This paper contributes to the research on how prospective teachers make sense of students' mathematical thinking in several ways. While previous research has identified different stances that teachers adopt when making sense of students' ideas (e.g., Davis, 1997; Scheiner, 2023; Sherin & van Es, 2009; Stockero, 2008; Walkoe et al., 2020) and highlighted the distribution and frequency of occurrence of these stances (e.g., Amador et al., 2022; Shin, 2021), our work extends this by exploring the extent or depth to which certain stances are used. We examine the specific extent to which prospective teachers adopt these stances, revealing new layers of complexity in how they make sense of students' ideas. This approach offers a more granular understanding of how prospective teachers make sense of students' ideas and the specific purposes for which they do so.

In addition, our research focuses specifically on how prospective teachers make sense of different valid student solutions to ambiguous mathematical problems. This is an important area of inquiry, as it highlights the complexity of making sense of students' ideas in situations where there is no single 'right' answer. By exploring how prospective teachers respond to this ambiguity, this research provides insights into the types of sense-making that are important for teachers to develop to effectively interpret students' mathematical thinking. This research helps to further highlight the cognitive demands of teachers' sense-making of students' ideas in complex situations.

[1] The notion of 'stance' has been used to refer to the interpretative framework that comes into play when considering students' thinking (e.g., Amador et al., 2022; Scheiner, 2023; Sherin & van Es, 2009; Walkoe et al., 2020). Following this line of research, a stance is understood here as the way of making sense or the position from which one makes sense of students' ideas.

Making sense of students' mathematical ideas

An important prerequisite for effective instruction is sensitivity toward students and their thinking. This requires teachers to purposefully analyse students' mathematical reasoning and work in ways that elucidate their thinking (Borko & Putnam, 1996; Franke & Kazemi, 2001). To achieve this goal, teachers need to engage in knowledge-based reasoning (van Es & Sherin, 2002), draw on knowledge about the subject matter and the way students think about the subject matter (e.g., Even & Tirosh, 2002; Fennema & Franke, 1992), as well as consider the context in which the instructional event in question takes place (Jarry-Shore & Borko, 2023; van Es & Sherin, 2002). Recent research has shown how teachers' mathematical thinking and conceptions shape their ability to make sense of students' thinking and decision-making (Kooloos et al., 2022a; Liang, 2023).

Further research has shown that with targeted support, prospective teachers can move beyond simply assessing student responses as correct or incorrect to understanding the underlying reasoning, deconstructing student strategies, and formulating subsequent questions to delve deeper into student thought processes, thereby achieving a deeper attunement to student thinking (see e.g., Sherin & van Es, 2005; Stockero, 2008; Walkoe et al., 2015).

Making sense of students' mathematical ideas, however, can involve very different cognitive demands, from describing students' reasoning and work to actively seeking multiple, even contradictory, ways of interpreting students' thinking. Although the latter approach requires much greater cognitive effort on the part of teachers, it promises to allow teachers to view future instructional events as novel interactions to which they can respond freshly rather than out of habit (Mason, 2002).

There is indeed a spectrum of manners or positions that teachers can take to make sense of students' ideas—what are generally referred to as *stances* in making sense of students' thinking (e.g., Amador et al., 2022; Davis, 1997; Kooloos et al., 2022b; Scheiner, 2023; Sherin & van Es, 2009; Walkoe et al., 2020). One of the most common stances teachers adopt when making sense of students' ideas is the *descriptive stance*. A descriptive stance is adopted by teachers when they focus on observable aspects or features of a particular object of attention (Sherin & van Es, 2009). For example, describing or documenting students' reasoning in their own words is indicative of a descriptive stance. Descriptive stances are usually uncommitted, meaning that teachers tend to assume non-biased positions when trying to make sense of students' ideas (Scheiner, 2023).

On the other hand, when teachers judge the quality of an object of attention, they adopt an *evaluative stance* (Sherin & van Es, 2009). An evaluative stance focuses on looking at and listening to students' ideas to determine their correctness (Davis, 1997). This may involve considering students' ideas against the background of an expected or anticipated response. For example, teachers pointing out what was correct or incorrect in students' reasoning or work indicates an evaluative stance (Crespo, 2000). An evaluative stance is particularly evident when teachers are primarily concerned with identifying and correcting students' mistakes or failures (Scheiner, 2023). However, such an evaluative stance can lead to disregarding students' ideas and their potential as a source for further learning (Kalinec-Craig et al., 2021).

An *interpretive stance* is adopted by teachers when they interpret the substance (rather than just the correctness) of students' thinking or make inferences from students' ideas (Sherin & van Es, 2009). For example, when listening to students' ideas with the aim of understanding their thinking, teachers adopt an interpretive stance (Davis, 1997). Teachers typically have difficulty developing a deep understanding of students' ideas and value them when these ideas deviate from what teachers anticipate or expect to see or hear (Morgan & Watson, 2002). However, when teachers purposefully interpret students' ideas as assets or resources, they adopt a *strengths-based stance* (Scheiner, 2023).

Teachers adopt a *hermeneutic stance* when they engage in the negotiation of meanings with students and revise their own understanding of the situation they are observing (Davis, 1997). This approach is crucial for developing a deep understanding of students' thinking and can involve the co-construction of meaning with students by actively participating in a process of inquiry into their ideas, as similarly noted by van Es and Sherin (2021) and referred to as a *stance of inquiry*. An inquiry-based stance, as we construe it here, reflects a (hypothetical) engagement in which teachers consider different approaches to digging deeper into students' thinking. This may involve the teacher imagining ways of clarifying or extending students' ideas, for example, hypothesizing about creating opportunities to clarify a student's reasoning or asking the student to explain their thinking in a different way. This emphasis is informed by the importance of such stances in creating opportunities to build on students' thinking (Franke et al., 2001; Steinberg et al., 2004).

When teachers engage in analyzing students' ideas and subsequently connect these ideas to broader mathematical concepts or topics, they adopt a *connective stance* (Walkoe et al., 2020). This process entails comprehending how students' ideas are interwoven with the overall progression of mathematical topics. Adopting a connective stance represents a deeper, more insightful engagement with students' mathematical thinking, emphasizing the importance of context and interconnectedness in learning. In this approach, the teacher's focus transcends mere correctness or basic understanding of students' ideas; it extends to an exploration of how these ideas mirror students' grasp of broader mathematical principles.

This variety of stances that teachers may adopt in making sense of students' ideas can be considered along different dimensions. Walkoe et al. (2020), for example, examined different stances along two levels of cognitive demand. The first level includes stances with lower cognitive demands, such as describing, evaluating, and restating students' ideas. The second level includes stances with higher cognitive demands, such as interpreting the substance of students' thinking, analyzing, and connecting students' thinking to larger mathematical ideas.

The distinction between lower-level and higher-level stances can also be considered in light of Mason's (2002) distinction between an *account of* and an *account for*. In particular, teachers' descriptive and evaluative stances suggest that teachers look for particular ideas, details and aspects in students' thinking (that is, they are seeking an account of students' thinking), which is different from interpretive and inquiry-based stances, which are arguably more concerned with teachers looking into students' thinking (that is, they are seeking an account for students' thinking).

The cognitive effort required for higher-level stances that involve accounting for students' thinking should not be underestimated, as an account for students' thinking requires a greater degree of *decentering*, i.e., increasingly taking the perspective of others (e.g., Arcavi & Isoda, 2007; Baş-Ader & Carlson, 2022). This poses an even greater cognitive challenge when teachers are confronted with a variety of student ideas that may develop in very different ways as students approach and solve complex or ambiguous mathematical problems (Doerr, 2006).

Although significant progress has been made in understanding the different stances teachers take when making sense of students' mathematical ideas, a nuanced exploration of the extent or depth of these stances in making sense of students' ideas remains underexplored. This gap highlights the critical need for more detailed insights into the complexity of prospective teachers' sense-making practices, particularly in contexts involving ambiguous mathematical problems where multiple valid student solutions emerge.

The following research questions guided the research design and analysis:

1. What stances do prospective teachers adopt, and to what extent or depth do they do so, in making sense of students' different ideas of an ambiguous mathematical problem?
2. What types of making sense emerge from prospective teachers' responses to students' different ideas based on the patterns of the extent of stances used?

Method

Qualitative and quantitative data analyses were used to examine the ways in which prospective teachers make sense of students' mathematical ideas. In particular, the focus was on the stances prospective teachers adopted, the extent or depth they used those stances, and the types of making sense that emerged from prospective teachers' responses to students' different ideas. In the following, the participants and context, the process of data collection and the data analysis are described.

Participants and context

The study included 123 prospective primary teachers who were enrolled in a mathematics teacher education course taught by the second author. The course was part of a four-year bachelor's teacher education program at a university in a regional area of Spain. The prospective teachers were at the end of the second year of study and had previously learning opportunities to gain mathematical content knowledge and mathematics didactic knowledge about numbers, algorithms, and arithmetic.[2] Participation in the study was voluntary and unpaid. Written consent was obtained from the participating teachers to use the data for research purposes.

Data collection

The study data consisted of participants' written responses to a purposefully designed task in which prospective teachers were asked to make sense of students' different ideas about an ambiguous mathematical problem. The task given to the prospective teachers was formulated in Spanish, and all the data were collected and analyzed in Spanish. Only the selected data presented in this paper were translated into English.

The mathematical problem statement was formulated in an 'ill-structured' way (Stoyanova & Ellerton, 1996), which invited participants to understand and approach it in different ways, leading to different possible solutions (Silver, 1997). Three student answers were given to the mathematical problem, demonstrating various approaches and ways of

[2] In this context, 'mathematics didactic knowledge' refers to the normative, principles-based knowledge within didactics of mathematics, distinct from 'mathematical knowledge for teaching' which is a practice-oriented knowledge used in actual mathematics teaching tasks (for a comparison, see Scheiner et al., 2023).

solving the problem. All three solutions were given under certain interpretations of the idea of 'equal distribution' and were compatible with the statement of the problem.

The prospective teachers who participated in this study were asked to respond in writing to how they made sense of the mathematical ideas of the three students. The participants were encouraged to think deeply about these mathematical ideas and to explain their reasoning in detail. To facilitate thorough and reflective responses, there was no limit to the length of their responses. In addition, prospective teachers were given a full week to complete this task. This ample time frame was designed to encourage in-depth reflection.

The following prompt was given to the prospective teachers: "Provide a detailed explanation of how you understand the mathematical ideas presented by the three students in the context of the given mathematical problem. Please be thorough in your reasoning." No further instructions or resources were given to assist prospective primary teachers in responding to the prompt. All the written responses from the prospective teachers were organized and blinded for analysis.

The use of an ambiguous mathematical problem in this research, along with the diverse answers from the students, serves several purposes. This approach creates a situation where there is no clear straightforward or single 'correct' answer, thereby inviting a variety of different solutions and approaches to the problem due to intentional linguistic ambiguity (Marmur & Zazkis, 2022). This ambiguity stems from the lack of clear criteria for distribution in division, creating an environment rich with multiple solution possibilities (Foster, 2011).

Such ambiguity, when applied to teacher education, can be beneficial because it stimulates prospective teachers to engage in complex, nuanced reasoning, encouraging them to deeply and critically consider various potential solutions. Moreover, it promotes an open-minded approach, urging them to pay close attention to the different ideas and strategies presented by each student. This approach to ambiguity can be productive, both mathematically and pedagogically (Foster, 2011; Marmur & Zazkis, 2022). It aids in the development of prospective teachers' sense-making skills and enhances their capacity to recognize and appreciate the diversity of students' thinking (Ribeiro et al., 2016).

Furthermore, using an ambiguous problem in this research allows for a more nuanced exploration of the different stances that prospective teachers adopt when making sense of students' ideas. By examining how prospective teachers respond to the ambiguity of a mathematical problem, this research can provide a deeper understanding of the thought processes and sense-making strategies that underpin teachers' interpretations of students' thinking.

Table 1 shows the mathematical problem statement and the responses of the three students.

Data analysis

We used content analysis to develop the coding scheme illuminating the manners and positions from which prospective teachers made sense of students' mathematical ideas. Content analysis is "a research method for subjective interpretation of the content of text data through the systematic classification process of coding and identifying themes or patterns" (Hsieh & Shannon, 2005, p. 1278). This method requires coders to stay close to participants' responses and allows for the use of inductive and deductive coding strategies (Cho & Lee, 2014).

Table 1 Task used in the study for prospective teachers' making sense of students' ideas

Spanish (original)	English (translation)
Profesor: "Tengo 22 margaritas, 31 amapolas, y 34 tulipanes. Si los reparto equitativamente en tres jarrones, ¿cuántas flores de cada tipo sobran?"	Teacher: "I have 22 daisies, 31 poppies, and 34 tulips. If I distribute them equally between three vases, how many flowers of each type are left over?"
Luis: "Yo sumo 22 + 31 + 34, me da 87, y eso lo divido entre tres. Me da que pongo en cada jarrón 29 flores, y no me sobra ninguna."	Luis: "I would add 22 + 31 + 34 which gives 87 and divide that by three. That means I put 29 flowers in each vase and there are none left."
Ana: "Yo divido cada cantidad de flores entre tres. Cojo el cociente más pequeño, 7, el de dividir 22 entre 3. Pongo 7 de cada tipo en cada jarrón, y me sobran 1 margarita, 10 amapolas, y 13 tulipanes."	Ana: "I would divide the total number of each type of flower by three. I would take the lowest denominator, which is 7, from dividing 22 by 3. So I would put 7 flowers of each type in each vase; so I would have 1 daisy, 10 poppies and 13 tulips."
Juan: "Yo divido cada cantidad de flores entre tres, y cojo el resto de cada división. Me sobran 1 margarita, 1 amapola, y 1 tulipán."	Juan: "I would divide the total for each type of flower by three and leave the remainder for each type. This leaves me with 1 daisy, 1 poppy and 1 tulip."

The data analysis took place in several phases. First, we began with a content analysis of approximately 20% of the prospective teachers' responses (25 of the 123 responses) using inductive and deductive approaches to develop a set of stances (categories) in making sense of students' ideas. The unit of analysis was a prospective teacher's entire written response, and the set of stances (and their descriptions) emerged from adopting, adapting, and adding to the stances identified in the literature (see the section "Making sense of students' mathematical ideas") based on the data. Through this approach, we found two additional stances that have not been previously described in the literature: a comparative stance and a projective stance (for details, see the Results section).

We were also interested in the extent or depth to which a particular stance was used by a prospective teacher. Previous research has highlighted the distribution and frequency of occurrence of certain stances (e.g., Amador et al., 2022; Shin, 2021) but not the extent or depth to which they were used. For this reason, we also coded the level of evidence for the use of a particular stance. Following Jacobs et al. (2010), we used three codes to capture the extent or depth of prospective teachers' use of a particular stance: no evidence (0), limited evidence (1) and strong evidence (2). Therefore, we used a 3-point scale that reflected the evidence we had of each stance in the prospective teachers' responses. A response coded with 0 for a stance indicates no evidence for that stance; that is, the response did not show any indications of the stance being considered. When coded with 1, the response offered limited evidence for a stance, that is, displaying the stance but only to a minimal extent and without depth. This might involve a few, possibly superficial, indications of the stance. On the other hand, a response coded with 2 for a stance signifies strong evidence. This response has several clear and detailed indications, suggesting a more extensive expression of the stance.

Table 2 shows the coding scheme developed to code prospective teachers' responses in terms of what stances are used and the extent or depth to which they are used. The coding scheme included seven stances (descriptive, evaluative, comparative, interpretive, inquiry-based, connective, and projective) and three levels of evidence at which the stances were

Table 2 Scheme for coding stances used and the extent and depth to which they are used in making sense of student ideas

Stance (category)	Description of level of evidence (code)		
	No evidence (0)	Limited evidence (1)	Strong evidence (2)
Descriptive	No indications of description or documentation of student ideas	Few indications of description or documentation of student ideas or representation of student thinking	Several indications of description or documentation of student ideas and detailed representation of student thinking
Evaluative	No indications of evaluation of the quality of student ideas	Few indications of evaluation of the quality of student ideas or judgment of their correctness	Several indications of evaluation of the quality of student ideas and judgment of their correctness
Comparative	No indications of comparison or juxtaposition of student ideas	Few indications of comparison or juxtaposition of student ideas or weighting of their quality against each other	Several indications of comparison or juxtaposition of student ideas and weighting of their quality against each other
Interpretive	No indications of interpretation of student ideas	Few indications of interpretation of student ideas or inferences based on them	Several indications of interpretation of student ideas and inferences based on them
Inquiry-based	No indications of exploration or analysis of student ideas to explain student thinking	Few indications of exploration or analysis of student ideas to explain student thinking	Several indications of detailed exploration and analysis of student ideas to explain student thinking or (hypothetically) consider different ways of engaging with student thinking
Connective	No indications of connections being made between student ideas and larger mathematical ideas	Few indications of connections being made between student ideas and larger mathematical ideas	Several indications of connections being made between student ideas and larger mathematical ideas
Projective	No indication of outlook on the future status or relevance of student ideas	Few indications of outlook on the future status or relevance of student ideas or anticipation of possible implications for student learning	Several indications of outlook on the future status or relevance of student ideas and anticipation of possible implications for student learning

used (no evidence, limited evidence and strong evidence).[3] The responses of the prospective teachers were coded using this coding scheme, which allowed us to track not only whether and how many prospective teachers adopted particular stances but also the extent or depth to which they did so in making sense of students' ideas. Because the ways in which prospective teachers made sense of students' ideas could be complex, we allowed for the coding of multiple stances in a single response given by a prospective teacher.

To check the reliability of the coding process, two coders double-coded about one-third of the prospective teachers' responses (40 out of 123 responses). The interrater reliability of the coding of the responses was above 90%, indicating a high degree of coding rigor and reliability. Disagreements were discussed and resolved through a process of achieving consensus.

To evaluate whether there were differences in the application of the identified stances, one-way analysis of variance (ANOVA) was conducted. This method was chosen because it determines whether there are statistically significant differences in the means of the stances, enabling us to draw conclusions regarding the distinctiveness of the stances in terms of their application.

We then explored whether there were groups of prospective teachers who shared similar patterns in the application of the identified stances. First, we conducted a hierarchical cluster analysis using the increasing squared Euclidean distance to determine broad clusters based on similarity or dissimilarity between the coded responses (Everitt, 1993). Using this technique, two distinct clusters were identified, one comprising 75 of the 123 responses and the other comprising the remaining 48 responses.[4] Second, we conducted *t*-tests to identify whether there were statistically significant differences between the two clusters regarding the means in the application of the identified stances.

Third, we looked in depth into the similarities and differences between the responses in each cluster and plotted the level of evidence of stances (no evidence, limited evidence and strong evidence) for each prospective teacher's response. This approach facilitated the identification of distinct types of making sense of students' ideas, derived from the patterns that emerged in the two clusters. These types of making sense could then be used to classify responses that conformed to these patterns, thereby enabling the formulation of a broader description applicable to each case, despite variations in specific details.

It is important to note that both techniques (cluster analysis and plotting the extent of stances) were exploratory rather than confirmatory, as the purpose was to generate types of making sense of students' ideas rather than to test them.

[3] As prospective teachers in this study could not engage directly with students, we consider the inquiry-based stance here as a reflective and hypothetical approach that represents an active engagement with student thinking, albeit in a simulated environment, and underscores a commitment to understanding and addressing the complexities of students' mathematical reasoning.

[4] Hierarchical cluster analysis begins by assuming that all participants' responses are in a single cluster, and then gradually divides participants' responses into smaller clusters. The analysis uses a stopping rule to determine the number of clusters appropriate for the data (although theoretically the analysis could be run until each response represents a different cluster). Since the cluster analysis is a hierarchical analysis, the number of clusters selected is somewhat arbitrary and depends largely on the desired level of generality of the clusters. The results of this hierarchical cluster analysis led to the identification of two clusters.

Results

In presenting the results of our analysis, we first report on the stances that prospective teachers used to make sense of students' mathematical ideas and the extent and depth to which they used these stances before reporting on the emerging types of making sense of students' ideas.

Stances used and extent to which they were used in prospective teachers' responses

Table 3 provides an overview of which stances were used by how many prospective teachers and to what extent.

The majority of prospective teachers adopted a descriptive stance in making sense of students' ideas, with 87% of prospective teachers' responses indicating that they were describing or documenting students' ideas and/or representing students' thinking. The extent and depth to which prospective teachers adopted a descriptive stance varied widely, with 69% of prospective teachers' responses showing strong evidence of a descriptive stance compared to 18% of prospective teachers' responses showing limited evidence. As an example of using a descriptive stance, consider the following excerpt of one of the prospective teachers' responses:

> …Ana's reasoning was that if you divide each number of flowers among the three vases, that is, 22: 3, 31: 3 and 34: 3, you get the results 7, 10 and 11, respectively, since she took the smallest quotient, 7. (PT1)

As seen in this response, the prospective teacher merely described one of the students' reasoning in their own words.

Almost all the prospective teachers' responses showed evidence of an evaluative stance, with approximately two-thirds (67%) of the responses showing several indications of evaluating the quality of students' ideas and judging their accuracy, while one-third of the prospective teachers' responses showed few indications (25%) or no indications (8%) of evaluating the quality of students' ideas. Responses showing strong evidence of an evaluative stance included statements such as,

> In my opinion, the correct answer to the problem … is given by Ana. She is the only one who solves the problem … in an equal way … Luis' way of solving the problem is incorrect because he does not distribute the flowers equally in each vase … Juan's answer would not be correct either since he also does not say how many flowers of each type are in each vase … (PT72)
> First of all, the solution Luis provides … is incorrect because he makes a sum of all types of flowers and divides the result of the sum among the three vases, so it is not done equally because there are a different number of daisies, poppies and tulips in each vase. (PT105)

As can be seen here, the focus was on assessing the accuracy of students' ideas and pointing out any errors or mistakes students have made in their reasoning.

A comparative stance was adopted by far fewer prospective teachers in their making sense of students' ideas. Only approximately one in six (16%) prospective teachers' responses showed indications of comparison or juxtaposition of students' ideas and/or weighting of their quality against each other. Interestingly, those taking a comparative

Table 3 Level of extent and depth prospective teachers adopted particular stances in making sense of student ideas

Stance	Description	Level of evidence		
		No evidence	Limited evidence	Strong evidence
Descriptive	Indications of description or documentation of student ideas and/or representation of student thinking	16 (13%)	22 (18%)	85 (69%)
Evaluative	Indications of evaluation of the quality of student ideas and/or judgment of their correctness	9 (8%)	31 (25%)	83 (67%)
Comparative	Indications of comparison or juxtaposition of student ideas and/or weighting of their quality against each other	104 (84%)	12 (10%)	7 (6%)
Interpretive	Indications of interpretation of student ideas and/or inferences based on them	33 (27%)	43 (35%)	47 (38%)
Inquiry-based	Indications of exploration and/or analysis of student ideas to explain student thinking	96 (78%)	24 (20%)	3 (2%)
Connective	Indications of connections being made between student ideas and larger mathematical ideas	119 (97%)	3 (2%)	1 (1%)
Projective	Indications of outlook on the future status or relevance of student ideas and/or anticipation of possible implications for student learning	113 (92%)	8 (6%)	2 (2%)

The table shows the number (and percentage) of responses from prospective teachers showing different levels of evidence using each stance

stance also adopted an evaluative stance in their response, except for one prospective teacher (PT18), who did not compare the students' reasoning in terms of mathematical quality but rather what they called 'levels of equality' adopted by the students. A response indicating a comparative stance was given, for example, by one prospective teacher, who stated,

> As for the solution proposed by Juan, we can see that it is the most correct of the three proposed and the correct one. … As for Ana's solution to the problem, we can see that the first step, which coincides with Juan's solution, is correct. … However, in the following steps, she chooses the smallest of the three quotients that result from the divisions …. This would be incorrect because she does not respond to the equal distribution as Juan does. (PT79)

In this response, it is evident that the prospective teacher not only evaluated but also compared the students' responses, considering Juan's solution to be correct and taking it as a reference for correctness.

An interpretive stance and an inquiry-based stance were not as common as a descriptive and an evaluative stance; however, approximately three-quarters (74%) of the prospective teachers used either interpretive and/or inquiry-based stances. A total of 73% of all the responses showed indications of interpreting students' ideas and/or making inferences based on them (interpretive stance), and 22% of the responses showed indications of exploring and analyzing students' ideas to explain students' thinking (inquiry-based stance).

As an example of using an interpretative stance, consider the following response by one of the prospective teachers:

> Our student Luis tries to clarify with his answer that, in each of the vases, there must be the same number of flowers, regardless of the kind or type of flower, reaching the conclusion that each vase contains a total of 29 flowers, including daisies, tulips and poppies. Our student, Ana, tries to show us with her answer that, in each of the vases, there must be an equal number of flowers of each type. ... This leads her to conclude that each of the 3 vases must have 7 daisies, 7 poppies and 7 tulips, since she takes the smallest quotient. Our student Juan, with his answer, tries to teach us that each type must be equally distributed in each vase, that is, that each vase must contain the same quantity of daisies, poppies and tulips ... Therefore, Juan comes to the conclusion that each vase should contain 7 daisies, 10 poppies and 11 tulips. (PT63)

In this response, the prospective teacher focused on the reasoning behind each student's answer, evident as the teacher explained how each student approached the problem differently, indicating a deep dive into the substance of their thinking. The prospective teacher also made inferences about students' mathematical understanding based on their approaches, drawing conclusions from the students' ideas, beyond their surface-level answers.

Interestingly, all prospective teachers using an inquiry-based stance also used an interpretive stance, except for one prospective teacher (PT3), who looked for possible explanations for the students' different answers without engaging in the interpretation of the students' ideas. In contrast, this prospective teacher problematised the mathematical problem itself as the focal point of the 'dilemma'.

As an example of the use of an inquiry-based stance, consider one of the prospective teachers who, after interpreting the students' ideas, stated,

> After analyzing the students' answers, I would mainly foster a joint discussion once all the students have tried to solve the problem. In this teacher-led debate, I would like to show them the multitude of possible answers that the problem could have, since it does not specify some details that are considered of great importance. I wouldn't dismiss any of the answers as incorrect because they all have a logical reasoning behind them. By this I mean that I myself did not see any other solutions when I solved the problem, and by seeing the students' answers and reflecting on them, my interpretation has been considerably expanded. … Furthermore, in this intervention I would try to get them to debate among themselves how the approach could be modified (adding or eliminating some things) so that each of the solutions would make more sense according to the constructed statement. (PT95)

In this response, it is evident that the prospective teacher adopted an inquiry-based stance. The prospective teacher planned to foster a joint discussion to explore multiple solutions, reflecting active engagement in meaning-making with students. Their recognition of expanded understanding after considering the students' ideas also shows a commitment to revising their own perspective. Additionally, encouraging student debates to refine approaches demonstrates an active role in the inquiry process, aligning with the central aspects of an inquiry-based stance.

Connective and projective stances were the least common stances displayed in prospective teachers' responses. Only four (3%) of the prospective teachers' responses showed indications of making connections between students' ideas and larger mathematical ideas (connective stance), and ten (8%) responses showed indications of giving an outlook on the future status or relevance of student ideas and/or using foresight for possible implications in students' thinking (projective stance). As an example of a connective stance, one prospective teacher connected the individual students' ideas to larger mathematical ideas of division:

> In Juan's example, we use partitive division because the dividend and the divisor are of the same nature. We divide the total number of flowers (dividend) by the number of vases (divisor) so we get the flowers to be distributed in each vase. In Luis' example, we use the quantitative division as it groups the different types of flowers, because it divides each quantity of flowers (dividend) by the types of flowers (divisor), so we obtain the number of flowers of each type and the flowers left over of each one. And in the last example of Ana, we also use the quantitative division, but in this case, making 3 divisions of each quantity of flowers (dividend) divided between 3 vases (divisor) and we group the flowers taking the smallest quotient of the 3 divisions made (daisy 22:3=7) distributing to each vase 7 flowers of each type (daisies, poppies and tulips). (PT2)

As an example of adopting a projective stance, one prospective teacher responded,

> In my view, mathematical problems have multiple solutions depending on the perspective from which you look at them. There are no right or wrong answers but different points of view. If the problem states precisely that they must take into account an equal distribution, the correct solution would be different, but what the problem is really looking for is not the solution. What this problem poses is that students should be able to think of different creative ways of solving it, giving them freedom of response and approach. Mathematics is not an exact subject with all the solutions written down and predefined, but a subject with multiple ways of approaching it and which should not be pigeonholed. The mathematical reasoning of the three students

Table 4 Overview of cluster differences regarding stance means

		M	*SD*	*t*-test
Descriptive*	Cluster 1	1.45	0.78	$t(121) = -2.12$, $p = 0.0362$
	Cluster 2	1.73	0.57	
Evaluative*	Cluster 1	1.93	0.25	$t(121) = 9.86$, $p = <0.001$
	Cluster 2	1.08	0.68	
Comparative	Cluster 1	0.24	0.59	$t(121) = 0.75$, $p = 0.4579$
	Cluster 2	0.17	0.43	
Interpretive*	Cluster 1	0.80	0.77	$t(121) = -6.21$, $p = <0.001$
	Cluster 2	1.60	0.57	
Inquiry-based*	Cluster 1	0.01	0.12	$t(121) = -8.18$, $p = .<0.001$
	Cluster 2	0.60	0.61	
Connective*	Cluster 1	0	0	$t(121) = -2.44$, $p = 0.0163$
	Cluster 2	0.10	0.37	
Projective*	Cluster 1	0.01	0.12	$t(121) = -3.50$, $p = <0.001$
	Cluster 2	0.23	0.52	

The table shows the mean of stances (*M*), the standard deviation (*SD*), and the results of the *t*-test. Cluster 1 had $N = 75$, and cluster 2 had $N = 48$

and their numerical flexibility make it possible for there to be more than one answer to the problem, and as teachers we need to continue to encourage these situations to occur in the classroom. (PT49)

This response shows how this prospective teacher went beyond analyzing the students' solutions and foresaw the potential of this type of open-ended task for improving students' mathematical flexibility, adopting a projective stance from an educational perspective.

One-way ANOVA (analysis of variance) revealed statistically significant differences among the means in the application of the various stances ($F(6, 854) = 190.45$, $p < 0.001$).

Emerging patterns in the use of stances in making sense of students' ideas

To explore patterns in the use and extent of stances in prospective teachers' responses, we first conducted a hierarchical cluster analysis and then plotted the level of evidence of stances (no evidence, limited evidence and strong evidence) for each prospective teacher response to identify types of making sense that offer possibilities for generalization.

The results of the hierarchical cluster analysis revealed the presence of two distinct clusters. The first cluster comprised a majority of the prospective teachers, representing almost two-thirds of all participants (75 out of 123 prospective teachers). The second cluster encompassed more than one-third of the participants (48 out of 123 prospective teachers).

To gain a deeper understanding of how the two groups of teachers differed based on the application of the stances, separate *t*-tests were conducted. The *t*-test results indicated statistically significant differences in the application of the stances between the two clusters, with the exception of the application of the comparative stance (see Table 4).

Notably, the mean score in regard to the evaluative stance is significantly greater for the first cluster than for the second cluster, whereas the mean scores regarding the interpretive, inquiry-based, connective, and projective stances are significantly greater for the second cluster than for the first cluster (see Table 4). This finding suggested that teachers in the first cluster were more concerned with the observable aspects of students' ideas (indicated by higher levels of evidence for using an evaluative stance), while teachers in the second cluster were more concerned with the substantive aspects of students' ideas (indicated by higher levels of evidence using interpretive, inquiry-based, connective and/or projective stances). This result supports Walkoe et al. (2020), who considered stances at two levels of cognitive demands (i.e., stances with lower cognitive demands and stances with higher cognitive demands).

To explore patterns in the use of stances in these two clusters in more detail, we plotted the extent or depth of stances used for each prospective teacher response. An example of this can be found in Table 5, which shows the level of evidence (no evidence, limited evidence and strong evidence) of the stances used in the responses of four different prospective teachers. With this technique, we were able to develop four types of making sense of students' ideas and match prospective teachers' responses to one of these four types (based on the patterns of the extent of stances used). In the following, these four types of making sense of students' ideas are outlined and illustrated by selected responses from prospective teachers.

Type 1: Making observations

In the first type, prospective teachers' responses in general displayed a high degree of a descriptive stance in their responses, while they also displayed a lower degree of evaluative and comparative stances as well as interpretive and inquiry-based stances and a lower degree of connective and projective stances. This pattern suggested that some prospective teachers were particularly concerned with making observations of students' thinking. As a case in point, consider the following response by one of the prospective teachers (see Teacher A in Table 5):

> … Luis has added the three types and then divided it by 3, which are the types of flowers there are, without taking into account that the problem says to distribute the flowers equally, so he has added all the types of flowers without taking into account the quantities of each type. … Juan divided the quantity of each type by 3 because there are 3 vases and he has distributed the flowers in each vase: 7 daisies, 10 poppies and 11 tulips. … Ana gives a different reasoning than Juan but it would be a correct reasoning for solving the problem. Ana has thought of dividing the quantities of flowers by 3 because there are 3 vases and then she has thought that she should take the smallest quotient because if she takes the largest quotient one of the types of flowers would not have enough flowers to distribute the same quantity among the three vases. Therefore, taking the smallest quotient gives her enough to distribute 7 flowers of each type in each vase and this would be the correct answer to the problem as it is distributed equally as indicated in the statement. (PT30)

This response provides several indications that students' mathematical ideas were described and documented in detail. However, there is also evidence that the quality of students' ideas was evaluated, but there is no evidence that the quality of their ideas was weighed against each other and no evidence that their ideas were interpreted or explored to explain their thinking. There is also no evidence that connections were made between

Table 5 Strong, limited and no evidence of the use of stances in the responses of four prospective teachers

			Descriptive	Evaluative	Comparative	Interpretive	Inquiry-based	Connective	Projective
Cluster 1	Teacher A	Strong evidence	•						
	PT30	Limited evidence		•					
		No evidence			•	•	•	•	•
	Teacher B	Strong evidence	•	•	•				
	PT8	Limited evidence							
		No evidence				•	•	•	•
Cluster 2	Teacher C	Strong evidence	•			•			
	PT16	Limited evidence					•		
		No evidence		•	•			•	•
	Teacher D	Strong evidence	•		•			•	
	PT18	Limited evidence				•	•		
		No evidence		•					•

students' ideas and larger mathematical ideas, and no evidence that the relevance of students' ideas to further learning was considered.

Type 2: Making assessments

In the second type, prospective teachers' responses in general displayed a high degree of evaluative and comparative stances, while they also displayed a lower degree of interpretive and inquiry-based stances and a lower degree of connective and projective stances. This pattern suggested that some prospective teachers were particularly concerned with making assessments of students' thinking. As a case in point, consider the following response by one of the prospective teachers (see Teacher B in Table 5):

> Luis … misunderstood the problem statement. … Luis' solution is not correct because he is not dividing the flowers equally between the 3 vases, but he has understood that the total flowers should be divided equally between the 3 vases, mixing the flowers of each type regardless. … Juan, unlike Luis, has interpreted the problem differently. … Juan, however, is more on the right track than Luis in the solution, but he does not fully understand the essence of the problem, as he does not use the term equitable in the correct way to solve the problem because it is based more on the concept of equality. … Ana's interpretation of the problem, for me, is the correct one. … (PT8)

There are several indications in this response of evaluating the quality of students' ideas and judging their accuracy, as well as indications of comparing or contrasting their ideas and weighing their quality against each other, but no indications of interpreting or examining their ideas to explain their thinking. There is also no evidence of connections being made between students' ideas and larger mathematical ideas, and there is no evidence of considering the relevance of students' ideas in further learning.

Type 3: Developing an understanding

In the third type, prospective teachers' responses in general displayed a high degree of interpretive and inquiry-based stances, a lower degree of evaluative and comparative stances, and a lower degree of connective and projective stances. This pattern suggested that some prospective teachers were particularly concerned with developing a (deep) understanding of the students' thinking. An example of this is the following response from one of the prospective teachers (see Teacher C in Table 5):

> … We can affirm that the idea of equal distribution that this student [Luis] has, is that in each vase there must be the same amount of flowers regardless of the type; that is to say, the equal distribution of the total of the flowers. Secondly, Juan has interpreted this concept in a different way, as can be seen in his answer. … He solves this problem by dividing the flowers equally in each vase, that is, he divides the same number of flowers of each type in each vase. Therefore, Juan divides 22 by 3, giving in the quotient 7 and in the remainder 1, thus obtaining the daisies, then he divides 31 by 3, giving in the quotient 10 and in the remainder 1, thus obtaining the poppies and, finally, he divides 34 by 3, thus obtaining 11 in the quotient and in the remainder 1. Therefore, I can conclude that the interpretation of the 'equitable concept' for Juan is that each kind of flowers has to be equally distributed in each vase, that is, that

> each vase has to have the same number of daisies, poppies and tulips (7 daisies, 10 poppies and 11 tulips). Next, let's analyse Ana's answer. … What she has actually done is to divide each quantity of flowers by three, and she has chosen the smallest quotient, in this case 7, which belongs to the daisies. Subsequently, in each vase, she has put 7 of each type of flower, that is to say, in the first vase she has put 7 daisies, 7 poppies and 7 tulips, in the second vase the same quantity as in the third one. If we add up the total number of flowers she has put in a vase, there are 21, so there are 21 flowers in each vase. Then she has subtracted the daisies that were 22–21 (what is in a vase) leaving 1 left over, then subtracted 31–21 leaving 10 poppies and, finally, she has subtracted 34–21 leaving 13 tulips left over. In this way, we can see that she makes an equitable distribution and thus obtains the possible solution to the problem. Therefore, the interpretation of this concept for Ana is that each vase should have the same amount of each type of flower. (PT16)

In this response, there are several indications of interpreting the substance of students' mathematical ideas and making some inferences from students' reasoning. Although to a lesser extent, this response also shows inquiry into students' ideas to develop a deeper understanding of their thinking. However, there is no evidence of assessing the quality of students' ideas or weighing their quality against each other, making connections between students' ideas and wider mathematical ideas, or considering the relevance of students' ideas to subsequent learning.

Type 4: Making connections or projections

In the fourth type, prospective teachers' responses in general displayed a high degree of connective and/or projective stances while potentially also displaying varying degrees of interpretive and inquiry-based stances as well as evaluative and comparative stances. This pattern suggested that some prospective teachers were particularly concerned with.

making connections or projections of the students' thinking. For example, consider the following response from one of the prospective teachers (see Teacher D in Table 5):

> …What I would do is to show the reasoning followed by each of them [the students], and then open a debate about the meaning of the concept of 'equitable' in mathematics. 'Equitable' in mathematics refers to equality (fairness), so equality can be understood at different levels: Equal number of flowers in each vase, as Luis understands; same number of flowers in each vase and same number within each type of flowers in each vase, as Juan understands; and same number of flowers in each vase, same number within each type of flowers in each vase and same number between the types of flowers in each vase, as Ana understands. This brings us to a deeper analysis of equity: For Louis equity is achieved only at the level of the number of flowers in each vase. According to Juan, there is equity in the number of total flowers in each vase (28) and within the number of each type of flowers in each vase (7 daisies in each vase, 10 poppies in each vase and 11 tulips in each vase). For Ana, equity is achieved at the level of total number of flowers in each vase (21) of each type of flowers in each vase (7 daisies in each vase, 7 poppies in each vase and 7 tulips in each vase). (PT18)

There are several indications in this response of making connections between students' mathematical ideas and broader or larger ideas, namely, 'equality in mathematics', as well as indications of comparing or contrasting their ideas. There is also some evidence, albeit to a lesser extent, of interpreting the content of students' ideas and exploring their ideas to develop a deeper understanding of their thinking.

Discussion

Many of the stances that prospective teachers in this study adopted are similar to the stances that previous research has identified, including a descriptive, evaluative, interpretive, inquiry-based and connective stance, with the first three stances being adopted quite frequently by teachers (e.g., Amador et al., 2022; Scheiner, 2023; Sherin & van Es, 2009; Walkoe et al., 2020). This study revealed two other stances not yet identified in previous research: a comparative stance adopted by prospective teachers who compare or juxtapose students' ideas and/or weigh their qualities against each other and a projective stance expressed by providing an outlook of the future status or relevance of students' ideas and/or anticipating possible implications for students' learning. While a projective stance was adopted by only a few prospective teachers, a comparative stance was adopted by 16% of the prospective teachers participating in this study. The fact that almost one in six prospective teachers adopted a comparative stance may be partly due to the nature of the task given to the prospective teachers, which involved three different approaches and ways in which students solved a mathematical problem statement, allowing them to consider similarities and differences in different students' ideas, which is not possible when only one student reasoning is given. However, we argue that being confronted with a variety of student ideas, which can develop in very different ways when thinking about a mathematical problem, is more closely related to teachers' practices in the classroom.

Previous research has distinguished between lower- and higher-order stances (Amador et al., 2022; Walkoe et al., 2020). The two clusters identified in this paper support the idea that the use of certain stances indicates a spectrum of lower- and higher-order thinking in regard to making sense of students' ideas. That is, some stances are more cognitively demanding than others. Lower-order stances are more concerned with the observable aspects of students' thinking, while higher-order stances are more concerned with the substantive aspects of students' thinking. In addition, lower-order stances can be seen more as involved in accounting of students' thinking, while higher-order stances are more involved in accounting for students' thinking (Mason, 2002).

However, the types of making sense of students' ideas identified in this paper also suggest that stances are used for different kinds of attribution and for different purposes. In particular, stances can be used to attribute value, meaning and/or significance to students' ideas.

- **Attributing value:** Stances can be used to attribute *value* to students' ideas. This can be done using an evaluative stance or a comparative stance. An evaluative stance involves making a judgment about whether an idea is correct or incorrect, productive or unproductive, or viable or inviable. A comparative stance, on the other hand, involves comparing and contrasting ideas to determine which one is better, more useful, or more appropriate in a given context. The purpose of these stances is to *assess* the quality of students' ideas.
- **Attributing meaning:** Stances can be used to attribute *meaning* to students' ideas. This can be done using an interpretative stance or an inquiry-based stance. An interpretative stance involves making inferences about students' thought processes and reason-

Table 6 Framework of the use of stances in making sense of student ideas

Order	Attribution	Purpose	Stance
Lower-order	None	Observation	Descriptive
	Value	Assessment	Evaluative
			Comparative
Higher-order	Meaning	Understanding	Interpretive
			Inquiry-based
	Significance	Projection/prediction	Connective
			Projective

ing based on their ideas. An inquiry-based stance, on the other hand, involves exploring and analyzing students' ideas to explain them or identify patterns in their thinking. The purpose of these stances is to *understand* the thinking behind students' ideas.

- **Attributing significance:** Stances can be used to attribute *significance* to students' ideas. This can be done using a connective or projective stance. A connective stance involves linking students' ideas to related or larger mathematical ideas. A projective stance, on the other hand, involves anticipating how those ideas might be useful or relevant in future learning. The purpose of these stances is to *project* or *predict* how students' ideas fit into the larger context of mathematical concepts and the learning trajectory.

As such, the different stances for attributing value, meaning and significance to students' ideas can be used for different purposes, including observation, assessment, understanding and projection/prediction. A descriptive stance is then used merely to document and describe students' ideas without attributing any value, meaning or significance to them. This stance is used for the purpose of observing students' reasoning. Table 6 outlines an emerging framework for the use of stances that includes their order of thought, kind of attribution and purpose.

This emerging framework extends previous work distinguishing between lower-order and high-order stances (Walkoe et al., 2020). Lower-order stances involve an account of students' thinking either without attributing value, meaning or significance to them for the purpose of observing students' reasoning (in the case of a descriptive stance) or with attributing value to students' ideas for the purpose of assessing the quality of students' ideas (in the case of evaluative and comparative stances). On the other hand, higher-order stances involve an account for students' thinking, either by attributing meaning to them for the purpose of understanding students' thinking (in the case of interpretive and inquiry-based stances) or attributing significance to them for the purpose of projecting/predicting students' ideas (in the case of connective and projective stances).

Conclusion

Making sense of students' ideas is central to developing responsive instruction that focuses on their thinking. In this study, we investigated the ways in which prospective teachers make sense of students' different mathematical ideas about an ambiguous mathematical problem. In particular, we explored the stances prospective teachers adopted, the extent or depth to which they did so, and the types of prospective teachers making sense of students' mathematical ideas.

The findings of this study extend previous research examining the various stances used in making sense of students' mathematical ideas. This has been done by first identifying two additional stances that have not been considered in the literature, namely, a comparative stance (i.e., comparing or contrasting student ideas and/or weighing their quality against each other) and a projective stance (i.e., foreseeing the future status or relevance of student ideas and/or anticipating possible impacts on student learning).

Second, the use of an ambiguous mathematical problem with three different students' mathematical approaches and solutions may have encouraged prospective teachers to use different stances for different purposes, such as assessing the quality of students' ideas, developing a deeper understanding of them, or projecting/predicting their relevance in relation to larger mathematical ideas or future learning. The design of such ill-structured mathematical problems with various students' ideas thus becomes an important focal point in promoting deep reflection on the part of prospective teachers (see e.g., Sánchez-Matamoros et al., 2019). While we acknowledge the central role of the ambiguous nature of the mathematical problem in this study, we also believe that further research is needed to determine whether certain characteristics of such problems influence the adoption of particular stances.

Third, the study builds upon previous related research by not only focusing on what stances were used in making sense of students' ideas but also on the extent and depth to which these stances were used. This approach, in particular, allowed us to explore patterns and identify different types of making sense of students' ideas.

Finally, the present study also extends existing related research by providing initial evidence that stances are used for different purposes, including observation, assessment, understanding and prediction. It is hoped that this expanded understanding of the use of stances will prove useful in future research and practice, particularly by drawing the attention of researchers and practitioners to the attribution of value, meaning and significance in teachers' making sense of students' mathematical ideas.

The study suggests a framework for different types of sense-making and their purposes. By categorizing the ways in which prospective teachers make sense of students' ideas into different types of sense-making, this research can provide a useful tool for understanding and analyzing the sense-making processes that teachers engage in. This framework can be used to support teacher education and professional development, helping teachers become more reflective and intentional practitioners who are able to support the diverse thinking of their students.

Further research is encouraged to consider the nuanced role that context plays in shaping the stances teachers adopt in the classroom. The dynamic nature of classroom situations requires a flexible approach to understanding how teachers' attributions (e.g., value, meaning, significance) can shift fluidly in response to different purposes (e.g., observation, assessment, understanding and projection/prediction). This complexity highlights the importance of developing research methodologies that capture the fluidity of stances used, allowing for a deeper exploration of how stances are influenced by and adapted to specific classroom contexts. As well as adding to the knowledge base, such research could help teacher educators develop effective ways of engaging teachers to better understand and expand their ways of making sense of students' ideas.

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Declarations

Conflict of interest The authors declare no competing interests.

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