Exploiting Torus Actions: Immaculate Line Bundles on Toric Varieties and Parametrizations of Gröbner Cells

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Introduction

Algebraic torus actions form a fundamental concept in algebraic geometry, providing a powerful tool to study the structure of algebraic varieties.

A split algebraic torus T over the field $\mathbf{k}$ is defined as a group isomorphic to the product of a finite number of copies of the multiplicative group $\mathbb{G}_m \cong \mathbf{k}^*$ of $\mathbf{k}$. This means that T is an abelian algebraic group, where multiplication is a well-defined algebraic map and invertible for its elements.

When an algebraic torus T acts on an algebraic variety X, it induces a group action on the points of that variety in a way that preserves the algebraic structure of the variety. This action often has important consequences on the geometry of the variety, revealing its geometric properties and being a powerful tool in its study.

One of the primary areas of interest in algebraic torus actions is the study of toric varieties. Toric varieties are algebraic varieties equipped with a torus action that has a dense orbit. Toric varieties have rich connections with discrete geometry. A normal toric variety can be defined by a polyhedral fan Σ and many of its properties can be translated into properties of the fan Σ.

In chapter 2 we make use of these connections to study a special class of line bundles on toric varieties. A line bundle on a toric variety X is given by a divisor class on X, so up to linear equivalence, it can be described by a torus invariant Weil divisor that is a formal sum of torus invariant prime divisors which are in one-to-one corresponds with the rays $\Sigma(1)$ of the fan. We will recapitulate how the cohomology of a line bundle can also be calculate discrete geometrically with the help of the fan and use properties of the fan to describe immaculate line bundles on X, see section 1.1.

In the case of toric varieties the dimension of the torus T equals the dimension of the variety X that T acts on. But even the action of a one-dimensional torus T on a variety (or scheme) X can reveal valuable insights into the structure of X. An important result is the paper by Bialynicki-Birula [Bia73] where such a torus action is used to define a decomposition of a scheme X into T-invariant subschemes. In chapter 3 we apply this to $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$, the Hilbert scheme of n points of a plane for $n \in \mathbb{Z}_{>0}$. The points of $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ correspond to zero-dimensional ideals I in the polynomial ring $\mathbf{k}[x,y]$ with $\dim_{\mathbf{k}} \mathbf{k}[x,y]/I = n$. We can define an action of a one-dimensional torus T on $\mathbf{k}[x,y]$ and this induces an action on the ideals of $\mathbf{k}[x,y]$ that restricts to an action on $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$. When the action is general enough, the only fixed points of this action in $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ are the zero-dimensional monomial ideals. These ideals can be identified with partitions of n. Under the given action all ideals in $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ specialize to monomial ideals. This process can also be understood in the language of leading term ideals. The subset of ideals specializing to a given monomial ideal or in other words with given leading term ideal are the so-called Gröbner cells. In chapter 3 we parametrize them by means of canonical Hilbert-Burch matrices, see section 1.2.

The computational component played a big role in both parts of this thesis. The computer algebra systems `polymake` [GJ00], Singular [DGPS23] and OSCAR [23] via `Julia` [BEKS17] were used to generate examples, and to find and test hypotheses. Both parts lead to extensions/modules for the computer algebra systems `polymake` and `Julia`. The developed software comes partially shipped with `polymake`, the rest is available on GitHub.

We will proceed by giving more detailed introductions to both main chapters with a focus on the author's contributions.

1.1 Immaculate Line Bundles on Toric Varieties, chapter 2

Most results presented in chapter 2 have been published in [ABKW20], co-authored with Klaus Altmann, Jarosław Buczyński, and Lars Kastner. For a detailed comparison between the paper and this chapter, please refer to the beginning of the chapter.

Exceptional sequences play an important role in the study of the derived category of a scheme X. An exceptional sequence is a sequence of exceptional elements of the derived category such that there are no backwards morphisms. A full exceptional sequence generates the derived category, or more precisely, it gives a semiorthogonal decomposition of the derived category of X. There has been a lot of research about the existence of full exceptional sequences, lengths of full exceptional sequences (especially in the context of phantom categories), the possibility of expanding a given exceptional sequence to a full exceptional sequence, classification of full exceptional sequences and many more, see e.g. [AA22; AW21; BGKS15; BH09; CM04; Cra11; Efi14; HP06; HP11; Kaw06; Kaw13; Mic11]. When there exist full exceptional sequences for schemes X, it has been a question whether one can build an exceptional sequence only out of sheaves or even of line bundles. For toric varieties Efimov has given a negative answer to this question in [Efi14]. Nevertheless, there are still many open problems in this research field.

On a smooth projective toric variety X, a sequence $(L_0, ..., L_n)$ of line bundles is exceptional if $\mathrm{H}^k(X, L_i \otimes L_j^{-1}) = 0$ for all $i > j$ and all $k \in \mathbb{Z}$. This motivates the following definition.

Definition 1.1 (Definition 2.3)**.** A line bundle L on a variety X is called *immaculate* if $\mathrm{H}^k(X, L) = 0$ for all $k \in \mathbb{Z}$.

In this sense, immaculate line bundles can be seen as building blocks for exceptional sequences of such. Immaculate line bundles also have a connection to Physics: they are contained in the vanishing sets studied in [BMW17; Bie18]. These sets are relevant in the studies of F-Theory, which describes a special set of solutions in string theory.

Although the study of full exceptional sequences was our motivation, in chapter 2 we focus on the structure of immaculate line bundles for (often projective) toric varieties.

The normal toric variety associated to a polyhedral fan $\Sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, will be denoted by $X = \mathbb{TV}(\Sigma)$, where N is the lattice of one-parameter subgroups of T and M is the dual lattice to N, the lattice of characters of T. Line bundles on X are given as $\mathcal{O}(D)$ with D being a divisor class in $\mathrm{Cl}(X)$. Whenever X has no torus factors and the rays of Σ are not contained in a proper sub-vector space, the class group of X can be obtained by the following well-known exact sequence

$$0 \to M \to^{\rho^*} \mathrm{Div}_T(X) = (\mathbb{Z}^{\Sigma(1)})^* \to^{\pi} \mathrm{Cl}(X) \to 0. \tag{1.1}$$

We will fix the notation and remind some basic results about divisors on toric varieties and their cohomologies in section 2.1.

In our strategy to study immaculate line bundles of a toric variety $X = \mathbb{TV}(\Sigma)$ the exact sequence 1.1 plays an important role. We will divide the task of finding immaculate line bundles into two steps. Firstly, we study subsets of the rays $\Sigma(1)$. A subset $\mathcal{R} \subset \Sigma(1)$ will be called tempting whenever the set $V^{>}(\mathcal{R}) := \mathbb{R}_{>0} \cdot \bigcup_{\sigma\in\Sigma} \operatorname{conv}(\mathcal{R} \cap \sigma(1))$ is not $\mathbf{k}$-acyclic, see Definition 2.4. The aim of this step is to identify all tempting subsets of the rays of Σ. The second step is to study the associated "maculate" subsets of the class group.

Definition 1.2 (Definition 2.6)**.** Let $\mathcal{R} \subset \Sigma(1)$ be a tempting set. We define the maculate set $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ and the maculate region $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ as

$$\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) := \pi(\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}) \text{ and } \mathcal{M}_{\mathbb{R}}(\mathcal{R}) := \pi(\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq -1}^{\mathcal{R}}).$$

We show in Proposition 2.9 that a line bundle $\mathcal{O}(D)$ is immaculate if and only if $D \notin \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ for any tempting subset $\mathcal{R} \subset \Sigma(1)$. For the maculate regions only one of the directions is true. It might happen that a divisor is immaculate, but belongs to a maculate region, see Example 2.10 and Example 2.39. Nevertheless in many of our examples we get the other implication as well. This is the case if the rays of the tail cone of the maculate region form a Hilbert basis, which is guaranteed by total unimodularity of the matrix π. The methods used are similar to [BH09] and [Efi14], but different to their approach of detecting full exceptional sequences, we will focus on studying the immaculate locus inside of the class group $\operatorname{Cl}(X)$.

The first step of the task is treated in section 2.2: deciding whether a subset $\mathcal{R} \subset \Sigma(1)$ is tempting or not. First we notice that monomials and subsets of rays that span a cone $\sigma \in \Sigma$, do not lead to temptation. On the other hand, a guaranteed source for temptations are primitive collections. A primitive collection P of Σ is a minimal non-face, in the sense that each proper subset of P spans a cone of the fan. The main characterization is the following theorem that is not contained in [ABKW20].

Theorem 1.3 (Proposition 2.28)**.** *Let $X = \mathbb{TV}(\Sigma)$ be a projective toric variety. If $\mathcal{R} \subset \Sigma(1)$ is tempting, then $\mathcal{R}$ and its complement $\Sigma(1) \setminus \mathcal{R}$ can be written as the union of primitive collections.*

The main point in the proof is that whenever $\mathcal{R}$ is not a union of primitive collections, it can be obtained as the rays of some cones of Σ that intersect in a common face and thus $V^{>}(\mathcal{R})$ can be retracted to this face and is $\mathbf{k}$-acyclic. This result was also proven by Efimov in [Efi14, Lemma 4.4] with different methods. In Example 2.30 we see that this is not a complete characterization, since primitive collections cannot identify all the cases where $\mathcal{R}$ (or its complement) consists of the rays of a subfan with convex support.

The rest of the chapter is devoted to the study of more concrete classes of examples of smooth projective toric varieties: we start with varieties of Picard rank 2 in section 2.3. Here the situation is simple, there are only four tempting subsets and the structure of the immaculate locus can be described nicely. The varieties of Picard rank 2 are a subclass of the next class, varieties with splitting fans that we study in section 2.4. The focus in this chapter lies on projective toric varieties of Picard rank 3 in section 2.5. This section differs from the equivalent section [ABKW20, Section VIII.3]. It contains the complete proofs and studies the structure of the immaculate locus in more detail. In subsection 2.5.1 we use the classification by Batyrev and see that the varieties of Picard rank 3 are either varieties with splitting fans and three primitive collections, so a subset of the class studied in section 2.4, or they have 5 primitive collections. By Proposition 2.47 the rays of a fan of a projective toric variety with exactly five primitive collections can be partitioned into five sets $X_0, X_1, \ldots, X_4$ of cardinalities $p_0, p_1, \ldots, p_4$. The primitive collections then consist of unions of two consecutive $\mathrm{P}_\alpha = X_\alpha \cup X_{\alpha+1}$. Now the variety is determined by two parameter vectors b and c. We identify the immaculate line bundles for this class of varieties.

Theorem 1.4 (Proposition 2.54 – Proposition 2.57)**.** *Let $X = \mathbb{TV}(\Sigma)$ be a projective toric variety of Picard rank 3 that has exactly five primitive collections. Then the immaculate locus of X contains the following types of line bundles:*

- *type (F): line bundles $(*,\ x,\ y)$ in full lines with $(x,\ y) \in Q_1 \cup Q_2$, see Proposition 2.54,*
- *type (A): line bundles $(*,\ y,\ -y)$ in line segments for $y \in [-p_1 - p_2 + 1, p_3 + p_4 - 1]$, see Table 2.2 and Proposition 2.55,*
- *type (B): if $p_2, p_3 \geq 2$: $p_0 - 1$ line bundles in a line line segment in $(*,\ -p_1 - p_2,\ p_1)$, see Proposition 2.56 (also for the statements for $p_2 = 1$ or $p_3 = 1$),*

where Q_1 and Q_2 denote two planar parallelograms with vertices depending on $p_0, \ldots, p_4$, see Definition 2.53 and Figures 2.4 – 2.6.

Whenever the parameters b and c are sufficiently general, then there are no other immaculate line bundles than the ones stated above and their Serre duals (Proposition 2.57).

In subsection 2.5.4 we study how the set of immaculate line bundles of different varieties of Picard rank 3 behave. The special case with vanishing parameters b and c is studied in subsection 2.5.5. Here a second class of lines of immaculates exists, similar to the lines in Proposition 2.54 in another direction.

Proposition 1.5 (Proposition 2.62)**.** Let $X = \mathbb{TV}(\Sigma)$ be a toric variety of Picard rank 3 with exactly 5 primitive collections, and let b and c from Proposition 2.47 be zero. Then the line bundles $(x,\ *,\ z)$ are immaculate for $(x,\ z) \in \tilde{Q}_1 \cup \tilde{Q}_2$, where $\tilde{Q}_1$ and $\tilde{Q}_2$ are two planar parallelograms with vertices depending on $p_0, \ldots, p_4$.

The last section, section 2.6, is devoted to the description of the computational aspects of the problem. In particular, we implemented an extension for `polymake` to calculate the immaculate locus for toric varieties.

1.2 Canonical Hilbert-Burch Matrices, chapter 3

The first part of chapter 3 is joint work with Roser Homs, published in [HW21] and [HW23]. For a detailed comparison between the papers and this chapter, please refer to the beginning of the chapter.

Hilbert schemes parametrize subschemes of a scheme with a fixed Hilbert function. The most basic examples are Hilbert schemes of points of affine spaces. The points of these schemes parametrize n points of the affine space, counted with multiplicities. Even the simplest among those, the Hilbert scheme of points in the affine plane $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x}, \mathrm{y}])$, has been studied for a long time, [ES87; ES88; Gal74; Gra83; Iar77; Yam89]. The points of $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x}, \mathrm{y}])$ parametrize n points in $\mathbb{A}^2$ counted with multiplicities, or with a more algebraic view, Artinian $\mathbf{k}$-algebras with two generators of vector space dimension n, so ideals I of $\mathbf{k}[x, y]$ such that the quotient of $\mathbf{k}[x, y]$ by I is an n-dimensional vector space. One strategy to study those spaces has been to decompose them into smaller spaces.

A term order τ on a polynomial ring defines an ordering of the monomials, and enables the definition of leading term of a polynomial f, denoted by $\mathrm{Lt}_\tau(f)$, as the biggest monomial occurring as a term of f. For an ideal I, we define the leading term ideal $\mathrm{Lt}_\tau(I)$ as the ideal generated by all leading terms $\mathrm{Lt}_\tau(f)$ for all $f \in I$. In [CV08] and [Con11] the authors parametrize subsets of $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ where the ideals share a common leading term ideal E with respect to the term order τ, for τ the lexicographic (lex) and the degreelexicographic (deglex) term order, respectively. This set will be denoted by $V_\tau(E)$. By [Bia73; ES87; Yam89] the sets $V_\tau(E)$ are affine spaces and form a cellular decomposition of $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$. By analogy to Schubert cells in Grassmanians they are called Gröbner cells. The idea of the parametrizations of $V_\tau(E)$ by [CV08] and [Con11] is to use the Hilbert-Burch theorem. Since each ideal in $V_\tau(E)$ is of codimension two, it can be generated by the maximal minors of a matrix, called Hilbert-Burch matrix. Obviously, this matrix is not at all unique. Both papers describe ways to pick a canonical representative. One aspect of their constructions is that the minors of their "canonical Hilbert-Burch matrix" of an ideal I do not only generate the ideal, but form a Gröbner basis with respect to the considered term order. That means that the leading terms of the minors also generate the leading term ideal $\mathrm{Lt}_\tau(I)$. We will review these results and fix notations in section 3.1.

In chapter 3 our aim is to generalize these results in two different directions. In the first part (section 3.2 – section 3.6) we study those ideals of $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ that correspond to points of multiplicity n at the origin, so local Artinian $\mathbf{k}$-algebras of length n. This subscheme of $\mathrm{Hilb^n}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ is called the punctual Hilbert scheme. The punctual Hilbert scheme (not only of the plane) has also been of great interest, see for example [Bri77; BG74; Iar77; Poo08; Göt90]. By Cohen's structure theorem, a local Artinian $\mathbf{k}$-algebra A is a quotient R/J, where R is the ring of formal power series and $J \subset R$ is an $\mathfrak{m}$-primary ideal. For this reason, we denote the punctual Hilbert scheme by $\mathrm{Hilb^n}(\mathbf{k}[\![\mathrm{x},\mathrm{y}]\!])$. Since J is $\mathfrak{m}$-primary, it can be generated by polynomials and the quotient R/J is naturally isomorphic to $P/(J \cap P)$, $J \cap P$ is an $\mathfrak{m}$-primary ideal of the polynomial ring P and $\mathfrak{m}$ denotes the maximal ideal in R and by a slight abuse of notation also the homogeneous maximal ideal of P. In [CV08] a parametrization of the sub-cell of (x, y)-primary ideals of $V_\tau(E)$ is given, so we obtain a cellular decomposition of the punctual Hilbert scheme. But this decomposition has a major drawback. For a local Artinian $\mathbf{k}$-algebra A one can define the Hilbert function of A as the Hilbert function of its associated graded ring. The associated graded ring of a local algebra $A = R/J$ can be recovered as the quotient of the polynomial ring by the initial ideal J^* of J, see Definition 3.10. The initial form of a polynomial f is the part of f of lowest degree, and similar to the leading term ideal, the initial ideal J^* is defined as the ideal generated by the initial forms of all $f \in J$. A natural wish for a decomposition of the punctual Hilbert scheme is that all ideals in a cell have the same Hilbert function. This is not the case in the sub-cell of (x, y)-primary ideals in the Gröbner cell $V_{\mathrm{lex}}(E)$, see Example 3.14. Roughly speaking, the reason is that taking the initial ideal and the leading term ideal with respect to a term order do not commute, see Remark 3.11. This gives the motivation to define a different decomposition by *local* Gröbner cells such that all ideals in the cell share the same Hilbert function. To achieve this we work in the power series ring $R = \mathbf{k}[\![x, y]\!]$ with so-called *local term orders* instead of considering (x, y)-primary ideals in $P = \mathbf{k}[x, y]$ with *global* term orders. Local term orders, and the analogue notions to leading term ideals, Buchberger division and Gröbner basis for the power series ring are introduced and some crucial results needed for our setting will be recalled in subsection 3.2.1. Working towards a parametrization of the "local" Gröbner cell $\mathcal{V}(E)$, in section 3.3, we firstly give a surjection to the cell.

Proposition 1.6 (Theorem 3.21)**.** Let $E = (x^t, x^{t-1}y^{m_1}, \dots, x^{t-i}y^{m_i}, \dots, y^{m_t}) \subset \mathbf{k}[\![x, y]\!]$ be a monomial ideal with canonical Hilbert-Burch matrix H and degree matrix U as in Definition 3.2. Let $\mathcal{N}(E) \subset \mathbf{k}[\![y]\!]^{(t+1)\times t}$ such that the non-zero entries of $N = (n_{i,j}) \in \mathcal{N}(E)$ satisfy that the order, $\operatorname{ord}(n_{i,j}) \geq u_{i,j} + 1$ for $i \leq j$ and $\operatorname{ord}(n_{i,j}) \geq u_{i,j}$ for $i > j$, with $\operatorname{ord}(n_{i,j}) = \deg(n^*_{i,j})$. Then the map

$$\varphi : \mathcal{N}(E) \to \mathcal{V}(E)$$

defined by

$$N \mapsto I_t(H + N)$$

is surjective. In particular, the minors of $H + N$ form a $\overline{\text{lex}}$-enhanced standard basis of $J \in \mathcal{V}(E)$.

This surjection is from an infinite dimensional vector space and determines Hilbert-Burch matrices of the ideals in $\mathcal{V}(E)$ such that the minors do not only generate the ideals, but form a $\overline{\text{lex}}$-enhanced standard basis (see Definition 3.9) of a specific form. In Proposition 3.26 a finite dimensional sub-vector space is given that also surjects onto $\mathcal{V}(E)$. Since the map is still not injective, this map cannot be used to define canonical Hilbert-Burch matrices for the ideals yet. In section 3.4 we define a subset of ideals in $\mathcal{V}(E)$ for which canonical Hilbert-Burch matrices can be chosen. It consists of ideals for which there exists a generating set that behaves well with respect to the local structure as well as with respect to the lexicographic order. More precisely, those ideals have a $\overline{\text{lex}}$-enhanced standard basis that is also a Gröbner basis with respect to lex. In Lemma 3.36 we show that for many monomial ideals this subset is already the whole Gröbner cell. Combining these results the following theorem gives a parametrization of the Gröbner cells of lex-segment ideals.

Theorem 1.7 (Theorem 3.38). *Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, x^{t-i}y^{m_i}, \ldots, y^{m_t}) \subset \mathbf{k}[\![x, y]\!]$ be a lex-segment ideal with canonical Hilbert-Burch matrix H and degree matrix U as in Definition 3.2 and $\mathcal{M}(E) \subset \mathcal{N}(E)$ such that $\deg(n_{i,j}) < d_{\min(i,j)}$ where $d_i = m_i - m_{i-1}$. Then*

$$\varphi : \mathcal{M}(E) \to \mathcal{V}(E)$$

by $N \mapsto I_t(H + N)$ is a bijection.

In the case that E is a lex-segment ideal (or under the slightly more general assumption of being a *relax-segment ideal*, see Definition 3.35) the set $\mathcal{M}(E)$ only contains strictly lower triangular matrices, in the sense that $n_{i,j} = 0$ for all $i \leq j$. One can define canonical Hilbert-Burch matrices for all ideals in the cell. In Lemma 3.46 we give a parametrization of the sub-cell $\mathcal{V}_{\text{hom}}(E)$ of $\mathcal{V}(E)$ consisting of all homogeneous ideals of the cell and calculate its dimension for general zero-dimensional monomial ideals E. In Conjecture 3.48 we propose that the whole cell $\mathcal{V}(E)$ should be parametrized by the set $\mathcal{N}_{<\underline{d}}(E)$, see Definition 3.47. In section 3.5 we investigate subsets of $\mathcal{V}(E)$ with a given number of generators and see that those subsets are quasi-affine varieties in the affine space $\mathcal{V}(E)$. We use our construction to investigate which minimal numbers of generators can be realized.

Section 3.6 focuses on the fact that the given Gröbner cells provide a cellular decomposition of the punctual Hilbert scheme $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\![\mathrm{x}, \mathrm{y}]\!])$. Cellular decompositions of schemes over $\mathbb{C}$ can be used to calculate their Betti numbers. We study our cellular decomposition for small values of n and compare the results to known results by [ES87; ES88] and [Bri77]. We compare the results about Betti numbers of the punctual Hilbert scheme of [ES87] to the numbers obtained by our proposed parametrization and verify that they coincide for $n \leq 50$. The fact that the Gröbner cells with respect to a local order restrict to a cellular decomposition of the subspace with fixed Hilbert function is used to calculate the Betti numbers of those subspaces for some values of n. We finish section 3.6 by giving another evidence for Conjecture 3.48. Using [Iar77, Theorem 2.11, Theorem 3.14] we conclude that the difference of dimensions of $\mathcal{V}(E)$ and $\mathcal{V}_{\text{hom}}(E)$ has to be the same for all E with the same Hilbert function h. We use this to verify that the dimensions of $\mathcal{N}_{<\underline{d}}(E)$ and $\mathcal{V}(E)$ coincide for all $E \in \mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\![\mathrm{x}, \mathrm{y}]\!])$ for $n \leq 50$. We also give a small outline on the use of our Julia module.

In the second part (section 3.7 – section 3.9), we return to $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x}, \mathrm{y}])$, so not necessarily (x, y)-primary ideals in the polynomial ring $\mathbf{k}[x, y]$, and study Gröbner cells for general term orders on $\mathbf{k}[x, y]$. Varying term orders is not a new idea: in [MR88] Gröbner fans of ideals are studied – term orders can be described by vectors and the fans of the Gröbner fan are those regions where the leading term ideal of I stays the same for all orders, see also [Stu96] for an introduction to the topic. In [AS05] the authors start from a different perspective. They define a graph that has a vertex for every monomial ideal, and two vertices corresponding to monomial ideals E and E' are joined by an edge whenever there exists an ideal I such that E and E' are the only possible occurring initial ideals with respect to any given term order. In Theorem 3.71 we give a surjection from a finite dimensional vector space to the Gröbner cell $V_\tau(E)$ with respect to a general term order τ and conjecture how a parametrization should look like in Conjecture 3.79. In section 3.9 intersections of two Gröbner cells $V_\tau(E)$ and $V_{\tau'}(E)$ are studied, so ideals for which the leading term ideal is the same with respect to two (or more) term orders. In [JS19] it was shown that for two orderings those intersections are affine spaces. In the case that one of the term orders is lex or deglex we give a parametrization of these sets. So in particular, we have parametrizations of some subsets of the Gröbner cells and can define canonical Hilbert-Burch matrices with respect to general term orders for the ideals in those subsets. The parametrizations of subsets also give evidence for Conjecture 3.79. In Corollary 3.92, we give a parametrization of those ideals that have a given monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ as leading term ideal with respect to all term orders τ. This set is isomorphic to a $(t + m_t)$-dimensional affine space. In section 3.10 we collect some more evidence for the correctness of Conjecture 3.79. We finish the chapter with the description of Gröbner cells for $n = 6$ and different term orders.

Immaculate Line Bundles on Toric Varieties

The results of this chapter were previously published by International Press in the paper "Immaculate line bundles on toric varieties" [ABKW20] by Klaus Altmann, Jarosław Buczyński, Lars Kastner, and me. The paper is published in "Pure and Applied Mathematics Quarterly", Volume 16 (2020), Number 4, Special Issue: In Honor of Prof. Gert-Martin Greuel's 75th Birthday, Pages 1147 – 1217. The publication is available at https://dx.doi.org/10.4310/PAMQ.2020.v16.n4.a12.

The present chapter contains a selection of adapted sections from [ABKW20]. The results of [ABKW20] about calculating the cohomology of a line bundle by differences of polytopes, so called p-immaculacy, and all statements connected only to these subtopics are omitted. Of the presented parts of the paper, subsection 2.2.2 differs from the published version and contains more statements about sources or absence of "temptations" (see Definition 2.4), where the main result is Proposition 2.28. This statement makes it possible to shorten the proofs of the statements identifying the "tempting subsets" in the case of splitting fans and Picard rank 3, Lemma 2.36 and Lemma 2.49. The main contribution in this thesis is in section 2.5: the detailed proofs of the statements about the structure of immaculate line bundles on projective toric varieties of Picard rank 3, as well as more statements concerning the structure of the immaculate sets when changing parameters in subsection 2.5.4, and about the structure in the special case of vanishing parameters in subsection 2.5.5.

The collaboration started at the Fields Institute in Toronto, and lead to a visit of Jarosław Buczyński in Berlin, where the other three authors worked at that time and a visit in Warsaw. A lot of the conceptual work was done during those visits. Of the presented material of [ABKW20] in this chapter, the results of section 2.5 were obtained almost entirely be me.

A more detailed comparison:

- section 2.1: An adapted version of [ABKW20, Chapter III], mostly shortened to contain only the relevant notions for the present chapter (including subsection 2.1.1 a shortened version of [ABKW20, section III.2] with Remark 2.2 added).

- section 2.2: a small part of [ABKW20, Chapter IV] (e.g. the definition of immaculate) and mostly [ABKW20, Chapter V] (without the non-toric example, Figure 4)
  - subsection 2.2.1 is a shortened version of [ABKW20, section V.1] Proposition 2.9 and its proof are a slight variation of [ABKW20, Proposition V.6]
  - The first part of subsection 2.2.2 is slightly adapted from [ABKW20, section V.2]. The following parts are new: Proposition 2.21, Example 2.22 and Example 2.23.

– The first part of subsubsection 2.2.2 (Primitive collections deluded) is [ABKW20, subsection V.2.3]. Proposition 2.27, Proposition 2.28, Example 2.29 and Example 2.30 are new.

- section 2.3 is [ABKW20, Chapter VI]
- section 2.4 is an adapted version of [ABKW20, Chapter VII].
  – subsection 2.4.1 is a slightly shortened version of [ABKW20, section VII.1] with two extra examples: Example 2.34 and Example 2.35.
  – subsection 2.4.2 [ABKW20, section VII.2] proof of Lemma 2.36 ([ABKW20, Lemma VII.2]) shortened with use of Proposition 2.28.
  – subsection 2.4.3 is a variant of [ABKW20, sections VII.3+VII.4]. The proof of the main result of this section Theorem 2.44 ([ABKW20, Theorem VII.12]) is only summarized, instead some examples were added: Example 2.41, Example 2.43 and subsubsection 2.4.3 for the splitting fans of Picard rank 3.
- section 2.5 contains the results of [ABKW20, Chapter VIII] but with the detailed proofs and added subsection 2.5.4 and subsection 2.5.5.
- section 2.6 is a slightly adapted version of [ABKW20, Chapter IX].

2.1 Toric geometry

The main objective in this chapter is to investigate a toric variety X and its immaculate locus within $\mathrm{Cl}(X)$. For this we will make use of the classical method of calculating the cohomology of equivariant line bundles from the fan Σ in $N_{\mathbb{R}}$.

In the following we will assume that $\mathbf{k}$ is an algebraically closed field. All our toric varieties are normal. Our main references for dealing with toric varieties are [CLS11; Ful93; KKMS73]. We denote by N the lattice of one-parameter subgroups of the torus acting on the toric variety, and by M the character lattice. Throughout Σ denotes a polyhedral fan in N and $X = \mathbb{TV}(\Sigma)$ the corresponding toric variety. For a cone σ in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ or $M_{\mathbb{R}} = M \otimes \mathbb{R}$ we denote the dual cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$, respectively, by $\sigma^{\vee}$.

The set of all cones of dimension k of a fan Σ is denoted $\Sigma(k)$. Similarly, for a cone σ, by $\sigma(k)$ we mean the set of all faces of dimension k. In order to reduce the notation, we will follow the standard convention to denote rays (one-dimensional polyhedral cones) and their primitive lattice generators by the same letter, usually ρ.

Most toric varieties occurring in this chapter will be complete, many even projective. For simplifying the notation in the proofs we will also assume that X has no torus factors. In particular, the support of Σ, $\mathrm{Supp}(\Sigma)$, is the whole $N_{\mathbb{R}}$.

Every Weil divisor on X is linearly equivalent to a torus invariant divisor $D = \sum_{\rho\in\Sigma(1)} \lambda_\rho \cdot D_\rho$ with $D_\rho := \overline{\mathrm{orb}(\rho)}$.

A fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^d$ gives rise to a map $\rho : \mathbb{Z}^{\Sigma(1)} \to N$, which takes the basis element indexed by a ray of Σ to the corresponding primitive element on that ray in N. Since by assumption the toric variety $X = \mathbb{TV}(\Sigma)$ has no torus factors, the cokernel of ρ is finite. If X is smooth, ρ is surjective. We denote the kernel by K, and we obtain an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{\ \rho\ } N.$$

It is well known that the dual of this sequence yields

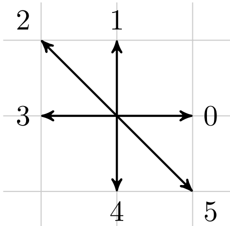
$$0 \longleftarrow \mathrm{Cl}(X) \xleftarrow{\pi} \mathrm{Div}_T(X) \xleftarrow{\rho^*} M \longleftarrow 0, \tag{2.1}$$

where $\mathrm{Div}_T(X) = \left(\mathbb{Z}^{\Sigma(1)}\right)^*$ denotes the group of torus invariant Weil divisors on X. Note that $\mathrm{Cl}(X)$ may have torsion, which corresponds to the torsion of the cokernel of ρ. The anticanonical class of X is $-K_X = \pi(\underline{1})$. The set of effective divisor classes is $\mathrm{Eff}_{\mathbb{Z}}(X) = \pi\left(\mathbb{Z}^{\Sigma(1)}_{\geq 0}\right)$, although often we really consider the effective cone $\mathrm{Eff}_{\mathbb{R}}(X) = \pi\left(\mathbb{R}^{\Sigma(1)}_{\geq 0}\right)$, where π is now considered as the map $\mathbb{R}^{\Sigma(1)} \to \mathrm{Cl}(X) \otimes \mathbb{R}$.

Example 2.1. Throughout the text we will regularly come back to the example of the del Pezzo surface of degree 6, which is the blow up of $\mathbb{P}^2$ in three points, also referred to as a *hexagon* due to the shapes of its fan and the polytopes of sections of ample divisors. This is also a smooth projective toric variety of Picard rank 4, which illustrates that our methods go beyond the main results presented in this chapter (splitting fans and Picard rank 3 cases). The exact sequence (2.1) in this example is given by the matrices

$$\rho^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The rows of ρ^* form the rays of our fan Σ, meaning we work with the following 2-dimensional fan:



With this choice of ρ^* and π the Nef cone is generated by the following 5 rays:

$$\begin{array}{lll} (1\ 1\ 0\ 0), & (1\ 0\ 1\ 1), & (0\ 1\ 1\ 0), \\ (1\ 1\ 1\ 0), & (1\ 1\ 1\ 1). & \end{array}$$

2.1.1 Toric cohomology

Let us review the classical method of calculating the cohomology groups of toric divisors.

If $D = \sum_{\rho \in \Sigma(1)} \lambda_\rho \cdot D_\rho$ is a Weil divisor on a toric variety $X = \mathbb{TV}(\Sigma)$, then for every $m \in M$ we define

$$V_{D,m} := \bigcup_{\sigma \in \Sigma} \mathrm{conv}\{\rho \mid \rho \in \sigma(1), \langle \rho, m \rangle < -\lambda_\rho\} \subseteq N_{\mathbb{R}}. \tag{2.2}$$

It is a classical result [CLS11, Theorem 9.1.3], that one obtains the m-th homogeneous piece of the sheaf cohomology of $\mathcal{O}_X(D)$ as

$$\mathrm{H}^i\left(\mathbb{TV}(\Sigma), \mathcal{O}_X(D)\right)_m = \widetilde{\mathrm{H}}^{i-1}(V_{D,m}, \mathbf{k}) \text{ for all } i \geq 0.$$

Recall that the reduced cohomology of a topological space S is defined via the cochain double complex of S mapping to a point. In particular, there arises a (-1)-st reduced cohomology, and

$$\widetilde{\mathrm{H}}^i(S,\mathbf{k}) = 0 \text{ for } i < -1 \qquad \widetilde{\mathrm{H}}^{-1}(S,\mathbf{k}) = \begin{cases} \mathbf{k} & \text{if } S = \emptyset \\ 0 & \text{if } S \neq \emptyset, \end{cases}$$
$$\mathrm{H}^0(S,\mathbf{k}) \twoheadrightarrow \widetilde{\mathrm{H}}^0(S,\mathbf{k}) \text{ with kernel } \mathbf{k}, \quad \text{and}$$
$$\widetilde{\mathrm{H}}^i(S,\mathbf{k}) = \mathrm{H}^i(S,\mathbf{k}) \text{ for } i > 0.$$

Here $\mathrm{H}^i(S,\mathbf{k})$ are the classical singular cohomology groups of the topological space S (with coefficients $\mathbf{k}$). See [Spa66, §4.3, §5.4] and [Hat02, §2.1, §3.1] for more details about singular and reduced (co)homology groups. See also a brief but relevant summary at the end of [CLS11, §9.0].

Since $0 \notin V_{D,m}$, one might retract $V_{D,m}$ onto a subset of the sphere $S^{d-1} \subseteq N_{\mathbb{R}}$ (where d is the dimension of X, and hence also of $N_{\mathbb{R}}$) without changing their cohomology. Alternatively, we can replace $V_{D,m}$ with $V^{>}_{D,m} := \mathbb{R}_{>0} \cdot V_{D,m}$. If Σ is simplicial, then we can also consider the "full" or "induced" subcomplexes $V^{\geq}_{D,m}$ of Σ, defined as $V^{\geq}_{D,m} := \left\{\sigma \in \Sigma \mid \sigma \setminus \{0\} \subset V^{>}_{D,m}\right\}$. Both sets are closely related, i.e. $V^{>}_{D,m} = \operatorname{Supp} V^{\geq}_{D,m} \setminus \{0\}$.

Remark 2.2. Note that $V_{D,m} = V_{D+\mathrm{div}(x^m),0}$. The equation $D + \mathrm{div}(x^m) = \sum_{\rho\in\Sigma(1)}(\lambda_\rho + \langle\rho, m\rangle)D_\rho$ yields that the resulting inequalities in Equation 2.2 in the two cases are equivalent. The set $V_{D,0}$ is thus of special interest and we can easily see that

$$V_{D,0} = \bigcup_{\sigma\in\Sigma} \mathrm{conv}\{\rho \mid \rho \in \sigma(1), \lambda_\rho < 0\}.$$

2.2 The immaculate locus in $\mathrm{Pic}(X)$

In this section we introduce the notion of an immaculate sheaf, concentrating on the case of line bundles.

Recall, that a sheaf is called *acyclic*, if it has all higher cohomology groups equal to zero. We will also say that for a field $\mathbf{k}$, a topological space V is $\mathbf{k}$-acyclic, if it is non-empty, arc-wise connected, and its singular cohomologies $\mathrm{H}^i(V,\mathbf{k}) = 0$ vanish for all $i > 0$. Note that in such case $\mathrm{H}^0(V,\mathbf{k}) = \mathbf{k}$. For example, all non-empty contractible spaces are $\mathbf{k}$-acyclic (for any $\mathbf{k}$). Spheres S^k are never $\mathbf{k}$-acyclic.

Definition 2.3. We call a sheaf $\mathcal{F}$ on a variety X *immaculate* if all cohomology groups $\mathrm{H}^p(X,\mathcal{F})$ ($p \in \mathbb{Z}$) vanish. The difference from the usual notion of acyclic sheaves is that we ask for the vanishing of H^0, too.

In particular, a toric sheaf of a T-invariant Weil divisor $\mathcal{O}_X(D)$ is immaculate if and only if all sets $V_{D,m}$ are $\mathbf{k}$-acyclic.

Let us compare two examples of smooth projective toric varieties with Picard rank 2: the product projective space $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hirzebruch surface $\mathbb{F}_1$. The first is a homogeneous space (for the semisimple group $\mathrm{SL}_2 \times \mathrm{SL}_2$). Figures 2.1 and 2.2 illustrate the Picard lattices of these examples, indicating the regions of line bundles with non-trivial cohomologies.

For homogeneous spaces, the regions for various H^i are disjoint, that is, for every line bundle L there is at most one value of i, such that $\mathrm{H}^i(L) \neq 0$, see for instance [Kos61, Theorem 5.14].

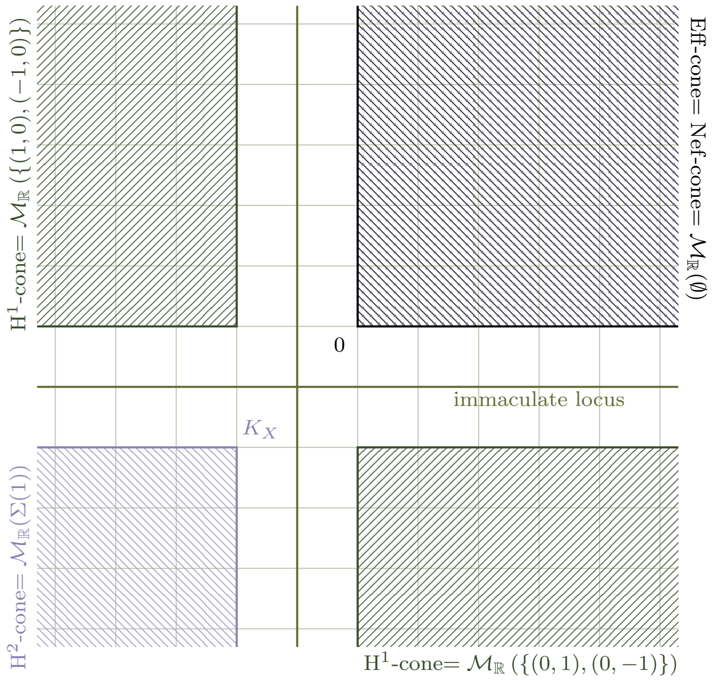


Figure 2.1: The Picard lattice of the surface $\mathbb{P}^1 \times \mathbb{P}^1$. The effective cone Eff is the cone of divisors with non-zero H^0 and it coincides with the Nef-cone. There are two cones of divisors with non-zero H^1, and one cone with non-zero H^2. The remaining line bundles are immaculate, and the immaculate locus consist of two lines parallel to the common facets of the Nef- and Eff-cones. The notation $\mathcal{M}_{\mathbb{R}}(\bullet)$ is explained in subsection 2.2.1.

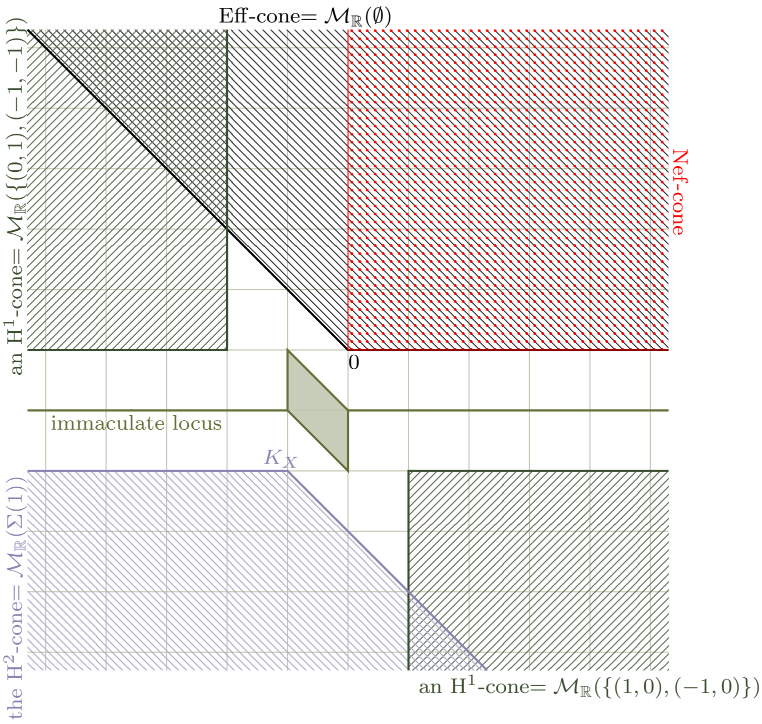


Figure 2.2: The Picard lattice of the Hirzebruch surface $\mathbb{F}_1 = \mathbb{TV}(\Sigma)$, where Σ has rays $\Sigma(1) = \{(0,1),(-1,-1),(1,0),(-1,0)\}$. The effective cone Eff is the cone of divisors with non-zero H^0. There are two cones of divisors with non-zero H^1, and one cone with non-zero H^2. In addition the Nef-cone is marked; it is a proper subset of the Eff-cone. The remaining line bundles are immaculate, and the immaculate locus consist of a bounded polytope and a line parallel to the unique common facet of the Nef- and Eff-cones.

For toric varieties this is not necessarily the case. As illustrated by the $\mathbb{F}_1$ example, the regions may intersect. The goal of this section is to show how to obtain these regions of line bundles with various cohomologies for any toric variety.

2.2.1 Temptations

Let $X = \mathbb{TV}(\Sigma)$ be a toric variety with no torus factors. For any subset $\mathcal{R} \subseteq \Sigma(1)$ we define $V^{>}(\mathcal{R}) \subset N_{\mathbb{R}}$, similar to $V^{>}_{D,0}$ as in subsection 2.1.1:

$$V^{>}(\mathcal{R}) := \mathbb{R}_{>0} \cdot \left(\bigcup_{\sigma \in \Sigma} \operatorname{conv}(\mathcal{R} \cap \sigma(1)) \right).$$

Moreover define $V^{\geq}(\mathcal{R})$ as the complex of cones $\{\operatorname{cone}(\mathcal{R} \cap \sigma(1)) \mid \sigma \in \Sigma\}$ in $N_{\mathbb{R}}$, so that

$$\operatorname{Supp} V^{\geq}(\mathcal{R}) = V^{>}(\mathcal{R}) \cup \{0\}.$$

In fact, $V^{>}(\mathcal{R}) = V^{>}_{-\sum_{\rho \in \mathcal{R}} D_\rho, 0}$ and analogously for $V^{\geq}$. Thus, as in subsection 2.1.1, if Σ is in addition simplicial, then $V^{\geq}(\mathcal{R})$ is the full ("induced") subcomplex of Σ generated by $\mathcal{R}$. This notion has an analogous function as that of "Supp($\mathbf{r}$)" in [BH09, section 4].

Definition 2.4. We call $\mathcal{R} \subseteq \Sigma(1)$ *tempting* if the geometric realization $V^{>}(\mathcal{R})$ of $V^{\geq}(\mathcal{R}) \setminus \{0\}$ admits some reduced cohomology, that is if it is not $\mathbf{k}$-acyclic.

Example 2.5. Following with our "hexagon" example (see notation in Example 2.1), the fan Σ of this surface has the following 34 tempting subsets $\mathcal{R} \subseteq \Sigma(1)$:

$$\begin{gathered}\emptyset, \{0,2\}, \{0,3\}, \{0,4\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\}, \{0,1,3\}, \{0,1,4\},\\ \{0,2,3\}, \{0,2,4\}, \{0,2,5\}, \{0,3,4\}, \{0,3,5\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\},\\ \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \{0,1,2,4\}, \{0,1,3,4\}, \{0,1,3,5\}, \{0,2,3,4\},\\ \{0,2,3,5\}, \{0,2,4,5\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{0,1,2,3,4,5\}.\end{gathered}$$

As in section 2.1 we denote both natural maps $\mathbb{Z}^{\Sigma(1)} \to \operatorname{Cl}(X)$ and $\mathbb{R}^{\Sigma(1)} \to \operatorname{Cl}(X) \otimes \mathbb{R}$ by π.

Definition 2.6. Let $\mathcal{R} \subseteq \Sigma(1)$ be a tempting subset. Then, we define $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$, the *$\mathcal{R}$-maculate set* of $\operatorname{Cl}(X)$, and $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$, the *$\mathcal{R}$-maculate region* of $\operatorname{Cl}(X) \otimes \mathbb{R}$, as

$$\begin{aligned}\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) &:= \pi\left(\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}\right) \text{ and}\\ \mathcal{M}_{\mathbb{R}}(\mathcal{R}) &:= \pi\left(\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq -1}^{\mathcal{R}}\right).\end{aligned}$$

Remark 2.7. The *forbidden sets* K_I, defined in [Efi14, Proposition 4.1] correspond to the $\mathcal{R}$-maculate sets $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ in our language.

Remark 2.8. When the fan Σ is complete. The empty set $\mathcal{R} = \emptyset$ yields $\mathcal{M}_{\mathbb{R}}(\emptyset) = \operatorname{Eff}(X)$. Moreover, Alexander duality implies that switching between $\mathcal{R}$ and $\Sigma(1) \setminus \mathcal{R}$ does not change the temptation status. After applying $\mathcal{M}$, the relation between the subsets $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ and $\mathcal{M}_{\mathbb{Z}}(\Sigma(1)\setminus\mathcal{R})$ of $\operatorname{Cl}(X)$ becomes Serre duality in $X = \mathbb{TV}(\Sigma)$. The reinterpretation in terms of Serre duality in the projective case is the following: the canonical divisor on X is $K_X = -\sum_\rho D_\rho$, and the Serre dual divisor to $D := \sum_\rho a_\rho D_\rho$ is $K_X - D = \sum_\rho (-1 - a_\rho) D_\rho$. Thus the duality swaps the sets $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ and $\mathcal{M}_{\mathbb{Z}}(\Sigma(1) \setminus \mathcal{R})$ and their cohomologies are dual to each the other.

The integral sets $\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) \subseteq \operatorname{Cl}(X)$ reflect more precisely the properties we need, but the real regions $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ are easier to control and they already contain a lot of information. Note that under the natural map $\kappa\colon \operatorname{Cl}(X) \to \operatorname{Cl}(X) \otimes \mathbb{R}$, $[D] \mapsto [D] \otimes 1$, the *$\mathcal{R}$-maculate* set is mapped into the $\mathcal{R}$-*maculate* region, that is $\kappa\colon \mathcal{M}_{\mathbb{Z}}(\mathcal{R}) \to \mathcal{M}_{\mathbb{R}}(\mathcal{R})$. In other words, the preimage $\kappa^{-1}\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ in $\mathrm{Cl}(X)$ contains $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$, or, slightly incorrect, $\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) \subseteq \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \cap \mathrm{Cl}(X)$. We will encounter several situations when $\kappa^{-1}\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ and $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ are either equal or not equal, depending on the saturation of respective cones.

Proposition 2.9. Suppose $X = \mathbb{TV}(\Sigma)$ is a toric variety with no torus factors.

(i) If $\mathcal{R} \subseteq \Sigma(1)$ is tempting, then for any i such that $\widetilde{\mathrm{H}}^{i-1}(V^{>}(\mathcal{R}), \mathbf{k}) \neq 0$ and any Weil divisor $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$, we have $\mathrm{H}^i(\mathcal{O}_X(D)) \neq 0$.

(ii) A line bundle $\mathcal{O}_X(D)$ for $[D] \in \mathrm{Cl}(X)$ is immaculate if and only if $D \notin \bigcup_{\mathcal{R}=\text{tempting}} \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$.

(iii) A line bundle $\mathcal{O}_X(D)$ such that $[D]_{\mathbb{R}} \notin \bigcup_{\mathcal{R}=\text{tempting}} \mathcal{M}_{\mathbb{R}}(\mathcal{R})$ is immaculate.

This statement is comparable with [BH09, Proposition 4.3 and 4.5] and [Efi14, Proposition 4.2].

Proof. Let $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R}) = \pi(\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}})$, then by definition D is linearly equivalent to some $D' = \sum_{\rho\in\Sigma(1)} \lambda_\rho D_\rho$, with $\mathcal{R} = \{\rho \in \Sigma(1) \mid \lambda_\rho < 0\}$. If $\mathcal{R} \subset \Sigma(1)$ is tempting, then $V^{>}(\mathcal{R})$ is not $\mathbf{k}$-acyclic, so there exists i with $\widetilde{\mathrm{H}}^{i-1}(V^{>}(\mathcal{R}), \mathbf{k}) \neq 0$. By [CLS11, Theorem 9.1.3] the cohomology group of the line bundle associated to D is $\mathrm{H}^i(\mathcal{O}_X(D))_m = \widetilde{\mathrm{H}}^{i-1}(V_{D,m}, \mathbf{k})$ and with Remark 2.2 $\mathrm{H}^i(\mathcal{O}_X(D))_m = \mathrm{H}^i(\mathcal{O}(D'))_0 = \widetilde{\mathrm{H}}^{i-1}(V_{D',0}, \mathbf{k})$. It is easy to see that $V^{>}_{D',0} = V^{>}(\mathcal{R})$, and thus $\mathrm{H}^i(\mathcal{O}_X(D))_m = \widetilde{\mathrm{H}}^{i-1}(V^{>}(\mathcal{R}), \mathbf{k}) \neq 0$, which proves (i).

If D is immaculate, then it is not in $\bigcup_{\mathcal{R}=\text{tempting}} \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ by (i). For the other direction, suppose that $[D]$ is not immaculate, then there exist $i \in \mathbb{Z}$ and $m \in M$ such that $\mathrm{H}^i(\mathcal{O}_X(D))_m \neq 0$. Set $D' = D + \mathrm{div}(x^m) = \sum_{\rho\in\Sigma(1)} \lambda_\rho D_\rho$ and $\mathcal{R} = \{\rho \in \Sigma(1) \mid \lambda_\rho < 0\}$. With the same arguments as before $\mathcal{R}$ is tempting and thus $[D]$ is in the maculate set $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$.

Finally, (iii) follows from (ii), since $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ implies $[D]_{\mathbb{R}} \in \mathcal{M}_{\mathbb{R}}(\mathcal{R})$. □

It is not always true, that $[D]_{\mathbb{R}} \in \mathcal{M}_{\mathbb{R}}(\mathcal{R})$ implies $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ as the following example shows.

Example 2.10. Let $X = \mathbb{TV}(\Sigma) = \mathbb{P}(2,3,5)$, the weighted projective plane with weights 2, 3, 5. Consider the divisor $D = D_{\rho_2} - D_{\rho_1}$. Notice that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(1)$. Then D is immaculate, but $[D]_{\mathbb{R}} \in \mathcal{M}_{\mathbb{R}}(\mathcal{R})$ for $\mathcal{R} = \emptyset$ (corresponding to the $\mathrm{Eff}_{\mathbb{R}}$-cone).

This leads to the following definition:

Definition 2.11. A divisor D is $\mathbb{R}$-*immaculate*, if

$$[D]_{\mathbb{R}} \in \mathrm{Cl}(X) \otimes \mathbb{R} \setminus \bigcup_{\mathcal{R}=\text{tempting}} \mathcal{M}_{\mathbb{R}}(\mathcal{R}).$$

Thus Example 2.10 shows a simple case of an immaculate Weil divisor that is not $\mathbb{R}$-immaculate. In Example 2.39 we construct a line bundle on a smooth toric projective variety with the same property. Up to the zero-th cohomology group, the concept of $\mathbb{R}$-immaculate divisor here is an analogue of the *strongly acyclic* line bundle in [BH09, Def. 4.4].

Definition 2.12. The *immaculate loci* of X are

$$\mathrm{Imm}_{\mathbb{Z}}(X) = \mathrm{Cl}(X) \setminus \bigcup_{\mathcal{R}\subset\Sigma(1),\ \mathcal{R} \text{ is tempting}} \mathcal{M}_{\mathbb{Z}}(\mathcal{R}), \text{ and}$$

$$\mathrm{Imm}_{\mathbb{R}}(X) = \kappa^{-1}\left((\mathrm{Cl}(X)\otimes\mathbb{R}) \setminus \bigcup_{\mathcal{R}\subset\Sigma(1),\ \mathcal{R} \text{ is tempting}} \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \right) \subset \mathrm{Cl}(X),$$

where $\kappa\colon \mathrm{Cl}(X) \to \mathrm{Cl}(X)\otimes\mathbb{R}$ is the natural map $[D] \mapsto [D]\otimes 1 = [D]_{\mathbb{R}}$.

Thus $\mathrm{Imm}_{\mathbb{Z}}(X)$ is the collection of all immaculate divisors. By Proposition 2.9(iii) all the divisors in $\mathrm{Imm}_{\mathbb{R}}(X)$ are immaculate, that is $\mathrm{Imm}_{\mathbb{R}}(X) \subset \mathrm{Imm}_{\mathbb{Z}}(X)$. More precisely, $\mathrm{Imm}_{\mathbb{R}}(X)$ is the set of all $\mathbb{R}$-immaculate divisors as in Definition 2.11.

Example 2.13. In contrast to Example 2.10, we can see that in the case of the hexagon (Example 2.1), all immaculate line bundles are $\mathbb{R}$-immaculate. This follows since the matrix π defining the map $(\mathbb{Z}^{\Sigma(1)})^* \to \mathrm{Pic}(X)$ is totally unimodular.

Example 2.14. We illustrate Proposition 2.9 with the example of the Hirzebruch surface $\mathbb{F}_a = \mathbb{TV}(\Sigma_a)$. The special cases $a = 0$ and $a = 1$ are presented in the Figures 2.1 and 2.2, respectively. More general cases will be explained in subsection 2.3.2.

The Gale transform, that is the map π, is given by the matrix

$$\pi = \left(\begin{array}{cc|cc} 1 & 1 & 0 & -a \\ 0 & 0 & 1 & 1 \end{array}\right).$$

The associated rays of the fan Σ_a are given by the matrix

$$\rho = \left(\begin{array}{cc|cc} 0 & -a & 1 & -1 \\ 1 & -1 & 0 & 0 \end{array}\right).$$

If we denote the four columns, that is the rays, by $\rho_1, \ldots, \rho_4$, then the tempting subsets of $\Sigma_a(1)$ are just $\emptyset$, $\Sigma_a(1)$, $\mathcal{R}_1 = \{\rho_1, \rho_2\}$, and $\mathcal{R}_2 = \{\rho_3, \rho_4\}$. The corresponding maculate regions are

$$\begin{aligned} \mathcal{M}_{\mathbb{R}}(\emptyset) &= \mathrm{cone}\big\langle (1,0), (0,1), (-a,1)\big\rangle = \mathrm{cone}\big\langle (1,0),\ (-a,1)\big\rangle, \\ \mathcal{M}_{\mathbb{R}}(\Sigma_a(1)) &= (a-2,-2) + \mathrm{cone}\big\langle (-1,0), (a,-1)\big\rangle, \\ \mathcal{M}_{\mathbb{R}}(\mathcal{R}_1) &= (-2,0) + \mathrm{cone}\big\langle (-1,0), (0,1), (-a,1)\big\rangle = (-2,0) + \mathrm{cone}\big\langle (-1,0), (0,1)\big\rangle, \\ \mathcal{M}_{\mathbb{R}}(\mathcal{R}_2) &= (a,-2) + \mathrm{cone}\big\langle (1,0), (0,-1)\big\rangle. \end{aligned}$$

The lattice points within the complement of the union of these four regions consist of the line $(*,-1)$ and, if $a \geq 1$, the two isolated points $(-1,0)$ and $(a-1,-2)$. In the degenerate case of $a = 0$, there is an additional line $(-1,*)$, see Figure 2.1. Here, all immaculate divisors are $\mathbb{R}$-immaculate.

2.2.2 Conditions on presence or absence of temptations

In this section we describe straightforward criteria that imply that a given subset of rays is tempting or it is non-tempting. The upshot is that, for all sets $\mathcal{R} \subseteq \Sigma(1)$ covered by one of these claims, one does not need to look at the topology of $V^{>}(\mathcal{R}) = \mathrm{Supp}V^{\geq}(\mathcal{R}) \setminus \{0\}$. In many examples theses conditions will already completely determine all tempting subsets, in other cases it reduces the number of candidates drastically. This is especially helpful when dealing with classes of toric varieties as we will do in section 2.3 – 2.5.

Monomials do not lead into temptation

The first criterion is similar to the boundedness condition in [HKP06, Proposition 2].

Proposition 2.15. Suppose $X = \mathbb{TV}(\Sigma)$ is a complete toric variety and $\mathcal{R} \subset \Sigma(1)$ is a tempting subset. Denote by $\rho^*\colon M_{\mathbb{R}} \to \mathbb{R}^{\Sigma(1)}$ the natural embedding of the principal torus invariant divisors into all torus invariant divisors. Then

$$\rho^*(M_{\mathbb{R}}) \cap \left(\mathbb{R}^{\Sigma(1)\setminus\mathcal{R}}_{\geq 0} \times \mathbb{R}^{\mathcal{R}}_{\leq 0}\right) = \{0\}\,.$$

Proof. Suppose on the contrary, that $(\rho^*)^{-1}\left(\mathbb{R}^{\Sigma(1)\setminus\mathcal{R}}_{\geq 0} \times \mathbb{R}^{\mathcal{R}}_{\leq 0}\right)$ is a positive dimensional cone $\tau \subset M_{\mathbb{R}}$. Consider the divisor $D = \sum_{\varrho\in\mathcal{R}} -D_\varrho$. Since $\mathcal{R}$ is tempting, the divisor has non-zero cohomologies in degree $-m$ for all $m \in \tau \cap M$. Thus, the cohomology groups $\bigoplus_{i=0}^{\dim X} \mathrm{H}^i(D)$ are infinitely dimensional, a contradiction to the completeness of X. □

Example 2.16. Consider the Hirzebruch surface $\mathbb{F}_a$ as in Example 2.14, and suppose $a > 0$. Then out of 16 subsets of $\{\rho_1, \rho_2, \rho_3, \rho_4\}$, only six survive the test provided by Proposition 2.15. Namely, these are the four tempting subsets as listed in Example 2.14, and $\{\rho_4\}$ and its complement $\{\rho_1, \rho_2, \rho_3\}$ having the property of the associated cone intersecting M in just $\{0\}$.

Example 2.17. In the "hexagon" case (see Examples 2.1 and 2.5), Proposition 2.15 shows that the following 18 out of $64 = 2^6$ subsets of $\Sigma(1)$ are non-tempting:

$$\{0,1\}\,,\{0,5\}\,,\{1,2\}\,,\{2,3\}\,,\{3,4\}\,,\{4,5\}\,,\{0,1,2\}\,,\{0,1,5\}\,,\{0,4,5\}\,,\{1,2,3\}\,,$$
$$\{2,3,4\}\,,\{3,4,5\}\,,\{0,1,2,3\}\,,\{0,1,2,5\}\,,\{0,1,4,5\}\,,\{0,3,4,5\}\,,\{1,2,3,4\}\,,\{2,3,4,5\}\,.$$

Faces are not tempting

Proposition 2.18. Suppose $X = \mathbb{TV}(\Sigma)$ is a complete toric variety and $\sigma \in \Sigma$ is any cone (or a proper subfan with strictly convex support). Then the subsets $\mathcal{R} = \sigma(1) \subset \Sigma(1)$ and $\Sigma(1) \setminus \mathcal{R}$ are not tempting.

Proof. The complex $V^{>}(\mathcal{R})$ is equal to the convex set $\sigma \setminus \{0\}$, hence it is contractible. By Alexander duality (see Remark 2.8) the complement is also not tempting. □

Example 2.19. For the Hirzebruch surface $\mathbb{F}_a$, only the four tempting subsets fail this test. All the other subsets are either faces or complements of faces.

Example 2.20. According to Proposition 2.18, in the "hexagon" case (see Examples 2.1, 2.5), the following 24 subsets of $\Sigma(1)$ are non-tempting: all single element subsets $\{i\}$, all consecutive two elements subsets $\{i, i+1\}$, and their complements (which have either four or five elements), which are all faces or their complements. Moreover, considering also three consecutive elements $\{i, i+1, i+2\}$ (which are rays of a subfan with a strictly convex support), we obtain 30 subsets, which are all the non-tempting subsets of $\Sigma(1)$. Alternatively, the three element subsets can be understood from Example 2.17.

The first part of Proposition 2.18 can be generalized to certain unions of faces.

Proposition 2.21. Suppose $X = \mathbb{TV}(\Sigma)$ is a complete toric variety and $\mathcal{R}$ such that the maximal cones $\sigma_1, \ldots, \sigma_l$ of $V^{\geq}(\mathcal{R})$ (with respect to inclusion) intersect in a common non-zero face $\tau \in \Sigma$.

Then $\mathcal{R}$ and $\Sigma(1) \setminus \mathcal{R}$ are not tempting.

Proof. All σ_i can be contracted to their face τ. Thus also $\sigma_i \setminus \{0\}$ can be contracted to $\tau \setminus \{0\}$. The set $V^{>}(\mathcal{R})$ is equal to the union of the $\sigma_i \setminus \{0\}$ and we can simultaneously contract all $\sigma_i \setminus \{0\}$ to the convex set $\tau \setminus \{0\}$, which is contractible.

Again Alexander duality implies that the complement is not tempting, either. □

Although this is not completely correct, we will say that $\mathcal{R}$ is the *union of cones meeting in a common face* when $\mathcal{R}$ satisfies the conditions from Proposition 2.21. This condition includes the case of Proposition 2.18 where $\mathcal{R} = \sigma(1)$ for a cone $\sigma \in \Sigma$ and the case that $\mathcal{R} = \Sigma'(1)$ where $\Sigma' \subset \Sigma$ is a subfan with convex support consisting of only two cones, or a subdivision of its support obtained by inserting a ray. But in dimensions greater than two, it captures also situations that are not covered by Proposition 2.18.

Example 2.22. Let Σ be a fan in $N_{\mathbb{R}} \cong \mathbb{R}^3$ with the rays

$$\{\rho_1 = (1,\ 0,\ 1),\ \rho_2 = (0,\ 1,\ 1),\ \rho_3 = (-1,\ -1,\ -2),\ \rho_4 = (1,\ 1,\ 1),\ \rho_5 = (0,\ 0,\ 1)\}$$

and the maximal cones be given by the following subsets of the rays (for better readability we only write the indices of the associated rays)

$$\{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\} \text{ and } \{2,4,5\}.$$

Then for $\mathcal{R} = \{1,2,5\}$, the maximal elements of the support of $V^{\geq}(\mathcal{R})$ are the cones $\operatorname{cone}(\rho_1, \rho_5)$ and $\operatorname{cone}(\rho_2, \rho_5)$ that intersect in the ray ρ_5. Thus by Proposition 2.21, it is not tempting, but neither $\mathcal{R}$ nor its complement correspond to a subfan of Σ with convex support, so this situation is not explained by Proposition 2.18.

Example 2.23. For the Hirzebruch surface $\mathbb{F}_a$ and the hexagon example, the condition from Proposition 2.21 is equivalent to Proposition 2.18.

Primitive collections delude

A *primitive collection* of a simplicial fan Σ is a "minimal non-face", that is, a subset of rays $\mathcal{R} \subset \Sigma(1)$, such that the cone spanned by $\mathcal{R}$ is not in Σ, but the cone spanned by each proper subset $\mathcal{R}' \subsetneq \mathcal{R}$ is in Σ. In other words, a subset $\mathcal{R} \subset \Sigma(1)$ of any fan is a primitive collection, if $\mathcal{R}$ is not contained in any single cone of Σ, but every proper subset is. This notion is particularly useful in the classification of projective toric varieties, see [Bat91], [CR09] for more details, see also subsection 2.4.1.

Proposition 2.24. Suppose $X = \mathbb{TV}(\Sigma)$ is a complete simplicial toric variety with no torus factors. Let $\mathcal{R} \subset \Sigma(1)$ be either empty or a primitive collection. Then $\mathcal{R}$ and its complement are tempting.

Proof. If $\mathcal{R} = \emptyset$ then the claim is clear, so suppose $\mathcal{R}$ is a primitive collection, that is, a subset which is does not generate a cone of Σ, but all its proper subsets do generate such cones. By Alexander duality it is enough to prove that $\mathcal{R} = \{\rho_1, \ldots, \rho_k\}$ is tempting. Since every ray belongs to Σ, we have $k \geq 2$. We distinguish between two cases: either $\mathcal{R}$ is linearly independent or not.

If $\mathcal{R}$ is linearly independent, then $\mathcal{V} := \operatorname{span}_{\mathbb{R}} \mathcal{R}$ is k-dimensional, and $\mathcal{R}^+ := \sum_{j=1}^k \mathbb{R}_{\geq 0} \cdot \rho_j$ is a k-dimensional simplicial cone in $\mathcal{V}$ which does not belong to Σ. On the other hand, its boundary $\partial\mathcal{R}^+$ is a subcomplex of Σ; it is exactly the complex $V^{\geq}(\mathcal{R})$ as in subsection 2.2.1.

Thus, $\operatorname{Supp} V^{\geq}(\mathcal{R}) \setminus \{0\} = \operatorname{Supp} \partial\mathcal{R}^{+} \setminus \{0\}$ is homotopy equivalent to a sphere S^{k-2}. In particular, it is not **k**-acyclic.

On the other hand, suppose $\mathcal{R}$ is linearly dependent. Since $\mathcal{R}$ is a primitive collection, all the cones generated by $\mathcal{R} \setminus \{\rho_j\}$ are necessarily simplicial. In particular, $\mathcal{V} := \operatorname{span}_{\mathbb{R}} \mathcal{R}$ is $(k-1)$-dimensional, and each $\mathcal{R} \setminus \{\rho_j\}$ spans a full-dimensional cone in $\mathcal{V}$ that belongs to Σ. Thus, these cones generate $V^{\geq}(\mathcal{R})$, and this is a complete fan in $\mathcal{V}$ which (up to $\mathbb{R}$-linear change of coordinates) looks like the $\mathbb{P}^{k-1}$-fan in $\mathbb{R}^{k-1}$. Again, $V^{>}(\mathcal{R}) = \operatorname{Supp} V^{\geq}(\mathcal{R}) \setminus \{0\}$ is homotopy equivalent to S^{k-2}, hence it is not **k**-acyclic. □

Example 2.25. For the Hirzebruch surface $\mathbb{F}_a$, all tempting subsets are predicted by Proposition 2.24. That is all four of them are either empty, or $\Sigma(1)$, or a primitive collection.

Example 2.26. Proposition 2.24 applied to the hexagon example (see Examples 2.1, 2.5), implies that the following 20 subsets are tempting:

$$\emptyset, \{0,2\}, \{0,3\}, \{0,4\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\}, \{0,1,2,4\}, \{0,1,3,4\},$$
$$\{0,1,3,5\}, \{0,2,3,4\}, \{0,2,3,5\}, \{0,2,4,5\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \Sigma(1).$$

With the notion of primitive collections, we can give a different characterization of the condition from Proposition 2.21.

Proposition 2.27. Suppose $X = \mathbb{TV}(\Sigma)$ is a complete simplicial toric variety with no torus factors. Let $\mathcal{R} \subset \Sigma(1)$ and $\sigma_1, \ldots, \sigma_l$ be the (inclusion) maximal cones of $V^{\geq}(\mathcal{R})$. Then $\sigma_1, \ldots, \sigma_l$ intersect in a common non-zero face $\tau \in \Sigma$ if and only if $\mathcal{R}$ is not the union of primitive collections.

Proof. With the notation from above $\mathcal{R} = \bigcup_{i=1}^{l} \sigma_i(1)$ and $\operatorname{Supp} V^{\geq}(\mathcal{R}) = \bigcup_{i=1}^{l} \operatorname{Supp} \sigma_i$. Let $\tau = \bigcap_{i=1}^{l} \sigma_i$ be the common face of the cones. Since $\tau \in \Sigma$ (and Σ is simplicial), $\tau(1)$ does not contain a primitive collection.

Now we will show that for $\rho \in \tau(1)$ there does not exists a primitive collection P with $\rho \in P \subset \mathcal{R}$. So $\mathcal{R}$ cannot be the union of primitive collections.

Suppose there exists a primitive collection P with $\rho \in P \subset \mathcal{R}$. Now since $P \subset \mathcal{R}$, the geometric realization

$$\operatorname{Supp} V^{\geq}(P) = \bigcup_{i=1}^{l} \operatorname{Supp} V^{\geq}(\sigma_i(1) \cap P)$$

is a union of faces, all containing the ray ρ, since $\rho \in \sigma_i$ and $\rho \in P$. Thus by Proposition 2.21 $V^{>}(P)$ is contractible and P not tempting, but P is primitive and hence by Proposition 2.24 it is tempting.

For the converse, consider the set

$$P(\mathcal{R}) := \{P \mid P \text{ primitive collection with } P \subset \mathcal{R}\}.$$

We define $B := \cup_{P \in P(\mathcal{R})} P \subset \mathcal{R} \subset \Sigma(1)$ and $F := \mathcal{R} \setminus B \subset \Sigma(1)$, which by assumption is non-empty. Notice that for $P \in P(\mathcal{R})$, it holds that $P \cap F = \emptyset$. By construction F does not contain any primitive collection, hence F consists of the rays of a cone $\tau \in \Sigma$.

For $P \in P(\mathcal{R})$ and $\rho \in P$ consider $S := F \cup P \setminus \rho$. Suppose that S is not the set of rays of a cone of Σ, which we will denote now as $S \notin \Sigma$, then there exists a primitive collection P' with

$P' \subset S$. Since $P \setminus \rho$ is a face, P' is not contained in it, thus $P' \cap F \neq \emptyset$. But $P' \subset S \subset \mathcal{R}$, so $P' \in P(\mathcal{R})$, but then $P' \cap F = \emptyset$, thus $S \in \Sigma$.

Now we define $S(P, \rho) := F \cup P \setminus \rho$ and we see that

$$\mathcal{R} = \cup_{P \in P(\mathcal{R})}(\cup_{\rho \in P} S(P, \rho))$$

and Supp $V^{\geq}(\mathcal{R})$ is the union of the supports of the cones $S(P, \rho)$, meeting in the common face τ with $\tau(1) = F$. □

This leads to the following statement which can be found with a different proof in [Efi14, Lemma 4.4]

Proposition 2.28. If $\mathcal{R}$ is tempting, then $\mathcal{R}$ and $\Sigma(1) \setminus \mathcal{R}$ are unions of primitive collections.

Proof. If $\mathcal{R}$ is not the union of primitive collections, then it is a union of cones meeting in a common face by Proposition 2.27 and by Proposition 2.21 $\mathcal{R}$ and $\Sigma(1) \setminus \mathcal{R}$ are not tempting. Applying the same argument to $\Sigma(1) \setminus \mathcal{R}$ shows that also $\Sigma(1) \setminus \mathcal{R}$ has to be the union of primitive collections. □

Example 2.29. We continue with Example 2.22. The primitive collections in this case are $\{1, 2\}$ and $\{3, 4, 5\}$. The subset $\{1, 2, 5\}$ that we already discussed before cannot be written as a union of primitive collections. Actually the only subsets that can be written as union of primitive collections are $\emptyset, \{1, 2\}, \{3, 4, 5\}$ and $\{1, 2, 3, 4, 5\}$. All of them are tempting by Proposition 2.24, since they are empty, a primitive collection or the set of all rays.

Thus for finding all tempting subsets of $\Sigma(1)$, we only have to investigate subsets that are unions of primitive collections and whose complements are also unions of primitive collections. However, not all sets with this property are tempting. In the hexagon example (see Examples 2.1 and 2.5) and in the previous example all subsets satisfying this condition are tempting, but in general this condition does not assure that $\mathcal{R}$ is tempting.

Example 2.30. Let $\Sigma \subset \mathbb{R}^2$ be a complete fan that consists of eight rays $\{0, 1, 2, 3, 4, 5, 6, 7\}$ ordered clockwise. All primitive collections are given by two non-adjacent rays. If $\#\mathcal{R} = 4$, then it is always the union of two primitive collections and $\Sigma(1) \setminus \mathcal{R}$ is, too. But if $\mathcal{R} = \{0, 1, 2, 3\}$, then Supp$V^{\geq}(\mathcal{R})$ is the union of three adjacent cones, where the three cones are the ones corresponding to the rays $\{0, 1\}, \{1, 2\}$ and $\{2, 3\}$. This set is contractible and hence not tempting. Notice that this was already implied by Proposition 2.18, since at least one of $V^{>}(\mathcal{R})$ and $V^{>}(\Sigma(1) \setminus \mathcal{R})$ is convex.

The condition of Proposition 2.28 can spot whether the associated $V^{\geq}(\mathcal{R})$ is a union of cones meeting in a common non-zero face. But it is not possible to spot whether $V^{\geq}(\mathcal{R})$ is a subdivision of a bigger cone that does not satisfy this condition. Clearly if the cones $V^{\geq}(\mathcal{R})$ form a subdivision of a bigger cone, $\mathcal{R}$ is not tempting, by Proposition 2.18. As a result any situation where the subset $\mathcal{R}$ is non-tempting, but contains a subdivided cone, can also not be judged by Proposition 2.28. This makes sense, since primitive collections do not provide the whole information about the convex structure of the fan. For this purpose one also needs the information about primitive relations, see subsection 2.4.1.

2.3 Toric varieties with Picard rank 2

We commence this section with recalling a well-known fact about smoothness of toric varieties in terms of Gale duality. Then we study our first family of examples, that is smooth complete toric varieties of Picard rank 2. Such varieties are described in [Kle88], and we can classify all the immaculate line bundles on them. While the case of Picard rank 2 is a special case of section 2.4, it will be helpful to spend some time on this. Here we are in a situation where all loci can be completely described and depicted, and this will be very helpful for understanding the general situation.

2.3.1 Spotting smoothness via Gale duality

When working with fans having only few generators, the Gale transform becomes the essential tool to investigate their combinatorial structure. We recall an argument showing that this instrument, considered for abelian groups instead of vector spaces, can spot smoothness, too. Let

$$0 \longrightarrow K \xrightarrow{\iota} \mathbb{Z}^n \xrightarrow{\rho} N \longrightarrow 0$$

be an exact sequence of free abelian groups with $d := \operatorname{rk} N$. This situation gives rise to the Gale transform being just the dual sequence

$$0 \longleftarrow K^* \xleftarrow{\iota^*} \mathbb{Z}^{n*} \longleftarrow N^* \longleftarrow 0.$$

Denote by $\mathbb{Z}^d \subseteq \mathbb{Z}^n$ and $\mathbb{Z}^{(n-d)*} \subseteq \mathbb{Z}^{n*}$ the orthogonal subgroups being generated by $\{e_1, \ldots, e_d\}$ and $\{e^{d+1}, \ldots, e^n\}$, respectively.

Proposition 2.31. The determinant of $\{\rho(e_1), \ldots, \rho(e_d)\}$ equals, maybe up to sign, the determinant of $\{\iota^*(e^{d+1}), \ldots, \iota^*(e^n)\}$.

Proof. Assuming that the restriction $\rho|_{\mathbb{Z}^d} : \mathbb{Z}^d \to N$ has a finite cokernel C (which is equivalent to $\rho|_{\mathbb{Z}^d}$ being injective or to $\mathbb{Q}^d \xrightarrow{\rho} N \otimes \mathbb{Q}$ being an isomorphism), we obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}^d & = & \mathbb{Z}^d & & \\
 & & & & \downarrow & & \downarrow \rho & & \\
0 & \longrightarrow & K & \xrightarrow{\iota} & \mathbb{Z}^n & \xrightarrow{\rho} & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{\bar{\iota}} & \mathbb{Z}^{n-d} & \xrightarrow{\rho} & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Dualizing the bottom row yields $\operatorname{coker}\left(\mathbb{Z}^{(n-d)*} \xrightarrow{\iota^*} K^*\right) = \operatorname{Ext}^1_{\mathbb{Z}}(C, \mathbb{Z})$. That is, the cokernels of $\mathbb{Z}^d$ in N and of $\mathbb{Z}^{(n-d)*}$ in K^* have the same order. □

2.3.2 Immaculate locus for Picard rank 2

After illustrating the general method for classifying immaculate line bundles in section 2.2, we describe now the immaculate loci in the specific case of smooth (complete) toric varieties of Picard rank 2.

Investigating Gale duals leads to the well-known classification of the combinatorial type of d-dimensional, simple, convex polytopes with $d+2$ vertices – they are $(\triangle^{\ell_1-1} \times \triangle^{d-\ell_1+1})^\vee$ for some $\ell_1 = 2, \dots, d$, where $\triangle^r$ means the r-dimensional simplex and $(\dots)^\vee$ denotes the dual of a polytope. This is a special case of the situation we will meet in subsection 2.4.2.

Explicitly, in [Kle88, Theorem 1], this classification was refined to find all complete smooth d-dimensional fans with $d+2$ rays, that is, all smooth complete toric varieties with Picard rank two. They are parametrized by the following data:

(i) a decomposition $d+2 = \ell_1 + \ell_2$ with $\ell_1, \ell_2 \geq 2$ and

(ii) a choice of non-positive integers $0 = c^1 \geq \dots \geq c^{\ell_2}$ which are jointly denoted by $c \in \mathbb{Z}^{\ell_2}_{\leq 0}$.

These data provide the $2 \times (\ell_1 + \ell_2)$-matrix

$$\left(\begin{array}{ccc|cccc} 1 & \dots & 1 & 0 & c^2 & \dots & c^{\ell_2} \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{array}\right)$$

encoding $\pi\colon (\mathbb{Z}^{(d+2)})^* \twoheadrightarrow \mathbb{Z}^2 = \operatorname{Cl} X$ (compare with Example 2.14, where we had $a = -c^2$). That is, the rays of the associated fan Σ_c are $u_i = \rho(e_i)$ and $v_j = \rho(f_j)$ in

$$N := \mathbb{Z}^{\ell_1+\ell_2} / \big(\mathbb{Z} \cdot (\underline{1}, c) + \mathbb{Z} \cdot (\underline{0}, \underline{1})\big) \cong \mathbb{Z}^d$$

where $\{e_1, \dots, e_{\ell_1}, f_1, \dots, f_{\ell_2}\}$ denotes the canonical basis in $\mathbb{Z}^{d+2} = \mathbb{Z}^{\ell_1+\ell_2}$ and $\rho : \mathbb{Z}^{d+2} \to N$ is the canonical projection. The fan structure is easy – the d-dimensional cones are σ_{ij} which are generated by $\Sigma_c(1) \setminus \{u_i, v_j\}$ $(i = 1, \dots, \ell_1,\ j = 1, \dots, \ell_2)$. That is, $\#\Sigma_c(d) = \ell_1 \ell_2$. Comparing with (2.3.1), one sees that the corresponding cross-frontier (2×2)-minors of the above matrix are $\det \begin{pmatrix} 1 & c^i \\ 0 & 1 \end{pmatrix} = 1$, that is, by Proposition 2.31, the cones σ_{ij} are indeed smooth. We will denote $\overline{c} := \sum_{\nu=1}^{\ell_2} c^\nu$. In [Kle88, Theorem 2] it is shown that X_c is Fano if and only if $-\overline{c} \leq \ell_1 - 1$.

Theorem 2.32. *Suppose $X = \mathbb{TV}(\Sigma_c)$ is a smooth complete toric variety of Picard rank 2. Then $\operatorname{Imm}_{\mathbb{Z}}(X) = \operatorname{Imm}_{\mathbb{R}}(X)$. Moreover, the line bundle represented by $(x, y) \in \mathbb{Z}^2 = \operatorname{Cl} X$ is immaculate if and only if one of the following holds:*

- $-\ell_2 < y < 0$ *or*
- $y \geq 0$ *and* $-\ell_1 < x < c^{\ell_2} y$ *or*
- $y \leq -\ell_2$ *and* $0 > x + \overline{c} > c^{\ell_2}(y + \ell_2) - \ell_1$.

Note that the second and the third case in the theorem are Serre dual to one another, while the first item is self-dual. The first item consists of $\ell_2 - 1$ (horizontal) affine lines. If $c = \underline{0}$, that is, if $X \simeq \mathbb{P}^{\ell_1-1} \times \mathbb{P}^{\ell_2-1}$, then the divisors appearing in the second and third item (plus parts of the first item) are the $\ell_1 - 1$ lines $-\ell_1 < x < 0$. Otherwise, $c^{\ell_2} < 0$, and there are only finitely many line bundles in the second and third items (the inequalities define triangles). The special case of $\ell_1 = \ell_2 = 2$ is illustrated on Figure 2.2, and another case of a 5-fold is on Figure 2.3. Points of the form $(-1, 0), \dots, (-\ell_1 + 1, 0)$ are always contained in the second item (independently of c). Later, in the more general setup of section 2.4, these points, together with the lines from the first item, will form the "generating seeds" in the sense of Definition 2.40.

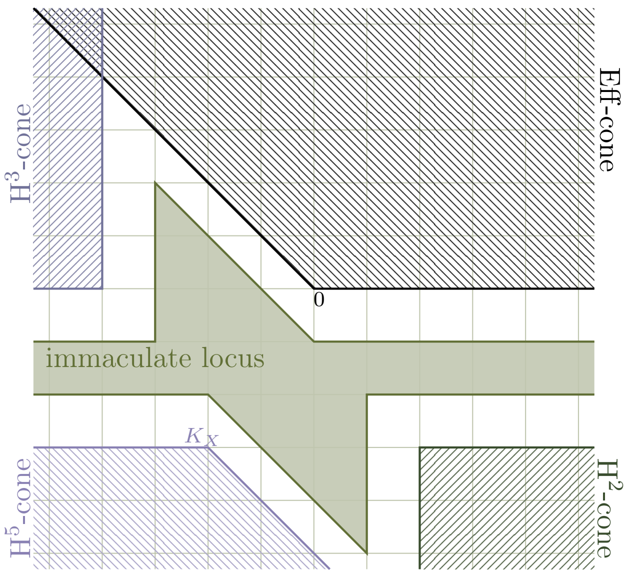


Figure 2.3: The Picard lattice and immaculate locus of a smooth projective toric 5-fold X with $\operatorname{Cl} X = \mathbb{Z}^2$ and the matrix $\pi = \left(\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$, that is, $c = (0,\ -1,\ -1)$.

Proof of Theorem 2.32. The only tempting subsets of $\Sigma_c(1)$ are $\emptyset$, $U = \{u_1, \ldots, u_{\ell_1}\}$, $V = \{v_1, \ldots, v_{\ell_2}\}$ and $\Sigma_c(1) = U \sqcup V$. We proceed along the lines of Example 2.14 and calculate the maculate loci:

$$\begin{aligned}
\mathcal{M}_{\mathbb{R}}(\emptyset) &= \operatorname{cone}\big\langle(1,0),\ (c^{\ell_2},1)\big\rangle,\\
\mathcal{M}_{\mathbb{R}}(\Sigma_c(1)) &= (-\bar{c} - \ell_1, -\ell_2) + \operatorname{cone}\big\langle(-1,0),\ (-c^{\ell_2},-1)\big\rangle,\\
\mathcal{M}_{\mathbb{R}}(U) &= (-\ell_1, 0) + \operatorname{cone}\big\langle(-1,0),\ (0,1)\big\rangle,\\
\mathcal{M}_{\mathbb{R}}(V) &= (-\bar{c}, -\ell_2) + \operatorname{cone}\big\langle(1,0),\ (0,-1)\big\rangle.
\end{aligned}$$

For every maculate $\mathcal{R} \subset \Sigma_c(1)$, the tail cone in the above locus is smooth and the primitive generators of rays are all in the image of the set $\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}$. Thus, the map $\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}} \to \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \cap \operatorname{Cl}(X)$ is surjective, i.e. $\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) = \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \cap \operatorname{Cl}(X)$. It follows that $\operatorname{Imm}_{\mathbb{Z}}(X) = \operatorname{Imm}_{\mathbb{R}}(X)$ and the explicit description of the immaculate locus follows by an explicit calculation of the inequalities of the cones above, and by taking the complement in $\operatorname{Cl}(X)$. □

Proposition 2.33. Suppose as above that $X = \mathbb{TV}(\Sigma_c)$ is a smooth complete toric variety of Picard rank 2. If L is a line bundle on X such that $\mathrm{H}^i(X, L) \neq 0$, then $i \in \{0, \ell_1 - 1, \ell_2 - 1, \dim X\}$.

Proof. As in the previous proof, the only tempting subsets are $\emptyset$, U, V and $U \sqcup V = \Sigma(1)$. In the proof of Proposition 2.24 we have seen that $V^{>}(U) \simeq S^{l_1 - 1}$ thus by (i) in Proposition 2.9, we obtain line bundles L with $\mathrm{H}^{\ell_1 - 1}(X, L) \neq 0$. The other tempting sets $\emptyset$, V and $\Sigma(1)$ lead to line bundles with $\mathrm{H}^i(X, L) \neq 0$ for $i = \{0, \ell_2 - 1, \dim X\}$, respectively. By (ii) of the same proposition, other cohomologies cannot occur. □

2.4 Toric varieties with splitting fans

In this section we apply the theory of section 2.2 to the case of splitting fans and calculate the essential part of the immaculate locus of line bundles in this setup. Let $X = \mathbb{TV}(\Sigma)$ be a smooth complete toric variety. Recall from subsubsection 2.2.2 that a primitive collection of a (smooth, hence simplicial) fan Σ is another word for a "minimal non-face". We say Σ is a *splitting fan*, if the primitive collections of Σ are pairwise disjoint. This is equivalent to an existence of a chain $\Sigma = \Sigma_k, \dots, \Sigma_1$ of fans such that $\mathbb{TV}(\Sigma_1) = \mathbb{P}^n$ and $\mathbb{TV}(\Sigma_{i+1}) \to \mathbb{TV}(\Sigma_i)$ is a toric split bundle, that is a projectivization of a direct sum of toric line bundles (see [Bat91, Cor. 4.4]). In particular, all such X are projective. Note that every smooth complete toric variety with Picard rank two satisfies this property with $k = 2$, see subsection 2.3.2.

2.4.1 Primitive relations

In this subsection we recall the notion of the primitive relation associated to a primitive collection and express all such relations for a splitting fan. Having in mind the application to splitting fans, which are smooth by definition, we restrict our presentation of primitive relations to the smooth case, following [Bat91]. See [CR09, §1.3] for a more general treatment.

Let Σ be a fan of a smooth complete toric variety X. For every primitive collection $\mathrm{P} \subseteq \Sigma(1)$ we denote $e_{\mathrm{P}} := \sum_{\rho \in \mathrm{P}} e_\rho$, where the $e_\rho \in \mathbb{Z}^{\Sigma(1)}$ is the basis element corresponding to ρ, which under the natural map $\mathbb{Z}^{\Sigma(1)} \to N$ is mapped to the primitive generator of the corresponding ray. The *focus of* P denoted by $\sigma(\mathrm{P})$ is the unique cone $\sigma \in \Sigma$ such that the image of e_{P} in N is contained in $\operatorname{int} \sigma \subset N_{\mathbb{R}}$. It leads to a unique element $f(\mathrm{P}) \in \mathbb{Z}_{\geq 1}^{\sigma(\mathrm{P})(1)}$ with $e_{\mathrm{P}} - f(\mathrm{P}) \in \ker(\mathbb{Z}^{\Sigma(1)} \to N)$. (Here, by convention, $\mathbb{Z}_{\geq 1}^{\emptyset} = \{0\}$.) The expression $e_{\mathrm{P}} - f(\mathrm{P})$ is called the *primitive relation* associated to P. As an element of $\mathrm{Cl}(X)^*$, it represents a class of 1-cycles.

In [Bat91, Proposition 3.1] it is shown that $\mathrm{P} \cap \sigma(\mathrm{P}) = \emptyset$, that is the elements of P are not among the generators of $\sigma(\mathrm{P})$. Moreover, if Σ is projective, then there exists a primitive collection P with $\sigma(\mathrm{P}) = 0$, see [Bat91, Proposition 3.2 and Theorem 4.3]. For complete fans Σ, all rays $\rho \in \Sigma(1)$ are contained in at least one primitive collection. If Σ is simplicial, then the number of primitive collections is at least the rank of $\mathrm{Cl}(\mathbb{TV}(\Sigma))$ and for smooth fans Σ equality holds only for splitting fans.

Example 2.34. Let us calculate the primitive relations for Example 2.22 and its continuation 2.29. The toric variety $X = \mathbb{TV}(\Sigma)$ in this example is a 3-dimensional, projective toric variety of Picard rank 2. We have seen that the two primitive collections are $P_1 = \{\rho_1 = (1,\ 0,\ 1), \rho_2 = (0,\ 1,\ 1)\}$ and $P_2 = \{\rho_3 = (-1,\ -1,\ -2), \rho_4 = (1,\ 1,\ 1), \rho_5 = (0,\ 0,\ 1)\}$. To determine the focus and primitive relations we first calculate $e_1 := e_{P_1} = \rho_1 + \rho_2 = (1,\ 1,\ 2)$ and $e_2 := e_{P_2} = 0$. Thus $\sigma(P_1) = \operatorname{cone}(\rho_4, \rho_5)$ and $e_1 = 1 \cdot \rho_4 + 1 \cdot \rho_5$. We obtain the following $f(P_1) = (1,\ 1)$ and the primitive relation is $(1,\ 1,\ 0,\ -1,\ -1)$. For P_2, the focus $\sigma(P_2) = 0$ and the primitive relation is $(0,\ 0,\ 1,\ 1,\ 1)$. When we write it as a matrix, we see how Example 2.22 fits into the description of subsection 2.3.2:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Example 2.35. In the hexagon example from Example 2.1 the primitive collections are not disjoint, so it is not an example of a splitting fan. Nevertheless, we want to calculate the primitive relations in this case.

Remember that all subsets with two non-consecutive entries form a primitive collection for the fan Σ of the hexagon. If $\mathrm{P} = \{i,\ i+3\}$, the focus is $\sigma(\mathrm{P}) = 0$ and we obtain the first three primitive relations in the matrix below.

For $\mathrm{P} = \{i,\ i+2\}$, the focus is $\sigma(\mathrm{P}) = \rho_{i+1}$ and this leads to the last six primitive relations in the matrix. For the sake of readability, we denote the primitive collection P in front of the row of the matrix containing its associated primitive relation.

$$\begin{array}{c} \{0,3\} \\ \{1,4\} \\ \{2,5\} \\ \{0,2\} \\ \{1,3\} \\ \{2,4\} \\ \{3,5\} \\ \{0,4\} \\ \{1,5\} \end{array} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix π from the example is obtained by choosing the first four primitive relations.

2.4.2 Temptation for splitting fans

Here we assume that X is a smooth projective toric variety of dimension d whose fan Σ is a splitting fan. We will first identify all of the tempting subsets $\mathcal{R} \subset \Sigma(1)$. Then in subsection 2.4.3 we describe the associated π-images $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ or $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ as introduced in Definition 2.6.

Let $\Sigma(1) = \mathrm{P}_1 \sqcup \ldots \sqcup \mathrm{P}_k$ be the decomposition into primitive collections of lengths $\ell_1, \ldots, \ell_k \geq 2$. To understand the tempting subsets of $\Sigma(1)$ we first want to investigate the combinatorial structure of the fan Σ. In particular, we want to determine the maximal dimensional cones $\sigma \in \Sigma(d)$. The maximal cones correspond to maximal subsets of $\Sigma(1)$ not containing any entire P_i. Since the P_i are disjoint, we obtain the bijection

$$\mathrm{P}_1 \times \ldots \times \mathrm{P}_k \xrightarrow{\sim} \Sigma(d), \quad (p_1, \ldots, p_k) \mapsto \Sigma(1) \setminus \{p_1, \ldots, p_k\}.$$

In particular, Σ is combinatorially equivalent to the normal fan of

$$\square = \square(\{1, \ldots, k\}) := \triangle^{\ell_1 - 1} \times \ldots \times \triangle^{\ell_k - 1}$$

where $\triangle^{\ell-1}$ denotes the $(\ell-1)$-dimensional simplex with ℓ vertices. Thus, we know that $\#\Sigma(1) = \sum_{i=1}^k \ell_i$, $d = \sum_{i=1}^k (\ell_i - 1)$, $\#\Sigma(d) = \prod_{i=1}^k \ell_i$, and, as noticed above $\mathrm{rk}(\mathrm{Cl}\, X) = \#\Sigma(1) - d = k$.

Now, the essential point is that in the case of splitting fans the temptation of a subset $\mathcal{R} \subseteq \Sigma(1)$ depends only on the combinatorial structure of Σ. The finer structure, the true shape of the fan reflected by the maps $\rho : \mathbb{Z}^{\Sigma(1)} \to N$, the primitive relations of Σ, or $\pi \colon \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X)$, does matter only for the second step of turning the tempting sets $\mathcal{R}$ into the maculate regions $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$.

Lemma 2.36. If Σ is a splitting fan with the decomposition $\Sigma(1) = \mathrm{P}_1 \sqcup \ldots \sqcup \mathrm{P}_k$ into primitive collections P_i, then the tempting subsets of $\Sigma(1)$ are $\mathcal{R}(J) := \bigcup_{j \in J} \mathrm{P}_j$ with $J \subseteq \{1, \ldots, k\}$.

Proof. By Proposition 2.28 the $\mathcal{R}(J)$ are the only candidates for tempting subsets. We will show that all of them are indeed tempting. Instead of the complex $(\mathrm{Supp}V^{\geq}(\mathcal{R}))\setminus\{0\} \subseteq \mathrm{Supp}\Sigma\setminus\{0\} \sim S^{d-1}$ we consider its dual version $G(\mathcal{R})$ built as the union of all (closed) facets $G(\rho) < \Box$ dual to $\rho \in \mathcal{R}$. Clearly, $\mathrm{Supp}V^{\geq}(\mathcal{R}) \setminus \{0\}$ is homotopy equivalent to $G(\mathcal{R})$, thus one is $\mathbf{k}$-acyclic if and only if the other is. A subset $J \subseteq \{1,\dots,k\}$ defines a splitting $\Box = \Box(J) \times \Box(\{1,\dots,k\} \setminus J)$ and accordingly we have $G(\mathcal{R}(J)) = \partial\Box(J) \times \Box(\{1,\dots,k\} \setminus J)$, which is not $\mathbf{k}$-acyclic. Thus every set $\mathcal{R}(J)$ is tempting. □

Knowing this we can already determine which cohomological degrees can occur.

Proposition 2.37. Let $X = \mathbb{TV}(\Sigma)$ with Σ a splitting fan with k primitive collections of lengths l_i for $i = 1,\dots,k$, and L be a line bundle on X such that $\mathrm{H}^i(X,L) \neq 0$, then $i \in \left\{\sum_{j\in J}(\ell_j - 1)\right\}_{J\subset\{1,\dots,k\}}$.

Proof. In the previous proof we have seen that $\mathcal{R}(J)$ leads to the non-$\mathbf{k}$-acyclic $G(\mathcal{R}(J)) = \partial\Box(J) \times \Box(\{1,\dots,k\} \setminus J)$. For cohomological considerations we can focus on the first factor $\partial\Box(J) = \partial\left(\prod_{j\in J} \triangle^{\ell_j - 1}\right)$. Thus we have the boundary of a polytope of dimension $\sum_{j\in J}(\ell_j - 1)$, so $\mathcal{R}(J)$ is homotopy equivalent to a $(\sum_{j\in J}(\ell_j - 1) - 1)$-dimensional sphere. The claim then follows by Proposition 2.9. □

2.4.3 Immaculate locus of splitting fans

The 2^k different sets $J \subseteq \{1,\dots,k\}$ yield 2^k tempting sets $\mathcal{R}(J)$, hence 2^k maculate regions $\mathcal{M}_{\mathbb{R}}(\mathcal{R}(J))$ within the k-dimensional space $\mathrm{Cl}(X)\otimes\mathbb{R} \cong \mathbb{R}^k$. This looks a little like the structure of 2^k octants in this space, but we will see in this subsection that typically the octants are "leaning", and they may intersect as illustrated on Figures 2.2 and 2.3. For this we first need to determine the structure of the map $\pi : \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X)$.

Write $\Sigma(1) = \bigsqcup_{i=1}^k \mathrm{P}_i$ the decomposition of the rays into the disjoint sets of primitive collections. In [Bat91, Corollary 4.4], Batyrev has proved that X can be obtained via a sequence of projectivizations of decomposable bundles. Within the fan language this means that we can assume that there is a sequence of fans $\Sigma = \Sigma_k,\dots,\Sigma_1,\Sigma_0 = 0$ in abelian groups $N = N_k \twoheadrightarrow \dots \twoheadrightarrow N_1 \twoheadrightarrow N_0 = 0$ such that the focus $\sigma(\mathrm{P}_j) = 0$ in N_j and $N_{j-1} = N_j/\operatorname{span}\mathrm{P}_j$. The fans Σ_j in N_j are splitting with $\Sigma_j(1) = \sqcup_{i=1}^j \mathrm{P}_i$, and they admit subfans $\widetilde{\Sigma}_{j-1} \subset \Sigma_j$ such that $\psi_j : N_j \twoheadrightarrow N_{j-1}$ induces an isomorphism $\widetilde{\Sigma}_{j-1} \xrightarrow{\sim} \Sigma_{j-1}$ (piecewise linear on the geometric realizations) and Σ_j consist of the sums of cones from $\widetilde{\Sigma}_{j-1}$ and proper subsets of P_j.

With $\ell_i = \#\,\mathrm{P}_i$, this explicit structure of Σ can be translated into the fact that π is a triangular block matrix

$$\pi = \begin{pmatrix} \underline{1} & c_{12} & \dots & c_{1k} \\ \underline{0} & \underline{1} & \dots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \dots & \underline{1} \end{pmatrix} \tag{2.3}$$

with k rows and k blocks of $1 \times \ell_j$-matrices ($j = 1,\dots,k$). While $\underline{1}$ denotes $(1,1,\dots,1)$ with ℓ_j entries, we have $c_{ij} \in \mathbb{Z}_{\leq 0}^{\ell_j}$. If needed, its entries will be denoted by $c_{ij}^{\nu} \in \mathbb{Z}_{\leq 0}$ ($i < j$, $\nu = 1,\dots,\ell_j$). Every row encodes a primitive relation, hence each c_{ij} has at least one zero entry (the support of $c_{i\bullet}$ is supposed to be a face, that is to not contain any full P_j). All matrices of the form (2.3) give rise to a smooth splitting fan of dimension $\sum_{i=1}^k(\ell_i - 1)$.

The smoothness of the variety $\mathbb{TV}(\Sigma)$ associated to the matrix π can also be derived directly from the method of subsection 2.3.1. The co-facets of the fan Σ give rise to choosing one column of π in every block. But this yields an upper triangular matrix with only 1 as the diagonal entries. Hence, the determinant equals 1, too.

Example 2.38. A simple case to have in mind is $k = 2$. The matrix of π is

$$\left(\begin{array}{ccc|cccc} 1 & \dots & 1 & 0 & c^2 & \dots & c^{\ell_2} \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{array}\right) = \begin{pmatrix} \underline{1} & c \\ \underline{0} & \underline{1} \end{pmatrix}$$

It covers the case of Hirzebruch surfaces. In subsection 2.3.2 we have discussed the immaculate locus of this matrix in detail.

Example 2.39. Consider the following smooth projective three dimensional toric variety $X = \mathbb{TV}(\Sigma) = \mathbb{P}(\mathcal{O}_Y(-2,0) \oplus \mathcal{O}_Y(0,-2))$, where $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{O}_Y(i,j) := \mathcal{O}_{\mathbb{P}^1}(i) \boxtimes \mathcal{O}_{\mathbb{P}^1}(j)$. Then the fan Σ is a splitting fan with matrix

$$\pi = \begin{pmatrix} 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The line bundle represented by

$$\pi\left((0,0,0,0,\tfrac{1}{2},\tfrac{1}{2})\right) = (-1,-1,1) \in \mathrm{Cl}(X)$$

is immaculate but not $\mathbb{R}$-immaculate (in the sense of Definition 2.11), since it is in the effective cone

$$\mathcal{M}_{\mathbb{R}}(\emptyset) = \operatorname{cone}\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The reason for the existence of an immaculate, but not $\mathbb{R}$-immaculate line bundle is that the rays of the effective cone are not a Hilbert basis.

We fix a format $\ell := (\ell_1, \dots, \ell_k)$ of splitting fans, that is a block format of the associated matrix π. We interpret c, that is the entries $c_{ij} \in \mathbb{Z}^{\ell_j}_{\leq 0}$ of π, as coordinates of the "moduli space" of splitting fans $\Sigma(\ell, c)$ of this fixed format ℓ. All these fans share the same combinatorial type – that of the normal fan of $\square := \triangle^{\ell_1 - 1} \times \dots \times \triangle^{\ell_k - 1}$, see subsection 2.4.2. Similarly, the associated toric varieties share the same Picard group. Since we use the primitive relations for the rows of π, we have even distinguished coordinates leading to a simultaneous identification $\mathrm{Cl}\,\mathbb{TV}(\Sigma(\ell, c)) = \mathbb{Z}^k$. This makes it possible to compare the immaculate loci of different $\Sigma(\ell, c)$ sharing the same ℓ.

Now, the basic idea is simple: For special c, e.g. $c = \underline{0}$, the immaculate locus is large – but it becomes smaller for growing $|c| := -c$. Roughly speaking, we will show that this shrinking of the immaculate locus becomes stationary, and we are going to calculate the limit.

There is, however, a technical obstacle. The center of symmetry $K_X/2$ arising from Serre duality moves with c. Thus, it is not the whole immaculate locus that becomes stationary – this works only for some generating seed. That is, there is a certain subset of $\mathbb{Z}^k$ which is immaculate for all $\Sigma(\ell, c)$ and which generates (via some operations/reflections corresponding to successive Serre dualities) the full immaculate locus if $-c$ is sufficiently large.

Definition 2.40. We call $\operatorname{Seed}(\ell) := \bigcup_{j=1}^{k} \left(\mathbb{Z}^{j-1} \times \{-1, \dots, -(\ell_j - 1)\} \times \underline{0}^{k-j}\right)$ the *generating immaculate seed* for ℓ in $\mathbb{Z}^k$.

Example 2.41. In the simple case of $k = 2$ as in Example 2.38, the generating immaculate seed consists of the lattice points in $\ell_2 - 1$ lines and $\ell_1 - 1$ further lattice points.

$$\operatorname{Seed}((\ell_1,\ \ell_2)) = \{(-1,\ 0),\ (-2,\ 0), \dots, (-(\ell_1 - 1),\ 0),\ (*,\ -1),\ (*,\ -2), \dots, (*,\ -(\ell_2 - 1))\}$$

where $* \in \mathbb{Z}$.

Depending on c, we can now define the operator enlarging a given seed in $\mathbb{Z}^k$. For fixed $i, j \in \{1, \dots, k\}$ we set $\overline{c}_{ij} := \sum_{\nu=1}^{\ell_j} c_{ij}^{\nu} \in \mathbb{Z}_{\leq 0}$. Moreover, denote

$$v_j := (\overline{c}_{1j}, \dots, \overline{c}_{j-1,\,j}, \ell_j, \underline{0}) \in (\mathbb{Z}^j \times \underline{0}) \subseteq \mathbb{Z}^k.$$

Definition 2.42. For a given subset $G \subseteq \mathbb{Z}^k$ we define its *c-hull* as the smallest set $\langle G \rangle_c \supseteq G$ satisfying the following recursive property: If $a \in \langle G \rangle_c \cap (\mathbb{Z}^j \times \underline{0}^{k-j})$ for some $j = 0, \dots, k-1$, then so is the shift $a - v_{j+1} \in \langle G \rangle_c$.

Note that $a - v_{j+1} \in \mathbb{Z}^j \times \{-\ell_{j+1}\} \times \underline{0}^{k-j-1}$. Hence, to obtain $\langle G \rangle_c$ out of G one can enlarge $\langle G \rangle_c$ successively: set $\langle G \rangle_c^{-1} := G$ and let $\langle G \rangle_c^{j+1} := \left(\langle G \rangle_c^{j} \cap (\mathbb{Z}^j \times \underline{0}^{k-j})\right) - v_{j+1}$. Then $\langle G \rangle_c = \bigcup_{j=0}^{k} \langle G \rangle_c^{j}$.

Example 2.43. The c-hull of $\operatorname{Seed}((\ell_1,\ \ell_2))$ in Example 2.41 can be calculated in one step. We check which $a \in \operatorname{Seed}(\ell)$ have last coordinate zero. Then for all of them we add $a - v_2 = (a_1,\ 0) - (\overline{c}_{12},\ \ell_2)$ to the set. This way we obtain

$$\langle \operatorname{Seed}((\ell_1,\ \ell_2)) \rangle_c = \left\{ \begin{array}{c} (-1,\ 0),\ (-2,\ 0), \dots, (-(\ell_1 - 1),\ 0), \\ (-1 - \overline{c}_{12},\ -\ell_2),\ (-2 - \overline{c}_{12},\ -\ell_2), \dots, (-(\ell_1 - 1) - \overline{c}_{12},\ -\ell_2), \\ (*,\ -1),\ (*,\ -2), \dots, (*,\ -(\ell_2 - 1)) \end{array} \right\}.$$

These definitions of $\operatorname{Seed}(\ell)$ and the hull operations allow to describe the locus of immaculate line bundles for "general" c. Recall the notions of maculate regions from Definition 2.6, and immaculate loci from Definition 2.12.

Theorem 2.44. *Fix ℓ and let c be a parameter leading to a matrix $\pi = \pi(\ell, c)$ with the associated splitting fan $\Sigma = \Sigma(\ell, c)$. Then:*

(i) $\operatorname{Seed}(\ell) \subseteq \operatorname{Imm}_{\mathbb{R}}(\Sigma(\ell, c))$ *for all c, that is the generating immaculate seeds are $\mathbb{R}$-immaculate.*

(ii) Both the loci $\operatorname{Imm}_{\mathbb{Z}}(\Sigma)$ *and* $\operatorname{Imm}_{\mathbb{R}}(\Sigma)$ *are closed under the c-hull operation.*

(iii) For "general" c, the immaculate loci are both equal to the minimal set satisfying the above. That is, $\operatorname{Imm}_{\mathbb{Z}}(\Sigma(\ell, c)) = \operatorname{Imm}_{\mathbb{R}}(\Sigma(\ell, c)) = \langle \operatorname{Seed}(\ell) \rangle_c$.

More precisely, a sufficient condition for "general" in (iii) is that for each $j = 1, \dots, k-1$ the vector $c_{j,j+1} \in \mathbb{Z}^{\ell_{j+1}}$ has at least two entries differing by more than ℓ_j.

We remark that for non-general values of c the conclusion of (iii) needs not to hold, see for example Figure 2.3 and compare to Example 2.43.

The proof of Theorem 2.44 can be found in [ABKW20]. It uses induction on the number of primitive collections and the fact that there exists a sequence of maps $X = X_k \to^{p_k} X_{k-1} \to$

$\ldots X_1 = \mathbb{P}^{l_1-1}$ with corresponding fans Σ_i such that $X_{i+1} = \mathbb{P}(\mathcal{E}_i)$ where $\mathcal{E}_i$ is a split vector bundle over $X_i = \mathbb{TV}(\Sigma_i)$. An important ingredient is [ABKW20, Corollary IV.6] which guarantees that for the maps $p_i : X_i \to X_{i-1}$ a line bundle L on X_{i-1} is immaculate if and only if its pullback $p_i^*(L)$ is immaculate. The usage of Serre duality on each X_i explains the hull operation from Definition 2.42.

Splitting fan varieties of Picard rank 3

We will now briefly discuss the structure of the immaculate locus of splitting fan varieties of Picard rank 3. That is, we will apply Theorem 2.44 to the specific case that $X = \mathbb{TV}(\Sigma)$ is a smooth projective toric variety with $\Sigma(1) = \mathrm{P}_1 \sqcup \mathrm{P}_2 \sqcup \mathrm{P}_3$, $\#\mathrm{P}_i = \ell_i$ for $i = 1, 2, 3$. The matrix π of primitive relations is the following matrix with three blocks of sizes ℓ_1, ℓ_2 and ℓ_3 and $c_{ij} \in \mathbb{Z}^{l_j}_{\leq 0}$:

$$\pi = \begin{pmatrix} \underline{1} & c_{12} & c_{13} \\ \underline{0} & \underline{1} & c_{23} \\ \underline{0} & \underline{0} & \underline{1} \end{pmatrix}.$$

Thus the parameters determining our variety are $\ell = (\ell_1,\ \ell_2,\ \ell_3)$ and $c = (c_{12},\ c_{13},\ c_{23})$. By Lemma 2.36 the tempting subset of $\Sigma(1)$ are the following eight:

$$\emptyset,\ \mathrm{P}_1,\ \mathrm{P}_2,\ \mathrm{P}_3,\ \mathrm{P}_2 \cup \mathrm{P}_3,\ \mathrm{P}_1 \cup \mathrm{P}_3,\ \mathrm{P}_1 \cup \mathrm{P}_2 \text{ and } \Sigma(1).$$

The latter four tempting subsets are complements of the first four.

There are eight maculate sets/regions, each of the regions being a shifted cone. The vertices of the regions – also referred to as maculate vertices – are

$$\begin{aligned}
v_{\emptyset} &= (0,\ 0,\ 0) & v_{\Sigma(1)} &= (-\ell_1 - \bar{c}_{12} - \bar{c}_{13},\ -\ell_2 - \bar{c}_{23},\ -\ell_3) \\
v_{\mathrm{P}_1} &= (-\ell_1,\ 0,\ 0) & v_{\mathrm{P}_2 \cup \mathrm{P}_3} &= (-\bar{c}_{12} - \bar{c}_{13},\ -\ell_2 - \bar{c}_{23},\ -\ell_3) \\
v_{\mathrm{P}_2} &= (-\bar{c}_{12},\ -\ell_2,\ 0) & v_{\mathrm{P}_1 \cup \mathrm{P}_3} &= (-\ell_1 - \bar{c}_{13},\ -\bar{c}_{23},\ -\ell_3) \\
v_{\mathrm{P}_3} &= (-\bar{c}_{13},\ -\bar{c}_{23},\ -\ell_3) & v_{\mathrm{P}_1 \cup \mathrm{P}_2} &= (-\ell_1 - \bar{c}_{12},\ -\ell_2,\ 0).
\end{aligned}$$

The sum of two vertices in one row is always $v_{\Sigma(1)} = -K_X$ as expected, since the maculate regions of P and $\Sigma(1) \setminus \mathrm{P}$ are related to each other by Alexander/Serre duality.

The tail cones of the regions are generated by

$$\begin{array}{cc|cc}
\emptyset & \begin{pmatrix} 1 & c_{12} & c_{13} \\ 0 & \underline{1} & c_{23} \\ 0 & \underline{0} & \underline{1} \end{pmatrix} & \Sigma(1) & \begin{pmatrix} -1 & -c_{12} & -c_{13} \\ 0 & -\underline{1} & -c_{23} \\ 0 & -\underline{0} & -\underline{1} \end{pmatrix} \\
\mathrm{P}_1 & \begin{pmatrix} -1 & c_{12} & c_{13} \\ 0 & \underline{1} & c_{23} \\ 0 & \underline{0} & \underline{1} \end{pmatrix} & \mathrm{P}_2 \cup \mathrm{P}_3 & \begin{pmatrix} 1 & -c_{12} & -c_{13} \\ 0 & -\underline{1} & -c_{23} \\ 0 & -\underline{0} & -\underline{1} \end{pmatrix} \\
\mathrm{P}_2 & \begin{pmatrix} 1 & -c_{12} & c_{13} \\ 0 & -\underline{1} & c_{23} \\ 0 & \underline{0} & \underline{1} \end{pmatrix} & \mathrm{P}_1 \cup \mathrm{P}_3 & \begin{pmatrix} -1 & c_{12} & -c_{13} \\ 0 & \underline{1} & -c_{23} \\ 0 & \underline{0} & -\underline{1} \end{pmatrix} \\
\mathrm{P}_3 & \begin{pmatrix} 1 & c_{12} & -c_{13} \\ 0 & \underline{1} & -c_{23} \\ 0 & \underline{0} & -\underline{1} \end{pmatrix} & \mathrm{P}_1 \cup \mathrm{P}_2 & \begin{pmatrix} -1 & -c_{12} & c_{13} \\ 0 & -\underline{1} & c_{23} \\ 0 & \underline{0} & \underline{1} \end{pmatrix}
\end{array}$$

The cones are in general not simplicial, e.g. in Example 2.39 the effective cone is generated by four rays.

The most prominent immaculate line bundles in the immaculate locus of X are the generating immaculate seed:

$$\mathrm{Seed}((\ell_1,\ \ell_2,\ \ell_3)) = \left\{ \begin{array}{l} (-1,\ 0,\ 0), (-2,\ 0,\ 0), \ldots, (-(\ell_1-1),\ 0,\ 0) \\ (*,\ -1,\ 0), (*,\ -2,\ 0), \ldots, (*,\ -(\ell_2-1),\ 0) \\ (*,\ *,\ -1), (*,\ *,\ -2), \ldots, (*,\ *,\ -(\ell_3-1)) \end{array} \right\}.$$

It consists of $\ell_1 - 1$ lattice points, the lattice points in $\ell_2 - 1$ lines and the lattice points of $\ell_3 - 1$ planes.

Especially for the last class, it is easy to see that they do not belong to any maculate set, since the last coordinate of the maculate vertices is always either 0 or $-\ell_3$ and we see that for those maculate regions with last coordinate zero, the rays of the tail cone of the maculate region have either zero or 1 as last coordinate, while for those with last coordinate $-\ell_3$ it is -1.

Example 2.45. When $\ell = (2,\ 2,\ 2)$ as in Example 2.39

$$\mathrm{Seed}((2,\ 2,\ 2)) = \{(-1,\ 0,\ 0), (*,\ -1,\ 0), (*,\ *,\ -1)\}.$$

the generating immaculate seed consists of one lattice point, the lattice points in one line and the lattice points of one plane.

Now we want to calculate the c-hull of $\mathrm{Seed}(\ell)$ as defined in Definition 2.42. First we determine the shifting vectors v_1, v_2, v_3:

$$v_1 = (2,\ 0,\ 0) \quad v_2 = (0,\ 2,\ 0) \quad v_3 = (-2,\ -2,\ 2).$$

In the first step of constructing $\langle \mathrm{Seed}(\ell) \rangle_c$ out of $\mathrm{Seed}(\ell)$, we check if $(*,\ 0,\ 0) \in \mathrm{Seed}(\ell)$, that is the case for $(-1,\ 0,\ 0)$, so we have to add $(-1,\ 0,\ 0) - v_2 = (-1,\ -2,\ 0)$ to $\langle \mathrm{Seed}(\ell) \rangle_c$.

In the next step, we look for all $a = (*,\ *,\ 0)$ in $\mathrm{Seed}(\ell) \cup (-1,\ -2,\ 0)$, and add $a - v_3$ to our set. Obviously, the element we have added in the previous step is of this form and we add the new element $(1,\ 0,\ -2)$ to $\langle \mathrm{Seed}(\ell) \rangle_c$. Another element of this form is $(-1,\ 0,\ 0)$, which leads to the element $(1,\ 2,\ -2)$. Then also the lattice points in the line $(*,\ -1,\ 0)$ are of this form and leads to a line $(*,\ 1,\ -2)$. We obtain that

$$\langle \mathrm{Seed}(\ell) \rangle_c = \{(-1,\ 0,\ 0),\ (-1,\ -2,\ 0),\ (1,\ 2,\ -2),\ (*,\ -1,\ 0),\ (*,\ 1,\ -2),\ (*,\ *,\ -1)\}.$$

Taking the c-hull has added two points and one line to the generating immaculate seed. In this case the condition for c being sufficiently general is not satisfied, because $c_{12} = (0,\ 0)$. Thence $\langle \mathrm{Seed}(\ell) \rangle_c$ is a proper subset of the immaculate locus. For example, the immaculate, but not $\mathbb{R}$-immaculate divisor $(-1,\ -1,\ 1) \notin \langle \mathrm{Seed}(\ell) \rangle_c$.

We will calculate the c-hull $\langle \mathrm{Seed}(\ell) \rangle_c$ of $\mathrm{Seed}(\ell)$ for general $\ell = (\ell_1,\ \ell_2,\ \ell_3)$ and $c = (c_{12},\ c_{13},\ c_{23})$. For this we first determine the points v_1, v_2, v_3 .

$$v_1 = (\ell_1,\ 0,\ 0) \quad v_2 = (\bar{c}_{12},\ \ell_2,\ 0) \quad v_3 = (\bar{c}_{13},\ \bar{c}_{23},\ \ell_3)$$

We calculate the c-hull step by step, first we have the seed $\mathrm{Seed}(\ell)$:

$$\mathrm{Seed}(\ell) = \{(a,\ 0,\ 0),\ (*,\ b,\ 0),\ (*,\ *,\ c)\}$$

with $a \in [-(\ell_1 - 1), \ldots, -1]$, $b \in [-(\ell_2 - 1), \ldots, -1]$ and $c \in [-(\ell_3 - 1), \ldots, -1]$.

In the first step we add the points $(a,\ 0,\ 0) - v_2 = (a - \bar{c}_{12},\ -\ell_2,\ 0)$. In the second step we add the points

- $(a,\ 0,\ 0) - v_3 = (a - \overline{c}_{13},\ -\overline{c}_{23},\ -\ell_3)$
- $(a - \overline{c}_{12},\ -\ell_2,\ 0) - v_3 = (a - \overline{c}_{12} - \overline{c}_{13},\ -\ell_2 - \overline{c}_{23},\ -\ell_3)$
- $(*,\ b,\ 0) - v_3 = (*,\ b - \overline{c}_{23},\ -\ell_3)$

$$\langle \mathrm{Seed}(\ell)\rangle_c = \left\{ \begin{array}{c} (a,\ 0,\ 0), (a - \overline{c}_{12},\ -\ell_2,\ 0), \\ (a - \overline{c}_{13},\ -\overline{c}_{23},\ -\ell_3),\ (a - \overline{c}_{12} - \overline{c}_{13},\ -\ell_2 - \overline{c}_{23},\ -\ell_3), \\ (*,\ b,\ 0), (*,\ b - \overline{c}_{23},\ -\ell_3), \\ (*,\ *,\ c) \end{array} \right\},$$

with as above $a \in [-(\ell_1 - 1), \ldots, -1]$, $b \in [-(\ell_2 - 1), \ldots, -1]$ and $c \in [-(\ell_3 - 1), \ldots, -1]$. That gives $4(\ell_1 - 1)$ points, $2(\ell_2 - 1)$ lines and $\ell_3 - 1$ planes.

By (iii) from Theorem 2.44 there are no further immaculate line bundles for c sufficiently general.

2.5 Toric varieties of Picard rank 3

In this section we finally make everything concrete in the case of Picard rank 3. We first review the classification of Batyrev and describe the tempting subsets of rays. We list a lot of immaculate line bundles and prove (similarly to Theorem 2.44) that for sufficiently general parameters the listed ones are all immaculate line bundles. In subsection 2.5.4 we study the behavior of the immaculate sets when changing the parameters. Finally in subsection 2.5.5 the special case of vanishing parameters of Batyrev's classification is considered.

2.5.1 Classification by Batyrev

In [Bat91] a classification of smooth, projective toric varieties of Picard rank 3 is given by using its primitive collections. See also [CR09].

Proposition 2.46 ([Bat91, Thm 5.7]). If Σ is a complete, regular d-dimensional fan with $d+3$ generators, then the number of primitive collections of its generators is equal to 3 or 5.

The number of primitive collections is always greater or equal to the Picard rank and equality is only attained for splitting fans. So in the case that there are exactly three primitive collections the fan Σ is a splitting fan and Theorem 2.44 provides a description of the immaculate locus in this case. We have discussed it in subsubsection 2.4.3.

Therefore, in the rest of the section we are going to assume that $\mathbb{TV}(\Sigma)$ is a smooth projective toric variety of Picard rank 3, which has exactly five primitive collections. Following [Bat91] we give a more precise description of the fan. There is a decomposition of the rays $\Sigma(1)$ into five disjoint subsets X_α and the primitive collections are given by $X_\alpha \cup X_{\alpha+1}$ for $\alpha \in \mathbb{Z}/5\mathbb{Z}$.

Proposition 2.47 ([Bat91, Thm 6.6]). Let us denote $\mathrm{P}_\alpha = X_\alpha \cup X_{\alpha+1}$, where $\alpha \in \mathbb{Z}/5\mathbb{Z}$,

$$X_0 = \{v_1, \ldots, v_{p_0}\},\ X_1 = \{y_1, \ldots, y_{p_1}\},\ X_2 = \{z_1, \ldots, z_{p_2}\},$$

$$X_3 = \{t_1, \ldots, t_{p_3}\},\ X_4 = \{u_1, \ldots, u_{p_4}\},$$

and $p_0 + \cdots + p_4 = d + 3$. Then any complete regular d-dimensional fan Σ with the set of generators $\Sigma(1) = \bigcup X_\alpha$ and five primitive collections P_α can be described up to a symmetry of the pentagon by the following primitive relations with non-negative integral coefficients $c_2, \dots, c_{p_2}$, $b_1, \dots b_{p_3}$:

$$\begin{aligned}
\sum_{i=1}^{p_0} v_i + \sum_{i=1}^{p_1} y_i - \sum_{i=2}^{p_2} c_i z_i - \sum_{i=1}^{p_3} (b_i+1) t_i &= 0, \\
\sum_{i=1}^{p_1} y_i + \sum_{i=1}^{p_2} z_i - \sum_{i=1}^{p_4} u_i &= 0, \\
\sum_{i=1}^{p_2} z_i + \sum_{i=1}^{p_3} t_i &= 0, \\
\sum_{i=1}^{p_3} t_i + \sum_{i=1}^{p_4} u_i - \sum_{i=1}^{p_1} y_i &= 0, \\
\sum_{i=1}^{p_4} u_i + \sum_{i=1}^{p_0} v_i - \sum_{i=2}^{p_2} c_i z_i - \sum_{i=1}^{p_3} b_i t_i &= 0.
\end{aligned}$$

It looks less scary if we write those equations as a matrix whose rows indicate the five primitive relations. This matrix consists of five blocks of columns of sizes $p_0, \dots, p_4$. By $\underline{0} = (0, 0, \dots, 0)$ and $\underline{1} = (1, 1, \dots, 1)$ we mean row vectors of the appropriate size to fit into the indicated block. Denoting $c = (0, c_2, \dots, c_{p_2}) \in \mathbb{Z}_{\geq 0}^{p_2}$ and $b = (b_1, \dots, b_{p_3}) \in \mathbb{Z}_{\geq 0}^{p_3}$, the primitive relation matrix looks like

$$\begin{pmatrix}
\underline{1} & \underline{1} & -c & -(b+\underline{1}) & \underline{0} \\
\underline{0} & \underline{1} & \underline{1} & \underline{0} & -\underline{1} \\
\underline{0} & \underline{0} & \underline{1} & \underline{1} & \underline{0} \\
\underline{0} & -\underline{1} & \underline{0} & \underline{1} & \underline{1} \\
\underline{1} & \underline{0} & -c & -b & \underline{1}
\end{pmatrix}.$$

Lemma 2.48. For all parameters $b = (b_1, \dots, b_{p_3}) \in \mathbb{Z}_{\geq 0}^{p_3}$ and $c = (0, c_2, \dots, c_{p_2}) \in \mathbb{Z}_{\geq 0}^{p_2}$ we obtain a smooth fan of Picard rank 3. This means the converse of Proposition 2.47.

Proof. This follows from subsection 2.3.1: We chose a submatrix of of the primitive relation matrix that forms a lattice basis, e.g. the first, second and fourth row. Then all the 3-minors with respect to the columns chosen from the blocks $(\alpha, \alpha+1, \alpha+3)$ for $\alpha \in \mathbb{Z}/5\mathbb{Z}$ are always 1. □

2.5.2 Tempting Subsets

As above we suppose $\mathbb{TV}(\Sigma)$ is a smooth projective toric variety of dimension d and Picard rank 3, whose fan Σ has five primitive relations.

For determining the immaculate locus the first step is to find the tempting subsets of $\Sigma(1)$. We have seen in Proposition 2.24 that the primitive collections, their complements, the empty set and the full subset $\Sigma(1)$ are tempting and we will see now that those are indeed the only tempting subsets. This is a special situation. Remember that out of the 34 tempting subsets for the hexagon only 20 are of this shape (see Example 2.5 and Example 2.26). And also for splitting fan varieties of Picard rank greater than 3 not all tempting subsets are of this shape, see Lemma 2.36.

Lemma 2.49. The only tempting subsets are primitive collections, their complements, the empty set and the full subset $\Sigma(1)$.

Proof. By Proposition 2.28 we know that if $\mathcal{R} \subset \Sigma(1)$ is tempting, then $\mathcal{R}$ and its complement are unions of primitive collections. If $\mathcal{R}$ is the union of three (or more) primitive collections, then $\mathcal{R} = \Sigma(1)$. In the case that $\mathcal{R}$ is the union of two primitive collections, it can be the union of two consecutive P_α. Then $\mathcal{R} = X_\alpha \cup X_{\alpha+1} \cup X_{\alpha+2}$ and $\Sigma(1) \setminus \mathcal{R} = X_{\alpha-1} \cup X_{\alpha-2} = \mathrm{P}_{\alpha-2}$ is a primitive collection, and by Proposition 2.24 it is tempting, thence $\mathcal{R}$ is tempting, too. But in particular, in this case $\mathcal{R}$ is the complement of a primitive collection.

If it is the union of two non-consecutive primitive collections, then $\mathcal{R}$ is the union of four X_α and the complement cannot be the union of primitive collections and thus is not tempting. □

With this knowledge we can already narrow down the degrees in which non-vanishing cohomology can occur.

Proposition 2.50. Suppose as above that $X = \mathbb{TV}(\Sigma)$ is a smooth projective toric variety of Picard rank 3 with five primitive collections P_α and a decomposition of $\Sigma(1)$ into five disjoint sets X_α of lengths p_α. If L is a line bundle on X such that $\mathrm{H}^i(X, L) \neq 0$, then $i \in \{0,\ p_\alpha + p_{\alpha+1} - 1,\ p_{\alpha-1} + p_{\alpha-2} + p_{\alpha-3} - 2,\ \dim X\}_{\alpha \in \mathbb{Z}/5\mathbb{Z}}$.

Proof. The tempting subset $\emptyset$ and $\Sigma(1)$ lead to line bundles with non-trivial cohomology in degrees 0 and $\dim X$ respectively.

By Proposition 2.24 $V^{>}(\mathrm{P}_\alpha)$ is homotopy equivalent to S^{l-2} with $l = \#\,\mathrm{P}_\alpha = p_\alpha + p_{\alpha+1}$, with (i) from Proposition 2.9 this gives line bundles L with $H^{l-1}(X, L) \neq 0$. The Serre dual $L^\vee$ of L is then a line bundle with $H^{\dim X-(l-1)}(X, L^\vee) \neq 0$ with $L^\vee = \mathcal{O}_X(D)$ and $[D] \in \mathcal{M}_\mathbb{Z}(\Sigma(1) \setminus \mathrm{P}_\alpha)$. Since $\dim X = \sum_{\alpha=0}^{4} p_\alpha - 3$, $\dim X - (l-1) = p_{\alpha-1} + p_{\alpha-2} + p_{\alpha-3} - 2$. Since there are no other tempting subsets, there cannot occur other degrees. □

2.5.3 Immaculate locus for Picard rank 3

We can calculate the immaculate line bundles as described in Proposition 2.9. For this we have to consider $\pi(\mathbb{Z}^{\Sigma(1)\setminus\mathcal{R}}_{\geq 0} \times \mathbb{Z}^{\mathcal{R}}_{\leq -1})$ for all maculate $\mathcal{R}$ where π is given as the transposed of the map embedding the kernel of the ray map into $\mathbb{Z}^{\Sigma(1)}$. This can be realized by selecting a $\mathbb{Z}$-basis out of the rows of the matrix of primitive relations presented at the end of subsection 2.5.1. Picking its first, second and fourth row, we obtain

$$\pi = \begin{pmatrix} \underline{1} & \underline{1} & -\underline{c} & -(b+\underline{1}) & \underline{0} \\ \underline{0} & \underline{1} & \underline{1} & \underline{0} & -\underline{1} \\ \underline{0} & -\underline{1} & \underline{0} & \underline{1} & \underline{1} \end{pmatrix}.$$

These are the primitive relations that, being understood as classes of 1-cycles, correspond to the rays of the Mori cone which in this case is a three-dimensional simplicial cone.

Let us start by investigating the maculate regions $\mathcal{M}_\mathbb{R}(\mathcal{R})$ for the maculate $\mathcal{R}$. They are all shifted cones, first we will have a look at the vertices. Denote $\overline{c} := \sum_{i=2}^{p_2} c_i$ and $\overline{b} = \sum_{i=1}^{p_3} b_i$.

$\mathcal{R}$	sign pattern	vertex
$\Sigma(1)$	$-\,-\,-\,-\,-$	$(-p_0 - p_1 + p_3 + \overline{c} + \overline{b},\ -p_1 - p_2 + p_4,\ p_1 - p_3 - p_4)$
$\emptyset$	$+\,+\,+\,+\,+$	$(0,\ 0,\ 0)$
P_0	$-\,-\,+\,+\,+$	$(-p_0 - p_1,\ -p_1,\ p_1)$
P_0^c	$+\,+\,-\,-\,-$	$(p_3 + \overline{c} + \overline{b},\ -p_2 + p_4,\ -p_3 - p_4)$
P_1	$+\,-\,-\,+\,+$	$(-p_1 + \overline{c},\ -p_1 - p_2,\ p_1)$
P_1^c	$-\,+\,+\,-\,-$	$(-p_0 + p_3 + \overline{b},\ p_4,\ -p_3 - p_4)$
P_2	$+\,+\,-\,-\,+$	$(p_3 + \overline{c} + \overline{b},\ -p_2,\ -p_3)$
P_2^c	$-\,-\,+\,+\,-$	$(-p_0 - p_1,\ -p_1 + p_4,\ p_1 - p_4)$
P_3	$+\,+\,+\,-\,-$	$(p_3 + \overline{b},\ p_4,\ -p_3 - p_4)$
P_3^c	$-\,-\,-\,+\,+$	$(-p_0 - p_1 + \overline{c},\ -p_1 - p_2,\ p_1)$
P_4	$-\,+\,+\,+\,-$	$(-p_0,\ p_4,\ -p_4)$
P_4^c	$+\,-\,-\,-\,+$	$(-p_1 + p_3 + \overline{c} + \overline{b},\ -p_1 - p_2,\ p_1 - p_3)$

In the following we will denote the vertex of P_i and P_i^c by v_i and v_{i^c}, respectively.

Next, we will have a look at the cones. The red entries indicate those generators of the cones that are a negative multiple of a column of π.

$\mathcal{R}$	generators							reduced generators			
	1	1	0	$-c_{p_2}$	$-(b_1+1)$	$-(b_{p_3}+1)$	0	1	$-c_{p_2}$	$-(b_{p_3}+1)$	0
$\emptyset$	0	1	1	1	0	0	-1	1	1	0	-1
	0	-1	0	0	1	1	1	-1	0	1	1
	-1	-1	0	$-c_{p_2}$	$-(b_1+1)$	$-(b_{p_3}+1)$	0	-1	0	0	
P_0	0	-1	1	1	0	0	-1	0	1	-1	
	0	1	0	0	1	1	1	0	0	1	
	1	-1	0	c_{p_2}	$-(b_1+1)$	$-(b_{p_3}+1)$	0	1	0	$-(b_{p_3}+1)$	
P_1	0	-1	-1	-1	0	0	-1	0	-1	0	
	0	1	0	0	1	1	1	0	0	1	
	1	1	0	c_{p_2}	b_1+1	$b_{p_3}+1$	0	1	0	0	
P_2	0	1	-1	-1	0	0	-1	1	-1	-1	
	0	-1	0	0	-1	-1	1	-1	0	1	
	1	1	0	$-c_{p_2}$	b_1+1	b_{p_3}	0	1	$-c_{p_2}$	b_1+1	0
P_3	0	1	1	1	0	0	1	0	1	0	1
	0	-1	0	0	-1	-1	-1	0	0	-1	-1
	-1	1	0	$-c_{p_2}$	$-(b_1+1)$	$-(b_{p_3}+1)$	0	-1	1	0	$-(b_1+1)$
P_4	0	1	1	1	0	0	1	0	1	1	0
	0	-1	0	0	1	1	-1	0	-1	0	1

Summing this up, we obtain the following vertices, and Hilbert basis of the cones in Table 2.1. With this we deduce that all immaculate line bundles are also $\mathbb{R}$-immaculate.

Lemma 2.51. For all twelve tempting subsets $\mathcal{R} \subset \Sigma(1)$ the rays of the tail cone of the respective maculate region $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ also form its Hilbert basis.

Proof. In Table 2.1 we see the vertices and the rays of the maculate regions. Most of the cones are simplicial cones, checking the determinants shows that they are smooth. In the other cases the cone is over a square which can be subdivided into two smooth cones. □

Table 2.1: The maculate regions

$\mathcal{R}$	vertex	condition	rays / Hilbert basis
$\emptyset$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$b_{p_3} \geq c_{p_2}$	$\begin{matrix} 1 & -(b_{p_3}+1) & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{matrix}$
		$b_{p_3} < c_{p_2}$	$\begin{matrix} 1 & -c_{p_2} & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{matrix}$
P_0	$\begin{pmatrix} -p_0 - p_1 \\ -p_1 \\ p_1 \end{pmatrix}$		$\begin{matrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}$
P_1	$\begin{pmatrix} -p_1 + \overline{c} \\ -p_1 - p_2 \\ p_1 \end{pmatrix}$		$\begin{matrix} 1 & 0 & -(b_{p_3}+1) \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$
P_2	$\begin{pmatrix} p_3 + \overline{c} + \overline{b} \\ -p_2 \\ -p_3 \end{pmatrix}$		$\begin{matrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{matrix}$
P_3	$\begin{pmatrix} p_3 + \overline{b} \\ p_4 \\ -p_3 - p_4 \end{pmatrix}$	$c_{p_2} \geq b_1 + 1$	$\begin{matrix} 1 & -c_{p_2} & b_1+1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{matrix}$
		$c_{p_2} < b_1 + 1$	$\begin{matrix} 1 & -c_{p_2} & b_1+1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{matrix}$
P_4	$\begin{pmatrix} -p_0 \\ p_4 \\ -p_4 \end{pmatrix}$	$b_1 = 0$	$\begin{matrix} -1 & 1 & -(b_1+1) \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}$
		$b_1 > 0$	$\begin{matrix} -1 & 1 & 0 & -(b_1+1) \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{matrix}$

Proposition 2.52. From 2.51 it follows that, independently of the parameters b, c, we always have that $\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) = \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \cap \operatorname{Pic} X$ and thus $\operatorname{Imm}_{\mathbb{Z}}(X) = \operatorname{Imm}_{\mathbb{R}}(X)$.

We will distinguish three classes (F), (A), (B) of line bundles which will become the main components for the immaculate locus. To locate these classes in $\mathbb{Z}^3$ we will use the horizontal projection $(x, y, z) \mapsto (y, z)$ and start with some geography on the target space.

Definition 2.53. Denote by Q_1 and Q_2 the following two planar parallelograms:

$$Q_1 = \operatorname{conv}\begin{pmatrix} (-p_1 - p_2 - p_3 + 2,\ p_1 - 1), (-p_1,\ p_1 - 1), \\ (-p_2 + p_4,\ -p_3 - p_4 + 1), (p_3 + p_4 - 2,\ -p_3 - p_4 + 1) \end{pmatrix},$$
$$Q_2 = \operatorname{conv}\begin{pmatrix} (-p_1 - p_2 + 1,\ p_1 + p_2 - 2), (p_4 - 1,\ -p_4), \\ (-p_1 - p_2 + 1,\ p_1 - p_3), (p_4 - 1,\ -p_2 - p_3 - p_4 + 2) \end{pmatrix}.$$

They are depicted in blue and red in Figure 2.4, and we will be interested in their union. Note the following two and a half special cases:

- If $p_2 = 1$, then $Q_2 \subset Q_1$ and the simplified vertices of Q_1 are:

$$(-p_1 - p_3 + 1, p_1 - 1), (-p_1, p_1 - 1), (p_4 - 1, -p_3 - p_4 + 1), (p_3 + p_4 - 2, -p_3 - p_4 + 1)$$

- If $p_3 = 1$, then $Q_1 \subset Q_2$ and the simplified vertices of Q_2 are:

$$(-p_1 - p_2 + 1, p_1 + p_2 - 2), (-p_1 - p_2 + 1, p_1 - 1), (p_4 - 1, -p_4), (p_4 - 1, -p_2 - p_4 + 1)$$

- If $p_2 = p_3 = 1$, then $Q_1 = Q_2$ is only a line segment with vertices

$$(-p_1, p_1 - 1), (p_4 - 1, -p_4).$$

Now we can describe the three classes of our immaculates. They consist of entire "horizontal" lines or line segments, that is, being always parallel to the x-axis.

Proposition 2.54 (Full horizontal lines (F).)**.** Let $X = \mathbb{TV}(\Sigma)$ be smooth, projective, toric variety of Picard rank 3 with exactly 5 primitive collections. Independent of the parameters c and b from Proposition 2.47, the line bundles $(*,\ x,\ y)$ with $(x,\ y) \in Q_1 \cup Q_2$ are immaculate.

Proof. For all temptings $\mathcal{R}$ consider $\overline{\mathcal{M}(\mathcal{R})}$, the projection of $\mathcal{M}(\mathcal{R})$ to the $(y,\ z)$-plane by omitting the x-coordinate. The projected maculate regions $\overline{\mathcal{M}(\mathcal{R})}$ do not depend on the parameters c and b.

If a line bundle D is maculate, then there exists an $\mathcal{R}$ such that $D \in \mathcal{M}(\mathcal{R})$, and thus $\overline{D} \in \overline{\mathcal{M}(\mathcal{R})}$.

So if $\overline{D} \in \mathbb{Z}^2 \setminus \bigcup_{\mathcal{R} \text{ tempting}} \overline{\mathcal{M}(\mathcal{R})}$, all D' in the line through D parallel to the kernel of the projection are immaculate. Thus, the problem of finding lines in x-direction reduces to investigating $\mathbb{Z}^2$ without the union of twelve shifted cones.

We have $\overline{\mathcal{M}(\mathcal{R})} = \overline{v_\mathcal{R}} + \overline{\sigma_\mathcal{R}}$ and $\overline{\mathcal{M}(\Sigma(1) \setminus \mathcal{R})} = (\overline{K} - \overline{v_\mathcal{R}}) - \overline{\sigma_\mathcal{R}}$

$\mathcal{R}$	$\overline{v_\mathcal{R}}$	$\overline{v_{\Sigma(1)\setminus\mathcal{R}}}$	rays of $\overline{\sigma_\mathcal{R}}$
$\emptyset$	$(0,0)$	$(-p_1 - p_2 + p_4, p_1 - p_3 - p_4)$	$(1,-1), (-1,1)$
P_0	$(-p_1, p_1)$	$(-p_2 + p_4, -p_3 - p_4)$	$(-1,1), (1,0)$
P_1	$(-p_1 - p_2, p_1)$	$(p_4, -p_3 - p_4)$	$(-1,0), (0,1)$
P_2	$(-p_2, -p_3)$	$(-p_1 + p_4, p_1 - p_4)$	$(1,-1), (-1,1)$
P_3	$(p_4, -p_3 - p_4)$	$(-p_1 - p_2, p_1)$	$(1,0), (0,-1)$
P_4	$(p_4, -p_4)$	$(-p_1 - p_2, p_1 - p_3)$	$(1,-1), (0,1)$

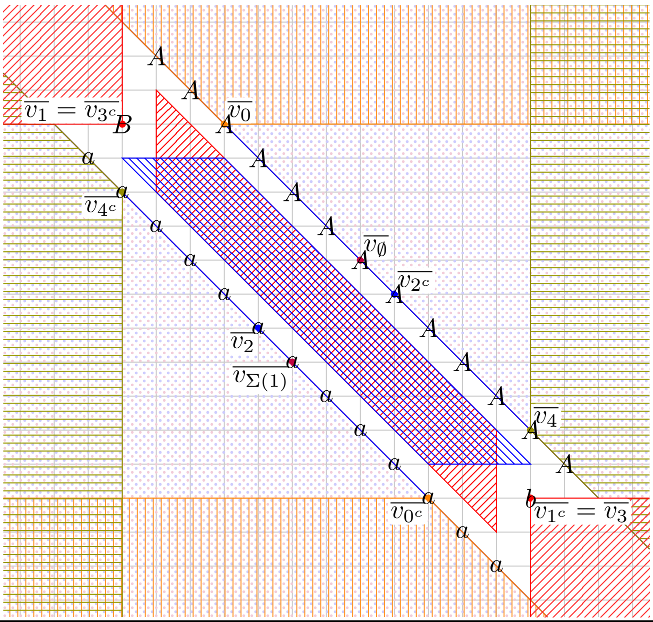
The following containment relations

- $\overline{\mathcal{M}(\Sigma(1))} = \overline{\mathcal{M}(\mathrm{P}_2)}$, implying $\overline{\mathcal{M}(\emptyset)} = \overline{\mathcal{M}(\mathrm{P}_2^c)}$
- $\overline{\mathcal{M}(\mathrm{P}_1)} = \overline{\mathcal{M}(\mathrm{P}_3^c)}$, implying $\overline{\mathcal{M}(\mathrm{P}_1^c)} = \overline{\mathcal{M}(\mathrm{P}_3)}$, and
- $\overline{\mathcal{M}(\mathrm{P}_0^c)}, \overline{\mathcal{M}(\mathrm{P}_4^c)} \subset \overline{\mathcal{M}(\Sigma(1))}$,

lead to

$$\mathbb{Z}^2 \setminus \bigcup_{\mathcal{R} \text{ tempting}} \overline{\mathcal{M}(\mathcal{R})} = \mathbb{Z}^2 \setminus (\overline{\mathcal{M}(\Sigma(1))} \cup \overline{\mathcal{M}(\emptyset)} \cup \overline{\mathcal{M}(\mathrm{P}_1)} \cup \overline{\mathcal{M}(\mathrm{P}_1^c)}) = (Q_1 \cup Q_2) \cap \mathbb{Z}^2.$$

□



Point	Coordinates	Point	Coordinates
$\overline{v_{\Sigma(1)}} = -\overline{K}$	$(-p_1 - p_2 + p_4,\ p_1 - p_3 - p_4)$	$\overline{v_\emptyset}$	$(0,\ 0)$
$\overline{v_{0^c}}$	$(-p_2 + p_4,\ -p_3 - p_4)$	$\overline{v_0}$	$(-p_1,\ p_1)$
$\overline{v_{1^c}} = \overline{v_3}$	$(p_4,\ -p_3 - p_4)$	$\overline{v_1} = \overline{v_{3^c}}$	$(-p_1 - p_2,\ p_1)$
$\overline{v_{2^c}}$	$(-p_1 + p_4,\ p_1 - p_4)$	$\overline{v_2}$	$(-p_2,\ -p_3)$
$\overline{v_{4^c}}$	$(-p_1 - p_2,\ p_1 - p_3)$	$\overline{v_4}$	$(p_4,\ -p_4)$

Figure 2.4: $p_2, p_3 > 1$
The projected maculate regions to the $(y,\ z)$-plane for the example $(p_1, p_2, p_3, p_4) = (4, 3, 2, 5)$ and a table with the general coordinates of the projected vertices of the maculate regions, where $\overline{v_i}$ and $\overline{v_{i^c}}$ denotes the projected vertex of the maculate region $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ for $\mathcal{R} = \mathrm{P}_i$ respectively $\mathcal{R} = \mathrm{P}_i^c$. The polyhedra Q_1 and Q_2 from Definition 2.53 are depicted in blue and red. The letters A and B indicate where the line segments of immaculate line bundles are located in the projection, and the letters a, b denote the location of their Serre duals.

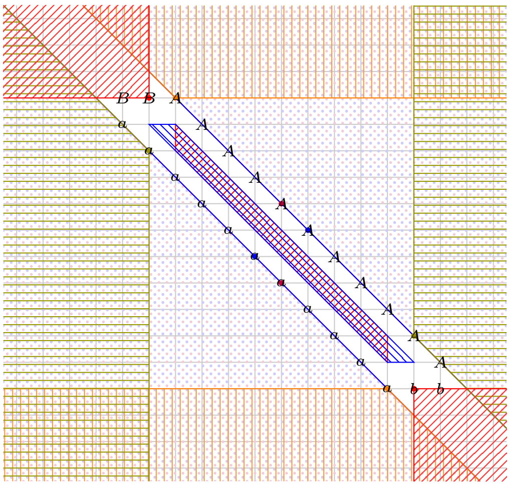


Figure 2.5: $p_2 = 1$
$(p_1, p_2, p_3, p_4) = (4, 1, 2, 5)$.

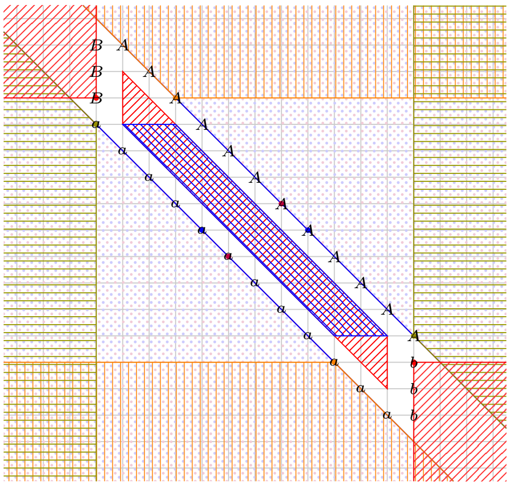


Figure 2.6: $p_3 = 1$
$(p_1, p_2, p_3, p_4) = (4, 3, 1, 5)$.

Proposition 2.55 (Line Segements of Type (A).)**.** There are line segments located over the diagonal $(*,\ y,\ -y)$ containing immaculate line bundles for $y \in [-p_3 - p_4 + 1, p_1 + p_2 - 1]$. Denote $D_{x,y} = (x,\ y,\ -y)$, and for any $y \in [-p_1 - p_2 + 1, p_3 + p_4 - 1]$ let

$$I_y \ := \ \{D_{x,y} \mid x_0(y) \le x \le x_1(y)\}$$

be the set of lattice points on the segment with x coordinate varying from $x_0(y)$ to $x_1(y)$. It holds that

$$x_0(y) = \begin{cases} -p_0 - p_1 + 1 & \text{for } y \in [-p_1 - p_2 + 1, -p_1] \\ -p_0 - p_4 + y + 1 & \text{for } y \in [-p_1 + 1, -p_1 + p_4 - 1] \\ -p_0 - p_1 + 1 & \text{for } y \in [-p_1 + p_4, p_4 - 1] \\ -p_0 - p_4 + y + 1 & \text{for } y \in [p_4, p_3 + p_4 - 1] \end{cases}$$

and

$$x_1(y) = \begin{cases} -1 & \text{for } y \in [-p_1 - p_2 + 1, -1] \\ y - 1 & \text{for } y \in [0, p_3 + p_4 - 1] \end{cases}.$$

Depending on p_1 and p_4 we obtain different combinations and different numbers of lattice points on the segments. The values of $x_0(y), x_1(y)$ and the number of elements of I_y is in Table 2.2. Notice that they do not depend on b or c, as in the case of type (F).

Proof. The divisor $D_{x,y}$ is immaculate if and only if it is not in any maculate region $\mathcal{M}(\mathcal{R})$ for any tempting subset $\mathcal{R} \subset \Sigma(1)$. By considering $\overline{D_{x,y}}$ we will narrow down the possible $\mathcal{R}$ that could make $D_{x,y}$ maculate.

Independent on y it holds $\overline{D_{x,y}} \in \overline{\mathcal{M}(\Sigma(1))}$ and $\overline{D_{x,y}} \in \overline{\mathcal{M}(\mathrm{P}_2^c)}$. For $y \ge p_4$ we have $\overline{D_{x,y}} \in \overline{\mathcal{M}(\mathrm{P}_4)}$. For $y \le -p_1$ we get $\overline{D_{x,y}} \in \overline{\mathcal{M}(\mathrm{P}_0)}$.

So we obtain the following:

I_1 $(-p_1 - p_2, -p_1]$	$I_2 \cup I_3 \cup I_4$ $[-p_1 + 1, p_4 - 1]$	I_5 $[p_4, p_3 + p_4)$
$\emptyset$	$\emptyset$	$\emptyset$
P_2^c	P_2^c	P_2^c
P_0		P_4

We will analyze now when $D_{x,y}$ belongs to any of the possible maculate regions:

- $\emptyset$: The Hilbert basis of $\sigma_\emptyset$ is

  $$\rho_1 = (1,\ 1,\ -1),\ \rho_{2,1} = (-(b_{p_3} + 1),\ 0,\ 1),\ \rho_{2,2} = (-c_{p_2},\ 0,\ 1),\ \rho_3 = (0,\ -1,\ 1),$$

  where $\rho_{2,1}$ is part of the Hilbert basis if $c_{p_2} \le b_{p_3}$ and $\rho_{2,2}$ otherwise. The vertex is $v_\emptyset = (0,\ 0,\ 0)$. We get $D_{x,y}$ as linear combination of the Hilbert basis in the following way:

  $$D_{x,y} = x \cdot \rho_1 + (x - y) \cdot \rho_3$$

  In order to be in the cone, we want it to be a positive linear combination, so $x \ge 0$ and $x - y \ge 0 \iff x \ge y$, that is for $y \le 0$ the two conditions reduce to $x \ge 0$ and for $y \ge 0$, they reduce to $x \ge y$.

Table 2.2: Isolated immaculate line bundles type A.

$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_1(y)$	$\#I_y$
Case $p_1 < p_4$				
$-p_1 - p_2 + 1$	$-p_1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$
$-p_1 + 1$	-1	$-p_0 - p_4 + y + 1$	-1	$p_0 + p_4 + \|y\| - 1$
0	$-p_1 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$
$-p_1 + p_4$	$p_4 - 1$	$-p_0 - p_1 + 1$	$y - 1$	$p_0 + p_1 + \|y\| - 1$
p_4	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$
Case $p_1 > p_4$				
$-p_1 - p_2 + 1$	$-p_1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$
$-p_1 + 1$	$-p_1 + p_4 - 1$	$-p_0 - p_4 + y + 1$	-1	$p_0 + p_4 + \|y\| - 1$
$-p_1 + p_4$	-1	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$
0	$p_4 - 1$	$-p_0 - p_1 + 1$	$y - 1$	$p_0 + p_1 + \|y\| - 1$
p_4	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$
Case $p_1 = p_4$				
$-p_1 - p_2 + 1$	$-p_1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$
$-p_1 + 1$	-1	$-p_0 - p_4 + y + 1$	-1	$p_0 + p_4 + \|y\| - 1$
0	$p_4 - 1$	$-p_0 - p_1 + 1$	$y - 1$	$p_0 + p_1 + \|y\| - 1$
p_4	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$

Thus:

$$D_{x,y} \in \mathcal{M}(\emptyset) \Leftrightarrow x \geq \begin{cases} 0 & \text{for } y \leq 0 \\ y & \text{for } y \geq 0 \end{cases}.$$

- P_2^c : The Hilbert basis of the tail cone of this maculate region is the following:

  $$\rho_1 = (-1,\ -1,\ 1),\ \rho_2 = (0,\ 1,\ 0),\ \rho_3 = (0,\ 1,\ -1).$$

  And we have the vertex $v_{2^c} = (-p_0 - p_1,\ -p_1 + p_4,\ p_1 - p_4)$.

  $$D_{x,y} = v_{2^c} + (-p_0 - p_1 - x) \cdot \rho_1 + (-p_0 - p_4 + y - x) \cdot \rho_3$$

  We want this to be a positive linear combination, so: $-p_0 - p_1 - x \geq 0 \iff -p_0 - p_1 \geq x$ and $-p_0 - p_4 + y - x \geq 0 \iff -p_0 - p_4 + y \geq x$. For both inequalities to hold we get $x \leq \min(-p_0 - p_1, -p_0 - p_4 + y)$. If $y \geq -p_1 + p_4$, the minimum is obtained in the first argument, otherwise in the second.

  Thus:

  $$D_{x,y} \in \mathcal{M}(\mathrm{P}_2^c) \Leftrightarrow x \leq \begin{cases} -p_0 - p_4 + y & \text{for } y \leq -p_1 + p_4 \\ -p_0 - p_1 & \text{for } y \geq -p_1 + p_4 \end{cases}.$$

- P_0 : Here the Hilbert basis consists of the following elements

  $$\rho_1 = (-1,\ 0,\ 0),\ \rho_2 = (0,\ 1,\ 0),\ \rho_3 = (0,\ -1,\ 1).$$

  $$v_0 = (-p_0 - p_1,\ -p_1,\ p_1)$$

  $y \leq -p_1$: It is clear that we have to use the last element of the Hilbert basis:

  $$D_{x,y} = v_0 + (-p_0 - p_1 - x) \cdot \rho_1 + (-p_1 - y) \cdot \rho_3$$

  Thus:

  $$D_{x,y} \in \mathcal{M}(\mathrm{P}_0) \Leftrightarrow x \leq -p_0 - p_1 \text{ and } y \leq -p_1.$$

- P_4 : In this case the Hilbert basis consists of the following four elements (and if $b_1 = 0$ only the first two and the last of them):

  $$\rho_1 = (-1,\ 0,\ 0),\ \rho_2 = (1,\ 1,\ -1),\ \rho_3 = (0,\ 1,\ 0),\ \rho_4 = (-(b_1 + 1),\ 0,\ 1).$$

  Again we are left with no choice how to create $D_{x,y}$, we have to use the second element of the Hilbert basis:

  $$D_{x,y} = v_4 + (-p_0 - p_4 + y - x) \cdot \rho_1 + (-p_4 + y) \cdot \rho_2$$

  Thus:

  $$D_{x,y} \in \mathcal{M}(\mathrm{P}_4) \Leftrightarrow x \leq -p_0 - p_4 + y \text{ and } y \geq p_4.$$

So we get

- $$D_{x,y} \in \mathcal{M}(\emptyset) \Leftrightarrow \begin{array}{c} y \le 0 \\ x \ge 0 \end{array} \text{ or } \begin{array}{c} y \ge 0 \\ x \ge y. \end{array}$$

- $$D_{x,y} \in \mathcal{M}(\mathrm{P}_2^c) \Leftrightarrow \begin{array}{c} y \le -p_1 + p_4 \\ x \le -p_0 - p_4 + y. \end{array} \text{ or } \begin{array}{c} y \ge -p_1 + p_4 \\ x \le -p_0 - p_1 \end{array}$$

- $$D_{x,y} \in \mathcal{M}(\mathrm{P}_0) \Leftrightarrow \begin{array}{c} y \le -p_1 \\ x \le -p_0 - p_1. \end{array}$$

- $$D_{x,y} \in \mathcal{M}(\mathrm{P}_4) \Leftrightarrow \begin{array}{c} y \ge p_4 \\ x \le -p_0 - p_4 + y. \end{array}$$

On the other hand it means:

- $$D_{x,y} \notin \mathcal{M}(\emptyset) \Leftrightarrow \begin{array}{c} y \le 0 \\ x \le -1 \end{array} \text{ or } \begin{array}{c} y \ge 0 \\ x \le y - 1. \end{array}$$

- $$D_{x,y} \notin \mathcal{M}(\mathrm{P}_2^c) \Leftrightarrow \begin{array}{c} y \le -p_1 + p_4 \\ x \ge -p_0 - p_4 + y + 1. \end{array} \text{ or } \begin{array}{c} y \ge -p_1 + p_4 \\ x \ge -p_0 - p_1 + 1 \end{array}$$

- $$D_{x,y} \notin \mathcal{M}(\mathrm{P}_0) \Leftrightarrow \begin{array}{c} y \le -p_1 \\ x \ge -p_0 - p_1 + 1 \end{array} \text{ or } \; y > -p_1$$

- $$D_{x,y} \notin \mathcal{M}(\mathrm{P}_4) \Leftrightarrow \; y < p_4 \; \text{ or } \begin{array}{c} y \ge p_4 \\ x \ge -p_0 - p_4 + y + 1. \end{array}$$

In conclusion:

- For $y \in I_1 = [-p_1 - p_2 + 1, p_1]$:

  $$\begin{array}{ll} x \le -1 & (\emptyset) \\ x \ge -p_0 - p_4 + y + 1 & (\mathrm{P}_2^c) \\ x \ge -p_0 - p_1 + 1 & (\mathrm{P}_0) \end{array}$$

  With

  $$-y \ge p_1 - p_4 \Leftrightarrow -p_1 - y \ge -p_4 \Leftrightarrow -p_0 - p_1 + 1 - y \ge -p_0 - p_4 + 1 \Leftrightarrow -p_0 - p_1 + 1 \ge -p_0 - p_4 + y + 1$$

  we obtain

  $$x \in [-p_0 - p_1 + 1, -1].$$

- For $y \in I_2 \cup I_3 \cup I_4 = [-p_1 + 1, p_4 - 1]$, the candidates are only $\emptyset, \mathrm{P}_2^c$. We will subdivide this case into sub-cases that depend on p_1 and p_4:

(i) For $-p_1 + p_4 \le y \le p_4 - 1$:

$$\begin{aligned} x &\le y - 1 & (\emptyset) \\ x &\ge -p_0 - p_1 + 1 & (\mathrm{P}_2^c) \end{aligned}$$

So $x \in [-p_0 - p_1 + 1, y - 1]$.

(ii) For $0 \le y \le -p_1 + p_4$:

$$\begin{aligned} x &\le y - 1 & (\emptyset) \\ x &\ge -p_0 - p_4 + y + 1 & (\mathrm{P}_2^c) \end{aligned}$$

So $x \in [-p_0 - p_4 + y + 1, y - 1]$.

(iii) For $-p_1 + 1 \le y \le 0$

$$\begin{aligned} x &\le -1 & (\emptyset) \\ x &\ge -p_0 - p_4 + y + 1 & (\mathrm{P}_2^c) \end{aligned}$$

So $x \in [-p_0 - p_4 + y + 1, -1]$.

(iv) For $-p_1 + p_4 \le y \le 0$:

$$\begin{aligned} x &\le -1 & (\emptyset) \\ x &\ge -p_0 - p_1 + 1 & (\mathrm{P}_2^c) \end{aligned}$$

$x \in [-p_0 - p_1 + 1, -1]$.

(v) $-p_1 + 1 \le y \le -p_1 + p_4$:

$$\begin{aligned} x &\le -1 & (\emptyset) \\ x &\ge -p_0 - p_4 + y + 1 & (\mathrm{P}_2^c) \end{aligned}$$

$x \in [-p_0 - p_4 + y + 1, -1]$.

- For $y \in I_5 = [p_4, p_3 + p_4 - 1]$: the candidate temptings were $\emptyset, \mathrm{P}_2^c, \mathrm{P}_4$:

$$\begin{aligned} x &\le y - 1 & (\emptyset) \\ x &\ge -p_0 - p_1 + 1 & (\mathrm{P}_2^c) \\ x &\ge -p_0 - p_4 + y + 1 & (\mathrm{P}_4) \end{aligned}$$

Since $y \ge p_4$:

$$-p_0 - p_4 + y + 1 \ge -p_0 - p_4 + p_4 + 1 = -p_0 + 1 \ge -p_0 - p_1 + 1$$

So

$$x \in [-p_0 - p_4 + y + 1, y - 1].$$

These are exactly the values of Table 2.2. □

Proposition 2.56 (Line segments of type (B).)**.** The line bundles in the line segments mentioned below are immaculate. Line bundles on the line that do not belong to the mentioned segment are not immaculate. The segments of this type depend on p_2 and p_3 via the parallelograms Q_1 and Q_2 elaborated in Definition 2.53.

- If $p_2, p_3 \geq 2$, then this type consist of just one horizontal segment whose projection to the $(y,\ z)$-plane is located left/above the intersection of the upper edges of the parallelograms Q_1 and Q_2, see the point marked as B on Figure 2.4. The line segment contains $p_0 - 1$ immaculate line bundles with coordinates

$$([-p_0 - p_1 + \overline{c} + 1, -p_1 + \overline{c} - 1],\ -p_1 - p_2,\ p_1),$$

where $\overline{c} := \sum c_i$.

- If $p_2 = 1$, then the points of Type (B) consist of p_3 horizontal line segments, each containing $p_0 - 1$ immaculate line bundles. The coordinates are

$$([-p_0 - p_1 + 1, -p_1 - 1],\ -p_1 - p_2 - y,\ p_1)$$

for $y \in [0, p_3 - 1]$. On Figure 2.5 their projections onto (y, z) plane are indicated by the letter B. Roughly speaking, their projections are at each lattice point directly above the upper edge of the parallelogram Q_1,

- For $p_3 = 1$, there are p_2 horizontal line segments of Type (B) each containing $p_0 - 1$ immaculate line bundles. The coordinates are

$$([-p_0 - p_1 + \overline{c} - y(\overline{b} + 1) + 1, -p_1 + \overline{c} - y(\overline{b} + 1) - 1],\ -p_1 - p_2,\ p_1 + y)$$

for $y \in [0, p_2 - 1]$. On Figure 2.6 their locations in the projection are indicated by the letter B, as for the previous case. In this case the projections are located directly left of Q_2.

Proof.

- $p_2, p_3 \geq 2$ The divisor $D_x = (x,\ -p_1 - p_2,\ p_1))$ is immaculate, if there is no $\mathcal{R}$ such that $D_x \in \mathcal{M}(\mathcal{R})$. Since D_x is only in $\mathcal{M}(\mathcal{R})$, for $\mathcal{R} = \mathrm{P}_1, \mathrm{P}_3^c$, we only have to check those two temptings.

  $v_1 = (-p_1 + \overline{c},\ -p_1 - p_2,\ p_1)$ and $(1,\ 0,\ 0) \in \sigma_1$ (and $(-1,\ 0,\ 0) \notin \sigma_1$), thus $D_x \in \mathcal{M}(\mathrm{P}_1)$ if and only if $x \in [-p_1 + \overline{c}, \infty)$.

  $v_{3^c} = (-p_0 - p_1 + c,\ -p_1 - p_2,\ p_1)$ and $(-1,\ 0,\ 0) \in \sigma_{3^c}$ (and $(1,\ 0,\ 0) \notin \sigma_{3^c}$), thus $D_x \in \mathcal{M}(\mathrm{P}_3^c)$ if and only if $x \in (-\infty, -p_0 - p_1 + \overline{c}]$.

- $p_2 = 1$ Now we consider the divisor $D_{x,y} = (x,\ -p_1 - p_2 - y,\ p_1))$ for $y \in [0, p_3 - 1]$. This divisor can only be in the maculate regions for the tempting sets P_1 or P_3^c. With $v_1 = (-p_1 + \overline{c},\ -p_1 - p_2,\ p_1)$ and

$$\rho_1 = (1,\ 0,\ 0),\ \rho_2 = (0,\ -1,\ 0),\ \rho_3 = (-(b_{p_3} + 1),\ 0,\ 1),$$

we have

$$\mathcal{M}(\mathrm{P}_1) = v_1 + \mathrm{cone}\,(\rho_1,\ \rho_2,\ \rho_3)\,.$$

Now

$$D_{x,y} = v_1 + (p_1 - \overline{c} + x) \cdot \rho_1 + y \cdot \rho_2.$$

There is no other possible way of expressing $D_{x,y}$ with the rays (since we have only these two rays we can use - the others would increase the third coordinate), so $D_{x,y} \in \mathcal{M}(\mathrm{P}_1) \Leftrightarrow x \geq -p_1 + \overline{c}$.

With $v_{3^c} = (-p_0 - p_1 + \overline{c},\ -p_1 - p_2,\ p_1)$ and

$$\rho_1 = (-1,\ 0,\ 0),\ \rho_2 = (c_{p_2},\ -1,\ 0),\ \rho_3 = (-(b_1+1),\ 0,\ 1),\ \rho_4 = (0,\ -1,\ 1),$$

we have

$$\mathcal{M}(\mathrm{P}_1) = v_{3^c} + \operatorname{cone}(\rho_1,\ \rho_2,\ \rho_3,\ \rho_4),$$

where ρ_4 only forms a ray if $c_{p_2} < b_1 + 1$.

Now

$$D_{x,y} = v_{3^c} + (-p_0 - p_1 + \overline{c} + yc_{p_2} - x) \cdot \rho_1 + y \cdot \rho_2.$$

Again, this is our only chance of expressing $D_{x,y}$ as an element in the shifted cone, so $D_{x,y} \in \mathcal{M}(\mathrm{P}_3^c) \Leftrightarrow -p_0 - p_1 + \overline{c} + yc_{p_2} \geq x$.

Thus $D_{x,y}$ is immaculate if and only if

$$-p_0 - p_1 + \overline{c} + yc_{p_2} + 1 \leq x \leq -p_1 + \overline{c} - 1$$

So for each $y \in [0, p_3 - 1]$ we get $p_0 - yc_{p_2} - 1$ immaculates.

In the case that $p_2 = 1$, $\overline{c} = c_{p_2} = 0$ and we obtain $D_{x,y}$ immaculate for $y \in [0, p_3 - 1]$ and $x \in [-p_0 - p_1 + 1, -p_1 - 1]$.

If $p_2 \geq 1$ and $c_{p_2} \geq p_0 - 1$, we only obtain immaculate $D_{x,y}$ for $y = 0$.

(Let $c_{p_2} \geq 1$, then $D_{x,y}$ is maculate for $y \geq \frac{p_0 - 1}{c_{p_2}}$ and any x.)

- $p_3 = 1$ Now let us consider $D_{x,y} = (x, -p_1 - p_2, p_1 + y)$ for $y \in [0, p_2 - 1]$. Again, P_1 and P_3^c are the only tempting sets we have to investigate further.

$$D_{x,y} = v_1 + (p_1 - \overline{c} + y + yb_{p_3} + x) \cdot \rho_1 + y \cdot \rho_3$$

So $D_{x,y} \in \mathcal{M}(\mathrm{P}_1) \Leftrightarrow x \geq -p_1 + \overline{c} - y - yb_{p_3}$.

($\Leftarrow$ is clear, the other one follows from the fact, that using other rays would increase the second coordinate.)

$$D_{x,y} = v_{3^c} + (-p_0 - p_1 + \overline{c} - y - yb_1 - x) \cdot \rho_1 + y \cdot \rho_3$$

So $D_{x,y} \in \mathcal{M}(\mathrm{P}_3^c) \Leftrightarrow -p_0 - p_1 + \overline{c} - y - yb_1 \geq x$. ($\Leftarrow$ is clear, the other one follows from the fact, that using other rays would increase the second coordinate.)

Thence, $D_{x,y}$ is immaculate if and only if

$$-p_0 - p_1 + \overline{c} - y - yb_1 + 1 \leq x \leq -p_1 + \overline{c} - y - yb_{p_3} - 1$$

which gives $p_0 - 1 - y(b_{p_3} - b_1)$ immaculates for each $y \in [0, p_2 - 1]$.

If now $p_3 = 1$, $b_{p_3} = b_1$ and we obtain $p_0 - 1$ immaculates for each $y \in [0, p_2 - 1]$.

But if $p_3 > 1$ and $b_{p_3} - b_1 \geq p_0 - 1$, the only immaculate $D_{x,y}$ occur for $y = 0$.

(Let $b_{p_3} > b_1$, then $D_{x,y}$ is maculate for $y \geq \frac{p_0 - 1}{b_{p_3} - b_1}$ and any x.)

$\square$

We have now seen that the types (F), (A), (B) are always immaculate. Moreover, for sufficiently "general" parameters the listed line bundles and their Serre duals are all immaculate line bundles.

Proposition 2.57. Let b, c be large enough, in the sense that the following conditions are satisfied:

- $\max(b_{p_3}, c_{p_2}) \geq p_0 + p_1 + \max(p_2, p_3) + p_4$,
- $c_{p_2} \geq p_0 - 1$ if $p_2 \geq 2$,
- $b_{p_3} - b_1 \geq p_0 - 1$ if $p_3 \geq 2$.

Then the only immaculate line bundles are the previously mentioned and their Serre duals.

Proof. We will divide the proof into two parts.

First we show that for D with $\overline{D} \in \overline{\mathcal{M}(\emptyset)} \setminus \operatorname{int}(\overline{\mathcal{M}(\mathrm{P}_1)} \cup \overline{\mathcal{M}(\mathrm{P}_1^c)})$ the line bundles of type A are the only immaculate ones.

Then we consider D with $\overline{D} \in \overline{\mathcal{M}(\mathrm{P}_1)}$, and show that the line bundles of type B are the only immaculates among those.

Using Serre duality one can conclude that there are no further immaculate line bundles D with $\overline{D} \in \cup_{\mathcal{R} \text{ tempting}} \overline{\mathcal{M}(\mathcal{R})}$.

- Let D with $\overline{D} \in \overline{\mathcal{M}(\emptyset)} \setminus \operatorname{int}(\overline{\mathcal{M}(\mathrm{P}_1)} \cup \overline{\mathcal{M}(\mathrm{P}_1^c)})$. We will restrict our considerations to the maculate regions for the tempting subset $\emptyset$ and P_2^c.

  We consider the divisors $D_{x,y,a} = (x,\ -y,\ y + a)$ and $E_{x,y,a} = (x,\ -y + a,\ y)$. The condition from above reduces to $y \in [-p_3 - p_4, p_1 + p_2]$. By writing the divisors as positive linear combinations of the rays (which are also the Hilbert basis) of the maculate regions, we can give inequalities for x which "decide" whether $D_{x,y,a}$ or $E_{x,y,a}$ is in the given macualte region.

  So we will have that $D_{x,y,a} \in \mathcal{M}(\emptyset) \iff x \leq d_\emptyset$ and $D_{x,y,a} \in \mathcal{M}(\mathrm{P}_2^c) \iff x \geq d_{\mathrm{P}_2^c}$. Then if $d_\emptyset + 1 \geq d_{\mathrm{P}_2^c}$ all $D_{x,y,a}$ are maculate.

  First we will consider P_2^c, since its Hilbert basis does not depend on the values of b, c. The vertex of the maculate region is $v_{2^c} = (-p_0 - p_1,\ -p_1 + p_4,\ p_1 - p_4)$. And the' Hilbert basis is
  $$\rho_1 = (-1,\ 1,\ 1),\ \rho_2 = (0,\ 1,\ 0),\ \rho_3 = (0,\ 1,\ -1).$$

  Then we obtain
  $$D_{x,y,a} = v_{2^c} + (-x - p_0 - p_1) \cdot \rho_1 + a \cdot \rho_2 + (-x - y - p_0 - p_4 - a) \cdot \rho_3.$$

  For this to be a positive linear combination we need to require
  $$x \leq \min(-p_0 - p_1, -y - p_0 - p_4 - a) = \begin{cases} -p_0 - p_1 & y \leq p_1 - p_4 - a \\ -y - p_0 - p_4 - a & y \geq p_1 - p_4 - a \end{cases}.$$

  We also get
  $$E_{x,y,a} = v_{2^c} + (-x - p_0 - p_1) \cdot \rho_1 + a \cdot \rho_2 + (-x - y - p_0 - p_4) \cdot \rho_3.$$

Here we require

$$x \leq \min(-p_0 - p_1, -y - p_0 - p_4) = \begin{cases} -p_0 - p_1 & y \leq p_1 - p_4 \\ -y - p_0 - p_4 & y \geq p_1 - p_4 \end{cases} .$$

Now let us consider the tempting subset $\emptyset$. Here the vertex is $(0,\ 0,\ 0)$. We have

$$\rho_1 = (1,\ 1,\ -1),\ \rho_{2,1} = (-(b_{p_3}+1),\ 0,\ 1),\ \rho_{2,2} = (-c_{p_2},\ 1,\ 0) \text{ and } \rho_3 = (0,\ -1,\ 1),$$

where $\rho_{2,1}$ is a ray when $b_{p_3} \geq c_{p_2}$ and $\rho_{2,2}$ if $b_{p_3} < c_{p_2}$.

- We will start with the case $b_{p_3} \geq c_{p_2}$:
  Here we get $D_{x,y,a}$ as the following linear combination

  $$D_{x,y,a} = (x + a \cdot (b_{p_3}+1)) \cdot \rho_1 + a \cdot \rho_{2,1} + (x + y + a \cdot (b_{p_3}+1)) \cdot \rho_3.$$

  For containment in the maculate region, we obtain the following condition

  $$x \geq \max(-a \cdot (b_{p_3}+1), -y - a \cdot (b_{p_3}+1)) = \begin{cases} -y - a \cdot (b_{p_3}+1) & y \leq 0 \\ -a \cdot (b_{p_3}+1) & y \geq 0 \end{cases} .$$

  Similarly for $E_{x,y,a}$,

  $$E_{x,y,a} = (x + a \cdot (b_{p_3}+1)) \cdot \rho_1 + a \cdot \rho_{2,1} + (x + y + a \cdot b_{p_3}) \cdot \rho_3$$

  we obtain the condition

  $$x \geq \max(-a \cdot (b_{p_3}+1), -y - a \cdot b_{p_3}) = \begin{cases} -y - a \cdot b_{p_3} & y \leq a \\ -a \cdot (b_{p_3}+1) & y \geq a \end{cases} .$$

- Now let us consider $b_{p_3} < c_{p_2}$:
  The following linear combination

  $$D_{x,y,a} = (x + a \cdot c_{p_2}) \cdot \rho_1 + a \cdot \rho_{2,2} + (x + y + a \cdot (c_{p_2}+1))) \cdot \rho_3$$

  leads to these inequalities:

  $$x \geq \max(-a \cdot c_{p_2}, -y - a \cdot (c_{p_2}+1)) = \begin{cases} -y - a \cdot (c_{p_2}+1) & y \leq a \\ -a \cdot c_{p_2} & y \geq a \end{cases} .$$

  And once again:

  $$E_{x,y,a} = (x + a \cdot c_{p_2}) \cdot \rho_1 + a \cdot \rho_{2,2} + (x + y + a \cdot c_{p_2}) \cdot \rho_3$$

  we get the condition on x:

  $$x \geq \max(-a \cdot c_{p_2}, -y - a \cdot c_{p_2}) = \begin{cases} -y - a \cdot c_{p_2} & y \leq 0 \\ -a \cdot c_{p_2} & y \geq 0 \end{cases} .$$

Now, we put the cases together. First for $b_{p_3} \geq c_{p_2}$:

$y \leq 0$	$y \leq p_1 - p_4 - a$	$-p_0 - p_1 + 1 \geq -y - a \cdot (b_{p_3} + 1)$
$y \leq 0$	$y \geq p_1 - p_4 - a$	$-y - p_0 - p_4 - a + 1 \geq -y - a \cdot (b_{p_3} + 1)$
$y \geq 0$	$y \leq p_1 - p_4 - a$	$-p_0 - p_1 + 1 \geq -a \cdot (b_{p_3} + 1)$
$y \geq 0$	$y \geq p_1 - p_4 - a$	$-y - p_0 - p_4 - a + 1 \geq -a \cdot (b_{p_3} + 1)$
$y \leq a$	$y \leq p_1 - p_4$	$-p_0 - p_1 + 1 \geq -y - a \cdot b_{p_3}$
$y \leq a$	$y \geq p_1 - p_4$	$-y - p_0 - p_4 + 1 \geq -y - a \cdot b_{p_3}$
$y \geq a$	$y \leq p_1 - p_4$	$-p_0 - p_1 + 1 \geq -a \cdot (b_{p_3} + 1)$
$y \geq a$	$y \geq p_1 - p_4$	$-y - p_0 - p_4 + 1 \geq -a \cdot (b_{p_3} + 1)$

$y \leq 0$	$y \leq p_1 - p_4 - a$	$a \cdot (b_{p_3} + 1) \geq -y + p_0 + p_1 - 1$
$y \leq 0$	$y \geq p_1 - p_4 - a$	$a \cdot b_{p_3} \geq p_0 + p_4 - 1$
$y \geq 0$	$y \leq p_1 - p_4 - a$	$a \cdot (b_{p_3} + 1) \geq p_0 + p_1 - 1$
$y \geq 0$	$y \geq p_1 - p_4 - a$	$a \cdot b_{p_3} \geq y + p_0 + p_4 - 1$
$y \leq a$	$y \leq p_1 - p_4$	$a \cdot b_{p_3} \geq -y + p_0 + p_1 - 1$
$y \leq a$	$y \geq p_1 - p_4$	$a \cdot b_{p_3} \geq p_0 + p_4 - 1$
$y \geq a$	$y \leq p_1 - p_4$	$a \cdot (b_{p_3} + 1) \geq p_0 + p_1 - 1$
$y \geq a$	$y \geq p_1 - p_4$	$a \cdot (b_{p_3} + 1) \geq y + p_0 + p_4 - 1$

By putting the boundary values for y and $a = 1$, we can make sure that the inequalities are satisfied for all other possible values:

$y \leq 0$	$y \leq p_1 - p_4 - a$	$b_{p_3} \geq p_0 + p_1 + p_3 + p_4 - 2$
$y \leq 0$	$y \geq p_1 - p_4 - a$	$b_{p_3} \geq p_0 + p_4 - 1$
$y \geq 0$	$y \leq p_1 - p_4 - a$	$b_{p_3} \geq p_0 + p_1 - 2$
$y \geq 0$	$y \geq p_1 - p_4 - a$	$b_{p_3} \geq p_0 + p_1 + p_2 + p_4 - 1$
$y \leq a$	$y \leq p_1 - p_4$	$b_{p_3} \geq p_0 + p_1 + p_3 + p_4 - 1$
$y \leq a$	$y \geq p_1 - p_4$	$b_{p_3} \geq p_0 + p_4 - 1$
$y \geq a$	$y \leq p_1 - p_4$	$b_{p_3} \geq p_0 + p_1 - 2$
$y \geq a$	$y \geq p_1 - p_4$	$b_{p_3} \geq p_0 + p_1 + p_2 + p_4 - 2$

So $b_{p_3} \geq p_0 + p_1 + \max(p_2, p_3) + p_4 - 1$.

Now we do the same steps for the case $b_{p_3} < c_{p_2}$:

$y \leq a$	$y \leq p_1 - p_4 - a$	$-p_0 - p_1 + 1 \geq -y - a \cdot (c_{p_2} + 1)$
$y \leq a$	$y \geq p_1 - p_4 - a$	$-y - p_0 - p_4 - a + 1 \geq -y - a \cdot (c_{p_2} + 1)$
$y \geq a$	$y \leq p_1 - p_4 - a$	$-p_0 - p_1 + 1 \geq -a \cdot c_{p_2}$
$y \geq a$	$y \geq p_1 - p_4 - a$	$-y - p_0 - p_4 - a + 1 \geq -a \cdot c_{p_2}$
$y \leq 0$	$y \leq p_1 - p_4$	$-p_0 - p_1 + 1 \geq -y - a \cdot c_{p_2}$
$y \leq 0$	$y \geq p_1 - p_4$	$-y - p_0 - p_4 + 1 \geq -y - a \cdot c_{p_2}$
$y \geq 0$	$y \leq p_1 - p_4$	$-p_0 - p_1 + 1 \geq -a \cdot c_{p_2}$
$y \geq 0$	$y \geq p_1 - p_4$	$-y - p_0 - p_4 + 1 \geq -a \cdot c_{p_2}$

$y \leq 0$	$y \leq p_1 - p_4 - a$	$a \cdot (c_{p_2} + 1) \geq -y + p_0 + p_1 - 1$
$y \leq 0$	$y \geq p_1 - p_4 - a$	$a \cdot c_{p_2} \geq p_0 + p_4 - 1$
$y \geq 0$	$y \leq p_1 - p_4 - a$	$a \cdot c_{p_2} \geq p_0 + p_1 - 1$
$y \geq 0$	$y \geq p_1 - p_4 - a$	$a \cdot c_{p_2} \geq y + p_0 + p_4 + a - 1$
$y \leq a$	$y \leq p_1 - p_4$	$a \cdot c_{p_2} \geq -y + p_0 + p_1 - 1$
$y \leq a$	$y \geq p_1 - p_4$	$a \cdot c_{p_2} \geq p_0 + p_4 - 1$
$y \geq a$	$y \leq p_1 - p_4$	$a \cdot c_{p_2} \geq p_0 + p_1 - 1$
$y \geq a$	$y \geq p_1 - p_4$	$a \cdot c_{p_2} \geq y + p_0 + p_4 - 1$

Putting the boundary values for y and $a = 1$.

$y \leq 0$	$y \leq p_1 - p_4 - a$	$c_{p_2} \geq p_0 + p_1 + p_3 + p_4 - 2$
$y \leq 0$	$y \geq p_1 - p_4 - a$	$c_{p_2} \geq p_0 + p_4 - 1$
$y \geq 0$	$y \leq p_1 - p_4 - a$	$c_{p_2} \geq p_0 + p_1 - 1$
$y \geq 0$	$y \geq p_1 - p_4 - a$	$c_{p_2} \geq p_0 + p_1 + p_2 + p_4$
$y \leq a$	$y \leq p_1 - p_4$	$c_{p_2} \geq p_0 + p_1 + p_3 + p_4 - 1$
$y \leq a$	$y \geq p_1 - p_4$	$c_{p_2} \geq p_0 + p_4 - 1$
$y \geq a$	$y \leq p_1 - p_4$	$c_{p_2} \geq p_0 + p_1 - 1$
$y \geq a$	$y \geq p_1 - p_4$	$c_{p_2} \geq p_0 + p_1 + p_2 + p_4 - 1$

So if $c_{p_2} \geq p_0 + p_1 + \max(p_2, p_3 - 1) + p_4$ we are safe.

In conclusion, if we require $\max(c_{p_2}, b_{p_3}) \geq p_0 + p_1 + \max(p_2, p_3 - 1) + p_4$, $D_{*,y,a}$ and $E_{*,y,a}$ are maculate for $y \in [-p_3 - p_4, p_1 + p_2]$ and $a \geq 1$.

So now we have shown that $(*,\ -y + a,\ y)$ and $(*,\ -y,\ y + a)$ are always maculate for sufficiently general c and b, for $y \in [-p_3 - p_4, p_1 + p_2]$.

We are left to show that $(*,\ g,\ h)$ is always maculate for $g > p_3 + p_4$ and $h > p_1 + p_2$. But we can write $(*,\ g,\ h) = (*,\ -y,\ y + (h - y)) + (g - y) \cdot (0,\ 1,\ 0) = D_{*,y,(h-y)} + (g - y) \cdot (0,\ 1,\ 0)$ with $y \in [-p_3 - p_4, p_1 + p_2]$, $h - y \geq 1$ and $g - y \geq 1$ and we know that $D_{*,y,(h-y)} \in \mathcal{M}(\emptyset) \cup \mathcal{M}(\mathrm{P}_2^c)$ and $(0,\ 1,\ 0) \in \sigma_\emptyset$ and also $(0,\ 1,\ 0) \in \sigma_{2^c}$, so $(*,\ g,\ h)$ is always maculate as well.

- We will now consider D with $\overline{D} \in \overline{\mathcal{M}(\mathrm{P}_1)}$, that is divisors $D_{x,y,z} = (x,\ -p_1 - p_2 - y,\ p_1 + z)$ for $y, z \geq 0$. The only candidate maculate regions come form the tempting sets P_1 and P_3^c.

  For $\mathcal{M}(\mathrm{P}_1)$ we have the vertex $v_1 = (-p_1 + \overline{c},\ -p_1 - p_2,\ p_1)$ and the cone σ_1 has the Hilbert basis

  $$\rho_1 = (1,\ 0,\ 0),\ \rho_2 = (0,\ -1,\ 0) \text{ and } \rho_3 = (-(b_{p_3} + 1),\ 0,\ 1).$$

  We obtain $D_{x,y,z}$ as the following linear combination:

  $$D_{x,y,z} = v_1 + (x + p_1 - \overline{c} + z \cdot (b_{p_3} + 1)) \cdot \rho_1 + y \cdot \rho_2 + z \cdot \rho_3.$$

  So $D_{x,y,z} \in \mathcal{M}(\mathrm{P}_1) \Leftrightarrow x \geq -p_1 + \overline{c} - z \cdot (b_{p_3} + 1)$.

  For $\mathcal{M}(\mathrm{P}_3^c)$ we have the vertex $v_{3^c} = (-p_0 - p_1 + \overline{c},\ -p_1 - p_2,\ p_1)$ and the cone σ_{3^c} has the Hilbert basis

  $$\rho_1 = (-1,\ 0,\ 0),\ \rho_2 = (c_{p_2},\ -1,\ 0),\ \rho_3 = (-(b_1 + 1),\ 0,\ 1)$$

and if $c_{p_2} < b_1 + 1$ we get the additional element $\rho_4 = (0,\ -1,\ 1)$.

We obtain $D_{x,y,z}$ as the following linear combination:

$$D_{x,y,z} = v_{3^c} + (-x - p_0 - p_1 + \overline{c} + y \cdot c_{p_2} - z \cdot (b_1 + 1)) \cdot \rho_1 + y \cdot \rho_2 + z \cdot \rho_3$$

So here we want $x \leq -p_0 - p_1 + \overline{c} + y \cdot c_{p_2} - z \cdot (b_1 + 1)$.

Hence to get that all $D_{*,y,z}$ are maculate we want

$$-p_0 - p_1 + \overline{c} + y \cdot c_{p_2} - z \cdot (b_1 + 1) + 1 \geq -p_1 + \overline{c} - z \cdot (b_{p_3} + 1).$$

$$y \cdot c_{p_2} + z \cdot (b_{p_3} - b_1) \geq p_0 - 1$$

For the general case, we get $c_{p_2} \geq p_0 - 1$ and $b_{p_3} - b_1 \geq p_0 - 1$. If $p_3 = 1$, then the $D_{x,0,z}$ are immaculate for $z \in [0, p_2 - 1]$ and the values of x as shown in Proposition 2.56. The case of $p_2 = 1$ is a sub-case of the following case.

Now if $c_{p_2} < b_1 + 1$, we get the following linear combination:

$$D_{x,y,z} = \begin{cases} v_{3^c} + (-x - p_0 - p_1 + \overline{c}) \cdot \rho_1 + y \cdot \rho_4 & \text{for } y = z \\ v_{3^c} + (-x - p_0 - p_1 + \overline{c} + (y - z)c_{p_2}) \cdot \rho_1 + (y - z) \cdot \rho_2 + z \cdot \rho_4 & \text{for } y > z \\ v_{3^c} + (-x - p_0 - p_1 + \overline{c} - (z - y)(b_1 + 1)) \cdot \rho_1 + (z - y) \cdot \rho_3 + y \cdot \rho_4 & \text{for } y < z \end{cases}$$

And we require:

$$x \leq \begin{cases} -p_0 - p_1 + \overline{c} & \text{for } y = z \\ -p_0 - p_1 + \overline{c} + (y - z) \cdot c_{p_2} & \text{for } y > z \\ -p_0 - p_1 + \overline{c} - (z - y) \cdot (b_1 + 1) & \text{for } y < z \end{cases}$$

So by putting together the inequalities we obtain

$$\begin{array}{ll} -p_0 - p_1 + \overline{c} + 1 \geq -p_1 + \overline{c} - z \cdot (b_{p_3} + 1) & \text{for } y = z \\ -p_0 - p_1 + \overline{c} + (y - z) \cdot c_{p_2} + 1 \geq -p_1 + \overline{c} - z \cdot (b_{p_3} + 1) & \text{for } y > z \\ -p_0 - p_1 + \overline{c} - (z - y) \cdot (b_1 + 1) + 1 \geq -p_1 + \overline{c} - z \cdot (b_{p_3} + 1) & \text{for } y < z \\ \hline z(b_{p_3} + 1) \geq p_0 - 1 & \text{for } y = z \\ (y - z) \cdot c_{p_2} + z \cdot (b_{p_3} + 1) \geq p_0 - 1 & \text{for } y > z \\ z \cdot (b_{p_3} - b_1) + y \cdot (b_1 + 1) \geq p_0 - 1 & \text{for } y < z \end{array}$$

So we get $b_{p_3} + 1 \geq p_0 - 1$, but since $c_{p_2} \leq b_1$ we have $b_{p_3} = \max(c_{p_2}, b_{p_3}) \geq p_0 + p_1 + \max(p_2, p_3 - 1) + p_4$, thus this is satisfied anyways. The case $y > z$ gives $c_{p_2} \geq p_0 - 1$. If $p_2 = 1$, then there are $D_{x,y,0}$ with $y \in [0, p_3 - 1]$ which are immaculate, as shown in Proposition 2.56. The case $y < z$ gives us that the difference $b_{p_3} - b_1 \geq p_0 - 1$. If $p_3 = 1$, the $b_{p_3} = b_1$ and we get the additional condition that $b_1 \geq p_0 - 2$, but here $b_1 = b_{p_3}$, thus as above, this is already satisfied by the condition from the first part of the proof.

□

2.5.4 Changing parameters

When changing to another (dimensional) Picard rank 3 variety, we can study how the immaculate locus changes. This could be interesting when trying to construct exceptional sequences inductively.

We will start with the behavior of the two parallelograms from Definition 2.53 that determine the full lines of immaculate line bundles described in Proposition 2.54.

Lemma 2.58. Let $\overline{p} = (p_0, p_1, p_2, p_3, p_4)$ and $\overline{p}' = (p_0', p_1', p_2', p_3', p_4')$, and $\overline{p} \leq \overline{p}'$ componentwise. We denote $Q_i := Q_i(\overline{p})$ and $Q_i' := Q_i(\overline{p}')$ for $i = 1, 2$. Then it holds that

$$Q_i \subseteq Q_i'.$$

Proof. Both Q_1 and Q_2 each have two vertices on the line $\{y + z = -1\}$. These are: $(-p_1, p_1 - 1), (p_3 + p_4 - 2, -p_3 - p_4 + 1)$ for Q_1 and $(-p_1 - p_2 + 1, p_1 + p_2 - 2), (p_4 - 1, -p_4)$ for Q_2. It is obvious that increasing the p_i componentwise will lead to line segments which are including the previous ones. Now the rest of the parallelograms is defined by their width (either in y- or in z-direction), which in both cases is $p_2 + p_3 - 2$. Increasing the parameters here will lead to the inclusion of the parallelograms. □

Lemma 2.59. Now we will look at the containment of the parallelograms in more detail, when we only increase one parameter by one.

- p_0: If we increase p_0 the Q_1 and Q_2 stay the same.
- p_1: We get the following new points:
  - $(-p_1 - p_2,\ p_1)$ in both Q_1 and Q_2,
  - $(-p_1 - p_2,\ p_1 + 1), \ldots, (-p_1 - p_2,\ p_1 + p_2 - 1)$ in Q_1, and
  - $(-p_1 - p_2 - p_3 + 1,\ p_1), \ldots, (-p_1 - p_2 - 1,\ p_1)$ in Q_2.

  That is in total $p_2 + p_3 - 1$ more points.
- p_2: We get the following new points:
  - $(-p_1 - p_2,\ p_1), \ldots, (-p_1 - p_2,\ p_1 + p_2 - 1)$ in Q_1,
  - $(-p_1 - p_2 - p_3 + 1,\ p_1 - 1), \ldots, (-p_1 - p_2 - 1,\ p_1 - p_3 + 1)$ in Q_2,
  - $(-p_1 - p_2,\ p_1 - p_3), \ldots, (-p_2 + p_4 - 1,\ -p_3 - p_4 + 1)$ in Q_1 and Q_2 and
  - $(-p_2 + p_4,\ -p_3 - p_4), \ldots, (p_4 - 1,\ -p_2 - p_3 - p_4 + 1)$ in Q_1.

  That is in total $p_1 + 2p_2 + p_3 + p_4 - 1$ more points.
- p_3: We get the following new points:
  - $(-p_1 - p_2 - p_3 + 1,\ p_1 - 1), \ldots, (-p_1 - p_2,\ p_1 - p_3)$ in Q_2,
  - $(-p_1 - p_2 + 1,\ p_1 - p_3 - 1), \ldots, (-p_2 + p_4,\ -p_3 - p_4)$ in Q_1 and Q_2,
  - $(-p_2 + p_4 + 1,\ -p_3 - p_4 - 1), \ldots, (p_4 - 1,\ -p_2 - p_3 - p_4 + 1)$ in Q_1, and
  - $(p_4,\ -p_3 - p_4), \ldots, (p_3 + p_4 - 1,\ -p_3 - p_4)$ in Q_2.

  That is in total $p_1 + p_2 + 2p_3 + p_4 - 1$ more points.
- p_4: We get the following new points:

– $(p_4,\ -p_3-p_4)$ in both Q_1 and Q_2,
– $(p_4,\ -p_3-p_4-1),\dots,(p_4,\ -p_2-p_3-p_4+1)$ in Q_1, and
– $(p_4+1,\ -p_3-p_4),\dots,(p_3+p_4-1,\ -p_3-p_4)$ in Q_2.

That is in total p_2+p_3-1 more points.

Proof. The proof is a simple calculation. □

Now, we can see how the immaculate line bundles of type (A) from Proposition 2.55 behave.

Lemma 2.60. If $p \le p'$ componentwise, then the immaculate line bundles of type (A) for p are contained in the line bundles of type (A) for p'.

Proof. Since if $y \le -1$ the value of $x_1(y) = -1$ and if $y \ge 0$ $x_1(y) = y-1$, we will only have to investigate $x_0(y)$ or $\#I_y$. If only p_0 varies, then it is obvious that the claim holds, since $\#I_y = p_0 - 1 + *$ with $*(y)$ being a positive integer, and the intervals staying the same.

For p_2 and p_3 it is also easy, since it only makes the intervals J_1 respectively J_5 longer, but inside of the others nothing is changed - also not the number of line bundles.

So the interesting cases are those, when p_1 and p_4 vary.

Showing that $\mathcal{M}(\mathcal{R})(p') \subset \mathcal{M}(\mathcal{R})(p)$ for $\mathcal{R} \in \{\emptyset, \mathrm{P}_2^c, \mathrm{P}_4, \mathrm{P}_0\}$ implies that $\mathrm{Imm}_A(p) \subset \mathrm{Imm}_A(p^{'})$

Remember that the maculate regions are shifted cones. The cones do not depend on p, but the vertices v_i do.

- $\emptyset$:

  The vertex $v_\emptyset = 0$, so it does not depend on p and it is clear that $\mathcal{M}(\mathcal{R})(p') = \mathcal{M}(\mathcal{R})(p)$

- P_2^c:

  The vertex $v_2^c(p) = (-p_0-p_1,\ -p_1+p_4,\ p_1-p_4)$, so $v_2^c(p') = (-p_0'-p_1',\ -p_1'+p_4',\ p_1'-p_4') = (-p_0-p_1-d_0-d_1,\ -p_1+p_4-d_1+d_4,\ p_1-p_4+d_1-d_4) = v_2^c(p)+(d_0+d_1)\cdot(-1,\ -1,\ 1)+(d_0+d_4)\cdot(0,\ 1,\ -1)$, where $d_i := p_i' - p_i \ge 0$ for $p' \ge p$ componentwise.

- P_0:

  The vertex $v_0(p) = (-p_0-p_4,\ -p_1,\ p_1)$ and $v_0(p') = v_0(p) + (d_0+d_4)\cdot(-1,\ 0,\ 0) + d_1 \cdot (0,\ -1,\ 1)$.

- P_4:

  The vertex $v_4(p) = (-p_0,\ p_4,\ -p_4)$ and $v_4(p') = v_4(p)+(d_0+d_4)\cdot(-1,\ 0,\ 0)+d_4\cdot(1,\ 1,\ -1)$.

□

Let us look at this in greater detail:

Lemma 2.61.
- When increasing p_0 by one we obtain $p_1+p_2+p_3+p_4-1$ more immaculate line bundles of type (A), namely
  (i) for $y \in [-p_1-p_2+1, -p_1]$: $(-p_0-p_1,\ y,\ -y)$,
  (ii) for $y \in [-p_1+1, -p_1+p_4-1]$: $(-p_0-p_4+y,\ y,\ -y)$,
  (iii) for $y \in [-p_1+p_4, p_4-1]$: $(-p_0-p_1,\ y,\ -y)$,

(iv) and for $y \in [p_4, p_3 + p_4 - 1]$: $(-p_0 - p_4 + y,\ y,\ -y)$.

- p_1: There are $p_0 + 2p_1 + p_2 + p_4 - 1$ more line bundles of type (A), namely
  (i) $([-p_0 - p_1, -1],\ -p_1 - p_2,\ p_1 + p_2)$,
  (ii) for $y \in [-p_1 - p_2 + 1, -p_1 - 1]$: $(-p_0 - p_1,\ y,\ -y)$,
  (iii) $([-p_0 - p_1 - p_4 + 1, -p_0 - p_1],\ -p_1,\ p_1)$,
  (iv) and for $y \in [-p_1 + p_4, p_4 - 1]$: $(-p_0 - p_1,\ y,\ -y)$.
- p_2: There are $p_0 + p_1 - 1$ more line bundles of type (A), namely $([-p_0 - p_1 + 1, -1],\ -p_1 - p_2,\ p_1 + p_2)$.
- p_3: There are $p_0 + p_4 - 1$ more line bundles of type (A), namely $([-p_0 + p_3 + 1, p_3 + p_4 - 1],\ p_3 + p_4,\ -p_3 - p_4)$.
- p_4: There are $p_0 + p_1 + p_3 + 2p_4 - 1$ more line bundles of type (A), namely
  (i) for $y \in [-p_1 + 1, -p_1 + p_4]$: $(-p_0 - p_4 + y,\ y,\ -y)$,
  (ii) $([-p_0 - p_1 + 1, -p_0],\ p_4,\ -p_4)$,
  (iii) for $y \in [p_4 + 1, p_3 + p_4 - 1]$: $(-p_0 - p_4 + y,\ y,\ -y)$,
  (iv) and $([-p_0 + p_3, p_3 + p_4 - 1],\ p_3 + p_4,\ -p_3 - p_4)$.

Proof.
- p_0: When changing p_0, the bounds for y are not changed. Also the upper bound $x_1(y)$ stays the same. Only the lower bound $x_0(y)$ is different. $x_0(y) = -p_0 + *(y)$, so for $p_0 + 1$ we obtain $x_0'(y) = x_0(y) - 1$.

  So in each line segment we have one line bundle more and there are $p_1 + p_2 + p_3 + p_4 - 1$ line segments.

  The new line bundles are:
  (i) for $y \in [-p_1 - p_2 + 1, -p_1]$: $(-p_0 - p_1,\ y,\ -y)$,
  (ii) for $y \in [-p_1 + 1, -p_1 + p_4 - 1]$: $(-p_0 - p_4 + y,\ y,\ -y)$,
  (iii) for $y \in [-p_1 + p_4, p_4 - 1]$: $(-p_0 - p_1,\ y,\ -y)$,
  (iv) and for $y \in [p_4, p_3 + p_4 - 1]$: $(-p_0 - p_4 + y,\ y,\ -y)$.

- p_1:
  (i) $p_1 > p_4$:

	$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_0'(y)$	$x_1(y) = x_1'(y)$	$\#I_y$	$\#I_y'$	$\#I_y' - \#I_y$
1	$-p_1 - p_2$		0	$-p_0 - p_1$	-1	0	$p_0 + p_1$	$p_0 + p_1$
2	$-p_1 - p_2 + 1$	$-p_1 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
3	$-p_1$		$-p_0 - p_1 + 1$	$-p_0 - p_1 - p_4 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 + p_4 - 1$	p_4
4	$-p_1 + 1$	$-p_1 + p_4 - 2$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y - 1$	0
5	$-p_1 + p_4 - 1$		$-p_0 - p_1$	$-p_0 - p_1$	-1	$p_0 + p_1$	$p_0 + p_1$	0
6	$-p_1 + p_4$	-1	$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
7	0	$p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	$y - 1$	$p_0 + p_1 + y - 1$	$p_0 + p_1 + y$	1
8	p_4	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4 - 1$	0

  (ii) $p_1 = p_4$:

	$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_0'(y)$	$x_1(y)$	$\#I_y$	$\#I_y'$	$\#I_y' - \#I_y$
1	$-p_1 - p_2$		0	$-p_0 - p_1$	-1	0	$p_0 + p_1$	$p_0 + p_1$
2	$-p_1 - p_2 + 1$	$-p_1 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
3	$-p_1$		$-p_0 - p_1 + 1$	$-p_0 - p_1 - p_4 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 + p_4 - 1$	p_4
4	$-p_1 + 1$	-2	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y - 1$	0
5	-1		$-p_0 - p_1$	$-p_0 - p_1$	-1	$p_0 + p_1$	$p_0 + p_1$	0
6	0	$p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	$y - 1$	$p_0 + p_1 + y - 1$	$p_0 + p_1 + y$	1
7	p_4	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4 - 1$	0

(iii) $p_1 < p_4$:

a) $p_1 + 1 < p_4$:

	$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_0'(y)$	$x_1(y) = x_1'(y)$	$\#I_y$	$\#I_y'$	$\#I_y' - \#I_y$
1	$-p_1 - p_2$		0	$-p_0 - p_1$	-1	0	$p_0 + p_1$	$p_0 + p_1$
2	$-p_1 - p_2 + 1$	$-p_1 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
3	$-p_1$		$-p_0 - p_1 + 1$	$-p_0 - p_1 - p_4 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 + p_4 - 1$	p_4
4	$-p_1 + 1$	-1	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y - 1$	0
5	0	$-p_1 + p_4 - 2$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_1$	$p_0 + p_1$	0
6	$-p_1 + p_4 - 1$		$-p_0 - p_1$	$-p_0 - p_1$	$y - 1$	$p_0 + p_1 - 1$	$p_0 + p_1$	0
7	$-p_1 + p_4$	$p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	$y - 1$	$p_0 + p_1 + y - 1$	$p_0 + p_1 + y$	1
8	p_4	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4 - 1$	0

b) $p_1 + 1 = p_4$:

	$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_0'(y)$	$x_1(y) = x_1'(y)$	$\#I_y$	$\#I_y'$	$\#I_y' - \#I_y$
1	$-p_1 - p_2$		0	$-p_0 - p_1$	-1	0	$p_0 + p_1$	$p_0 + p_1$
2	$-p_1 - p_2 + 1$	$-p_1 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
3	$-p_1$		$-p_0 - p_1 + 1$	$-p_0 - p_1 - p_4 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 + p_4 - 1$	p_4
4	$-p_1 + 1$	$-p_1 + p_4 - 2 = -1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y - 1$	0
5	$-p_1 + p_4 - 1 = 0$		$-p_0 - p_1$	$-p_0 - p_1$	$y - 1$	$p_0 + p_1$	$p_0 + p_1$	0
6	$-p_1 + p_4 = 1$	$p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1$	$y - 1$	$p_0 + p_1 - 1$	$p_0 + p_1$	1
7	p_4	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y + 1$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4 - 1$	0

In all cases we have in total $p_0 + 2p_1 + p_2 + p_4 - 1$ more immaculate line bundles than before. Those are

(i) $([-p_0 - p_1, -1],\ -p_1 - p_2,\ p_1 + p_2)$,

(ii) for $y \in [-p_1 - p_2 + 1, -p_1 - 1]$: $(-p_0 - p_1,\ y,\ -y)$,

(iii) $([-p_0 - p_1 - p_4 + 1, -p_0 - p_1],\ -p_1,\ p_1)$,

(iv) and for $y \in [-p_1 + p_4, p_4 - 1]$: $(-p_0 - p_1,\ y,\ -y)$.

- p_2: When p_2 is increased by one, we get one more line segment of line bundles. This is the line segment $([-p_0 - p_1 + 1, -1],\ -p_1 - p_2,\ p_1 + p_2)$. This are $p_0 + p_1 - 1$ more line bundles of type A.

- p_3: By increasing p_3 by one, we obtain also one more line segment. It is at the other end of the type A points, with the following coordinates: $([-p_0 + p_3 + 1, p_3 + p_4 - 1],\ p_3 + p_4,\ -p_3 - p_4)$. This are $p_0 + p_4 - 1$ more line bundles of type A.

- p_4:

(i) $p_1 > p_4$:

a) $p_1 = p_4 + 1$:

	$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_0'(y)$	$x_1(y) = x_1'(y)$	$\#I_y$	$\#I_y'$	$\#I_y' - \#I_y$
1	$-p_1 - p_2 + 1$	$-p_1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 - 1$	0
2	$-p_1 + 1$	$-p_1 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y$	1
3	$-p_1 + p_4 = -1$		$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
4	0	$-p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	$y - 1$	$p_0 + p_1 + y - 1$	$p_0 + p_1 + y - 1$	0
5	p_4		$-p_0 + 1$	$-p_0 - p_1 + 1$	$p_4 - 1$	$p_0 + p_4 - 1$	$p_0 + p_1 + p_4 - 1$	p_1
6	$p_4 + 1$	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4$	1
7	$p_3 + p_4$		$p_3 + p_4$	$-p_0 + p_3$	$p_3 + p_4 - 1$	0	$p_0 + p_4$	$p_0 + p_4$

b) $p_1 > p_4$:

	$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_0'(y)$	$x_1(y) = x_1'(y)$	$\#I_y$	$\#I_y'$	$\#I_y' - \#I_y$
1	$-p_1 - p_2 + 1$	$-p_1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 - 1$	0
2	$-p_1 + 1$	$-p_1 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y$	1
3	$-p_1 + p_4$		$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
4	$-p_1 + p_4 + 1$	-1	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 - 1$	0
5	0	$-p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	$y - 1$	$p_0 + p_1 + y - 1$	$p_0 + p_1 + y - 1$	0
6	p_4		$-p_0 + 1$	$-p_0 - p_1 + 1$	$p_4 - 1$	$p_0 + p_4 - 1$	$p_0 + p_1 + p_4 - 1$	p_1
7	$p_4 + 1$	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4$	1
8	$p_3 + p_4$		$p_3 + p_4$	$-p_0 + p_3$	$p_3 + p_4 - 1$	0	$p_0 + p_4$	$p_0 + p_4$

(ii) $p_1 = p_4$:

	$\bullet \leq y$	$y \leq \bullet$	$x_0(y)$	$x'_0(y)$	$x_1(y) = x'_1(y)$	$\#I_y$	$\#I'_y$	$\#I'_y - \#I_y$
1	$-p_1 - p_2 + 1$	$-p_1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 - 1$	0
2	$-p_1 + 1$	-1	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y$	1
3	0		$-p_0 - p_1 + 1$	$-p_0 - p_1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1$	1
4	1	$-p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	$y - 1$	$p_0 + p_1 + y - 1$	$p_0 + p_1 + y - 1$	0
5	p_4		$-p_0 + 1$	$-p_0 - p_1 + 1$	$p_4 - 1$	$p_0 + p_4 - 1$	$p_0 + p_1 + p_4 - 1$	p_1
6	$p_4 + 1$	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4$	1
7	$p_3 + p_4$		$p_3 + p_4$	$-p_0 + p_3$	$p_3 + p_4 - 1$	0	$p_0 + p_4$	$p_0 + p_4$

(iii) $p_1 < p_4$:

	$\bullet \leq y$	$y \leq \bullet$	$x_0(y)$	$x'_0(y)$	$x_1(y) = x'_1(y)$	$\#I_y$	$\#I'_y$	$\#I'_y - \#I_y$
1	$-p_1 - p_2 + 1$	$-p_1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	-1	$p_0 + p_1 - 1$	$p_0 + p_1 - 1$	0
2	$-p_1 + 1$	-1	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	-1	$p_0 + p_4 - y - 1$	$p_0 + p_4 - y$	1
3	0	$-p_1 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4$	1
4	$-p_1 + p_4$		$-p_0 - p_1 + 1$	$-p_0 - p_4 + y$	$-p_1 + p_4 - 1$	$p_0 + p_4 - 1$	$p_0 + p_4$	1
5	$-p_1 + p_4 + 1$	$-p_4 - 1$	$-p_0 - p_1 + 1$	$-p_0 - p_1 + 1$	$y - 1$	$p_0 + p_1 + y - 1$	$p_0 + p_1 + y - 1$	0
6	p_4		$-p_0 + 1$	$-p_0 - p_1 + 1$	$p_4 - 1$	$p_0 + p_4 - 1$	$p_0 + p_1 + p_4 - 1$	p_1
7	$p_4 + 1$	$p_3 + p_4 - 1$	$-p_0 - p_4 + y + 1$	$-p_0 - p_4 + y$	$y - 1$	$p_0 + p_4 - 1$	$p_0 + p_4$	1
8	$p_3 + p_4$		$p_3 + p_4$	$-p_0 + p_3$	$p_3 + p_4 - 1$	0	$p_0 + p_4$	$p_0 + p_4$

Here we have in total $p_0 + p_1 + p_3 + 2p_4 - 1$ more immaculate line bundles than before.

Those are exactly:

(i) for $y \in [-p_1 + 1, -p_1 + p_4]$: $(-p_0 - p_4 + y,\ y,\ -y)$,

(ii) $([-p_0 - p_1 + 1, -p_0],\ p_4,\ -p_4)$,

(iii) for $y \in [p_4 + 1, p_3 + p_4 - 1]$: $(-p_0 - p_4 + y,\ y,\ -y)$,

(iv) and $([-p_0 + p_3, p_3 + p_4 - 1],\ p_3 + p_4,\ -p_3 - p_4)$.

□

2.5.5 Vanishing parameters

When the parameters b and c from Proposition 2.47 are set to zero, we obtain an additional class of lines of immaculate line bundles. The lines in this class are in y-direction. Let us have a look at the situation in this set-up.

The map $\pi : \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X)$ simplifies to

$$\pi = \begin{pmatrix} \underline{1} & \underline{1} & \underline{0} & -\underline{1} & \underline{0} \\ \underline{0} & \underline{1} & \underline{1} & \underline{0} & -\underline{1} \\ \underline{0} & -\underline{1} & \underline{0} & \underline{1} & \underline{1} \end{pmatrix}.$$

Proposition 2.62 (Lines in y-direction)**.** Let $X = \mathbb{TV}(\Sigma)$ be a projective toric variety of Picard rank 3 with exactly 5 primitive collections, and let b and c from Proposition 2.47 be zero. Then the line bundles $(x,\ *,\ z)$ are immaculate for $(x,\ z) \in \tilde{Q}_1 \cup \tilde{Q}_2$, with

$$\tilde{Q}_1 = \mathrm{conv}\begin{pmatrix} (p_3 - p_0,\ -p_3 - p_4 + 1),\ (p_3 + p_4 - 2,\ -p_3 - p_4 + 1), \\ (-p_0 - p_1 - p_4 + 2,\ p_1 - 1),\ (-p_1,\ p_1 - 1) \end{pmatrix}$$

and

$$\tilde{Q}_2 = \mathrm{conv}\begin{pmatrix} (p_0 - p_1 + 1,\ p_1 - p_4),\ (p_0 - p_1 + 1,\ p_0 + p_1 - 2), \\ (p_3 - 1,\ -p_0 - p_3 - p_4 + 2),\ (p_3 - 1,\ -p_3) \end{pmatrix}.$$

Table 2.3: The maculate regions for $b = 0, c = 0$

$\mathcal{R}$	vertex	rays / Hilbert basis
$\emptyset$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{matrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{matrix}$
P_0	$\begin{pmatrix} -p_0 - p_1 \\ -p_1 \\ p_1 \end{pmatrix}$	$\begin{matrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}$
P_1	$\begin{pmatrix} -p_1 \\ -p_1 - p_2 \\ p_1 \end{pmatrix}$	$\begin{matrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$
P_2	$\begin{pmatrix} p_3 \\ -p_2 \\ -p_3 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{matrix}$
P_3	$\begin{pmatrix} p_3 \\ p_4 \\ -p_3 - p_4 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{matrix}$
P_4	$\begin{pmatrix} -p_0 \\ p_4 \\ -p_4 \end{pmatrix}$	$\begin{matrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}$

Proof. We look at the projection of the maculate regions $\mathcal{M}(\mathcal{R})$ in Table 2.3 to the $(x,\ z)$-plane. Remember that the maculate region can be written as $\mathcal{M}(\mathcal{R}) = v_{\mathcal{R}} + \sigma_{\mathcal{R}}$, and $\mathcal{M}(\mathcal{R}^c) = v_{\mathcal{R}^c} - \sigma_{\mathcal{R}}$. Denote the projections by $\overline{v_{\mathcal{R}}}$ and $\overline{\sigma_{\mathcal{R}}}$.

$\mathcal{R}$	$\overline{v_{\mathcal{R}}}$	$\overline{v_{\mathcal{R}^c}}$	rays of $\overline{\sigma_{\mathcal{R}}}$
$\emptyset$	$(0,\ 0)$	$(-p_0 - p_1 + p_3,\ p_1 - p_3 - p_4)$	$(1,\ -1), (0,\ 1)$
P_0	$(-p_0 - p_1,\ p_1)$	$(p_3,\ -p_3 - p_4)$	$(-1,\ 0), (0,\ 1)$
P_1	$(-p_1,\ p_1)$	$(-p_0 + p_3,\ -p_3 - p_4)$	$(1,\ 0), (-1,\ 1)$
P_2	$(p_3,\ -p_3)$	$(-p_0 - p_1,\ p_1 - p_4)$	$(1,\ -1), (0,\ 1)$
P_3	$(p_3,\ -p_3 - p_4)$	$(-p_0 - p_1,\ p_1)$	$(1,\ 0), (0,\ -1)$
P_4	$(-p_0,\ -p_4)$	$(-p_1 + p_3,\ p_1 - p_3)$	$(-1,\ 0), \pm(1,\ -1)$

We see that $\overline{\mathcal{M}(\mathrm{P}_0)} = \overline{\mathcal{M}(\mathrm{P}_3^c)}$ and with $v_2 = v_\emptyset + p_3 \cdot (1,\ -1)$ and $v_\emptyset = v_{4^c} + (p_1 - p_3) \cdot (1,\ -1)$, we get the inclusions $\overline{\mathcal{M}(\mathrm{P}_2)} \subset \overline{\mathcal{M}(\emptyset)} \subset \overline{\mathcal{M}(\mathrm{P}_4^c)}$. Since $v_1 = v_{4^c} + p_3 \cdot (-1,\ 1)$, it also holds that $\overline{\mathcal{M}(\mathrm{P}_1)} \subset \overline{\mathcal{M}(\mathrm{P}_4^c)}$,

In conclusion, the line through D is immaculate if and only if

$$\overline{D} \in \mathbb{Z}^2 \setminus \left(\overline{\mathcal{M}(\mathrm{P}_0)} \cup \overline{\mathcal{M}(\mathrm{P}_0^c)} \cup \overline{\mathcal{M}(\mathrm{P}_4)} \cup \overline{\mathcal{M}(\mathrm{P}_4^c)} \right).$$

□

Remark 2.63. There are two (and a half) special cases.

- $p_0 = 1$: In this case $\tilde{Q}_2 \subset \tilde{Q}_1$ and the simplified vertices of $\tilde{Q}_1$ are

  $(p_3-1,\ -p_3-p_4+1),\ (p_3+p_4-2,\ -p_3-p_4+1),\ (-p_1-p_4+1,\ p_1-1)$ and $(-p_1,\ p_1-1)$.

- $p_4 = 1$: In this case $\tilde{Q}_1 \subset \tilde{Q}_2$ and the simplified vertices of $\tilde{Q}_2$ are

  $(p_0-p_1+1,\ p_1-1),\ (p_0-p_1+1,\ p_0+p_1-2),\ (p_3-1,\ -p_0-p_3+1)$ and $(p_3-1,\ -p_3)$.

- $p_0 = p_4 = 1$: In this case $\tilde{Q}_1 \subset \tilde{Q}_2 \subset \tilde{Q}_1$, thus we have a segment with vertices

$$(p_3-1,\ -p_3),\ (-p_1,\ p_1-1).$$

To determine all immaculate line bundles one could proceed in a similar manner as in the previous section. For specific examples, that is for fixed values of $p_0, \ldots, p_4$, there are `polymake` scripts in the GitHub repository that can be used to calculate the immaculate locus. In the next section we are going to speak about the computational aspects.

2.6 Computational aspects

In this section we want to highlight the computational advantages of immaculate line bundles and maculate regions. All of these objects and conditions give rise to nice combinatorial algorithms. Throughout the development of this paper we have implemented these as a `polymake` [GJ00] extension. The combinatorial nature of these algorithms makes them very fast, as opposed to many algorithms from commutative algebra. This stresses the main computational advantage of working with toric varieties. We will give a short sketch of the resulting algorithms. The `polymake` extension itself can be found at `https://github.com/lkastner/immaculatePolymake`.

Table 2.4: Lines of immaculate line bundles for the hexagon

unbounded direction	basepoint
1 1 0 0	0 0 -1 -1
	1 0 -1 0
	0 0 -1 0
	-1 0 -1 -1
1 0 1 1	0 -1 -1 0
	0 -1 0 0
	-1 -1 0 0
	-1 -1 -1 0
0 1 1 0	-1 0 0 0
	-1 0 1 0
	-1 0 0 -1
	-1 0 -1 -1

To compute the immaculate locus of a projective toric variety $\mathbb{TV}(\Sigma)$, we need to find the tempting $\mathcal{R} \subseteq \Sigma(1)$. For this we need to check if $V^{>}(\mathcal{R})$ is $\mathbf{k}$-acyclic. The easiest way is to brute force this by checking any subset of rays and then compute the homology. The

Table 2.5: Isolated immaculate line bundles for the hexagon

Pic(X) coordinates			
-2	-2	-2	-2
-2	-2	-2	0
0	0	0	-1
0	0	0	1

homology computation is already built in `polymake` and many other software frameworks for combinatorial software as well. One can also imagine a more sophisticated approach by considering sub-diagrams of the Hasse diagram of Σ or using the different characterizations obtained in subsection 2.2.2. So far this has never been a bottleneck in our examples, though in case this happens, results of subsection 2.2.2 might be of use.

From the collection of all tempting $\mathcal{R}$ we can finally compute the immaculate locus $\mathrm{Imm}_{\mathbb{R}}(X)$, or rather the lattice points thereof. We only need to compute the intersection of all complements of the $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$. It is not difficult to see that this is a union of polyhedra. Since $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ is a rational polyhedral cone, we can write it as a finite intersection of halfspaces. Taking the complement of this cone means taking the union of the complementary halfspaces. Since we are only interested in the lattice points of $\mathrm{Imm}_{\mathbb{R}}(X)$, we just move the bounding hyperplane by one away from $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ and do not worry about openness of the complement. Now we get the polyhedra giving the lattice points of $\mathrm{Imm}_{\mathbb{R}}(X)$ by picking one complementary halfspace for every $\mathcal{R}$ and then intersecting these. Consider any possible combination and take the union of the resulting polyhedra.

We now restrict our attention to the hexagon example (see Examples 2.1 and 2.5). We immediately see that the main bottleneck of the algorithm for $\mathrm{Imm}_{\mathbb{R}}(X)$ is the amount of intersections to compute. There are 34 tempting $\mathcal{R}$'s and if every $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ was bounded by only two hyperplanes, we would have to compute 2^{34} intersections. In fact, all $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ are actually bounded by more than two hyperplanes. This issue can be overcome by building the intersections step by step and eliminating trivial intersections in between. We start by building the complementary halfspaces of $\mathcal{M}_{\mathbb{R}}(\mathcal{R}_1)$ and $\mathcal{M}_{\mathbb{R}}(\mathcal{R}_2)$, then we consider any intersection. If an intersection is empty already, we eliminate it. Furthermore, we choose the inclusion maximal intersections. Then we intersect the resulting polyhedra with the complementary halfspaces of $\mathcal{M}_{\mathbb{R}}(\mathcal{R}_3)$ and so on.

Thus we have computed the immaculate loci $\mathrm{Imm}_{\mathbb{Z}}(X) = \mathrm{Imm}_{\mathbb{R}}(X)$. They are equal to a union of three unbounded polyhedra and four isolated lattice points that are listed in Table 2.5. Each unbounded polyhedron consists of four parallel lines, that is lattice lines. The exact lines are collected in Table 2.4. Each pair of quadruples of lines intersects in four points.

Now it is easy to compute all exceptional sequences that are contained in the projection of the cube $\pi([-1,0]^6) \subseteq \mathrm{Cl}(X)$. One just collects the lattice points in the projected cube and then runs a depth first search. There are 228 exceptional sequences of length six in the projected cube. Under the group action on the hexagon these 228 exceptional sequences correspond to 19 orbits of size 12. In Table 2.6 we list one representative from each orbit. Note that we do not need to use the four isolated points for these exceptional sequences. This is different than in the case of the splitting fans, for example the Picard rank 2 case (see [CM04]).

Table 2.6: Exceptional sequences of line bundles for the hexagon

$$D^0 = [0, 0, 0, 0]$$

D^1				D^2				D^3				D^4				D^5			
-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1	-2	-2	-1	0	-1	-1	-2	-1
-1	-1	-1	-1	-2	-2	-1	-1	-1	-1	-2	-1	-2	-1	-2	-2	-1	-2	-2	-1
-1	-1	-1	-1	-2	-1	-1	-1	-2	-2	-1	-1	-1	-1	-2	-1	-2	-1	-2	-2
-1	-1	-1	-1	-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1	-1	-1	-2	-1
-1	-1	-1	-1	-1	-1	-1	0	-2	-1	-1	-1	-1	-2	-1	0	-1	-1	-2	-1
-1	-1	0	0	-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1	-2	-2	-1	0
-1	-1	0	0	-1	-1	-1	-1	-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1
-1	-1	0	0	-1	-1	-1	-1	-1	-1	-1	0	-2	-1	-1	-1	-1	-2	-1	0
-1	-1	0	0	-1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	-2	-1	-1	-1
-1	-1	0	0	-1	0	-1	-1	0	-1	-1	0	-1	-1	-1	-1	-1	-1	-1	0
-1	-1	0	0	0	0	-1	-1	-1	0	-1	-1	0	-1	-1	0	-1	-1	-1	-1
-1	-1	0	0	0	0	-1	-1	0	0	-1	0	-1	0	-1	-1	0	-1	-1	0
-1	0	0	-1	-1	-1	-1	-1	-2	-1	-1	-1	-2	-2	-1	-1	-2	-1	-2	-2
-1	0	0	-1	-1	-1	0	0	-1	-1	-1	-1	-2	-1	-1	-1	-2	-2	-1	-1
-1	0	0	-1	-1	-1	0	0	-1	0	-1	-1	-1	-1	-1	-1	-2	-1	-1	-1
-1	0	0	-1	-1	-1	0	0	0	0	-1	-1	-1	0	-1	-1	-1	-1	-1	-1
-1	0	0	-1	-1	0	0	0	-1	-1	0	0	-1	0	-1	-1	-2	-1	-1	-1
-1	0	0	-1	0	-1	0	0	-1	-1	0	0	-1	-1	-1	-1	-2	-2	-1	-1
-1	0	0	-1	0	-1	0	0	-1	-1	0	0	0	0	-1	-1	-1	-1	-1	-1

Canonical Hilbert-Burch Matrices

The following chapter is on canonical Hilbert-Burch matrices. The first part of this chapter (in particular section 3.1 – section 3.4) is adapted from the paper “Canonical Hilbert-Burch matrices for power series” [HW21] by Roser Homs and me that was published in “Journal of Algebra”. The publication is available at https://doi.org/10.1016/j.jalgebra.2021.04.021. The results of section 3.5 and a shorter version of section 3.6 will be published in the follow-up paper “Deformations of local Artin rings via Hilbert-Burch matrices” [HW23] by Roser Homs and me that is accepted for publication in the proceedings of “Grenoble Deformations AMS-SMF-EMS, meeting 18-22 July, 2022.” in “Contemporary Mathematics (CONM)”. A preprint version of this paper is available at https://arxiv.org/abs/2309.06871.

The project started as part of my PhD thesis with a visit in Genova at Maria Evilina Rossi’s working group suggested by Alexandru Constantinescu. During the stay I had some preliminary results and conjectures that I continued to work on, but then left the project for a while. Roser Homs made a research stay later on with Maria Evilina Rossi’s working group and was suggested to look into my preliminary results, since they were relevant for her research concerning Gorenstein rings. She continued to work on it and used some of the results in her thesis, [Hom19]. After a while, we continued to work on the project together, resulting in [HW21]. In the process of writing the paper we had a very intense collaboration, rethinking/rechecking many of the results, and writing and rewriting many parts of the paper. This makes it not possible to attribute every single part of the paper to only one of us. The structure and presentation of the first part of the paper [HW21] is similar to [Hom19]. However, one can say that the observation that for ideals in the power series ring with lex-segment leading term ideal the reduced $\overline{\text{lex}}$-enhanced standard basis is a Gröbner basis with respect to the lexicographic order and the idea to use this for the parametrization of the Gröbner cell was by me. The application to the construction of Gorenstein rings [HW21, section 6] was done by Roser Homs and is not presented in this thesis.

In 2022 we decided to continue to work on the topic resulting in [HW23]. The ideas and concepts were mostly established while working together, making it difficult to attribute them to only one of us. The writing of section 3 of that paper was done mostly by Roser Homs and the writing of section 4 mostly by me.

The following list contains a detailed comparison between the chapter and the papers [HW21; HW23]:

- section 3.1 slightly adapted from [HW21, section 2] expanded Theorem 3.4, added Example 3.5 and smaller changes
- section 3.2 is slightly adapted from [HW21, section 3], added Example 3.14
- section 3.3 is [HW21, section 4]

- section 3.4 is adapted from [HW21, section 5]
  - changed the proof of Proposition 3.29, by adding Definition 3.30 and Lemma 3.31
  - adapted Remark 3.33
  - added Definition 3.35 and adapted the statements and notations in the rest of the chapter accordingly
  - adapted the proof of Lemma 3.36
  - added Remark 3.37
  - small correction in Corollary 3.39
  - other example in Example 3.41
  - correction in Example 3.43 and added a remark about Betti numbers
  - changed the paragraph before Lemma 3.44
  - added Lemma 3.46 (this is a version of [HW23, Proposition 3.11, Lemma 4.3]) + paragraph afterwards,
  - added Remark 3.49,
- section 3.5 contains some results of [HW23, section 3]
- section 3.6 is a longer version of [HW23, section 4]

3.1 Parametrization of ideals in $\mathbf{k}[x, y]$

In the present section we review the parametrization of Gröbner cells in $P = \mathbf{k}[x, y]$ in terms of Hilbert-Burch matrices given by Conca-Valla in [CV08] and Constantinescu in [Con11]. Let $\mathbf{k}$ be an arbitrary field. Given a term ordering τ on a polynomial ring P over $\mathbf{k}$ and an ideal $I \subset P$, the leading term ideal of I is defined as follows.

Definition 3.1. The *leading term ideal* $\mathrm{Lt}_\tau(I)$ of the ideal $I \subset P$ with respect to a term ordering τ in the polynomial ring P is the monomial ideal generated by all leading terms of elements in I, i.e. $\mathrm{Lt}_\tau(I) = (\mathrm{Lt}_\tau(f) : f \in I)$.

A subset $\{f_0, \dots f_t\} \subset I$ such that $\mathrm{Lt}_\tau(I) = (\mathrm{Lt}_\tau(f_0), \dots, \mathrm{Lt}_\tau(f_t))$ is called *Gröbner basis* of I with respect to the term order τ.

In this section we will consider the lexicographical term ordering (lex) and the degree-lexicographical term ordering (deglex) on $P = \mathbf{k}[x, y]$. Recall that with the former we first compare the exponents of x of two monomials, whereas with the latter we first compare their degree. Note that in a polynomial ring in two variables the lexicographical term ordering is equivalent to the reverse lexicographical term ordering.

Consider a zero-dimensional monomial ideal E in P. By taking the smallest integer t such that $x^t \in E$ and the smallest integers m_i such that $x^{t-i}y^{m_i} \in E$ for any $1 \leq i \leq t$, we can always express such a monomial ideal as

$$E = (x^t, x^{t-1}y^{m_1}, \dots, x^{t-i}y^{m_i}, \dots, y^{m_t}), \tag{3.1}$$

where $0 = m_0 < m_1 \leq \cdots \leq m_t$ is an increasing sequence. If all the inequalities are strict, we call E a *lex-segment ideal*.

After fixing a term order τ, we can ask for all ideals I in P with leading term ideal E, that is the Gröbner cell of E with respect to τ, denoted by $V_\tau(E)$. Reduced Gröbner bases provide a parametrization of this set of ideals. However, explicitly describing such a parametrization is not always straightforward. In [CV08], Conca and Valla consider a different approach: instead of focusing on the generators of I, they study the relations or syzygies among the generators. A Hilbert-Burch matrix of the ideal I encodes these relations. Therefore, giving such a parametrization is equivalent to choosing a canonical Hilbert-Burch matrix for each ideal I.

Definition 3.2. Let E be the monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$. The *canonical Hilbert-Burch matrix* H of E is the Hilbert-Burch matrix of E of the form

$$H = \begin{pmatrix} y^{d_1} & 0 & \cdots & 0 \\ -x & y^{d_2} & \cdots & 0 \\ 0 & -x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & y^{d_t} \\ 0 & 0 & \cdots & -x \end{pmatrix},$$

where $d_i = m_i - m_{i-1}$ for any $1 \le i \le t$.

The *degree matrix* U of E is the $(t+1) \times t$ matrix with integer entries $u_{i,j} = m_j - m_{i-1} + i - j$, for $1 \le i \le t+1$ and $1 \le j \le t$.

It follows from the definition that $u_{i,i} = d_i$ and $u_{i+1,i} = 1$, for $1 \le i \le t$.

Conca-Valla parametrize the set $V_0(E) = V_{\mathrm{lex}}(E)$ of all ideals I in P that share the same zero-dimensional leading term ideal E with respect to the lexicographical term ordering. They give a set of matrices that deform the canonical Hilbert-Burch matrix of the monomial ideal E into Hilbert-Burch matrices of each I. We use the same notation as in [CV08].

Definition 3.3. We denote by $T_0(E)$ the set of matrices $N = (n_{i,j})$ of size $(t+1) \times t$ with entries in $\mathbf{k}[y]$ such that

- $n_{i,j} = 0$ for any $i < j$,
- $\deg(n_{i,j}) < d_j$ for any $i \ge j$.

The subset $T_2(E) \subset T_0(E)$ is defined as the set of matrices $N \in T_0(E)$ where the entries additionally satisfy that

- the diagonal entries $n_{i,i} = 0$ for all $i = 1, \ldots, t$
- the entry $n_{i,j}$ has no constant term for every $i = j+1, \ldots, k+1$ where $k = \max\{\nu : j \le \nu \le t \mid m_\nu = m_j\}$.

Notice that when E is a lex-segment ideal the second condition reduces to $n_{j+1,j}$ not having a constant term.

Theorem 3.4. *[CV08, Theorem 3.3] Given a zero-dimensional monomial ideal E in $P = \mathbf{k}[x, y]$ with canonical Hilbert-Burch matrix H, the map*

$$\begin{array}{rccc} \Phi : & T_0(E) & \to & V_0(E) \\ & N & \mapsto & I_t(H+N) \end{array}$$

is a bijection that restricts to a bijection

$$\Phi|_{T_2(E)}: \quad T_2(E) \quad \to \quad V_2(E)$$

on the subset of (x, y)-primary ideals $V_2(E) \subset V_0(E)$.

Additionally, the t-minors of $H + N$ form a Gröbner basis of $I_t(H + N)$ with respect to the lexicographic order.

This theorem allows us to define a canonical Hilbert-Burch matrix of any zero-dimensional ideal I of P as $H + \Phi^{-1}(I)$, where H is the canonical Hilbert-Burch matrix of the monomial ideal $\mathrm{Lt}_{\mathrm{lex}}(I)$.

Example 3.5. We will consider a small example with $n = 3$. Let $E = (x^2, xy, y^2)$, so $m = (1, 2)$, $d = (1, 1)$ and the canonical Hilbert-Burch matrix H of E as below. The set $T_0(E)$ consists of matrices with constant polynomials at each position, except for a zero at position $(1, 2)$. So $V_0(E)$ is a 5-dimensional affine space.

$$H = \begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix} \qquad N = \begin{pmatrix} c_1 & 0 \\ c_2 & c_3 \\ c_4 & c_5 \end{pmatrix}$$

The (x, y)-primary ideals among those in $V_0(E)$ – $V_2(E)$ – are parametrized by $T_2(E)$. The first condition of $T_2(E)$ implies $c_1 = c_3 = 0$ and the second condition that $c_2 = c_5 = 0$. Thus, $V_2(E)$ is a 1-dimensional affine space. The canonical Hilbert-Burch matrix of an ideal $I \in V_2(E)$ is

$$H + \Phi^{-1}(I) = \begin{pmatrix} y & 0 \\ -x & y \\ c & -x \end{pmatrix},$$

and $\{x^2 - cy, xy, y^2\}$ forms a lex-Gröbner basis of I.

In [Con11], Constantinescu parametrizes the Gröbner cell of lex-segment ideals E with respect to the degree-lexicographic term ordering

$$V_{\mathrm{deglex}}(E) = \{I \subset P : \mathrm{Lt}_{\mathrm{deglex}}(I) = E\}.$$

Definition 3.6. Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t}) \subset P$ be a monomial ideal, and $U = (u_{i,j})$ its associated degree matrix. Denote by $\mathcal{A}(E)$ the set of $(t + 1) \times t$ matrices $A = (a_{i,j})$ with entries in $\mathbf{k}[y]$ such that all its non-zero entries satisfy

$$\deg(a_{i,j}) \leq \begin{cases} \min(u_{i,j} - 1, d_i - 1), & i \leq j; \\ \min(u_{i,j}, d_j - 1), & i > j. \end{cases}$$

Theorem 3.7. *[Con11, Theorem 3.1] Given a zero-dimensional lex-segment ideal L in $P = \mathbf{k}[x, y]$ with canonical Hilbert-Burch matrix H, the map*

$$\begin{array}{rccc} \Phi: & \mathcal{A}(L) & \to & V_{\mathrm{deglex}}(L) \\ & A & \mapsto & I_t(H + A) \end{array}$$

is a bijection and the t-minors form a Gröbner basis of $I_t(H + A)$ with respect to deglex.

The proofs of well-definedness and surjectivity of Φ in [Con11] hold for any monomial ideal and, although the lex-segment hypothesis is needed in his proof of injectivity, the author conjectures that Φ is a proper parametrization in the general case.

3.2 From polynomials to power series

We are interested in a construction of Gröbner cells for $\mathfrak{m}$-primary ideals of $P = \mathbf{k}[x, y]$ in the same spirit as the ones presented in section 3.1 but compatible with the local structure in the sense that all ideals in the same Gröbner cell have the same Hilbert function. So from now on we will work in the ring of formal power series $R = \mathbf{k}[\![x, y]\!]$.

The goal of this section is to provide the necessary tools to extend the strategies in the proofs by Conca-Valla and Constantinescu to the local setting. In the first part, we define a local term ordering $\overline{\tau}$ and the notion of $\overline{\tau}$-enhanced standard basis, the local analogous to Gröbner basis and revisit results about the lifting of syzygies in the second subsection. Although our focus is on the polynomial and the power series ring with two variables, we will state those definitions and results for the general case of polynomial and power series rings in n variables.

3.2.1 Enhanced standard basis and Grauert's division

Definition 3.8. A term ordering τ in the polynomial ring $P = \mathbf{k}[x_1, \ldots, x_n]$ induces a reverse-degree ordering $\overline{\tau}$ in $R = \mathbf{k}[\![x_1, \ldots, x_n]\!]$ such that for any monomials M, M' in R,

$$M >_{\overline{\tau}} M' \quad \text{if and only if} \quad \begin{array}{ll} \deg(M) < \deg(M') & \text{or} \\ \deg(M) = \deg(M') \text{ and } M >_{\tau} M'. & \end{array}$$

We call $\overline{\tau}$ the *local term ordering* induced by the global term ordering τ.

Note that the local term orderings induced by the lexicographical and the degree lexicographical term orderings are the same.

Analogously to the notion of leading term ideal with respect to a global term ordering and Gröbner bases in the polynomial ring P, we define the leading term ideal with respect to a local term ordering and $\overline{\tau}$-enhanced standard bases.

Definition 3.9. Let $J \subset R$ be an ideal, we define the *leading term ideal* of J as the monomial ideal generated by the leading terms with respect to the local term ordering $\overline{\tau}$, i.e.

$$\mathrm{Lt}_{\overline{\tau}}(J) = (\mathrm{Lt}_{\overline{\tau}}(f) : f \in J) \subset P.$$

A subset $\{f_1, \ldots, f_m\}$ of J is a *$\overline{\tau}$-enhanced standard basis of J* if the leading terms of the elements generate the leading term ideal $\mathrm{Lt}_{\overline{\tau}}(J)$, i.e. $\mathrm{Lt}_{\overline{\tau}}(J) = (\mathrm{Lt}_{\overline{\tau}}(f_1), \ldots, \mathrm{Lt}_{\overline{\tau}}(f_m))$.

Definition 3.10. The *initial form* f^* of an element $f \in R$ is the homogeneous polynomial consisting of the terms in f of lowest degree – called the *order of* f, denoted by $\mathrm{ord}(f)$.

The *initial ideal* $J^* \subset P$ is the homogeneous ideal generated by the initial forms of elements in $J \subset R$.

Remark 3.11. By definition of local term ordering, the leading terms $\mathrm{Lt}_{\overline{\tau}}(f) = \mathrm{Lt}_{\tau}(f^*)$ agree. Therefore, $\mathrm{Lt}_{\overline{\tau}}(J) = \mathrm{Lt}_{\tau}(J^*)$. Let $\mathrm{HF}_{R/J} = h$ denote the Hilbert function of R/J. Then

$$\mathrm{HF}_{R/J} = \mathrm{HF}_{P/J^*} = \mathrm{HF}_{P/\,\mathrm{Lt}_{\tau}(J^*)} = \mathrm{HF}_{P/\,\mathrm{Lt}_{\overline{\tau}}(J)} = \mathrm{HF}_{P/\,\mathrm{Lex}(h)},$$

where $\mathrm{Lex}(h)$ is the unique lex-segment ideal with the same Hilbert function.

Remark 3.12. The term standard basis was first used by Hironaka in [Hir64, Definition 3] to refer to systems of generators of J such that their initial forms generate the initial ideal J^*. However, this terminology is not consistent in literature and in other sources standard basis refers to what we here define as $\overline{\tau}$-enhanced standard basis, e.g. [GP07]. The notation used in this paper is the same as in [Ber09]. Notice that a $\overline{\tau}$-enhanced standard basis of an ideal in particular forms a standard basis in the sense of [Hir64].

Example 3.13. *Comparison between leading terms w.r.t. global and local term orderings.* Consider the lex-segment ideal $L = (x^3, x^2y, xy^3, y^5) \subset \mathbf{k}[x, y]$, H its canonical Hilbert-Burch matrix and U its degree matrix from Definition 3.2:

$$H = \begin{pmatrix} y & 0 & 0 \\ -x & y^2 & 0 \\ 0 & -x & y^2 \\ 0 & 0 & -x \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}.$$

Consider the matrix $M = H + N$, where N is a 4×3 matrix with all zero entries except for 1 in the $(4, 3)$-entry. From Conca-Valla parametrization in Theorem 3.4, we know that $I = I_3(M) \subset \mathbf{k}[x, y]$ is an ideal in $V_0(L)$. Indeed, the maximal minors of M give the lex-Gröbner basis $\{x^3 - x^2, x^2y - xy, xy^3 - y^3, y^5\}$ of I and $\mathrm{Lt}_{\mathrm{lex}}(I) = L$.

However, the 3×3-minors of M are not a $\overline{\mathrm{lex}}$-enhanced standard basis of the ideal $J = I\mathbf{k}[\![x, y]\!]$, namely the extension of I in the power series ring. Since $N \notin T_2(L)$, the ideal I is not (x, y)-primary, so $J \cap \mathbf{k}[x, y] \neq I$. In fact, $J = (x^2, xy, y^3)$ is itself a monomial ideal. The reason why the leading term ideal changes when computed with respect to $\overline{\mathrm{lex}}$ is that $n_{4,3} = 1$ has a term of degree lower than $u_{4,3} = 1$. Finally, note that $\mathrm{Lt}_{\mathrm{lex}}(I) \neq \mathrm{Lt}_{\mathrm{lex}}(I^*) = \mathrm{Lt}_{\overline{\mathrm{lex}}}(J) = J$.

Example 3.14. *(x, y)-primary ideal with changing leading terms* In the previous example, the ideal of maximal minors was not (x, y)-primary, so we might have already suspected that the leading term ideals would be different. We continue Example 3.5 with $E = (x^2, xy, y^2)$. Remember that the canonical Hilbert-Burch matrix of an (x, y)-primary ideal I in $V_0(E)$ was

$$H + \Phi^{-1}(I) = \begin{pmatrix} y & 0 \\ -x & y \\ c & -x \end{pmatrix},$$

and the signed minors $\{x^2 - cy, xy, y^2\}$ form a lex-Gröbner basis of I. But if $c \neq 0$, $\mathrm{Lt}_{\overline{\tau}}(x^2 - cy) = y \notin E = \mathrm{Lt}_{\mathrm{lex}}(I)$, thence $\mathrm{Lt}_{\overline{\tau}}(I\mathbf{k}[\![x, y]\!]) \neq \mathrm{Lt}_{\mathrm{lex}}(I)$. The reason is again that an entry has a term of degree lower than the corresponding entry in the degree matrix of E,

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Buchberger division is the essential tool in Buchberger's algorithm to calculate Gröbner bases. In the power series ring it can be replaced by Grauert's division, see [Gra72]. Later on, Mora gave an analogous method to Buchberger's algorithm to calculate $\overline{\tau}$-enhanced standard bases in the local case: the tangent cone algorithm, see [Mor82]. We reproduce next a modern formulation of Grauert's division theorem in $\mathbf{k}[\![x_1, \ldots, x_n]\!]$ from [GP07, Theorem 6.4.1]:

Theorem 3.15. *[Grauert's Division Theorem] Let $f, f_0, \dots, f_t$ be in R. Then there exist $q_0, \dots, q_t, r \in R$ such that*

$$f = \sum_{i=0}^{t} q_i f_i + r$$

satisfying the following properties:

(i) No monomial of r is divisible by any $\mathrm{Lt}_{\overline{\tau}}(f_i)$, for $0 \le i \le t$.

(ii) If $q_i \neq 0$, $\mathrm{Lt}_{\overline{\tau}}(q_i f_i) \le_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(f)$.

These techniques can be used to extend results that are well-understood for graded algebras to the local case. In [ERV14], they have been successfully applied to characterize the Hilbert function of one dimensional quadratic complete intersections.

3.2.2 Lifting of syzygies in local rings

The connection between the lifting of syzygies and Gröbner bases has been widely studied in polynomial rings, see [KR00, Theorem 2.4.1]. Analogous results hold for rings of formal power series.

Let $\mathcal{F}$ be a subset $\{f_0, \dots, f_t\}$ of R and set $\mathrm{Lt}_{\overline{\tau}}(\mathcal{F}) = \{\mathrm{Lt}_{\overline{\tau}}(f_0), \dots, \mathrm{Lt}_{\overline{\tau}}(f_t)\}$. By a slight abuse of notation, $\mathcal{F}$ and $\mathrm{Lt}_{\overline{\tau}}(\mathcal{F})$ will be regarded as $(t+1)$-tuples of R^{t+1} when convenient. Mora, Pfister and Traverso prove in [MPT89, Theorem 3] that $\mathcal{F}$ is a $\overline{\tau}$-enhanced standard basis of an ideal of R if and only if any homogeneous syzygy of $\mathrm{Lt}_{\overline{\tau}}(\mathcal{F})$ can be lifted to a syzygy of $\mathcal{F}$.

For the sake of completeness, we will now give a precise definition of lifting in this setting following the notation of [Ber09, Definition 1.7]. We define the *degree* of $m = (m_1, \dots, m_{t+1}) \in R^{t+1}$ with respect to the $(t+1)$-tuple $\mathcal{F} \in R^{t+1}$ and the local term ordering $\overline{\tau}$ as

$$\deg_{(\overline{\tau},\mathcal{F})}(m) = \max_{\overline{\tau}}\{\mathrm{Lt}_{\overline{\tau}}(m_i f_{i-1}) : 1 \le i \le t+1 \text{ and } m_i \neq 0\}.$$

An element $\sigma = \{\sigma_1, \dots, \sigma_{t+1}\} \in R^{t+1}$ is homogeneous with respect to $(\overline{\tau}, \mathcal{F})$-degree if all its non-zero components reach the maximum leading term, namely $\mathrm{Lt}_{\overline{\tau}}(\sigma_i f_{i-1}) = \deg_{(\overline{\tau},\mathcal{F})}(\sigma)$ for any $i \in \{1, \dots, t+1\}$ such that $\sigma_i \neq 0$.

Definition 3.16. We call $m \in R^{t+1}$ a $(\overline{\tau}, \mathcal{F})$-*lifting* of a $(\overline{\tau}, \mathcal{F})$-homogeneous element $\sigma \in R^{t+1}$ if $m = \sigma + n$, where $n = (n_1, \dots, n_{t+1}) \in R^{t+1}$ satisfies

$$\mathrm{Lt}_{\overline{\tau}}(n_i f_{i-1}) <_{\overline{\tau}} \deg_{(\overline{\tau},\mathcal{F})}(\sigma) \tag{3.2}$$

for any $1 \le i \le t+1$ such that $n_i \neq 0$. Conversely, we call σ the $(\overline{\tau}, \mathcal{F})$-*leading form* of m and denote it by $\mathrm{LF}_{(\overline{\tau},\mathcal{F})}(m) = \sigma \in R^{t+1}$.

If both $\overline{\tau}$ and $\mathcal{F}$ are clear from the context, we will just say that m is a lifting of σ, which in its turn is the leading form of m. The shift on the indices of n and $\mathcal{F}$ in (3.2) is convenient for our specific setting, as we will see in the following example.

Example 3.17. *Liftings of homogeneous elements in R-free modules.* Consider a monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t}) \subset \mathbf{k}[\![x, y]\!] =: R$ and take $\mathcal{F} = (f_0, \dots, f_t) \in R^{t+1}$ such that $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$ for any $0 \le i \le t$. The columns $\sigma^1, \dots, \sigma^t$ of the canonical Hilbert-Burch matrix H of E are $(\overline{\tau}, \mathcal{F})$-homogeneous elements with $\deg_{(\overline{\tau},\mathcal{F})}(\sigma^j) = x^{t-j+1}y^{m_j}$ for any $1 \le j \le t$. We can build liftings m^j of σ^j by taking $m^j = \sigma^j + n^j$, where $n^j = (n_{1,j}, \dots, n_{t+1,j})$ is a $(t+1)$-tuple of R^{t+1} such that either $n_{i,j} = 0$ or $\mathrm{Lt}_{\overline{\tau}}(n_{i,j})x^{t-i+1}y^{m_{i-1}} <_{\overline{\tau}} x^{t-j+1}y^{m_j}$.

As in the polynomial case, Bertella proves in [Ber09, Theorem 1.10] that the module of syzygies of $\mathcal{F}$ is generated by liftings of homogeneous generators of the module of syzygies of $\mathrm{Lt}_{\overline{\tau}}(\mathcal{F})$. Recall that the fact that syzygies lift is equivalent to the existence of a flat family I_t where $I_0 = \mathrm{Lt}_{\overline{\tau}}(\mathcal{F})$ and $I_1 = (\mathcal{F})$, see [Ste03, Chapter 1] and [MS05, Lemma 18.8].

In the same paper, Bertella provides a very explicit characterization of $\overline{\tau}$-enhanced standard bases in codimension two in terms of matrices that encode leading forms of the generators of the module of syzygies of the ideal:

Theorem 3.18. *[Ber09, Theorem 1.11] Let M be a $(t+1)\times t$ matrix with entries in R. For $0 \le i \le t$, let f_i be the determinant of M after removing row $i+1$ and set $\mathcal{F} = (f_0, \dots, f_t)$. Let H be the matrix whose columns are the $(\overline{\tau}, \mathcal{F})$-leading forms of the columns of M. Assume that:*

- $\mathrm{ht}(f_0, \dots, f_t) = 2$,
- $I_t(H) = (\mathrm{Lt}_{\overline{\tau}}(f_0), \dots, \mathrm{Lt}_{\overline{\tau}}(f_t))$.

Then the following are equivalent:

(i) $\{f_0, \dots, f_t\}$ *is a $\overline{\tau}$-enhanced standard basis of the ideal $I_t(M)$.*

(ii) $\mathrm{ht}(\mathrm{Lt}_{\overline{\tau}}(f_0), \dots, \mathrm{Lt}_{\overline{\tau}}(f_t)) = 2$.

In other words, for zero-dimensional ideals J in $R = \mathbf{k}[\![x,y]\!]$, a $\overline{\tau}$-enhanced standard basis $\mathcal{F}$ arises from maximal minors of a Hilbert-Burch matrix M that encodes liftings of syzygies of $\mathrm{Lt}_{\overline{\tau}}(\mathcal{F})$.

3.3 Towards a parametrization of ideals in $\mathbf{k}[\![x,y]\!]$

From now on we turn our attention to the case of two variables. By τ we denote the lexicographical term ordering on $P = \mathbf{k}[x,y]$, and $R = \mathbf{k}[\![x,y]\!]$.

Definition 3.19. Given a zero-dimensional monomial E ideal in P, we denote by $\mathcal{V}(E)$ the set of ideals $J \subset R$ such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$.

Let us start by defining a set of matrices whose maximal minors generate all the ideals with the same leading term ideal with respect to the local term ordering $\overline{\tau}$.

Definition 3.20. Let E be a zero-dimensional monomial ideal with canonical Hilbert-Burch matrix H and associated degree matrix $U = (u_{i,j})$. We define the set $\mathcal{N}(E)$ of $(t+1)\times t$ matrices $N = (n_{i,j})$ with entries in $\mathbf{k}[\![y]\!]$ such that all its non-zero entries satisfy

$$\mathrm{ord}(n_{i,j}) \ge \begin{cases} u_{i,j}+1, & i \le j; \\ u_{i,j}, & i > j \end{cases},$$

where $\mathrm{ord}(n_{i,j})$ denotes the degree of the initial form of $n_{i,j}$.

Theorem 3.21. *Given a monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ in P with canonical Hilbert-Burch matrix H and degree matrix U, let $\mathcal{V}(E)$ be the set of ideals in Definition 3.19 and let $\mathcal{N}(E)$ be the set of matrices in Definition 3.20. The map*

$$\begin{array}{rccl} \varphi: & \mathcal{N}(E) & \to & \mathcal{V}(E) \\ & N & \mapsto & I_t(H+N) \end{array}$$

is surjective.

We prove Theorem 3.21 in two steps: well-definedness in Lemma 3.22 and surjectivity in Lemma 3.23.

Lemma 3.22. The map φ is well-defined.

Proof. We need to prove that the leading term ideal $\mathrm{Lt}_{\overline{\tau}}(I_t(H+N))$ is the monomial ideal E for any matrix $N = (n_{i,j}) \in \mathcal{N}(E)$.

Consider the matrix $M = H + N$. The order bounds on the entries of N yield

$$\mathrm{ord}(m_{i,j}) \geq \begin{cases} u_{i,j}+1, & i<j; \\ u_{i,j}, & i \geq j. \end{cases}$$

Set $f_i = \det[M]_{i+1}$, for any $0 \leq i \leq t$, where $[M]_{i+1}$ is the square matrix that we get after removing row $i+1$ of M. Since

$$f_i = \sum_{\sigma \in S_t} \mathrm{sgn}(\sigma) \prod_{1\leq k\leq t+1,\, k\neq i+1} m_{k,\sigma(k)},$$

we study the leading terms of polynomials of the form $h = \prod_{1\leq k\leq t+1,\, k\neq i+1} m_{k,\sigma(k)}$.

If h is the product of all elements in the main diagonal of $[M]_{i+1}$, then $\mathrm{Lt}_{\overline{\tau}}(h) = x^{t-i}y^{m_i}$. We claim that any other $h \neq 0$ satisfies $\mathrm{Lt}_{\overline{\tau}}(h) <_{\overline{\tau}} x^{t-i}y^{m_i}$. Indeed, since

$$\mathrm{Lt}_{\overline{\tau}}(h) = \prod_{1\leq k\leq t+1,\, k\neq i+1} \mathrm{Lt}_{\overline{\tau}}(m_{k,\sigma(k)}),$$

then

$$\mathrm{ord}(h) = \sum_{1\leq k\leq t+1,\, k\neq i+1} \mathrm{ord}(m_{k,\sigma(k)}) \geq \sum_{1\leq k\leq t+1,\, k\neq i+1} u_{k,\sigma(k)}.$$

Equality can only be reached if (i,j) satisfy $i \geq j$, namely

$$h = \prod_{k=1}^{i}(y^{d_k} + n_{k,k}) \prod_{k=i+1}^{t+1} m_{k,\sigma(k)},$$

hence the maximal power of x is only reached at the main diagonal. Thus, any $h \neq 0$ away from the main diagonal satisfies $\mathrm{Lt}_{\overline{\tau}}(h) <_{\overline{\tau}} x^{t-i}y^{m_i}$ and, therefore, $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$.

Now we need to show that $\{f_0, \ldots, f_t\}$ forms a $\overline{\tau}$-enhanced standard basis of $I_t(M)$. From the order bounds on the entries $n_{i,j}$ of N, it follows that the columns of M are liftings of the columns of H. See Example 3.17 for more details. By Theorem 3.18, it is enough to show that $\mathrm{ht}\left((\mathrm{Lt}_{\overline{\tau}}(f_0), \ldots, \mathrm{Lt}_{\overline{\tau}}(f_t))\right) = 2$, which is clear because this ideal contains the pure powers $x^t = \mathrm{Lt}_{\overline{\tau}}(f_0)$ and $y^{m_t} = \mathrm{Lt}_{\overline{\tau}}(f_t)$. Therefore, $\mathrm{Lt}_{\overline{\tau}}(I_t(M)) = E$. □

Lemma 3.23. The map φ is surjective.

Proof. Consider a $\overline{\tau}$-enhanced standard basis $\{f_0, \ldots, f_t\}$ of $J \in \mathcal{V}(E)$ such that $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$. We can assume that the monomials in the support of the f_i's are not divisible by x^t, except for $\mathrm{Lt}_{\overline{\tau}}(f_0)$.

For any $1 \leq j \leq t$, consider the S-polynomials $S_j := S(f_{j-1}, f_j) = y^{d_j}f_{j-1} - xf_j$. Note that no monomial in $\mathrm{Supp}(S_j)$ is divisible by x^{t+1} for any $1 \leq j \leq t$. By Theorem 3.15 we have

$$S_j = \sum_{i=0}^{t} q_{i,j} f_i,$$

for some $q_{i,j} \in \mathbf{k}[\![x,y]\!]$ such that $\mathrm{Lt}_{\overline{\tau}}(q_{i,j}f_i) \leq \mathrm{Lt}_{\overline{\tau}}(S_j)$. We claim that $q_{i,j} \in \mathbf{k}[\![y]\!]$.

In fact, we will prove that this holds for any $f \in J$ such that x^{t+1} does not divide any monomial in $\mathrm{Supp}(f)$. Assume $\mathrm{LC}_{\overline{\tau}}(f) = 1$. Consider such an f, then $\mathrm{Lt}_{\overline{\tau}}(f) = x^s y^r$ for some $0 \leq s \leq t$. On the other hand, from the fact that $\mathrm{Lt}_{\overline{\tau}}(f)$ belongs to $\mathrm{Lt}_{\overline{\tau}}(J)$, it follows that $x^{t-i}y^{m_i}$ must divide $\mathrm{Lt}_{\overline{\tau}}(f)$ for some $0 \leq i \leq t$. Then $t-i \leq s$ and $m_i \leq r$, hence $m_{t-s} \leq m_i \leq r$. Define

$$g = f - y^{r-m_{t-s}} f_{t-s}.$$

The new element g still belongs to J and satisfies again that none of its monomials is divisible by x^{t+1}. In this way we can define a sequence $(g_i)_{i\in\mathbb{N}}$, starting by $g_0 = f$, whose elements have decreasing leading terms with respect to $\overline{\tau}$. As in the proof of Grauert's division theorem in [GP07, Theorem 6.4.1], $\sum_{i\in\mathbb{N}} g_k$ converges with respect to the $\mathfrak{m}$-adic topology and

$$f = \sum_{k\in\mathbb{N}} (g_k - g_{k+1}) = \sum_{i=0}^{t} \left(\sum_{k\in\mathbb{N}, s_k = t-i} y^{r_k - m_{t-s_k}} \right) f_i.$$

Therefore, for any $1 \leq j \leq t$, the S-polynomial S_j provides a relation between generators of J

$$y^{d_j} f_{j-1} - x f_j + \sum_{i=1}^{t+1} n_{i,j} f_{i-1} = 0,$$

where $n_{i,j} = -q_{i-1,j} \in \mathbf{k}[\![y]\!]$. This expression can be encoded in the matrix $M = H + N$, where $N = (n_{i,j})$. From $\mathrm{Lt}_{\overline{\tau}}(n_{i,j}f_{i-1}) \leq_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(S_j)$ it follows that any column m^i of M is a lifting of a column σ^i of H. The columns $\sigma^1, \ldots, \sigma^t$ of H constitute a homogeneous system of generators of $\mathrm{Syz}(\mathrm{Lt}_{\overline{\tau}}(J))$. Then, by [Ber09, Theorem 1.10], $m^1, \ldots, m^t$ generate $\mathrm{Syz}(J)$. The Hilbert-Burch theorem ensures that J is generated by the maximal minors of M.

Finally, the order bounds on the entries of N are obtained again from $\mathrm{Lt}_{\overline{\tau}}(n_{i,j}f_{i-1}) \leq_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(S_j)$. Indeed, $x^{t-i+1}y^{m_{i-1}+\beta_{i,j}} <_{\overline{\tau}} x^{t-j+1}y^{m_j}$, where $\mathrm{Lt}_{\overline{\tau}}(n_{i,j}) = y^{\beta_{i,j}}$. Since

$$\beta_{i,j} + t - i + 1 + m_{i-1} \geq t - j + 1 + m_j, \tag{3.3}$$

we have $\beta_{i,j} \geq i - j + m_j - m_{i-1} = u_{i,j}$. If $\beta_{i,j} = u_{i,j}$, then equality holds in (3.3) and hence $t - i + 1 < t - j + 1$. In other words, $\beta_{i,j} \geq u_{i,j}$ and equality is only reachable when $i > j$. □

The proof of Lemma 3.23 provides a constructive method to obtain a matrix $N \in \mathcal{N}(E)$ from any $\overline{\tau}$-enhanced standard basis $\{f_0, f_1, \ldots, f_t\}$ of $J \in \mathcal{V}(E)$ such that $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$ and x^t does not divide any term of any f_i except for $\mathrm{Lt}_{\overline{\tau}}(f_0)$.

Example 3.24. *Matrices in $\mathcal{N}(E)$ with power series entries and how to avoid them.* Set $J = (x^4 + x^3y, y^2 + x^3 + x^2y)$ and consider the $\overline{\tau}$-enhanced standard basis

$$\begin{aligned}
f_0 &= x^4 + x^3y,\\
f_1 &= x^3y^2 + y^5,\\
f_2 &= x^2y^2,\\
f_3 &= xy^2,\\
f_4 &= y^2 + x^3 + x^2y.
\end{aligned}$$

It can be checked that it satisfies the conditions of Lemma 3.23. The first S-polynomial is $y^2 f_0 - x f_1 = \left(\sum_{i\geq 1} y^i\right) f_1 + \left(\sum_{i\geq 3} y^i\right) f_2 - y^3 f_3 - \left(\sum_{i\geq 4} y^i\right) f_4$. The matrix $N \in \mathcal{N}(E)$ with proper power series as entries, where the 4-minors are $f_0, \ldots, f_4$ from above is:

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -\sum_{i\geq 1} y^i & \sum_{i\geq 1} y^i & 0 & 0 \\ -\sum_{i\geq 3} y^i & \sum_{i\geq 2} y^i & 0 & 0 \\ y^3 & 0 & 0 & 0 \\ \sum_{i\geq 4} y^i & -\sum_{i\geq 3} y^i & 0 & 0 \end{pmatrix}.$$

By removing all the terms of degree larger than 3 we get the matrix

$$\overline{N} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -y-y^2-y^3 & y+y^2+y^3 & 0 & 0 \\ -y^3 & y^2+y^3 & 0 & 0 \\ y^3 & 0 & 0 & 0 \\ 0 & -y^3 & 0 & 0 \end{pmatrix}$$

with polynomial entries. Check that $J = \varphi(N) = \varphi(\overline{N})$. Observe that, although the behavior with respect to the syzygies is much better, the $\overline{\tau}$-enhanced standard basis of J given by the minors of $H + \overline{N}$ is less simple, for example $\bar{f}_0 = x^4 + x^3 y + y^4 - xy^4 + y^5 - x^2 y^4 - xy^5 + y^6 - x^2 y^5 - xy^6$.

Next we will see that for any ideal $J \in \mathcal{V}(E)$, we can always find a matrix $N \in \mathcal{N}(E)$ with polynomial entries such that $\varphi(N) = J$. This is achieved by removing the terms in the entries of N with degree strictly higher than the socle degree of R/J, namely the largest integer s such that $\mathfrak{m}^{s+1} \subset J$.

Definition 3.25. Let E be a monomial ideal and let s be the socle degree of R/E. We define the set of matrices $\mathcal{N}(E)_{\leq s} := \mathcal{N}(E) \cap (\mathbf{k}[\![y]\!]_{\leq s})^{(t+1)\times t}$.

Proposition 3.26. The restriction of φ to $\mathcal{N}(E)_{\leq s}$ is surjective.

Proof. Consider $J \in \mathcal{V}(E)$, by Lemma 3.23 we know that $J = I_t(H + N)$ for some $N \in \mathcal{N}(E)$. Recall that J has the same Hilbert function as E, hence the socle degree of J is also s. We express N as $N = \overline{N} + \widetilde{N}$, where $\overline{N} \in \mathcal{N}(E)_{\leq s}$ and $\widetilde{N} \in (\mathbf{k}[\![y]\!]_{\geq s+1})^{(t+1)\times t}$. We decompose $\widetilde{N}$ into matrices $\widetilde{N}_{i,j}$ with at most one non-zero entry at position (i,j) such that $\widetilde{N} = \sum_{i=1,\ldots t+1, j=1,\ldots,t} \widetilde{N}_{i,j}$.

By definition, $J = (f_0, \ldots, f_t)$, where $f_k = \det([H+N]_{k+1})$. Our goal is to prove that $J = (\bar{f}_0, \ldots, \bar{f}_t)$, where $\bar{f}_k = \det([H + \overline{N}]_{k+1})$.

Let us use the Laplacian rule to rewrite the determinant. We denote by $[M]_{(l,m),n}$ the (square) submatrix of M that is obtained by deleting the l-th and m-th rows and the n-th column. Then

$$\begin{aligned} f_k &= \det\left(\left[H + \overline{N} + \textstyle\sum_{i,j} \widetilde{N}_{i,j}\right]_{k+1}\right) \\ &= \det\left(\left[H + \overline{N}\right]_{k+1}\right) + \textstyle\sum_{i,j} \pm \tilde{n}_{i,j} \cdot \det\left(\left[H + \overline{N}\right]_{(k+1,i),j}\right) \\ &= \bar{f}_k + \textstyle\sum_{i,j} \pm \tilde{n}_{i,j} \cdot \det\left(\left[H + \overline{N}\right]_{(k+1,i),j}\right). \end{aligned}$$

Since $\tilde{n}_{i,j} \in \mathbf{k}[\![y]\!]_{\geq s+1}$, it is clear that $f_k - \bar{f}_k \in (x,y)^{s+1} \subset J$. Then $J' = I_t(H + \overline{N}) = (\bar{f}_0, \ldots, \bar{f}_t) \subset J$ and, because $\mathrm{Lt}_{\overline{\tau}}(J') = \mathrm{Lt}_{\overline{\tau}}(J)$, we deduce that $J = (\bar{f}_0, \ldots, \bar{f}_t)$. □

It is important to note that Proposition 3.26 does not provide a parametrization of $\mathcal{V}(E)$. In general, the map φ is not injective even when we restrict it to $\mathcal{N}(E)_{\leq s}$.

Example 3.27. *The restriction of φ is not injective.* Continuing Example 3.24, note that $\overline{N} \in \mathcal{N}_{\leq 4}(E)$ but also

$$N' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{N}_{\leq 4}(E),$$

with $\varphi(N') = \varphi(\overline{N}) = J$.

The corresponding associated $\overline{\tau}$-enhanced standard basis of J is $\{x^4+x^3y,\ x^3y^2,\ x^2y^2,\ xy^2,\ y^2+x^3+x^2y\}$.

Remark 3.28. We have seen that $\varphi : \mathcal{N}(E) \to \mathcal{V}(E)$ as well as its restriction $\varphi : \mathcal{N}_{\leq s}(E) \to \mathcal{V}(E)$ are not injective. Although an ideal J can be obtained from different matrices of the form $H+N$, the systems of polynomial generators $\{f_0, \dots, f_t\}$ of J that arise as maximal minors of any such matrices are all different. In other words, the map $\mathcal{N}(E) \to R^{t+1}$, that sends N to the maximal minors of $H+N$, is injective.

Indeed, if two matrices $N, N' \in \mathcal{N}(E)$ satisfy that the maximal minors of $H+N$ and $H+N'$ coincide, it follows that $N = N'$. The argument is the same as in the first paragraph of [Con11, Section 3.2] and we reproduce it here. Let $\{f_0, \dots, f_t\}$ be the maximal minors of $H+N$ and $H+N'$. The columns of both matrices are syzygies of $\{f_0, \dots, f_t\}$, thence the columns of their difference $H+N-(H+N') = N-N' \in \mathbf{k}[\![y]\!]^{(t+1)\times t}$ are also syzygies, but since the leading terms of the f_i involve different powers of x, it follows that $N = N'$.

3.4 Parametrization for lex-segment leading term ideals

A special situation occurs when a $\overline{\tau}$-enhanced standard basis of J and a Gröbner basis of the ideal $I = J \cap P$ with respect to the lexicographical term ordering, shortly denoted lex-Gröbner basis, coincide. In this setting, we can overcome the lack of injectivity of $\varphi : \mathcal{N}(E) \to \mathcal{V}(E)$ by using Conca-Valla's parametrization of $V_0(E)$.

Proposition 3.29. Let $J \in \mathcal{V}(E)$ be an ideal that admits a $\overline{\tau}$-enhanced standard basis $\{f_0, \dots, f_t\}$ that is also a lex-Gröbner basis of $I = J \cap P$ with $\mathrm{Lt}_{\overline{\tau}}(f_i) = \mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-i}y^{m_i}$. Then there exists a unique matrix $N \in \mathcal{N}(E) \cap T_0(E)$ such that $J = I_t(H+N)$.

Proof. Let $\{f_0, \dots, f_t\}$ be a $\overline{\tau}$-enhanced standard basis of J that is also a lex-Gröbner basis with $\mathrm{Lt}_{\overline{\tau}}(f_i) = \mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-i}y^{m_i}$. Then the f_i are the signed maximal minors of $H+N$ for some $N \in \mathcal{N}(E)$ that is a strictly lower triangular matrix with polynomial entries. Here by strictly lower triangular, we mean that $n_{i,j} = 0$ for all $i \leq j$. If there were non-zero entries $n_{i,j}$ with $i \leq j$, the minors $\{f_0, \dots, f_t\}$ of $H+N$ could not form a lex-Gröbner basis.

Assume that N is not yet in $T_0(E)$, namely there exist (i,j) with $\deg(n_{i,j}) \geq d_j$. In that case we will perform an (i,j)-reduction move defined in Definition 3.30 and obtain a matrix $\widetilde{N}$.

By Lemma 3.31 the matrix $\widetilde{N}$ will still be in $\mathcal{N}(E)$, thus the maximal minors of $H+\widetilde{N}$ will form a $\overline{\tau}$-enhanced standard basis of J.

After performing finitely many reduction steps from the last to the first column, we will obtain a matrix $N_0 \in T_0(E) \cap \mathcal{N}(E)$ with $J = I_t(H+N_0)$. By Theorem 3.4, N_0 is unique. □

Definition 3.30. [Con11, Proof of 3] Let E be a monomial ideal with canonical Hilbert-Burch matrix H, $N \in \mathcal{N}(E)$ strictly lower triangular with polynomial entries, (i,j) such that $\deg(n_{i,j}) \geq d_j$, and $r_{i,j}, q_{i,j} \in \mathbf{k}[y]$ with $n_{i,j} = r_{i,j} + y^{d_j} q_{i,j}$.

The (i,j)-reduction move on N is defined by the following two steps. Note that since N is strictly lower triangular, it corresponds to the second type of reduction moves from [Con11]:

Step 1. Add the j-th row multiplied by $-q_{i,j}$ to the i-th row of $H + N$.

Step 2. Add the $(i-1)$-th column multiplied by $q_{i,j}$ to the $(j-1)$-th column of the matrix resulting from Step 1.

The matrix $\mathrm{Red}_{i,j}(N)$ is the difference of the matrix resulting from Step 2 and H.

The reduction move does not change the ideal of maximal minors – $I_t(H+N) = I_t(H + \mathrm{Red}_{i,j}(N))$ – and the (i,j)-entry of $\mathrm{Red}_{i,j}(N)$ has degree strictly less than d_j. In the following lemma we show that the order bounds on the entries are preserved.

Lemma 3.31. Let $N \in \mathcal{N}(E)$ be a strictly lower triangular matrix with polynomial entries.

Then the matrix we obtain after an (i,j)-reduction move, $\mathrm{Red}_{i,j}(N)$ still satisfies the order bounds defining $\mathcal{N}(E)$.

Proof. Let i,j such that $e = \deg(n_{i,j}) \geq d_j$. That is $n_{i,j} = \sum_{k=u_{i,j}}^{e} c_{i,j,k} y^k$ with $c_{i,j,k} \in \mathbf{k}$. We rewrite

$$\begin{aligned} n_{i,j} &= \textstyle\sum_{k=u_{i,j}}^{e} c_{i,j,k} y^k \\ &= \textstyle\sum_{k=u_{i,j}}^{d_j-1} c_{i,j,k} y^k + \sum_{k=d_j}^{e} c_{i,j,k} y^k \\ &= r_{i,j} + y^{d_j} q_{i,j} \end{aligned}$$

with

- $\mathrm{ord}(r_{i,j}) \geq u_{i,j}$ and $\deg(r_{i,j}) \leq d_j - 1$,
- $\mathrm{ord}(q_{i,j}) \geq \max(u_{i,j} - d_j, 0) = \max(m_j - m_{i-1} + i - j - (m_j - m_{j-1}), 0) = \max(m_{j-1} - m_{i-1} + i - j, 0)$ and $\deg(q_{i,j}) \leq e - d_j$.

Then we will use the (i,j)-reduction move defined in Definition 3.30.

Let $M = H + N$, M' be the matrix that is obtained from Step 1, M'' the matrix obtained from Step 2, and $\mathrm{Red}_{i,j}(N) = M'' - H$.

Only the i-th row and the $(j-1)$-th column of $\widetilde{N} := \mathrm{Red}_{i,j}(N)$ are different from the entries of N.

Remember that

$$m_{l,k} = h_{l,k} + n_{l,k} = \begin{cases} 0 & l < k \\ h_{k,k} = y^{d_k} & l = k \\ h_{k+1,k} + n_{k+1,k} = -x + n_{k+1,k} & l = k+1 \\ n_{l,k} & l - k \geq 2 \end{cases}$$

Let us first examine the i-th row, and $k \neq j-1$:

$$\begin{aligned} \tilde{n}_{i,k} &= m''_{i,k} - h_{i,k} \\ &= m'_{i,k} - h_{i,k} \\ &= (m_{i,k} - q_{i,j} m_{j,k}) - h_{i,k} \\ &= (h_{i,k} + n_{i,k} - q_{i,j} m_{j,k}) - h_{i,k} \\ &= n_{i,k} - q_{i,j} m_{j,k} \end{aligned}$$

- $k > j$, then $m_{j,k} = 0$, and $\tilde{n}_{i,k} = n_{i,k}$.
- $k = j$, then $m_{j,j} = y^{d_j}$, and $\tilde{n}_{i,j} = n_{i,j} - q_{i,j}y^{d_j} = r_{i,j}$. It holds $\operatorname{ord}(r_{i,j}) \geq u_{i,j}$.
- $k < j - 1$, then $m_{j,k} = n_{j,k}$, so $\tilde{n}_{i,k} = n_{i,k} - q_{i,j}n_{j,k}$.

$$\begin{aligned}\operatorname{ord}(q_{i,j}n_{j,k}) &= \operatorname{ord}(q_{i,j}) + \operatorname{ord}(n_{j,k}) \\ &\geq \max(u_{i,j} - d_j + u_{j,k}, u_{j,k}) \\ &= \max(m_{j-1} - m_{i-1} + i - j + m_k - m_{j-1} + j - k, u_{j,k}) \\ &= \max(u_{i,k}, u_{j,k}) \geq u_{i,k}\end{aligned}$$

Now we look at the $(j-1)$-th column, and $k \neq i$:

$$\begin{aligned}\tilde{n}_{k,j-1} &= m''_{k,j-1} - h_{k,j-1} \\ &= (m_{k,j-1} + q_{i,j}m_{k,i-1}) - h_{k,j-1} \\ &= ((h_{k,j-1} + n_{k,j-1}) + q_{i,j}m_{k,i-1}) - h_{k,j-1} \\ &= n_{k,j-1} + q_{i,j}m_{k,i-1}\end{aligned}$$

- $k < i - 1$, then $m_{k,i-1} = 0$.
- $k = i - 1$, then $m_{i-1,i-1} = y^{d_{i-1}}$
- $k > i$, then $m_{k,i-1} = n_{k,i-1}$

In our considerations about the order we combine the last two cases, since the order of the product satisfy the same bounds. Then

$$\begin{aligned}\operatorname{ord}(q_{i,j}m_{k,i-1}) &\geq \max(u_{i,j} - d_j + u_{k,i-1}, u_{k,i-1}) \\ &= \max(m_{j-1} - m_{i-1} + i - j + m_{i-1} - m_{k-1} + k - (i-1), u_{k,i-1}) \\ &= \max(m_{j-1} - m_{k-1} + k - (j-1), u_{k,i-1}) \\ &= \max(u_{k,j-1}, u_{k,i-1}) \geq u_{k,j-1}\end{aligned},$$

and $\tilde{n}_{k,j-1}$ satisfies the order bounds for $k \neq i$.

Now the only entry left to check is

$$\begin{aligned}\tilde{n}_{i,j-1} &= m''_{i,j-1} - h_{i,j-1} \\ &= (m'_{i,j-1} + q_{i,j}\bar{m}_{i,i-1}) \\ &= (m_{i,j-1} - q_{i,j}m_{j,j-1}) + q_{i,j}(m_{i,i-1} - q_{i,j}m_{j,i-1}) \\ &= (m_{i,j-1} - q_{i,j}(-x + n_{j,j-1})) + q_{i,j}(-x + n_{i,i-1} - q_{i,j}m_{j,i-1}) \\ &= n_{i,j-1} - q_{i,j}n_{j,j-1} + q_{i,j}(n_{i,i-1} - q_{i,j}m_{j,i-1})\end{aligned}$$

$i - (j-1) \geq 2$, so $m_{i,j-1} = n_{i,j-1}$ gives the last equality.

$$\begin{aligned}\operatorname{ord}(q_{i,j}n_{j,j-1}) &= \operatorname{ord}(q_{i,j}) + \operatorname{ord}(n_{j,j-1}) \\ &\geq \operatorname{ord}(q_{i,j}) + 1 \\ &\geq \max(u_{i,j} - d_j + 1, 1) \\ &= \max(m_{j-1} - m_{i-1} + i - j + 1, 1) \\ &= \max(u_{i,j-1}, 1) \geq u_{i,j-1}\end{aligned}$$

$$\begin{aligned}\operatorname{ord}(q_{i,j}n_{i,i-1}) &= \operatorname{ord}(q_{i,j}) + \operatorname{ord}(n_{i,i-1}) \\ &\geq \operatorname{ord}(q_{i,j}) + 1 \geq u_{i,j-1}\end{aligned}$$

Since $i > j$,

$$m_{j,i-1} = \begin{cases} 0 & j < i-1 \\ y^{d_j} & j = i-1 \end{cases}$$

If $j < i-1$, we are done, since the last summand is 0. If $j = i-1$, then $n_{i,i-1} - q_{i,j}m_{j,i-1} = \tilde{n}_{i,i-1} = r_{i,j}$, still satisfying the same order bounds, so $\operatorname{ord}(q_{i,j}\tilde{n}_{i,i-1}) = \operatorname{ord}(q_{i,j}n_{i,i-1}) \geq u_{i,j-1}$.

This finishes the proof. □

Proposition 3.29 allows us to extend the definition of canonical Hilbert-Burch matrix to any ideal that has a $\overline{\tau}$-enhanced standard basis $\{f_0, \ldots, f_t\}$ that satisfies $\mathrm{Lt}_{\overline{\tau}}(f_i) = \mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-i}y^{m_i}$. Moreover, the proof of Proposition 3.29 gives an algorithm to construct the canonical matrix from the matrix that encodes the S-polynomials of $\{f_0, \ldots, f_t\}$ via reduction moves.

Definition 3.32. Set $\mathcal{M}(E) := \mathcal{N}(E) \cap T_0(E)$. Let $J \in \mathcal{V}(E)$ be an ideal that admits a $\overline{\tau}$-enhanced standard basis which is also a lex-Gröbner basis of $I = J \cap P$. We define the *canonical Hilbert-Burch matrix* of J as $H + N$, where N is the unique matrix in $\mathcal{M}(E)$ such that $J = I_t(H+N)$.

Remark 3.33. The subset $V_2(E) \subset V_0(E)$ of $(x,y)-$primary ideals I such that $\mathrm{Lt}_{\mathrm{lex}}(I) = E$ is parametrized by the set of matrices $T_2(E)$ (see Definition 3.3). It is not difficult to check that $\mathcal{M}(E) = \mathcal{N}(E) \cap T_0(E) = \mathcal{N}(E) \cap T_2(E)$. Since the ideal $J \cap P$ is (x,y)-primary, it is also clear that the matrix N_0 obtained in the proof of Proposition 3.29 is not only an element of $T_0(E)$, but from $T_2(E)$.

Example 3.34. *Canonical Hilbert-Burch matrix.* Consider $J = (x^6, xy^2 - y^5, y^8)$ and $E = \mathrm{Lt}_{\overline{\tau}}(J) = (x^6,\ x^5y^2,\ x^4y^2,\ x^3y^2,\ x^2y^2,\ xy^2,\ y^8)$. Set $f_0 = x^6$, $f_i = x^{t-i}y^2$ for $i = 1, \ldots, 4$, $f_5 = xy^2 - y^5$ and $f_6 = y^8$. Note that $\{f_0, \ldots, f_6\}$ is a $\overline{\tau}$-enhanced standard basis of J with $\mathrm{Lt}_{\mathrm{lex}}(f_i) = \mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$. The matrix $H+N$ associated to $\{f_0, \ldots, f_6\}$ is the following:

$$\begin{pmatrix} y^2 & 0 & 0 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & -x-y^3 & y^6 \\ 0 & 0 & 0 & 0 & -1 & -x+y^3 \end{pmatrix}.$$

The matrix $N \in \mathcal{N}(E)$ is strictly lower triangular, but since $\deg(n_{6,5}) = 3 \geq d_5 = 0$ and $\deg(n_{7,5}) = 0 \geq d_5 = 0$, we see that $N \notin T_0(E)$. By performing the reduction moves $(6,5)$ and $(7,5)$, we obtain the canonical Hilbert-Burch matrix $H + N_0$ of J, with $N_0 \in \mathcal{M}(E)$:

$$M_0 = H + N_0 = \begin{pmatrix} y^2 & 0 & 0 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & y^6 \\ 0 & 0 & 0 & 0 & 0 & -x+y^3 \end{pmatrix}.$$

There is a class of monomial ideals E such that any ideal in $\mathcal{V}(E)$ satisfies the hypothesis of Proposition 3.29:

Definition 3.35. We call a monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ *relax-segment ideal* if it satisfies

$$m_j - j - 1 \leq m_i - i \text{ for all } j < i. \tag{3.4}$$

Lemma 3.36. Let $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ be a relax-segment ideal, then the reduced $\overline{\tau}$-enhanced standard basis of $J \in \mathcal{V}(E)$ is a Gröbner basis of $I = J \cap P$ with respect to the lexicographical term ordering and $\mathrm{Lt}_{\mathrm{lex}}(I) = E$.

Proof. Let $\{f_i\}_{i\in\mathcal{I}}$ with $\mathcal{I} \subset \{0, \dots, t\}$ be the unique reduced $\overline{\tau}$-enhanced standard basis of J with $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$. We need to show that the leading terms with respect to lex are the right ones, and that $\{f_i\}_{i\in\mathcal{I}}$ forms a lex-Gröbner basis.

(i) $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-i}y^{m_i}$ for any $i \in \mathcal{I}$:

Let us suppose that $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^k y^l \neq x^{t-i}y^{m_i}$. Since $x^{t-i}y^{m_i} \in \mathrm{Supp}(f_i)$, then

$$x^k y^l >_{\mathrm{lex}} x^{t-i}y^{m_i}$$

and hence there are two possible situations:

a) $k = t - i$ and $l > m_i$:

Then $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-i}y^l$ is in the support of $\mathrm{tail}_{\overline{\tau}}(f_i)$ and $x^{t-i}y^l \in E$. But $\{f_j\}_{j\in\mathcal{I}}$ is reduced by assumption.

b) $k > t - i$:

Then we can set $k = t - j$ for some $0 < j < i$. Since $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-j}y^l$ and $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$, the following holds

$$t - i + m_i = \deg(x^{t-i}y^{m_i}) < \deg(x^{t-j}y^l) = t - j + l.$$

If there was an equality on the degree, the local term ordering and lex, would provide the same leading terms. If $l \geq m_j$, we get a contradiction again to the reducedness of $\{f_j\}_{j\in\mathcal{I}}$. Thus, we obtain the following sequence of strict inequalities

$$t - i + m_i < t - j + l < t - j + m_j.$$

It is equivalent to

$$m_i - i + 1 \leq l - j \leq m_j - j - 1$$

But by assumption $m_j - j - 1 \leq m_i - i$, which leads to a contradiction.

(ii) $\{f_i\}_{i\in\mathcal{I}}$ is a Gröbner basis of I with respect to lex:

Since $\{f_i\}_{i\in\mathcal{I}}$ is a subset of I, $E = (\mathrm{Lt}_{\mathrm{lex}}(f_i))_{i\in\mathcal{I}} \subset \mathrm{Lt}_{\mathrm{lex}}(I)$. We can check the equality $\mathrm{Lt}_{\mathrm{lex}}(I) = E$ by looking at the dimensions. From $R/J \cong P/I$, it follows that

$$\dim_{\mathbf{k}}(P/\mathrm{Lt}_{\mathrm{lex}}(I)) = \dim_{\mathbf{k}}(P/I) = \dim_{\mathbf{k}}(R/J) = \dim_{\mathbf{k}}(P/\mathrm{Lt}_{\overline{\tau}}(J)) = \dim_{\mathbf{k}}(P/E)$$

and hence the inclusion $E \subset \mathrm{Lt}_{\mathrm{lex}}(I)$ becomes an equality.

□

Remark 3.37. Since for lex-segment ideals the sequence $(m_i - i)_i$ is strictly increasing, lex-segment ideals are an instance of relax-segment ideals. But the class of ideals is bigger. For example ideals with equality $m_i = m_{i+1}$ for exactly one i satisfy this condition, too. The condition (3.4) for being a relax-segment ideal can be reformulated to

$$u_{i+1,j} \leq 2 \text{ for all } i < j \iff u_{i,j} \geq 0 \text{ for all } i \leq j.$$

Theorem 3.38. *Let $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ be a relax-segment ideal with canonical Hilbert-Burch matrix H. Then the restriction of the map φ from Theorem 3.21 to $\mathcal{M}(E)$*

$$\begin{array}{rccc} \varphi: & \mathcal{M}(E) & \to & \mathcal{V}(E) \\ & N & \mapsto & I_t(H+N) \end{array}$$

is a bijection.

Proof. The map φ is well-defined by Lemma 3.22. Lemma 3.36 and Proposition 3.29 ensure the existence of a unique matrix $N \in \mathcal{M}(E)$ such that $J = I_t(H+N)$ for any ideal $J \in \mathcal{V}(E)$. □

When E is a relax-segment ideal, then the set $\mathcal{M}(E)$ has a simple description. It is formed by strictly lower triangular matrices of size $(t+1) \times t$ with entries in $\mathbf{k}[y]$ such that

$$n_{i,j} = \begin{cases} 0, & i \leq j; \\ c_{i,j}^{v_{i,j}} y^{v_{i,j}} + c_{i,j}^{v_{i,j}+1} y^{v_{i,j}+1} + \dots + c_{i,j}^{d_j-1} y^{d_j-1}, & i > j; \end{cases}$$

where $v_{i,j} := \max(u_{i,j}, 0)$.

Corollary 3.39. Let E be the relax-segment ideal $(x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ with degree matrix $U = (u_{i,j})$, $v_{i,j} = \max(u_{i,j}, 0)$ and $d_j = m_j - m_{j-1}$ for any $1 \leq i \leq t+1$ and $1 \leq j \leq t$. Then $\mathcal{V}(E) \cong \mathcal{M}(E)$ is an affine space of dimension $\mathbf{N}$, where

$$\mathbf{N} = \sum_{\substack{2 \leq j+1 \leq i \leq t+1 \\ v_{i,j} < d_j}} (d_j - v_{i,j}).$$

Remark 3.40. Note that when L is the lex-segment ideal with Hilbert function h, a detailed study of the degree matrix U (as we will do in section 3.5) allows us to rewrite the formula in terms of h. We obtain

$$\dim(\mathcal{V}(L)) = n - t - \sum_{l \geq 2} n_l \binom{l}{2},$$

where n_l denotes the number of jumps of height l in the Hilbert function h, see [HW23, Proposition 3.9]. This recovers the result on the dimension of the stratum $\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$ of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$ with prescribed Hilbert function h by [Bri77, Theorem III.3.1] and [Iar77, Theorem 2.12].

Let us show the details of the parametrization of the Gröbner cell $\mathcal{V}(E)$ as an affine space $\mathbb{A}_{\mathbf{k}}^{\mathbf{N}}$ with an example:

Example 3.41. *Gröbner cell of a lex-segment ideal.* Consider the lex-segment ideal $L = (x^3, x^2y, xy^4, y^9)$ with canonical Hilbert-Burch matrix H. By Theorem 3.38, any canonical Hilbert-Burch matrix of an ideal $J \in \mathcal{V}(L)$ is of the form $M = H + N$ with $N \in \mathcal{M}(L)$:

$$M = \begin{pmatrix} y & 0 & 0 \\ -x & y^3 & 0 \\ c_1 & -x + c_2y + c_3y^2 & y^5 \\ c_4 & c_5 + c_6y + c_7y^2 & -x + c_8y + c_9y^2 + c_{10}y^3 + c_{11}y^4 \end{pmatrix}.$$

Any ideal in $J \in \mathcal{V}(L)$ can be identified with the point $(c_1, c_2, \dots, c_{11}) \in \mathbb{A}^{11}_{\mathbf{k}}$ such that $J = I_3(M)$. The monomial ideal L can be obtained as $I_3(H)$, so it corresponds to point $0 \in \mathbb{A}^{11}_{\mathbf{k}}$.

Corollary 3.42. Assume $\mathrm{char}(\mathbf{k}) = 0$ and let h be an admissible Hilbert function. Let $L = \mathrm{Lex}(h)$ be the unique lex-segment ideal such that $\mathrm{HF}_{R/L} = h$. Then any ideal $J \subset R$ such that $\mathrm{HF}_{R/J} = h$ is of the form $I_t(H + N)$, for some $N \in \mathcal{M}(L)$, after a generic change of coordinates.

Proof. It follows from Theorem 3.38 and the fact that for any $J \subset R$ such that $\mathrm{HF}_{R/J} = h$ it holds $\mathrm{Lex}(h) = \mathrm{Gin}_{\overline{\tau}}(J)$. Here $\mathrm{Gin}_{\overline{\tau}}(J)$ is the extension to the local case defined in [Ber09, Theorem–Definition 1.14] of the usual notion of generic initial ideal. □

Example 3.43. *Two cellular decompositions of* $\mathrm{Hilb}^3(\mathbf{k}[\![x, y]\!])$. There are three monomial ideals of colength 3 in two variables: $E_1 = (x, y^3)$, $E_2 = (x^2, xy, y^2)$ (see Example 3.5 and Example 3.14) and $E_3 = (x^3, y)$. The punctual Hilbert scheme $\mathrm{Hilb}^3(\mathbf{k}[\![x, y]\!])$ can be decomposed into the three corresponding Gröbner cells that depend on the term ordering that we choose. The following table describes the ideals that we find in each Gröbner cell with respect to the lexicographical term ordering, namely $V_2(E_i)$, and the induced local term ordering, namely $\mathcal{V}(E_i)$, with $i = 1, 2, 3$. Recall that $V_2(E_i)$ is the affine space in Conca-Valla parametrization introduced in Theorem 3.4 that only considers $\mathfrak{m}$-primary ideals in the polynomial ring, hence it provides a proper stratification of $\mathrm{Hilb}^3(\mathbf{k}[\![x, y]\!])$.

E_i	$E_1 = (x, y^3)$	$E_2 = (x^2, xy, y^2)$	$E_3 = (x^3, y)$
HF_{R/E_i}	$(1, 1, 1)$	$(1, 2)$	$(1, 1, 1)$
$\overline{\tau}$	$J = (x + c_1y + c_2y^2, y^3)$	$J = (x^2, xy, y^2)$	$J = (x^3, y + cx^2)$
$\tau = \mathrm{lex}$	$I = (x, y^3 + c_2y^2 + c_1y)$	$I = (x^2 + cy, xy, y^2)$	$I = (x^3, y)$

The second row of the table displays the Hilbert function of the local ring $(R/E_i, \mathfrak{n})$ as a sequence of natural numbers such that the element in position t (starting at position 0) is the dimension of the $\mathbf{k}$-vector space $\mathfrak{n}^t/\mathfrak{n}^{t+1}$ and zero-dimensional vector spaces are omitted.

Consider an ideal $I \in V_2(E_2)$ with $c \neq 0$ and note that y belongs to the initial ideal I^*. Therefore $E_2 = \mathrm{Lt}_\tau(I) \neq \mathrm{Lt}_\tau(I^*)$, hence the Hilbert functions of the local rings P/I and P/E_2 differ. In other words, the Gröbner cell $V_2(E_2)$ contains ideals with different Hilbert functions.

On the other hand, the local Gröbner cell $\mathcal{V}(E_2)$ consists of a single point. By construction, such cells will always preserve the Hilbert function. In this sense we say that the Gröbner cells $\mathcal{V}(E)$ respect the Hilbert function stratification of $\mathrm{Hilb}^n(\mathbf{k}[\![x, y]\!])$.

We can also observe in this example that in both cellular decompositions there is exactly one cell of dimensions zero, one and two. This is no coincidence. For a scheme X and over $\mathbb{C}$ the vector of these dimensions is an invariant:

$$\#\{\text{cells of dimension } i \text{ in a cellular decomposition of } X\} = b_{2i}(X),$$

where $b_{2i}(X)$ denotes the $2i$-th Betti number. We will discuss this in more detail in section 3.6.

In the general case, we have a surjective map $\varphi : \mathcal{N}(E)_{\leq s} \to \mathcal{V}(E)$. Restricting to $\mathcal{M}(E)$ we get an injection to $\mathcal{V}(E)$ that is surjective on the subset of ideals whose reduced standard basis also form a lex-Gröbner basis (including for example all homogeneous ideals $J \in \mathcal{V}(E)$). If E is a relax-segment ideal, it is the whole Gröbner cell $\mathcal{V}(E)$. The restriction of φ to $\mathcal{M}(E)$ is not surjective onto $\mathcal{V}(E)$ anymore, if E is not a relax-segment ideal.

Lemma 3.44. If E is not a relax-segment ideal, i.e. it does not satisfy (3.4), then there exists $J \in \mathcal{V}(E)$ such that $\mathrm{Lt}_{\mathrm{lex}}(J \cap P) \neq E$.

Proof. Since condition (3.4) is not satisfied, there exist $k < l$ such that

$$m_k - k - 1 > m_l - l. \tag{3.5}$$

Take $i = \max\{i \mid m_l = m_i\}$ and $j = \min\{j \mid m_k = m_j\}$, then

$$m_j - j - 1 > m_k - k - 1 > m_l - l > m_i - i.$$

Note that (3.5) still holds and now additionally $d_j \geq 1$ and $d_{i+1} \geq 1$.

Set $f_k = x^{t-k}y^{m_k}$ for $k \in \{0, \ldots, t\} \setminus \{i\}$ and $f_i = x^{t-i}y^{m_i} + x^{t-j}y^{m_j-1}$. Consider the ideal $J = (f_0, \ldots, f_t)$ of R. Clearly, $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-j}y^{m_j-1} \notin E$, thus $\mathrm{Lt}_{\mathrm{lex}}(J \cap P) \neq E$.

Now we need to prove that $\mathrm{Lt}_{\overline{\tau}}(J) = E$. From (3.5) we have $t - i + m_i < t - j + m_j - 1$, so $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$. The polynomial f_i cannot be reduced by the other (monomial) generators.

The S-polynomials are

$$S_l = \begin{cases} -x^{t-j+1}y^{m_j-1}, & l = i; \\ x^{t-j}y^{m_j-1+d_{i+1}}, & l = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

If $i < t$, check that $S_i = y^{d_j-1}f_{j-1}$ and $S_{i+1} = y^{d_{i+1}-1}f_j$. Then the matrix N has only two non-zero entries $n_{j,i} = y^{d_j-1}$ and $n_{j+1,i+1} = -y^{d_{i+1}-1}$. If $i = t$ there is only one non-zero S-polynomial. In any case, one can check that $N \in \mathcal{N}(E)$. Thence, $\{f_0, \ldots, f_t\}$ forms a $\overline{\tau}$-enhanced standard basis and $J \in \mathcal{V}(E)$. ☐

Example 3.45. $\mathcal{M}(E) \to \mathcal{V}(E)$ *not surjective.* Consider $E = (x^6, xy^2, y^8)$ as in Example 3.34. E is not a relax-segment ideal because for $(i, j) = (5, 1)$ we have $m_1 - 1 - 1 = 0 > m_5 - 5 = 2 - 5 = -3$. The ideal J from Lemma 3.44 in this case is generated by the monomials $x^{6-k}y^{m_k}$ for $k = 0, \ldots, 4, 6$ and $xy^2 + x^5y$. $J \cap P \notin V_0(E)$ because $\mathrm{Lt}_{\mathrm{lex}}(J \cap P) = (x^6, x^5y, x^2y^2, xy^3, y^8)$. Therefore, $J \notin \varphi(\mathcal{M}(E))$.

Even though we have now seen that for general E the map is not surjective, we can still parametrize the sub-cell of $\mathcal{V}(E)$ of homogeneous ideals.

Lemma 3.46. Let $E \in P$ be a zero-dimensional monomial ideal. The homogeneous sub-cell $\mathcal{V}_{hom}(E)$ of $\mathcal{V}(E)$ consisting of all homogeneous ideals of $V(E)$ is an affine space of dimension $\#\{(i, j) \mid i > j, 0 \leq u_{i,j} < d_j\}$.

Moreover, for any $J \in \mathcal{V}(E)$ and any $N \in \mathcal{N}(E)$ with $J = I_t(H + N)$ its initial ideal J^* can be computed by projecting N into that affine space.

Proof. Homogeneous ideals J in $\mathcal{V}(E)$ are inside the set of ideals in $V(E)$ admitting a canonical Hilbert-Burch matrix. This is clear since for homogeneous f, the leading terms $\mathrm{Lt}_{\overline{\tau}}(f) = \mathrm{Lt}_{\mathrm{lex}}(f)$ coincide, so a homogeneous $\overline{\tau}$-enhanced standard basis of J will also form a lex-Gröbner basis of $J \cap \mathbf{k}[x, y]$. Thence by Proposition 3.29 $\mathcal{V}_{\mathrm{hom}}(E) \subset \varphi(\mathcal{M}(E))$, and there exists a unique $N_0 \in \mathcal{M}(E)$ such that $J = I_t(H + N_0)$.

We claim that an ideal $J = I_t(H + N)$, for some $N \in \mathcal{M}(E)$, is homogeneous if and only if the maximal minors $f_0, \dots, f_t$ of $H + N$ are homogeneous. Obviously, homogeneity of a system of generators yields homogeneity of J. For the other direction, note that the maximal minors $f_0, \dots, f_t$ of $H + N$ are a $\overline{\tau}$-enhanced standard basis of $J = I_t(H + N)$ by construction for any $N \in \mathcal{N}(E)$. By Remark 3.11, they also form a standard basis of J, namely $J^* = (f_0^*, \dots, f_t^*)$. Let us call N^* the matrix obtained by only keeping the terms of degree $u_{i,j}$ in each entry $n_{i,j}$. Clearly, the maximal minors of $H + N^*$ are $f_0^*, \dots, f_t^*$ and $J^* = I_t(H + N^*)$, asserting the second statement of Lemma 3.46. Note that $N^* \in \mathcal{M}(E)$. So if J is homogeneous then $J = I_t(H + N) = I_t(H + N^*)$ with $N, N^* \in \mathcal{M}(E)$, implies $N = N^*$ by Proposition 3.29.

In conclusion, $\mathcal{V}_{\mathrm{hom}}(E)$ is parametrized by the subset of $\mathcal{M}(E)$ where $n_{i,j} = c_\bullet y^{u_{i,j}}$ for $i > j$ and $0 \le u_{i,j} < d_j$. So it is an affine space of dimension $\#\{(i,j) \mid i > j, 0 \le u_{i,j} < d_j\}$. □

The dimension formula in this result agrees with [CV08, Corollary 3.1] and shows that even though their Gröbner cells differ – as we have seen in Example 3.43 – the sub-cells of homogeneous ideals agree. This is not a big surprise, since for homogeneous f the leading terms $\mathrm{Lt}_{\overline{\tau}}(f) = \mathrm{Lt}_{\mathrm{lex}}(f)$ agree by the definition of local term order induced by lex. When E is a lex-segment ideal with Hilbert function h the dimension formula given in Lemma 3.46 can be rewritten in terms of the Hilbert function as

$$\dim(\mathcal{V}_{\mathrm{hom}}(E)) = t + \sum_{i=t-1}^{s} (h_{i-1} - h_i)(h_i - h_{i+1}),$$

see [HW23, Proposition 3.11]. This dimension formula is equivalent to the one for the homogeneous subscheme of $\mathrm{Hilb}^h(\mathbf{k}[\![x, y]\!])$ given and proven by different methods in [Iar77, Theorem 2.12].

Coming back to the Gröbner cell $\mathcal{V}(E)$ several computations, comparison to [Con11], considerations about the reduction moves and a detailed study of complete intersections suggest us how to replace $\mathcal{N}(E)$ in Theorem 3.21 in order to obtain a bijection.

Definition 3.47. We define the subset $(\mathbf{k}[y]_{<\underline{d}})^{(t+1)\times t} \subset \mathbf{k}[y]^{(t+1)\times t}$ as matrices where the non-zero entries satisfy the following degree conditions:

$$\deg(n_{i,j}) < \begin{cases} d_i, & i \le j; \\ d_j, & i > j. \end{cases}$$

We define the subset $\mathcal{N}(E)_{<\underline{d}}$ of $\mathcal{N}(E)$ as $\mathcal{N}(E)_{<\underline{d}} := \mathcal{N}(E) \cap (\mathbf{k}[y]_{<\underline{d}})^{(t+1)\times t}$.

Conjecture 3.48. *Let E be a monomial ideal. Then the set $\mathcal{N}(E)_{<\underline{d}}$ parametrizes $\mathcal{V}(E)$.*

For any relax-segment ideal E, the sets $\mathcal{N}(E)_{<\underline{d}}$ and $\mathcal{M}(E)$ coincide. By Theorem 3.38, the conjecture is true for such E, which includes lex-segment ideals. For general E, we have an inclusion $\mathcal{M}(E) \subset \mathcal{N}(E)_{<\underline{d}}$. Moreover, the matrix N constructed in the proof of Lemma 3.44, which is not in $\mathcal{M}(E)$, can also be transformed to a matrix in $\mathcal{N}(E)_{<\underline{d}}$ via reduction moves.

Remark 3.49. An approach analogous to Proposition 3.29 and [Con11, Proof of 3] with reduction moves does not work in general. It can be verified that if we start with any $N \in \mathcal{N}(E)$ the matrix obtained by a reduction move is in $\mathcal{N}(E)$, see Lemma 3.31 where the case $i > j$ is shown. If the matrix is additionally strictly upper or strictly lower triangular, there is an obvious order in which one can perform the reduction moves to obtain a matrix in $\mathcal{N}(E)_{<\underline{d}}$.

For a general matrix $N \in \mathcal{N}(E)$ this order is not so clear. This problem already arises in [Con11, Proof of 3] and is solved by considering reduction moves that are maximal for an element, namely those producing a maximal increase of the degree of the element. In our setting the situation is worse, since even when starting with a matrix with polynomial entries, an (i,j)-reduction move can create entries that are proper power series. This happens if the entry on the diagonal is not zero. And even when starting with a matrix with only zeros on the diagonal, general reduction moves will create non-zero diagonal entries.

Additionally, reduction moves do not give a way of reducing entries on the diagonal. In Example 3.24 and its continuation Example 3.27 we have two matrices $\overline{N}, N' \in \mathcal{N}_{\leq s}(E)$ with $J = I_t(H + \overline{N}) = I_t(H + N')$. The matrix $N' \in \mathcal{N}(E)_{<\underline{d}}$ is our desired matrix, but it cannot be obtained from $\overline{N}$ by this type of reduction moves.

Example 3.50. *Non-relax-segment ideal E where Conjecture 3.48 holds.* Consider the monomial ideal $E = (x^4, y^2)$. Using reduced $\overline{\tau}$-enhanced standard bases, it can be proved that any $J \in \mathcal{V}(E)$ is of the form $J = (x^4 + ax^3y, y^2 + bx^3 + cx^3y + dx^2y)$. The S-polynomials of the standard basis

$$\begin{aligned}
f_0 &= x^4 + ax^3y\\
f_1 &= x^3y^2\\
f_2 &= x^2y^2\\
f_3 &= xy^2 + (d-ab)x^3y + (ad-a^2b)x^2y^2\\
f_4 &= y^2 + bx^3 + cx^3y + dx^2y + (ad-a^2b)xy^2 + (a^3b^2 - 2a^2bd + ad^2 + c)x^3y
\end{aligned}$$

of J give the matrix $M = H + N$, with $N \in \mathcal{N}(E)_{<\underline{d}}$ and $I_4(M) = J$, satisfying the conjecture:

$$M = \begin{pmatrix} y^2 & 0 & (d-ab)y & b + (a^3b^2 - 2a^2bd + ad^2 + c)y \\ -x-ay & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{pmatrix}.$$

3.5 Betti strata of Gröbner cells

The Gröbner cells $\mathcal{V}(E)$ form a stratification of the punctual Hilbert scheme $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\![\mathrm{x},\mathrm{y}]\!])$. The cells can be further stratified by considering subsets of ideals with a given minimal number of generators. For an ideal J, we denote the minimal number of generators by $\mu(J)$. The results and methods of this section are similar to the results obtained in [Ber09, Section 2].

Definition 3.51. For a monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t}) \subset P$, we denote the *$d$-Betti stratum of* $\mathcal{V}(E)$ by

$$\mathcal{V}_d(E) = \{J \in \mathcal{V}(E) \mid \mu(J) = d\}.$$

If $\mathcal{V}_d(E) \neq \emptyset$, we will say that d is an *admissible number of generators*. When $d = 2$ we will sometimes refer to $\mathcal{V}_2(E)$ as $\mathcal{V}_{\mathrm{CI}}(E)$, since the ideals generated by two elements are complete intersection ideals. We denote

$$\mathcal{V}_{\leq d}(E) = \{J \subset \mathcal{V}(E) \mid \mu(J) \leq d\}.$$

The following lemma describes how to calculate the minimal number of generators of an ideal $J \subset R$ with the help of its Hilbert-Burch matrix. It follows easily by Nakayama's lemma.

Lemma 3.52. [Ber09, Lemma 2.1] Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ be a monomial ideal in P with canonical Hilbert-Burch matrix H and $J = I_t(H + N)$ with $N \in \mathcal{N}(E)$. Then $\mu(J) = t + 1 - \operatorname{rk} \overline{H + N}$, where $\overline{H + N}$ is the image of $H + N$ in $(H + \mathcal{N}(E)) \otimes_R {}^{R}\!/\!_{\mathfrak{m}}$, so the matrix that consists only of the constant terms of $H + N$.

If E is a lex-segment ideal, then $\operatorname{rk} \overline{H + N} = \operatorname{rk} \bar{N}$.

Whenever we have a parametrization of $\mathcal{V}(E) \cong \mathbb{A}^D$ in terms of canonical Hilbert-Burch matrices, Lemma 3.52 implies a description of the d-Betti stratum $\mathcal{V}_d(E)$ of $\mathcal{V}(E)$. The notation $I_\bullet(-)$ in the following proposition will refer to an ideal in $\mathbf{k}[c_1, \ldots, c_D]$ and $\mathbb{V}(I_\bullet(-))$ denotes the set of vanishing of the ideal inside $\mathbb{A}^D$.

Proposition 3.53. Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ with canonical Hilbert-Burch matrix H and $\varphi : \mathcal{T}(E) \to \mathcal{V}(E)$ by $N \mapsto I_t(H + N)$ be a parametrization, with $\mathcal{T}(E) \cong \mathbb{A}^D$. Then the d-Betti stratum $\mathcal{V}_d(E)$ of $\mathcal{V}(E)$ can be parametrized as

$$\mathcal{V}_d(E) \cong \mathbb{V}(I_{t+2-d}(\overline{H + N})) \setminus \mathbb{V}(I_{t+1-d}(\overline{H + N})) \subset \mathcal{V}(E).$$

When E is a lex-segment ideal, $\varphi : \mathcal{M}(E) \to \mathcal{V}(E)$ from Theorem 3.38 is a parametrization, and

$$\mathcal{V}_d(E) \cong \mathbb{V}(I_{t+2-d}(\overline{N})) \setminus \mathbb{V}(I_{t+1-d}(\overline{N})) \subset \mathcal{V}(E).$$

Proof. Since $\varphi : \mathcal{T}(E) \to \mathcal{V}(E)$ is a parametrization, the ideal J can be identified with the matrix N such that $J = I_t(H + N)$. By Lemma 3.52, $J \in \mathcal{V}_d(E)$ if and only if $\operatorname{rk} \overline{H + N} = t + 1 - d$. That means that N is in the set of matrices with $\operatorname{rk} \overline{H + N} \leq t + 1 - d$, but not in the set where $\operatorname{rk} \overline{H + N} \leq t - d$. The rank of $\overline{H + N}$ is $\leq t + 1 - d$ if and only if all its minors of size $t + 2 - d$ vanish. So

$$N \in \mathbb{V}(I_{t+2-d}(\overline{H + N})) \setminus \mathbb{V}(I_{t+1-d}(\overline{H + N})).$$

□

A parametrization of the affine space $\mathcal{V}(E)$ in terms of canonical Hilbert-Burch matrices allows us to see $\mathcal{V}_{\leq d}(E)$ as a determinantal variety inside the affine space $\mathcal{V}(E)$. When we only start with the surjection $\varphi : \mathcal{N}(E) \to \mathcal{V}(E)$, as in Theorem 3.21, we can still obtain a similar statement.

Proposition 3.54. Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ and $\varphi : \mathcal{N}(E) \to \mathcal{V}(E)$ by $N \mapsto I_t(H + N)$, from Theorem 3.21. Then the restriction of φ to

$$\mathbb{V}(I_{t+2-d}(\overline{H + N})) \setminus \mathbb{V}(I_{t+1-d}(\overline{H + N})) \to \mathcal{V}_{\leq d}(E) \setminus \mathcal{V}_{\leq (d-1)}(E) \cong \mathcal{V}_d(E)$$

is well-defined and surjective.

Proof. All the arguments of the proof of Proposition 3.53 can be directly applied to this setting. The only difference is that here there is no unique matrix $N \in \mathcal{N}(E)$, so the restriction of φ is not injective. □

Remark 3.55 (Upper bound for admissible d). For all ideals $J \in \mathcal{V}(E)$, J has a $\overline{\tau}$-enhanced standard basis consisting of $\mu(E)$ elements, so $\mu(J) \leq \mu(E)$. This can also be seen with Lemma 3.52, for the canonical Hilbert-Burch matrix H of the monomial ideal E, the rank $\operatorname{rk} \overline{H} = \#\{i \mid d_i = 0\} =: l \geq 0$ and $\operatorname{rk} \overline{H + N} \geq l$ for all $N \in \mathcal{N}(E)$, so $\mu(J) = t+1-\operatorname{rk} \overline{H+N} \leq t+1-l = \mu(E)$. For a lex-segment ideal L, the minimal number of generators $\mu(J)$ of $J \in \mathcal{V}(L)$ is equal to $t+1$ if and only if $\bar{N}$ is the zero matrix. So all the constant terms have to vanish, so $\mathcal{V}_{t+1}(E) = \mathbb{A}^{D'}$, where $D' = D - \#\{\text{allowed constant terms in } N\}$.

This affine subspace $\mathcal{V}_{t+1}(E) \cong \mathbb{A}^{D'}$ of $\mathcal{V}(E) \cong \mathbb{A}^{D}$ is actually untouched by the conditions on the number of generators. All the polynomial equations of $\mathcal{V}_d(E)$ do not involve the parameters of this subspace, so $\mathcal{V}_d(E)$ can be understood as the fiber product of $\mathbb{V}(I_{t+2-d}(\bar{N})) \setminus \mathbb{V}(I_{t+1-d}(\bar{N})) \subset \mathbb{A}^{D-D'}$ and $\mathcal{V}_{t+1}(E) \cong \mathbb{A}^{D'}$.

We will now address the question whether a minimal number of generators is admissible in $\mathcal{V}(E)$. From Proposition 3.54 and Proposition 3.53 we can see that $\mathcal{V}_d(E) \neq \emptyset$ if and only if $\emptyset \neq \mathbb{V}(I_{t+2-d}(\overline{H+N})) \subsetneq \mathbb{V}(I_{t+1-d}(\overline{H+N}))$. So this question highly depends on the structure of $\bar{N}$ or in other words the allowed constant coefficients of matrices $N \in \mathcal{N}(E)$ or $N \in \mathcal{M}(E)$. And by the definitions of $\mathcal{N}(E)$ and $\mathcal{M}(E)$ this is a question about non-positive entries of the degree matrix U associated to E.

From now on we will restrict to lex-segment ideals, i.e. $L = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ with $0 = m_0 < m_1 < \dots < m_t$. By Theorem 3.38 we know that $\mathcal{M}(L)$ parametrizes $\mathcal{V}(L)$ and the matrices $N \in \mathcal{M}(L)$ are all lower triangular matrices, so we are interested in the non-positive entries in the lower triangle of the degree matrix U associated to L.

Lemma 3.56. Let $L = (x^t, x^{t-1}y^{m_1}, \dots, x^1 y^{m_{t-1}}, x^{m_t})$ be a lex-segment ideal and U its associated degree matrix. Let $\mathcal{J}$ be a set of pairs of positive integers defined by

$$\mathcal{J} := \{(j,k) \mid j \leq t,\ d_{j+1} = d_{j+2} = \dots = d_{j+k-1} = 1\}.$$

Then $u_{j+k,j} = 1$ if and only if $(j,k) \in \mathcal{J}$. All other entries $u_{j+k,j} \leq 0$.

Proof. The following equations can easily be verified:

(i) $m_{j+s} = m_j + \sum_{l=1}^{s} d_{j+l}$

(ii) $u_{j+k,j} = m_j + k - m_{j+k-1} = k - \sum_{l=1}^{k-1} d_{j+l}$.

Since L is a lex-segment ideal, $d_i > 0$ and with point (ii), $u_{j+k,j} \leq 1$. The value 1 is achieved if and only if $d_{j+1} = d_{j+2} = \dots = d_{j+k-1} = 1$. Thus $u_{j+k,j} = 1$ if and only if $(j,k) \in \mathcal{J}$. For all $(j,k) \notin \mathcal{J}$, the entry $u_{j+k,j} \leq 0$. □

Notice that d_1 does not play a role in the pattern of 1s in the lower left triangle of U.

Remark 3.57. The set $\mathcal{J}$ describes the pattern of 1s and non-positive entries of the lower left triangle of U. It is determined by the consecutive $d_\bullet = 1$. Obviously, the longest sequences of equations $d_{j+1} = d_{j+2} = \dots = d_{j+k-1} = 1$ determine the set $\mathcal{J}$ completely. Denote by $\mathcal{I}$ the subset of $\mathcal{J}$ that corresponds to such maximal sequences. Then if $(j,k) \in \mathcal{I}$, $d_{j+1} = d_{j+2} = \dots = d_{j+k-1} = 1$, thus clearly $d_{j+1} = d_{j+2} = \dots = d_{j+l-1} = 1$, for all $l \leq k$ and $(j,l) \in \mathcal{J}$. Also $d_{i+1} = \dots = d_{j+k-1}$ for $i \geq j$, and $(i,l) \in \mathcal{J}$ for $i \geq j$ and $l \leq j+k-i$. On the other hand, if $(i,l) \in \mathcal{J}$, then $d_{i+1} = \dots = d_{i+l-1} = 1$ and this sequence of equations is a sub-sequence of a maximal sequence $d_{j+1} = d_{j+2} = \dots = d_{j+k-1}$ with $j \leq i$ and $l \leq k$.

If $(j,k) \in \mathcal{I}$, then $u_{j+k,j} = 1$ (indicated in bold) is a *lower corner in the pattern of* 1*s in* U in the following sense:

$$U = \left(\begin{array}{cccccccccccccccc}
d_1 &&&&&&&&&&&&&&& \\
\mathbf{1} & \neq 1 &&&&&&&&&&&&&& \\
& 1 & 1 &&&&&&&&&&&&& \\
& \vdots & \ddots & \ddots &&&&&&&&&&&& \\
& \vdots && \ddots & 1 &&&&&&&&&&& \\
& \mathbf{1} & \dots & \dots & 1 & \neq 1 &&&&&&&&&& \\
&&&&& \mathbf{1} & \ddots &&&&&&&&& \\
&&&&&& \ddots & \ddots &&&&&&&& \\
&&&&&&& \mathbf{1} & \neq 1 &&&&&&& \\
&&&&&&&& 1 & 1 &&&&&& \\
&&&&&&&& \vdots & \ddots & \ddots &&&&& \\
&&&&&&&& \vdots && \ddots & 1 &&&& \\
&&&&&&&& \mathbf{1} & \dots & \dots & 1 & \neq 1 &&& \\
&&&&&&&&&&&& 1 & 1 && \\
&&&&&&&&&&&& \vdots & \ddots & \ddots & \\
&&&&&&&&&&&& \vdots && \ddots & 1 \\
&&&&&&&&&&&& \mathbf{1} & \dots & \dots & 1
\end{array}\right),$$

where all entries in the lower left triangle of U that are left out are non-positive.

The indicated matrix U already indicates how we can construct the pattern of 1s just from $d_1, \dots, d_t$. When d_j is the last $d_\bullet$ different from 1, we go down to the first column where d_{j+k} is the first $d_\bullet$ different from 1, and set $u_{j+k,j} = 1$. Then all entries to the right and up from $u_{j+k,j}$ up to the diagonal are also equal to 1.

Lemma 3.58. Let $L = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ be a lex-segment ideal with associated degree matrix U, $\mathcal{J}$ as in Lemma 3.56 and $k_0 := \max_{(j,k)\in\mathcal{J}} k$. Then

(i) $\operatorname{rk} \bar{N} \leq t - k_0$ for all $N \in \mathcal{M}(L)$,

(ii) for all $0 \leq r \leq t - k_0$, there are matrices $N \in \mathcal{M}(L)$ with $\operatorname{rk} \bar{N} = r$.

Proof. By the definition of k_0, we know that there exists j_0 such that $u_{j_0+k_0,j_0} = 1$ is a corner in the pattern of 1s of the lower left triangle in the U-matrix as in Remark 3.57, and thus all the entries in the submatrix $u_{i,j}$ with $i \leq j_0 + k_0$ and $j \geq j_0$ have positive entries, and for the corresponding submatrix of $\bar{N}$ this yields a zero submatrix of size $(j_0 + k_0) \times (t - j_0 + 1)$ as indicated here:

$$\bar{N} = \left(\begin{array}{ccc|ccc} * & \dots & * & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ & & & 0 & \dots & 0 \\ \hline & & & * & \dots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \dots & * & * & \dots & * \end{array}\right).$$

The first $j_0 - 1$ columns of $\bar{N}$ have rank at most $j_0 - 1$, and in the columns j_0 to t the only allowed non-zero entries are in rows $j_0 + k_0 + 1$ to $t + 1$, so the rank of $\bar{N}$ is at most $j_0 - 1 + ((t + 1) - (j_0 + k_0 + 1) + 1) = t - k_0$. This proves the first point.

When $k_0 = \max_{(j,k)\in\mathcal{J}} k$, then $u_{i,j} \leq 0$ for all i, j with $i - j \geq k_0 + 1$. In particular, for $1 \leq l \leq t - k_0$ and $j = 1, \dots, t + 1 - k_0 - l$, the entries $u_{j+k_0+l,j} \leq 0$ since the inequality $j + k_0 + l - j = k_0 + l \geq k_0 + 1$ holds. Thus, there exists $N \in \mathcal{M}(L)$ with $n_{j+k_0+l,j} = 1$ for $j = 1, \dots, t + 1 - k_0 - l$ and all other entries vanishing. Then the ranks are $\operatorname{rk} \bar{N} = \operatorname{rk} N = t + 1 - k_0 - l$. For $l = 1$ we achieve the maximal rank $\operatorname{rk} \bar{N} = t - k_0$ and for $l = t - k_0$ the constructed matrix is the matrix with only non-vanishing entry $n_{t+1,t} = 1$ of rank one. The zero matrix is clearly an element of $\mathcal{M}(L)$ of rank zero. □

The constructed matrices $N \in \mathcal{M}(L)$ in the proof of Lemma 3.58 lead to the same matrices $M = H + N$ as in the proof of [Ber09, Theorem 2.4]. They are obviously not the only matrices in $\mathcal{M}(L)$ with $\operatorname{rk} \bar{N} = r$. As already discussed in Remark 3.55 all the coefficients of monomials of positive degree can be chosen freely. This gives an affine space of matrices $N' \in \mathcal{M}(L)$ with $\bar{N}' = N$. Another way of obtaining a matrix $N' \in \mathcal{M}(L)$ with $\operatorname{rk} N' = \operatorname{rk} \bar{N}' = r$ is to set $n_{j+t+1-r,j}$ to any non-zero constant, and the entries below $n_{j+t+1-r+i,j}$ with $1 \leq i \leq r - j$ can be chosen freely. The following corollary makes it more precise.

Corollary 3.59. Let L be a lex-segment ideal and k_0 as in Lemma 3.58, then the d-Betti stratum $\mathcal{V}_d(L) = \mathbb{V}(I_{t+2-d}(\bar{N})) \setminus \mathbb{V}(I_{t+1-d}(\bar{N}))$ of $\mathcal{V}(L)$ is non-empty if and only if $k_0 + 1 \leq d \leq t + 1$.

The $(t + 1)$-Betti stratum $\mathcal{V}_{t+1}(L)$ is an affine space in $\mathcal{V}(L)$.

The $(k_0 + 1)$-Betti stratum $\mathcal{V}_{k_0+1}(L)$ is the full dimensional quasi-affine variety

$$\mathcal{V}(L) \setminus \mathbb{V}(I_{t-k_0}(\bar{N})).$$

Proof. By Lemma 3.58 there exists $N \in \mathcal{M}(L)$ with $\operatorname{rk} \bar{N} = r$, if and only if $0 \leq r \leq t - k_0$. By Lemma 3.52 $\mu(I_t(H + N)) = t + 1 - \operatorname{rk} \bar{N}$. With $0 \leq r \leq t - k_0$, this means that there exist ideals with minimal number of generators between $k_0 + 1$ and $t + 1$, and all ideals J in $\mathcal{V}(L)$ have $\mu(J)$ in this range.

The set $\mathcal{V}_{t+1}(L)$ is parametrized by all the matrices $N \in \mathcal{M}(L)$ with $\bar{N} = 0$. So all parameters of constant coefficients of polynomials $n_{i,j}(y)$ vanish and all other coefficients can be chosen arbitrarily.

The $(k_0 + 1)$-Betti stratum is

$$\mathcal{V}_{k_0+1}(L) = \mathbb{V}(I_{t-k_0+1}(\bar{N})) \setminus \mathbb{V}(I_{t-k_0}(\bar{N})) = \mathcal{V}(L) \setminus \mathbb{V}(I_{t-k_0}(\bar{N})).$$

□

Corollary 3.60. The Gröbner cell $\mathcal{V}(L)$ of the lex-segment ideal L contains complete intersection ideals if and only if $d_i > 1$ for all $i = 2, \dots, t$.

Proof. By Corollary 3.59 $\mathcal{V}(L)$ contains complete intersection ideals if and only if $k_0 = 1$. That means for all $j = 1, \dots, t$, $d_{j+1} > 1$, otherwise $(j, 2) \in \mathcal{J}$ from Lemma 3.56. □

For the remainder of the section, we will rewrite k_0 in terms of the Hilbert function h of L. Let $\Delta(h) := \max\{|h_i - h_{i-1}|\}$ be the maximal jump of the Hilbert function h.

Lemma 3.61. Let $L = (x^t, x^{t-1}y_1^m, \dots, y^{m_t})$ be a lex-segment ideal, $h := h(L)$ its Hilbert function, and k_0 as in Lemma 3.58. Then $k_0 = \Delta(h)$.

Proof. An easy but helpful observation is that the Hilbert function h of a lex-segment ideal L, can be calculated as follows:

$$h_i = \begin{cases} i+1 & i < t \\ t+1-\#\{f_j \mid \deg(f_j) = t\} & i = t \\ h_{i-1} - \#\{f_j \mid \deg(f_j) = i\} & t+1 \leq i \leq m_t - 1 \\ 0 & i \geq m_t, \end{cases}$$

where $f_j = x^{t-j}y^{m_j}$.

Then the jump at position i $|h_i - h_{i-1}|$ is the following:

$$|h_i - h_{i-1}| = \begin{cases} 1 & i < t \\ \#\{f_j \mid \deg(f_j) = t\} - 1 & i = t \\ \#\{f_j \mid \deg(f_j) = i\} & t+1 \leq i \leq m_t \\ 0 & i \geq m_t + 1. \end{cases}$$

Since L is a lex-segment ideal, $\deg(f_j) = t - j + m_j \leq \deg(f_{j+1}) = t - (j+1) + m_{j+1}$ and equality occurs if and only if $d_{j+1} = m_{j+1} - m_j = 1$.

Let j be minimal with $\deg(f_j) = i$. Then

$$\begin{aligned} \#\{f_l \mid \deg(f_l) = i\} &= \max\{k \mid \deg(f_j) = \deg(f_{j+1}) = \dots = \deg(f_{j+k-1})\} \\ &= \max\{k \mid d_{j+1} = d_{j+2} = \dots = d_{j+k-1} = 1\} \end{aligned}$$

Let $t+1 \leq i \leq m_t$ and j minimal with $\deg(f_j) = i$, then

$$|h_i - h_{i-1}| = \max\{k \mid d_{j+1} = d_{j+2} = \dots = d_{j+k-1}\},$$

so $(j, k) \in \mathcal{J}$ from Lemma 3.56 and $(j, k+1) \notin \mathcal{J}$. For $i = t$, $\deg(f_0) = t$ and

$$|h_t - h_{t-1}| = \max\{k \mid d_1 = d_2 = \dots = d_{k-1} = 1\} - 1$$

we conclude $(1, k-1) \in \mathcal{J}$ and $(1, k) \notin \mathcal{J}$. So with k_0 as in Lemma 3.58

$$k_0 = \max\{k \mid (j, k) \in \mathcal{J}\} = \max\{|h_i - h_{i-1}|\} = \Delta(h).$$

□

Whenever the cell $\mathcal{V}(L)$ is dense in $\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$ we can reformulate Corollary 3.59 with Lemma 3.61 and obtain the following proposition. The condition for complete intersection ideals is due to [Mac27] and was reproved with different methods by [Bri77; Iar77; RS10]. The more general result was proven by [Ber09, Theorem 2.4].

Proposition 3.62. Let $\mathrm{char}(\mathbf{k}) = 0$ and h be an admissible Hilbert function. There exists an ideal $J \in \mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$ with $\mu(J) = d$ if and only if $d \in [\Delta(h)+1, t+1]$.

In particular, there are complete intersection ideals with Hilbert function h if and only if $\Delta(h) = 1$.

Proof. This follows by combining Corollary 3.59 and Lemma 3.61 and the fact that up to a generic change of coordinates all ideals in $\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$ have leading term ideal L. The minimal number of generators of an ideal is invariant under a change of coordinates. □

3.6 A cellular decomposition of the punctual Hilbert scheme

The collection of all Gröbner cells $\mathcal{V}(E)$ forms a cellular decomposition of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$. A cellular decomposition which additionally respects the Hilbert function stratification of $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\![\mathrm{x},\mathrm{y}]\!])$ induces a cellular decomposition of $\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$, the stratum of $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$ with a prescribed Hilbert function h. When working with $\mathbf{k} = \mathbb{C}$, this can be used to calculate the Betti numbers of those spaces. We now give some detailed examples and evidence for Conjecture 3.48.

Example 3.63 (A cellular decomposition of $\mathrm{Hilb}^6(\mathbf{k}[\![x,y]\!])$)**.** Let us consider $n = 6$. There are eleven partitions of 6, so the punctual Hilbert scheme $\mathrm{Hilb}^6(\mathbf{k}[\![x,y]\!])$ has a cellular decomposition into eleven cells. There are four possible Hilbert functions $[1,2,3]$, $[1,2,2,1]$, $[1,2,1,1,1]$ and $[1,1,1,1,1,1]$.

Note that this is significantly different from Briançon's table in [Bri77, Section IV.2] because there the author provides a representative of all possible analytic types of ideals in $\mathrm{Hilb}^6(\mathbf{k}[\![x,y]\!])$, whereas our cells contain ideals with a common leading term ideal but in general different analytic types coexist in the same cell (for example, ideals with different number of generators as discussed in section 3.5).

The Hilbert function $[1,2,3]$ is only attained by the ideal $(x^3, x^2y, xy^2, y^3) = (x,y)^3$. The Gröbner cell consists only of the single point corresponding to the monomial ideal itself. It is minimally generated by four elements.

There are six cells with Hilbert function $[1,2,2,1]$, see Figure 3.1. Admissible parameters to obtain a homogeneous ideal, i.e. those of $\mathcal{V}_{\mathrm{hom}}(E)$, are indicated in bold. General ideals in this cell will be minimally generated by two elements. If $c_2 = 0$ the ideal is generated by three elements.

The other five cells with this Hilbert function are not lex-segment. The cells $[2,2,2]$ and $[3,3]$, corresponding to monomial complete intersection ideals (x^3, y^2) and (x^2, y^3), consist only of complete intersection ideals. The two cells $[1,1,2,2]$ and $[1,1,1,3]$ only contain ideals that are minimally generated by 3 elements while the cell $[1,1,4]$ contains ideals I, J with $\mu(I) = 2$ and $\mu(J) = 3$. Notice that the difference $\dim(\mathcal{V}(E)) - \dim(\mathcal{V}_{\mathrm{hom}}(E)) = 1$ for all E with this Hilbert function.

There are two cells with Hilbert function $[1,2,1,1,1]$, see Figure 3.2. Both contain ideals minimally generated by three elements – in case $c_3 = 0$ for $E_1 = (x^5, xy, y^2)$ or $c_1 = 0$ for

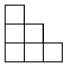
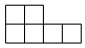
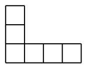
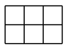
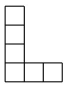
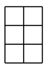
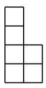
m	boxes	H+N	dimension	μ
		[1,2,3]		
$[1,2,3]$		$\begin{pmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & -x & y \\ 0 & 0 & -x \end{pmatrix}$	0	4
		[1,2,2,1]		
$[1,1,2,2]$		$\begin{pmatrix} y & 0 & 0 & c_1 \\ -x & 1 & 0 & 0 \\ 0 & -x & y & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{pmatrix}$	1	3
$[1,1,1,3]$		$\begin{pmatrix} y & 0 & c_1 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y^2 \\ 0 & 0 & 0 & -x+\boldsymbol{c_2}y \end{pmatrix}$	2	3
$[2,2,2]$		$\begin{pmatrix} y^2 & 0 & c_1y \\ -x+\boldsymbol{c_2}y & 1 & 0 \\ 0 & -x & 1 \\ 0 & 0 & -x \end{pmatrix}$	2	2
$[1,1,4]$		$\begin{pmatrix} y & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^3 \\ \boldsymbol{c_1} & 0 & -x+\boldsymbol{c_2}y+c_3y^2 \end{pmatrix}$	3	2,3
$[3,3]$		$\begin{pmatrix} y^3 & 0 \\ -x+\boldsymbol{c_1}y+c_2y^2 & 1 \\ \boldsymbol{c_3}y^2 & -x \end{pmatrix}$	3	2
$[2,4]$		$\begin{pmatrix} y^2 & 0 \\ -x+\boldsymbol{c_1}y & y^2 \\ \boldsymbol{c_2}+c_3y & -x+\boldsymbol{c_4}y \end{pmatrix}$	4	2,3

Figure 3.1: Gröbner cells for Hilbert functions [1,2,3] and [1,2,2,1]

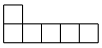
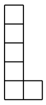
m	boxes	H+N	dimension	μ
		$[1,2,1,1,1]$		
$[1,1,1,1,2]$		$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	3	2,3
$[1,5]$		$\begin{pmatrix} y & 0 \\ -x & y^4 \\ c_1 & -x+\mathbf{c_2}y+c_3y^2+c_4y^3 \end{pmatrix}$	4	2,3

Figure 3.2: Gröbner cells with Hilbert function [1,2,1,1,1]

$E_2 = (x^2, xy, y^5)$ – and complete intersection ideals. The difference of dimension of the cell and the homogeneous sub-cell is

$$\dim(\mathcal{V}(E_1)) - \dim(\mathcal{V}_{\text{hom}}(E_1)) = \dim(V(E_2)) - \dim(\mathcal{V}_{\text{hom}}(E_2)) = 3.$$

The Hilbert function $[1,1,1,1,1,1]$ is attained by the monomial ideals (x^6, y) and the lex-segment monomial ideal (x, y^6). Both cells consist only of complete intersection ideals and have dimensions 4 and 5, see Figure 3.3. The cell of $m = [6]$ is the dense open subset of the punctual Hilbert scheme $\text{Hilb}^6(\mathbf{k}[\![x,y]\!])$. The difference of dimension of the cell and the homogeneous sub-cell is

$$\dim(\mathcal{V}((x^6,y))) - \dim(\mathcal{V}_{\text{hom}}((x^6,y))) = \dim(\mathcal{V}((x,y^6))) - \dim(\mathcal{V}_{\text{hom}}((x,y^6))) = 4.$$

As stated before, for non-relax-segment ideals E the set $\mathcal{N}_{<\underline{d}}(E)$ contains matrices with non-zero entries above the diagonal. This is the case for $m = (1,1,2,2), (1,1,1,3), (2,2,2), (1,1,1,1,2)$ and $(1,1,1,1,1,1)$. For those ideals we checked by comparing to the reduced standard basis that the map from Conjecture 3.48 gives a parametrization of the Gröbner cell.

We can investigate the dimensions of the occurring cells. When we define a_i as the number of cells of dimension i, we obtain

$$a = (a_0, a_1, a_2, a_3, a_4, a_5) = (1,1,2,3,3,1).$$

This vector is an invariant of the space.

When a scheme X over $\mathbb{C}$ has a cellular decomposition with dimension vector a as defined in Example 3.63, then all other cellular decompositions of X will have the same dimension vector. It holds that $a_i = b_{2i}(X)$, the $2i$-th Betti number of X, and that the Betti numbers $b_{2i+1}(X) = 0$, see [Bia73, Theorem 4.4/4.5] and [Ful98, Chapter 19.1].

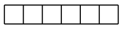
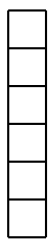
m	boxes	H+N	dimension	μ
		$[1,1,1,1,1,1]$		
$[1]^6$		$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	4	2
$[6]$		$\begin{pmatrix} y^6 \\ -x + \boldsymbol{c_1} y + c_2 y^2 + c_3 y^3 + c_4 y^4 + c_5 y^5 \end{pmatrix}$	5	2

Figure 3.3: Gröbner cells with Hilbert function [1,1,1,1,1,1]

In [ES87] this method is used to calculate the Betti numbers of the Hilbert scheme of points of the projective plane, the affine plane and, of most interest to us, the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$. In [Göt90] the same methods were used to calculate the Betti numbers of the stratum $\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$. These results were obtained by representation theoretical methods without giving explicit parametrizations of the cells.

To state the theorem of [ES87], we need the following definition.

Definition 3.64. Let $l, n \in \mathbb{Z}_{>0}$, then we define $P(n,l)$ as the number of partitions of n bounded by l, i.e. as the number of sequences $0 = m_0 < m_1 \leq \cdots \leq m_t \leq l$ such that $\sum_{i=1}^t m_i = n$.

Theorem 3.65. *[ES87, Theorem 1.1 (iv)] The non-zero Betti numbers of the punctual Hilbert scheme* $\mathrm{Hilb^n}(\mathbf{k}[\![\mathrm{x},\mathrm{y}]\!])$ *are*

$$b_{2i}(\mathrm{Hilb^n}(\mathbf{k}[\![\mathrm{x},\mathrm{y}]\!])) = P(i, n-i).$$

This result gives us a way of checking Conjecture 3.48 for plausibility by checking whether $\#\{E \mid \dim(\mathcal{N}_{<\underline{d}}) = i\} = P(i, n-i)$.

We created a Julia [BEKS17] module that uses Oscar.jl [23] to calculate examples and perform this plausibility check. The module and Jupyter-notebooks with experiments are available at https://github.com/anelanna/LocalHilbertBurch.jl.

The function `sorted_celllist(n)` of the module can calculate the (conjectural) cellular decomposition for a given n. For all partitions of n it creates a `Cell` that has the following properties:

- `m` – the partition,
- `E` – its associated monomial ideal,
- `d` – the vector of differences,
- `U` – the degree matrix,

- `hilb` – the Hilbert function of the ideals in the cell,
- `H` – the canonical Hilbert-Burch matrix of the monomial ideal,
- `M` – the general canonical Hilbert-Burch matrix $H + N$,
- `N` – the corresponding element in $\mathcal{N}_{<\underline{d}}(E)$,
- `I` – the maximal minors of $M = H + N$,
- `dim` – the dimension of the cell,
- `M_hom` – the general canonical Hilbert-Burch matrix of a homogeneous ideal,
- `N_hom` – the corresponding matrix N, and
- `dim_hom` – the dimension of the homogeneous sub-cell cell.

The output of the function `sorted_celllist(n)` is a dictionary mapping an integer i to a vector with the cells of dimension i. We use dictionaries rather than vectors, since Julia indexes from 1 and we wanted to avoid confusion by having to subtract 1 in various places to get to the correct entry. To avoid having to store a lot of matrices, we also introduced a structure `SmallCell` that only has the properties `m`, `hilb`, `dim` and `dim_hom`. The function `sorted_celllist(n)` can be called as `sorted_celllist(SmallCell, n)`, then the output will be a dictionary mapping i to a vector of `SmallCell`s of dimension i. By recording the sizes of the vectors in the dictionary, we obtain a dictionary of number of cells of dimension i. We can understand it as the vector a.

The numbers of bounded partitions can be found in the On-Line Encyclopedia of Integer Sequences, [Inc23], as (diagonals in) A008284, or relabeled, and thus serving our purposes a bit better, as (diagonals in) A058398 (see https://oeis.org/A008284 and https://oeis.org/A058398).

For $n \leq 50$ we calculated all dimension vectors of our proposed cellular decomposition and checked that they are the correct ones. We have included them for $n \leq 30$ in Table 3.1.

Example 3.66 (Example 3.63 continued)**.** We can use the connection between cellular decompositions and Betti numbers described above to calculate the Betti numbers of $\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$ for h a Hilbert function in the $n = 6$ case. We obtain the following Betti numbers:

$$b_i(\mathrm{Hilb}^{(1,2,3)}(\mathbf{k}[\![x,y]\!])) = \begin{cases} 1, & i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$b_i(\mathrm{Hilb}^{(1,2,2,1)}(\mathbf{k}[\![x,y]\!])) = \begin{cases} 1, & i = 2, 8; \\ 2, & i = 4, 6; \\ 0, & \text{otherwise.} \end{cases}$$

$$b_i(\mathrm{Hilb}^{(1,1,1,2,1)}(\mathbf{k}[\![x,y]\!])) = \begin{cases} 1, & i = 6, 8; \\ 0, & \text{otherwise.} \end{cases}$$

$$b_i(\mathrm{Hilb}^{(1,1,1,1,1,1)}(\mathbf{k}[\![x,y]\!])) = \begin{cases} 1, & i = 8, 10; \\ 0, & \text{otherwise.} \end{cases}$$

n	the Betti numbers // dimensions vector
1	1
2	1, 1
3	1, 1, 1
4	1, 1, 2, 1
5	1, 1, 2, 2, 1
6	1, 1, 2, 3, 3, 1
7	1, 1, 2, 3, 4, 3, 1
8	1, 1, 2, 3, 5, 5, 4, 1
9	1, 1, 2, 3, 5, 6, 7, 4, 1
10	1, 1, 2, 3, 5, 7, 9, 8, 5, 1
11	1, 1, 2, 3, 5, 7, 10, 11, 10, 5, 1
12	1, 1, 2, 3, 5, 7, 11, 13, 15, 12, 6, 1
13	1, 1, 2, 3, 5, 7, 11, 14, 18, 18, 14, 6, 1
14	1, 1, 2, 3, 5, 7, 11, 15, 20, 23, 23, 16, 7, 1
15	1, 1, 2, 3, 5, 7, 11, 15, 21, 26, 30, 27, 19, 7, 1
16	1, 1, 2, 3, 5, 7, 11, 15, 22, 28, 35, 37, 34, 21, 8, 1
17	1, 1, 2, 3, 5, 7, 11, 15, 22, 29, 38, 44, 47, 39, 24, 8, 1
18	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 40, 49, 58, 57, 47, 27, 9, 1
19	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 41, 52, 65, 71, 70, 54, 30, 9, 1
20	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 54, 70, 82, 90, 84, 64, 33, 10, 1
21	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 55, 73, 89, 105, 110, 101, 72, 37, 10, 1
22	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 75, 94, 116, 131, 136, 119, 84, 40, 11, 1
23	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 76, 97, 123, 146, 164, 163, 141, 94, 44, 11, 1
24	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 99, 128, 157, 186, 201, 199, 164, 108, 48, 12, 1
25	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 100, 131, 164, 201, 230, 248, 235, 192, 120, 52, 12, 1
26	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 133, 169, 212, 252, 288, 300, 282, 221, 136, 56, 13, 1
27	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 134, 172, 219, 267, 318, 352, 364, 331, 255, 150, 61, 13, 1
28	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 174, 224, 278, 340, 393, 434, 436, 391, 291, 169, 65, 14, 1
29	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 175, 227, 285, 355, 423, 488, 525, 522, 454, 333, 185, 70, 14, 1
30	1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 229, 290, 366, 445, 530, 598, 638, 618, 532, 377, 206, 75, 15, 1

Table 3.1: The number of cells of dimension i in the cellular decomposition of $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$ for $n = 1, \ldots, 30$. As discussed before these are the Betti numbers of $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$.

More generally, we can describe the Betti numbers of the subscheme $\mathrm{Hilb}^{(1,1,\dots,1)}(\mathbf{k}[\![x,y]\!])$ of the punctual Hilbert scheme for any n. Note that this was also stated in [Göt90, Remark 2.2a]

Lemma 3.67. Set $h = (1,\dots,1)$ with $\sum_{i=0}^{s} h_i = n$. Then $\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$ has only two non-vanishing Betti numbers:

$$b_{2(n-2)}(\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])) = b_{2(n-1)}(\mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])) = 1.$$

Proof. There are only two cells with this Hilbert function, namely $L = (x, y^n)$ and $E = (x^n, y)$. The ideal L is the lex-segment ideal with this Hilbert function and its cell $\mathcal{V}(L)$ has dimension $n-1$. By considering the reduced $\overline{\tau}$-enhanced standard basis of E we can verify that Conjecture 3.48 holds in this case or just directly show that $\dim(\mathcal{V}(E)) = n-2$. So the cellular decomposition has one cell of dimension $n-2$ and one of dimension $n-1$. □

We study another example in more detail, and refer to the appendix for similar tables with the cellular decompositions of $\mathrm{Hilb^n(\mathbf{k}[\![x, y]\!])}$ for $n = 1,2,3,4,5,8,9$, in Table 3.9 – Table 3.17.

Example 3.68. (A cellular decomposition of $\mathrm{Hilb}^7(\mathbf{k}[\![x,y]\!])$) Let us now consider the case of $\mathrm{Hilb}^7(\mathbf{k}[\![x,y]\!])$. There are 15 cells corresponding to the 15 partitions of the number 7 and five admissible Hilbert functions $(1,2,3,1),(1,2,2,2),(1,2,1,1,1,1)$ and $(1,1,1,1,1,1,1)$. By Conjecture 3.48, we get the following parametrizations of the corresponding cells. By comparing to reduced standard bases we checked that this is really a parametrization of the Gröbner cells.

- $(1,2,3,1)$: The four cells with this Hilbert function have dimensions zero to three. Hence $\mathrm{Hilb}^{(1,2,3,1)}(\mathbf{k}[\![x,y]\!])$ has the following non-vanishing Betti numbers $b_0 = b_2 = b_4 = b_6 = 1$.

  Notice that all ideals with this Hilbert function are homogeneous, so $\dim(\mathcal{V}(E)) - \dim(\mathcal{V}_{\mathrm{hom}}(E)) = 0$ for all $E \in \mathrm{Hilb}^h(\mathbf{k}[\![x,y]\!])$.

m	M	dimension
$[1,1,2,3]$	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & y & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -x \end{pmatrix}$	0
$[2,2,3]$	$\begin{pmatrix} y^2 & 0 & 0 \\ -x+\mathbf{c_1}y & 1 & 0 \\ 0 & -x & y \\ 0 & 0 & -x \end{pmatrix}$	1
$[1,3,3]$	$\begin{pmatrix} y & 0 & 0 \\ -x & y^2 & 0 \\ \mathbf{c_1} & -x+\mathbf{c_2}y & 1 \\ 0 & 0 & -x \end{pmatrix}$	2
$[1,2,4]$	$\begin{pmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & -x & y^2 \\ \mathbf{c_1} & \mathbf{c_2} & -x+\mathbf{c_3}y \end{pmatrix}$	3

- $(1,2,2,2)$: The three cells with this Hilbert functions have dimensions two to four, so $\mathrm{Hilb}^{(1,2,2,2)}(\mathbf{k}[\![x,y]\!])$ has the following non-zero Betti numbers $b_4 = b_6 = b_8 = 1$. The difference of dimension is $\dim(\mathcal{V}(E)) - \dim(\mathcal{V}_{\mathrm{hom}}(E)) = 2$.

m	M	dimension
$[1,2,2,2]$	$\begin{pmatrix} y & 0 & 0 & c_1 \\ -x & y & 0 & c_2 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{pmatrix}$	2
$[1,1,1,4]$	$\begin{pmatrix} y & 0 & c_1 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y^3 \\ 0 & 0 & 0 & -x+\mathbf{c_2}y+c_3y^2 \end{pmatrix}$	3
$[3,4]$	$\begin{pmatrix} y^3 & 0 \\ -x+\mathbf{c_1}y+c_2y^2 & y \\ \mathbf{c_3}y+c_4y^2 & -x \end{pmatrix}$	4

- $(1,2,2,1,1)$: This is the Hilbert function of four cells with dimensions three to five. The non-vanishing Betti numbers of $\mathrm{Hilb}^{(1,2,2,1,1)}(\mathbf{k}[\![x,y]\!])$ are $b_6 = b_{10} = 1$ and $b_8 = 2$. The difference of dimension in this stratum is $\dim(\mathcal{V}(E)) - \dim(\mathcal{V}_{\mathrm{hom}}(E)) = 3$.

m	M	dimension
$[1,1,1,2,2]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & y & 0 \\ 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	3
$[1,1,1,1,3]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & y^2 \\ 0 & 0 & 0 & 0 & -x+\mathbf{c_4}y \end{pmatrix}$	4
$[1,1,5]$	$\begin{pmatrix} y & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^4 \\ c_1 & 0 & -x+\mathbf{c_2}y+c_3y^2+c_4y^3 \end{pmatrix}$	4
$[2,5]$	$\begin{pmatrix} y^2 & 0 \\ -x+\mathbf{c_1}y & y^3 \\ c_2+c_3y & -x+\mathbf{c_4}y+c_5y^2 \end{pmatrix}$	5

- $(1,2,1,1,1,1)$: The non-vanishing Betti numbers of $\mathrm{Hilb}^{(1,2,1,1,1,1)}(\mathbf{k}[\![x,y]\!])$ are $b_8 = b_{10} = 1$.

The difference of dimensions in this stratum is $\dim(\mathcal{V}(E)) - \dim(\mathcal{V}_{\text{hom}}(E)) = 4$.

m	M	dimension
$[1,1,1,1,1,2]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	4
$[1,6]$	$\begin{pmatrix} y & 0 \\ -x & y^5 \\ c_1 & -x + \mathbf{c_2} y + c_3 y^2 + c_4 y^3 + c_5 y^4 \end{pmatrix}$	5

- $(1,1,1,1,1,1,1)$: The stratum $\text{Hilb}^{(1,1,1,1,1,1,1)}(\mathbf{k}[\![x,y]\!])$ has two cells. They correspond to the ideal (x^7, y) and (x, y^7) which is the maximal cell of this punctual Hilbert scheme. The non-vanishing Betti numbers are $b_{10} = b_{12} = 1$, as expected by Lemma 3.67. The difference of dimensions in this stratum is $\dim(\mathcal{V}(E)) - \dim(\mathcal{V}_{\text{hom}}(E)) = 5$.

m	M	dimension
$[1,1,1,1,1,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	5
$[7]$	$\begin{pmatrix} y^7 \\ -x + \mathbf{c_1} y + c_2 y^2 + c_3 y^3 + c_4 y^4 + c_5 y^5 + c_6 y^6 \end{pmatrix}$	6

The associated dimension vector of the punctual Hilbert scheme is $a = (1,1,2,3,4,3,1)$.

We have seen in Examples 3.63 and 3.68 that for each stratum $\text{Hilb}^h(\mathbf{k}[\![x,y]\!])$ of the punctual Hilbert scheme $\text{Hilb}^n(\mathbf{k}[\![x,y]\!])$ the difference of dimensions of $\mathcal{V}(E)$ and $\mathcal{V}_{\text{hom}}(E)$ is constant for $E \in \text{Hilb}^h(\mathbf{k}[\![x,y]\!])$. This is due to a result by Iarrobino:

Theorem 3.69. *[Iar77, Theorem 2.11, Theorem 3.14] The stratum* $\text{Hilb}^h(\mathbf{k}[\![x,y]\!])$ *is a locally trivial bundle over* $\text{Hilb}^h_{\text{hom}}(\mathbf{k}[\![x,y]\!])$ *having fiber, an affine space, and a global section.*

The Gröbner cells $\mathcal{V}(E)$ where E are the monomial ideals with Hilbert function h form a cellular decomposition of $\text{Hilb}^h(\mathbf{k}[\![x,y]\!])$, and the homogeneous sub-cells $\mathcal{V}_{\text{hom}}(E)$ form a cellular decomposition of $\text{Hilb}^h_{\text{hom}}(\mathbf{k}[\![x,y]\!])$. By Lemma 3.46 the fibration restricts to $\mathcal{V}(E) \to \mathcal{V}_{\text{hom}}(E)$.
Since we also know the dimension of $\mathcal{V}_{\text{hom}}(E)$ by Lemma 3.46, we can check (at least in examples) whether the difference $\dim(\mathcal{N}_{<\underline{d}}(E)) - \dim(\mathcal{V}_{\text{hom}}(E))$ is constant for all E with the same Hilbert function. If this is the case, it implies that $\dim(\mathcal{N}_{<\underline{d}}(E)) = \dim(\mathcal{V}(E))$. Using our `Julia` module we have checked that for all strata $\text{Hilb}^h(\mathbf{k}[\![x,y]\!])$ of $\text{Hilb}^n(\mathbf{k}[\![x,y]\!])$ with $n \le 50$ these differences are constant providing strong evidence for Conjecture 3.48.

3.7 Back to the polynomial ring: A surjection to the Gröbner cell

Another way to generalize the results by [CV08] and [Con11] is to keep the polynomial ring $P = \mathbf{k}[x, y]$ but change the term ordering τ.

One application of parametrizations of Gröbner cells with other term orders could be to study the intersection of two Gröbner cells with respect to different term orderings, i.e. to parametrize all ideals I that have initial ideal E with respect to two different orderings, we do this in section 3.9. Similar to the results in [Con11, Chapter 6], where the parametrization of Gröbner cells with respect to deglex is used to parameterize homogeneous ideals in $\mathbf{k}[x, y, z]$, one could use the parametrization of Gröbner cells with respect to other orders to study ideals in $\mathbf{k}[x, y, z]$ that are homogeneous with respect to some non-standard grading.

Varying term orders is not a new idea: in [MR88] all possible initial ideals for a given ideal I are studied. The cones in the Gröbner fan of an ideal describe the term orders for which the initial ideal stays the same, see also [Stu96] for an introduction to the topic. In [AS05] the authors start from a different perspective. They define a graph that has all monomial ideal as vertices, and two vertices corresponding to monomial ideals E and E' are joined by an edge whenever there exists an ideal I such that E and E' are the only possible occurring initial ideals with respect to any given term order. These ideals I are called *edge providing ideals*. Inside the paper they define the Schubert scheme $\Omega_c(E)$ of E in direction of c and then the edge providing ideals are parametrized by the intersection $\Omega_c(E) \cap \Omega_{-c}(E')$. The Schubert schemes in direction c are affine schemes, but not affine spaces as our Gröbner cells. The parametrization is done by considering reduced Gröbner bases.

For a general term order τ on $\mathbf{k}[x, y]$ we want to parametrize the set $V_\tau(E)$ of all ideals with leading term ideal E with respect to τ. Remember that by [Bia73, Theorem 4.4] the $V_\tau(E)$ are affine spaces. A term order defines an action of $\mathbf{k}^*$ on the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbf{k}[x, y])$. The fixed points of the action are the monomial ideals, and thence isolated, and $\mathrm{Hilb}^n(\mathbf{k}[x, y])$ is smooth. The sets of ideals specializing to a given fixed point E, are exactly the ideals in $V_\tau(E)$. As before H will denote the canonical Hilbert-Burch matrix of E. We want to find a set of matrices $\mathcal{N}_\tau(E) \subset k[y]^{t+1\times t}$, such that

$$\begin{array}{rccl} \Phi_{\tau,E}: & \mathcal{N}_\tau(E) & \to & V_\tau(E) \\ & N & \mapsto & I_t(H+N) \end{array}$$

is a parametrization.

First we recall what term orders on $\mathbf{k}[x, y]$ look like. Given a weight vector $\omega = (a, b) \in \mathbb{Z}^2_{\geq 0}$, we define $\leq_\omega$ on $\mathbf{k}[x, y]$ as

$$x^\alpha y^\beta \leq_\omega x^{\alpha'} y^{\beta'} \quad \text{if and only if} \quad \deg_\omega(x^\alpha y^\beta) = a\alpha + b\beta \leq \deg_\omega(x^{\alpha'} y^{\beta'}) = a\alpha' + b\beta',$$

where $\deg_\omega(M)$ denotes the weighted degree of the monomial M. The relation given by $\leq_\omega$ does not define a term order on $\mathbf{k}[x, y]$, since different monomials will have the same weighted degree. On a finite set of monomials all different term orders can be represented by an $\leq_\omega$ for a suitable ω, see for example [Stu96, Proposition 1.11]. Notice that ω defines the $\mathbf{k}^*$-action that was mentioned in the previous paragraph. Since we do not want to make any restrictions to guarantee that $\leq_\omega$ is a term order, we let $\tau := \tau_\omega$ be the term order on $\mathbf{k}[x, y]$ where

$$M <_\tau M' \quad \text{if and only if} \quad \begin{array}{l} \deg_\omega(M) < \deg_\omega(M') \text{ or} \\ \deg_\omega(M) = \deg_\omega(M') \text{ and } M <_{\text{lex}} M'. \end{array}$$

The facts that in $\mathrm{Hilb}^n(\mathbf{k}[x,y])$ only finitely many monomials play a role, and on a finite set of monomials all term orders can be represented by a $\leq_\omega$ for a suitable ω ensure that by considering the Gröbner cells with respect to these τ we really describe Gröbner cells with respect to all possible term orders. Whenever $b = 0$, then the term order induced by ω is lex. Since the lexicographic term order is also induced by $\omega = (a,\ b)$ with $a \gg b > 0$, we will assume from now on that $b \neq 0$. We will allow $a = 0$. In that case we obtain the lexicographic order with $y > x$ which we denote as $\mathrm{lex}_{y>x}$.

Definition 3.70. For a zero-dimensional monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \ldots, xy^{m_{t-1}}, y^{m_t}) \subset \mathbf{k}[x,y]$ we define $W(E) = (w_{i,j})$ *the weighted degree matrix of* E by

$$w_{i,j} := m_j - m_{i-1} + \frac{a}{b}(i-j).$$

Different to the usual degree matrix $U(E)$ that only has integer values, the weighted degree matrix $W(E)$ can have non-integer entries.

We define the subset $\mathcal{N}_\tau(E) \subset \mathbf{k}[y]^{t+1\times t}$ as the set of matrices $N = (n_{i,j})$ whose non-zero entries satisfy the following degree bounds:

$$\deg(n_{i,j}) \begin{cases} < w_{i,j} & i \leq j \\ \leq w_{i,j} & i > j, \end{cases} \tag{3.6}$$

or the equivalent version with integer values:

$$\deg(n_{i,j}) \leq \begin{cases} \lceil w_{i,j} \rceil - 1 & i \leq j \\ \lfloor w_{i,j} \rfloor & i > j. \end{cases} \tag{3.7}$$

While condition 3.7 looks better for specific values of a and b, condition 3.6 is sometimes easier to work with, so in some cases we will use one and in other cases the other description.

Note that whenever $w_{i,j} \notin \mathbb{Z}$ all the degree bounds become the same, i.e.

$$\deg(n_{i,j}) < w_{i,j} \iff \deg(n_{i,j}) \leq w_{i,j} \iff \deg(n_{i,j}) \leq \lceil w_{i,j} \rceil - 1 = \lfloor w_{i,j} \rfloor.$$

Remember that $w_{i,i} = d_i \in \mathbb{Z}$.

Theorem 3.71. *Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t}) \subset \mathbf{k}[x,y]$ be a monomial ideal with canonical Hilbert-Burch matrix H and $\mathcal{N}_\tau(E)$ be the set of matrices from Definition 3.70. Then the map*

$$\begin{aligned} \Phi_{\tau,E} :\ \mathcal{N}_\tau(E) &\to V_\tau(E) \\ N &\mapsto I_t(H+N) \end{aligned}$$

is surjective.

In particular, for $N \in \mathcal{N}_\tau(E)$ the t-minors of $H+N$ form a Gröbner basis with respect to τ.

We will prove this theorem in two steps, well-definedness in Lemma 3.72 and surjectivity in Lemma 3.73.

Lemma 3.72. The map $\Phi_{\tau,E}$ is well defined.

Proof. Let $I := I_t(H+N)$ be the ideal of maximal minors of $H+N$. We will show that $\mathrm{Lt}_\tau(I) = E$ by first showing that the leading terms of the minors are the correct ones and then showing that they form a τ-Gröbner basis.

(i) Let f_i be the signed i-th minor of $H+N$, i.e. $f_i = (-1)^i \det([H+N]_{i+1})$ where $[A]_j$ denotes the matrix obtained of A by deleting its j-th row. By construction of the weighted degree matrix $W(E)$ the minors are homogeneous with respect to the weight vector ω if and only if the entries of N have exactly degree $w_{i,j} = m_j - m_{i-1} + \frac{a}{b}(i-j)$. Thus, if we have a summand of $n_{i,j}$ of lower degree, the resulting summand of the minor will have a smaller weighted degree as well. The cases where equality occurs are the most interesting, since $\mathrm{Lt}_\tau(f_i) = \mathrm{Lt}_\tau(\mathrm{in}_\omega(f_i))$. So let $M := x^\alpha y^\beta$ be a monomial in the support of f_i, then we have to show that $M \leq_\tau x^{t-i}y^{m_i}$. If $M \notin \mathrm{Supp}(\mathrm{in}_\omega(f_i))$, then $\deg_\omega(M) < \deg_\omega(x^{t-i}y^{m_i})$ and thence $M <_\tau x^{t-i}y^{m_i}$.

   So let $M = x^\alpha y^\beta \in \mathrm{Supp}(\mathrm{in}_\omega(f_i))$, then $M <_\tau x^{t-i}y^{m_i}$ if and only if $\alpha < t-i$, and this is guaranteed by requiring that $\deg(n_{i,j}) \leq \lceil w_{i,j} \rceil - 1$ for $i \leq j$.

(ii) We show that the set $\{f_0, \ldots, f_t\}$ of signed minors of $H+N$ forms a Gröbner basis of I.

   The argument is the same as in the proof of well-definedness in [Con11, Theorem 3.1] and reproduced here. Since the syzygy-module of E is generated by the columns of its Hilbert-Burch matrix H, we only have to consider the S-polynomials of the form $y^{d_i} f_{i-1} - x f_i$ for $i = 1, \ldots, t$ and show that they can be written as $\sum_{j=0}^t Q_j f_j$ with $\mathrm{Lt}_\tau(Q_j f_j) \leq_\tau \mathrm{Lt}_\tau(y^{d_i} f_{i-1} - x f_i)$, see [KR00, Remark 2.5.6].

   By construction, we know that

$$y^{d_i} f_{i-1} - x f_i + \sum_{j=0}^{t} n_{j+1,i} f_j = 0$$

   The $n_{i,j}$ are polynomials in $\mathbf{k}[y]$ and all the leading terms of the f_j are products of different powers of x with a monomial in y, so the same applies for the leading terms of the $n_{i,j} f_j$ and they cannot cancel each other. Thence $\max_j \{\mathrm{Lt}_\tau(n_{i,j} f_j) \mid n_{i,j} \neq 0\} = \mathrm{Lt}_\tau(y^{d_i} f_{i-1} - x f_i)$ and $\{f_0, \ldots, f_t\}$ forms a Gröbner basis of I. So the map $\Phi_{\tau,E}$ is well-defined.

□

Lemma 3.73. The map $\Phi_{\tau,E}$ is surjective.

Proof. Let I be an ideal in $V_\tau(E)$, that is $\mathrm{Lt}_\tau(I) = E$. So there exists a Gröbner basis $\{f_0, \ldots, f_t\} \subset I$ such that $\mathrm{Lt}_\tau(f_i) = x^{t-i}y^{m_i}$. We can additionally assume that no monomial in the support of $f_0, \ldots, f_t$ is divisible by x^t, except for $\mathrm{Lt}_\tau(f_0)$. Then no monomial in the support of $S_i := y^{d_i} f_{i-1} - x f_i$ is divisible by x^{t+1}. Since $S_i \in I$ and $\{f_0, \ldots, f_t\}$ forms a Gröbner basis the reduction via the Buchberger algorithm will yield zero. To see that the polynomials obtained by the Buchberger algorithm satisfy the conditions we will study the reduction procedure in more detail.

The reduction proceeds in the following way: We determine $\mathrm{Lt}_\tau(S_i) = x^\alpha y^\beta$. By assumption $\alpha < t+1$ and find the minimal j such that $\mathrm{Lt}_\tau(f_j) = x^{t-j}y^{m_j}$ divides $\mathrm{Lt}_\tau(S_i)$. Then we set $Q = \frac{\mathrm{Lt}_\tau(S_i)}{\mathrm{Lt}_\tau(f_j)}$. Since we chose the minimal j, $Q = y^{\beta - m_j}$. We continue the process with $S_i - Q f_j$. This polynomial has lower degree and no monomial in its support is divisible by x^{t+1}, so we can apply the same argument. Since $S_i \in I$ and $\{f_0, \ldots, f_t\}$ is a Gröbner basis we will reach 0 after a finite number of steps. The polynomial $n_{j+1,i}$ is the sum of the different Q from the steps where the initial term was $x^{t-j}y^\bullet$, so in particular, it holds $n_{j+1,i} \in \mathbf{k}[y]$.

It remains to show the degree condition. Since

$$\mathrm{Lt}_\tau(S_i) = x^{t-j}y^\beta <_\tau x^{t-i+1}y^{m_i} = \mathrm{Lt}_\tau(y^{d_i}f_{i-1}) = \mathrm{Lt}_\tau(xf_i),$$

it holds that $\deg_\omega(S_i) = a(t-j) + b\beta \le a(t-i+1) + bm_i$. This yields $\beta \le \frac{a}{b}(j-i+1) + m_i$ and thus

$$\deg(n_{j+1,i}) = \deg(y^{\beta - m_j}) = \beta - m_j \le m_i - m_j - \frac{a}{b}(j+1-i) = w_{j+1,i}.$$

But since we have that $\mathrm{Lt}_\tau(S_i)$ is strictly smaller than $x^{t-i+1}y^{m_i}$ equality can only happen when $t-j < t-i+1$ and that is the case for $i \le j$.

Renaming the indices, we obtain the desired degree bounds for $n_{k,l}$:

$$\deg(n_{k,l}) \begin{cases} < w_{k,l} & k \le l \\ \le w_{k,l} & k > l \end{cases}.$$

□

As in Lemma 3.23 the proof of surjectivity gives an algorithm to construct the matrix N from a given τ-Gröbner basis $\{f_0, f_1, \dots, f_t\}$, where $\mathrm{Lt}_\tau(f_i) = x^{t-i}y^{m_i}$ and no term in the support of the f_i is divisible by x^t except for $\mathrm{Lt}_\tau(f_0)$.

Example 3.74. Let us consider $m = (2, 3, 5, 7)$. The associated monomial ideal is $E = (x^4, x^3y^2, x^2y^3, xy^5, y^7)$ with canonical Hilbert-Burch matrix

$$H = \begin{pmatrix} y^2 & 0 & 0 & 0 \\ -x & y & 0 & 0 \\ 0 & -x & y^2 & 0 \\ 0 & 0 & -x & y^2 \\ 0 & 0 & 0 & -x \end{pmatrix}.$$

Let τ be the term order induced by $\omega = (3,\ 2)$ and $I \in V_\tau(E)$ be the following ideal

$$I = (\mathbf{x^4} - x^2y^3 - 2x^2y^2,\ \mathbf{x^3y^2} - xy^5 - 2xy^4,\ \mathbf{x^2y^3},\ \mathbf{xy^5},\ \mathbf{y^7} + 2x^3y + 2xy^4 + x^3 - 3xy^3 - 2xy^2).$$

The given generators of I form a Gröbner basis with respect to τ with $\mathrm{Lt}_\tau(f_i) = x^{4-i}y^{m_i}$ and no monomial in the support of the f_i is divisible by x^4 for $i = 1, 2, 3, 4$. By the procedure described in the proof of Lemma 3.73 we obtain the following preimage of I under the map $\Phi_{\tau,E}$:

$$N = \begin{pmatrix} 0 & 0 & 0 & 2y+1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4y+2 \\ 0 & y+2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{N}_\tau(E).$$

The degrees of the non-zero entries of N are

- $\deg(n_{1,4}) = 1 < w_{1,4} = m_4 - m_0 + \frac{3}{2}(1-4) = \frac{5}{2}$,
- $\deg(n_{3,4}) = 1 < w_{3,4} = m_4 - m_2 + \frac{3}{2}(3-4) = \frac{3}{2}$,

- $\deg(n_{4,2}) = 1 \leq w_{4,2} = m_2 - m_3 + \frac{3}{2}(4-2) = 1$.

All of them satisfy the degree bounds of $W(3,2)$ from Equation 3.6 and Equation 3.7. To not overload the notation we just write $w_{i,j}$ for the entries of $W(a,b)$ whenever the corresponding $(a,\ b)$ is clear from context.

The given generators of I do not form a Gröbner basis with respect to the term order induced by $(1,\ 1)$ – the degreelexicographic order deglex. We can see this already by looking at the degrees of the non-zero entries of N and comparing them to $w_{1,4} = u_{1,4} = 4$, $w_{3,4} = u_{3,4} = 3$ and $w_{4,2} = u_{4,2} = 0$. It holds that $\deg(n_{4,2}) = 1 \not\leq u_{4,2} = 0$. Indeed, the leading terms of the τ-Gröbner basis $\{f_0, \ldots, f_4\}$ are not the right ones with respect to deglex: $\mathrm{Lt}_{\mathrm{deglex}}(f_0) = x^2y^3 \neq x^4$ and $\mathrm{Lt}_{\mathrm{deglex}}(f_1) = xy^5 \neq x^3y^2$. We can reduce f_0 by f_2, and f_1 with f_3 and obtain a Gröbner basis of I with respect to deglex:

$$\{x^4 - 2x^2y^2,\ x^3y^2 - 2xy^4,\ x^2y^3,\ xy^5,\ y^7 + 2x^3y + 2xy^4 + x^3 - 3xy^3 - 2xy^2\}.$$

Those generators give another preimage of I under $\Phi_{\tau,E}$:

$$\tilde{N} = \begin{pmatrix} 0 & 0 & 0 & 2y+1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2y+1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{N}_\tau(E).$$

Different to the preimage $N = \Phi_{\tau,E}^{-1}(I) \notin \mathcal{N}_{\mathrm{deglex}}(E)$, the preimage $\tilde{N}$ also satisfies the degree bound $\deg(\tilde{n}_{4,2}) \leq u_{4,2}$, and consequently $\tilde{N} \in \mathcal{N}_{\mathrm{deglex}}(E)$. In the following section – section 3.8 – we will discuss a procedure how to obtain $\tilde{N}$ from N without considering the associated Gröbner basis.

Let us consider I with respect to term orders induced by some other $(a,\ b)$. In the following table, the matrix $W(a,b)$ and the (integer) degree bounds that are induced by it (see Equation 3.7) can be found for different values of $(a,\ b)$. For the term orders τ induced by the first four values of $(a,\ b)$ $x >_\tau y$. The matrix W for $(1,\ 1)$ is just the usual degree matrix U. For $(1,\ 18)$ and $(1,\ 100)$ the degree bounds induced by $W(a,b)$ are equal. This is in compliance with the fact that both values of $(a,\ b)$ induce $\mathrm{lex}_{y>x}$ – the lexicographic order with x being smaller than y – on $\mathrm{Hilb}^{17}(\mathbf{k}[x,y])$.

$(a,\ b)$	$W(a,b)$	degree bounds induced by $W(a,b)$
$(100,\ 1)$	$\begin{pmatrix} 2 & -97 & -195 & -293 \\ 100 & 1 & -97 & -195 \\ 199 & 100 & 2 & -96 \\ 297 & 198 & 100 & 2 \\ 395 & 296 & 198 & 100 \end{pmatrix}$	$\begin{pmatrix} 1 & -98 & -196 & -294 \\ 100 & 0 & -98 & -196 \\ 199 & 100 & 1 & -97 \\ 297 & 198 & 100 & 1 \\ 395 & 296 & 198 & 100 \end{pmatrix}$
$(17,\ 1)$	$\begin{pmatrix} 2 & -14 & -29 & -44 \\ 17 & 1 & -14 & -29 \\ 33 & 17 & 2 & -13 \\ 48 & 32 & 17 & 2 \\ 63 & 47 & 32 & 17 \end{pmatrix}$	$\begin{pmatrix} 1 & -15 & -30 & -45 \\ 17 & 0 & -15 & -30 \\ 33 & 17 & 1 & -14 \\ 48 & 32 & 17 & 1 \\ 63 & 47 & 32 & 17 \end{pmatrix}$
$(3,\ 2)$	$\begin{pmatrix} 2 & 3//2 & 2 & 5//2 \\ 3//2 & 1 & 3//2 & 2 \\ 2 & 3//2 & 2 & 5//2 \\ 3//2 & 1 & 3//2 & 2 \\ 1 & 1//2 & 1 & 3//2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$
$(1,\ 1)$	$\begin{pmatrix} 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \end{pmatrix}$
$(5,\ 8)$	$\begin{pmatrix} 2 & 19//8 & 15//4 & 41//8 \\ 5//8 & 1 & 19//8 & 15//4 \\ 1//4 & 5//8 & 2 & 27//8 \\ -9//8 & -3//4 & 5//8 & 2 \\ -5//2 & -17//8 & -3//4 & 5//8 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ -2 & -1 & 0 & 1 \\ -3 & -3 & -1 & 0 \end{pmatrix}$
$(1,\ 18)$	$\begin{pmatrix} 2 & 53//18 & 44//9 & 41//6 \\ 1//18 & 1 & 53//18 & 44//9 \\ -8//9 & 1//18 & 2 & 71//18 \\ -17//6 & -17//9 & 1//18 & 2 \\ -43//9 & -23//6 & -17//9 & 1//18 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 4 \\ -1 & 0 & 1 & 3 \\ -3 & -2 & 0 & 1 \\ -5 & -4 & -2 & 0 \end{pmatrix}$
$(1,\ 100)$	$\begin{pmatrix} 2 & 299//100 & 249//50 & 697//100 \\ 1//100 & 1 & 299//100 & 249//50 \\ -49//50 & 1//100 & 2 & 399//100 \\ -297//100 & -99//50 & 1//100 & 2 \\ -124//25 & -397//100 & -99//50 & 1//100 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 4 \\ -1 & 0 & 1 & 3 \\ -3 & -2 & 0 & 1 \\ -5 & -4 & -2 & 0 \end{pmatrix}$

Comparing the degrees of the entries of N to the ones in the tabular shows that I is in the Gröbner cell of E for the term order induced by $(3,\ 2)$. For term orders τ induced by $(100,\ 1)$ and $(17,\ 1)$ the entries $n_{1,4}$ and $n_{3,4}$ of degree 1 lead to $N \notin \mathcal{N}_\tau(E)$ and indeed $I \notin V_\tau(E)$. In both cases the leading term ideal is $\mathrm{Lt}_\tau(I) = (x^3, x^2y^3, xy^5, y^9) = E'$ with associated partition $m' = (3, 5, 9)$. The matrix describing the syzygies of the reduced Gröbner basis of I with respect to those orders is

$$N' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -y+2 & 0 & 0 \\ 2y^2 - y & 0 & 0 \end{pmatrix},$$

with $\deg(n'_{3,1}) = 1$ and $\deg(n'_{4,1}) = 2$. The associated degree bounds for those entries are $w_{3,1} \in \{198, 32\}$ and $w_{4,1} \in \{294, 45\}$, which are obviously satisfied, thus $N' \in \mathcal{N}_\tau(E')$.

With respect to the order by $(5,\ 8)$, the leading term ideal is again $E'' = (x^6, x^4y, x^2y^2, xy^4, y^7)$ with associated partition $m'' = (1, 1, 2, 2, 4, 7)$, reduced Gröbner basis

$$\{x^6,\ x^4y,\ x^2y^2 - 1/2x^4,\ xy^4 - 1/4x^5,\ y^7 + 1/2x^5 - 3xy^3 + 2x^3y - 2xy^2 + x^3\}$$

and

$$N'' = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4}y^2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}y & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3y-2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{N}_\tau(E).$$

The same E'', m'' and N'' are obtained also for $(1,\ 18)$ and $(1,\ 100)$.

3.8 Towards a parametrization

For all $I \in V_\tau(E)$ Theorem 3.71 already gives a matrix $N \in \mathcal{N}_\tau(E)$, but this matrix is not unique – we have seen this in Example 3.74 where we found the preimages N and $\tilde{N}$ of I under $\Phi_{\tau,E}$, for τ being the term order induced by $(3,\ 2)$. As we have discussed in Remark 3.28, the matrix we obtain when we start with a given Gröbner basis $\{f_0, \ldots, f_t\}$ is unique. That means that the map $\Phi_{\tau,E} : \mathcal{N}_\tau(E) \to \mathbf{k}[x,y]^{t+1}$ defined by $N \mapsto (f_0, \ldots, f_t)$, where the f_i are the signed maximal minors of $H + N$, is injective. But as a system of generators of an ideal, and also its Gröbner basis, is not unique, we do not obtain injectivity when we consider the map $\Phi_{\tau,E} : \mathcal{N}_\tau(E) \to V_\tau(E)$.

In the proof of Proposition 3.29 we already encountered the reduction moves $\mathrm{Red}_{i,j}$ that were defined in [Con11]. For a monomial ideal E with canonical Hilbert-Burch matrix E, and a matrix $N \in \mathbf{k}[y]^{(t+1)\times t}$ representing the syzygies of $I \in V_\tau(E)$, it holds that the maximal minors of $H + N$ and $H + \mathrm{Red}_{i,j}(N)$ are different, but the ideals of maximal minors will be equal.

Definition 3.75 (Reduction moves $\mathrm{Red}_{i,j}$)**.** Let $N \in \mathcal{N}_\tau(E)$, then we define the polynomial $q_{i,j} \in k[y]$ in the following way:

$$n_{i,j}(y) = \begin{cases} (y^{d_i} + n_{i,i}(y)) \cdot q_{i,j}(y) + r_{i,j}(y) & i \leq j \\ (y^{d_j} + n_{j,j}(y)) \cdot q_{i,j}(y) + r_{i,j}(y) & i > j, \end{cases}$$

with $\deg(r_{i,j}) < d_i$ respectively $\deg(r_{i,j}) < d_j$.

Then for $i < j$ *the reduction move* $\mathrm{Red}_{i,j}$ is defined as follows: ($M = H + N$, M' is the matrix obtained after the first part of the reduction move and M'' is the matrix obtained in the end.)

- Add the i-th column of M multiplied with $-q_{i,j}$ to the j-th column of M.
- Add the $j+1$-th row of M' multiplied with $q_{i,j}$ to the $i+1$-th row of M'.

The reduction move for $i > j$ is defined similarly:

- Add the j-th row of M multiplied with $-q_{i,j}$ to the i-th row of M.
- Add the $i-1$-th column of M' multiplied with $q_{i,j}$ to the $j-1$-th column of M'.

If $j = 1$ only do the first part of this move.

We define $\mathrm{Red}_{i,j}(N)$ as the difference of matrix obtained at the end and H, $\mathrm{Red}_{i,j}(N) := M'' - H$. Note that $\mathrm{Red}_{i,j}(N) \in \mathbf{k}[y]^{t+1\times t}$.

Example 3.76. In the following example, we will show how to obtain $\tilde{N}$ from N by a series of reduction moves.

Remember that $m = (2,3,5,7)$, $d = (2,1,2,2)$,

$$N = \begin{pmatrix} 0 & 0 & 0 & 2y+1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4y+2 \\ 0 & y+2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H+N = \begin{pmatrix} y^2 & 0 & 0 & 2y+1 \\ -x & y & 0 & 0 \\ 0 & -x & y^2 & 4y+2 \\ 0 & y+2 & -x & y^2 \\ 0 & 0 & 0 & -x \end{pmatrix}.$$

The entry $(4,2)$ of the matrix N satisfies $1 = \deg(n_{4,2}) \geq d_2 = 1$. We perform the reduction move $\mathrm{Red}_{4,2}$ on N by subtracting the second row of $H+N$ from the fourth row, and then adding the third column from the first column.

$$\text{Step 1:} \quad \begin{pmatrix} y^2 & 0 & 0 & 2y+1 \\ -x & y & 0 & 0 \\ 0 & -x & y^2 & 4y+2 \\ x & 2 & -x & y^2 \\ 0 & 0 & 0 & -x \end{pmatrix} \quad \text{Step 2:} \quad \begin{pmatrix} y^2 & 0 & 0 & 2y+1 \\ -x & y & 0 & 0 \\ y^2 & -x & y^2 & 4y+2 \\ 0 & 2 & -x & y^2 \\ 0 & 0 & 0 & -x \end{pmatrix}.$$

Now $\deg(\mathrm{Red}_{4,2}(N)_{3,1}) = 2 \geq d_1$, and we can perform the $(3,1)$ reduction move on $\mathrm{Red}_{4,2}(N)$, i.e. subtract the first row of $H + \mathrm{Red}_{4,2}(N)$ from the third row, and obtain

$$\mathrm{Red}_{3,1}(\mathrm{Red}_{4,2}(N)) = \tilde{N}.$$

We will show now that if $N \in \mathcal{N}_\tau(E)$, also $\mathrm{Red}_{i,j}(N) \in \mathcal{N}_\tau(E)$. Therefore, a matrix $N \in \mathcal{N}_\tau(E)$ and a matrix that is obtained from N by performing a (sequence of) reduction move(s) are a source of non-injectivity of $\Phi_{\tau,E} : \mathcal{N}_\tau(E) \to V_\tau(E)$.

Lemma 3.77. If we start with a matrix $N \in \mathcal{N}_\tau(E)$, i.e. a matrix satisfying the degree conditions Equation 3.6, $\widetilde{N} := \mathrm{Red}_{i,j}(N)$ will also satisfy the degree conditions:

$$\deg(\tilde{n}_{k,l}) \begin{cases} < w_{k,l} & k \leq l \\ \leq w_{k,l} & k > l \end{cases}$$

Proof. Let $\tilde{N} := \mathrm{Red}_{i,j}(N)$. Notice that $w_{i,i} = m_i - m_{i-1} + \frac{a}{b}(i-i) = d_i$, thus the diagonal entry of N satisfies $\deg(n_{i,i}) < d_i$ already. So there are the two remaining cases:

- $(i < j)$:

  Remember that $n_{i,j} = q_{i,j}(y^{d_i} + n_{i,i}) + r_{i,j}$. From the degree bound $\deg(n_{i,j}) < w_{i,j}$, we obtain the following degree bound for $q_{i,j}$

  $$\deg(q_{i,j}) < w_{i,j} - d_i = w_{i+1,j} - \frac{a}{b}.$$

  For the following calculations this observation will be useful:

  $$w_{i+1,j} - \frac{a}{b} + w_{k,l} = \begin{cases} w_{k,j} & l = i \\ w_{i+1,l} & k = j+1. \end{cases}$$

Here are the calculations:

$$\begin{aligned} w_{i+1,j} - \tfrac{a}{b} + w_{k,i} &= m_j - m_i + \tfrac{a}{b}((i+1)-j) - \tfrac{a}{b} + m_i - m_{k-1} + \tfrac{a}{b}(k-i) \\ &= m_j - m_{k-1} + \tfrac{a}{b}(k-j) = w_{k,j} \end{aligned}$$

$$\begin{aligned} w_{i+1,j} - \tfrac{a}{b} + w_{j+1,l} &= m_j - m_i + \tfrac{a}{b}((i+1)-j) - \tfrac{a}{b} + m_l - m_j + \tfrac{a}{b}((j+1)-l) \\ &= m_l - m_i + \tfrac{a}{b}((i+1)-l) = w_{i+1,l} \end{aligned}$$

We will now check the degree bounds for the entries of $\tilde{N}$. Most of the entries of $\tilde{N}$ are just the entries of N, only the once in j-th column and in the $i+1$-th row are changed. These entries of $\tilde{N}$ are just the sum of an entry of N with a product of $q_{i,j}$ and a second factor. For the degree bounds we will only have to check the new summand. Since $q_{i,j}$ satisfies a strict degree bound, we can be a bit less careful with the degree bounds of the second factor in the product and will just always assume the non-strict inequality, since in any case we obtain a strict inequality for the product.

(i) $k = i; l = j$:

$\tilde{n}_{k,l} = r_{i,j}$ satisfies the degree bounds by construction.

(ii) $k \neq i, i+1; l = j$:

$\tilde{n}_{k,l} = n_{k,j} - q_{i,j} n_{k,i}$

$$\begin{aligned} \deg(q_{i,j} n_{k,i}) &< w_{i+1,j} - \tfrac{a}{b} + \deg(n_{k,i}) \\ &\leq w_{i+1,j} - \tfrac{a}{b} + w_{k,i} = w_{k,j} \end{aligned}$$

(iii) $k = i+1; l = j+1$:

$\tilde{n}_{k,l} = n_{i+1,j+1} + q_{i,j}(y^{d_{j+1}} + n_{j+1,j+1})$

$$\begin{aligned} \deg(q_{i,j}(y^{d_{j+1}} + n_{j+1,j+1})) &< w_{i+1,j} - \tfrac{a}{b} + \deg(y^{d_{j+1}}) \\ &\leq w_{i+1,j} - \tfrac{a}{b} + w_{j+1,j+1} = w_{i+1,j+1} \end{aligned}$$

(iv) $k = i+1; l \neq j, j+1$:

$\tilde{n}_{k,l} = n_{i+1,l} + q_{i,j} n_{j+1,l}$

$$\begin{aligned} \deg(q_{i,j} n_{j+1,l}) &< w_{i+1,j} - \tfrac{a}{b} + \deg(n_{j+1,l}) \\ &\leq w_{i+1,j} - \tfrac{a}{b} + w_{j+1,l} = w_{i+1,l} \end{aligned}$$

(v) $k = i+1; l = j$:

$\tilde{n}_{k,l} = n_{i+1,j} - q_{i,j} n_{i+1,i} + q_{i,j} \tilde{n}_{j+1,j}$

$$\begin{aligned} \deg(q_{i,j} n_{i+1,i}) &< w_{i+1,j} - \tfrac{a}{b} + \deg(n_{i+1,i}) \\ &\leq w_{i+1,j} - \tfrac{a}{b} + w_{i+1,i} = w_{i+1,j} \end{aligned}$$

$$\begin{aligned} \deg(q_{i,j} \tilde{n}_{j+1,j}) &< w_{i+1,j} - \tfrac{a}{b} + \deg(n_{j+1,j}) \\ &\leq w_{i+1,j} - \tfrac{a}{b} + w_{j+1,j} = w_{i+1,j} \end{aligned}$$

The matrix $\tilde{N}$ satisfies all necessary degree bounds.

- $(i > j)$: Remember that $n_{i,j} = q_{i,j}(y^{d_j}+n_{j,j})+r_{i,j}$. From the degree bound $\deg(n_{i,j}) \leq w_{i,j}$, we get the following degree bound for $q_{i,j}$:

$$\deg(q_{i,j}) \leq w_{i,j} - d_j = w_{i,j-1} - \frac{a}{b}.$$

For the following calculations, this observation will be useful.

$$w_{i,j-1} - \frac{a}{b} + w_{k,l} = \begin{cases} w_{i,l} & k = j \\ w_{k,j-1} & l = i-1 \end{cases}$$

Here are the calculations:

$$\begin{aligned} w_{i,j-1} - \tfrac{a}{b} + w_{j,l} &= m_{j-1} - m_{i-1} + \tfrac{a}{b}(i-(j-1)-1) + m_l - m_{j-1} + \tfrac{a}{b}(j-l) \\ &= m_l - m_{i-1} + \tfrac{a}{b}(i-l) = w_{i,l} \end{aligned}$$

$$\begin{aligned} w_{i,j-1} - \tfrac{a}{b} + w_{k,i-1} &= m_{j-1} - m_{i-1} + \tfrac{a}{b}(i-(j-1)-1) + m_{i-1} - m_{k-1} + \tfrac{a}{b}(k-(i-1)) \\ &= m_{j-1} - m_{k-1} + \tfrac{a}{b}(k-(j-1)) = w_{k,j-1} \end{aligned}$$

As in the previous case, we have to check the degree bounds for the new summand. Since in this case the degree bound that $q_{i,j}$ satisfies is non-strict, we have to be a bit more careful what degree bounds the second factor satisfies.

(i) $k = i; l = j$:

$\tilde{n}_{k,l} = r_{i,j}$ satisfies the degree bounds by construction.

(ii) $k = i; l \neq j, j-1$:

$\tilde{n}_{k,l} = n_{i,l} - q_{i,j}n_{j,l}$

$$\begin{aligned} \deg(q_{i,j}n_{j,l}) &\leq w_{i,j-1} - \tfrac{a}{b} + \deg(n_{j,l}) \\ &\begin{cases} < w_{i,j-1} - \frac{a}{b} + w_{j,l} = w_{i,l} & j \leq l \\ \leq w_{i,j-1} - \frac{a}{b} + w_{j,l} = w_{i,l} & j > l \end{cases} \\ &\begin{cases} < w_{i,j-1} - \frac{a}{b} + w_{j,l} = w_{i,l} & i \leq l \\ < w_{i,j-1} - \frac{a}{b} + w_{j,l} = w_{i,l} & j \leq l < i \\ \leq w_{i,j-1} - \frac{a}{b} + w_{j,l} = w_{i,l} & i > j > l \end{cases} \end{aligned}$$

For $i \leq l$, we get the strict inequality. For $i > l$ the non-strict inequality is enough. But for some of the entries we even get the strict one.

(iii) $k = i-1; l = j-1$:

$n\tilde{n}_{k,l} =_{i-1,j-1} + q_{i,j}(y^{d_{i-1}} + n_{i-1,i-1})$

$$\begin{aligned} \deg(q_{i,j}(y^{d_{i-1}} + n_{i-1,i-1})) &\leq w_{i,j-1} - \tfrac{a}{b} + \deg(y^{d_{i-1}}) \\ &\leq w_{i,j-1} - \tfrac{a}{b} + w_{i-1,i-1} = w_{i-1,j-1} \end{aligned}$$

Since $i-1 > j-1$ this (non-strict) inequality is the one we hoped for.

(iv) $k \neq i, i-1, l = j-1$:

$\tilde{n}_{k,l} = n_{k,j-1} + q_{i,j} n_{k,i-1}$

$$\begin{aligned} \deg(q_{i,j} n_{k,i-1}) &\leq w_{i,j-1} - \tfrac{a}{b} + \deg(n_{k,i-1}) \\ &\begin{cases} < w_{i,j-1} - \frac{a}{b} + w_{k,i-1} = w_{k,j-1} & k \leq i-1 \\ \leq w_{i,j-1} - \frac{a}{b} + w_{k,i-1} = w_{k,j-1} & k > i-1 \end{cases} \\ &\begin{cases} < w_{i,j-1} - \frac{a}{b} + w_{k,i-1} = w_{k,j-1} & k \leq j-1 \\ < w_{i,j-1} - \frac{a}{b} + w_{k,i-1} = w_{k,j-1} & j \leq k \leq i-1 \\ \leq w_{i,j-1} - \frac{a}{b} + w_{k,i-1} = w_{k,j-1} & k > i-1 \end{cases} \end{aligned}$$

For $k \leq j-1$ we get the strict inequality that we need. For the cases $k > j-1$ we only need the non-strict inequality, but in some of the cases we even get the strict one.

(v) $k = i, l = j-1$:

$\tilde{n}_{k,l} = n_{i,j-1} - q_{i,j} n_{j,j-1} + q_{i,j} \tilde{n}_{i,i-1}$

$$\begin{aligned} \deg(q_{i,j} n_{j,j-1}) &\leq w_{i,j-1} - \tfrac{a}{b} + \deg(n_{j,j-1}) \\ &\leq w_{i,j-1} - \tfrac{a}{b} + w_{j,j-1} = w_{i,j-1} \end{aligned}$$

$$\begin{aligned} \deg(q_{i,j} \tilde{n}_{i,i-1}) &\leq w_{i,j-1} - \tfrac{a}{b} + \deg(n_{i,i-1}) \\ &\leq w_{i,j-1} - \tfrac{a}{b} + w_{i,i-1} = w_{i,j-1} \end{aligned}$$

Since $i > j > j-1$ this is exactly the inequality we need.

The matrix $\tilde{N}$ satisfies all necessary degree bounds.

□

As in Definition 3.47, $(\mathbf{k}[y]_{<\underline{d}})^{(t+1)\times t} \subset \mathbf{k}[y]^{(t+1)\times t}$ will denote the subset of matrices N where the entries satisfy the following degree conditions:

$$\deg(n_{i,j}) < \begin{cases} d_i, & i \leq j; \\ d_j, & i > j \end{cases}$$

Definition 3.78. We define the set $\mathcal{N}_\tau(E)_{<\underline{d}} := \mathcal{N}_\tau(E) \cap \mathbf{k}[y]_{<\underline{d}}$.

We state the following conjecture:

Conjecture 3.79. *Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ be a monomial ideal and τ be a term order on $\mathbf{k}[x, y]$. Then restriction of $\Phi_{\tau,E}$ from Theorem 3.71 to $\mathcal{N}_\tau(E)_{<\underline{d}}$*

$$\Phi_{\tau,E} : \mathcal{N}_\tau(E)_{<\underline{d}} \to V_\tau(E)$$

is a bijection.

It is clear from the construction that it is not possible to perform further reduction moves on matrices from $\mathcal{N}_\tau(E)_{<\underline{d}}$. This does not necessarily proof injectivity of the map, but it eliminates one obvious source of non-injectivity. However, it is not clear that the map is even surjective, as we do not know if it is possible to perform a series of reduction moves on a given matrix $N \in \mathcal{N}_\tau(E)$ until it reaches the desired form.

For $\omega = (1,1)$ the weighted degree matrix W is just the usual degree matrix U and the set of matrices from Conjecture 3.79 is the set of matrices of [Con11, Theorem 3.1] and thence Conjecture 3.79 holds when E is lex-segment. Whenever $\omega = (a,b)$ with $a \gg b$ the resulting term order will be equivalent to the lexicographic term order. Then the structure of $W(a,b)$ is the following:

$$w_{i,j} = \begin{cases} d_i & i = j \\ m_j - m_{i-1} + \frac{a}{b}(i-j) < 0 & i < j \\ m_j - m_{i-1} + \frac{a}{b}(i-j) \gg 0 & i > j \end{cases}$$

So for $N \in \mathcal{N}_\tau(E)_{<\underline{d}}$ the entries are $n_{i,j} = 0$ for $i < j$, and for $i \geq j$ they satisfy $\deg(n_{i,j}) \leq d_j - 1$. This is exactly the description by [CV08, Definition 3.2], so by [CV08, Theorem 3.3] the conjecture also holds for lex.

Remark 3.80. We easily see that Conjecture 3.79 holds for the trivial case of $E = (x, y^n)$. Well-definedness of the map is clear, since $\mathcal{N}_\tau(E)_{<\underline{d}} \subset \mathcal{N}_\tau(E)$. By surjectivity of Theorem 3.71, we obtain for all $I \in V_\tau(E)$, a matrix $N \in \mathcal{N}_\tau(E)$. The additional degree condition of $\mathcal{N}_\tau(E)_{<\underline{d}}$ is trivially satisfied for the diagonal entry $n_{1,1}$. And after at most one reduction move $n_{2,1}$ also satisfies the additional degree bound $\deg(n_{2,1}) < d_1$.

Injectivity is easily shown, too: Suppose there exist $N, N' \in \mathcal{N}_\tau(E)_{<\underline{d}}$ with $I_t(H+N) = I_t(H+N')$. Then we denote the minors of $H+N$ and $H+N'$ as f_0, f_1 and f_0', f_1' respectively. $f_0' \in (f_0, f_1)$, so $f_0' - f_0 = (x - n_{2,1}') - (x - n_{2,1}) = n_{2,1} - n_{2,1}' \in I$. But the difference is a polynomial in y with all terms of degree less than $d_1 = n$, so unless the difference is zero, $f_0' - f_0 \notin I$. The same argument can be applied for $f_1' - f_1 = (y^n + n_{1,1}') - (y^n + n_{1,1})$. So $N = N'$, and the map is injective.

A strategy to prove surjectivity of $\mathcal{N}_\tau(E)_{<\underline{d}} \to V_\tau(E)$ of Conjecture 3.79 for general E could be a similar one as in [Con11], by induction on t. Then $E = (x, y^n)$ is the base case.

Example 3.81. Continuing Example 3.74, we describe different $\mathcal{N}_\tau(E)_{<\underline{d}}$ as in Definition 3.78 for $E = (x^4, x^3y^2, x^2y^3, xy^5, y^7) \in \mathrm{Hilb}^{17}(\mathbf{k}[x,y])$. The corresponding partition of 17 is $m = (2,3,5,7)$ with vector of differences $d = (2,1,2,2)$.

(a, b)	$M = H + N$ with $N \in \mathcal{N}_\tau(E)$	degree bounds by W
(100, 1)	$\begin{pmatrix} y^2+yc_2+c_1 & 0 & 0 & 0 \\ -x+yc_4+c_3 & y+c_5 & 0 & 0 \\ yc_7+c_6 & -x+c_8 & y^2+yc_{10}+c_9 & 0 \\ yc_{12}+c_{11} & c_{13} & -x+yc_{15}+c_{14} & y^2+yc_{17}+c_{16} \\ yc_{19}+c_{18} & c_{20} & yc_{22}+c_{21} & -x+yc_{24}+c_{23} \end{pmatrix}$	$\begin{pmatrix} 1 & -98 & -196 & -294 \\ 100 & 0 & -98 & -196 \\ 199 & 100 & 1 & -97 \\ 297 & 198 & 100 & 1 \\ 395 & 296 & 198 & 100 \end{pmatrix}$
(17, 1)	$\begin{pmatrix} y^2+yc_2+c_1 & 0 & 0 & 0 \\ -x+yc_4+c_3 & y+c_5 & 0 & 0 \\ yc_7+c_6 & -x+c_8 & y^2+yc_{10}+c_9 & 0 \\ yc_{12}+c_{11} & c_{13} & -x+yc_{15}+c_{14} & y^2+yc_{17}+c_{16} \\ yc_{19}+c_{18} & c_{20} & yc_{22}+c_{21} & -x+yc_{24}+c_{23} \end{pmatrix}$	$\begin{pmatrix} 1 & -15 & -30 & -45 \\ 17 & 0 & -15 & -30 \\ 33 & 17 & 1 & -14 \\ 48 & 32 & 17 & 1 \\ 63 & 47 & 32 & 17 \end{pmatrix}$
(3, 2)	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 & yc_6+c_5 & yc_8+c_7 \\ -x+yc_{10}+c_9 & y+c_{11} & c_{12} & c_{13} \\ yc_{15}+c_{14} & -x+c_{16} & y^2+yc_{18}+c_{17} & yc_{20}+c_{19} \\ yc_{22}+c_{21} & c_{23} & -x+yc_{25}+c_{24} & y^2+yc_{27}+c_{26} \\ yc_{29}+c_{28} & c_{30} & yc_{32}+c_{31} & -x+yc_{34}+c_{33} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$
(1, 1)	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 & yc_6+c_5 & yc_8+c_7 \\ -x+yc_{10}+c_9 & y+c_{11} & c_{12} & c_{13} \\ yc_{15}+c_{14} & -x+c_{16} & y^2+yc_{18}+c_{17} & yc_{20}+c_{19} \\ c_{21} & c_{22} & -x+yc_{24}+c_{23} & y^2+yc_{26}+c_{25} \\ 0 & 0 & c_{27} & -x+yc_{29}+c_{28} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \end{pmatrix}$
(5, 8)	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 & yc_6+c_5 & yc_8+c_7 \\ -x+c_9 & y+c_{10} & c_{11} & c_{12} \\ c_{13} & -x+c_{14} & y^2+yc_{16}+c_{15} & yc_{18}+c_{17} \\ 0 & 0 & -x+c_{19} & y^2+yc_{21}+c_{20} \\ 0 & 0 & 0 & -x+c_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ -2 & -1 & 0 & 1 \\ -3 & -3 & -1 & 0 \end{pmatrix}$
(1, 18)	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 & yc_6+c_5 & yc_8+c_7 \\ -x+c_9 & y+c_{10} & c_{11} & c_{12} \\ 0 & -x+c_{13} & y^2+yc_{15}+c_{14} & yc_{17}+c_{16} \\ 0 & 0 & -x+c_{18} & y^2+yc_{20}+c_{19} \\ 0 & 0 & 0 & -x+c_{21} \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 4 \\ -1 & 0 & 1 & 3 \\ -3 & -2 & 0 & 1 \\ -5 & -4 & -2 & 0 \end{pmatrix}$
(1, 100)	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 & yc_6+c_5 & yc_8+c_7 \\ -x+c_9 & y+c_{10} & c_{11} & c_{12} \\ 0 & -x+c_{13} & y^2+yc_{15}+c_{14} & yc_{17}+c_{16} \\ 0 & 0 & -x+c_{18} & y^2+yc_{20}+c_{19} \\ 0 & 0 & 0 & -x+c_{21} \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 4 \\ -1 & 0 & 1 & 3 \\ -3 & -2 & 0 & 1 \\ -5 & -4 & -2 & 0 \end{pmatrix}$

We see that (100, 1) and (17, 1) give the same sets $\mathcal{N}_\tau(E)_{<\underline{d}}$. This is clear, since both induce $\mathrm{lex}_{x>y}$ on $\mathrm{Hilb}^{17}(\mathbf{k}[x,y])$. They agree with the description of the Gröbner cell with respect to the lexicographic order given by [CV08], as already elaborated for the general case above. The dimension of $\mathcal{N}_\tau(E)_{<\underline{d}}$ in both cases is $24 = 17 + 7 = n + m_t$.

For the term orders induced by (3, 2), (1, 1), (5, 8) and (1, 18), we used `Singular` to confirm that $\Phi_{\tau,E} : \mathcal{N}_\tau(E)_{<\underline{d}} \to V_\tau(E)$ is surjective. We created a general $N \in \mathcal{N}_\tau(E)$ and performed series of reduction moves to finally obtain an $N \in \mathcal{N}_\tau(E)_{<\underline{d}}$. The number of reduction moves needed was 9, 8, 11 and 12, respectively.

We have already see in Example 3.74 that (1, 18) and (1, 100) both induce $\mathrm{lex}_{y>x}$ on $\mathrm{Hilb}^{17}(\mathbf{k}[x,y])$ and the degree bounds induced by W as in Equation 3.7 agree. Thus also the sets $\mathcal{N}_\tau(E)_{<\underline{d}}$ are equal. They have dimension $21 = 17 + 4 = n + t$.

For the term orders by (1, 1), and (5, 8) the dimensions of $\mathcal{N}_\tau(E)_{<\underline{d}}$ are 29 and 21, for (3, 2) the dimension is $34 = 2n$. If $\Phi_{\tau,E} : \mathcal{N}_\tau(E)_{<\underline{d}} \to V_\tau(E)$ is really a parametrization, then it is the (unique) cell of maximal dimension. And a general ideal of $\mathrm{Hilb}^{17}(\mathbf{k}[x,y])$ will have E as leading term ideal with respect to τ (at least whenever $\mathrm{char}(\mathbf{k}) = 0$). Computer experiments with the software `Singular` suggest that this is true.

3.8.1 Lexicographic order with $y > x$

In this subsection we study the term order $\text{lex}_{y>x}$ which corresponds to $(a,\ b) = (0,\ 1)$ or $a \ll b$. We start with a more detailed description of $V_{\text{lex}_{y>x}}(E)_{<\underline{d}}$.

Proposition 3.82. Let $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ be a monomial ideal. The set $\mathcal{N}_{\text{lex}_{y>x}}(E)_{<\underline{d}}$ consists of matrices $N \in \mathbf{k}[y]^{(t+1)\times t}$ where

(i) for $i \le j$, $n_{i,j}$ is a polynomial of degree less than d_i,

(ii) for $i > j$, $n_{i,j}$ can be a non-zero constant polynomial if and only if $m_{j-1} < m_j = m_{i-1}$.

For lex-segment ideals E, (ii) can be reformulated to

- $n_{i+1,i}$ is a constant polynomial and all other $n_{i,j}$ with $i + 1 > j$ are zero.

Proof. The proof boils down to comparing the degree bounds by W as in Equation 3.7 and those by d.

The entries of the weighted degree matrix are $w_{i,j} = m_j - m_{i-1} + \frac{a}{b}(i - j) = m_j - m_{i-1}$. For $i \le j$, $m_i \le m_j$ and thus $m_j - m_{i-1} \le d_i$, so the bounds by W are always already imposed by the degree bounds of d, $\min(m_j - m_{i-1} - 1, d_i - 1) = d_i - 1$ and $n_{i,j}$ is a polynomial of degree at most $d_i - 1$.

In the lower left corner, that is for $i > j$, the degree bounds by W are $\deg(n_{i,j}) \le m_j - m_{i-1}$. Since $m_j \le m_{i-1}$, $m_j - m_{i-1} \le 0$ with equality if and only if $m_j = m_{i-1}$. Thus, the entry $n_{i,j}$ of $N \in \mathcal{N}_{\text{lex}_{y>x}}(E)_{<\underline{d}}$ can be at most of degree 0, and this is allowed if and only if $d_j = m_j - m_{j-1} > 0$. If E is lex-segment, then the entries of m are strictly increasing and $m_j = m_{i-1}$ is only possible when $i - 1 = j$. Since all $m_{j-1} < m_j$, $\deg(n_{i+1,i}) = 0$ is always allowed. □

Corollary 3.83. The dimension of $\mathcal{N}_{\text{lex}_{y>x}}(E)_{<\underline{d}}$ is

$$\dim(\mathcal{N}_{\text{lex}_{y>x}}(E)_{<\underline{d}}) = n + t.$$

Proof. We will determine the dimension by counting the number of admissible coefficients in $\mathcal{N}_{\text{lex}_{y>x}}(E)_{<\underline{d}}$. We start by counting those from the upper right corner diagonal by diagonal, starting with the main diagonal. The entries $n_{i,i}$ for $i = 1, \dots, t$ are all polynomials of degree at most $d_i - 1$, so for each of them we have d_i coefficients. This leads to $\sum_{i=1}^{t} d_i = m_t$.

For the entries on the next diagonal, $n_{i,i+1}$ is a polynomial of degree at most $d_i - 1$ for $i = 1, \dots, t - 1$, so $\sum_{i=1}^{t-1} d_i = m_{t-1}$. In conclusion, the upper right corner gives

$$\sum_{l=0}^{t-1} \left(\sum_{i=1}^{t-l} d_i \right) = \sum_{l=0}^{t-1} m_{t-l} = n$$

coefficients.

It remains to count the number of admissible coefficients in the lower left corner. If E is lex-segment, then the only non-zero entries there are the constant polynomials $n_{i+1,i}$ for $i = 1, \dots, t$. Therefore, the lower left corner adds another t admissible coefficients and

$$\dim(\mathcal{N}_{\text{lex}_{y>x}}(E)_{<\underline{d}}) = n + t.$$

In general, we have to determine $\#\{(i,j) \mid i > j,\ m_{j-1} < m_j = m_{i-1}\}$. Let us split the set

$$\#\{(i,j) \mid i > j,\ m_{j-1} < m_j = m_{i-1}\} = \#\{j \mid d_j > 0\} + \sum_{k=1,\ d_k>0}^{t} \#\{l \mid l > 0,\ m_k = m_{k+l}\}.$$

That is, the positive d_j are counted in the first summand, and the sum in the second summand counts all $d_j = 0$, so in total $\#\{(i,j) \mid i > j,\ m_{j-1} < m_j = m_{i-1}\} = \#\{j \mid d_j > 0\} + \#\{j \mid d_j = 0\} = t$. □

Since the dimension of $V_{\mathrm{lex}_{y>x}}(E)$ can be calculated by using [CV08] and renaming x and y, we see that $\dim(\mathcal{N}_{\mathrm{lex}_{y>x}}(E)_{<\underline{d}}) = \dim(V_{\mathrm{lex}_{y>x}}(E))$, providing additional evidence for Conjecture 3.79.

3.9 Intersecting Gröbner cells

One application of parametrizations of Gröbner cells is to study the intersection of two different Gröbner cells. For τ, τ' being two term orders on $\mathbf{k}[x,y]$ we denote the intersection of two Gröbner cells as $V_{\tau,\tau'}(E) = \{I \subset \mathbf{k}[x,y] \mid \mathrm{Lt}_\tau(I) = \mathrm{Lt}_{\tau'}(I) = E\}$. In [JS19, Example 7.9] it was shown that this is an affine space by using a generalization of the Białynicki-Birula-decomposition for actions of $\mathbb{G}_m$ to actions of general reductive groups, in this specific case $\mathbb{G}_m \times \mathbb{G}_m$. We are interested in a concrete parametrization of the space.

Definition 3.84. Let $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$, $\omega = (a,b)$, $\omega' = (a', b')$ as above with $\frac{a}{b} > \frac{a'}{b'}$ and τ,τ' the associated term orders. Then we define the set of matrices $\mathcal{N}_{\tau,\tau'}(E) \subset \mathbf{k}[y]^{t+1\times t}$, as those $N = (n_{i,j})$ where its non-zero entries satisfy the following degree condition:

$$\deg(n_{i,j}) \begin{cases} < w_{i,j} = m_j - m_{i-1} + \frac{a}{b}(i-j) & i < j \\ < w_{i,j} = d_i & i = j \\ \leq w'_{i,j} = m_j - m_{i-1} + \frac{a'}{b'}(i-j) & i > j \end{cases}. \tag{3.8}$$

Lemma 3.85. With the same notation as in Definition 3.84, the map

$$\begin{array}{rccc} \Phi_{\tau,\tau',E}: & \mathcal{N}_{\tau,\tau'}(E) & \to & V_{\tau,\tau'}(E) \\ & N & \mapsto & I_t(H+N) \end{array}$$

is well-defined.

Proof. Since $\mathcal{N}_{\tau,\tau'}(E) \subset \mathcal{N}_\tau(E), \mathcal{N}_{\tau'}(E)$ well-definedness follows by Lemma 3.72. □

It is natural to assume that the map is also surjective, because the two maps ϕ_τ and $\phi_{\tau'}$ are surjective by Theorem 3.71. The ideal I is in $V_\tau(E)$, so there exists $N \in \mathcal{N}_\tau(E)$ such that $I = I_t(H+N)$. And also $I \in V_{\tau'}(E)$, so there exists $N' \in \mathcal{N}_{\tau'}(E)$ with $I = I_t(H+N')$. But the map in Theorem 3.71 is not injective, so in general $N \neq N'$, even if we start with $\tau = \tau'$. So it is also not clear that $N \in \mathcal{N}_{\tau'}(E)$. That would be equivalent to the fact that the signed t-minors of $H+N$ also form a τ'-Gröbner basis.

Example 3.86. Let $a = 2$ and $a' = b' = b = 1$. Then $\{f_0 = x - y^2, f_1 = y\}$ is a τ-Gröbner basis of $I = (x - y^2, y)$, and $\{f_0, f_1\}$ are the 1-minors of $H + N = (y^{m_1}, -x + y^2)^t$. Since

$2 \leq w_{2,1} = \frac{a}{b} = 2$, $N \in \mathcal{N}_\tau(E)$, but $2 > w'_{2,1} = \frac{a'}{b'} = 1$, so $N \notin \mathcal{N}_{\tau'}(E)$. Indeed $\{x - y^2, y\}$ does not form a Gröbner basis of I with respect to the degreelexicographic order.

The choice of the τ-Gröbner basis or equivalently of N is not good. The ideal I is the monomial ideal itself and a better Gröbner basis would have been the monomial generators.

In Example 3.74 we have also seen an instance of this. The Gröbner basis of I with respect to the term order induced by (3, 2) lead to the matrix N that was not an element of $\mathcal{N}_{\text{deglex}}(E)$, even though $I \in V_{\text{deglex}}(E)$.

We can overcome this problem in general by considering reduced Gröbner bases. The following lemma will be the key ingredient.

Lemma 3.87. Let τ, τ' be two term orders on a polynomial ring P and $I \subset P$ such that $\text{Lt}_\tau(I) = \text{Lt}_{\tau'}(I)$. Then a reduced Gröbner basis of I with respect to τ is a reduced Gröbner basis of I with respect to τ'.

Proof. Let $\{f_0, \dots, f_t\}$ be a reduced Gröbner basis of I with respect to τ. Then no monomial in the support of the tail of f_i, i.e. in the support of $f_i - \text{Lt}_\tau(f_i)$ is in $\text{Lt}_\tau(I)$. Clearly the monomial $\text{Lt}_{\tau'}(f_i)$ is one of the monomials in the support of f_i, and an element of the ideal $\text{Lt}_{\tau'}(I)$. By assumption $\text{Lt}_{\tau'}(I) = \text{Lt}_\tau(I)$, but $\text{Lt}_\tau(f_i)$ is the only monomial in the support of f_i that is in $\text{Lt}_\tau(I)$. Thence $\text{Lt}_{\tau'}(f_i) = \text{Lt}_\tau(f_i)$ and thus $\{f_0, \dots, f_t\}$ forms a Gröbner basis of I with respect to τ' as well. □

Remark 3.88. Note that this is generally known. Algorithms for computing universal Gröbner bases of ideals – a set of polynomials that forms a Gröbner basis of an ideal with respect to any term order – as described for example in [Stu96, Chapter 3], determine the universal Gröbner basis by taking the union of the reduced Gröbner bases for one term order in each cone of the Gröbner fan. A cone inside the Gröbner fan is exactly the region where the leading term ideal of a given ideal does not change with varying term order.

Lemma 3.89. The map $\Phi_{\tau,\tau',E}$ of Lemma 3.85 is surjective.

Proof. Let $I \in V_{\tau,\tau'}(E)$, then by Lemma 3.87 the reduced Gröbner bases of I with respect to τ and τ' coincide. If E is not lex-segment, i.e. the sequence $0 = m_0 \leq m_1 \leq \dots \leq m_t$ is not strictly increasing, we add some generators to the reduced Gröbner basis, relabel such that $\text{Lt}_\tau(f_i) = \text{Lt}_{\tau'}(f_i) = x^{t-i}y^{m_i}$ and possibly reduce with f_0 such that no term of the f_i other than $\text{Lt}_\tau(f_0)$ is divisible by x^t.

By Theorem 3.71 there is a matrix $N \in \mathcal{N}_\tau(E)$ and a matrix $N' \in \mathcal{N}_{\tau'}(E)$ such that the signed maximal minors of $H + N$ and $H + N'$ are $\{f_0, \dots, f_t\}$. By Remark 3.28 $N = N'$. □

The parametrizations in [CV08] and [Con11] help us to overcome the lack of injectivity of the general case, whenever we intersect with the Gröbner cell of the lexicographic or degreelexicographic order.

Proposition 3.90. Let τ be any term order on $\mathbf{k}[x, y]$ and τ' be the lexicographic or the degreelexicographic ordering. Let $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ be a monomial ideal (or in the case of deglex a lex-segment ideal). Then we have a parametrization of $V_{\tau,\tau'}(E)$ by $\mathcal{N}_{\tau,\tau'}(E)_{<\underline{d}}$.

Proof. Let $I \in V_{\tau,\tau'}(E)$, then by Lemma 3.89 we obtain $N \in \mathcal{N}_{\tau,\tau'}(E)$. Now by [CV08, Theorem 3.3] or [Con11, Theorem 3.1] we can perform a sequence of reduction moves on N such that the resulting matrix $\tilde{N}$ satisfies the additional degree conditions $\deg(n_{i,j}) < d_{\min(i,j)}$. By the injectivity part of the above cited theorems the matrix $\tilde{N}$ is unique. By Lemma 3.77 the matrix $\tilde{N}$ still satisfies the relevant degree bounds for τ, thus $\tilde{N} \in \mathcal{N}_{\tau,\tau'}(E)_{<\underline{d}}$. □

Notice that Proposition 3.90 gives us a parametrization of certain subsets of $V_\tau(E)$. Namely of those ideals $I \in V_\tau(E)$ that satisfy $\mathrm{Lt}_{\mathrm{lex}}(I) = E$ or $\mathrm{Lt}_{\mathrm{deglex}}(I) = E$. Thence we can define canonical Hilbert-Burch matrices with respect to τ of those ideals as $H + \Phi^{-1}_{\tau,\tau',E}(I)$, where $\tau' \in \{\mathrm{lex}, \mathrm{deglex}\}$. The notion of canonical Hilbert-Burch matrix highly depends on the term order. In general the canonical Hilbert-Burch matrix of I with respect to lex and with respect to deglex will be different, even the sizes of them will often not coincide. It depends on the leading term ideals $\mathrm{Lt}_{\mathrm{lex}}(I)$ and $\mathrm{Lt}_{\mathrm{deglex}}(I)$.

Proposition 3.90 also gives us a way of describing the set of all ideals I that have the same leading term ideal E with respect to all term orders.

Definition 3.91. Let $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t}) \subset \mathbf{k}[x, y]$ be a monomial ideal, then we define

$$\mathrm{V}_{\mathrm{uni}}(E) = \{I \subset \mathbf{k}[x, y] \mid \mathrm{Lt}_\tau(I) = E \text{ for all term orders } \tau\}$$

Let $\mathcal{N}_{\mathrm{uni}}(E) \subset \mathbf{k}[y]^{t+1\times t}$ be the set of matrices such that

- $\deg(n_{i,j}) \leq d_i - 1$ for $i = j$
- $\deg(n_{i,j}) = 0$ for $i > j$ and $m_{j-1} < m_j = m_{i-1}$,
- $n_{i,j} = 0$ else.

Corollary 3.92. Let $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t}) \subset \mathbf{k}[x, y]$ be a monomial ideal with canonical Hilbert-Burch matrix H. Then the set of ideals that has E as leading term ideal with respect to all possible term orders $\mathrm{V}_{\mathrm{uni}}(E)$ can be parametrized by

$$\begin{array}{rccc} \Phi_{\mathrm{uni},E}: & \mathcal{N}_{\mathrm{uni}}(E) & \to & \mathrm{V}_{\mathrm{uni}}(E) \\ & N & \mapsto & I_t(H + N). \end{array}$$

Proof. An ideal I has the same leading term ideal with respect to all term orders if and only if I has the same leading term ideal with respect to the lexicographic orders $\mathrm{lex}_{x>y}$ and $\mathrm{lex}_{y>x}$.

One direction is obvious. The other follows by the fact that the region where the leading term ideal stays constant is a cone. See literature about the Gröbner fan, e.g. [Stu96; Bay82; BM88]. The order $\mathrm{lex}_{x>y}$ corresponds to $\omega = (1, 0)$ and $\mathrm{lex}_{y>x}$ to $\omega = (0, 1)$. Thus if the leading term of I is equal with respect to both lex orders, it has to be the same for all term orders.

Thus $\mathrm{V}_{\mathrm{uni}}(E) = V_{\tau,\tau'}(E)$ with $\tau = \mathrm{lex}_{x>y}$ and $\tau' = \mathrm{lex}_{y>x}$. By Proposition 3.90 we have a parametrization $\Phi_{\tau,\tau',E} : \mathcal{N}_{\tau,\tau'}(E)_{<\underline{d}} \to V_{\tau,\tau'}(E) = \mathrm{V}_{\mathrm{uni}}(E)$. The vanishing of $n_{i,j}$ with $i < j$ is guaranteed by $I \in V_\tau(E)$. In Proposition 3.82 we have studied the structure of $\mathcal{N}_{\mathrm{lex}_{y>x}}(E)_{<\underline{d}}$ in detail and obtained the condition for $i > j$. ☐

Example 3.93. When considering E form Example 3.74 with $m = (2, 3, 5, 7)$, then $\mathrm{V}_{\mathrm{uni}}(E) \cong \mathbb{A}^{11}$ with canonical Hilbert-Burch matrix of $I \in \mathrm{V}_{\mathrm{uni}}(E)$ being

$$\begin{pmatrix} y^2 + yc_2 + c_1 & 0 & 0 & 0 \\ -x + c_3 & y + c_4 & 0 & 0 \\ 0 & -x + c_5 & y^2 + yc_6 + c_7 & 0 \\ 0 & 0 & -x + c_8 & y^2 + yc_9 + c_{10} \\ 0 & 0 & 0 & -x + c_{11} \end{pmatrix}.$$

We see that the only non-zero entries are on the first two diagonals. The ideal E is lex-segment, so let us also investigate the set $\mathrm{V}_{\mathrm{uni}}(E_2)$ for a monomial ideal E_2 that is not lex-segment.

Let $E_2 = (x^4, x^2y^2, xy^6, y^7)$ with partition $(2,2,6,7)$, then $\dim(\mathrm{V}_{\mathrm{uni}}(E_2)) = 11$, too, and the canonical Hilbert-Burch matrix of an ideal in $\mathrm{V}_{\mathrm{uni}}(E_2)$ is

$$\begin{pmatrix} y^2+yc_2+c_1 & 0 & 0 & 0 \\ -x+c_3 & 1 & 0 & 0 \\ c_4 & -x & y^4+y^3c_6+y^2c_5+yc_8+c_7 & 0 \\ 0 & 0 & -x+c_9 & y+c_{10} \\ 0 & 0 & 0 & -x+c_{11} \end{pmatrix}.$$

Here, the entry $(3,1)$ below the second diagonal is allowed to be non-zero.

For the monomial ideal $E_3 = (x^6, x^3y, xy^4, y^6) \in \mathrm{Hilb}^{17}(\mathbf{k}[x,y])$ with partition $(1,1,1,4,4,6)$ the canonical Hilbert-Burch matrix of ideals in $\mathrm{V}_{\mathrm{uni}}(E_3)$ is

$$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 \\ c_4 & 0 & -x & y^3+y^2c_7+yc_6+c_5 & 0 & 0 \\ 0 & 0 & 0 & -x+c_8 & 1 & 0 \\ 0 & 0 & 0 & c_9 & -x & y^2+yc_{11}+c_{10} \\ 0 & 0 & 0 & 0 & 0 & -x+c_{12} \end{pmatrix}$$

and $\dim(\mathrm{V}_{\mathrm{uni}}(E_3)) = 12$.

We can determine the dimension of $\mathrm{V}_{\mathrm{uni}}(E)$ for general monomial ideals E.

Corollary 3.94. Let $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$, and $\mathrm{V}_{\mathrm{uni}}(E)$ as in Definition 3.91, then the dimension of $\mathrm{V}_{\mathrm{uni}}(E)$ is

$$\dim(\mathrm{V}_{\mathrm{uni}}(E)) = m_t + t.$$

Proof. By Corollary 3.92 we have to count the number of coefficients in $\mathcal{N}_{\mathrm{uni}}(E)$, thus

$$\begin{aligned} \dim(\mathrm{V}_{\mathrm{uni}}(E)) &= \textstyle\sum_{i=1}^t d_i + \#\{(i,j) \mid i > j,\ m_{j-1} < m_j = m_{i-1}\} \\ &= m_t + t, \end{aligned}$$

where the last equality follows by the proof of Corollary 3.83. □

Remark 3.95. If one only cares about all term orders τ such that $x >_\tau y$, then the intersection of the Gröbner cells with respect to those orders τ can be obtained as $V_{\mathrm{lex,deglex}}(E)$. By Proposition 3.90 this set is parametrized by $\mathcal{N}_{\mathrm{lex,deglex}}(E)$, that is the set of $(t+1) \times t$ matrices N with entries in $\mathbf{k}[y]$ where

- $n_{i,j} = 0$ for $i < j$
- for $i \geq j$ the non-zero entries satisfy $\deg(n_{i,j}) \leq \min(u_{i,j}, d_j - 1)$.

For $E = (x, y^n)$ the following holds

$$V_{\mathrm{lex}}(E) = V_{\mathrm{deglex}}(E) = V_{\mathrm{lex,deglex}}(E) = V_{\mathrm{lex}_{x>y},\mathrm{lex}_{y>x}}(E) \subset V_{\mathrm{lex}_{y>x}}(E).$$

In this section we have studied intersections of Gröbner cells $V_\tau(E) \cap V_{\tau'}(E)$. Another interesting object of study would be intersections of Gröbner cells not only of different term order τ and τ' of the same ideal E, but to vary the monomial ideal as well, i.e. $V_\tau(E) \cap V_{\tau'}(E')$. In [AS05] a subset of those ideals is studied, the edge-providing ideals. Those are ideals such that E and E' are the only possible leading term ideals. They provide an edge in the so-called *graph of monomial ideals*. It is not clear whether general intersections of Gröbner cells are still affine spaces, see [JS19, Example 7.9]. The main point in studying the intersections $V_\tau(E) \cap V_{\tau'}(E)$ with (canonical) Hilbert-Burch matrices was that the minors of $H + N$, with N being a matrix in the intersection $\mathcal{N}_{\tau,\tau'}(E)$, form a Gröbner basis with respect to τ and τ', and that a reduced Gröbner basis of $I \in V_\tau(E) \cap V_{\tau'}(E)$ with respect to τ is a reduced Gröbner basis of I also with respect to τ'. When changing the ideal E to E' even the formats of the Hilbert-Burch matrices are most often not the same, so a new strategy for studying this situation would be needed.

3.10 Cellular decompositions of $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}])$

For each term order τ on $\mathbf{k}[x, y]$ the Gröbner cells $\{V_\tau(E)\}$ of all monomial ideals $E \in \mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ form a cellular decomposition of $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}])$. As in section 3.6 we can check Conjecture 3.79 for plausibility by using the results of [ES87] about the Betti numbers of $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ and compare them to the numbers that would result from Conjecture 3.79.

Theorem 3.96. *[ES87, Theorem 1.1 (iii)] The non-zero Betti numbers of the Hilbert scheme of points* $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ *are*

$$b_{2i}(\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}])) = P(2n - i, i - n),$$

where $P(n, l)$*, defined in Definition 3.64, is the number of partitions of* n *bounded by* l*.*

Remember that for a scheme X over $\mathbb{C}$ with a cellular decomposition, the Betti numbers can be recovered by the number of cells of a given dimension, more precisely $b_{2i}(X) = \#\{\text{cells of dimension } i\}$.

We have checked for $n \leq 23$ and the term orders τ induced by $(a,\ b)$ for $a \leq 23, b \leq 24$ with $\gcd(a, b) = 1$ that for each of these cases

$$\#\{\text{monomial ideals } E \in \mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}]) \mid \dim(\mathcal{N}_\tau(E)_{<\underline{d}}) = i\} = P(2n - i, i - n).$$

The module and a Jupyter-notebook with the calculations is available on GitHub, `https://github.com/anelanna/LocalHilbertBurch.jl`. We give a brief description of the parts of the module relevant for term orders, similar to the one in section 3.6.

For this case – the so-called graded case – the module can calculate the (conjectural) cellular decomposition from Conjecture 3.79 for a given n, a, b. For all partitions of n it creates a `GradedCell` that has the following properties:

- `a` – the weight a,
- `b` – the weight b,
- `m` – the partition,
- `E` – its associated monomial ideal,

n	the Betti numbers // dimensions vector
1	1
2	1, 1
3	1, 1, 1
4	1, 2, 1, 1
5	1, 2, 2, 1, 1
6	1, 3, 3, 2, 1, 1
7	1, 3, 4, 3, 2, 1, 1
8	1, 4, 5, 5, 3, 2, 1, 1
9	1, 4, 7, 6, 5, 3, 2, 1, 1
10	1, 5, 8, 9, 7, 5, 3, 2, 1, 1
11	1, 5, 10, 11, 10, 7, 5, 3, 2, 1, 1
12	1, 6, 12, 15, 13, 11, 7, 5, 3, 2, 1, 1
13	1, 6, 14, 18, 18, 14, 11, 7, 5, 3, 2, 1, 1
14	1, 7, 16, 23, 23, 20, 15, 11, 7, 5, 3, 2, 1, 1
15	1, 7, 19, 27, 30, 26, 21, 15, 11, 7, 5, 3, 2, 1, 1
16	1, 8, 21, 34, 37, 35, 28, 22, 15, 11, 7, 5, 3, 2, 1, 1
17	1, 8, 24, 39, 47, 44, 38, 29, 22, 15, 11, 7, 5, 3, 2, 1, 1
18	1, 9, 27, 47, 57, 58, 49, 40, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
19	1, 9, 30, 54, 70, 71, 65, 52, 41, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
20	1, 10, 33, 64, 84, 90, 82, 70, 54, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
21	1, 10, 37, 72, 101, 110, 105, 89, 73, 55, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
22	1, 11, 40, 84, 119, 136, 131, 116, 94, 75, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
23	1, 11, 44, 94, 141, 163, 164, 146, 123, 97, 76, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
24	1, 12, 48, 108, 164, 199, 201, 186, 157, 128, 99, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
25	1, 12, 52, 120, 192, 235, 248, 230, 201, 164, 131, 100, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
26	1, 13, 56, 136, 221, 282, 300, 288, 252, 212, 169, 133, 101, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
27	1, 13, 61, 150, 255, 331, 364, 352, 318, 267, 219, 172, 134, 101, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
28	1, 14, 65, 169, 291, 391, 436, 434, 393, 340, 278, 224, 174, 135, 101, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
29	1, 14, 70, 185, 333, 454, 522, 525, 488, 423, 355, 285, 227, 175, 135, 101, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
30	1, 15, 75, 206, 377, 532, 618, 638, 598, 530, 445, 366, 290, 229, 176, 135, 101, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1

Table 3.2: The vectors of number of cells of dimension i for $i = n+1, \dots, 2n$. There are no cells of lower dimensions, so for a nicer presentation we left those out. As discussed before these numbers are also the Betti numbers of $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x}, \mathrm{y}])$ for $n = 1, \dots, 30$.

- `d` – the vector of differences,
- `W` – the weighted degree matrix,
- `rW` – the (integer) degree bounds arising by the matrix W as in Equation 3.7,
- `hilb` – the Hilbert function of the monomial ideal,
- `H` – the canonical Hilbert-Burch matrix of the monomial ideal,
- `M` – the general canonical Hilbert-Burch matrix $H + N$,
- `N` – the corresponding element in $\mathcal{N}_\tau(E)_{<\underline{d}}$,
- `I` – the maximal minors of $M = H + N$, and
- `dim` – the dimension of the cell.

The vector of dimensions can be computed with `graded_sorted_celllist(n)`, a function that returns a dictionary mapping an integer i to a vector with the cells of dimension i.

In Table 3.2 we see these dimension vectors for $n = 1, \dots, 30$. The minimal dimension of a cell is $n + 1$. We left out the occurring zeros for nicer presentation. For the calculation of these numbers we used the weights $a = b = 1$. For $n \leq 23$ we have checked that those are also the dimension vectors for any other $a \leq 23, b \leq 24$.

Example 3.97 (Five cellular decompositions of $\mathrm{Hilb}^6(\mathbf{k}[x, y])$)**.** We finish this section by studying five cellular decompositions of $\mathrm{Hilb}^6(\mathbf{k}[x, y])$ for term orders τ given by different values of $(a,\ b)$. In Table 3.3 – Table 3.7 we find the (conjectural) parametrizations of the cells corresponding to the term orders τ given by $(a,\ b) \in \{(6,\ 1), (3,\ 2), (1,\ 1), (2,\ 3), (1,\ 7)\}$. For each partition $m = (m_1, \ldots, m_t)$ of 6, we obtain a monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ and its associated cell $V_\tau(E)$ – a partition gives one row in each table. Since there are 11 partitions of 6, we obtain 11 cells and 11 rows in each table. The dimension of $\mathcal{N}_\tau(E)_{<\underline{d}}$ is found in the first column, the second column gives the partition m, the next column contains the (conjectural) canonical Hilbert-Burch matrix of a general ideal in $V_\tau(E)$. In the last column we find the (integer) matrix that describes the degree bounds corresponding to W from Equation 3.7. The cells in the tables are ordered by increasing dimension. Using `Singular` we have checked that in all cases all matrices $N \in \mathcal{N}_\tau(E)$ can be transformed into matrices of $\mathcal{N}_\tau(E)_{<\underline{d}}$. This guarantees surjectivity of $\Phi_{\tau,E} : \mathcal{N}_\tau(E)_{<\underline{d}} \to V_\tau(E)$ and thus $\dim(V_\tau(E)) \leq \dim(\mathcal{N}_\tau(E)_{<\underline{d}})$. With the additional knowledge that

$$\#\{E \mid \dim(V_\tau(E)) = i\} = \#\{E \mid \dim(\mathcal{N}_\tau(E)_{<\underline{d}}) = i\}$$

for all i, we can conclude that $\dim(V_\tau(E)) = \dim(\mathcal{N}_\tau(E)_{<\underline{d}})$.

In Table 3.3, we start with $(a,\ b) = (6,\ 1)$. In $\mathrm{Hilb}^6(\mathbf{k}[x, y])$ this corresponds to the lexicographic order with $x > y$. The cells we obtain are as a consequence the same as in [CV08, Theorem 3.3], see Theorem 3.4, and thus the dimension of $V_\tau(E)$ is $\dim(V_\tau(E)) = n + m_t$, in particular, the cell of minimal dimension is $V_\tau((x^6, y))$ and the cell of maximal dimension is $V_\tau((x, y^6))$. As long as $x >_\tau y$, the minimal dimensional cell is $V_\tau((x^6, y))$, this is the case in Tables 3.3, 3.4 and 3.5, when $a \geq b$. When $x <_\tau y$, that is $a < b$, as in Tables 3.6 and 3.7, then $V_\tau((x, y^6))$ is the cell of minimal dimension.

The cell with maximal dimension is

- $V_\tau((x, y^6))$ for $(a,\ b) = (6,\ 1)$ – it is the cell corresponding to $m = (6)$,
- $V_\tau((x^2, xy^2, y^4))$ for $(a,\ b) = (3,\ 2)$ – it is the cell corresponding to $m = (2, 4)$,
- $V_\tau((x^3, x^2y, xy^2, y^3))$ for $(a,\ b) = (1,\ 1)$ and $(a,\ b) = (2,\ 3)$ – it is the cell corresponding to $m = (1, 2, 3)$,
- $V_\tau((x^6, y))$ for $(a,\ b) = (1,\ 7)$ – it is the cell corresponding to $m = (1, 1, 1, 1, 1, 1)$.

In Table 3.8 a comparison of the dimensions of all cells can be found.

Notice that the term order in Table 3.7 given by $(a,\ b) = (1,\ 7)$ corresponds to $\mathrm{lex}_{y>x}$. The dimension of the cell corresponding to the partition $m = (m_1, \ldots, m_t)$ is $6 + t$, analogous to the case of $\mathrm{lex}_{x>y}$ – the roles of t and m_t are interchanged. This holds also for general E as we have seen in Corollary 3.83. The canonical Hilbert-Burch matrix of an ideal in a cell, may have non-zero entries in the lower-left corner of degree zero, namely for all $(i < j)$ such that $m_{j-1} < m_j = m_{i-1}$. In particular, $n_{j+1,j}$ can be a non-zero constant as long as $d_j > 0$. The dimension of the cells has to be the same as we would obtain by renaming the variables and then considering $\mathrm{lex}_{x>y}$, but the canonical Hilbert-Burch matrices look different, even their sizes are different. The canonical Hilbert-Burch matrices in the maximal cell in $\mathrm{lex}_{x>y}$ (as in Table 3.3) are 2×1 matrices, whereas the canonical Hilbert-Burch matrices in the maximal dimensional cell in $\mathrm{lex}_{x<y}$ are 7×6 matrices.

dimension	m	$M = H + N$	degree bounds by W
7	$[1,1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & 0 & -x & 1 \\ c_7 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -12 & -18 & -24 & -30 \\ 6 & -1 & -7 & -13 & -19 & -25 \\ 12 & 6 & -1 & -7 & -13 & -19 \\ 18 & 12 & 6 & -1 & -7 & -13 \\ 24 & 18 & 12 & 6 & -1 & -7 \\ 30 & 24 & 18 & 12 & 6 & -1 \\ 36 & 30 & 24 & 18 & 12 & 6 \end{pmatrix}$
8	$[1,1,1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & y+c_6 \\ c_7 & 0 & 0 & 0 & -x+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -12 & -18 & -23 \\ 6 & -1 & -7 & -13 & -18 \\ 12 & 6 & -1 & -7 & -12 \\ 18 & 12 & 6 & -1 & -6 \\ 24 & 18 & 12 & 6 & 0 \\ 29 & 23 & 17 & 11 & 6 \end{pmatrix}$
8	$[1,1,2,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & y+c_4 & 0 \\ c_5 & 0 & -x+c_6 & 1 \\ c_7 & 0 & c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -11 & -17 \\ 6 & -1 & -6 & -12 \\ 12 & 6 & 0 & -6 \\ 17 & 11 & 6 & -1 \\ 23 & 17 & 12 & 6 \end{pmatrix}$
8	$[2,2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & 0 & 0 \\ -x+yc_4+c_3 & 1 & 0 \\ yc_6+c_5 & -x & 1 \\ yc_8+c_7 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & -5 & -11 \\ 6 & -1 & -7 \\ 12 & 6 & -1 \\ 18 & 12 & 6 \end{pmatrix}$
9	$[1,1,1,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y^2+yc_6+c_5 \\ c_7 & 0 & 0 & -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -12 & -16 \\ 6 & -1 & -7 & -11 \\ 12 & 6 & -1 & -5 \\ 18 & 12 & 6 & 1 \\ 22 & 16 & 10 & 6 \end{pmatrix}$
9	$[1,2,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & y+c_3 & 0 \\ c_4 & -x+c_5 & y+c_6 \\ c_7 & c_8 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -5 & -10 \\ 6 & 0 & -5 \\ 11 & 6 & 0 \\ 16 & 11 & 6 \end{pmatrix}$
9	$[3,3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 & 0 \\ -x+y^2c_6+yc_5+c_4 & 1 \\ y^2c_9+yc_8+c_7 & -x \end{pmatrix}$	$\begin{pmatrix} 2 & -4 \\ 6 & -1 \\ 12 & 6 \end{pmatrix}$
10	$[1,1,4]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & y^3+y^2c_6+yc_5+c_4 \\ c_7 & 0 & -x+y^2c_{10}+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -9 \\ 6 & -1 & -4 \\ 12 & 6 & 2 \\ 15 & 9 & 6 \end{pmatrix}$
10	$[2,4]$	$\begin{pmatrix} y^2+yc_2+c_1 & 0 \\ -x+yc_4+c_3 & y^2+yc_6+c_5 \\ yc_8+c_7 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 1 & -3 \\ 6 & 1 \\ 10 & 6 \end{pmatrix}$
11	$[1,5]$	$\begin{pmatrix} y+c_1 & 0 \\ -x+c_2 & y^4+y^3c_6+y^2c_5+yc_4+c_3 \\ c_7 & -x+y^3c_{11}+y^2c_{10}+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 6 & 3 \\ 8 & 6 \end{pmatrix}$
12	$[6]$	$\begin{pmatrix} y^6+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+y^5c_{12}+y^4c_{11}+y^3c_{10}+y^2c_9+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Table 3.3: $n = 6$, $a = 6$, $b = 1$

dimension	m	$M=H+N$	degree bounds by W
7	$[1,1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & 0 & -x & 1 \\ c_7 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -4 & -6 & -7 \\ 1 & -1 & -2 & -4 & -5 & -7 \\ 3 & 1 & -1 & -2 & -4 & -5 \\ 4 & 3 & 1 & -1 & -2 & -4 \\ 6 & 4 & 3 & 1 & -1 & -2 \\ 7 & 6 & 4 & 3 & 1 & -1 \\ 9 & 7 & 6 & 4 & 3 & 1 \end{pmatrix}$
8	$[1,1,1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & y+c_6 \\ c_7 & 0 & 0 & 0 & -x+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -4 & -5 \\ 1 & -1 & -2 & -4 & -4 \\ 3 & 1 & -1 & -2 & -3 \\ 4 & 3 & 1 & -1 & -1 \\ 6 & 4 & 3 & 1 & 0 \\ 6 & 5 & 3 & 2 & 1 \end{pmatrix}$
8	$[1,1,2,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & y+c_4 & 0 \\ c_5 & 0 & -x+c_6 & 1 \\ c_7 & 0 & c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & -1 & -1 & -3 \\ 3 & 1 & 0 & -1 \\ 3 & 2 & 1 & -1 \\ 5 & 3 & 3 & 1 \end{pmatrix}$
8	$[6]$	$\begin{pmatrix} y^6+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 1 \end{pmatrix}$
9	$[1,1,1,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y^2+yc_6+c_5 \\ c_7 & 0 & 0 & -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -2 \\ 1 & -1 & -2 & -2 \\ 3 & 1 & -1 & 0 \\ 4 & 3 & 1 & 1 \\ 4 & 2 & 1 & 1 \end{pmatrix}$
9	$[1,5]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^4+y^3c_7+y^2c_6+yc_5+c_4 \\ 0 & -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 1 & 3 \\ -1 & 1 \end{pmatrix}$
9	$[2,2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & c_3 & 0 \\ -x+yc_5+c_4 & 1 & 0 \\ yc_7+c_6 & -x & 1 \\ yc_9+c_8 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & -2 \\ 3 & 1 & -1 \\ 4 & 3 & 1 \end{pmatrix}$
10	$[1,1,4]$	$\begin{pmatrix} y+c_1 & 0 & c_2 \\ -x+c_3 & 1 & 0 \\ c_4 & -x & y^3+y^2c_7+yc_6+c_5 \\ c_8 & 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$
10	$[3,3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 & yc_5+c_4 \\ -x+yc_7+c_6 & 1 \\ y^2c_{10}+yc_9+c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 1 \end{pmatrix}$
11	$[1,2,3]$	$\begin{pmatrix} y+c_1 & c_2 & 0 \\ -x+c_3 & y+c_4 & c_5 \\ c_6 & -x+c_7 & y+c_8 \\ c_9 & c_{10} & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$
12	$[2,4]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+yc_6+c_5 & y^2+yc_8+c_7 \\ yc_{10}+c_9 & -x+yc_{12}+c_{11} \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

Table 3.4: $n=6$, $a=3$, $b=2$

dimension	m	$M = H + N$	degree bounds by W
7	$[1,1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & 0 & -x & 1 \\ c_7 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 \\ 1 & -1 & -2 & -3 & -4 & -5 \\ 2 & 1 & -1 & -2 & -3 & -4 \\ 3 & 2 & 1 & -1 & -2 & -3 \\ 4 & 3 & 2 & 1 & -1 & -2 \\ 5 & 4 & 3 & 2 & 1 & -1 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$
8	$[1,1,1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & y+c_6 \\ c_7 & 0 & 0 & 0 & -x+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 & -3 \\ 1 & -1 & -2 & -3 & -3 \\ 2 & 1 & -1 & -2 & -2 \\ 3 & 2 & 1 & -1 & -1 \\ 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 1 \end{pmatrix}$
8	$[1,1,2,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & y+c_4 & 0 \\ c_5 & 0 & -x+c_6 & 1 \\ c_7 & 0 & c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 & -2 \\ 1 & -1 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 2 & 1 & 1 & -1 \\ 3 & 2 & 2 & 1 \end{pmatrix}$
8	$[6]$	$\begin{pmatrix} y^6+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 1 \end{pmatrix}$
9	$[1,1,1,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y^2+yc_6+c_5 \\ c_7 & 0 & 0 & -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -1 \\ 1 & -1 & -2 & -1 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}$
9	$[1,5]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^4+y^3c_7+y^2c_6+yc_5+c_4 \\ 0 & -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 1 & 3 \\ -2 & 1 \end{pmatrix}$
9	$[2,2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & c_3 & 0 \\ -x+yc_5+c_4 & 1 & 0 \\ yc_7+c_6 & -x & 1 \\ yc_9+c_8 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & -2 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$
10	$[1,1,4]$	$\begin{pmatrix} y+c_1 & 0 & c_2 \\ -x+c_3 & 1 & 0 \\ c_4 & -x & y^3+y^2c_7+yc_6+c_5 \\ c_8 & 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$
10	$[3,3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 & yc_5+c_4 \\ -x+yc_7+c_6 & 1 \\ y^2c_{10}+yc_9+c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$
11	$[2,4]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+yc_6+c_5 & y^2+yc_8+c_7 \\ c_9 & -x+yc_{11}+c_{10} \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$
12	$[1,2,3]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & y+c_5 & c_6 \\ c_7 & -x+c_8 & y+c_9 \\ c_{10} & c_{11} & -x+c_{12} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Table 3.5: $n = 6$, $a = 1$, $b = 1$

dimension	m	$M = H + N$	degree bounds by W
7	[6]	$\begin{pmatrix} y^6 + y^5c_6 + y^4c_5 + y^3c_4 + y^2c_3 + yc_2 + c_1 \\ -x + c_7 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
8	[1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y + c_1 & c_2 & 0 & 0 & 0 & 0 \\ -x + c_3 & 1 & 0 & 0 & 0 & 0 \\ c_4 & -x & 1 & 0 & 0 & 0 \\ c_5 & 0 & -x & 1 & 0 & 0 \\ c_6 & 0 & 0 & -x & 1 & 0 \\ c_7 & 0 & 0 & 0 & -x & 1 \\ c_8 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 & -2 & -3 \\ 0 & -1 & -1 & -2 & -3 & -3 \\ 1 & 0 & -1 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 & -1 & -2 \\ 2 & 2 & 1 & 0 & -1 & -1 \\ 3 & 2 & 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 2 & 1 & 0 \end{pmatrix}$
8	[1, 5]	$\begin{pmatrix} y + c_1 & c_2 \\ -x + c_3 & y^4 + y^3c_7 + y^2c_6 + yc_5 + c_4 \\ 0 & -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 0 & 3 \\ -3 & 0 \end{pmatrix}$
8	[2, 4]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 \\ -x + c_5 & y^2 + yc_7 + c_6 \\ 0 & -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$
9	[1, 1, 1, 1, 2]	$\begin{pmatrix} y + c_1 & c_2 & 0 & 0 & 0 \\ -x + c_3 & 1 & 0 & 0 & 0 \\ c_4 & -x & 1 & 0 & 0 \\ c_5 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & -x & y + c_7 \\ c_8 & 0 & 0 & 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 & -1 \\ 0 & -1 & -1 & -2 & -2 \\ 1 & 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & -1 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{pmatrix}$
9	[1, 1, 4]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & 1 & 0 \\ c_5 & -x & y^3 + y^2c_8 + yc_7 + c_6 \\ 0 & 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}$
9	[3, 3]	$\begin{pmatrix} y^3 + y^2c_3 + yc_2 + c_1 & y^2c_6 + yc_5 + c_4 \\ -x + c_7 & 1 \\ yc_9 + c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$
10	[1, 1, 1, 3]	$\begin{pmatrix} y + c_1 & c_2 & 0 & c_3 \\ -x + c_4 & 1 & 0 & 0 \\ c_5 & -x & 1 & 0 \\ c_6 & 0 & -x & y^2 + yc_8 + c_7 \\ c_9 & 0 & 0 & -x + c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$
10	[2, 2, 2]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 & c_5 \\ -x + c_6 & 1 & 0 \\ yc_8 + c_7 & -x & 1 \\ yc_{10} + c_9 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$
11	[1, 1, 2, 2]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & 0 \\ -x + c_4 & 1 & 0 & 0 \\ c_5 & -x & y + c_6 & c_7 \\ c_8 & 0 & -x + c_9 & 1 \\ c_{10} & 0 & c_{11} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$
12	[1, 2, 3]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & y + c_5 & c_6 \\ c_7 & -x + c_8 & y + c_9 \\ c_{10} & c_{11} & -x + c_{12} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Table 3.6: $n = 6$, $a = 2$, $b = 3$

dimension	m	$M = H + N$	degree bounds by W
7	$[6]$	$\begin{pmatrix} y^6+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+c_7 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
8	$[1,5]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^4+y^3c_7+y^2c_6+yc_5+c_4 \\ 0 & -x+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 0 & 3 \\ -4 & 0 \end{pmatrix}$
8	$[2,4]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+c_5 & y^2+yc_7+c_6 \\ 0 & -x+c_8 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -2 & 0 \end{pmatrix}$
8	$[3,3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 & y^2c_6+yc_5+c_4 \\ -x+c_7 & 1 \\ c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$
9	$[1,1,4]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & 1 & 0 \\ c_5 & -x & y^3+y^2c_8+yc_7+c_6 \\ 0 & 0 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ -3 & -3 & 0 \end{pmatrix}$
9	$[1,2,3]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & y+c_5 & c_6 \\ 0 & -x+c_7 & y+c_8 \\ 0 & 0 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ -2 & -1 & 0 \end{pmatrix}$
9	$[2,2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 & yc_6+c_5 \\ -x+c_7 & 1 & 0 \\ c_8 & -x & 1 \\ c_9 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
10	$[1,1,1,3]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 & c_4 \\ -x+c_5 & 1 & 0 & 0 \\ c_6 & -x & 1 & 0 \\ c_7 & 0 & -x & y^2+yc_9+c_8 \\ 0 & 0 & 0 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ -2 & -2 & -2 & 0 \end{pmatrix}$
10	$[1,1,2,2]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 & c_4 \\ -x+c_5 & 1 & 0 & 0 \\ c_6 & -x & y+c_7 & c_8 \\ 0 & 0 & -x+c_9 & 1 \\ 0 & 0 & c_{10} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 \end{pmatrix}$
11	$[1,1,1,1,2]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 & c_4 & c_5 \\ -x+c_6 & 1 & 0 & 0 & 0 \\ c_7 & -x & 1 & 0 & 0 \\ c_8 & 0 & -x & 1 & 0 \\ c_9 & 0 & 0 & -x & y+c_{10} \\ 0 & 0 & 0 & 0 & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}$
12	$[1,1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ -x+c_7 & 1 & 0 & 0 & 0 & 0 \\ c_8 & -x & 1 & 0 & 0 & 0 \\ c_9 & 0 & -x & 1 & 0 & 0 \\ c_{10} & 0 & 0 & -x & 1 & 0 \\ c_{11} & 0 & 0 & 0 & -x & 1 \\ c_{12} & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Table 3.7: $n = 6$, $a = 1$, $b = 7$

m	(6, 1)	(3, 2)	(1, 1)	(2, 3)	(1, 7)
$[1,1,1,1,1,1]$	7	7	7	8	12
$[1,1,1,1,2]$	8	8	8	9	11
$[2,2,2]$	8	9	9	10	9
$[1,1,2,2]$	8	8	8	11	10
$[1,1,1,3]$	9	9	9	10	10
$[1,2,3]$	9	11	12	12	9
$[3,3]$	9	10	10	9	8
$[1,1,4]$	10	10	10	9	9
$[2,4]$	10	12	11	8	8
$[1,5]$	11	9	9	8	8
$[6]$	12	8	8	7	7

Table 3.8: Comparison between the dimension of the cells $V_\tau(E)$ for term orders given by different values of $(a,\ b)$, where $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$.

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Summary

This dissertation contains two chapters on the use of torus actions in algebraic geometry.

In chapter 2 we study "immaculate line bundles" on projective toric varieties. The cohomology groups of those line bundles vanish in all degrees, including the 0-th degree. Immaculate line bundles can be seen as building blocks of full exceptional sequences of line bundles of the variety. All the immaculate line bundles of a toric variety $X = \mathbb{TV}(\Sigma)$ can be identified in two steps. First identify those subsets of the rays $\Sigma(1)$ whose geometric realization is not $\mathbf{k}$-acyclic, they will be called tempting. Those subsets of the rays give "maculate sets/regions" in the class group of the variety. A line bundle is immaculate, if it is not in any of those maculate sets. So the first step in finding immaculate line bundles is to find all tempting subsets. When X is projective, the main result for this is that primitive collections – subsets of the rays that do not span a cone, but each proper subset spans a cone – are always tempting. And a subset of rays can only be tempting if it is the union of primitive collections. The same has to hold for the complement, too. We give descriptions of the immaculate line bundles for different examples. In particular, we describe the immaculate locus for projective toric varieties of Picard rank 3. Most of the results have been published in [ABKW20].

In chapter 3 we study the Hilbert scheme of n points in affine plane. It describes all ideals in the polynomial ring of two variables whose quotient is an n-dimensional vector space. The Hilbert scheme can be decomposed into so called Gröbner cells. They consist of those ideals that have a prescribed leading term ideal with respect to a given term order. The Gröbner cells for the lexicographic and the degree-lexicographic order are parametrized in [CV08] and [Con11], respectively, by canonical Hilbert-Burch matrices. A Hilbert-Burch matrix of an ideal is a matrix generating the syzygies of the ideal. Its maximal minors also generate the ideal. These results are generalized in two directions. Firstly, we consider the ring of formal power series. Here we give a parametrization of the cells that respects the Hilbert function stratification of the punctual Hilbert scheme. In particular, this cellular decomposition restricts to a cellular decomposition of the subscheme consisting of ideals with a prescribed Hilbert function. We use the parametrization to describe subsets of the Gröbner cells associated to lex-segment ideals with a given minimal number of generators. These subsets are quasi-affine varieties inside the cell. Most of these results have been published in [HW21] and [HW23]. The second way of changing the setting is to consider a general term order on the polynomial ring. We give a surjection to the Gröbner cell with respect to this ordering and parametrizations of subsets of the cell, as well as a conjecture how the parametrization of the whole cell should look like. We also study intersections of Gröbner cells with respect to different term orders.

Zusammenfassung

Die vorliegende Dissertation besteht aus zwei Kapiteln zu zwei unterschiedlichen Anwendungen von Toruswirkungen in der algebraischen Geometrie.

Die wichtigsten Objekte des Kapitels 2 sind *unbefleckte Geradenbündel* auf projektiven torischen Varietäten $X = \mathbb{TV}(\Sigma)$, Geradenbündel, deren Kohomologiegruppen alle verschwinden. Unbefleckte Geradenbündel können als Bausteine für exzeptionelle Sequenzen aus Geradenbündeln dienen und somit die derivierte Kategorie der Varietät beschreiben. Die Bestimmung von unbefleckten Geradenbündeln lässt sich in zwei Schritte aufteilen. Es lassen sich Teilmengen der Strahlen $\Sigma(1)$ des die torische Varietät beschreibenden Fächers Σ identifizieren, deren geometrische Realisierungen nicht $\mathbf{k}$-azyklisch sind. Diese *verlockenden* Teilmengen der Strahlen definieren *befleckte* Teilmengen der Klassengruppe $\mathrm{Cl}(X)$. Ein Geradenbündel ist genau dann unbefleckt, wenn es in keiner befleckten Teilmenge von $\mathrm{Cl}(X)$ liegt. Die Bestimmung aller unbefleckten Geradenbündel lässt sich also in zwei Schritte aufteilen. Das Bestimmen der verlockenden Teilmengen der Strahlen und das Bestimmen der zugehörigen befleckten Regionen. Primitive Kollektionen – Teilmengen der Strahlen, die selbst keinen Kegel des Fächers aufspannen, aber jede ihrer Teilmenge spannt einen Kegel des Fächers auf – sind verlockend und außerdem ist eine Teilmenge nur dann verlockend, wenn sie eine Vereinigung von primitiven Kollektionen ist. Dies muss auch für das Komplement gelten. Wir geben die Beschreibung für die unbefleckten Geradenbündel für verschiedene Beispielklassen von projektiven torischen Varietäten. Insbesondere beschreiben wir die unbefleckten Geradenbündel für projektive torische Varietäten von Picardrang 3. Die meisten dieser Ergebnisse sind in [ABKW20] erschienen.

In Kapitel 3 geht es um das Hilbertschema von n Punkten in der affinen Ebene. Seine Punkte sind Ideale im Polynomenring $\mathbf{k}[x, y]$, deren Quotient ein n-dimensionaler $\mathbf{k}$-Vektorraum ist. Das Hilbertschema kann in sogenannte Gröbnerzellen unterteilt werden. Sie umfassen Ideale, die bezüglich einer Termordnung τ ein festgelegtes Leitideal haben. In [CV08] und [Con11] werden für die lexikographische und gradlexikographische Termordnung Parametrisierung der Gröbnerzellen durch kanonische Hilbert-Burch Matrizen angegeben. Hilbert-Burch Matrizen beschreiben die Syzygien des Ideals und ihre maximalen Minoren erzeugen das Ideal. Die Ergebnisse werden in zwei Richtungen verallgemeinert. Zunächst betrachten wir Ideale im Ring der formalen Potenzreihen. Wir geben eine Parametrisierung der Zellen, bei der die lokale Struktur der Ideale berücksichtigt wird. Insbesondere lässt sich diese zelluläre Unterteilung des lokalen Hilbertschemas auf eine zelluläre Unterteilung des Unterschemas einschränken, das nur Ideale mit einer gegebenen Hilbertfunktion beinhaltet. Durch diese Parametrisierung lassen sich für Ideale in diesen Zellen kanonische Hilbert-Burch Matrizen definieren. Diese benutzen wir um Teilmengen der Gröbnerzellen mit einer vorgegebenen minimalen Anzahl von Erzeugern zu beschreiben. Diese Teilmengen sind quasi-affine Varietäten in der Gröbnerzelle. Die meisten der Resultate sind in [HW21] und [HW23] erschienen. Die zweite Möglichkeit das Setting zu ändern, ist beliebige Termordnungen auf dem Polynomenring zu betrachten. Im zweiten Teil von Kapitel 3 geben wir eine Surjektion auf diese Gröbnerzellen, sowie Parametrisierungen von Teilmengen und geben eine Vermutung, wie eine Parametrisierung der ganzen Zelle aussieht. Außerdem untersuchen wir Schnitte von Gröbnerzellen bezüglich verschiedener Termordnungen.

Appendix

In Tables 3.9 – 3.17 show the cellular decompositions of $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\![\mathrm{x},\mathrm{y}]\!])$ that are compatible with its Hilbert function stratification, for $n = 1, 2, 3, 4, 5, 8, 9$. The ones for $n = 6, 7$ can be found with a detailed explanation in Example 3.63 and Example 3.68, respectively.

Tables 3.18 – 3.43 show the (conjectural) cellular decompositions in terms of canonical Hilbert-Burch matrices for $n = 1, 2, 3, 4, 5, 7$, and term orders induced by different values of $(a,\ b)$. More precisely, for all monomial ideals in $\mathrm{Hilb}^{\mathrm{n}}(\mathbf{k}[\mathrm{x},\mathrm{y}])$ we give the dimension of the associated cell $V_\tau(E)$ and $M = H + N$, where H is the canonical Hilbert-Burch matrix of E and $N \in \mathcal{N}_\tau(E)_{<\underline{d}}$. As in Example 3.97, where we have studied five different cellular decompositions of $\mathrm{Hilb}^6(\mathbf{k}[x,y])$, we have used `Singular` to assert that $\Phi_{\tau,E} : \mathcal{N}_\tau(E)_{<\underline{d}} \to V_\tau(E)$ is surjective.

m	Hilbert function	M	dimension
[1]	[1]	$\begin{pmatrix} y \\ -x \end{pmatrix}$	0

Table 3.9: The Gröbner cells for $\mathrm{Hilb}^1(\mathbf{k}[\![x,y]\!])$

m	Hilbert function	M	dimension
$[1,1]$	$[1,1]$	$\begin{pmatrix} y & 0 \\ -x & 1 \\ 0 & -x \end{pmatrix}$	0
$[2]$	$[1,1]$	$\begin{pmatrix} y^2 \\ -x+yc_1 \end{pmatrix}$	1

Table 3.10: The Gröbner cells for $\text{Hilb}^2(\mathbf{k}[[x,y]])$

m	Hilbert function	M	dimension
$[1,2]$	$[1,2]$	$\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}$	0
$[1,1,1]$	$[1,1,1]$	$\begin{pmatrix} y & 0 & c_1 \\ -x & 1 & 0 \\ 0 & -x & 1 \\ 0 & 0 & -x \end{pmatrix}$	1
$[3]$	$[1,1,1]$	$\begin{pmatrix} y^3 \\ -x+y^2c_2+yc_1 \end{pmatrix}$	2

Table 3.11: The Gröbner cells for $\text{Hilb}^3(\mathbf{k}[[x,y]])$

m	Hilbert function	M	dimension
$[1,1,2]$	$[1,2,1]$	$\begin{pmatrix} y & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y \\ 0 & 0 & -x \end{pmatrix}$	0
$[2,2]$	$[1,2,1]$	$\begin{pmatrix} y^2 & 0 \\ -x+yc_1 & 1 \\ 0 & -x \end{pmatrix}$	1
$[1,3]$	$[1,2,1]$	$\begin{pmatrix} y & 0 \\ -x & y^2 \\ c_1 & -x+yc_2 \end{pmatrix}$	2
$[1,1,1,1]$	$[1,1,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 \\ -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{pmatrix}$	2
$[4]$	$[1,1,1,1]$	$\begin{pmatrix} y^4 \\ -x+y^3c_3+y^2c_2+yc_1 \end{pmatrix}$	3

Table 3.12: The Gröbner cells for $\text{Hilb}^4(\mathbf{k}[[x,y]])$

m	Hilbert function	M	dimension
$[1,2,2]$	$[1,2,2]$	$\begin{pmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & -x & 1 \\ 0 & 0 & -x \end{pmatrix}$	0
$[1,1,3]$	$[1,2,2]$	$\begin{pmatrix} y & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^2 \\ 0 & 0 & -x+yc_1 \end{pmatrix}$	1
$[2,3]$	$[1,2,2]$	$\begin{pmatrix} y^2 & 0 \\ -x+yc_1 & y \\ yc_2 & -x \end{pmatrix}$	2
$[1,1,1,2]$	$[1,2,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 \\ -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -x \end{pmatrix}$	2
$[1,4]$	$[1,2,1,1]$	$\begin{pmatrix} y & 0 \\ -x & y^3 \\ c_1 & -x+y^2c_3+yc_2 \end{pmatrix}$	3
$[1,1,1,1,1]$	$[1,1,1,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	3
$[5]$	$[1,1,1,1,1]$	$\begin{pmatrix} y^5 \\ -x+y^4c_4+y^3c_3+y^2c_2+yc_1 \end{pmatrix}$	4

Table 3.13: The Gröbner cells for $\mathrm{Hilb}^5(\mathbf{k}[\![x,y]\!])$

m	Hilbert function	M	dimension
$[1,2,2,3]$	$[1,2,3,2]$	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & y & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -x \end{pmatrix}$	0
$[1,1,3,3]$	$[1,2,3,2]$	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & y^2 & 0 \\ 0 & 0 & -x+yc_1 & 1 \\ 0 & 0 & 0 & -x \end{pmatrix}$	1
$[1,1,2,4]$	$[1,2,3,2]$	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & y & 0 \\ 0 & 0 & -x & y^2 \\ 0 & 0 & c_1 & -x+yc_2 \end{pmatrix}$	2
$[2,3,3]$	$[1,2,3,2]$	$\begin{pmatrix} y^2 & 0 & 0 \\ -x+yc_1 & y & 0 \\ yc_2 & -x & 1 \\ 0 & 0 & -x \end{pmatrix}$	2
$[1,1,1,2,3]$	$[1,2,3,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & y & 0 \\ 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	3
$[1,1,2,2,2]$	$[1,2,2,2,1]$	$\begin{pmatrix} y & 0 & 0 & c_1 & c_2 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & y & 0 & c_3 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	3
$[2,2,4]$	$[1,2,3,2]$	$\begin{pmatrix} y^2 & 0 & 0 \\ -x+yc_1 & 1 & 0 \\ 0 & -x & y^2 \\ yc_2 & 0 & -x+yc_3 \end{pmatrix}$	3
$[1,1,1,1,2,2]$	$[1,2,2,1,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & y & 0 \\ 0 & 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	4
$[1,1,1,1,4]$	$[1,2,2,2,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & 0 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & y^3 \\ 0 & 0 & 0 & 0 & -x+y^2c_4+yc_3 \end{pmatrix}$	4
$[1,2,5]$	$[1,2,3,1,1]$	$\begin{pmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & -x & y^3 \\ c_1 & c_2 & -x+y^2c_4+yc_3 \end{pmatrix}$	4
$[1,3,4]$	$[1,2,3,2]$	$\begin{pmatrix} y & 0 & 0 \\ -x & y^2 & 0 \\ c_1 & -x+yc_2 & y \\ c_3 & yc_4 & -x \end{pmatrix}$	4
$[2,2,2,2]$	$[1,2,2,2,1]$	$\begin{pmatrix} y^2 & 0 & yc_1 & yc_3+c_2 \\ -x+yc_4 & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{pmatrix}$	4

Table 3.14: The Gröbner cells for $\mathrm{Hilb}^8(\mathbf{k}[\![x,y]\!])$,part 1

m	Hilbert function	M	dimension
$[1,1,1,1,1,1,2]$	$[1,2,1,1,1,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	5
$[1,1,1,1,3]$	$[1,2,2,1,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & y^2 \\ 0 & 0 & 0 & 0 & 0 & -x+yc_5 \end{pmatrix}$	5
$[1,1,1,5]$	$[1,2,2,2,1]$	$\begin{pmatrix} y & 0 & c_1 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y^4 \\ c_2 & 0 & 0 & -x+y^3c_5+y^2c_4+yc_3 \end{pmatrix}$	5
$[1,1,6]$	$[1,2,2,1,1,1]$	$\begin{pmatrix} y & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^5 \\ c_1 & 0 & -x+y^4c_5+y^3c_4+y^2c_3+yc_2 \end{pmatrix}$	5
$[4,4]$	$[1,2,2,2,1]$	$\begin{pmatrix} y^4 & 0 \\ -x+y^3c_3+y^2c_2+yc_1 & 1 \\ y^3c_5+y^2c_4 & -x \end{pmatrix}$	5
$[1,1,1,1,1,1,1,1]$	$[1,1,1,1,1,1,1,1]$	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	6
$[1,7]$	$[1,2,1,1,1,1,1]$	$\begin{pmatrix} y & 0 \\ -x & y^6 \\ c_1 & -x+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2 \end{pmatrix}$	6
$[2,6]$	$[1,2,2,1,1,1]$	$\begin{pmatrix} y^2 & 0 \\ -x+yc_1 & y^4 \\ yc_3+c_2 & -x+y^3c_6+y^2c_5+yc_4 \end{pmatrix}$	6
$[3,5]$	$[1,2,2,2,1]$	$\begin{pmatrix} y^3 & 0 \\ -x+y^2c_2+yc_1 & y^2 \\ y^2c_5+yc_4+c_3 & -x+yc_6 \end{pmatrix}$	6
$[8]$	$[1,1,1,1,1,1,1,1]$	$\begin{pmatrix} y^8 \\ -x+y^7c_7+y^6c_6+y^5c_5+y^4c_4+y^3c_3+y^2c_2+yc_1 \end{pmatrix}$	7

Table 3.15: The Gröbner cells for $\mathrm{Hilb}^8(\mathbf{k}[\![x,y]\!])$,part 2

m	Hilbert function	M	dimension
[1, 2, 3, 3]	[1, 2, 3, 3]	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & y & 0 & 0 \\ 0 & -x & y & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{pmatrix}$	0
[1, 2, 2, 4]	[1, 2, 3, 3]	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & y & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y^2 \\ 0 & 0 & 0 & -x+yc_1 \end{pmatrix}$	1
[1, 1, 2, 2, 3]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y & 0 & 0 & c_1 & c_2 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & y & 0 & 0 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	2
[1, 1, 3, 4]	[1, 2, 3, 3]	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & y^2 & 0 \\ 0 & 0 & -x+yc_1 & y \\ 0 & 0 & yc_2 & -x \end{pmatrix}$	2
[1, 1, 1, 3, 3]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y & 0 & c_1 & 0 & c_2 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & y^2 & 0 \\ 0 & 0 & 0 & -x+yc_3 & 1 \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	3
[2, 2, 2, 3]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y^2 & 0 & yc_1 & yc_2 \\ -x+yc_3 & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -x \end{pmatrix}$	3
[2, 3, 4]	[1, 2, 3, 3]	$\begin{pmatrix} y^2 & 0 & 0 \\ -x+yc_1 & y & 0 \\ yc_2 & -x & y \\ yc_3 & 0 & -x \end{pmatrix}$	3
[1, 1, 1, 1, 2, 3]	[1, 2, 3, 1, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & y & 0 \\ 0 & 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	4
[1, 1, 1, 2, 4]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & 0 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & y & 0 \\ 0 & 0 & 0 & -x & y^2 \\ 0 & 0 & 0 & c_3 & -x+yc_4 \end{pmatrix}$	4
[1, 1, 2, 5]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & y & 0 \\ 0 & 0 & -x & y^3 \\ c_1 & 0 & c_2 & -x+y^2c_4+yc_3 \end{pmatrix}$	4
[1, 2, 2, 2, 2]	[1, 2, 2, 2, 2]	$\begin{pmatrix} y & 0 & 0 & c_1 & c_2 \\ -x & y & 0 & c_3 & c_4 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & -x \end{pmatrix}$	4
[3, 3, 3]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y^3 & 0 & y^2c_1 \\ -x+y^2c_3+yc_2 & 1 & 0 \\ y^2c_4 & -x & 1 \\ 0 & 0 & -x \end{pmatrix}$	4
[1, 1, 1, 1, 1, 2, 2]	[1, 2, 2, 1, 1, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & y & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	5
[1, 1, 1, 1, 5]	[1, 2, 2, 2, 2]	$\begin{pmatrix} y & 0 & c_1 & c_2 & 0 \\ -x & 1 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & -x & y^4 \\ 0 & 0 & 0 & 0 & -x+y^3c_5+y^2c_4+yc_3 \end{pmatrix}$	5
[1, 1, 1, 2, 2, 2]	[1, 2, 2, 2, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & y & 0 & c_5 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	5

Table 3.16: The Gröbner cells for $\mathrm{Hilb}^9(\mathbf{k}[\![x, y]\!])$,part 1

m	Hilbert function	M	dimension
[1, 2, 6]	[1, 2, 3, 1, 1, 1]	$\begin{pmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & -x & y^4 \\ c_1 & c_2 & -x+y^3c_5+y^2c_4+yc_3 \end{pmatrix}$	5
[1, 4, 4]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y & 0 & 0 \\ -x & y^3 & 0 \\ c_1 & -x+y^2c_3+yc_2 & 1 \\ c_4 & y^2c_5 & -x \end{pmatrix}$	5
[2, 2, 5]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y^2 & 0 & 0 \\ -x+yc_1 & 1 & 0 \\ 0 & -x & y^3 \\ yc_3+c_2 & 0 & -x+y^2c_5+yc_4 \end{pmatrix}$	5
[1, 1, 1, 1, 1, 1, 1, 2]	[1, 2, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	6
[1, 1, 1, 1, 1, 1, 3]	[1, 2, 2, 1, 1, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x+yc_6 \end{pmatrix}$	6
[1, 1, 1, 1, 1, 4]	[1, 2, 2, 2, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 \\ -x & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & y^3 \\ 0 & 0 & 0 & 0 & 0 & -x+y^2c_6+yc_5 \end{pmatrix}$	6
[1, 1, 1, 6]	[1, 2, 2, 2, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & 0 \\ -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & y^5 \\ c_2 & 0 & 0 & -x+y^4c_6+y^3c_5+y^2c_4+yc_3 \end{pmatrix}$	6
[1, 1, 7]	[1, 2, 2, 1, 1, 1, 1]	$\begin{pmatrix} y & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^6 \\ c_1 & 0 & -x+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2 \end{pmatrix}$	6
[1, 3, 5]	[1, 2, 3, 2, 1]	$\begin{pmatrix} y & 0 & 0 \\ -x & y^2 & 0 \\ c_1 & -x+yc_2 & y^2 \\ c_3 & yc_5+c_4 & -x+yc_6 \end{pmatrix}$	6
[4, 5]	[1, 2, 2, 2, 2]	$\begin{pmatrix} y^4 & 0 \\ -x+y^3c_3+y^2c_2+yc_1 & y \\ y^3c_6+y^2c_5+yc_4 & -x \end{pmatrix}$	6
[1, 1, 1, 1, 1, 1, 1, 1, 1]	[1, 1, 1, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	7
[1, 8]	[1, 2, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y & 0 \\ -x & y^7 \\ c_1 & -x+y^6c_7+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2 \end{pmatrix}$	7
[2, 7]	[1, 2, 2, 1, 1, 1, 1]	$\begin{pmatrix} y^2 & 0 \\ -x+yc_1 & y^5 \\ yc_3+c_2 & -x+y^4c_7+y^3c_6+y^2c_5+yc_4 \end{pmatrix}$	7
[3, 6]	[1, 2, 2, 2, 1, 1]	$\begin{pmatrix} y^3 & 0 \\ -x+y^2c_2+yc_1 & y^3 \\ y^2c_5+yc_4+c_3 & -x+y^2c_7+yc_6 \end{pmatrix}$	7
[9]	[1, 1, 1, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y^9 \\ -x+y^8c_8+y^7c_7+y^6c_6+y^5c_5+y^4c_4+y^3c_3+y^2c_2+yc_1 \end{pmatrix}$	8

Table 3.17: The Gröbner cells for $\mathrm{Hilb}^9(\mathbf{k}[\![x, y]\!])$,part 2

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
2	[1]	$\begin{pmatrix} y + c_1 \\ -x + c_2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$

Table 3.18: $n = 1$, $a = 6$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
2	[1]	$\begin{pmatrix} y + c_1 \\ -x + c_2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Table 3.19: $n = 1$, $a = 1$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
2	[1]	$\begin{pmatrix} y + c_1 \\ -x + c_2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Table 3.20: $n = 1$, $a = 1$, $b = 7$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
3	[1, 1]	$\begin{pmatrix} y + c_1 & 0 \\ -x + c_2 & 1 \\ c_3 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$
4	[2]	$\begin{pmatrix} y^2 + yc_2 + c_1 \\ -x + yc_4 + c_3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Table 3.21: $n = 2$, $a = 2$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
3	[1, 1]	$\begin{pmatrix} y + c_1 & 0 \\ -x + c_2 & 1 \\ c_3 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$
4	[2]	$\begin{pmatrix} y^2 + yc_2 + c_1 \\ -x + yc_4 + c_3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Table 3.22: $n = 2$, $a = 1$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
3	[2]	$\begin{pmatrix} y^2 + yc_2 + c_1 \\ -x + c_3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
4	[1, 1]	$\begin{pmatrix} y + c_1 & c_2 \\ -x + c_3 & 1 \\ c_4 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$

Table 3.23: $n = 2$, $a = 1$, $b = 3$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
4	$[1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & 1 \\ c_4 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -3 & -6 \\ 3 & -1 & -4 \\ 6 & 3 & -1 \\ 9 & 6 & 3 \end{pmatrix}$
5	$[1,2]$	$\begin{pmatrix} y+c_1 & 0 \\ -x+c_2 & y+c_3 \\ c_4 & -x+c_5 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 3 & 0 \\ 5 & 3 \end{pmatrix}$
6	$[3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 \\ -x+y^2c_6+yc_5+c_4 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Table 3.24: $n = 3$, $a = 3$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
4	$[1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & 1 \\ c_4 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 \\ 1 & -1 & -2 \\ 3 & 1 & -1 \\ 4 & 3 & 1 \end{pmatrix}$
5	$[3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 \\ -x+yc_5+c_4 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
6	$[1,2]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y+c_4 \\ c_5 & -x+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$

Table 3.25: $n = 3$, $a = 3$, $b = 2$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
4	$[1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & 1 \\ c_4 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 \\ 1 & -1 & -2 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$
5	$[3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 \\ -x+yc_5+c_4 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
6	$[1,2]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y+c_4 \\ c_5 & -x+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

Table 3.26: $n = 3$, $a = 1$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
4	$[3]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 \\ -x+c_4 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
5	$[1,2]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y+c_4 \\ 0 & -x+c_5 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$
6	$[1,1,1]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & 1 & 0 \\ c_5 & -x & 1 \\ c_6 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

Table 3.27: $n = 3$, $a = 1$, $b = 4$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
5	$[1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & 1 \\ c_5 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -4 & -8 & -12 \\ 4 & -1 & -5 & -9 \\ 8 & 4 & -1 & -5 \\ 12 & 8 & 4 & -1 \\ 16 & 12 & 8 & 4 \end{pmatrix}$
6	$[1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & y+c_4 \\ c_5 & 0 & -x+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & -4 & -7 \\ 4 & -1 & -4 \\ 8 & 4 & 0 \\ 11 & 7 & 4 \end{pmatrix}$
6	$[2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & 0 \\ -x+yc_4+c_3 & 1 \\ yc_6+c_5 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & -3 \\ 4 & -1 \\ 8 & 4 \end{pmatrix}$
7	$[1,3]$	$\begin{pmatrix} y+c_1 & 0 \\ -x+c_2 & y^2+yc_4+c_3 \\ c_5 & -x+yc_7+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 4 & 1 \\ 6 & 4 \end{pmatrix}$
8	$[4]$	$\begin{pmatrix} y^4+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+y^3c_8+y^2c_7+yc_6+c_5 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Table 3.28: $n = 4$, $a = 4$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
5	$[1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & 1 \\ c_5 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -4 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & -1 & -2 \\ 4 & 3 & 1 & -1 \\ 6 & 4 & 3 & 1 \end{pmatrix}$
6	$[1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & y+c_4 \\ c_5 & 0 & -x+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 \\ 1 & -1 & -1 \\ 3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$
6	$[4]$	$\begin{pmatrix} y^4+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_6+c_5 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
7	$[2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & c_3 \\ -x+yc_5+c_4 & 1 \\ yc_7+c_6 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 3 & 1 \end{pmatrix}$
8	$[1,3]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^2+yc_5+c_4 \\ c_6 & -x+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

Table 3.29: $n = 4$, $a = 3$, $b = 2$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
5	$[1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & 1 \\ c_5 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & -1 & -2 & -3 \\ 2 & 1 & -1 & -2 \\ 3 & 2 & 1 & -1 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
6	$[1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & y+c_4 \\ c_5 & 0 & -x+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$
6	$[4]$	$\begin{pmatrix} y^4+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_6+c_5 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
7	$[2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & c_3 \\ -x+yc_5+c_4 & 1 \\ yc_7+c_6 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$
8	$[1,3]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^2+yc_5+c_4 \\ c_6 & -x+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

Table 3.30: $n = 4$, $a = 1$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
5	$[4]$	$\begin{pmatrix} y^4+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+c_5 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$
6	$[1,1,1,1]$	$\begin{pmatrix} y+c_1 & c_2 & 0 & 0 \\ -x+c_3 & 1 & 0 & 0 \\ c_4 & -x & 1 & 0 \\ c_5 & 0 & -x & 1 \\ c_6 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & -1 & -1 & -2 \\ 1 & 0 & -1 & -1 \\ 2 & 1 & 0 & -1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$
6	$[1,3]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^2+yc_5+c_4 \\ 0 & -x+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$
7	$[2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+c_5 & 1 \\ yc_7+c_6 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$
8	$[1,1,2]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & 1 & 0 \\ c_5 & -x & y+c_6 \\ c_7 & 0 & -x+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Table 3.31: $n = 4$, $a = 2$, $b = 3$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
5	$[4]$	$\begin{pmatrix} y^4+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+c_5 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$
6	$[1,3]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^2+yc_5+c_4 \\ 0 & -x+c_6 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ -2 & 0 \end{pmatrix}$
6	$[2,2]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+c_5 & 1 \\ c_6 & -x \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$
7	$[1,1,2]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & 1 & 0 \\ c_5 & -x & y+c_6 \\ 0 & 0 & -x+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}$
8	$[1,1,1,1]$	$\begin{pmatrix} y+c_1 & c_2 & c_3 & c_4 \\ -x+c_5 & 1 & 0 & 0 \\ c_6 & -x & 1 & 0 \\ c_7 & 0 & -x & 1 \\ c_8 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Table 3.32: $n = 4$, $a = 1$, $b = 5$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
6	$[1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & 1 \\ c_6 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -5 & -10 & -15 & -20 \\ 5 & -1 & -6 & -11 & -16 \\ 10 & 5 & -1 & -6 & -11 \\ 15 & 10 & 5 & -1 & -6 \\ 20 & 15 & 10 & 5 & -1 \\ 25 & 20 & 15 & 10 & 5 \end{pmatrix}$
7	$[1,1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y+c_5 \\ c_6 & 0 & 0 & -x+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & -5 & -10 & -14 \\ 5 & -1 & -6 & -10 \\ 10 & 5 & -1 & -5 \\ 15 & 10 & 5 & 0 \\ 19 & 14 & 9 & 5 \end{pmatrix}$
7	$[1,2,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & y+c_3 & 0 \\ c_4 & -x+c_5 & 1 \\ c_6 & c_7 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -4 & -9 \\ 5 & 0 & -5 \\ 9 & 5 & -1 \\ 14 & 10 & 5 \end{pmatrix}$
8	$[1,1,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & y^2+yc_5+c_4 \\ c_6 & 0 & -x+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & -5 & -8 \\ 5 & -1 & -4 \\ 10 & 5 & 1 \\ 13 & 8 & 5 \end{pmatrix}$
8	$[2,3]$	$\begin{pmatrix} y^2+yc_2+c_1 & 0 \\ -x+yc_4+c_3 & y+c_5 \\ yc_7+c_6 & -x+c_8 \end{pmatrix}$	$\begin{pmatrix} 1 & -3 \\ 5 & 0 \\ 9 & 5 \end{pmatrix}$
9	$[1,4]$	$\begin{pmatrix} y+c_1 & 0 \\ -x+c_2 & y^3+y^2c_5+yc_4+c_3 \\ c_6 & -x+y^2c_9+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 5 & 2 \\ 7 & 5 \end{pmatrix}$
10	$[5]$	$\begin{pmatrix} y^5+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+y^4c_{10}+y^3c_9+y^2c_8+yc_7+c_6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Table 3.33: $n = 5$, $a = 5$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
6	$[1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & 1 \\ c_6 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -4 & -6 \\ 1 & -1 & -2 & -4 & -5 \\ 3 & 1 & -1 & -2 & -4 \\ 4 & 3 & 1 & -1 & -2 \\ 6 & 4 & 3 & 1 & -1 \\ 7 & 6 & 4 & 3 & 1 \end{pmatrix}$
7	$[1,1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y+c_5 \\ c_6 & 0 & 0 & -x+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -3 \\ 1 & -1 & -2 & -3 \\ 3 & 1 & -1 & -1 \\ 4 & 3 & 1 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}$
7	$[5]$	$\begin{pmatrix} y^5+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_7+c_6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$
8	$[1,1,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & y^2+yc_5+c_4 \\ c_6 & 0 & -x+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$
8	$[1,2,2]$	$\begin{pmatrix} y+c_1 & c_2 & 0 \\ -x+c_3 & y+c_4 & 0 \\ c_5 & -x+c_6 & 1 \\ c_7 & c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \\ 3 & 3 & 1 \end{pmatrix}$
9	$[1,4]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^3+y^2c_6+yc_5+c_4 \\ c_7 & -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$
10	$[2,3]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+yc_6+c_5 & y+c_7 \\ yc_9+c_8 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$

Table 3.34: $n = 5$, $a = 3$, $b = 2$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
6	$[1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & 1 \\ c_6 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 & -4 \\ 1 & -1 & -2 & -3 & -4 \\ 2 & 1 & -1 & -2 & -3 \\ 3 & 2 & 1 & -1 & -2 \\ 4 & 3 & 2 & 1 & -1 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$
7	$[1,1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y+c_5 \\ c_6 & 0 & 0 & -x+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -2 \\ 1 & -1 & -2 & -2 \\ 2 & 1 & -1 & -1 \\ 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix}$
7	$[5]$	$\begin{pmatrix} y^5+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_7+c_6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$
8	$[1,2,2]$	$\begin{pmatrix} y+c_1 & c_2 & 0 \\ -x+c_3 & y+c_4 & 0 \\ c_5 & -x+c_6 & 1 \\ c_7 & c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix}$
8	$[1,4]$	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^3+y^2c_6+yc_5+c_4 \\ 0 & -x+yc_8+c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}$
9	$[1,1,3]$	$\begin{pmatrix} y+c_1 & 0 & c_2 \\ -x+c_3 & 1 & 0 \\ c_4 & -x & y^2+yc_6+c_5 \\ c_7 & 0 & -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
10	$[2,3]$	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+yc_6+c_5 & y+c_7 \\ yc_9+c_8 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

Table 3.35: $n = 5$, $a = 1$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
6	$[5]$	$\begin{pmatrix} y^5 + y^4c_5 + y^3c_4 + y^2c_3 + yc_2 + c_1 \\ -x + c_6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
7	$[1,1,1,1,1]$	$\begin{pmatrix} y + c_1 & c_2 & 0 & 0 & 0 \\ -x + c_3 & 1 & 0 & 0 & 0 \\ c_4 & -x & 1 & 0 & 0 \\ c_5 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & -x & 1 \\ c_7 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & -2 & -3 \\ 1 & 0 & -1 & -1 & -2 \\ 2 & 1 & 0 & -1 & -1 \\ 2 & 2 & 1 & 0 & -1 \\ 3 & 2 & 2 & 1 & 0 \end{pmatrix}$
7	$[1,4]$	$\begin{pmatrix} y + c_1 & c_2 \\ -x + c_3 & y^3 + y^2c_6 + yc_5 + c_4 \\ 0 & -x + c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 0 & 2 \\ -2 & 0 \end{pmatrix}$
8	$[1,1,1,2]$	$\begin{pmatrix} y + c_1 & c_2 & 0 & 0 \\ -x + c_3 & 1 & 0 & 0 \\ c_4 & -x & 1 & 0 \\ c_5 & 0 & -x & y + c_6 \\ c_7 & 0 & 0 & -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$
8	$[2,3]$	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 \\ -x + c_5 & y + c_6 \\ c_7 & -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
9	$[1,1,3]$	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & 1 & 0 \\ c_5 & -x & y^2 + yc_7 + c_6 \\ c_8 & 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
10	$[1,2,2]$	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & y + c_5 & c_6 \\ c_7 & -x + c_8 & 1 \\ c_9 & c_{10} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$

Table 3.36: $n = 5$, $a = 2$, $b = 3$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
6	$[5]$	$\begin{pmatrix} y^5 + y^4c_5 + y^3c_4 + y^2c_3 + yc_2 + c_1 \\ -x + c_6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
7	$[1,4]$	$\begin{pmatrix} y + c_1 & c_2 \\ -x + c_3 & y^3 + y^2c_6 + yc_5 + c_4 \\ 0 & -x + c_7 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 0 & 2 \\ -3 & 0 \end{pmatrix}$
7	$[2,3]$	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 \\ -x + c_5 & y + c_6 \\ 0 & -x + c_7 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$
8	$[1,1,3]$	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & 1 & 0 \\ c_5 & -x & y^2 + yc_7 + c_6 \\ 0 & 0 & -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ -2 & -2 & 0 \end{pmatrix}$
8	$[1,2,2]$	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & y + c_5 & c_6 \\ 0 & -x + c_7 & 1 \\ 0 & c_8 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$
9	$[1,1,1,2]$	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 \\ -x + c_5 & 1 & 0 & 0 \\ c_6 & -x & 1 & 0 \\ c_7 & 0 & -x & y + c_8 \\ 0 & 0 & 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix}$
10	$[1,1,1,1,1]$	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 & c_5 \\ -x + c_6 & 1 & 0 & 0 & 0 \\ c_7 & -x & 1 & 0 & 0 \\ c_8 & 0 & -x & 1 & 0 \\ c_9 & 0 & 0 & -x & 1 \\ c_{10} & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Table 3.37: $n = 5$, $a = 1$, $b = 6$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
8	$[1,1,1,1,1,1,1]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 & 0 \\ c_6 & 0 & 0 & 0 & -x & 1 & 0 \\ c_7 & 0 & 0 & 0 & 0 & -x & 1 \\ c_8 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -7 & -14 & -21 & -28 & -35 & -42 \\ 7 & -1 & -8 & -15 & -22 & -29 & -36 \\ 14 & 7 & -1 & -8 & -15 & -22 & -29 \\ 21 & 14 & 7 & -1 & -8 & -15 & -22 \\ 28 & 21 & 14 & 7 & -1 & -8 & -15 \\ 35 & 28 & 21 & 14 & 7 & -1 & -8 \\ 42 & 35 & 28 & 21 & 14 & 7 & -1 \\ 49 & 42 & 35 & 28 & 21 & 14 & 7 \end{pmatrix}$
9	$[1,1,1,1,1,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & 0 & -x & y+c_7 \\ c_8 & 0 & 0 & 0 & 0 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -7 & -14 & -21 & -28 & -34 \\ 7 & -1 & -8 & -15 & -22 & -28 \\ 14 & 7 & -1 & -8 & -15 & -21 \\ 21 & 14 & 7 & -1 & -8 & -14 \\ 28 & 21 & 14 & 7 & -1 & -7 \\ 35 & 28 & 21 & 14 & 7 & 0 \\ 41 & 34 & 27 & 20 & 13 & 7 \end{pmatrix}$
9	$[1,1,1,2,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & y+c_5 & 0 \\ c_6 & 0 & 0 & -x+c_7 & 1 \\ c_8 & 0 & 0 & c_9 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -7 & -14 & -20 & -27 \\ 7 & -1 & -8 & -14 & -21 \\ 14 & 7 & -1 & -7 & -14 \\ 21 & 14 & 7 & 0 & -7 \\ 27 & 20 & 13 & 7 & -1 \\ 34 & 27 & 20 & 14 & 7 \end{pmatrix}$
9	$[1,2,2,2]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & y+c_3 & 0 & 0 \\ c_4 & -x+c_5 & 1 & 0 \\ c_6 & c_7 & -x & 1 \\ c_8 & c_9 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -13 & -20 \\ 7 & 0 & -7 & -14 \\ 13 & 7 & -1 & -8 \\ 20 & 14 & 7 & -1 \\ 27 & 21 & 14 & 7 \end{pmatrix}$
10	$[1,1,1,1,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & y^2+yc_7+c_6 \\ c_8 & 0 & 0 & 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -7 & -14 & -21 & -26 \\ 7 & -1 & -8 & -15 & -20 \\ 14 & 7 & -1 & -8 & -13 \\ 21 & 14 & 7 & -1 & -6 \\ 28 & 21 & 14 & 7 & 1 \\ 33 & 26 & 19 & 12 & 7 \end{pmatrix}$
10	$[1,1,2,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & y+c_4 & 0 \\ c_5 & 0 & -x+c_6 & y+c_7 \\ c_8 & 0 & c_9 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & -7 & -13 & -19 \\ 7 & -1 & -7 & -13 \\ 14 & 7 & 0 & -6 \\ 20 & 13 & 7 & 0 \\ 26 & 19 & 13 & 7 \end{pmatrix}$
10	$[1,3,3]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & y^2+yc_4+c_3 & 0 \\ c_5 & -x+yc_7+c_6 & 1 \\ c_8 & yc_{10}+c_9 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -5 & -12 \\ 7 & 1 & -6 \\ 12 & 7 & -1 \\ 19 & 14 & 7 \end{pmatrix}$
10	$[2,2,3]$	$\begin{pmatrix} y^2+yc_2+c_1 & 0 & 0 \\ -x+yc_4+c_3 & 1 & 0 \\ yc_6+c_5 & -x & y+c_7 \\ yc_9+c_8 & 0 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 1 & -6 & -12 \\ 7 & -1 & -7 \\ 14 & 7 & 0 \\ 20 & 13 & 7 \end{pmatrix}$
11	$[1,1,1,4]$	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y^3+y^2c_7+yc_6+c_5 \\ c_8 & 0 & 0 & -x+y^2c_{11}+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -7 & -14 & -18 \\ 7 & -1 & -8 & -12 \\ 14 & 7 & -1 & -5 \\ 21 & 14 & 7 & 2 \\ 25 & 18 & 11 & 7 \end{pmatrix}$
11	$[1,2,4]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & y+c_3 & 0 \\ c_4 & -x+c_5 & y^2+yc_7+c_6 \\ c_8 & c_9 & -x+yc_{11}+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -11 \\ 7 & 0 & -5 \\ 13 & 7 & 1 \\ 18 & 12 & 7 \end{pmatrix}$
11	$[3,4]$	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 & 0 \\ -x+y^2c_6+yc_5+c_4 & y+c_7 \\ y^2c_{10}+yc_9+c_8 & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 2 & -4 \\ 7 & 0 \\ 13 & 7 \end{pmatrix}$
12	$[1,1,5]$	$\begin{pmatrix} y+c_1 & 0 & 0 \\ -x+c_2 & 1 & 0 \\ c_3 & -x & y^4+y^3c_7+y^2c_6+yc_5+c_4 \\ c_8 & 0 & -x+y^3c_{12}+y^2c_{11}+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -7 & -10 \\ 7 & -1 & -4 \\ 14 & 7 & 3 \\ 17 & 10 & 7 \end{pmatrix}$
12	$[2,5]$	$\begin{pmatrix} y^2+yc_2+c_1 & 0 \\ -x+yc_4+c_3 & y^3+y^2c_7+yc_6+c_5 \\ yc_9+c_8 & -x+y^2c_{12}+yc_{11}+c_{10} \end{pmatrix}$	$\begin{pmatrix} 1 & -3 \\ 7 & 2 \\ 11 & 7 \end{pmatrix}$
13	$[1,6]$	$\begin{pmatrix} y+c_1 & 0 \\ -x+c_2 & y^5+y^4c_7+y^3c_6+y^2c_5+yc_4+c_3 \\ c_8 & -x+y^4c_{13}+y^3c_{12}+y^2c_{11}+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 7 & 4 \\ 9 & 7 \end{pmatrix}$
14	$[7]$	$\begin{pmatrix} y^7+y^6c_7+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+y^6c_{14}+y^5c_{13}+y^4c_{12}+y^3c_{11}+y^2c_{10}+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 7 \end{pmatrix}$

Table 3.38: $n = 7$, $a = 7$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
8	[1, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 & 0 \\ c_6 & 0 & 0 & 0 & -x & 1 & 0 \\ c_7 & 0 & 0 & 0 & 0 & -x & 1 \\ c_8 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -4 & -6 & -7 & -9 \\ 1 & -1 & -2 & -4 & -5 & -7 & -8 \\ 3 & 1 & -1 & -2 & -4 & -5 & -7 \\ 4 & 3 & 1 & -1 & -2 & -4 & -5 \\ 6 & 4 & 3 & 1 & -1 & -2 & -4 \\ 7 & 6 & 4 & 3 & 1 & -1 & -2 \\ 9 & 7 & 6 & 4 & 3 & 1 & -1 \\ 10 & 9 & 7 & 6 & 4 & 3 & 1 \end{pmatrix}$
9	[1, 1, 1, 1, 1, 2]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & 0 & -x & y+c_7 \\ c_8 & 0 & 0 & 0 & 0 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -4 & -6 & -6 \\ 1 & -1 & -2 & -4 & -5 & -6 \\ 3 & 1 & -1 & -2 & -4 & -4 \\ 4 & 3 & 1 & -1 & -2 & -3 \\ 6 & 4 & 3 & 1 & -1 & -1 \\ 7 & 6 & 4 & 3 & 1 & 0 \\ 8 & 6 & 5 & 3 & 2 & 1 \end{pmatrix}$
9	[1, 1, 1, 2, 2]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & y+c_5 & 0 \\ c_6 & 0 & 0 & -x+c_7 & 1 \\ c_8 & 0 & 0 & c_9 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -3 & -5 \\ 1 & -1 & -2 & -3 & -4 \\ 3 & 1 & -1 & -1 & -3 \\ 4 & 3 & 1 & 0 & -1 \\ 5 & 3 & 2 & 1 & -1 \\ 6 & 5 & 3 & 3 & 1 \end{pmatrix}$
9	[7]	$\begin{pmatrix} y^7+y^6c_7+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 1 \end{pmatrix}$
10	[1, 1, 1, 1, 3]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & y^2+yc_7+c_6 \\ c_8 & 0 & 0 & 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -4 & -4 \\ 1 & -1 & -2 & -4 & -3 \\ 3 & 1 & -1 & -2 & -2 \\ 4 & 3 & 1 & -1 & 0 \\ 6 & 4 & 3 & 1 & 1 \\ 5 & 4 & 2 & 1 & 1 \end{pmatrix}$
10	[1, 1, 1, 4]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & 1 & 0 \\ c_4 & 0 & -x & y^3+y^2c_7+yc_6+c_5 \\ c_8 & 0 & 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -3 & -1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & -1 & 1 \\ 4 & 3 & 1 & 2 \\ 3 & 1 & 0 & 1 \end{pmatrix}$
10	[1, 2, 2, 2]	$\begin{pmatrix} y+c_1 & c_2 & 0 & 0 \\ -x+c_3 & y+c_4 & 0 & 0 \\ c_5 & -x+c_6 & 1 & 0 \\ c_7 & c_8 & -x & 1 \\ c_9 & c_{10} & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -2 & -3 \\ 1 & 0 & -1 & -3 \\ 2 & 1 & -1 & -2 \\ 3 & 3 & 1 & -1 \\ 5 & 4 & 3 & 1 \end{pmatrix}$
10	[1, 6]	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^5+y^4c_8+y^3c_7+y^2c_6+yc_5+c_4 \\ 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 4 \\ -2 & 1 \end{pmatrix}$
11	[1, 1, 2, 3]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & y+c_4 & c_5 \\ c_6 & 0 & -x+c_7 & y+c_8 \\ c_9 & 0 & c_{10} & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -2 \\ 1 & -1 & -1 & -2 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 2 & 2 & 1 \end{pmatrix}$
11	[1, 1, 5]	$\begin{pmatrix} y+c_1 & 0 & c_2 \\ -x+c_3 & 1 & 0 \\ c_4 & -x & y^4+y^3c_8+y^2c_7+yc_6+c_5 \\ c_9 & 0 & -x+yc_{11}+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 3 \\ 0 & -1 & 1 \end{pmatrix}$
11	[2, 2, 3]	$\begin{pmatrix} y^2+yc_2+c_1 & c_3 & 0 \\ -x+yc_5+c_4 & 1 & 0 \\ yc_7+c_6 & -x & y+c_8 \\ yc_{10}+c_9 & 0 & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & -1 \\ 3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$
12	[1, 3, 3]	$\begin{pmatrix} y+c_1 & c_2 & 0 \\ -x+c_3 & y^2+yc_5+c_4 & c_6 \\ c_7 & -x+yc_9+c_8 & 1 \\ c_{10} & yc_{12}+c_{11} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix}$
12	[2, 5]	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+yc_6+c_5 & y^3+y^2c_9+yc_8+c_7 \\ c_{10} & -x+yc_{12}+c_{11} \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$
13	[3, 4]	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 & y^2c_6+yc_5+c_4 \\ -x+yc_8+c_7 & y+c_9 \\ y^2c_{12}+yc_{11}+c_{10} & -x+c_{13} \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$
14	[1, 2, 4]	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & y+c_5 & c_6 \\ c_7 & -x+c_8 & y^2+yc_{10}+c_9 \\ c_{11} & c_{12} & -x+yc_{14}+c_{13} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Table 3.39: $n = 7$, $a = 3$, $b = 2$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
8	[1, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 & 0 \\ c_6 & 0 & 0 & 0 & -x & 1 & 0 \\ c_7 & 0 & 0 & 0 & 0 & -x & 1 \\ c_8 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 \\ 1 & -1 & -2 & -3 & -4 & -5 & -6 \\ 2 & 1 & -1 & -2 & -3 & -4 & -5 \\ 3 & 2 & 1 & -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 & -1 & -2 & -3 \\ 5 & 4 & 3 & 2 & 1 & -1 & -2 \\ 6 & 5 & 4 & 3 & 2 & 1 & -1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$
9	[1, 1, 1, 1, 1, 2]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 & 0 \\ c_5 & 0 & 0 & -x & 1 & 0 \\ c_6 & 0 & 0 & 0 & -x & y+c_7 \\ c_8 & 0 & 0 & 0 & 0 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -4 \\ 1 & -1 & -2 & -3 & -4 & -4 \\ 2 & 1 & -1 & -2 & -3 & -3 \\ 3 & 2 & 1 & -1 & -2 & -2 \\ 4 & 3 & 2 & 1 & -1 & -1 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 & 1 \end{pmatrix}$
9	[1, 1, 1, 2, 2]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & y+c_5 & 0 \\ c_6 & 0 & 0 & -x+c_7 & 1 \\ c_8 & 0 & 0 & c_9 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -2 & -3 \\ 1 & -1 & -2 & -2 & -3 \\ 2 & 1 & -1 & -1 & -2 \\ 3 & 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 1 & -1 \\ 4 & 3 & 2 & 2 & 1 \end{pmatrix}$
9	[7]	$\begin{pmatrix} y^7+y^6c_7+y^5c_6+y^4c_5+y^3c_4+y^2c_3+yc_2+c_1 \\ -x+yc_9+c_8 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 1 \end{pmatrix}$
10	[1, 1, 1, 1, 3]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 & 0 \\ c_3 & -x & 1 & 0 & 0 \\ c_4 & 0 & -x & 1 & 0 \\ c_5 & 0 & 0 & -x & y^2+yc_7+c_6 \\ c_8 & 0 & 0 & 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & -3 & -2 \\ 1 & -1 & -2 & -3 & -2 \\ 2 & 1 & -1 & -2 & -1 \\ 3 & 2 & 1 & -1 & 0 \\ 4 & 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & 0 & 1 \end{pmatrix}$
10	[1, 1, 5]	$\begin{pmatrix} y+c_1 & 0 & c_2 \\ -x+c_3 & 1 & 0 \\ c_4 & -x & y^4+y^3c_8+y^2c_7+yc_6+c_5 \\ 0 & 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \\ -1 & -2 & 1 \end{pmatrix}$
10	[1, 2, 2, 2]	$\begin{pmatrix} y+c_1 & c_2 & 0 & 0 \\ -x+c_3 & y+c_4 & 0 & 0 \\ c_5 & -x+c_6 & 1 & 0 \\ c_7 & c_8 & -x & 1 \\ c_9 & c_{10} & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 \\ 1 & 0 & -1 & -2 \\ 1 & 1 & -1 & -2 \\ 2 & 2 & 1 & -1 \\ 3 & 3 & 2 & 1 \end{pmatrix}$
10	[1, 6]	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^5+y^4c_8+y^3c_7+y^2c_6+yc_5+c_4 \\ 0 & -x+yc_{10}+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 4 \\ -3 & 1 \end{pmatrix}$
11	[1, 1, 1, 4]	$\begin{pmatrix} y+c_1 & 0 & 0 & c_2 \\ -x+c_3 & 1 & 0 & 0 \\ c_4 & -x & 1 & 0 \\ c_5 & 0 & -x & y^3+y^2c_8+yc_7+c_6 \\ c_9 & 0 & 0 & -x+yc_{11}+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -2 & 0 \\ 1 & -1 & -2 & 0 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 0 & -1 & 1 \end{pmatrix}$
11	[1, 1, 2, 3]	$\begin{pmatrix} y+c_1 & 0 & 0 & 0 \\ -x+c_2 & 1 & 0 & 0 \\ c_3 & -x & y+c_4 & c_5 \\ c_6 & 0 & -x+c_7 & y+c_8 \\ c_9 & 0 & c_{10} & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix}$
11	[2, 5]	$\begin{pmatrix} y^2+yc_2+c_1 & yc_4+c_3 \\ -x+yc_6+c_5 & y^3+y^2c_9+yc_8+c_7 \\ 0 & -x+yc_{11}+c_{10} \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}$
12	[2, 2, 3]	$\begin{pmatrix} y^2+yc_2+c_1 & c_3 & c_4 \\ -x+yc_6+c_5 & 1 & 0 \\ yc_8+c_7 & -x & y+c_9 \\ yc_{11}+c_{10} & 0 & -x+c_{12} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$
12	[3, 4]	$\begin{pmatrix} y^3+y^2c_3+yc_2+c_1 & y^2c_6+yc_5+c_4 \\ -x+yc_8+c_7 & y+c_9 \\ yc_{11}+c_{10} & -x+c_{12} \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$
13	[1, 3, 3]	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & y^2+yc_6+c_5 & c_7 \\ c_8 & -x+yc_{10}+c_9 & 1 \\ c_{11} & yc_{13}+c_{12} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$
14	[1, 2, 4]	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & y+c_5 & c_6 \\ c_7 & -x+c_8 & y^2+yc_{10}+c_9 \\ c_{11} & c_{12} & -x+yc_{14}+c_{13} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Table 3.40: $n = 7$, $a = 1$, $b = 1$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
8	[7]	$\begin{pmatrix} y^7 + y^6c_7 + y^5c_6 + y^4c_5 + y^3c_4 + y^2c_3 + yc_2 + c_1 \\ -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 0 \end{pmatrix}$
9	[1, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y+c_1 & c_2 & 0 & 0 & 0 & 0 & 0 \\ -x+c_3 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_4 & -x & 1 & 0 & 0 & 0 & 0 \\ c_5 & 0 & -x & 1 & 0 & 0 & 0 \\ c_6 & 0 & 0 & -x & 1 & 0 & 0 \\ c_7 & 0 & 0 & 0 & -x & 1 & 0 \\ c_8 & 0 & 0 & 0 & 0 & -x & 1 \\ c_9 & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 & -2 & -3 & -4 \\ 0 & -1 & -1 & -2 & -3 & -3 & -4 \\ 1 & 0 & -1 & -1 & -2 & -3 & -3 \\ 2 & 1 & 0 & -1 & -1 & -2 & -3 \\ 2 & 2 & 1 & 0 & -1 & -1 & -2 \\ 3 & 2 & 2 & 1 & 0 & -1 & -1 \\ 4 & 3 & 2 & 2 & 1 & 0 & -1 \\ 4 & 4 & 3 & 2 & 2 & 1 & 0 \end{pmatrix}$
9	[1, 6]	$\begin{pmatrix} y+c_1 & c_2 \\ -x+c_3 & y^5 + y^4c_8 + y^3c_7 + y^2c_6 + yc_5 + c_4 \\ 0 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ 0 & 4 \\ -4 & 0 \end{pmatrix}$
9	[2, 5]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 \\ -x+c_5 & y^3 + y^2c_8 + yc_7 + c_6 \\ 0 & -x+c_9 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 2 \\ -2 & 0 \end{pmatrix}$
10	[1, 1, 1, 1, 1, 2]	$\begin{pmatrix} y+c_1 & c_2 & 0 & 0 & 0 & 0 \\ -x+c_3 & 1 & 0 & 0 & 0 & 0 \\ c_4 & -x & 1 & 0 & 0 & 0 \\ c_5 & 0 & -x & 1 & 0 & 0 \\ c_6 & 0 & 0 & -x & 1 & 0 \\ c_7 & 0 & 0 & 0 & -x & y+c_8 \\ c_9 & 0 & 0 & 0 & 0 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 & -2 & -2 \\ 0 & -1 & -1 & -2 & -3 & -2 \\ 1 & 0 & -1 & -1 & -2 & -2 \\ 2 & 1 & 0 & -1 & -1 & -1 \\ 2 & 2 & 1 & 0 & -1 & 0 \\ 3 & 2 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{pmatrix}$
10	[1, 1, 1, 4]	$\begin{pmatrix} y+c_1 & c_2 & 0 & c_3 \\ -x+c_4 & 1 & 0 & 0 \\ c_5 & -x & 1 & 0 \\ c_6 & 0 & -x & y^3 + y^2c_9 + yc_8 + c_7 \\ 0 & 0 & 0 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & 2 \\ -1 & -1 & -2 & 0 \end{pmatrix}$
10	[1, 1, 5]	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & 1 & 0 \\ c_5 & -x & y^4 + y^3c_9 + y^2c_8 + yc_7 + c_6 \\ 0 & 0 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$
10	[3, 4]	$\begin{pmatrix} y^3 + y^2c_3 + yc_2 + c_1 & y^2c_6 + yc_5 + c_4 \\ -x+c_7 & y+c_8 \\ c_9 & -x+c_{10} \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
11	[1, 1, 1, 1, 3]	$\begin{pmatrix} y+c_1 & c_2 & 0 & 0 & c_3 \\ -x+c_4 & 1 & 0 & 0 & 0 \\ c_5 & -x & 1 & 0 & 0 \\ c_6 & 0 & -x & 1 & 0 \\ c_7 & 0 & 0 & -x & y^2 + yc_9 + c_8 \\ c_{10} & 0 & 0 & 0 & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -2 & 0 \\ 0 & -1 & -1 & -2 & -1 \\ 1 & 0 & -1 & -1 & 0 \\ 2 & 1 & 0 & -1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}$
11	[1, 1, 1, 2, 2]	$\begin{pmatrix} y+c_1 & c_2 & 0 & 0 & 0 \\ -x+c_3 & 1 & 0 & 0 & 0 \\ c_4 & -x & 1 & 0 & 0 \\ c_5 & 0 & -x & y+c_6 & c_7 \\ c_8 & 0 & 0 & -x+c_9 & 1 \\ c_{10} & 0 & 0 & c_{11} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -2 \\ 1 & 0 & -1 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & 1 & 0 \end{pmatrix}$
11	[1, 2, 4]	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & y+c_5 & c_6 \\ c_7 & -x+c_8 & y^2 + yc_{10} + c_9 \\ 0 & 0 & -x+c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$
12	[1, 2, 2, 2]	$\begin{pmatrix} y+c_1 & c_2 & c_3 & 0 \\ -x+c_4 & y+c_5 & c_6 & 0 \\ c_7 & -x+c_8 & 1 & 0 \\ c_9 & c_{10} & -x & 1 \\ c_{11} & c_{12} & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$
12	[1, 3, 3]	$\begin{pmatrix} y+c_1 & c_2 & c_3 \\ -x+c_4 & y^2 + yc_6 + c_5 & yc_8 + c_7 \\ 0 & -x+c_9 & 1 \\ c_{10} & yc_{12} + c_{11} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
13	[2, 2, 3]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 & yc_6 + c_5 \\ -x+c_7 & 1 & 0 \\ yc_9 + c_8 & -x & y+c_{10} \\ yc_{12} + c_{11} & 0 & -x+c_{13} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
14	[1, 1, 2, 3]	$\begin{pmatrix} y+c_1 & c_2 & c_3 & c_4 \\ -x+c_5 & 1 & 0 & 0 \\ c_6 & -x & y+c_7 & c_8 \\ c_9 & 0 & -x+c_{10} & y+c_{11} \\ c_{12} & 0 & c_{13} & -x+c_{14} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Table 3.41: $n = 7$, $a = 2$, $b = 3$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
8	[7]	$\begin{pmatrix} y^7 + y^6c_7 + y^5c_6 + y^4c_5 + y^3c_4 + y^2c_3 + yc_2 + c_1 \\ -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 0 \end{pmatrix}$
9	[1, 6]	$\begin{pmatrix} y + c_1 & c_2 \\ -x + c_3 & y^5 + y^4c_8 + y^3c_7 + y^2c_6 + yc_5 + c_4 \\ 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ 0 & 4 \\ -5 & 0 \end{pmatrix}$
9	[2, 5]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 \\ -x + c_5 & y^3 + y^2c_8 + yc_7 + c_6 \\ 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 2 \\ -3 & 0 \end{pmatrix}$
9	[3, 4]	$\begin{pmatrix} y^3 + y^2c_3 + yc_2 + c_1 & y^2c_6 + yc_5 + c_4 \\ -x + c_7 & y + c_8 \\ 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$
10	[1, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & 0 & 0 & 0 & 0 \\ -x + c_4 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_5 & -x & 1 & 0 & 0 & 0 & 0 \\ c_6 & 0 & -x & 1 & 0 & 0 & 0 \\ c_7 & 0 & 0 & -x & 1 & 0 & 0 \\ c_8 & 0 & 0 & 0 & -x & 1 & 0 \\ c_9 & 0 & 0 & 0 & 0 & -x & 1 \\ c_{10} & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 & -1 & -1 & -2 \\ 0 & -1 & -1 & -1 & -2 & -2 & -2 \\ 0 & 0 & -1 & -1 & -1 & -2 & -2 \\ 1 & 0 & 0 & -1 & -1 & -1 & -2 \\ 1 & 1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & -1 & -1 \\ 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 2 & 2 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$
10	[1, 1, 5]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & 1 & 0 \\ c_5 & -x & y^4 + y^3c_9 + y^2c_8 + yc_7 + c_6 \\ 0 & 0 & -x + c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \\ -3 & -4 & 0 \end{pmatrix}$
10	[1, 2, 4]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & y + c_5 & c_6 \\ 0 & -x + c_7 & y^2 + yc_9 + c_8 \\ 0 & 0 & -x + c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ -1 & 0 & 1 \\ -2 & -2 & 0 \end{pmatrix}$
10	[1, 3, 3]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & y^2 + yc_6 + c_5 & yc_8 + c_7 \\ 0 & -x + c_9 & 1 \\ 0 & c_{10} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -2 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$
11	[1, 1, 1, 1, 3]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & 0 & c_4 \\ -x + c_5 & 1 & 0 & 0 & 0 \\ c_6 & -x & 1 & 0 & 0 \\ c_7 & 0 & -x & 1 & 0 \\ c_8 & 0 & 0 & -x & y^2 + yc_{10} + c_9 \\ 0 & 0 & 0 & 0 & -x + c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -2 & 0 \end{pmatrix}$
11	[1, 1, 1, 4]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 \\ -x + c_5 & 1 & 0 & 0 \\ c_6 & -x & 1 & 0 \\ c_7 & 0 & -x & y^3 + y^2c_{10} + yc_9 + c_8 \\ 0 & 0 & 0 & -x + c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 2 \\ -2 & -2 & -3 & 0 \end{pmatrix}$
11	[2, 2, 3]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 & yc_6 + c_5 \\ -x + c_7 & 1 & 0 \\ c_8 & -x & y + c_9 \\ c_{10} & 0 & -x + c_{11} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
12	[1, 1, 1, 1, 1, 2]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & 0 & 0 & c_4 \\ -x + c_5 & 1 & 0 & 0 & 0 & 0 \\ c_6 & -x & 1 & 0 & 0 & 0 \\ c_7 & 0 & -x & 1 & 0 & 0 \\ c_8 & 0 & 0 & -x & 1 & 0 \\ c_9 & 0 & 0 & 0 & -x & y + c_{10} \\ c_{11} & 0 & 0 & 0 & 0 & -x + c_{12} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & -2 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$
12	[1, 1, 2, 3]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 \\ -x + c_5 & 1 & 0 & 0 \\ c_6 & -x & y + c_7 & c_8 \\ c_9 & 0 & -x + c_{10} & y + c_{11} \\ 0 & 0 & 0 & -x + c_{12} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix}$
13	[1, 2, 2, 2]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 \\ -x + c_5 & y + c_6 & c_7 & c_8 \\ 0 & -x + c_9 & 1 & 0 \\ c_{10} & c_{11} & -x & 1 \\ c_{12} & c_{13} & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
14	[1, 1, 1, 2, 2]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 & c_5 \\ -x + c_6 & 1 & 0 & 0 & 0 \\ c_7 & -x & 1 & 0 & 0 \\ c_8 & 0 & -x & y + c_9 & c_{10} \\ c_{11} & 0 & 0 & -x + c_{12} & 1 \\ c_{13} & 0 & 0 & c_{14} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Table 3.42: $n = 7$, $a = 1$, $b = 3$

dimension	m	canonical Hilbert-Burch matrix $M = H + N$	degree bounds by W
8	[7]	$\begin{pmatrix} y^7 + y^6c_7 + y^5c_6 + y^4c_5 + y^3c_4 + y^2c_3 + yc_2 + c_1 \\ -x + c_8 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 0 \end{pmatrix}$
9	[1, 6]	$\begin{pmatrix} y + c_1 & c_2 \\ -x + c_3 & y^5 + y^4c_8 + y^3c_7 + y^2c_6 + yc_5 + c_4 \\ 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ 0 & 4 \\ -5 & 0 \end{pmatrix}$
9	[2, 5]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 \\ -x + c_5 & y^3 + y^2c_8 + yc_7 + c_6 \\ 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 2 \\ -3 & 0 \end{pmatrix}$
9	[3, 4]	$\begin{pmatrix} y^3 + y^2c_3 + yc_2 + c_1 & y^2c_6 + yc_5 + c_4 \\ -x + c_7 & y + c_8 \\ 0 & -x + c_9 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$
10	[1, 1, 5]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & 1 & 0 \\ c_5 & -x & y^4 + y^3c_9 + y^2c_8 + yc_7 + c_6 \\ 0 & 0 & -x + c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \\ -4 & -4 & 0 \end{pmatrix}$
10	[1, 2, 4]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & y + c_5 & c_6 \\ 0 & -x + c_7 & y^2 + yc_9 + c_8 \\ 0 & 0 & -x + c_{10} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ -1 & 0 & 1 \\ -3 & -2 & 0 \end{pmatrix}$
10	[1, 3, 3]	$\begin{pmatrix} y + c_1 & c_2 & c_3 \\ -x + c_4 & y^2 + yc_6 + c_5 & yc_8 + c_7 \\ 0 & -x + c_9 & 1 \\ 0 & c_{10} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -2 & 0 & -1 \\ -2 & 0 & 0 \end{pmatrix}$
10	[2, 2, 3]	$\begin{pmatrix} y^2 + yc_2 + c_1 & yc_4 + c_3 & yc_6 + c_5 \\ -x + c_7 & 1 & 0 \\ c_8 & -x & y + c_9 \\ 0 & 0 & -x + c_{10} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}$
11	[1, 1, 1, 4]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 \\ -x + c_5 & 1 & 0 & 0 \\ c_6 & -x & 1 & 0 \\ c_7 & 0 & -x & y^3 + y^2c_{10} + yc_9 + c_8 \\ 0 & 0 & 0 & -x + c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 \\ -3 & -3 & -3 & 0 \end{pmatrix}$
11	[1, 1, 2, 3]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 \\ -x + c_5 & 1 & 0 & 0 \\ c_6 & -x & y + c_7 & c_8 \\ 0 & 0 & -x + c_9 & y + c_{10} \\ 0 & 0 & 0 & -x + c_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & -1 & 0 \end{pmatrix}$
11	[1, 2, 2, 2]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 \\ -x + c_5 & y + c_6 & c_7 & c_8 \\ 0 & -x + c_9 & 1 & 0 \\ 0 & c_{10} & -x & 1 \\ 0 & c_{11} & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$
12	[1, 1, 1, 1, 3]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 & c_5 \\ -x + c_6 & 1 & 0 & 0 & 0 \\ c_7 & -x & 1 & 0 & 0 \\ c_8 & 0 & -x & 1 & 0 \\ c_9 & 0 & 0 & -x & y^2 + yc_{11} + c_{10} \\ 0 & 0 & 0 & 0 & -x + c_{12} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & -2 & -2 & 0 \end{pmatrix}$
12	[1, 1, 1, 2, 2]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 & c_5 \\ -x + c_6 & 1 & 0 & 0 & 0 \\ c_7 & -x & 1 & 0 & 0 \\ c_8 & 0 & -x & y + c_9 & c_{10} \\ 0 & 0 & 0 & -x + c_{11} & 1 \\ 0 & 0 & 0 & c_{12} & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \end{pmatrix}$
13	[1, 1, 1, 1, 1, 2]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ -x + c_7 & 1 & 0 & 0 & 0 & 0 \\ c_8 & -x & 1 & 0 & 0 & 0 \\ c_9 & 0 & -x & 1 & 0 & 0 \\ c_{10} & 0 & 0 & -x & 1 & 0 \\ c_{11} & 0 & 0 & 0 & -x & y + c_{12} \\ 0 & 0 & 0 & 0 & 0 & -x + c_{13} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$
14	[1, 1, 1, 1, 1, 1, 1]	$\begin{pmatrix} y + c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ -x + c_8 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_9 & -x & 1 & 0 & 0 & 0 & 0 \\ c_{10} & 0 & -x & 1 & 0 & 0 & 0 \\ c_{11} & 0 & 0 & -x & 1 & 0 & 0 \\ c_{12} & 0 & 0 & 0 & -x & 1 & 0 \\ c_{13} & 0 & 0 & 0 & 0 & -x & 1 \\ c_{14} & 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Table 3.43: $n = 7$, $a = 1$, $b = 8$