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Bianchi VIII and IX vacuum cosmologies: Almost every solution forms particle horizons and converges to the Mixmaster attractor

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Abstract Bianchi models are an essential building block, in the BKL picture, towards understanding generic cosmological singularities (due to Belinskii, Khalatnikov, and Lifshitz, [BKL70, BKL82]). We study the behaviour of spatially homogeneous anisotropic vacuum spacetimes of Bianchi type VIII and IX, as they approach the big bang singularity.

It is known since 2001 that typical Bianchi IX spacetimes converge towards the so-called Mixmaster attractor as time goes towards the singularity. We extend this result to the case of Bianchi VIII vacuum.

The BKL picture suggests that particle horizons should form, i.e. spatially separate regions should causally decouple. We prove that this decoupling indeed occurs, for Lebesgue almost every Bianchi VIII and IX vacuum spacetime.

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Zusammenfassung Bianchi Modelle sind, im BKL Bild, ein essentieller Bestandteil für das Verständnis generischer kosmologischer Singularitäten (nach Belinskii, Khalatnikov und Lifshitz, [BKL70, BKL82]).

Wir studieren das Verhalten räumlich homogener anisotroper Vakuum Raumzeiten der Bianchi-Typen VIII und IX nahe der Urknall Singularität.

Seit 2001 ist bekannt, dass generische Bianchi IX Raumzeiten gegen den sogenannten Mixmaster-Attraktor konvergieren, in Zeitrichtung zum Urknall. Wir erweitern dieses Resultat auf den Fall von Bianchi VIII Vakuum.

Das BKL-Bild legt nahe, dass Partikel-Horizonte entstehen sollten, d.h. räumlich getrennte Raumzeit-Regionen kausal entkoppeln. Wir zeigen dass diese Entkopplung tatsächlich stattfindet, für Lebesgue fast alle Bianchi VIII und IX Vakuum Raumzeiten.

Selbstständigkeitserklärung Hiermit versichere ich, Bernhard Brehm, dass ich diese Arbeit selbstständig verfasst habe, alle Hilfen und Hilfsmittel angegeben habe und diese Arbeit nicht in einem früheren Dissertationsverfahren eingereicht habe.

Contents

1 Introduction

Spatially homogeneous cosmological models. The behaviour of cosmological models is governed by the Einstein field equations, coupled with equations describing the presence of matter. Simpler models are obtained under symmetry assumptions. The class of models studied in this work, Bianchi models, assume spatial homogeneity, i.e. "every point looks the same". Then, one only needs to describe the behaviour over time of any single point, and the partial differential Einstein field equations become a system of ordinary differential equations.

Directional isotropy assumes that "every spatial direction looks the same". This leads to the well-known FLRW (Friedmann-Lemaitre-Robertson-Walker) models. These models describe an initial ("big bang") singularity, followed by an expansion of the universe, slowed down by ordinary and dark matter and accelerated by a competing positive cosmological constant ("dark energy").

Bianchi models assume spatial homogeneity, but relax the assumption of directional isotropy. Spatial homogeneity assumes that there is a Lie-group G of spacetime isometries, which foliates the spacetime into three-dimensional space-like hypersurfaces on which G acts transitively: For every two points $\boldsymbol{x}, \boldsymbol{y}$ in the same hypersurface there is a group element $g \in G$ such that $g \cdot \boldsymbol{x} = \boldsymbol{y}$. The resulting ordinary differential equations depend on the Lie-algebra of G, the so-called Killing fields. The three-dimensional Lie-algebras have been classified by Luigi Bianchi in 1898, hence the name "Bianchi-models"; for a commented translation see [Bia01, Jan01], and for a modern treatment see [WE05, Section 1].

The two most studied classes of spatially homogeneous anisotropic cosmological models are the Bianchi-types IX and VIII, which are the focus of this work. Both of these models exhibit a big bang like singularity in at least one time-direction, and a universe that initially expands from this singularity, until, in the case of Bianchi IX, it recollapses into a time-reversed big bang ("big crunch"). The big bang singularity is present even in the vacuum case, where matter is absent and only gravity self-interacts. According to conventional wisdom, "matter does not matter" near the singularity. For this reason, we simplify our analysis by considering only the vacuum case.

Note that the symmetry assumptions already restrict the global topology of the space-like hypersurfaces, and that the isotropic FLRW-models are not contained as a special case: The only homogeneous isotropic vacuum model is flat Minkowski space.

For a detailed introduction to Bianchi models, we refer to [WE05]. A short derivation of the governing ordinary differential Wainwright-Hsu equations (2.3.2) is given in Appendix A.5, and physical interpretations of some of our results are given in Section 2.2. For an excellent survey on Bianchi cosmologies, we refer to [HU09a], and for further physical questions we refer to [UVEWE03, HUR$^+$09].

The Wainwright-Hsu equations The dynamical behaviour of Bianchi VIII and IX spacetimes is governed by the Wainwright-Hsu equations. This system of four ordinary differential equations, which we will just call Bianchi system, describes the dynamics of anisotropic homogeneous vacuum spacetimes of several Bianchi types, where each Bianchi type corresponds to an invariant subset; see Table 1.

The Wainwright-Hsu equations employ a time-dependent rescaling, called Hubble-normalization, that eliminates one degree of freedom. For this reason, equilibria of the Bianchi system do not correspond to static (time-independent) spacetimes, but instead to spacetimes that are self-similar expanding towards the singularity, i.e. relative equilibria under spatial rescaling.

The Bianchi system is equivariant under a group action by $\mathbb{Z}_2 \times S_3$; this corresponds to a simultaneous sign-reversal and permutations of the three Killing fields. The Lie-algebra, i.e. the commutators of the three Killing fields are given by $[e_i, e_j] = \hat{n}_k \epsilon_{ijk} e_k$, where $\hat{n}_i \in \{+1, -1, 0\}$; its Bianchi type does not change if we simultaneously reverse the signs of all three $\hat{n}_i$, or permute them. The decomposition of the phase-space of the Bianchi system according to Bianchi type (and connected components) can be viewed as a cell-decomposition. This viewpoint is used extensively in [HU09b], under the name "Lie contraction hierarchy".

The equivariance gives us an invariant set $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, where $\mathcal{T}_i$ corresponds to the fixed points under the index permutation that exchanges the other two indices. This invariant set is called Taub-space, and has dimension and codimension two. Solutions in this set are also called Taub spacetimes, or LRS (locally rotationally symmetric) spacetimes. The latter name is descriptive, in the sense that these spacetimes have additional (partial, local) isotropy. Taub spacetimes behave, in several ways, different from general (i.e. non-Taub) Bianchi spacetimes. For a detailed description, we refer to [Rin01, HU09b, HU09a].

It is well-known and relatively easy to prove that solution of Bianchi type $VIII$ and IX converge to the sets describing lower Bianchi-types. It is noteworthy that $\overline{M_{\mathrm{IX}}} \cap M_{\mathrm{VI}_0} = \emptyset$, i.e. the boundary of the region of phase-space describing Bianchi IX does not intersect the region of phase-space describing Bianchi VI_0. For this reason, Bianchi IX is considered simpler than Bianchi VIII.

The Mixmaster attractor. The invariant set $\mathcal{A}$, consisting of Bianchi types I and II, is called the Mixmaster attractor. There are good heuristic arguments that $\mathcal{A}$ really is an attractor for time approaching the singularity, and that the dynamics on and near $\mathcal{A}$ can be considered chaotic (sometimes also called "oscillatory").

Let us give a short description of the Mixmaster attractor. The Mixmaster attractor consists of three two-dimensional spheres that intersect in a circle of equilibria, called the Kasner-circle $\mathcal{K}$. The Kasner-circle is the set of Bianchi I solutions; the corresponding spacetimes are called Kasner-solutions, and describe a universe that evolves via time-dependent rescaling. In one time-direction, a singularity is reached in finite eigen-time; towards the singularity, spatial volumes and two spatial directions shrink, while the third spatial direction expands.

The remaining part of the Mixmaster attractor, i.e. the remaining six hemispheres, consist of heteroclinic orbits, i.e. of solutions that converge to one point on the Kasner-circle in one time-direction, and a different point on the Kasner-circle in the other time-

Bianchi Type	$\hat{n}_1$	$\hat{n}_2$	$\hat{n}_3$	$\dim M$	$\#M$	$\dim M\cap\mathcal{T}$	$\#M\cap\mathcal{T}$	Group
I	0	0	0	1	1	0	6	$\mathbb{R}^3$
II	$+$	0	0	2	3	1	3	$H_3(\mathbb{R})$
VI$_0$	0	$+$	$-$	3	6	$\emptyset$	0	$E(1,1)$
VII$_0$	0	$+$	$+$	3	6	1	6	$E(2)$
VIII	$-$	$+$	$+$	4	6	2	1	$SL(2,\mathbb{R})$
IX	$+$	$+$	$+$	4	2	2	3	$SO(3,\mathbb{R})$.

Table 1: Bianchi types described by the Wainwright-Hsu equations. Column 2 gives a representative of the structure constants, modulo permutations and simultaneous sign-reversal. Column 3 gives the dimension of the set in phase-space corresponding to the Bianchi type, and column 4 counts its number of connected components. Column 5 gives the dimension of the intersection of this set with the Taub-spaces, and column 6 counts the number of connected components of this intersection. The last column gives a Lie-group corresponding to the Lie-algebra of this Bianchi type. Here, $H_3(\mathbb{R})$ is the three-dimensional Heisenberg group, $E(1,1)$ is the group of affine isometries of $1+1$ dimensional Minkowski-space, and $E(2)$ is the group of affine isometries of the Euclidean plane.

direction.

Except for three special points, called Taub-points $\mathbf{T}_i$, out of the six points in $\mathcal{K}\cap\mathcal{T}$, all other equilibria in $\mathcal{K}$ are normally hyperbolic, i.e. the linearization of the Bianchi system has one kernel direction, corresponding to the Kasner-circle, and one unstable and two stable eigenvalues (with respect to time going towards the singularity). All four eigenvectors are tangential to $\mathcal{A}$ (this is possible because $\mathcal{A}$ is not an embedded manifold; it can be viewed as a smooth compact immersed two-dimensional manifold, with self-intersections).

This gives rise to the following heuristic description of solutions near $\mathcal{A}$: Solutions will stay near one equilibrium $\boldsymbol{p}_n$ for a long time; then, they follow the unstable direction, i.e. the heteroclinic orbit in $\mathcal{A}$ emanating from $\boldsymbol{p}_n$, until they are near an equilibrium $\boldsymbol{p}_{n+1}$; then, this process continues. It is possible to describe this behaviour by a map, $\boldsymbol{p}_{n+1}=K(\boldsymbol{p}_n)$, called the Kasner-map $K:\mathcal{K}\to\mathcal{K}$. It is an orientation reversing double cover of S^1, and it is C^0-conjugate to $[z]_{\mathbb{Z}}\to[-2z]_{\mathbb{Z}}$. This is the source of the heuristic chaoticity (since double covers of S^1 are chaotic, when viewed as time-discrete dynamical systems). For a graphical description, see Figure 1.

From the Wainwright-Hsu equations, one can see by direct calculation that $\mathcal{A}$ is linearly attracting, away from the three Taub-points $\mathbf{T}_i$ (we verify this in Section 3). This is the heuristic reason for expecting $\mathcal{A}$ to be an attractor.

The only difficulty are the Taub-points, where two eigenvalues pass through zero, giving a three-dimensional center-manifold. This prevents one from directly turning this heuristic into a proof. This heuristic description is known since at least [Mis69, BKL70].

For a modern description, we refer to Section 3, or [HU09a, HU09b].

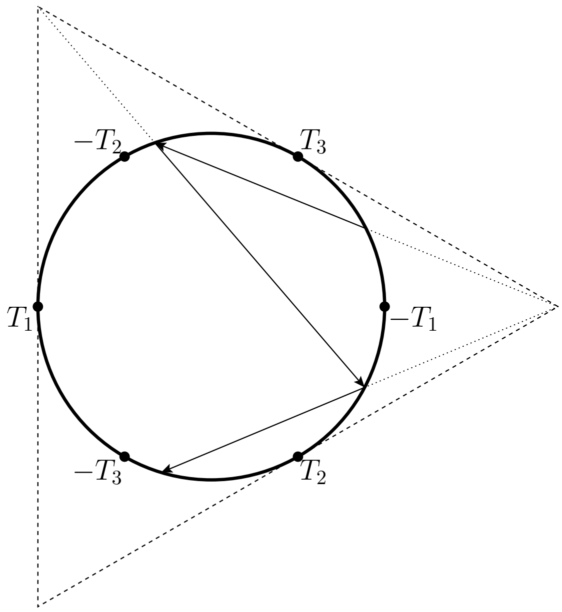


Figure 1: The Kasner-circle, and a short heteroclinic chain on $\mathcal{A}$, in a suitable projection. The dashed and dotted lines are not part of the attractor, but hint at the geometric construction underlying the Kasner-map. The six points in $\mathcal{K} \cap \mathcal{T}$ are marked, such that $\mathcal{T}_i \cap \mathcal{K} = \{+\mathbf{T}_i, -\mathbf{T}_i\}$; hyperbolicity fails at the three Taub-points $\mathbf{T}_i$. The arrows are oriented for time going towards the singularity.

Attractor Theorems: Previous and novel results Even though these heuristics have been known for a long time, a rigorous proof that $\mathcal{A}$ is actually attracting is comparatively recent. In the case of Bianchi IX, this was originally proven by Ringström in [Rin01]; a shorter proof is by [HU09b], and another new proof is provided in this work:

Theorem (Bianchi IX attractor theorem)**.** *In Bianchi* IX *vacuum, all non-Taub solutions converge to* $\mathcal{A}$*, as time goes towards the singularity.*

These solutions have at least one non-Taub ω*-limit point on the Kasner-circle, as time goes towards the singularity. The set of Taub-solutions is an embedded submanifold of codimension two; hence, this theorem applies for generic solutions, both in the sense of Lebesgue and Baire.*

The first main result of our work shows that $\mathcal{A}$ is also an attractor for Bianchi VIII solutions. A comparable result was previously unknown.

Theorem (Bianchi VIII attractor theorem; summarized from Theorems 2, 4 and 5)**.** *In Bianchi* VIII *vacuum, all solutions fall into one of the following three classes:*

Attract. *The solution converges to* $\mathcal{A}$*, and has at least one non-Taub* ω*-limit point on the Kasner-circle, as time goes towards the singularity.*

Taub. *The solution is contained in a Taub-space.*

Except. *The solution has exactly two ω-limit points on the Kasner-circle, both of which lie in the same Taub-space. All other ω-limit points lie on heteroclinic orbits connecting these two points, and are of Bianchi type* II *or* VI_0*.*

The set where Attract *applies is generic, both in the sense of Baire (it is open and dense) and Lebesgue (it has full Lebesgue measure). It is an open neighbourhood of $\mathcal{A} \setminus \mathcal{T}$.*

The above formulation is a synthesis of Theorems 2, 4, and 5, which are proven in Sections 6 and 7.

We make no claim on whether the case Except is possible at all; we believe it is possible, but are unable to prove this. This question is further discussed in Section 6.

This result is made possible by new estimates near the Taub-spaces, that are derived in Section 5. These estimates allow us to describe Bianchi VIII and IX solutions in a unified framework; hence, we also give a new proof of the Bianchi IX attractor theorem. Along the way, we show some useful estimates about the convergence, that are collected in Theorem 2. These are of independent interest, even if one is only interested in Bianchi IX solutions.

Stable Foliations. One may ask for a more precise description of how solutions get attracted to $\mathcal{A}$. That is, one may ask for the relation between solutions to the Bianchi system that converge to $\mathcal{A}$, and heteroclinic chains in $\mathcal{A}$, i.e. orbits of the Kasner map. Naively, one might hope for a foliation over $\mathcal{K}$ into invariant hypersurfaces of codimension one, that each contain the solutions following a specific heteroclinic chain.

Results in this area are [Bég10, LHWG11, RT10, LRT12, Buc13]; for an excellent, though outdated on this question, survey we recommend [HU09a]. Let us grossly paraphrase some of these results:

Theorem [Bég10, LHWG11, LRT12, Buc13] (grossly paraphrased)**.** *Heteroclinic chains in $\mathcal{A}$ that stay bounded away from the Taub-points attract stable manifolds of codimension one.*

[LHWG11, LRT12] use estimates "by hand", and really apply to all heteroclinic chains that stay bounded away from the Taub-points, and the resulting stable manifolds are of Lipschitz regularity. [Bég10, Buc13] use Takens-linearization and abstract theory of hyperbolic systems; they provide more regularity of the stable manifolds, but are only applicable to a subset of heteroclinic orbits, that fulfill certain non-resonance condition in addition to staying bounded away from the Taub-points. Regardless, the set of heteroclinic chains that stay bounded away from the Taub-points is dense but non-generic, in the sense that it is both meagre[4] and has Lebesgue-measure zero.

[4] A set is called generic in the sense of Baire, if it is *co-meagre*, i.e. it contains a countable intersection of open and dense sets. Then its complement is called *meagre*. By construction, countable intersections of co-meagre sets are co-meagre and countable unions of meagre sets are meagre. Baire's Theorem states that co-meagre subsets of complete metric spaces (or locally compact Hausforff spaces) are always dense and especially nonempty.

Reiterer and Trubowitz [RT10] claim related results, for a much wider class of heteroclinic chains that have full (one-dimensional) Lebesgue measure, but with less focus on the regularity or codimension of the attracted sets.

This general class of constructions and questions, i.e. partial stable foliations over specific hetroclinic chains in $\mathcal{A}$, is not the focus of this work. Instead, we describe and estimate solutions and the flow directly, without explicitly focussing on the Kasner-map and heteroclinic chains in $\mathcal{A}$.

The question of particle horizons. One of the most salient features of relativity is *causality*: The state of the world at some point in spacetime is only affected by states in its past light-cone and can only affect states in its future light-cone.

The past light-cone $J^-(\boldsymbol{p})$ of a spacetime point $\boldsymbol{p} \in M$, sometimes also called the causal past of $\boldsymbol{p}$, is the set of points $\boldsymbol{q}$ that are reachable from $\boldsymbol{p}$ by a time-like past-directed curve γ, with $g(\dot\gamma, \dot\gamma) \le 0$. The future light-cone $J^+(\boldsymbol{p})$ is defined analogously. Two points $\boldsymbol{p}, \boldsymbol{q} \in M$ are said to *causally decouple* towards the past if their past light-cones are disjoint, $J^-(\boldsymbol{p}) \cap J^-(\boldsymbol{q}) = \emptyset$, i.e. if there is no past event which causally influences both $\boldsymbol{p}$ and $\boldsymbol{q}$. Consider the communicating region of $\boldsymbol{p} \in M$, i.e. the set $J^+(J^-(\boldsymbol{p}))$, i.e. the set of points $\boldsymbol{q}$ that do not decouple from $\boldsymbol{p}$. The boundary of this set, $\partial J^+(J^-(\boldsymbol{p}))$, is the cosmic horizon of $\boldsymbol{p}$, sometimes also called particle horizon; hence, everything beyond the horizon is causally decoupled.

In Figure 2, we illustrate the formation of horizons. An example where no horizons form is given by the (flat, connected) Minkowski-space $M = \mathbb{R} \times \mathbb{R}^3$: There is no singularity, and $J^+(J^-(\boldsymbol{p})) = M$ for all $\boldsymbol{p}$, and hence $\partial J^+(J^-(\boldsymbol{p})) = \emptyset$. An example where, at least formally, past horizons form, is given by the (flat, connected) Minkowski-space $M = (0, \infty) \times \mathbb{R}^3$ (where we simply removed half of the spacetime). This example demonstrates that we should only ask about horizons for inextendible spacetimes.

Apart from the question of convergence to $\mathcal{A}$, the next important physical question in the context of Bianchi cosmologies is that of locality of light-cones:

1. Do nonempty horizons $\partial J^+(J^-(\boldsymbol{p})) \neq \emptyset$ form towards the (past) singularity? Does this happen, at least, if $\boldsymbol{p}$ is sufficiently near to the singularity?

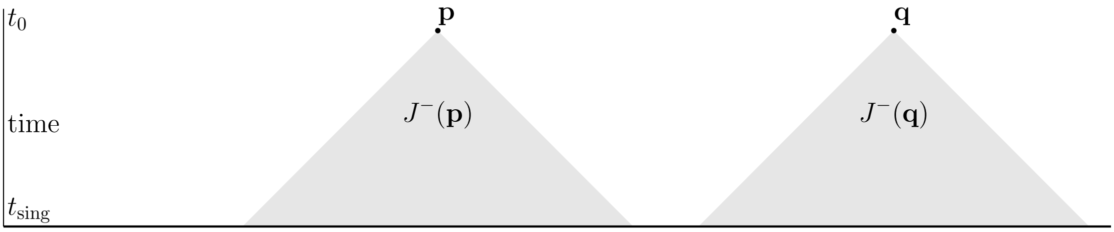
2. Are the spatial hypersurfaces $\{t = t_0 = \text{const}\} \cap (J^+(J^-(\boldsymbol{p})), \partial J^+(J^-(\boldsymbol{p}))$, considered as three-dimensional manifolds with boundary, diffeomorphic to the three dimensional unit ball $[B^3, \partial B^3]$?

3. Are the past light-cones $J^-(\boldsymbol{p})$, and the communicating regions $J^+(J^-(\boldsymbol{p}))$ spatially bounded? Do they shrink down to a point, as $\boldsymbol{p}$ and t_0 approach the singularity?

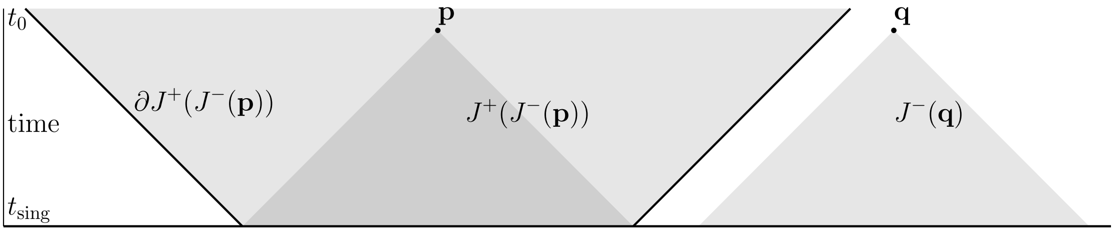
The first question is formulated completely independently of the foliation of the spacetime into space-like hypersurfaces. The second question depends on the foliation, but is at least easy to clearly state. The third question is not clearly stated here, because it compares the spatial extents of the light-cones and communicating regions for different times. This subtlety is discussed in Section 2.2.

We say that a solution forms a particle horizon if all these questions are answered with “yes”; we say that particle horizons definitively fail to form if $J^+(J^-(\boldsymbol{p})) = M$ for all $\boldsymbol{p} \in M$; and otherwise, we say that particle horizons partially fail to form. The latter should be viewed as an umbrella term for “it’s complicated”. All sensible definitions agree on the definite cases; for the complicated case, the answer depends on the precise definition that the author in question decides to use.

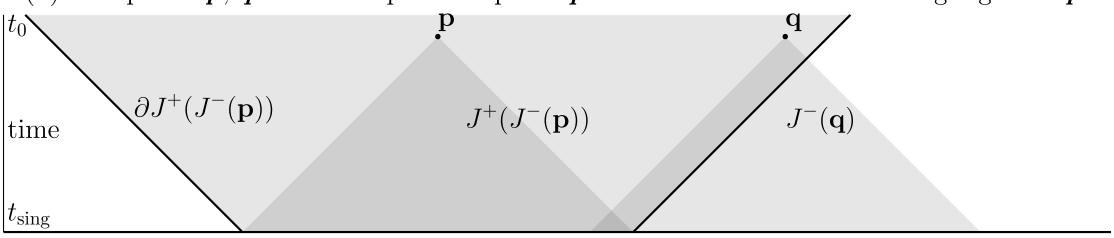
This work is only concerned with showing that (some, almost all) solutions definitely form particle horizons; for this reason, it is unnecessary for us to discuss the subtleties of the complicated case.



(a) Two points $\boldsymbol{p}$, $\boldsymbol{q}$ that decouple towards the singularity. Their past light-cones are disjoint.



(b) Two points $\boldsymbol{p}$, $\boldsymbol{q}$ that decouple. The point $\boldsymbol{q}$ lies outside the communicating region of $\boldsymbol{p}$.



(c) Two points $\boldsymbol{p}$, $\boldsymbol{q}$ that do not decouple towards the singularity. Their past light-cones have nonempty intersection (the darkest shaded region). The point $\boldsymbol{q}$ lies inside the communicating region of $\boldsymbol{p}$.

Figure 2: Examples of decoupling and non-decoupling towards the singularity at $t = t_{\text{sing}}$, and of horizons.

Originally, Misner [Mis69] suggested that typically no horizons should form in Bianchi IX. This was proposed as a possible explanation of the observed approximate homogeneity of the universe: If the homogeneity is due to past mixing, then different observed points in our current past light-cone must themselves have a shared causal past.

The basis for this belief is the following: It is known that Taub-solutions fail to form particle horizons (at least partially); and due to the chaoticity of the Kasner-map, it is expected that typical solutions will come arbitrarily close to Taub-solutions, as time goes towards the singularity.

Misner later changed his mind to the current consensus intuition that typical Bianchi VIII and IX solutions should form particle horizons. Some more details on this are given in Section 2.2. Further discussion of these questions can be found in e.g. [Wal84, Chapter 5], [HE73, Chapter 5].

The BKL picture. Spatially homogeneous spacetimes, and especially the question of particle horizons, play an essential role in the so-called BKL picture (often also called BKL-conjecture). The BKL picture is due to Belinskii, Khalatnikov and Lifshitz ([BKL70]), and describes generic cosmological singularities in terms of homogeneous spacetimes. This picture roughly claims the following:

1. Generic cosmological singularities "are curvature-dominated", i.e. behave like the vacuum case. More succinctly, "matter does not matter".
2. Generic cosmological singularities are chaotic and "oscillatory", which means that the directions that get stretched or compressed switch over time.
3. Generic cosmological singularities "locally behave like" spatially homogeneous ones[5]. By this, one means that:
    (a) Different regions causally decouple towards the singularity, i.e. particle horizons form.
    (b) If one restricts attention to a single communicating region, then, as time goes towards the singularity, the spacetime can be well approximated by a homogeneous one.
    (c) Different spatial regions may have different geometry towards the singularity (since they decouple). This kind of behaviour has been described as "foam-like".

Boundedness of the communicating regions, i.e. formation of particle horizons, in spatially homogeneous models is a necessary condition for the consistency of the BKL-picture: (3.a) claims that different spatial regions of an inhomogeneous spacetime M causally decouple towards the big bang, and (3.b) claims that such decoupled regions behave "like they were homogeneous", i.e. like a homogeneous spacetime $\tilde{M}$. Then, $\tilde{M}$ must itself form horizons, in order to not contradict (3.a) and (3.b).

Previous results on particle horizons. One way of viewing the formation of particle horizons is as a race between the collapse of space and the end of time, towards the singularity. If the curvature blow-up is sufficiently faster than the spatial collapse (expansion

[5]Even more; they are supposed to behave like solutions of Bianchi type VIII, IX or $\mathrm{VI}^*_{-\frac{1}{9}}$

in physical time, where the singularity is in the past), then particle horizons form, because there is not enough time for communication between far points. Otherwise, no horizons form.

In the context of the Wainwright-Hsu equations, the question boils down to: *Do solutions converge to $\mathcal{A}$ sufficiently fast? If yes, then particle horizons form. If no, then the questions of particle horizons may have subtle answers.*

The aforementioned solutions constructed in [LHWG11, Bég10], with initial conditions on certain hypersurfaces of codimension one, all converge essentially uniformly exponentially to $\mathcal{A}$. This is definitely fast enough for particle horizons to form.

Reiterer and Trubowitz claim in [RT10] that the solutions constructed therein also converge to $\mathcal{A}$ fast enough for this to happen. The claimed results in [RT10] are somewhat nontrivial to parse. In short, [RT10] construct solutions which converge rapidly to certain parts of the Mixmaster attractor $\mathcal{A}$. These parts of the Mixmaster attractor have full (one-dimensional) volume, and all these constructed solutions form particle horizons. Claims about the extent, in phase-space, of these constructed solutions like e.g. Hausdorff-dimension or Baire-category, are not made in [RT10].

The solutions constructed in [LHWG11, Bég10] were the first known nontrivial solutions that could be proven to form particle horizons. It is still unknown whether there exist nontrivial counterexamples, i.e. non-Taub solutions that fail to form particle horizons.

Novel results on particle horizons. The most important result of our work is the following:

Theorem 6 (Almost sure formation of particle horizons)**.** *Lebesgue almost every solution in Bianchi* VIII *and* IX *vacuum forms particle horizons towards the big bang singularity. These shrink to a point.*

In fact, we show even stronger bounds than $\mathrm{diam} J^+(J^-(\boldsymbol{p})) < \infty$ in Theorem 7; however, these lack a direct physical interpretation.

The restriction to Lebesgue almost every solution in genuine, i.e. the methods of this work cannot point out any specific solution that forms a particle horizon. The corresponding question for Baire-generic solutions remains open (see Question 7.1).

We believe that Baire-generic solutions should fail to form particle horizons, in the strongest possible sense $J^+(J^-(\boldsymbol{p})) = M$. Our reasons for making this conjecture will be explained in future work.

On the other hand, from a purely physical standpoint, measure theoretic genericity tends to trump topological genericity; this is because we want statistical physics, and especially the second law of thermodynamics to be applicable.

These measure-theoretic estimates are based on an unbounded volume-form ω_4 on Bianchi phase-space; this volume-form expands towards the singularity, with the remarkably short equation[6] $\mathrm{D}_t\omega_4 = 2N^2\omega_4$. This formula might be considered as the third

[6]The term N^2 can be physically interpreted as the Hubble-normalized scalar curvature of the space-

main result of this work, and is derived and used in Section 7. To the best of our knowledge, and to our surprise, this equation $\mathrm{D}_t\omega_4 = 2N^2\omega_4$ appears to be a novel discovery, even though it can be verified by direct calculation.

Now, for a volume-expanding flow, every forward invariant set must either have infinite or vanishing volume; this allows us to show various inequalities for a.e. solution and sufficiently late times towards the singularity. Combined with the estimates near the Taub-spaces (shown in Section 5), that lead to Theorem 2, these can be used to prove Theorem 6.

Structure of this work and strategy. Let us summarize the structure and contents of the various sections of this work, as well as the strategy leading to the main results. This summary omits some minor subtleties that are addressed in the main text; as such, some of the statements are not necessarily literally true, without the qualifications in the main text.

Section 2. We present the Wainwright-Hsu equations, i.e. the Bianchi system, as well as some transformations that will be needed later on. The most important equations are also summarized in Appendix A.1, for easier reference. The relation between solutions to the Wainwright-Hsu equations and actual homogeneous spacetimes is explained in Section 2.1, and physical properties of these spacetimes are related to dynamical properties of solutions in Section 2.2. A derivation of the Wainwright-Hsu equations from the Einstein field equations of general relativity, for vacuum solutions, is given in Appendix A.5.

The version of the Wainwright-Hsu equations for vacuum used in this work differs slightly from the versions used in [Rin01, HU09a, HU09b, LHWG11, Bég10]. Firstly, we rescaled, in order to set various numerical constants to one; this change is purely cosmetic. Secondly, and most substantially, we reverted the direction of time, such that the singularity lies in the future, at $t \to +\infty$; this change is purely cosmetic, but very convenient for dynamical systems language.

The third change is most important, but entirely insubstantial: The Bianchi-system lives on a four-manifold; it is possible, customary and convenient to smoothly embed this four-manifold as a hypersurface $\mathcal{M} \subset \mathbb{R}^5$, and extend the vectorfield to all of $\mathbb{R}^5$. Then, $\mathcal{M}$ is of course invariant under the extended vectorfield, and only the behaviour on $\mathcal{M}$ matters for the study of homogeneous vacuum spacetimes.

We use the same embedding $\mathcal{M} \subset \mathbb{R}^5$, but choose a different extension of the vectorfield to $\mathbb{R}^5$. The most customary extension, used in [Rin01, HU09a, HU09b, LHWG11, Bég10], has the physical interpretation of describing homogeneous spacetimes with perfect fluid matter; thus, one can discuss, essentially for free, certain classes of non-vacuum spacetimes within the same phase-space and using the same equations (the equations of state of the perfect fluid are a parameter of the extension of the vectorfield). This work only considers vacuum spacetimes, and extends the vectorfield in a way that is more convenient for measure-theoretic considerations, especially for proving $\mathrm{D}_t\omega_4 = 2N^2\omega_4$.

like hypersurfaces of homogeneity. The volume-form ω_4 on phase-space should have a physical interpretation in terms of symplectic volume; unfortunately, we are unable to give such an interpretation.

Even though this extension was chosen for computational convenience only, it was pointed out that this extension has a physical interpretation in the context of Horava-Lifshitz modified gravity [HU].

Let us point out that particle horizons form for a solution $\boldsymbol{x}(t)$ if $\int_0^\infty \delta(t)\mathrm{d}t < \infty$, where the quantity $\delta(\boldsymbol{x}) \sim d(\boldsymbol{x}, \mathcal{A})$ roughly corresponds to the distance to the Mixmaster attractor, and the singularity is placed at $t \to \infty$. This fact has been known virtually forever, c.f. e.g. [Mis69, HR09] and the references therein.

Section 3. Now armed with actual equations to discuss, we give another overview of the dynamical behaviour. This will discuss the Mixmaster attractor $\mathcal{A}$, the Kasner map, the dynamics far from $\mathcal{A}$, and give an overview of the proof of the Bianchi IX attractor theorem in [Rin01]. The proofs and structure of this section are quite similar to [Rin01]; the main difference that we extensively use the more quantitative Lemma 3.6 instead of [Rin01, Lemma 5.2], which simplifies many arguments.

Section 4. We give a rigorous version of the heuristic arguments that solutions near $\mathcal{A}$ converge exponentially to $\mathcal{A}$, as long as they stay bounded away from the Taub-points. We could not find any previous work giving such a rigorous treatment of these arguments; nevertheless, the contents of this section should be considered as known for more than 30 years.

Section 5. We discuss the behaviour near the Taub-spaces $\mathcal{T}$. Our discussion centers around the quotient $\frac{\delta}{r}$, and its differential equations, (2.4.2d) for Bianchi IX, and (2.4.7d) for Bianchi VIII, where $r(\boldsymbol{x}) \sim d(\boldsymbol{x}, \mathcal{T})$ roughly corresponds to the distance to the Taub-spaces. Similar quotients in Bianchi IX also play a prominent role, as Z_{-1} in [Rin01], and as ζ in [HU09b].

The main novel insight in this section is that we can show a-priori estimates on $\frac{\delta}{r}$ for solutions that approach $\mathcal{T}$ from other parts of phase-space. Using these a-priori bounds, we can show much stronger and easier averaging estimates, that hold in both Bianchi VIII and IX. Previously, [Rin01] managed, in a heroic effort, to show averaging estimates in Bianchi IX, independently of such a-priori bounds; in Bianchi VIII, such estimates are impossible without a-priori bounds on $\frac{\delta}{r}$. For the sake of completeness, we repeat this effort in Lemma 5.6, but note that this lemma is never used in this work, outside of the literature review.

These averaging estimates, Proposition 5.3, prove an average exponential convergence to $\mathcal{A}$, with non-uniform rate $\mathrm{D}_t \log \delta(t) \sim -r^2(t)$.

Section 6. We synthesize the results of the previous sections to prove that $\mathcal{A}$ is an attractor. Combining the estimates from Section 4 and 5, we prove a local attractor theorem 2, that is entirely novel in the case of Bianchi VIII, and in the case of Bianchi IX adds the more detailed estimates $\frac{\delta}{r} \to 0$ and $\int_0^\infty \delta^2(t)\mathrm{d}t < \infty$. Unfortunately, this integral estimate is insufficient to show that particle horizons form.

Combining this with the results from Section 3, we obtain a new proof of the global Bianchi IX attractor theorem 1 and 3, as well as the entirely novel global Bianchi VIII attractor theorem 4, minus the genericity claims. The hypothetical exceptions in the Bianchi VIII attractor theorem must have $\frac{\delta}{r}(t) > \epsilon > 0$ for all sufficiently large times, and some fixed ϵ.

Section 7. We introduce the unbounded volume-form ω_4, which roughly corresponds to the Lebesgue-measure in logarithmic coordinates, and show the remarkably simple equation $\mathrm{D}_t\omega_4 = 2N^2\omega_4$. This equation shows that the flow of the Bianchi system is volume-expanding with respect to ω_4.

Forward invariant sets under volume expanding dynamical systems must either have vanishing or infinite volume. One can show, by direct calculation, that the set $\textsc{Bad} = \{\delta > r^4\}$ has finite ω_4-volume; using the volume expansion, we then show that ω_4-almost every solution $\boldsymbol{x}(t)$ must have $\delta(t) < r^4(t)$ for all sufficiently late times. Since ω_4 and the usual Lebesgue measure have the same zero-sets, this estimate holds for Lebesgue a.e. solution.

This immediately implies the genericity claims, Theorem 5, of the global Bianchi VIII attractor theorem. Upon integrating the averaging estimates from Section 5, the inequality $\delta < r^4$ yields $\int_0^\infty \delta(t)\mathrm{d}t < \infty$ for almost every solution, i.e. the a.e. formation of particle horizons, which is the main result of this work, Theorem 6.

A more detailed look allows us to also estimate certain phase-space averages of the diameter of communicating regions; these estimates are given in Theorem 7.

2 Setting, Notation and the Wainwright-Hsu equations

The subject of this work, i.e. the behaviour of homogeneous anisotropic vacuum spacetimes with Bianchi Class A homogeneity under the Einstein field equations of general relativity, can be described by a system of ordinary differential equations, called the Wainwright-Hsu equations (2.3.4).

In Section 2.1, we will introduce the Wainwright-Hsu ordinary differential equations and various auxiliary quantities and definitions, and provide a rough summary of their dynamics. Then we transform the Wainwright-Hsu equations into polar coordinates in Section 2.4, which are essential for the analysis in Section 5.

There are multiple equivalent formulations of the Wainwright-Hsu equations in use by different authors, which differ in sign and scaling conventions, most importantly the direction of time. This work uses reversed time, such that the big bang singularity is at $t = +\infty$. The relation of the Wainwright-Hsu equations to the Einstein equations of general relativity will be relegated to Section A.5. The relation between properties of solutions to the Wainwright-Hsu equations and physical properties of the corresponding spacetimes is discussed in Section A.5.

General Notations. In this work, we will often use the notation $\boldsymbol{x} = (x_1, \ldots, x_n)$ in order to emphasize that a variable $\boldsymbol{x}$ refers to a point and not to a scalar quantity. If we consider a curve $\boldsymbol{x}(t)$ into a space where different coordinates have names, e.g. $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^5 = \{(\Sigma_+, \Sigma_-, N_1, N_2, N_3)\}$, then we will in an abuse of notation write $N_1(t) = N_1(\boldsymbol{x}(t))$ in order to refer to the N_1-coordinate of $\boldsymbol{x}(t)$.

We use $\pm$ to refer to either $+1$ or -1, and different occurrences of $\pm$ are always unrelated, such that e.g. $(\pm, \pm, \pm) \in \{(+,+,+), (-,+,+), (+,-,+), (-,-,+), \ldots\}$. We will use $*$ to refer to either $+1$,-1 or 0, also such that different occurrences of $*$ are unrelated.

We use $C > 0$ to refer to large unspecified constants and $0 < c$ to refer to small unspecified constants. These are "PDE-style" constants, that should be interpreted as $\mathcal{O}(1)$; so statements like $C + 1 \leq C$ are true.

If the constants are later recalled, or we want to specify their dependencies, then we number them within the section of their definition, like e.g. $C_{2.1}$, or $c_{2.2}$. Numbered constants are real numbers; so statements like $C_{2.1} + 1 \leq C_{2.1}$ are false.

We will often encounter differential equations and inequalities like e.g. $\mathrm{D}_t x(t) = f(t)x(t)$ with $f(t) \leq h(t)$. In an abuse of notation, we will often write this as $\frac{x'}{x} = \mathrm{D}_t \log|x|(t) < h(t)$, regardless of whether we might have $x(t) = 0$ for some times t. Most of the time, we will integrate this anyways to $|x|(t_2) \leq \exp(\int_{t_1}^{t_2} h(s)\mathrm{d}s)\,|x|(t_1)$ for $t_2 \geq t_1$, which is unquestionably well-defined and true.

This prompts the following general convention: Equalities and inequalities involving time-derivatives of logarithms, like $\frac{x'}{x} = \mathrm{D}_t \log|x| \leq h$ should be interpreted as $\mathrm{D}_t x = f\,x$ and $f \leq h$. Other equalities and inequalities involving formally meaningless expressions are to be ignored, if safely possible.

As we will see in Section , backwards existence of solutions might fail in the Bianchi system. Following the general convention of ignoring formally meaningless expressions if safely possible, we define $\phi(M, -t) = \phi(\cdot, t)^{-1}[M] = \{\boldsymbol{x} : \phi(\boldsymbol{x}, t) \in M\}$, where ϕ denotes the flow to the Bianchi system and M is a subset of phase-space. This is explained in more verbosity in Appendix A.3.

Integrals with boundary are oriented, such that $\int_a^b f(t)\mathrm{d}t = -\int_b^a f(t)\mathrm{d}t$; integrals over sets and integrals of forms over manifolds are unoriented. For sets and manifolds, we only need to integrate non-negative quantities; hence, we consider infinite integrals as well-formed expressions, and $\int_M f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ is well-defined if $f : M \to [0, \infty]$ is measurable and a.e. defined. We adopt the convention that $0 \cdot \infty = 0$ (in order to restrict domains of integration by multiplication with indicator functions), and consider $\frac{1}{\infty}$ as undefined.

The end of nested proofs, or especially relevant sub-proofs, is signified by the symbol $\blacksquare$ instead of $\square$, in order to avoid confusion about whether the proof is concluded or not.

2.1 Spatially Homogeneous Spacetimes

We study the behaviour of homogeneous spacetimes, also called Bianchi-models. These are Lorentz four-manifolds, foliated by space-like hypersurfaces on which a group of isometries acts transitively, subject to the vacuum Einstein Field equations. That is, we assume that we have a frame of four linearly independent vectorfields $e_0 = \partial_t, e_1, e_2, e_3$, where e_1, e_2, e_3 are Killing fields, with dual co-frame $\mathrm{d}t, \omega_1, \omega_2, \omega_3$, such that the metric has the form

$$g = g_{00}(t)\mathrm{d}t \otimes \mathrm{d}t + g_{11}(t)\omega_1 \otimes \omega_1 + g_{22}(t)\omega_2 \otimes \omega_2 + g_{33}(t)\omega_3 \otimes \omega_3,$$

and the commutators (i.e. the Lie-algebra of the spatial homogeneity) has the form

$$[e_i, e_j] = \sum_k \gamma_{ij}^k e_k \qquad \gamma_{ij}^k = \hat{n}_k \epsilon_{ijk},$$

where ϵ_{ijk} is the usual Levi-Civita symbol ($\epsilon_{ijk} = +1$ if $(ijk) \in \{(123), (231), (312)\}$, $\epsilon_{ijk} = -1$ if $(ijk) \in \{(132), (321), (213)\}$ and $\epsilon_{ijk} = 0$ otherwise). The signs $\hat{n}_i \in \{+1, -1, 0\}$ determine the Bianchi Type of the cosmological model, according to Table 1, page 8.

The metric is described by the seven Hubble-normalized variables $H, \widetilde{N}_i, \Sigma_i$, with $i \in \{1, 2, 3\}$, according to

$$g_{00} = -\frac{1}{4}H^{-2} \qquad g_{ii} = \frac{1}{48}\frac{H^{-2}}{\widetilde{N}_j \widetilde{N}_k}, \tag{2.1.1}$$

where (i, j, k) is always assumed to be a permutation of $\{1, 2, 3\}$, and subject to the linear and sign constraints

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0, \qquad H < 0, \qquad \widetilde{N}_i > 0 \quad \text{for all } i \in \{1, 2, 3\}. \tag{2.1.2}$$

The variable H corresponds to the Hubble scalar, i.e. the expansion speed of the cosmological model, i.e. the mean curvature of the surfaces $\{t = \text{const}\}$ of spatial homogeneity. The "shears" Σ_i correspond to the trace-free Hubble-normalized principal curvatures (hence, the linear "trace-free" constraint). The condition $H < 0$ corresponds to our choice of the direction of time: We choose to orient time such that the universe is shrinking, i.e. the singularity (big bang) lies in the future; this unphysical choice of time-direction is just for convenience of notation.

The vacuum Einstein Field equations state that the spacetime is Ricci-flat. If we express the normalized trace-free principal curvatures Σ_i as time-derivatives of the metric variables $\widetilde{N}_i$, then the the Einstein Field equations become the Wainwright-Hsu equations (2.1.3), which are a system of seven ordinary differential equations, subject to one linear constraint equation ($\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$) and one algebraic equation, called the Gauss-constraint $G = 1$ (2.1.2)). The calculations leading to (2.1.3) are given in Section A.5; alternatively, we recommend [WE05].

$$\begin{aligned}
H' &= \frac{1}{2}(1 + 2\Sigma^2)H \\
\Sigma_i' &= (1 - \Sigma^2)\Sigma_i + \frac{1}{2}S_i \\
\widetilde{N}_i' &= -(\Sigma^2 + \Sigma_i)\widetilde{N}_i \\
1 &\stackrel{!}{=} \Sigma^2 + N^2 := G,
\end{aligned} \tag{2.1.3}$$

where we used the shorthands

$$
\begin{aligned}
N_i &:= \hat{n}_i \widetilde{N}_i \\
\Sigma^2 &:= \frac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) \\
N^2 &:= N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1) \\
S_i &:= 4\left(N_i(2N_i - N_j - N_k) - (N_j - N_k)^2\right).
\end{aligned}
$$

2.2 Physical Properties of Solutions for Bianchi VIII and IX

We will now use the results of this work in order to describe some physical properties of Bianchi spacetimes. We restrict our attention to the case of Bianchi IX and VIII, where all $\hat{n}_i \neq 0$.

Bounded life-time. Since the mean curvature H corresponds to the time-derivative of the spatial volume form $\sqrt{g_{11}g_{22}g_{33}}$, the universe described by such a metric is contracting for $H < 0$. Physically, we are interested in the behaviour towards the initial (big bang) singularity; this setting is time-reversed to physical time-variables, and we should look at the behaviour of solutions for $t \to +\infty$. Since $|H|$ is at least uniformly exponentially growing, we can immediately see $\textsc{EigenFuture} = \int_0^\infty \sqrt{-g_{00}}\mathrm{d}t < \infty$, that is, the universe has only a finite (eigen-) lifetime until H blows up and a singularity occurs. In our coordinates, this singularity is placed at $t = +\infty$.

A priori, we do not know whether this singularity is a physical singularity (with curvature blow-up), or whether instead it is just our coordinate system that blows up. It has been proven in [Rin00] that, in Bianchi VIII and IX, the singularity is physical and curvature blows up. This is done by considering the so-called Kretschmann scalar $\kappa = \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha,\beta}^{\;\;\delta,\gamma} R_{\delta,\gamma}^{\alpha,\beta}$ and showing that $\lim_{t\to\infty}\kappa(t) = \infty$. We refer to [Rin00] for the details.

Bounded spatial metric coefficients. The coefficients $g_{ii} = \frac{1}{48}\frac{|N_i|}{H^2|N_1N_2N_3|}$ stay bounded: The global attractor Theorems 3 and 4, pages 60 and 61, imply that the $|N_i|$ stay bounded for $t \to +\infty$. For the denominator, we can compute, using (2.3.10), page 27,

$$
\mathrm{D}_t \log|H^2N_1N_2N_3| = -3\Sigma^2 + 1 + 2\Sigma^2 = N^2 \geq -4|N_1N_2N_3|^{\frac{2}{3}}.
$$

Lemma 3.6, page 36, shows that $|H^2N_1N_2N_3|$ stays bounded away from zero. Hence all three g_{ii} stay bounded for $t \to +\infty$. Indeed, outside of the Taub-spaces $\mathcal{T}_i = \{N_j = N_k, \Sigma_j = \Sigma_k\}$, we have $N^2 > \epsilon > 0$ for large amounts of time, as follows trivially from the proof of Theorems 3 and 4, and we can conclude $\lim_{t\to+\infty} g_{ii}(t) = 0$.

Particle Horizons. Recall question of particle horizons from the introduction, and the definition of communicating regions, which we here adjust to match our convention that the big bang singularity is situated in the future:

Definition 2.1. Let $\gamma : [0,1] \to M$ be a differentiable curve. We say that the curve is *time-like*, if $g(\dot\gamma, \dot\gamma) \le 0$ and $\dot\gamma_0 \neq 0$ for all $t \in [0,1]$. We say that it is *future-directed*, if $\dot\gamma_0 > 0$, and *past-directed* if $\dot\gamma_0 < 0$.

Let $\boldsymbol{p} \in M$ a spacetime point. Then we define its past light-cone $J^-(\boldsymbol{p})$ and its future light-cone $J^+(\boldsymbol{p})$ by

$$J^-(\boldsymbol{p}) = \{\boldsymbol{q} : \text{ there is } \gamma : [0,1] \to M \ \text{ with } \gamma(0) = \boldsymbol{p}, \gamma(1) = \boldsymbol{q}, \ \text{ time-like past directed}\}$$
$$J^+(\boldsymbol{p}) = \{\boldsymbol{q} : \text{ there is } \gamma : [0,1] \to M \ \text{ with } \gamma(0) = \boldsymbol{p}, \gamma(1) = \boldsymbol{q}, \ \text{ time-like future directed}\}.$$

We define its past communicating region as $J^+(J^-(\boldsymbol{p})) = \bigcup_{\boldsymbol{q}\in J^-(\boldsymbol{p})} J^+(\boldsymbol{q})$, and its future communicating region as $J^-(J^+(\boldsymbol{p})) = \bigcup_{\boldsymbol{q}\in J^+(\boldsymbol{p})} J^-(\boldsymbol{q})$. We define its past cosmic horizon, also called particle horizon, as $\partial J^+(J^-(\boldsymbol{p}))$, i.e. as the boundary of its past communicating region. We define its future cosmic horizon, also called particle horizon, as $\partial J^-(J^+(\boldsymbol{p}))$, i.e. as the boundary of its future communicating region.

Since we inverted physical time such that the big bang singularity lies in the future, we are concerned with proving that the following horizon $\partial J^-(J^+(\boldsymbol{p})) \neq \emptyset$ is nonempty.

Particle horizons are determined by estimates on $\int \sqrt{|N_iN_j|}(t)\mathrm{d}t$. This gives the physical interpretation of Theorem 6, page 65, that states that almost every solution has $\int_{t_0}^{\infty} \sqrt{|N_iN_j|}(t)\mathrm{d}t < \infty$:

Lemma 2.2. *There is a constant $C > 0$, such that, for Bianchi* IX *and* VIII *vacuum spacetimes M, we can estimate for $\boldsymbol{p} \in M$ and $t_0 \ge t(\boldsymbol{p})$*

$$\begin{aligned}\operatorname{diam}_h \left[J^+(\boldsymbol{p})) \cap \{\boldsymbol{q} \in M : t(\boldsymbol{q}) = t_0\}\right] &\le C \int_{t(\boldsymbol{p})}^{t_0} \max_{j\neq k} \sqrt{|N_jN_k|}(t)\mathrm{d}t \\ \operatorname{diam}_h \left[J^-(J^+(\boldsymbol{p})) \cap \{\boldsymbol{q} \in M : t(\boldsymbol{q}) = t(\boldsymbol{p})\}\right] &\le C \int_{t(\boldsymbol{p})}^{\infty} \max_{j\neq k} \sqrt{|N_jN_k|}(t)\mathrm{d}t,\end{aligned} \tag{2.2.1}$$

where the diameter is measured with the symmetry metric h given on the surfaces of homogeneity $\{t = \text{const}\}$ by $h = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3$.

Proof. Any time-like singularity-directed curve γ starting in $\boldsymbol{p}$ must fulfil $g(\dot\gamma, \dot\gamma) \le 0$. Parametrize γ over the time t; then, $|\dot\gamma_i| \le \sqrt{-g_{00}g^{ii}} = \sqrt{12}\sqrt{|N_jN_k|}$. It is clear that the h-length of such a curve must be bounded by $C \max_{j\neq k} \int_{t(\boldsymbol{p})}^{\infty} \sqrt{|N_jN_k|}(t)\mathrm{d}t$, if the curve only accesses times later than t_0. This proves the estimate on $J^+(\boldsymbol{p})$. The estimate on $J^-(J^+(\boldsymbol{p}))$ follows. □

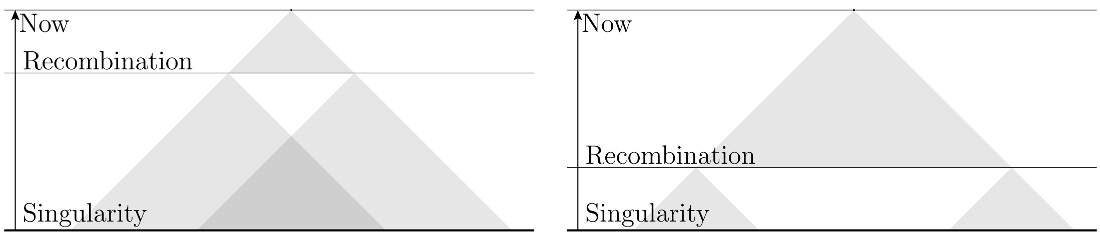
Lemma 2.3. *Suppose that M is a spacetime corresponding to a Bianchi* VIII *or* IX *solution with $\int_0^\infty \sqrt{|N_jN_k|}(t)\mathrm{d}t < \infty$ for all $j \neq k$. Use the shorthands $M_{>t_0} = \{\boldsymbol{p} \in M : t(\boldsymbol{p}) > t_0\}$ and $J^-(J^+(t_0)) = J^-(J^+(\boldsymbol{p}))$ for some $\boldsymbol{p} \in M$ with $t(\boldsymbol{p}) = t_0$. Then the following holds:*

1. $\lim_{t_0\to\infty} \operatorname{diam}_h \left[J^-(J^+(t_0)) \cap \{\boldsymbol{q} \in M : t(\boldsymbol{q}) = t_0\}\right] = 0.$
2. $\lim_{t_0\to\infty} \operatorname{diam}_g \left[J^-(J^+(t_0)) \cap \{\boldsymbol{q} \in M : t(\boldsymbol{q}) = t_0\}\right] = 0.$

3. *For $t_0 > 0$ large enough, $\partial J^-(J^+(t_0)) \neq \emptyset$.*

4. *For $t_0 > 0$ large enough, $[J^-(J^+(t_0)) \cap M_{t_0}, (\partial J^-(J^+(t_0))) \cap M_{t_0}]$ is diffeomorphic to $[(0,1) \times B^3, (0,1) \times \partial B^3]$, where B^3 is the three dimensional unit ball, and the sets are diffeomorphic as manifolds with boundary.*

Proof. If $\int_{t_0}^{\infty} \max_{j \neq k} \sqrt{|N_j N_k|}(t)\mathrm{d}t < \infty$, then $\lim_{t_0 \to \infty} \int_{t_0}^{\infty} \max_{j \neq k} \sqrt{|N_j N_k|}(t)\mathrm{d}t = 0$, which proves (1). (2) follows because the metric coefficients g_{ij} are bounded. (4) follows because the injectivity radius $\mathrm{inj}_h(t)$ of the hypersurfaces of homogeneity is independent of the time t, if we measure it with respect to the (time-independent) h-metric. (3) follows trivially from (4). $\square$



(a) A sketched spacetime, where homogeneity of the observable universe could be explained by mixing between the big bang and recombination.

(b) A sketched spacetime, where homogeneity of the observable universe cannot be explained by mixing between the big bang and recombination.

Figure 3: Whether observed homogeneity could be explained by mixing depends on the relation of the conformal distance between recombination and singularity versus the conformal distance between the present time and recombination.

Particle Horizons and Homogenization Astronomical observations show that the universe appears to be mostly homogeneous at large scales. A possible explanation for the observed homogeneity, c.f. e.g. [Mis69], might be that the universe, i.e. matter, radiation, etc, mixed in the early universe. This is only possible if the outer parts of our optical past light-cone have a joint causal past (see Figure 3), and is superficially at odds with the formation of particle horizons.

However, optical astronomical observations go only back to the recombination, the moment where the primordial plasma condensed to a gas and became transparent to light. Hence, there is only a bounded region of spacetime in our past, which is optically accessible, and mixing may have happened in the time-frame between the initial singularity and recombination. This is a reason that the formation of particle horizons does not necessarily spell doom for attempts to explain the observed homogeneity through mixing (apart from the actual universe not being a homogeneous, anisotropic vacuum spacetime of Bianchi type VIII or IX).

Indeed, the same problem is present in the "standard model" of cosmology, which is a (homogeneous isotropic) FLRW model. There, the observed homogeneity is typically explained by inflation, i.e. one postulates a phase of rapid expansion that increases the conformal distance between recombination and singularity, mediated by an exotic, hitherto unobserved matter field.

2.3 The Wainwright-Hsu equations

Recall (2.1.3). The equation for H is decoupled from the remaining equations. Thus, we can drop the equation for H, solve the remaining equations, and afterwards integrate to obtain H. Likewise, we can stick with the equations for N_i instead of $\widetilde{N}_i$, such that $\hat{n}_i = \operatorname{sign} N_i$. For Bianchi-types VIII and IX this already determines the metric, and for the lower Bianchi types we can again integrate afterwards. This yields a standard form of the vacuum Wainwright-Hsu equations from (2.1.3), as used in e.g. [Rin01, HU09a, HU09b, LHWG11, Bég10], up to constant factors. The most useful equations are also summarized in Section A.1.

It is useful to solve for the linear constraint $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$, introducing $\mathbf{\Sigma} = (\Sigma_+, \Sigma_-)$ by

$$\begin{aligned} &\mathbf{T}_1 = (-1, 0) \qquad \mathbf{T}_2 = \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right) \qquad \mathbf{T}_3 = \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right) \\ &\Sigma_i = 2\langle \mathbf{T}_i, \mathbf{\Sigma}\rangle \qquad \Sigma_+ = -\frac{1}{2}\Sigma_1 \qquad \Sigma_- = \frac{1}{2\sqrt{3}}(\Sigma_3 - \Sigma_2), \end{aligned} \tag{2.3.1}$$

which turns the vacuum Wainwright-Hsu differential equations into a system of five ordinary differential equations on $\mathbb{R}^5 = \{(\Sigma_+, \Sigma_-, N_1, N_2, N_3)\} = \{\mathbf{\Sigma}, \boldsymbol{N}\}$, with one algebraic constraint equation (2.3.3). The three points $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ are called Taub-points. We will, in an abuse of notation, consider the Taub-points both as points in $\mathbb{R}^2$, and as points in $\mathbb{R}^5$ (where all three N_i vanish). The Wainwright-Hsu equations are then given by the differential equations

$$N_i' = -(\Sigma^2 + 2\langle \mathbf{T}_i, \mathbf{\Sigma}\rangle)N_i \tag{2.3.2a}$$

$$= -\left(|\mathbf{\Sigma} + \mathbf{T}_i|^2 - 1\right) N_i \tag{2.3.2b}$$

$$\mathbf{\Sigma}' = N^2\mathbf{\Sigma} + 2\left(N_1^2\mathbf{T}_1 + N_2^2\mathbf{T}_2 + N_3^2\mathbf{T}_3 + N_1N_2\mathbf{T}_3 + N_2N_3\mathbf{T}_1 + N_3N_1\mathbf{T}_2\right) \tag{2.3.2c}$$

$$= N^2\mathbf{\Sigma} + 2\begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_3 & \mathbf{T}_2 \\ & \mathbf{T}_2 & \mathbf{T}_1 \\ & & \mathbf{T}_3 \end{pmatrix} [\boldsymbol{N}, \boldsymbol{N}], \tag{2.3.2d}$$

and the Gauss constraint equation

$$1 \stackrel{!}{=} \Sigma^2 + N^2 =: G(\boldsymbol{x}), \tag{2.3.3}$$

where we used the shorthands

$$\Sigma^2 = \Sigma_+^2 + \Sigma_-^2, \qquad N^2 = N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1).$$

We can unpack these equations with unambiguous notation into

$$N_1' = -(\Sigma^2 - 2\Sigma_+)N_1 \tag{2.3.4a}$$
$$N_2' = -(\Sigma^2 + \Sigma_+ - \sqrt{3}\Sigma_-)N_2 \tag{2.3.4b}$$
$$N_3' = -(\Sigma^2 + \Sigma_+ + \sqrt{3}\Sigma_-)N_3 \tag{2.3.4c}$$
$$\Sigma_+' = N^2\Sigma_+ - 2N_1^2 + N_2^2 + N_3^2 + N_1N_2 - 2N_2N_3 + N_1N_3 \tag{2.3.4d}$$
$$\Sigma_-' = N^2\Sigma_- + \sqrt{3}\left(-N_2^2 + N_3^2 + N_1N_2 - N_1N_3\right), \tag{2.3.4e}$$

which is, up to constant factors, the form of the vacuum Wainwright-Hsu equations used in e.g. [Rin01, LHWG11, Bég10].

It is occasionally useful to fully tensorize the Wainwright-Hsu equations, yielding the form

$$\begin{aligned} \boldsymbol{N}' &= -\langle \boldsymbol{\Sigma}, \boldsymbol{\Sigma}\rangle \boldsymbol{N} - \boldsymbol{D}[\boldsymbol{\Sigma}, \boldsymbol{N}] \\ \boldsymbol{\Sigma}' &= Q[\boldsymbol{N}, \boldsymbol{N}]\boldsymbol{\Sigma} + \boldsymbol{T}[\boldsymbol{N}, \boldsymbol{N}] \\ G(\boldsymbol{\Sigma}, \boldsymbol{N}) &= \langle \boldsymbol{\Sigma}, \boldsymbol{\Sigma}\rangle + Q[\boldsymbol{N}, \boldsymbol{N}] \overset{!}{=} 1, \end{aligned} \tag{2.3.5}$$

where $Q : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and $\boldsymbol{T} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^2$ and $\boldsymbol{D} : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$. We write Q as a 3×3-matrix with entries in $\mathbb{R}$ such that $Q[\boldsymbol{N}, \boldsymbol{M}] = \boldsymbol{N}^T Q \boldsymbol{N} = N^2$; we write $\boldsymbol{T}$ as a similar 3×3-matrix with entries in $\mathbb{R}^2$. We write $\boldsymbol{D}$ as a 3×3-matrix with entries in $\mathbb{R}^2$ such that $\boldsymbol{D}[\boldsymbol{\Sigma}, \boldsymbol{N}] = (\boldsymbol{D}\boldsymbol{N}) \cdot \boldsymbol{\Sigma}$, where $\boldsymbol{D}\boldsymbol{N}$ is the usual matrix product (with entries in $\mathbb{R}^2$) and the dot-product is evaluated component wise. Then the tensors $Q, \boldsymbol{T}, \boldsymbol{D}$ can be written as

$$Q = \begin{pmatrix} 1 & -2 & -2 \\ & 1 & -2 \\ & & 1 \end{pmatrix}, \quad \boldsymbol{T} = \begin{pmatrix} 2\mathbf{T}_1 & 2\mathbf{T}_3 & 2\mathbf{T}_2 \\ & 2\mathbf{T}_2 & 2\mathbf{T}_1 \\ & & 2\mathbf{T}_3 \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} 2\mathbf{T}_1 & & \\ & 2\mathbf{T}_2 & \\ & & 2\mathbf{T}_3 \end{pmatrix} \tag{2.3.6}$$

Permutation Equivariance. The equations (2.1.3) are equivariant under permutations $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ of the three indices. This permutation invariance also applies to (2.3.2) as

$$(\boldsymbol{\Sigma}, N_1, N_2, N_3) \to (A_\sigma \boldsymbol{\Sigma}, N_{\sigma(1)}, N_{\sigma(2)}, N_{\sigma(3)}),$$

where $A_\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear isometry with $A_\sigma^T \mathbf{T}_i = \mathbf{T}_{\sigma(i)}$. The equations are also equivariant under $(\boldsymbol{\Sigma}, \boldsymbol{N}) \to (\boldsymbol{\Sigma}, -\boldsymbol{N})$, as can be seen directly from (2.3.5).

Invariance of the Constraint. The signs of the N_i are preserved under the flow, because N_i' is a multiple of N_i and therefore $N_i = 0$ implies $N_i' = 0$.

Claim 2.3.1. The quantity G is preserved under the flow.

$\mathcal{M} = \{\boldsymbol{x} \in \mathbb{R}^5 : \ G(\boldsymbol{x}) = 1\}$	the physically relevant Phase-space
$\mathcal{M}_{\hat{n}} = \{\boldsymbol{x} \in \mathcal{M} : \ \operatorname{sign} N_i = \hat{n}_i\}$	a specific octant of the Phase-space
$\mathcal{K} = \mathcal{M}_{000} = \{\boldsymbol{x} \in \mathcal{M} : \ \boldsymbol{N} = 0\}$	the Kasner circle
$\mathcal{A} = \{\boldsymbol{x} \in \mathcal{M} : \ \text{at most one } N_i \neq 0\}$	the Mixmaster attractor
$\mathcal{A}_{\hat{n}} = \overline{\mathcal{M}_{\hat{n}}} \cap \mathcal{A}$	a specific octant of $\mathcal{A}$
$\mathcal{T}_i = \{\boldsymbol{x} \in \mathcal{M} : N_j = N_k,\ \langle \mathbf{T}_j, \boldsymbol{\Sigma}\rangle = \langle \mathbf{T}_k, \boldsymbol{\Sigma}\rangle\}$	a Taub-space
$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$	all three Taub-spaces
$\mathcal{TL}_i = \{\boldsymbol{x} \in \mathcal{M} : N_j = N_k,\ N_i = 0,\ \boldsymbol{\Sigma} = \mathbf{T}_i\} \subseteq \mathcal{T}_i$	a Taub-line
$\mathcal{T}_i^G = \{\boldsymbol{x} \in \mathcal{M} : \lvert N_j\rvert = \lvert N_k\rvert,\ \langle \mathbf{T}_j, \boldsymbol{\Sigma}\rangle = \langle \mathbf{T}_k, \boldsymbol{\Sigma}\rangle\}$	a generalized Taub-space; only invariant if $\operatorname{sign} N_j = \operatorname{sign} N_k$

Table 2: Named subsets. Here (i, j, k) stands for a permutation of $\{1, 2, 3\}$ and $\hat{n} \in \{+, 0, -\}^3$. All of these sets, except for $\mathcal{T}_i^G$, are invariant.

Proof. This can best be seen from (2.3.5):

$$\begin{aligned}
\mathrm{D}_t G &= 2\langle \boldsymbol{\Sigma}, \boldsymbol{\Sigma}'\rangle + Q[\boldsymbol{N}, \boldsymbol{N}'] + Q[\boldsymbol{N}', \boldsymbol{N}] \\
&= 2Q[\boldsymbol{N}, \boldsymbol{N}]\langle \boldsymbol{\Sigma}, \boldsymbol{\Sigma}\rangle + 2\boldsymbol{\Sigma} \cdot \boldsymbol{T}[\boldsymbol{N}, \boldsymbol{N}] - 2\langle \boldsymbol{\Sigma}, \boldsymbol{\Sigma}\rangle Q[\boldsymbol{N}, \boldsymbol{N}] \\
&\quad - Q[\boldsymbol{D}[\boldsymbol{\Sigma}, \boldsymbol{N}], \boldsymbol{N}] - Q[\boldsymbol{N}, \boldsymbol{D}[\boldsymbol{\Sigma}, \boldsymbol{N}]] \\
&= \Sigma \cdot \boldsymbol{N}^T \left[2\boldsymbol{T} - \boldsymbol{D}^T Q - Q\boldsymbol{D}\right] \boldsymbol{N}.
\end{aligned}$$

Using $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 = 0$ and (2.3.6), it is a simple matter of matrix multiplication to verify that $2\boldsymbol{T} - \boldsymbol{D}^T Q - Q\boldsymbol{D} = 0$ and hence $\mathrm{D}_t G = 0$. Therefore, sets of the form $\{\boldsymbol{x} \in \mathbb{R}^5 : G(\boldsymbol{x}) = c\}$ are invariant for any $c \in \mathbb{R}$ and especially for the physical $c = 1$. ☐

Claim 2.3.2. The set $\mathcal{M} = \{\boldsymbol{x} \in \mathbb{R}^5 : G(\boldsymbol{x}) = 1\}$ is a smooth embedded submanifold.

Proof. This is apparent from the implicit function theorem, since (if $\boldsymbol{x} \neq 0$)

$$\begin{aligned}
\frac{1}{2}\mathrm{d}G &= \Sigma_+ \mathrm{d}\Sigma_+ + \Sigma_- \mathrm{d}\Sigma_- \\
&\quad + (N_1 - N_2 - N_3)\mathrm{d}N_1 + (N_2 - N_3 - N_1)\mathrm{d}N_2 + (N_3 - N_1 - N_2)\mathrm{d}N_1 \neq 0.
\end{aligned}$$

☐

Named invariant sets. There are several recurring important sets, which require names and are listed in Table 2.

The set $\mathcal{M}$ is invariant because G is a constant of motion. The Taub-space $\mathcal{T}_i$ is invariant because of the equivariance under exchange of the two other indices j and k. The invariance of the Taub-lines $\mathcal{TL}_i$ can be seen by considering (2.3.4) for $i = 1$ and applying the permutation invariance for $\mathcal{TL}_2$ and $\mathcal{TL}_3$. The generalized Taub-spaces $\mathcal{T}_i^G$ are not invariant if $\hat{n}_j \neq \hat{n}_k$. The other sets are invariant because the signs $\hat{n}_i = \operatorname{sign} N_i$ are fixed.

Recall that the signs of the N_i correspond to the Bianchi Type of the Lie-algebra associated to the homogeneity of the cosmological model and are given in Table 1, page 8 (up to index permutations).

Auxiliary Quantities. The following quantities turn out to be useful later on (where (i,j,k) is a permutation of $(1,2,3)$):

$$\Delta = |N_1 N_2 N_3|^{\frac{1}{3}} \tag{2.3.7a}$$

$$\delta_i = 2\sqrt{|N_j N_k|} \tag{2.3.7b}$$

$$r_i = \sqrt{(|N_j| - |N_k|)^2 + \frac{1}{3}\langle \mathbf{T}_j - \mathbf{T}_k, \mathbf{\Sigma}\rangle^2} \tag{2.3.7c}$$

ψ_i such that:

$$r_1 \cos\psi_1 = \frac{1}{\sqrt{3}}\langle \mathbf{T}_3 - \mathbf{T}_2, \mathbf{\Sigma}\rangle = \Sigma_- \qquad r_1 \sin\psi_1 = |N_2| - |N_3| \tag{2.3.7d}$$

$$r_2 \cos\psi_2 = \frac{1}{\sqrt{3}}\langle \mathbf{T}_1 - \mathbf{T}_3, \mathbf{\Sigma}\rangle = -\frac{\sqrt{3}}{2}\Sigma_+ - \frac{1}{2}\Sigma_- \qquad r_2 \sin\psi_2 = |N_3| - |N_1| \tag{2.3.7e}$$

$$r_3 \cos\psi_3 = \frac{1}{\sqrt{3}}\langle \mathbf{T}_2 - \mathbf{T}_1, \mathbf{\Sigma}\rangle = \frac{\sqrt{3}}{2}\Sigma_+ - \frac{1}{2}\Sigma_- \qquad r_3 \sin\psi_3 = |N_1| - |N_2|. \tag{2.3.7f}$$

The auxiliary products δ_i^2 can be used to measure the distance from the Mixmaster attractor $\mathcal{A} = \{\boldsymbol{x} : \max_i \delta_i(\boldsymbol{x}) = 0\}$. The r_i and can be used to measure the distance from the generalized Taub-space $\mathcal{T}_i^G = \{\boldsymbol{x} : r_i(\boldsymbol{x}) = 0\}$, and the (r_i, ψ_i)-pairs form polar coordinates around the generalized Taub-spaces.

The products Δ and δ_i obey an especially geometric differential equation, similar to (2.3.2b):

$$\Delta' = -|\mathbf{\Sigma}|^2 \Delta \tag{2.3.8a}$$

$$\delta_i' = -\left(\left|\mathbf{\Sigma} - \frac{\mathbf{T}_i}{2}\right|^2 - \frac{1}{4}\right)\delta_i. \tag{2.3.8b}$$

Volume Expansion In the following, we will give a short overview of the behaviour of logarithmic volumes under the flow defined by (2.3.2). More details and proofs are provided in Section 7, and some general facts about the interplay between flows and volumes are given in Appendix A.4.

Set $\beta_i = -\log|N_i|$, and consider the Lebesgue-measure on these coordinates, i.e. the volume-form $\omega_5 = \mathrm{d}\Sigma_+ \wedge \mathrm{d}\Sigma_- \wedge \mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2 \wedge \mathrm{d}\beta_3$. It is trivial to see from (2.3.2) that $\mathrm{D}_t\omega_5 = 2N^2\omega_5$. Define the four-form ω_4 such that $\omega_5 = \omega_4 \wedge \mathrm{d}G$, e.g. by $\omega_4 = \iota_X\omega_5$ where $X = \frac{\nabla G}{\langle \nabla G, \nabla G\rangle}$. Then ω_4 is a volume-form on $\mathcal{M}$. Because G is a constant of motion, we have $\mathrm{D}_t\mathrm{d}G = 0$ and $\mathrm{D}_t\omega_4 = 2N^2\omega_4$. This formula is remarkable for its simplicity and simple proof, and is extensively used in Section 7.

Indeed, the formula $\mathrm{D}_t\omega_4 = 2N^2\omega_4$ is one of the major insights of this work. To the best of our knowledge, this insight is novel.

Preliminary Estimates We immediately see from (2.3.8a) that Δ is nonincreasing. The following condenses some basic estimates on $\mathcal{M}$, using the constraint $G = 1$ only. The appearing constants do not substantially matter, and could be replaced by an unspecific C; mostly non-sharp choices are nevertheless given explicitly, for easier verification:

Lemma 2.4. *Let $\boldsymbol{x} \in \mathcal{M}$. The following holds, using $\Delta^3 = |N_1N_2N_3|$:*

1. *If $\boldsymbol{x} \in \mathcal{M}$ is not of Bianchi type* IX, *i.e. $\boldsymbol{x} \notin \mathcal{M}_{+++} \cup \mathcal{M}_{---}$, then $N^2 \geq 0$ and $|\boldsymbol{\Sigma}|^2 \leq 1$.*
2. *If $\boldsymbol{x} \in \mathcal{M}$ is of Bianchi type* IX, *i.e. $\boldsymbol{x} \in \mathcal{M}_{+++} \cup \mathcal{M}_{---}$, then $N^2 \geq -4\Delta^2$ and $|\boldsymbol{\Sigma}|^2 \leq 1 + 4\Delta^2$, and $|\boldsymbol{\Sigma}| \leq 1 + 2\Delta^2$.*
3. *If $\Delta \leq 1$ and $|N_3| \geq |N_2| \geq |N_1| \geq 0$, then $\delta_3 \leq \delta_2 \leq 5\sqrt{\Delta}$ and $|N_2 - N_3| \leq 1 + \frac{7}{2}\Delta$.*
4. *If $\Delta \leq 1$ and $|\boldsymbol{x}| \geq 10$ and $|N_3| \geq |N_2| \geq |N_1| \geq 0$, then $\hat{n}_2 = \hat{n}_3$ and $\delta_1/2 \geq |N_2| \geq \frac{1}{\sqrt{2}}|\boldsymbol{x}| - 6 > 1 > |N_1|$, and $|N_1| \leq \Delta^3|N_2|^{-2}$, and $\delta_3 \leq \delta_2 \leq 2\Delta^{\frac{3}{2}}|N_2|^{-\frac{1}{2}}$. For $|\boldsymbol{x}| \gg 10$, if $N_2 > 0$, we will describe such points colloquially with $N_2 \approx N_3 \gg 1 \gg |N_1|$.*
5. *If $\Delta \geq 1$ and $|N_3| \geq |N_2| \geq |N_1| \geq 0$, then $\delta_3 \leq \delta_2 \leq 5\Delta$ and $|N_2 - N_3| \leq \frac{9}{2}\Delta^2$.*
6. *If $\Delta \geq 1$ and $|\boldsymbol{x}| \geq 10\Delta^2$ and $|\boldsymbol{x}| \geq 5$ and $|N_3| \geq |N_2| \geq |N_1| \geq 0$, then $\hat{n}_2 = \hat{n}_3$ and $\delta_1/2 \geq |N_2| \geq \frac{1}{\sqrt{3}}|\boldsymbol{x}| - 6\Delta^2 > \Delta^2 \geq 1 > |N_1|$, and $|N_1| \leq \Delta^3|N_2|^{-2} \leq |N_2|^{-\frac{1}{2}}$, and $\delta_3 \leq \delta_2 \leq 2\Delta^{\frac{3}{2}}|N_2|^{-\frac{1}{2}}$. For $|\boldsymbol{x}| \gg 10\Delta^2$, if $N_2 > 0$, we will describe such points colloquially with $N_2 \approx N_3 \gg 1 \gg |N_1|$.*
7. *Suppose $|N_1N_2N_3| \leq 10^{-6}$, i.e. $\Delta \leq 10^{-2}$. Then it is impossible to have $r_i \leq 0.1$ for all three $i \in \{1, 2, 3\}$. Furthermore, if $\delta_i \geq 1$, then $r_i \leq 2$.*

Proof. Claim (1) works by assuming without loss of generality $N_1 < 0 < N_2, N_3$; we get

$$N^2 = (N_1 + N_2 - N_3)^2 - 4N_1N_2 > 0. \tag{2.3.9}$$

Then, one applies $|\boldsymbol{\Sigma}|^2 + N^2 = 1$.

Claim (2) works by assuming without loss of generality $N_3 \geq N_2 \geq N_1 > 0$; we get

$$\begin{aligned} N^2 &= (N_1 + N_2 - N_3)^2 - 4N_1N_2 \geq -4N_1N_2 \\ &\geq -4(N_1N_2N_3)^{\frac{2}{3}} \left(\frac{N_1N_2}{N_3N_3}\right)^{\frac{1}{3}} \geq -4(N_1N_2N_3)^{\frac{2}{3}}. \end{aligned} \tag{2.3.10}$$

By the permutation symmetry, the above inequality holds regardless of which N_i is largest. Then, one applies $|\boldsymbol{\Sigma}|^2 + N^2 = 1$.

Claims (3) and (5) work by seeing $|N_1| \leq \Delta$ and $|N_1N_2| \leq \Delta^2$. Use $1 \geq N^2 \geq -4|N_1N_2| - 2|N_1|\,|N_2 - N_3| + |N_2 - N_3|^2$ in order to see $|N_2 - N_3| \leq \Delta + \sqrt{1 + 4\Delta^2 + \Delta}$.

This yields $|N_1N_3| \le |N_1N_2| + |N_1(N_2 - N_3)| \le 2\Delta^2 + \Delta\sqrt{1+4\Delta^2+\Delta}$. Considering $\Delta \le 1$ and $\Delta \ge 1$, the claims (3) and (5) follow.

Claim (4) follows directly from $|\boldsymbol{x}|^2 \le 2N_3^2 + |\boldsymbol{\Sigma}^2| + 1$, applying (1) or (2) to see $|N_3| \ge \frac{1}{\sqrt{2}}\sqrt{|\boldsymbol{x}|^2 - 5} \ge \frac{1}{\sqrt{2}}|\boldsymbol{x}| - 1$, applying (3) to see $|N_2 - N_3| \le 5$ and hence $\hat{n}_2 = \hat{n}_3$ and $|N_2| \ge \frac{1}{\sqrt{2}}|\boldsymbol{x}| - 6 > 1 > |N_1|$. The other inequalities are simple reformulations of $|N_3| \ge |N_3| \ge |N_1|$.

Claim (6) follows directly from $|\boldsymbol{x}|^2 \le 3N_3^2 + |\boldsymbol{\Sigma}^2| + 1$, applying (1) or (2) to see $|N_3| \ge \frac{1}{\sqrt{3}}\sqrt{|\boldsymbol{x}|^2 - 5\Delta^2} \ge \frac{1}{\sqrt{3}}|\boldsymbol{x}| - \Delta^2$, applying (5) to see $|N_2 - N_3| \le 5\Delta^2$ and hence $\hat{n}_2 = \hat{n}_3$ and $|N_2| \ge \frac{1}{\sqrt{3}}|\boldsymbol{x}| - 6\Delta^2 > 1$. The other inequalities are simple reformulations of $|N_3| \ge |N_3| \ge |N_1|$.

In order to see claim (7), note that if all three $r_i \le 0.1$, then all three $||N_j|-|N_k|| \le 0.1$ are roughly equal and $\max_i |N_i| \le \Delta^3 + 0.1 \le 0.2$, which yields $N^2 \le 0.36$. However, $|\boldsymbol{\Sigma}| \le \sqrt{3}/10$, and hence $N^2 = 1 - |\boldsymbol{\Sigma}|^2 \ge 0.9$, yielding the desired contradiction.

In order to see the last part of claim (8), assume without loss of generality that $|N_3| \ge |N_2| \ge |N_1| \ge 0$. By (4), using $\Delta < 10^{-2}$, we have $\delta_2 \le 5\sqrt{\Delta} \le 0.2$; hence, we only need to show that $r_1 \le 3$. We have $1 \ge r_1^2 - 2|N_1N_2| - 2|N_1N_3| \ge r_1^2 - 15\sqrt{\Delta}$ and hence $r_1^2 \le 3$. □

2.4 The Wainwright-Hsu equations in polar coordinates

Near the generalized Taub-spaces $\mathcal{T}_i^G$, it is possible to use polar coordinates (2.3.7). Without loss of generality we will only transform (2.3.2) into these coordinates around the Taub-space $\mathcal{T}_1^G$ (the other ones can be obtained by permuting the indices and rotating or reflecting $\boldsymbol{\Sigma}$).

The use of polar coordinates near the Taub-spaces $\mathcal{T}_i$ for Bianchi IX, i.e. $\mathcal{M}_{+++}$, is by no means novel (c.f. e.g. [Rin01], [HU09a]). However, to the best of our knowledge, polar coordinates around the generalized Taub-spaces $\mathcal{T}_i^G$ have not been used previously in the case where the generalized Taub-space fails to be invariant.

We only use polar coordinates on $\mathcal{M} = \{\boldsymbol{x} : \; G(\boldsymbol{x}) = 1\}$.

Polar Coordinates around the invariant Taub-spaces. Consider the case $\mathcal{M}_{*++}$, where $N_2, N_3 > 0$ are positive and we are interested in a neighbourhood of $\mathcal{T}_1 = \{\boldsymbol{x} : \Sigma_- = 0,\ N_2 - N_3 = 0\}$. The sign of N_1 does not significantly matter. The best form of the Wainwright-Hsu equations for transformation into polar coordinates is (2.3.4).

We use the additional shorthands

$$N_- = N_2 - N_3 \qquad N_+ = N_2 + N_3, \tag{2.4.1a}$$

such that (with (2.3.7)):

$$r_1 \ge 0 : \qquad r_1^2 = \Sigma_-^2 + N_-^2 \tag{2.4.1b}$$

$$\psi_1 : \qquad N_- = r_1 \sin\psi_1 \qquad \Sigma_- = r_1 \cos\psi_1 \tag{2.4.1c}$$

$$N_+^2 = N_-^2 + \delta_1^2 \qquad N^2 = N_-^2 + N_1(N_1 - 2N_+). \tag{2.4.1d}$$

This gives us the differential equations (using $\Sigma^2 + N^2 = 1$):

$$N_-' = (N^2 - 1 - \Sigma_+)N_- + \sqrt{3}\Sigma_- N_+ \tag{2.4.1e}$$
$$N_+' = (N^2 - 1 - \Sigma_+)N_+ + \sqrt{3}\Sigma_- N_- \tag{2.4.1f}$$
$$\Sigma_-' = N^2\Sigma_- - \sqrt{3}N_- \left(N_+ - N_1\right), \tag{2.4.1g}$$

allowing us to further compute

$$\frac{r_1'}{r_1} = \frac{\Sigma_-\Sigma_-' + N_- N_-'}{r_1^2} = N^2 - (\Sigma_+ + 1)\frac{N_-^2}{r_1^2} + \sqrt{3}N_1\frac{\Sigma_- N_-}{r_1^2} \tag{2.4.2a}$$
$$\psi_1{}' = \frac{\Sigma_- N_-' - N_-\Sigma_-'}{r_1^2} = \sqrt{3}N_+ - (\Sigma_+ + 1)\frac{N_-\Sigma_-}{r_1^2} - \sqrt{3}N_1\frac{N_-^2}{r_1^2} \tag{2.4.2b}$$
$$\frac{\delta_1'}{\delta_1} = N^2 - (\Sigma_+ + 1) \tag{2.4.2c}$$
$$\mathrm{D}_t \log\frac{\delta_1}{r_1} = -(\Sigma_+ + 1)\frac{\Sigma_-^2}{r_1^2} - \sqrt{3}N_1\frac{\Sigma_- N_-}{r_1^2}. \tag{2.4.2d}$$

Near $\mathbf{T}_1$, i.e. for $\Sigma_+ \approx -1$, we can rearrange some terms in (2.4.2), using

$$1 + \Sigma_+ = \frac{\Sigma_-^2 + N^2}{1 - \Sigma_+} = \frac{r_1^2 + N_1(N_1 - 2N_+)}{1 - \Sigma_+}. \tag{2.4.3}$$

This yields

$$\frac{r_1'}{r_1} = r_1^2 \sin^2\psi_1 \frac{-\Sigma_+}{1 - \Sigma_+} + N_1\, h_r \tag{2.4.4a}$$
$$\frac{\delta_1'}{\delta_1} = \frac{-1}{1 - \Sigma_+} r_1^2 \cos^2\psi_1 + \frac{-\Sigma_+}{1 - \Sigma_+} r_1^2 \sin^2\psi_1 + N_1\, h_\delta \tag{2.4.4b}$$
$$\mathrm{D}_t \log\frac{\delta_1}{r_1} = \frac{-1}{1 - \Sigma_+} r_1^2 \cos^2\psi_1 + N_1(h_\delta - h_r) \tag{2.4.4c}$$
$$\psi_1{}' = \sqrt{3}r_1\sqrt{\sin^2\psi_1 + \frac{\delta_1^2}{r_1^2}} - \frac{r_1^2}{1 - \Sigma_+}\sin\psi_1\cos\psi_1 + N_1 \sin\psi_1\, h_\psi, \tag{2.4.4d}$$

where

$$h_r = +\sqrt{3}\frac{\Sigma_- N_-}{r_1^2} + (N_1 - 2N_+)\left(1 - \frac{N_-^2}{(1 - \Sigma_+)r_1^2}\right) \tag{2.4.5a}$$
$$h_\delta = (N_1 - 2N_+)\frac{-\Sigma_+}{1 - \Sigma_+} \tag{2.4.5b}$$
$$h_\psi = -\sqrt{3}\sin\psi_1 - \cos\psi_1 \frac{N_1 - 2N_+}{1 - \Sigma_+}. \tag{2.4.5c}$$

Let us recall the convention that formally meaningless equations are to be ignored, e.g. (2.4.4d) if $r_1 = 0$; time-derivatives of logarithms, like e.g. (2.4.2a), are to be interpreted as $r_1' = \left[N^2 - (\Sigma_+ + 1)\ldots\right] r_1$.

Claim 2.4.1. We have $|h_r|, |h_\delta|, |h_\psi| \le 5$ if $|N_1|, |N_+|, |N_-| \le \frac{1}{2}$ and $\Sigma_+ < 0$.

Proof. The equations (2.4.4) and (2.4.5) follow directly from (2.4.2) by plugging in (2.4.3), and collecting the "morally higher order" terms into (2.4.5). The estimates can be verified, and improved upon, by direct calculation. □

Polar Coordinates around the non-invariant generalized Taub-spaces. Without loss of generality, assume that we are in $\mathcal{M}_{*+-}$, i.e. $N_2 > 0 > N_3$, and that we are interested in a neighbourhood of $\mathcal{T}_1^G = \{\boldsymbol{x} : \Sigma_- = 0, |N_2| - |N_3| = 0\}$.

The assumption $N_2 > 0 > N_3$ is incompatible with $N_3 = N_2$; hence, $\mathcal{M}_{*+-} \cap \mathcal{T}_1 = \emptyset$, which is why we have to work with $\mathcal{T}_1^G$. Since $\mathcal{T}_1^G$ is not invariant, we expect the corresponding equations for r_1' and $\psi_1{}'$ to become singular near $r_1 = 0$.

We will proceed analogous to the case of $\mathcal{T}_1$, using the same names for quantities which fulfil the same function in this work, such that the definitions of e.g. N_+, N_- will depend on the signs $\hat{n}_2, \hat{n}_3$. Hence, we introduce shorthands

$$N_- = |N_2| - |N_3| = N_2 + N_3, \qquad N_+ = |N_2| + |N_3| = N_2 - N_3, \tag{2.4.6a}$$

such that (with (2.3.7)):

$$r_1 \ge 0 : \qquad r_1^2 = \Sigma_-^2 + N_-^2 \tag{2.4.6b}$$

$$\psi_1 : \qquad N_- = r_1 \sin\psi_1 \qquad \Sigma_- = r_1 \cos\psi_1 \tag{2.4.6c}$$

$$N_+^2 = N_-^2 + \delta_1^2 \qquad N^2 = N_+^2 + N_1(N_1 - 2N_-). \tag{2.4.6d}$$

This gives us the differential equations (using $\Sigma^2 + N^2 = 1$):

$$N_-' = (N^2 - 1 - \Sigma_+)N_- + \sqrt{3}\Sigma_- N_+ \tag{2.4.6e}$$

$$N_+' = (N^2 - 1 - \Sigma_+)N_+ + \sqrt{3}\Sigma_- N_- \tag{2.4.6f}$$

$$\Sigma_-' = N^2\Sigma_- - \sqrt{3}\left(N_- N_+ - N_+ N_1\right), \tag{2.4.6g}$$

allowing us to further compute

$$\frac{r_1'}{r_1} = \frac{\Sigma_-\Sigma_-' + N_- N_-'}{r_1^2} = N^2 - (\Sigma_+ + 1)\frac{N_-^2}{r_1^2} + \sqrt{3}N_1\frac{\Sigma_- N_+}{r_1^2} \tag{2.4.7a}$$

$$\psi_1{}' = \frac{\Sigma_- N_-' - N_-\Sigma_-'}{r_1^2} = \sqrt{3}N_+ - (\Sigma_+ + 1)\frac{N_-\Sigma_-}{r_1^2} - \sqrt{3}N_1\frac{N_- N_+}{r_1^2} \tag{2.4.7b}$$

$$\frac{\delta_1'}{\delta_1} = N^2 - (\Sigma_+ + 1) \tag{2.4.7c}$$

$$\mathrm{D}_t \log\frac{\delta_1}{r_1} = -(\Sigma_+ + 1)\frac{\Sigma_-^2}{r_1^2} - \sqrt{3}N_1\frac{\Sigma_- N_+}{r_1^2}. \tag{2.4.7d}$$

Near $\mathbf{T}_1$, i.e. for $\Sigma_+ \approx -1$, we can rearrange some terms in (2.4.7), using

$$1 + \Sigma_+ = \frac{\Sigma_-^2 + N^2}{1 - \Sigma_+} = \frac{r_1^2 + \delta_1^2 + N_1(N_1 - 2N_-)}{1 - \Sigma_+}. \tag{2.4.8}$$

$$\frac{r_1'}{r_1} = \frac{-\Sigma_+}{1-\Sigma_+} r_1^2 \sin^2\psi_1 + \delta_1^2 \frac{\cos^2\psi_1 - \Sigma_+}{1-\Sigma_+} + N_1\, h_r \tag{2.4.9a}$$

$$\frac{\delta_1'}{\delta_1} = \frac{-1}{1-\Sigma_+} r_1^2 \cos^2\psi_1 + \frac{-\Sigma_+}{1-\Sigma_+} r_1^2 \sin^2\psi_1 + \frac{-\Sigma_+}{1-\Sigma_+}\delta_1^2 + N_1\, h_\delta \tag{2.4.9b}$$

$$\mathrm{D}_t \log\frac{\delta_1}{r_1} = \frac{-1}{1-\Sigma_+} r_1^2 \cos^2\psi_1 - \delta_1^2 \frac{\cos^2\psi_1}{1-\Sigma_+} + N_1(h_\delta - h_r) \tag{2.4.9c}$$

$$\psi_1{}' = \sqrt{3} r_1 \sqrt{\sin^2\psi_1 + \frac{\delta_1^2}{r_1^2}} - \frac{r_1^2+\delta_1^2}{1-\Sigma_+}\cos\psi_1 \sin\psi_1 + N_1 \sin\psi_1\, h_\psi, \tag{2.4.9d}$$

where

$$h_r = +\sqrt{3}\frac{\Sigma_- N_+}{r_1^2} + (N_1 - 2N_-)\left(1 - \frac{N_-^2}{(1-\Sigma_+)r_1^2}\right) \tag{2.4.10a}$$

$$h_\delta = (N_1 - 2N_-)\frac{-\Sigma_+}{1-\Sigma_+} \tag{2.4.10b}$$

$$h_\psi = -\sqrt{3}\sqrt{\sin^2\psi_1 + \frac{\delta_1^2}{r_1^2}} - \cos\psi_1 \frac{N_1 - 2N_-}{1-\Sigma_+}. \tag{2.4.10c}$$

Claim 2.4.2. If $|N_1|, |N_+|, |N_-| \leq \frac{1}{2}$ and $\Sigma_+ < 0$ and $r_1 > 0$, and $\frac{\delta_1}{r_1} \leq 1$, then $|h_r|, |h_\delta|, |h_\psi| \leq 5$.

Proof. The equations (2.4.9) and (2.4.10) follow directly from (2.4.7) by plugging in (2.4.8), and collecting the "morally higher order" terms into (2.4.10). In order to see the inequalities, note that $N_+ = \sqrt{N_-^2 + \delta_1^2}$ and hence $N_+ \leq \sqrt{2} r_1$. Then, the estimates can be verified, and improved upon, by direct calculation. □

3 Description of the Dynamics

We will now give an overview of the behaviour of trajectories of (2.3.2). This overview will contain most of the classic results about Bianchi type A vacuum cosmological models.

Our overview will be organized by first describing the simplest subsets named in Table 2 and then progressing to the higher dimensional subsets, finally describing Bianchi Type IX ($\mathcal{M}_{+++}$) and Bianchi Type VIII ($\mathcal{M}_{++-}$) solutions. Our approach in this section is very similar to [Rin01] and [HU09b]; unless explicitly otherwise stated, all observations in this section can be found therein.

A very short summary of relevant dynamics. The Kasner circle $\mathcal{K}$ is actually a circle and consists entirely of equilibria. The so-called Mixmaster attractor $\mathcal{A}$ consists of three 2-spheres $\{\boldsymbol{x} : \Sigma_+^2 + \Sigma_-^2 + N_i^2 = 1,\ N_j = N_k = 0\}$, which intersect in $\mathcal{K}$. Only half of these spheres are accessible for any trajectory, since the sign N_i are fixed. For this reason, these half-spheres are also called "Kasner-caps", i.e. $\mathcal{A}_{+00}$ is the $N_1 > 0$-cap.

The dynamics on the Kasner-caps will be discussed in Section 3.1; each orbit in a Kasner-cap is a heteroclinic orbit connecting two equilibria on the Kasner-circle $\mathcal{K}$.

The long-time behaviour of the lower dimensional Bianchi-types (at least one $N_i = 0$) is well-understood: All such solutions converge to an equilibrium $\boldsymbol{p} \in \mathcal{K}$ as $t \to \infty$. The behaviour in the highest-dimensional Bianchi Types IX and VIII is not yet fully understood, and is therefore of most interest in this work.

It is known (c.f. [Rin01]) that Bianchi Type IX solutions that do not lie in a Taub-space converge to the Mixmaster attractor as $t \to +\infty$, i.e. towards the big bang singularity. It has been conjectured that generic Bianchi Type VIII solutions share this behaviour; this will be proven in this work (Theorem 4 and 5).

The question of particle horizons was mentioned in the introduction and further discussed from a physical viewpoint in Section 2.2. In terms of the Wainwright-Hsu equations, the question can be formulated as (see Section 2.2, or c.f. e.g. [HR09]):

$$\text{Is} \quad I(\boldsymbol{x}) = \max_i \int_0^\infty \delta_i(\phi(\boldsymbol{x}, t))\mathrm{d}t = 2 \max_i \int_0^\infty \sqrt{|N_j N_k|}(t)\mathrm{d}t < \infty\,?$$

Here ϕ is the flow to (2.3.2). The spacetime associated to the solution $\phi(\boldsymbol{x}, \cdot)$ forms a particle horizon if and only if $I < \infty$.

It is known that there exist solutions in Bianchi IX and VIII, where $I < \infty$ (c.f. [LHWG11]). It is not known, whether there exist any nontrivial solutions with $I = \infty$ (it is known that solutions starting in $\mathcal{T}_1$ in Bianchi VIII and IX converge to an equilibrium outside of $\mathcal{A}$, and hence have $I = \infty$). We prove that, in both Bianchi IX and VIII and for Lebesgue almost every initial condition, particle horizons develop ($I < \infty$) (Theorem 6).

3.1 Lower Bianchi Types

Bianchi Type I: The Kasner circle. The smallest, i.e. lowest dimensional, Bianchi-type is Type I, $\mathcal{M}_{000} = \mathcal{K}$, where all three N_i vanish (see Table 2). By the constraint $N^2 + \Sigma^2 = 1$, we can see that $\mathcal{K}$ is the unit circle in the (Σ_+, Σ_-)-plane, and consists entirely of equilibria.

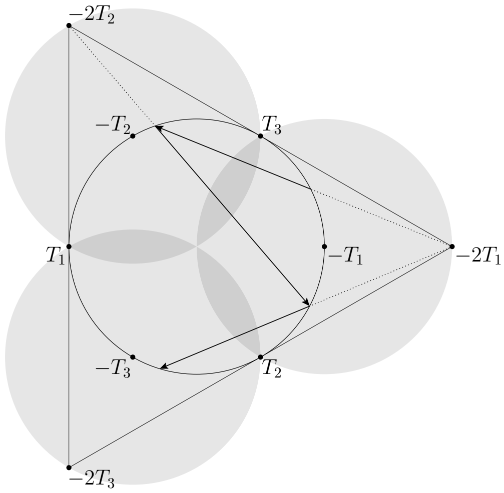
The linear stability of these equilibria is given by the following

Lemma 3.1. *Let $\boldsymbol{p} = (\Sigma_+, \Sigma_-, 0, 0, 0) \in \mathcal{K}$; first consider the case $\boldsymbol{p} \neq \mathbf{T}_i$ and without loss of generality $|\boldsymbol{p} + \mathbf{T}_1| < 1$. Then the vectorfield has one central direction given by $\partial_{\mathcal{K}} = (-\Sigma_-, \Sigma_+, 0, 0, 0)$, one unstable direction given by $\partial_{N_1} = (0, 0, 1, 0, 0)$, and two stable directions given by $\partial_{N_2} = (0, 0, 0, 1, 0)$ and $\partial_{N_3} = (0, 0, 0, 0, 1)$.*

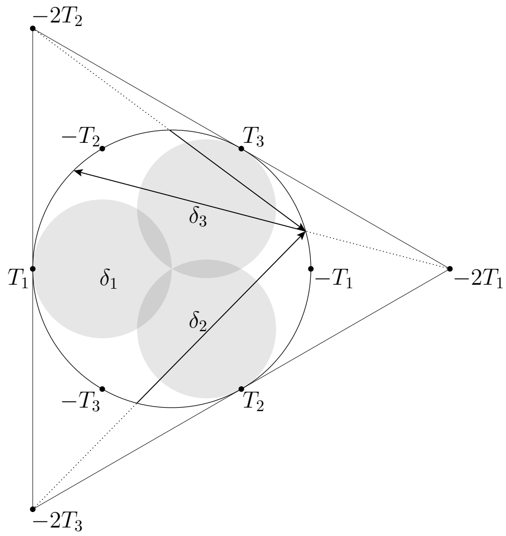
The three Taub-points $\mathbf{T}_i$ have each one stable direction given by ∂_{N_i} and three center directions given by $\partial_{\mathcal{K}}$, ∂_{N_j} and ∂_{N_k}.

Proof. We first note that the four vectors $\partial_{\mathcal{K}}$ and ∂_{N_i} form a basis of the tangent space $T_{\boldsymbol{p}}\mathcal{M} = \ker \mathrm{d}G$.

The stability of an equilibrium is determined by the eigenvalues and eigenspaces of the Jacobian of the vector field; a generalized eigenspace is *central*, if its eigenvalue has vanishing real part, it is *stable* if its eigenvalue has negative real part, and it is *unstable* if its eigenvalue has positive real part. The Jacobian $\mathrm{D}f$ of the vector field f given by (2.3.2) at $\boldsymbol{p} \in \mathcal{K}$ is diagonal with three entries of the form $\mathrm{D}f = \lambda_1 \partial_{N_1} \otimes \mathrm{d}N_1 + \lambda_{N_2} \partial_2 \otimes \mathrm{d}N_2 + \lambda_3 \partial_{N_3} \otimes \mathrm{d}N_3$; we can read off the stability from (2.3.2b) and Figure 4a. $\square$



(a) The three discs, where $N_i'/N_i > 0$ and a short heteroclinic chain.



(b) The three discs, where $\delta_i'/\delta_i > 0$. Also, the Kasner-map K is a double cover.

Figure 4: Stability properties and the Kasner map. All figures are in (Σ_+, Σ_-)-projection.

The Taub-line. There exists another structure of equilibria, given by $\mathcal{TL}_i := \{\boldsymbol{x} \in \mathcal{M} : N_j = N_k,\, N_i = 0,\, \boldsymbol{\Sigma} = \mathbf{T}_i\}$. Up to index permutations and $\boldsymbol{N} \to -\boldsymbol{N}$, this set has the form $\mathcal{TL}_1 \cap \mathcal{M}_{0++} = \{\boldsymbol{p} : \boldsymbol{p} = (-1, 0, 0, n, n),\, n > 0\}$. We observe that this is a line of equilibria. Each such equilibrium has one stable direction (corresponding to N_1) and three center directions.

Bianchi Type II: The Kasner caps. Consider without loss of generality the set $\mathcal{M}_{+00}$, i.e. $N_1 > 0 = N_2 = N_3$. The constraint $G = 1$ then reads

$$1 = N_1^2 + \Sigma_+^2 + \Sigma_-^2,$$

i.e. the so-called "Kasner-cap" $\mathcal{M}_{+00}$ forms a half-sphere with $\mathcal{K}$ as its boundary. Considering (2.3.2), we can see that $\boldsymbol{\Sigma}' = (\boldsymbol{\Sigma} + 2\mathbf{T}_1)N_1^2$ is a scalar multiple of $\boldsymbol{\Sigma} + 2\mathbf{T}_1$; hence, the $\boldsymbol{\Sigma}$-projection of the trajectory stays on the same line through $-2\mathbf{T}_1$. Since $N_1^2 \geq 0$, this trajectory is heteroclinic and converges in forward and backward times to the two intersections of this line with the Kasner circle $\mathcal{K}$, where the α-limit is closer to $-2\mathbf{T}_1$. Such a trajectory is depicted in Figure 4a.

The Mixmaster-Attractor $\mathcal{A}$. The most relevant set for the long-time behaviour of the Bianchi system is the Mixmaster attractor $\mathcal{A}$. It consists of the union of all six Bianchi II pieces and the Kasner circle. Since the signs of the N_i stay constant along trajectories, it is useful to only study a piece of $\mathcal{A}$, given without loss of generality by:

$$\mathcal{A}_{+++} = \mathcal{M}_{+00} \cup \mathcal{M}_{0+0} \cup \mathcal{M}_{00+} \cup \mathcal{K}.$$

The set $\mathcal{A}_{+++}$ is given by the union of three perpendicular half-spheres (the three Kasner-caps), which intersect in the Kasner-circle. The dynamics in the Mixmaster-Attractor now consists of the Kasner circle $\mathcal{K}$ of equilibria and the three caps, consisting of heteroclinic orbits to $\mathcal{K}$. It is described in detail by the so-called Kasner-map.

The Kasner-map $K : \mathcal{K} \to \mathcal{K}$. We wish to describe which equilibria in $\mathcal{K}$ are connected by heteroclinic orbits. We can collect this in a relation $K \subseteq \mathcal{K} \times \mathcal{K}$, i.e. we write $\boldsymbol{p}_- K \boldsymbol{p}_+$ if either $\boldsymbol{p}_- = \boldsymbol{p}_+ = \mathbf{T}_i$ or there exists a heteroclinic orbit $\gamma : \mathbb{R} \to \mathcal{A}$ such that $\boldsymbol{p}_- = \lim_{t\to-\infty} \gamma(t)$ and $\boldsymbol{p}_+ = \lim_{t\to+\infty} \gamma(t)$.

We can see by Figure 4a (or Lemma 3.1) that each non-Taub point $\boldsymbol{p}_-$ has a one-dimensional unstable manifold, i.e. one trajectory in $\mathcal{A}_{\hat{n}}$, which converges to $\boldsymbol{p}_-$ in backwards time. Therefore, the relation K can be considered as a (single-valued, everywhere defined) map.

This map is depicted in Figure 4b, and has a simple geometric description in the (Σ_+, Σ_-)-projection: Given some $\boldsymbol{p}_- \in \mathcal{K}$, we draw a straight line through $\boldsymbol{p}_-$ and the nearest of the three points $-2\mathbf{T}_i$. This line has typically two intersections $\boldsymbol{p}_-$ and $\boldsymbol{p}_+ \in \mathcal{K}$ with the Kasner-circle, one of which is nearer to $-2\mathbf{T}_i$, which is $\boldsymbol{p}_-$, and one which is further away, which is $\boldsymbol{p}_+$. At a $\mathbf{T}_i$, there are two possible choices of nearest $-2\mathbf{T}_j$ and $-2\mathbf{T}_k$ and the lines through these points are tangent to $\mathcal{K}$; we just set $K(\mathbf{T}_i) = \mathbf{T}_i$. We see from Figure 4b that this map $K : \mathcal{K} \to \mathcal{K}$ is continuous and a double cover (i.e. each point $\boldsymbol{p}_+$ has two preimages $\boldsymbol{p}_-^1$ and $\boldsymbol{p}_-^2$, which both depend continuously on $\boldsymbol{p}_+$).

Looking at Figure 4a, we can also see that the Kasner-map is expanding. Hence it is C^0-conjugate to either $[z]_\mathbb{Z} \to [2z]_\mathbb{Z}$ or $[z]_\mathbb{Z} \to [-2z]_\mathbb{Z}$; since it has three fixed points the latter case must apply. Hence we have

Proposition 3.2. *There exists a homeomorphism $\Psi : \mathcal{K} \to \mathbb{R}/3\mathbb{Z}$, such that $\Psi(\mathbf{T}_i) = [i]_{3\mathbb{Z}}$ and*

$$\Psi(K(\boldsymbol{p})) = [-2\Psi(\boldsymbol{p})]_{3\mathbb{Z}} \qquad \forall \boldsymbol{p} \in \mathcal{K}.$$

A formal proof of Proposition 3.2 is a digression; for this reason, it is deferred until Section A.2, where we give a more detailed description of the Kasner map.

Basic Heuristics near the Mixmaster Attractor. Heuristically, the Kasner-map determines the behaviour of solutions near $\mathcal{A}$: Consider an initial condition $\boldsymbol{x}_0 \in \mathcal{M}_{\pm\pm\pm}$ near $\mathcal{A}$, i.e. an initial condition where none of the N_i vanish. Then the trajectory $\boldsymbol{x}(t)$ will closely follow the heteroclinic solution γ_1 passing near $\boldsymbol{x}_0$. Let $\boldsymbol{p}_1$ be the end-point of this heteroclinic; $\boldsymbol{x}(t)$ will follow γ_1 and stay for some time near $\boldsymbol{p}_1$ (since it is an equilibrium). However, if $\boldsymbol{p}_1 \neq \mathbf{T}_i$, then one of the N_i directions is unstable; therefore, $\boldsymbol{x}(t)$ will leave the neighbourhood of $\boldsymbol{p}_1$ along the unique heteroclinic emanating from $\boldsymbol{p}_1$, and follow it until it is near $\boldsymbol{p}_2 = K(\boldsymbol{p}_1)$. This should continue until $\boldsymbol{x}(t)$ leaves the vicinity of $\mathcal{A}$, which should not happen at all (at least if the name "Mixmaster Attractor" is well deserved).

The expansion of the Kasner-map is the source of the (so far heuristic) chaoticity of the dynamics of the Bianchi system: The expansion along the Kasner-circle supplies the sensitive dependence on initial conditions, while the remaining two directions are contracting. The last direction is neutral and corresponds to the time-evolution.

Bianchi-Types VII_0 and VI_0. There are two Bianchi-types, where exactly one of the three N_i vanishes: Types VII_0 and VI_0. In these Bianchi-Types, we have monotone functions (Lyapunov functions), which suffice to almost completely determine the long-time behaviour of trajectories. Without loss of generality, we focus on the case where $N_1 = 0$. Then we can write

$$\Sigma_+' = (1-\Sigma^2)(\Sigma_+ + 1), \qquad 1 = \Sigma^2 + (N_2 - N_3)^2. \tag{3.1.1}$$

We can immediately see that Σ_+ is non-decreasing along trajectories; indeed, we must have $\lim_{t\to\pm\infty} \Sigma^2(t) = 1$ for all trajectories. Considering (2.3.2b) and Figure 4a, we can see that, for $t \to +\infty$ we must either have $\Sigma_+ \geq \frac{1}{2}$ and $N_2, N_3 \to 0$ (since $(N_2 - N_3)^2 \to 0$) or $\Sigma_+ = -1$ and $N_2 = N_3$ all along; then the trajectory lies in the Taub-line $\mathcal{TL}_1$. In backward time, we must have $\Sigma_+ \to -1$: All other points on the Kasner-circle have one of the two N_2, N_3 unstable. These statements can be formalized as

Lemma 3.3. *Consider an initial condition $\boldsymbol{x}_0$ with $\boldsymbol{x}_0 \in \mathcal{M}_{0+-}$, i.e. $N_2 > N_1 = 0 > N_3$. Then, in forward time, the trajectory $\boldsymbol{x}(t) = \phi(\boldsymbol{x}_0, t)$ converges to a point on the Kasner-circle $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{p}_+ \in \mathcal{K}$ with $\Sigma_+(\boldsymbol{p}_+) > \frac{1}{2}$. In backwards time, the trajectory converges to the Taub-point $\mathbf{T}_1 = (-1,0,0,0,0) = \lim_{t\to-\infty} \boldsymbol{x}(t)$.*

Lemma 3.4. *Consider an initial condition $\boldsymbol{x}_0$ with $\boldsymbol{x}_0 \in \mathcal{M}_{0++} \setminus \mathcal{TL}_1$. Then, in forward time, the trajectory $\boldsymbol{x}(t) = \phi(\boldsymbol{x}_0, t)$ converges to a point on the Kasner-circle $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{p}_+ \in \mathcal{K}$ with $\Sigma_+(\boldsymbol{p}_+) > \frac{1}{2}$. In backwards time, the $\boldsymbol{\Sigma}$-projection of the trajectory converges to the Taub-point $\mathbf{T}_1 = (-1, 0) = \lim_{t\to-\infty} \boldsymbol{\Sigma}(\boldsymbol{x}(t))$. No claim about the dynamics of the $N_i(t)$ for $t \to -\infty$ is made.*

Proof of Lemma 3.3 and 3.4. Most of the proof is contained in the preceding paragraph; we only need to exclude the possibiliy that $\Sigma_+(\boldsymbol{p}_+) = \frac{1}{2}$, i.e. $\lim_{t\to\infty} \boldsymbol{x}(t) \in \{\mathbf{T}_2, \mathbf{T}_3\}$. In order to exclude this case, we permute indices such that $\boldsymbol{x}_0 \in \mathcal{M}_{\pm 0 \pm}$ and exclude $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{T}_1$. Recall the polar coordinates (2.4.4a); using $N_2 = 0$ we have $N_- = N_+$ and can simplify to

$$\mathrm{D}_t \log r_1 = r_1^2 \sin^2\psi_1 \frac{-\Sigma_+}{1-\Sigma_+} + \sqrt{3} N_1 \frac{\Sigma_- N_-}{r_1^2} + N_1(N_1 - 2N_+)\left(1 - \frac{N_-^2}{(1-\Sigma_+) r_1^2}\right) \geq -C|N_1|.$$

However, $|N_1|' \leq -|N_1|$ near $\mathbf{T}_1$; hence, integrating this inequality shows that we cannot have $\lim_{t\to\infty} r_1(t) = 0$. $\square$

It can be shown that, in the case of Bianchi VII_0, i.e. $\boldsymbol{x}_0 \in \mathcal{M}_{0++} \setminus \mathcal{TL}_1$, the limit $\lim_{t\to-\infty} \boldsymbol{x}(t) = \boldsymbol{p} \in \mathcal{TL}_1 \setminus \{\mathbf{T}_1\}$ exists and does not lie on the Kasner circle. This claim follows directly from Lemma 5.6; however, the proof of Lemma 5.6 is rather lengthy and not required for our main results.

3.2 Bianchi-Types VIII and IX for large N

As we have seen, the lower Bianchi types do not support recurrent dynamics. This is different in the two top-dimensional Bianchi-types VIII and IX. This section is devoted to first describing the behaviour far from $\mathcal{A}$, where $|N|$ may be large.

First, recall the definition and differential equation (2.3.8a), $\mathrm{D}_t\Delta = -|\boldsymbol{\Sigma}|^2\Delta$ for $\Delta = |N_1N_2N_3|^{\frac{1}{3}}$ and Lemma 2.4. This tells us that the triple product $|N_1N_2N_3|$ is monotonically decreasing. This decrease is strict, because, by elementary calculation, $\boldsymbol{\Sigma} = 0$ implies $N^2 = 1$ and $\boldsymbol{\Sigma}' \neq 0$.

Also, observe the following equation, easily derived from (2.3.2) and (2.3.3):

$$\mathrm{D}_t\Sigma_+ = (1-\Sigma^2)(\Sigma_+ + 1) + 3N_1(N_2 + N_3 - N_1). \tag{3.2.1}$$

Using these observations, we can easily see the following:

Lemma 3.5 (Long-Time Existence)**.** *Every solution $\boldsymbol{x} : [0,T) \to \mathcal{M}$ of* (2.3.2) *has unbounded forward existence time (i.e. no finite-time blow-up occurs towards the future, i.e. towards the big bang singularity).*

If $\boldsymbol{x} \notin \mathcal{M}_{+++} \cup \mathcal{M}_{---}$ is not of Bianchi type IX*, then the solution also has unbounded backward existence time.*

Proof. The only way that long-term existence can fail is finite-time blow-up, i.e. $\lim_{t\to T_{\max}} |\boldsymbol{x}(t)| = \infty$ for some $0 < T_{\max} < \infty$. We cannot exclude this possibility a priori, since the vectorfield given by (2.3.2) is polynomial.

From (2.3.2) we see that $|\mathrm{D}_tN| \leq C(1+\Sigma^2)|N|$. The claim follows from $\Delta(t) \leq \Delta(0)$ for all $t \geq 0$, apparent from (2.3.8a), and Lemma 2.4.

$\square$

Finite-time blow-up in backwards time for Bianchi IX is not excluded, and can indeed occur ("recollapse" of the described universe, c.f. e.g. [Rin01]). In Appendix A.3, we address the notational problems posed by the failure of long-time backwards existence.

We have seen in (2.3.8a) that the triple product $|N_1N_2N_3|$ is strictly decreasing; we will now show that this decrease is exponential. In the following, it is useful to imagine $C_{\Delta,3.1} = 1$:

Lemma 3.6 (Essentially exponential convergence of $|N_1N_2N_3|$)**.** *For every $C_{\Delta,3.1} > 0$, there exists $c_{3.1} = c_{3.1}(C_{\Delta,3.1}) > 0$ such that for all trajectories $\boldsymbol{x} : [t_1, t_2] \to \mathcal{M}$ with $\Delta(t_1) \leq C_{\Delta,3.1}$ we have*

$$\Delta(t_2) \leq 2e^{-c_{3.1}(t_2-t_1)}\Delta(t_1).$$

Proof. We consider the averaged rate of decay $s : \mathcal{M} \to [0, \infty)$, given by

$$s(\boldsymbol{y}) = \int_0^1 \Sigma^2(\phi(\boldsymbol{y}, t))\mathrm{d}t.$$

Suppose we had shown a lower bound $s(\boldsymbol{y}) > c_{3.1} \in (0, 0.5)$ for all $\boldsymbol{y} \in \mathcal{M}$ with $\Delta(\boldsymbol{y}) \leq C_{\Delta,3.1}$. Then, for $t_2 = t_1 + k + t$ with $k \in \mathbb{N}$ and $t \in [0, 1)$:

$$-\log \frac{\Delta(t_2)}{\Delta(t_1)} = \int_{t_1}^{t_2} \Sigma^2(t)\mathrm{d}t \geq kc_{3.1} \geq (t_2 - t_1 - 1)c_{3.1}$$

and the assertion would follow. The remainder of this proof is devoted to showing such a bound.

At first we note that $s(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathcal{M}$, since $\boldsymbol{\Sigma} = 0$ implies $N^2 = 1$ and $\boldsymbol{\Sigma}' \neq 0$, as we remarked at the beginning of Section 3.2. Since s is clearly continuous it is uniformly bounded above zero on all compact subsets. Hence, the only potential problems are for $\boldsymbol{x} \to \infty$, with bounded $\Delta(\boldsymbol{x}) \leq C_{\Delta,3.1}$; there, we must exclude the possibility that $s(\boldsymbol{x}) \to 0$.

In view of Lemma 2.4 we have, without loss of generality with respect to index permutations, $|N_1| \ll 1 \ll |N_2| \approx |N_3|$, and $|N_1|(|N_2| + |N_3|) \ll 1$ for $\boldsymbol{x} \to \infty$ with bounded $\Delta(\boldsymbol{x}) \leq C_{\Delta,3.1}$. These inequalities are preserved for one unit of time, since $\mathrm{D}_t|N_i| \leq C(1 + |\boldsymbol{\Sigma}|^2)|N_i|$.

Then, (3.2.1) yields $\mathrm{D}_t\Sigma_+(\boldsymbol{x}) \approx (1-|\boldsymbol{\Sigma}|^2)(\Sigma_+ +1)$. This clearly contradicts $s(\boldsymbol{x}) \to 0$: Assume $s(\boldsymbol{x}_0) \approx 0$; then, integration shows $\Sigma_+(t) \approx t + \Sigma_+(0)$ for $t \in [0, 1]$.

In order to see this more rigorously, assume $s(\boldsymbol{x}_0) < \epsilon_1$ and $|N_1|(|N_2| + |N_2|) < \epsilon_2$, for all $\boldsymbol{x}(t) = \phi(\boldsymbol{x}, t)$ with $t \in [0, 1]$, and estimate $\Sigma_+(t) - \Sigma_+(0) \geq t - 3\epsilon_2 t - \epsilon_1 - \sqrt{t\epsilon_1} - 2\epsilon_1(1 + 2\Delta^2(0))$. Here, we used that generally $\int_0^a |f|(b)\mathrm{d}b \leq \sqrt{a \int_0^a f^2(b)\mathrm{d}b}$, and that $|\boldsymbol{\Sigma}| \leq 1+2\Delta^2$. This implies, for $t \in [0.5, 1]$, that $\Sigma_+(t)-\Sigma_+(0) \geq 0.5-c\epsilon_2-C\epsilon_1(1+c_{3.1})$. Adjusting ϵ_2 (from $\boldsymbol{x} \to \infty$) and ϵ_1 (our claim) yields the desired contradiction, and concludes the proof.

□

This result, i.e. Lemma 3.6, is not as explicitly stated in the previous works [Rin01, HU09b], and certainly not as extensively used, but is not a novel insight either. It directly proves that metric coefficients stay bounded, see Section 2.2.

Using Lemma 3.6, we can relatively quickly see the following:

Lemma 3.7 (Existence of ω-limits)**.** *For every initial condition $\boldsymbol{x}_0 \in \mathcal{M}$, the ω-limit set is nonempty, $\omega(\boldsymbol{x}_0) \neq \emptyset$, i.e. there exists a sequence of times $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ such that the limit $\lim_{n\to\infty} \boldsymbol{x}(t_n)$ exists.*

Proof. The only way of avoiding the existence of an ω-limit is to have $\lim_{t\to\infty} |\boldsymbol{x}(t)| = \infty$. Assume that we had such a solution $\boldsymbol{x} = \boldsymbol{x}(t)$, where without loss of generality $|N_1N_2N_3|(t) \leq 1$ for all $t \geq 0$; we will derive a contradiction. Recalling Lemma 2.4, we see that $|N_1| \to 0$ and $|N_2|, |N_3| \to \infty$, without loss of generality with respect to index permutations. Recall and estimate, using Lemma 2.4 and Lemma 3.6:

$$\mathrm{D}_t \Sigma_+(t) = (1-\Sigma^2)(\Sigma_+ + 1) + 3N_1(N_2 + N_3 - N_1) \tag{3.2.1}$$
$$\geq |(1-\Sigma^2)(\Sigma_+ + 1)| - Ce^{-ct} \tag{3.2.2}$$
$$\mathrm{D}_t \log \delta_1(t) = -(\Sigma^2 + \Sigma_+). \tag{3.2.3}$$

Suppose that $\Sigma_+ \to -1$ as $t \to +\infty$. Then, by (3.2.2), this convergence must be exponentially fast. This means that the right-hand side of (3.2.3) decays exponentially, contradicting the assumption $\delta_1 = 2\sqrt{|N_2 N_3|} \to \infty$.

Suppose on the other hand that $\Sigma_+ \not\to -1$ as $t \to \infty$. Since $|\boldsymbol{\Sigma}|^2 \leq 4\Delta^2$, we then must have $1 + \Sigma_+(t) > \epsilon$ for some $\epsilon > 0$ for all sufficiently large times $t \geq T_0$. Informally, we can see from Figure 4b that, with $1 + \Sigma_+ > \epsilon$, either $\mathrm{D}_t\Sigma_+ \gg 0$ or $\mathrm{D}_t \log(\delta_1) \ll 0$, and therefore $C > \Sigma_+ > -1 + \epsilon$ contradicts $\delta_1 \to \infty$.

Formally, since Σ_+ is bounded, $\int_{T_0}^{\infty}(1-\Sigma^2)\epsilon \mathrm{d}t < \int_{T_0}^{\infty}\Sigma_+' \mathrm{d}t + C < \infty$, and hence $\int_{T_0}^{\infty}(1-\Sigma^2)\mathrm{d}t < \infty$. However, $\mathrm{D}_t \log(\delta_1) = -\Sigma^2 - \Sigma_+ \leq 1 - \Sigma^2 - \epsilon$ and integration shows $\lim_{t\to\infty} \delta_1(t) = 0$, contradicting the assumption $\delta_1 = 2\sqrt{|N_2 N_3|} \to \infty$.

□

3.3 The Bianchi IX Attractor Theorem

The Mixmaster Attractor was known at least since [Mis69]. However, the first proof that $\mathcal{A}$ actually is an attractor was given comparatively recently in [Rin01, Theorem 19.2, page 65], and simplified in [HU09b]. This important result is the following:

Theorem 1 (Classical Bianchi IX Attractor Theorem)**.** *Let $\boldsymbol{x}_0 \in \mathcal{M}_{+++} \setminus \mathcal{T}$. Then*

$$\lim_{t\to\infty} \mathrm{dist}(\boldsymbol{x}(t), \mathcal{A}) = 0.$$

Also, the ω-limit set $\omega(\boldsymbol{x}_0)$ contains a point $\boldsymbol{p} \in \mathcal{K} \setminus \mathcal{T}$.

The proofs of Theorem 1 given in [Rin01] and [HU09b] require some subtle averaging arguments (summarized as Lemma 5.6), which are lengthy and fail to generalize to the case of Bianchi VIII initial data. We will now give the first steps leading to the proof of Theorem 1, up to the missing averaging estimates for Bianchi IX solutions. Then, we will state the missing estimates and show how they prove Theorem 1. Afterwards, we will give a high-level overview of how we replace Lemma 5.6 in this work. Nevertheless, for the sake of completeness, we provide a proof of Lemma 5.6 in Section 5.3.

Rigorous steps leading to Theorem 1. We first show that solutions cannot converge to the Taub-line $\mathcal{TL}_i$, if they do not start in the Taub-space $\mathcal{T}_i$:

Lemma 3.8 (Taub Space Instability)**.** *There exists constants $\rho_{3.1} = 0.1$ and $C_{3.2} > 0$, such that the following holds:*

For any piece of trajectory $\boldsymbol{x} : [t_1, t_2] \to \{\boldsymbol{x} \in \mathcal{M} : |\boldsymbol{\Sigma}(\boldsymbol{x}) - \mathbf{T}_1| \leq \rho_{3.1}\}$ with $N_2, N_3 \geq 0$, the following estimate holds:

$$r_1(\boldsymbol{x}(t_2)) \geq e^{-C_{3.2} h(\boldsymbol{x}(t_1))} r_1(\boldsymbol{x}(t_1)), \qquad \textit{where} \tag{3.3.1a}$$

$$h(\boldsymbol{x}) = |N_1| + |N_1|^2 + |N_1 N_2| + |N_1 N_3|. \tag{3.3.1b}$$

Proof. This Lemma uses the polar coordinates introduced in Section 2.4, i.e. (2.4.4a) to see that:

$$\begin{aligned}
\mathrm{D}_t \log r_1 = & \frac{-\Sigma_+}{1-\Sigma_+} r_1^2 \sin^2 \psi_1 + \sqrt{3} N_1 \sin \psi_1 \cos \psi_1 \\
& \qquad + N_1 (N_1 - 2N_2 - 2N_3) \left(1 - \frac{\sin^2 \psi_1}{1 - \Sigma_+}\right) &(3.3.2a)\\
& \geq -C|N_1| - C|N_1|^2 - C|N_1 N_2| - C|N_1 N_3| &(3.3.2b)\\
\mathrm{D}_t \log |N_1| \approx & -3 \leq -1, &(3.3.2c)\\
\mathrm{D}_t \log N_2 \approx & \ \mathrm{D}_t \log N_3 \approx 0. &(3.3.2d)
\end{aligned}$$

The desired estimate then follows by integration.

Since this is the first of many proofs of the same style, let us be more verbose, once. Following the general convention explained on page 17, (3.3.2c) is to be interpreted as $\mathrm{D}_t N_1 = f_{N_1} N_1$ and $f_{N_1} \leq -1$, and similarly for the other equations. Hence, there is no problem of ill-definedness with (3.3.2a) in the case that one or more of r_1, N_1, N_2, N_3 are zero. The actual inequalities (3.3.2a) are easily verified by direct calculation.

Now, let us integrate. First, integrating (3.3.2c) shows $|N_1|(s) \leq e^{-(s-t_1)} |N_1|(t_1)$ for all $s \in [t_1, t_2]$, and an analogue inequality for $|N_1|^2$. Using the analog (3.3.2d), we see $h(s) \leq e^{-(s-t_1)} h(t_1)$ for all $s \in [t_1, t_2]$, and, using this, the claim follows from integrating (3.3.2b). □

Together with Lemma 3.7, this allows us to see that there are ω-limit points on $\mathcal{K}$. Even more, we can give quite a detailed description of how solutions might fail to have ω-limit points on $\mathcal{K}$. The following Lemma and its proof require a little bit more background knowledge from the theory of dynamical systems; this is all summarized in Appendix A.3, see especially Lemma A.2 for basic properties of ω-limit sets. The heteroclinic trajectory connecting $-\mathbf{T}_i \to \mathbf{T}_i$ is denoted by $W^u(-\mathbf{T}_i) = W^s(\mathbf{T}_i)$, i.e. it is the unstable manifold of $-\mathbf{T}_i$, and at the same time the stable manifold of $W^s(\mathbf{T}_i)$. Of course, there are two such trajectories, one with $N_i > 0$ and one with $N_i < 0$. Which one is meant will always be apparent from the context, and in an abuse of notation we suppress the implicit subscript $\hat{n} \in \{+1, -1, 0\}^3$ to $W^u_{\hat{n}}(\mathbf{T}_i)$.

Lemma 3.9. *Let $\boldsymbol{x} : [0, \infty) \to \in \mathcal{M}_{\pm\pm\pm}$ be a solution. Then, one of the following cases applies (for some i):*

1. *The* Attract *case. There exists a non-Taub ω-limit point $\boldsymbol{p} \in \omega(\boldsymbol{x}_0) \cap (\mathcal{K} \setminus \mathcal{T})$.*

2. *The* Taub$_i$ *case. We have $\hat{n}_j = \hat{n}_k$ and $\boldsymbol{x}_0 \in \mathcal{T}_i$.*

3. *The* EXCEPT$_i$ *case. We have* $\hat n_j = -\hat n_k$ *and* $\boldsymbol{x}$ *converges to the heteroclinic cycle* $-\mathbf{T}_i \to \mathbf{T}_i \to -\mathbf{T}_i$. *By this, we mean* $W^u(-\mathbf{T}_i) \subsetneq \omega(\boldsymbol{x}_0) \subseteq W^u(-\mathbf{T}_i) \cup W^s(-\mathbf{T}_i) \cup \{\mathbf{T}_i, -\mathbf{T}_i\}$. *Then,* $\limsup_{t\to\infty} \delta_i(t) > 0$.

4. *The* TAUBEXCEPT$_i$ *case. We have* $\hat n_j = \hat n_k$ *and* $\boldsymbol{x}$ *converges to the heteroclinic cycle* $-\mathbf{T}_i \to \overline{\mathcal{TL}_i} \to -\mathbf{T}_i$. *By this, we mean* $W^u(-\mathbf{T}_1) \subsetneq \omega(\boldsymbol{x}_0) \subseteq W^u(-\mathbf{T}_i) \cup W^s(-\mathbf{T}_i) \cup \{-\mathbf{T}_i\} \cup \mathcal{TL}_i$. *Then,* $\limsup_{t\to\infty} \delta_i(t) > 0$.

5. *The* TAUBCONVERGE$_i$ *case. We have* $\hat n_j = -\hat n_k$ *and* $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{T}_i$.

Let us remark that the cases TAUBCONVERGE and TAUBEXCEPT are actually impossible, due to Lemma 5.4 and Lemma 6.3. We do not know whether the case EXCEPT is possible.

Proof. The proof proceeds by indirection: We assume that neither ATTRACT nor TAUB applies for $\boldsymbol{x}$, and will then describe the possible limit sets (nonempty due to Lemma 3.7). Assume without loss of generality that $|N_1N_2N_3| < 1$ for all $t \geq 0$ (otherwise, wait some time; recall Lemma 3.6).

Let us first exclude a case that is prominently absent from the above list: We cannot have $\omega(\boldsymbol{x}_0) \subseteq \overline{\mathcal{TL}_i}$ and $\hat n_j = \hat n_k$. This is evident from Lemma 3.8.

Let $\boldsymbol{y} \in \omega(\boldsymbol{x}_0)$; we will now discuss possible cases for $\boldsymbol{y}$, before putting them all together in the end. We know from Lemma 3.6 that $\boldsymbol{y}$ has $N_1N_2N_3 = 0$, i.e. is of lower Bianchi type.

Assume that $\boldsymbol{y} \in \mathcal{K}$. Then, by our assumption that ATTRACT does not apply, we must have $\boldsymbol{y} \in \{+\mathbf{T}_i, -\mathbf{T}_i : i \in \{1,2,3\}\}$.

Assume that $\boldsymbol{y} \in \mathcal{A}$. Then, by our assumptions, we must have $\boldsymbol{y} \in W^s(-\mathbf{T}_i)$ for some $i \in \{1,2,3\}$, in order to not contradict the assumption that ATTRACT does not apply. This is because $\alpha(\boldsymbol{y}) \subseteq \omega(\boldsymbol{x}_0)$ and $\omega(\boldsymbol{y}) \subseteq \omega(\boldsymbol{x}_0)$, due to the abstract theory of dynamical systems.

Assume that $\boldsymbol{y} \in \mathcal{M}_{0+-}$, without loss of generality. Then, by Lemma 3.3, we must have $\boldsymbol{y} \in W^s(-\mathbf{T}_1) \cap \mathcal{M}_{0+-}$.

Assume that $\boldsymbol{y} \in \mathcal{M}_{0++} \setminus \overline{\mathcal{TL}_1}$, without loss of generality. Then, by Lemma 3.4, we must have $\boldsymbol{y} \in W^s(-\mathbf{T}_1) \cap \mathcal{M}_{0++}$.

The last possible case is that $\boldsymbol{y} \in \overline{\mathcal{TL}_1}$, without loss of generality. This is an exhaustive description of all possible ω-limit points $\boldsymbol{y} \in \omega(\boldsymbol{x}_0)$.

We have therefore shown, for $\boldsymbol{x}_0 \in \mathcal{M}_{\hat n} \setminus \mathcal{T}$, where ATTRACT does not apply, that

$$\omega(\boldsymbol{x}_0) \subseteq M^*_{\hat n} := \overline{\mathcal{M}_{\hat n}} \cap \bigcup_i (\overline{\mathcal{T}L_i} \cup W^s(-\mathbf{T}_i) \cup \{\mathbf{T}_i\} \cup W^u(-\mathbf{T}_i)).$$

Assume that we are in Bianchi VIII, without loss of generality $\boldsymbol{x}_0 \in \mathcal{M}_{++-} \setminus \mathcal{T}$. Then, due to Lemma 2.4 and the constraint $1 = |\boldsymbol{\Sigma}|^2 + N^2$, the set M^*_{-++} has three connected components; two of them, associated to $\mathcal{M}_{-0+}$ and $\mathcal{M}_{-+0}$, are bounded, and one, associated to $\mathcal{M}_{0++}$, is unbounded. Due to the abstract theory of dynamical systems (see Lemma A.2), ω-limit sets are either compact and connected, or non-compact. Therefore, only one of the indices i from the definition of M^*_{-++} can appear in $\omega(\boldsymbol{x}_0)$. Morally, this means that exactly on of the mutually exclusive cases TaubExcept$_1$, Except$_2$, Except$_3$, TaubConverge$_2$ or TaubConverge$_3$ must apply; formally, we still need to prove some additional claimed properties of the solution in the cases Except and TaubExcept. The claimed lower bound $W^u(-\mathbf{T}_i) \subseteq \omega(\boldsymbol{x}_0)$ in the cases Except$_i$ and TaubExcept$_i$ from the fact that $\boldsymbol{x}(t)$ cannot converge to $-\mathbf{T}_i$, since $|N_i| > 0$ is unstable (and any hypothetical "creeping along the center manifold $\mathcal{K}$" would contradict the assumption that Attract does not apply). The same argument shows $W^u(-\mathbf{T}_i) \subsetneq \omega(\boldsymbol{x}_0)$, and hence $\omega(\boldsymbol{x}_0) \cap (W^s(-\mathbf{T}_i) \setminus \mathcal{A}) \neq \emptyset$, and hence the lower bound $\limsup_{t\to\infty} \delta_i(t) > 0$.

Let us next consider the case of Bianchi IX, without loss of generality $\boldsymbol{x}_0 \in \mathcal{M}_{+++} \setminus \mathcal{T}$. Then, due to Lemma 2.4 and the constraint $1 = |\boldsymbol{\Sigma}|^2 + N^2$, the set M^*_{+++} has three connected components, all three of them unbounded.

We would like to apply the same argument as in Bianchi VIII; however, this does not directly work, because non-compact ω-limit sets can generally be disconnected. Deferring this problem for a moment, we see that the conclusions would follow if we could show that only one index i appears in $\omega(\boldsymbol{x}_0) \subset M^*_{+++}$: Then, one of the three mutually exclusive cases TaubExcept$_1$, TaubExcept$_2$ and TaubExcept$_3$ would apply.

However, we can see from Lemma 2.4 that this problem is cosmetic only, since the three unbounded parts of M^*_{+++} "do not touch at infinity".

Let us make this rigorous; we will modify the standard proof that compact ω-limit sets are connected, given in Lemma A.2. Consider $X := \overline{\mathcal{M}_{+++} \cap \{N_1N_2N_3 \leq 1\}}$; we split $X = X_0 \cup (X_1 \cup X_2 \cup X_3)$, where X_1, X_2, X_3 are open and mutually disjoint, and X_0 is open and bounded (as a subset of $\mathbb{R}^5$). In view of Lemma 2.4, take e.g. $X_0 = \{\boldsymbol{x} \in X : |x| < 11\}$ and $X_i = \{\boldsymbol{x} \in X : |x| > 10,\ N_i < 1\}$. Then, find three mutually disjoint open sets $U_i \subset X_0$ that cover $M^*_{+++} \cap X_0$, with the set corresponding to $\mathcal{M}_{0++} \subseteq U_1$ and analogously for the other indices. Now assume that the solution $\boldsymbol{x} : [0, T) \to X$ is not eventually contained in $U_i \cup X_i$ for some i; we will derive a contradiction. Indeed, recalling the proof of Lemma A.2, we immediately find a sequence $t_n \to \infty$ such that $\boldsymbol{x}(t_n) \notin \bigcup_i U_i \cup X_i$. However, we must have $\boldsymbol{x}(t_n) \in X_0$ for all n; taking a convergent subsequence, possible because X_0 is bounded, yields the desired contradiction. $\square$

Therefore, we know that $\liminf_{t\to\infty} \max_i \delta_i(t) = 0$ and hence $\liminf_{t\to\infty} \operatorname{dist}(\boldsymbol{x}(t), \mathcal{A}) = 0$ for initial conditions in $\mathcal{M}_{\pm\pm\pm} \setminus \mathcal{T}$. Furthermore, we know that, in the case of Bianchi IX, there must be multiple ω-limit points. In the case of Bianchi VIII, with $N_1 < 0 < N_2, N_3$, it is still imaginable that $\omega(\boldsymbol{x}_0) \subseteq \{\mathbf{T}_2, \mathbf{T}_3\}$; this will be excluded in Lemma 5.4. While we presently lack the necessary estimates to prove the missing part of the attractor theorem, $\limsup_{t\to\infty} \max_i \delta_i(t) = 0$, we can at least describe how this may fail: Each δ_i can only grow by a meaningful factor in the vicinity of a Taub-point $\mathbf{T}_i$. This fact has, unfortunately, a relatively technical formulation in Lemma 3.10. This statement may become clearer when Lemma 3.10 is used in the proof of Theorem 1, page 42.

Lemma 3.10. *Given $\rho_{3.2} > 0$, there exist $C_{3.3}, c_{3.4} = C_{3.3}, c_{3.4}(\rho_{3.2}) > 0$ such that the following holds:*

Suppose we have a piece of trajectory $\boldsymbol{x} : [t_1, t_2] \to \mathcal{M}_{\pm\pm\pm}$, such that:

1. *We have the product bound $|N_1N_2N_3|(t_1) < c_{3.4}$ (and hence for all $t \in [t_1, t_2]$).*
2. *The first point of the partial trajectory is bounded away from the Taub-line $\mathcal{TL}_1$, i.e. $1 + \Sigma_+(t_1) > \rho_{3.2}$.*
3. *The piece of trajectory has $1 \geq \delta_1^4(t) \geq |N_1N_2N_3|(t)$ for all $t \in [t_1, t_2]$, i.e. $|N_1| \leq 4\delta_1^2 = 16|N_2N_3|$ for all $t \in [t_1, t_2]$, i.e. δ_1 is large compared to N_1, but bounded.*

Then δ_1 can only increase by a bounded factor along this piece of trajectory, i.e. $\delta_1(t_2) \leq C_{3.3}\delta_1(t_1)$.

Proof. Recall the proof of Lemma 3.7. From $\delta_1^4 \geq |N_1N_2N_3|$, we know that $|N_1| \leq 4\delta_1^2$ and $|N_1| \leq 2\sqrt{|N_1|}\delta_1 = 2\sqrt{|N_1N_2N_3|}$. Therefore, equation (3.2.2) holds.

If $c_{3.4} > 0$ is small enough, this allows us to see that $1 + \Sigma_+(t) > \frac{1}{2}\rho_{3.2}$ for all $t \in [t_0, t_1]$. Since $|\Sigma_+| \leq 2$ is bounded, integrating (3.2.2) shows $\int_{t_1}^{t_2} |1 - \Sigma^2|\rho_{3.2}\mathrm{d}t < C$ and hence $\int_{t_1}^{t_2} |1 - \Sigma^2|\mathrm{d}t < \frac{C}{\rho_{3.2}}$. Integrating (3.2.3), i.e. $\mathrm{D}_t \log \delta_1 = (1 - \Sigma^2) - (1 + \Sigma_+) \leq (1 - \Sigma^2) - \frac{1}{2}\rho_{3.2}$, yields the claim. $\square$

Sketch of classic proofs of Theorem 1. The previous proofs of Theorem 1, both in [Rin01] and [HU09b], rely on the following estimate (Lemma 3.1 in [HU09b], Section 15 in [Rin01]):

Lemma 5.6. *We consider the neighbourhood of $\mathcal{T}_1$ with $N_2, N_3 > 0$. There exist constants $\rho_{5.3} > 0$ small enough and $C_{5.14} > 1$ large enough, such that, for any piece of trajectory $\boldsymbol{x} : [0, T] \to \{\boldsymbol{y} \in \mathcal{M}_{*++} : |\boldsymbol{\Sigma}(\boldsymbol{y}) - \mathbf{T}_1| \leq \rho_{5.3}\}$, the following estimate holds:*

$$\begin{aligned} \delta_1(\boldsymbol{x}(T)) &\leq C_{5.14}e^{C_{5.14}h(\boldsymbol{x}(0))}\delta_1(\boldsymbol{x}(0)), \qquad \textit{where} \\ h(\boldsymbol{x}) &= |N_1| + |N_1|^2 + |N_1N_2| + |N_1N_3|. \end{aligned} \tag{5.3.2}$$

The proof of this Lemma 5.6 requires some lengthy averaging arguments and will be deferred until Section 5.3, page 56. We stress that Lemma 5.6 is not actually needed for any of the results in this work, and will be proven only for the sake of completeness of this literature review.

Proof of Theorem 1 using Lemma 5.6. We will first proof the following

Claim 3.3.1. For solutions $\boldsymbol{x} : [0, \infty) \to \mathcal{M}_{\pm++} \setminus \mathcal{T}_1$, we have $\lim_{t\to\infty} \delta_1(t) = 0$.

Proof of Claim 3.3.1. Make Lemma 3.10 and Lemma 5.6 compatible by choosing $\rho_{3.2} = \rho_{5.3}$. Assume that, without loss of generality, $|N_1N_2N_3|(t) < c_{3.4}$ for all $t \geq 0$.

By Lemma 3.9, we have $\liminf_{t\to\infty} \delta_1(t) = 0$. For any sub-interval $[T^L, T^R] \subseteq [0, \infty)$ with $1 > \delta_1^4 > |N_1N_2N_3|$ for all $t \in [T^L, T^R]$, we can only have a bounded increase $\delta_1(T^R) \leq C_{3.3}C_{5.14}\delta_1(T^L)$: Large relative increases of δ_1 are neither possible away from $\mathbf{T}_1$ (Lemma 3.10), nor near $\mathbf{T}_1$ (Lemma 5.6). Now, since $\lim_{t\to\infty} |N_1N_2N_3| = \liminf_{t\to\infty} \delta_1 = 0$, this implies $\limsup_{t\to\infty} \delta_1(t) = 0$. $\blacksquare$

This clearly proves $\lim_{t\to\infty} \delta_i(t) = 0$ for all three $i \in \{1,2,3\}$ for solutions in $\mathcal{M}_{+++} \setminus \mathcal{T}$. In order to see that an ω-limit point $\boldsymbol{p} \in \mathcal{K} \setminus \mathcal{T}$ exists, consider Lemma 3.9. The possibilities EXCEPT and TAUBCONVERGE are a priori impossible, since we are in $\mathcal{M}_{+++}$. The possibility TAUBEXCEPT is excluded by Claim 3.3.1. $\square$

Remark 3.11. Lemma 5.6 includes the Bianchi VIII case of $N_2, N_3 > 0 > N_1$. The versions stated in [Rin01, HU09b] only consider the Bianchi IX case $N_1, N_2, N_3 > 0$; however, their proofs extend to this case virtually unchanged.

This allows us to show Claim 3.3.1, that proves $N_2N_3 \to 0$ in the Bianchi VIII case $N_1 < 0 < N_2, N_3$, also keeping the proofs of [Rin01, HU09b] virtually unchanged.

To the best of our knowledge, Claim 3.3.1 has not been explicitly stated in the literature before; nevertheless, since it is a trivial corollary of the previously known (nontrivial!) proofs for Bianchi IX, it should not be considered a novel result of this work.

The above proof also shows that in Bianchi VII_0, i.e. for any $\boldsymbol{x}_0 \in \mathcal{M}_{0++} \setminus \mathcal{T}_1$, we must have $\lim_{t\to-\infty} \boldsymbol{x}(t) = \boldsymbol{p}_-$ with $\boldsymbol{p}_- = (-1,0,0,N,N)$ for some $N > 0$. This is false in the case of Bianchi VI_0: Any $\boldsymbol{x}_0 \in \mathcal{M}_{0+-}$ has $\lim_{t\to-\infty} \boldsymbol{x}(t) = (-1,0,0,0,0)$. Hence, δ_1 can grow by an arbitrarily large factor near $\mathbf{T}_1$ in $\mathcal{M}_{*+-}$, and no analogue of Lemma 5.6 can hold in the Bianchi VIII models $\mathcal{M}_{*+-}$ and $\mathcal{M}_{*-+}$. In other words, attempts to trivially adapt the proof of Claim 3.3.1 to show $\delta_2, \delta_3 \to 0$ are doomed.

This difficulty is partially responsible for the fact that, for Bianchi the VIII case $\boldsymbol{x}_0 \in \mathcal{M}_{-++}$, it was previously unknown whether $\lim_{t\to\infty} N_1N_2(t) = 0$ and $\lim_{t\to\infty} N_1N_3(t) = 0$.

Sketch of our replacement for Lemma 5.6. In this work, we will replace the rather subtle averaging estimates from Lemma 5.6 by the program outlined the introduction and in this paragraph. Let us first repeat the reasons, why we want to avoid Lemma 5.6:

1. The analogue statement of Lemma 5.6 in Bianchi VIII is wrong. Lemma 3.3 shows that counterexamples to such a generalization can be found by taking any sequence $\{\boldsymbol{x}_n\}$ of initial data converging to any point in $\mathcal{M}_{0-+}$. Therefore, any argument relying on Lemma 5.6 has no chance of carrying over to the Bianchi VIII case.

2. The proof of Lemma 5.6 is lengthy and requires subtle averaging arguments.

3. The complexity of the proof of Lemma 5.6 is not unavoidable: Most of the effort is spent trying to understand asymptotic regimes *that do not occur anyway.*

Our replacement is described by the following program:

1. At first, we study pieces of trajectories $\boldsymbol{x} : [0,T] \to \mathcal{M}_{\pm\pm\pm}$, which start near $\mathcal{A}$ and stay bounded away from the generalized Taub-spaces $\mathcal{T}_i^G$, i.e. have all $r_i > \rho_{3.3}$. Along such partial solutions, all three δ_i decay essentially exponentially (Proposition 4.1).

2. Next, consider how solutions near $\mathcal{A}$ can enter the neighbourhood of the generalized Taub-spaces. This can only happen near some $-\mathbf{T}_i$ (Proposition 4.2).

3. For such solutions entering the vicinity of $-\mathbf{T}_i$, the quotient $\frac{\delta_i}{r_i}$ is initially small, and stays small near $-\mathbf{T}_i$ and along the heteroclinic leading to $+\mathbf{T}_i$ (Proposition 5.1).

4. Next, we study solutions near $\mathbf{T}_i$ for which $\frac{\delta_i}{r_i}$ is initially small. Then, $\frac{\delta_i}{r_i}$ stays small. This additional condition ($\delta_i \ll r_i$) allows us to describe solutions with easier averaging arguments and stronger conclusions than Lemma 5.6. Bianchi VIII solutions can be analysed in the same way. This is done in Section 5.2, leading to the conclusion that δ_i decays essentially exponentially, with nonuniform rate $\sim cr_i^2$, which is, up to constant factors, the same rate as on the Kasner circle (Proposition 5.3).

5. Finally, we combine the previous steps in Section 6 in order to prove Theorems 2, 3 and 4. These extend Theorem 1 with somewhat finer control over solutions and provide an analogue in Bianchi VIII.

4 Dynamics near the Mixmaster-Attractor $\mathcal{A}$

Our previous arguments in Section 3 about the dynamics of (2.3.2) have been of a rather qualitative and global character. We have established that there exist ω-limit points on the Mixmaster-attractor $\mathcal{A}$.

We have also sketched the classical proof that trajectories converge to $\mathcal{A}$ in the case of Bianchi Type IX (Theorem 1) (where we deferred the proof of the crucial estimate Lemma 5.6 to a later point).

In this section, we will give a more precise description of the behaviour near $\mathcal{A}$. The goal of this section is to show that pieces of trajectories $\boldsymbol{x} : [0, T] \to \mathcal{M}$ near $\mathcal{A}$ converge to $\mathcal{A}$ essentially exponentially, at least as long as they stay bounded away from the Taub-points $\mathbf{T}_i$.

The goal of this section is to prove the following two Propositions 4.1 and 4.2:

Proposition 4.1 (Essentially uniform exponential convergence to $\mathcal{A}$ away from the Taub points)**.** *For any $0 < \rho_{4.1} \le 0.1$ small enough, there exist constants, $c_{4.1}, C_{4.2}, \epsilon_{d,4.1} = c_{4.1}, C_{4.2}, \epsilon_{d,4.1}(\rho_{4.1}) > 0$, such that the following holds:*

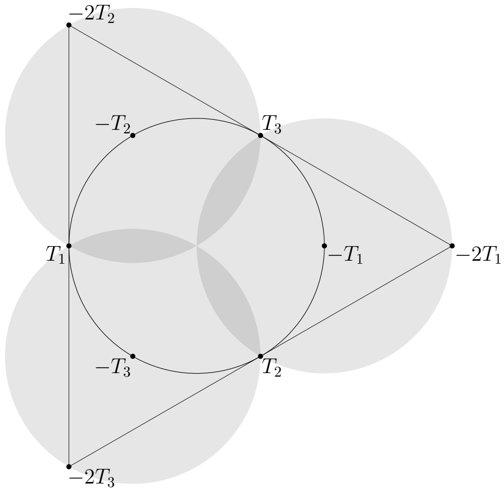
Consider a trajectory $\boldsymbol{x} : [0, T^*) \to \mathcal{M}_{\pm\pm\pm}$, such that, for all $t \in [0, T^*)$ and $i \in \{1, 2, 3\}$ the following inequalities hold:

$$\max_i \delta_i(t) < \epsilon_{d,4.1} \tag{4.1a}$$

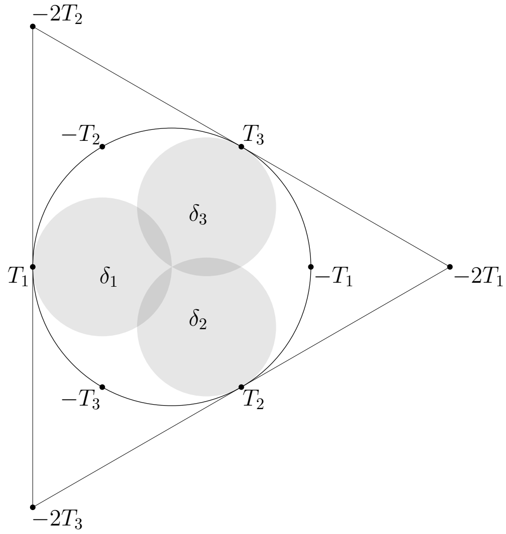
$$\min_i d(\boldsymbol{x}(t), \mathbf{T}_i) > \rho_{4.1}. \tag{4.1b}$$

Then, each δ_i is essentially uniformly exponentially decreasing in $[0, T^*)$, i.e. for all $0 \le t_1 \le t_2 < T^*$ and $i \in \{1, 2, 3\}$:

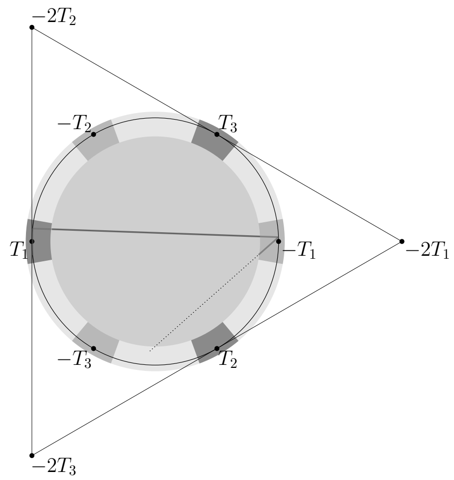
$$\delta_i(t_2) \le C_{4.2} e^{-c_{4.1}(t_2 - t_1)} \delta_i(t_1). \tag{4.2a}$$



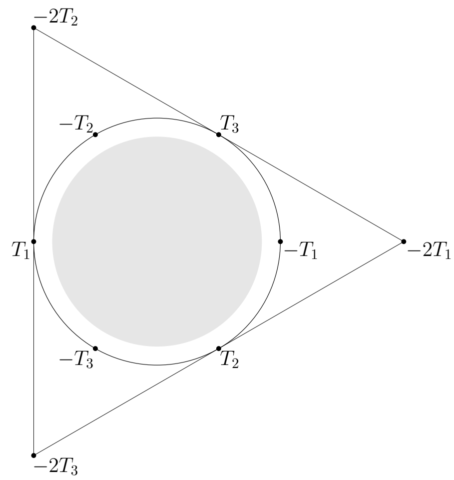
(a) The stability defining discs for the N_i



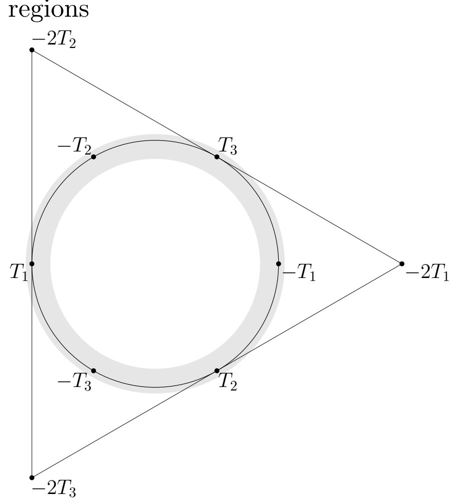
(b) The stability defining discs for the δ_i



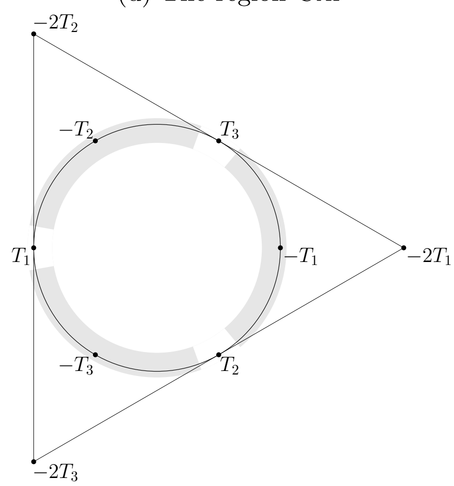
(c) A trajectory leaving the controlled regions



(d) The region Cap



(e) The region Circle



(f) The region Hyp

Figure 5: The relevant regions are colored in gray and are not up to scale (except for Fig.4a and Fig.4b). See also Fig.4.

We can replace the assumption (4.1a) *by* $\max_i \delta_i(\boldsymbol{x}_0) < C_{4.2}\epsilon_{d,4.1}$. *Then, if* $T^* < \infty$ *and one of the inequalities* (4.1b), (4.1a) *is violated at time* T^*, *it must be* (4.1b), *and* (4.1a) *must still hold at* T^*.

Furthermore, given $c_{4.3}, \epsilon_{N,4.2} > 0$ *arbitrarily small, we can adjust* $\epsilon_{d,4.1} > 0$ *downwards, such that, if* $0 \leq t_1 < t_2 < T^*$, *such that* $|N_j|(t_1) > \epsilon_{N,4.2}$ *and* $|N_k|(t_2) \geq \epsilon_{N,4.2}$ *with* $j \neq k$, *then for all* $t \in [t_2, T^*)$*:*

$$\delta_i(t) \leq c_{4.3}\delta_i(t_1). \tag{4.2b}$$

First, note that the part of the Proposition about replacing the assumption (4.1a) by $\max_i \delta_i(\boldsymbol{x}_0) < C_{4.2}\epsilon_{d,4.1}$. is really a trivial corollary of the remaining claims. Informally, Proposition 4.1 states that trajectories near $\mathcal{A}$ converge exponentially to $\mathcal{A}$, as long as they stay bounded away from the Taub points. The following Proposition 4.2 extends this by describing that the only way for trajectories near $\mathcal{A}$ to reach the vincinity of the Taub-points is via the heteroclinic connection $-\mathbf{T}_i \to \mathbf{T}_i$:

Proposition 4.2. *Assume the setting of Proposition 4.1. It is possible to choose* $\epsilon_{N,4.2}$, $\epsilon_{d,4.1} > 0$ *small enough such that additionally the following estimate holds, with* $C_{4.4} = 5$*:*

Assume that $T^* = \inf\{t > 0 : d(\boldsymbol{x}(t), \mathbf{T}_\ell) \leq \rho_{4.1}\} < \infty$, *and that initially, for all* $i \in \{1, 2, 3\}$

$$d(\boldsymbol{x}_0, \mathcal{T}_i) > C_{4.4}\rho_{4.1}. \tag{4.3a}$$

Then the final part of the trajectory preceding T^* *must have the form depicted in Figure 5c, i.e. there is* $\ell \in \{1, 2, 3\}$ *and* $\ell \neq j \in \{1, 2, 3\}$ *and there are times* $0 < T^A < T^B < T^C \leq T^*$ *(typically* $T^C = T^*$*) such that*

$$|N_j|(T^A) \geq \epsilon_{N,4.2} \tag{4.4a}$$
$$d(\boldsymbol{x}(t), -\mathbf{T}_\ell) \leq C_{4.4}\rho_{4.1} \qquad \forall t \in [T^A, T^B] \tag{4.4b}$$
$$|N_\ell(t)| \geq \epsilon_{N,4.2} \qquad \forall t \in [T^B, T^C] \tag{4.4c}$$
$$d(\boldsymbol{x}(t), \mathbf{T}_\ell) \leq C_{4.4}\rho_{4.1} \qquad \forall t \in [T^C, T^*]. \tag{4.4d}$$

Informal Outline of Proofs

Informal proof of Proposition 4.1. We split the trajectory into time intervals where it is either near $\mathcal{K}$ (i.e. $\max_i |N_i| \leq \epsilon_N$) or away from $\mathcal{K}$ (i.e. $\max_i |N_i| \geq \epsilon_N$).

Near $\mathcal{K}$, we can see from (2.3.2b) and (2.3.8b) that each $\mathrm{D}_t \log |N_i|$ and $\mathrm{D}_t \log \delta_i$ depends only on the $\boldsymbol{\Sigma}$-coordinates and is positive only on some disc in $\mathbb{R}^2$. These six discs are plotted in Figures 5a and 5b. We can observe that these discs only touch and intersect $\mathcal{K}$ at the three Taub-points and that near each point $\boldsymbol{p} \in \mathcal{K} \setminus \{\mathbf{T}_i\}$, exactly one of the $\log |N_i|$ is increasing and the two remaining $\log |N_j|$ and all three $\log \delta_i$ are decreasing. Under our assumption $\min_i d(\boldsymbol{x}(t), \mathbf{T}_i) > \rho$, this increase and decrease is uniform if $\epsilon_N = \epsilon_N(\rho) > 0$ is small enough and $\max_i |N_i| \leq \epsilon_N$.

Hence, for any small piece of trajectory $\boldsymbol{x} : [t_1, t_2] \to \{\boldsymbol{x} \in \mathcal{M} : \min_i d(\boldsymbol{x}, \mathbf{T}_i) \geq \rho, \max_i |N_i| \leq \epsilon_N\}$, one of the $|N_i|$ is uniformly exponentially increasing, while all three

δ_i and the remaining two $|N_j|, |N_k|$ are uniformly exponentially decreasing, with some rate $2c_{4.1} = 2c_{4.1}(\rho_{4.1}, \epsilon_{N,4.2}) > 0$.

Eventually the trajectory will leave the neighbourhood of $\mathcal{K}$; since we assumed that we are near $\mathcal{A}$, i.e. $\max_i \delta_i < \epsilon_d$, this can only happen near one of the Kasner caps (Bianchi type II). By continuity of the flow, the trajectory will follow a heteroclinic orbit until it is near $\mathcal{K}$ again, and will spend only bounded amount of time $T < C(\epsilon_N)$ for this transit. Hence, all $|N_i|$ and δ_i can only change by a bounded factor, independent of ϵ_d, during such a heteroclinic transit.

The time spent near $\mathcal{K}$ between two heteroclinic transits is bounded below by $\mathcal{O}(|\log \epsilon_d|)$ (for fixed ρ, ϵ_N): Consider an interval $[t_1, t_2]$ spent near $\mathcal{K}$, where $t_1 > 0$. Suppose without loss of generality that initially $|N_1|(t_1) = \epsilon_N$ and that $|N_2|$ is uniformly exponentially increasing, such that $|N_2|(t_2) = \epsilon_N$. Then, we must have $|N_2|(t_1) = \frac{\delta_3^2}{4|N_1|}(t_1) \leq C{\epsilon_N}^{-1}{\epsilon_d}^2 \ll |N_2|(t_2) = \epsilon_N$, and we must have $t_2 - t_1 \geq c|\log \epsilon_d|$. Hence, if $\epsilon_d \ll \epsilon_N$ is small enough, the exponential decrease of the three δ_i will dominate all contributions from the heteroclinic transits and we obtain an estimate of the form (4.2a). The estimate 4.2b follows if we have at least one heteroclinic transit. $\square$

Informal proof of Proposition 4.2. We again use continuity of the flow: Each small heteroclinic "bounce" near one of the $\mathbf{T}_i$ must increase the distance from $\mathbf{T}_i$ by at least some $c_{4.5} = c_{4.5}(\rho) > 0$. By continuity of the flow, each episode with $\max_i |N_i| \geq \epsilon_N$ therefore must increase the distance from $\mathbf{T}_i$ by $c_{4.5}/2$; near $\mathcal{K}$, the trajectory is almost constant and $d(\boldsymbol{x}_0, \mathbf{T}_i)$ cannot shrink by more that $c_{4.5}/3$. Hence the only way to reach the vicinity of a Taub point is by following the heteroclinic $-\mathbf{T}_\ell \to \mathbf{T}_\ell$. $\square$

Formal Proofs The remainder of this section is devoted to making these informal proofs rigorous, i.e. filling all the gaps and replacing the hand-wavy arguments by formal ones. We begin by naming the regions of the phase-space, where the various estimates hold:

Definition 4.3. Given $\rho, \epsilon_N, \epsilon_d > 0$ (later chosen in this order) we define:

$$\begin{aligned}
\textsc{Cap}[\epsilon_N, \epsilon_d] &= \{\boldsymbol{x} \in \mathcal{M} : \max |N_i| \geq \epsilon_N, \max_i \delta_i \leq \epsilon_d\} \\
\textsc{Circle}[\epsilon_N, \epsilon_d] &= \{\boldsymbol{x} \in \mathcal{M} : \max |N_i| \leq \epsilon_N, \max_i \delta_i \leq \epsilon_d\} \\
\textsc{Hyp}[\rho, \epsilon_N, \epsilon_d] &= \textsc{Circle}[\epsilon_N, \epsilon_d] \setminus [B_\rho(\mathbf{T}_1) \cup B_\rho(\mathbf{T}_2) \cup B_\rho(\mathbf{T}_3)]\,.
\end{aligned} \tag{4.5}$$

These sets are sketched in Figure 5 (not up to scale). They are constructed such that for appropriate parameter choices:

1. The union $\textsc{Cap} \cup \textsc{Circle}$ contains an entire neighbourhood of $\mathcal{A}$ (by construction).
2. The region $\textsc{Circle}$ is a small neighbourhood of the Kasner circle. This is because of the constraint $1 = \Sigma^2 + N^2$ and $|N^2| < C{\epsilon_N}^2$ (see Figure 5e).

3. The region $\textsc{Cap}$ has three connected components, where one of the three $|N_i| \gg 0$, because by $\max_i |N_i| \geq \epsilon_N$ at least one N_i must be bounded away from zero and by $\max_i 2\sqrt{|N_j N_k|} \leq \epsilon_d$ at most one N_i can be bounded away from zero (this only works if ϵ_d is small enough, depending on ϵ_N).

   The region $\textsc{Cap}$ is bounded away from the Kasner circle (see Figure 5d). By continuity of the flow, the dynamics in $\textsc{Cap}$ can be approximated by pieces of heteroclinic orbits in $\mathcal{A}$, up to uniformly small errors (Lemma 4.5).

4. The region $\textsc{Hyp}$ has three connected components. In each connected component, one of the $|N_i|$ is uniformly exponentially increasing and the remaining two $|N_j|$, $|N_k|$ are uniformly exponentially decreasing. All three products δ_i are uniformly exponentially decreasing in $\textsc{Hyp}$ (Lemma 4.4; this only works if ϵ_N is small enough, depending on ρ).

5. The remaining part of the neighbourhood of $\mathcal{A}$, i.e. $\textsc{Circle} \setminus \textsc{Hyp}$, consists of the neighbourhoods of the three Taub points. The analysis of the dynamics in these neighbourhoods is deferred until Section 5.

Lemma 4.4 (Uniform Hyperbolicity Estimates). *Given any $\rho_{4.2} > 0$ small enough, we find $0 < \epsilon_{N,4.3} = \epsilon_{N,4.3}(\rho_{4.2}) < 0.1$ and $c_{4.6} = c_{4.6}(\rho_{4.2}) > 0$ small enough, such that for any $\boldsymbol{x} \in \textsc{Hyp}[\rho_{4.2}, \epsilon_{N,4.3}, \infty]$, we find one $i \in \{1,2,3\}$ such that $\mathrm{D}_t \log|N_i| > c_{4.6}$, and the remaining two $\mathrm{D}_t \log|N_j| < -c_{4.6}$ and all three $\mathrm{D}_t \log \delta_j < -2c_{4.6}$.*

Let $\rho_{4.2}, \epsilon_{N,4.3}, c_{4.6} > 0$ as above, and $C_{4.7} = 2$. For any piece of trajectory $\boldsymbol{x} : (t_1, t_2) \to \textsc{Hyp}[\rho_{4.2}, \epsilon_{N,4.3}, \infty]$, we can conclude

$$\begin{aligned} &\int_{t_1}^{t_2} |N_i| \mathrm{d}t < \frac{\epsilon_{N,4.3}}{c_{4.6}} \\ \operatorname{diam} \boldsymbol{x}(t_1,t_2) \leq &\int_{t_1}^{t_2} |\boldsymbol{x}'(t)| \mathrm{d}t \leq C_{4.7} \int_{t_1}^{t_2} \max_i |N_i|(t) \mathrm{d}t \leq C_{4.7} \frac{\epsilon_{N,4.3}}{c_{4.6}}. \end{aligned} \tag{4.6}$$

Proof. The first part of the lemma consists of choosing $c_{4.6}$ and $\epsilon_{N,4.3}$, depending on $\rho_{4.2}$. From Equations (2.3.2b) and (2.3.8b), we see that each $\mathrm{D}_t \log|N_i|$ and $\mathrm{D}_t \log \delta_i$ depends only on the $\boldsymbol{\Sigma}$-coordinates and is positive only on some disc in $\mathbb{R}^2$. These six discs are plotted in Figures 5a and 5b. The three δ_i-discs (Fig. 5b) do not intersect $\mathcal{K}$, and touch only near the three Taub-points; the three N_i-discs (Fig. 5a) intersect $\mathcal{K}$ only at the three Taub-points. Hence, when we exclude a neighbourhood of the Taub-points, and consider a small neighbourhood of $\mathcal{K}$, all six logarithmic derivatives are bounded away from zero.

By the constraint $1 - \Sigma^2 = N^2$ and $|N^2| \leq 9\epsilon_{N,4.3}^2$, the set $\textsc{Hyp}$, depicted in Figure 5, is near $\mathcal{K}$ (as described), and the desired uniformity estimates hold.

The second part follows from the uniform hyperbolicity in $\textsc{Hyp}$: In each component of $\textsc{Hyp}$, exactly one N_i is unstable (see Figure 5 and Figure 4a). Suppose without loss of generality that N_1 is the unstable direction; then we can estimate for $t \in (t_1, t_2)$:

$$|N_1(t)| \leq e^{-c_{4.6}(t_2 - t)} |N_1(t_2)|, \qquad |N_2(t)| \leq e^{-c_{4.6}(t - t_1)} |N_2(t_1)|.$$

For $|N_3|$, the analogous estimate as for N_2 holds. Using $|N_1(t_2)| \leq \epsilon_{N,4.3}$ and $|N_2(t_1)| \leq \epsilon_{N,4.3}$ and integrating yields the claim about $\int |N|\mathrm{d}t$. From (2.3.2), we see $|\boldsymbol{x}'| \leq C_{4.7} \max_i |N_i|$ (if $\max_i |N_i| \leq 0.1$). $\square$

Continuity of the flow allows us to approximate solutions in Cap by heteroclinic solutions in $\mathcal{A}$, up to any desired precision c, if we only chose the distance from $\mathcal{A}$ (i.e. $\epsilon_d > 0$) small enough. More precisely:

Lemma 4.5 (Continuity of the flow near Cap)**.** *Let $\epsilon_{N,4.4} > 0$ and $c_{4.8} > 0$. Then there exists $\epsilon_{d,4.5} = \epsilon_{d,4.5}(\epsilon_{N,4.4}, c_{4.8}) > 0$ small enough and $C_{4.9} = C_{4.9}(\epsilon_{N,4.4}) > 0$ large enough, such that the following holds:*

Let $\boldsymbol{x} : (t_1, t_2) \to \text{Cap}[\epsilon_{N,4.4}, \epsilon_{d,4.5}]$ be a piece of a trajectory. Then $t_2 - t_1 < C_{4.9}$ and there exists $\boldsymbol{y} \in \mathcal{A}$ such that

$$d(\boldsymbol{x}(t), \phi(\boldsymbol{y}, t - t_1)) < c_{4.8} \qquad \text{for all} \quad t \in (t_1, t_2). \tag{4.7}$$

Proof. Follows from continuity of the flow and the fact that all trajectories in $\mathcal{A}$ are heteroclinic and must leave Cap at some time. $\square$

We now have collected all the ingredients to formally prove the two main results from this section. At first, we combine Lemma 4.4 and Lemma 4.5 in order to show that each δ_i is uniformly essentially exponentially decreasing in Cap $\cup$ Hyp:

Formal proof of Proposition 4.1. We are given $\rho_{4.1} > 0$. We will first care only about $\rho_{4.1}$, afterwards adjust constants in order to prove the "furthermore" claim involving $c_{4.3}, \epsilon_{N,4.2}$. We first apply Lemma 4.4 with $\rho_{4.2} = \rho_{4.1}$. We fix $c_{4.1} = c_{4.6}$. Next, we apply Lemma 4.4, with $\epsilon_{N,4.4} = \epsilon_{N,4.3}$ and $c_{4.8} = 1$, and will later choose $\epsilon_{d,4.1} \leq \epsilon_{d,4.5}$), and $C_{4.2} > 0$.

Set $\mu_i = \delta_i'/\delta_i + c_{4.6}$; in order to prove (4.2a), it suffices to show that $\int_{t_1}^{t_2} \mu_i(t)\mathrm{d}t < \log C_{4.2}$ is bounded above, independently of the partial trajectory $\boldsymbol{x} = \boldsymbol{x}(t)$, $i \in \{1, 2, 3\}$ and t_1,t_2. Fix $\boldsymbol{x} : [t_1, t_2] \to \mathcal{M}$ and $t_1 < t_2$.

Decompose $[t_1, t_2]$ into intervals $S_k < T_k < S_{k+1}$ corresponding to the preimages of the regions Cap and Hyp, i.e. such that $\boldsymbol{x}([S_k, T_k]) \subseteq \text{Hyp}$ and $\boldsymbol{x}([T_k, S_{k+1}]) \subseteq \text{Cap}$.

We begin by considering the contribution from Cap, i.e., an interval $[T_k, S_{k+1}]$. By Lemma 4.5, we have $S_{k+1} - T_k < C_{4.9}$; since $\mu_i < 5$ is bounded above, we get $\int_{T_k}^{S_{k+1}} \mu_i(t)\mathrm{d}t < 5C_{4.9}$.

Next, we consider the contributions from Hyp. In this region, $\mu_i < -c_{4.6}$. Take an interval $[S_k, T_k]$, which is not the initial or final interval, i.e. $t_1 < S_k < T_k < t_2$. Assume without loss of generality that $|N_1(S_k)| = |N_2(T_k)| = \epsilon_{N,4.3}$. Then $|N_2(S_k)| = 0.25\delta_3^2/|N_1|(S_k) < \epsilon_{d,4.1}^2 \epsilon_{N,4.3}^{-1}$. Since $|N_2'/N_2| < 3$, we obtain $T_k - S_k > -\frac{2}{3} \log \frac{\epsilon_{d,4.1}}{\epsilon_{N,4.3}}$. Adjust $\epsilon_{d,4.1} > 0$ to be so small, that $c_{4.1}(T_k - S_k) > 5C_{4.9}$. Then such an interval gives us a contribution of $\int_{S_k}^{T_k} \mu_i(t)\mathrm{d}t < -5C_{4.9}$.

For the complete interval (t_1, t_2), sum over k; two disjoint intervals in the Cap-region must always enclose an interval in the Hyp-section, which cancels the contribution of its preceding CAP-region. Therefore, at most the last CAP-region stays unmatched and we obtain $\int_{t_1}^{t_2} \mu_i(t)\mathrm{d}t < \log C_{4.2} = 5C_{4.9}$.

The remaining "furthermore" claim (4.2b) is trivial: Its assumption ensures that we have at least one $t_1 \leq S_k < T_k \leq t_2$, and we can make its contribution arbitrarily large by decreasing $\epsilon_{d,4.1}$, depending on $\epsilon_{N,4.2}, c_{4.3}$. □

Next, we adjust the constants from Lemma 4.5 in order to show that trajectories near $\mathcal{A}$ can only enter the vicinity of Taub-points via the heteroclinic $-\mathbf{T}_\ell \to \mathbf{T}_\ell$:

Formal proof of Proposition 4.2. We find some $0 < c_{4.10} < \rho_{4.1}$ such that $d(K(p), T_i) > d(p, T_i) + c_{4.10}$ for every $p \in \mathcal{K}$ with $d(p, \mathbf{T}_i) \in (\rho_{4.1}, 0.5]$. It is evident from Figure 4a (or, formally, Proposition 3.2) that this is possible.

By Lemma 4.4, we can choose $\epsilon_{N,4.2}$ small enough that $\operatorname{diam} \boldsymbol{x}([t_1, t_2]) < c_{4.10}/8$ for pieces of trajectories $\boldsymbol{x} : (t_1, t_2) \to$ HYP. Using the continuity of the flow, i.e. Lemma 4.5, we can make $\epsilon_{d,4.1}$ small enough such that pieces $\boldsymbol{x} : (t_1, t_2) \to$ CAP are approximated by heteroclinic orbits up to distance $c_{4.8} = c_{4.10}/8$.

Now, we know by assumption that $d(\boldsymbol{x}(T^*), \mathbf{T}_\ell) \leq \rho_{4.1}$ for some $\ell \in \{1, 2, 3\}$. There are two cases: If $\boldsymbol{x}(T^*) \in \textsc{Cap}[\epsilon_{N,4.2}, \epsilon_{d,4.1}]$, then we set $T^C = T^*$. Otherwise, we take

$$T^C = \inf \left\{ t \leq T^* : \boldsymbol{x}([t, T^*]) \subset \overline{B_{1.5\rho_{4.1}}(\mathbf{T}_\ell)} \cap \textsc{Circle}[\epsilon_{N,4.2}, \epsilon_{d,4.1}\right\}.$$

By the assumption (4.3a), we have $T^C > 0$. Now, $\boldsymbol{x}(T^C) \in \textsc{Cap}[\epsilon_{N,4.2}, \epsilon_{d,4.1}]$: We cannot have $d(\mathbf{T}_\ell, \boldsymbol{x}(T^C)) = 1.5\rho_{4.1}$, since we already know $\operatorname{diam} \boldsymbol{x}([T^C, T^*]) \leq c_{4.10}/8 < 0.5\rho_{4.1}$ (since, by construction, $\boldsymbol{x}([T^C, T^*]) \subseteq \textsc{Hyp}(\rho_{4.1}, \epsilon_{N,4.2}, \epsilon_{d,4.1})$). This proves (4.4d), as well as

$$\boldsymbol{x}(T^C) \in \partial\textsc{Cap}[\epsilon_{N,4.2}, \epsilon_{d,4.1}] \cap \partial\textsc{Circle}[\epsilon_{N,4.2}, \epsilon_{d,4.1}] \cap B_{1.5\rho_{4.1}}(\mathbf{T}_\ell).$$

Next, we set

$$T^B = \inf \left\{ t \in [0, T^C) : \boldsymbol{x}([t, T^C)) \subseteq \textsc{Cap}[\epsilon_{N,4.2}, \epsilon_{d,4.1}] \right\}.$$

In the interval $t \in [T^B, T^C]$, the trajectory is in one of the three CAP regions; this must be the $|N_\ell| \geq \epsilon_N$ cap, since otherwise $d(\boldsymbol{x}(t), \mathbf{T}_\ell)$ would be increasing (see Figure 4a). We set

$$T^A = \inf \left\{ t \in [0, T^B) : \boldsymbol{x}([t, T^B)) \subseteq \textsc{Circle}[\epsilon_{N,4.2}, \epsilon_{d,4.1}] \right\}.$$

Similar arguments yield the claim (4.4b). □

Remark 4.6. The constants generated in this section are sub-optimal (at least exponentially so). If one cared at all about their numerical values, then one would need to replace Lemma 4.5 and Proposition 3.2 by explicit estimates.

5 Analysis near the generalized Taub-spaces $\mathcal{T}_i^G$

In this section, we will study the dynamics in the vicinity of the generalized Taub-spaces, without loss of generality $\mathcal{T}_1^G$, using the polar coordinates from Section 2.4. Throughout this section, we will work under the assumption that $r_1 \leq \rho_{5.1} = 0.1$. The results in this section do not directly rely on Section 4; nevertheless, we think that it is useful to read Section 4 first. This section is structured in the following way:

In Section 5.1, we study the behaviour of trajectories near the heteroclinic orbit $-\mathbf{T}_1 \to \mathbf{T}_1$, which come from either the N_2-cap or the N_3-cap. In Section 5.2, we study the further behaviour near $\mathbf{T}_1$ of such trajectories.

In Section 5.3, we will study the behaviour of trajectories near $\mathbf{T}_1$ which do *not* necessarily have the prehistory described in Section 5.1, and especially provide the deferred proof of Lemma 5.6. This section is mostly optional for our main results: Any trajectory which ever leaves the region where Section 5.3 is necessary will never revisit this region, a fact which is proven without referring to any results from Section 5.3. Therefore, Section 5.3 is not required for the proofs of Theorems 2 and 7; it is required for the proofs of Theorems 3, 4, 5 and 6, as well as the completion of the literature review.

5.1 Analysis near $-\mathbf{T}_1$ and near the heteroclinic $-\mathbf{T}_1 \to \mathbf{T}_1$

The behaviour of trajectories away from $\mathbf{T}_1$ is already partially described by Proposition 4.1; we only need to additionally estimate the quotient $\frac{\delta_1}{r_1}$ in this region. The necessary estimates can be summarized in the following. Note that, up to renaming of constants, this is the setting described in the conclusion of Proposition 4.2, only that we allow to start the analysis at a later time $0 \in [T^A, T^C]$.

Proposition 5.1. *Let* $\rho_{5.1} = 0.1$. *For any* $0 < \epsilon_{N,5.1} \leq 0.1$, *we find* $C_{5.1}, C_{5.2}, c_{5.3}, C_{5.4}, \epsilon_{d,5.2} = C_{5.1}, C_{5.2}, c_{5.3}, C_{5.4}, \epsilon_{d,5.2}(\epsilon_{N,5.1}) > 0$, *such that the following holds:*

Let $0 \leq T^B \leq T^C$ *and* $\boldsymbol{x} : [0, T^C) \to \mathcal{M}_{\pm\pm\pm} \setminus \mathcal{T}_1$ *be a trajectory, such that:*

$$r_1(t) \leq \rho_{5.1} \qquad \forall\, t \in [0, T^C) \tag{5.1.1a}$$

$$|N_1|(t) \leq \epsilon_{N,5.1} \quad \text{and} \quad d(\boldsymbol{x}(t), -\mathbf{T}_1) \leq 2\rho_{5.1} \qquad \forall\, t \in [0, T^B) \tag{5.1.1b}$$

$$|N_1|(t) \geq \epsilon_{N,5.1} \qquad \forall\, t \in [T^B, T^C) \tag{5.1.1c}$$

$$\delta_i(t) \leq \epsilon_{d,5.2} \qquad \forall\, t \in [0, T^C), i \in \{1,2,3\} \tag{5.1.1d}$$

$$\frac{\delta_1}{r_1}(t) \leq 1 \qquad \forall\, t \in [0, T^C). \tag{5.1.1e}$$

Then, the following estimates hold:

$$\frac{\delta_1}{r_1}(t_2) \leq C_{5.1} \frac{\delta_1}{r_1}(t_1) \qquad \forall 0 \leq t_1 \leq t_2 < T^C \tag{5.1.2a}$$

$$\delta_i(t_2) \leq C_{5.4} e^{-c_{5.3}(t_2 - t_1)} \delta_i(t_1) \qquad \forall 0 \leq t_1 \leq t_2 < T^C, i \in \{1,2,3\} \tag{5.1.2b}$$

$$r_1(t_2) \geq C_{5.2} r_1(t_1) \qquad \forall T^B \leq t_1 \leq t_2 < T^C. \tag{5.1.2c}$$

We can replace the assumption (5.1.1d) *by* $\delta_i(0) < C_{5.4}\epsilon_{d,5.2}$*, and assumption* (5.1.1e) *by* $\frac{\delta_1}{r_1}(0) < C_{5.1}$*. If* $\epsilon_{d,5.2} > 0$ *is small enough, then the latter follows from* $|N_2|(0) = \epsilon_{N,5.1}$*.*

Note that the cases $T^B = 0$ is admissible; then, some of the assumptions and conclusions are vacuous. Note that the last claim about replacing assumptions is exactly what we want to do if we start from $0 = T^A$, provided by Proposition 4.2.

Proof. Let us first prove the part about replacing assumptions. It is obvious from (5.1.2b) that we can replace the assumption (5.1.1d) by $\delta_i(0) < C_{5.4}\epsilon_{d,5.2}$, and, using (5.1.2a), assumption (5.1.1e) by $\frac{\delta_1}{r_1}(0) < C_{5.1}$.

We can replace the assumption (5.1.1e) by $|N_2|(0) = \epsilon_{N,5.1}$ (or, equivalently $|N_3|(0) = \epsilon_{N,5.1}$): Assuming that $\epsilon_{d,5.2}$ is small enough compared to $\epsilon_{N,5.1}$, we have $|N_2|(0) = \frac{\delta_1^2(0)}{4|N_3(0)|} \leq \epsilon_{d,5.2}^2\epsilon_{N,5.1}^{-1} \leq 0.5\epsilon_{N,5.1}$ and hence $r_1(0) \geq |N_3| - |N_2| \geq 0.5\epsilon_{N,5.1}$. Then, we can adjust $\epsilon_{d,5.2}$.

$T^C - T^B < C$ is bounded, by continuity of the flow. For $t \in [0, T^B]$, all three $\mathrm{D}_t \log\delta_i < -0.5$, by direct calculation. This implies the decay of the δ_i, i.e. (5.1.2b).

The only remaining claims are that r_1 cannot significantly shrink in $[T^B, T^C]$, i.e. (5.1.2c), and that $\frac{\delta_1}{r_1}$ cannot significantly grow in $[0, T^C]$, i.e. (5.1.2a).

Let us first handle the case where $N_2 > 0 > N_3$. Note that $\delta_1 \leq r_1$, i.e. (5.1.1e), implies $N_+ = \sqrt{N_-^2 + \delta_1^2} \leq \sqrt{2}r_1$. Recall and estimate, using $N^2 \leq 2$, $|\mathbf{\Sigma}| \leq 1$:

$$\mathrm{D}_t \log r_1 = N^2 - (\Sigma_+ + 1)\frac{N_-^2}{r_1^2} + \sqrt{3}N_1\frac{\Sigma_- N_+}{r_1^2} \geq -4. \tag{2.4.7a}$$

Hence, (5.1.2c), using $T^C - T^B < C$. Next, recall and estimate

$$\mathrm{D}_t \log\frac{\delta_1}{r_1} = -(\Sigma_+ + 1)\frac{\Sigma_-^2}{r_1^2} - \sqrt{3}N_1\frac{\Sigma_- N_+}{r_1^2} \leq \sqrt{6}|N_1|. \tag{2.4.7d}$$

Hence, (5.1.2b), using $T^C - T^B < C$, and $\mathrm{D}_t \log|N_1| \geq 0.5$ for $t \in [0, T^A]$.

The proof for $N_2, N_3 > 0$ works almost the same way. We do not need to invoke $N_+ \leq \sqrt{2}r_1$, but instead need $|\mathbf{\Sigma}| \leq 1 + 2\Delta^2 \leq 1 + \sqrt{|N_1|}$. Recall and estimate, using $N^2 \leq 2$

$$\mathrm{D}_t \log r_1 = N^2 - (\Sigma_+ + 1)\frac{N_-^2}{r_1^2} + \sqrt{3}N_1\frac{\Sigma_- N_-}{r_1^2} \geq -4 \tag{2.4.2a}$$

Hence, (5.1.2c), using $T^C - T^B < C$. Next, recall and estimate

$$\mathrm{D}_t \log\frac{\delta_1}{r_1} = -(\Sigma_+ + 1)\frac{\Sigma_-^2}{r_1^2} - \sqrt{3}N_1\frac{\Sigma_- N_-}{r_1^2} \leq \sqrt{3}|N_1|. \tag{2.4.2d}$$

Hence, (5.1.2b), using $T^C - T^B < C$, and $\mathrm{D}_t \log|N_1| \geq 0.5$ for $t \in [0, T^A]$.

This concludes the proof, since all other combinations of signs of N_2, N_3 can be reduced to these two cases by permutations. ∎

Remark 5.2. In Proposition 5.1 relied on (5.1.1e). This means geometrically that we excluded a set of the form INACCESSIBLE$[\epsilon_q]$ from our analysis, given by

$$\textsc{Inaccessible}[\epsilon_q] = \{\boldsymbol{x} \in \mathbb{R}^5 : \delta_1 \geq \epsilon_q r_1\},$$

i.e. we described the dynamics outside of INACCESSIBLE and showed that the set INACCESSIBLE cannot reached by initial conditions described by Proposition 4.2. We will next show that INACCESSIBLE cannot be reached by solutions starting near $\mathbf{T}_1$, but outside INACCESSIBLE.

Even though Bianchi VIII, i.e. $\mathcal{M}_{*+-}$, lacks an explicit invariant Taub-space $\mathcal{T}_1$, the INACCESSIBLE-set around the generalized Taub-space $\mathcal{T}_1^G$ is a suitable "morally backwards invariant" replacement.

The set INACCESSIBLE looks like a cone times $\mathbb{R}^2$, since both r_1 and δ_1 are homogeneous of first order in N_2, N_3, Σ_- and independent of N_1 and Σ_+. Of course, $\mathcal{M} \cap$ INACCESSIBLE is not a cone, since the constraint $G = 1$ is nonlinear.

5.2 Analysis near $\mathbf{T}_1$

Our analysis of the neighbourhood of $\mathbf{T}_1$ can be summarized in the following, where one could mentally replace $0 = T^C$ for the time-interval where the solution $\boldsymbol{x}$ is described:

Proposition 5.3. *Let* $\rho_{5.1} = 0.1$. *There exist constants* $\epsilon_{q,5.3}, C_{5.5}, c_{5.6}, c_{5.7}, C_{5.8}, C_{5.9}, C_{5.10} > 0$ *(depending on no other constants), such that the following holds:*

Let $\boldsymbol{x} : [0, T^*) \to B_{\rho_{5.1}}(\mathbf{T}_1) \cap \mathcal{M}_{*\pm\pm} \setminus \mathcal{T}_1$ *be a partial trajectory, such that for all* $t \in [0, T^*)$*:*

$$\frac{\delta_1}{r_1}(t) \leq \epsilon_{q,5.3}. \tag{5.2.1}$$

Then, for all $0 \leq t_1 \leq t_2 < T^*$*, the following estimates hold:*

$$\begin{aligned}
(|N_1|, \delta_2, \delta_3)(t_2) &\leq C_{5.5} \exp\left[-c_{5.6}(t_2 - t_1)\right] (|N_1|, \delta_2, \delta_3)(t_1) && \text{(5.2.2a)}\\
\frac{\delta_1}{r_1}(t_2) &\leq C_{5.10} \frac{\delta_1}{r_1}(t_1) && \text{(5.2.2b)}\\
r_1(t_2) &\geq C_{5.9} r_1(t_1) && \text{(5.2.2c)}\\
\delta_1(t_2) &\leq C_{5.8} \exp\left[-c_{5.7} r_1^2(t_1)(t_2 - t_1)\right] \delta_1(t_1) && \text{(5.2.2d)}\\
T^* &< \infty. && \text{(5.2.2e)}
\end{aligned}$$

The assumption (5.2.1) *can be replaced by* $\frac{\delta_1}{r_1}(0) \leq \epsilon_{q,5.3} C_{5.10}$.

First, note that it is obvious from the (5.2.2b) that the assumption (5.2.1) can be replaced by $\frac{\delta_1}{r_1}(0) < \epsilon_{q,5.3} C_{5.10}$. We begin by proving the first three of the claims, in a way analogous to the proof of Proposition 5.1:

Proof of Proposition 5.3, conclusions (5.2.2a)*,* (5.2.2b)*,* (5.2.2c)*.* The exponential decay (5.2.2a) follows trivially from (2.3.2b), see e.g. Figures 5a and 4a.

The claims (5.2.2b) and (5.2.2c) follow almost exactly like in the proof of Proposition 5.1. First, for $N_2 > 0 > N_3$, recall (2.4.7d), use $\delta_1 \leq r_1$ to see $N_+ \leq \sqrt{2}r_1$, and use $|\mathbf{\Sigma}| \leq 1$ to see estimate $\mathrm{D}_t \log \frac{\delta_1}{r_1} \leq \sqrt{6}|N_1|$. Using $\mathrm{D}_t \log |N_1| \leq -1$, the claim (5.2.2b) follows by integration. In order to see (5.2.2c), recall (2.4.9a) and Claim 2.4.2, estimate $\mathrm{D}_t \log r_1 \geq -C|N_1|$, and integrate.

Let us next handle the case of $N_2, N_3 > 0$. In order to see (5.2.2b), recall (2.4.9c) and Claim 2.4.1, estimate $\mathrm{D}_t \log r_1 \geq -C|N_1|$, and integrate. In order to see (5.2.2c), recall (2.4.4a) and Claim 2.4.1, estimate $\mathrm{D}_t \log r_1 \geq -C|N_1|$, and integrate.

and use $1 + \Sigma_+ \geq -C\Delta$ to see from (2.4.7d) that $\mathrm{D}_t \log \frac{\delta_1}{r_1} \leq C\Delta + C|N_1|$. Then the claim follows by integration.

Using $\frac{\delta_1}{r_1} < 1$, we can look at (2.4.9a) and (2.4.4a) to estimate $\mathrm{D}_t \log r_1 > -C|N_1|$, which upon integration yields the claim (5.2.2c). □

The next estimate (5.2.2d) requires a slightly more involved averaging-style argument, similar to the proof of Proposition 4.1:

Proof of Theorem 5.3, conclusion (5.2.2d). We set

$$\mu = \mathrm{D}_t \log \delta_1 + 0.1 r_1^2.$$

It suffices to prove that $\int_{t_1}^{t_2} \mu(t)\mathrm{d}t \leq \log C_{5.8}$ is bounded above. The cases of Bianchi VIII and IX work the same way; we will here only explicitly discuss the Bianchi VIII-case $N_2 > 0 > N_3$. Recall

$$\frac{\delta_1'}{\delta_1} = \frac{-1}{1-\Sigma_+} r_1^2 \cos^2 \psi_1 + \frac{-\Sigma_+}{1-\Sigma_+} r_1^2 \sin^2 \psi_1 + \frac{-\Sigma_+}{1-\Sigma_+} \delta_1^2 + N_1 h_\delta. \tag{2.4.9b}$$

Strategy. We will first consider times where $|N_1| \not\ll r_1$; the integral $\int \mu \mathrm{d}t$ over these times will be bounded by $\int |N_1| \mathrm{d}t$. Next, we consider later times, where $|N_1| \ll r_1$. We will split μ into a nonpositive and a nonnegative part; the nonnegative (bad) part will have a contribution for every ψ_1 rotation that is bounded by Cr_1, while the nonpositive (good) part will have a negative contribution for every ψ_1-rotation which scales with $r_1 \log \frac{\delta_1}{r_1}$. Adjusting $\epsilon_{q,5.3}$ will then yield the desired estimate (after summing over ψ_1-rotations).

Estimates for large $|N_1|$. Choose $\widetilde{T}$ (possibly $\widetilde{T} = 0$) such that $|N_1(t)| \geq 0.1 r_1^2(t)$ for $t \in (0, \widetilde{T}]$ and $|N_1(t)| \leq 0.1 r_1^2(t)$ for $t \in [\widetilde{T}, T^*]$. This is possible, since $\mathrm{D}_t \log |N_1| \approx -3 < 2\mathrm{D}_t \log r_1$. Then $|\mu(t)| \leq C\sqrt{|N_1(t)|}$ for all $t \in [0, \widetilde{T}]$ and hence $\int_0^{\widetilde{T}} |\mu(t)| \mathrm{d}t < C_{5.11}$.

Averaging Estimates. Consider without loss of generality only times larger than $\widetilde{T}$. Using $\Sigma_+ \approx -1$ and $\delta_1 \leq 0.1 r_1$, we can estimate $h_\delta < 1$ and

$$\mu \leq -0.4 r_1^2 \cos^2 \psi_1 + 0.6 r_1^2 \sin^2 \psi_1 + 0.01 r_1^2 + 0.1 r_1^2 + |N_1| \leq -0.25 r_1^2 + r_1^2 \sin^2 \psi_1. \tag{5.2.3}$$

Recall

$$\psi_1{}' = \sqrt{3}r_1\sqrt{\sin^2\psi_1 + \frac{\delta_1^2}{r_1^2}} - \frac{r_1^2+\delta_1^2}{1-\Sigma_+}\cos\psi_1\sin\psi_1 + N_1\sin\psi_1\, h_\psi. \tag{2.4.9d}$$

We can estimate:

$$r_1|\sin\psi_1| \le \psi_1{}' \le 2r_1\sqrt{\sin^2\psi_1+\frac{\delta_1^2}{r_1^2}}. \tag{5.2.4}$$

Let $\mu_+ = r_1^2\sin^2\psi_1$ be the positive (bad) part of μ; take times $t_1 < t_L < t_R < t_2$ with $|\psi_1(t_R) - \psi_1(t_L)| \le 2\pi$. Then

$$\int_{t_L}^{t_R}\mu_+(t)\mathrm{d}t \le \int_{\psi_1(t_L)}^{\psi_1(t_R)}\frac{\mu_+(t)}{\psi_1{}'(t)}\mathrm{d}\psi_1 \le C_{5.9}^{-1}r_1(t_R)\int_0^{2\pi}|\sin\psi_1|\mathrm{d}\psi_1 = 4C_{5.9}^{-1}r_1(t_R). \tag{5.2.5}$$

On the other hand, let $\mu_- = -0.25r_1^2$ be the negative (good) part of μ. Take times $t_1 < t_L < t_R < t_2$ with $\psi_1(t_L) = k\pi - 0.1$ and $\psi_1(t_R) = k\pi + 0.1$ for some $k\in\mathbb{Z}$; then we can estimate

$$\begin{aligned}\int_{t_L}^{t_R}\mu_-(t)\mathrm{d}t &\le -\frac{1}{4}\int_{-0.1}^{+0.1}\frac{r_1^2}{\psi_1{}'}\mathrm{d}\psi_1 \le -\frac{1}{8}\int_{-0.1}^{+0.1}\frac{r_1}{\sqrt{\sin^2\psi_1+\frac{\delta_1^2}{r_1^2}}}\mathrm{d}\psi_1\\ &\le -\frac{1}{8}r_1(T^L)C_{5.9}^{-1}\int_{-0.1}^{+0.1}\frac{1}{\sqrt{\sin^2\psi_1+\epsilon_{q,5.3}^2}}\mathrm{d}\psi_1\\ &\le -cr_1(T^L)\log\epsilon_{q,5.3}.\end{aligned}$$

We can immediately see that, if $\epsilon_{q,5.3} > 0$ is small enough, then $\int_{t^L}^{t^R}\mu_-(t)\mathrm{d}t < -4C_{5.9}^{-1}$. Hence, by summing over rotations, we can conclude the assertion (5.2.2d), with error $C_{5.8} = \exp\left[4C_{5.9}^{-1}\rho_{5.1} + C_{5.11}\right]$ and rate $c_{5.7} = 0.1C_{5.9}^{-2}$.

□

Proof of Proposition 5.3, conclusion (5.2.2e). We need to show that solutions with small quotient $\frac{\delta_1}{r_1}$ cannot stay near $\mathbf{T}_1$ forever.

Assuming without loss of generality $|N_1| \le 0.1r_1^2$ we have $\mathrm{D}_{\psi_1}\log r_1 \ge cr_1|\sin\psi_1| - C|N_1|$. This shows that the only way never leaving the vicinity of $\mathbf{T}_1$ is for the angle $\psi_1\in\mathbb{R}$ to stay bounded, i.e. $\lim_{t\to\infty}\psi_1(t) = \psi_1{}^{**}$ and $\lim_{t\to\infty}\delta_1(t) = 0$ (since otherwise r_1 increases by a too large amount during each rotation). This is impossible, since the possible limit-points lie on the Kasner-circle $\mathcal{K}\setminus\{\mathbf{T}_1\}$ and are not $\mathbf{T}_1$; hence, either N_2 or N_3 is unstable and since initially $N_2\neq 0\neq N_3$, the trajectory cannot converge to such a point. □

5.3 Analysis in the Inaccessible-cones

Our whole approach aims at avoiding the much more tricky analysis of the dynamics in the Inaccessible-cones, where possibly $\delta_1 \geq r_1$: Since trajectories starting outside of these cones never enter them, it is unnecessary to know what happens in the Inaccessible-cones. However, for various global questions, it is useful to collect at least some results inside of these cones.

We already know that solutions in Bianchi IX cannot converge to the Taub-points; the same holds in Bianchi VIII, even for solutions in Inaccessible:

Lemma 5.4. *For an initial condition* $\boldsymbol{x}_0 \in \mathcal{M}_{++-}$*, it is impossible to have* $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{T}_1$.

Proof. Suppose we have such a solution. Using equation (2.4.7d), we can write near $\mathbf{T}_1$:

$$\mathrm{D}_t \log |N_1| \frac{\delta_1}{r_1} \leq \sqrt{3}\frac{|\Sigma_-|}{r_1}|N_1|\frac{\sqrt{N_-^2 + \delta_1^2}}{r_1} - 2.5 \leq 2|N_1|\frac{\delta_1}{r_1} - 2.5.$$

Hence, if ever $|N_1|\frac{\delta_1}{r_1} < 1$, this inequality is preserved and $|N_1|\frac{\delta_1}{r_1}$ decays exponentially. Then we can estimate, using (2.4.9a), $\mathrm{D}_t \log r_1 \geq -C|N_1| - C|N_1|\frac{\delta_1}{r_1}$; all the terms on the right hand side have bounded integral and $r_1 \to 0$ is impossible.

On the other hand, if $r_1 < |N_1|\delta_1$ for all sufficiently large times, we can estimate, using (2.4.9b), $\mathrm{D}_t \log \delta_1 \geq -Cr_1^2 - C|N_1| \geq -C\sqrt{|N_1|}$, which has bounded integral and thus contradicts $\delta_1 \to 0$. $\square$

Unfortunately, this is all we can presently say in the Inaccessible cone in Bianchi VIII.
In the case of Bianchi IX, we can still average over ψ_1-rotations, even if $\delta_1 \gg r_1$, leading to the following two results:

Lemma 5.5. *We consider the neighbourhood of* $\mathcal{T}_1$ *with* $N_2, N_3 > 0$*. There exist constants* $\rho_{5.2}, c_{5.12}, C_{5.13} > 0$*, such that, for any piece of trajectory* $\boldsymbol{x} : [0,T] \to \{\boldsymbol{y} \in \mathcal{M}_{*++} : |\boldsymbol{\Sigma}(\boldsymbol{y}) - \mathbf{T}_1| \leq \rho_{5.2}\}$*, the following estimate holds:*

$$\log \frac{\delta_1(T)}{\delta_1(0)} \leq C_{5.13}(1 + h(\boldsymbol{x}(0))) + (1 - c_{5.12}) \log \frac{r_1(T)}{r_1(0)}, \qquad \textit{where} \tag{5.3.1}$$
$$h(\boldsymbol{x}) = |N_1| + |N_1|^2 + |N_1N_2| + |N_1N_3|.$$

Lemma 5.6. *We consider the neighbourhood of* $\mathcal{T}_1$ *with* $N_2, N_3 > 0$*. There exist constants* $\rho_{5.3} > 0$ *small enough and* $C_{5.14} > 1$ *large enough, such that, for any piece of trajectory* $\boldsymbol{x} : [0,T] \to \{\boldsymbol{y} \in \mathcal{M}_{*++} : |\boldsymbol{\Sigma}(\boldsymbol{y}) - \mathbf{T}_1| \leq \rho_{5.3}\}$*, the following estimate holds:*

$$\delta_1(\boldsymbol{x}(T)) \leq C_{5.14} e^{C_{5.14} h(\boldsymbol{x}(0))} \delta_1(\boldsymbol{x}(0)), \qquad \textit{where} \tag{5.3.2}$$
$$h(\boldsymbol{x}) = |N_1| + |N_1|^2 + |N_1N_2| + |N_1N_3|.$$

Lemma 5.5 follows trivially from Lemma 5.6. The required statement from Lemma 5.5 is only that for a piece of trajectory near $\mathbf{T}_1$, if r_1 increases by a large factor, then $\frac{\delta_1}{r_1}$ must decrease by a large factor. Lemma 5.6 is not actually needed, except in the Literature review in Section 3.3, and only given for the sake of completeness.

Proof of Lemma 5.5.

Assumptions without loss of generality. We can assume, without loss of generality, that $\delta_1 > \epsilon_{q,5.4} r_1$, for all $t \in [0,T]$, for some small $\epsilon_{q,5.4} > 0$, because otherwise (i.e. for later times) Proposition 5.3 applies. Likewise, we can assume without loss of generality that $h(t) < 1$, because h is exponentially decreasing and $\log \delta_1' < C(1+h)$. We can as well assume $|N_1| \leq C_{5.15} r_1^4$ for all $t \in [0,T]$, for arbitrarily large $C_{5.15} > 0$, because of a similar argument as in the proof of Proposition 5.3. The claim that $c_{5.12}, C_{5.13} > 0$ are independent of other constants remains true, even if formally $c_{5.12}, C_{5.13} = c_{5.12}, C_{5.13}(\rho_{5.3}, \epsilon_{q,5.4}, C_{5.15})$.

Differential Estimates. Let us now collect some differential estimates. We use the auxilliary quantity $\zeta = \frac{\delta_1}{r_1} > \epsilon_{q,5.4}$; using (2.4.4), we can easily see

$$\begin{array}{rcl} c r_1 \zeta & \leq & \mathrm{D}_t \psi_1 \quad \leq C r_1 \zeta \\ c r_1^2 \zeta^{-1} \sin^2 \psi_1 - C|N_1| \frac{1}{{\psi_1}'} & \leq & \mathrm{D}_{\psi_1} r_1 \quad \leq C r_1^2 \zeta^{-1} \\ c r_1 \cos^2 \psi_1 - C|N_1| \frac{1}{{\psi_1}'} & \leq & -\mathrm{D}_{\psi_1} \zeta \quad \leq c r_1. \end{array} \tag{5.3.3}$$

We will use ψ_1 as a new time variable.

Proof. Since $|\mathrm{D}_{\psi_1} \log \zeta| < C$ and $|\mathrm{D}_{\psi_1} \log r_1|$ are bounded, it suffices to prove an estimate of the form

$$\int_0^{2\pi} \mathrm{D}_{\psi_1} \log \zeta \mathrm{d}\psi_1 \overset{!}{\leq} -c \int_0^{2\pi} \mathrm{D}_{\psi_1} \log r_1 \mathrm{d}\psi_1.$$

Using (5.3.3), and the fact that terms of order $|N_1|$ are negligible, it suffices to show

$$\int_0^{2\pi} r_1 \zeta^{-1} \cos^2 \psi_1 \mathrm{d}\psi_1 \overset{!}{\geq} c \int_0^{2\pi} r_1 \zeta^{-1} \sin^2 \psi_1 \mathrm{d}\psi_1.$$

This estimate is obvious, since $r_1(\psi_1)$ and $\zeta(\psi_1)$ can only vary by bounded factors along a single rotation. □

Proof of Lemma 5.6. We continue at the end of the proof of Lemma 5.5.

Integral Estimates. Clearly, r_1 is almost non-decreasing and bounded, and therefore $\int_{\psi_1(0)}^{\psi_1(T)} |\mathrm{D}_{\psi_1} r_1| \mathrm{d}\psi_1 < C$. Using the previously stated averaging, we see that this implies $\int_{\psi_1(0)}^{\psi_1(T)} r_1^2 \zeta^{-1} \mathrm{d}\psi_1 < C$.

Program Our goal is to show

$$\int_{\psi_1(0)}^{\psi_1(T)} \frac{\mathrm{D}_t \log \delta_1}{\mathrm{D}_t \psi_1} \mathrm{d}\psi_1 < C.$$

We will introduce a parametrized family $F_\theta = F_\theta(\boldsymbol{x})$ of functions, $\theta \in [0,1]$, that interpolates between $F_1 = \frac{\mathrm{D}_t \log \delta_1}{\mathrm{D}_t \psi_1}$ and some better behaved F_0. Then, we will estimate

$$\int_{\psi_1(0)}^{\psi_1(T)} F_1(\boldsymbol{x}(\psi_1))\mathrm{d}\psi_1 \leq \int_{\psi_1(0)}^{\psi_1(T)} F_0(\boldsymbol{x}(\psi_1))\mathrm{d}\psi_1 + \int_{\psi_1(0)}^{\psi_1(T)} \max_{\theta\in[0,1]} |\mathrm{D}_\theta F_\theta(\boldsymbol{x}(\psi_1))|\mathrm{d}\psi_1.$$

Estimates of pertubative terms We define

$$\begin{aligned} F_\theta(\boldsymbol{x}) = &\frac{r_1}{\sqrt{3}(2 - \theta\frac{r_1^2 + N_1(N_1-2N_+)}{1-\Sigma_+})} \\ &\cdot \frac{\sin^2\psi_1 - \cos^2\psi_1}{\sqrt{\sin^2\psi_1 + \zeta^2} + \theta\frac{r_1}{1-\Sigma_+}\cos\psi_1\sin\psi_1 + \theta N_1 \sin\psi_1 h_\psi} \\ &- \frac{\theta}{\mathrm{D}_t\psi_1} r_1^2 \cos^2\psi_1 \frac{r_1^2 + N_1(N_1 - 2N_+)}{(1-\Sigma_+)^2} + \frac{\theta}{\mathrm{D}_t\psi_1} N_1 h_\delta. \end{aligned}$$

Using $(1+\Sigma_+)(1-\Sigma_+) = r_1^2 + N_1(N_1 - 2N_+)$, we can easily verify that $\frac{\mathrm{D}_t \log\delta_1}{\mathrm{D}_t\psi_1} = F_1$. We will estimate for $\theta \in [0,1]$:

$$|\mathrm{D}_\theta F_\theta| \leq C\left(r^2\zeta^{-1} + r_1^{-1}\zeta|N_1|\right).$$

We have already estimated the resulting integrals at the beginning of the proof; this will therefore yield

$$\int_{\psi_1(0)}^{\psi_1(T)} \max_{\theta\in[0,1]} |\mathrm{D}_\theta F_\theta(\psi_1, \boldsymbol{x}(\psi_1))|\mathrm{d}\psi_1 < C.$$

Now, in order to generate the estimates $\mathrm{D}_\theta F_\theta$, we just differentiate and estimate for each occurrence of θ separately, using $\mathrm{D}_t\psi_1 > cr_1\zeta$ for the last two additive terms. The explicit calculations and estimates are trivial.

Time-advanced Estimate. We will now estimate $\int F_0 \mathrm{d}\psi_1$. We have

$$F_0(\psi_1, r_1, \zeta) = \frac{1}{2\sqrt{3}} \frac{r_1}{\sqrt{\sin^2\psi_1 + \zeta^2}}\left(\sin^2\psi_1 - \cos^2\psi_1\right).$$

Let $\widetilde{\psi_1} = \psi_1 + \pi/2$ and $(\widetilde{r}_1, \widetilde{\zeta}) = (r_1, \zeta)(\psi_1 + \pi/2)$ be time-advanced variables. Clearly, it suffices to estimate

$$
\begin{aligned}
\int_{\psi_1(0)}^{\psi_1(T)-\frac{\pi}{2}} & F_0(\psi_1, r_1, \zeta)\mathrm{d}\psi_1 + \int_{\psi_1(0)+\frac{\pi}{2}}^{\psi_1(T)} F_0(\psi_1, r_1, \zeta)\mathrm{d}\psi_1 \\
&= \int_{\psi_1(0)}^{\psi_1(T)-\frac{\pi}{2}} \left[F_0(\psi_1, r_1, \zeta) + F_0(\widetilde{\psi_1}, \widetilde{r}_1, \widetilde{\zeta})\right] \mathrm{d}\psi_1 \\
&= \int_{\psi_1(0)}^{\psi_1(T)-\frac{\pi}{2}} \left[F_0(\psi_1, r_1, \zeta) + F_0(\widetilde{\psi_1}, r_1, \zeta)\right] \mathrm{d}\psi_1 \qquad (5.3.4) \\
&\quad + \left[\int_{\psi_1(t_1)}^{\psi_1(t_2)-\frac{\pi}{2}} |\partial_\zeta F_0|\,|\zeta - \widetilde{\zeta}| + |\partial_{r_1} F_0|\,|r_1 - \widetilde{r}_1| \mathrm{d}\psi_1\right] \quad \overset{!}{\leq} C.
\end{aligned}
$$

Now, the critical estimate is that the term (5.3.4) has a sign:

$$
\begin{aligned}
F_0(\psi_1, & r_1, \zeta) + F_0(\widetilde{\psi_1}, r_1, \zeta) \\
&= \frac{r_1}{2\sqrt{3}} \left(\frac{1}{\sqrt{\sin^2\psi_1 + \zeta^2}} - \frac{1}{\sqrt{\cos^2\psi_1 + \zeta^2}} \right) \left(\sin^2\psi_1 - \cos^2\psi_1\right) \leq 0.
\end{aligned}
$$

This can be trivially seen by separately considering the cases $\sin^2\psi_1 \geq \cos^2\psi_1$ and $\sin^2\psi_1 \leq \cos^2\psi_1$.

In order to estimate the remaining terms, we easily see $|\mathrm{D}_{r_1} F_0| \leq C$ and $|r - \widetilde{r}| \leq Cr^2\zeta^{-1}$, as well as $|\mathrm{D}_\zeta F_0| \leq C r_1 \zeta^{-1}$, and $|\zeta - \widetilde{\zeta}| \leq C\frac{|N_1|}{\mathrm{D}_t\psi_1} + Cr_1$. All the resulting integrals have been estimated at the beginning of the proof. □

6 Attractor Theorems

The goal of this section is to prove that typical initial conditions converge to $\mathcal{A}$. We have already seen Theorem 1, which is however somewhat unsatisfactory: It tells nothing about the speed and the details of the convergence; it relies on Lemma 5.6, which has a rather lengthy proof (page 56f) mainly discussing the case $\delta_1 \gg r_1$, *which is not supposed to happen anyway*; lastly, the proof of Theorem 1 has no chance of generalizing to the case of Bianchi VIII.

In this section, we will combine the analysis of the previous Sections 4 and 5 in order to prove a *local* attractor result, holding both in Bianchi VIII and IX. Together with the results from Section 3 and some minor calculation, this will yield a "global attractor theorem", i.e. a classification of solutions failing to converge to $\mathcal{A}$, which happen to be rare; in the case of Bianchi IX, this recovers and extends Theorem 1, and in the case of Bianchi VIII, this answers a longstanding conjecture.

6.1 Statement of the attractor Theorems.

The local attractor theorem is given by the following (proof on page 63):

Theorem 2 (Local Attractor Theorem)**.** *There exist constants* $\epsilon_{q,6.1}, \epsilon_{d,6.2}, C_{6.1}, C_{6.2}$, $C_{6.3} > 0$ *such that the following holds:*

Let $\boldsymbol{x}_0 \in \mathcal{M}_{\pm\pm\pm}$ *be an initial condition in either Bianchi* VIII *or* IX*, with*

$$\frac{\delta_i}{r_i} < \epsilon_{q,6.1} \quad \textit{and} \quad \delta_i < \epsilon_{d,6.2} \qquad \forall i \in \{1,2,3\}. \tag{6.1.1}$$

Then, for all $i \in \{1,2,3\}$ *and* $t_2 \geq t_1 \geq 0$*:*

$$\delta_i(t_2) \leq C_{6.2}\delta_i(t_1) \tag{6.1.2a}$$

$$\frac{\delta_i}{r_i}(t_2) \leq C_{6.1}\frac{\delta_i}{r_i}(t_1). \tag{6.1.2b}$$

Furthermore, for all $i \in \{1,2,3\}$,

$$\lim_{t\to\infty} \delta_i(t) = 0 \tag{6.1.2c}$$

$$\lim_{t\to\infty} \frac{\delta_i}{r_i}(t) = 0 \tag{6.1.2d}$$

$$\int_0^\infty \delta_i^2(t)\mathrm{d}t < C_{6.3}\frac{\delta_i^2}{r_i^2}(\boldsymbol{x}_0), \tag{6.1.2e}$$

and the ω*-limit set* $\omega(\boldsymbol{x}_0)$ *must contain at least one point* $\boldsymbol{p} \in \mathcal{K} \setminus \mathcal{T}$*.*

Note that the constants appearing in Theorem 2 are really constants, i.e. depend on nothing. The name "local attractor theorem" is descriptive: We describe a subset of the basin of attraction, i.e. an open neighbourhood of $\mathcal{A} \setminus \mathcal{T}$ which is attracted to $\mathcal{A}$ and given by

$$\textsc{Basin}_{\hat{n}}[\epsilon_d, \epsilon_q] = \left\{ \boldsymbol{x} \in \mathcal{M}_{\hat{n}} : \delta_i \leq \epsilon_d, \frac{\delta_i}{r_i} \leq \epsilon_q \quad \forall i \in \{1,2,3\} \right\}. \tag{6.1.3}$$

The integral estimate (6.1.2e) tells us that the convergence to $\mathcal{A}$ must be reasonably fast.

We can combine the local attractor Theorem 2 with the discussion in Section 3 in order to prove a global attractor theorem. Since some trajectories fail to converge to $\mathcal{A}$, most notably trajectories in the Taub-spaces, a global attractor theorem must necessarily take the form of a classification of all exceptions. In this view, the global result for the case of Bianchi Type IX models is the following (proof on page 65):

Theorem 3 (Bianchi IX global attractor Theorem)**.** *Consider* $\mathcal{M}_{+++}$*, i.e. Bianchi* IX*. Then, for any initial condition* $\boldsymbol{x}_0 \in \mathcal{M}_{+++}$*, the long-time behaviour of* $\boldsymbol{x}(t)$ *falls into exactly one of the following mutually exclusive classes (*$i \in \{1,2,3\}$*):*

Attract. *For large enough times, Theorem 2 applies.*

Taub$_i$**.** *We have* $\boldsymbol{x}_0 \in \mathcal{T}_i$*, and hence* $\boldsymbol{x}(t) \in \mathcal{T}_i$ *for all times.*

The set of initial conditions for which ATTRACT *applies is "generic" in the following sense: It is open and dense in $\mathcal{M}_{+++}$ and its complement has Lebesgue-measure zero (evident from the fact that $\mathcal{T}_i$ are embedded lower dimensional submanifolds, of both dimension and codimension two).*

The analogous, novel result for the case of Bianchi VIII is the following (proof on page 65):

Theorem 4 (Bianchi VIII global attractor Theorem). *Consider $\mathcal{M}_{++-}$, i.e. Bianchi* VIII *with $N_1, N_2 > 0 > N_3$. Then, for any initial condition $\boldsymbol{x}_0 \in \mathcal{M}_{++-}$, the long-time behaviour of $\boldsymbol{x}(t)$ falls into exactly one of the following mutually exclusive classes:*

ATTRACT. *For large enough times, Theorem 2 applies.*

TAUB$_3$. *We have $\boldsymbol{x}_0 \in \mathcal{T}_3$, and hence $\boldsymbol{x}(t) \in \mathcal{T}_3$ for all times.*

EXCEPT$_1$. *For large enough times, the trajectory follows the heteroclinic object*

$$-\mathbf{T}_1 \to \mathbf{T}_1 \to -\mathbf{T}_1.$$

In other words, let $W^s(-\mathbf{T}_1)$ be the two-dimensional stable manifold of $-\mathbf{T}_1$ and let $W^u(-\mathbf{T}_1) = W^s(\mathbf{T}_1)$ be the one-dimensional unstable manifolds of $-\mathbf{T}_1$. Then, $W^u(-\mathbf{T}_1) \subsetneq \omega(\boldsymbol{x}_0) \subsetneq W^u(-\mathbf{T}_1) \cup W^s(-\mathbf{T}_1) \cup \{\mathbf{T}_1, -\mathbf{T}_1\}$. Furthermore, we have $\lim_{t\to\infty} \max(\delta_2, \delta_3)(t) = 0$ and $\limsup_{t\to\infty} \delta_1(t) > 0 = \liminf_{t\to\infty} \delta_1(t)$.

EXCEPT$_2$. *The analogue of* EXCEPT$_1$ *applies, with the indices* 1 *and* 2 *exchanged.*

This theorem should be read in conjunction with the following, the proof of which will be deferred until page 70 in Section 7:

Theorem 5 (Bianchi VIII global attractor Theorem genericity). *In Theorem 4, the set of initial conditions $\boldsymbol{x}_0$ for which* ATTRACT *applies is "generic" in the following sense: It is open and dense in $\mathcal{M}_{++-}$ and its complement has Lebesgue-measure zero.*

Question 6.1. *It is currently unknown, whether the case* EXCEPT *in Bianchi* VIII *is possible at all. Is it?*

We expect that solutions in EXCEPT$_1$ *actually converge to the heteroclinic cycle, where $\mathbf{T}_1 \to -\mathbf{T}_1$ is realized by the unique connection in $\mathcal{T}_1^G$, i.e. $\lim_{t\to\infty} r_1(t) = 0$, instead of following any other heteroclinic in $W^u(\mathbf{T}_1) \cap W^s(-\mathbf{T}_1)$. Can this be shown?*

For topological reasons, we naively expect that the set of initial conditions, where EXCEPT *applies, is nonempty, and is of dimension and codimension two.*

6.2 Proof of the local attractor Theorem.

Let us now prove the local attractor theorem. In order to simplify the proof, we will first prove the following:

Lemma 6.2 (Un-bootstrapped local attractor theorem)**.** *There exist constants $\epsilon_{q,6.3}$, $\epsilon_{d,6.4}, C_{6.1}, C_{6.2}, C_{6.3} > 0$ such that the following holds:*

Let $\boldsymbol{x}_0 : [0,T] \to \mathcal{M}_{\pm\pm\pm}$ be a solution in either Bianchi VIII *or* IX*, such that the following holds for all $t \in [0,T]$:*

$$\frac{\delta_i}{r_i} < \epsilon_{q,6.3} \quad \textit{and} \quad \delta_i < \epsilon_{d,6.4} \qquad \forall i \in \{1,2,3\}. \tag{6.2.1}$$

Then, for all $i \in \{1,2,3\}$ and $0 \le t_1 \le t_2 < T$:

$$\delta_i(t_2) \le C_{6.2}\delta_i(t_1) \tag{6.1.2a}$$

$$\frac{\delta_i}{r_i}(t_2) \le C_{6.1}\frac{\delta_i}{r_i}(t_1). \tag{6.1.2b}$$

Furthermore, for all $i \in \{1,2,3\}$,

$$\int_0^T \delta_i^2(t)\mathrm{d}t < C_{6.3}\frac{\delta_i^2}{r_i^2}(\boldsymbol{x}_0). \tag{6.1.2e}$$

This is almost equivalent to Theorem 2.

Informal proof. Let us first give the ingredients of the proof, where the problem of how to choose appropriate constants is ignored. Recall Section 5, especially Propositions 5.1 and 5.3, as well as Section 4, and Section 4, especially Propositions 4.1 and 4.2.

These results tell us that δ_i can only grow by bounded amounts, and must then shrink again when they are in Hyp, and the literally same estimates hold for the quotients $\frac{\delta_i}{r_i}$. More precisely: We split $[0,T]$ into a finite sequence $0 < \ldots < T_n^A < T_n^B < T_n^C < T_{n+1}^A < \ldots < T$, such that Proposition 4.1 applies in $[T_n^A, T_n^B]$ (i.e. we are away from the Taub-spaces), Proposition 5.1 applies in $[T_n^B, T_n^C]$ (i.e. we are near $-\mathbf{T}_i$ or near the heteroclinic $-\mathbf{T}_i \to \mathbf{T}_i$) and Proposition 5.3 applies in $[T_n^C, T_{n+1}^A]$ (i.e. we are near $\mathbf{T}_i$). This splitting is made possible by Proposition 4.2.

In every such interval, each δ_i and each quotient can only grow by a bounded factor:

1. In $[T_n^A, T_n^B]$, away from the Taub-spaces, (4.2a) shows that δ_i can only grow by $C_{4.2}$, and by boundedness of r_i, $\frac{\delta_i}{r_i}$ can only grow by $C_{4.2}\frac{2}{\rho_{4.1}}$.
2. In $[T_n^B, T_n^C]$, near the heteroclinic $-\mathbf{T}_\ell \to \mathbf{T}_\ell$, (5.1.2b) shows that δ_i can only grow by $C_{5.4}$, and (5.1.2a) shows that $\frac{\delta_i}{r_i}$ can only grow by $C_{5.1}$.
3. In $[T_n^C, T_{n+1}^A]$, near $\mathbf{T}_\ell$, (5.2.2d) shows that δ_i can only grow by $C_{5.8}$, and (5.2.2b) shows that $\frac{\delta_i}{r_i}$ can only grow by $C_{5.10}$.

Of course, the analogue estimates hold for the first time-interval starting at 0, and the last time-interval ending in T.

However, due to (4.2b), each δ_i must decay by a total factor of $c_{4.3}$ over the course of every $[T_n^A, T_n^B]$ interval. Hence, each quotient $\frac{\delta_i}{r_i}$ must decay by a total factor of $c_{4.3}\frac{2}{\rho_{4.1}}$ over such an interval.

If $c_{4.3}$ is sufficiently small, then this decay can cancel the growth from adjacent intervals, and we obtain (6.1.2a) and (6.1.2b), similar as in the proof of Proposition 4.1. In fact, this shows the following

Claim 6.2.1. $\frac{\delta_i}{r_i}(T_n^C)$ and $\delta_i(T_n^C)$ decay uniformly exponentially in n (not in time T_n^C!).
Proof. Contained in preceding paragraph. ■

The claim (6.1.2e) then follows from integrating (5.2.2d) (and (5.1.2b) and (4.2a).

Choice of constants. Let us now give the deferred choice of appropriate constants.

First, $\rho_{4.1} = 0.01$ and $\epsilon_{N,4.2} > 0$, such that Proposition 4.2 holds; this fixes $C_{4.2}, c_{4.1} > 0$. Now, the choice of $c_{4.3} > 0$ is still open, as is $\epsilon_{d,4.1} = \epsilon_{d,4.1}(c_{4.3}) > 0$, and will be deferred for a moment.

Next, we fix $\rho_{5.1} = 0.1$ and $\epsilon_{N,5.1} = \epsilon_{N,4.2} > 0$ in Propositions 5.1 and 5.3. This is possible, since we are allowed to adjust both $\epsilon_{N,5.1}$ and $\epsilon_{N,4.2}$ downwards. This fixes the remaining constants in the used Propositions, except for $c_{4.3} > 0$.

Now, let us choose $c_{4.3} > 0$. We choose $c_{4.3} > 0$ so small, that the following two inequalities hold:

$$c_{4.3}C_{4.2}C_{5.4}C_{5.8} < \frac{1}{2}, \qquad c_{4.3}C_{4.2}\frac{2}{\rho_{4.1}}C_{5.1}C_{5.10} < \frac{1}{2}.$$

This fixes, finally, $\epsilon_{d,4.1} > 0$, and all these constants are now "constant constants", i.e. depend on nothing.

Now, let us state the constants appearing in the statement of the Lemma; they are:

$$C_{6.2} = C_{4.2}C_{5.4}C_{5.8}, \qquad C_{6.1} = C_{4.2}\frac{2}{\rho_{4.1}}C_{5.1}C_{5.10},$$
$$C_{6.3} = 2\min\left(\frac{C_{4.2}}{c_{4.1}}, \frac{C_{5.8}}{c_{5.7}}\right), \qquad \epsilon_{q,6.3} = \min(1, \epsilon_{q,5.3}),$$
$$\epsilon_{d,6.4} = \min(\epsilon_{d,4.1}, \epsilon_{d,5.2}, \epsilon_{q,5.3}).$$

Rigorous proof. Now we know precisely what to prove. With these constants fixed, the referenced Propositions are applicable and the first paragraph constitutes a rigorous proof of Lemma 6.2. □

We claimed that Lemma 6.2 was almost equivalent to Theorem 2. This works the following way:

Proof of the local attractor Theorem 2. There are two differences between Theorem 2 and Lemma 6.2: Firstly, the assumptions differ, and secondly, we get new conclusions that we need to show.

Let us adapt the assumptions first. We set $\epsilon_{d,6.2} = C_{6.2}\epsilon_{q,6.3}$ and $\epsilon_{q,6.1} = C_{6.1}\epsilon_{q,6.3}$. Then, under the assumptions of Theorem 2, we can see that the assumptions of Lemma 6.2 are satisfied for all times $t > 0$.

Apart from this, we need to show the additional claims (6.1.2c) and (6.1.2d), and that the ω-limit set $\omega(\boldsymbol{x}_0)$ must contain at least one point in $\mathcal{K} \setminus \mathcal{T}$.

Let us begin with the claims (6.1.2c) and (6.1.2d). Consider again the sequence $T_n^A < T_n^B < T_n^C < T_{n+1}^A$ from the proof of Lemma 6.2. If this sequence is infinite, the claims (6.1.2c) and (6.1.2d) follow from the exponential decay in n. If this sequence ends in some T_n^A, then we eventually avoid the neighbourhood of the Taub-spaces, and (4.2a) proves the conclusions. The sequence cannot end in some T_n^B, by construction. It also cannot end in some T_n^C, due to (5.2.2e).

If the sequence $T_n^A < T_n^B < T_n^C < T_{n+1}^A$ ends in some T_n^A, then the claim on $\omega(\boldsymbol{x}_0)$ follows by taking an arbitrary $\boldsymbol{y} \in \omega(\boldsymbol{x}_0)$, and $\boldsymbol{p} = \alpha(\boldsymbol{y})$. If the sequence is infinite instead, we use the same construction with an arbitrary accumulation point of the sequence $\boldsymbol{x}(T_n^A)$. □

6.3 Proof of the global attractor theorems

Next, we prove the global results for Bianchi IX and VIII. This is simply a matter of combining Theorem 2, Lemma 3.9, and Lemma 5.5 and 5.4. We view this as a novel and simpler proof of the global attractor theorem, compared to the one given on page 42, since Lemma 5.5 replaces Lemma 5.6.

Recall Lemma 3.9. We will first exclude the case TaubExcept: In order to reuse parts of the proof of the global Bianchi IX attractor theorem for the analogue in Bianchi VIII, we will split it off into the following First, exlcude the case TaubExcept from Lemma 3.9:

Lemma 6.3. *The case* TaubExcept *is impossible in Lemma 3.9.*

Proof of Lemma 6.3. The proof is by contradiction; assume we had a solution $\boldsymbol{x} : [0,\infty) \to \mathcal{M}_{\pm++} \setminus \mathcal{T}_1$, where TaubExcept_1 applies.

Without loss of generality, assume that $\frac{\delta_1}{r_1}(t) \geq \epsilon_{q,6.5}$ for all $t > 0$ (by Lemma 2.4 and Theorem 2).

We will now take a look at the dynamics of the quotient $\frac{\delta_1}{r_1}$, and will show that $\frac{\delta_1}{r_1} \to 0$, contradicting our assumptions and concluding the proof.

Recall

$$\mathrm{D}_t \log \frac{\delta_1}{r_1} = -(\Sigma_+ + 1)\frac{\Sigma_-^2}{r_1^2} - \sqrt{3}N_1\frac{\Sigma_- N_-}{r_1^2}. \tag{2.4.2d}$$

As we have seen several times already, in the neighbourhood of $\{-\mathbf{T}_1\} \cup W^u(\mathbf{T}_1) \cup \{\mathbf{T}_1\} \cup \mathcal{TL}_1$, the integral $\int |N_1| \mathrm{d}t$ is bounded, and the quotient can only grow by a bounded factor. Using $(1-\Sigma_+)(1+\Sigma_+) = r_1^2 + N_1(N_1 - 2N_+)$, and the fact that $|N_1|$ and $|N_1|N_+$ are arbitrarily small and exponentially decreasing near $\{\mathbf{T}_1\} \cup \mathcal{TL}_1$, we see that r_1 must increase up to some finite value $r_1 \geq \epsilon > 0$ near $\{\mathbf{T}_1\} \cup \mathcal{TL}_1$ (otherwise, we could not leave the neighbourhood again to visit $-\mathbf{T}_1$). Applying Lemma 5.5, we see that the quotient $\frac{\delta_1}{r_1}$ must therefore decrease by an arbitrarily large factor.

Therefore, the last necessary step is to show that the quotient can only increase by a bounded factor on the return trip $\{\mathbf{T}_1\} \cup \mathcal{TL}_1 \to -\mathbf{T}_1$, along $W^s(-\mathbf{T}_1)$. In other words, we want to show that along any time-interval $[T^L, T^R]$ connecting from $\{\boldsymbol{y} : |\boldsymbol{\Sigma} - \mathbf{T}_1| < \epsilon\}$ to $\{\boldsymbol{y} : |\boldsymbol{y} + \mathbf{T}_1| < \epsilon\}$ while staying bounded away from Basin, we have $\int_{T^L}^{T^R} |N_1(t)| \mathrm{d}t < C$.

We will now show that $|N_1| \leq \delta_1$ for all $t \in [T^L, T^R]$. If, at some $t \in [T^L, T^R]$, we had $|N_1| = \delta_1$, then we get at time t that $0 \approx N_1 N_2 N_3 = N_1^3/4 = 2\sqrt{3}\delta_1^3$. Then, the only way of having $\delta_1 / r_1(t) \geq \epsilon_{q,6.1}$ is $r_1(t) \approx 0$. Therefore, $(1 - \Sigma_+)(1 + \Sigma_+) = r_1^2 + N_1(N_1 - 2N_+) \approx 0$, contradicting our assumptions on $[T_n^L, T_n^R]$.

Therefore, $|N_1| \leq \delta_1$ for all $t \in [T^L, T^R]$, and hence $|N_1| \leq 2\sqrt{|N_1|}\sqrt{|N_1 N_2 N_3|}$, which has bounded integral.

This shows that the quotient $\frac{\delta_1}{r_1}$ can only grow by a bounded factor on the return-trip from $\boldsymbol{T}_1 \cup \mathcal{TL}_1$ to $-\mathbf{T}_1$. Together with the arbitrarily large decrease when leaving the neighbourhood of $\boldsymbol{T}_1 \cup \mathcal{TL}_1$, and the bounded growth near $\{-\mathbf{T}_1\} \cup W^u(\mathbf{T}_1) \cup \{\mathbf{T}_1\} \cup \mathcal{TL}_1$, we see that $\frac{\delta_1}{r_1} \to 0$, contradicting the assumptions. □

Proof of the global Bianchi IX *attractor Theorem 3.* Follows trivially from Theorem 2, Lemma 3.9 and Lemma 6.3. □

Proof of the global Bianchi VIII *attractor Theorem 4.* The proof works by contradiction; let $\boldsymbol{x} : [0, \infty) \to \mathcal{M}_{++-} \setminus \mathcal{T}_3$, such that Attract does not apply.

Recall Lemma 3.9; the case TaubExcept$_3$ is excluded by Lemma 6.3 and the cases TaubConverge$_{2,3}$ are excluded by Lemma 5.4.

The only remaining claim to prove is that, without loss of generality in Except$_1$, $\delta_2, \delta_3 \to 0$. This follows from the fact that $\omega(\boldsymbol{x}_0) \subset \mathcal{M}_{0+-} \cup \mathcal{A}$ is compact.

□

7 Phase-Space Volume and Integral Estimates

This section is devoted to proving the last three main Theorems of this work. The first one is the already stated genericity of the attracting case in Theorem 4:

Theorem 5 (Bianchi VIII global attractor genericity Theorem). *In Theorem 4, the set of initial conditions $\boldsymbol{x}_0$ for which* Attract *applies is "generic" in the following sense: It is open and dense in $\mathcal{M}_{++-}$ and its complement has Lebesgue-measure zero.*

The second theorem of this section answers affirmatively the "locality" part of the longstanding BKL conjecture for spatially homogeneous Bianchi class A vacuum space-times, for measure theoretic notions of genericity, and can be considered the main result of this work:

Theorem 6 (Almost sure formation of particle horizons). *For Lebesgue almost every initial condition in $\mathcal{M}$, with respect to the induced measure from $\mathbb{R}^5 = \{(\Sigma_+, \Sigma_-, N_1, N_2, N_3)\}$, the following holds:*

$$\int_0^\infty \max_i \delta_i(t) \mathrm{d}t = 2 \int_0^\infty \max_{i \neq j} \sqrt{|N_i N_j|}(t) < \infty. \tag{7.1}$$

This means that almost every Bianchi VIII *and* IX *vacuum spacetime forms particle horizons towards the big bang singularity. This physical interpretation is described in Section 2.2, Lemma 2.2 or [HR09].*

This result about Lebesgue a.e. solutions immediately raises the question for counter-examples:

Question 7.1. *It is currently not known, whether there exist any solutions, which are attracted to $\mathcal{A}$ and have infinite integral $\int_0^\infty \delta_i(t) = \infty$.*

Do such solutions exist? Is it possible to describe an example of such a solution?

We very strongly expect that such solutions do *exist, for reasons which will be explained in future work.*

The third and last theorem of this section strengthens and extends the previous result:

Theorem 7 (L^p estimates for the generalized localization integral)**.** *Let $\alpha \in (0, 2)$ and $p \in (0, 1]$ such that $\alpha p > 2p - 1$, i.e. $p < \frac{1}{2-\alpha}$. Let $M \subset \mathcal{M}_{\pm\pm\pm}$ be a compact subset such that Theorem 2 holds for every initial condition in $\boldsymbol{x}_0 \in M$, i.e. $M \subset$ Basin. Then $I_\alpha \in L^p(M)$, where*

$$I_\alpha(\boldsymbol{x}_0) := \int_0^\infty \max_i \delta_i^\alpha(t)\mathrm{d}t, \tag{7.2}$$

i.e., using ϕ for the flow to (2.3.2) *and* $\mathrm{d}^4\boldsymbol{x}$ *for the (four dimensional) Lebesgue measure on $\mathcal{M}_{\pm\pm\pm}$,*

$$\int_M \left[\int_0^\infty \delta_i^\alpha(\phi(\boldsymbol{x}, t))\mathrm{d}t\right]^p \mathrm{d}^4\boldsymbol{x} < \infty \qquad \forall\, i \in \{1, 2, 3\}. \tag{7.3}$$

If instead $\alpha \geq 2$, we already know from Theorem 2 that $I_\alpha \in L^\infty(M)$.

Theorem 7 makes a much stronger claim than Theorem 6, even for $\alpha = 1$: Local L^p-integrability is a sufficient condition for a.e. finiteness, but not necessary. On the other hand, we are aware of no immediate physical interpretation of Theorem 7.

The proof of Theorem 7 does not rely on Theorem 6. Even though the proof of Theorem 6 is therefore entirely optional, we nevertheless choose to state and prove Theorem 6 separately, because we view it as the more important result, and can give a more geometric proof than for Theorem 7.

Question 7.2. *Unfortunately, the case $\alpha = 1 = p = 1$, i.e. $I_1 \in L^1_{loc}$, is out of reach of Theorem 7, which only provides $I_1 \in L^{1-\epsilon}_{loc}$ for any $\epsilon > 0$. An extension to $\alpha = p = 1$ would imply* finite expectation *of the conformal size of particle horizons.*

Is it possible to say something about $\alpha = p = 1$?

Outline of the proofs. Let us now give a short overview over the remainder of this section. Our primary tool will be a volume-form ω_4, which is *expanded* under the flow ϕ of (2.3.2). An alternative description would be to say that we construct a density function, such that ϕ is volume-expanding. The volume-form will be constructed and discussed in Section 7.1. We will use some very basic facts and definitions from the intersection of differential forms, measure theory and dynamical systems theory, which are given in Appendix A.4 for the convenience of the reader.

We will use this expanding volume-form in order to prove our three Theorems in Section 7.2.

7.1 Volume Expansion

This section studies the evolution of phase-space volumes. Using logarithmic coordinates $\beta_i = -\log|N_i|$, the equations differential equations (2.3.2) *without* the constraint (2.3.3) yield the remarkably simple and controllable formula $\mathrm{D}_t\omega_5 = 2N^2\omega_5$ for the evolution of the five-dimensional Lebesgue-measure ω_5 (with respect to $\Sigma_\pm, \beta_i$). This formula shows that the flow ϕ expands the volume ω_5. Such a volume expansion is impossible for systems living on a manifold with finite volume; it is possible with logarithmic coordinates because these coordinates have pushed the attractor to infinity, and typical solutions escape to infinity in these coordinates.

Volume expansion for the extended system (without constraint $G = 1$). Consider coordinates β_i given by

$$\beta_i = -\log|N_i| \qquad \mathrm{d}\beta_i = -\frac{\mathrm{d}N_i}{N_i} \qquad \partial_{\beta_i} = -N_i\partial_{N_i} \tag{7.1.1a}$$

$$N_i = \hat{n}_i e^{-\beta_i} \qquad \mathrm{d}N_i = -\hat{n}_i e^{-\beta_i}\mathrm{d}\beta_i \qquad \partial_{N_i} = -\hat{n}_i e^{\beta_i}\partial_{\beta_i}. \tag{7.1.1b}$$

and consider the Lebesgue-measure with respect to the β_i coordinates:

$$\omega_5 = |\mathrm{d}\Sigma_+ \wedge \mathrm{d}\Sigma_- \wedge \mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2 \wedge \mathrm{d}\beta_3|. \tag{7.1.2}$$

Let $\lambda(\boldsymbol{x}, t)$ denote the volume expansion for $\phi(\boldsymbol{x}, t)$, i.e. $\phi^*(\boldsymbol{x}, t)\omega_5 = \lambda(\boldsymbol{x}, t)\omega_5$, where ϕ^* is the pull-back acting on differential forms.

Claim 7.1.1. We can compute $\mathrm{D}_t\omega_5 = 2N^2\omega_5$, and hence

$$\lambda(\boldsymbol{x}, t) = \exp\left[2\int_0^t N^2(\phi(\boldsymbol{x}, s))\mathrm{d}s\right]. \tag{7.1.3}$$

Proof. Write the vectorfield f corresponding to (2.3.2) in β_i coordinates. The volume expansion is given by the trace $\mathrm{D}_t\lambda = \operatorname{tr} f$. We see that f_{β_i} is independent of β_i, and $\operatorname{tr}\partial_{\boldsymbol{\Sigma}} f_{\boldsymbol{\Sigma}} = 2N^2$ is the only occuring term. □

Volume expansion on $\mathcal{M}$. We are not really interested in the behaviour of ϕ on $\mathbb{R}^5$, and the measure ω_5. Instead, we are interested in dynamics and measures on the set $\mathcal{M} = \{\boldsymbol{x} \in \mathbb{R}^5 : G(\boldsymbol{x}) = 1\}$. We can get an induced measure on $\mathcal{M}$ by choosing a vectorfield $X : \mathbb{R}^5 \to T\mathbb{R}^5$ such that $\mathrm{D}_X G = 1$ in a neighbourhood of $\mathcal{M}$. Then we set

$$\begin{aligned} \omega_4 &= \iota_X \omega_5, \quad \text{i.e.} \\ \omega_4[X_1, \dots, X_4] &= \omega_5[X, X_1, \dots X_4] \qquad \text{for} \quad X_1, \dots, X_4 \in T\mathcal{M}. \end{aligned} \tag{7.1.4}$$

Claim 7.1.2. The induced volume ω_4 is independent of the choice of X (as long as $\mathrm{D}_X G = 1$). Since G is a constant of motion, we have $\mathrm{D}_t \mathrm{d}G = 0$ and ω_4 is transformed by the same λ as ω_5, i.e. $\mathrm{D}_t \omega_4 = 2N^2 \omega_4$.

Proof. The independence of the choice of X follows from abstract theory. We have seen in Claim 2.3.1 that G is a constant of motion on $\mathbb{R}^5$. By abstract theory, this implies that ω_4 is transported by the same equation as ω_5. $\square$

The volume ω_4 is really expanding: In most of the phase space, $N^2 > 0$, and always $N^2 > 4\Delta^2 = -4|N_1 N_2 N_3|^{\frac{2}{3}}$, which has bounded integral by Lemma 3.6. Let us state this more precisely:

Lemma 7.3. *For every $\epsilon_{N,7.1} > 0$, there exist $\epsilon_{d,7.2}, c_{7.1} = \epsilon_{d,7.2}, c_{7.1}(\epsilon_{N,7.1}) > 0$, such that the following holds:*

For every $\boldsymbol{x}_0 \in \mathcal{M}$ with $\max_i |N_i| \geq \epsilon_{N,7.1}$ and $\max_i \delta_i \leq \epsilon_{d,7.2}$, and every time $t \geq 1$, we have $\lambda(\boldsymbol{x}_0, t) \geq 1 + c_{7.1}$, where λ is defined in (7.1.3).

Proof. Assume first that $\boldsymbol{x}_0$ is not of Bianchi type IX. Then $N^2 \geq 0$, and if $\epsilon_{d,7.2} > 0$ is small enough, then we clearly have $|N_1| \gg |N_2|, |N_3|$, without loss of generality, and hence $N^2(\boldsymbol{x}_0) \geq \epsilon_{N,7.1}/2$. This clearly implies $\lambda(\boldsymbol{x}_0, 1) > 1 + c$, and by $N^2 \geq 0$ outside of Bianchi IX, $\lambda(\boldsymbol{x}_0, t) \geq \lambda(\boldsymbol{x}_0, 1) > 1 + c$, for some small $c_{7.1} > 0$, and all constants independent of $\boldsymbol{x}_0$.

If we are in Bianchi type IX, then the same proof shows $\lambda(\boldsymbol{x}_0, 1) > 1 + 2c_{7.1}$. In view of Lemma 3.6, we can choose $\epsilon_{d,7.2}(\epsilon) > 0$ small enough that $\int_0^\infty |\min(N^2, 0)|(t)\mathrm{d}t < \epsilon$ for all $\boldsymbol{x}_0$ fulfilling our assumptions (since $|N_1 N_2 N_3| \leq \epsilon_{d,7.2}^{\frac{2}{3}}$). Adjusting constants yields the conclusion for Bianchi type IX. $\square$

Volume expansion for Poincaré-maps. We consider the volume-form $\omega_3 = \iota_f \omega_4$, where f is the vectorfield (2.3.2). By invariance of f under the flow ϕ, we again have $\phi^*(\boldsymbol{x}, t)\omega_3 = \lambda(\boldsymbol{x}, t)\omega_3$.

If $S \subseteq \mathcal{M}$ is a Poincaré-section and $K \subseteq S$ is a set with $|K|_{\omega_3} = \int_S \chi_K |\omega_3| = 0$, then, by Fubini's Theorem, $|\phi(K, \mathbb{R})|_{\omega_4} = 0$. The same applies for the (manifold-, not topological) boundary ∂S. The measure ω_4 is absolutely bi-continuous with respect to the ordinary Lebesgue measure, i.e. the notions of sets of measure zero coincide for the ordinary Lebesgue-measure and ω_4.

At some point, we need to actually compute integrals, in coordinates. For this, we use the following:

Lemma 7.4. *Let $S \subseteq \mathcal{M}_{+\pm\pm}$ be a Poincaré-section intersecting the heteroclinic orbit $-\mathbf{T}_1 \to \mathbf{T}_1$, of the form:*

$$S \subseteq \{\boldsymbol{x} \in \mathcal{M} : N_1 = \mathit{const} = h,\, r_1 \leq \epsilon,\, \delta_1 \leq \epsilon\},$$

with $\epsilon > 0$ small enough. For the sake of concreteness, one can take $h = 0.1$ and $\epsilon = 0.05$. Then, S can be written as a smooth graph $\Sigma_+ = \Sigma_+(\Sigma_-, N_2, N_3)$, and we can write $\omega_3 = \rho(\boldsymbol{x})\mathrm{d}\Sigma_- \wedge \mathrm{d}\beta_2 \wedge \mathrm{d}\beta_3$ with a bounded density function $\rho(\boldsymbol{x}) < C_{7.2} = C_{7.2}(h, \epsilon)$.

Proof. It is obvious that S is a smooth graph with $|\partial_{\Sigma_-,\beta_2,\beta_3}\Sigma_+| \leq \mathcal{O}(\epsilon)$. Note $\partial_{\beta_1}G = 2N_1^2 - 2N_1N_- \geq C > 0$ if $\epsilon > 0$ is small enough, and $\Sigma_+' = f_{\Sigma_+} > C > 0$. We can then write

$$\begin{aligned}\omega_3 &= |\mathrm{d}\Sigma_- \wedge \mathrm{d}\beta_2 \wedge \mathrm{d}\beta_3| \\ &\quad \cdot \left|\omega_5\left[f,\, |\partial_{\beta_1}G|^{-1}\partial_{\beta_1},\, \partial_{\Sigma_-} + (\partial_{\Sigma_-}\Sigma_+)\partial_{\Sigma_+},\, \partial_{\beta_2} + (\partial_{\beta_2}\Sigma_+)\partial_{\Sigma_+},\, \partial_{\beta_3} + (\partial_{\beta_3}\Sigma_+)\partial_{\Sigma_+}\right]\right| \\ &\leq C|\mathrm{d}\Sigma_- \wedge \mathrm{d}\beta_2 \wedge \mathrm{d}\beta_3|.\end{aligned}$$

□

7.2 Proofs of the main Theorems

We will now use the ω_3-expansion between Poincaré-sections in order give proofs of the main results. Note that the return-times to the Poincaré-sections are very similar the times T^C, used in Sections 4, 5 and 6.

The proofs of Theorems 5 and 6 rely on the following:

Lemma 7.5. *Consider a small Poincaré-section S, as in Lemma 7.4, intersecting the heteroclinic $-\mathbf{T}_1 \to \mathbf{T}_1$:*

$$S = \{\boldsymbol{x} \in \mathcal{M} : N_1 = \mathit{const} = h,\, r_1 \leq \epsilon,\, \delta_1 \leq \epsilon\}.$$

Let $g : [0, \epsilon) \to [0, 1)$ be continuous and increasing. Define $\textsc{Bad}_g = \{\boldsymbol{x} \in S \cap \mathcal{M}_{\pm\pm\pm} : \delta_1 \geq g(r_1)\}$. If $\int_0^{0.1} |\log g(r)|^2\mathrm{d}r < \infty$.

Assume that $\int_0^{0.1} |\log g(r)|^2\mathrm{d}r < \infty$. Then, $|\textsc{Bad}_g|_{\omega_3} < \infty$, and Lebesgue almost every solution eventually avoids $\textsc{Bad}_g$.

Proof. Fix the signs $\hat{n} \in \pm^3$. In an abuse of notation, we suppress this in the following.

Claim 7.2.1. $|\textsc{Bad}_g|_{\omega_3} < \infty$.

Proof. Take logarithms in the defining inequality for $\textsc{Bad}_g$, in order to see $\textsc{Bad}_g = \{\boldsymbol{x} \in S : \beta_2 + \beta_3 \leq \log 4 + 2|\log g(r_1)|\} \subset \{\boldsymbol{x} \in S : \beta_2 + \beta_3 \leq \log 4 + 2|\log g(|\Sigma_-|)|\}$. Then, estimate

$$|\textsc{Bad} \cap S|_{\omega_3} \leq |\{\boldsymbol{x} \in S : \beta_2 + \beta_3 \leq C + C|\log g(|\Sigma_-|)|\}|_{\omega_3} \tag{7.2.1a}$$
$$= \int_{\{\boldsymbol{x}\in S:\,\beta_2+\beta_3\leq C+C|\log g(|\Sigma_-|)|\}} |\omega_3| \tag{7.2.1b}$$
$$\leq C \int_{\{\boldsymbol{x}\in S:\,\beta_2+\beta_3\leq C+C|\log g(|\Sigma_-|)|\}} |\mathrm{d}\Sigma_- \wedge \mathrm{d}\beta_2 \wedge \mathrm{d}\beta_3| \tag{7.2.1c}$$
$$\leq C + C\int_{-\epsilon}^{\epsilon} |\log g(|\Sigma_-|)|^2 \mathrm{d}\Sigma_- < \infty. \tag{7.2.1d}$$

From (7.2.1c) to (7.2.1d), we integrated β_2, β_3, using $\beta_2, \beta_3 \geq 0$. In order to go from (7.2.1b) to (7.2.1c), we used Lemma 7.4. ■

Define $\textsc{BadRecurrent}_g = \{\boldsymbol{x} \in S : \boldsymbol{\Phi}_S^k(x) \in \textsc{Bad}\,\text{for infinitely many } k\}$, where Φ_S is the Poincaré-map to S.

Claim 7.2.2. $|\textsc{BadRecurrent}_g|_{\omega_3} = 0$.

Proof. We can write

$$|\textsc{BadRecurrent}_g|_{\omega_3} = \left|\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} \Phi_S^{-k}(\textsc{Bad}_g)\right|_{\omega_3}$$
$$\leq \lim_{n\to 0} \sum_{k\geq n} \left|\Phi_S^{-k}(\textsc{Bad}_g)\right|_{\omega_3} \leq \lim_{n\to 0} \sum_{k\geq n} q^{-k}\,|\textsc{Bad}_g|_{\omega_3} = 0.$$

where $q = \inf\{\lambda(\boldsymbol{x}, T_S(\boldsymbol{x})) : \boldsymbol{x} \in S\} > 1$. ■□

Proof of Theorem 5 (Bianchi VIII *global attractor genericity).* Attract holds for an open set of initial conditions (by virtue of continuity of the flow). Since Taub can only happen on an embedded submanifold of lower dimension, it suffices to prove that $\textsc{Except}_1$ happens only for a set of initial conditions with Lebesgue measure zero, without loss of generality with respect to index permuations.

Recall Theorem 2 and apply Lemma 7.5, with $g(r) = \frac{\epsilon_{q,6.1}}{2} r$. □

Proof of Theorem 6 (almost sure formation of particle horizons). Without loss of generality, we will concentrate on showing $\int_0^\infty \delta_1(t)\mathrm{d}t < \infty$. Let $S \subset \mathcal{M}$ be a Poincaré-section, as in the proof of Lemma 7.5. For a solution $\boldsymbol{x} : [0,\infty) \to \textsc{Basin}$, i.e. for a solution where Theorem 2 applies, let $T_1(\boldsymbol{x}) < T_2(\boldsymbol{x}) < \ldots$, and $\boldsymbol{x}_n = \boldsymbol{x}(T_n(\boldsymbol{x})) \in S$ be the (possibly finite) sequence of recurrence times and points to S.

If the sequence of recurrence times is finite (or empty), ending in some T_k, then $\delta_1(t) \leq e^{-ct}$ for all $t > T_k$ (see Claim 6.2.1; alternatively, consider this obvious from the preciding sections). Then, we have $\int_0 \infty\delta_1(t)\mathrm{d}t < \infty$; hence, we will assume that the sequence of recurrence times is infinite, for the rest of the proof.

We know that $\delta_1(T_n) \le e^{-cn}$ for all $n \ge 0$; for a proof, see Claim 6.2.1; alternatively, consider this obvious from the preceding sections, or alternatively, note that $|h|\delta_1^2 = \Delta^3$, where $h = N_1$ was the defining equation for S, and $\Delta(t) \le e^{-ct}$.

Consider $g(r) = r^4$, and apply Lemma 7.5, in order to see that almost every solution $\boldsymbol{x}$ has $\delta_1 \le r_1^4$, for all sufficiently late recurrence times T_n, $n > 0$ large enough.

Let $\boldsymbol{x}$ be such a solution, starting in $\boldsymbol{x}_0 = \boldsymbol{x}(0)$, that avoids $\mathrm{B}\textsc{ad}_g$ for returns later than T_{k_0}. Assuming that the sequence of recurrence times in infinite, we can estimate:

$$\begin{aligned}
\int_0^\infty \delta_1(\phi(\boldsymbol{x}_0, t))\mathrm{d}t &= \int_0^{T_{k_0}} \delta_1(t)\mathrm{d}t + \sum_{n=k_0}^{\infty} \int_{T_n}^{T_{n+1}} \delta_1(t)\mathrm{d}t \\
&\le C(\boldsymbol{x}_0) + \sum_{n=k_0}^{\infty} \int_{T_n}^{T_{n+1}} C_{5.8} \exp\left[-c_{5.7} r_1^2(T_n)(t - T_n)\right] \delta_1(T_n)\mathrm{d}t \\
&\le C(\boldsymbol{x}_0) + \sum_{n=k_0}^{\infty} C \frac{\delta_1}{r_1^2}(T_n) \\
&\le C(\boldsymbol{x}_0) + \sum_{n=k_0}^{\infty} C \sqrt{\delta_1}(T_n) < \infty,
\end{aligned}$$

where we used Propositions 5.3, 4.1 and $\delta_1 < r_1^4$ and the fact that the sequence $\delta_1(T_n)$ decays uniformly exponentially in n. For solutions where the sequence of recurrence times is finite, the above series becomes a finite sum; then, the same estimate holds. □

Proof of Theorem 7 (L^p estimates for the generalized localization integral). We want to show that $I_\alpha \in L^p_{\mathrm{loc}}(\mathrm{B}\textsc{asin})$. The claim for $\alpha \in [2, \infty)$ and $p = \infty$ follows trivially from Theorem 2.

Without loss of generality, we restrict our attention to δ_1, and $p \le 1$. Consider the known Poincaré-Section S through $-\mathbf{T}_1 \to \mathbf{T}_1$; set $Q = S \cap \mathrm{B}\textsc{asin}$. We will first estimate I_α in Q, and will extend this to the rest of phase-space afterwards.

Claim 7.2.3. $I_\alpha \in L^p(Q, \omega_3)$, if α, p are as in the assumptions, and $p \le 1$.

Proof. Let $Q_n = \Phi_S^{-1}(Q_{n-1})$, and $Q_0 = Q$ be the sets of points that return to Q at least n times, and let $T_n : Q \to [0, \infty]$ be the recurrence times, such that $T_0 = 0$, $T_n(\boldsymbol{x}) < \infty$ if and only if $\boldsymbol{x} \in Q_n$, and $\phi(\boldsymbol{x}, T_n(\boldsymbol{x})) \in Q$ for $\boldsymbol{x} \in Q_n$. We can estimate, for some positive

$s \in (2p-1, \alpha p)$, $p \leq 1$:

$$
\begin{aligned}
\int_{\boldsymbol{x}\in Q}\left[\int_0^\infty \delta_i^\alpha(\phi(\boldsymbol{x},t))\mathrm{d}t\right]^p |\omega_3| &= \int_{\boldsymbol{x}\in Q}\left[\sum_n \int_{T_n}^{T_{n+1}} \delta_i^\alpha(\phi(\boldsymbol{x},t))\mathrm{d}t\right]^p |\omega_3| \\
&\leq \sum_n \int_{\boldsymbol{x}\in Q_n}\left[\int_{T_n}^{T_{n+1}} \delta_i^\alpha(\phi(\boldsymbol{x},t))\mathrm{d}t\right]^p |\omega_3| \qquad (7.2.2a)\\
&\leq C\sum_n \int_{\boldsymbol{x}\in Q_n}\left[\frac{\delta_i^\alpha}{r_i^2}(T_n)\right]^p |\omega_3| \\
&= C\sum_n \int_{\boldsymbol{x}\in Q_n}\left[\delta_i^{\alpha p-s} r_i^{-2p+s}\left(\frac{\delta_i}{r_i}\right)^s\right](T_n)\,|\omega_3| \\
&\leq C\sum_n \sup_{\boldsymbol{x}\in Q_n}\left(\frac{\delta_i}{r_i}\right)^s(T_n)\ \cdot\ \int_{\boldsymbol{x}\in Q_n}\left[\delta_i^{\alpha p-s} r_i^{-2p+s}\right](T_n)\,|\omega_3|. \qquad (7.2.2b)
\end{aligned}
$$

Here, the first inequality is meant only informally. We have used $p \leq 1$ in order to split the integral in (7.2.2a) and the Hölder inequality in (7.2.2b). We continue the estimates by noting that $\sup_{\boldsymbol{x}\in C}\frac{\delta_i}{r_i}(T_n)$ decreases exponentially in n. Hence we only need to bound the second factor. This can be done by using $\alpha p - s > 0$ and $-2p+s > -1$ in order to see

$$
\begin{aligned}
\int_{\boldsymbol{x}\in Q_n}\left[\delta_i^{\alpha p-s} r_i^{-2p+s}\right](\Phi_S^n\boldsymbol{x})\,|\omega_3| &= \int_{\boldsymbol{y}\in\Phi_S(Q_n)}\left[\delta_i^{\alpha p-s} r_i^{-2p+s}\right](\boldsymbol{y})\,|(\Phi_S^n)^*\omega_3| \\
&\leq C\int_S\left[\delta_i^{\alpha p-s} r_i^{-2p+s}\right]|\omega_3| \\
&\leq C\int_{-0.1}^{0.1}\left[\int_{\beta_2,\beta_3\geq 0} e^{-\frac{(\beta_2+\beta_3)(\alpha p-s)}{2}}|\Sigma_-|^{-2p+s}|\mathrm{d}\beta_2\wedge\mathrm{d}\beta_3|\right]\mathrm{d}\Sigma_- \\
&\leq C\int_{-0.1}^{0.1}|\Sigma_-|^{-2p+s}\mathrm{d}\Sigma_- < \infty.
\end{aligned}
$$

■

Therefore, we have shown $I_\alpha \in L^p(Q,\omega_3)$, if either $0 < \alpha \leq 1$ and $p < (2-\alpha)^{-1}$, or if $1 < \alpha < 2$ and $p \in (0,1]$.

In order to get a more global estimate, let $\boldsymbol{p} \in \textsc{Basin}\setminus S$. Let $R \subset \textsc{Basin}$ be a small Poincaré-section with $\boldsymbol{p} \in R$, such that $\overline{R} \subset \textsc{Basin}$ is compact, and solutions starting in R never return to R (this can be constructed by e.g. using the strictly decreasing $|N_1N_2N_3|$).

Claim 7.2.4. *$I_\alpha \in L^p(R,\omega_3)$.*

Proof. Set $R_Q = \Phi^{-1}(Q) \subseteq R$ and $R_\infty = R\setminus R_Q$. Then, by construction, $\Phi_S : R_Q \to Q$

is injective, and we can estimate:

$$\int_{\boldsymbol{x}\in R}\left[\int_0^\infty \delta_i^\alpha(\phi(\boldsymbol{x},t))\mathrm{d}t\right]^p|\omega_3| \le \int_{\boldsymbol{x}\in R_\infty} I_\alpha|\omega_3| + \int_{\boldsymbol{x}\in R_Q}\left[\int_0^{T_S(\boldsymbol{x})} \delta_i^\alpha(\phi(\boldsymbol{x},t))\mathrm{d}t\right]^p\omega_3$$
$$+\int_{\boldsymbol{y}=\Phi_S(\boldsymbol{x})}\left[\int_0^\infty \delta_i^\alpha(\phi(\boldsymbol{y},t))\mathrm{d}t\right]^p|\Phi_S^*\omega_3| < \infty.$$

■

Now, we are done, since open sets of the form $\phi(R_{\boldsymbol{p}},(0,1))$ form an open cover of Basin; hence, for any compact $K \subset$ Basin, we only need to consider finitely many such sets and add up the L^p-integrals of I_α. The property of being L^p_{Loc} is independent of the underlying volume-form, as long as it has a continuous positive density function. □

Note that the proof of Theorem 7 does not depend on the volume expansion; it only needs volume non-contraction.

A Appendix

A.1 Glossary of Equations and Notations

For easier reference, we compressed the most frequently referenced equations and notations on few pages.

Wainwright-Hsu equations The Wainwright-Hsu equations are given by:

$$N_i' = -(\Sigma^2 + 2\langle \mathbf{T}_i, \mathbf{\Sigma}\rangle)N_i \tag{2.3.2a}$$

$$= -\left(|\mathbf{\Sigma} + \mathbf{T}_i|^2 - 1\right) N_i \tag{2.3.2b}$$

$$\mathbf{\Sigma}' = N^2\mathbf{\Sigma} + 2\begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_3 & \mathbf{T}_2 \\ & \mathbf{T}_2 & \mathbf{T}_1 \\ & & \mathbf{T}_3 \end{pmatrix}[\boldsymbol{N}, \boldsymbol{N}], \quad \text{where} \tag{2.3.2c}$$

$$\mathbf{T}_1 = (-1, 0) \qquad \mathbf{T}_2 = \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right) \qquad \mathbf{T}_3 = \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right) \tag{2.3.1}$$

$$1 \overset{!}{=} \Sigma^2 + N^2 = \Sigma_+^2 + \Sigma_-^2 + N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1). \tag{2.3.3}$$

Auxiliary quantities are given by:

$$\delta_i = 2\sqrt{|N_jN_k|} \tag{2.3.7b}$$

$$r_i = \sqrt{(|N_j| - |N_k|)^2 + \frac{1}{3}\langle \mathbf{T}_j - \mathbf{T}_k, \mathbf{\Sigma}\rangle^2} \tag{2.3.7c}$$

$$\delta_i' = -\left(\left|\mathbf{\Sigma} - \frac{\mathbf{T}_i}{2}\right|^2 - \frac{1}{4}\right)\delta_i. \tag{2.3.8b}$$

Named Sets We use $\mathcal{M} = \{\boldsymbol{x} \in \mathbb{R}^5 : G(\boldsymbol{x}) = 1\}$, and $\mathcal{M}_{\hat{n}} \subset \mathcal{M}$ to denote the signs of the three N_i, with $\hat{n} \in \{+, -, 0\}^3$. If we use $\pm$ in subscripts, the repeated occurences are unrelated, such that $\mathcal{M}_{\pm\pm\pm} = \{\boldsymbol{x} \in \mathcal{M} : \text{all three } N_i \neq 0\}$. We use the notation $\mathcal{T}_i = \{\boldsymbol{x} \in \mathcal{M} : \langle \mathbf{T}_j \mathbf{\Sigma}\rangle = \langle \mathbf{T}_k, \Sigma\rangle,\, N_j = N_k\}$ for the Taub-spaces, where i, j, k are a permutation of $\{1, 2, 3\}$.

We frequently use the following subsets of $\mathcal{M}$ (with the obvious definition for subscripts $\hat{n} \in \{+, -, 0\}^3$):

$$\textsc{Cap}[\epsilon_N, \epsilon_d] = \{\boldsymbol{x} \in \mathcal{M} : \max |N_i| \geq \epsilon_N, \max_i \delta_i \leq \epsilon_d\} \tag{4.5}$$

$$\textsc{Circle}[\epsilon_N, \epsilon_d] = \{\boldsymbol{x} \in \mathcal{M} : \max |N_i| \leq \epsilon_N,\ \max_i \delta_i \leq \epsilon_d\} \tag{4.5}$$

$$\textsc{Hyp}[\rho, \epsilon_N, \epsilon_d] = \textsc{Circle}[\epsilon_N, \epsilon_d] \setminus [B_\rho(\mathbf{T}_1) \cup B_\rho(\mathbf{T}_2) \cup B_\rho(\mathbf{T}_3)] \tag{4.5}$$

$$\textsc{Basin}_{\hat{n}}[\epsilon_d, \epsilon_q] = \left\{\boldsymbol{x} \in \mathcal{M}_{\hat{n}} : \delta_i \leq \epsilon_d,\ \frac{\delta_i}{r_i} \leq \epsilon_q \quad \forall i \in \{1, 2, 3\}\right\}. \tag{6.1.3}$$

Polar Coordinates for Bianchi IX In polar coordinates, the equations around $r_1 \ll 1$ become for $N_2, N_3 > 0$:

$$\Sigma_- = r_1 \cos\psi_1, \quad N_- = r_1 \sin\psi_1 = N_3 - N_2, \quad N_+ = N_3 + N_2$$

$$\mathrm{D}_t \log r_1 = N^2 - (\Sigma_+ + 1)\frac{N_-^2}{r_1^2} + \sqrt{3}N_1 \frac{\Sigma_- N_-}{r_1^2} \tag{2.4.2a}$$

$$= r_1^2 \sin^2\psi_1 \frac{-\Sigma_+}{1-\Sigma_+} + N_1\, h_r \tag{2.4.4a}$$

$$\mathrm{D}_t \log \delta_1 = N^2 - (\Sigma_+ + 1) \tag{2.4.2c}$$

$$= \frac{-1}{1-\Sigma_+} r_1^2 \cos^2\psi_1 + \frac{-\Sigma_+}{1-\Sigma_+} r_1^2 \sin^2\psi_1 + N_1\, h_\delta \tag{2.4.4b}$$

$$\mathrm{D}_t \log \frac{\delta_1}{r_1} = -(\Sigma_+ + 1)\frac{\Sigma_-^2}{r_1^2} - \sqrt{3}N_1 \frac{\Sigma_- N_-}{r_1^2} \tag{2.4.2d}$$

$$= \frac{-1}{1-\Sigma_+} r_1^2 \cos^2\psi_1 + N_1(h_\delta - h_r) \tag{2.4.4c}$$

$$\psi_1{}' = \sqrt{3} r_1 \sqrt{\sin^2\psi_1 + \frac{\delta_1^2}{r_1^2}} - \frac{r_1^2}{1-\Sigma_+} \sin\psi_1 \cos\psi_1 + N_1 \sin\psi_1\, h_\psi, \tag{2.4.4d}$$

where the terms $|h_r|, |h_\delta|, |h_\psi|$ are bounded (if $|N_i|$, $\Sigma_+ < 0$, and Σ_- are bounded) and given in (2.4.5), page 29.

Polar Coordinates for Bianchi VIII In polar coordinates, the equations around $r_1 \ll 1$ become for $N_2 > 0$, $N_3 < 0$:

$$\Sigma_- = r_1 \cos\psi_1, \quad N_- = r_1 \sin\psi_1 = N_2 + N_3 \quad N_+ = N_2 - N_3$$

$$\mathrm{D}_t \log r_1 = N^2 - (\Sigma_+ + 1)\frac{N_-^2}{r_1^2} + \sqrt{3}N_1 \frac{\Sigma_- N_+}{r_1^2} \tag{2.4.7a}$$

$$= \frac{-\Sigma_+}{1-\Sigma_+} r_1^2 \sin^2\psi_1 + \delta_1^2 \frac{\cos^2\psi_1 - \Sigma_+}{1-\Sigma_+} + N_1\, h_r \tag{2.4.9a}$$

$$\mathrm{D}_t \log \delta_1 = N^2 - (\Sigma_+ + 1) \tag{2.4.2c}$$

$$= \frac{-1}{1-\Sigma_+} r_1^2 \cos^2\psi_1 + \frac{-\Sigma_+}{1-\Sigma_+} r_1^2 \sin^2\psi_1 + \frac{-\Sigma_+}{1-\Sigma_+}\delta_1^2 + N_1\, h_\delta \tag{2.4.9b}$$

$$\mathrm{D}_t \log \frac{\delta_1}{r_1} = -(\Sigma_+ + 1)\frac{\Sigma_-^2}{r_1^2} - \sqrt{3}N_1 \frac{\Sigma_- N_+}{r_1^2} \tag{2.4.7d}$$

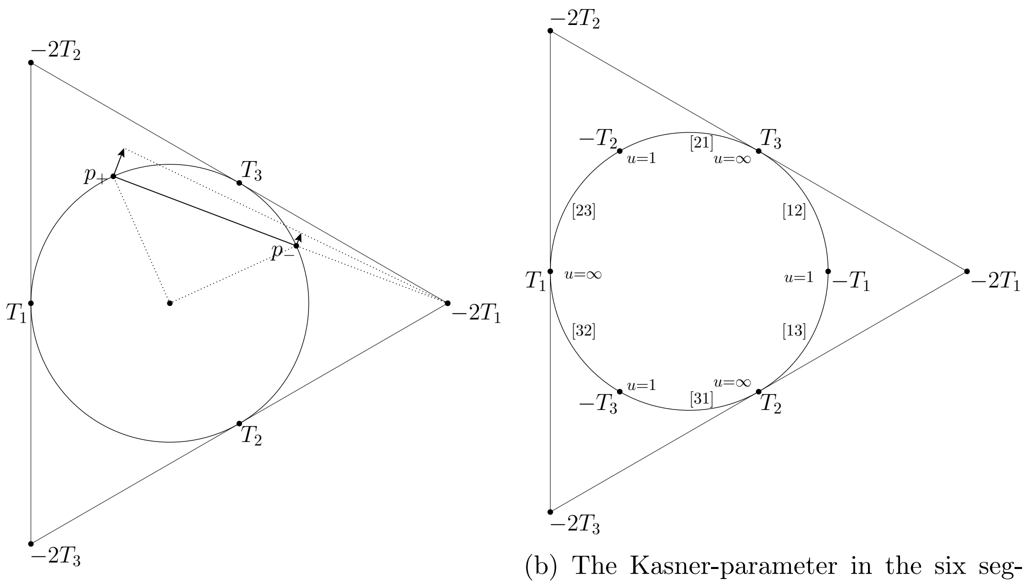
$$= \frac{-1}{1-\Sigma_+} r_1^2 \cos^2\psi_1 - \delta_1^2 \frac{\sin^2\psi_1}{1-\Sigma_+} + N_1(h_\delta - h_r) \tag{2.4.9c}$$

$$\psi_1{}' = \sqrt{3} r_1 \sqrt{\cos^2\psi_1 + \frac{\delta_1^2}{r_1^2}} - \frac{r_1^2 + \delta_1^2}{1-\Sigma_+} \cos\psi_1 \sin\psi_1 + N_1 \cos\psi_1\, h_\psi, \tag{2.4.9d}$$

where the terms $|h_r|, |h_\delta|, |h_\psi|$ are bounded (if $|N_i|$, $\Sigma_+ < 0$, Σ_-, and $\frac{\delta_1}{r_1}$ are bounded) and given in (2.4.10), page 31.

A.2 Properties of the Kasner Map

We deferred a detailed discussion of the Kasner-map K in Section 3.1, especially the proof of Proposition 3.2. We will first give a simple proof of Proposition 3.2, and then discuss classical ways of describing the Kasner map.



(a) A graphical proof of Lemma A.1.

(b) The Kasner-parameter in the six segments.

Figure 6: Expansion of the Kasner-map

Expansion of the Kasner-map. We can see from Figure 4b (p. 33) that the Kasner-map is a double-cover, has three fixed points and reverses orientation. From Figure 6a, we can see that K is non-uniformly expanding:

Lemma A.1. *Consider the vectorfield $\partial_{\mathcal{K}}(\boldsymbol{p}) = (-\Sigma_-(\boldsymbol{p}), +\Sigma_+(\boldsymbol{p}))$. Assume without loss of generality that $\boldsymbol{p}_-$ is such that the Kasner-map $\boldsymbol{p}_+ = K(\boldsymbol{p}_-)$ proceeds via the N_1-cap, i.e. $d(\boldsymbol{p}_-, -\mathbf{T}_1) < 1$ (see Figure 4).*

Then the Kasner-map K is differentiable at $\boldsymbol{p}_-$ and we have

$$K'(\boldsymbol{p}_-) = -\frac{|\boldsymbol{p}_+ + 2\mathbf{T}_1|}{|\boldsymbol{p}_- + 2\mathbf{T}_1|} < -1, \qquad \textit{where} \qquad K'(\boldsymbol{p}') : \quad K_*\partial_{\mathcal{K}}(\boldsymbol{p}_-) = K'(\boldsymbol{p}_-)\partial_{\mathcal{K}}(\boldsymbol{p}_+).$$

Proof of Lemma A.1. Informally, differentiability is evident from the construction in Figure 6a. The component of $\partial_{\mathcal{K}}$ which is normal to the line through $\boldsymbol{p}_+$, $\boldsymbol{p}_-$ and $-\mathbf{T}_1$ gets just elongated by a factor $\lambda = \frac{|\boldsymbol{p}_+ + 2\mathbf{T}_1|}{|\boldsymbol{p}_- + 2\mathbf{T}_1|}$. The angle between this line and the Kasner-circle, i.e. $\partial_{\mathcal{K}}$, is constant; therefore, the length of the component tangent to $\mathcal{K}$ must get elongated by the same factor. The negative sign is evident from Figure 6a.

Formally, the relation between $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$ is described by

$$|\boldsymbol{p}_-| = |\boldsymbol{p}_+| = 1 \qquad \boldsymbol{p}_+ + 2\mathbf{T}_1 = \lambda(\boldsymbol{p}_- + 2\mathbf{T}_1).$$

Setting $\boldsymbol{p}_- = \boldsymbol{p}_-(t)$, we obtain a differentiable function $\lambda = \lambda(t)$ by the implicit function theorem (as long as $\langle \boldsymbol{p}_+, \boldsymbol{p}_- + 2\mathbf{T}_1\rangle \neq 0$) and obtain (where we use $'$ to denote derivatives with respect to t):

$$\boldsymbol{p}_+' = \lambda'(\boldsymbol{p}_- + 2\mathbf{T}_1) + \lambda \boldsymbol{p}_-'.$$

Assuming $\boldsymbol{p}_+ \neq \boldsymbol{p}_-$, we can set

$$\boldsymbol{v} = \frac{\boldsymbol{p}_+ + 2\mathbf{T}_1}{|\boldsymbol{p}_+ + 2\mathbf{T}_1|} = \frac{\boldsymbol{p}_- + 2\mathbf{T}_1}{|\boldsymbol{p}_- + 2\mathbf{T}_1|} = \frac{\boldsymbol{p}_+ - \boldsymbol{p}_-}{|\boldsymbol{p}_+ - \boldsymbol{p}_-|},$$

and compute the projection to the normal component of $\boldsymbol{v}$: $(1-\boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{p}_+' = \lambda(1-\boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{p}_-'$, since the vector coefficient of λ' is parallel to $\boldsymbol{v}$. Now the vectors $\boldsymbol{p}_+'$ and $\boldsymbol{p}_-'$ are tangent to the Kasner-circle; letting $J(\Sigma_+, \Sigma_-) = (-\Sigma_-, \Sigma_+)$ be the unit rotation we can see that $\boldsymbol{p}_+' = \pm|\boldsymbol{p}_+'| J\boldsymbol{p}_+$ and $\boldsymbol{p}_-' = \pm|\boldsymbol{p}_-'| J\boldsymbol{p}_-$. Hence

$$\begin{aligned}
|(1-\boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{p}_+'|^2 &= \left(1 - \langle v, J\boldsymbol{p}_+\rangle^2\right)|\boldsymbol{p}_+'|^2 &&= \left(1 - \frac{\langle \boldsymbol{p}_-, J\boldsymbol{p}_+\rangle^2}{|\boldsymbol{p}_+ - \boldsymbol{p}_-|^2}\right)|\boldsymbol{p}_+'|^2\\
|(1-\boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{p}_-'|^2 &= \left(1 - \langle v, J\boldsymbol{p}_-\rangle^2\right)|\boldsymbol{p}_-'|^2 &&= \left(1 - \frac{\langle \boldsymbol{p}_+, J\boldsymbol{p}_-\rangle^2}{|\boldsymbol{p}_+ - \boldsymbol{p}_-|^2}\right)|\boldsymbol{p}_-'|^2.
\end{aligned}$$

By antisymmetry of the matrix J, i.e. $\langle \boldsymbol{p}_-, J\boldsymbol{p}_+\rangle = -\langle J\boldsymbol{p}_-, \boldsymbol{p}_+\rangle$, we therefore have $|\boldsymbol{p}_+'| = \lambda|\boldsymbol{p}_-'|$. □

Symbolic Description. For a given $\boldsymbol{p}_0 \in \mathcal{K}$, we can symbolically encode the trajectory $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$ (with $\boldsymbol{p}_{n+1} = K(\boldsymbol{p}_n)$) under the Kasner-map. The easiest way to do so is to encode it by $(s_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$, where $s_n = i$ if $\boldsymbol{p}_n \to \boldsymbol{p}_{n+1}$ occurs via the $|N_i| > 0$-cap. Then $(s_n)_{n\in\mathbb{N}}$ has the property that no symbol repeats, i.e. $s_{n+1} \neq s_n$ for all $n \in \mathbb{N}$. We have, however, an ambiguity if $\boldsymbol{p}_N = \mathbf{T}_i$ for some $N > 0$. If this occurs, then also all later points have $\boldsymbol{p}_{N+n} = \mathbf{T}_i$. We chose to allow both encodings $\boldsymbol{p}_N = j$ and $\boldsymbol{p}_N = k$, as long as the property that no symbol repeats is preserved. Factoring out this ambiguity gives us a map

$$\begin{aligned}
&\Psi : \mathcal{K} \to \{(s_n)_{n\in\mathbb{N}}\{1,2,3\}^{\mathbb{N}} : \text{no sumbol repeats}\} \,/\, \{(*\overline{ij}) = (*\overline{ji})\},\\
&\Psi(p_0) = (s_n)_{n\in\mathbb{N}}, \qquad \text{such that } d(\boldsymbol{p}_n, -\mathbf{T}_{s_n}) \leq 1 \text{ and no symbol repeats,}
\end{aligned}$$

where $*$ stands for an arbitrary initial piece and $\overline{jk}$ stands for a periodic tail $(*\overline{jk}) = (*jkjkjk\ldots)$. This map Ψ is continuous (since the Kasner-map is continuous), where we endow the target space $\{1,2,3\}^{\mathbb{N}}/\sim$ with the quotient topology of the product topology. Note that, by construction, Ψ semi-conjugates the Kasner-map K to the shift-map σ:

$$\Psi \circ K = \sigma \circ \Psi, \quad \text{where} \quad \sigma : (s_0 s_1 s_2 \ldots) \to (s_1 s_2 \ldots).$$

In order to see that Ψ is a homeomorphism, we construct a continuous inverse. Denote the three segments of $\mathcal{K}$ as $\mathcal{K}_i = \{\boldsymbol{p} \in \mathcal{K} : d(\boldsymbol{p}, -\mathbf{T}_i) \leq 1\}$. We can construct inverse maps $K_{ij}^{-1} : \mathcal{K}_j \to \mathcal{K}_i$, such that $K \circ K_{ij}^{-1} : \mathcal{K}_i \to \mathcal{K}_i = \mathrm{id}$. Then we get an inverse map

$$\Psi^{-1} : (s_n)_{n\in\mathbb{N}} \to \bigcap_{\ell\in\mathbb{N}} K_{s_0 s_1}^{-1} K_{s_1 s_2}^{-1} \ldots K_{s_\ell s_{\ell+1}}^{-1}(\mathcal{K}_{s_{\ell+1}}).$$

We now need to show that $\Psi^{-1}((s_n)_{n\in\mathbb{N}}) = \{\boldsymbol{p}\}$ is a single point, which depends continuously on $(s_n)_{n\in\mathbb{N}}$, and is actually the inverse of Ψ.

We first consider a sequence $(s_n)_{n\in\mathbb{N}}$ which does not end up in a Taub-point.

In order to see that $\Psi^{-1}((s_n)_{n\in\mathbb{N}})$ is nonempty, note that it is the intersection of a descending sequence of nonempty compact sets. In order to see that it contains only a single point, note that K_{ij}^{-1} is (nonuniformly) contracting by Lemma A.1; hence the length $\textsc{len}_\ell = |K^{-1}_{s_0 s_1} K^{-1}_{s_1 s_2} \dots K^{-1}_{s_\ell s_{\ell+1}}(\mathcal{K}_{s_{\ell+1}})|$ is decreasing. The length $\textsc{len}_\ell$ also cannot converge to some $\textsc{len}_\infty$ as $\ell \to \infty$, since we have $|K_{ij}^{-1}(I)| < |I|$ for any interval I with $|I| > 0$. In order to see that Ψ^{-1} is continuous, we need to show that $\operatorname{diam} \Psi^{-1}\{(r_n)_{n\in\mathbb{N}} : r_n = s_n \,\forall n \le N\} \to 0$ as $N \to \infty$. This also follows from the previous argument of decreasing lengths.

Next, we consider a sequence $(s_n)_{n\in\mathbb{N}}$ which does end up in a Taub-point $\mathbf{T}_i$ at $n = N$. Let $(s'_n)_{n\in\mathbb{N}}$ denote the other representative of $(s_n)_{n\in\mathbb{N}}$, i.e. we changed $(s_0, \dots, s_{N-1}, \overline{jk}) \leftrightarrow ((s_0, \dots, s_{N-1}, \overline{kj}))$. The previous arguments about single-valuedness and continuity still apply for each of the two representatives; we only need to show that Ψ^{-1} coincides for both. This is obvious.

It is also obvious by our construction that Ψ and Ψ^{-1} are inverse to each other. Hence Ψ and Ψ^{-1} C^0-conjugate K to the shift-map

$$\Psi \circ K \circ \Psi^{-1} = \sigma : (s_0 s_1 s_2 \dots) \to (s_1 s_2 \dots).$$

The exact same arguments apply in order to conjugate the map $D : \mathbb{R}/3\mathbb{Z} \to \mathbb{R}/3\mathbb{Z}$, $D : [x]_{3\mathbb{Z}} \to [-2x]_{3\mathbb{Z}}$ to the same shift, where we replaced $\mathbf{T}_i = i$ and $\mathcal{K}_i = [i+1, i+2]$. Hence, the Kasner-map is C^0 conjugate to D:

Proposition 3.2. *There exists a homeomorphism $\Psi : \mathcal{K} \to \mathbb{R}/3\mathbb{Z}$, such that $\Psi(\mathbf{T}_i) = [i]_{3\mathbb{Z}}$ and*

$$\Psi(K(\boldsymbol{p})) = [-2\Psi(\boldsymbol{p})]_{3\mathbb{Z}} \qquad \forall \boldsymbol{p} \in \mathcal{K}.$$

Kasner Eras and Epochs. A useful and customary description of the symbolic dynamics of K is obtained by distinguishing between "small" bounces around a Taub-point, called "Kasner epochs" and denoted by the letter S in this work, and "long" bounces, called "Kasner eras" and denoted by the letter L in this work. The S and L encoding of an orbit can be obtained from previous $\{1,2,3\}$-encoding by the map

$$\textsc{EpochEra} : \{s \in \{1,2,3\}^{\mathbb{N}} : \, s_n \neq s_{n+1}\} \to \{(s_0 s_1 | r_0 r_1 \dots) : \, s_0, s_1 \in \{1,2,3\}, \, r_n \in \{\mathrm{S}, \mathrm{L}\}\}$$

$$(s_0 s_1 s_2 \dots) \to (s_0 s_1 | r_0 r_1 \dots), \quad \text{where} \quad \begin{cases} r_n = \mathrm{S} & \text{if } s_n = s_{n+2}, \\ r_n = \mathrm{L} & \text{if } s_n \neq s_{n+2}. \end{cases}$$

We can remember the value of s_n as a sub-index of r_n, such that e.g.

$$(132121213231\dots) \to (13|\mathrm{L}_1\mathrm{L}_3\mathrm{L}_2\mathrm{S}_1\mathrm{S}_2\mathrm{S}_1\mathrm{S}_2\mathrm{L}_1\mathrm{L}_3\mathrm{S}_2\mathrm{L}_3 *_1 \dots).$$

Then the Kasner-map becomes

$$(s_0 s_1|r_0 r_1 r_2 r_3 \ldots) \to \begin{cases} (s_1 i|r_1 r_2 \ldots) & \text{if } s_0 = i \text{ and } r_0 = \mathrm{S} \\ (s_1 i|r_1 r_2 \ldots) & \text{if } s_0 \neq i \neq s_1 \text{ and } r_0 = \mathrm{L}. \end{cases}$$

Note that the first two indices in $\{1, 2, 3\}$ describe in which of the six symmetric segments of $\mathcal{K}$ a point lies, see Figure 6b.

Another customary way of writing such sequences is to write every symbol L as a semicolon ";" and abbreviate the S symbols in between by just their number, such that the previous example becomes

$$(13|\mathrm{L}_1\mathrm{L}_3\mathrm{L}_2\mathrm{S}_1\mathrm{S}_2\mathrm{S}_1\mathrm{S}_2\mathrm{L}_1\mathrm{L}_3\mathrm{S}_2\mathrm{L}_3 *_1 \ldots) \to (13|0;0;0;4;0;1;\ldots).$$

The Kasner-Parameter. There exists an explicit coordinate transformation, related to the continued fraction expansion, which realizes the conjugacy to the shift-space. This is done via the so-called *Kasner-parameter* u, and is the most standard way of discussing the Kasner-map.

In order to introduce the Kasner-parameter, it is useful to use the coordinates which make the permutation symmetry of the indices more apparent. This is done via

$$\Sigma_i = 2\langle \mathbf{T}_i, \mathbf{\Sigma}\rangle \qquad \Sigma_+ = -\frac{1}{2}\Sigma_1 \qquad \Sigma_- = \frac{1}{2\sqrt{3}}(\Sigma_3 - \Sigma_2).$$

These variables are constrained by $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ and have $\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 6(\Sigma_+^2 + \Sigma_-^2)$.

The parametrization of $\mathcal{K}$ depends not only on a real parameter $u \in \mathbb{R}$, but also on a permutation (i, j, k) of $\{1, 2, 3\}$ and is given by $\mathbf{\Sigma} = \mathbf{\Sigma}(u, (ijk))$ such that

$$\Sigma_i = -1 + 3\frac{-u}{u^2 + u + 1} \quad \Sigma_j = -1 + 3\frac{u + 1}{u^2 + u + 1} \quad \Sigma_k = -1 + 3\frac{u^2 + u}{u^2 + u + 1}.$$

We can immediately observe that $\langle \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3, \mathbf{\Sigma}\rangle = 0$, as it should be; hence, the above really defines a function $\Psi : \mathbb{R} \times \textsc{sym}_3 \to \mathbb{R}^2$, where $\textsc{sym}_3$ is the set of permutations of $\{1, 2, 3\}$. Direct calculation shows that $\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 6$ for all $u \in \mathbb{R}$. Hence, the u coordinates do actually parametrize the Kasner circle. We have the noteworthy symmetry properties

$$\begin{aligned} \mathbf{\Sigma}(u, (ijk)) &= \mathbf{\Sigma}\left(u^{-1}, (ikj)\right) &&= \mathbf{\Sigma}\left(-(u+1), (jik)\right) &&= \mathbf{\Sigma}\left(\tfrac{-1}{u+1}, (jki)\right) \\ &= \mathbf{\Sigma}\left(-\tfrac{u}{u+1}, (kji)\right) &&= \mathbf{\Sigma}\left(-\tfrac{u+1}{u}, (kij)\right). \end{aligned}$$

We can use the symmetry to normalize u to a value $u \in [1, \infty]$, thus giving a parametrization of the Kasner circle as in Figure 6b.

The Kasner-map in u-coordinates. We consider without loss of generality a heteroclinic orbit $\gamma \subseteq \mathcal{M}_{+00}$. In the $\mathbf{\Sigma}$-projection, this heteroclinic orbit is a straight line through $-2\mathbf{T}_1$; hence, the quotient $\frac{\Sigma_-}{2-\Sigma_+}$ stays constant. It is given by

$$\frac{\Sigma_-}{2-\Sigma_+}(u) = \sqrt{3}\frac{u^2-1}{4(u^2+u+1)-(u^2+4u+1)} = \frac{\sqrt{3}}{3}\frac{u^2-1}{u^2+1}.$$

If we write the α-limit of γ with respect to the 123 permutation, then we must have $u \in [0,\infty]$ (because N_1 would not be unstable otherwise). Then the Kasner-map must be given by $(u,(123)) \to (-u, 123)$. Assume that $u \in [2,\infty]$; then we can renormalize $K(u)$, such that $K(u,123) = (u-1, 213)$. If instead $u \in [1,2]$, then we can renormalize such that $K(u,123) = \left(\frac{1}{u-1}, 231\right)$. Applying symmetrical arguments for the other caps yields the following way of writing the Kasner map:

$$\begin{aligned}
&\mathcal{K} = \text{SYM}_3 \times [1,\infty] \,/\, \{(1,ijk)=(1,ikj), (\infty,ijk)=(\infty,jik)\}\\
&K:\mathcal{K}\to\mathcal{K} \quad (u,ijk) \to \begin{cases} (u-1, jik) & \text{if } u\in[2,\infty]\\ \left(\dfrac{1}{u-1}, jki\right) & \text{if } u \in [1,2].\end{cases}
\end{aligned}$$

Note that the Kasner-map is actually well-defined and continuous at $u=2$, due to the identification $(1,ijk)=(1,ikj)$. It is also well-defined at $u=1$, since the identifications $(1,ijk)=(1,ikj)$ and $(\infty,ijk)=(\infty,jik)$ are compatible. It is continuous at $u=1$, since we use the usual compactification at $u=\infty$, such that a neighbourhood basis of (∞,ijk) is given by $\{([R,\infty],ijk)\cup([R,\infty],jik)\}_{R\gg 0}$. Likewise, the Kasner-map is well-defined and continuous at the three fixed-points $u=\infty$, due to the identification $(\infty,ijk)=(\infty,jik)$ and the compactification at $u=\infty$.

Symbolic description in u-coordinates. The two local inverses of the Kasner-map in u-coordinates are given by

$$\begin{aligned}
&\mathrm{S}:(u,ijk)\to(u+1,jik)\\
&\mathrm{L}:(u,ijk)\to\left(1+\frac{1}{u}, kij\right)
\end{aligned}$$

Using the same construction as in the proof of Proposition 3.2, we see that the inverse coding map CFE^{-1} given by the continued fraction expansion

$$\text{CFE}^{-1}:(ij|a_0;a_1;a_2;\ldots)\to\left(1+a_0+\frac{1}{1+a_1+\frac{1}{1+a_2+\ldots}}\,,\,ijk\right),$$

and the Kasner-map is given by

$$\text{CFE}\circ K\circ\text{CFE}^{-1}:(ij|a_0;a_1;a_2;\ldots)\to\begin{cases}(ji\,|\,a_0-1;a_1;a_2;\ldots) & \text{if } a_0>0\\ (jk\,|\,a_1;a_2;\ldots) & \text{if } a_0=0.\end{cases}$$

A.3 General Properties of Dynamical Systems

This section contains some basic facts and definitions about dynamical systems. These are common knowledge in the dynamical systems community; readers who familiar with this background need only skim paragraphs on invariant sets, long-time existence in the Bianchi system, and invariant manifolds, where some slightly non-standard notation is explained. For reference, we recommend e.g.[Wig03].

Local flows A local flow is a continuous map $\phi : U \to X$, where $U \subset X \times \mathbb{R}$ is open, X is a topological space, and $X \times \{0\} \subset U$, with the properties $\phi(\boldsymbol{x}, 0) = \boldsymbol{x}$ and, whenever all expressions are defined, $\phi(\boldsymbol{x}, t+s) = \phi(\phi(\boldsymbol{x}, t), s)$. The domain U is called maximal if additionally, for $\boldsymbol{x} \in X$ and $t, s \in \mathbb{R}$ with $ts > 0$, $(\boldsymbol{x}, t+s) \in U$ if and only if $(\phi(\boldsymbol{x}, s), t) \in U$. In the following, we will always assume that local flows are defined on their maximal domain.

The local flow is called global, if $U = X \times \mathbb{R}$ (for the maximal domain); otherwise, we say that it has finite-time blow-up. Note that local flows on compact spaces are always global. Also note that the only way that global forward existence can fail is by points $\boldsymbol{x}_0 \in X$, with $T^* = \sup\{t \geq 0 : (\boldsymbol{x}_0, t) \in U\} < \infty$, such that $\lim_{t \to T^*} \phi(\boldsymbol{x}_0, t)$ does not exist. Global backward existence can only fail only in the analogous way.

The Picard-Lindelöf Theorem states that locally Lipschitz vectorfields f on finite-dimensional differentiable manifolds X (without boundary, but possibly non-compact) define a unique local flow solving $\mathrm{D}_t \phi(\boldsymbol{x}, t) = f(\phi(\boldsymbol{x}, t))$, and that this local flow is as regular as f.

For some technical points it is more convenient to work with global flows. For flows coming from vectorfields, we can conveniently associate a global flow via an Euler-multiplier, i.e. via a loc. Lipschitz function $\mu : X \to (0, \infty)$. Then, we discuss the flow to μf instead of f. Many dynamical properties are invariant under Euler-multipliers, since these correspond to a simple reparametrization of time; for these, we can avoid the discussion of local flows.

Claim A.3.1. Euler-multipliers correspond to a time-rescaling.

Proof. To the system $\mathrm{D}_t \boldsymbol{x} = f(\boldsymbol{x})$, add the equation $\mathrm{D}_t \tau = \mu(\boldsymbol{x})$. Then, $\mathrm{D}_t \phi(\boldsymbol{x}, \tau(t)) = \mu(\phi(\boldsymbol{x}, \tau(t))) f(\phi(\boldsymbol{x}, \tau(t)))$. $\square$

One way of obtaining a suitable Euler-multiplier is the following: Fix some metric d on X, such that all coordinate charts are locally Lipschitz; then, we set $\rho(\boldsymbol{x}) = \sup\{0 < r \leq 1 : \{\boldsymbol{y} : d(\boldsymbol{x}, \boldsymbol{y}) \leq r\} \text{ is compact}\}$, and $\mu = 0.1 \frac{\rho}{1+|f|_d}$, where $|f(\boldsymbol{x})|_d := \limsup_{t \to 0,\, t \neq 0} |t|^{-1} d(\boldsymbol{x}, \phi(\boldsymbol{x}, t))$.

Claim A.3.2. Using the above construction, μf is associated to a global flow.

Proof. Let $(\boldsymbol{x}_0, T)$ be in the maximal domain of the local flow ϕ to μf, and use the shorthand $\boldsymbol{x}(t) = \phi(\boldsymbol{x}_0, t)$. Then, we can estimate $\operatorname{diam}_d \boldsymbol{x}([0, T]) \leq \int_0^T |\mu f|_d(\phi(\boldsymbol{x}(t))\mathrm{d}t \leq 0.1 \max\{\rho(\boldsymbol{x}(t)) : t \in [0, T]\} \leq \frac{1}{9}\rho(\boldsymbol{x}_0)$. Since $|\mu f|_d < 1$ is bounded, this excludes (forward) finite time blow-up. Backward finite time blow-up is similarly impossible. $\square$

Invariant sets Assume that ϕ is a global flow. Then, we define a set $M \subseteq X$ to be *forward invariant* if $\phi(M,t) \subseteq M$ for all $t \geq 0$, we define it to be *backward invariant* if $\phi(M,t) \subseteq M$ for all $t \leq 0$, and we define it to be *invariant* if it is both forward and backward invariant; then, $\phi(M,t) = M$ for all $t \in \mathbb{R}$. The notions of invariance are clearly invariant under Euler-multipliers, as long as we are considering global flows.

For local flows, there are multiple contenders for a sensible definition of (forward, backward) invariant sets. We choose the following: Take a global flow that is associated by Euler-multiplier, and take the definitions for global flows.

This means that we have only given a definition for invariant sets for local flows that are associated to a locally Lipschitz vectorfield on a differentiable manifold. This is sufficiently general in the context of this work.

Long-time existence in the Bianchi system Let ϕ denote the local flow on $\mathcal{M}$, such that $t \to \phi(\boldsymbol{x},t)$ are solutions to (2.3.2). In Lemma 3.5, we prove that ϕ is defined for all positive times, and remark (without proof) that backwards finite time blow-up can occur in Bianchi IX, and remark (with proof) that ϕ is defined for all negative times for all other relevant Bianchi types than IX.

We mostly discuss the behaviour of solutions for positive times; hence, we very rarely encounter situations where global backward existence of solutions is a possible issue. For this reason, we decided that applying a global Euler-multiplier that ensures global existence is not worth the hassle, and the clutter in (2.3.2). For a work that decides to modify the equations in order to get rid of this problem, see e.g. [HU09b].

An unfortunate side-effect is that the effective definitions of invariant sets are superficially asymmetric in time: A set $M \subseteq \mathcal{M}$ is forward invariant, if $\{\phi(\boldsymbol{x},t) : t \geq 0,\ \boldsymbol{x} \in M\} \subseteq M$, and backward invariant if $\{\boldsymbol{x} : \exists t \geq 0 : \phi(\boldsymbol{x},t) \in M\} \subseteq M$, and invariant if it is both forward and backward invariant.

We can consider preimages of sets or pull-backs of differential forms, without having to invert any maps.

In an abuse of notation, we nevertheless sometimes write $\phi(M,(-\infty,0]) := \{\boldsymbol{x} : \exists t \geq 0 : \phi(\boldsymbol{x},t) \in M\}$ for the generated backward invariant set, and $\phi(M,\mathbb{R}) := \phi(M,\mathbb{R}_-) \cup \phi(M,\mathbb{R}_+)$ for the generated invariant set.

Invariant Manifolds Let $f : X \to TX$ be a vectorfield that is at least C^1, and let ϕ be its associated local flow. Let $\boldsymbol{p} \in X$ be a fixed point, i.e. $f(\boldsymbol{p}) = 0$, and let $A = \mathrm{D}f(\boldsymbol{p})$ be its Jacobian. We can split its spectrum, i.e. its set of eigenvalues $\operatorname{spec} A \subseteq \mathbb{C}$, into three parts, depending on the signs of their real parts. An eigenvalue $\lambda_i \in \operatorname{spec} A$ is called *stable* if its real part is negative; it is called *central* if its real part is zero, and it is called *unstable* if its real part is positive. Let π_s, π_c, π_u be the spectral projections corresponding to this splitting.

Theorem (Invariant Manifold Theorem)**.** *Let* $W^s(\boldsymbol{p}) = \{\boldsymbol{x} \in X : \lim_{t\to\infty} \phi(\boldsymbol{x},t) = \boldsymbol{p}$ *exponentially*$\}$*. Then,* $W^s(\boldsymbol{p})$ *is an immersed submanifold, with an immersion* $\Psi : \pi_s T_{\boldsymbol{p}} X \to X$ *that has* $\mathrm{D}\Psi^s(0) = \mathrm{id}$ *(i.e.* $\mathrm{D}\Psi^s(0)$ *is the embedding* $\pi_s T_{\boldsymbol{p}} X \subseteq X$*). This* *immersion is as smooth as the vectorfield. $W^s(\boldsymbol{p})$ is called the stable manifold of $\boldsymbol{p}$; it is invariant and unique.*

Let $W^u(\boldsymbol{p}) = \{\boldsymbol{x} \in X : \lim_{t\to-\infty} \phi(\boldsymbol{x},t) = \boldsymbol{p}$ exponentially$\}$. Then, $W^u(\boldsymbol{p})$ is an immersed submanifold, with an immersion $\Psi^u : \pi_u T_{\boldsymbol{p}}X \to X$ that has $\mathrm{D}\Psi^u(0) = \mathrm{id}$ (i.e. $\mathrm{D}\Psi(0)$ is the embedding $\pi_u T_{\boldsymbol{p}}X \subseteq X$). This immersion is as smooth as the vectorfield. $W^u(\boldsymbol{p})$ is called the stable manifold of $\boldsymbol{p}$; it is invariant and unique.

There exists an immersion $\Psi^c : \pi_c T_{\boldsymbol{p}}X \to X$ that has $\mathrm{D}\Psi(0) = \mathrm{id}$ (i.e. $\mathrm{D}\Psi^c(0)$ is the embedding $\pi_c T_{\boldsymbol{p}}X \subseteq X$), with the following properties. $W^c(\boldsymbol{p}) := \mathrm{Range}\,\Psi^c$ is called the center manifold; it is invariant. Furthermore, for all sufficiently small $\epsilon > 0$, there exists $\epsilon' > 0$, such that $\Psi^c(B^c_{\epsilon'}(0))$ is a graph over $\pi_c T_{\boldsymbol{p}}X$ and contains all points $\boldsymbol{x} \in X$ with the property that $\phi(\boldsymbol{x},t) < \epsilon$ for all $t \in \mathbb{R}$. Here, $B^c_r(0)$ denotes the ball of radius r in $\pi_c T_{\boldsymbol{p}}X$, centred at the origin, with respect so some arbitrary norm. The center manifold is in general non-unique, so is should rather be called "a center manifold". It can be arranged that Ψ^c is C^k, $k < \infty$, if $f \in C^k$.

In this work, we never really apply this theorem; however, we use its notation, and the intuitions it implies.

When we use this notation, we suppress an index $\hat{n} \in \{+1, -1, 0\}$. For example, $W^s(-\mathbf{T}_1)$ has the eight parts $W^s(-\mathbf{T}_1) \cap \mathcal{M}_{0**}$, and in an abuse of notation, we sometimes write $W^s(-\mathbf{T}_1)$ when we rather mean $W^s_{\hat{n}}(-\mathbf{T}_1) := W^s(-\mathbf{T}_1) \cap \mathcal{M}_{\hat{n}}$, trusting that the Bianchi type is apparent from the context.

Elementary Properties of ω-limit sets Let us recall some basic properties of ω-limit sets. In the following, we assume that ϕ is a global flow on a locally compact (possibly incomplete and non-compact) metric space (X, d). In the Bianchi system, we apply the definitions to a global flow that is associated by Euler-multiplier.

Let $\boldsymbol{x}_0 \in X$. We define its ω-limit set $\omega(\boldsymbol{x}_0)$ as the set of all $\boldsymbol{y} \in X$, such that there exists a sequence of times $t_n \to \infty$, with $\lim_{n\to\infty} \phi(\boldsymbol{x}_0, t_n) = \boldsymbol{y}$. Analogously, we define its α-limit set $\alpha(\boldsymbol{x}_0)$ as the set of all $\boldsymbol{y} \in X$, such that there exists a sequence of times $t_n \to -\infty$, with $\lim_{n\to\infty} \phi(\boldsymbol{x}_0, t_n) = \boldsymbol{y}$.

Claim A.3.3. The notion of α and ω-limit sets is invariant under Euler-multipliers.

Proof. Obvious. □

Lemma A.2. *Let $\boldsymbol{x}_0 \in X$. If X is compact, then $\omega(\boldsymbol{x}_0) \neq \emptyset$. The set $\omega(\boldsymbol{x}_0) \subseteq X$ is closed and invariant. If $\omega(\boldsymbol{x}_0)$ is compact, then it is connected. If $\boldsymbol{y} \in \omega(\boldsymbol{x}_0)$, then $\omega(\boldsymbol{y}) \subseteq \omega(\boldsymbol{x}_0)$ and $\alpha(\boldsymbol{y}) \subseteq \omega(\boldsymbol{x}_0)$.*

The same statements hold if we exchange α and ω limit sets, i.e. reverse time.

Proof. First, note that the claims about invariance under Euler-multipliers and reversing time are obvious. If X is compact, then it is sequentially compact, and clearly $\omega(\boldsymbol{x}_0) \neq \emptyset$. The claim that $\omega(\boldsymbol{x}_0)$ is (relatively) closed is obvious when considering the meaning of $\boldsymbol{y} \notin \omega(\boldsymbol{x}_0)$. The claim that $\omega(\boldsymbol{x}_0)$ is invariant is obvious from its definition.

The claims that $\omega(\boldsymbol{y}) \subseteq \omega(\boldsymbol{x}_0)$ and $\alpha(\boldsymbol{y}) \subseteq \omega(\boldsymbol{x}_0)$ follow directly from invariance and closedness of $\omega(\boldsymbol{x}_0)$.

Let us now show that $\omega(\boldsymbol{x}_0)$ is connected if it is compact. Assume otherwise: Let $U, V \subseteq X$ be two disjoint open sets with $\omega(\boldsymbol{x}_0) \subseteq U \cup V$, and neither $\omega(\boldsymbol{x}_0) \subseteq U$ nor $\omega(\boldsymbol{x}_0) \subseteq V$. There exists a sequence of times $t_n \to \infty$, such that $\phi(\boldsymbol{x}_0, t_n) \in X \setminus (U \cup V)$: Otherwise, we would have $\phi(\boldsymbol{x}_0, [T, \infty)) \subseteq U$ (or $\subseteq V$), for some sufficiently late time $T \geq 0$, since $[T, \infty)$ is connected for all $T \in \mathbb{R}$, and ϕ is continuous. This is however excluded by our setup.

Now, we can use the assumed compactness of $\omega(\boldsymbol{x}_0)$ to find a convergent subsequence, i.e. a sequence $t_n \to \infty$ such that $\phi(\boldsymbol{x}_0, t_n) \in X \setminus (U \cup V)$ converges. This is a contradiction. □

A.4 Transformation of Volumes on Manifolds

This section contains some basic facts about the transformation of volumes under flows. These facts are common knowledge; for reference, we recommend one of [Kön13, Lan06, Rud64]. Recall the general conventions about integrals from the beginning of Section 2.

Volume and integral transformation in $\mathbb{R}^n$. If $U \subseteq \mathbb{R}^n$ is open, and $\Phi : U \to \Phi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism onto its image, then integrals transform by

$$\int_{\boldsymbol{y} \in \Phi(U)} f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \int_{\boldsymbol{x} \in U} f(\Phi(\boldsymbol{x})) \lambda(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \qquad \lambda(\boldsymbol{x}) = |\det \mathrm{D}\Phi(\boldsymbol{x})|,$$

for any a.e. defined measurable function $f : \Phi(U) \to [0, \infty]$, henceforth simply called *a.e.-function*. (if one side of the integral is infinite, then so is the other). If we want to integrate with respect to a density, i.e. an a.e. function $\rho : \mathbb{R}^n \to (0, \infty)$, then we get

$$\int_{\boldsymbol{y} \in \Phi(U)} f(\boldsymbol{y}) \rho(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \int_{\boldsymbol{x} \in U} f(\Phi(\boldsymbol{x})) \lambda(\boldsymbol{x}) \rho(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \qquad \lambda(\boldsymbol{x}) = \frac{\rho(\Phi(\boldsymbol{x}))}{\rho(\boldsymbol{x})} |\det \mathrm{D}\Phi(\boldsymbol{x})|.$$

Denote indicator functions of sets K by χ_K, i.e. $\chi_K(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in K$ and $\chi_K(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin K$. Then, this equation tells us how to transform volumes $\mathrm{vol}_\rho(K) := |K|_\rho := \int_{\mathbb{R}^n} \chi_K(\boldsymbol{x}) \rho(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$ of measurable sets $K \subseteq U$:

$$\mathrm{vol}_\rho(\Phi(K)) = \int_{\Phi(U)} \chi_K \rho |\mathrm{d}\boldsymbol{x}| = \int_U \chi_{\Phi^{-1}(K)} \lambda \rho \mathrm{d}\boldsymbol{x}.$$

Note that $|K|_\rho = 0$ if and only if $|K| := |K|_1 = 0$, and $\int_{\mathbb{R}^n} f \mathrm{d}\boldsymbol{x} = 0$ if and only if $\int_{\mathbb{R}^n} f \rho \mathrm{d}\boldsymbol{x} = 0$, for non-negative a.e.-functions f, and a.e. densities $\rho : \mathbb{R}^n \to (0, \infty)$ (the notion of sets with finite measure, however, is not independent of ρ). If $U \subseteq \mathbb{R}^n$ is open and $p \in (0, \infty)$, then we say that an a.e. function f is in $L^p_{\mathrm{Loc}}(U)$ if $\int_K f^p \mathrm{d}\boldsymbol{x} < \infty$ for every compact $K \subset U$. If $\rho : U \to (0, \infty)$ is continuous, then this is equivalent to $\int_K f^p \rho \mathrm{d}\boldsymbol{x} < \infty$ for every compact $K \subset U$.

The same transformations of integrals apply for piecewise diffeomorphisms $\Phi : U \to V$, in the following sense: Suppose that we have a countable family $\Omega_i \subseteq U_i \subseteq U$, such that each U_i is open, each Ω_i is measurable, and $\Phi_i : U_i \to \Phi_i(U_i)$ is a family of diffeomorphisms onto their image. Suppose that the sets $(\Omega_i)_{i\in\mathbb{N}}$ are pairwise disjoint, as are the sets $(\Phi_i(\Omega_i))_{i\in\mathbb{N}}$, and suppose that $|U \setminus \bigcup_i \Omega_i| = 0$ and $|V \setminus \bigcup_i \Phi_i(\Omega_i)| = 0$. Then, set $\Phi(\boldsymbol{x}) = \Phi_i(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega_i$, and $\lambda(\boldsymbol{x}) = \lambda_i(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega_i$. Then, Φ and λ are a.e. defined, and for every a.e. function $f : \Phi(U) \to [0,\infty]$

$$\int_{\Phi(U)} f(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \int_U f(\Phi(\boldsymbol{x}))\lambda(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Volume transformation under flows in $\mathbb{R}^n$ Suppose that we have a flow $\phi : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, corresponding to a C^1 vectorfield $f : \mathbb{R}^n \to \mathbb{R}^n$. Then, we want to compute $\lambda : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}_+$, given by $\lambda(\boldsymbol{x},t) = \det \mathrm{D}_{\boldsymbol{x}}\phi(\boldsymbol{x},t)$. Clearly, $\lambda(\boldsymbol{x}, t_1 + t_2) = \lambda(\boldsymbol{x}, t_1)\lambda(\phi(\boldsymbol{x}, t_1), t_2)$ for all $t_1, t_2 \geq 0$; this allows us to compute $\mathrm{D}_t\lambda(\boldsymbol{x},t) = \lambda(\boldsymbol{x},t)\mathrm{tr}\,\partial_{\boldsymbol{x}} f(\phi(\boldsymbol{x},t))$, using $\mathrm{D}_t \det A(t)_{|t=0} = \mathrm{tr}\,\mathrm{D}_t A(t)_{|t=0}$, for any differentiable family of matrices with $A(0) = \mathrm{id}$. Integrating this yields $\log\lambda(\boldsymbol{x},t) = \int_0^t \mathrm{tr}\partial_{\boldsymbol{x}} f(\phi(\boldsymbol{x},s))\mathrm{d}s$.

If we instead want to transform with respect to a differentiable density $\rho : \mathbb{R}^n \to (0,\infty)$, i.e. are interested in $\lambda(\boldsymbol{x},t) = \frac{\rho(\phi(\boldsymbol{x},t))}{\rho(\boldsymbol{x})} \det \mathrm{D}_{\boldsymbol{x}}\phi(\boldsymbol{x},t)$, then we get $\log\lambda(\boldsymbol{x},t) = \int_0^t \partial_{\boldsymbol{x}} \log(\rho) \cdot f(\phi(\boldsymbol{x},t)) + \mathrm{tr}\partial_{\boldsymbol{x}} f(\phi(\boldsymbol{x},s))\mathrm{d}s$.

Volume transformation on manifolds The language of differential forms gives a supremely convenient framework for transforming integrals. Let $\Phi : M \to \Phi(M) \subset N$ be a diffeomorphism between n-differential manifolds, and let ω_N, ω_M be volume-forms. Then, the above transformation of integrals reads, for a.e. functions f on $\Phi(M)$, as

$$\int_{\Phi(M)} f|\omega_N| = \int_M f \circ \Phi\, |\Phi^*\omega_N| = \int_M f \circ \phi\, \lambda|\omega_M|,$$

where $\lambda : M \to [0,\infty)$ is such that $|\omega_M| = \lambda|\Phi^*\omega_N|$, and the pull-back is given by $\Phi^*\omega_N[X_1,\ldots,X_n] = \omega_N[\Phi_* X_1, \ldots, \Phi_* X_n]$ for vectors $X_1, \ldots X_n \in T_{\boldsymbol{p}}M$, with $\Phi_* X_i = \mathrm{D}\Phi(\boldsymbol{p})X_i$.

Volume transformation under flows on manifolds If we have an n-dimensional differentiable manifold M, with volume form $\omega = \rho$, and a flow $\phi : M \times [0,\infty) \to M$ associated to a differentiable vectorfield f, then we can again write $\phi(\cdot,t)^*\omega = \lambda(\cdot,t)\omega$ and have $\lambda(\boldsymbol{x}, t_1 + t_2) = \lambda(\boldsymbol{x}, t_1)\lambda(\phi(\boldsymbol{x}, t_1), t_2)$ for $t_1, t_2 \geq 0$. In practice, we will compute λ in coordinates.

Restriction to Iso-Surfaces. Let $\Phi : M \to M$ be a diffeomorphism on an n-dimensional manifold M with volume form ω, which gets transported by $\lambda = \frac{|\Phi^*\omega|}{|\omega|}$. Suppose that $G : M \to \mathbb{R}$ is a preserved quantity under Φ, i.e. $G(\Phi(\boldsymbol{x})) = G(\boldsymbol{x})$ and suppose that 1 is a regular value of G. Now we are interested in the iso-surface $M_1 = \{\boldsymbol{x} \in M : G(\boldsymbol{x}) = 1\}$ and in a volume form $\widetilde{\omega}$ on M_1, which gets transported by the same λ. This can be realized by choosing some vectorfield $X : M_1 \to TM$ with $\mathrm{D}_X G = \mathrm{const} = 1$ and setting $\widetilde{\omega} = \iota_X\omega$. Since G is preserved, we have $\Phi_* X - X \in TM_1$ and hence for a basis

$X_1, \ldots, X_{n-1}$ of TM_1:

$$\begin{aligned}\lambda\widetilde{\omega}[X_1, \ldots X_n] &= \lambda\omega[X, X_1, \ldots, X_n] = \omega[\Phi_* X, \Phi_* X_1, \ldots, \Phi_* X_{n-1}] \\ &= \omega[X, \Phi_* X_1, \ldots, \Phi_* X_{n-1}] + \omega[\Phi_* X - X, \Phi_* X_1, \ldots, \Phi_* X_{n-1}] \\ &= \Phi^*\widetilde{\omega}[X_1, \ldots, X_{n-1}].\end{aligned}$$

It is clear that the induced volume $\widetilde{\omega}$ does not depend on the choice of X, up to possibly a sign.

Contraction with invariant vectorfields. Let $\Phi : M \to M$ be a diffeomorphism on an n-dimensional manifold M with volume form ω, which gets transported by $\lambda = \frac{\Phi^*\omega}{\omega}$. Suppose that the vectorfield $Y : M \to TM$ is preserved quantity under Φ, i.e. $\Phi_* Y = Y$. Then the contracted volume-form $\widetilde{\omega} = \iota_Y \omega$ has $\Phi^*\widetilde{\omega} = \lambda\widetilde{\omega}$:

$$\begin{aligned}\Phi^*\widetilde{\omega}[X_2, \ldots, X_n] &= \omega[Y, \Phi_* X_2, \ldots, \Phi_* X_n] = \omega[\Phi_* Y, \Phi_* X_2, \ldots, \Phi_* X_n] \\ &= \Phi^*\omega[Y, X_2, \ldots, X_n] = \lambda\omega[Y, X_2, \ldots, X_n] = \lambda\widetilde{\omega}[X_2, \ldots, X_n].\end{aligned}$$

Poincaré-Sections Suppose that $\phi : M \times \mathbb{R} \to M$ is a local flow with forward long-time existence, corresponding to the vectorfield f on an n-dimensional manifold M with volume form ω, such that $\phi(\boldsymbol{x}, t)^*\omega = \lambda(\boldsymbol{x}, t)$. Set $\widetilde{\omega} = \iota_f \omega$. Clearly, $\phi(\cdot, t)_* f = f$.

Let $S \subset M$ be a Poincaré-section, i.e. an embedded submanifold with boundary of codimension one, that is transverse to f (i.e. $f(S) \cap TS = \emptyset$). Let $K \subset S$ be a set with $|K|_{\widetilde{\omega}} = 0$. Then, $|\phi(K, (-\epsilon, \epsilon))|_\omega = 0$, by Fubini's theorem, and $|\phi(K, \mathbb{R})|_\omega = 0$ by absolute continuity of the flow. For technical reasons, we do not consider the boundary ∂S to be a subset of S, but nevertheless $|\phi(\partial S, \mathbb{R})|_\omega = 0$, for the same reasons.

Let $\boldsymbol{x}_0 \in M$, and $t > 0$ such that $\phi(\boldsymbol{x}_0, t) \in S$. Then, by the implicit function theorem, we find an open neighbourhood $U \ni \boldsymbol{x}$ and differentiable function $T_S : U \to \mathbb{R}_+$, such that $\Phi_S(\boldsymbol{x}) := \phi(\boldsymbol{x}, T_S(\boldsymbol{x}))$ is a differentiable map $\Phi_S : U \to S$. We call Φ_S a branch of the Poincaré-map. We can compute $\Phi_S^*\widetilde{\omega} = \lambda_S\widetilde{\omega}$, where $\lambda_S(\boldsymbol{x}) = \lambda(\boldsymbol{x}, T_S(\boldsymbol{x}))$. This is because, for any vectorfield X on U, $\Phi_{S*} X = \phi_*(\cdot, T_S(\cdot))X + f\,\mathrm{D}_X T_S$, and hence

$$\begin{aligned}\Phi_S^*\widetilde{\omega}[X_2, \ldots, X_n] &= \omega[f, \phi_*(\cdot, T(\cdot))X_2 + f\mathrm{D}_{X_2} T_S, \ldots, \phi_*(\cdot, T_S(\cdot))X_n + f\mathrm{D}_{X_n} T_S] \\ &= \omega[f, \phi_*(\cdot, T(\cdot))X_2, \ldots, \phi_*(\cdot, T_S(\cdot))X_n] \\ &= \omega[\phi_*(\cdot, T_S(\cdot))f, \phi_*(\cdot, T_S(\cdot))X_2, \ldots, \phi_*(\cdot, T_S(\cdot))X_n] = \lambda_S\widetilde{\omega}[X_2, \ldots, X_n].\end{aligned}$$

Next, we consider the first-return map, i.e. we (partially, discontinuously) define $T_S(\boldsymbol{x}) = \inf\{t > 0 : \phi(\boldsymbol{x}, t) \in S\}$, and $\Phi_S(\boldsymbol{x}) = \phi(\boldsymbol{x}, T_S(\boldsymbol{x}))$, and $\lambda_S(\boldsymbol{x}) = \lambda(\boldsymbol{x}, T_S(\boldsymbol{x}))$, and $\Phi_S^{-1}(S) := \{\boldsymbol{x} \in M : T_S(\boldsymbol{x}) < \infty\} = \phi(S, (-\infty, 0))$. We want to apply the first-return map on S; note that the restriction on S, i.e. $\Phi_S|S$, is injective. Then, recalling the initial discussion of the transformation under piecewise diffeomorphisms, we get for any measurable set $M \subseteq S$, and for a.e. functions $f : S \to [0, \infty]$:

$$|\Phi_S(M \cap \Phi_S^{-1}(S))|_{\widetilde{\omega}} = \int_{M \cap \Phi_S^{-1}(S)} \lambda_S |\widetilde{\omega}|$$

$$|\Phi_S^{-1}(M) \cap S|_{\widetilde{\omega}} = \int_{M \cap \Phi_S(S)} \frac{1}{\lambda_S(\Phi_S^{-1}(\boldsymbol{x}))} |\widetilde{\omega}|$$

$$\int_{\boldsymbol{x} \in \Phi_S^{-1}(S) \cap S} f(\Phi_S(\boldsymbol{x}))\lambda_S(\boldsymbol{x})|\widetilde{\omega}| = \int_{\boldsymbol{y} \in \Phi_S(S) \cap S} f(\boldsymbol{y})|\widetilde{\omega}|.$$

Note that we only need countably many pieces because $\Phi_S^{-1}(S) \cap S$ is open. Also, the set of points that hit ∂S before hitting S is a set of measure zero, so it is ultimately irrelevant whether we consider ∂S to be a subset of S in the definition of Φ_S.

A.5 Derivation of the Wainwright-Hsu equations

The goal of this section is to connect the Einstein field equations of general relativity to the Wainwright-Hsu equations (2.3.2) discussed in this work. Alternatively, we recommend [WE05].

A.5.1 Spatially Homogeneous Spacetimes

We are interested in spatially homogeneous spacetimes. We assume that (M^4, g) is a Lorentz-manifold, and we have a symmetry adapted co-frame: $\{\omega_1, \omega_2, \omega_3, \mathrm{d}t\}$, corresponding to a frame of vectorfields $\{e_0, e_1, e_2, e_3\}$, such that e_1, e_2, e_3 are Killing, i.e. the metric depends only on t. We assume that the metric has the form

$$g = g_{00}(t)\mathrm{d}t \otimes \mathrm{d}t + g_{11}(t)\omega_1 \otimes \omega_1 + g_{22}(t)\omega_2 \otimes \omega_2 + g_{33}(t)\omega_3 \otimes \omega_3,$$

where $g_{00} < 0$ and the other three $g_{ii} > 0$. The spatial homogeneity is described by the commutators of the three Killing fields e_1, e_2, e_3; we assume that it is given (for positive permutations (i, j, k) of $(1, 2, 3)$) by:

$$[e_i, e_j] = \gamma_{ij}^k e_k = \hat{n}_k e_k \qquad \mathrm{d}\omega_i = -\hat{n}_i \omega_j \wedge \omega_k,$$

where $\hat{n}_i \in \{-1, 0, +1\}$ describe the Bianchi type of the surfaces $\{t = \text{const}\}$ of spatial homogeneity.

General equations for the Christoffel symbols. The general equations for Christoffel symbols are given by:

$$\begin{aligned}
\nabla_{e_i} e_j &= \sum_k \Gamma^k_{ij} e_k \\
[e_i, e_j] &= \nabla_{e_i} e_j - \nabla_{e_j} e_i \quad \Rightarrow \quad \Gamma^k_{ij} - \Gamma^k_{ji} = \gamma^k_{ij} \\
\mathrm{D}_{e_i} g(e_j, e_k) &= \partial_{e_i} g_{jk} = \sum_\ell g_{\ell k} \Gamma^\ell_{ij} + g_{\ell j} \Gamma^\ell_{ik} \\
\partial_{e_i} g_{jk} + \partial_{e_j} g_{ik} - \partial_{e_k} g_{ij} &= \sum_\ell g_{\ell k} \Gamma^\ell_{ij} + g_{\ell j} \Gamma^\ell_{ik} + g_{\ell k} \Gamma^\ell_{ji} + g_{\ell i} \Gamma^\ell_{jk} - g_{\ell i} \Gamma^\ell_{kj} - g_{\ell j} \Gamma^\ell_{ki} \\
&= \sum_\ell g_{\ell i} \gamma^\ell_{jk} + g_{\ell j} \gamma^\ell_{ik} + g_{\ell k} \gamma^\ell_{ji} + 2 g_{\ell k} \Gamma^\ell_{ij}
\end{aligned}$$

We can solve this for the Christoffel symbols by multiplying with the inverse metric:

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2} g^{k\ell} \left(\partial_{e_i} g_{j\ell} + \partial_{e_j} g_{i\ell} - \partial_{e_\ell} g_{ij} - \sum_n g_{ni} \gamma^n_{j\ell} - g_{nj} \gamma^n_{i\ell} - g_{n\ell} \gamma^n_{ji} \right) \\
&= \frac{1}{2} g^{kk} \left(\partial_{e_i} g_{jk} + \partial_{e_j} g_{ik} - \partial_{e_k} g_{ij} - g_{ii} \gamma^i_{jk} - g_{jj} \gamma^j_{ik} - g_{kk} \gamma^k_{ji} \right),
\end{aligned}$$

where we used the fact that the metric is diagonal in the last equation.

Christoffel symbols for spatially homogeneous space times. If we insert the indices into this equation, we obtain up to index permutations the following non-vanishing Christoffel symbols:

$$\begin{aligned}
&\nabla_{e_0} e_0 = \Gamma^0_{00} e_0 && \Gamma^0_{00} = \frac{1}{2} g^{00} \partial_{e_0} g_{00} \\
&\nabla_{e_1} e_2 = \Gamma^3_{12} e_3 && \Gamma^3_{12} = -\frac{1}{2} g^{33} \left(g_{11} \gamma^1_{23} + g_{22} \gamma^2_{13} + g_{33} \gamma^3_{21} \right) \\
& && \phantom{\Gamma^3_{12}} = \frac{1}{2} g^{33} \left(g_{11} \hat{n}_1 - g_{22} \hat{n}_2 - g_{33} \hat{n}_3 \right) \\
&\nabla_{e_0} e_1 = \Gamma^1_{01} e_1 = \Gamma^1_{10} e_1 && \Gamma^1_{10} = \frac{1}{2} g^{11} \partial_{e_0} g_{11} \\
&\nabla_{e_1} e_1 = \Gamma^0_{11} e_0 && \Gamma^0_{11} = -\frac{1}{2} g^{00} \partial_{e_0} g_{11}
\end{aligned}$$

Extrinsic Curvature of surfaces of homogeneity. The Weingarten-map ${K_i}^j$ and second fundamental form K_{ij} of the surfaces of spatial homogeneity given, up to permutation, by:

$$\begin{aligned}
&K(e_1) = \nabla_{e_1} \frac{1}{\sqrt{-g_{00}}} e_0 = \sqrt{-g^{00}} \Gamma^1_{10} e_1 \qquad K^j_i = \delta^j_i \sqrt{-g^{00}} \Gamma^i_{i0} \\
&\quad K_{ij} = g(e_i, \nabla_{e_j} \sqrt{-g^{00}} e_0) = \sqrt{-g_{00}} \Gamma^0_{ji} = \sum_k g_{ki} \sqrt{-g^{00}} \Gamma^k_{j0}, \quad \text{i.e.,} \\
&\quad \Gamma^1_{10} = \sqrt{-g_{00}} K^1_1, \qquad\qquad \Gamma^0_{11} = \sqrt{-g^{00}} g_{11} K^1_1
\end{aligned}$$

The extrinsic curvature corresponds to the normalized time-derivative of the spatial coefficients of the metric:

$$\sqrt{-g^{00}}\nabla_{e_0}\sqrt{g_{ii}} = \sqrt{-g^{00}}\frac{1}{2}\sqrt{g_{ii}}g^{jj}\nabla_{e_0}g_{kk} = \sqrt{-g^{00}}\Gamma^i_{i0}\sqrt{g_{ii}} = K^i_i.$$

Riemannian Curvature. The Riemannian curvature tensor is given by the equation

$$\begin{aligned}\sum_\ell R_{ijk}{}^\ell e_\ell &= \left(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]}\right) e_k \\ &= \sum_{\ell,n}\left(\Gamma^n_{jk}\Gamma^\ell_{in} - \Gamma^n_{ik}\Gamma^\ell_{jn} + \nabla_{e_i}\Gamma^\ell_{jk} - \nabla_{e_j}\Gamma^\ell_{in} - \gamma^n_{ij}\Gamma^\ell_{nk}\right) e_\ell.\end{aligned}$$

If we lower the last index, we have $R_{ijk\ell} = g((\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})e_k, e_\ell)$. Together with $(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})g(e_k, e_\ell) = 0$, this makes apparent the anti-symmetries $R_{ijk\ell} = -R_{jik\ell} = R_{ji\ell k} = -R_{ij\ell k}$.

Inserting the indices gives us the following potentially non-vanishing terms of the Riemann tensor, up to permutation and anti-symmetry, which are relevant for the Ricci curvature:

$$\begin{aligned}R_{121}{}^2 &= \Gamma^3_{21}\Gamma^2_{13} - \Gamma^0_{11}\Gamma^2_{20} - \gamma^3_{12}\Gamma^2_{31} \\ R_{010}{}^1 &= \Gamma^1_{10}\Gamma^1_{01} - \Gamma^0_{00}\Gamma^1_{10} + \nabla_{e_0}\Gamma^1_{10}.\end{aligned}$$

Raising and using $\widetilde{R}$ for the intrinsic curvature of the surfaces of homogeneity gives us

$$\begin{aligned}R_{12}{}^{12} &= g^{11}\Gamma^3_{21}\Gamma^2_{13} - \gamma^3_{12}\Gamma^2_{31}g^{11} - g^{11}\Gamma^0_{11}\Gamma^2_{20} \\ &= \widetilde{R}_{12}{}^{12} - K^1_1K^2_2 \\ R_{01}{}^{01} &= g^{00}\Gamma^1_{10}\Gamma^1_{01} - g^{00}\Gamma^0_{00}\Gamma^1_{10} + g^{00}\nabla_{e_0}\Gamma^1_{10} \\ &= -K^1_1K^1_1 + \sqrt{-g^{00}}\Gamma^0_{00}K^1_1 + g^{00}\nabla_{e_0}\sqrt{-g_{00}}K^1_1 \\ &= -K^1_1K^1_1 - \sqrt{-g^{00}}\nabla_{e_0}K^1_1.\end{aligned}$$

Setting

$$n_i = \hat{n}_i\sqrt{g^{11}g^{22}g^{33}}g_{ii} = \hat{n}_i\widetilde{n}_i, \quad \text{i.e.} \quad \widetilde{n}_i = \sqrt{g^{11}g^{22}g^{33}}g_{ii},$$

the spatial curvature is given by

$$\begin{aligned}\widetilde{R}_{12}{}^{12} &= g^{11}\Gamma^3_{21}\Gamma^2_{13} - g^{11}\gamma^3_{12}\Gamma^2_{31} = g^{11}\left(\Gamma^3_{21}\Gamma^2_{13} - \Gamma^3_{12}\Gamma^2_{31} + \Gamma^3_{21}\Gamma^2_{31}\right) \\ &= \frac{1}{4}\left(-n_1^2 - n_2^2 + 3n_3^2 + 2n_1n_2 - 2n_2n_3 - 2n_3n_1\right) \\ \widetilde{R}_1{}^1 &= \widetilde{R}_{12}{}^{12} + \widetilde{R}_{13}{}^{13} = \frac{1}{2}\left(-n_1^2 + n_2^2 + n_3^2 - 2n_2n_3\right) \\ \widetilde{R} &= \widetilde{R}_1{}^1 + \widetilde{R}_2{}^2 + \widetilde{R}_3{}^3 = \frac{1}{2}(n_1^2 + n_2^2 + n_3^2 - 2(n_1n_2 + n_2n_3 + n_3n_1)).\end{aligned}$$

A.5.2 The Einstein Field equation

The Einstein equations in vacuum state that the spacetime is Ricci-flat, i.e.

$$\begin{aligned}
{R_0}^0 &= {R_{01}}^{01} + {R_{02}}^{02} + {R_{02}}^{02} = 0\\
{R_1}^1 &= {R_{01}}^{01} + {R_{12}}^{12} + {R_{13}}^{13} = 0\\
{R_2}^2 &= {R_{02}}^{02} + {R_{21}}^{21} + {R_{23}}^{23} = 0\\
{R_3}^3 &= {R_{03}}^{03} + {R_{31}}^{31} + {R_{32}}^{32} = 0.
\end{aligned}$$

Adding the last three equations and subtracting the first gives us an equation which does not contain time-derivatives of K and hence is a constraint equation, called the "Gauss constraint". It is given by

$$0 = {R_{12}}^{12} + {R_{23}}^{23} + {R_{31}}^{31} = \frac{1}{2}\widetilde{R} - K^1_1 K^2_2 - K^2_2 K^3_3 - K^3_3 K^1_1.$$

The evolution equations are given by

$$\sqrt{-g^{00}}\nabla_{e_0}\widetilde{n}_i = \sqrt{-g^{00}}\nabla_{e_0}\sqrt{g_{ii}g^{jj}g^{kk}} = \sqrt{-g^{00}}(\Gamma^i_{i0} - \Gamma^j_{j0} - \Gamma^k_{k0})\widetilde{n}_i = (K^i_i - K^j_j - K^k_k)\widetilde{n}_i,$$

and

$$\begin{aligned}
{R_{0i}}^{0i} &= -{R_{ij}}^{ij} - {R_{ik}}^{ik} = {R_{jk}}^{jk}\\
\sqrt{-g^{00}}\nabla_{e_0}K^i_i &= -K^i_i K^i_i + K^j_j K^k_k - {\widetilde{R}_{jk}}^{\ jk}.
\end{aligned}$$

Trace-free formulation. It is useful to split the variables into their trace and trace-free parts:

$$\begin{aligned}
H &= \frac{1}{3}(K^1_1 + K^2_2 + K^3_3), \qquad \text{i.e. } H \text{ is the mean curvature of } \{t = \text{const}\}\\
\sigma_i &= K^i_i - H\\
\frac{1}{6}\widetilde{R} &= \frac{1}{3}\left({\widetilde{R}_{12}}^{\ 12} + {\widetilde{R}_{23}}^{\ 23} + {\widetilde{R}_{31}}^{\ 31}\right) = \frac{1}{12}\left(n_1^2 + n_2^2 + n_3^2 - 2(n_1n_2 + n_2n_3 + n_3n_1)\right)\\
s_i &= {\widetilde{R}_{jk}}^{\ jk} - \frac{1}{6}\widetilde{R} = \frac{1}{3}\left(2n_i^2 - n_j^2 - n_k^2 - n_in_j + 2n_jn_k - n_kn_i\right),
\end{aligned}$$

where we note that $(\sigma_1+\sigma_2+\sigma_3)^2 = 0 = \sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)$. We then obtain

$$\begin{aligned}
0 &= \frac{1}{2}\widetilde{R} - (\sigma_1+H)(\sigma_2+H) - (\sigma_2+H)(\sigma_3+H) - (\sigma_1+H)(\sigma_3+H) \\
&= \frac{1}{2}\widetilde{R} - 3H^2 - (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) \\
&= \frac{1}{2}\widetilde{R} - 3H^2 + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\
\sqrt{-g^{00}}\nabla_{e_0}\widetilde{n}_i &= (2\sigma_i - H)\widetilde{n}_i \\
\sqrt{-g^{00}}\nabla_{e_0}H &= \frac{1}{3}\Big[-(\sigma_1+H)^2 - (\sigma_2+H)^2 - (\sigma_3+H)^2 \\
&\qquad + (\sigma_1+H)(\sigma_2+H) + (\sigma_2+H)(\sigma_3+H) + (\sigma_3+H)(\sigma_1+H) \\
&\qquad - \widetilde{R}_{23}{}^{23} - \widetilde{R}_{13}{}^{13} - \widetilde{R}_{12}{}^{12}\Big] \\
&= -\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2) - \frac{1}{6}\widetilde{R} = -\frac{1}{3}(\sigma_1^2+\sigma_2^2+\sigma_3^2) - H^2 \\
\sqrt{-g^{00}}\nabla_{e_0}\sigma_1 &= -(\sigma_1+H)^2 + (\sigma_2+H)(\sigma_3+H) - \widetilde{R}_{23}{}^{23} - \sqrt{-g^{00}}\nabla_{e_0}H \\
&= \sigma_2\sigma_3 - \sigma_1^2 + (\sigma_2+\sigma_3-2\sigma_1)H - \widetilde{R}_{23}{}^{23} - \sqrt{-g^{00}}\nabla_{e_0}H \\
&= (\sigma_1+\sigma_3)(\sigma_1+\sigma_2) - \sigma_1^2 - 3\sigma_1 H + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2) + \frac{1}{6}\widetilde{R} - \widetilde{R}_{23}{}^{23} \\
&= -3\sigma_1 H - s_i.
\end{aligned}$$

Hubble Normalization. We can further simplify by Hubble-normalizing to $\overline{\Sigma}_i = \frac{\sigma_i}{H}$ and $\overline{N}_i = \frac{n_i}{H}$, and introducing a shorthand for Σ^2 and N^2

$$\begin{aligned}
\overline{N}_i &= \widetilde{\overline{N}}_i \hat{n}_i \\
\Sigma^2 &= \frac{1}{6}\left(\Sigma_1^2+\Sigma_2^2+\Sigma_3^2\right) \\
N^2 &= \frac{1}{12}\left[\overline{N}_1^2+\overline{N}_2^2+\overline{N}_3^2 - 2(\overline{N}_1\overline{N}_2+\overline{N}_2\overline{N}_3+\overline{N}_3\overline{N}_1)\right] = \frac{1}{6}\widetilde{R}H^{-2} \\
S_i &= -\frac{1}{3}\left[-2\overline{N}_1^2+\overline{N}_2^2+\overline{N}_3^2+\overline{N}_1\overline{N}_2 - 2\overline{N}_2\overline{N}_3+\overline{N}_3\overline{N}_1\right] = s_i H^{-2}
\end{aligned}$$

which yields equations

$$\begin{aligned}
1 &= \Sigma^2 + N^2 \\
\sqrt{-g^{00}}\nabla_{e_0}H &= -\left(2\Sigma^2+1\right)H^2 \\
\sqrt{-g^{00}}\nabla_{e_0}\widetilde{\overline{N}}_i &= (2\Sigma_i - 1 + 2\Sigma^2 + 1)\widetilde{\overline{N}}_i H = 2(\Sigma^2+\Sigma_i)\widetilde{\overline{N}}_i H \\
\sqrt{-g^{00}}\nabla_{e_0}\overline{\Sigma}_1 &= -3\overline{\Sigma}_1 H - s_i H^{-1} + (2\overline{\Sigma}^2+1)H = 2(\Sigma^2-1)\overline{\Sigma}_1 H - S_i H.
\end{aligned}$$

The Wainwright-Hsu equations as used in this work. Assuming $H < 0$ for an initial surface (which can be obtained if $H \neq 0$ by choosing the direction of the unit normal $\sqrt{-g^{00}}e_0$ and reverting the direction of time), we can set $\sqrt{-g^{00}} = -\frac{1}{2}H$. Setting $\widetilde{N}_i = \sqrt{12}\,\widetilde{\widetilde{N}}_i$, we obtain the variant of the Wainwright-Hsu equations used in this work, (2.1.3), corresponding to the metric (2.1.1).

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