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Arbitrage and non-linear taxes

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Abstract

Incorporating a progressive income tax into an economic decision problem raises
the question whether this tax does not create arbitrage opportunities. We investigate
this problem in a riskless (multi-period) economy. With a convex tax function we
identify a particular kind of arbitrage (called bounded arbitrage): In this case the
gain achievable through arbitrage trade is limited and cannot reach infinity.We are
able to give a complete characterisation based on prizes of the traded assets as to
whether bounded as well as unbounded arbitrage opportunities will exist.

Keywords No-arbitrage with taxation · Fundamental theorem of asset pricing · Non-
constant tax rates · Application of convex optimization problems

JEL Classification C61 · E62 · G12 · H24

1 Introduction

1.1 Nonlinear taxes and decision making

One cannot ignore taxes when making economic decisions. This is true in particular
for taxes at the personal level that will be the subject of our paper. If we want to con-
ceive individual behavior when a personal income tax is present we need to under-
stand the impact of this levy. Typically, the literature assumes a linear income tax

(a) US American Federal income tax rates regulated by 26 U.S. Code §1c.

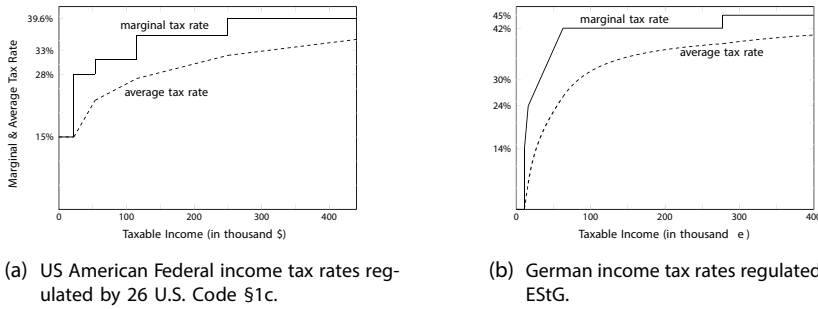(b) German income tax rates regulated by §32a EStG.

Fig. 1 Comparison of marginal and average tax rates for the US American and German income tax code 2022 (for single households)

(i.e., the tax liability being a constant multiple of the tax base) since it is straightforward to handle.[1] Exactly this simplification is the focus of our paper because we do not know any personal income tax worldwide that is linear.

If a tax is nonlinear then the first step can be to linearize the tax—resulting in a range of tax rates that might be applicable in the decision making process. One could argue that the variation of these rates is not that sizeable and hence tax nonlinearity might be neglected. That is not true, national tax rates are typically progressive—and tax rates vary within a wide range.

The tax code of two large economies illustrates this point. The current statutory federal tax rate in the US on interest income increases from 0 % to 39.6 % (cf. Fig. 1a) and had reached a peak of 90 % between 1953 and 1963.[2] The German income tax offers a progressive tax rate going from 0 % to 45 % (cf. Fig. 1b). It is very difficult to find empirical evidence for some "natural tax rate" that could be used in a linearization.[3] Thus the impact of tax nonlinearity becomes pertinent.

Individual economic decisions are typically formulated as a (utility) maximization problem. We specifically investigate the impact of a nonlinear tax on this maximization problem. Of course, the first and most important question is whether a nonlinear income tax does not generate arbitrage opportunities, making utility maximization impossible (i.e., the maximization problem would have no optimal solution). Such a tax arbitrage can arise for various reasons

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[1] Two examples shall illustrate that such linear transformations do not change the analysis. Brennan (1970) writes "...we assume for simplicity that each investor has marginal tax rates on dividend and capital gains income t_{di}, and t_{gi} which are constant and independent of their portfolio choice" Similar to this, Bradford (2000) states "linearity is a desideratum of a tidy tax system."

[2] See taxfoundation.org for an overview of the statutory federal tax rate between 1913 and 2013.

[3] Sialm (2009) tried to identify marginal investment tax rates for US companies. The Institute of Auditors in Germany agreed on a standardized tax rate of 35 % for valuation purposes without reasoning, for details see Heintzen et al. (2008, in German).

(and often discussed under the keywords timing or qualification arbitrage, see Rosembuj 2011 or Brennan and McDonald 2015).

It is important to point out an immediate consequence of our approach. By looking at individual economic decisions, we have to assume that investors take prices as given. An equilibrium model that endogenizes these prices would only play a role in a second step, after we have fully determined the individual demand of all investors. Thus, in our paper prices will be given as exogenous quantities and we consider only one investor, not several different investors, or even an equilibrium.

To this end, we need to simplify our analysis, as for now we can only answer this question for the case under certainty; uncertainty raises (technical) issues that further complicate the investigation. Therefore, we will limit ourselves here only to the case of secure assets and it will be seen that the investigation is extensive enough.

Besides personal income taxes also corporate taxes play an important role. But we will disregard corporate taxes in our paper because they raise completely different issues and would unnecessarily complicate our analysis without gaining new knowledge.

1.2 Intuition of the main result

We start with an illustration of our main result. Whether a market is arbitrage-free or whether it allows arbitrage can be determined exclusively by analyzing prices of the traded securities. The typical procedure is as follows: One fixes some securities (the starting point) and then checks whether, relative to this starting point, other securities make it possible to realize a gain. If this is the case, there is an arbitrage; otherwise, the market is arbitrage-free.

The choice of the starting point seems arbitrary. In a multi-period model, for example, the starting point could be a zero-coupon bond or a coupon-paying bond. However, it is clear that at the end of the analysis, the property of the pricing system (arbitrage-free or allowing arbitrage) does not depend on the choice of this starting point. Rather, it follows from mathematical logic that the starting point determines, at best, whether the analysis becomes cumbersome or simple.

We will choose as the starting point of our investigation a coupon bond that pays a fixed interest rate of r_f throughout its life. The rate r_f corresponds to "the interest rate" of the model so that the price of the bond is normalized to 1. Then we will raise the question to what extent another risk-free asset generates arbitrage opportunities. We do not restrict the structure of the payments of the other asset.

In order to exemplify our result, we will assume the simplest conceivable case in which a nonlinear income tax occurs. We consider the case of one period ($t = 0$ and $t = 1$) as well as only two different assets. Furthermore, the investor has an endowment that pays w in $t = 1$ (for the sake of readability, we will already use the symbols from the latter part of our paper here).

We turn to the tax base (tax is paid only in $t = 1$). For the coupon bond the tax base will be the interest payment r_f. For the second asset its tax base can (and will) be different from its cash flow due to periodization. In order to be as generic as possible we assume that the tax basis is known, it will be denoted by a number b whereby we will not place any further restrictions in this number. If, for example, an asset is tax exempt its tax base will be zero, $b_t = 0$. Furthermore, the taxable amount of the endowment payment is given by $\bar{w}$.

Given the tax base the tax due is stipulated by a tax liability function $T(\cdot)$. To exemplify our main result, we assume that this function is differentiable and satisfies the usual Lagrange conditions.[4] Furthermore, we assume that T is strictly convex. This allows us to present an intuitive depiction of our main result.

Let h^0 be the amount of coupon bond that pays $1 + r_f$ in one period. The coupon bond costs 1 today. The investor holds h^1 of the second bond. Its price today is p and this bond pays x one period later. p determines whether the market is free of arbitrage or not.

Arbitrage is given iff a self-financing portfolio today earns a gain tomorrow, i.e., for some numbers h^0, h^1 we have

$$h^0 + h^1 p = 0$$
$$h^0(1 + r_f) + h^1 x + w - T(h^0 r_f + h^1 b + \bar{w}) > w - T(\bar{w}).$$

This condition is handled in a straightforward manner by substituting h^0 and solving the maximization problem

$$\sup_{h^1} \quad -h^1 p(1 + r_f) + h^1 x - T(-h^1 p r_f + h^1 b + \bar{w}) + T(\bar{w}).$$

By looking at the optimal h^1, we can immediately identify whether an arbitrage exists. The case $h^1 = 0$ corresponds to an objective function (gain of the self-financing portfolio) being zero or a market free of arbitrage. But for an optimal $h^1 \neq 0$ an arbitrage opportunity exists since the objective function must have a value above zero. By evaluating the optimal h^1 and using the usual FOC, we arrive at the characterisation

$$p = \frac{x - bT'(z)}{1 + r_f(1 - T'(z))}, \qquad \text{with } z := \bar{w} - h^1 p r_f + h^1 b. \tag{1}$$

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[4] Since we have worked out all mathematical details meticulously in the appendix, we will be faintly negligent with the formal elements in this section.

Equation (1) now clearly elucidates our intuition.

Without taxes (i.e., $T(\cdot) = 0$) it follows that only $p = \frac{x}{1+r_f}$ is compatible with the (well-known) no-arbitrage condition. This is straightforward.

If the tax is linear (i.e., $T(y) = \tau \cdot y$) we see that only $p = \frac{x-\tau b}{1+r_f(1-\tau)}$ is compatible with the no-arbitrage condition. This relation is also well-known in the literature.[5]

If the tax is nonlinear, the condition of no-arbitrage can be interpreted in the following manner. We use the just stated equation from the linear case for the observed price p as *definition* of an implied tax rate τ, i.e.,

$$p =: \frac{x - \tau b}{1 + r_f(1 - \tau)} \quad \text{or} \quad \tau := \frac{p(1 + r_f) - x}{p r_f - b}.$$

If this implied tax rate τ is equal to the marginal tax rate $T'(\bar{w})$ the optimal solution must be $h^1 = 0$ (because the tax liability is strictly convex) and hence the market is free of arbitrage. If the implied tax rate is not equal the marginal tax rate at the endowment $T'(\bar{w})$, then $h^1 \neq 0$ and hence an arbitrage opportunity must exist. In summary, it can be stated that a market is free of arbitrage if and only if the implied tax rate is equal to the marginal tax rate at the endowment. This is intuitive: If both rates are not equal, the investor can increase her taxable income and with it the marginal tax rate until the discrepancy between the tax rate implied by the prices and her own marginal tax rate is eliminated.

1.3 The outline of the paper

Once the intuition of the result is clear it should be realized that at least two assumptions made in the previous section are challenging at best:

- We assumed a differentiable and strictly convex tax liability function. We do not know of any country in the world where this function satisfies only differentiability, let alone strict convexity. Already having an allowance (as it is typically the case in any national tax code) with an otherwise linear tax violates strict convexity and leads to a kink and it is not clear what a "marginal tax rate" at this kink should be. Also, it is by no means straightforward that the mathematical technique that we used in our introductory example is still applicable in this case.

  Therefore, another assumption is necessary. To this end, we presuppose a tax liability function that comprises a very broad class of existing national tax codes. Apart from other minor technical conditions, we will rely on convex tax liabilities as a function of the tax base. Examples of convex tax liabilities include piecewise linear functions with two or more different tax rates (cf. the American federal income tax in 2022 as shown in Fig. 1a) or tax rates that are affine as in the case of German income tax system in 2022, see Fig. 1b. Tax allowances, as

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[5] For an example, see Ross (1987, p. 380).

is the case with the German capital income tax 2022 ("Abgeltungsteuer"), ensure convexity as well.

In this context we will clarify the relation between progressive tax rates and convexity. Both notions are not equivalent.

Furthermore, convexity in the tax liability forces us to generalize the classical arbitrage approach by augmenting the terminology. Typically an arbitrage opportunity is a riskfree gain that can be increased to an arbitrary scale. Once we find a trading strategy paying out a positive amount today without any expenses in the future, we can repeat this strategy over and over again and therefore become endlessly rich. When tax liabilities are non-linear, the situation is different. It may be the case that we find an arbitrage opportunity with a positive payment today and no cash outflow tomorrow, but the maximum amount of money obtained using this strategy multiple times is still limited to a constant $K < \infty$. Therefore, we call such an arbitrage opportunity *bounded* in contrast to the classic (unbounded) arbitrage opportunity. Such a bounded arbitrage is not possible in our intuitive example above.

- Arbitrage is typically considered in a setup with more than one period. Since we already provide all mathematical details using convex analysis it is only a minor step to include multiple periods. And it will also allow us to investigate problems of periodization, i.e., what effect the taxation of capital gains has on the asset's value.

Our main result will be a complete characterisation of prices that are compliant with the no-arbitrage principle for bounded as well as unbounded arbitrage opportunities. Moreover, we show that the absence of arbitrage opportunities is closely connected to the properties of the tax liability function—a link that has not been made in the literature so far.

1.4 Literature

The impact of non-linear taxation on asset prices (and therefore on investment decisions) has been examined to some extent.[6]

Miller (1977) is probably the first author who investigated personal and corporate income taxes. Although a progressive income tax is mentioned in his paper (p. 268) only constant tax rates are considered.

Schaefer (1982) is one of the first to analyze a non-linear tax liability. Concentrating on riskfree assets he uses the example of two investors; equilibrium prices necessarily exists if the marginal tax rates of both investors overlap. If net discount factors are not unique, arbitrage opportunities will exist and therefore an equilibrium is unattainable. To overcome this dilemma Schaefer assumes constraints on short sales showing that the introduction of non-linear tax liabilities enables clientele effects, i.e., allowing investors in different tax brackets to hold different amounts of assets and/or pay different prices for the same assets.

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[6] There is a long list of literature focusing on linear taxation which we will disregard here. For recent works on this topic see, e.g., Gallmeyer and Srivastava (2011).

From our point of view Schaefer's model reveals a fundamental weakness: marginal tax rates "to avoid pointless complication" are unique and for two different tax bases there are two different marginal tax rates, see Schaefer (1982, p. 168). Using this assumption Schaefer requires that marginal tax rates are strictly monotone which we prove to be equivalent to strict convexity—a property that does not seem to hold for any tax code in the world.

One important work combining arbitrage theory and taxes is Ross (1987). He develops a fundamental theorem for riskless as well as risky assets. The tax base comprises cash flows and he assumes a convex tax liability function. Ross differentiates between local and global arbitrage. An arbitrage is local (abbreviated LAO, see his definition 1 on p. 376) if it offers a classical arbitrage in the sense of a positive income with non-positive costs given a particular initial portfolio (endowment). For another investor with a different initial endowment, this arbitrage does not necessarily have to result in a positive income. An arbitrage is said to be global if it offers a local arbitrage at every endowment.

This concept is related to our approach, but there are important differences. What we call a bounded arbitrage is not defined in Ross (1987), but it resembles a LAO that is not "extendable" (see his definition 3 on p. 376). Ross (1987) is focused on the question whether conditions on LAOs imply the existence of a risk-neutral probability measure. He does not examine conditions for non-extendable LAOs and therefore cannot ascertain the connection between arbitrage opportunities and particular properties of the tax code as we do.

Furthermore, the definition of a local arbitrage seems to be more general and less restrictive than the usual arbitrage condition. And because Ross forbids such an opportunity, the absence of a local arbitrage is a stronger assumption than the classical one. Also, without proof Ross claims that his results can be transferred into the multi-period or continuous framework. But the associated problems are not easy to deal with. For example, Ross' assumption of a piecewise linear tax (his Lemma 5 on p. 378) rules out the case of the German income tax of 2022 having quadratic tax liability functions for small- to medium-sized income or any (partially) strictly convex tax function, see Ross (1987, p. 392).

Notice that the concept of bounded arbitrage is nothing new. A significant body of research is dedicated to the phenomenon that capital flows only slowly between markets to exploit existing arbitrage opportunities (see, for example, Duffie 2010; Oehmke 2011; Gromb and Vayanos 2018). In these models the arbitrageurs face illiquidity frictions between segmented markets. Hence, there arbitrage opportunities effectively turn out to be bounded. The main driver of boundedness in those models are illiquidity affects as well as market segmentation, while in our paper the arbitrage bound arises from progressive taxation.

Prisman (1986) investigates convex tax liabilities and includes transaction costs in the model (which we will neglect here). By using a standard dual theory of convex optimization, he is able to derive a non-linear and investor-specific pricing operator. Unlike us, Prisman does not work out the exact link between arbitrage-free asset prices and the structure of tax liability functions.

Dybvig and Ross (1986) generalize the model of Schaefer (1982) for arbitrary risky assets revealing that marginal tax rates may influence asset prices under the

premise of short sale constraints. Dammon and Green (1987) extend the model of Dybvig and Ross by permitting short sales and proving an existence theorem for a competitive equilibrium. The definition of no-tax arbitrage that Dammon and Green use is closely connected (in the one-period model) to our definition of bounded arbitrage. But again the authors fail to give a clear statement of the structure of prices that prevent tax-arbitrage.

Dermody and Rockafellar (1991) analyze non-linear taxes in a multi-period bond framework, including transaction costs. They assume future tax payments depend solely on prices rather than the specific amount that an investor holds. This assumption is critical. Investors can easily avoid paying taxes, which Dermody and Rockafellar have to rule out. Because long and short prices do not coincide in their model, the authors conclude that net discount factors (term structure) are not unique, and further "[t]here are strong mathematical reasons for believing that the non-uniqueness indicates underlying non-linearities in the behavior of value that cannot be captured by a single term structure", see Dermody and Rockafellar (1991, p. 32).

2 Arbitrage-free asset pricing under certainty

2.1 Convex tax liability function and its properties

In the analysis of income taxation, we have to rely on two basic concepts: tax base and tax liability. Let x denote the tax base. Then the tax liability is a function $T(x) : \mathbb{R} \longrightarrow \mathbb{R}$ that is also applied for a negative tax base, thus including the possibility of taxable losses. Real world tax codes rarely pay negative taxes back but define complicated carryback or -forward mechanisms. These mechanisms are hardly tractable mathematically, in particular because they often involve caps. On the other hand, if the investor has sufficient positive income from endowments, a negative tax base actually corresponds to a lower tax payment. With this in mind, we refrain from modeling a tax carryforward mechanism.

The following is a reasonable definition of a tax rate or tax scale:

Definition 1 (*Average tax rate*) An average tax rate (or also tax scale) is the function $t(\cdot) : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ for which

$$T(x) = t(x) \cdot x \tag{2}$$

holds.

At $x = 0$ the average tax rate is undefined. We will later ensure that $t(x) \in [0, 1)$ holds.

Marginal tax rates display the coefficient of incremental tax over incremental income. Without further assumptions, we cannot guarantee the differentiability of $T(\cdot)$ at all x. This happens to be the case when there are jumps in the tax rates or

in the presence of tax allowances, see Fig. 1. Thus the following definition only holds for bases in which the limits below exist.

Definition 2 (*Marginal tax rate, left- and right-derivative*) The marginal tax rate $T'(\cdot) : \mathbb{R} \to \mathbb{R}$ at a base x_0 is

$$T'(x_0) := \lim_{x \to x_0} \frac{T(x) - T(x_0)}{x - x_0},$$

whenever the limit exists and is unique. Further we call

$$T'_{-}(x_0) := \lim_{x \uparrow x_0} \frac{T(x) - T(x_0)}{x - x_0} = \sup_{x < x_0} \frac{T(x) - T(x_0)}{x - x_0}$$

and

$$T'_{+}(x_0) := \lim_{x \downarrow x_0} \frac{T(x) - T(x_0)}{x - x_0} = \inf_{x > x_0} \frac{T(x) - T(x_0)}{x - x_0}$$

left- and right-derivative respectively.

We call a tax function *differentiable* if the marginal tax rate exists at all $x_0 \in \mathbb{R}$ or equivalently if we have $T'_{-}(x_0) = T'_{+}(x_0)$ for all x_0. As stated above, differentiability clearly does not hold for a variety of tax functions.

We focus on *convex* tax liability functions.

Definition 3 (*Convex tax function*) A tax liability function $T(\cdot)$ is convex if for all $0 \le \lambda \le 1$ and $x, y \in \mathbb{R}$ we have

$$T(\lambda x + (1 - \lambda)y) \le \lambda T(x) + (1 - \lambda)T(y). \tag{3}$$

Many tax systems have the property of monotonically increasing average tax rates $t(x)$ in x. Such average tax rates will be denoted as *progressive*. The two concepts of convexity and progression are not logically equivalent. Convexity of a tax implies progression due to the *increasing slope property* since

$$\frac{T(x) - T(0)}{x - 0} = t(x)$$

is increasing on $\mathbb{R} \setminus \{0\}$, see Hiriart-Urruty and Lemaréchal (2001, proposition 6.1). But the opposite implication is not true; we have provided an example in the Appendix 1 where a progressive tax is not convex. In this respect the assumption of

convexity of the tax liability is somewhat stronger than the assumption of a progressive tax scale.

And it is perfectly reasonable to assume convexity: From an economic perspective this means that tax payers with higher taxable income are charged at higher marginal tax rates in order to ensure the so called "vertical tax justice".[7] Therefore, many tax codes are convex by law. As Graham and Smith Jr. (1999) point out, investors try to obtain the highest degree of convexity by exploiting the variety of interdependent tax laws and thereby flattening their tax payments which may result in strict convex tax liability functions. Vice versa, if we allow for non-convex tax functions, we allow for cases in which higher taxable income is marginally taxed at a lower rate than for smaller taxable income, leading to tax injustice.

Tax functions that are convex and defined on $\mathbb{R}$ are necessarily continuous, see Rockafellar (1997, corollary 10.1.1). This rules out the possibility of an exemption limit, where at a certain income level of F the whole income is taxed and not just the excess amount over F.[8]

For continuous tax functions convexity is equivalent the weaker statement of condition (3) only for $\lambda = \frac{1}{2}$ or

$$2T\left(\frac{x+y}{2}\right) \leq T(x) + T(y).$$

This inequality is quite intuitive when we look at the regulation of splitting income as it can be found in many European countries as well as in the US: Instead of taxing each spouse, the average income of the couple is taxed, thus resulting in tax savings.

We will furthermore assume that there shall be no tax paid without income, or conversely

$$T(0) = 0. \tag{4}$$

The condition is appropriate insofar as for typical tax functions with tax scales below one the property $T(x) \leq x$ holds for all $x \geq 0$. If $T(0) < 0$, we have an arbitrage for doing nothing. This obvious arbitrage opportunity cannot hold in our analysis.[9]

In the appendix (see subsection 2) we have shown a very general result covering convex tax liability functions that involves some mathematical concepts that require considerable effort (the so-called subdifferential). We believe that for almost every

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[7] Notice however, we observe particular (minor) cases in national tax law in which the above assumption is not satisfied and the tax liability function is not convex. One example is the German tax code. The reason is (see OECD 2006, p. 65, in German) that individuals (single parent or single-income families with two children) after an increase in income will face a marginal tax rate above 100 % (all transfer payment included) due to an *increase* of the average tax rate from 50 % to 55 %.

[8] A typical tax with exemption limit has the structure

$$T(x) = \begin{cases} 0 & \text{if } x \leq F, \\ \tau x & \text{otherwise.} \end{cases}$$

Note that the function is neither continuous at $x = F$ nor is it convex, but marginal tax equals average tax at all $x \neq 0$.

[9] A negative income tax would be a counterexample, see Friedman (1962, chapter XII).

application using existing national tax codes we can restrict ourselves to a subset of convex tax liability functions. To understand what we mean we look at the following example of a tax liability function:

$$T(x) = \begin{cases} x + 1 - \sqrt{x+1} & x \geq 0, \\ \frac{x}{2} & x < 0. \end{cases}$$

For positive income the above tax function has marginal tax rates of $1 - \frac{1}{2\sqrt{x+1}}$, thus monotone increasing towards 100 % without ever reaching its limit. For any national tax system such a behavior is atypical. To the contrary, we observe that all national income tax systems have constant tax rates at higher income, i.e., the tax function is an affine function on the boundaries. Including this property will allow for a much simpler characterisation of all arbitrage opportunities then the more general result shown in the appendix.[10] We assume this property from now on:

Definition 4 A tax function $T(\cdot)$ is *affine on the boundaries* iff there is a sufficiently large tax base $x_0 \gg 0$ and two numbers T^+, T^- such that for all $x > x_0$ we have

$$T(x) = T^+ + \tau_{max} \cdot x$$

and for all $x < -x_0$

$$T(x) = T^- + \tau_{min} \cdot x$$

with $0 \leq \tau_{min} \leq \tau_{max} < 1$ denoting the minimal and the maximal marginal tax rate.

Given these elements of the tax we can now turn to the description of our main model.

2.2 Assets and trading strategies

Our presentation of the model will follow the standard procedure in a multi-period setup. We consider a framework with different points in time denoted by $s = 0, 1 \ldots, S$. The investor's initial endowments at time $s = 1, \ldots, S$ are given by w_s. Investors are subject to payment of taxes on their taxable income. Given their endowment w_s, at time $s = 1, \ldots, S$ the amount $\bar{w}_s$ is taxable.

Tax liability comprises of a (time independent) convex function $T(\cdot)$ which is due to a yet-to-be-defined tax base. Furthermore, $T(0) = 0$ holds. Without trading the investor's net income stream is $w_s - T(\bar{w}_s)$.

In $s = 0, \ldots, S - 1$ it is possible to trade securities on the capital market. The theory of arbitrage-free asset pricing is characterised by a relative valuation: the price of one asset or trading strategy is determined by the price of another asset or trading strategy with known market prices, offering the same cash flows as the asset to be

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[10] We are grateful to Tyrrell Rockafellar who suggested these boundary conditions.

valued (also called duplication or replication principle). Both prices have to be identical; otherwise an arbitrage opportunity can be realized.

For this reason we will presume the existence of a coupon bond. The theory of arbitrage then explains how to compute prices of other assets. If we allow that the coupon bond can be sold and bought at any point in time (dynamic trading strategy), we can achieve any arbitrary (riskfree) payout structure after taxes. Therefore, under certainty it is sufficient to assume one single asset with given prices.

There exists a standard coupon bond with constant interest payments of the risk-free rate $r_f > -100\%$; the price for one coupon bond with taxes is normalized to one. If the investor holds the bond she receives cash flows which are given by a vector $x^0 = (r_f, \ldots, r_f, 1 + r_f) \in \mathbb{R}^S$. The price vector for one unit of the coupon bond is $p^0 = (1, \ldots, 1, 0) \in \mathbb{R}^{S+1}$. The terminal price, for reasons of convenience, is set to zero since there is no trade carried out at the last point in time. The tax base of the coupon bond is equal the interest payment r_f at any point in time.

We introduce a second asset with arbitrary payout structure and investigate which properties of prices lead to the no-arbitrage condition in our market. Cash flows of the second bond are denoted by $x^1 = (x_1^1, \ldots, x_S^1) \in \mathbb{R}^S$ where the index states the time of capital inflow. Prices for one unit are $p^1 = (p_0^1, \ldots, p_{S-1}^1, 0) \in \mathbb{R}^{S+1}$.

In most countries throughout the world, taxable income is not equal to actual cash flows because very detailed legal provisions must be considered (periodisation). Owners of securities might receive payments that do not have the character of dividends or (in this case) interest, such as repayment of capital. Such payments do not incur income taxes. So as not to mix such payments with taxable income, we introduce another variable b_s^1 that comprises the income tax base for the second asset. The only requirement is that at time s, the asset possesses an income tax base of

$$b_s^1.$$

This tax base can be the cash flow of the second asset, part of or the entire capital gain or even zero if the second asset is tax exempt. Which value is to be used for b_s^1 is determined by the tax law. Notice that the tax base of a standard coupon bond are given by $b^0 = (r_f, \ldots, r_f)$.

The total tax base of a portfolio amounts to the sum of all tax bases, weighted by portfolio holdings which we denote by $h^0 = (h_0^0, h_1^0, \ldots, h_{S-1}^0) \in \mathbb{R}^S$ for the first bond and, analogously, $h^1 \in \mathbb{R}^S$ for the second. We simplify $h_{-1}^0 = h_S^0 = 0$. After trading the securities our investor can implement an after-tax-withdrawal for consumption purposes of

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\begin{aligned}
\Delta_s(h, w, \bar{w}) := & -p_s^1 h_s^1 - p_s^0 h_s^0 + \left(p_s^1 + x_s^1\right) h_{s-1}^1 + \left(p_s^0 + x_s^0\right) h_{s-1}^0 + w_s \\
& - T\left(b_s^1 h_{s-1}^1 + b_s^0 h_{s-1}^0 + \bar{w}_s\right) \quad s = 1, \ldots, S.
\end{aligned}
\tag{5}
$$

Note that this definition differs from the usual model without taxation only in terms of the introduction of endowment-specific tax payments $T(\cdot)$. The cost of the trading strategy h is independent of w_s and given by

$$-\Delta_0(h) := p_0^1 h_0^1 + p_0^0 h_0^0$$

We define arbitrage as follows.

Definition 5 (*Arbitrage*) A trading strategy h is called arbitrage opportunity iff $\Delta_s(h, w, \bar{w}) \geq w_s - T(\bar{w}_s)$ for all $s = 1, \dots, S$ and $-\Delta_0(h) \leq 0$ where at least one inequality is strict. If no arbitrage opportunities exist, we call the market free of arbitrage.

In the theory of arbitrage without taxation, the initial endowments w_s are canceled out of the above definition and the investor trades as if her initial income is zero. The same holds true for linear taxes. Each arbitrage opportunity h then can be multiplied by a constant $\lambda > 0$ so that the arbitrage profits of h are multiplied by λ—arbitrage gains are theoretically unbounded. In the presence of convex taxes it is possible to realize arbitrage profits that cannot be increased on an arbitrary scale, i.e., the optimal arbitrage gain remains constant at a certain level.

To see this we focus on the following minimization problem (P) given by

$$\inf_{h \in \mathbb{R}^S} -\Delta_0(h)$$
$$\text{s.t.} \quad \Delta_s(h, w, \bar{w}) \geq w_s - T(\bar{w}_s) \quad s = 1, \dots, S, \tag{P}$$

where p^* denotes the optimal value (withdrawal) of (P). This problem determines the lowest cost for a strategy that pays at least as much as doing nothing. Since $h \equiv 0$ is feasible, the optimal value must be non-positive ($p^* \leq 0$). If an arbitrage opportunity exists we can assume without loss of generality that the optimal value is negative ($p^* < 0$).[11] By definition the arbitrage gain is $-p^*$. Arbitrage-free markets correspond to an optimal value of $p^* = 0$.

If taxes do not exist arbitrage opportunities always imply $p^* = -\infty$. If a convex tax function is present, it might be that the optimal solution of (P) is finite ($-\infty < p^*$). This gives rise to the following definition.

Definition 6 (*Bounded and Unbounded Arbitrage*) The market has *bounded* arbitrage opportunities iff for the optimal value p^* in (P) $p^* > -\infty$ holds. The market has *unbounded* arbitrage opportunities iff $p^* = -\infty$ in (P).

In the presence of linear taxes, bounded arbitrage opportunities do not exist.

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[11] Suppose there are riskfree profits in a subsequent period s with $\Delta_s(h, w, \bar{w}) > w_s - T(\bar{w}_s)$ and zero price today ($-\Delta_0(h) = 0$). By selling short some amounts of the coupon bond in $s-1$ we can realize a strategy with net payoffs of zero in s and positive net-payoffs in $s-1$. We can repeat our strategy and defer money in the prior period $s-2$ and so on, resulting in a positive withdrawal in $s = 0$. This is the case whenever average tax rates are lower than 100 % which we explicitly presupposed beforehand. For a more recent analysis on the issue of local versus global tax arbitrage in an after tax setting, see Kühn (2019).

2.3 Characterization of all possible arbitrage opportunities

We now turn to the main result of our paper. To formulate necessary and sufficient conditions for arbitrage-free markets we introduce the concept of implied tax rates deduced from current market prices. There are s points in time with $s = 0, \dots, S - 1$. Let p_s be the market price of an asset at time s with cash flow payments x_s^1. The tax due is determined by the income tax function $T(b_s^1)$. If the tax is linear with a constant tax rate τ_s at time s, it is a well-known fact that the market is free of arbitrage if and only if for all s[12]

$$p_{s-1}^1 = \frac{p_s^1 + x_s^1 - \tau_s \cdot b_s^1}{1 + r_f \cdot (1 - \tau_s)}. \tag{6}$$

If $p_{s-1}^1 \neq \frac{b_s^1}{r_f}$ (we will look at this case in a later section), we can solve for τ_s in the above formula and get

$$\tau_s = \frac{(1 + r_f)p_{s-1}^1 - p_s^1 - x_s^1}{r_f p_{s-1}^1 - b_s^1}. \tag{7}$$

That gives reason for the following definition.

Definition 7 (*Implied tax rates*) Suppose $b_s^1 \neq r_f p_{s-1}^1$ for all s. The numbers τ_s that satisfy (7) are called implied tax rates of prices p_s^1 in time s.

The following statement gives a clear characterisation whether a market is free of arbitrage in the presence of a nonlinear tax where the tax liability is affine on the boundary. The proof is in the appendix.

Theorem 8 *Let τ_s be the implied tax rates for given market prices p_s which satisfy equality (7). Then the following results hold true.*

1. *The market is free of arbitrage iff $T_-'(\bar{w}_s) \leq \tau_s \leq T_+'(\bar{w}_s)$ for all s.*
2. *There are unbounded arbitrage opportunities iff for at least one implied tax rate we have $\tau_s < \tau_{min}$ or $\tau_s > \tau_{max}$.*
3. *There are bounded arbitrage opportunities iff $\tau_{min} \leq \tau_s \leq \tau_{max}$ for all s and for at least one s' the condition $\tau_{s'} < T_-'(\bar{w}_s)$ or $T_+'(\bar{w}_s) < \tau_{s'}$ holds.*

We have illustrated the results in Fig. 2. Whether the market is free of arbitrage or not depends on how the left- and right-derivatives at the endowment are made up. Because we are dealing here with a model under certainty,

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[12] Note that this is the same fundamental property that Ross (1985, S. 379–380) deduces from his fundamental theorem by replacing the actual tax base b_s^1 by x_s^1 in the above formula. Ross does not differentiate between actual cash flows and taxable cash flows.
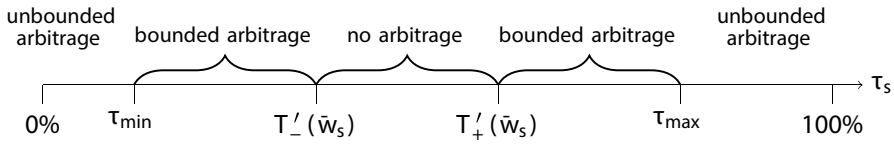
Fig. 2 Illustration of theorem 8. The figure shows a "tax line" that all values of an implicit tax rate τ_s can attain. If the actual value is in the interval $[T'_-(\bar{w}_s), T'_+(\bar{w}_s)]$ prices are free of arbitrage. If the value is in the interval $[\tau_{min}, T'_-(\bar{w}_s))$ or $(T'_+(\bar{w}_s), \tau_{max}]$ there are bounded arbitrage opportunities. In all other cases there exist an unbounded arbitrage opportunity
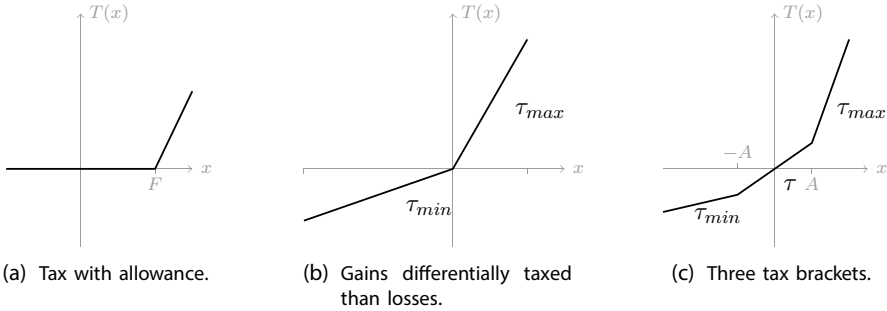

(a) Tax with allowance. (b) Gains differentially taxed than losses. (c) Three tax brackets.

Fig. 3 Three examples of tax liability functions

an endowment with other securities can be transformed into a corresponding endowment without further ado. We will discuss three applications of this theorem using typical tax functions in the next section.

Investors in asset markets are highly heterogeneous in terms of their personal tax schedules: they could invest through their tax-free retirement plans, some investors do pay personal taxes, but a subset of them may carry forward tax losses from previous periods. Different investors will thus determine different prices according to our considerations. Within our arbitrage theory, this is unproblematic because only one investor is considered. In an equilibrium model with several heterogeneous investors, however, there must be a uniform price for each asset. This problem can no longer be solved with the help of our arbitrage theory; for this, other methods are needed which go beyond our paper. In any case, however, an equilibrium theory must take into account what the individual calculus of an investor is, and our work makes an important contribution to this.

2.4 Three explanatory examples with zero endowment

We assume in the following that $\bar{w}_s = 0$ holds for all s. We analyze three different tax liability functions, see Fig. 3.

In order to highlight the relative evaluation with other bonds, we would like to focus on the special case of a tax-exempt zero-coupon bond. We assume that the second asset is a so-called zero-coupon bond, which makes no further payments until maturity. In addition, let us assume that the capital gains of this zero-coupon bond are tax-free.[13] All these conditions lead to the following parameters of our second asset:

$$x_s^1 = \begin{cases} 0 & \text{if } s < S, \\ 1 & \text{if } s = S, \end{cases}$$
$$b_s^1 = 0 \quad \text{if } s = 1, \dots, S. \tag{8}$$

We want to determine those prices for the zero-coupon bond in the three examples that do not generate arbitrage. For this purpose we use Theorem 8. If prices cannot generate arbitrage, we can exploit the relation with the implicit tax rates (6) and use (8). If $s < S$ we have

$$p_{s-1}^1 = \frac{p_s^1 + 0 - \tau_s \cdot 0}{1 + r_f(1 - \tau_s)} \qquad \Longrightarrow \qquad p_{s-1}^1 = \frac{p_s^1}{1 + r_f(1 - \tau_s)}$$

and for $s = S$ we have

$$p_{S-1}^1 = \frac{0 + 1 - \tau_s \cdot 0}{1 + r_f(1 - \tau_S)} \qquad \Longrightarrow \qquad p_{S-1}^1 = \frac{1}{1 + r_f(1 - \tau_S)}.$$

The following applies together for all $s = 1, \dots, S$

$$p_{s-1}^1 = \frac{1}{1 + r_f(1 - \tau_s)} \cdots \frac{1}{1 + r_f(1 - \tau_S)} \tag{9}$$

and it now only remains to characterise the tax rates τ_s in more detail.

Tax with allowance

We start analyzing a tax with allowance (see Fig. 3a). The corresponding tax liability function is

$$T(x) = \tau \max(x - F, 0), \qquad \tau \in [0, 1).$$

Tax payments apply when income exceeds $F > 0$, otherwise the tax is zero. Based on the notation of Theorem 8 we have

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[13] We have introduced the second asset on the understanding that we will not make any further assumptions on the tax base b_s. However, it is important to realize that this is a consequence of the fact that only one second asset is examined in the arbitrage theory. If not one but several other securities are introduced (for example, tax-exempt zero-coupon bonds with different maturities), there will likely be tax interactions among the securities. If, for example, a coupon bond can be replicated by the zero-coupon bonds, then the tax base of the coupon bond is zero as well—which contradicts our assumption about its tax base equal to the interest payment. We will not go into the problems involved because in the framework of arbitrage theory, we do not analyze such overdetermined models.

$$\tau_{min} = 0, \quad \tau_{max} = \tau,$$

and $T'(0) = 0$. Theorem 8 states that the only prices compatible with no-arbitrage are those for which $\tau_s = 0$ holds so that taxes could be neglected in pricing. Note that the result remains the same for tax functions which are not necessarily linear beyond F but, for example, quadratic (as is the case in Germany). For all arbitrage opportunities to be bounded, one must have an implied tax rate $\tau_s \in (0, \tau]$ for all s. For all other tax rates, the arbitrage opportunity is unbounded.

The only prices for the zero-coupon bond that exclude arbitrage are given by (9) and hence $p_s^1 = \frac{1}{(1+r_f)^{S-s}}$.

Different taxation of gains and losses

The second example represents a tax in which gains and losses are taxed differently (see Fig. 3b). We have

$$T'_-(0) = \tau_{min}, \quad T'_+(0) = \tau_{max}.$$

There are no arbitrage opportunities if the implied tax rates are contained in $[\tau_{min}, \tau_{max}]$. For all implied tax rates that do not fall into this interval we have unbounded arbitrage opportunities. Note that there are no bounded arbitrage opportunities.

Again the price of the zero-coupon bond that does not generate an arbitrage opportunity can be determined by (9). If p_s^1 is from the interval $p_s^1 \in \left[\frac{1}{(1+r_f(1-\tau_{max}))^{S-s}}, \frac{1}{(1+r_f(1-\tau_{min}))^{S-s}} \right]$, the market is free of arbitrage.

Three tax brackets

The third example shows a tax function with three different values of marginal rates $\tau_{min} < \tau < \tau_{max}$ (see Fig. 3c):

$$T(x) = \begin{cases} \tau_{min} \cdot x + (\tau_{min} - \tau) \cdot A & \text{if } x < -A \\ \tau \cdot x & \text{if } -A \le x \le A \\ \tau_{max} \cdot x + (\tau - \tau_{max}) \cdot A & \text{if } A < x. \end{cases}$$

Then

$$T'(0) = \tau.$$

Thus, the only implied tax rate offering no arbitrage is $\tau_s = \tau$. For all $\tau_s \in [\tau_{min}, \tau_{max}] \setminus \{\tau\}$ there are bounded arbitrage opportunities. For all other values of τ_s the arbitrage opportunity is unbounded.

In this example, it is surprising that the upper tax bracket does not seem to play a role. This is simply due to our assumption that the investor has no initial endowment, i.e., she is in the zero tax bracket. If we assume a sufficiently high initial endowment, the upper marginal tax rate would determine whether an arbitrage exists or not.

Once again we can determine the price of the zero-coupon bond using (9): p_s^1 has to equal to $\frac{1}{(1+r_f(1-\tau))^{S-s}}$ to prevent an arbitrage opportunity.

2.5 The case of an investment neutral tax

Finally, we turn to a case that has so far remained open. In order to define implied tax rates, a condition that seems technical was necessary. In order to convert to τ, the denominator of the expression (7) must not become zero. But what happens when prices satisfy the equation $b_s^1 = r_f p_{s-1}^1$?

In a one-period model $(S = 1)$ it is straightfoward to show that this assumption is always satisfied if the tax base of an asset is given by the difference of cash flows and prices. But it turns out that in a multi-period model this case is far more significant than a technical note. To understand this, we assume that at each time point this equality holds. Then we can show that only prices that are as in a market without taxes ensure the absence of arbitrage opportunities.

Theorem 9 *Let $b_s^1 = r_f p_{s-1}^1$ for all s. Then the market is free of arbitrage iff*

$$p_{s-1}^1 = \frac{p_s^1 + x_s^1}{1 + r_f} \quad s = 1, \dots, S.$$

If there is at least one point in time s for which the above equality does not hold, unbounded arbitrage profits exist.

For this result to hold it is necessary that the tax base at each point in time has a certain form, which is known as economic gain in the literature.[14] Assuming this, only prices as in an economy without taxes are arbitrage-free. This result is well known for the case of linear taxes, but not for the case of a nonlinear tax. That this result also holds under non-linear taxation is an additional contribution of this work.

3 Conclusion

We investigate arbitrage opportunities in a capital market model under certainty with multiple trading periods and convex tax liability functions. By deducing period-specific implied tax rates from prevailing market prices of bonds, we can give a complete characterisation of arbitrage depending on the investor's initial taxable income. If all implied tax rates equal marginal tax rates at the tax base of the investor's initial taxable income, then the no-arbitrage condition holds. This condition is necessary as well as sufficient.

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[14] This was first introduced by Preinreich (1951) and Samuelson (1964).

In addition, we show that arbitrage opportunities can be bounded or unbounded. Arbitrage opportunities are unbounded iff there is at least one point in time s where the implied tax rate τ_s is not equal to any possible marginal tax rate $T'(\cdot)$. In all other circumstances, bounded arbitrage opportunities exist.

Appendix 1: A progressive tax that is not convex

Consider two tax rates $\tau_0 > \tau_1$ at income level x_0 and x_1 that satisfy $\frac{\tau_1}{\tau_0} + \frac{x_0}{x_1} > 1$. The following example $T : \mathbb{R} \to \mathbb{R}$ of a tax liability

$$T(x) = \begin{cases} 0, & x \le x_0, \\ \tau_0 \cdot (x - x_0), & x_0 < x \le x_1, \\ \tau_1 \cdot (x - x_1) + \tau_0(x_1 - x_0), & x_1 < x, \end{cases}$$

is obviously continuous. It is not convex: This follows from $T'_+(x_0) = \tau_0 > \tau_1 = T'_+(x_1)$ and $x_0 < x_1$. But the tax rate

$$t(x) = \begin{cases} 0, & 0 < x \le x_0, \\ \tau_0 - \frac{\tau_0 x_0}{x}, & x_0 < x \le x_1, \\ \tau_1 - \frac{\tau_1 x_1 - \tau_0(x_1 - x_0)}{x}, & x_1 < x, \end{cases}$$

is always monotonically increasing and hence the tax is progressive: This is the case because from $\frac{\tau_1}{\tau_0} + \frac{x_0}{x_1} > 1$ it follows that $\tau_1 x_1 - \tau_0(x_1 - x_0) > 0$.

Appendix 2: Subdifferential, subgradient, and conjugate tax function

In many countries we observe tax systems that are piecewise linear with different tax brackets so that the marginal tax rate takes different values for different income.[15] The simplest case is when profits are taxed differently from losses,

$$T(x) = \tau_- \cdot \min(x, 0) + \tau_+ \cdot \max(x, 0) \tag{10}$$

Figure 4 shows an income tax of type (10) where losses are taxed at a rate of $\tau_- = 5\,\%$ and gains are taxed by $\tau_+ = 25\,\%$. The tax function is convex and not differentiable at $x = 0$.[16]

As we observe in Fig. 4a, there is not one but multiple straight lines forming a tangent of $T(\cdot)$ at $x = 0$. The set of all possible slopes is given by the interval $[\tau_-, \tau_+]$. We call the union of all possible tangent slopes *subdifferential* of $T(\cdot)$ at

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[15] This is formally not the same as linear although in some literature these terms are used interchangeably.

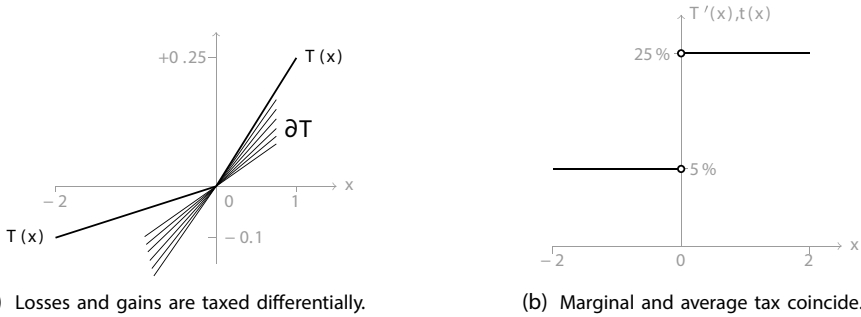[16] Since the kink lies in the origin, we can observe the identity of marginal and average tax rate for all $x \ne 0$.

(a) Losses and gains are taxed differentially.   (b) Marginal and average tax coincide.

Fig. 4 Example of a piecewise linear income tax with $\tau_- = 5\,\%$ and $\tau_+ = 25\,\%$

x denoted by $\partial T(x)$, which represents a generalization of the first derivative in the context of convex functions. Only if the tax function is differentiable at x, the subdifferential is a single element given by $\partial T(x) = \{T'(x)\}$. In the case of non-differentiability, $\partial T(x)$ is not a single number but rather a set of numbers, and in our one-dimensional case it is always an interval. In the setting of Fig. 4a, $\partial T(0)$ is equal to $[T'_-(0), T'_+(0)] = [\tau_-, \tau_+]$.

We refer to the formal definition of a subdifferential from the theory of convex analysis.[17]

Definition 10 (*Subdifferential and Subgradient*) The subdifferential of a convex tax function $T(\cdot)$ at x is given by the following set of real numbers

$$\partial T(x) := \{g \in \mathbb{R} \mid \forall y \quad T(y) \geq T(x) + g(y - x)\}. \tag{11}$$

An element $g \in \partial T(x)$ is called a subgradient of $T(\cdot)$ at x.

Finally we need the definition of a conjugate tax function in order to explain possible arbitrage opportunities. In particular, we are interested in the domain, the set of numbers for which the conjugate is finite.

Definition 11 (*Conjugate tax function*) The conjugate tax function $T^*(\cdot)$ of a tax liability function $T(\cdot)$ is given by

$$T^*(\tau) := \sup_{x \in \mathbb{R}} \tau x - T(x).$$

The domain is given by $\mathrm{dom}\,(T^*) = \{\tau \mid T^*(\tau) < \infty\}$.

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[17] For standard literature see (Rockafellar, 1997, p. 214-215), as well as Hiriart-Urruty and Lemaréchal, (2001, p. 165). There, one can find the proof that the subdifferential of a convex function, defined on $\mathbb{R}$, always exists and is represented by a closed interval.

The conjugate tax function is convex on its domain (see Rockafellar, 1997, p. 104). Lastly, we need another general result from the theory of conjugates.

Lemma 12 *For all x the relation $\partial T(x) \subset \mathrm{dom}\,(T^*)$ holds.*

Proof In Hiriart-Urruty and Lemaréchal, (2001, Theorem 1.4.1, p. 220) it is shown that $\tau \in \partial T(x)$ is equivalent to

$$T^*(\tau) + T(x) - \tau \cdot x = 0.$$

Therefore, $T^*(\tau)$ is finite and it holds $\tau \in \mathrm{dom}\,(T^*)$. $\square$

Appendix 3: Proof of Theorem 8

First we prove a general theorem that does not rely on taxes with affine boundaries.

Let us consider a single implied tax rate of τ_s. We will look at three different (mutually exclusive) possibilities for τ_s to depend on the tax function. The sets of interest involve $\partial T(\bar{w}_s)$ and $\mathrm{dom}\,(T^*)$. Since $\partial T(\bar{w}_s) \subset \mathrm{dom}\,(T^*)$, by using Lemma 12, it is only possible for τ_s to lie in $\partial T(\bar{w}_s)$ or in $\mathrm{dom}\,(T^*) \setminus \partial T(\bar{w}_s)$ or in neither of the two sets. Using this observation, we can establish a complete characterisation of all existing arbitrage opportunities in the market that are liable to income taxes.

Theorem 13 *Let τ_s denote the implied tax rates of prices p_s in $s = 0, \dots, S - 1$ which satisfy equality (7). Then the following is true.*

1. *The market is free of arbitrage iff $\tau_s \in \partial T(\bar{w}_s)$ for all s.*
2. *There are unbounded arbitrage opportunities iff for at least one implied tax rate $\tau_s \notin \mathrm{dom}\,(T^*)$.*
3. *In all other cases, there are bounded arbitrage opportunities.*

The third case is true iff $\tau_s \in \mathrm{dom}\,(T^*)$ for all s, and for at least one s' the condition $\tau_{s'} \in \mathrm{dom}\,(T^*) \setminus \partial T(\bar{w}_s)$ holds.

In this special case, the domain of the conjugate is easy to compute since it coincides with the union of all possible marginal tax rates. Because $\partial T(w_s)$ is convex and closed (on $\mathbb{R}$), it represents an interval given by $\partial T(\bar{w}_s) = [T'_-(\bar{w}_s), T'_+(\bar{w}_s)]$. If $T(\cdot)$ is differentiable at $\bar{w}_s$, the interval reduces down to a single element, i.e., $\partial T(\bar{w}_s) = \{T'(\bar{w}_s)\}$.

The proof of Theorem 13 is based on standard arguments from dual convex analysis. We start with the following Lemma.

Lemma 14 *For the conjugate tax function $T^*(\cdot)$ holds*

(i) $T^*(\tau) \geq \tau x - T(x)$ *for all* $x \in \mathbb{R}$ (*Fenchel's inequality*).
(ii) $T^*(\tau) = \tau x^* - T(x^*)$ *iff* $\tau \in \partial T(x^*)$.
(iii) $0 < T^*(\tau) - \tau x^* + T(x^*) < \infty$ *iff* $\tau \in \mathrm{dom}\,(T^*) \setminus \partial T(x^*)$.

Proof

(i)   follows directly from the definition of the conjugate function.
(ii) follows immediately by Corollary 1.4.4 in Hiriart-Urruty and Lemaréchal, (2001, p. 221).
(iii) by definition of dom (T^*) as well as property (i) and (ii).

$\square$

Starting with the primal problem (P) we analyze the dual Lagrange-Problem (D) given by

$$\sup_{\lambda \in \mathbb{R}^S} \inf_{h \in \mathbb{R}^S} \; L(h, \lambda)$$
$$\text{s.t. } \lambda_s \geq 0 \quad s = 1, \ldots, S, \tag{12}$$

with $\lambda = (\lambda_1, \ldots, \lambda_S)'$ and the Lagrangian[18]

$$L(h, y, \lambda) := (p_0)'h_0 - \sum_{s=1}^{S} \lambda_s \big(\Delta_s(h, w, \bar{w}) - w_s + T(\bar{w}_s) \big)$$

$$= (p_0)'h_0 - \sum_{s=1}^{S} \lambda_s \big\{ (p_s + x_s)'h_{s-1} + T(\bar{w}_s) - T\big((b)'_s h_{s-1} + \bar{w}_s\big) - (p_s)'h_s \big\},$$

using vector notation for h_s, p_s and x_s as well as b_s for both assets (standard coupon bond and bond to be valued).

Let d^* be the optimal value of (D). It is well known that (D) is a concave optimization problem with $d^* \leq p^*$ (weak duality). Equality holds if the strong Slater condition is satisfied which we will show:

Lemma 15 *There exists a strategy h such that all constraint inequalities are strict for this strategy, i.e., h is an inner point of (P).*

Proof To show that such a strategy exists consider h where only the standard coupon bond is involved, $(h_s, 0)$. The tax function T is concave and the supremum of the marginal tax rates was assumed to be below 1, i.e., there exists a $\tau < 1$ such that

$$\forall x > y \qquad T(x) - T(y) < \tau \cdot (x - y).$$

The withdrawal at time s for strategy h amounts to

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[18] $(\cdot)'$ denotes the transpose of a vector. For the definition of $\Delta_s(\cdot, w)$ see equation (5) respectively.

$$\Delta_s(h, w, \bar{w}) - \left(w_s - T(\bar{w}_s)\right) = (1 + r_f)h_{s-1} - h_s - \left(T(r_f h_{s-1} + \bar{w}_s) - T(\bar{w}_s)\right)$$
$$> (1 + r_f)h_{s-1} - h_s - \tau(r_f h_{s-1} + \bar{w}_s - \bar{w}_s)$$
$$= \left(1 + r_f(1 - \tau)\right)h_{s-1} - h_s.$$

Any strategy h with $\left(1 + r_f(1 - \tau)\right)h_{s-1} - h_s = 0$ represents an inner point of (P) or $\Delta_s(h, w, \bar{w}) > w_s - T(\bar{w}_s)$. ☐

From lemma 15 it follows that strong duality holds and we have $d^* = p^*$. We transform the variable in the dual problem

$$y_s := (b_s)' h_{s-1} + \bar{w}_s$$

and rewrite (D) which leads to the following maximization problem

$$\sup_{(\lambda, v)} \inf_{h, y} \quad \hat{L}(h, y, \lambda, v)$$
$$\text{s.t.} \quad \lambda_s \geq 0 \quad s = 1, \dots, S,$$

where $v = (v_1, \dots, v_S)'$ are new Lagrange multipliers with Lagrangian

$$\hat{L}(h, y, \lambda, v) := (p_0)' h_0 - \sum_{s=1}^{S} \lambda_s \left((p_s + x_s)' h_{s-1} + T(\bar{w}_s) - T(y_s) - (p_s)' h_s\right)$$
$$+ \sum_{s=1}^{S} v_s \left((b_s)' h_{s-1} + \bar{w}_s - y_s\right).$$

Setting $\lambda_0 = 1$ and using that $(p_S)' h_S = 0$ for both bonds, we can combine sums to

$$\hat{L}(h, y, \lambda, v) = \sum_{s=1}^{S} \lambda_{s-1} (p_{s-1})' h_{s-1} - \sum_{s=1}^{S} \left(\lambda_s(p_s + x_s)' h_{s-1} - v_s(b_s)' h_{s-1}\right)$$
$$- \sum_{s=1}^{S} \left(v_s y_s - \lambda_s T(y_s) - v_s \bar{w}_s + \lambda_s T(\bar{w}_s)\right),$$

which results in

$$\hat{L}(h, y, \lambda, v) = \sum_{s=1}^{S} \left(\lambda_{s-1} p_{s-1} - \left(\lambda_t(p_s + x_s) - v_s b_s\right)\right)' h_{s-1}$$
$$- \sum_{s=1}^{S} \left(v_s y_s - \lambda_s T(y_s) - v_s \bar{w}_s + \lambda_s T(\bar{w}_s)\right).$$

Note that some infima and suprema in the above problem can be simplified such that (D) is written as

$$\sup_{(\lambda,\nu)} \quad \sum_{s=1}^{S} \inf_{h_{s-1}} \left(\lambda_{s-1}p_{s-1} - \lambda_s(p_s + x_s) + \nu_s b_s \right)' h_{s-1}$$

$$- \sum_{s=1}^{S} \sup_{y_s} \left(\nu_s y_s - \lambda_s T(y_s) - \nu_s \bar{w}_s + \lambda_s T(\bar{w}_s) \right) \tag{13}$$

$$\text{s.t.} \quad \lambda_s \geq 0 \quad s = 1, \dots, S.$$

Looking at the objective of (13) one has

$$\inf_{h_{s-1}} \left(\lambda_{s-1}p_{s-1} - \lambda_s(p_s + x_s) + \nu_s b_s \right)' h_{s-1}$$

$$= \begin{cases} 0, & \lambda_{s-1}p_{s-1} - \lambda_t(p_s + x_s) + \nu_t b_s = 0, \\ -\infty, & \text{else.} \end{cases}$$

Hence, (13) is equivalent to

$$\sup_{(\lambda,\nu)} - \sum_{s=1}^{S} \sup_{y_s} \left(\nu_s y_s - \lambda_s T(y_s) - \nu_s \bar{w}_s + \lambda_s T(\bar{w}_s) \right)$$

$$\text{s.t.} \quad \lambda_s \geq 0, \quad \lambda_{s-1}p_{s-1} - \lambda_s(p_s + x_s) + \nu_s b_s = 0 \qquad s = 1, \dots, S. \tag{14}$$

We now prove that all feasible λ_s in (14) are strictly positive. Suppose there is a s' with $\lambda_{s'} = 0$ in (14). Then $d^* = -\infty$ unless $\nu_{s'} = 0$ in (14). Due to the recursion property in the constraints we get for all $s < s'$ immediately $\lambda_s = \nu_s = 0$ and especially for $s = 1$ we have

$$\lambda_0 p_0 = p_0 = 0$$

which contradicts the existence of a riskfree coupon bond with $p_0^0 = 1$. Thus, we conclude positivity of all feasible Lagrange multipliers in (13).

In this case

$$\sup_{y_s} \left(\nu_s y_s - \lambda_s T(y_s) - \nu_s \bar{w}_s + \lambda_s T(\bar{w}_s) \right) = \sup_{y_s} \left[\lambda_s \frac{\nu_s}{\lambda_s} y_s - \lambda_s T(y_s) - \frac{\nu_s}{\lambda_s} \bar{w}_s + T(\bar{w}_s) \right]$$

$$= \lambda_s \sup_{y_s} \left[\frac{\nu_s}{\lambda_s} y_s - T(y_s) \right] - \frac{\nu_s}{\lambda_s} \bar{w}_s + T(\bar{w}_s)$$

$$= \lambda_s \left[T^* \left(\frac{\nu_s}{\lambda_s} \right) - \frac{\nu_s}{\lambda_s} \bar{w}_s + T(\bar{w}_s) \right].$$

Plugging in this term into (14) we get an equivalent problem

$$- \inf_{(\lambda,\nu)} \sum_{s=1}^{S} \lambda_s \left[T^* \left(\frac{\nu_s}{\lambda_s} \right) - \frac{\nu_s}{\lambda_s} \bar{w}_s + T(\bar{w}_s) \right]$$

$$\text{s.t. } \lambda_s > 0, \quad p_{s-1} = \frac{\lambda_s}{\lambda_{s-1}} (p_s + x_s) - \frac{\nu_s}{\lambda_{s-1}} b_s, \qquad s = 1, \dots, S, \tag{15}$$

with usual convention $\inf \emptyset = \infty$. Thus infeasibility of (15) results in $p^* = d^* = -\infty$.

By substituting $\tau_s := \frac{\nu_s}{\lambda_s}$ and looking at the standard coupon bond it follows for the optimal Lagrange multipliers

$$\lambda_s = \frac{\lambda_{s-1}}{1 + r_f(1 - \tau_s)} \quad s = 1, \dots, S.$$

Iterating the above expression we get

$$\lambda_s = \frac{1}{\prod_{u=1}^{s} \left(1 + r_f(1 - \tau_u) \right)} \quad s = 1, \dots, S.$$

We conclude that (15) is equivalent to

$$- \inf_{\substack{\tau_s \in \text{dom}(T^*), \\ s = 1, \dots, S}} \sum_{s=1}^{S} \frac{T^*(\tau_s) - \tau_s \bar{w}_s + T(\bar{w}_s)}{\prod_{u=1}^{s} \left(1 + r_f(1 - \tau_u) \right)}$$

$$\text{s.t. } p_{s-1} = \frac{p_s + x_s - \tau_s b_s}{1 + r_f(1 - \tau_s)} \quad s = 1, \dots, S. \tag{16}$$

Using (16) we can finish the proof of Theorem 13. Obviously, τ_s are implied tax rates of the second bond. Then the following holds[19]:

1. By using Fenchel's inequality (see property (i) in 14) all summands in (16) are nonnegative and $p^* = d^* = 0$ iff $T^*(\tau_s) = \tau_s \bar{w}_s - T(\bar{w}_s)$ for all s being equivalent, after using (ii) in lemma 14, to $\tau_s \in \partial T(\bar{w}_s)$ for all s.
2. Since the sum in (16) is finite with nonnegative elements it follows $p^* = \infty$ iff for at least one s we have $\tau_s \notin \text{dom}(T^*)$.
3. By using (iii) in lemma 14 it follows $-\infty < p^* < 0$ iff $\tau_s \in \text{dom}(T^*)$ for all s and for at least one s' it holds $\tau_{s'} \in \text{dom}(T^*) \backslash \partial T(\bar{w}_{s'})$.

Having established Theorem 13 we can finally verify Theorem 8.

Proof First we proof that under piecewise linearity $\text{dom}(T^*) = [\tau_{min}, \tau_{max}]$. Since

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[19] Notice that the constraint in (16) is a vector equation and must hence hold for both assets. But looking at the coupon bond it can be rearranged to the trivial formulation $r_f(1 - \tau_s) = (1 - \tau_s) r_f$ which does not place a restriction in the implied tax rate. Only the second bond restricts τ_s.

$$T^*(\tau) := \sup_x \tau x - T(x) \quad \Longrightarrow \quad \forall x \ T^*(\tau) \geq \tau x - T(x)$$

Fenchel's inequality holds. Now suppose $\tau > \tau_{max}$ and $T^*(\tau)$ finite, then by definition $\tau \in \mathrm{dom}\,(T^*)$. Using Fenchel's inequality for sufficiently large x, exploiting piecewise linearity, we have

$$T^*(\tau) \geq (\tau - \tau_{max}) \cdot x - T.$$

Since this inequality holds for all sufficiently large x and by assumption $\tau - \tau_{max} > 0$ we get $T^*(\tau) = +\infty$ which contradicts our assumptions. We conclude for all $\tau \in (\tau_{max}, \infty)$ that $\tau \notin \mathrm{dom}\,(T^*)$.

Analogously, it is shown that for all $\tau < \tau_{min}$ we have $T^*(\tau) = \infty$ resulting in

$$\mathrm{dom}\,(T^*) \subset [\tau_{min}, \tau_{max}]. \tag{17}$$

Further from lemma 12 we have

$$\bigcup_x \partial T(x) \subset \mathrm{dom}\,(T^*)$$

and because of Rockafellar, (1997, Theorem 24.1, p. 227-229 i.c.w. Theorem 24.3, p. 232) accounting for differentiability at the boundaries we have

$$[\tau_{min}, \tau_{max}] = \bigcup_x \partial T(x).$$

Together with inequality (17) this results in

$$[\tau_{min}, \tau_{max}] \subset \mathrm{dom}\,(T^*) \subset [\tau_{min}, \tau_{max}].$$

and finally in $\mathrm{dom}\,(T^*) = [\tau_{min}, \tau_{max}]$.

The statement follows now directly from theorem 13.   □

Appendix 4: Proof of Theorem 9

Proof As shown in the proof of theorem 13, our primal problem (P) can be solved by using the dual problem (18) given by

$$-\inf_{\tau_{t_1},\dots,\tau_{t_S} \in \mathrm{dom}\,(T^*)} \sum_{s=1}^{S} \frac{T^*(\tau_s) - \tau_s \bar{w}_s + T(\bar{w}_s)}{\prod_{s=1}^{t}\left(1 + r_f(1 - \tau_s)\right)} \tag{D*}$$

$$\text{s.t.} \quad p_{s-1} = \frac{p_s + x_s - \tau_s b_s}{1 + r_f(1 - \tau_s)} \quad s = 1, \dots, S.$$

Let now $b_s^1 = r_f p_{s-1}^1$ hold for all s. By construction, for the coupon bond we also have

$$b_s^0 = r_f = r_f p_{s-1}^0 \quad s = 1, \dots, S - 1,$$

and

$$b_S^0 = 1 + r_f - 1 = r_f p_{S-1}^0.$$

Straightforward algebra shows that the constraints are independent of τ_s. Since the objective function in (18) is non-negative and we have for all s the relation $\partial T(\bar{w}_s) \subset \mathrm{dom}\,(T^*)$, (18) is equivalent to

$$\sup \quad 0$$
$$\text{s.t.} \quad p_{s-1} = \frac{p_s + x_s}{1 + r_f} \quad s = 1, \dots, S$$

with $p^* = d^* = -\infty$ in case of infeasibility and null otherwise. $\qquad\square$

References

Bradford DF (2000) Taxation, wealth, and saving. MIT Press, Cambridge

Brennan MJ (1970) Taxes, market valuation and corporate financial policy. Natl Tax J 23(4):417–427

Brennan TJ, McDonald RL (2015) Deconstructing the taxation of packaged financial strategies. SSRN eLibrary (available at http://ssrn.com/paper=2602621)

Dammon RM, Green RC (1987) Tax arbitrage and the existence of equilibrium prices for financial assets. J Finance 42(5):1143–1166. https://doi.org/10.2307/2328519

Dermody JC, Rockafellar RT (1991) Cash stream valuation in the face of transaction costs and taxes. Math Finance 1(1):31–54. https://doi.org/10.1111/j.1467-9965.1991.tb00003.x

Duffie D (2010) Presidential address: asset price dynamics with slow-moving capital. J Finance 65(4):1237–1267

Dybvig P, Ross SA (1986) Tax clienteles and asset pricing. J Finance 41:751–62

Friedman M (1962) Capital freedom. University of Chicago Press, Chicago

Gallmeyer M, Srivastava S (2011) Arbitrage and the tax code. Math Finance Econ 4:183–221

Graham JR, Smith CW Jr (1999) Tax incentives to hedge. J Finance 54(6):2241–2262

Gromb D, Vayanos D (2018) The dynamics of financially constrained arbitrage. J Finance 73(4):1713–1750

Heintzen M, Kruschwitz L, Löffler A et al (2008) Die typisierende Berücksichtigung der persönlichen Steuerbelastung des Anteilseigners beim squeeze-out. Z Betriebswirt 78:275–287

Hiriart-Urruty JB, Lemaréchal C (2001) Fundamentals of convex analysis. Grundlehren Text Editions, Springer, Berlin

Kühn C (2019) How local in time is the no-arbitrage property under capital gains taxes? Math Finance Econ 13(3):329–358

Miller MH (1977) Debt and taxes. J Finance 32:261–275

OECD (2006) Oecd-beschäftigungsausblick https://doi.org/10.1787/empl_outlook-2006-de, /content/book/empl_outlook-2006-de

Oehmke M (2011) Gradual arbitrage. SSRN eLibrary (available at http://ssrn.com/paper=1364126)

Preinreich GA (1951) Models of taxation in the theory of the firm. Economia Internazionale 4:372–397

Prisman EZ (1986) Valuation of risky assets in arbitrage free economies with frictions. J Finance 41(3):545–557

Rockafellar RT (1997) Convex analysis, 10th edn. Princeton, Princeton landmarks an mathematics and physics, Princeton University Press

Rosembuj T (2011) International tax arbitrage. Intertax 39(4):158–168

Ross SA (1985) Debt and taxes and uncertainty. J Finance 40:637–657

Ross SA (1987) Arbitrage and martingales with taxation. J Polit Econ 95:371–393

Samuelson PA (1964) Tax deductibility of economic depreciation to insure invariant valuations. J Polit Econ 72:604–606

Schaefer SM (1982) Taxes and security market equilibrium. In: Sharpe WF, Cootner CM (eds) Financial Economics: Essays in Honor of Paul Cootner. Prentice-Hall, Englewood Cliffs (NJ), pp 159–178

Sialm C (2009) Tax changes and asset pricing. Am Econ Rev 99(4):1356–1383