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Minimum degree conditions for containing an r-regular r-connected spanning subgraph

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ABSTRACT

We study optimal minimum degree conditions when an n-vertex graph G contains an r-regular r-connected spanning subgraph. We prove for r fixed and n large the condition to be $\delta(G) \geqslant \frac{n+r-2}{2}$ when $nr \equiv 0 \pmod 2$. This answers a question of M. Kriesell.

1. Introduction

A typical question in extremal graph theory is to determine (asymptotically) optimal minimum degree conditions for a graph G on n vertices to contain a given copy of some spanning graph. The classical theorem of Dirac [5] asserts the optimal minimum degree condition to contain a Hamilton cycle to be $\frac{n}{2}$. There are numerous generalisations of this result to higher connected cycles (powers of Hamilton cycles) [9], which in turn generalise the theorems of Corrádi and Hajnal [4] and Hajnal and Szemerédi [6] about clique factors in graphs. The most comprehensive result which asymptotically subsumes all of the mentioned results is the bandwidth theorem of Böttcher, Schacht and Taraz [3]. This theorem states that for $\gamma > 0$ a large enough n-vertex graph G with minimum degree $\delta(G) \geqslant (\frac{r-1}{r} + \gamma)n$ contains any n-vertex graph H that has chromatic number at most r, maximum degree bounded by a constant and bandwidth[4] $o(n)$. We also refer to the excellent survey by Kühn and Osthus [12] for more results.

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2 OP is supported by DFG, Germany grant PA 3513/1-1.
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4 The *bandwidth* of an n-vertex graph H is the minimum b such that there exists a labelling of the vertices with $[n]$ for which $|i-j| \leqslant b$ for all edges ij of H.

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The present work is motivated by a question of Matthias Kriesell [11] about the optimal minimum degree condition sufficient to assert the existence of a 4-regular 4-connected spanning subgraph.

> What is the minimum $\delta_4(n) > 0$ such that every graph G on n vertices with $\delta(G) \geqslant \delta_4(n)$ does contain a 4-regular 4-connected spanning subgraph?

This question in turn was motivated by the work of Bang-Jensen and Kriesell on good acyclic orientations of 4-regular 4-connected graphs [1]. We answer Kriesell's question by providing an exact answer to the more general question about the minimum degree condition $\delta_r(n)$ that guarantees an r-connected r-regular spanning subgraph provided that n is sufficiently large. Note that $nr \equiv 0 \pmod 2$ is a necessary condition for such a subgraph to exist. The aforementioned bandwidth theorem [3] asymptotically answers the question, assuming there exists a bipartite r-connected r-regular subgraph of sublinear bandwidth, and in that case implies $\delta_r(n) \leqslant (1/2 + \gamma)n$ for sufficiently large n. We determine $\delta_r(n)$ precisely provided that n is sufficiently large.

Theorem 1. *For any $r \geqslant 2$ there exists an n_0 such that any n-vertex graph G with minimum degree $\delta(G) \geqslant \frac{n+r-2}{2}$, $n \geqslant n_0$, and $nr \equiv 0 \pmod 2$ contains a spanning r-regular r-connected subgraph.*

Note that when r is odd $nr \equiv 0 \pmod 2$ implies that n is even and actually $\delta(G) \geqslant \frac{n+r-1}{2}$ as $\delta(G)$ is an integer. This theorem asserts that there are r-connected spanning subgraphs of G which are minimal in terms of the number of edges and, in fact, r-regular. Observe that for $r = 2$ this follows immediately from Dirac's theorem [5] with $n_0 = 3$, as a Hamilton cycle is 2-regular and 2-connected. Owing to the use of the regularity lemma, the n_0 given by Theorem 1 will be very large, more precisely a tower function with height being some polynomial in r. Note that for $r \geqslant 2$ an n-vertex graph G with minimum degree $\delta(G) \geqslant \frac{n+r-2}{2}$ is always r-connected, whereas the union of a copy of $K_{\lfloor (n+r-1)/2 \rfloor}$ and $K_{\lceil (n+r-1)/2 \rceil}$ sharing $r-1$ vertices has minimum degree $\left\lfloor \frac{n+r-1}{2} \right\rfloor - 1 = \left\lceil \frac{n+r-2}{2} \right\rceil - 1$. This certifies that Theorem 1 is optimal and gives $\delta_r(n)$. We remark that to find an r-regular spanning subgraph already $\delta(G) \geqslant n/2$ is sufficient and this is best possible. We will briefly discuss this in Section 3.6 after we have sketched the proof of Theorem 1.

In fact we will prove something stronger than just the containment of some spanning r-connected r-regular subgraph. In the following we briefly introduce some notation and the type of r-connected r-regular subgraphs that will be found in G by Theorem 1. For an integer t the *t-blow-up of a graph F* is obtained by replacing every vertex by t vertices and every edge by a complete bipartite graph $K_{t,t}$. Let C_n be the cycle on n vertices and P_n the n-vertex path. We denote by $C_n(t)$ and $P_n(t)$ the t-blow-up of C_n and P_n, respectively. Note that $C_n(t)$ has nt vertices and is $(2t)$-regular.

We also need a similar construction that is $(2t-1)$-regular for an integer t. We denote by $C_n(t-\frac{1}{2})$ the t-blow-up of C_n for n even, where every other edge only gets a $K_{t,t}$ minus a perfect matching. Similarly, for any n, $P_n(t - \frac{1}{2})$ is the t-blow-up of the n-vertex path, where every other edge only gets a $K_{t,t}$ minus a perfect matching. We refer to the collection of these constructions (for cycles and paths) as the *$(t - \frac{1}{2})$-blow-ups*. Note that $C_n(t - \frac{1}{2})$ has nt vertices and is $(2t - 1)$-regular. The purpose of this unusual notation is that now for any integer $r \geqslant 2$ we have that $C_n(\frac{r}{2})$ is r-regular, all internal vertices of $P_n(\frac{r}{2})$ have degree r and we do not need to distinguish between the odd and even case each time.

Often in our proof of Theorem 1 we will be able to find a spanning copy of an $\frac{r}{2}$-blow-up of a cycle, but this is not always possible, for example, when n is not divisible by $\lceil \frac{r}{2} \rceil$. Moreover, when n is even and not divisible by $r = 4$, the graph G obtained by taking the disjoint union of two cliques $K_{n/2-2}$ and adding four additional vertices that are connected to all previous $n - 4$ vertices cannot contain a copy of $C_{n/2}(4)$. Therefore, we also need to find other structures with all but a small proportion of vertices contained in $\frac{r}{2}$-blow-ups of paths (see Section 3.5 for more details).

At least for even n and r, the graphs $C_n(\frac{r}{2})$ can also be interpreted as n copies of the same small graph glued together on a specific set of vertices in a cyclic order. It would be interesting to study the minimum degree threshold for other spanning structures that can be obtained by a similar operation from other graphs that are not $K_{t,t}$ (or $K_{t,t}$ minus a perfect matching).

In particular, when the graph is not bipartite there is still a lot to explore, for example, one could glue copies of C_5 on independent sets of size two in a cyclic way.[5] This might be interesting, as for a C_5-factor ($n/5$ pairwise disjoint copies of C_5) the minimum degree threshold $\delta(G) \geqslant \frac{3}{5}n$ is determined by the critical chromatic number[6] rather then the chromatic number, which would give $\delta(G) \geqslant \frac{2}{3}n$. Which of the two is guiding the threshold is known in general (see [12]), but it is not clear how this might translate to connected structures. This can be seen as a generalisation of factors to a connected cyclic structure in the same way as perfect matchings (respectively clique factors) generalise to Hamilton cycles (respectively powers of Hamilton cycles).

1.1. Organisation of the paper

The paper is structured as follows. In Section 2 we collect the essential tools we will use (regularity and blow-up lemmas), while Section 3 provides a proof overview, which consists of three cases (Extremal Case I, Extremal Case II and the Non-Extremal Case). These cases are dealt with subsequently in Sections 4, 5 and 6.

2. Tools and notation

For standard graph theoretic definitions we refer to Bollobás [2]. The main tools we will use are Szemerédi's regularity lemma [13] and the blow-up lemma by Komlós, Sárközy, and Szemerédi [8]. Let $G = (V, E)$ be a graph. For any two sets $A, B \subseteq V$, we denote by $e_G(A, B)$ the number of edges of G with one endpoint in A and one in B (edges contained in the intersection $A \cup B$ are counted twice) and define the *density* $d(A, B)$ between these sets to be $\frac{e_G(A,B)}{|A|\,|B|}$.

Definition 2. The pair (A, B) is ε*-regular* if for all $X \subseteq A$, $Y \subseteq B$ with $|X| \geqslant \varepsilon|A|$, $|Y| \geqslant \varepsilon|B|$ we have $|d(X, Y) - d(A, B)| \leqslant \varepsilon$.

The following two lemmas are standard and easily follow from the definition (c.f. Fact 1.3 and 1.5 in [10]). The first states that in ε-regular pairs most vertices have large degree into any (not too small) set on the other side.

Lemma 3. *Let (A, B) be an ε-regular pair with $d(A, B) \geqslant d$ and let $B' \subseteq B$ with $|B'| > \varepsilon|B|$. Then*

$$|\{a \in A \colon \deg(a, B') \leqslant (d - \varepsilon)|B'|\}| \leqslant \varepsilon|A|\,.$$

The second lemma guarantees that (not too small) induced subgraphs of ε-regular pairs are still regular (although with a slightly worse parameter).

Lemma 4 (*Slicing Lemma*)**.** *Let (A, B) be an ε-regular pair with $d(A, B) = d$, let $\frac{1}{2} \geqslant \gamma > \varepsilon$, and $A' \subseteq A$ and $B' \subseteq B$ be of size $|A'| \geqslant \gamma|A|$ and $|B'| \geqslant \gamma|B|$. Then (A', B') is an ε/γ-regular pair with $d(A, B) \geqslant d'$, where $|d - d'| \leqslant \varepsilon$.*

We will use the following degree form of Szemerédi's regularity lemma by Komlós and Simonovits [10].

Lemma 5 (*Regularity Lemma, Degree Version*)**.** *For every $\varepsilon > 0$ and integer ℓ_0 there exists an integer L such that for any graph G on at least L vertices and any $d \in [0, 1]$ there is a partition of $V(G)$ into $\ell_0 < \ell + 1 \leqslant L$ clusters $V_0, \ldots, V_\ell$ and a subgraph G' of G such that*

(P1) $|V_0| \leqslant \varepsilon|V(G)|$ *and* $|V_i| = T \leqslant \varepsilon|V(G)|$ *for all* $1 \leqslant i \leqslant \ell$.
(P2) $\deg_{G'}(v) \geqslant \deg_G(v) - (d + \varepsilon)|V|$ *for all* $v \in V$.

[5] More precisely, take $n/3$ copies of C_5 with vertices $v_{i,1}, \ldots, v_{i,5}$ in circular order for $i = 1, \ldots, n/3$ and identify $v_{i,1}$ with $v_{i+1,2}$ and $v_{i,3}$ with $v_{i+1,4}$ for $i = 1, \ldots, n/3$.

[6] The *critical chromatic number* of a graph H is $(\chi(H) - 1)v(H)/(v(H) - \sigma(H))$, where $\chi(H)$ is the chromatic number and $\sigma(H)$ denotes the minimum possible size of a colour class amongst all colourings of H with $\chi(H)$ colours.

(P3) *For* $1 \leqslant i \leqslant \ell$ *the set* V_i *is independent in* G'*.*
(P4) *For* $1 \leqslant i < j \leqslant \ell$ *the pair* (V_i, V_j) *is* ε*-regular in* G' *and has density* 0 *or at least* d*.*

Szemerédi's regularity lemma goes in tandem with the so-called *cluster graph*. After an application of Lemma 5 to a graph G this is a graph R on vertex set $[\ell]$, and ij is an edge if and only if (V_i, V_j) is ε-regular in G' with density at least d. It is a well known fact that the linear minimum degree is inherited by the cluster graph.

Fact 6. *If* $\delta(G) \geqslant \gamma n$ *and* $\varepsilon \leqslant d$*, then the cluster graph* R *has minimum degree* $\delta(R) \geqslant (\gamma - 2d)\ell$*.*

Proof. Indeed, otherwise with (P3) in G' there would be vertices with degree less than $(\gamma - 2d)\ell \cdot \frac{n}{\ell} + \varepsilon n \leqslant \gamma n - (d + \varepsilon)n$ contradicting (P2). □

When working with the regular pairs, one often needs a somewhat stronger concept of super-regularity.

Definition 7. The pair (A, B) is an (ε, δ)-*super-regular* pair if $d(A', B') \geqslant \delta$ for all $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geqslant \varepsilon|A|$, $|B'| \geqslant \varepsilon|B|$, and $\deg(a, B) \geqslant \delta|B|$, $\deg(b, A) \geqslant \delta|A|$ for all $a \in A$, $b \in B$.

The next lemma asserts that every ε-regular pair contains an almost spanning super-regular pair and directly follows from the definition and the two previous lemmas.

Lemma 8. *Let* (A, B) *be an* ε*-regular pair with* $d(A, B) = d$*. Then there exists* $A' \subseteq A$ *and* $B' \subseteq B$ *with* $|A'| \geqslant (1 - \varepsilon)|A|$ *and* $|B'| \geqslant (1 - \varepsilon)|B|$ *such that* (A', B') *is a* $(2\varepsilon, d - 3\varepsilon)$*-super-regular pair.*

The blow-up lemma of Komlós, Sárközy and Szemerédi [8] allows us to embed spanning subgraphs with bounded degree. We will use the following special case deduced from Komlos, Sárközy and Szemerédi [8, Remark 13].

Lemma 9 (*Bipartite Blow-Up Lemma*)**.** *For each* $d, c > 0$ *and integer* $\Delta \geqslant 0$ *there exist* $\varepsilon > 0$, $\alpha > 0$ *and an integer* n_0 *such that the following holds for any* $n \geqslant n_0$*. Let* H *be a bipartite graph on classes* A *and* B *with* $|A| = |B| = n$ *such that* (A, B) *is an* (ε, d)*-super-regular pair and let* G *be a bipartite graph on classes* X *and* Y *with* $|X| = |Y| = n$ *that has maximum degree bounded by* Δ*. Moreover, for any* $X' \subseteq X$ *and* $Y' \subseteq Y$ *with* $|X'|, |Y'| \leqslant \alpha n$*, let* $A_x \subseteq A$ *and* $B_y \subseteq B$ *for each* $x \in X'$ *and* $y \in Y'$ *with* $|A_x|, |B_y| \geqslant cn$*. Then there exists an embedding of* G *into* H *such that all* $x \in X'$ *and* $y \in Y'$ *are embedded into* A_x *and* B_y*, respectively.*

We remark that in our application X' and Y' will be of constant size, hence we can ignore the α.

3. Proof overview

The proof of Theorem 1 will be split into three cases. We now explain this case distinction and then give an overview of the proof for each of these cases. Let G be a graph with minimum degree $\frac{n+r-2}{2}$ and $V = V(G)$ throughout the rest of the paper. For $\alpha > 0$ we call G α-*extremal* if there are two sets $A, B \subseteq V(G)$ of size $(\frac{1}{2} - \alpha)n \leqslant |A|, |B| \leqslant \frac{n}{2}$ such that $d(A, B) < \alpha$. With the help of the regularity lemma we will cover the case that G is not α-extremal for any $\frac{1}{32} > \alpha > 0$ in Section 6.

3.1. Pinning down the extremal cases

So we can assume that G is α-extremal for some $\alpha > 0$. Using the minimum degree condition in G we can show that in this case either G contains a large set that is 'almost' independent – Extremal Case I – or G is 'close' to the disjoint union of two cliques $K_{n/2}$ – Extremal Case II. For this we first argue that A and B have to be almost disjoint or almost the same. Indeed, if $2\sqrt{\alpha}n \leqslant |A \cap B| \leqslant (\frac{1}{2} - 2\sqrt{\alpha})n$, then $|A \setminus B| \geqslant \sqrt{\alpha}n$ and using that G is α-extremal we arrive at a contradiction by double counting the edges between A and V

$$\begin{aligned}|A|\cdot\tfrac{1}{2}n &\leqslant e(A,V)=e(A,B)+e(A\cap B,V\setminus(A\cup B))+e(A\setminus B,V\setminus B)\\&\leqslant \alpha|A|\cdot|B|+|A\cap B|\cdot|V\setminus(A\cup B)|+|A\setminus B|\cdot|V\setminus B|\\&=\alpha|A|\cdot|B|+|A|\cdot|V\setminus B|-|A\cap B|\cdot|A\setminus B|\\&\leqslant \alpha|A|\cdot\tfrac{1}{2}n+|A|\cdot(\tfrac{1}{2}+\alpha)n-2\sqrt{\alpha}n\cdot\sqrt{\alpha}n\\&\leqslant |A|(\tfrac{1}{2}+\tfrac{1}{2}\alpha+\alpha-2\alpha)n\leqslant|A|(\tfrac{1}{2}-\tfrac{1}{2}\alpha)n\,.\end{aligned}$$

In the case when $|A\cap B|\leqslant 2\sqrt{\alpha}n$ we want to remove vertices that have too small degree into their own part. Using that $|V\setminus(A\cup B)|\leqslant 3\sqrt{\alpha}n$ we get

$$\begin{aligned}|A|\cdot\tfrac{1}{2}n &\leqslant e(A,V)\leqslant e(A,B)+e(A,V\setminus(A\cup B))+2e(A)\\&\leqslant \alpha|A|\cdot|B|+|A|\cdot 3\sqrt{\alpha}n+|A|^2-|\{v\in A\colon \deg(v,A)\leqslant(\tfrac{1}{2}-2\sqrt[4]{\alpha})n\}|\cdot\sqrt[4]{\alpha}n\end{aligned}$$

and rearranging gives that there are at most $6\sqrt[4]{\alpha}n$ vertices in A with degree at most $(\frac{1}{2}-2\sqrt[4]{\alpha})n$ into A and analogously for B. We remove all these vertices from A and B, also make them disjoint by removing at most $2\sqrt{\alpha}n$ vertices from one of them, and then we iteratively add vertices from $V\setminus(A\cup B)$ to A (or B) that have degree at least $(\frac{1}{2}-140r\sqrt[4]{\alpha})n$ into A (or B). With $\alpha'=14\sqrt[4]{\alpha}$ we obtain the following, which will be the first extremal case treated in Section 4.

Extremal Case I. There are two disjoint sets $A,B\subseteq V(G)$ with $(\frac{1}{2}-\alpha')n\leqslant|A|,|B|\leqslant(\frac{1}{2}+\alpha')n$ such that $G[A]$ and $G[B]$ have minimum degree $(\frac{1}{2}-10r\alpha')n$ and any vertex from $V\setminus(A\cup B)$ has degree at least $9r\alpha'n$ into A and B.

Otherwise, $|A\cap B|\geqslant(\frac{1}{2}-2\sqrt{\alpha})n$, then $|A\setminus B|\leqslant 2\sqrt{\alpha}n$ and

$$e(A)\leqslant e(A,B)+|A\setminus B|^2<\alpha|A||B|+4\alpha n^2\leqslant 5\alpha n^2\,.$$

As $|\{v\in A\colon \deg(v,A)\geqslant 5\sqrt{\alpha}n\}|\cdot 5\sqrt{\alpha}n\leqslant 2e(A)$ this implies that there are at most $2\sqrt{\alpha}n$ vertices in A that have degree at least $5\sqrt{\alpha}n$ into A. Similarly, rearranging

$$|V\setminus A|\cdot|A|-5\sqrt{\alpha}n\cdot|\{v\in V\setminus A\colon \deg(v,A)\leqslant(\tfrac{1}{2}-6\sqrt{\alpha})n\}|\geqslant e(A,V\setminus A)\geqslant|A|\tfrac{1}{2}n-e(A)$$

we get that there are at most $2\sqrt{\alpha}n$ vertices in $V\setminus A$ that have degree at most $(\frac{1}{2}-5\sqrt{\alpha})n$ into A. We remove the former vertices from A, add the latter to A, and denote by B the complement of A. With $\alpha'=3\sqrt{\alpha}$ we obtain the following which will be the second extremal case treated in Section 5.

Extremal Case II. There is a partition of V into two sets A and B with $(\frac{1}{2}-\alpha')n\leqslant|A|,|B|\leqslant(\frac{1}{2}+\alpha')n$ with minimum degree $\alpha'n$ between these sets and such that all but $\alpha'n$ vertices in A (B respectively) have degree at least $(\frac{1}{2}-3\alpha')n$ into B (A respectively).

We will prove the assertion of the theorem in both extremal cases for any sufficiently small $\alpha'>0$ and in the non-extremal regime for sufficiently small $\alpha>0$. Then, Theorem 1 follows, as for some $\alpha>0$ we can find a spanning r-regular r-connected subgraph in G regardless of whether G is α-extremal or not. In the remainder of this section we sketch the argument for each of the three cases and afterwards explain why our constructions are indeed r-connected. For simplicity, we will use α in each of the cases. We remark that the exact minimum degree condition is only necessary in the extremal cases and that the non-extremal case goes through with $\delta(G)\geqslant(\frac{1}{2}-\gamma)n$ for a sufficiently small $\gamma>0$.

3.2. Non-extremal case

Let $k=v(G)/\lceil r/2\rceil$. We would like to find a spanning copy of $C_k(\frac{r}{2})$ in G, but an obvious necessary condition for this is, for odd r, that $v(G)\equiv 0 \pmod{2\lceil\frac{r}{2}\rceil}$. If r is even, the slightly weaker condition $v(G)\equiv 0 \pmod{\frac{r}{2}}$ already suffices. If this condition is satisfied, we will succeed; otherwise, we will find a slightly locally modified version of $C_k(\frac{r}{2})$. For the proof we will have constants

$$0<\varepsilon\ll\nu\ll d\ll\beta\ll\alpha<\frac{1}{32}$$

and $s = \lceil \frac{r}{2} \rceil$. We follow similar arguments as in [7], which can be summarised by the following procedure in which the index $\ell' + 1$ will correspond to 1.

Step 1 Apply the regularity lemma (Lemma 5) with ε and d to obtain a regular partition of G with leftover set V_0 with $|V_0| \leqslant \varepsilon n$.
Step 2 Find ℓ' disjoint ε-regular pairs (X_i, Y_i) and add the leftover to V_0 such that $|V_0| \leqslant 20dn$.
Step 3 For $i = 1, \dots, \ell'$ connect Y_i to X_{i+1} (index $\ell' + 1 = 1$) with the $\frac{r}{2}$-blow-up of a path that we denote by P_i.
Step 4 For $i = 1, \dots, \ell'$ turn (X_i, Y_i) into an $(\varepsilon, d - \varepsilon)$-super-regular pair (X_i', Y_i') with $|X_i'| = |Y_i'|$ by adding leftover vertices to V_0 such that $|V_0| \leqslant 23dn$.
Step 5 Repeatedly take νn vertices from V_0' and append them to the paths P_i.
Step 6 For $i = 1, \dots, \ell'$ use the blow-up lemma (Lemma 9) to find a spanning copy of an $\frac{r}{2}$-blow-up of a path in (X_i', Y_i') connecting P_{i-1} with P_i.

Step 1 is natural and for **Step 2** it is enough to find a large matching in a graph with minimum degree close to $\frac{n}{2}$. During the performance of **Step 5** the degree of some vertices might get too small. In this case we add them to a set Q that we take care of before the next round. However, this will not be hard, as there will be at most $3\varepsilon n \ll \nu n$ vertices added to Q. Apart from this, **Step 5** is very similar to **Step 3**, which we now sketch with more details. Let X, Y be the clusters that we want to connect with the $\frac{r}{2}$-blow-up of a path P. If there is a cluster Z such that (X, Z) and (Z, Y) are ε-regular pairs with density at least d then we can easily find this path. Otherwise, let A be the union of all clusters Z such that (X, Z) is an ε-regular pair with density at least d and B the union of all clusters Z for (Y, Z) analogously. By the minimum degree property in the cluster graph we get $|A|, |B| \geqslant (\frac{1}{2} - \alpha)n$. As G is not α-extremal we have $d(A, B) \geqslant \alpha$. Therefore, there exist two clusters $Z_1 \in A$ and $Z_2 \in B$ with $d(Z_1, Z_2) \geqslant \alpha$ and then (X, Z_1), (Z_1, Z_2), and (Z_2, Y) are ε-regular pairs with density at least d. Then it will be again easy to find the path that we are interested in.

We have to ensure that the end vertices of the paths always have high degree into the other cluster of the respective super-regular pair, because we want to connect them later and keep them through **Step 4**. Furthermore, we have to ensure that in **Step 5** the sizes of the $(\varepsilon, d - \varepsilon)$-super-regular pairs remain balanced. We will give the details in Section 6.

3.3. Extremal Case I

In this extremal case we will not use the regularity lemma, but the blow-up lemma will be helpful. Recall that in this case G is 'close' to the union of two disjoint cliques of size roughly $\frac{n}{2}$ on vertex sets A and B. The main challenge is to find a *bridge* that connects both these cliques. It is then easy to find the desired structure using the high degrees.

Step 1 In the case when r is even the bridge will be a matching of size r between A and B such that the end-vertices are well connected on their side. The odd case is a little more delicate and we will find a matching of size $r + 1$ or r depending on the size of $V(G) \setminus (A \cup B)$ and the parity of A and B. We then build a path-like structure on both sides that contain the end-vertices of the bridge.
Step 2 Absorb all vertices that do not belong to A or B by extending both ends of the path-like structures. We can ensure that the left-over on each side has size divisible by $2r$.
Step 3 It will be easy to see that the left-over on each side can be split into a super-regular pair which can be covered with the $\frac{r}{2}$-blow-up of a path using Lemma 9.

If we are careful with the end-tuples between each of the steps this gives an r-regular r-connected path-like structure covering G. In Section 4 we will give the details of the even and odd case separately.

3.4. Extremal Case II

Again, we will not use the regularity lemma in this part, but the blow-up lemma will still be helpful. We can assume that we have a partition of $V(G)$ into A and B of size $(\frac{1}{2} \pm \alpha)n$ such that between these sets we have minimum degree αn, all but at most αn vertices from A (or B) have degree at least $(\frac{1}{2} - 3\alpha)n$ into B (or A). W.l.o.g. assume that $|A| + m = \frac{1}{2}n = |B| - m$, where $0 \leqslant m \leqslant \alpha n$. Note that in $G[B]$ we have minimum degree at least $m + \frac{r-2}{2}$. Let $s = \lceil \frac{r}{2} \rceil$.

Step 1 If $\Delta(G[B]) \leqslant 4s\alpha n$ find m vertex-disjoint copies of $K_{1,s}$ in $G[B]$, such that all vertices are well connected to A. Otherwise, separate the vertices with higher degrees, then find vertex-disjoint copies of $K_{1,s}$ as before, and afterwards find additional vertex-disjoint copies of $K_{1,r}$, such that the leaves are well connected to A.

Step 2 Absorb these copies of $K_{1,s}$ and $K_{1,r}$ into an r-regular path-structure and then connect these together into one longer path-structure. After removing the path that we constructed we are left with sets $A_1 \subseteq A$ and $B_1 \subseteq B$ with $|A_1| = |B_1|$.

Step 3 Absorb all vertices that do not have large degree to the other side into the path by alternating between both sides. After removing these vertices we are left with sets $A_2 \subseteq A_1$ and $B_2 \subseteq B_1$ with $|A_2| = |B_2|$ and the property that all vertices have large degree to the other side.

Step 4 It is easy to see that (A_2, B_2) is a super-regular pair and that we can cover it with the $\frac{r}{2}$-blow-up of a path using Lemma 9.

If we are careful with the end-tuples between each of the steps this gives an r-regular r-connected path-structure covering G. For the first step we use the following.

Lemma 10. *For any integer s there exists $\alpha > 0$ such that the following holds. Let G be an n vertex graph with maximum degree $\Delta(G) \leqslant 4s\alpha n$ and minimum degree $\delta(G) \geqslant m + s - 1$, where $1 \leqslant m \leqslant \alpha n$. Then there are $2m$ pairwise vertex-disjoint copies of $K_{1,s}$ in G.*

The proof of this lemma and the second extremal case will be given in Section 5.

3.5. Constructions

First recall that the $\frac{r}{2}$-blow-up of a cycle is r-regular and also r-connected. It will not always be possible to construct this, but it will be the basic building block. We might need to absorb some exceptional vertices, for example, when n is not divisible by r. In the case when r is even we then remove a perfect matching from one $K_{s,s}$ and add one vertex that is connected to all $2s = r$ vertices that just lost one neighbour (c.f. Figs. 2, 5, and 10). The resulting graph is still r-connected, because we cannot disconnect this part of the cycle by removing less than $\frac{r}{2}$ vertices. A similar construction will be used in the case when r is odd (c.f. Figs. 4, 5 and 11) that also preserves r-connectivity. Apart from this, in the first extremal case, we also have to connect two $\frac{r}{2}$-blow-ups of cycles by using at most r edges between them (c.f. **Step 1** of Section 3.3). In general, we only need to cover a small linear fraction of the vertices from G in this way and, therefore, almost all vertices are contained in the $\frac{r}{2}$-blow-up of a path.

3.6. Regular spanning subgraphs

In this section we would like to briefly discuss how our arguments can be used to find a spanning r-regular subgraph even with a smaller minimum degree condition. First note that $K_{\lceil (n+1)/2 \rceil, \lfloor (n-1)/2 \rfloor}$ has minimum degree $\lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n}{2} \rceil - 1$ and does not contain a spanning r-regular subgraph for any $r \geqslant 1$. On the other hand, for $r \geqslant 1$ there exists an n_0 such that any graph G on $n \geqslant n_0$ vertices with minimum degree $\delta(G) \geqslant \frac{n}{2}$ contains a spanning r-regular subgraph. This can be proved along the lines of our argument and we will now briefly explain the changes that are necessary to adapt the overview given above to this easier question. We recommend to revisit this part after reading the proof.

In the first extremal case it suffices to find a single edge, which serves as a *bridge* between A and B to get rid of divisibility issues (c.f. **Step 1**). This is the only place in which the exact minimum degree condition is necessary in this extremal case even for r-connected subgraphs. Then, we only use edges inside of $G[A]$ and $G[B]$ to finish the spanning subgraph (c.f. **Step 2** and **Step 3**). In the

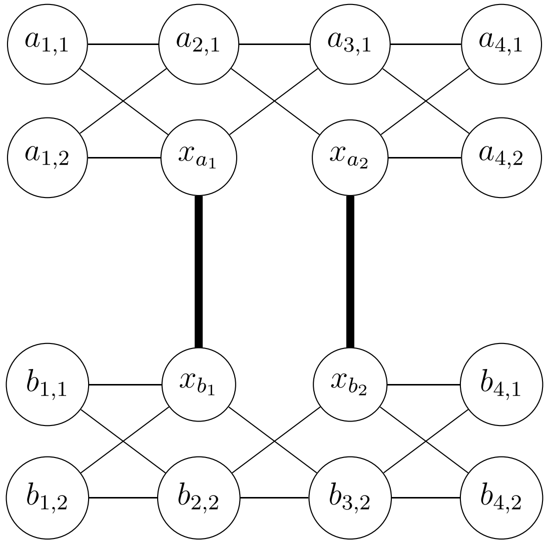

Fig. 1. Bridge between the sets A and B in the special case $r = 4$.

second extremal case, when $|A| + m = \frac{1}{2}n = |B| - m$ for some $0 \leqslant m \leqslant \alpha n$, using $\delta(G[B]) \geqslant m$ it is possible to find mr edges inside of $G[B]$ such that no vertex in B is incident to more than r of them (c.f. **Step 1**), which is again the only place in which the exact minimum degree condition is necessary in this extremal case. With these mr edges the larger size of B is compensated and we complete this to a spanning r-regular subgraph using only edges between A and B (c.f. **Step 2**–**Step 4**). As pointed out above, in the non-extremal case of Theorem 1 we anyway only need $\delta(G) \geqslant (\frac{1}{2} - \gamma)n$ for some small $\gamma > 0$.

4. Extremal Case I

In this section we deal with the first extremal case. We will not use the regularity lemma in this part, but the blow-up lemma will still be helpful.

Proof of Extremal Case I. Let $r \geqslant 3$ be an integer, let $\varepsilon > 0$ be given by Lemma 9 on input $\frac{1}{2}$, $\frac{1}{2}$, and r and without loss of generality we may assume $0 < 1000r^2\alpha \leqslant \varepsilon$. Let G be an n-vertex graph with $\delta(G) \geqslant \frac{n+r-2}{2}$ and let $A, B \subseteq V(G)$ with $(\frac{1}{2} - \alpha)n \leqslant |A| \leqslant |B| \leqslant (\frac{1}{2} + \alpha)n$ such that $G[A]$ and $G[B]$ have minimum degree $(\frac{1}{2} - 10r\alpha)n$, every vertex in $C = V(G) \setminus (A \cup B)$ has degree at least $9r\alpha n$ into each of A and B, and $|C| \leqslant 2\alpha n$. Our goal is to find an r-regular, r-connected spanning subgraph in G provided that n is large enough.

4.1. The even case

Assume that r is even. We begin by constructing $\frac{r}{2}$ bridges of size 2 between A and B (**Step 1** of Section 3.3). A visualisation can be found in Fig. 1.

The next result will allows us to transition between the sets A and B. The minimum-degree condition is crucial for the result to be true.

Claim 11. *Suppose $\delta(G) \geqslant \frac{n+r-2}{2}$ and $|A| \leqslant |B|$. There is a matching $(x_{a_1}x_{b_1}, \ldots, x_{a_r}x_{b_r})$ such that $\left|N(x_{a_i}) \cap A\right| \geqslant \frac{n}{5}$ and $\left|N(x_{b_j}) \cap B\right| \geqslant \frac{n}{5}$ for all $i, j \leqslant r$.*

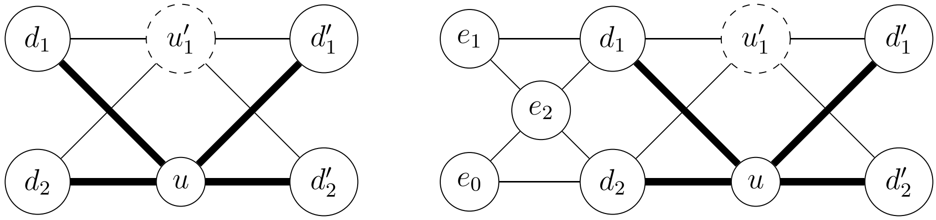

Fig. 2. Absorbers $\xi_4(u)$ (left) and $\xi'_4(u)$ (right) when $r = 4$.

Proof. In order to construct the matching, it suffices to find r vertex-disjoint edges from A to $V \setminus A$. Indeed, suppose we have r pairwise disjoint edges $a_1c_1, \dots, a_rc_r$. If $c_i \in B$, we take this edge. If $c_i \in V \setminus (A \cup B)$, c_i has either $\frac{n}{5}$ edges into B (in this case, take edge a_ic_i), or it has $\frac{n}{5}$ edges into A. Let $i_1, \dots, i_l$ be the indices such that c_{i_j} does not have $\frac{n}{5}$ edges into B. By assumption, each c_{i_j} has αn neighbours in B. Select $b_{i_1}, \dots, b_{i_l}$ s.t. $b_{i_j} \in N(c_{i_j}) \cap B$ and $b_{i_j} \neq b_{i_k}$ for all $j \neq k$ (and being disjoint from those $c_i \in B$, which is clearly possible since n is sufficiently large). We replace the edges $a_{i_j}c_{i_j}$ in the matching by the edges $c_{i_j}b_{i_j}$ and call the substructure *bridge* (see Fig. 1).

It remains to show that these edges exist. First, suppose that n is even. If $|A| \leqslant \frac{n-r}{2}$, the minimum degree of $\frac{n+r-2}{2}$ guarantees that each vertex of A has at least $\frac{n+r-2}{2} - (\frac{n-r}{2} - 1) = r$ neighbours outside of A, hence the assertion follows. Suppose $|A| = \frac{n-r}{2} + i$ with $i \in [\frac{r}{2}]$. In this case $|V \setminus A| = \frac{n+r}{2} - i$ and every vertex from A has at least $r - i$ neighbours in $V \setminus A$. On the other hand every vertex from $V \setminus A$ has at least $\frac{n+r-2}{2} - (\frac{n+r}{2} - i - 1) = i$ neighbours in A. It follows that there cannot be a vertex cover of $G[A, V \setminus A]$ of size $r - 1$ and hence there is a matching of size r by Kőnigs Theorem.[7]

Now, if n is odd, because the minimum degree needs to be an integer, it is at least $\frac{n+1+r-2}{2}$, hence upon removal of one vertex, we are left with a graph on $n' = n - 1$ vertices and minimum degree at least $\frac{n+1+r-2}{2} - 1 = \frac{n'+r-2}{2}$ (and n' being even). Hence the assertion follows from the previous discussion. $\square$

Therefore, Claim 11 gives us the green sub-structure of Fig. 1. Now, we take two of those matching edges (think of them as being $\frac{r}{2}$ pairs of 2 edges). Denote the vertices of these edges that are connected to at least $\frac{n}{5}$ vertices in A as x_{a_1}, x_{a_2}. We next prove that the black structure around x_{a_1}, x_{a_2} shown in Fig. 1 exists.

Claim 12. *There is a completely disjoint selection of distinct vertices $a_{1,1}, \dots, a_{1,r/2}, a_{4,1}, \dots, a_{4,r/2} \in A$ and $a_{2,1}, \dots, a_{2,r/2-1}, a_{3,1}, \dots, a_{3,r/2-1} \in A$ for each $i = 1, \dots, r/2$ with the following properties:*

(1) The edges $a_{j,k}a_{j+1,\ell}$ for $j = 1, 2, 3$ and $k, \ell \in [\frac{r}{2}]$ (or $[\frac{r}{2} - 1]$, respectively) exist,
(2) the edges $x_{a_{2i-1}}a_{1,j}$ and $x_{a_{2i}}a_{4,j}$ exist for $j \in [\frac{r}{2}]$,
(3) the edges $x_{a_{2i-1}}a_{3,j}$ and $x_{a_{2i}}a_{2,j}$ exist for $j \in [\frac{r}{2} - 1]$.

The same holds in B.

Proof. Let $i = 1, \dots, r/2$ and X be the set of vertices selected so far. We initialise $X = \{x_{a_1}, \dots, x_{a_r}\}$ and note that throughout we will have $|X| \leqslant r^2$. We now select vertices $a_{1,j}, a_{3,j} \in N(x_{a_1}) \cap (A \setminus X)$ for $j = 1, \dots, r/2$ arbitrarily, but no $a_{3,r/2}$, and add them to X. These exist as $x_{a_{2i-1}}$ has at least $\frac{n}{5} - r^2$ neighbours in $A \setminus X$. Each of these vertices is adjacent to at least $(\frac{1}{2} - 10r\alpha)n$ vertices in A, hence each vertex has at least $(\frac{1}{2} - 10r\alpha)n - r^2 \geqslant (\frac{1}{2} - 11r\alpha)n$ neighbours in $A \setminus X$. Therefore, the

[7] Kőnigs Theorem states that in a bipartite graph the size of a maximum matching equals the size of the smallest vertex cover.

joint neighbourhood

$$N := \left(A \cap \bigcap_{j=1}^{r/2} N(a_{1,j}) \cap \bigcap_{j=1}^{r/2-1} N(a_{3,j}) \right) \setminus X$$

has size at least $(\frac{1}{2} - 11r^2\alpha)n$. Therefore, we find

$$\left| N(x_{a_2}) \cap N \right| \geqslant \frac{n}{100},$$

thus the claim follows as the same holds in B as well. □

We denote the subgraphs that we found by $A_1, \dots, A_{r/2}, B_1, \dots, B_{r/2}$ and keep the notion X for the union of their vertices. Clearly, each vertex in A stays connected to at least $(\frac{1}{2} - 11r\alpha)n$ vertices in $A \setminus X$ and analogously for B. Next, we introduce a *gluing operation* GE.

Claim 13 (*Gluing Operation GE*)**.** *Given two disjoint sets $D_1, D_2 \subset A$ of size exactly $\frac{r}{2}$ and a set $X \subseteq A$ of size at most $6r\alpha n$, we find two disjoint sets $D, D' \subset A \setminus X$ of size $\frac{r}{2}$ such that*

$$G[D_1, D] \equiv K_{r/2,r/2}, \quad G[D, D'] \equiv K_{r/2,r/2} \quad \text{and} \quad G[D', D_2] \equiv K_{r/2,r/2}\,.$$

Proof. As the joint $A \setminus X$ - neighbourhood of D_1 and D_2 has size at least $(\frac{1}{2} - 14r^2\alpha)n$, the assertion follows. □

This gluing operation GE is now used to connect the subgraphs induced by Claims 11–12. More precisely, for $A_1, A_2 \subset A$ two vertex disjoint such subgraphs, each containing $2r$ vertices, then GE can be used to connect the outer left vertices of D_1 with the outer right vertices of D_2 (the outer left of D_1 correspond to $a_{1,1}, a_{1,2}$ in Fig. 1 and the outer right of D_2 to $a_{4,1}, a_{4,2}$). To glue all subgraphs in A into a path like subgraph and, independently all subgraphs in B, together, we apply GE repeatedly with parts of $A_1, \dots, A_{r/2}$ and $B_1, \dots .B_{r/2}$, always adding the new vertices $D \cup D'$ to X. Note that $|X| \leqslant 2r^2$ throughout. This results in path-like structures P_A and P_B, we define $A' = A \setminus X$, $B' = B \setminus X$, and note $(\frac{1}{2} - 2\alpha)n \leqslant |A'| \leqslant |B'| \leqslant (\frac{1}{2} + \alpha)n$.

In the next step we need to absorb left-over vertices (**Step 2** of Section 3.3). To this end define two absorber-graphs for a vertex u: $\xi_r(u)$ and $\xi'_r(u)$ (see Fig. 2).

Definition 14. Let $D \in \{A', B'\}$ and X be a set of vertices and u a vertex such that $|N(u) \cap D| \geqslant \frac{n}{6}$. Let

- $D_1 = \{d_1, \dots, d_{r/2}\}, D_2 = \{d'_1, \dots, d'_{r/2}\} \subset N(u) \cap D \setminus X$, hence r pairwise disjoint vertices, and
- $D' = \{u'_1, \dots, u'_{r/2-1}\} \subset N(D_1) \cap N(D_2) \cap D \setminus (X \cup D_1 \cup D_2 \cup \{u\})$.

The absorber $\xi_r(u)$ is the graph containing D_1, D_2, D' and u as well as all the edges from D_1 to $D' \cup \{u\}$ and from D_2 to $D' \cup \{u\}$. Now let

- $\frac{r}{2}$ vertices $E_0 = \{e_0, \dots, e_{r/2-1}\}$ and an additional disjoint vertex $e_{r/2}$ from $D \setminus (D_1 \cup D_2 \cup D' \cup E_0)$ such that $G[E_0, D_1]$ contains $K_{r/2,r/2}$ minus a perfect matching and $G[E_0 \cup D_1, e_{r/2}] = K_{r,1}$.

The absorber $\xi'_r(u)$ is $\xi_r(u)$ together with the vertices $E_0 \cup \{e_{r/2}\}$ and all the edges in $E_0 \cup \{e_{r/2}\}$ and between this and D_1.

As long as $|X| \leqslant 6r\alpha n$ the existence of these absorbers follows with $\delta(G[A']), \delta(G[B']) \geqslant (\frac{1}{2} - 11r\alpha)n$ as every vertex has degree at least $8r\alpha n$ into A' or B'. Therefore, we are now in position to absorb the exceptional set $C = V(G) \setminus (A \cup B)$ (of course, without the bridging vertices on P_A and P_B), which has size at most αn (**Step 2** of Section 3.3). Of these vertices from C we denote those by $u_1, \dots, u_t$ that have $\frac{n}{5}$ neighbours in A and by $u'_1, \dots, u'_{t'}$ the others. We create pairwise disjoint absorbers $\xi_r(u_1), \ldots, \xi_r(u_t)$ and $\xi_r(u'_1), \ldots, \xi_r(u'_{t'})$, where D is either A' or B', respectively. Next, for $i = 1, \ldots, t-1$ (or $i = 1, \ldots, t'-1$), we use the gluing operation GE to glue D_2 of $\xi_r(u_i)$ and D_1 of $\xi_r(u_i+1)$ ($\xi_r(u'_i)$ and $\xi_r(u'_i+1)$) and denote the resulting path like structures by P'_A and P'_B, respectively. During these at most $2\alpha n$ gluing operations we always set X to be the vertices used in all previous gluing steps and the absorbers, which satisfies $|X| \leqslant 4r\alpha n$. Finally, use GE again to glue P'_A and P'_B to P_A and P_B, respectively, and denote these by P''_A and P''_B, respectively. After the repetitive gluing, we are left with sets $A'' = A' \setminus V(P''_A)$ and $B'' = B' \setminus V(P''_B)$ and the path-like subgraphs P''_A, P''_B of size at most $5r\alpha n$, hence $\left|A''\right|, \left|B''\right| \geqslant (\frac{1}{2} - 6r\alpha)n$.

Before closing the path in both sets (hence, creating a cycle like structure which contains all vertices that are not part of P''_A or P''_B), which is a standard application of the blow-up lemma, we need to make sure that certain divisibility conditions hold. As we wish to close the cycle by concatenating an even number of copies of $K_{r/2,r/2}$, we require that $\left|A''\right| \equiv \left|B''\right| \equiv 0 \pmod r$. We now consider $|A''|$, the argument for $|B''|$ is analogous. If $\left|A''\right| \equiv 0 \pmod r$, set $A''' = A''$ and proceed. Otherwise, if there is $0 < i < r$ such that $\left|A''\right| \equiv i \pmod r$, arbitrarily pick i vertices $a_1, \ldots, a_i \in A''$ and obtain disjoint absorbers $\xi'_r(a_1), \ldots, \xi'_r(a_i)$ with $D = A'$ and X all vertices in P''_A and all absorbers constructed so far. Further, each absorber consumes $2r+1$ vertices of A'', hence after removing them from A'', the divisibility condition holds.

Now, glue the absorbers sequentially to P''_A using GE. This is possible, as the set X of all vertices used in P''_A, in the absorbers, and for the gluing here has size at most $6r\alpha n$. As each gluing operation consumes r vertices, the divisibility does not change hence we are left with a set $A''' = A'' \setminus X$.

The end-tuples of the structure P''_A, thus the outer-left and outer-right vertices of this "path", have at least $\frac{1}{2}|A'''|$ common neighbours in A''' by construction. It is left to prove that there is a $r/2-$blowup of the path inside of A''' which starts at the outer-left vertices of P''_A and ends at the outer-right vertices of P''_A (**Step 3** of Section 3.3). This will be found by means of the Blow-Up-Lemma (Lemma 9). Recall that $\left|A'''\right|$ is a multiple of $2r$ and partition A''' into 2 sets M_1, M_2 each of size $\left|A'''\right|/2$ (divisible by r). As $G[A]$ has minimum degree $(1/2 - 10r\alpha)n$ and, so far, for the bridges and the connections at most $2r^2$ many vertices were used and for absorbing of the set C and fixing the parity at most $6r\alpha n$ vertices were necessary, we have $\left|A'''\right| \geqslant (1/2-7r\alpha)n$ and the minimum degree of $G[A''']$ is at least, $(1/2 - 16r\alpha)n$. From this it follows that (M_1, M_2) is $(\varepsilon, \frac{1}{2})$-super-regular Indeed, for any $X \subseteq M_1$ and $Y \subseteq M_2$ with $|X| \geqslant \varepsilon|M_1|$, $|Y| \geqslant \varepsilon|M_2|$ we have $d(X, Y) \geqslant (|Y| - 17r\alpha n)/|Y| \geqslant \frac{1}{2}$, where the last inequality holds as $\varepsilon \geqslant 170r\alpha$. The bipartite Blow-Up Lemma (Lemma 9) now directly implies the existence of the desired structure. We analogously proceed with B'' to get B''' and find the corresponding structure there.

We are left to argue that the constructed subgraph is r-connected and r-regular.

Claim 15. *The constructed subgraph is r-connected and r-regular.*

Proof. The r-regularity essentially follows by construction and we only point out that in the absorber $\xi'_r(u)$ a vertex $e_{r/2}$ is inserted into $K_{r/2,r/2}$ while a perfect matching is removed. For the r-connectedness we argue as follows. Upon removal of up to $r-1$ bridge-vertices, the parts do not fall apart. Furthermore, removing up to $r-1$ vertices in the $\frac{r}{2}$-blow-up of the path part of the subgraph does not disconnect the structure. Moreover, the absorbing structure $\xi_r(u)$ itself is isomorphic to an $\frac{r}{2}$-blowup of the path on three vertices. Finally, disconnecting the graph by removing up to $r-1$ vertices in $\xi'_r(r)$ is not possible. □

4.2. The odd case

Assume that r is odd and recall that n is even and $\delta(G) \geqslant \frac{n+r-1}{2}$ ($nr \equiv 0 \pmod 2$). The argument in the odd case is a bit more delicate than in the even case. Indeed, while in the process above all divisibility conditions could be easily established, in the odd case, we might end up with two sets of vertices of odd size. If there is a set C, we can easily absorb those vertices in a way that after absorbing both parts of the graph contain an even number of vertices — which we require to embed a regular graph. If on the other hand there is no such set C, we need to be much more

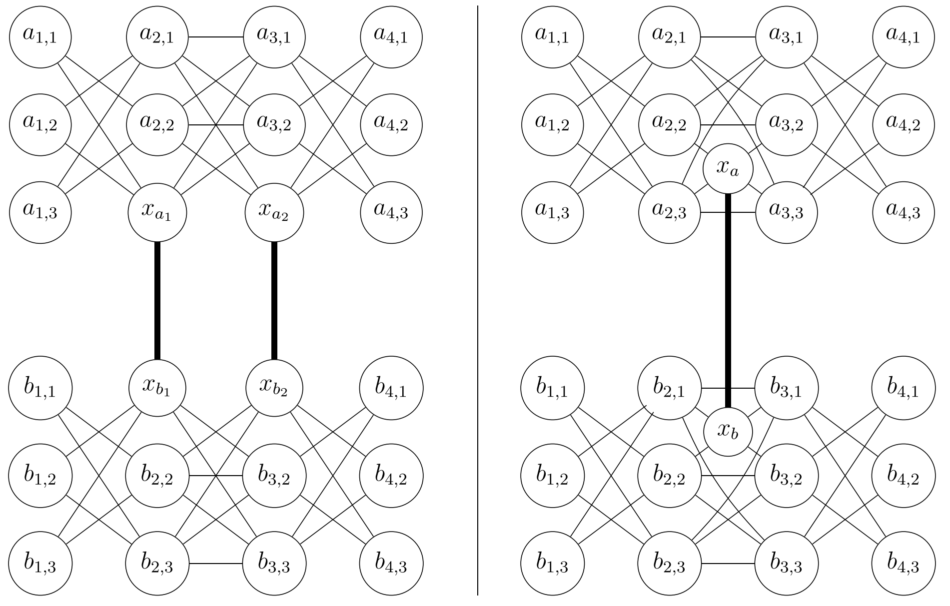

Fig. 3. The two types of connections between the sets A and B in the special case $r = 5$.

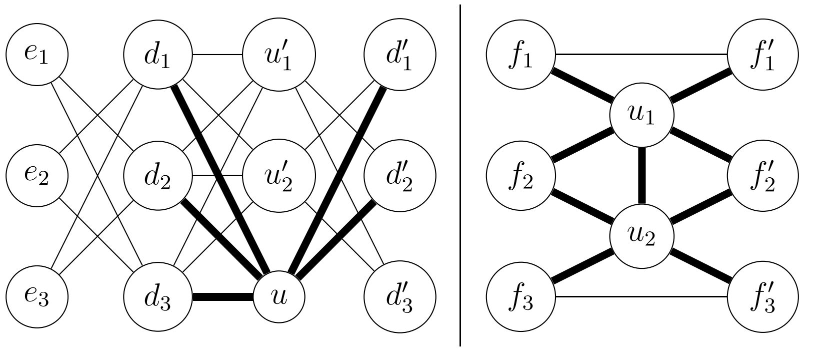

Fig. 4. Absorbers $\xi_5(u)$ (left) and $\xi'_5(u_1, u_2)$ (right) with $r = 5$.

careful. We will tackle this problem by having two different types of *bridge graphs* between A and B, one consuming an even number of vertices of each set, one consuming an odd number — thus, depending on the size of C and the parity of A and B, we need to use two different constructions. The two types of *bridge graphs* are visualised in Fig. 3 for the special case $r = 5$. Formally, the base for both are two copies of $P_3(r/2)$. For the first type, the *even bridge graph*, we then pick four vertices, x_{a_1}, x_{a_2} on each side of the first $K_{(r+1)/2,(r+1)/2}$ and x_{b_1}, x_{b_2} on each side of the second $K_{(r+1)/2,(r+1)/2}$, and then we remove edges $x_{a_1}x_{a_2}$, $x_{b_1}x_{b_2}$ and add edges $x_{a_1}x_{b_1}$, $x_{a_2}x_{b_2}$. For the second type, the *odd bridge graph*, we add additional vertices x_a, x_b, add an edge between them, remove a matching of size $(r-1)/2$ from both $K_{(r+1)/2,(r+1)/2}$ and connect x_a, x_b to the vertices that lost an edge in the first $K_{(r+1)/2,(r+1)/2}$, respectively second. We call the edges, which have one vertex in A and one in B *bridges*. We begin by showing that we can find $r+1$ bridges (**Step 1** of Section 3.3).

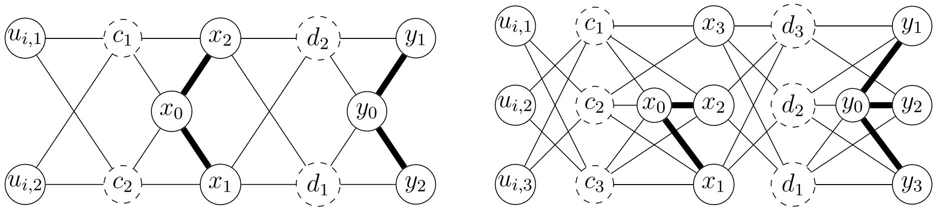

Fig. 5. Including two copies of $K_{1,s}$ into an r-regular path structure in the even ($r = 4$) and odd case ($r = 5$) with solid vertices in A, dashed vertices in B, and thick edges from $K_{1,s}$.

Claim 16. *Suppose $\delta(G) \geqslant \frac{n+r-2}{2}$ and $|A| \leqslant |B|$. Furthermore, let n be large enough. Then there is a matching $(x_{a_1}x_{b_1}, \dots, x_{a_{r+1}}x_{b_{r+1}})$ such that we have $\left|N(x_{a_i}) \cap A\right| \geqslant \frac{n}{5}$ and $\left|N(x_{b_j}) \cap B\right| \geqslant \frac{n}{5}$ for all $i, j \leqslant r+1$.*

Proof. As in the proof of Claim 11, it suffices to find $r+1$ vertex disjoint edges from A to $V \setminus A$. Recall that n is even.

If $|A| \leqslant \frac{n-r-1}{2}$, the minimum degree of $\frac{n+r-1}{2}$ guarantees that each vertex of A needs to find at least $\frac{n+r-1}{2} - (\frac{n-r-1}{2} - 1) = r + 1$ neighbours outside of A, hence the assertion follows.

Suppose $|A| = \frac{n-r-1}{2} + i$ with $i \in [\lfloor \frac{r}{2} \rfloor]$. In this case, $|V \setminus A| = \frac{n+r+1}{2} - i$ and every vertex from A has at least $\frac{n+r-1}{2} - (\frac{n-r-1}{2} + i - 1) = r - i + 1$ neighbours in $V \setminus A$. On the other hand every vertex from $V \setminus A$ has at least $\frac{n+r-1}{2} - (\frac{n+r+1}{2} - i - 1) = i$ neighbours in A. It follows that there cannot be a vertex cover of $G[A, V \setminus A]$ of size r and hence there is a matching of size $r + 1$ by Kőnigs Theorem. □

Similarly as in the even case, Claim 16 gives us $\frac{r+1}{2}$ pairs of bridge-edges as in Fig. 3 (the thick edges). Clearly, the rest of the bridge graph, even or odd, with $\frac{r+1}{2}$ instead of $\frac{r}{2}$ can be created completely analogously to Claim 12. Next, we re-define the gluing operation GE to GO as follows noting that it analogously holds in B.

Claim 17 (*Gluing Operation GO*)**.** *Given two disjoint sets $D_1, D_2 \subset A$ of sizes exactly $\frac{r+1}{2}$ and a set $X \subseteq A$ of size at most $5r\alpha n$, we find two disjoint sets $D, D' \subset A \setminus X$ of size $\frac{r+1}{2}$ such that*

$$G[D_1, D] \equiv G[D, D'] \equiv G[D', D_2] \equiv K_{(r+1)/2,(r+1)/2} .$$

Proof. This follows directly from the fact that each vertex in A is connected to at least $(\frac{1}{2} - 4\alpha)n$ vertices in A and GO uses only finitely many vertices of the neighbourhoods. □

We stress at this point that GO can be applied in the case when the vertices in D_1 and D_2 currently have degree $\frac{r-1}{2}$ (then we exclude a matching of size $\frac{r+1}{2}$ between D and D') or degree $\frac{r+1}{2}$ (then we exclude matching of size $\frac{r+1}{2}$ between D_1, D and D', D_2). Furthermore observe, that gluing consumes $r + 1$ vertices from the underlying set.

We then proceed building the bridge graphs as follows to ensure that either we have an even number of vertices on both sides left or $|C| > 0$. If $|A|, |B| \equiv 1 \pmod 2$ we build $\frac{r-1}{2}$ even bridge graphs and one odd bridge graph. Otherwise, we have $|A|, |B| \equiv 0 \pmod 2$ or $|C| > 0$, and we build $\frac{r+1}{2}$ even bridge graphs.

With GO we glue the bridge graphs in A (and B respectively) together using mutually disjoint vertex sets $D_1^A, \dots, D_{(r-1)/2}^A$ and $D_1^B, \dots, D_{(r-1)/2}^B$ constructing the path structures P_A, P_B of constant size. Then, we set $A' = A \setminus V(P_A)$, $B' = B \setminus V(P_B)$.

Next, we define absorbing structures for the left-over vertices and for absorbing vertices in order to guarantee divisibility (**Step 2** of Section 3.3). They need to be defined slightly differently as in the even case (Fig. 4).

Definition 18. Let $D \in \{A', B'\}$ and X be a set of vertices and u a vertex such that $|N(u) \cap D| \geqslant \frac{n}{6}$. Let

- $D_1 = \{d_1, \ldots, d_{(r+1)/2}\}$, $D_2 = \{d'_1, \ldots, d'_{(r+1)/2}\} \subset N(u) \cap D \setminus X$, hence $r+1$ pairwise disjoint vertices,
- $D' = \{u'_1, \ldots, u'_{(r-1)/2}\} \subset N(D_1) \cap N(D_2) \cap D \setminus (X \cup D_1 \cup D_2 \cup \{u\})$, and
- $E_0 = \{e_1, \ldots, e_{(r+1)/2}\} \subset N(D_1) \cap D \setminus (X \cup D_1 \cup D_2 \cup D' \cup \{u\})$.

The absorber $\xi_r(u)$ has vertex set $E_0 \cup D_1 \cup D_2 \cup D' \cup \{u\}$ and all the edges from D_1 to $D' \cup \{u\}$, from D_2 to $D' \cup \{u\}$ except a perfect matching, and from E_0 to D_1 except a perfect matching.

Furthermore, for two adjacent vertices $u_1, u_2 \in D$ let

- $F = \{f_1, f_2, \ldots, f_{(r+1)/2}\}$, $F' = \{f'_1, \ldots, f'_{(r+1)/2}\} \subset N(u_1) \cap N(u_2) \cap D \setminus X$, hence $r+1$ different vertices in the joint neighbourhood of u_1 and u_2, such that $f_1 f'_1$ and $f_{(r+1)/2} f'_{(r+1)/2}$ are edges and $G[F \cup F']$ contains an $r-2$ regular subgraph.

The absorber $\xi'_r(u_1, u_2)$ has vertex set $F \cup F' \cup \{u_1, u_2\}$ and all the edges from the $r-2$ regular subgraph, the edges $f_1 f'_1$ and $f_{(r+1)/2} f'_{(r+1)/2}$, the edge $u_1 u_2$, and all edges between $\{u_1, u_2\}$ except for $u_2 f_1$, u_2, f'_1, $u_1 f_{(r+1)/2}$, and $u_2 f'_{(r+1)/2}$.

As long as $|X| \leqslant 5r\alpha n$ the existence of these absorbers follows with $\delta(G[A']), \delta(G[B']) \geqslant (\frac{1}{2} - 11r\alpha)n$ as every vertex has degree $8r\alpha n$ into A' or B' and both absorbers have size at most $2r+2$. Observe that absorbing a vertex $u \notin D$ consumes $2r+1$ vertices from D (an odd number) while absorbing $u_1, u_2 \in D$ consumes $r+3$ vertices (including u_1, u_2) in D (an even number that is congruent to $2 \pmod{r+1}$).

Subsequently, we absorb C using independent copies of $\xi_r(\cdot)$ to ensure that the parity of the remaining vertices in the almost cliques is even, if that does not already hold. Here we need that every vertex from C has degree at least $8r\alpha n$ into A' and B', so we can choose to build $\xi_r(\cdot)$ within any of the two sets. Then, as above, we glue the absorbers together by GO and extend P_A to P'_A and construct P'_B, both of size at most $(3r+4)|C| \leqslant 7r\alpha n$. We are left with sets $A'' = A \setminus V(P'_A)$ and $B'' = B \setminus V(P'_B)$ and $|A''| \geqslant (\frac{1}{2} - 8r\alpha)n$, $|B''| \geqslant (\frac{1}{2} - 8r\alpha)n$. Furthermore, each vertex in A'' is connected to at least $(\frac{1}{2} - 16r\alpha)n$ vertices in A'' and the same applies to B''.

Again, as in the even case, we need to make sure that $|A''| \equiv 0 \pmod{r+1}$, as we want to close the cycle structure with $\frac{r}{2}$-blow-ups of paths that cover everything. If the divisibility condition holds, set $A''' = A''$ and proceed. Otherwise, there is $0 < i < r/2$ such that $|A''| \equiv 2i \pmod{r+1}$ as $|A''|$ is even. Select $2i$ different vertices $a_{1,1}, a_{1,2}, \ldots, a_{i,1}, a_{i,2}$ in A'' and absorb them using disjoint instances $\xi'_r(a_{1,1}, a_{1,2}), \ldots, \xi'_r(a_{i,1}, a_{i,2})$ with $D = A''$. As each absorber consumes $r+3$ vertices, after removing these vertices from A'', the divisibility condition now holds. Finally, as in the even case, glue the absorbed parts together with GO which does not change the divisibility by $r+1$. Thus, we are left with a set A''' which consists of the vertices of A'' without the absorbed vertices and the gluing structures. Analogously, the same applies for B'''. Now, as in the even case, the result directly follows from Lemma 9 and the following claim (**Step 3** of Section 3.3).

Claim 19. *The constructed subgraph is r-connected and r-regular.*

Proof. As in the even case, r-regularity as well as r-connectivity on the $\frac{r+1}{2}$-blow-up of the path part is obvious. The first type of bridge (see Fig. 3 on the left) does not harm connectivity as before. In the second type of bridge (see Fig. 3 on the right) only the special vertices x_a and x_b need our attention. But as they are of degree r and connected to $\frac{r-1}{2}$ vertices on both sides of the $K_{(r+1)/2,(r+1)/2}$, isolating a part of the graph is not possible either. The absorbing structures clearly sustain the connectivity property. □

This finishes the proof of the first extremal case. □

5. Extremal Case II

In this section we deal with the second extremal case and follow **Step 1**–**Step 4** as outlined in Section 3.4. We start by proving the auxiliary lemma for finding stars.

Proof of Lemma 10. Let s be an integer and choose $0 < \alpha < \frac{1}{32s^2(s+1)}$. Let G be an n-vertex graph with maximum degree $\Delta(G) \leqslant 4s\alpha n$ and minimum degree $\delta(G) \geqslant m+s-1$, where $1 \leqslant m \leqslant \alpha n$. Assume we have already found $0 \leqslant t < 2m$ copies of $K_{1,s}$ and let V' be the remaining vertices. Then, by the maximum degree condition in G,

$$e(G[V']) \geqslant \tfrac{1}{2}n(m+s-1) - t(s+1)4s\alpha n.$$

If $m \geqslant s+1$ this is at least $\frac{1}{4}n(m+s-1) \geqslant \frac{1}{2}ns$ and gives a vertex of degree at least s in $G[V']$. On the other hand, if $m \leqslant s$ the above is at least $(\frac{1}{2}s - \frac{1}{4})n > \frac{s-1}{2}n$ and again this gives a vertex of degree at least s in $G[V']$. $\square$

Proof of Extremal Case II. Let $r \geqslant 2$, $s = \lceil \frac{r}{2} \rceil \geqslant 1$, and $r' = 2s$. Let $\varepsilon > 0$ be given by Lemma 9 on input $\frac{1}{2}$, $\frac{1}{2}$, and r. We obtain $\alpha > 0$ from Lemma 10 and additionally assume that $100s\alpha \leqslant \varepsilon$. Let G be an n-vertex graph with minimum degree $\delta(G) \geqslant \frac{n+r-2}{2}$ and $nr \equiv 0 \pmod 2$. Further, assume that there is a partition of $V(G)$ into A and B with $|A| + m = \frac{1}{2}n = |B| - m$, where $0 \leqslant m \leqslant \alpha n$ such that between these sets we have minimum degree αn and all but at most αn vertices from A (or B) have degree at least $(\frac{1}{2} - 3\alpha)n$ into B (or A).

Step 1. Note $\delta(G[B]) \geqslant m+s-1$. Let $B' \subseteq B$ be the vertices of degree at most $2s\alpha n$ in $G[B]$ and let $m' = |B \setminus B'|$. If $m' < m$, then $\delta(G[B']) \geqslant (m-m') + s - 1$ and we apply Lemma 10 to find $2(m-m')$ copies of $K_{1,s}$. By choice of B' each vertex from these copies of $K_{1,s}$ has degree at least $\frac{n}{2} - 2s\alpha n \geqslant |A| - 2s\alpha n$ into A. Let W be the union of the vertices from these copies of $K_{1,s}$

For $i = 1, \dots, \min\{m', m\}$ we can iteratively pick a vertex $x_i \in B \setminus (B' \cup \{x_1, \dots, x_{i-1}\})$ and neighbours $y_{i,1}, \dots, y_{i,r'}$ of x_i from $B \setminus (W \cup \bigcup_{j=1}^{i-1}\{y_{j,1}, \dots, y_{j,r'}\})$ such that $y_{i,1}, \dots, y_{i,r'}$ have degree at least $|A| - 2s\alpha n$ into A. This is possible, because any such x_i has at least $2s\alpha n$ neighbours in B and all but αn vertices in B have degree $(\frac{1}{2} - 3\alpha)n$ into A. Add the vertices of these copies of $K_{1,r'}$ to W. We have obtained $\min\{m', m\}$ copies of $K_{1,r'}$ and $\max\{2(m-m'), 0\}$ copies of $K_{1,s}$ such that all vertices in the copies of $K_{1,s}$ and the leaves in the copies of $K_{1,r'}$ have degree at least $|A| - 2s\alpha n$ into A.

Step 2. We will now iteratively absorb these copies of $K_{1,s}$ and $K_{1,r'}$ into an r-regular path-like structure. We start with s vertices $\mathbf{v}' = (v'_1, \dots, v'_s)$ from $A \setminus W$ that have degree at least $|B| - 4\alpha n$ into B and s vertices $\mathbf{u}_0 = (u_{0,1}, \dots, u_{0,s})$ from $B \setminus W$ that have degree at least $|A| - 4\alpha n$ into A such that $v_j u_{0,j'}$ is an edge for $1 \leqslant j, j' \leqslant s$. This will be the base of our structure and we note that all vertices have degree s, so when extending this we need to add s or $s-1$ vertices depending on the parity of r. We add the vertices in $\mathbf{v}'$ and $\mathbf{u}_0$ to W. Now we want to extend this by alternating between A and B while absorbing the copies of $K_{1,s}$ and $K_{1,r'}$ found earlier.

To do this, for $k = 0, \dots, m$ let P_k be the structures that we build and W_k be the vertices used in P_k and in the remaining copies of $K_{1,s}$ and $K_{1,r'}$, we will see that

$$|W_k| \leqslant 2s(k+1) + 2\,m(s+1) \leqslant 8\alpha n,$$

and let $\mathbf{u}_i = (u_{i,1}, \dots, u_{i,s})$ be the end of P_k in B (will be defined precisely below) and assume that $u_{i,j}$ has at least $|A| - 2s\alpha n$ neighbours in A for $j = 1, \dots, s$. We will now repeatedly use the fact that, as $40s\alpha < \varepsilon$, for up to $2s+1$ vertices $b_1, \dots, b_{2s+1}$ from B that have degree at least $|A| - 2s\alpha n$ into A there are at least $|A|/2$ common neighbours of $b_1, \dots, b_{2s+1}$ in A.

If there are two copies of $K_{1,s}$ left, whose vertices we denote by $x_0, x_1, \dots, x_s$ and $y_0, y_1, \dots, y_s$ (x_0, y_0 are the centres), then by choosing vertices from B we want to extend our structure from $\mathbf{u}_i$ to cover both of them. We connect the first $K_{1,s}$ to $\mathbf{u}_i$ by picking s vertices $\{c_1, \dots, c_s\}$ in the common neighbourhood of all $\{u_{i,1}, \dots, u_{i,s}, x_0, x_1, \dots, x_s\}$. Then we connect the second $K_{1,s}$ to the leaves of the first by picking s vertices $\{d_1, \dots, d_s\}$ in the common neighbourhood of

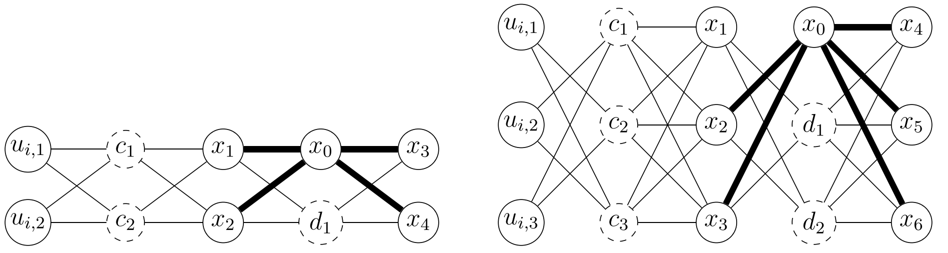

Fig. 6. Including one copy of $K_{1,r'}$ into an r-regular path structure in the even ($r = 4$) and odd case ($r = 5$) with solid vertices in A, dashed vertices in B, and thick edges from $K_{1,r'}$.

$\{x_1, \dots, x_s, y_0, y_1, \dots, y_s\}$. We let $\mathbf{u}_{i+1} = (u_{i+1,1}, \dots, u_{i+1,s}) = (y_1, \dots, y_s)$. If r is even, we ignore the edges c_jx_j and d_jy_j for $j = 1, \dots, s$. This extends our structure in the desired way and all vertices have degree r, except those in $\mathbf{v}'$ and $\mathbf{u}_{i+1}$, which have degree s (see Fig. 5). If r is odd, we achieve the same by ignoring the edges $u_{i,j}c_j$, c_jx_j and d_jy_j for $j = 1, \dots, s$, x_0x_s, d_sy_0 and x_jd_j for $j = 1, \dots, s-1$, which reduces the degree of all vertices

$$u_{i,1}, \dots, u_{i,s}, c_1, \dots, c_s, x_0, \dots, x_s, d_1, \dots, d_s, y_0$$

from $2s = r + 1$ to r. This extends P_k to P_{k+1} and $\mathbf{u}_{i+1}$ is the end of P_{k+1}.

Otherwise, we pick a copy of $K_{1,r'}$, whose vertices we denote by $x_0, x_1, \dots, x_{r'}$ (x_0 is the centre), and extend our structure with the $\frac{r}{2}$-blow-up of a path on five vertices by choosing some vertices from B as follows. First we connect the vertices from $\mathbf{u}_i$ with s leaves of the $K_{1,r'}$ by choosing s common neighbours $c_1, \dots, c_s$ of $\{u_{i,1}, \dots, u_{i,s}, x_1, \dots, x_s\}$ in A. Then we choose $s-1$ vertices $d_1, \dots, d_{s-1}$ from the common neighbourhood of the $x_1, \dots, x_{r'}$. We define $\mathbf{u}_{i+1} = (u_{i+1,1}, \dots, u_{i+1,s}) = (x_{s+1}, \dots, x_{r'})$. As before this already is the $\frac{r}{2}$-blow-up of a path extending our structure in the desired way (see Fig. 6) if r is even. If r is odd, to achieve this, we ignoring two matchings given by the edges $u_{i,j}c_j$ for $j = 1, \dots, s$, x_jd_{j-1} for $j = 2, \dots, s$, and x_1x_0. This extends P_k to P_{k+1} and $\mathbf{u}_{i+1}$ is the end of P_{k+1}.

We add all new vertices in the structure to W. We can repeat this until all copies of $K_{1,s}$ and $K_{1,r}$ are covered, because in each step W will increase by at most $6r$ vertices and all vertices have degree at least $(\frac{1}{2} - 3\alpha)n$ to the other side. We let $\mathbf{u}' = (u_{m,1}, \dots, u_{m,s})$ be the final end of this construction. Now let $A_1 = A \setminus W$ and $B_1 = B \setminus W$ and note that $|A_1| = |B_1|$ because our construction uses $2m$ vertices more from B than from A, exactly the centres of the stars $K_{1,s}$, $K_{1,r'}$, where the latter count twice as they not just add a vertex, but replace a vertex from A.

Step 3. For the next step, let $A' \subseteq A_1$ be the vertices of degree at most $|B_1| - 12s\alpha n$ into B_1 and $B' \subseteq B_1$ be the vertices of degree at most $|A_1| - 12s\alpha n$ into A_1. Note that $|A'|, |B'| \leqslant \alpha n$, because we removed at most $8s\alpha n$ vertices from each of A and B to get A_1 and B_1. By using an $\frac{r}{2}$-blow-up of a path (exactly as in Fig. 6) we cover the vertices in A' and B' with our r-regular path structure covering in total at most $4s\alpha n$ additional vertices. To do this, for each $a \in A'$, we pick a copy of $K_{1,r}$ centred at a with leaves $b_1, \dots, b_{r'}$ in $B_1 \setminus B'$. Then we iteratively extend the path structure from $\mathbf{u}'$ exactly as in the second part of **Step 2**, where a corresponds to x_0 and the leaves $b_1, \dots, b_{r'}$ to $x_1, \dots, x_{r'}$. Here the parity remains intact as a comes from A, whereas above x_0 came from B. We proceed analogously for $b \in B'$, where the roles are reversed and we extend from $\mathbf{v}'$ instead.

Add the vertices used for covering A' and B' to W, let $\mathbf{u} = (u_1, \dots, u_s)$ and $\mathbf{v} = (v_1, \dots, v_s)$ be the last vertices of this construction in B and A, and note that we can assume that they have degree at least $|A| - 20s\alpha n$ into A. We let $A_2 = (A_1 \setminus W)$ and $B_2 = (B_1 \setminus W)$ and note that $|A_2| = |B_2|$.

Step 4. We have that every vertex from A_2 (or B_2) has degree at least $|A_2| - 16s\alpha n$ into A_2 (or $|B_2| - 16s\alpha n$ into B_2). Then it is easy to see that (A_2, B_2) is $(\varepsilon, \frac{1}{2})$-super-regular as $100s\alpha \leqslant \varepsilon$. Indeed, for any $A' \subseteq A_2$ and $B' \subseteq B_2$ with $|A'| \geqslant \varepsilon|A_2|$, $|B'| \geqslant \varepsilon|B_2|$ we have $d(A', B') \geqslant (|B'| - 16s\alpha n)/|B'| \geqslant \frac{1}{2}$, where the last inequality follows from $\frac{1}{2}\varepsilon|B_2| \geqslant 16s\alpha n$.

Moreover, the vertices in $\mathbf{v}$ have $|B_2|-26s\alpha n \geqslant \frac{1}{2}|B_2|$ common neighbours in B_2 and similarly for $\mathbf{u}$ with A_2. We apply Lemma 9 to cover the remaining vertices of A_2 and B_2 with the $\frac{r}{2}$-blow-up of a path such that the ends connect to $\mathbf{v}$ and $\mathbf{u}$. This completes the construction of a spanning r-regular structure in G. To see that it is also r-connected it suffices to note that we have the $\frac{r}{2}$-blow-up of a path except for some parts replaced by the graphs from Fig. 5, which do not harm this property. □

6. Non-extremal case

In this section we deal with the case that G is not α-extremal. Recall, that the assumption implies that for any two sets $A, B \subseteq V(G)$ of size $(\frac{1}{2}-\alpha)n \leqslant |A|, |B| \leqslant \frac{n}{2}$ we have $d(A,B) \geqslant \alpha$. We will follow **Step 1**–**Step 6** as outlined in Section 3.2.

Proof of Non-Extremal Case. Given $r \geqslant 3$, let $0 < \alpha < \frac{1}{32}$ and $s = \lceil \frac{r}{2} \rceil$ and we choose constants such that

$$\varepsilon \ll \nu \ll d \ll \beta \ll \alpha,$$

where, in particular,

$$\varepsilon \leqslant \nu, \qquad \nu \leqslant d^{2s}2^{-s-1}, \qquad d \leqslant \beta, \qquad 500s\beta \leqslant \alpha$$

and 2ε is smaller than the output of Lemma 9 with input $\frac{d}{2}$, $\frac{1}{4}d^s$, and r. Let L be given by Lemma 5 on input ε.

Step 1. Let G be an n-vertex graph with minimum degree $\delta(G) \geqslant (\frac{1}{2}-\beta)n$. Note that we prove the stronger variant as remarked earlier. From Lemma 5 we get a partition of the vertex set $V(G)$ into $\ell+1 \leqslant L$ clusters V_0 with $|V_0| \leqslant \varepsilon n$ and $V_1, \dots, V_\ell$ of size T and a subgraph $G' \subseteq G$ such that (P1)–(P4) hold. We denote by R the graph on vertex set $[\ell]$ with edges ij if and only if the pair (V_i, V_j) is ε-regular with density at least d. In R we have minimum degree $\delta(R) \geqslant (\frac{1}{2}-\beta-2d)\ell$ by Fact 6. Similarly, we can deduce that R is not $\frac{\alpha}{2}$-extremal. Otherwise, there would be two sets of vertices $\mathcal{A}, \mathcal{B}$ in R such that $(\frac{1}{2}-\frac{\alpha}{2})\ell \leqslant |\mathcal{A}|, |\mathcal{B}| \leqslant \frac{1}{2}\ell$ and $d(\mathcal{A},\mathcal{B}) < \frac{\alpha}{2}$. Then $A = \bigcup_{i\in\mathcal{A}} V_i$ and $B = \bigcup_{i\in\mathcal{B}} V_i$ both have size at most $\frac{\ell}{2}\cdot\frac{n}{\ell} = \frac{n}{2}$ and at least $(\frac{1}{2}-\frac{\alpha}{2})\ell\cdot(1-\varepsilon)\frac{n}{\ell} \geqslant (\frac{1}{2}-\alpha)n$ and we have

$$d(A,B) = \frac{e(A,B)}{|A|\,|B|} \leqslant \frac{\frac{\alpha}{2}\cdot(\frac{n}{\ell})^2\cdot|\mathcal{A}|\,|\mathcal{B}| + |A|(d+\varepsilon)n}{|A||B|} \leqslant \frac{\frac{\alpha}{2}}{(1-\alpha)^2} + \frac{(d+\varepsilon)}{\frac{1}{2}-\alpha} < \alpha,$$

which contradicts our assumption that G is not α-extremal. We will repeatedly use the following fact that holds as R is not $\frac{\alpha}{2}$-extremal.

Fact 20. *For any two sets $\mathcal{A}, \mathcal{B} \subseteq V(R)$ of size at least $(\frac{1}{2}-\frac{\alpha}{2})\ell$ there is an edge $AB \in E(R)$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

In the following we will treat the clusters as vertices of R.

Step 2. Next let M be a maximum matching in R and $D \subseteq [\ell]$ be the clusters not covered by M. Naturally, D is an independent set in R and if there are at least two vertices u and v in D then no neighbour of u is connected to a neighbour of v by an edge of M. Therefore, $2|M| \geqslant \deg_R(u)+\deg_R(v)$ and so $|M| \geqslant (\frac{1}{2}-\beta-2d)\ell$ by the minimum degree in R. We let $\ell' = |M|$ and denote the regular pairs corresponding to edges of M by (X_i, Y_i) for $i = 1, \dots, \ell'$. In the following, we will sometimes find one class of such a regular pair not knowing if it is X_i or Y_i and, for simplicity, just assume that it is one of them.

Step 3. For any clusters Z and W we call an s-tuple $\mathbf{z} = (z_1, \dots, z_s)$ from Z *well-connected into* W if the vertices $z_1, \dots, z_s$ have at least $\frac{1}{2}d^sT$ common neighbours in W and individually have degree at least $(d-\varepsilon)T$ into W. Now fix any $i \in [\ell']$. We want to connect Y_i and X_{i+1} by the $\frac{r}{2}$-blow-up of a path (index $\ell'+1 = 1$). For this we consider the set of neighbours $\mathcal{W}$ and $\mathcal{Z}$ in R of X_{i+1} and Y_i, respectively. It follows from the minimum degree in R that $\mathcal{W}$ and $\mathcal{Z}$ have size at least $(\frac{1}{2}-\frac{\alpha}{2})\ell$. By Fact 20 there is $W \in \mathcal{W}$ and $Z \in \mathcal{Z}$ such that WZ is an edge in R.

By repeatedly applying Lemma 3 and using $\varepsilon \leqslant d/(2s)$ we want to argue that all but at most $2s\varepsilon T^s$ s-tuples $\mathbf{x}=(x_1,\dots,x_s)$ from X_{i+1} are well connected into W and Y_{i+1}. For this we first use Lemma 3 to get that all but $2\varepsilon T$ vertices of X_{i+1} have degree at least $(d-\varepsilon)T$ into W and Y_{i+1}. Then we assume that, for some $j=1,\dots,s-1$, we have that all but $2j\varepsilon T^j$ j-tuples from X_{i+1} that have at least $(d-\varepsilon)^jT$ common neighbours in W and Y_{i+1}. For any such j-tuple $(x_1,\dots,x_j)$ we then apply Lemma 3 with the common neighbourhood of $x_1,\dots,x_j$ in W and Y_{i+1} to get that all but $2j\varepsilon T^j\cdot T+(1-2j\varepsilon)T^j\cdot 2\varepsilon T \leqslant 2(j+1)\varepsilon T^{j+1}$ $(j+1)$-tuples from X_{i+1} that have at least $(d-\varepsilon)^{j+1}T$ common neighbours in W and Y_{i+1}. For $j+1=s$ with $(d-\varepsilon)^sT \geqslant \frac{1}{2}d^sT$ this proves the statement, where the inequality holds when $\varepsilon \leqslant d/(2s)$.

The same holds for tuples from Y_i, W, and Z with respect to their neighbouring clusters. We fix tuples $\mathbf{x}=(x_1,\dots,x_s)$, $\mathbf{w}=(w_1,\dots,w_s)$, $\mathbf{z}=(z_1,\dots,z_s)$, and $\mathbf{y}=(y_1,\dots,y_s)$ from X_{i+1}, W, Z, and Y_i, respectively, such that $\mathbf{xwzy}$ gives the $\frac{r}{2}$-blow up of a path on 4 vertices and $\mathbf{x}$ and $\mathbf{y}$ are well-connected into Y_{i+1} and X_i, respectively. We obtain this by first picking $\mathbf{x}$ that is well connected into Y_{i+1} and W, which is possible as $2s\varepsilon T^s < \frac{1}{2}\binom{T}{s}$. Then we observe that in the common neighbourhood of $\mathbf{x}$ in W there is a choice for $\mathbf{w}$ that is well connected into Z, because there are at least $\binom{d^sT/2}{s} > 2s\varepsilon T^s$ s-tuples in the common neighbourhood of $\mathbf{x}$ in W, where the inequality follows from $\varepsilon < d^{2s}2^{-s-1}$ and n sufficiently large. We denote this path by P_i and remove any internal vertices (those in $\mathbf{w}$ and $\mathbf{z}$) from the clusters. We can repeat this for all i, because we need only $4s\ell' \leqslant 2sL$ vertices in total.

Step 4. To make the matching edges super-regular, we let $i \in [\ell']$ and apply Lemma 8 to the pair (Y_i, X_i). We make sure that the end-tuple $\mathbf{x}$ of P_{i-1} remains in X_i and the end-tuple $\mathbf{y}$ of P_i remains in Y_i, which is fine as they have degree at least $(d-3\varepsilon)T$ to the other side. After removing a few additional vertices we arrive at sets Y_i and X_i such that $|Y_i|=|X_i|=T' \geqslant (1-2\varepsilon)T$, where $T' \equiv 0 \pmod{s}$, and the pair (Y_i,X_i) is $(2\varepsilon, d-4\varepsilon)$-super-regular, where we also appeal to Lemma 4. We add the vertices removed during this procedure and also the vertices that belong to clusters of D to V_0 and note that $|V_0| \leqslant \varepsilon n+\ell'4\varepsilon T+\beta n+2dn \leqslant 2\beta n$.

Step 5. **Setup**. We want to absorb V_0 by extending the $\frac{r}{2}$-blow-up of paths P_i. After each extension we need to maintain the location of the end-tuples and also ensure that they are well-connected. During the procedure we will have to deal with sets of already covered vertices, which we denote by W_0 and W. Here W_0 only contains vertices that where used during recent iterations of our procedure, which will be moved to W later. We collect some properties and definitions involving W_0 and W that we will need later. If $|W_0| \leqslant 2s\nu n$ and $|W| \leqslant 20s\beta n$ there are at most $2s\nu n/(\frac{1}{8}d^s\frac{n}{\ell}) = 16s\nu d^{-s}\ell \leqslant 16s\beta\ell$ clusters that intersect W_0 in at least $\frac{1}{8}d^s\frac{n}{\ell}$ vertices and at most $20s\beta n/(\frac{1}{4}\cdot\frac{n}{\ell}) = 80s\beta\ell$ clusters that intersect W in at least $\frac{1}{4}\cdot\frac{n}{\ell}$ vertices. We denote by H the set of all clusters that do not have both properties and note $|H| \geqslant (1-100s\beta)\ell$.

Now consider a vertex $v \in V_0$. There are at least $(\frac{1}{2}-150s\beta)\ell$ clusters in H that intersect $N_G(v) \setminus (W \cup W_0)$ in at least $d\frac{n}{\ell}$ vertices. We denote this set of clusters by $H(v)$. Similarly, let $H_M(v)$ be the clusters from H, which share an edge of M with another cluster from $H(v)$ and note that we have the same lower bound as M is a matching. Summing up we have $|H| \geqslant (1-100s\beta)\ell$ and $|H(v)|, |H_M(v)| \geqslant (\frac{1}{2}-150s\beta)\ell$ for all $v \in V_0$. Note that $H(v)$ and $H_M(v)$ are large enough for Fact 20.

Covering $2s$ vertices. We will use the following claim to repeatedly cover $2s$ vertices. To start our procedure we let $W=\varnothing$ and W_0 be all internal vertices (not in end-tuples) of the paths $P_1,\dots,P_{\ell'}$.

Claim 21. *Assume that $|W_0| \leqslant 2s\nu n$, $|W| \leqslant 20s\beta n$ and that $W_0 \cup W$ contains all internal vertices (not in end-tuples) of the $\frac{r}{2}$-blow-up of paths $P_1,\dots,P_{\ell'}$. Moreover, for $i \in [\ell']$, assume that X_i and Y_i are disjoint from $W \cup W_0$, that (X_i,Y_i) is $(4\varepsilon, \frac{1}{2}d)$-super-regular, and $|X_i|=|Y_i| \equiv 0 \pmod{s}$.*

Then we can cover $2s$ vertices $v_1,\dots,v_{2s}$ from V_0 by extending the $\frac{r}{2}$-blow-up of paths within clusters of H using at most $20s^2$ vertices. Moreover, for $i \in [\ell']$, after removing the vertices that are not in end-tuples of the paths from X_i, Y_i we still have $|X_i|=|Y_i| \equiv 0 \pmod{s}$.

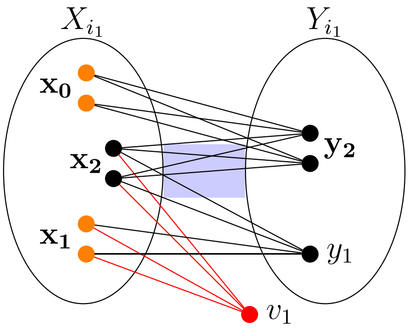

Fig. 7. Absorbing vertex v_1 in the case $r = 2s = 4$ if $\mathbf{x_0}$ is the current end of path P_i and $\mathbf{x_1}$ is the new end.

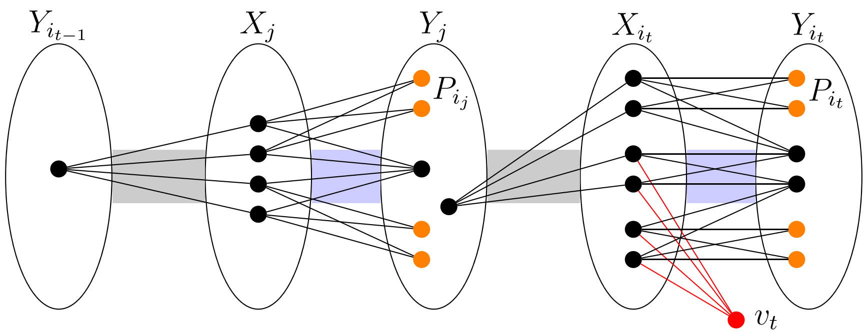

Fig. 8. Absorbing vertex v_t in the case $r = 2s = 4$. The blue connection between classes indicates that those edges belong to the fixed matching in the cluster graph while gray connections indicate using additional edges. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

After applying the claim, we also remove the $2s$ vertices $v_1, \ldots, v_{2s}$ from V_0. We will apply the claim at most νn times and, thus, $|W_0| \leqslant 2s\nu n$ throughout. With the bound on ν it follows from the fact that they were $(2\varepsilon, d - 4\varepsilon)$-super-regular and also ε-regular even earlier that pairs in M remain $(2\varepsilon, \frac{1}{2}d)$-super-regular. We will ensure that new end-tuples of paths are always well-connected. Moreover, the end-tuples of a path that were well-connected into a cluster X at some point during these applications of Claim 21 still have at least $\frac{1}{4}d^sT$ common neighbours in the new $X' \subseteq X$, because they had $\frac{1}{2}d^sT$ common neighbours in X and clusters only remain active in H if the intersection with W_0 is less than $\frac{1}{8}d^sT$, i.e. $|X \setminus X'| \leqslant \frac{1}{4}d^sT$.

Proof of Claim 21. Let i_1 be such that $X_{i_1} \in H(v_1)$ and $Y_{i_1} \in H$ and $\mathbf{x_0}$ be the end-tuple of P_{i_1} in X_{i_1}. As $\mathbf{x_0}$ is well-connected into Y_{i_1} and $(N_G(v_1) \cap X_{i_1}) \setminus (W \cup W_0)$ is of size at least $d\frac{n}{\ell}$, we can greedily pick tuples $\mathbf{x}_1$, $\mathbf{x}_2$ in X_{i_1} and $\mathbf{y}_1$, $\mathbf{y}_2$ in Y_{i_1} with the exception that $\mathbf{y}_1$ contains v_1 and such that $\mathbf{x}_1\mathbf{y}_1\mathbf{x}_2\mathbf{y}_2\mathbf{x_0}$ gives the $\frac{r}{2}$-blow-up of a path and $\mathbf{x_1}$ is well-connected into Y_{i_1}. Now we remove the internal vertices $(\mathbf{x_2}, \mathbf{y_1}, \mathbf{y_2}, \mathbf{x_0})$ from the clusters and add them to W_0 (this adds $4s$ vertices; see Fig. 7). We note that $|Y_{i_1}| - 1 = |X_{i_1}| \equiv 0 \pmod s$ and let P_{i_1} be the longer path.

We continue in a similar fashion to cover $v_2, \ldots, v_s$, see Fig. 8 for an illustration. For this let $t = 2, \ldots, s$ and assume that i_{t-1} is such that $|Y_{i_{t-1}}| - t + 1 = |X_{i_{t-1}}| \equiv 0 \pmod s$. Let $\mathcal{A}$ be the neighbours of $Y_{i_{t-1}}$ in R and let $\mathcal{B}$ be those clusters which share an edge of M with a cluster from $\mathcal{A}$. By Fact 20 applied to $\mathcal{B}$ and $H(v_t)$ there are indices i_t, j such that $X_{i_t} \in H(v_t)$, $Y_{i_t}, X_j, Y_j \in H$ and $Y_{i_{t-1}}X_j$ and $Y_jX_{i_t}$ are edges of R. This gives the path $Y_{i_{t-1}}, X_j, Y_j, X_{i_t}, Y_{i_t}$ in R. We let $\mathbf{y_0}$ and $\mathbf{y_0'}$ be the end-tuples of P_j and P_{i_t} in Y_j and Y_{i_t}, respectively. Similarly as for P_{i_1} above we find tuples $\mathbf{x_1}, \mathbf{x_2}$ in X_j, $\mathbf{y_1}$ with $t-1$ vertices in $Y_{i_{t-1}}$ and $s-t+1$ vertices in Y_j, and $\mathbf{y_2}$ in Y_j, such that $\mathbf{y_0}, \mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}$ is

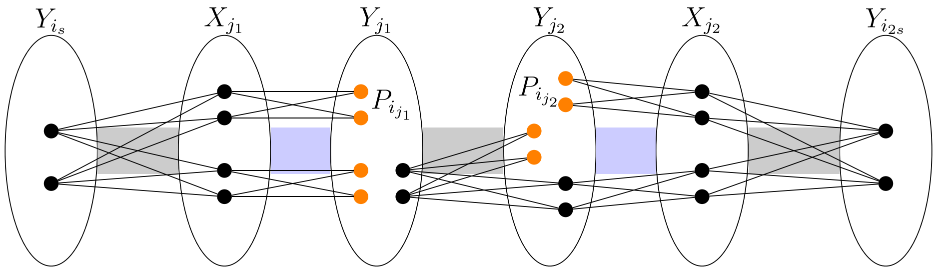

Fig. 9. Balancing Y_{i_s} and $Y_{i_{s+1}}$ in the case $r = 2s = 4$. As before, we indicate the fixed matching in the cluster graph by blue connections and additional edges by gray connections. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

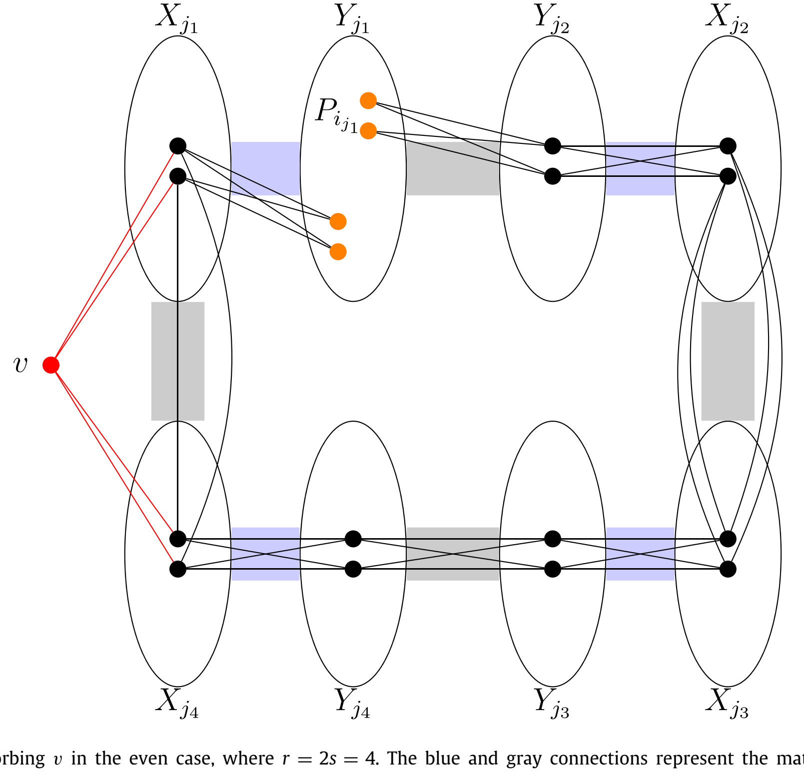

Fig. 10. Absorbing v in the even case, where $r = 2s = 4$. The blue and gray connections represent the matching edges and non-matching edges in the cluster graph again. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the $\frac{r}{2}$-blow-up of a path extending P_j and $\mathbf{y_2}$ is well-connected into X_j. We also find tuples $\mathbf{x'_1}, \mathbf{x'_2}, \mathbf{x'_3}$ in X_{i_t}, $\mathbf{y'_1}$ with $t-1$ vertices in Y_j and $s-t+1$ vertices in Y_{i_t}, $\mathbf{y'_2}$ in Y_{i_t} also containing v_t, and $\mathbf{y'_3}$ in Y_{i_t}, such that $\mathbf{y'_0}, \mathbf{x'_1}, \mathbf{y'_1}, \mathbf{x'_2}, \mathbf{y'_2}, \mathbf{x'_3}, \mathbf{y'_3}$ is the $\frac{r}{2}$-blow-up of a path extending P_{i_t} and $\mathbf{y'_3}$ is well-connected into X_{i_t}. After removing the internal vertices $(\mathbf{y_0}, \mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y'_0}, \mathbf{x'_1}, \mathbf{y'_1}, \mathbf{x'_2}, \mathbf{y'_2}, \mathbf{x'_3})$ we have $|Y_{i_{t-1}}| = |X_{i_{t-1}}| \equiv 0 \pmod s$, $|Y_j| = |X_j| \equiv 0 \pmod s$, and $|Y_{i_t}| - t = |X_{i_t}| \equiv 0 \pmod s$ and we add the removed vertices to W_0 (this adds $10s$ vertices).

We repeat the same procedure to cover $v_{s+1}, \ldots, v_{2s}$ and are left to deal with $|Y_{i_s}| - s = |X_{i_s}| \equiv 0 \pmod s$ and $|Y_{i_{2s}}| - s = |X_{i_{2s}}| \equiv 0 \pmod s$. For this, see Fig. 9 for an illustration, let $\mathcal{A}$ be the clusters

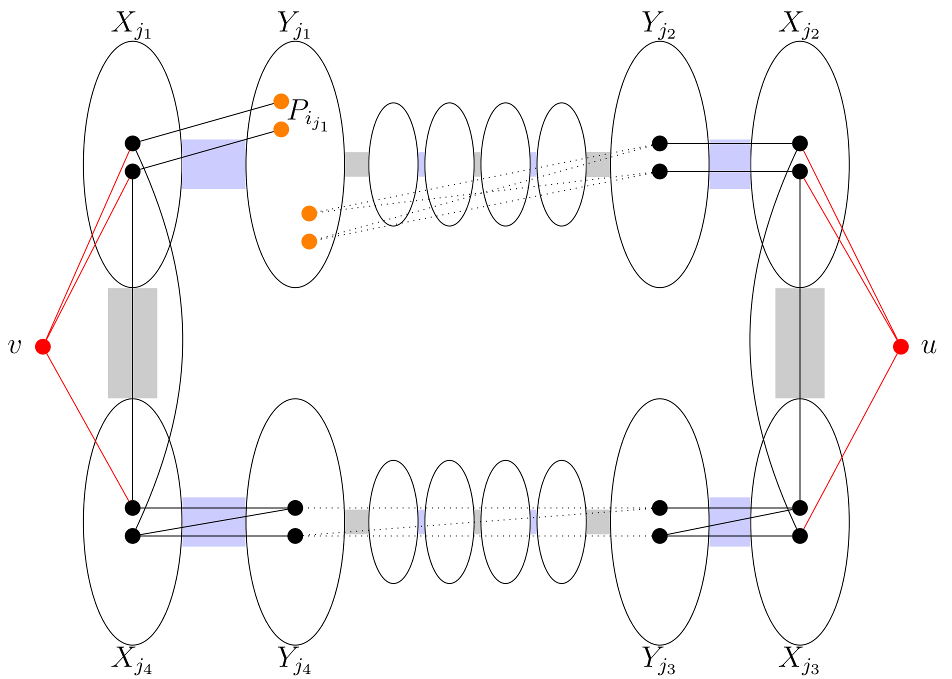

Fig. 11. Absorbing v and u in the odd case, where $r = 3, s = 2$. The blue and gray connections represent the matching edges and non-matching edges in the cluster graph again. The dotted edges indicate an r-regular r-connected path structure. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

in R that share an edge in M with a neighbour of the cluster Y_{i_s} and, similarly, let $\mathcal{B}$ be the clusters in R that share an edge in M with a neighbour of the cluster $Y_{i_{2s}}$. By Fact 20 applied to $\mathcal{A}$ and $\mathcal{B}$ we find indices j_1, j_2 such that $X_{j_1}, Y_{j_1}, X_{j_2}, Y_{j_2} \in H$ and $Y_{i_s}X_{j_1}$, $Y_{j_1}Y_{j_2}$, and $X_{j_2}Y_{i_{s+1}}$ are edges of R. We let $\mathbf{y_0}$ and $\mathbf{y'_0}$ be the end-tuples of $P_{i_{j_1}}$ and $P_{i_{j_2}}$ in Y_{j_1} and Y_{j_2}, respectively. As before we greedily find tuples $\mathbf{x_1}, \mathbf{x_2}$ in X_{j_1}, $\mathbf{y_1}$ in Y_{i_s}, and $\mathbf{y_2}$ in Y_{j_1}, such that $\mathbf{y_0}, \mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}$ is the $\frac{r}{2}$-blow-up of a path extending $P_{i_{j_1}}$ and $\mathbf{y_2}$ is well-connected into X_{j_1}. Also we find tuples $\mathbf{x'_1}, \mathbf{x'_2}$ in X_{j_2}, $\mathbf{y'_1}$ in $Y_{i_{2s}}$, $\mathbf{y'_2}, \mathbf{y'_4}$ in Y_{j_2}, and $\mathbf{y'_3}$ in Y_{j_1}, such that $\mathbf{y'_0}, \mathbf{x'_1}, \mathbf{y'_1}, \mathbf{x'_2}, \mathbf{y'_2}, \mathbf{y'_3}, \mathbf{y'_4}$ is the $\frac{r}{2}$-blow-up of a path extending $P_{i_{j_2}}$ and $\mathbf{y'_4}$ is well-connected into X_{j_2}. After removing the internal vertices $(\mathbf{y_0}, \mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y'_0}, \mathbf{x'_1}, \mathbf{y'_1}, \mathbf{x'_2}, \mathbf{y'_2}, \mathbf{x'_3}, \mathbf{y'_3})$ we have $|X_{j_1}| = |Y_{j_1}| \equiv 0 \pmod{s}$, $|X_{j_2}| = |Y_{j_2}| \equiv 0 \pmod{s}$, $|Y_{i_s}| = |X_{i_s}| \equiv 0 \pmod{s}$, and $|Y_{i_{s+1}}| = |X_{i_{s+1}}| \equiv 0 \pmod{s}$. We add the internal vertices to W_0 (this adds $10s$ vertices). In total we add at most $2 \cdot (4s + (s-1)10s) + 10s \leqslant 20s^2$ vertices to W_0.

Reset after νn iterations. After νn applications of Claim 21 we want to reestablish the original super-regularity condition for the pairs of clusters in M so that we can continue for another νn rounds. For $i \in [\ell']$, by Lemma 4, the pairs (X_i, Y_i) are 2ε-regular with density at least $d - \varepsilon$, as less than half of the vertices from each part were removed by definition of H. Then, by Lemma 8, for any $i \in [\ell']$ we need to remove at most $2\varepsilon\frac{n}{\ell}$ vertices from each of X_i and Y_i to get that (X_i, Y_i) is $(4\varepsilon, d - 5\varepsilon)$-super-regular again and $|Y_i| = |X_i| \equiv 0 \pmod{s}$. We will greedily absorb these vertices into the path P_i without any degree dropping below $\frac{3d}{4} \cdot \frac{n}{\ell}$ using the following claim.

Claim 22. *Let (X, Y) be a $(2\varepsilon, \frac{1}{2}d)$-super-regular pair and $X' \subseteq X$, $Y' \subseteq Y$ be sets with $|X'| \leqslant 2\varepsilon|X|$, $|Y'| \leqslant 2\varepsilon|Y|$ such that $(X \setminus X', Y \setminus Y')$ is $(4\varepsilon, d - 5\varepsilon)$-super-regular. Then we can extend the path P from the end-tuple in Y such that it contains X', Y', and has a new well-connected end-tuple in Y using at most $2\varepsilon\frac{n}{\ell} \cdot 4s$ vertices from $X \cup Y$.*

Proof. The proof goes exactly as the first case in the proof of Claim 21, illustrated in Fig. 7, except that v_1 now is the vertex from $X' \cup Y'$. $\square$

We ensure that, for each $i \in [\ell']$, both ends of the path P_i are extended at least two steps so that the ends are well-connected again. We add the vertices used in these paths to W, also move the vertices from W_0 to W, and reset $W_0 = \varnothing$. If $|V_0| \geqslant 2s$ we can now continue by covering another $2s$ vertices from V_0 using Claim 21.

We still need to estimate the number of vertices added to W throughout the whole procedure of covering V_0, which were at most $2\beta n$ vertices in the beginning. There are at most $\lceil \frac{2\beta n}{2s\nu n} \rceil \leqslant \frac{\beta}{s\nu} + 1$ iterations of the argument for covering $2s\nu n$ vertices and for covering $2s$ vertices of V_0 we need at most $20s^2$ vertices. During these iterations we will always have

$$|W| \leqslant \frac{20s^2}{2s} 2\beta n + \left(\frac{\beta}{s\nu} + 1\right) \cdot \ell' 2\varepsilon \frac{n}{\ell} \cdot 4s \leqslant 40s\beta n,$$

where the second term comes from the vertices we need to absorb after each iteration. Therefore, we can indeed repeat this until $|V_0| < 2s$.

Covering the last vertices. If $|V_0| = 0$ we are done with this step. Otherwise, we have $0 < |V_0| = t < 2s$ and $n \not\equiv 0 \pmod{2s}$, because the structures we build have size congruent to $0 \pmod{2s}$, and there cannot be the $\frac{r}{2}$-blow-up of a cycle in G. If r is odd $nr \equiv 0 \pmod 2$ implies that n and also t are even. We need to absorb the last t vertices in a different way. If r is even, let $v \in V_0$ and with Fact 20 pick j_1, j_4 such that $X_{j_1}, X_{j_4} \in H(v)$, $Y_{j_1}, Y_{j_4} \in H$, and $X_{j_1}X_{j_4}$ is an edge of R. Then we consider those clusters that share an edge of M with a neighbour of Y_{j_1} and Y_{j_4} respectively, i.e. $N_M(N_R(Y_{j_1}))$ and $N_M(N_R(Y_{j_4}))$. We apply Fact 20 to these sets and get indices j_2, j_3 such that $X_{j_2}X_{j_3}$ is an edge of R. By construction we then have $X_{j_2}, Y_{j_2}, X_{j_3}, Y_{j_3} \in H$ and also that $Y_{j_1}Y_{j_2}$, $Y_{j_4}Y_{j_3}$ are edges of R. We extend the path P_{j_1} as before by following $Y_{j_1}, X_{j_1}, X_{j_4}, Y_{j_4}, Y_{j_3}, X_{j_3}, X_{j_2}, Y_{j_2}, Y_{j_1}$ such that the vertex v is in the neighbourhood of the new vertices from X_{j_1} and X_{j_4}. This allows us to include v into the path. Note that this is no longer the $\frac{r}{2}$-blow-up of a path (see Fig. 10), but instead resembles the absorbing structure from the second extremal case (see Fig. 5). If r is odd let $u, v \in V_0$ (using that t is even) and we proceed similarly to include both vertices (see Fig. 11). Here we find $X_{j_1}, X_{j_4} \in H(v)$ and $X_{j_2}, X_{j_3} \in H(u)$ such that $X_{j_1}X_{j_4}$ and $X_{j_2}X_{j_3}$ are edges of R and then connect Y_{j_1} to Y_{j_2} and Y_{j_4} to Y_{j_3} as in the even case by using four additional clusters for each connection. Note that here the path structure from X_{j_4} to X_{j_3} including u and v also is not an $\frac{r}{2}$-blow-up (alternating $K_{s,s}$ and $K_{s,s}$ minus a perfect matching), but needs to have one edge shifted (alternating $K_{s,s}$ minus an edge and $K_{s,s}$ minus a matching of size $s-1$), because v and u only have $s-1$ neighbours in X_{j_4} and X_{j_3}, respectively. Put differently, the structure that we build is an $\frac{r}{2}$-blow-up of a path from X_{j_4} to X_{j_3} with end-tuples connected to v and u, respectively, but then we remove an edge vx_0 for $x_0 \in X_{j_4}$, add an edge x_0x_1 for $x_1 \in Y_{j_4}$, and so on until we reach u.

Step 6. We fully absorbed V_0 into the connecting paths such that the end-tuples are well-connected to the other side of the matching edge. Let $i \in [\ell']$ and denote by $\mathbf{x} = (x_1, \dots, x_s)$ and $\mathbf{y} = (y_1, \dots, y_s)$ the end-tuples of the paths P_{i-1} and P_i, respectively. Remove $\mathbf{x}$ from X_i and $\mathbf{y}$ from Y_i, note that $|X_i| = |Y_i| \equiv 0 \pmod s$ and that (X_i, Y_i) is $(2\varepsilon, \frac{d}{2})$-super-regular. Denote the common neighbours of $\mathbf{x}$ in Y_i by Y', the common neighbours of $\mathbf{y}$ in X_i by X', and note that $|X'|, |Y'| \geqslant \frac{1}{4}d^s|X|$. Therefore, we can apply Lemma 9 to cover X_i and Y_i with the $\frac{r}{2}$-blow-up of a path and end-tuples within X' and Y', which then connects P_{i-1} to P_i.

Together this gives an r-regular subgraph in G. In the case when $n \equiv 0 \pmod{2s}$ we have constructed the $\frac{r}{2}$-blow-up of a path, which is r-connected. To see that it is also r-connected in the other cases it suffices to observe that in the case when r is even removing a perfect matching from a $K_{s,s}$ and adding a vertex v to all these r vertices (see Fig. 10) preserves this property (as in the second extremal case). Similarly, in the case when r is odd, removing a perfect matching from two copies of $K_{s,s}$ in an $\frac{r}{2}$-blow-up of a path, connecting u and v each to r of these vertices, and shifting the path in between accordingly as described above (also see Fig. 11) still preserves this property. $\square$

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