Non-local Boundary Value Problems for Mixed Type Equations with Non-smooth Line of Changing Type with Spectral Parameter

DISSERTATION

des Fachbereichs Mathematik und Informatik

der Freien Universität Berlin

zur Erlangung des Grades eines

Doktors der Naturwissenschaften

Erster Gutachter: **Prof. Dr. M. S. Salakhitdinov**

Zweiter Gutachter: **Prof. Dr. H. Begehr**

Tag der Disputation: 19. Juli 2007

Vorgelegt von

Jahongir Dehqonovich Mamajonov

2007

Table of contents

Introduction

Urgency of the theme. Many problems of mechanics, physics, geophysics are reduced to solutions of partial differential equations, which do not belong to known classes of elliptic, parabolic or hyperbolic equations. As a rule, these equations are called non-classic equations of mathematical physics.

Apparently, for the first time non-classic equations of mathematical physic have appeared in S.A.Chapligin's works [9] at investigating transonic current, where they were introduced as so-called mixed type equations.

Equations are called mixed type equations, which in the one part of the domain of definition, they are of elliptic type, in the other of hyperbolic type. The investigation of boundary value problems for mixed type equations were started in F.Tricomi's [41] and S. Hellerstedt's [13] works, in 20-30s of the twentieth century, where for the first time there were stated boundary value problems for the model mixed type equations

$$yu_{xx} + u_{yy} = 0,$$

$$\mathrm{sign} y|y|^m u_{xx} + u_{yy} = 0, \; m > 0.$$

Nowadays these boundary value problems are called Tricomi and Hellerstedt problems. In a new stage of the development of boundary value problems for mixed type equations they appeared in the works of M.A. Lavrent'ev, I.A. Vekua, S.A. Xristianovich, F.I. Frankl, K.G. Guderle and so on. In these works the importance of study problems for mixed type equations were indicated, in particular, the Tricomi problem, which is connected with transonic gas dynamics, magnetohydrodynamic currents with passage by the speed of sound, with theory of infinitesimal bending surfaces, and also with many other questions of mechanics.

Nowadays many considered problems for mixed type equations are widened significantly and also denoted by "mixed type equations". Boundary value problems for mixed type equations were studied intensively from 1970s. It can be explained, that non-local problems contain a wide class of local boundary value problems and during their study different questions of applied nature appear, for example, questions of mathematical biology [25], mathematical simulation processes, study of laser [8], problems of plasma physics [4], [26], [17] and so on.

In previous years many works are devoted to studying boundary value problems with non-local conditions, among them we can note works of A.V. Bitsadze [6], M.S. Salakhitdinov [33],

[36], [34], T.D. Djurayev [10], [11], A.M. Naxushyev [24-26] and their students.

In 1969, A.V. Bitsadze and A.A. Samarskiy [5] have formulated and investigated a new problem for uniformly elliptic equations, which differ from other problem. These boundary conditions connect values of the desired solution on the boundary with inner points of the domain.

After this works, many works have appeared in different formulation, which are devoted to problems of Bitsadze-Samarskiy type for partial differential equations. Among them we can note works of V.A. Ilin and E.I. Moiseev [14], M.S. Salakhitdinov and A.K. Urinov [36], M.M. Smirnov [38] and so on.

Boundary-value problems for mixed type equations with spectral parameter were studied intensively from the second half of the seventies of the last century. It can be explained, on the one hand, that some multivariate analogues of basic boundary value problems for mixed type equations can be studied by their reduction (by the help of the Fourier transformation) to problems for equations with spectral parameter. On the other hand, commonly known methods, which are powerful tools for studying elliptic operators, were found a little adjusted for applying to boundary value problems for mixed type equation, as a spectral theory of mixed type equations is relative in principal, to a spectral theory of non self-conjugate operators. Therefore, not only the problem of full description of the spectrum of these problems is very interesting, but also the characterization of eigenvalues, i.e. those values of the spectral parameter for which the uniqueness theorem is not valid.

Many works are devoted to study these questions. In connection with this we note a work by T.S.Kal'menov [15] in which he proved the existence of even one eigenvalue of the Tricomi problem for the Lavrent'ev - Bitsadze equation, a work by E.I. Moiseev [21], [20] in which sectors are found where there is no eigenvalue of the Tricomi problem for series of mixed type equations and a work by S.M.Ponomarev [27] where eigenfunctions and eigenvalues of the Tricomi problem are found for the equation $U_{xx} + \mathrm{sign} y U_{yy} - \lambda U$ in a special domain.

Many non-local problems for two equations of mixed elliptic-hyperbolic type with spectral parameter, from which, in part, follows no spectrums of this problems exists in some sectors of the complex plane, were investigated in the works of M.S. Salakhitdinov and A.K.Urinov [36].

That is why the natural question appears: can one investigate principal classic and non-local problems for mixed type equations with two lines of changing type of the equations with spectral parameter and questions on the spectrum of these problems.

The aim of the work. The principal aim of this dissertation is to investigate non-local boundary value problems for mixed type equations with two lines of changing type with spectral parameter.

For achievement of the formulated aim:

1. Many problems are were formulated and investigated for equations of elliptic-hyperbolic type with spectral parameter with two lines of changing type with non-local condition in the hyperbolic part of the mixed domain, and in the elliptic part, when the boundary is a quarter circle, with the Dirichlet conditions and the third boundary condition.

2. Non-local problem is formulated with conditions as Bitsadze-Samarskiy type in the elliptic part and non-local conditions in the hyperbolic part of the mixed domain for the elliptic-hyperbolic type equation with spectral parameter with two lines of changing type and it's unique solvability is proved.

3. In the case, when the boundary of the elliptic part is a quarter ring two non-local problems are formulated for the elliptic-hyperbolic type equation with spectral parameter with two lines of changing type and their unique solvability are proved.

4. In the class L_2 spectral properties of solution of a non-local problem is investigated, in particular, completeness is proved and a basis system of eigenfunctions is given.

Method of investigation. Investigating problems equivalently reduced to the system of integral and sometimes integro-differential equation. In solving the obtained systems a method from the theory of partial differential equations, a spectral theory of linear operators, the theory of singular integral equations, a method of complex analysis, the energy integral, and also the extremum principle are applied. For finding a system of eigenfunctions the method of separation of variables is used.

Scientific news of the dissertation. In this work the following new results are obtained:

1. Conditions are found for the complex parameter λ, ensuring uniqueness of the solution of considered problems. Further, in the plane of the complex parameter a domain of the values λ is given, outside of which the considered non-local problems can have eigenvalues.

2. Sufficient conditions are found for uniqueness and existence of solutions of the formulated problems.

3. At first eigenvalues and corresponding to them eigenfunctions are found for one of the most general mixed problem, in which is given the third boundary condition on the boundary of the elliptic part of the mixed domain and in the hyperbolic part a non-local condition is given by some integral operator, and the completeness of eigenfunctions are proved in the class L_2.

4. A new method is developed to prove the existence of the solution of the considering problem, i.e. by applying eigenfunctions in evident form as solutions of the formulated problem in the case, when the Theorem of uniqueness of the solution of the considering problem is given.

Approbation of the dissertation. Results of the dissertation were discussed in the republican seminar “Modern problems of the theory of partial differential equations” (Institute of mathematics, Uzbek Academy of Sciences, heads are academicians M.S.Salakhitdinov and T.D.Djuraev). The main results were also discussed in the republican seminars “Modern problems of computational mathematics and mathematical physics” (National University of Uzbekistan, head is academician Sh.A. Alimov), "Differential equations and spectral analysis"(National University of Uzbekistan, heads is academician of Academy of Science of Republic of Uzbekistan M.S.Salakhitdinov and doctor of physic-mathematical science R.R.Ashurov) and in the seminar "Real and complex analysis"of Professor H.Begehr (Freie Universitaet Berlin). Some parts of the dissertation were reported in international scientific conferences on the theme "Mixed type equations and contiguous problems of analysis and informatics"in 2004, city Nal'chik, Russia, "Differential equations with partial derivative and contiguous problems of analysis and informatics"in 2004 city Tashkent and also in the republican conference of young mathematics scholars in Uzbekistan, which was devoted to 125 years of academician V.I.Romanovskiy.

Publication. Principal results of the dissertation where published in the works [48-55].

Structure of the work. The dissertation consist of introduction, three chapters and references. Numbering of formulas are double: the first number indicate on the number of chapter, the second number is the number of the formula in this chapter. Numbering of statements are threefold: the first number indicate on the number of chapter, the second number is the number of the paragraph, the third number is the number of the statement in this paragraph.

We pass to describe the substance of the dissertation.

The first chapter consist of two paragraphs. In the first paragraph is formulated and investigated one non-local problem for the equation

$$\text{sign}y u_{xx} + \text{sign}x u_{yy} - \lambda^2 u = 0, \qquad (L.B)$$

in mixed domain, in which on the boundary of the elliptic part is given a Dirichlet condition, and in the hyperbolic part a non-local condition with a determined integral operator for this equation, where λ is a given complex number, moreover $\lambda = \lambda_1$ at $x > 0$, $y > 0$, $\lambda = \lambda_2$ at $x > 0$, $y < 0$ and $x < 0$, $y > 0$.

Let Ω be a finite simply-connected domain in the plane of the variables x and y, bounded at $x>0,\ y>0$ by the line $\sigma_0: x^2+y^2=1$, at $x>0,\ y<0$ by the characteristics $OD: x+y=0$, $AD: x-y=1$ and at $x<0,\ y>0$ by the characteristics $OC: x+y=0$, $BC: x-y=-1$ of the equation $(L.B)$, and let

$$\Omega_0=\Omega\cap(x>0,\ y>0),\ \ \Omega_1=\Omega\cap(x>0,\ y<0),\ \ \Omega_2=\Omega\cap(x<0,\ y>0),$$

$$OA=\{(x,y):0<x<1,\ y=0\},\quad OB=\{(x,y):x=0,\ 0<y<1\}.$$

Further, let $\theta_{x0},\ \theta_{x1}$ and $\theta_{0y},\ \theta_{1y}$ be points of intersection of the characteristics of the equation $(L.B)$, outgoing from points $(x,0)\in OA$ and $(0,y)\in OB$ with characteristics OD, AD and OC, BC, respectively, i.e.

$$\theta_{x0}=\left(\frac{x}{2},-\frac{x}{2}\right),\theta_{x1}=\left(\frac{x+1}{2},\frac{x-1}{2}\right)\ \text{ and }\ \ \theta_{0y}=\left(-\frac{y}{2},\frac{y}{2}\right),\theta_{1y}=\left(\frac{y-1}{2},\frac{y+1}{2}\right).$$

We call a function $u(x,y)$ a regular solution of the equation $(L.B)$ in the domain Ω if: $u(x,y)\in C(\overline{\Omega})\cap C^1(\Omega)\cap C^2(\Omega_0\cup\Omega_1\cup\Omega_2)$ satisfies the equation $(L.B)$ in $\Omega\backslash(OA\cup OB)$; derivatives $u_x(x,y)$ and $u_y(x,y)$ can become infinity of the order less than one in points $A(1,0)$, $B(0,1)$ and $O(0,0)$.

Problem A_λ**.** Find a regular solution of the equation $(L.B)$ in the domain Ω satisfying the conditions

$$u(x,y)=\varphi(x,y),\qquad (x,y)\in\overline{\sigma}_0,\tag{1.1}$$

$$a_1(x)A^{0,\lambda_2}_{0x}\left[u(\theta_{x0})\right]+b_1(x)A^{0,\lambda_2}_{1x}\left[u(\theta_{x1})\right]$$
$$+c_1(x)u(x,0)=d_1(x),\qquad (x,0)\in\overline{OA},\tag{1.2$_1$}$$

$$a_2(y)A^{0,\lambda_2}_{0y}\left[u(\theta_{0y})\right]+b_2(y)A^{0,\lambda_2}_{1y}\left[u(\theta_{1y})\right]$$
$$+c_2(y)u(0,y)=d_2(y),\qquad (0,y)\in\overline{OB}.\tag{1.2$_2$}$$

Here $a_j(t),\ b_j(t),\ c_j(t)$ are given real-valued functions, moreover $a_j^2(t)+b_j^2(t)\neq 0$, $t\in[0,1]$, $j=1,2$, and $\varphi(x,y)$, $d_j(t)$ are in general, complex-valued functions, $A^{n,\lambda}_{mx}$ an operator witch has been introduced and studied in the monographs [36],

$$A_{mx}^{n,\lambda}\left[f(x)\right] \equiv f(x) - \int\limits_{m}^{x} f(t)\left(\frac{t-m}{x-m}\right)^{n} \frac{\partial}{\partial t} J_0\left[\lambda\sqrt{(x-m)(x-t)}\right] dt.$$

For the given functions we require that $a_j(t)$, $b_j(t)$, $c_j(t)$, $d_j(t) \in C^{(1,r)}[0,1]$, $\varphi(x,y) \in C(\overline{\sigma}_0)$, where $0 < r = \text{const} < 1$.

A uniqueness theorem for the solution of the problem A_λ is proved. Conditions are found for the complex parameter λ, ensuring uniqueness of the solution of the considered problem. Further, a domain for the parameter value λ is indicated, outside of witch the problem A_λ can have eigenvalues.

Principal results of this paragraph are the following theorems:

Theorem 1.1.1. *Let* $\alpha_j(t) \equiv 0$, $(j = 1,2)$, $\text{Re}\lambda_1^2 \geq 0$ *and one of the following conditions: a)* $b_j(t) \equiv 0$*; b)* $a_j(t) \equiv 0$*; c)* $a_j(t) \not\equiv 0$, $b_j(t) \not\equiv 0$, $a_j(t) \neq b_j(t)$ *and*

$$\int\limits_0^1\int\limits_0^1 \left|K_j(t,z,\lambda_2)\right|^2 dtdz < 1, \quad (j = 1,2)$$

be fulfilled, where

$$\alpha_j(t) = a_j(t) + b_j(t) + 2c_j(t), \qquad t = \begin{cases} x, & \text{if} \quad j = 1, \\ y, & \text{if} \quad j = 2, \end{cases}$$

$$K_j(t,z,\lambda_2) = \begin{cases} \dfrac{\partial}{\partial t}\left\{a_j(t)J_0\left[\lambda_2(t-z)\right]\right\} / \left[a_j(t) - b_j(t)\right], & t \geq z, \\ \dfrac{\partial}{\partial t}\left\{b_j(t)J_0\left[\lambda_2(z-t)\right]\right\} / \left[a_j(t) - b_j(t)\right], & z \geq t. \end{cases}$$

Then if a solution of the problem A_λ *exists, then it is unique.*

Theorem 1.1.2. *If conditions*

$$\alpha_j(t) \neq 0, \quad b_j(t) \equiv 0, \quad \frac{c_j(t)}{a_j(t)} > -\frac{1}{2}, \quad \left(\frac{c_j(t)}{a_j(t)}\right)' \geq 0, \quad t \in [0,1], \tag{1.3}$$

$$\text{Re}\lambda_1^2 \geq \left(\text{Im}\lambda_2\right)^2, \tag{1.4}$$

are fulfilled, then the problem A_λ *cannot have more than one solution.*

Theorem 1.1.3. *If conditions (1.4) and*

$$\alpha_j(t) \neq 0, \quad a_j(t) \equiv 0, \quad \frac{c_j(t)}{b_j(t)} > -\frac{1}{2}, \quad \left(\frac{c_j(t)}{b_j(t)}\right)' \leq 0, \quad t \in [0,1], \tag{1.5}$$

are fulfilled, then the problem A_λ cannot have more than one solution.

Theorem 1.1.4. *If conditions* $\mathrm{Re}\lambda_1^2 \geq 0$, $\mathrm{Im}\lambda_2 = 0$ and

$$\left(\frac{a_j(t)}{\alpha_j(t)}\right)' \leq 0, \quad \left(\frac{b_j(t)}{\alpha_j(t)}\right)' \geq 0, \quad t \in [0,1]; \quad \frac{a_j(1)}{\alpha_j(1)} + \frac{b_j(0)}{\alpha_j(0)} \geq 0,$$

are fulfilled, then the problem A_λ cannot have more than one solution.

Proposition. If $\lambda_1 = \lambda_2 = \lambda$ and one of the conditions (1.3) or (1.5) is fulfilled, then the problem A_λ can have eigenvalues only outside the domain $D_1 = \left\{\lambda : \ |\mathrm{Re}\lambda| \geq \sqrt{2}\,|\mathrm{Im}\lambda|\right\}$.

As usually, we call *eigenvalues* of the problem those values λ, for which non-trivial solutions of the corresponding homogenous problem exist. This non-trivial solutions are called *eigenfunctions.*

Existence of the solution of the problem A_λ is installed by the method of integral equations, as this investigated problem becomes a system of singular integral equations and this system in equivalent form is brought to a system of Fredholm integral equations of the second kind. Unconditional solvability of the system follows from the uniqueness of the solution of the problem.

In the second paragraph for the convenience we introduce the following notation $-\lambda_1^2 = \mu^2$, $\lambda_2 = \mu$ in equation $(L.B)$. Then equation $(L.B)$ becomes

$$Ku \equiv \mathrm{sign}x\, u_{xx} + \mathrm{sign}y\, u_{yy} + \mu^2 u = 0. \tag{$L.B_1$}$$

For equation $(L.B_1)$ one most general mixed problem is formulated and investigated in a finite simply connected mixed domain Ω, which is described in paragraph 1. In the boundary of the elliptic part of the mixed domain the condition $\alpha u + \beta\dfrac{\partial u}{\partial n} = 0$, and in the hyperbolic part non-local conditions with definite integral operators are given.

A uniqueness theorem for the solution of the formulated problem is proved. When a domain of the ellipticity equation is the sector $\pi/2$ with the center in the beginning of the coordinate, by the method of separation of variables eigenvalues $\mu_{n,m}$ are found and in evident form corresponding to them eigenfunctions are constructed. The question about completeness of the system of eigenfunctions is studied in elliptic, hyperbolic and on the whole of the mixed domains, and also for the structure of solution of the given problem is shown by applying a system of eigenfunctions.

Problem A_μ^0. Find a regular solution $u(x,y) \in C^1(\Omega \cup \sigma_0)$ of the equation $(L.B_1)$, in the domain Ω satisfying the conditions:

$$\alpha u(x,y) + \beta \frac{\partial u(x,y)}{\partial n} = \psi(x,y), \quad (x,y) \in \sigma_0,$$
$$A_{0x}^{0,\mu}\left[u(\theta_{x0})\right] + \gamma_1 u(x,0) = 0, \quad (x,0) \in \overline{OA},$$
$$A_{0y}^{0,\mu}\left[u(\theta_{0y})\right] + \gamma_2 u(0,y) = 0, \quad (0,y) \in \overline{OB},$$

where α, β, γ_1, γ_2 are given real numbers, moreover $\alpha^2 + \beta^2 \neq 0$; n is the outer normal to σ_0 and $\psi(x,y)$ is a given, in general, complex-valued function.

The validity of the following theorem is proved.

Theorem 1.2.1. *If conditions $\alpha \cdot \beta \geq 0$, $\alpha^2 + \beta^2 \neq 0$, $\gamma_j \geq -1/2$ $(j = 1,2)$ and $\mathrm{Re}\mu = 0$ are fulfilled, then the homogeneous problem A_μ^0 has only the trivial solution.*

Further, we assume that the condition of the Theorem 1.2.1 is fulfilled and we go over to polar coordinates $r = \sqrt{x^2+y^2}$, $\varphi = arctg\dfrac{y}{x}$, $(0 \leq r \leq 1$, $0 \leq \varphi \leq \dfrac{\pi}{2})$, the eigenvalues $\mu_{n,m} = \alpha_m^{(\nu_n)}$ $(m,n = 1,2,...)$ of the problem A_μ^0 are found by the method of separation of variables, where $\alpha_m^{(\nu_n)}$ is the m-th root of the equation $\alpha J_{\nu_n}(\mu) + \beta\mu J'_{\nu_n}(\mu) = 0$ (at $\beta = 0$ or $\beta \neq 0$, $\alpha/\beta + \nu_n \geq 0$, $\nu_n > 0$, $n \in N$ this equation has only real roots), where

$$\nu_n = \begin{cases} 2n-1, & \text{if} \quad \gamma_1+\gamma_2+2\gamma_1\gamma_2 = 0, \\ 2n - \dfrac{2}{\pi} arctg\gamma, & \text{if} \quad \gamma_1+\gamma_2+2\gamma_1\gamma_2 \neq 0, \quad \gamma \geq 0, \\ 2(n-1) - \dfrac{2}{\pi} arctg\gamma, & \text{if} \quad \gamma_1+\gamma_2+2\gamma_1\gamma_2 \neq 0, \quad \gamma < 0, \end{cases}$$

$\gamma = (1+\gamma_1+\gamma_2)/(\gamma_1+\gamma_2+2\gamma_1\gamma_2)$, $n \in N$. The corresponding system of eigenfunctions are determined by

$$u_{n,m}(x,y)=\begin{cases} c_{n,m}J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2+y^2}\right]\ sin(\nu_n\varphi+\varphi_0), & (x,y)\in\Omega_0,\\ k_{n,m}^{(1)}\left[(1+\gamma_1)\left(\dfrac{x+y}{x-y}\right)^{\nu_n/2}-\gamma_1\left(\dfrac{x-y}{x+y}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2-y^2}\right], & (x,y)\in\Omega_1,\\ (-1)^n k_{n,m}^{(2)}\left[(1+\gamma_2)\left(\dfrac{y+x}{y-x}\right)^{\nu_n/2}-\gamma_2\left(\dfrac{y-x}{y+x}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{y^2-x^2}\right], & (x,y)\in\Omega_2,\end{cases}\tag{1.6}$$

where $\varphi_0=arcctg(1+2\gamma_1)$ and $c_{n,m},\ k_{n,m}^{(1)},\ k_{n,m}^{(2)}\ \neq 0$ are arbitrary real constants.

If we go over to polar coordinates $r=\sqrt{x^2+y^2}$, $\varphi_1=arctg\dfrac{x}{y}$, $(0\leq r\leq 1,\ 0\leq\varphi_1\leq\dfrac{\pi}{2})$, then we obtain the second form of the system of eigenfunctions

$$u_{n,m}(x,y)=\begin{cases} (-1)^n c_{n,m}J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2+y^2}\right]\ sin(\nu_n\varphi_1+\varphi_0^1), & (x,y)\in\Omega_0,\\ (-1)^n k_{n,m}^{(1)}\left[(1+\gamma_1)\left(\dfrac{x+y}{x-y}\right)^{\nu_n/2}-\gamma_1\left(\dfrac{x-y}{x+y}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2-y^2}\right], & (x,y)\in\Omega_1,\\ k_{n,m}^{(2)}\left[(1+\gamma_2)\left(\dfrac{y+x}{y-x}\right)^{\nu_n/2}-\gamma_2\left(\dfrac{y-x}{y+x}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{y^2-x^2}\right], & (x,y)\in\Omega_2.\end{cases}\tag{1.7}$$

Let

$$G_0=\left\{(\gamma_1,\gamma_2):\ \gamma_1>-1,\ \gamma_2>-1,\ \gamma_1+\gamma_2>-1\right\},$$

$$G_{ks}=\left\{(\gamma_1,\gamma_2):\ \gamma_k>-1,\ \gamma_s<-1,\ \gamma_1+\gamma_2<-1\right\},\quad k,s=1,2,\quad k\neq s.$$

Then the following theorems are valid.

Theorem 1.2.2. *If $(\gamma_1,\gamma_2)\in G_0\cup G_{12}$ ($(\gamma_1,\gamma_2)\in G_0\cup G_{21}$), then the system of eigenfunctions (1.6) ((1.7)) of the problem A_μ^0 is complete in $L_2(\Omega_0)$.*

Theorem 1.2.3. *If $\gamma_1=\gamma_2=0$, then the system of eigenfunctions (1.6) ((1.7)) of the problem A_μ^0 is complete in $L_2(\Omega_1)$ and $L_2(\Omega_2)$.*

Theorem 1.2.4. *If $\gamma_1 = \gamma_2 = 0$, then the system of eigenfunctions (1.6) ((1.7)) of the problem A_μ^0 is not complete in $L_2(\Omega)$.*

At the end of this paragraph we assume, that the conditions of the Theorem 1.2.1 are fulfilled. Then, taking the propositions of the Theorems 1.2.2-1.2.4 for those values of the parameter $\mu \neq \alpha_m^{(\nu_n)}$ into account, the solution of the problem A_μ^0 is found in the evident form

$$u(x,y,\mu) = \begin{cases} \sum\limits_{n=1}^{\infty} f_n \dfrac{J_{\nu_n}(\mu r)}{\alpha J_{\nu_n}(\mu) + \beta\mu J'_{\nu_n}(\mu)} sin(\nu_n\varphi + \varphi_0), & (x,y) \in \Omega_0, \\ \sum\limits_{n=1}^{\infty} \dfrac{f_n}{\sqrt{2+4\gamma_1+4\gamma_1^2}} \left[(1+\gamma_1)\left(\dfrac{x+y}{x-y}\right)^{\nu_n/2} - \gamma_1\left(\dfrac{x-y}{x+y}\right)^{\nu_n/2}\right] \times & \\ \qquad \times \dfrac{J_{\nu_n}\left[\mu\sqrt{x^2-y^2}\right]}{\alpha J_{\nu_n}(\mu) + \beta\mu J'_{\nu_n}(\mu)}, & (x,y) \in \Omega_1, \\ \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n f_n}{\sqrt{2+4\gamma_2+4\gamma_2^2}} \left[(1+\gamma_2)\left(\dfrac{x+y}{x-y}\right)^{\nu_n/2} - \gamma_2\left(\dfrac{x-y}{x+y}\right)^{\nu_n/2}\right] \times & \\ \qquad \times \dfrac{J_{\nu_n}\left[\mu\sqrt{y^2-x^2}\right]}{\alpha J_{\nu_n}(\mu) + \beta\mu J'_{\nu_n}(\mu)}, & (x,y) \in \Omega_2. \end{cases} \tag{1.8}$$

One essential result of this paragraph is the following theorem.

Theorem 1.2.5. *If $\alpha\beta \geq 0$, $\alpha^2 + \beta^2 \neq 0$, $\gamma_j \geq -1/2$ $(j = 1, 2)$,* $\mathrm{Re}\mu = 0$, *$f(\varphi) \in C^\delta[0, \pi/2]$, $\delta \in (0, 1]$, then the problem A_μ^0 has a unique solution and it is presented in the form (1.8), where $f(\varphi) = f(r,\varphi)|_{r=1} = \psi(x,y)|_{\sigma_0}$.*

The second chapter consists of four paragraphs and in it the Bitsadze-Samarskiy type problem is investigated for the equation $(L.B)$. Here it is also assumed, that λ is a given complex number, moreover $\lambda = \lambda_0$ at $x > 0,\ y > 0,\ \lambda = \lambda_1$ at $x > 0,\ y < 0$ and $\lambda = \lambda_2$ at $x < 0,\ y > 0$. In this chapter those notations are used which we were used in the first chapter.

§ 2.1 of the second chapter is devoted to preliminary information, which is needed for proving the theorem of uniqueness of the solution of the studied problem.

In § 2.2 of the second chapter the formulation of the problem for equation $(L.B)$ is given.

Problem BS_λ. Find a regular solution of the equation $(L.B)$, in the domain Ω satisfying the following conditions

$$u(x,y) = \sum_{k=1}^{n} \alpha_k(x,y)u(r_k x, r_k y) + g(x,y), \qquad (x,y) \in \bar{\sigma}_0,$$

$$A_{0x}^{0,\lambda_2}\left[u(\theta_{x0})\right] + c_1(x)u(x,0) = d_1(x), \qquad (x,0) \in \overline{OA},$$

$$A_{0y}^{0,\lambda_2}[u(\theta_{0y})] + c_2(y)u(0,y) = d_2(y), \qquad (0,y) \in \overline{OB},$$

where $c_j(t)$ $(j = 1,2)$ are given a real-valued functions, and $\alpha_k(x,y)$ $(k = 1,...,n)$, $g(x,y)$, $d_j(t)$ are given, in general, complex-valued functions, $r_1,\ldots,r_n$ are given real numbers, moreover $0 < r_1 < r_2 < \ldots < r_n < 1$. From the given functions we require, that $c_j(t),\ d_j(t) \in C^{(2,r)}[0,1]$, where $0 < r = const < 1$, $\alpha_k(x,y)$, $g(x,y) \in C(\bar{\sigma}_0)$, $k = 1,...,n$.

In § 2.3 of the second chapter the theorem of uniqueness of the solution of the problem BS_λ is proved. Conditions for the complex parameter λ are found, ensuring uniqueness of the solution of the considering problem. Further, the domain of the values of the parameter λ is shown, outside of this domain the problem BS_λ can have eigenvalues.

Principal result of § 2.3 is the following theorem.

Theorem 2.3.1. *Let*

$$\mathrm{Re}\lambda_0^2 \geq \delta_0^2 = max(|\lambda_1^2|, |\lambda_2^2|), \tag{2.1}$$

$$c_j(t) \geq -\frac{1}{2}, \quad c_j'(t) \geq 0, \quad 0 \leq t \leq 1; \quad j = 1,2 \tag{2.2}$$

and for some $\delta \in [\delta_0^2, \mathrm{Re}\lambda_0^2]$ the inequality

$$\sum_{k=1}^{n} |\alpha_k(x,y)| \left[\frac{e^{\delta r_k x} + e^{\delta r_k y}}{e^{\delta x} + e^{\delta y}}\right] \leq 1, \qquad (x,y) \in \overline{\sigma}_0 \tag{2.3}$$

is fulfilled.

Then, if the solution of the problem BS_λ exists, then it is unique.

Proposition. If $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$ and suppose the conditions (2.2), (2.3) are fulfilled. Then the problem BS_λ can have eigenvalues only outside of $\mathrm{Im}\lambda = 0$.

In § 2.4 of the second chapter sufficient conditions for the given functions are determined, under which the existence of a solution of the problem BS_λ is investigated by the help of integral equations. The investigated problem becomes a system of three Fredholm equation of second kind, the solvability at which follows from the uniqueness of the solution.

The third chapter consist of three paragraphs, and in it two non-local problems for equation $(L.B)$ in a mixed domain are investigated, the boundary of the elliptic part of which is a quarter ring.

§ 3.1 of the third chapter is devoted to the formulation of the problems Γ_0^λ and Γ_1^λ.

Let Δ be a finite simply-connected domain of the plane of variables xOy, bounded at $x>0$, $y>0$ by the lines $\sigma_{01}: x^2+y^2=1$, $\sigma_{02}: x^2+y^2=p^2$, $(0<p<1)$ and for $xy<0$ by the characteristics $x+y=p$, $x-y=\pm 1$ of the equation $(L.B)$.

Let us introduce the notations:

$$\Delta_0=\Delta\cap(x>0,y>0),\ \ \Delta_1=\Delta\cap(x>0,y<0),\ \ \Delta_2=\Delta\cap(x<0,y>0),$$

$$I_1=\{(x,y): p<x<1, y=0\},\qquad I_2=\{(x,y): x=0,\, p<y<1\},$$

$$\theta_{px}(x)=((p+x)/2;\ (p-x)/2),\qquad \theta_{py}(y)=((p-y)/2;\ (p+y)/2),$$

$$\theta_{x1}(x)=((x+1)/2;\ (x-1)/2),\qquad \theta_{1y}(y)=((y-1)/2;\ (y+1)/2).$$

We call a function $u(x,y)$ a regular solution of the equation $(L.B)$ in the domain $\Delta\setminus(I_1\cup I_2)$, and the derivatives $u_x(x,y)$, $u_y(x,y)$ can become infinity of order less than one in the points $A(p,0)$, $B(1,0)$, $C(0,1)$ and $D(0,p)$.

Problem Γ_0^{λ}. Find a regular solution of the equation $(L.B)$, in the domain Δ satisfying the conditions

$$u(x,y)\in C(\overline{\Delta})\cap C^1(\Delta)\cap C^2(\Delta\backslash I_1\backslash I_2);$$

$$u(x,y)=\varphi_j(x,y),\qquad (x,y)\in\bar{\sigma}_{0j},\quad (j=1,2.);\tag{3.1}$$

$$a_1(x)A_{px}^{0,\lambda_2}\left[u(\theta_{px})\right]+b_1(x)A_{1x}^{0,\lambda_2}\left[u(\theta_{x1})\right]$$
$$+c_1(x)u(x,0)=d_1(x),\qquad (x,0)\in\overline{I_1};\tag{3.2$_1$}$$

$$a_2(y)A_{py}^{0,\lambda_2}\left[u(\theta_{py})\right]+b_2(y)A_{1y}^{0,\lambda_2}\left[u(\theta_{1y})\right]$$
$$+c_2(y)u(0,y)=d_2(y),\qquad (0,y)\in\overline{I_2}.\tag{3.2$_2$}$$

Problem Γ_1^{λ}. Find a regular solution of the equation $(L.B)$, in the domain Δ satisfying the boundary condition (3.1) and the conditions

$$u(x,y)\in C(\overline{\Delta})\cap C^1(\overline{\Delta}\backslash\sigma_{0j})\cap C^2(\Delta\backslash I_1\backslash I_2),\quad (j=1,2.);$$

$$a_1(x)A_{px}^{1,\lambda_2}\left[\frac{d}{dx}u(\theta_{px})\right]+b_1(x)A_{1x}^{1,\lambda_2}\left[\frac{d}{dx}u(\theta_{x1})\right]$$
$$+c_1(x)\frac{\partial}{\partial y}u(x,0)=d_1(x),\qquad (x,0)\in I_1;\tag{3.3$_1$}$$

$$a_2(y)A_{py}^{1,\lambda_2}\left[\frac{d}{dy}u(\theta_{py})\right]+b_2(y)A_{1y}^{1,\lambda_2}\left[\frac{d}{dy}u(\theta_{1y})\right]$$

$$+c_2(y)\frac{\partial}{\partial x}u(0,y)=d_2(y), \qquad (0,y)\in I_2. \qquad (3.3_2)$$

Here $a_j(t)$, $b_j(t)$, $c_j(t)$ are given real-valued functions, moreover $a_j^2(t)+b_j^2(t)\neq 0$, $t\in[p,1]$, $j=1,2$, and $\varphi_j(x,y)$ and $d_j(t)$ are given, in general, complex-valued functions.

For the given functions we require that $a_j(t)$, $b_j(t)$, $c_j(t)$, $d_j(t)\in C^1\left(\overline{I_j}\right)\cap C^{(1+k,r)}\left(I_j\right)$, $\varphi_j(x,y)\in C(\bar{\sigma}_{0j})$, where $0<r=const<1$ and $k=1$ in problem Γ_0^λ and $k=0$ in problem Γ_1^λ.

In the second paragraph of the third chapter the problem Γ_0^λ is investigated. A theorem of uniqueness of the solution is proved. Conditions for the complex parameter λ are found, ensuring uniqueness of the solution of the considered problem. Further, the domain of the values of parameter λ is show, so that outside of this domain the non-local problem can have eigenvalues.

The principal results of this paragraph are the following theorems.

Theorem 3.2.1. *Let $\alpha_j(t)\equiv 0$, $\mathrm{Re}\lambda_1^2\geq 0$ and one of the following group of conditions is fulfilled: a) $b_j(t)\equiv 0$; b) $a_j(t)\equiv 0$; c) $a_j(t)\not\equiv 0$, $b_j(t)\not\equiv 0$, $a_j(t)\neq b_j(t)$ and*

$$\int_p^1\int_p^1 \left|K_j(t,z,\lambda_2)\right|^2 dtdz<1, \quad j=1,2,$$

where

$$\alpha_j(t)=a_j(t)+b_j(t)+2c_j(t), \quad t=\begin{cases} x, & if \quad j=1,\\ y, & if \quad j=2,\end{cases}$$

$$K_j(t,z,\lambda_2)=\begin{cases}\dfrac{\partial}{\partial t}\left\{a_j(t)J_0\left[\lambda_2(t-z)\right]\right\}/\left[a_j(t)-b_j(t)\right], & t\geq z,\\[2mm] \dfrac{\partial}{\partial t}\left\{b_j(t)J_0\left[\lambda_2(z-t)\right]\right\}/\left[a_j(t)-b_j(t)\right], & z\geq t.\end{cases}$$

Then, the solution of the problem Γ_0^λ exists and is unique.

Theorem 3.2.2. *Let one of the following group of conditions is fulfilled*

1) $$\mathrm{Re}\lambda_1^2\geq\left(\mathrm{Im}\lambda_2\right)^2;$$

$$\alpha_j(t)\neq 0,\;\; b_j(t)\equiv 0,\;\; \frac{c_j(t)}{a_j(t)}>-\frac{1}{2}, \quad \left(\frac{c_j(t)}{a_j(t)}\right)'\geq 0, \quad t\in[p,1]; \qquad (3.4)$$

2) $$\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2\,;$$

$$\alpha_j(t) \neq 0,\;\; a_j(t) \equiv 0,\quad \frac{c_j(t)}{b_j(t)} > -\frac{1}{2},\quad \left(\frac{c_j(t)}{b_j(t)}\right)' \leq 0,\quad t \in [p,1]; \tag{3.5}$$

3) $$\mathrm{Re}\lambda_1^2 \geq 0,\qquad \mathrm{Im}\lambda_2 = 0;$$

$$\alpha_j(t) \neq 0,\quad a_j(t) \not\equiv 0,\quad b_j(t) \not\equiv 0, \tag{3.6}$$

$$\left(\frac{a_j(t)}{\alpha_j(t)}\right)' \leq 0,\quad \left(\frac{b_j(t)}{\alpha_j(t)}\right)' \geq 0,\quad t \in [p,1];\qquad \frac{a_j(1)}{\alpha_j(1)} + \frac{b_j(p)}{\alpha_j(p)} \geq 0. \tag{3.7}$$

Then the problem Γ_0^λ cannot have more than one solution.

Proposition. If $\lambda_1 = \lambda_2 = \lambda$ and one of the conditions (3.4) or (3.5) are fulfilled, then the problem Γ_0^λ (consequently, the Tricomi problem) can have eigenvalues only outside of the domain $D_1 = \left\{\lambda : |\mathrm{Re}\lambda| \geq \sqrt{2}\,|\mathrm{Im}\lambda|\right\}$.

Under the determined sufficient conditions for the given functions, the existence of the solution of the problem Γ_0^λ is proved.

In the third paragraph of the third chapter problem Γ_1^λ is investigated. The theorem for uniqueness of the solution is proved. And here conditions for the parameter λ also found, ensuring uniqueness of the solution of the problem Γ_1^λ. Further, the domain of the values of the parameter λ is shown, outside of which problem Γ_1^λ can have eigenvalues.

Principal results of this paragraph are the following theorems.

Theorem 3.3.1. *Let $\beta_j(t) \equiv 0$, $j = 1,2$. Then if $\mathrm{Re}\lambda_1^2 \geq 0$ and one of the following group of conditions: a) $b_j(t) \equiv 0$, $j = 1,2$, b) $a_j(t) \equiv 0$, $j = 1,2$, c) $a_j(t) \equiv 0$, $b_k(t) \neq 0$, $j, k = 1,2$, $j \neq k$ is fulfilled, then the solution of the problem Γ_1^λ exists and is unique, where $\beta_j(t) = a_j(t) - b_j(t) - 2c_j(t)$.*

Theorem 3.3.2. *Let one of the following group of conditions be fulfilled*

1) $$\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2\,,$$

$$\beta_j(t) \neq 0,\;\; b_j(t) \equiv 0,\quad \frac{c_j(t)}{a_j(t)} < \frac{1}{2},\quad \left(\frac{c_j(t)}{a_j(t)}\right)' \leq 0,\quad t \in [p,1]; \tag{3.8}$$

2) $$\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2\,,$$

$$\beta_j(t) \neq 0,\;\; a_j(t) \equiv 0,\quad \frac{c_j(t)}{b_j(t)} > \frac{1}{2},\quad \left(\frac{c_j(t)}{b_j(t)}\right)' \leq 0,\quad t \in [p,1]; \tag{3.9}$$

3) $$\mathrm{Re}\lambda_1^2 \geq 0, \quad \mathrm{Im}\lambda_2 = 0,$$

$$\beta_j(t) \neq 0, \quad a_j(t) \not\equiv 0, \quad b_j(t) \not\equiv 0, \tag{3.10}$$

$$\left(\frac{a_j(t)}{\beta_j(t)}\right)' \leq 0, \qquad \left(\frac{b_j(t)}{\beta_j(t)}\right)' \leq 0, \quad t \in [p, 1]; \qquad \frac{a_j(1)}{\beta_j(1)} - \frac{b_j(p)}{\beta_j(p)} \geq 0. \tag{3.11}$$

Then for the problem Γ_1^λ there cannot exist more than one solution.

Proposition. If $\lambda_1 = \lambda_2 = \lambda$ and one conditions of group (3.8) or (3.9) is fulfilled, then the problem Γ_1^λ (consequently the Tricomi problem) can have eigenvalues only outside of the domain $D_1 = \left\{\lambda : |\mathrm{Re}\lambda| \geq \sqrt{2}\,|\mathrm{Im}\lambda|\right\}$.

Under the determined sufficient conditions for the given functions the existence of the solution of the problem Γ_1^λ is proved.

Acknowledgements. The author is grateful to his scientific supervisors, to doctor of physic-mathematical sciences, academician of Uzbek Academy Sciences Mahmud Salakhitdinov Salakhitdinovich for offering the theme of this work and for his constant control at fulfilling this work.

And to doctor of physic-mathematical sciences of Uzbek Academy Sciences Ahmadjon Kushakovich O'rinov for useful remarks.

And also to professor of FU Berlin, faculty of mathematic and informatics Heinrich Begehr for useful remarks.

Chapter 1

NON-LOCAL PROBLEMS AND THEIR SPECTRAL PROPERTIES FOR MIXED TYPE EQUATION, WHEN THE DOMAIN OF ELLIPTICITY IS A QUARTER CIRCLE

In this chapter, the existence and uniqueness of the solution to the boundary value problems with displacement for the elliptic-hyperbolic equation with spectral parameter with two perpendicular lines of changing type of the equation in a finite simply connected mixed domain are studied, i.e. for the following equation

$$Mu \equiv \mathrm{sign}y \, u_{xx} + \mathrm{sign}x \, u_{yy} - \lambda^2 u = 0. \qquad (L.B)$$

In the case when the uniqueness theorem does not hold eigenvalues and corresponding eigenfunctions are found for one most general mixed problem for the equation

$$Ku \equiv \mathrm{sign}x \, u_{xx} + \mathrm{sign}y \, u_{yy} + \mu^2 u = 0. \qquad (L.B_1)$$

Completeness of the system of eigenfunctions is investigated in the class L_2, where λ, μ are given complex numbers, moreover $\lambda = \lambda_1$ at $x > 0$, $y > 0$, $\lambda = \lambda_2$ at $x > 0$, $y < 0$ and $x < 0$, $y > 0$.

§ 1.1. Non-local problem with Dirichlet condition on the boundary of the elliptic part of the domain

1.1.1. Formulation of the problem A_λ

Let Ω be a finite simply-connected domain in the plane of the variables x and y, bounded at $x>0,\ y>0$ by the line $\sigma_0 : x^2+y^2=1$, at $x>0,\ y<0$ by the characteristics $OD : x+y=0$, $AD : x-y=1$ and at $x<0,\ y>0$ by the characteristics $OC : x+y=0,\ BC : x-y=-1$ of the equation $(L.B)$, and let

$$\Omega_0=\Omega\cap(x>0,\ y>0),\ \ \Omega_1=\Omega\cap(x>0,\ y<0),\ \ \Omega_2=\Omega\cap(x<0,\ y>0),$$

$$OA=\{(x,y):0<x<1,\ y=0\},\quad OB=\{(x,y):x=0,\ 0<y<1\}.$$

Further, let $\theta_{x0},\ \theta_{x1}$ and $\theta_{0y},\ \theta_{1y}$ be the points of intersection of the characteristics of the equation $(L.B)$, outgoing from the points $(x,0)\in OA$ and $(0,y)\in OB$ with characteristics $OD,\ AD$ and $OC,\ BC$, respectively, i.e.

$$\theta_{x0}=\left(\frac{x}{2},-\frac{x}{2}\right),\theta_{x1}=\left(\frac{x+1}{2},\frac{x-1}{2}\right)\quad\text{and}\quad\theta_{0y}=\left(-\frac{y}{2},\frac{y}{2}\right),\theta_{1y}=\left(\frac{y-1}{2},\frac{y+1}{2}\right).$$

We call a function $u(x,y)$ a regular solution of the equation $(L.B)$ in the domain Ω if:

a) $u(x,y)\in C(\overline{\Omega})\cap C^1(\Omega)\cap C^2(\Omega_0\cup\Omega_1\cup\Omega_2)$ satisfies the equation $(L.B)$ in $\Omega\backslash(OA\cup OB)$;

b) the derivatives $u_x(x,y)$ and $u_y(x,y)$ can become infinite of order less than one of the points $A(1,0)$, $B(0,1)$ and $O(0,0)$.

Problem A_λ. Find a regular solution of the equation $(L.B)$ in the domain Ω satisfying the conditions

$$u(x,y)=\varphi(x,y),\qquad (x,y)\in\overline{\sigma}_0, \tag{1.1}$$

$$a_1(x)A_{0x}^{0,\lambda_2}\left[u(\theta_{x0})\right]+b_1(x)A_{1x}^{0,\lambda_2}\left[u(\theta_{x1})\right]$$
$$+c_1(x)u(x,0)=d_1(x),\qquad (x,0)\in\overline{OA}, \tag{1.2$_1$}$$

$$a_2(y)A_{0y}^{0,\lambda_2}\left[u(\theta_{0y})\right]+b_2(y)A_{1y}^{0,\lambda_2}\left[u(\theta_{1y})\right]$$
$$+c_2(y)u(0,y)=d_2(y),\qquad (0,y)\in\overline{OB}. \tag{1.2$_2$}$$

Here $a_j(t)$, $b_j(t)$, $c_j(t)$ are given real-valued functions, moreover $a_j^2(t) + b_j^2(t) \neq 0$, $t \in [0,1]$, $j = 1,2$, and $\varphi(x,y)$, $d_j(t)$ are, in general, complex-valued functions, $A_{mx}^{n,\lambda}$ is an operator which has been introduced and studied in the monograph [36],

$$A_{mx}^{n,\lambda}\left[f(x)\right] \equiv f(x) - \int\limits_m^x f(x)\left(\frac{t-m}{x-m}\right)^n \frac{\partial}{\partial t} J_0\left[\lambda\sqrt{(x-m)(x-t)}\right] dt.$$

From given functions we require that $a_j(t)$, $b_j(t)$, $c_j(t)$, $d_j(t) \in C^{(1,r)}[0,1]$, $\varphi(x,y) \in C(\overline{\sigma}_0)$, where $0 < r = const < 1$.

Besides, at $a_j(t) \neq 0$ we require that

$$b_k(0) = 0, \qquad a_k(0) + c_k(0) \neq 0, \tag{1.3}$$

even for one value of k $(k = 1,2)$ and the compatibility condition

$$d_1(0)\left[\, a_2(0) + c_2(0)\right] = d_2(0)\left[\, a_1(0) + c_1(0)\right] \tag{1.4}$$

is fulfilled, if (1.3) takes place for $k = 1,2$.

1.1.2. Uniqueness of the solution of the problem A_λ

The main functional relation in Ω_1 and Ω_2.

We consider the equation $(L.B)$ in the domains Ω_1 and Ω_2.

As in [39], it is not difficult to prove, that, if $\tau_1(x) = u(x,0)$, $\nu_1(x) = u_y(x,0)$, $\tau_2(y) = u(0,y)$, $\nu_2(y) = u_x(0,y)$ and $\tau_j(x) \in C[0,1] \cap C^2(0,1)$, $\nu_j(x) \in C^1(0,1)$, moreover $\tau_j'(t)$, $\nu_j(t) \in L[0,1]$, $j = 1,2$, then any twice continuously differentiable solution of equation $(L.B)$ in the domain Ω_j $(j = 1,2)$ can be represented as

$$u(x,y) = \frac{1}{2}\left[\tau_j(x+y) + \tau_j(x-y)\right] +$$

$$+\frac{1}{2}\int\limits_{x-y}^{x+y}\nu_j(t)J_0\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]dt+\frac{\lambda_2}{2}y\int\limits_{x-y}^{x+y}\tau_j(t)\frac{J_1\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]}{\sqrt{(x-t)^2-y^2}}dt. \qquad (1.5_j)$$

Using formulas (1.5_j) and conditions (1.2_j), after some calculations we have the following equality

$$\alpha_j(t)\tau_j(t)=2d_j(t)-\tau_j(0)a_j(t)J_0\left[\lambda_2 t\right]-\tau_j(1)b_j(t)J_0\left[\lambda_2(1-t)\right]+$$
$$+a_j(t)\int\limits_0^t\nu_j(z)J_0\left[\lambda_2(t-z)\right]dz+b_j(t)\int\limits_t^1\nu_j(z)J_0\left[\lambda_2(z-t)\right]dz, \quad j=1,2, \qquad (1.6_j)$$

where $0\le t\le 1$,

$$\alpha_j(t)=a_j(t)+b_j(t)+2c_j(t), \quad t=\begin{cases} x, & \text{if} \quad j=1,\\ y, & \text{if} \quad j=2.\end{cases}$$

Equalities (1.6_j) provide a basic functional relation between $\tau_j(t)$ and $\nu_j(t)$ on the segments OA and OB attained from the hyperbolic part of the mixed domain Ω.

By virtue of conditions (1.3) and (1.4) the constant $\tau_1(0)=\tau_2(0)=u(0,0)$ is identically defined from conditions (1.2_j) $j=1,2$.

As one can see from the relations (1.6_j) there are the following cases: $\alpha_j(t)\equiv 0$, $j=1,2$; $\alpha_j(t)\ne 0$, $j=1,2$; $\alpha_j(t)\equiv 0$, $\alpha_k(t)\ne 0$, $j,k=1,2$, $j\ne k$.

Note, that in [40] the uniqueness of the solution to the Tricomi problem for the equation $(L.B)$ is proved by the method of energy integral by using the Laplace transformation. But this method is inapplicable for the non-local problems on the hyperbolic part of the domain of equation $(L.B)$. The problem A_λ was considered in [44] for the case, when λ is a real number. We shall prove the uniqueness and existence of the solution to the problem A_λ, when λ is a complex number, using the method which is described in [36].

The following Lemma plays the essential role in proving the uniqueness theorem.

Lemma 1.1.1. *Let $u(x,y)$ be a regular solution of the equation $(L.B)$ in the domain Ω_0 which is equal to zero on $\overline{\sigma}_0$. Then the equality*

$$(\mathrm{Re}\lambda_1^2-\delta^2)\left[\iint\limits_{\Omega_0'}|\vartheta|^2dxdy+\iint\limits_{\Omega_0''}|\omega|^2dxdy\right]+\iint\limits_{\Omega_0'}|\nabla\vartheta|^2dxdy+\iint\limits_{\Omega_0''}|\nabla\omega|^2dxdy+$$

$$+\mathrm{Re}\int\limits_0^1 e^{2\delta x}\tau_1(x)\nu_1(x)dx+\mathrm{Re}\int\limits_0^1 e^{2\delta y}\tau_2(y)\nu_2(y)dy=0, \tag{1.7}$$

is valid, where $\vartheta(x,y)=e^{\delta x}u(x,y)$ *in* $\Omega_0'=\Omega_0\cap(x>y)$; $\omega(x,y)=e^{\delta y}u(x,y)$ *in* $\Omega_0''=\Omega_0\cap(x<y)$, $\forall\delta\in R$, $\nabla\equiv\frac{\partial}{\partial x}\overrightarrow{i}+\frac{\partial}{\partial y}\overrightarrow{j}$ *is the nabla operator. Moreover* $\vartheta(x,y)=\omega(x,y)$ *on* $OK:y=x$; $\overline{\tau}_1(x)=\overline{u}(x,0)$, $\nu_1(x)=u_y(x,0)$, $\overline{\tau}_2(y)=\overline{u}(0,y)$, $\nu_2(y)=u_x(0,y)$, *where* $\overline{u}(x,y)$ *denotes the complex conjugation to* $u(x,y)$.

Proof. Let $u(x,y)$ be a function satisfying the conditions of Lemma 1.1.1. We divide the domain Ω_0 into two parts by drawing the straight line $y=x$ from the origin of the coordinates to the point of intersection with the curve σ_0. Denote this point by K. Assume that Ω_0' is the domain adjacent to the axis Ox, and Ω_0'' is the domain adjacent to the axis Oy. Consider the function $\vartheta(x,y)=e^{\delta x}u(x,y)$ and $\omega(x,y)=e^{\delta y}u(x,y)$ in domains Ω_0' and Ω_0'', respectively, where $\delta\in\mathrm{R}$. Then, equation $(L.B)$ becomes the forms

$$L\vartheta\equiv\vartheta_{xx}+\vartheta_{yy}-2\delta\vartheta_x+(\delta^2-\lambda_1^2)\vartheta=0\quad\text{in}\quad\Omega_0'$$
$$L\omega\equiv\omega_{xx}+\omega_{yy}-2\delta\omega_y+(\delta^2-\lambda_1^2)\omega=0\quad\text{in}\quad\Omega_0''.$$

Multiplying $L\vartheta=0$ and $L\omega=0$ with the functions $\overline{\vartheta}(x,y)$ and $\overline{\omega}(x,y)$, which are the complex conjugate to $\vartheta(x,y)$ and $\omega(x,y)$ respectively, then we rewrite them in this forms

$$(\overline{\vartheta}\vartheta_x)_x+(\overline{\vartheta}\vartheta_y)_y-2\delta\overline{\vartheta}\vartheta_x-\overline{\vartheta}_x\vartheta_x-\overline{\vartheta}_y\vartheta_y-(\lambda_1^2-\delta^2)\left|\vartheta\right|^2=0\quad\text{in}\quad\Omega_0',$$
$$(\overline{\omega}\omega_x)_x+(\overline{\omega}\omega_y)_y-2\delta\overline{\omega}\omega_y-\overline{\omega}_x\omega_x-\overline{\omega}_y\omega_y-(\lambda_1^2-\delta^2)\left|\omega\right|^2=0\quad\text{in}\quad\Omega_0''.$$

Let $\sigma_0^\varepsilon=\{(x,y):x^2+y^2=(1-\varepsilon)^2\}$, $\Omega_0^{\prime(\varepsilon,\delta_1)}$ and $\Omega_0^{\prime\prime(\varepsilon,\delta_1)}$ be the domains restricted by curves σ_0^ε, $y=\delta_1$, $y=x$ and σ_0^ε, $x=\delta_1$, $y=x$ respectively (ε and δ_1 are sufficiently small positive numbers). If we integrate the last equalities on the domains $\Omega_0^{\prime(\varepsilon,\delta_1)}$ and $\Omega_0^{\prime\prime(\varepsilon,\delta_1)}$, then using the Green formula and taking the real part of the obtained equalities, we obtain

$$(Re\lambda_1^2-\delta^2)\iint\limits_{\Omega_0^{\prime(\varepsilon,\delta_1)}}|\vartheta|^2dxdy+\iint\limits_{\Omega_0^{\prime(\varepsilon,\delta_1)}}|\nabla\vartheta|^2dxdy+$$
$$+Re\int\limits_{\partial\Omega_0^{\prime(\varepsilon,\delta_1)}}\overline{\vartheta}(\vartheta_xdy-\vartheta_ydx)+\delta Re\int\limits_{\partial\Omega_0^{\prime(\varepsilon,\delta_1)}}\left|\vartheta\right|^2dy=0,$$

$$(Re\lambda_1^2-\delta^2)+\iint\limits_{\Omega_0''^{(\varepsilon,\delta_1)}}|\omega|^2dxdy+\iint\limits_{\Omega_0''^{(\varepsilon,\delta_1)}}|\nabla\omega|^2dxdy+$$
$$+Re\int\limits_{\partial\Omega_0''^{(\varepsilon,\delta_1)}}\overline{\omega}(\omega_x dy-\omega_y dx)+\delta Re\int\limits_{\partial\Omega_0''^{(\varepsilon,\delta_1)}}|\omega|^2\,dx=0.$$

If we take the limit when $\delta_1\to 0$, $\varepsilon\to 0$ and combine the obtained equalities taking into account that $u|_{\bar{\sigma}_o}=0$ and $\vartheta=\omega$ on $OK: x=y$, we get the equality (1.7).

Now we investigate the uniqueness of the solution of the problem A_λ.

Case 1. Let $\alpha_j(t)\equiv 0,\ \ j=1,2.$

Theorem 1.1.1. *Let* $\alpha_j(t)\equiv 0$, $(j=1,2)$, $\mathrm{Re}\lambda_1^2\geq 0$ *and one of the following group conditions: a)* $b_j(t)\equiv 0$*; b)* $a_j(t)\equiv 0$*; c)* $a_j(t)\not\equiv 0$, $b_j(t)\not\equiv 0$, $a_j(t)\neq b_j(t)$ *and*

$$\int\limits_0^1\int\limits_0^1|K_j(t,z,\lambda_2)|^2\,dtdz<1,\quad (j=1,2)\tag{1.8}$$

be fulfilled, where

$$\alpha_j(t)=a_j(t)+b_j(t)+2c_j(t),\qquad t=\begin{cases}x, & if \quad j=1,\\ y, & if \quad j=2,\end{cases}$$

$$K_j(t,z,\lambda_2)=\begin{cases}\dfrac{\partial}{\partial t}\left\{a_j(t)J_0\left[\lambda_2(t-z)\right]\right\}/\left[a_j(t)-b_j(t)\right], & t\geq z,\\ \dfrac{\partial}{\partial t}\left\{b_j(t)J_0\left[\lambda_2(z-t)\right]\right\}/\left[a_j(t)-b_j(t)\right], & z\geq t.\end{cases}$$

Then if a solution of the problem A_λ *exists, then it is unique.*

Proof. Let $u(x,y)$ be a solution of the homogeneous problem A_λ and $\alpha_j(t)\equiv 0$ $(j=1,2)$. Then, the relation (1.6_j) becomes

$$a_j(t)\int\limits_0^t\nu_j(z)J_0\left[\lambda_2(t-z)\right]dz+b_j(t)\int\limits_t^1\nu_j(z)J_0\left[\lambda_2(z-t)\right]dz=\psi_j(t),\tag{1.9$_j$}$$

where $\psi_j(t)=\tau_j(0)a_j(t)J_0\left[\lambda_2 t\right]+\tau_j(1)b_j(t)J_0\left[\lambda_2(1-t)\right],\ \ j=1,2.$

a) Let $b_j(t)\equiv 0$. If we substitute $t=0$ in (1.9_j) and taking into account the condition $a_j(t)\neq 0$, we find that $\tau_j(0)=0$. Then, (1.9_j) has the form

$$a_j(t)\int\limits_0^t\nu_j(z)J_0\left[\lambda_2(t-z)\right]dz=0,\qquad j=1,2.$$

From these integral equations, by virtue of $a_j(t) \neq 0$ it follows, that $\nu_j(t) \equiv 0$, $j = 1, 2$.

b) Let $a_j(t) \equiv 0$. Then, by the same way as in a), from (1.9_j) we have

$$b_j(t) \int_t^1 \nu_j(z) J_0 \left[\lambda_2(z-t)\right] dz = 0,$$

by virtue of $b_j(t) \neq 0$, we obtain $\nu_j(t) \equiv 0$, $j = 1, 2$.

c) Let now $a_j(t) \not\equiv 0$, $b_j(t) \not\equiv 0$, $a_j(t) \neq b_j(t)$. Taking into account $\tau_j(0) = 0$, $\tau_j(1) = 0$, and differentiate the equality (1.9_j) with respect to t, we obtain the Fredholm integral equation of the second kind

$$\nu_j(t) + \int_0^1 \nu_j(z) K_j(t, z, \lambda_2) dz = 0, \quad j = 1, 2.$$

By virtue of the condition (1.8) this equation has the unique solution $\nu_j(t) \equiv 0$ [19].

From above follows that the functions $\nu_j(t) \equiv 0$ ($j = 1, 2$) are uniquely determined in the case, when the functions $a_1(x)$, $b_1(x)$ satisfy one of the conditions a), b), c), and the functions $a_2(y)$, $b_2(y)$ satisfy the rest of these conditions (for example $a_1(x)$, $b_1(x)$ satisfy b) and $a_2(y)$, $b_2(y)$ satisfy a) or c)).

Consequently, at fulfilling each of the group conditions of Theorem 1.1.1 we have $\nu_j(t) \equiv 0$ $j = 1, 2$. On the other hand, by virtue of $u|_{\overline{\sigma}_o} = 0$ the equality (1.7) is valid. If we put $\delta = 0$ in (1.7), with regard to $\mathrm{Re}\lambda_1^2 \geq 0$ and $\nu_j(t) \equiv 0$, $j = 1, 2$, we obtain $\vartheta(x, y) \equiv 0$, in $\overline{\Omega}_0'$, $\omega(x, y) \equiv 0$ in $\overline{\Omega}_0''$, i.e. $u(x, y) \equiv 0$ in $\overline{\Omega}_0$. From here the statement of Theorem 1.1.1 follows.

Case 2. Let $\alpha_j(t) \neq 0$, $j = 1, 2$.

We consider the following cases a) $b_j(t) \equiv 0, j = 1, 2$, b) $a_j(t) \equiv 0, j = 1, 2$ and c) $a_j(t) \not\equiv 0$, $b_j(t) \not\equiv 0$, $j = 1, 2$.

a) Let $b_j(t) \equiv 0$, $j = 1, 2$.

Theorem 1.1.2. *If the conditions*

$$\alpha_j(t) \neq 0, \quad b_j(t) \equiv 0, \quad \frac{c_j(t)}{a_j(t)} > -\frac{1}{2}, \quad \left(\frac{c_j(t)}{a_j(t)}\right)' \geq 0, \quad t \in [0, 1], \tag{1.10}$$

$$\mathrm{Re}\lambda_1^2 \geq \left(\mathrm{Im}\lambda_2\right)^2, \tag{1.11}$$

are fulfilled, then the problem A_λ cannot have more than one solution.

To prove Theorem 1.1.2, we shall use

Lemma 1.1.2. *Let $(-\delta) \geq |\mathrm{Im}\lambda_2|$ and the condition (1.10) be fulfilled. Then the inequality*

$$\mathrm{Re}\int_0^1 e^{2\delta t}\overline{\tau}_j(t)\ \nu_j(t)dt \geq 0. \tag{1.12}$$

is valid for $d_j(t) \equiv 0$.

Lemma 1.1.2 can be proved by a similar method as it was used in [36].

Proof of Theorem 1.1.2. Let $u(x,y)$ be a solution of the homogeneous problem A_λ. Then the inequality (1.12) is true for $\delta = -|\mathrm{Im}\lambda_2|$ by (1.10) according to Lemma 1.1.2. On the other hand the equality (1.7) is valid because of $u|_{\overline{\sigma}_o} = 0$ according to Lemma 1.1.1. If we set $\delta = -|\mathrm{Im}\lambda_2|$ in (1.7) and taking into account the inequality (1.12) we have

$$\left(\mathrm{Re}\lambda_1^2 - (\mathrm{Im}\lambda_2)^2\right)\left[\iint\limits_{\Omega_0'} |\vartheta|^2 dxdy + \iint\limits_{\Omega_0''} |\omega|^2 dxdy\right] +$$

$$+\iint\limits_{\Omega_0'} |\nabla\vartheta|^2 dxdy + \iint\limits_{\Omega_0''} |\nabla\omega|^2 dxdy + \sum_{j=1}^{2}\mathrm{Re}\int_0^1 e^{2\delta t}\overline{\tau}_j(t)\nu_j(t)dt = 0.$$

From here and by virtue of the condition (1.11) we obtain that $\vartheta(x,y) \equiv 0$ in $\overline{\Omega}_0'$, $\omega(x,y) \equiv 0$ in $\overline{\Omega}_0''$, i. e. $u(x,y) \equiv 0$ in $\overline{\Omega}_0$, from which the statement of Theorem 1.1.2 follows.

b) Let $a_j(t) \equiv 0$, $j = 1, 2$.

Theorem 1.1.3. *If condition (1.11) and*

$$\alpha_j(t) \neq 0, \quad a_j(t) \equiv 0, \quad \frac{c_j(t)}{b_j(t)} > -\frac{1}{2}, \quad \left(\frac{c_j(t)}{b_j(t)}\right)' \leq 0, \quad t \in [0,1], \tag{1.13}$$

are fulfilled, then the problem A_λ can not have more than one solution.

The proposition of this theorem follows from Lemma 1.1.1 and from the following lemma.

Lemma 1.1.3. [36] *Let $\delta \geq |\mathrm{Im}\lambda_2|$ and the conditions (1.13) be fulfilled. Then (1.12) is true for $d_j(t) \equiv 0$.*

The proof of Theorem 1.1.3 is similar to the proof of Theorem 1.1.2.

c) Now let $a_j(t) \not\equiv 0$, $b_j(t) \not\equiv 0$.

Lemma 1.1.4. *Let $\tau_j(0) = 0$, $\tau_j(1) = 0$, $|\delta| \geq |\mathrm{Im}\lambda_2|$;*

$$\alpha_j(t) \neq 0, \quad a_j(t) \not\equiv 0, \quad b_j(t) \not\equiv 0; \tag{1.14}$$

$$a_{j1}(1) \geq 0, \quad \delta a_{j1}(t) \leq 0, \quad a'_{j1}(t) \leq 0, \quad t \in [0,1]; \tag{1.15}$$

$$b_j(0) = 0, \quad \delta b_{j1}(t) \geq 0, \quad b'_{j1}(t) \geq 0, \quad t \in [0,1], \tag{1.16}$$

where $a_{j1}(t) = a_j(t)/\alpha_j(t)$, $b_{j1}(t) = b_j(t)/\alpha_j(t)$, $j = 1,2$. *Then the inequality (1.12) is valid for* $d_j(t) \equiv 0$, $j = 1,2$.

The proof of this lemma is similar to the proof of Lemmas 1.1.2 and 1.1.3.

Remark 1.1. The conditions (1.14)-(1.16) and $|\delta| \geq |\mathrm{Im}\lambda_2|$ are fulfilled simultaneously only at $\delta = \mathrm{Im}\lambda_2 = 0$.

Indeed, if $\delta > 0$, then by the second condition from (1.15) the inequality $a_{j1}(t) \leq 0$ is valid. Hence, by virtue of $a_{j1}(1) \geq 0$ follows that $a_{j1}(1) = 0$. Thus, $a_{j1}(1) = 0$, $a_{j1}(t) \leq 0$, $a'_{j1}(t) \leq 0$, $t \in [0,1]$. It is easy to see that the function possessing these conditions is identically equal to zero, i.e. $a_{j1}(t) \equiv 0$, which is impossible.

If $\delta < 0$, then by conditions (1.16) the relation $b_{j1}(t) \leq 0$ is valid. From here, taking into account $b'_{j1}(t) \geq 0$, $\forall t \in [0,1]$, we have $b_{j1}(t) \equiv 0$, which contradicts the condition $b_{j1}(t) \not\equiv 0$. Consequently, $\delta = 0$, but then $\mathrm{Im}\lambda_2 = 0$.

Thus, it is necessary to set $\delta = 0$ in order to obtain the inequality (1.12). Then the following lemma is valid

Lemma $1.1.4_1$. *Let* $\tau_j(0) = 0$, $\tau_j(1) = 0$, $\mathrm{Im}\lambda_2 = 0$ *and the conditions*

$$\left(\frac{a_j(t)}{\alpha_j(t)}\right)' \leq 0, \qquad \left(\frac{b_j(t)}{\alpha_j(t)}\right)' \geq 0, \quad t \in [0,1]; \qquad \frac{a_j(1)}{\alpha_j(1)} + \frac{b_j(0)}{\alpha_j(0)} \geq 0 \tag{1.17}$$

are fulfilled. Then the inequality

$$\mathrm{Re} \int_0^1 \bar{\tau}_j(t)\nu_j(t)dt \geq 0 \tag{1.18}$$

is valid for $d_j(t) \equiv 0$, $j = 1,2$.

From Lemmas 1.1.1 and $1.1.4_1$ follows the following result.

Theorem 1.1.4. *If conditions* (1.14), (1.17) *and* $\mathrm{Im}\lambda_2 = 0$, $\mathrm{Re}\lambda_1^2 \geq 0$ *are fulfilled, then the problem* A_λ *cannot have more than one solution.*

The proposition of this theorem generalizing the results of [44], which has been obtained for $\lambda_1 = \lambda_2 = \lambda \in R$.

Case 3. Now let $\alpha_j(t) \equiv 0$, $\alpha_k(t) \neq 0$, $j \neq k$, $j, k = 1, 2$.

Let $\alpha_1(x) \equiv 0$, $\alpha_2(y) \neq 0$ and the functions $a_1(x)$, $b_1(x)$, $c_1(x)$ satisfy one of the conditions a), b), c) in the case 1 for $j = 1$, and the functions $a_2(y)$, $b_2(y)$, $c_2(y)$ satisfy one of the conditions a), b), c) in the case 2 for $j = 2$. Then from Theorems 1.1.1, 1.1.2, 1.1.3 and 1.1.4, we have $\nu_1(x) = 0$ and $\mathrm{Re}\int\limits_0^1 e^{2\delta y}\overline{\tau}_2(y)\nu_2(y)dy \geq 0$ for the homogeneous problem A_λ. Consequently, in this case when fulfilling the condition (1.11) from (1.7) follows $u(x, y) \equiv 0$, $(x, y) \in \overline{\Omega}_0$. From here follows uniqueness of the solution of the problem A_λ for $\alpha_1(x) \equiv 0$, $\alpha_2(y) \neq 0$.

Remark 1.2. The uniqueness of the solution of the Tricomi problem follows from Theorem 1.1.2 (1.1.3) for $c_j(t) \equiv 0$, $(j = 1, 2)$ for the equation $(L.B)$. When $c_j(t) \equiv 0$, $(j = 1, 2)$, $\lambda_1 = \lambda_2 = \lambda$ the theorem about uniqueness of the solution of the Tricomi problem for the equation $(L.B)$ was obtained in [40] by using the Laplace transformation.

Remark 1.3. For $\lambda_1 = \lambda_2 = \lambda$ condition (1.11) is equivalent to the inequality

$$|\mathrm{Re}\lambda| \geq \sqrt{2}\,|\mathrm{Im}\lambda|\,. \tag{1.19}$$

From this remark and Theorems 1.1.2 and 1.1.3 follows the following result.

Proposition. If $\lambda_1 = \lambda_2 = \lambda$ and one of the conditions (1.10) or (1.13) is fulfilled, then the problem A_λ can have eigenvalues only outside the domain $D_1 = \left\{\lambda : \ |\mathrm{Re}\lambda| \geq \sqrt{2}\,|\mathrm{Im}\lambda|\right\}$.

1.1.3. Existence of the solution of the problem A_λ

We consider five cases for the investigation of the existence of the solution of the problem A_λ in correspondence to Theorems 1.1.1 - 1.1.4 and supposing that

$$\varphi(x, y) = (xy)^\varepsilon \varphi^*(x, y), \qquad \varphi^*(x, y) \in C(\overline{\sigma}_0), \quad \varepsilon > 1. \tag{1.20}$$

1. Let the conditions of Theorem 1.1.1. be fulfill. Then the relation (1.6_j) becomes to form (1.9_j).

a) Let $b_j(t) \equiv 0,\ j = 1, 2$. We require in addition that $d_j(0) = 0$. Then by virtue of $a_j(t) \neq 0$ we have the integral equation

$$\int\limits_0^t \nu_j(z) J_0\left[\lambda_2(t-z)\right] dz = -2\frac{d_j(t)}{a_j(t)}.$$

From here we find

$$\nu_j(t) = -2C_{0t}^{0,\lambda_2}\left[\frac{d_j(t)}{a_j(t)}\right], \tag{1.21}$$

where [36]

$$C_{mx}^{0,\lambda}\left[f(x)\right] = sign(x-m)\left\{\frac{d}{dx}f(x) + \frac{1}{2}\lambda^2\int\limits_m^x f(t)\overline{J}_1\left[\lambda(x-t)\right] dt\right\}. \tag{1.22}$$

b) Let $a_j(t) \equiv 0,\ j = 1, 2$. If we require in addition $d_j(1) = 0$ and taking into account $b_j(t) \neq 0$, we obtain the integral equation

$$\int\limits_t^1 \nu_j(z) J_0\left[\lambda_2(z-t)\right] dz = -2\frac{d_j(t)}{b_j(t)}.$$

Then, we have

$$\nu_j(t) = -2C_{1t}^{0,\lambda_2}\left[\frac{d_j(t)}{b_j(t)}\right]. \tag{1.23}$$

c) Let conditions of c) of Theorem 1.1.1. be fulfilled. Then if we proceed similarly as in the proof of Theorem 1.1.1, we obtain the Fredholm integral equation of the second kind

$$\nu_j(t) + \int\limits_0^1 K_j(t, z, \lambda_2)\nu_j(z)dz = g_j(t),$$

where $g_j(t) = \psi_j(t)/[b_j(t) - a_j(t)]$. By virtue of the condition (1.8) the last equation has a unique solution [19]. It can be represented as

$$\nu_j(t) = g_j(t) - \int\limits_0^1 R_j(t, z, \lambda_2)g_j(z)dz, \tag{1.24}$$

where $R_j(t,z,\lambda_2)$ is the resolvent of the kernel $K_j(t,z,\lambda_2)$. Consequently, if the conditions of Theorem 1.1.1 are fulfilled, then unknown functions $\nu_j(x)$ $j=1,2$, are uniquely found by the corresponding additional conditions from the equation (1.9_j) and it has determined one of the formulas (1.21), (1.23), (1.24). Therefore in this case the problem A_λ is equivalent to the problem N for finding a regular solution of the equation $(L.B)$ in the domain Ω_0, satisfying boundary condition (1.1), and $u_y(x,0)=\nu_1(x)$, $u_x(0,y)=\nu_2(y)$, $0<x,y<1$, where $\nu_j(t)\in C^1(0,1)$ is a known function.

It is easy to prove, that the solution of this problem exists, is unique and can be represented as

$$u(x,y)=u_0(x,y)+\iint\limits_{\Omega_0} R(\xi,\eta;x,y)u_0(\xi,\eta)d\xi d\eta, \tag{1.25}$$

where [33]

$$u_0(x,y)=\int\limits_{\sigma_o}\varphi(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,y)ds$$

$$-\int\limits_0^1\nu_1(t)G(t,0;x,y)dt-\int\limits_0^1\nu_2(t)G(0,t;x,y)dt,$$

$G(\zeta,z)=\dfrac{1}{2\pi}\left(\ln\left|\dfrac{1-\zeta^2z^2}{\zeta^2-z^2}\right|-\ln\left|\dfrac{1-\zeta^2\bar z^2}{\zeta^2-\bar z^2}\right|\right)$ is the Green function of the problem N for the Laplace equation in Ω_0; $R(\xi,\eta;x,y)$ is the resolvent of the kernel $(-\lambda_1^2)G(\zeta,z)$; n is the inner normal to σ_0, and s is the length of the arc counted from the point A in the positive direction; $\zeta=\xi+i\eta$, $z=x+iy$.

The solution of the problem A_λ is determined in the domains Ω_1 and Ω_2 by the formulas (1.5_j), moreover, here $\tau_1(x)=u(x,0)$, $\tau_2(y)=u(0,y)$ is determined from (1.25).

2. Let the conditions of Theorem 1.1.4 be fulfilled. We assume in addition that $a_j(t)\neq b_j(t)$, $\forall t\in[0,1]$, $j=1,2$. Considering the problem N for the equation $(L.B)$ in the domain Ω_0, we obtain formula (1.25), from which we get the basic relations between $\tau_j(t)$ and $\nu_j(t)$, $j=1,2$, got from the elliptic part of the mixed domain Ω_0, setting first $y=0$ and then $x=0$:

$$\tau_j(t)+\frac{1}{\pi}\int\limits_0^1\nu_j(z)\ln\left|\frac{1-z^2t^2}{z^2-t^2}\right|dz+\frac{1}{\pi}\int\limits_0^1\nu_k(z)\ln\left|\frac{1+z^2t^2}{z^2+t^2}\right|dz$$

$$+\int_0^1 \nu_1(z)H_{j1}(t,z)dz+\int_0^1 \nu_2(z)H_{j2}(t,z)dz=f_j(t), \tag{1.26}$$

where $0\le t\le 1,\ j,k=1,2,\ j\ne k,$

$$\left.\begin{aligned}
H_{11}(x,z)&=\iint_{\Omega_0} R(\xi,\eta;x,0)G(z,0;\xi,\eta)d\xi d\eta;\\
H_{12}(x,z)&=\iint_{\Omega_0} R(\xi,\eta;x,0)G(0,z;\xi,\eta)d\xi d\eta;\\
H_{21}(y,z)&=\iint_{\Omega_0} R(\xi,\eta;0,y)G(z,0;\xi,\eta)d\xi d\eta;\\
H_{22}(y,z)&=\iint_{\Omega_0} R(\xi,\eta;0,y)G(0,z;\xi,\eta)d\xi d\eta;\\
f_1(x)&=\int_{\sigma_o}\varphi(\xi,\eta)\left[\frac{\partial}{\partial n}G(\xi,\eta;x,0)+\iint_{\Omega_0} R(\xi_1,\eta_1;x,0)\frac{\partial}{\partial n}G(\xi,\eta;\xi_1,\eta_1)d\xi_1 d\eta_1\right]ds;\\
f_2(y)&=\int_{\sigma_o}\varphi(\xi,\eta)\left[\frac{\partial}{\partial n}G(\xi,\eta;0,y)+\iint_{\Omega_0} R(\xi_1,\eta_1;0,y)\frac{\partial}{\partial n}G(\xi,\eta;\xi_1,\eta_1)d\xi_1 d\eta_1\right]ds.
\end{aligned}\right\} \tag{1.27}$$

Elimination $\tau_j(t)$ from (1.6_j) and (1.26), and differentiating them with respect to t, we obtain

$$A_j(t)\nu_j(t)+\frac{2t}{\pi}\int_0^1 \nu_j(z)\left(\frac{1}{z^2-t^2}-\frac{z^2}{1-z^2t^2}\right)dz$$

$$-\frac{2t}{\pi}\int_0^1 \nu_k(z)\left(\frac{1}{z^2+t^2}-\frac{z^2}{1+z^2t^2}\right)dz$$

$$+\int_0^1 \nu_1(z)M_{j1}(t,z)dz+\int_0^1 \nu_2(z)M_{j2}(t,z)dz=f'_{1j}(t), \tag{1.28}$$

where

$$\left.\begin{aligned}
M_{jj}(t,z)&=\begin{cases}\dfrac{\partial}{\partial t}\left[H_{jj}(t,z)+\dfrac{a_j(t)}{\alpha_j(t)}J_0[\lambda_2(t-z)]\right] & z\le t,\\[2ex] \dfrac{\partial}{\partial t}\left[H_{jj}(t,z)+\dfrac{b_j(t)}{\alpha_j(t)}J_0[\lambda_2(z-t)]\right] & z\ge t,\end{cases}\\
M_{jk}(t,z)&=\frac{\partial}{\partial t}H_{jk}(t,z), \qquad j,k=1,2,\quad j\ne k,
\end{aligned}\right\} \tag{1.29}$$

$$A_j(t)=\left\{a_j(t)-b_j(t)\right\}/\alpha_j(t),$$

$$f_{1j}(t) = f_j(t) + \{\tau_j(0)a_j(t)J_0[\lambda_2 t] + \tau_j(1)b_j(t)J_0[\lambda_2(1-t)] - 2d_j(t)\}/\alpha_j(t)$$

For simplicity we additionally assume, that $A_1(t) = A_2(t) = A(t)$.

If we introduce the notations

$$\nu_1(t) + \nu_2(t) = \mu_1(t), \qquad \nu_1(t) - \nu_2(t) = \mu_2(t), \tag{1.30}$$

we can rewrite the system (1.28) as

$$\begin{cases} A(t)\mu_1(t) + \dfrac{4t^3}{\pi}\displaystyle\int_0^1 \left(\frac{1}{z^4 - t^4} - \frac{z^4}{1 - z^4 t^4}\right)\mu_1(z)dz = F_1(t), \\ A(t)\mu_2(t) + \dfrac{4t}{\pi}\displaystyle\int_0^1 \left(\frac{z^2}{z^4 - t^4} - \frac{z^2}{1 - z^4 t^4}\right)\mu_2(z)dz = F_2(t), \end{cases} \tag{1.31}$$

where

$$F_1(t) = f_{11}(t) + f_{12}(t)$$
$$-\int_0^1 \nu_1(z)\left[M_{11}(t,z) + M_{21}(t,z)\right]dz - \int_0^1 \nu_2(z)\left[M_{12}(t,z) + M_{22}(t,z)\right]dz,$$

$$F_2(t) = f_{11}(t) - f_{12}(t)$$
$$-\int_0^1 \nu_1(z)\left[M_{11}(t,z) - M_{21}(t,z)\right]dz - \int_0^1 \nu_2(z)\left[M_{12}(t,z) - M_{22}(t,z)\right]dz.$$

Taking the identities

$$\tau = 2z^4(1+z^8)^{-1}, \qquad y = 2t^4(1+t^8)^{-1} \tag{1.32}$$

or

$$z = \tau^{1/4}\left(1 + \sqrt{1-\tau^2}\right)^{-1/4}, \qquad t = y^{1/4}\left(1 + \sqrt{1-y^2}\right)^{-1/4}$$

we can rewrite the system of equation (1.31) as

$$A(t)\rho_j(y) + \frac{1}{\pi}\int_0^1 \frac{\rho_j(\tau)}{\tau - y}d\tau = \Phi_j(y), \qquad j = 1, 2, \tag{1.33}$$

where

$$\rho_1(y) = t^{-3}(1+t^8)\mu_1(t), \quad \rho_2(y) = t^{-1}(1+t^4)^{-1}(1+t^8)\mu_2(t) \tag{1.34}$$

$$\Phi_1(y) = t^{-3}(1+t^8)F_1(t), \quad \Phi_2(y) = t^{-1}(1+t^4)^{-1}(1+t^8)F_2(t). \tag{1.35}$$

Taking (1.27), (1.29), (1.32), (1.35) and properties of the given functions into account, it is not difficult to verify, that $\Phi_j(y) \in C^{(1,r)}(0 < y < 1)$, $0 < r < 1$, $(j = 1, 2)$ and it can have singularities of order less than 1 and $1/2$ at $j = 1$, less $1/2$ and $1/2$ at $j = 2$, when $y \to 0$ and $y \to 1$, respectively.

By virtue of $A^2(t) + 1 \neq 0$, $\forall t \in [0, 1]$ (1.33) is a singular integral equation of normal kind.

From the formulation of the problem A_λ and the equalities (1.30), (1.34), by virtue of (1.32) follows, that the solution of the system of equations (1.33) is to be found in the class of functions, which are differentiable for $0 < y < 1$ and can be infinity of order less than 1 and of the order $1/2$ for $j = 1$, as $y \to 0$, when $A(t) > 0$, less than $1/2$ and $1/2$ for $j = 2$, as $y \to 1$, when $A(t) < 0$ (as in this case the index of the equation is equal to zero). The solution of the system of equations (1.33) exists in this class in both cases and it is given by the formula [23]

$$\rho_j(y) = \frac{1}{1 + A^2(t)} \left[A(t)\Phi_j(y) - \frac{z(y)}{\pi} \int_0^1 \frac{\Phi_j(\eta)}{z(\eta)(\eta - y)} d\eta \right], \quad j = 1, 2,$$

where

$$z(y) = \sqrt{1 + A^2(t)} \exp \left[-\frac{1}{\pi} \int_0^1 \frac{arctg 1/A(\eta)}{\eta - y} d\eta \right].$$

If we return to the variables t, z and the function $\nu_j(t)$, we obtain a system of Fredholm integral equations of the second kind with continuous kernels, moreover the right part of this system belongs to the class $C^{(1,r)}(0, 1)$ and it can have singularities of the order less than 1 as $t \to 0$ and $t \to 1$. Unconditional solvability of this system follows (absolutely) from the uniqueness of the solution of the problem A_λ.

3 (4). Let the conditions of Theorem 1.1.2 (1.1.3) be fulfilled. We require additionally that the condition $d_j(0) \equiv 0$ ($d_j(1) \equiv 0$) is fulfilled. Then analogously to the previous point, we obtain a singular integral equation in the form (1.33). Solving the obtained equation we determine the function $\nu_j(t)$, moreover it can have singularities of the order less than 1 as $t \to 0$ and $t \to 1$, as by virtue of the conditions (1.10) ((1.13)) $A(t) = a_j(t)/\alpha_j(t) > 0$ ($A(t) = -b_j(t)/\alpha_j(t) < 0$).

5. Let $\alpha_1(x) \equiv 0$**,** $\alpha_2(y) \neq 0$**.** The functions $a_1(x)$, $b_1(x)$, $c_1(x)$ satisfy one of the group conditions of Theorem 1.1.1 at $j=1$, and the functions $a_2(y)$, $b_2(y)$, $c_2(y)$ satisfy the conditions of the Theorems 1.1.2 or 1.1.3 at $j=2$. Then, as in the case 3 (4), for determination of the function $\nu_2(y)$ we have the singular integral equation

$$A(y)\nu_2(y) + \frac{2y}{\pi}\int_0^1 \nu_2(z)\left(\frac{1}{z^2-y^2} - \frac{z^2}{1-z^2y^2}\right)dz + \int_0^1 \nu_2(z)M_{22}(y,z)dz = g_2(y), \tag{1.36}$$

where

$$g_2(y) = f'_{12}(y) + \frac{2y}{\pi}\int_0^1 \nu_1(z)\left(\frac{1}{z^2+y^2} - \frac{z^2}{1+z^2y^2}\right)dz - \int_0^1 \nu_1(z)M_{21}(y,z)dz,$$

and the function $\nu_1(x)$ is determined by one of the formulas (1.21), (1.23), (1.24). After changing variables $\tau = 2z^2(1+z^4)^{-1}$, $y = 2t^2(1+t^4)^{-1}$ the equation (1.36) becomes the form (1.33) and it is investigated as the last one.

Consequently, the functions $\nu_j(t)$ $(j=1,2)$ are uniquely determined in all cases. If we know $\nu_j(t)$, we determine $\tau_j(t)$ by formula (1.26). After determination of $\tau_j(t)$ and $\nu_j(t)$, the solution of the problem A_λ is defined as the solution of the problem N and the Cauchy problem for the equation $(L.B)$ in the domains Ω_0 and Ω_1, Ω_2 and it is given by the formulas (1.25) and (1.5_j), respectively.

Remark 1.4. Existence of the solution of the problem A_λ can be proved by the method which was proposed in paragraph 1.2.

With this the proof of the existence and uniqueness of the solution of the problem A_λ is completed.

§ 1.2. Non-local problem with third boundary condition on the boundary of the elliptic part of the domain

In this paragraph for convenience we introduce the following notation $-\lambda_1^2 = \mu^2$, $\mu = \lambda_2$. Then equation $(L.B)$ becomes the form

$$Ku \equiv \mathrm{sign}x\, u_{xx} + \mathrm{sign}y\, u_{yy} + \mu^2 u = 0, \tag{$L.B_1$}$$

and for this equation we consider the most general mixed problem in a finite simply connected mixed domain Ω, which was described in paragraph 1.

Uniqueness of the solution is proved, the condition to the complex parameter μ ensuring uniqueness of the solution to the problem under consideration is found. Further, eigenvalues and corresponding to them eigenfunctions are found, the question about completeness of the system of eigenfunctions in the elliptic, hyperbolic and in the whole of the mixed domain are studied. A new method for proving the existence of the solution of the considering problem is proposed, i.e. in the case, when a theorem of uniqueness of the solution of the problem is valid, the structure of the solutions in evident form of the problem is shows by using a system of eigenfunctions.

1.2.1. Formulation of the problem A^0_μ

Problem A^0_μ. Find a regular solution $u(x,y) \in C^1(\Omega \cup \sigma_0)$ of the equation $(L.B_1)$, in the domain Ω satisfying the conditions

$$\alpha u(x,y) + \beta \frac{\partial u(x,y)}{\partial n} = \psi(x,y), \quad (x,y) \in \sigma_0, \tag{1.37}$$

$$A^{0,\mu}_{0x}\left[u(\theta_{x0})\right] + \gamma_1 u(x,0) = 0, \quad (x,0) \in \overline{OA}, \tag{1.38$_1$}$$

$$A^{0,\mu}_{0y}\left[u(\theta_{0y})\right] + \gamma_2 u(0,y) = 0, \quad (0,y) \in \overline{OB}, \tag{1.38$_2$}$$

where α, β, γ_1, γ_2 are given real numbers, moreover $\alpha^2 + \beta^2 \neq 0$; n is the outer normal to σ_0 and $\psi(x,y)$ is a given, in general, complex-valued function.

Note that in [27], [31], eigenvalues and eigenfunctions of the Tricomi problem are found by a different method and it is investigated with respect to completeness. In [31] applying a system of eigenfunctions for the structure of the solution of the Tricomi problem is show, for those values of the parameter μ, when uniqueness of the theorem holds.

1.2.2. Uniqueness of the solution of the problem A^0_μ

Let $u(x,y)$ be a solution of the problem A^0_μ and $\tau_1(x) = u(x,0)$, $\nu_1(x) = u_y(x,0)$, $\tau_2(y) = u(0,y)$, $\nu_2(y) = u_x(0,y)$. Then, using the formulas (1.5_1), (1.5_2), which gives the solutions of the Cauchy problem for the equation $(L.B_1)$ in the domains Ω_1, Ω_2, and conditions (1.38_1), (1.38_2), respectively, we obtain

$$(1 + 2\gamma_j)\tau_j(t) = \int\limits_0^t \nu_j(z) J_0[\mu(t-z)]dz, \quad 0 \leq t \leq 1. \tag{1.39$_j$}$$

Equalities (1.39_j), $(j = 1,2)$, are basic functional relations between $\tau_j(t)$ and $\nu_j(t)$ on the segments OA and OB got from hyperbolic parts of the mixed domain Ω.

So, we reduce the problem A^0_μ to the following equivalent elliptic problem in Ω_0 :

Problem C^0_μ. Find values of the complex parameter μ and corresponding to them nontrivial functions $u(x,y) \in C(\overline{\Omega}_o) \cap C^1(\overline{\Omega}_0) \cap C^2(\Omega_0)$ in $\overline{\Omega}_0$, satisfying the equation $(L.B_1)$ and the conditions (1.37), (1.39_1), (1.39_2).

Theorem 1.2.1. *If the conditions* $\alpha\cdot\beta\geq 0$, $\alpha^2+\beta^2\neq 0$, $\gamma_j\geq -1/2$ $(j=1,2)$ *and* $\mathrm{Re}\mu=0$ *are fulfilled, then the homogeneous problem* C^0_μ (A^0_μ) *has only the trivial solution.*

Proposition of the Theorem 1.2.1 follows from the following lemmas.

Lemma 1.2.1. *Let* $u(x,y)$ *be a solution of the homogeneous problem* C^0_μ. *Then the equality*

$$(-\mathrm{Re}\mu^2-\delta^2)\left[\iint\limits_{\Omega_0'}|\vartheta|^2dxdy+\iint\limits_{\Omega_0''}|\omega|^2dxdy\right]+\iint\limits_{\Omega_0'}|\nabla\vartheta|^2dxdy+\iint\limits_{\Omega_0''}|\nabla\omega|^2dxdy+$$

$$+\int\limits_{\sigma_0}w(x,y)ds+\mathrm{Re}\int\limits_0^1 e^{2\delta x}\tau_1(x)\nu_1(x)dx+\mathrm{Re}\int\limits_0^1 e^{2\delta y}\tau_2(y)\nu_2(y)dy=0$$

holds, where $w(x,y)\equiv 0$, *if* $\alpha\beta=0$ *and* $w(x,y)=(\alpha/\beta)\left(|\vartheta|^2+|\omega|^2\right)$, *if* $\alpha\cdot\beta\neq 0$; $\vartheta(x,y)=e^{\delta x}u(x,y)$, $(x,y)\in\Omega_0'=\Omega_0\cap(x\geq y)$; $\omega(x,y)=e^{\delta y}u(x,y)$, $(x,y)\in\Omega_0''=\Omega_0\cap(x\leq y)$; $\bar\tau_1(x)=\overline{u}(x,0)$, $\nu_1(x)=u_y(x,0)$, $\bar\tau_2(y)=\overline{u}(0,y)$, $\nu_2(y)=u_x(0,y)$.

Lemma 1.2.2. *Let* $(-\delta)\geq|\mathrm{Im}\mu|$ *and the conditions* $\gamma_j\geq -1/2$, $j=1,2$ *are fulfilled. Then the inequalities (1.12) are valid.*

From Theorem 1.2.1 we have the following corollary: if the conditions of the Theorem 1.2.1 are fulfilled, then the problem A^0_μ can have eigenvalues only outside of $\mathrm{Re}\mu=0$.

1.2.3. Determination of the system of eigenfunctions and investigation on completeness

Consider the homogeneous problem C^0_μ (A^0_μ). Then we go over to polar coordinates $r=\sqrt{x^2+y^2}$, $\varphi=arctg\dfrac{y}{x}$, $(0\leq r\leq 1,\ 0\leq\varphi\leq\dfrac{\pi}{2})$ and find the solution of the problem in the form $u(x,y)=R(r)\Phi(\varphi)$. It is not difficult to prove that the eigenvalues of the problem C^0_μ (A^0_μ) are $\mu_{n,m}=\alpha_m^{(\nu_n)}$ $(m,n=1,2,...)$, where $\alpha_m^{(\nu_n)}$ is the m-th root of the equation $\alpha J_{\nu_n}(\mu)+\beta\mu J'_{\nu_n}(\mu)=0$ (for $\beta=0$ or $\beta\neq 0$, $\alpha/\beta+\nu_n\geq 0$, $\nu_n>0$, $n\in N$, this equation has only real roots [46]), where

$$\nu_n=\begin{cases}2n-1, & \text{if}\quad \gamma_1+\gamma_2+2\gamma_1\gamma_2=0,\\ 2n-\dfrac{2}{\pi}arctg\gamma, & \text{if}\quad \gamma_1+\gamma_2+2\gamma_1\gamma_2\neq 0,\quad \gamma\geq 0,\\ 2(n-1)-\dfrac{2}{\pi}arctg\gamma, & \text{if}\quad \gamma_1+\gamma_2+2\gamma_1\gamma_2\neq 0,\quad \gamma<0,\end{cases}$$

$\gamma = (1+\gamma_1+\gamma_2)/(\gamma_1+\gamma_2+2\gamma_1\gamma_2)$, $n \in N$, and corresponding to them the eigenfunctions in Ω_0 are determined by the equality

$$u_{n,m}(x,y) = c_{n,m} J_{\nu_n}\left(\alpha_m^{(\nu_n)} r\right) sin(\nu_n \varphi + \varphi_0), \quad n,m = 1,2,..., \tag{1.40}$$

where $\varphi_0 = arcctg(1+2\gamma_1)$, and $c_{n,m} \neq 0$ are constants.

To find out eigenfunctions of the problem A_μ^0 in the domain Ω_1, we find from (1.40)

$$u_{n,m}(x,0) = k_{n,m}^{(1)} J_{\nu_n}\left[\alpha_m^{(\nu_n)} x\right], \tag{1.41}$$

$$\lim_{y\to 0} \frac{\partial}{\partial y} u_{n,m}(x,y) = k_{n,m}^{(1)}(1+2\gamma_1)\nu_n x^{-1} J_{\nu_n}\left[\alpha_m^{(\nu_n)} x\right], \tag{1.42}$$

where $k_{n,m}^{(1)} = c_{n,m}/\sqrt{2+4\gamma_1+4\gamma_1^2}$.

It is known [32], that the family of solutions of the equation $(L.B_1)$ in the domain Ω_1 has the form

$$u(x,y) = \left[\chi_1\left(\frac{x-y}{x+y}\right)^{\rho/2} + \chi_2\left(\frac{x+y}{x-y}\right)^{\rho/2}\right] J_\rho\left[\mu\sqrt{x^2-y^2}\right], \tag{1.43}$$

where $\mathrm{Re}\rho \geq 0$, χ_1 and χ_2 are arbitrary constants.

If we require for the function (1.43) to fulfill the conditions (1.41) and (1.42) we find that eigenfunctions of the problem A_μ^0 in the domain Ω_1 are determined by the formulas

$$u_{n,m}(x,y) = k_{n,m}^{(1)}\left[(1+\gamma_1)\left(\frac{x+y}{x-y}\right)^{\nu_n/2} - \gamma_1\left(\frac{x-y}{x+y}\right)^{\nu_n/2}\right] J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2-y^2}\right], \tag{1.44}$$

where $n,m = 1,2,....$

To obtain the general solution of the equation $(L.B_1)$ in Ω_2, we change x to y, y to x by virtue of the symmetry of the coefficients of the equation $(L.B_1)$:

$$u(x,y) = \left[\chi_3\left(\frac{y-x}{y+x}\right)^{\rho/2} + \chi_4\left(\frac{y+x}{y-x}\right)^{\rho/2}\right] J_\rho\left[\mu\sqrt{y^2-x^2}\right],$$

where $\mathrm{Re}\rho \geq 0$, χ_3 and χ_4 are arbitrary constants.

We obtain from (1.40)

$$u_{n,m}(0,y) = (-1)^n k_{n,m}^{(2)} J_{\nu_n}\left[\alpha_m^{(\nu_n)} y\right], \tag{1.45}$$

$$\lim_{x\to 0}\frac{\partial}{\partial x}u_{n,m}(x,y) = (-1)^n k_{n,m}^{(2)}(1+2\gamma_2)\nu_n y^{-1} J_{\nu_n}\left[\alpha_m^{(\nu_n)} y\right], \tag{1.46}$$

where $k_{n,m}^{(2)} = c_{n,m}/\sqrt{2+4\gamma_2+4\gamma_2^2}$.

Here we require the conditions (1.45) and (1.46) to be satisfied. In the domain Ω_2 we obtain eigenfunctions of the problem A_μ^0 determined by:

$$u_{n,m}(x,y) = (-1)^n k_{n,m}^{(2)}\left[(1+\gamma_2)\left(\frac{y+x}{y-x}\right)^{\nu_n/2} - \gamma_2\left(\frac{y-x}{y+x}\right)^{\nu_n/2}\right] J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{y^2-x^2}\right] \tag{1.47}$$

where $n,m = 1,2,\ldots.$

It is not difficult to prove that the functions (1.44) ((1.47)) satisfy the conditions (1.38_1) $((1.38_2))$.

Indeed, we rewrite the functions (1.44) in the form

$$u_{n,m}(x,y) = \frac{k_{n,m}^{(1)}}{\Gamma(\nu_n+1)}\left(\frac{\alpha_m^{(\nu_n)}}{2}\right)^{\nu_n}$$

$$\times\left[(1+\gamma_1)(x+y)^{\nu_n} - \gamma_1(x-y)^{\nu_n}\right]\overline{J}_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2-y^2}\right],$$

where $\overline{J}_\alpha(z) = \Gamma(\alpha+1)(z/2)^{-\alpha}J_\alpha(z)$.

We found $u_{n,m}(\theta_{x0})$ and substituting this in (1.38_1), we obtain

$$x^{\nu_n} - \int_0^x t^{\nu_n}\frac{\partial}{\partial t}J_0\left[\alpha_m^{(\nu_n)}\sqrt{x(x-t)}\right]dt = x^{\nu_n}\overline{J}_\alpha(\alpha_m^{(\nu_n)}x). \tag{1.48}$$

By the help of the expansion of the Bessel function into a power series, it is not difficult to verify the identity

$$\int_0^x t^\alpha\frac{\partial}{\partial t}J_0\left[\mu\sqrt{x(x-t)}\right]dt = x^\alpha\left[1-\overline{J}_\alpha(\mu x)\right],\quad 0\le\alpha\in R.$$

Taking this into account, then at once the correctness of equality (1.48) follows.

Composing the formulas (1.40), (1.44) and (1.47), we obtain a system of eigenfunctions of the problem A_μ^0 in the mixed domain Ω

$$u_{n,m}(x,y)=\begin{cases} c_{n,m}J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2+y^2}\right]\ sin(\nu_n\varphi+\varphi_0), & (x,y)\in\Omega_0,\\ k_{n,m}^{(1)}\left[(1+\gamma_1)\left(\dfrac{x+y}{x-y}\right)^{\nu_n/2}-\gamma_1\left(\dfrac{x-y}{x+y}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2-y^2}\right], & (x,y)\in\Omega_1,\\ (-1)^n k_{n,m}^{(2)}\left[(1+\gamma_2)\left(\dfrac{y+x}{y-x}\right)^{\nu_n/2}-\gamma_2\left(\dfrac{y-x}{y+x}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{y^2-x^2}\right], & (x,y)\in\Omega_2.\end{cases}\tag{1.49}$$

If we go over to polar coordinates by the formulas $r=\sqrt{x^2+y^2}$, $\varphi_1=arctg\dfrac{x}{y}$, $(0\le r\le 1$, $0\le\varphi_1\le\dfrac{\pi}{2})$, then it is not difficult to verify, that $sin(\nu_n\varphi+\varphi_0)=(-1)^n sin(\nu_n\varphi_1+\varphi_0^1)$, where $\varphi_0^1=arcctg(1+2\gamma_2)$. Then we obtain the second form of the system of eigenfunctions.

$$u_{n,m}(x,y)=\begin{cases} (-1)^n c_{n,m}J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2+y^2}\right]\ sin(\nu_n\varphi_1+\varphi_0^1), & (x,y)\in\Omega_0,\\ (-1)^n k_{n,m}^{(1)}\left[(1+\gamma_1)\left(\dfrac{x+y}{x-y}\right)^{\nu_n/2}-\gamma_1\left(\dfrac{x-y}{x+y}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{x^2-y^2}\right], & (x,y)\in\Omega_1,\\ k_{n,m}^{(2)}\left[(1+\gamma_2)\left(\dfrac{y+x}{y-x}\right)^{\nu_n/2}-\gamma_2\left(\dfrac{y-x}{y+x}\right)^{\nu_n/2}\right]\times & \\ \qquad\times J_{\nu_n}\left[\alpha_m^{(\nu_n)}\sqrt{y^2-x^2}\right], & (x,y)\in\Omega_2.\end{cases}\tag{1.50}$$

In the plane $\gamma_1 O\gamma_2$ we introduce the notation

$$G_0=\{(\gamma_1,\gamma_2):\ \gamma_1>-1,\ \gamma_2>-1,\ \gamma_1+\gamma_2>-1\}$$

$$G_{ks}=\{(\gamma_1,\gamma_2):\ \gamma_k>-1,\ \gamma_s<-1,\ \gamma_1+\gamma_2<-1\},\quad k,s=1,2,\quad k\neq s.$$

Lemma 1.2.3. *The system of sines* $\{sin\,(\nu_n\theta/2+\varphi_0)\}_{n=1}^{+\infty}$ *and* $\{sin\,(\nu_n\theta_1/2+\varphi_0^1)\}_{n=1}^{+\infty}$ *are forms the Riesz basis if* $(\gamma_1,\gamma_2)\in G_0\cup G_{12}$ *and* $(\gamma_1,\gamma_2)\in G_0\cup G_{21}$*, respectively.*

This lemma can be proved by Theorem 1 from [22].

We recall, that a Riesz basis φ_n in $L_2(a,b)$ is called complete in this space if for each function $f\in L_2(a,b)$

$$\sum_{n=1}^{\infty}\left|\int_a^b f(x)\overline{\varphi_n(x)}dx\right|^2<+\infty$$

and for each sequence of numbers c_1, c_2, ... with $\sum\limits_{n=1}^{\infty}|c_n|^2<\infty$, such function $f(x)\in L_2(a,b)$ exists such that

$$\int_a^b f(x)\overline{\varphi_n(x)}dx=c_n,\qquad n=1,2,....$$

If $\{\varphi_n\}_{n=1}^{+\infty}$ is a Riesz basis, then a unique sequence $\{\psi_n\}_{n=1}^{+\infty}$ exists, which generates together with $\{\varphi_n\}_{n=1}^{+\infty}$ a biorthogonal system

$$(\varphi_i,\psi_j)=\int_a^b\varphi_i(x)\overline{\psi_j(x)}dx=\begin{cases}0, & \text{if}\;\; i\neq j,\\ 1, & \text{if}\;\; i=j.\end{cases}$$

Then sequence $\{\psi_n\}_{n=1}^{+\infty}$ is also a Riesz basis and for each functions $f\in L_2(a,b)$ the equality

$$f(x)=\sum_{n=1}^{\infty}\alpha_n\varphi_n(x),\qquad \alpha_n=(f,\psi_n),\qquad n=1,2,\;...\;,$$

is valid, where the series is converge it in the mean quadratically. Besides, substituting the Perseval inequality the following two-sided estimate

$$m\int_a^b|f(x)|^2dx\le\sum_{n=1}^{\infty}|\alpha_n|^2\le M\int_a^b|f(x)|^2dx,\tag{1.51}$$

is valid, where m and M are positive numbers, independent from the function $f(x)$.

Lemma 1.2.4. *If one of the conditions* $\gamma_1+\gamma_2+2\gamma_1\gamma_2=0$, $\gamma\geq 0$ *and* $\gamma<-1$ *are satisfied, the system of functions* $\{x^{\nu_n-1}\}_{n=1}^{+\infty}$ *is complete in* $L_2\,[0,1]$, *where* $\gamma=(1+\gamma_1+\gamma_2)/(\gamma_1+\gamma_2+2\gamma_1\gamma_2)$.

This lemma follows from the Munts theorem about completeness of the system of functions $\{x^{m_n}\}_{n=1}^{+\infty}$ in $L_p\,[a,b]\,,\;\;0\leq a<b,\;\;p>1$, i.e. the condition

$$\sum_{k=1}^{+\infty}\frac{1}{m_k}=+\infty, \qquad -\frac{1}{p}<m_1<m_2<...$$

is necessary and sufficiently for the completeness of the system of functions $\{x^{m_n}\}_{n=1}^{+\infty}$ in $L_p\,[a,b]\,,$ $a\geq 0$, $p>1$.

Theorem 1.2.2. *If* $(\gamma_1,\gamma_2)\in G_0\cup G_{12}$ $((\gamma_1,\gamma_2)\in G_0\cup G_{21})$, *then the system of eigenfunctions (1.49) ((1.50)) of the problem* A_μ^0 *is complete in* $L_2(\Omega_0)$.

We prove the Theorem 1.2.2 for the system (1.49). We assume that a function $F_0(x,y)$ exists in $L_2(\Omega_0)$ such that

$$\iint\limits_{\Omega_0} F_0(x,y)u_{n,m}(x,y)dxdy=0 \tag{1.52}$$

for all $n,m\in N$. Let us show, that the function $F_0(x,y)=0$ almost everywhere in Ω_0.

If we go over to polar coordinates $x=rcos\varphi$, $y=rsin\varphi$ and taking (1.49) into account, from (1.52) we have

$$0=\int\limits_0^1\int\limits_0^{\pi/2} f_0(r,\varphi)J_{\nu_n}\left[\alpha_m^{(\nu_n)}r\right]\,[sin\,(\nu_n\varphi+\varphi_0)]\,rd\varphi dr$$

$$=\int\limits_0^1 F_n(r)J_{\nu_n}\left[\alpha_m^{(\nu_n)}r\right]rdr, \tag{1.53}$$

where

$$f_0(r,\varphi)=F_0(r\cos\varphi,r\sin\varphi), \qquad F_n(r)=\int\limits_0^{\pi/2} f_0(r,\varphi)\,sin\,(\nu_n\varphi+\varphi_0)\,d\varphi.$$

Using the Cauchy-Bunyakovskiy inequality, it is not difficult to prove, that the integral $\int\limits_0^1\sqrt{r}\,|F_n(r)|\,dr$ exists and converges absolutely. Then from (1.53) follows, that for the functions $F_n(r)$ all coefficients of the Fourier-Bessel series are equal to zero. Therefore, from the Young theorem [46] follows, that $F_n(r)\equiv 0$ $(n=1,2,...)$, i.e.

$$\int_0^{\pi/2} f_0(r,\varphi) sin\left(\nu_n\varphi+\varphi_0\right)d\varphi=0 \tag{1.54}$$

for all $n=1,2,...$ and at any $r\in(0,1)$.

If we replacing the variables $\theta=2\varphi$ in the integral (1.54), then when fulfilling the conditions of Lemma 1.2.3, the system of sines $\{sin\left(\nu_n\theta/2+\varphi_0\right)\}_{n=1}^{+\infty}$ is complete in $L_2(0,\pi)$. So from (1.54) follows, that for any r, the set of φ, where $f_0(r,\varphi)\neq 0$, has the measure zero. By virtue of the Fubini Theorem follows, that $f(r,\varphi)=0$ almost everywhere in Ω_0. From here the statement of the theorem follows.

Theorem 1.2.2 for the system (1.50) one can check analogously.

Theorem 1.2.3. *If* $\gamma_1=\gamma_2=0$*, then the system of eigenfunctions (1.49) ((1.50)) of the problem* A_μ^0 *is complete in* $L_2(\Omega_1)$ *and* $L_2(\Omega_2)$*.*

We give the proof for the domain Ω_1. One can analogously prove the statement for the domain Ω_2.

Let $\gamma_1=\gamma_2=0$ and suppose, that there exists a function $F_1(x,y)\in L_2(\Omega_1)$ such that

$$\iint\limits_{\Omega_1} F_1(x,y)u_{n,m}(x,y)dxdy=0,\quad n,m\in N. \tag{1.55}$$

Let us show, that $F_1(x,y)=0$ almost everywhere in Ω_1. Replacing the variables $\xi=x+y,\ \eta=x-y$ and taking (1.49) into account, from (1.55) we have

$$\int_0^1 d\eta\int_0^\eta f_1\left(\xi,\eta\right)\left(\frac{\xi}{\eta}\right)^{\nu_n/2} J_{\nu_n}\left(\alpha_m^{(\nu_n)}\sqrt{\xi\eta}\right)d\xi=0,\qquad n,m=1,2,...,$$

where $f_1(\xi,\eta)=F_1(x,y)$.

Setting $\xi=t\eta$ in the inner integral, changing the order of integration and then replacing the variables $\eta s=r,\ t=s^2$, we obtain

$$\int_0^1 J_{\nu_n}\left(\alpha_m^{(\nu_n)}r\right)rdr\int_r^1 s^{\nu_n-1}f_1\left(rs,\frac{r}{s}\right)ds=0.$$

It follows from here, that for function

$$F_n(r) = \int_r^1 s^{\nu_n - 1} f_1\left(rs, \frac{r}{s}\right) ds, \quad 0 \le r \le 1, \;\; n \in N,$$

all coefficients of the Fourier-Bessel series are equal to zero, therefore from the Young Theorem follows that $F_n(r) = 0$ for all $n \in N$ for any $r \in [0,1]$. Then by Lemma 1.2.4 at any r the set of those s, where $f_1(rs, r/s) \neq 0$, has measure zero. Therefore, according to the Fubini Theorem $f_1(rs, r/s) = 0$ almost everywhere in $\Omega_1^* = \{(s,r) : r < s < 1, \, 0 < r < 1\}$, consequently also in Ω_1. The Theorem is proved.

Theorem 1.2.4. *If $\gamma_1 = \gamma_2 = 0$, then the system of eigenfunctions (1.49) ((1.50)) of the problem A_μ^0 is not complete in $L_2(\Omega)$.*

Proof. In the domain Ω consider the function

$$F(x,y) = \begin{cases} F_0(x,y), & (x,y) \in \Omega_0, \\ F_1(x,y), & (x,y) \in \Omega_1, \\ F_2(x,y), & (x,y) \in \Omega_2 \end{cases}$$

from $L_2(\Omega)$ and the integral

$$P = \iint_{\Omega} F(x,y) u_{n,m}(x,y) dxdy = \iint_{\Omega_0} F_0(x,y) u_{n,m}(x,y) dxdy +$$

$$+ \iint_{\Omega_1} F_1(x,y) u_{n,m}(x,y) dxdy + \iint_{\Omega_2} F_2(x,y) u_{n,m}(x,y) dxdy.$$

Taking (1.49) into account and making replacements as in the proofs of Theorems 1.2.2 and 1.2.3, respectively, we obtain

$$P = \frac{c_{m,n}}{\sqrt{2}} \int_0^1 J_{\nu_n}\left[\alpha_m^{(\nu_n)} r\right] r \left[\int_0^\pi f_0\left(r, \frac{\theta}{2}\right) sin\left(\nu_n \frac{\theta}{2} + \frac{\pi}{4}\right) d\theta + \right.$$

$$\left. + \int_r^1 s^{\nu_n - 1} f_1\left(rs, \frac{r}{s}\right) ds + (-1)^n \int_r^1 s^{\nu_n - 1} f_2\left(rs, \frac{r}{s}\right) ds \right] dr. \tag{1.56}$$

Following [27], we consider the functions

$$f_0\left(r,\frac{\theta}{2}\right)=\sum_{k=1}^{\infty}\left[\frac{1}{2k(2k+1)}+\frac{(-1)^k}{(2k+2)(2k+3)}-\right.$$

$$\left.-\left(\frac{r^{2k}}{2k}-\frac{r^{2k+1}}{2k+1}\right)-(-1)^k\left(\frac{r^{2k+2}}{2k+2}-\frac{r^{2k+3}}{2k+3}\right)\right]h_k(\theta), \qquad (1.57)$$

$$f_1\left(rs,\frac{r}{s}\right)=-s(1-s), \qquad f_2\left(rs,\frac{r}{s}\right)=-s^3(1-s),$$

where $\{h_k(\theta)\}_{k=1}^{+\infty}$ is biorthogonal associated to the system of sines $\{sin\left[(\nu_n\theta/2+\pi/4\right]\}_{n=1}^{+\infty}$:

$$h_n(\theta)=\frac{2}{\pi}\frac{(2\cos\theta/2)^{-1}}{(tg\theta/2)^{1/2}}\sum_{k=1}^{n}(\sin k\theta)B_{n-k},$$

$$B_l=\sum_{m=0}^{l}C_{1/2}^{l-m}C_{1/2}^{m}(-1)^{l-m}, \quad C_l^n=\frac{l(l-1)\cdots(l-n+1)}{n!}.$$

Since $h_k(\theta)$ is uniformly bounded a constant [22], the series (1.57) at any $0\le r\le 1$, $0\le\theta\le\pi$ uniformly converges, and the function f_0 is continuous in Ω_0.

Substituting the functions $f_0,\ f_1,\ f_2$ in (1.56), we obtain that there exists a function $F\in L_2(\Omega)$ and $F(x,y)\ne 0$ in Ω such, that $P=0$. The theorem is proved.

1.2.4. Existence of the solution of the problem A_μ^0

Existence of the solution of the problem A_μ^0 can be proved as for the problem A_λ by the method of integral equations with potentials. But here when proving existence of the solution we use another method, i.e. by applying eigenfunctions of studying problem A_μ^0.

Let the conditions of Theorem 1.2.1 be fulfill. Then $\mu\ne\alpha_m^{(\nu_n)}$. For these values of μ, we find the solution of the problem A_μ^0 in the domain Ω_0 in the form of the series

$$u(x,y,\mu)=\sum_{n=1}^{+\infty}f_n\frac{J_{\nu_n}(\mu r)}{\alpha J_{\nu_n}(\mu)+\beta\mu J'_{\nu_n}(\mu)}sin(\nu_n\varphi+\varphi_0). \qquad (1.58)$$

We suppose that the series (1.58) admits term by term repeated differentiation with respect to the variables r and φ on the set $0 < r \leq 1$, $0 < \varphi < \pi/2$. Then it's sum satisfies the equation $(L.B_1)$ and conditions (1.39_1), (1.39_2). The coefficients f_n, $n \in N$ can be found and that the function (1.58) satisfies the condition (1.37).

Satisfying (1.58), for the boundary condition (1.37) at $r = 1$ we obtain

$$f(\varphi) = \sum_{n=1}^{+\infty} f_n sin(\nu_n \varphi + \varphi_0), \qquad 0 \leq \varphi \leq \pi/2, \tag{1.59}$$

where $f(\varphi) = f(r, \varphi)|_{r=1} = \psi(x, y)|_{\sigma_0}$.

If the function $f \in C^{\delta}[0, \pi/2]$, $\delta \in (0, 1]$, then by virtue of the result of [22], the series (1.58) converges uniformly on $[0, \pi/2]$ and the coefficients are determined by the equalities

$$f_n = \int_0^{\pi} f\left(\frac{\theta}{2}\right) h_n(\theta) d\theta, \qquad n = 1, 2, \dots\ . \tag{1.60}$$

As $h_n(\theta)$ are uniformly bounded by some constant, then $|f_n| \leq M$, $n = 1, 2, \dots$, $M = const > 0$. By virtue of the asymptotic formulas [42]

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^2, \quad n \to +\infty \quad \text{and} \quad xJ'_{\nu}(x) = \nu J_{\nu}(x) - xJ_{\nu+1}(x)$$

the series (1.58) converges uniformly for any $r \leq r_0 < 1$, since for large n the following estimation

$$\left| f_n \frac{J_{\nu_n}(\mu r) sin(\nu_n \varphi + \varphi_0)}{\alpha J_{\nu_n}(\mu) + \beta \mu J'_{\nu_n}(\mu)} \right| \leq r_0^{\nu_n} M_1, \qquad M_1 = const > 0$$

is true.

One can analogously show, that the series (1.58), for which the coefficients are determined by the formula (1.60), allows repeated differentiation on Ω_0 with respect to the variables r and φ, and $u \in C^1(\overline{\Omega}_0)$.

Using the series (1.58), for finding eigenfunctions in the domains Ω_1 and Ω_2, we find the solutions of the problem A_{μ}^0 in the domains Ω_1 and Ω_2 in evident form:

$$u(x, y, \mu) = \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{2 + 4\gamma_1 + 4\gamma_1^2}} \left[(1 + \gamma_1) \left(\frac{x+y}{x-y}\right)^{\nu_n/2} - \gamma_1 \left(\frac{x-y}{x+y}\right)^{\nu_n/2} \right] \times$$

$$\times \frac{J_{\nu_n}\left[\mu\sqrt{x^2-y^2}\right]}{\alpha J_{\nu_n}(\mu)+\beta\mu J'_{\nu_n}(\mu)} \quad \text{in} \quad \Omega_1, \tag{1.61}$$

$$u(x,y,\mu)=\sum_{n=1}^{\infty}\frac{(-1)^n f_n}{\sqrt{2+4\gamma_2+4\gamma_2^2}}\left[(1+\gamma_2)\left(\frac{x+y}{x-y}\right)^{\nu_n/2}-\gamma_2\left(\frac{x-y}{x+y}\right)^{\nu_n/2}\right]\times$$

$$\times \frac{J_{\nu_n}\left[\mu\sqrt{y^2-x^2}\right]}{\alpha J_{\nu_n}(\mu)+\beta\mu J'_{\nu_n}(\mu)} \quad \text{in} \quad \Omega_2. \tag{1.62}$$

It is not difficult to prove, that the series (1.61) in $\overline{\Omega}_1$ and the series (1.62) in $\overline{\Omega}_2$ converge uniformly, allow term by term repeated differentiation with respect to the variables x, y in the domains Ω_1, Ω_2 and satisfy the conditions (1.38_1) and (1.38_2) respectively.

Thus, here we proved the following result.

Theorem 1.2.5. *If* $\alpha\cdot\beta\geq 0$, $\alpha^2+\beta^2\neq 0$, $\gamma_j\geq -1/2$ *(*$j=1,2$*)*, $\mathrm{Re}\mu=0$, $f(\varphi)\in C^{\delta}[0,\pi/2]$, $\delta\in(0,1]$, *then the problem* A_{μ}^{0} *has a unique solution presented in the form*

$$u(x,y,\mu)=\begin{cases}\displaystyle\sum_{n=1}^{\infty} f_n\frac{J_{\nu_n}(\mu r)}{\alpha J_{\nu_n}(\mu)+\beta\mu J'_{\nu_n}(\mu)}sin(\nu_n\varphi+\varphi_0), & (x,y)\in\Omega_0,\\[2ex] \displaystyle\sum_{n=1}^{\infty}\frac{f_n}{\sqrt{2+4\gamma_1+4\gamma_1^2}}\left[(1+\gamma_1)\left(\frac{x+y}{x-y}\right)^{\nu_n/2}-\gamma_1\left(\frac{x-y}{x+y}\right)^{\nu_n/2}\right]\times & \\ \displaystyle\qquad\times\frac{J_{\nu_n}\left[\mu\sqrt{x^2-y^2}\right]}{\alpha J_{\nu_n}(\mu)+\beta\mu J'_{\nu_n}(\mu)}, & (x,y)\in\Omega_1,\\[2ex] \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n f_n}{\sqrt{2+4\gamma_2+4\gamma_2^2}}\left[(1+\gamma_2)\left(\frac{x+y}{x-y}\right)^{\nu_n/2}-\gamma_2\left(\frac{x-y}{x+y}\right)^{\nu_n/2}\right]\times & \\ \displaystyle\qquad\times\frac{J_{\nu_n}\left[\mu\sqrt{y^2-x^2}\right]}{\alpha J_{\nu_n}(\mu)+\beta\mu J'_{\nu_n}(\mu)}, & (x,y)\in\Omega_2.\end{cases}$$

With this existence and uniqueness of the solution of the problem A_{μ}^{0} is proved.

Chapter 2

Boundary value problem of Bitsadze-Samarskiy type for the equation $(L.B)$

A non-local boundary value problem for elliptic type equations was offered and investigated in the paper of A.B.Bitsadze and A.A.Samarskiy [5]. This problem appeared to generalize the well-known Dirichlet problem and is called the Bitsadze-Samarskiy problem. In this problem non-local conditions express the connection between the values of the unknown function on the boundary with those in inner points of the considered domain. Problems with such type of non-local conditions have been investigated in [34, 35] for some mixed type equations, where the elliptic part of the considered domain is a rectangle. Later many works are devoted to the problems of Bitsadze-Samarskiy type for partial differential equations in various formulations, e.g. in [38], [1], [11], [14], [43], [36].

The investigation of problems for elliptic-hyperbolic type of equations with spectral parameter, two lines of changing type, and non-local conditions, in which the border of the elliptic part of the considering domain is a quarter circle is far from being complete.

In this chapter in a finite simply connected mixed domain Ω, which is described in Chapter 1, we investigate the problem of Bitsadze-Samarskiy type for the equation $(L.B)$, and we also assume, that λ is a given complex number, moreover $\lambda = \lambda_0$ at $x > 0$, $y > 0$, $\lambda = \lambda_1$ at $x > 0$, $y < 0$ and $\lambda = \lambda_2$ at $x < 0$, $y > 0$. Besides, we use those notation from Chapter 1.

§ 2.1. Preliminary information

Firstly we formulate several statements, which will be useful in proving a theorem of uniqueness of the solution of the studied problem.

Consider the function

$$\omega(x,y)=\begin{cases}\operatorname{ch}\delta x+\operatorname{ch}\delta y & \text{in} \quad \Omega_0,\\ \operatorname{ch}\delta x+\cos\delta y & \text{in} \quad \Omega_1,\\ \operatorname{ch}\delta y+\cos\delta x & \text{in} \quad \Omega_2,\end{cases}$$

in the domain Ω, where $\delta\in R$. It is not difficult to verify, that $\omega(x,y)$ is positive in Ω and belongs to $C^2\left(\overline{\Omega}\right)$.

Let $u(x,y)\in C^2\left(\Omega_0\right)$ be a function, satisfying the equation $(L.B)$ in the domain Ω_0. Then if we introduce a new function $\vartheta(x,y)=\omega^{-1}(x,y)u(x,y)$ in the domain Ω_0, we obtain the equation

$$\Delta\vartheta+2\frac{\omega_x}{\omega}\vartheta_x+2\frac{\omega_y}{\omega}\vartheta_y+\left(\delta^2-\lambda_0^2\right)\vartheta=0 \qquad \text{in} \qquad \Omega_0 \tag{2.1}$$

for the function $\vartheta(x,y)$.

If $\vartheta(x_0,y_0)\neq 0$, where (x_0,y_0) is a point of Ω_0, then there exists a neighborhood $S\subset\Omega_0$ of this point, in which $\vartheta(x,y)\neq 0$. Multiplying the equation (2.1) with $\overline{\vartheta}/\left|\vartheta\right|$ in S and taking the real part of the obtained equality, we have

$$\Delta\left|\vartheta\right|+2\frac{\omega_x}{\omega}\left|\vartheta\right|_x+2\frac{\omega_y}{\omega}\left|\vartheta\right|_y-\left[\operatorname{Re}\lambda_0^2-\delta^2+\frac{1}{\left|\vartheta\right|^4}\left\{\left(\operatorname{Im}\overline{\vartheta}\vartheta_x\right)^2+\left(\operatorname{Im}\overline{\vartheta}\vartheta_y\right)^2\right\}\right]\left|\vartheta\right|=0. \tag{2.2}$$

From here we have the following proposition: if the function $\vartheta(x,y)$ satisfies the equation (2.1) in S, then $\left|\vartheta(x,y)\right|$ satisfies the equation (2.2).

Using this proposition, one can prove the theorems below in the same way as it was done in [3], [30].

Theorem 2.1.1. *Let $u(x,y)\in C^2(\Omega_0)$ be a function, which satisfies the equation $(L.B)$ in the domain Ω_0. Then the positive maximum $\left|\vartheta(x,y)\right|$ of the function $\vartheta(x,y)=\omega^{-1}(x,y)u(x,y)$ is not reached in any point of the domain Ω_0, if only $\vartheta(x,y)\not\equiv const$ in the domain Ω_0.*

Theorem 2.1.2. *Let $u(x,y)$ be a regular solution of the equation $(L.B)$, in the domain Ω_0 and $\vartheta(x,y)=\omega^{-1}(x,y)u(x,y)$ in Ω_0. Then, if*

$$\sup_{\overline{OA}} |\vartheta(x,y)| = |\vartheta(\xi,0)| > \sup_{\overline{\sigma_0}\cup\overline{OB}} |\vartheta(x,y)|$$

$$\left(\sup_{\overline{OB}} |\vartheta(x,y)| = |\vartheta(0,\eta)| > \sup_{\overline{\sigma_0}\cup\overline{OA}} |\vartheta(x,y)|\right),$$

then

$$\lim_{y\to+0} \frac{\partial}{\partial y}|\vartheta(\xi,y)| < 0, \quad 0 < \xi < 1 \quad \left(\lim_{x\to+0} \frac{\partial}{\partial x}|\vartheta(x,\eta)| < 0, \quad 0 < \eta < 1\right). \tag{2.3}$$

We can prove the existence of $\lim\limits_{y\to+0} \dfrac{\partial}{\partial y}|\vartheta(\xi,y)| < 0$, $0 < \xi < 1$ using the following lemma.

Lemma 2.1.1. *If $\vartheta(x,y) \in C(\overline{\Omega}_0) \cap C^1(\Omega_0 \cup OA)$ and $|\vartheta(\xi,0)| > 0$, $0 < \xi < 1$, then there exist $\lim\limits_{y\to+0} |\vartheta(\xi,y)|_y$ and the equality*

$$\lim_{y\to+0} \frac{\partial}{\partial y}|\vartheta(\xi,y)| = \mathrm{Re}\left\{\frac{\overline{\vartheta}(\xi,0)}{|\vartheta(\xi,0)|} \lim_{y\to+0} \frac{\partial}{\partial y}\vartheta(\xi,y)\right\} \qquad in \qquad \Omega_0 \tag{2.4}$$

is valid.

Proof. By virtue of $|\vartheta(\xi,0)| > 0$ and $\vartheta(\xi,0) \in C(\overline{\Omega}_0)$ some neighborhood $S \subset (\Omega_0 \cup OA)$ exists of points $(\xi,0)$, in which $|\vartheta(x,y)| > 0$. In this neighborhood the equality

$$\frac{\partial}{\partial y}|\vartheta(x,y)| = \mathrm{Re}\left\{\frac{\overline{\vartheta}(x,y)}{|\vartheta(x,y)|} \frac{\partial}{\partial y}\vartheta(x,y)\right\} \qquad \text{in} \qquad \Omega_0$$

is valid.

Substituting $x = \xi$, and taking a limit in the previous equality as $y \to +0$ and taking $\vartheta(x,y) \in C(\overline{\Omega}_0) \cap C^1(\Omega_0 \cup OA)$ into account, the statement of Lemma 2.1.1 is obtained.

In a similar way the existence of $\lim\limits_{x\to+0} \dfrac{\partial}{\partial x}|\vartheta(x,\eta)| < 0, \quad 0 < \eta < 1$, is proved.

Remark 2.1. If $\mathrm{Re}\lambda_0^2 \geq 0$ then the statement of Theorems 2.1.1 and 2.1.2 remain true for the function $u(x,y)$.

Consider the equation $(L.B)$ in Ω_1 and Ω_2. In a similar way as in Lemma 2.1.1 the following lemmas can be proved.

Lemma $2.1.2_1$. *If $\vartheta \in C(\overline{\Omega}_1) \cap C^1(\Omega_1 \cup OA)$ and $|\vartheta(\xi,0)| > 0$, $0 < \xi < 1$, then there exists $\lim\limits_{y\to-0} |\vartheta(\xi,y)|_y$ and the equality*

$$\lim_{y\to-0} \frac{\partial}{\partial y}|\vartheta(\xi,y)| = \mathrm{Re}\left\{\frac{\overline{\vartheta}(\xi,0)}{|\vartheta(\xi,0)|} \lim_{y\to-0} \frac{\partial}{\partial y}\vartheta(\xi,y)\right\} \qquad in \qquad \Omega_1 \tag{2.5}$$

is valid.

Lemma 2.1.2$_2$ is formulated, as Lemma 2.1.2$_1$, for the function $\vartheta(x,y) \in C(\overline{\Omega}) \cap C^1(\Omega_2 \cup OB)$ in the domain Ω_2.

§ 2.2. Formulation of the problem BS_λ

Consider the equation $(L.B)$ in the domain Ω.

Problem BS_λ. Find a regular solution of the equation $(L.B)$, in the domain Ω satisfying the conditions

$$u(x,y) = \sum_{k=1}^{n} \alpha_k(x,y) u(r_k x, r_k y) + g(x,y), \qquad (x,y) \in \overline{\sigma}_0; \tag{2.6}$$

$$A_{0x}^{0,\lambda_1}\left[u(\theta_{x0})\right] + c_1(x) u(x,0) = d_1(x), \qquad (x,0) \in \overline{OA}; \tag{2.7$_1$}$$

$$A_{0y}^{0,\lambda_2}\left[u(\theta_{y0})\right] + c_2(y) u(0,y) = d_2(y), \qquad (0,y) \in \overline{OB}. \tag{2.7$_2$}$$

Here $c_j(t)$ $(j=1,2)$ are given a real-valued functions, and $\alpha_k(x,y)$ $(k=\overline{1,n})$, $g(x,y)$, $d_j(t)$ are, in general, complex-valued functions, $r_1,\ldots,r_n$ is are given real numbers, moreover $0 < r_1 < r_2 < \ldots < r_n < 1$. For the given functions we require

$$c_j(t), \;\; d_j(t) \in C^{(2,r)}[0,1], \quad \text{where} \quad 0 < r = const < 1.$$

$$\alpha_k(x,y), \quad g(x,y) \in C(\overline{\sigma}_0), \quad k = \overline{1,n},$$

and assume, that

$$\sum_{k=1}^{n} |\alpha_k(x,y)| \neq 0, \qquad (x,y) \in \overline{\sigma}_0.$$

Further, it is not difficult to verify, that if the point $(x_0,\; y_0)$ is moving along $\overline{\sigma}_0$, then the points $(r_k x_0,\; r_k y_0)$, $k=\overline{1,n}$, will be moving along concentric semicircles $\overline{\sigma}_0 = \{(x,y) : x^2+y^2 = r_k^2,\; x \geq 0,\; y \geq 0\}$, $k=\overline{1,n}$, lying in $\overline{\Omega}_0$. Consequently, (2.6) is a condition, which connects the values of an unknown function on the boundary with inner points of the domain Ω_0. Therefore the problem BS_λ^0 is concerned to belong to the class of problems as offered in [5].

It should be noted, that problems with a condition of type (2.6) for uniformly elliptic equations are studied in [29], [36].

§ 2.3. Uniqueness of the solution of the problem BS_λ

Let $u(x, y)$ be a solution of the problem BS_λ^0. Then the basic functional relations between $\tau_j(t)$ and $\nu_j(t)$ on the segments OA and OB, defined from conditions (2.7_1), (2.7_2) for the function $u(x, y)$, are expressed by the formulas

$$P_j(t)\tau_j(t) = 2d_j(t) - \tau_j(0)J_0\left[\lambda_j t\right] + \int_0^t \nu_j(z)J_0\left[\lambda_j(t-z)\right]\, dz, \tag{2.9_j}$$

where $0 \leq t \leq 1, \quad j = 1, 2,$

$$P_j(t) = 1 + 2c_j(t), \qquad t = \begin{cases} x, & \text{if} \quad j = 1, \\ y, & \text{if} \quad j = 2. \end{cases}$$

The following lemma play an essential role in proving the uniqueness theorem.

Lemma $2.3.1_1$. *Let $u(x, y)$ be a regular solution of the equation $(L.B)$ in the domain $\overline{\Omega}_1$, satisfying the condition (2.7_1) with $d_1(x) \equiv 0$ and $\vartheta(x, y) = \omega^{-1}(x, y)u(x, y)$, $\delta \geq |\lambda_1|$, and let the conditions*

$$P_1(x) \geq 0, \qquad P_1'(x) \geq 0, \qquad 0 \leq x \leq 1. \tag{2.10_1}$$

$$P_1'(x) + P_1(x)\left[\frac{\delta\,\mathrm{sh}\,\delta x}{\mathrm{ch}\,\delta x + 1} - |\lambda_1|\int_0^x \frac{I_1[|\lambda_1|(x-t)]}{x-t}dt\right] \geq 0 \tag{2.11_1}$$

be fulfilled. Then, if $\sup\limits_{\overline{OA}} |\vartheta(x, y)| = |\vartheta(\xi, 0)| > 0$, $0 < \xi < 1$, the inequality

$$\lim_{y\to -0} \frac{\partial}{\partial y}|\vartheta(\xi, y)| \geq 0, \qquad 0 < \xi < 1 \tag{2.12_1}$$

is valid.

Proof. By virtue of $d_1(x) \equiv 0$ and taking $P_1(x) \geq 0$ in (2.9_1) into account, for $x = 0$ follows, that $\tau_1(0) = 0$. Then relation (2.9_1) becomes

$$P_1(x)\tau_1(x) = \int\limits_0^x \nu_1(z) J_0 \left[\lambda_1(x-z)\right] dz, \quad 0 \leq x \leq 1.$$

From this relation, we find the function $\nu_1(x)$ and rewrite the obtained equality as [36]

$$\lim_{y \to -0} \frac{\partial}{\partial y} u(x,y) = \left[P_1(x)u(x,0)\right]'_x + \lambda_1^2 \int\limits_0^x P_1(z)u(z,0) \frac{J_1\left[\lambda_1(x-z)\right]}{\lambda_1(x-z)} dz, \quad 0 \leq x \leq 1,$$

where $J_1[\cdot]$ is the first order Bessel function.

Taking $\vartheta(x,y) = \omega^{-1}(x,y)u(x,y)$ and (2.5) into account, and observing that $P_1(x)$ is a real-valued function, we have

$$\lim_{y \to -0} \frac{\partial}{\partial y} |\vartheta(\xi,y)| = P_1(\xi)|\widetilde{\tau}_1(\xi)|' + \left[P_1'(\xi) + \frac{\delta \operatorname{sh} \delta\xi}{\operatorname{ch} \delta\xi + 1} P_1(\xi)\right] |\widetilde{\tau}_1(\xi)|$$

$$+ \int\limits_0^\xi \frac{\operatorname{ch} \delta t + 1}{\operatorname{ch} \delta\xi + 1} P_1(t) Re \left\{ \lambda_1^2 \widetilde{\tau}_j(t) \frac{\overline{\widetilde{\tau}_1(\xi)}}{|\widetilde{\tau}_1(\xi)|} \frac{J_1[\lambda_1(\xi - t)]}{\lambda_1(\xi - t)} \right\} dt, \tag{2.13}$$

where $\widetilde{\tau}_1(x) = \vartheta(x,0)$.

We rewrite the equality (2.13) in the form

$$\lim_{y \to -0} \frac{\partial}{\partial y} |\vartheta(\xi,y)| = P_1(\xi)|\widetilde{\tau}_1(\xi)|'$$

$$+ \int\limits_0^\xi \left[\frac{\operatorname{ch} \delta z + 1}{\operatorname{ch} \delta\xi + 1} P_1(z) \operatorname{Re} \left\{ \lambda_1^2 \widetilde{\tau}_1(z) \frac{\overline{\widetilde{\tau}_1(\xi)}}{|\widetilde{\tau}_1(\xi)|} \frac{J_1[\lambda_1(\xi - z)]}{\lambda_1(\xi - z)} \right\} \right.$$

$$\left. + P_1(\xi)|\lambda_1^2||\widetilde{\tau}_1(\xi)| \frac{I_1[|\lambda_1|(\xi - z)]}{|\lambda_1|(\xi - z)} \right] dz$$

$$+ \left[P_1'(\xi) + P_1(\xi) \left\{ \frac{\delta \operatorname{sh} \delta\xi}{\operatorname{ch} \delta\xi + 1} - |\lambda_1^2| \int\limits_0^\xi \frac{I_1[|\lambda_1|(\xi - z)]}{|\lambda_1|(\xi - z)} dz \right\} \right] |\widetilde{\tau}_1(\xi)|. \tag{2.14}$$

From the condition (2.10_1) it is not difficult to verify the inequality

$$\frac{\operatorname{ch}\delta z+1}{\operatorname{ch}\delta\xi+1}P_1(z)\operatorname{Re}\left\{\lambda_1^2\widetilde{\tau}_1(z)\frac{\overline{\widetilde{\tau}_1(\xi)}}{|\widetilde{\tau}_1(\xi)|}\frac{J_1[\lambda_1(\xi-z)]}{\lambda_1(\xi-z)}\right\}\leq P_1(\xi)|\lambda_1^2||\widetilde{\tau}_1(\xi)|\frac{I_1[|\lambda_1|(\xi-z)]}{|\lambda_1|(\xi-z)}. \tag{2.15}$$

Now we return to equality (2.14). The first term on the right-hand side of (2.14) is equal to zero, as $x=\xi$ is a point of a positive maximum to the function $|\widetilde{\tau}_1(\xi)|$. By the inequality (2.15), the second term on the right-hand side of (2.14) is non-negative. Using (2.10_1), (2.11_1), $\delta\geq|\lambda_1|$, $|\widetilde{\tau}_1(\xi)|>0$, one can easily verify, that the third term on the right-hand side of (2.14) is also non-negative.

Consequently, the right-hand side of (2.14) is non-negative, i.e. the inequality (2.12_1) is true and Lemma $2.3.1_1$ is proved.

In the same way, the following lemma can be proved.

Lemma $2.3.1_2$**.** *Let $u(x,y)$ be a regular solution of the equation $(L.B)$ in the domain $\overline{\Omega}_2$ satisfying the condition (2.7_2) with $d_2(y)\equiv 0$, and $\vartheta(x,y)=\omega^{-1}(x,y)u(x,y)$, $\delta\geq|\lambda_2|$, and the conditions*

$$P_2(y)\geq 0,\qquad P_2'(y)\geq 0,\qquad 0\leq y\leq 1. \tag{2.10$_2$}$$

$$P_2'(y)+P_2(y)\left[\frac{\delta\operatorname{sh}\delta y}{\operatorname{ch}\delta y+1}-|\lambda_1|\int\limits_0^y\frac{I_1[|\lambda_1|(y-t)]}{y-t}dt\right]\geq 0 \tag{2.11$_2$}$$

be fulfilled. Then, if $\sup\limits_{\overline{OB}}|\vartheta(x,y)|=|\vartheta(0,\eta)|>0$, $0<\eta<1$, so the inequality

$$\lim_{x\to-0}\frac{\partial}{\partial x}|\vartheta(x,\eta)|\geq 0,\qquad 0<\eta<1 \tag{2.12$_2$}$$

is valid.

When λ_1 and λ_2 are a real numbers, we have more simple conditions on λ_1, λ_2. In this case in the domain Ω we substitute $v(x,y)=\varpi^{-1}(x,y)u(x,y)$ where $\varpi(x,y)=\exp(\delta x)+\exp(\delta y)$. Then (2.13) becomes

$$\lim_{y\to-0}\frac{\partial}{\partial y}|v(\xi,y)|=P_1(\xi)|\widetilde{\tau}_1(\xi)|'+\left[P_1'(\xi)+\delta\frac{P_1(\xi)e^{\delta\xi}-1}{e^{\delta\xi}+1}\right]|\widetilde{\tau}_1(\xi)|$$

$$+\int\limits_0^\xi\frac{e^{\delta z}+1}{e^{\delta\xi}+1}P_1(z)\left\{\lambda_1^2\widetilde{\tau}_j(z)\frac{J_1[\lambda_1(\xi-z)]}{\lambda_1(\xi-z)}\right\}dz, \tag{2.13$_1$}$$

where $\widetilde{\tau}_1(x) = \upsilon(x,0)$.

We rewrite the equality (2.13_1) in the form

$$\lim_{y\to -0} \frac{\partial}{\partial y} |\upsilon(\xi,y)| = P_1(\xi)|\widetilde{\tau}_1(\xi)|'$$

$$+\int\limits_0^\xi \frac{e^{\delta z}+1}{e^{\delta\xi}+1}\left[\frac{1}{2}|\lambda_1^2|P_1(\xi)|\widetilde{\tau}_1(\xi)| + P_1(z)\left\{\lambda_1^2\widetilde{\tau}_1(z)\frac{J_1[\lambda_1(\xi-z)]}{\lambda_1(\xi-z)}\right\}\right]dz$$

$$+\left[P_1'(\xi) + \delta\frac{P_1(\xi)e^{\delta\xi}-1}{e^{\delta\xi}+1} - \frac{1}{2}\frac{|\lambda_1^2|P_1(\xi)}{e^{\delta\xi}+1}\int\limits_0^\xi (e^{\delta z}+1)dz\right]|\widetilde{\tau}_1(\xi)|. \qquad (2.14_1)$$

Consider

$$l_0 = P_1'(\xi) + \delta\frac{P_1(\xi)e^{\delta\xi}-1}{e^{\delta\xi}+1} - \frac{1}{2}\frac{|\lambda_1^2|P_1(\xi)}{e^{\delta\xi}+1}\int\limits_0^\xi (e^{\delta z}+1)dz.$$

It is not difficult to verify, that if $P_1(\xi) \geq 1$, $P_1'(\xi) \geq 0$, $\delta \geq |\lambda_1|$, then $l_0 \geq 0$.

Now we prove the inequality

$$\pm P_1(z)\left\{\lambda_1^2\widetilde{\tau}_1(z)\frac{J_1[\lambda_1(\xi-z)]}{\lambda_1(\xi-z)}\right\} \leq \frac{1}{2}|\lambda_1^2|P_1(\xi)|\widetilde{\tau}_1(\xi)|. \qquad (2.15_1)$$

Indeed, if we take into account, that $0 \leq P_1(z) \leq P_1(\xi)$ (while $P_1'(z) \geq 0$, and consequently $P_1(z)$ is a non-decreasing function), $\pm\lambda_1^2 \leq |\lambda_1^2|$, $\left|\frac{J_1[x]}{x}\right| \leq \frac{1}{2}$ and $|\widetilde{\tau}_1(z)| \leq |\widetilde{\tau}_1(\xi)|$, we include, that the inequality (2.15_1) is valid.

Now we return to equality (2.14_1). The first term on the right-hand side of (2.14_1) is equal to zero, because $x = \xi$ is a point of a positive maximum to the function $|\widetilde{\tau}_1(\xi)|$. From inequality (2.15_1) we obtain, that the second term of (2.14_1) is non-negative. Using (2.10_1), $\delta \geq |\lambda_1|$, $|\widetilde{\tau}_1(\xi)| > 0$, the third term is also non-negativity.

Consequently, the right-hand side of (2.14_1) is non-negative, i.e. the inequality (2.12_1) is valid.

In the same way one can prove (2.12_1), when λ_2 is a real number.

Using Lemmas $2.3.1_1$ and $2.3.1_2$ the following theorem is proved.

Theorem 2.3.1. *Let*

$$\mathrm{Re}\lambda_0^2 \geq \delta_0^2 = max(|\lambda_1^2|, |\lambda_2^2|), \tag{2.16}$$

$$c_j(t) \geq -\frac{1}{2}, \quad c_j'(t) \geq 0, \quad 0 \leq t \leq 1; \quad j = \overline{1,2} \tag{2.17}$$

and even for one $\delta \in [\delta_0^2, \mathrm{Re}\lambda_0^2]$ *the inequality*

$$\sum_{k=1}^{n} |\alpha_k(x,y)| \left[\frac{e^{\delta r_k x} + e^{\delta r_k y}}{e^{\delta x} + e^{\delta y}} \right] \leq 1, \quad (x,y) \in \overline{\sigma}_0 \tag{2.18}$$

be fulfilled. Then, if the solution of the problem BS_λ *exists, it is unique.*

Proof. Let $u(x,y)$ be a solution of the homogenous problem BS_λ. We assume, that $\vartheta(x,y) \not\equiv$ *const* in the domain $\overline{\Omega}_0$. Then from the Theorem 2.1.1 follows, that

$$\sup_{\overline{\Omega}_0} |\vartheta(x,y)| = |\vartheta(\xi,\eta)| > 0, \quad \forall(\xi,\eta) \in \overline{OA} \cup \overline{OB} \cup \overline{\sigma}_0.$$

Let $(\xi,\eta) \in \overline{OA}$ or $(\xi,\eta) \in \overline{OB}$, i.e. $\sup_{\overline{\Omega}_0} |\vartheta(x,y)| = \sup_{\overline{OA}} |\vartheta(x,0)| = |\vartheta(\xi,0)| > 0$, $0 < \xi < 1$, or $\sup_{\overline{\Omega}_0} |\vartheta(x,y)| = \sup_{\overline{OB}} |\vartheta(0,y)| = |\vartheta(0,\eta)| > 0$, $0 < \eta < 1$. Then, taking propositions of Theorem 2.1.2 into account, we have inequality (2.3). Assuming in the Lemmas $2.3.1_1$ and $2.3.1_2$ $\delta = |\lambda_1|$ and $\delta = |\lambda_2|$, we obtain the inequalities (2.12_1) and (2.12_2), what is impossible by virtue of continuity of $|\vartheta(\xi,y)|_y$, $|\vartheta(x,\eta)|_x$ on the lines $y = 0$ and $x = 0$ respectively (this proposition follows from the characteristics of the functions $\omega(x,y)$ and $u(x,y)$). Consequently, $(\xi,\eta) \notin \overline{OA} \cup \overline{OB}$.

Now let $(\xi,\eta) \in \overline{\sigma}_0$. Then $|\vartheta(x,y)| < |\vartheta(\xi,\eta)|$, $\forall(x,y) \in \Omega_0 \cup \overline{OA} \cup \overline{OB}$. Taking this into account and the condition (2.18), from (2.6), and observing $g(x,y) \equiv 0$, we obtain $|\vartheta(\xi,\eta)| < |\vartheta(\xi,\eta)|$, which is impossible.

The obtained contradiction shows, that $\vartheta(x,y) \equiv$ *const* in $\overline{\Omega}_0$. Taking into account $\vartheta(0,0) = 0$ (this fact follows from $\vartheta(x,y) = \omega^{-1}(x,y)u(x,y)$ and by virtue of $u(0,0) = 0$, which is noted in Lemma $2.3.1_1$). From here, we conclude, that $\vartheta(x,y) \equiv 0$ in $\overline{\Omega}_0$, and consequently, $u(x,y) \equiv 0$ in $\overline{\Omega}_0$. The Theorem is proved.

Remark 2.2. In the case, when $\lambda_0 \in \mathrm{R}$, and λ_j $(j = 1,2)$ are real or pure imaginary number, then Theorem 2.3.1 can be proved by the extremal principle for the elliptic equations and the

Zaremba-Giro principle [7], when the solution $u(x,y)$ can be looked for in the class of real-valued function.

Remark 2.3. In the special case, when $\alpha_k(x,y) \equiv 0$, $k = 1, ..., n$, follows uniqueness of the solution of the problem A_λ, when $b_j(x) \equiv 0$, and when $\alpha_k(x,y) \equiv 0$, $(k = 1, ..., n)$, $c_j(x) \equiv 0$ $(j = 1, 2)$, the uniqueness of the solution of the Tricomi problem, which was obtained in [40] by the help of a Laplace transformation, which follows from Theorem 2.3.1.

Remark 2.4. At $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$ the condition (2.16) is equivalent to the equality $\mathrm{Im}\lambda = 0$.

From here and by the Theorem 2.3.1 we have the following proposition

Proposition. If $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$ and the conditions (2.17), (2.18) are fulfilled. Then the problem BS_λ can have eigenvalues only outside of $\mathrm{Im}\lambda = 0$.

§ 2.4. Existence of a solution of the problem BS_λ

Let the function $\varphi(x,y)$ be the value of the unknown solution $u(x,y)$ of the problem BS_λ on $\overline{\sigma}_0$ and denote $\nu_1(x) = \lim\limits_{y \to -0} u_y(x,y)$, $\nu_2(y) = \lim\limits_{x \to -0} u_x(x,y)$. If we find by the help of the given functions identically functions $\varphi(x,y)$, $\nu_1(x)$, $\nu_2(y)$ fulfilling the condition (1.20) and $\nu_j(t) \in C^1(0,1)$, $j = 1, 2$ (moreover $\nu_j(t)$ could have a singularity of order less than one at $t \to 0,\ t \to 1$), then in the domain Ω_0 the solution of the problem BS_λ is determined by the formula (1.25).

Therefore in the following we are engaged in finding the functions $\varphi(x,y)$, $\nu_1(x)$ and $\nu_2(y)$. In addition, we assume that

$$\alpha_k(x,y) = (xy)^\varepsilon \alpha_k^*(x,y), \quad k = \overline{1,n}; \qquad g(x,y) = (xy)^\varepsilon g^*(x,y) \tag{2.19}$$

$$\alpha_k^*(x,y),\ \ g^*(x,y) \ \in C(\overline{\sigma}_0), \qquad \varepsilon > 1.$$

Conditions (2.19) provide the fulfilling of (1.20).

Let the condition of Theorem 2.3.1 be fulfilled. By virtue of $c_j(t) \geq -1/2$ from (2.9_j) $(j = 1, 2)$ at $t = 0$ it follows, that $\tau_j(0) = 0$. Then functions $\nu_j(t)$ are identically found from (2.9_j), moreover we require in addition that $d_j(0) = 0$.

Further, substituting (1.25) in (2.6) and taking into account the notation $u(x,y)|_{\overline{\sigma}_0} = \varphi(x,y)$, we obtain

$$\varphi(x,y) - \int\limits_{\sigma_0} \varphi(\xi,\eta) N(\xi,\eta;x,y) ds = F(x,y), \qquad (x,y) \in \overline{\sigma}_0, \tag{2.20}$$

where

$$N(\xi,\eta;x,y) = \sum_{k=1}^{n} \alpha_k(x,y)$$

$$\times \left\{ \frac{\partial}{\partial n} G(\xi,\eta;r_k x, r_k y) + \iint\limits_{\Omega_0} R(\xi_1,\eta_1;r_k x,r_k y) \frac{\partial}{\partial n} G(\xi,\eta;\xi_1,\eta_1) d\xi_1 d\eta_1 \right\},$$

$$F(x,y) = g(x,y)$$

$$-\sum_{k=1}^{n} \alpha_k(x,y) \int\limits_0^1 \nu_1(t) \left\{ G(t,0;r_k x, r_k y) + \iint\limits_{\Omega_0} R(\xi,\eta;r_k x,r_k y) G(t,0;\xi,\eta) d\xi d\eta \right\} dt$$

$$-\sum_{k=1}^{n} \alpha_k(x,y) \int\limits_0^1 \nu_2(t) \left\{ G(0,t;r_k x, r_k y) + \iint\limits_{\Omega_0} R(\xi,\eta;r_k x,r_k y) G(0,t;\xi,\eta) d\xi d\eta \right\} dt.$$

By virtue of $(\xi,\eta) \neq (r_k x, r_k y)$, $\forall(\xi,\eta),(x,y) \in \overline{\sigma}_0$, $k = \overline{1,n}$, the function $N(\xi,\eta;x,y)$ continuously on $\overline{\sigma}_0 \times \overline{\sigma}_0$. Besides, it is not difficult to verify, that $F(x,y) \in C(\overline{\sigma}_0)$. Then, by changing the variables $x = \cos\theta$, $y = \sin\theta$, $\xi = \cos\theta_1$, $\eta = \sin\theta_1$, $0 \le \theta, \theta_1 \le \pi/2$, from (2.20) we get a Fredholm integral equation of second kind for $\varphi(\cos\theta, \sin\theta)$. Regarding the function $\varphi(x,y)$ is temporarily known and for the equation $(L.B)$ solving the problem BS_λ, we obtain a singular integral equation in the form (1.33).

By solving the obtained equation, identically we find the functions $\nu_j(t)$, $(j = 1,2)$ by $\varphi(x,y)$, which is the first and the second functional relation between $\nu_1(x)$, $\varphi(x,y)$ and $\nu_2(y)$, $\varphi(x,y)$ respectively. On the other hand, equality (2.20) is the third functional relation between $\nu_1(x)$, $\nu_2(y)$ and $\varphi(x,y)$, which is determined by the condition, that the solution $u(x,y)$ of the problem BS_λ^0 should satisfy (2.6).

Consequently, for the functions $\nu_1(x)$, $\nu_2(y)$ and $\varphi(x,y)$ we have a system of three Fredholm equations of second kind, unique solvability of this system follows from the uniqueness of the solution of the considered problems.

This ends existence and uniqueness of the solution of the problem BS_λ.

Chapter 3

NON-LOCAL PROBLEMS FOR MIXED TYPE EQUATION WITH SPECTRAL PARAMETER, WHEN THE DOMAIN OF ELLIPTICITY IS A QUARTER RING

The third chapter of this dissertation consists of four paragraphs and in it non-local problems for the mixed elliptic-hyperbolic equation $(L.B)$ with two lines of changing type with spectral parameter investigated. Moreover, here we assume that $\lambda = \lambda_1$ for $x > 0,\ \ y > 0$, $\lambda = \lambda_2$ at $x > 0,\ \ y < 0$ and $x < 0,\ \ y > 0$, where λ_1 and λ_2 are given complex parameters.

§ 3.1. Formulation of problems

Let Δ be a finite simply-connected mixed domain of the plane of the variables xOy, bounded for $x > 0$, $y > 0$ by the lines $\sigma_{01} : x^2 + y^2 = 1$, $\sigma_{02} : x^2 + y^2 = p^2$, $(0 < p < 1)$ and for $x \cdot y < 0$ by the characteristics $x + y = p$, $x - y = \pm 1$ of the equation $(L.B)$.

Let us introduce the notations:

$$\Delta_0 = \Delta \cap (x > 0, y > 0), \ \ \Delta_1 = \Delta \cap (x > 0, y < 0), \ \ \Delta_2 = \Delta \cap (x < 0, y > 0),$$

$$I_1 = \{(x, y) : p < x < 1, y = 0\}, \qquad I_2 = \{(x, y) : x = 0,\ p < y < 1\},$$

$$\theta_{px}(x) = ((p + x)/2;\ (p - x)/2), \qquad \theta_{py}(y) = ((p - y)/2;\ (p + y)/2),$$

$$\theta_{x1}(x) = ((x + 1)/2;\ (x - 1)/2), \qquad \theta_{1y}(y) = ((y - 1)/2;\ (y + 1)/2).$$

We call a function $u(x, y)$ a regular solution of the equation $(L.B)$ in the domain $\Delta \setminus (I_1 \cup I_2)$, the derivatives of $u_x(x, y)$, $u_y(x, y)$ which can become infinite of order less than one in the points $A(p, 0)$, $B(1, 0)$, $C(0, 1)$ and $D(0, p)$.

Problem Γ_0^λ. Find a regular solution of the equation $(L.B)$, in the domain Δ satisfying the conditions

$$u(x, y) \in C(\overline{\Delta}) \cap C^1(\Delta) \cap C^2(\Delta \backslash I_1 \backslash I_2);$$

$$u(x,y)=\varphi_j(x,y), \qquad (x,y)\in\overline{\sigma}_{0j}, \quad (j=1,2.); \tag{3.1}$$

$$a_1(x)A_{px}^{0,\lambda_2}\left[u(\theta_{px})\right]+b_1(x)A_{1x}^{0,\lambda_2}\left[u(\theta_{x1})\right]$$

$$+c_1(x)u(x,0)=d_1(x), \qquad (x,0)\in\overline{I_1}; \tag{3.2$_1$}$$

$$a_2(y)A_{py}^{0,\lambda_2}\left[u(\theta_{py})\right]+b_2(y)A_{1y}^{0,\lambda_2}\left[u(\theta_{1y})\right]$$

$$+c_2(y)u(0,y)=d_2(y), \qquad (0,y)\in\overline{I_2}. \tag{3.2$_2$}$$

Problem Γ_1^λ. Find a regular solution of the equation $(L.B)$, in the domain Δ satisfying the boundary condition (3.1) and the conditions

$$u(x,y)\in C(\overline{\Delta})\cap C^1(\overline{\Delta}\backslash\sigma_{0j})\cap C^2(\Delta\backslash I_1\backslash I_2), \quad (j=1,2.);$$

$$a_1(x)A_{px}^{1,\lambda_2}\left[\frac{d}{dx}u(\theta_{px})\right]+b_1(x)A_{1x}^{1,\lambda_2}\left[\frac{d}{dx}u(\theta_{x1})\right]$$

$$+c_1(x)\frac{\partial}{\partial y}u(x,0)=d_1(x), \qquad (x,0)\in I_1; \tag{3.3$_1$}$$

$$a_2(y)A_{py}^{1,\lambda_2}\left[\frac{d}{dy}u(\theta_{py})\right]+b_2(y)A_{1y}^{1,\lambda_2}\left[\frac{d}{dy}u(\theta_{1y})\right]$$

$$+c_2(y)\frac{\partial}{\partial x}u(0,y)=d_2(y), \qquad (0,y)\in I_2, \tag{3.3$_2$}$$

Here $a_j(t)$, $b_j(t)$, $c_j(t)$ are given real-valued functions, moreover $a_j^2(t)+b_j^2(t)\neq 0$, $t\in[p,1]$, $j=1,2$, and $\varphi_j(x,y)$, $d_j(t)$ are given, in general complex-valued functions.

From the given functions we require that $a_j(t)$, $b_j(t)$, $c_j(t)$, $d_j(t)\in C^1\left(\overline{I_j}\right)\cap C^{(1+k,r)}\ (I_j)$, $\varphi_j(x,y)\in C(\bar{\sigma}_{0j})$, where $0<r=const<1$. $k=1$ in the problem Γ_0^λ and $k=0$ in the problem Γ_1^λ and assume, that

$$\varphi_j(x,y)=(xy)^\varepsilon\varphi_j^*(x,y), \qquad \varphi_j^*(x,y)\in C(\overline{\sigma}_{0j}), \quad \varepsilon>1. \tag{3.4}$$

§ 3.2. Investigation of the problem Γ_0^λ

In the domains Ω_j, $j = 1, 2$, using the formulas (1.5_j) and conditions (3.2_j), after some calculation, we have

$$\alpha_j(t)\tau_j(t) = 2d_j(t) - \tau_j(p)a_j(t)J_0\left[\lambda_2(t-p)\right] - \tau_j(1)b_j(t)J_0\left[\lambda_2(1-t)\right]$$

$$+a_j(t)\int_p^t \nu_j(z)J_0\left[\lambda_2(t-z)\right]\,dz + b_j(t)\int_t^1 \nu_j(z)J_0\left[\lambda_2(z-t)\right]\,dz, \qquad (3.5_j)$$

where $p \le t \le 1$, $\ j = 1, 2$.

$$\alpha_j(t) = a_j(t) + b_j(t) + 2c_j(t) \qquad t = \begin{cases} x, & \text{if} \quad j = 1, \\ y, & \text{if} \quad j = 2. \end{cases}$$

Equalities (3.5_j) , $(j = 1, 2)$, provide a basic functional relation between $\tau_j(t)$ and $\nu_j(t)$ on the segments I_1 and I_2 respectively, attained from the hyperbolic part of the mixed domain Δ.

We consider by analysis of the relation (3.5_j) $(j = 1, 2)$ the following three cases: $\alpha_j(t) \equiv 0$, $j = 1, 2$; $\alpha_j(t) \not\equiv 0$, $j = 1, 2$; $\alpha_j(t) \equiv 0$, $\alpha_k(t) \neq 0$, $j, k = 1, 2$, $j \neq k$.

The following lemma plays the essential role in the proof of the uniqueness theorem.

Lemma 3.2.1. *If $u(x, y)$ is a regular solution of the equation $(L.B)$ in the domain Δ_0 and is equal to zero on $\overline{\sigma}_{0j}$ $(j = 1, 2.)$, then the equality*

$$(\mathrm{Re}\lambda_1^2 - \delta^2)\left[\iint\limits_{\Delta_0'} |\vartheta|^2 dxdy + \iint\limits_{\Delta_0''} |\omega|^2 dxdy\right] + \iint\limits_{\Delta_0'} |\nabla\vartheta|^2 dxdy + \iint\limits_{\Delta_0''} |\nabla\omega|^2 dxdy +$$

$$+\mathrm{Re}\int_p^1 e^{2\delta x}\tau_1(x)\nu_1(x)dx + \mathrm{Re}\int_p^1 e^{2\delta y}\tau_2(y)\nu_2(y)dy = 0 \qquad (3.6)$$

is valid, where $\vartheta(x, y) = e^{\delta x}u(x, y)$, $\ \bar{\tau}_1(x) = \bar{u}(x, 0)$, $\nu_1(x) = u_y(x, 0)$ in $\Delta_0' = \Delta_0 \cap (x > y)$; $\omega(x, y) = e^{\delta y}u(x, y)$, $\bar{\tau}_2(y) = \bar{u}(0, y)$, $\nu_2(y) = u_x(0, y)$ in $\Delta_0'' = \Delta_0 \cap (x < y)$; $\ \forall|\delta| \ge |\mathrm{Im}\lambda_2|$. Moreover $\vartheta(x, y) = \omega(x, y)$ on $K_1K_2 :\ y = x$.

This lemma is proved analogously as Lemma 1.1.1.

§ 3.2.1. Investigation of the problem Γ_0^λ for $\alpha_j(x) \equiv 0, \quad j = 1, 2$

Theorem 3.2.1. *Let $\alpha_j(t) \equiv 0$, $\mathrm{Re}\lambda_1^2 \geq 0$ and one of the following group of conditions is fulfilled: a) $b_j(t) \equiv 0$; b) $a_j(t) \equiv 0$; c) $a_j(t) \not\equiv 0$, $b_j(t) \not\equiv 0$, $a_j(t) \neq b_j(t)$ and*

$$\int_p^1 \int_p^1 |K_j(t, z, \lambda_2)|^2 \, dt dz < 1, \quad j = 1, 2, \tag{3.7}$$

where

$$K_j(t, z, \lambda_2) = \begin{cases} \dfrac{\partial}{\partial t} \{a_j(t) J_0 [\lambda_2(t - z)]\} / [a_j(t) - b_j(t)], & t \geq z, \\ \dfrac{\partial}{\partial t} \{b_j(t) J_0 [\lambda_2(z - t)]\} / [a_j(t) - b_j(t)], & z \geq t. \end{cases}$$

Then, the solution of the problem Γ_0^λ exists and is unique.

Fulfilling the conditions of Theorem 3.2.1 we obtain from (3.5_j), $j = 1, 2$, the equations

$$a_j(t) \int_p^t \nu_j(z) J_0 [\lambda_2(t - z)] \, dz + b_j(t) \int_t^1 \nu_j(z) J_0 [\lambda_2(z - t)] \, dz = -2d_j(t). \tag{3.8_j}$$

As in Theorem 1.1.1, fulfilling the conditions a), b) and c) identically from equation (3.8_j) we find the function $\nu_j(t)$ by the formulas

$$\nu_j(t) = -2C_{pt}^{0,\lambda_2} \left[\frac{d_j(t)}{a_j(t)} \right], \tag{3.9}$$

$$\nu_j(t) = -2C_{1t}^{0,\lambda_2} \left[\frac{d_j(t)}{b_j(t)} \right], \tag{3.10}$$

$$\nu_j(t) = g_j(t) - \int_p^1 R_j(t, z, \lambda_2) g_j(z) dz, \tag{3.11}$$

respectively, here $g_j(t) = -2d_j(t)/[b_j(t) - a_j(t)]$, $R_j(t, z, \lambda_2)$ is the resolvent of the kernel $K_j(t, z, \lambda_2)$, $C_{mx}^{0,\lambda}$ is the operator, determined by the formula (1.22).

It follows from the above, that the functions $\nu_j(t)$ $(j = 1, 2)$ are identically determined in the case when the functions $a_1(x)$, $b_1(x)$ satisfy one of the conditions a), b), c), and the functions $a_2(y)$, $b_2(y)$ satisfy the other of these conditions.

Consequently, at $\alpha_j(t) \equiv 0$, $j = 1, 2$, the problem Γ_0^λ is equivalent to the problem N for the equation $(L.B)$ in the domain Δ_0 with boundary conditions (3.1) and $u_y(x,0) = \nu_1(x)$, $u_x(0,y) = \nu_2(y)$, $p < x, y < 1$, where $\nu_j(t) \in C^1(p,1)$ is a well-known functions, determined by one of the formulas (3.9), (3.10) and (3.11).

Let $u(x,y)$ be a solution of the problem N for the equation $(L.B)$ in Δ_0. Then using the Green formula, it is not difficult to prove that the problem N for the equation $(L.B)$ in Δ_0 is equivalent (in the meaning of solvability) to the integral equation

$$u(x,y) + \lambda_1^2 \iint\limits_{\Delta_0} G(\xi,\eta;x,y)u(\xi,\eta)d\xi d\eta = u_0(x,y),$$

where [43]

$$u_0(x,y) = \sum_{j=1}^{2} \int\limits_{\sigma_{oj}} \varphi_j(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,y)ds$$

$$-\int\limits_p^1 \nu_1(t)G(t,0;x,y)dt - \int\limits_p^1 \nu_2(t)G(0,t;x,y)dt, \tag{3.12}$$

$$G(\xi,\eta;x,y) = P(\omega,z) + P(\overline{\omega},z) + P(-\omega,z) + P(-\overline{\omega},z), \tag{3.13}$$

$$P(\omega,z) = \frac{1}{2\pi}\ln\left|\vartheta_1\left(\frac{\ln\omega + \ln\overline{z}}{2\pi i\tau}\right)/\vartheta_1\left(\frac{\ln\omega - \ln z}{2\pi i\tau}\right)\right|.$$

Here n is the inner normal to σ_{0j} relative to Δ_0, s is the arc length; $\omega = \xi + i\eta$, $z = x + iy$, $\overline{\omega} = \xi - i\eta$, $\overline{z} = x - iy$; $\vartheta_1(\zeta) = \vartheta_1\left(\zeta| - \frac{1}{\tau}\right)$ is the theta function [2], $\tau = \frac{\ln p}{\pi i}$, $i = \sqrt{-1}$ is the imaginary unit.

To the obtained integral equation we adapt the Fredholm theorems. Then its solvability follows from the uniqueness of the solution of problem N. Therefore its solution, consequently, and the solution of the problem N exists, is unique and determined by

$$u(x,y) = u_0(x,y) + \iint\limits_{\Delta_0} R(\xi,\eta;x,y)u_0(\xi,\eta)d\xi d\eta. \tag{3.14}$$

Here $R(\xi,\eta;x,y)$ is the resolvent of the kernel $(-\lambda_1^2)\,G(\xi,\eta;x,y)$.

In the domains Δ_1 and Δ_2 the solution of the problem Γ_0^λ is determined by the formulas (1.5_1) and (1.5_2), moreover $\tau_1(x) = u(x,0)$, $\tau_2(y) = u(0,y)$ are found from (3.14).

§ 3.2.2. Investigation of the problem Γ_0^λ for $\alpha_j(x) \neq 0, \quad j = 1,2$

Theorem 3.2.2. *Let one of the following group of conditions be fulfilled:*

1)
$$\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2 ;$$

$$\alpha_j(t) \neq 0, \;\; b_j(t) \equiv 0, \quad \frac{c_j(t)}{a_j(t)} > -\frac{1}{2}, \quad \left(\frac{c_j(t)}{a_j(t)}\right)' \geq 0, \quad t \in [p,1]; \tag{3.15}$$

2)
$$\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2 ;$$

$$\alpha_j(t) \neq 0, \;\; a_j(t) \equiv 0, \quad \frac{c_j(t)}{a_j(t)} > -\frac{1}{2}, \quad \left(\frac{c_j(t)}{a_j(t)}\right)' \leq 0, \quad t \in [p,1]; \tag{3.16}$$

3)
$$\mathrm{Re}\lambda_1^2 \geq 0, \qquad \mathrm{Im}\lambda_2 = 0;$$

$$\alpha_j(t) \neq 0, \quad a_j(t) \not\equiv 0, \quad b_j(t) \not\equiv 0, \tag{3.17}$$

$$\left(\frac{a_j(t)}{\alpha_j(t)}\right)' \leq 0, \quad \left(\frac{b_j(t)}{\alpha_j(t)}\right)' \geq 0, \quad t \in [p,1]; \qquad \frac{a_j(1)}{\alpha_j(1)} + \frac{b_j(p)}{\alpha_j(p)} \geq 0. \tag{3.18}$$

Then the problem Γ_0^λ cannot have more than one solution.

The preposition of the theorem follows from the equality (3.6) and from the following lemma.

Lemma 3.2.2. *Let $\tau_j(p) = 0$, $(-\delta) \geq |Im\lambda_2|$ and the condition (3.15) be fulfilled. Then for $d_j(t) \equiv 0$ the inequality*

$$P_j \equiv \mathrm{Re} \int\limits_p^1 e^{2\delta t} \bar{\tau}_j(t) \nu_j(t) dt \geq 0 \tag{3.19}$$

is valid.

Proof. Let $d_j(t) \equiv 0$. Then by virtue of the second and the third from the conditions (3.15), from (3.5_j) for $t = 0$ it follows that $\tau_j(p) = 0$. Taking into account $d_j(t) \equiv b_j(t) \equiv 0$ and $\tau_j(p) = 0$, from (3.5_j) we find the functions $\tau_j(t)$ and with regard of $\mathrm{Re}[\bar{\tau}_j(t)\nu_j(t)] = \mathrm{Re}[\tau_j(t)\bar{\nu}_j(t)]$ we substitute it in (3.19); changing the function $J_0[\cdot]$ by the formula [16]

$$J_s(z) = \frac{2\left(\frac{z}{2}\right)^s}{\sqrt{\pi}\left(s + \frac{1}{2}\right)} \int\limits_0^1 \left(1 - \xi^2\right)^{s - \frac{1}{2}} \cos z\xi d\xi, \quad \mathrm{Re}s > -\frac{1}{2}, \tag{3.20}$$

and taking into account the fact, that $a_{j1}(t) = a_j(t)/\alpha_j(t)$ is a real function, we have

$$P_j \equiv \frac{2}{\pi} \int\limits_0^1 \left(1-\xi^2\right)^{-1/2} d\xi \int\limits_p^1 e^{2\delta t} a_{j1}(t) dt \int\limits_p^t \mathrm{Re}\left[\bar{\nu}_j(t)\nu_j(z)\cos\lambda_2\xi(t-z)\right] dz. \tag{3.21}$$

Let $\lambda_2 = \lambda_{21} + i\lambda_{22}$ and $\nu_j(t) = \nu_{j1}(t) + i\nu_{j2}(t)$.

It is not difficult to set [36], that

$$\mathrm{Re}\left[\bar{\nu}_j(t)\nu_j(z)\cos\lambda_2(t-z)\xi\right]$$

$$= \frac{1}{2} e^{-2\lambda_{22}\xi t} \sum_{n=1}^{2} \Phi_{jn}(t,\xi)\Phi_{jn}(z,\xi) + \frac{1}{2} e^{2\lambda_{22}\xi t} \sum_{n=3}^{4} \Phi_{jn}(t,\xi)\Phi_{jn}(z,\xi), \tag{3.22}$$

where

$$\left.\begin{array}{l} \Phi_{j1}(t,\xi) \\ \Phi_{j3}(t,\xi) \end{array}\right\} = \left[\nu_{j11}(t,\xi) \pm \nu_{j22}(t,\xi)\right] e^{\pm\lambda_2\xi t},$$

$$\left.\begin{array}{l} \Phi_{j2}(t,\xi) \\ \Phi_{j4}(t,\xi) \end{array}\right\} = \left[\nu_{j12}(t,\xi) \mp \nu_{j21}(t,\xi)\right] e^{\pm\lambda_2\xi t},$$

$$\nu_{jk1}(t,\xi) = \nu_{jk}(t)\cos\lambda_{21}t\xi, \qquad \nu_{jk2}(t,\xi) = \nu_{jk}(t)\sin\lambda_{21}t\xi, \quad k = 1, 2.$$

Taking (3.22) into account, after some calculations we have

$$\int\limits_p^t \mathrm{Re}\left[\bar{\nu}_j(t)\nu_j(z)\cos\lambda_2(t-z)\xi\right] dz$$

$$= \frac{1}{4} e^{-2\lambda_{22}\xi\, t} \sum_{n=1}^{2} \frac{d}{dt}\Phi_{jn1}^2(t,\xi) + \frac{1}{4} e^{2\lambda_{22}\xi\, t} \sum_{n=3}^{4} \frac{d}{dt}\Phi_{jn1}^2(t,\xi), \tag{3.23}$$

where

$$\Phi_{jn1}(t,\xi) = \int\limits_p^t \Phi_{jn}(z,\xi) dz, \quad n = \overline{1,4}.$$

Substituting (3.23) in (3.21) and integrating by parts, we obtain

$$P_j \equiv \frac{1}{2\pi}\int\limits_0^1 \left(1-\xi^2\right)^{-1/2} d\xi \left\{ a_{j1}(1)e^{2(\delta-\lambda_{22}\xi)}\sum_{n=1}^{2}\Phi_{jn1}^2(1,\xi)\right.$$

$$+\int\limits_p^1 \left[-a'_{j1}(t)-2a_{j1}(t)(\delta-\lambda_{22}\xi)\right]e^{2t(\delta-\lambda_{22}\xi)}\left[\sum_{n=1}^{2}\Phi_{jn1}^2(t,\xi)\right]dt$$

$$+a_{j1}(1)e^{2(\delta+\lambda_{22}\xi)}\sum_{n=3}^{4}\Phi_{jn1}^2(1,\xi)$$

$$\left.+\int\limits_p^1 \left[-a'_{j1}(t)-2a_{j1}(t)(\delta+\lambda_{22}\xi)\right]e^{2t(\delta+\lambda_{22}\xi)}\left[\sum_{n=3}^{4}\Phi_{jn1}^2(t,\xi)\right]dt\right\}. \qquad (3.24)$$

From here, taking $a_{j1}(t)>0$, $a'_{j1}(t)\le 0$, $\forall t\in[0,1]$ and $(-\delta\pm\lambda_{22}\xi)\ge 0$ at $-\delta\ge|\lambda_{22}|$, $|\xi|\le 1$ into account, we conclude that $P_j\ge 0$.

Lemma 3.2.3. *Let $\tau_j(1)=0$, $\delta\ge|Im\lambda_2|$ and the condition (3.16) be fulfilled. Then for $d_j(t)\equiv 0$ inequality (3.19) is valid.*

Proof. Let $d_j(t)\equiv 0$. Then, by virtue of the second and the third from the condition (3.16), from (3.5$_j$) at $t=1$ it follows, that $\tau_j(1)=0$. Taking into account $\tau_j(1)=0$ and $d_j(t)\equiv a_j(t)\equiv 0$, from (3.5$_j$), we find the function $\tau_j(t)$ and with regard to $\mathrm{Re}[\bar\tau_j(t)\nu_j(t)]=\mathrm{Re}[\tau_j(t)\bar\nu_j(t)]$ we substitute it in (3.19); changing the function $J_0[\cdot]$ by formula (3.20) and taking into account the fact, that $b_{j1}(t)=b_j(t)/\alpha_j(t)$ is a real function, we have

$$P_j \equiv \frac{2}{\pi}\int\limits_0^1 \left(1-\xi^2\right)^{-1/2} d\xi \int\limits_p^1 e^{2\delta t}b_{j1}(t)dt\int\limits_t^1 \mathrm{Re}\left[\bar\nu_j(t)\nu_j(z)\cos\lambda_2\xi(z-t)\right]dz. \qquad (3.25)$$

Using the equality (3.22), after some calculations we find

$$\int\limits_t^1 \mathrm{Re}\left[\bar\nu_j(t)\nu_j(z)\cos\lambda_2(t-z)\xi\right]dz$$

$$=-\frac{1}{4}e^{-2\lambda_{22}\xi t}\sum_{n=1}^{2}\frac{d}{dt}\Phi_{jn2}^2(t,\xi)-\frac{1}{4}e^{2\lambda_{22}\xi t}\sum_{n=3}^{4}\frac{d}{dt}\Phi_{jn2}^2(t,\xi), \qquad (3.26)$$

where

$$\Phi_{jn2}(t,\xi)=\int\limits_t^1 \Phi_{jn}(z,\xi)dz, \qquad n=\overline{1,4}.$$

Substituting (3.26) in (3.25) and integrating by parts we find

$$P_j \equiv \frac{1}{2\pi}\int\limits_0^1 \left(1-\xi^2\right)^{-1/2} d\xi \left\{ b_{j1}(p)e^{-2(\delta-\lambda_{22}\xi)} \sum_{n=1}^{2} \Phi^2_{jn2}(p,\xi) \right.$$

$$+\int\limits_p^1 \left[b'_{j1}(t)+2b_{j1}(t)(\delta-\lambda_{22}\xi)\right] e^{2t(\delta-\lambda_{22}\xi)} \left[\sum_{n=1}^{2} \Phi^2_{jn2}(t,\xi)\right] dt$$

$$+b_{j1}(p)e^{-2(\delta+\lambda_{22}\xi)} \sum_{n=3}^{4} \Phi^2_{jn2}(p,\xi)$$

$$\left. +\int\limits_p^1 \left[b'_{j1}(t)+2b_{j1}(t)(\delta+\lambda_{22}\xi)\right] e^{2t(\delta+\lambda_{22}\xi)} \left[\sum_{n=3}^{4} \Phi^2_{jn2}(t,\xi)\right] dt \right\}. \tag{3.27}$$

From here, taking $b_{j1}(t) > 0$, $b'_{j1}(t) \geq 0$, $\forall t \in [0,1]$ and $(\delta \pm \lambda_{22}\xi) \geq 0$ at $\delta \geq |\lambda_{22}|$, $|\xi| \leq 1$ into account, we conclude that $P_j \geq 0$.

Lemma 3.2.4. *Let $\tau_j(p) = 0$, $\tau_j(1) = 0$, $\delta = Im\lambda_2 = 0$ and the conditions (3.17) and (3.18) be fulfilled. Then for $d_j(t) \equiv 0$ the inequality*

$$\mathrm{Re}\int\limits_p^1 \bar{\tau}_j(t)\nu_j(t)dt \geq 0$$

is valid.

The correctness of the proposition of this lemma follows from Lemmas 3.2.2 and 3.2.3 by using formulas (3.23), (3.26) and the condition $\mathrm{Im}\lambda_2 = 0$.

Remark 3.1. From Theorem 3.2.2 for $b_j(t) \equiv 0$, $c_j(t) \equiv 0$ or $a_j(t) \equiv 0$, $c_j(t) \equiv 0$, $j = 1, 2$ the uniqueness of the solution of the Tricomi problem for the equation $(L.B)$ in the domain Δ follows.

From Remark 1.3 and Theorem 3.2.2 follows

Proposition. If $\lambda_1 = \lambda_2 = \lambda$ and one of the conditions (3.15) or (3.16) are fulfilled, then the problem Γ_0^λ (consequently, the Tricomi problem) con have eigenvalues only outside of the domain $D_1 = \left\{\lambda : |\mathrm{Re}\lambda| \geq \sqrt{2}\,|\mathrm{Im}\lambda|\right\}$.

We go over to proving existence of the solution of the problem Γ_0^λ. We assume that the conditions of group 3) of Theorem 3.2.2 are fulfilled, and for simplicity in addition we assume that

$$\left[a_1(t)-b_1(t)\right]/\alpha_1(t) = \left[a_2(t)-b_2(t)\right]/\alpha_2(t) = A(t) \neq 0.$$

Considering problem N for the equation $(L.B)$ in the domain Δ_0, we obtain formula (3.14), from which assuming at first $y=0$, and in the next $x=0$, we obtain the functional relation between $\tau_j(t)$ and $\nu_j(t)$ on the segments I_1 and I_2, attained from Δ_0

$$\tau_j(t)+\int\limits_p^1 \nu_j(z)G(z,0;t,0)dz+\int\limits_p^1 \nu_k(z)G(0,z;t,0)dz$$

$$+\int\limits_p^1 \nu_1(z)H_{j1}(t,z)dz+\int\limits_p^1 \nu_2(z)H_{j2}(t,z)dz=f_j\,(t),\quad p\le t\le 1, \tag{3.28}$$

where $j,k=1,2,\quad j\neq k,$

$$\begin{aligned}
H_{11}(x,z)&=\iint\limits_{\Delta_0} R(\xi,\eta;x,0)G(z,0;\xi,\eta)d\xi d\eta;\\
H_{12}(x,z)&=\iint\limits_{\Delta_0} R(\xi,\eta;x,0)G(0,z;\xi,\eta)d\xi d\eta;\\
H_{21}(y,z)&=\iint\limits_{\Delta_0} R(\xi,\eta;0,y)G(z,0;\xi,\eta)d\xi d\eta;\\
H_{22}(y,z)&=\iint\limits_{\Delta_0} R(\xi,\eta;0,y)G(0,z;\xi,\eta)d\xi d\eta;
\end{aligned}$$

$$\left.\begin{aligned}
f_1(x)=\sum_{j=1}^{2}\int\limits_{\sigma_{oj}}\varphi_j(\xi,\eta)\Bigg[&\frac{\partial}{\partial n}G(\xi,\eta;x,0)\\
&+\iint\limits_{\Delta_0} R(\xi',\eta';x,0)\frac{\partial}{\partial n}G(\xi,\eta;\xi',\eta')d\xi'd\eta'\Bigg]ds;\\
f_2(y)=\sum_{j=1}^{2}\int\limits_{\sigma_{oj}}\varphi_j(\xi,\eta)\Bigg[&\frac{\partial}{\partial n}G(\xi,\eta;0,y)\\
&+\iint\limits_{\Delta_0} R(\xi',\eta';0,y)\frac{\partial}{\partial n}G(\xi,\eta;\xi',\eta')d\xi'd\eta'\Bigg]ds.
\end{aligned}\right\} \tag{3.29}$$

By finding formula (3.28) takin into account that $G(z,0;t,0)=G(0,z;0,t)$ and $G(z,0;0,t)=G(0,z;t,0)$.

If we eliminate $\tau_j(t)$ from (3.28) and (3.5_j), and in the next differentiate them with respect to t, we obtain

$$A(t)\nu_j(t)+\int\limits_p^1 \nu_j(z)\frac{\partial}{\partial t}G(z,0;t,0)dz+\int\limits_p^1 \nu_k(z)\frac{\partial}{\partial t}G(0,z;t,0)dz$$

$$+\int_p^1 \nu_1(z)M_{j1}(t,z)dz + \int_p^1 \nu_2(z)M_{j2}(t,z)dz = f'_{1j}(t), \tag{3.30}$$

where

$$M_{jj}(t,z) = \begin{cases} \dfrac{\partial}{\partial t}\left[H_{jj}(t,z) + \dfrac{a_j(t)}{\alpha_j(t)}J_0[\lambda_2(t-z)]\right] & \text{at } z \le t, \\ \dfrac{\partial}{\partial t}\left[H_{jj}(t,z) + \dfrac{b_j(t)}{\alpha_j(t)}J_0[\lambda_2(z-t)]\right] & \text{at } z \ge t, \end{cases}$$

$$M_{jk}(t,z) = \frac{\partial}{\partial t}H_{jk}(t,z), \qquad j \ne k, \quad j,k = 1,2,$$

$$f_{1j}(t) = f_j(t)$$

$$+\left\{\tau_j(p)a_j(t)J_0\left[\lambda_2(t-p)\right] + \tau_j(1)b_j(t)J_0\left[\lambda_2(1-t)\right] - 2d_j(t)\right\}/\alpha_j(t).$$

Introducing the notation $\nu_1(x) - \nu_2(x) = \mu_1(x)$, $\nu_1(x) + \nu_2(x) = \mu_2(x)$, we rewrite the system (3.30) in the form

$$A(t)\mu_j(t) + \frac{1}{\pi}\int_p^1 K(t,z)\mu_j(z)dz$$

$$+\int_p^1 L_{j1}(t,z)\mu_1(z)dz + \int_p^1 L_{j2}(t,z)\mu_2(z)dz = F_j(t), \quad p < t < 1, \tag{3.31}$$

where

$$K(t,z) = \sum_{k=-\infty}^{+\infty}\left[\frac{p^{2k}}{z - p^{2k}t} - \frac{p^{2k}z}{1 - p^{2k}tz}\right], \tag{3.32}$$

$$L_{jj}(x,t) = \frac{\partial}{\partial x}\left[\frac{2\ln x \ln t}{\pi \ln p} + \frac{1}{\pi}\ln\left|\frac{1+xt}{x+t}\prod_{k=1}^{+\infty}\frac{(1+p^{2k}xt)(xt+p^{2k})}{(t+p^{2k}x)(x+p^{2k}t)}\right| \mp G(0,t;x,0)\right]$$

$$+\frac{1}{2}\left[M_{11}(t,z) + M_{22}(t,z) \mp M_{12}(t,z) \mp M_{21}(t,z)\right],$$

$$L_{jk}(t,z) = \frac{1}{2}\left[M_{11}(t,z) - M_{22}(t,z) \pm M_{12}(t,z) \mp M_{21}(t,z)\right],$$

$$F_j(t) = f_{11}(t) \mp f_{12}(t), \quad j,k = 1,2, \quad j \ne k.$$

From the form of the functions $L_{jk}(t,z)$ follows, that this functions are continuous in the rectangle $\{p \le t, z \le 1\}$ and $\dfrac{\partial}{\partial t}L_{jk}(t,z)$ exists for $t \neq z$, and have logarithmic singularities at $t = z$.

Further, using (3.4) and the condition on $a_j(t)$, $b_j(t)$, $c_j(t)$, and $d_j(t)$, it is not difficult to verify, that $F_j(t) \in C[p,1] \cap C^{(1,r)}(p,1)$, $j = 1,2$. From here and from the formulation of the problem Γ_0^λ follows, that the solution of the equation (3.31) must be found in the class of functions $\mu_j(t) \in C^{(1,r)}(p,1)$, which con have a singularity of order less than one as $t \to p$, if $A(t) < 0$ and at $t \to 1$, if $A(t) > 0$.

In this class the solution of the equation (3.31) exists and is determined by the formulas [23]

$$\mu_j(t) = \frac{A(t)Q_j(t)}{1+A^2(t)} - \frac{D(t)}{[1+A^2(t)]\,\pi}\int_p^1 \frac{Q_j(z)}{D(z)}K(t,z)dz, \qquad t \in (p,1), \tag{3.33}$$

where

$$Q_j(t) = F_j(t) - \sum_{k=1}^{2}\int_p^1 L_{jk}(t,z)\mu_k(z)dz, \qquad j,k = 1,2,$$

$$D(t) = \left[\prod_{n=-\infty}^{+\infty} \frac{(p^{2n-1}-t)(p^{2n+1}-t)}{(p^{2n}-t)^2}\right]^{\frac{1}{\pi} arctg \frac{1}{A(t)}}.$$

If we return to the functions $\nu_j(t)$, we obtain a system of Fredholm integral equations of the second kind. Unconditional solvability of the system follows from the uniqueness of the solution of the problem Γ_0^λ.

After having found $\tau_j(t)$ and $\nu_j(t)$, the solution of the problem Γ_0^λ in the domains Δ_0 and Δ_1, Δ_2 is defined as the solution of the problem N and the Cauchy problem for the equation $(L.B)$ and it is given by the formulas (3.14) and (1.5_j) respectively.

Let the conditions of group 1) (2)) of the Theorem 3.2.2 be fulfilled. We require in addition that the condition $d_j(p) = 0$ $(d_j(1) = 0)$ is fulfilled. Then analogously to the previous, we obtain a singular integral equation in the form (3.31). Solving the obtained equation we determine the function $\nu_j(t)$, moreover it can have a singularities of order less than one at $t \to p$ and $t \to 1$, as by virtue of the conditions (3.15) ((3.16)) $A(t) = a_j(t)/\alpha_j(t) > 0$ $(A(t) = -b_j(t)/\alpha_j(t) < 0)$.

With this ended our investigation of the problem Γ_0^λ for $\alpha_j(t) \neq 0$, $j = 1,2$.

3.2.3. Investigation of the problem Γ_0^λ for $\alpha_j(t) \equiv 0,\ \ \alpha_k(t) \neq 0,\ \ j,k = 1,2,\ \ j \neq k$

Let $\alpha_1(x) \equiv 0$, $\alpha_2(y) \neq 0$ and the functions $a_1(x)$, $b_1(x)$, $c_1(x)$ satisfy one of the conditions a), b), c) of Theorem 3.2.1 for $j = 1$, and the functions $a_2(y)$, $b_2(y)$, $c_2(y)$ satisfy one of the conditions 1), 2), 3) of Theorem 3.2.2 for $j = 2$. Then for the homogenous problem Γ_0^λ we have

$$\nu_1(x) \equiv 0 \qquad \text{and} \qquad \mathrm{Re}\int\limits_0^1 e^{2\delta y}\bar{\tau}_2(y)\nu_2(y)dy \geq 0,$$

that correspond to one of the cases of Theorems 3.2.1 and 3.2.2, respectively.

Consequently, and in this case fulfilling the condition $\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2$ from (3.6) follows that $\vartheta(x,y) \equiv 0$, $(x,y) \in \overline{\Delta}_0'$, $\omega(x,y) \equiv 0$, $(x,y) \in \overline{\Delta}_0''$, i.e. $u(x,y) \equiv 0$ in $\overline{\Delta}_0$. From here follows the uniqueness of the solution of the problem Γ_0^λ.

We go over to proving existence of the solution of the problem Γ_0^λ. For this by virtue of $\alpha_1(x) \equiv 0$, from Theorem 3.2.1, the function $\nu_1(x)$ is determined by one of the formulas (3.9)–(3.11), and for determination of $\nu_2(y)$ we have a singular integral equation analogously to (3.31):

$$A(y)\nu_2(y) + \frac{1}{\pi}\int\limits_p^1 K(y,z)\nu_2(z)dz + \int\limits_p^1 K_1(y,z)\nu_2(z)dz = \widetilde{F}(y), \tag{3.34}$$

where $K(y,z)$ is a kernel, determined by the formula (3.32) and

$$K_1(y,z) = \frac{\partial}{\partial y}\left[\frac{2\ln y \ln z}{\pi \ln p} - \frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\ln\frac{z + p^{2k}y}{1 + p^{2k}yz}\right] + M_{22}(y,z)$$

$$\widetilde{F}(y) = f_{22}(y) - \int\limits_p^1 \left[G_x(0,z;y,0) + M_{21}(y,z)\right]\nu_1(z)dz.$$

Turning to (3.34), by the formula (3.33) and using the uniqueness of the solution of the problem Γ_0^λ, we find analogously the function $\nu_2(y)$.

Further, solution of the problem Γ_0^λ in Δ_0 is determined by the formula (3.14), and in Δ_j $(j = 1,2)$ by the formulas $(1.5_j), j = 1,2$.

With this ended our investigation of the problem Γ_0^λ for $\alpha_1(x) \equiv 0,\ \alpha_2(y) \neq 0$.

§ 3.3. Investigation of the problem Γ_1^λ

Using the formula (1.5_j), ($j = 1,2$) in Δ_j and the conditions (3.3_j), ($j = 1,2$), after some operations we have

$$\beta_j(t)\nu_j(t) = -2d_j(t) + a_j(t)C_{pt}^{0,\lambda_2}\left[\tau_j(t)\right] - b_j(t)C_{1t}^{0,\lambda_2}\left[\tau_j(t)\right], \qquad (3.35_j)$$

where $\beta_j(t) = a_j(t) - b_j(t) - 2c_j(t)$, $p < t < 1$, $j = 1,2$, and $C_{mx}^{0,\lambda}$ is an integro-differential operator, determined by the formula (1.22).

Equalities (3.35_j), $j = 1,2$, provide a basic functional relation between $\tau_j(t)$ and $\nu_j(t)$ on the segments I_1 and I_2 respectively, attained from the hyperbolic part of the mixed domain Δ.

At proving uniqueness and existence of the solution of the problem Γ_1^λ we consider the cases, when $\beta_j(t) \equiv 0$, $j = 1,2$ and when $\beta_j(t) \neq 0$, $j = 1,2$.

3.3.1. Investigation of the problem Γ_1^λ for $\beta_j(t) \equiv 0, \;\; j = 1, 2$

Theorem 3.3.1. *Let $\beta_j(t) \equiv 0$, $j = 1, 2$. Then, if* $\mathrm{Re}\lambda_1^2 \geq 0$ *and one of the following group of conditions a) $b_j(t) \equiv 0$, $j = 1, 2$, b) $a_j(t) \equiv 0$, $j = 1, 2$, c) $a_j(t) \equiv 0$, $b_k(t) \neq 0$, $j, k = 1, 2$, $j \neq k$ is fulfilled, then the solution of the problem Γ_1^λ exists and is unique.*

Proof. Let $u(x, y)$ be a solution of the homogenous problem Γ_1^λ and $\beta_j(t) \equiv 0$. Then the relation (3.35_j) becomes

$$a_j(t) C_{pt}^{0,\lambda_2}\left[\tau_j(t)\right] - b_j(t) C_{1t}^{0,\lambda_2}\left[\tau_j(t)\right] = 0, \qquad p < t < 1.$$

From here fulfilling one group of conditions a), b), c) we obtain an integro-differential equation for the unknown functions $\tau_j(t)$. Further, taking $\tau_j(p) = 0$, $\tau_j(1) = 0$ into account, respectively, from the obtained equation we find that $\tau_j(t) \equiv 0$, $p < t < 1$.

On the other hand, by virtue of $u(x, y)|_{\overline{\sigma}_{0j}} = 0$, according to Lemma 3.2.1, the equality (3.6) is valid. Assuming $\delta = 0$ and taking into account $\mathrm{Re}\lambda_1^2 \geq 0$, $\tau_j(t) \equiv 0$, from (3.6) we obtain $\vartheta(x, y) \equiv 0$ in $\overline{\Delta}_0'$, $\omega(x, y) \equiv 0$ in $\overline{\Delta}_0''$, consequently $u(x, y) \equiv 0$ in $\overline{\Delta}_0$, from here follows uniqueness of the solution of the problem Γ_1^λ.

We go over to proving existence of the solution of the problem Γ_1^λ. We assume that the conditions of Theorem 3.3.1 are fulfill. Then relation (3.35_j) becomes

$$a_j(t) C_{pt}^{0,\lambda_2}\left[\tau_j(t)\right] - b_j(t) C_{1t}^{0,\lambda_2}\left[\tau_j(t)\right] = 2d_j(t), \quad p < t < 1. \tag{3.36_j}$$

a) Let $b_j(t) \equiv 0$, $j = 1, 2$. We assume in addition that

$$\int_p^1 \left[d_j(t)/a_j(t)\right] J_0[\lambda_2(1 - t)dt = 0, \quad j = 1, 2. \tag{3.37}$$

Then from (3.36_j) we obtain the equation

$$\tau_j(t) + \lambda_2^2 \int_p^t \tau_j(z) \frac{J_0[\lambda_2(t - z)]}{\lambda_2(t - z)} dz = 2\frac{d_j(t)}{a_j(t)}, \qquad j = 1, 2.$$

Taking into account $\tau_j(p) = 0$, from the last equation we find the function

$$\tau_j(t) = 2\int\limits_p^t [d_j(z)/a_j(z)]\, J_0[\lambda_2(t-z)]dz, \quad j = 1,2. \tag{3.38}$$

The condition (3.37) is an agreement condition in the point $t = 1$.

b) Let $a_j(t) \equiv 0$, $j = 1,2$. Then, analogous to the case a), assuming in addition

$$\int\limits_p^1 [d_j(t)/b_j(t)]\, J_0[\lambda_2(t-p)dt = 0, \quad j = 1,2, \tag{3.39}$$

and taking $\tau_j(1) = 0$, $j = 1,2$ into account, from (3.36_j), we find the function

$$\tau_j(t) = -2\int\limits_t^1 [d_j(z)/b_j(z)]\, J_0[\lambda_2(z-t)]dz, \quad j = 1,2. \tag{3.40}$$

(3.39) is an agreement condition in the point $t = p$.

c) Let $a_j(t) \equiv 0$, $b_k(t) \neq 0$, $j,k = 1,2$, $j \neq k$. Then the function $\tau_j(t)$ is found by one of the formulas (3.38) or (3.40), and $\tau_k(t)$ by the other of these formulas with the additional conditions (3.37) or (3.39), respectively.

Consequently, for $\beta_j(t) \equiv 0$, $j = 1,2$, and fulfilling one of the group of conditions a), b), c) the problem Γ_1^λ is equivalently reduced to the Direchlet problem for the equation $(L.B)$ in Δ_0. It is known that the solution of this problem exists, is unique and determined by the form (3.14), moreover [43]

$$u_0(x,y) = \sum_{j=1}^2 \int\limits_{\sigma_{oj}} \varphi_j(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,y)ds$$

$$+\int\limits_p^1 \tau_1(t)G_\eta(t,0;x,y)dt + \int\limits_p^1 \tau_2(t)G_\xi(0,t;x,y)dt, \quad (j = 1,2.), \tag{3.41}$$

$$G(\xi,\eta;x,y) = P(\omega,z) + P(-\omega,z) - P(\overline{\omega},z) - P(-\overline{\omega},z).$$

Here n is the inner normal to σ_{0j} relative to Δ_0, s is the arc length.

In the domains Δ_1 and Δ_2 the solution of the problem Γ_1^λ is determined by the formulas (1.5_1) and (1.5_2), moreover $\nu_1(x) = u_y(x,0)$ and $\nu_2(y) = u_x(0,y)$ are found from (3.14).

§ 3.3.2. Investigation of the problem Γ_1^λ for $\beta_j(t) \neq 0,\ j = 1, 2$

Theorem 3.3.2. *Let one of the following group of conditions be fulfilled*

1) $$\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2,$$

$$\beta_j(t) \neq 0,\ \ b_j(t) \equiv 0, \quad \frac{c_j(t)}{a_j(t)} < \frac{1}{2}, \quad \left(\frac{c_j(t)}{a_j(t)}\right)' \leq 0, \qquad t \in [p, 1]; \tag{3.42}$$

2) $$\mathrm{Re}\lambda_1^2 \geq (\mathrm{Im}\lambda_2)^2,$$

$$\beta_j(t) \neq 0,\ \ a_j(t) \equiv 0, \quad \frac{c_j(t)}{b_j(t)} > -\frac{1}{2}, \quad \left(\frac{c_j(t)}{b_j(t)}\right)' \leq 0, \qquad t \in [p, 1]; \tag{3.43}$$

3) $$\mathrm{Re}\lambda_1^2 \geq 0, \quad \mathrm{Im}\lambda_2 = 0,$$

$$\beta_j(t) \neq 0, \quad a_j(t) \not\equiv 0, \quad b_j(t) \not\equiv 0, \tag{3.44}$$

$$\left(\frac{a_j(t)}{\beta_j(t)}\right)' \leq 0, \qquad \left(\frac{b_j(t)}{\beta_j(t)}\right)' \leq 0, \quad t \in [p, 1]; \qquad \frac{a_j(1)}{\beta_j(1)} - \frac{b_j(p)}{\beta_j(p)} \geq 0. \tag{3.45}$$

Then for the problem Γ_1^λ cannot exist more than one solution.

The proposition of the Theorem follows from equality (3.6) and from the following lemma.

Lemma 3.3.1. *Let $\tau_j(p) = 0$, $(-\delta) \geq |\mathrm{Im}\lambda_2|$ and the condition (3.42) be fulfilled. Then for $d_j(t) \equiv 0$ inequality (3.19) is valid.*

Proof. Taking $b_j(x) \equiv d_j(x) \equiv 0$, and $a_j(x) \neq 0$ into account, from the equalities (3.35_j) we obtain the integro-differential equations

$$\tau_j'(t) + \lambda_2^2 \int\limits_p^t \tau_j(z) \frac{J_1[\lambda_2(t-z)]}{\lambda_2(t-z)} dz = \frac{\nu_j(t)}{a_{j2}(t)},$$

where $a_{j2}(t) = a_j(t)/\beta_j(t)$.

Turning into the last equation, with regard to $\tau_j(p) = 0$, we obtain $\tau_j(t)$ and substituting it in the integral (3.19), observing, that $a_{1j}(t)$ is a real valued function, as in Lemma 3.2.2, we reduce P_j to the form (3.24), moreover the function $a_{j1}(t)$ will change to $a_{j2}(t)$, and $\nu_j(t)$ to $\nu_j(t)/a_{j2}(t)$,

$j = 1, 2$. From here taking the condition $a_{j2}(t) > 0$, $a'_{j2}(t) \leq 0$, $\forall t \in [p, 1]$ and the inequality $(-\delta \pm \lambda_{22}\xi) \geq 0$ (as $(-\delta) \geq |\lambda_{22}|$, $|\xi| \leq 1$) into account, we conclude that the inequality (3.19) is valid.

Lemma 3.3.2. *Let* $\tau_j(1) = 0$, $\delta \geq |\text{Im}\lambda_2|$ *and the condition (3.43) be fulfilled. Then for* $d_j(t) \equiv 0$ *the inequality (3.19) is true.*

Proof. By virtue of $a_j(x) \equiv d_j(x) \equiv 0$ and $b_j(x) \neq 0$ we can rewrite the equality (3.35_j) in the form

$$-\tau'_j(t) + \lambda_2^2 \int_t^1 \tau_j(z) \frac{J_1[\lambda_2(z-t)]}{\lambda_2(z-t)} dz = \frac{\nu_j(t)}{b_{j2}(t)},$$

where $b_{j2}(t) = -b_j(t)/\beta_j(t)$. Taking $\tau_j(1) = 0$ into account, from the last integro-differential equation we find the function $\tau_j(t)$. Substituting it in the integral (3.19) and taking into account the fact, that $b_{j2}(t)$ is a real valued function, as in Lemma 3.2.3, we reduce it to the form (3.27), only here $b_{j1}(t)$ will change to $b_{j2}(t)$, and $\nu_j(t)$ to $\nu_j(t)/b_{j2}(t)$, $j = 1, 2$. Taking into account this and the conditions of lemma, we conclude that inequality (3.19) is valid.

Lemma 3.3.3. *Let* $\tau_j(p) = 0$, $\tau_j(1) = 0$, $\delta = \text{Im}\lambda_2 = 0$ *and the conditions (3.44) and (3.45) be fulfilled. Then for* $d_j(t) \equiv 0$ *the inequality*

$$\text{Re} \int_p^1 \overline{\tau}_j(t)\nu_j(t) \geq 0$$

is valid.

For proving this lemma simultaneously use the proof of Lemmas 3.3.1 and 3.3.2.

Remark 3.1. From the Theorem 3.3.2 for $b_j(t) \equiv 0$, $c_j(t) \equiv 0$, $(j = 1, 2)$ or $a_j(t) \equiv 0$, $c_j(t) \equiv 0$, $(j = 1, 2)$ the uniqueness of the solution of the Tricomi problem for the equation $(L.B)$ follows.

From Remark 1.3 and Theorem 3.3.2 follows the following

Proposition. If $\lambda_1 = \lambda_2 = \lambda$ and one of the conditions (3.42) or (3.43) is fulfilled, then the problem Γ_0^λ (consequently the Tricomi problem) con have eigenvalues only outside of the domain $D_1 = \left\{\lambda : |\text{Re}\lambda| \geq \sqrt{2}\,|\text{Im}\lambda|\right\}$.

We go over to prove existence of the solution of the problem Γ_1^λ. We assume that the condition 3) of Theorem 3.3.2 is fulfilled, and in addition we assume that the condition

$$a_j(t)+b_j(t)\neq 0,\ \ j=1,2,\qquad \frac{\beta_1(t)}{a_1(t)+b_1(t)}=\frac{\beta_2(t)}{a_2(t)+b_2(t)}=B(t)\neq 0 \tag{3.46}$$

or $a_j(t)+b_j(t)\equiv 0,\ \ j=1,2$ is fulfilled.

Let the condition (3.46) be fulfilled.

Consider problem N in the domain Δ_0 for the equation $(L.B)$ and obtain relation (3.28) between $\tau_j(t)$ and $\nu_j(t)$ $(j=1,2)$, attained from the elliptic part of the mixed domain Δ. Substituting (3.28) in (3.35_j), after some operations we obtain a system of integral equation for $\mu_j(t)=\nu_1(t)+(-1)^j\nu_2(t)$, which is equivalent to problem Γ_1^λ:

$$B(t)\mu_j(t)+\frac{1}{\pi}\int\limits_p^1 K(t,z)\mu_j(z)dz$$

$$+\int\limits_p^1 T_{j1}(t,z)\mu_1(z)dz+\int\limits_p^1 T_{j2}(t,z)\mu_2(z)dz=\Gamma_j(t), \tag{3.47}$$

where

$$\Gamma_j(t)=\gamma_1(t)+(-1)^j\gamma_2(t),\qquad j=1,2,\qquad t\in[p,1],$$

$$\gamma_j(t)=f_j'(t)+\lambda_2^2\frac{a_j(t)}{a_j(t)+b_j(t)}\int\limits_p^t f_j(z)\frac{J_1\left[\lambda_2(t-z)\right]}{\lambda_2(t-z)}dz$$

$$-\lambda_2^2\frac{b_j(t)}{a_j(t)+b_j(t)}\int\limits_t^1 f_j(z)\frac{J_1\left[\lambda_2(z-t)\right]}{\lambda_2(z-t)}dz-\frac{2d_j(t)}{a_j(t)+b_j(t)},$$

the functions $f_1(x)$, $f_2(y)$ and $K(t,z)$ are determined by the formulas (3.29) and (3.32) respectively. $T_{jk}(t,z)$ are expressed by the well-known function $a_j(t)$, $b_j(t)$, $c_j(t)$, $J_1[\lambda_2(t-z)]$, $G(\xi,\eta;x,y)$ and $R(\xi,\eta;x,y)$, moreover they are continuous in the rectangle $p\leq t$, $z\leq 1$, continuously differentiable for t if $t\neq z$ and $\dfrac{\partial}{\partial t}T_{jk}(t,z)$ have logarithmic singularities at $t=z$.

Using (3.4) and the condition on $a_j(t)$, $b_j(t)$, $c_j(t)$ and $d_j(t)$, we can prove that $\Gamma_j(t)\in C[p,1]\cap C^{(1,r)}(p,1),\ \ 0<r\leq 1,\ \ j=1,2.$

From the formulation of the problem Γ_1^λ and the properties of the function $\Gamma_j(t)$ follows that we must find the solution of the equation (3.47) in the class of functions

$\mu_j(t) \in C^{(1,r)}(p,1)$, $j = 1,2$, which can have singularities of order less than one at $t \to p$ and $t \to 1$.

Turning to equation (3.47) (analogously to (3.31)) by the formula (3.33) and if we return to the functions $\nu_1(x)$ and $\nu_2(y)$, we obtain a system of Fredholm integral equations of the second kind, solvability of which follows from Theorem 3.3.2. The found $\nu_1(x)$ and $\nu_2(y)$ depending on $B(t) > 0$ or $B(t) < 0$ can have singularities of order less than one as $t \to p$ or $t \to 1$, respectively.

In this case, when $a_j(t) + b_j(t) \equiv 0$ $(j = 1,2)$, substituting (3.28) in (3.35_j) respectively, we obtain a system of Fredholm integral equations of the second kind for $\nu_j(t)$ $(j = 1,2)$, the solvability of which follows from Theorem 3.3.2.

Remark 3.2. Analogously to the previous paragraph, solvability of the problem Γ_1^λ can be investigated in that case, when $\beta_j(t) \equiv 0$, $\beta_k(t) \neq 0$, $j,k = 1,2$, $j \neq k$.

References

1. **Alimov, Sh.A.** *About one spectral problem of Bitsadze-Samarskiy type.* Dokl. Acad. Nauk SSSR, 1986, Vol 287, No 6, pp. 1289-1290. (in Russian).
2. **Achieser, N.I.** *Elements of the theory of elliptic functions.* Moscow:, Nauka, 1970. (in Russian).
3. **Babenko, K.I.** *To the theory of mixed type equations.* Successes of mathematical sciences. 1953, Vol 80, No 2.(in Russian).
4. **Bers, L.** *Mathematical questions subsonic and transonic of gaseous dynamics.* Moscow:, IL, 1961, pp. - 208. (in Russian).
5. **Bitsadze, A.V., Samarskiy, A.A.** *About some simple generalizations of linear elliptic boundary value problems.* Dokl. Acad. Nauk SSSR, 1969, Vol 185, No 4, pp. 739-740. (in Russian).
6. **Bitsadze, A.V.** *To the problem of mixed type equation.* Works of MI Acad. Nauk SSSR, 1953, Vol. 41, pp. - 59. (in Russian).
7. **Bitsadze, A.V.** *Boundary value problems for elliptic equation of second kind.* Moscow:, Nauka, 1966, pp. - 203. (in Russian).
8. **Bassanini, P., Calaverni, M.** *Contrazioni multi systems iperbolici, eprobemia del laser.* Atti Semin. mat. e. fis. Univ. Madena:, 1982, Vol. 31, № 1, p. 32-50.
9. **Chapligin, S.A.** *About gaseous sprays.* Total essays collection, Moscow:, 1933, Vol. 2. (in Russian).
10. **Djuraev, T.D.** *Boundary value problems for mixed and mixed-composite type equations.* Tashkent:, FAN, 1979, pp. - 240. (in Russian).

11. **Djuraev, T.D., Sopuev, A., Mamajonov, M.** *Boundary value problems for parabolic-hyperbolic type equations.* Tashkent:, FAN, 1986. pp. - 220. (in Russian).

12. **Frankl, I.** *Selected works of gaseous dynamics.* Moscow:, Nauka, 1973, pp. - 711. (in Russian).

13. **Hellerstedt, S.** *Problems for the equation* $y^m u_{xx} + u_{yy} = 0$. Arkiv f. M. A. F. 26 A, - 1938, Vol. 3, pp. 1-32.

14. **Ilin, V.A., Moiseev, E.I.** *Non-local boundary value problems for Sturm-Liouvile operator in differential and difference of treatments.* DAN SSSR, 1986, Vol. 291, № 6, pp. 534-539. (in Russian).

15. **Kal'menov, T.S.** *About spectrum of the Tricomi problem for Lavrent'ev-Bitsadze equation.* Differential equation. 1977, Vol. 13, № 8, pp. 1418-1425. (in Russian).

16. **Kuznetsov, D.S.** *Special functions.* Visshaya shkola, Moscow:, 1965. pp 424. (in Russian).

17. **Kuzmin, A.G.** *Non-classic equation of mixed type and their applications to gas dynamics.* Leningrad:, Publisher LGU, 1990, pp. - 208. (in Russian).

18. **Lavrent'ev, M.A., Bitsadze, A.V.** *To problems of mixed type equations.* Dokl. Acad. Nauk SSSR, 1950, Vol. 70, № 3, pp. 373-376. (in Russian).

19. **Mikhlin, S.G.** *Lectures on linear integral equations.* Fizmatgiz, Moscow:, 1959. pp. 232. (in Russian).

20. **Moiseev, E.I.** *Mixed type equation with spectral parameter.* Izdatelstvo MGU, Moscow:, 1988, pp. 150. (in Russian).

21. **Moiseev, E.I.** *Some questions of spectral theory of mixed type equations.* Doctoral thesis, Moscow, 1979 (in Russian).

22. **Moiseev, E.I.** *On basisness of a system of sines.* Differential equation. 1987, V. 23, No. 1, pp. 177-179 (in Russian).

23. **Muskheleshvili, N.I.** *Singular integral equations.* Noordhoff, Groningen, 1953.

24. **Nakhushev, A.M.** *About one approximate method of solving boundary value problems for differential equations and it's applications to dynamics.* Differential equation, 1982, Vol. 18, № 1, pp. 72-81. (in Russian).

25. **Nakhushev, A.M.** *Equations of mathematical biology.* Moscow:, Visshaya shola, 1995, pp. - 301. (in Russian).

26. **Nakhushev, A.M.** *About some boundary value problems for hyperbolic and mixed type equations.* Differential equations, 1969, Vol. 5, № 1, pp. 44-59.

27. **Ponomarev, S.M.** *Spectral theory of the main boundary value problems for mixed type Lavrent'ev-Bitsadze equation.* Doctoral thesis, Moscow:, 1981

28. **Prudnikov, A.P., Brichkov, Y.A., Marichev, O.I.** *Integrals and series. Special functions.* Moscow:, Nauka, 1983, pp. - 752. (in Russian).

29. **Pulatov, A.K.** *About one non-local problem of Bitsadze-Samarskiy.* Dokl. AN UzSSR, 1985, № 8, pp. 4-5. (in Russian).

30. **Sabitov, K.B.** *Extremal property of module solution of one class of systems of mixed type equations.* Dokl AN SSSR, 1990, Vol 310, No 1, pp. 33-36. (in Russian).

31. **Sabitov,K.B., Karamova, A.A.** *Spectral properties of solutions of the Tricomi problem for the mixed type equation with two lines of changing type and their applications.* Izvestiya AN SSSR, ser. "Matematika", 2001, V.65, No. 4, pp. 133-150 (in Russian).

32. **Sabitov, K.B., Tichomirov, V.V.** *About the structure of eigenvalues and eigenfunctions of gas dynamics of the Frankl problem.* Mathematical simulation, 1990, Vol. 2, № 10, pp. 100-109.

33. Salakhitdinov, M.S. *Equations of mixed-composite type.* FAN, Tashkent:, 1974, pp.156. (in Russian).

34. Salakhitdinov, M.S., Tolipov, A. *About some boundary value problems for one class of mixed type equations.* Differential equation, 1973, Vol. 9, No 1, pp. 142-148. (in Russian).

35. Salakhitdinov, M.S., Tolipov, A. *About some boundary value problems for mixed type equations with one and two lines degeneration.* Differential equation, 1972, Vol. 8, No 1, pp. 134-142. (in Russian).

36. Salakhitdinov, M.S., Urinov, A.K. *Boundary value problems for mixed type equations with a spectral parameter.* FAN, Tashkent:, 1997, pp. 166.

37. Salakhitdinov, M.S., Urinov, A.K. *About one non-local boundary value problems for mixed type equation with non-smooth lines of degeneration.* Dokl. AN SSSR, 1982, Vol. 262, № 3, pp. 539-541. (in Russian).

38. Smirnov, M.M. *Mixed type equations.* Moscow:, Visshaya shkola, 1985. pp. 304. (in Russian).

39. Tikhonov, A.N., Samarskiy, A.A. *Equations of the mathematical physics.* Nauka, Moscow:, 1966, pp. - 736. (in Russian).

40. Toshmirzaev, Yu.U. *To the theory boundary value problems for mixed type equations with non-smooth lines of degeneration.* Ph. D. Thesis. Tashkent:, 1979. pp. 94. (in Russian).

41. Tricomi, Francesco. *Sulle equazoni lineari alle derivate parziali di 2 ordine di tipo misto.* Mem. Lincei. Ser. V, XIV. fasc. VII: Translated into Russian (About linear partial differential equations of second order of mixed type). Gostekhizdat, Moscow-Leningrad:, 1947. pp. - 140.

42. Whittaker E.T., Watson G.N. *Course of modern analysis.* Moscow:, Physic-mathematical literature, 1963, pp. - 516. (in Russian).

43. Urinov, A.K. *Non-local boundary value problems for mixed type equations in simple connected and double-connected domains.* Ph. Doc. Physical Mathematics Science, Tashkent:, 1993, pp.- 260. (in Russian).

44. Urinov, A.K. *Boundary value problems for mixed type equations with non-smooth lines of degeneration.* Ph. D. Thesis. Tashkent:, 1983, pp. 124.

45. Urinov, A.K. *Boundary value problems for mixed type equation with non-smooth lines of degeneration.* News of AN UzSSR, 1987, № 6, pp. 22-27 .

46. Watson, G.N. *A Treatise on the Theory of Bessel Functions.* 1945, Cambridge Univ. Press, Cambridge.

47. Zaynulabidov, M.M. *About some boundary value problems for mixed type equation with two perpendicular lines of degeneration.* Differential equations, 1969, Vol. 5, № 1, pp. 91-99. (in Russian).

48. Salakhitdinov, M.S., Mamajonov, J.D. *Uniqueness of the solution of the Tricomi problem and non-local problems for the Lavrent'ev-Bitsadze equation with two perpendicular lines of degeneration with spectral parameter.* Proceedings of intern. Conf. "Spectral theory of differential operators and related problems". Ufa:, 2003, pp. 107–113 (in Russian).

49. Salakhitdinov, M.S., Mamajonov, J.D. *About uniqueness of the solution of Tricomi problem for the Lavrent'ev-Bitsadze equation with two perpendicular lines of degeneration with spectral parameter.* Works of International Russian-Kazakhs symposium "Mixed type equation and relative problems of analysis and informatics", Nal'chik:, 22-26th of May, 2004, pp. 160-163. (in Russian).

50. **Mamajonov, J.D.** *Uniqueness of the solution of the boundary value problem with displacement for mixed type equation with non-smooth lines of degeneration with spectral parameter.* "Republican conference of young mathematics scholars, which was devoted to 125 years of academician V.I.Romanovskiy" Thesis lectures, 9th December 2004, Tashkent:, pp. 56-58. (in Russian).

51. **Mamajonov, J.D.** *Uniqueness of the solution of the boundary value problem with displacement for mixed type equation with non-smooth lines of changing type with spectral parameter.* Uzbek Mathematical Journal, 2005, № 2, pp. 20-26. (in Russian).

52. **Mamajonov, J.D.** *Spectral properties of the solution of non-local problems for mixed type equation with non-smooth lines of changing type.* Uzbek Mathematical Journal, 2005, № 4, pp. 61-71. (in Russian).

53. **Mamajonov, J.D.** *Problems on eigenvalues for Lavrent'ev-Bitsadze equations with two perpendicular lines of changing type.* Dok. AN RUz., 2005, № 6, pp. 3-6. (in Russian).

54. **Mamajonov, J.D.** *About one Bitsadze-Samarskiy problem for elliptic-hyperbolic equation with two lines of changing type with spectral parameter.* Uzbek Mathematical Journal, 2006, № 2, pp. 59-69. (in Russian).

55. **Mamajonov, J.D.** *Boundary value problems with displacement for the Lavrent'ev-Bitsadze equation with two perpendicular lines of changing type with spectral parameter.* Complex Variables and Elliptic Equations, An International Journal, Vol 51. № 7, 2006, p 683-699.

Zusammenfassung

Wesentliche nichtlokale Randwertprobleme werden für elliptisch-hyperbolische Gleichungen mit Spektralparameter im Viertelkreis und Viertelring untersucht, wobei der Gleichungstyp sich auf einer nicht glatten Kurve ändert. Die Probleme werden auf äquivalente Systeme von Integral- und gelegentlich Integrodifferentialgleichungen überführt. Methoden für partielle Differentialgleichungen, aus der Spektraltheorie linearer Operatoren, der Theorie der singulären Integralgleichungen, der komplexen Analysis, das Energieintegral und auch das Extremalprinzip werden zur Lösung des erhaltenen Systems angewandt. Zum Auffinden eines Systems von Eigenfunktionen wird die Methode der Trennung der Variablen verwandt. Folgende neue Ergebnise werden erzielt:

1. Für den komplexen Parameter λ werden Bedingungen angegeben, die die Eindeutigkeit der Lösung sichern. Ausserdem wird in der komplexen Ebene ein Gebiet für den Parameter λ angegeben, ausserhalb dessen die untersuchten nichtlokalen Probleme Eigenwerte haben können.

2. Für Eindeutigkeit und Exsistenz der Lösungen werden hinreichende Bedingungen formuliert.

3. Eigenwerte und zugehörige Eigenfunktionen werden für eines der allgemeinen gemischten Probleme gefunden, in dem auf dem Rand des elliptischen Teilgebiets die dritte Randbedingung und für den hyperbolischen Teil mittels Integraloperatoren ausgedrückte nichtlokale Bedingungen gegeben sind. Ausserdem wird die Vollständigkeit des Systems der Eigenfunktionen im Raum L_2 bewiesen.

4. Eine neue Methode zum Existenzbeweis für die Lösungen des betrachteten Problems wird entwickelt, indem Eigenfunktionen in offensichtlicher Form als Lösungen des formulierten Problems in dem Fall angewandt werden, in dem die Eindeutigkeit der Lösung des Problems gegeben ist.