Symmetry Groupoids in Dynamical Systems

Spatio-temporal Patterns and a Generalized Equivariant Bifurcation Theory

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quia ex hoc pene Nihilo pene Mundum ipsum, in quo omnia, efformaui: . . . iam ter maximi Animalis, globi telluris, animam in Niuis Atomo exhibeo?

Kepler, De Niue Sexangula, 1611

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1. Introduction

Going back to Henri Poincaré, the main concern of the theory of dynamical systems for differential equations is the *qualitative* characterization of solutions. *Symmetries*, described by group transformations, help immensely in this quest — providing that they exist, which is often the case only in very special dynamical systems.

In this thesis, we significantly enlarge the class of dynamical systems which can be studied by symmetry methods, moving our focus from groups to *groupoids* as the underlying algebraic structure describing symmetry. Building on the groupoid framework, we fundamentally generalize the notion of equivariance and equivariant bifurcation theory. In summary, we present a new unified theory of symmetric spatio-temporal patterns.

There are three main conceptual novelties in this thesis.

First, we redefine *symmetry* in dynamical systems and differential equations. Roughly speaking, symmetries are described by linear isomorphisms on flow-invariant linear subspaces which map solutions of dynamical systems to solutions. In contrast to the previous literature, we do *not* require that the linear isomorphisms map *all* solutions to solutions. Instead, we only ask for this property *in certain linear subspaces*, whose crucial property is their *flow-invariance*. Building upon this new definition, we find that the underlying algebraic structure of symmetries is a *groupoid*.

The main difference between a group and a groupoid is that not all (iso-)morphisms need to be composable, making the groupoid a more general algebraic object. The critical notions of identity, invertibility and less strict composition rules are preserved in groupoids, allowing us to construct a generalized equivariant bifurcation theory. In particular, we prove *equivaroid* (i.e., groupoid-equivariant) versions of the equivariant branching lemma and the equivariant Lyapunov–Schmidt reduction, as well as an extension of the equivaroid branching lemma which allows higher dimensional kernels.

Second, we take a new perspective on spatio-temporal patterns of time-periodic solutions of differential equations. Specifically, we define *spatio-temporal patterns* as a groupoid acting directly on the space of periodic solutions, more precisely as a pair (γ, ϑ), where γ is a linear isomorphism which acts on the phase space of the dynamical system and ϑ is a *component-wise time-shift*. This construction gives us a new classification of spatio-temporal patterns beyond rotating and discrete waves. To prove the existence of such patterns, we generalize the equivariant Hopf bifurcation theorem, including a version which allows for higher dimensional kernels and thereby multi-frequency patterns.

Lastly, we ask to *construct dynamical systems with groupoid symmetries*, where we focus on the rational design of networks with prescribed symmetries. Here we find that the relevant algebraic object is in fact the *symmetry monoid* paired to a given flow-invariant subspace. The elements of the symmetry monoid are then used as *coupling matrices* to generate networks with prescribed patterns and groupoid symmetries.

This introductory chapter is organized as follows: In Section 1.1 we give a short historical overview on symmetry in general aspects. It is the aim of this section to show that the concept of *symmetry* has been in constant change over the last two millennia. With this thesis, we intend to change and broaden the meaning of symmetry once more; namely in the context of dynamical systems and equivariance, where symmetry and *group theory* are nowadays mostly used as synonyms. To give us the necessary background knowledge, Section 1.2 contains a brief summary of group symmetry, group representations and equivariance in dynamical systems. These are immensely powerful tools in the qualitative study of dynamical systems. However, we also find patterns in systems which do not show any group symmetry, and additionally, we even find patterns which are not predicted by group theory in symmetric systems. This prompts us to search for a more general theory which includes these phenomena. The aims and goals as well as the guiding questions of this thesis are formulated in Section 1.3. We close Chapter 1 with a grasshopper's guide in Section 1.4.

1.1. A brief historical introduction to symmetry

What is *symmetry*? Going back to its roots in antiquity, we find two different meanings of symmetry, both of which differ from our use of the word today.

In mathematics, we are familiar with Euclid of Alexandria's (around 325 – 270 BC) use of the word symmetry in his monumental *"Elements"* (Book X, Definition 1) [4, 31]:

> **Σύμμετρα** μεγέθη λέγεται τὰ τῷ αὐτῷ μετρῳ μετρούμενα,
> **ἀσύμμετρα** δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.
>
> Those magnitudes measured by the same measure are said to be commensurable, but those of which no magnitude admits to be a common measure are said to be incommensurable.

While Euclid employs the greek word *symmetry* "σύμμετρα" (as well as asymmetry), it is nowadays usually translated with the latin equivalent *commensurable* (where "σύμ" is translated by "com" and "μετρα" by "mensurable"), and the same meaning still applies to "commensurable" to this day.

In arts and descriptions of natural phenomena however, symmetry had a different meaning which could best be translated with "well proportioned", as we can read from Vitruvius (ca. 80/70 BC – ca. 15 BC) in his influential work *"De Architectura Libri Decem"* [31, 54]:

> Item symmetria est ex ipsius operis membris conveniens consensus ex partibusque separatis ad universae figurae speciem ratae partis responsus. Uti in hominis corpore a cubito, pede, palmo, digito ceterisque particulis symmetros est eurythmiae qaalitas, sic est in operum perfectionibus.

> Symmetry is proportioned correspondence of the elements of the work itself, a response, in any given part, of the separate parts to the appearance of the entire figure as a whole. Just as in the human body there is a harmonious quality of shapeliness expressed in terms of the cubit, foot, palm, digit, and other small units, so it is in completing the work of architecture.

Vitruvius' definition of symmetry survived until the Renaissance, in fact, it is best known to us through the famous drawing "Le proporzioni del corpo umano secondo Vitruvio" by Leonardo da Vinci (1452 – 1519).

The first change in the concept of symmetry towards the modern meaning is seen in the following text by Claude Perrault (1613 – 1688), who is famous for designing the east facade of the Louvre in Paris [31, 50]:

> Symmetrie en François signifie seulement un rapport de parité & d'égalité [. . .]. Symmetrie en François est le rapport par exemple que des fenestres ont les unes aux autres quand elles sont toutes de hauteur & de largeur égale, & que leur nombre & leurs espaces sont pareils à droit & à gauche, en sorte que si les espaces sont inégaux d'un costé, une pareille inégalité se rencontre en l'autre.

> Symmetry, in French, signifies only, a relation of parity and equality [...]. Symmetry, in French, signifies the relation, for example, that windows have one to another, when they are all of an equal height and equal breadth; and that their number and distances are equal to the right and the left; so that if the distances be unequal of one side, the like inequality is to be found in the other.

To distinguish this new meaning of symmetry from the old one by Vitruvius, Perrault referred to the latter as "proportion". One could interpret Perrault's new definition already as what is known to us as "bilateral symmetry".

In the 17th century, the use of the word symmetry was mostly restricted to architecture [31]. We do not find it in the natural sciences; even Kepler, whose famous description of the snowflake has become one of the prime examples of a symmetric object in nature [65], does not use the word symmetry. This changed in the 18th century, when symmetry became a term employed in botany, zoology and cristallography.

In his *Élements de géométrie* from 1794 (a work still in sale today!), Adrien-Marie Legendre (1752 – 1833) introduces symmetry as a term in the discussion of solid angles [40, 39]:

> Cette sorte d'égalité, qui n'est pas absolue ou de superposition, mérite d'être distinguée par une dénomination particulière : nous l'appellerons *égalité par symétrie.* Ainsi les deux angles solides dont il s'agit, qui sont formés par trois angles plans égaux chacun à chacun, mais disposés dans un ordre inverse, s'appelleront angles *égaux par symétrie*, ou simplement *angles symétriques.*

> This kind of equality, which is not absolute, or does not admit of superposition, deserves to be distinguished by a particular denomination; we shall call it *equality by symmetry.*

> Thus the two solid angles under consideration, which are respectively contained by three plan angles equal, each to each, but disposed in a contrary order in the one from what they are in the other, we shall call angles *equal by symmetry*, or simply *symmetrical angles*.

In his text, Legendre allowed the order of the planes constituting solid angles to be switched. Contrary to previous works, he found that such a switching yields essentially the same solid angle and he treated it as such, resulting in his famous theory on symmetrical polyhedra.

The next milestone was the introduction of *symmetrical functions* by Sylvestre François Lacroix (1765 – 1843). In his *Traité du calcul différentiel et du calcul intégral* from 1797, we find the first mention of symmetry beyond geometry, as well as a notion of *invariance* under pemutations [31, 37]:

> Cependant certaines fonctions des racines d'une équation quelconque peuvent s'exprimer d'une manière rationnelle au moyen de ses coefficiens, et s'obtiennent par conséquent par des équations du premier degré; les fonctions dont je parle, sont celles qui renferment toutes les racines combinées d'une manière semblable, soit entr'elles, soit avec d'autres quantités, et que pour cela je nommerai fonctions symétriques: la somme des racines, celle de leurs produits deux à deux, trois à trois, etc. respectivement égales aux coefficiens du second, du troisième, du quatrième, etc. termes, sont de ce genre. [...] En effect, il est facile de voir qu'aucune des fonctions symétriques ... ne peut changer de valeur, de quelque manière qu' on permute entr'elles les lettres $\alpha, \beta, \gamma, \delta$, etc., et cette invariabilité est, comme nous l'avons fait remarquer plus haut, le caractère essentiel des fonctions symétriques.

> However, certain functions of an arbitrary equation may be expressed in a rational manner by means of its coefficients, and

> consequently they can be obtained by equations of the first degree; the functions of which I speak are those which contain all the roots combined in a similar manner, either among themselves or with other quantities, and for this [reason] I will call them symmetrical functions: the sum of the roots, those of their products taken two at a time, three at a time, etc., [which are] equal respectively to the coefficients of the second, third, or fourth, etc. terms [of the equation], are [also] of this kind. [...] None of the symmetrical functions ... changes its value no matter how one permutes the letters $\alpha, \beta, \gamma, \delta$, etc., and this invariability is, as we had occasion to remark above, the essential character of symmetrical functions.

The symmetries are expressed as permutations for Lacroix, and other mathematicians such as Lagrange, Abel, and Ruffini [8]. The main breakthrough was achieved by Évariste Galois (1811 – 1832), who classified the permutation operations into what we now call *groups*, (normal) subgroups and conjugated subgroups [8, 22, 23]:

> Grouper les opérations, les classer suivant leurs difficultés et non suivant leurs formes; telle est, suivant moi, la mission des géomètres futurs; telle est la voie où je suis entré dans cet ouvrage.

> Put operations into groups, class them according to their difficulty and not according to their form; that is, according to me, the mission of future geometers, that is the path that I have entered in this work.

Since then, automorphism groups and symmetry are virtually the same [65]. In recent years, however, the mathematical focus has started to shift to groupoids instead of groups, because, as Alan Weinstein (*1943) puts it [64],

> There are plenty of objects which exhibit what we clearly recognize as symmetry, but which admit few or no nontrivial automorphisms. It turns out that the symmetry, and hence much of the structure, of such objects can be characterized algebraically, if we use groupoids and not just groups.

In this thesis, we will encounter many dynamical systems which are not symmetric under any group automorphism (or *equivariant*, see below for a precise definition) but which clearly have some form of additional structure or symmetry, or which exhibit clear spatio-temporal patterns that are normally associated with equivariant systems, only. It seems that to study these phenomena we need to broaden our view on symmetry in the context of dynamical systems and differential equations.

1.2. Review on group symmetry and equivariance

But first, before offering a generalized viewpoint on symmetry, let us very briefly review the current treatment of symmetry and groups in dynamical systems.

Definition 1.2.1 (Group, [61]). A set G of elements $g, h, k, \ldots$ together with a binary operation $\cdot\colon G \times G \to G$ is called a *group* if the following conditions hold:

i) **Associativity:** $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ if g, h, k are in G;

ii) **Identity element:** There exists an element $e \in G$ such that for all $g \in G$, $e \cdot g = g \cdot e = g$ holds;

iii) **Inverse element:** For all $g \in G$, there exists an element $g' \in G$ such that $g' \cdot g = g \cdot g' = e$.

The connection between the abstract algebraic group structure and those dynamical systems which are given by differential equations on a vector space is as follows: The groups are *represented* on vector spaces via group homomorphisms.

Definition 1.2.2 (Group representation, [12])**.** Given a group G and a Banach space X, a linear *representation* of G on X is a continuous group homomorphism $\rho\colon G \to \mathrm{GL}(X)$ from G to the group of invertible linear maps in X.

Unless otherwise noted, all representations in this thesis are assumed to be bounded linear as well as strongly continuous, i.e., we require that the map

$$\begin{aligned} G \times X &\to X \\ (\gamma, x) &\mapsto \rho(\gamma)x \end{aligned} \tag{1.1}$$

is continuous.

In the following, let X, Y be Banach spaces, and let $f\colon X \to Y$. Let G be a group whose elements γ are represented by $\rho_X(\gamma)$ on the space X, and by $\rho_Y(\gamma)$ on the space Y.

Definition 1.2.3 (Group symmetry, [20, 27])**.** The group element $\gamma \in G$ is a *group symmetry* of

$$\dot{x} = f(x) \tag{1.2}$$

if for every solution $x(t)$ of (1.2), $\rho_X(\gamma)x(t)$ is also a solution of (1.2).

Often, group symmetries are simply called symmetries. We will keep the more precise name “group symmetry” in order to avoid confusion with the groupoid symmetries introduced in later chapters.

It turns out that all $\gamma \in G$ are group symmetries if and only if f is *G-equivariant*.

Definition 1.2.4 (Equivariant dynamical system, [12, 20, 27])**.** We say that a dynamical system generated by

$$\dot{x} = f(x) \tag{1.3}$$

is *G-equivariant* if for all $\gamma \in G$ and all $x \in X$ the following holds:

$$f(\rho_X(\gamma)x) = \rho_Y(\gamma)f(x). \tag{1.4}$$

To see the connection between Definitions 1.2.3 and 1.2.4, let $x(t)$ be a solution of $\dot{x} = f(x)$ and let $\gamma \in G$ be a group symmetry of system (1.3). By Definition 1.2.3, $z(t) := \rho_X(\gamma)x(t)$ is also a solution of (1.3). Then

$$\dot{z} = f(z) = f\left(\rho_X(\gamma)x\right) \qquad \text{and} \tag{1.5}$$
$$\dot{z} = \rho_Y(\gamma)\dot{x} = \rho_Y(\gamma)f(x) \tag{1.6}$$

holds. Conversely, suppose that f is G-equivariant and let $x(t)$ be a solution of $\dot{x} = f(x)$. Then

$$\rho_Y(\gamma)\dot{x} = \rho_Y(\gamma)f(x) = f(\rho_X(\gamma)x)), \tag{1.7}$$

which implies that $z(t) := \rho_X(\gamma)x(t)$ is also a solution.

Last, we introduce the following concepts, which — in their generalized versions in the context of symmetry groupoids instead of groups — will play a central role in this thesis.

Definition 1.2.5 (Space of K-fixed vectors, isotropy subgroup, [12, 20, 27])**.** Let ρ_X be a representation of the group G on the space X.

i) Let K be a subgroup of G. Then we call

$$X_K := \{x \in X \mid \rho_X(\gamma)x = x \text{ for all } \gamma \in K\} \tag{1.8}$$

the space of *K-fixed vectors* in X.

ii) We call the subgroup

$$G_x := \{\gamma \in G \mid \rho_X(\gamma)x = x\} \subseteq G \tag{1.9}$$

the *isotropy subgroup* of $x \in X$.

Note that $x \in X_K$ if and only if $K \subseteq G_x$. Moreover, we find the following *conjugacy* between the fixed vectors, $\rho_X(\gamma)X_K = X_{\gamma K \gamma^{-1}}$, and the isotropy subgroups, $G_{\rho_X(\gamma)x} = \gamma G_x \gamma^{-1}$. Another important consequence is that the space of K-fixed vectors is invariant under G-equivariant nonlinearities f. More precisely, let $f\colon X \to Y$ be G-equivariant, then $f\colon X_K \to Y_K$.

Symmetry and equivariance occur throughout the natural sciences: To only name a few, the motion of free rigid bodies in Newtonian mechanics is usually described by an SO(3)-equivariant system; most animals are (roughly) axisymmetric (or more: the starfish is a striking $\mathbb{Z}_5$-equivariant example); snowflakes possess a D_6-symmetry; and the different gaits of four-legged animals can be associated to different isotropy subgroups of the symmetry group D_4. There is an immense body of research on symmetry and symmetric solutions ("patterns"). We can not possibly attempt to mention all results here, instead we refer to the classical textbooks and reviews [12, 17, 27, 25, 33].

1.3. Main goal and guiding questions of this thesis

The theory of equivariant dynamical systems usually works under the assumption that a dynamical system is equivariant under a specific group action. Then the patterns are distilled from the isotropy subgroups.

But what if a system is not fully equivariant? Does that mean that pattern formation is impossible? Of course not! Or what if a system shows more intricate patterns than the equivariance suggests? Does that mean that the theory fails completely? Again, of course not!

However, the current theory of equivariant dynamical systems is not equipped to deal with such cases. This is mainly due to the limitations that groups offer as a rather simple algebraic object. This is not to say that we should dismiss groups entirely — on the contrary!

We should nevertheless review the old definitions of symmetry in dynamical systems with a fresh mind and see where that leads us.

It is our main goal to generalize
the existing definition of symmetry in dynamical systems
and create a refined but much more widely applicable theory
of symmetric dynamical systems.

From this goal the following guiding questions arise:

- *What is an appropriate generalized definition of symmetry in dynamical systems which captures the occurring patterns in systems **without** group symmetry?*
  In our minds, symmetry is so strongly linked to groups that it is easy to neglect that symmetry is also tightly linked to the existence of linear flow-invariant subspaces, as we have seen above. To free ourselves, just for a moment, from the group structure, we suggest to start with linear flow-invariant subspaces as the main building blocks of our new symmetry definition.

- *What is the algebraic structure implied by the new, generalized symmetry?*
  As we will see later on, our definition of symmetry will lead us to groupoids, which we can interpret as a collection of groups, each tied to a linear flow-invariant subspace, and connected via conjugating morphisms.

- *How can spatio-temporal patterns be described in terms of the generalized symmetry?*
  Such a definition should include frequently discussed phenomena, such as multi-frequency patterns, and also include patterns such as discrete and rotating waves for which a comprehensive bifurcation theory exists.

- *Is it possible to rationally design dynamical systems with prescribed symmetries of groupoid type?*
  To answer this question at least partially, we will stay in the realm of networks of coupled oscillators and study their coupling matrices. Interestingly, the algebraic object behind the design of such networks is a monoid, that is, while we use invertibility of symmetries in the main description and in bifurcation theory, it is not necessary for the construction of symmetric systems.

1.4. Grasshopper's guide

To answer the questions raised in the previous subsection, we will proceed as follows:

In Chapter 2 we will first present the new definition of symmetry in the context of ordinary differential equations (ODEs). We will show that the set of symmetries possesses the algebraic structure of a groupoid and we will formulate a generalized notion of equivariance. We purposely concentrate on ODEs here in order to focus entirely on groupoid symmetry without the technical difficulties which could arise in infinite-dimensional systems. Moreover, we illustrate each new concept with one or two examples which are as simple as possible but also instructive. These examples are also designed to demonstrate the ubiquity of the new generalized concept of groupoid symmetry as well as the fact that standard group symmetry has little or even nothing to say in these situations.

It is clear by now that we are dealing with symmetries which possess the underlying structure of a groupoid. This leads us to Chapter 3 where we will first study *equivaroid maps* (a newly coined term generalizing equivariant maps) between two Banach spaces. Compared to the case of standard group equivariance, there are new difficulties associated with restricting a system to a given subspace: In which sense is the restricted system still equivaroid? Which subspaces are left invariant under the groupoid action? To answer these questions, we also newly introduce the terms *subequivaroid* and *invaroid*. As a preparation for the bifurcation theory in the following chapters, we will collect properties of the linearization of equivaroid maps. Lastly, we will show how equivaroid generators imply groupoid symmetries on the level of the semigroup of a dynamical system. Again, all of these concepts are illustrated by simple and instructive examples highlighting the main features and novelties of groupoid symmetries.

Chapter 4 deals with steady-state bifurcation from the trivial equilibrium. We first prove equivaroid Lyapunov–Schmidt reduction, here the main question is: How is the groupoid inherited by the reduced bifurcation equation? Using the theorem of Crandall and Rabinowitz, we are then able to state and prove the *equivaroid branching lemma.* We are therefore able to prove bifurcation of patterns even if systems are not fully equivariant. In addition, we also find that the equivaroid branching lemma gives more precise information on the nature of the pattern than the equivariant branching lemma even if the system is fully equivariant. This is also shown in examples. But even though the equivaroid branching lemma is both more general and more precise than previous works, it does not explain all patterns. We therefore go one step further and prove the *iterated equivaroid branching lemma*, which allows higher dimensional kernels within flow-invariant subspaces which cannot be divided into smaller flow-invariant subspaces.

In Chapter 5 we prove Hopf bifurcation in equivaroid systems. We rely on the abstract Hopf bifurcation setting by Vanderbauwhede, and then define *groupoid spatio-temporal patterns.* To this end, we introduce an action of the N-dimensional torus as a *componentwise time-shift*, as a generalization of time-shifts on the circle S^1. We then state and prove *equivaroid Hopf bifurcation.* Last, as an extension, we prove *iterated equivaroid Hopf bifurcation*, which allows existence proofs of periodic orbits with even more elaborate spatio-temporal patterns than the equivaroid Hopf bifurcation allows, such as multi-frequency patterns.

In Chapter 6 we ask the question which dynamical systems allow for such groupoid symmetries and we deal with the rational design of dynamical systems with prescribed groupoid symmetries. We focus on the design of finite networks. The relevant algebraic objects used for the coupling between the cells of the network will be *monoids*, whose invertible elements double as symmetries of the network.

We close this thesis with a conclusion and short discussion in Chapter 7. We first present a summary of our aims, methods, and results. We then compare the symmetry groupoid to other recent generalizations of symmetry groups, in particular to quiver symmetries and the groupoid formalism. We end with an overview on open problems and further research.

As we cannot assume that the readers are familiar with groupoids, we also include a chapter in the Appendix on this topic, Chapter A. The first part of this chapter is dedicated to the definition of groupoids, and we give a total of nine examples to illustrate the variety of structures covered by the definition. We then turn to vertex groups and conjugating morphisms, both of which play a central role in our discussion of symmetry. We close with a short summary on groupoid representation theory.

2. Groupoid symmetries in ordinary differential equations

In this chapter we first introduce a new generalized definition of symmetry. For the sake of clarity, we restrict ourselves to the finite-dimensional case of ordinary differential equations here and postpone the more general discussion to Chapter 3.

This chapter is organized as follows: In Section 2.1 we redefine symmetry for ordinary differential equations. We investigate the underlying algebraic structure of symmetries, and note that it forms a groupoid. For the reader not familiar with groupoids, more information can be found in the appendix. In the following Section 2.2 we formulate the new symmetry definition in terms of a generalized equivariance — this provides us with a convenient condition for the generalization to the infinite-dimensional case and the bifurcation theorems in later chapters. Sections 2.3 and 2.4 contain a detailed exposition of the different types of symmetries within the groupoid framework.

2.1. The new generalized definition of symmetry

In this section we present a new generalized definition of symmetry in the context of dynamical systems, discuss its algebraic structure as well as its connections to the existing literature, and end with some examples with generalized symmetries but without group symmetries. Through our definition, we are able to vastly generalize the study of pattern formation beyond standard group symmetry.

For pedagogical reasons, we focus on the finite-dimensional case in this introduction in order not to be hindered by technical difficulties at first. We will treat the general case from Chapter 3 onwards.

Contrary to the case of group symmetry discussed above, we start with a definition of symmetry based on *linear isomorphisms*, that is, we do not assume any abstract algebraic conditions behind such an isomorphism or even assume that there exists more than one such isomorphism which meets the definition. This careful approach allows us to extract the main algebraic structure which we can henceforth use. At this point, it is extremely important not to be prejudiced about the type of algebraic structure behind the word "symmetry", but to see symmetry purely as the phenomenon that we want to describe, namely mapping solutions to solutions.

Definition 2.1.1 ((X_j, X_k)-symmetry in $\mathbb{R}^n$)**.** Consider

$$\dot{x} = f(x), \tag{2.1}$$

with $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Let X_j, X_k be linear flow-invariant subspaces of $X = \mathbb{R}^n$, that is, $f(X_j) \subseteq X_j$, $f(X_k) \subseteq X_k$.

We call a linear isomorphism

$$\gamma\colon X \to X \text{ with } \gamma X_j = X_k \tag{2.2}$$

an (X_j, X_k)-*symmetry* of system (2.1) if the following holds:

$$\begin{gathered} x(t) \text{ is a solution of (2.1) in } X_j \text{ if and only if} \\ \gamma x(t) \text{ is a solution of (2.1) in } X_k. \end{gathered}$$

Let us compare this definition to the existing definition of symmetry in dynamical systems, Definition 1.2.3.

First, note that we have lifted the restriction that a symmetry needs to hold for *all* elements $x \in X$. The new Definition 2.1.1 thereby enables us to study a much greater number of patterns in dynamical systems via the symmetry methods than Definition 1.2.3; see the examples below and throughout the thesis. On the other hand, in the special case $X_j = X_k = X$, Definition 2.1.1 reduces to Definition 1.2.3.

Second, as we have discussed in the introduction, the set of symmetries γ from Definition 1.2.3 forms a group [27]. This is not the case for the (X_j, X_k)-symmetries, as not all (X_j, X_k)-symmetries can be composed with one another. We will now explore their algebraic structure in detail.

As the new definition of symmetry is *strongly dependent on the flow-invariant subspaces*, it is to be expected that they play an important role in our further analysis. This motivates the need for a specific name for these spaces.

Definition 2.1.2 (Vertex space)**.** Let $X_j \subseteq X = \mathbb{R}^n$ be a linear flow-invariant subspace of the system $\dot{x} = f(x)$, that is, $X_j \subseteq X$ is linear and $f(X_j) \subseteq X_j$. Then we call X_j a *vertex space* of $\dot{x} = f(x)$.

We index the vertex spaces to obtain a family $\{X_j\}_{j \in I}$ with an index set I (not necessarily finite). For any fixed indexing, it is sometimes convenient to abbreviate "(X_j, X_k)-symmetry" by "(j, k)-symmetry". We also denote the set of (j, k)-symmetries by H_{jk}.

Next, we collect the following algebraic properties of (j, k)-symmetries.

Proposition 2.1.3 (Properties of (j, k)-symmetries)**.**

i) ***Identities for each vertex space:*** *For every vertex space $X_j \subseteq X$, the identity on X is a (j, j)-symmetry.*

ii) ***Inverses of symmetries are symmetries:*** *Let $\gamma\colon X \to X$ be a (j,k)-symmetry, then its inverse $\gamma^{-1}\colon X \to X$ is a (k,j)-symmetry.*

iii) ***Composing symmetries:*** *Let $\gamma_1\colon X \to X$ be a (j,k)-symmetry, and $\gamma_2\colon X \to X$ be a (k,m)-symmetry. Then the linear isomorphism $\gamma_3 := \gamma_2 \circ \gamma_1\colon X \to X$ is a (j,m)-symmetry.*

Remark 2.1.4. A word of caution: As Proposition 2.1.3 shows, it does not suffice to say "$\gamma = \mathrm{Id}$ is a symmetry" in the context of Definition 2.1.1. This is because the same linear isomorphism γ can give rise to *distinct* (X_j, X_k)-symmetries, such as in the case of the identities. We will very carefully distinguish the isomorphisms from the symmetries in the remainder of this thesis.

Proof. Item (i) is trivial. Concerning item (ii), since γ is an isomorphism, its inverse exists, and $X_j = \gamma^{-1}X_k$. Let $\tilde{x}(t)$ be a solution in X_k. Then, by definition, $x(t) := \gamma^{-1}\tilde{x}(t)$ is a solution in X_j if and only if $\gamma x(t) = \tilde{x}(t)$ is a solution in X_k.

For the last point, let $\gamma_1 : X \to X$ be a (j,k)-symmetry, then any $x(t)$ is a solution in X_j if and only if $\gamma_1 x(t)$ is a solution in X_k. Similarly, let $\gamma_2\colon X \to X$ be a (k,m)-symmetry, then any solution $\tilde{x}(t)$ is a solution in X_k if and only if $\gamma_2\tilde{x}(t)$ is a solution in X_m. In particular, this last statement holds if $\tilde{x}(t) := \gamma x(t) \in X_k$, which implies item (iii). $\square$

It turns out that the properties of the composable symmetries that we just showed are indeed characteristic for *groupoids*. We will now first review the definition for groupoids, and then apply this definition to the set of (j,k)-symmetries, $j,k \in I$, as defined above.

Definition 2.1.5 (Groupoid, [34, 38, 43, 47, 64])**.** Let B be a set. A *groupoid* is a set Γ of morphisms $\gamma : B \to B$, $\gamma \in \Gamma$, equipped with the following maps:

- a surjective *source map* $s\colon \Gamma \to B$, $\gamma \mapsto s(\gamma)$,
- a surjective *target map* $t\colon \Gamma \to B$, $\gamma \mapsto t(\gamma)$,

- an injective *identity map* $e\colon B \to \Gamma$, $b \mapsto e(b) =: e_b$,
- a *partial binary composition operation* defined on the set of composable morphisms $\Gamma \star \Gamma := \{(\gamma_2, \gamma_1) \in \Gamma \times \Gamma \mid t(\gamma_1) = s(\gamma_2)\}$:

$$\begin{aligned} \circ\colon \quad \Gamma \star \Gamma &\to \Gamma \\ (\gamma_2, \gamma_1) &\mapsto \gamma_2 \circ \gamma_1, \end{aligned} \tag{2.3}$$

which satisfy the following properties:

i) the partial binary operation is associative, that is, for all (γ_3, γ_2), $(\gamma_2, \gamma_1) \in \Gamma \star \Gamma$, the identity $(\gamma_3 \circ \gamma_2) \circ \gamma_1 = \gamma_3 \circ (\gamma_2 \circ \gamma_1)$ holds;

ii) the identity map defines a family of identity morphisms in the following sense:

   a) for all $b \in B$ it holds that $s(e_b) = t(e_b) = b$,

   b) for all γ such that $s(\gamma) = b$ it holds that $\gamma \circ e_b = \gamma$,

   c) for all γ such that $t(\gamma) = b$ it holds that $e_b \circ \gamma = \gamma$;

iii) each morphism $\gamma \in \Gamma$ has a two-sided inverse $\gamma^{-1} \in \Gamma$ such that

$$\begin{aligned} s(\gamma) = t(\gamma^{-1}), \quad t(\gamma) = s(\gamma^{-1}), \quad \text{and} \\ \gamma^{-1} \circ \gamma = e_{s(\gamma)}, \quad \gamma \circ \gamma^{-1} = e_{t(\gamma)}. \end{aligned} \tag{2.4}$$

We denote such a groupoid by $\Gamma \rightrightarrows B$. The set B is called the *base*, and its elements are called *objects* or *vertices*. We call the object $s(\gamma) \in B$ the *source* of the morphism γ, and the object $t(\gamma) \in B$ its *target*.

For more details on groupoids and many examples see Appendix A.

Our goal is to see how the new Definition 2.1.1 of symmetry is related to the abstract algebraic groupoid structure. To this end, we define the *symmetry groupoid*.

Definition 2.1.6 (The symmetry groupoid)**.** The (abstract) *symmetry groupoid*

$$\Gamma := H_{I*I} := \bigcup_{j\in I}\bigcup_{k\in I} H_{jk}. \tag{2.5}$$

of the system $\dot{x} = f(x)$ is defined as follows:

- The *base* of the symmetry groupoid is given by the index set I of the vertex spaces.
- To each $\gamma \in H_{jk}, \gamma\colon X \to X$, we faithfully associate an *(abstract) morphism* $\tilde{\gamma}\colon j \to k$. The *source* of a morphism $\tilde{\gamma}\colon j \to k$ is given by the index j, while its *target* is given by the index k.
- For each index $k \in I$, the *identity* matrix is a (k,k)-symmetry for all $k \in I$, to which we associate the abstract morphism e_k.
- The *partial binary operation* $\circ$ is defined on the set

$$\Gamma \star \Gamma = \{(\tilde{\gamma}_2, \tilde{\gamma}_1) \mid (\gamma_2, \gamma_1) \in H \star H\} \tag{2.6}$$

with $\widetilde{\gamma_2 \cdot \gamma_1} = \tilde{\gamma}_2 \circ \tilde{\gamma}_1$, where

$$\begin{aligned} H \star H := \{(\gamma_1, \gamma_2) \in H_{I*I} \times H_{I*I} \mid \gamma_1 \in H_{jk}, \gamma_2 \in H_{km} \text{ for} \\ j,k,m \in I \text{ with } H_{jk} \neq \emptyset, H_{km} \neq \emptyset\}. \end{aligned} \tag{2.7}$$

Theorem 2.1.7 (The symmetry groupoid is indeed a groupoid)**.** *The set $\Gamma = H_{I*I}$ over the base $\{X_k\}_{k\in I}$ together with composition of linear isomorphisms on $H \star H$ as in Proposition 2.1.3 forms a groupoid.*

Proof. The basic structure of the groupoid has been established in Proposition 2.1.3 and associativity is given through the general associativity of composition of functions. □

To close this section, let us consider two network examples which possess groupoid symmetries but no standard group symmetries. To see this, we will find the vertex spaces, and the symmetry groupoid for each example.

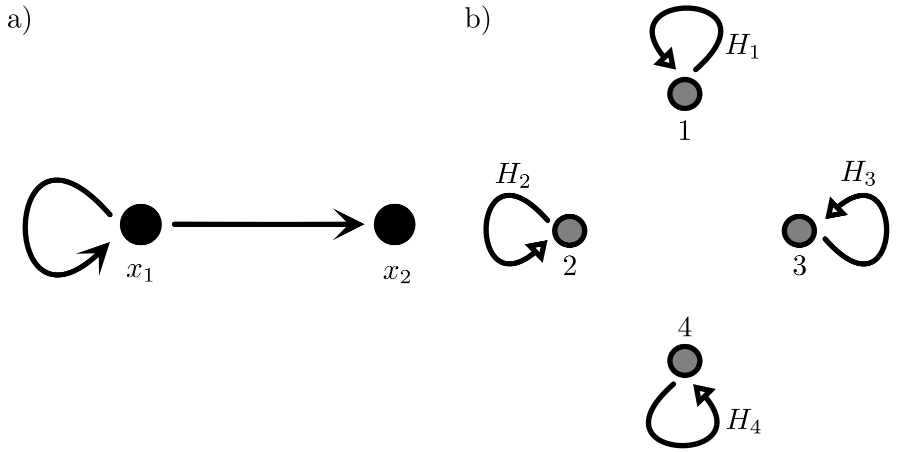

Figure 2.1.: A simple network of two coupled cells (left) and its symmetry groupoid (right). a) Sketch of system (2.8), the arrows depict the coupling between the individual cells x_1, x_2. b) Graphical representation of the symmetry groupoid of system (2.8). Here, the circles denote the objects (vertices), and the arrows denote the morphisms. For simplicity all the sets $H_j := H_{jj}$ are drawn with one arrow only.

Example 2.1.8 (Two coupled cells, I)**.** Let us consider the network given by the ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1) \\ \dot{x}_2 &= f(x_2, x_1), \end{aligned} \tag{2.8}$$

with $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(0,0) = 0$ and cells x_1, x_2. The network may be graphically represented as in Fig. 2.1.

This network does not possess any symmetries described by a group. However, it does possess symmetries in the groupoid sense.

Searching for linear flow-invariant subspaces, we find that the vertex

Vertex pair (j,k)	Set of (j,k)-symmetries
$(1,1)$	$H_{11} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{\mathrm{Id}\}$
$(2,2)$	$H_{22} = \left\{ \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix} \,\middle|\, a,b \in \mathbb{R}, a \neq b \right\}$
$(3,3)$	$H_{33} = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \,\middle|\, a,b \in \mathbb{R}, a \neq 0 \right\}$
$(4,4)$	$H_{44} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R}, ad-bc \neq 0 \right\}$
$(j,k), j \neq k$	$H_{jk} = \emptyset$

Table 2.1.: The symmetry groupoid of system (2.8).

spaces $X_k \subseteq \mathbb{R}^2$, $k = 1,2,3,4$, are given by

$$\begin{aligned} X_1 &= \mathbb{R}^2, \quad X_2 = \{(x_1,x_2) \,|\, x_1 = x_2\}, \\ X_3 &= \{(x_1,x_2) \,|\, x_1 = 0\}, \quad X_4 = \{(0,0)\}. \end{aligned} \tag{2.9}$$

For a clearer overview, we collect the (j,k)-symmetries in Table 2.1. For the one-dimensional subspaces X_2 and X_3, the set of (j,j)-symmetries consists of all the invertible matrices for which the invariant subspace X_j, $j = 2,3$, is an eigenvector with eigenvalue 1. In other words, the symmetries in this specific example are all *pointwise*, but still encode the system structure. Finally, note that the sets H_{jj} form subgroups of GL(2).

Moreover, in this example, symmetries from different sets H_{jk} can not be composed, i.e., while it is of course technically possible to perform a matrix multiplication, we will not necessarily obtain another groupoid-symmetry from this operation. Schematically, the symmetry groupoid

H_{I*I} of the system (2.8) is drawn in Fig. 2.1 b). We see that it consists not of one, but of four individual groups!

In the following example, we illustrate that a symmetry groupoid can also contain conjugating morphisms.

Example 2.1.9 (Two coupled cells, II)**.** Next, we consider the following two-cell network with nonlinear, *nonidentical* coupling between the cells:

$$\begin{aligned} \dot{x}_1 &= f(x_1) + \, x_1 x_2 \\ \dot{x}_2 &= f(x_2) + 2\, x_1 x_2, \end{aligned} \tag{2.10}$$

with $f\colon \mathbb{R} \to \mathbb{R}$, $f(0) = 0$, $x_1, x_2 \in \mathbb{R}$. **System** (2.10) **does not possess any group symmetries, but it exhibits a surprisingly rich set of groupoid symmetries.**

The vertex spaces of system (2.10) are given by

$$\begin{aligned} &X_1 = \mathbb{R}^2, \quad X_2 = \{(x_1, x_2) \,|\, x_1 = 0\}, \\ &X_3 = \{(x_1, x_2) \,|\, x_2 = 0\}, \quad X_4 = \{(0, 0)\}. \end{aligned} \tag{2.11}$$

The symmetry sets H_{jk} can be found in Table 2.2. Note that this two-cell network also possesses *conjugating* symmetries, that is, symmetries which conjugate between the two one-dimensional subspaces X_2 and X_3. In this specific example, the conjugating symmetries occur due to the fact that the nonidentical coupling vanishes on the subspaces X_2, X_3. Therefore, the existence of a solution in one of these subspaces implies the existence of the corresponding solution in the other subspace.

It is worthwhile noting that the inverse of each matrix in the set H_{23} is an element of H_{32} and vice versa. *Neither H_{23} nor H_{32} are groups!* In particular, they do not contain the identity operation. A composition of symmetries from H_{jk} and H_{lm} gives a symmetry element from H_{jm} if $k = l$. Schematically, the set of symmetries H_{I*I} of the system (2.8) is drawn in Fig. 2.2.

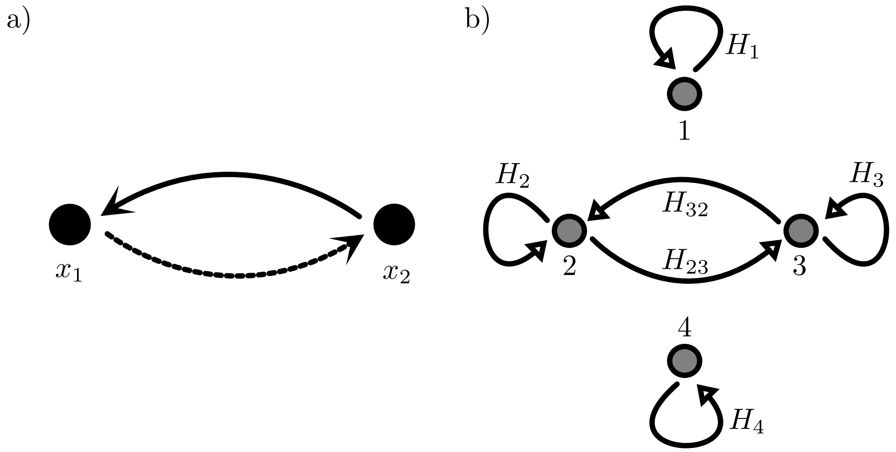

Figure 2.2.: A simple network of two coupled cells (left) and its symmetry groupoid (right). a) Sketch of system (2.10), the arrows depict the coupling between the individual cells x_1, x_2. Nonidentical coupling is indicated by the dotted arrow. b) Graphical representation of the symmetry groupoid of system (2.10). Here, the circles denote the objects (vertices), and the arrows denote the morphisms. For simplicity all the sets $H_j := H_{jj} <$ are drawn with one arrow only.

2.2. Equivaroid systems

So far, we have analyzed a system and then collected the groupoid symmetries. But what does that mean for the dynamical system? Can we find a condition on the level of the differential equations that characterizes the symmetries, similar to the condition of equivariance? As an answer this question, we need to introduce two new terms: *source equivariance*, and *equivaroid.*

To see how these terms emerge, let x^* be a solution of the ordinary differential equation

$$\dot{x} = f(x), \tag{2.12}$$

Vertex pair (j,k)	Set of (j,k)-symmetries
$(1,1)$	$H_{11} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{\mathrm{Id}\}$
$(2,2)$	$H_{22} = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle\vert\ a \neq 0 \right\}$
$(3,3)$	$H_{33} = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \middle\vert\ b \neq 0 \right\}$
$(4,4)$	$H_{44} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle\vert\ ad - bc \neq 0 \right\}$
$(2,3)$	$H_{23} = \left\{ \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \middle\vert\ b \neq 0 \right\}$
$(3,2)$	$H_{32} = \left\{ \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \middle\vert\ a \neq 0 \right\}$
else	$H_{jk} = \emptyset$

Table 2.2.: The symmetry groupoid of system (2.10). Here $a, b, c, d \in \mathbb{R}$.

and assume that x^* is confined to the flow-invariant subspace (vertex space) $X_j \subseteq \mathbb{R}^N$. Note that such a vertex space always exists. In the case that X_j is not a proper subspace but $X_j = \mathbb{R}^N$, the following argument reduces to the standard procedure in group symmetry; see e.g., [27]. In all other cases, we should pay special attention to the involved vertex spaces.

Suppose that $x^*(t) \in X_j$ for all $t \geq 0$ and let γ be a (j,k)-symmetry. Then for all $t \geq 0$, $z(t) := \gamma x(t)$ lies in the vertex space X_k. Moreover, $z(t)$ is also a solution of (2.12), more precisely,

$$\dot{z} = f(z) = f(\gamma x). \tag{2.13}$$

On the other hand,

$$\dot{z} = \gamma\dot{x} = \gamma f(x). \tag{2.14}$$

It follows that a necessary condition for γ to be a (j,k)-symmetry of $\dot{x} = f(x)$ is the commutativity of f and γ, but only on the subspace X_j. This is different from the well-known concept of *equivariance*, where the commutativity of f and γ is required to hold *for all* $x \in X$. In other words, we find that

$$f(\gamma x) = \gamma f(x) \quad \text{for all } x \in X_j, \gamma \in H_{jk}. \tag{2.15}$$

In general, the commutativity of f and γ does not hold for all $x \in X = \mathbb{R}^N$. This necessary condition for the existence of groupoid symmetries is therefore much less strict than the usual equivariance condition for group symmetries which implies that many more dynamical systems will fulfill it.

Definition 2.2.1 (Source equivariance)**.** We say that the system $\dot{x} = f(x)$ is X_j*-source equivariant* with respect to $H_{jI} := \bigcup_{k\in I} H_{jk}$ if

$$\gamma f(x) = f(\gamma x) \tag{2.16}$$

holds for all $x \in X_j$ and for all $\gamma \in H_{jI}$.

The naming "source equivariance" is influenced by the groupoid terminology: j is the source object for all morphisms $\gamma \in H_{jI}$, that is, all morphisms with source j.

But is the source equivariance of f and γ also a sufficient condition? Suppose that $f(\gamma x) = \gamma f(x)$ holds on a flow-invariant subspace X_j for a (j,k)-symmetry γ. Then, if γx is a solution, x is also a solution:

$$\gamma\dot{x} = f(\gamma x) = \gamma f(x). \tag{2.17}$$

By Proposition 2.1.3, γ is invertible, and it follows that the equivariance of f and γ on the vertex space X_j is also a sufficient condition. In summary, we have proven the following Proposition.

Proposition 2.2.2 (Source equivariance versus (j,k)-symmetries)**.**

i) Let $\gamma\colon X \to X$ be a (j,k)-symmetry. Then, for all $x \in X_j$, the following equality holds:

$$\gamma f(x) = f(\gamma x). \tag{2.18}$$

ii) Let X_j, X_k be flow-invariant subspaces of $\dot{x} = f(x)$. Let $\gamma\colon X \to X$ be a linear isomorphism which additionally fulfills $\gamma X_j = X_k$. If source equivariance

$$\gamma f(x) = f(\gamma x), \tag{2.19}$$

holds for all such γ and for all $x \in X_j$, then γ is a (j,k)-symmetry.

Remark 2.2.3. Note that the equations (2.18) and (2.19) are equalities in the vertex space X_k: Although $x \in X_j$, both $\gamma f(x)$ and $f(\gamma x)$ are elements of X_k.

The last proposition inspires the following new definition which is a generalization of group equivariance, now formulated for groupoids.

Definition 2.2.4 (Equivaroid systems)**.** Consider $\dot{x} = f(x)$, $f\colon X \to X$, $X = \mathbb{R}^n$. Let $\{X_j\}_{j\in I}$ be an indexed family of linear flow-invariant subspaces of X, i.e., $f(X_j) \subseteq X_j$ for all $j \in I$.

Let $(\Gamma \rightrightarrows I)$ be a groupoid Γ over the base I.

We say that the dynamical system generated by $\dot{x} = f(x)$ is $(\Gamma \rightrightarrows I)$-*equivaroid* if the following holds:

There exists a faithful representation ρ of the groupoid $\Gamma \rightrightarrows I$ on the space X, i.e., $\rho(j) = X$ for all $j \in I$ and $\rho(\tilde{\gamma})\colon X \to X$ for all $\tilde{\gamma} \in \Gamma$, such that

i) for all $\tilde{\gamma} \in \Gamma$ with $\tilde{\gamma}\colon j \to k$, we have $\rho(\tilde{\gamma})X_j = X_k$.

ii) for all $x \in X_j$ and for all morphisms $\tilde{\gamma}\colon j \to k$, source equivariance holds,
$$\rho(\tilde{\gamma})f(x) = f(\rho(\tilde{\gamma})x). \tag{2.20}$$

Note that the groupoid $(\Gamma \rightrightarrows I)$ is not necessarily identical to the symmetry groupoid, that is, it does not need to cover all groupoid symmetries in the sense of Definition 2.1.1. This leaves us some degree of freedom: It is possible to treat only a subgroupoid of the symmetry groupoid, if that is feasible from an application point of view. Additionally, since any group is a groupoid with only one object, it also means that a system which is equivariant with respect to a group G is also equivaroid. That is, Definition 2.2.4 covers both the well known case of equivariant systems, as well as many additional systems which are not equivariant but still possess a symmetry groupoid.

In the following example, we will check the source equivariance condition for a specific system, and we will see clearly how it distinguishes from standard equivariance.

Example 2.2.5 (Two asymmetrically, nonlinearly coupled cells II, continued from Example 2.1.9)**.** We consider again the dynamical system of

$$\begin{aligned} \dot{x}_1 &= f(x_1) + x_1x_2 \\ \dot{x}_2 &= f(x_2) + 2x_1x_2, \end{aligned} \tag{2.21}$$

where $f\colon \mathbb{R} \to \mathbb{R}$, $f(0) = 0$. **This system is not equivariant with respect to any group, but it is equivaroid with respect to its symmetry groupoid.**

The source equivariance on the space $X_2 = \{(x_1, x_2) \in \mathbb{R} \mid x_1 = 0\}$ for the elements of H_{22},

$$H_{22} = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R}, a \neq 0 \right\}, \tag{2.22}$$

holds because each element of H_{22} fixes the space X_2, pointwise. More interesting is the source equivariance for the conjugating (X_2, X_3)-symmetries

$$H_{23} = \left\{ \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R}, b \neq 0 \right\}. \tag{2.23}$$

Application of the elements of H_{23} to both sides of the equation (2.21) and all $(x_1, x_2) \in \mathbb{R}^2$ yields:

$$\begin{aligned} a\dot{x}_1 + \dot{x}_2 &= f(ax_1 + x_2) + (ax_1 + x_2)x_1 \\ b\dot{x}_1 &= f(bx_1) + 2(ax_1 + x_2)x_1. \end{aligned} \tag{2.24}$$

Obviously, this is not true for all $(x_1, x_2) \in \mathbb{R}^2$*. It is, however, true on the vertex space* X_2*:* Plugging in $(x_1, x_2) = (0, x_2) \in X_2$ gives

$$\begin{aligned} \dot{x}_2 &= f(x_2) \\ 0 &= 0, \end{aligned} \tag{2.25}$$

i.e., a true statement in the subspace X_3 and therefore, source equivariance on the subspace X_2 holds. The other cases can be shown analogously, for brevity, we will omit the calculations. In conclusion, the system (2.21) is $(H_{I*I} \rightrightarrows I)$-equivaroid.

2.3. Vertex groups — the building blocks of symmetry

To underline the connections between symmetries and groupoids, in the following two Sections 2.3 and 2.4, we investigate the most elementary properties and the algebraic structure of the symmetry groupoid. Now, in Section 2.3, we concentrate on those symmetries of the symmetry groupoid which act only on one vertex space, and will proceed with those symmetries which connect to vertex spaces in the following Section 2.4.

Throughout, consider $\dot{x} = f(x)$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$, with an indexed family of vertex spaces $\{X_j\}_{j \in I}$, and let $(H_{I*I} \rightrightarrows I)$ be its symmetry groupoid.

Proposition 2.3.1 (Group structure of (j, j)-symmetries)**.** *The set H_{jj} of all (j, j)-symmetries together with composition of linear maps forms a group.*

Proof. First, the set of (j, j)-symmetries is closed under composition; see Proposition 2.1.3. The other group axioms are also fulfilled since we already proved that $(H_{I*I} \rightrightarrows I)$ forms a groupoid. □

Proposition 2.3.1 suggests the following definition.

Definition 2.3.2 (Vertex symmetry group)**.** The group $H_j := H_{jj}$ of all (j, j)-symmetries is called the *vertex symmetry group* of X_j.

Proposition 2.3.3 (Invariance of X_j, as a set)**.** *The vertex symmetry group H_j leaves the vertex space X_j invariant, as a set, i.e., $\gamma X_j = X_j$ holds for all $\gamma \in H_j$.*

This proposition is clear by definition. The setwise invariance under the vertex symmetry group immediately raises the question of pointwise invariance.

Definition 2.3.4 (Vertex isotropy group)**.** We denote by K_j the set of all linear invertible isomorphisms $\kappa \colon X \to X$ which leave the vertex space X_j invariant, pointwise, i.e.,

$$K_j := \{\kappa \colon X \to X \text{ linear isomorphism} \mid \kappa x = x \text{ for all } x \in X_j\}. \tag{2.26}$$

Together with composition of isomorphisms, we call K_j the *vertex isotropy group* of X_j.

This definition is justified by the following proposition which tells us that the elements κ of K_j are indeed (j, j)-symmetries:

Proposition 2.3.5 (The vertex isotropy group is a subgroup of the vertex symmetry group)**.** *The vertex isotropy group K_j is a group. In particular, it is a subgroup of the vertex symmetry group H_j.*

Proof. First, note that all elements of the vertex isotropy group are elements of the vertex symmetry group, by definition. Moreover, the vertex isotropy group is trivially closed under composition, and associativity is also given. To see that the inverse is an element of K_j, multiplying $\kappa x = x$ by κ^{-1} on both sides yields $x = \kappa^{-1} x$. Lastly, K_j trivially contains the identity element. □

Corollary 2.3.6 (Flow-invariance implies symmetry)**.** *Let $X_j \subseteq X = \mathbb{R}^n$ be a vertex space of a system $\dot{x} = f(x)$. Then those linear isomorphisms which leave X_j invariant, pointwise, form (j, j)-symmetries.*

We now describe the relation between the vertex symmetry and the vertex isotropy group. To this end, we define the group homomorphism to the general linear group on the space X_j,

$$\Phi\colon H_j \to \mathrm{GL}(X_j) \tag{2.27}$$

$$h \mapsto \Phi(h), \tag{2.28}$$

via the identity

$$\Phi(h)(x|_{X_j}) := (hx)\,|_{X_j} \quad \text{for all } x \in X_j. \tag{2.29}$$

The kernel of this homomorphism is given by the group K_j, i.e., $\ker \Phi = K$. Remember that K_j leaves all $x \in X_j$ invariant, pointwise, therefore it acts as the identity on the vertex space X_j.

Then, by the homomorphism theorem [61], K_j is a normal subgroup of the group H_j and

$$H_j / K_j \cong \operatorname{Im} \Phi. \tag{2.30}$$

This motivates the following definition.

Definition 2.3.7 (Vertex quotient group)**.** We define the quotient

$$Q_j := H_j / K_j \tag{2.31}$$

as the *vertex quotient group.*

It depends on the context whether it is more useful to let the vertex quotient group act on the full space X or on the smaller space X_j. The importance of the vertex quotient group can easily be seen from the following lemma.

Lemma 2.3.8 (Q_j-equivariance)**.** *The system $\dot{x} = f(x)$ restricted to the space X_j,*

$$\dot{x}_j = f(x_j), \tag{2.32}$$

where we abbreviate $x_j := x|_{X_j}$, is Q_j-equivariant, i.e., for all $x_j \in X_j$ and for all $q \in Q_j$, we have

$$f(qx_j) = qf(x_j). \tag{2.33}$$

Proof. Note that q defines a standard symmetry of the *restricted* system (2.32) for all $q \in Q_j$. Since Q_j defines a group, source equivariance (2.33) follows from standard arguments; see e.g., [27]. □

As the last part of this section, we discuss the following question: If there exist several vertex spaces, can we conclude the existence of others?

Proposition 2.3.9. *Let $X_1, X_2, \ldots, X_k$ be arbitrary vertex spaces of a system $\dot{x} = f(x)$. Then*

$$X_m := \bigcap_{i=1}^{k} X_i \subseteq X \tag{2.34}$$

is a nonempty flow-invariant linear subspace and hence a vertex space with a vertex isotropy group K_m, defined as in Definition 2.3.4, where

$$K_i \subseteq K_m \quad (\forall i = 1, \ldots, k). \tag{2.35}$$

Proof. For all $i \in \{1, \ldots, k\}$, the flow-invariance of the vertex spaces X_i implies

$$f\left(\bigcap_{i=1}^{k} X_i\right) \subseteq X_i. \tag{2.36}$$

Therefore,

$$f(X_m) = f\left(\bigcap_{i=1}^{k} X_i\right) \subseteq \bigcap_{i=1}^{k} X_i = X_m. \tag{2.37}$$

To prove the symmetry claim, note that X_m is a subspace of all X_i, $i = 1, \ldots, k$. Therefore, all elements of the vertex isotropy groups K_i of the spaces X_i must also be in the vertex isotropy group K_m of X_m. □

In the following examples, we will see that indeed two cases can happen: Vertex isotropy groups can either be equal to vertex symmetry groups or they can be true subgroups and lead to vertex quotient groups.

Example 2.3.10 (Two coupled cells I, continued from Example 2.1.8)**.** Note that the set of (j, j)-symmetries forms a group H_j for all $j = 1, 2, 3, 4$. These vertex isotropy groups K_j are equal to the vertex symmetry groups H_j (remember that the symmetries are all pointwise), and consequently, the vertex quotient groups are all trivial. In other words, the symmetries all act as the identity on the respective vertex spaces.

Moreover, in accordance with Proposition 2.3.9, we note that the intersection of the vertex spaces X_2 and X_3 forms the smaller vertex space X_4, whose vertex isotropy group is $K_4 = \mathrm{GL}(2)$. In particular, it contains the smaller vertex isotropy groups K_2 and K_3.

Example 2.3.11 (Three coupled cells, I)**.** Let us consider the network given by

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1) \\ \dot{x}_3 &= f(x_3, x_2), \end{aligned} \tag{2.38}$$

with $f \colon \mathbb{R}^2 \to \mathbb{R}$ and no additional requirements on the nonlinearity f. The network is graphically represented in Fig. 2.3.

This is yet another example of a network which does not possess any symmetries described by a group, but plenty of (j, k)-symmetries. We list all groupoid symmetries in Table 2.3.

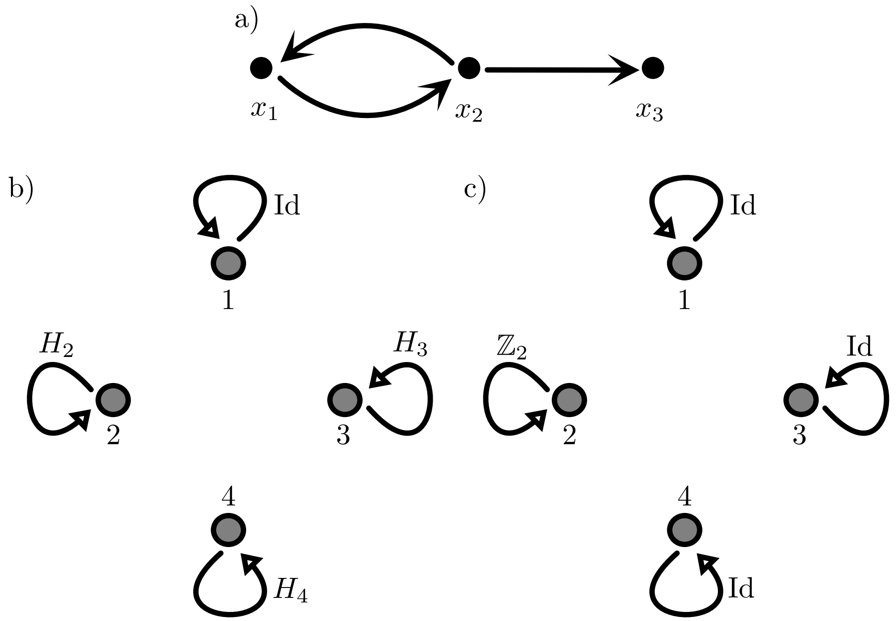

Figure 2.3.: A simple network of three coupled cells (top), its symmetry groupoid (bottom left) as well as its quotient groups (bottom right). a) Sketch of system (2.38), the arrows depict the coupling between the individual cells x_1, x_2, x_3. b) Graphical representation of the symmetry groupoid of system (2.38). Here, the gray dots denote the objects (vertices), and the arrows denote the morphisms. For simplicity all the sets H_j are drawn with one arrow only. c) Groupoid of system (2.38), reduced to the vertex quotient groups.

In this example not all vertex symmetry and isotropy groups are equal: Specifically, the vertex isotropy group of X_2 is given by

$$K_2 = \left\{ h = \begin{pmatrix} a & 0 & 1-a \\ b & 1 & -b \\ c & 0 & 1-c \end{pmatrix} \middle| \det h \neq 0 \right\} \subset H_2, \tag{2.39}$$

j	Vertex spaces $X_j \subset \mathbb{R}^3$	Vertex symmetry groups H_j
1	$\mathbb{R}^3$	$\{\mathrm{Id}\}$
2	$\{(x_1, x_2, x_3) \mid x_1 = x_3\}$	$\left\{ \begin{pmatrix} a & 1 & -a \\ b & 0 & 1-b \\ c & 1 & -c \end{pmatrix} \middle\vert\, a \neq -c \right\}$ $\bigcup \left\{ \begin{pmatrix} a & 0 & 1-a \\ b & 1 & -b \\ c & 0 & 1-c \end{pmatrix} \middle\vert\, a \neq c \right\}$
3	$\{(x_1, x_2, x_3) \mid x_1 = x_2\}$	$\left\{ \begin{pmatrix} a & 1-a & 0 \\ b & 1-b & 0 \\ c & -c & 1 \end{pmatrix} \middle\vert\, a \neq b \right\}$
4	$\{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$	$\left\{ \begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ e & f & 1-e-f \end{pmatrix} \middle\vert\, \det \neq 0, \right\}$

Table 2.3.: The symmetry groupoid of system (2.38).

which is a strict subgroup of H_2. Therefore, the corresponding vertex quotient group is given by $Q_2 = \mathbb{Z}_2$, acting on $\mathbb{R}^2$. Calculating the system restricted to X_2, we obtain

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1), \end{aligned} \tag{2.40}$$

where we have omitted the redundant (third) equation. The system (2.40) is indeed $\mathbb{Z}_2$-equivariant. This means that, surprisingly, whenever a solution is of the form (x_1^*, x_2^*, x_1^*), then also (x_2^*, x_1^*, x_2^*) is a solution. The vertex spaces X_1, X_3, X_4 all have $K_j = H_j$. It follows that $Q_j = \{\mathrm{Id}\}$ for $j = 1, 3, 4$.

Last, note that the intersection of the two vertex spaces X_2 and X_3 gives the smaller vertex space X_4 and that both $K_2 \subset K_4$ and $K_3 \subset K_4$.

2.4. Conjugating symmetries — connecting the building blocks

In this section we continue our quest to find out about the algebraic structure of the groupoid-symmetries. Here, we focus on (j,k)-symmetries which conjugate between different subspaces, i.e., $j \neq k$. As before, we denote the set of (j,k)-symmetries by H_{jk}.

Definition 2.4.1 (Conjugate vertex spaces)**.** Two vertex spaces X_j, X_k, $j \neq k$, are called *conjugate* if there exists a (j,k)-symmetry γ.

In other words, we say that X_j and X_k are conjugate if H_{jk} is nonempty. Then, also the set H_{kj} is nonempty and in particular, it contains the inverses of the elements of H_{jk}.

Proposition 2.4.2 (Inverse of conjugating symmetries)**.** *Let $\gamma \in H_{jk}$. Then $\gamma^{-1} \in H_{kj}$.*

By definition, all the symmetries are isomorphisms, which proves the following proposition.

Proposition 2.4.3 (Conjugate vertex spaces and symmetry groups)**.**

i) Conjugate vertex spaces are isomorphic. In particular, they have the same dimension.

ii) Let X_j and X_k be conjugate vertex spaces. Then for all $\gamma \in H_{jk}$ and for the associated vertex symmetry groups it holds that

$$\gamma H_j \gamma^{-1} = H_k, \tag{2.41}$$

i.e., conjugate vertex spaces lead to conjugated vertex symmetry groups.

Remark 2.4.4. Not all isomorphic vertex spaces are conjugate; see Example 2.1.8 above.

There is one type of conjugating symmetries that deserves particular attention, as it might happen that some of the conjugating symmetries are also elements of vertex groups for a differerent vertex space.

Definition 2.4.5 (Inherited vertex conjugacy)**.** Let X_j be a vertex space with a nontrivial vertex quotient group Q_j. Let $X_k, X_m \subset X_j$ be two conjugate vertex spaces. We call those $\gamma \in H_{km}$, which are also elements of Q_j, *j-inherited (k, m)-conjugating symmetries.* We define

$$Q^j_{km} := Q_j \cap H_{km}. \tag{2.42}$$

We also define the set of *inherited (k, m)-conjugating symmetries*

$$Q_{km} := \{\gamma \in H_{km} \mid \exists\, j \in I \text{ with } X_k, X_m \subseteq X_j \text{ such that } \gamma \in Q_j\}. \tag{2.43}$$

Proposition 2.4.6 (Conjugated vertex quotient groups)**.** *Let X_k and X_m be conjugate vertex spaces. If there exists j such that for all inherited conjugating symmetries $Q^j_{km} \neq \emptyset$, then for all $\gamma \in Q^j_{km}$ and for the associated vertex quotient groups, the following holds:*

$$\gamma Q_m \gamma^{-1} = Q_k, \tag{2.44}$$

i.e., inherited conjugacy leads to conjugated vertex quotient groups.

We now consider two contrasting examples concerning inheritance in conjugating symmetries: In Example 2.4.7, the conjugating symmetries are also inherited. This is not the case in Example 2.4.8.

Example 2.4.7 (Three coupled cells II)**.** Let us consider the following system:

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_2) \\ \dot{x}_3 &= f(x_3, x_2). \end{aligned} \tag{2.45}$$

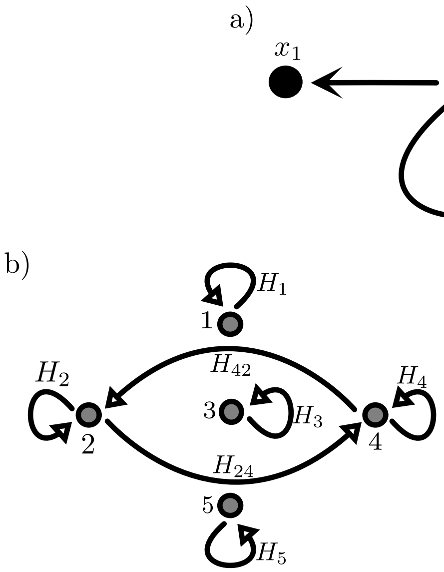

Figure 2.4.: A simple network of three coupled cells (top), its symmetry groupoid (bottom left) as well as its inherited subgroupoid (bottom right). a) Sketch of system (2.45), the arrows depict the coupling between the individual cells x_1, x_2, x_3. b) and c) Graphical representation of the symmetry groupoid and inherited subgroupoid of system (2.45), respectively. Here, the gray dots denote the objects (vertices), and the arrows denote the morphisms. For simplicity all the sets H_j are drawn with one arrow only.

Here $f\colon \mathbb{R}^2 \to \mathbb{R}$. **This system is $\mathbb{Z}_2$-equivariant, where $\mathbb{Z}_2$ acts by permutation on the variables x_1 and x_3. Moreover, system** (2.45) **possesses five vertex spaces X_k with corresponding vertex symmetry groups H_k, $k = 1, \dots, 5$; see Table 2.4, as well as some conjugating symmetries; see Table 2.5.** The only nontrivial quotient groups are $Q_1 = \mathbb{Z}_2$, which reflects the $\mathbb{Z}_2$-equivariance of (2.45), and $Q_3 = \mathbb{Z}_2$, which corresponds to the subspace X_3 where the

j	Vertex spaces $X_j \subset \mathbb{R}^3$	Vertex symmetry groups H_j
1	$\mathbb{R}^3$	$\left\{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\} = \mathbb{Z}_2$
2	$\{(x_1,x_2,x_3)\,\|\,x_1 = x_2\}$	$\left\{\begin{pmatrix}a&1-a&0\\b&1-b&0\\c&-c&1\end{pmatrix} \,\middle\|\, a \neq b\right\}$
3	$\{(x_1,x_2,x_3)\,\|\,x_1 = x_3\}$	$\left\{\begin{pmatrix}a&0&1-a\\b&1&-b\\c&0&1-c\end{pmatrix} \,\middle\|\, a \neq c\right\}$
4	$\{(x_1,x_2,x_3)\,\|\,x_2 = x_3\}$	$\left\{\begin{pmatrix}1&a&-a\\0&b&1-b\\0&c&1-c\end{pmatrix} \,\middle\|\, b \neq c\right\}$
5	$\{(x_1,x_2,x_3)\,\|\,x_1 = x_2 = x_3\}$	$\left\{\begin{pmatrix}a&b&1-a-b\\c&d&1-c-d\\e&f&1-e-f\end{pmatrix} \,\middle\|\, \det \neq 0\right\}$

Table 2.4.: Vertex spaces and vertex groups of system (2.45) with $a, b, c, d, e, f \in \mathbb{R}$. They form part of the symmetry groupoid, together with Table 2.5.

equivariance is not broken. In contrast, the $\mathbb{Z}_2$-equivariance has been broken in the vertex spaces X_2 and X_4, but the subspaces are now conjugated; see Table 2.5.

Example 2.4.8 (Two coupled cells II, continued from Example 2.1.9)**.** As an example where none of the conjugating morphisms are inherited,

(X_j, X_k)	(Inner) Conjugating (X_j, X_k)-symmetries H_{jk}, Q_{jk}
(X_2, X_4)	$H_{24} = \left\{ \begin{pmatrix} a & -a & 1 \\ b & 1-b & 0 \\ c & 1-c & 0 \end{pmatrix} \middle\vert\, b \neq c \right\}$
	$Q_{24}^1 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$
(X_4, X_2)	$H_{42} = \left\{ \begin{pmatrix} 0 & a & 1-a \\ 0 & b & 1-b \\ 1 & c & -c \end{pmatrix} \middle\vert\, a \neq b \right\}$
	$Q_{42}^1 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$

Table 2.5.: (Inner) Conjugating symmetries of system (2.45) with $a, b, c \in \mathbb{R}$. They form part of part of the symmetry groupoid, together with Table 2.4.

consider again the following two-cell network:

$$\begin{aligned} \dot{x}_1 &= g(x_1) + \, x_1 x_2 \\ \dot{x}_2 &= g(x_2) + 2\, x_1 x_2, \end{aligned} \tag{2.46}$$

with $g\colon \mathbb{R} \to \mathbb{R}$, $g(0) = 0$. From the vertex spaces and their vertex symmetry groups H_j, and the conjugating symmetries H_{jk}, already determined in Example 2.1.9, we obtain the groupoid. None of the conjugated morphisms is inherited, since there does not exist any j (in this case only $j = 1$ would be possible) such that any of the conjugating morphisms is an element of H_j.

3. General theory of equivaroid maps

After the finite-dimensional introduction to groupoid symmetries, we now set the ground for a general theory applicable to dynamical systems on Banach spaces and equivaroid mappings between non-equal spaces.

This chapter is organized as follows: We start in Section 3.1 with newly defining the concept of *equivaroid maps* on Banach spaces and illustrating it by some examples. In Section 3.2 we will see that particular care is necessary if an equivaroid system is restricted to a subspace: In which sense is it still equivaroid with respect to the same groupoid? To answer this question, we introduce the term *subequivaroid*. We also introduce *invaroid* subspaces, that is, special subspaces which are left invariant under the action of the full groupoid. Section 3.3 is devoted to the study of the linearization of an equivaroid map as a preparation for the equivaroid bifurcation theory in the following chapters. In Section 3.4 we define groupoid symmetries for strongly continuous semigroups and we show that an equivaroid infinitesimal generator implies symmetries on the semigroup and vice versa.

3.1. Equivaroid maps

After introducing the concept of equivaroid maps in the previous chapter, we need to generalize the term to the general case $\mathcal{F}\colon X \to Y$, where X, Y are Banach spaces. This allows for a broader applicability, particularly in the context of partial differential equations.

Definition 3.1.1 (Equivaroid maps on Banach spaces)**.** Consider the map $\mathcal{F}\colon X \to Y$, X, Y Banach spaces. Let $\{X_j\}_{j\in I}, \{Y_j\}_{j\in I}$ be indexed families of linear closed subspaces ("vertex spaces") of X, Y, such that $\mathcal{F}(X_j) \subseteq Y_j$ for all $j \in I$.

Let $(\Gamma \rightrightarrows I)$ be a groupoid Γ over the base I. We say that $\mathcal{F}$ is $(\Gamma \rightrightarrows I)$-equivaroid if the following holds:

There exist two representations ρ_X, ρ_Y of the groupoid $(\Gamma \rightrightarrows I)$ on the spaces X, Y such that

i) for all $\gamma \in \Gamma$ with $\gamma\colon j \to k$, we have

$$\rho_X(\gamma)X_j = X_k, \quad \rho_Y(\gamma)Y_j = Y_k; \tag{3.1}$$

ii) for all $x \in X_j$ and for all morphisms $\gamma\colon j \to k$, source equivariance holds, i.e.,

$$\rho_Y(\gamma)\mathcal{F}(x) = \mathcal{F}(\rho_X(\gamma)x). \tag{3.2}$$

There are two main situations where we use the above definition: First, we use it for examples of equivaroid partial differential equations, where we interpret $\mathcal{F}$ as the generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, on Y with dense domain X. We will treat this case in detail in Section 3.4.

Second, we will use this setting for the proof of the equivaroid Hopf

Vertex spaces $X_0 = Y_0, X_{j,i} = Y_{j,i}, j \in \mathbb{N}, i = 1, 2$
$Y_0 = \left\{ x(t) = \sum_{k=0}^{\infty} a_k \sin(kt) \,\middle|\, \forall\, k \in \mathbb{N} : a_k = 0 \right\} = \{0\}$
$Y_{j,1} = \left\{ x(t) = \sum_{k=0}^{\infty} a_k \sin(kt) \,\middle|\, \forall\, k \neq nj,\ n \in \mathbb{N} : a_k = 0 \right\}$
$Y_{j,2} = \left\{ x(t) = \sum_{k=0}^{\infty} a_k \sin(kx) \,\middle|\, \forall\, k \neq nj,\ n \in \mathbb{N} \text{ odd} : a_k = 0 \right\}$

Table 3.1.: Vertex spaces of the shift operator (3.3) (neglecting conditions on the coefficients a_k which ensure appropriate convergence).

bifurcation theorem; see Chapter 5, where X and Y are spaces of 2π-periodic functions.

Example 3.1.2 (Shift operators)**.** Consider the map

$$\begin{aligned} \mathcal{F}: \quad X &\to Y \\ x(\cdot) &\mapsto x(\cdot - \pi) + f(x(\cdot)) \end{aligned} \tag{3.3}$$

with $f\colon \mathbb{R} \to \mathbb{R}$ as an odd polynomial, and $X = Y = L^2_0 := \{u(\cdot) \in L^2([0,\pi],\mathbb{R}) \mid u(0) = u(\pi) = 0\}$. Whenever necessary for the calculations, we consider the 2π-periodic continuation $u(-x) := -u(x), u(x+2\pi) := u(x)$ to $x \in \mathbb{R}$. We find the vertex spaces as in Table 3.1 and consider groupoid symmetries as in Table 3.2, acting by convolution, where $h\colon \mathbb{R} \to \mathbb{R}$ has finite L^1-norm:

$$(h * x)(t) := \frac{1}{\pi} \int_0^{\pi} h(t-s)x(s)\mathrm{d}s. \tag{3.4}$$

Example 3.1.3 (Differential operators)**.** Consider the map

$$\begin{aligned} \mathcal{F}: X &\to Y \\ u &\mapsto u_{xx} + f(u) \end{aligned} \tag{3.5}$$

Vertex groups $H_0, H_{j,i}, j \in \mathbb{N}, i = 1, 2$
$H_0 = \left\{ h(t) = \sum\limits_{k=0}^{\infty} b_k \cos(kt) \right\}$
$H_{j,1} = \left\{ h(t) = \pm \sum\limits_{k=0}^{\infty} b_k \cos(kt) \,\middle|\, \forall\, k = nj,\, n \in \mathbb{N} : b_k = 1 \right\}$
$H_{j,2} = \left\{ h(t) = \pm \sum\limits_{k=0}^{\infty} b_k \cos(kt) \,\middle|\, \forall\, k = nj,\; n \in \mathbb{N} \text{ odd} : b_k = 1 \right\}$

Table 3.2.: Vertex groups of the shift operator (3.3). Strictly speaking, to obtain vertex symmetry groups, we would need to check invertibility as well as integrability, which sets additional conditions on the coefficients b_k. We will not get lost in these technical difficulties here.

with $f\colon \mathbb{R} \to \mathbb{R}$ as an odd polynomial, and

$$\begin{aligned} Y &= L^2_0 := \{u(\cdot) \in L^2([0,\pi], \mathbb{R}) \mid u(0) = u(\pi) = 0\}, \\ X &= \tilde{H}^2_0 := L^2_0 \cap H^2. \end{aligned} \tag{3.6}$$

The vertex spaces are given in Table 3.3, and the vertex groups in Table 3.4, where the groupoid symmetries act by convolution, and where $h : \mathbb{R} \to \mathbb{R}$ has finite L^1-norm:

$$\left(h * u\right)(x) := \frac{1}{\pi} \int\limits_0^\pi h(x-y)u(y)\mathrm{d}y. \tag{3.7}$$

It is also worth noting that the vertex spaces and symmetry groups in those last two examples are essentially the same. This can be used to control dynamical systems and obtain stable solutions with a desired spatio-temporal control as has been done by this author previously [56, 57].

Vertex spaces $X_0 = Y_0, X_{j,i} = Y_{j,i}, j \in \mathbb{N}, i = 1,2$
$Y_0 = \left\{ u(x) = \sum\limits_{k=0}^{\infty} a_k \sin(kx) \,\middle|\, \forall\, k \in \mathbb{N} : a_k = 0 \right\} = \{0\}$
$Y_{j,1} = \left\{ u(x) = \sum\limits_{k=0}^{\infty} a_k \sin(kx) \,\middle|\, \forall\, k \neq nj,\ n \in \mathbb{N} : a_k = 0 \right\}$
$Y_{j,2} = \left\{ u(x) = \sum\limits_{k=0}^{\infty} a_k \sin(kx) \,\middle|\, \forall\, k \neq nj,\ n \in \mathbb{N} \text{ odd} : a_k = 0 \right\}$

Table 3.3.: Vertex spaces of the differential operator (3.5) (again neglecting the conditions on the coefficients a_k which ensure appropriate convergence).

3.2. Subequivaroid maps and invaroid subspaces

As our goal is the development of an equivaroid bifurcation theory, we will often want to reduce a system to a smaller system. This raises the question whether some subspaces are invariant under the action of the groupoid. We will first introduce *subequivaroid maps* on subspaces, i.e., maps which are also equivaroid with respect to the same groupoid, but on smaller subspaces. This usually gives new vertex spaces and new representations.

Definition 3.2.1 (Subequivaroid maps)**.** Let $\mathcal{F}\colon X \to Y$ be $(\Gamma \rightrightarrows I)$-equivaroid. We say that $\mathcal{F}$ is *subequivaroid* on the closed linear spaces $U \subseteq X$, $V \subseteq Y$ if the following conditions hold:

i) $U_j := X_j \cap U$, $V_j := Y_j \cap V$ are vertex spaces of $\mathcal{F} : U \to V$ for all $j \in I$;

ii) There exist projections $\Pi_U : X \to U$, $\Pi_V : Y \to V$ to the subspaces

Vertex groups $H_0, H_{j,i}, j \in \mathbb{N}, i = 1, 2$
$H_0 = \left\{ h(x) = \sum\limits_{k=0}^{\infty} b_k \cos(kx) \right\}$
$H_{j,1} = \left\{ h(x) = \pm \sum\limits_{k=0}^{\infty} b_k \cos(kx) \,\middle|\, \forall\, k = nj,\, n \in \mathbb{N} : b_k = 1 \right\}$
$H_{j,2} = \left\{ h(x) = \pm \sum\limits_{k=0}^{\infty} b_k \cos(kx) \,\middle|\, \forall\, k = nj,\; n \in \mathbb{N} \text{ odd} : b_k = 1 \right\}$

Table 3.4.: Vertex groups of the differential operator (3.5) (also neglecting conditions on the coefficients b_k to ensure appropriate convergence, integrability, and invertibility).

U, V with $x_U := \Pi_U X$ and $y_V := \Pi_V y$ such that

$$\begin{aligned} \rho_U(\gamma)x_U &:= \Pi_U \rho_X(\gamma)x \\ \rho_V(\gamma)x_V &:= \Pi_V \rho_V(\gamma)y \end{aligned} \tag{3.8}$$

define subrepresentations on U and V, respectively, i.e., for all morphisms $\gamma \in \Gamma$ with $\gamma\colon j \to k$, we have

$$\rho_U(\gamma)U_j = U_k, \quad \rho_V(\gamma)V_j = V_k. \tag{3.9}$$

iii) for all $x_U \in U_j$ and for all morphisms $\gamma\colon j \to k$, source equivariance holds, i.e.,

$$\rho_V(\gamma)\mathcal{F}(x_U) = \mathcal{F}(\rho_U(\gamma)x_U). \tag{3.10}$$

Remark 3.2.2. Note that, in the above definition, some vertex spaces U_j, V_j might be the trivial zero space. This does not contradict the definition, as the zero space has arbitrary symmetry.

As a preparation for the equivaroid bifurcation theory which we will develop in the following chapters, we are also interested whether some specific subspaces are invariant under the action of the groupoid. To understand this concern, it is important to notice that a morphism $\gamma\colon j \to k$ will map a vertex space X_j to a *different* vertex space X_k. A vertex space X_j is therefore not necessarily invariant by the action of the full groupoid.

Definition 3.2.3 (Invaroid subspaces)**.** We consider a linear closed subspace $\overline{X}$ of a Banach space X. Let $\{X_j\}_{j\in I}$ be an indexed family of linear closed subspaces ("vertex spaces") of X and let $(\Gamma \rightrightarrows I)$ be a groupoid Γ over the base I with representation ρ_X. We say that the subspace $\overline{X}$ is $(\Gamma \rightrightarrows I)$-*invaroid* if the following holds:

For all $j \in I$ and for all morphisms $\gamma\colon j \to k$,

$$\rho_X(\gamma)\big(\overline{X} \cap X_j\big) \subseteq \big(\overline{X} \cap X_k\big). \tag{3.11}$$

We will use the following example to illustrate the new concepts *invaroid* and *subequivaroid.*

Example 3.2.4 (Three coupled cells, continued from Example 2.4.7)**.** We reconsider the following three-cell network:

$$\begin{aligned} \dot{x}_1 &= f(\lambda, x_1) + x_2 \\ \dot{x}_2 &= f(\lambda, x_2) + x_2 \\ \dot{x}_3 &= f(\lambda, x_3) + x_2, \end{aligned} \tag{3.12}$$

Here $x_1, x_2, x_3 \in \mathbb{R}$, $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. We additionally assume $\mathcal{L} := \mathrm{D}_x\, f(\lambda, 0) = \lambda$. The symmetry groupoid is found in Tables 3.5 and 3.6. Note that the symmetry groupoid is larger compared to that of system (2.45) in Tables 2.4 and 2.5 because of the additional assumption $f(\lambda, 0) \equiv 0$ (apart from that, it is the same network structure).

The linearization at the trivial equilibrium $x = 0$ is given by

$$\mathcal{L} := \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda + 1 & 0 \\ 0 & 1 & \lambda \end{pmatrix}. \tag{3.13}$$

j	Vertex spaces $X_j \subset \mathbb{R}^3$	Vertex symmetry groups H_j
1	$\mathbb{R}^3$	$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = \mathbb{Z}_2$
2	$\{(x_1, x_2, x_3) \mid x_1 = x_2\}$	$\left\{ \begin{pmatrix} a & 1-a & 0 \\ b & 1-b & 0 \\ c & -c & 1 \end{pmatrix} \middle\vert\, a \neq b \right\}$
3	$\{(x_1, x_2, x_3) \mid x_1 = x_3\}$	$\left\{ \begin{pmatrix} a & 0 & 1-a \\ b & 1 & -b \\ c & 0 & 1-c \end{pmatrix} \middle\vert\, a \neq c \right\}$
4	$\{(x_1, x_2, x_3) \mid x_2 = x_3\}$	$\left\{ \begin{pmatrix} 1 & a & -a \\ 0 & b & 1-b \\ 0 & c & 1-c \end{pmatrix} \middle\vert\, b \neq c \right\}$
5	$\{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$	$\left\{ \begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ e & f & 1-e-f \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$
6	$\{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$	$\left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$
7	$\{(x_1, x_2, x_3) \mid x_2 = x_3 = 0\}$	$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$
8	$\{(0,0,0)\}$	$\left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$

Table 3.5.: Vertex spaces and symmetry groups of system (3.12), with $a, b, c, d, e, f, g, h, i \in \mathbb{R}$. Note that $H_2 \subset H_6$ and $X_6 \subset X_2$, as well as $H_4 \subset H_7$ and $X_7 \subset X_4$.

(X_j, X_k)	Conjugating (X_j, X_k)-symmetries H_{jk}
(X_2, X_4)	$\left\{ \begin{pmatrix} a & -a & 1 \\ b & 1-b & 0 \\ c & 1-c & 0 \end{pmatrix} \middle\vert\ b \neq c \right\}$
(X_4, X_2)	$\left\{ \begin{pmatrix} 0 & a & 1-a \\ 0 & b & 1-b \\ 1 & c & -c \end{pmatrix} \middle\vert\ a \neq b \right\}$
(X_6, X_7)	$\left\{ \begin{pmatrix} a & b & 1 \\ c & d & 0 \\ e & f & 0 \end{pmatrix} \middle\vert\ \det \neq 0 \right\}$
(X_7, X_6)	$\left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 1 & e & f \end{pmatrix} \middle\vert\ \det \neq 0 \right\}$

Table 3.6.: Conjugating symmetries of system (3.12), with $a, b, c, d, e, f \in \mathbb{R}$. Neither of the sets H_{jk} are groups.

At $\lambda = 0$, $\mathcal{L}$ is clearly not invertible. In particular, we find the kernel

$$\ker \mathcal{L} = \operatorname{span}\{(1,0,0)^T, (0,0,1)^T\}, \tag{3.14}$$

and it turns out that $\ker \mathcal{L}$ is a good choice for a $(\Gamma \rightrightarrows I)$-invaroid subspace, that is, for all $j \in I$ and for all morphisms $\gamma \colon j \to k$,

$$\rho_X(\gamma)\big(\ker \mathcal{L} \cap X_j\big) \subseteq \big(\ker \mathcal{L} \cap X_k\big). \tag{3.15}$$

We start with the vertex groups:

i) $j = 1$. We only need to check

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ker \mathcal{L} \subseteq \mathcal{L}, \tag{3.16}$$

and this is obviously true.

ii) $j = 2$ and $j = 6$. All elements of the spaces $\ker \mathcal{L} \cap X_2 = \ker \mathcal{L} \cap X_6$ are of the form $(0, 0, x)^T$ with $x \in \mathbb{R}$. Since $H_2 \subset H_6$, it is sufficient to do the calculations for $j = 6$:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} \in \ker \mathcal{L} \cap X_6 = \ker \mathcal{L} \cap X_2. \tag{3.17}$$

iii) $j = 3$. In this case, all elements of the space $\ker \mathcal{L} \cap X_3$ are of the form $(x, 0, x)^T$ with $x \in \mathbb{R}$ and we find that

$$\begin{pmatrix} a & 0 & 1-a \\ b & 1 & -b \\ c & 0 & 1-c \end{pmatrix} \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} \in \ker \mathcal{L} \cap X_3. \tag{3.18}$$

iv) $j = 4$ and $j = 7$. The calculation is similar: All elements of the spaces $\ker \mathcal{L} \cap X_4 = \ker \mathcal{L} \cap X_7$ are of the form $(x, 0, 0)^T$ with $x \in \mathbb{R}$ and we find that

$$\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \in \ker \mathcal{L} \cap X_7 = \ker \mathcal{L} \cap X_4. \tag{3.19}$$

v) $j = 5$ and $j = 8$. There is nothing left to prove, since $\ker \mathcal{L} \cap X_5 = \ker \mathcal{L} \cap X_8 = (0, 0, 0)^T$.

Last, we need to see how the conjugating morphisms act on $\ker \mathcal{L}$:

vi) $j = 2$, $k = 4$ and $j = 6$, $k = 7$:

$$\begin{pmatrix} a & b & 1 \\ c & d & 0 \\ e & f & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \in \ker \mathcal{L} \cap X_7 = \ker \mathcal{L} \cap X_4. \tag{3.20}$$

vi) $j = 4$, $k = 2$ and $j = 7$, $k = 6$:

$$\begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 1 & e & f \end{pmatrix} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} \in \ker \mathcal{L} \cap X_6 = \ker \mathcal{L} \cap X_2. \tag{3.21}$$

It follows from the same calculations that system (3.12) is subequivaroid with respect to the subspace $\ker \mathcal{L} \subseteq X = Y = \mathbb{R}^3$.

3.3. Properties of the linearization of an equivaroid system

In this section we will look at the linearization of an equivaroid system from the groupoid point of view. In particular, we will study the linearization in different vertex spaces and find that each derivative is itself equivaroid with respect to a different groupoid. Surprisingly, even derivatives in vertex spaces which are completely void of internal symmetries often possess a nontrivial groupoid. We prove that the linearization around the trivial equilibrium of a Γ-equivaroid system is again Γ-equivaroid. These results have immediate consequences for bifurcation theory and pattern formation.

Lemma 3.3.1 (Properties of the linearization in equivaroid systems)**.** *Let $\mathcal{F} : X \to Y$ be continuously differentiable as well as $(\Gamma \rightrightarrows I)$-equivaroid. Let $\mathcal{L}_{x_0} := \mathrm{D}_x \mathcal{F}(x_0)$ for some $x_0 \in X_j$ for some $j \in I$.*

Then the following hold for all $v \in X_j$:

i) ***Isotropy:*** *For all $\gamma \colon j \to j$ such that $\rho_X(\gamma)x_0 = x_0$ and for all $v \in X_j$, commutativity holds:*

$$\rho_Y(\gamma)\mathcal{L}_{x_0}v = \mathcal{L}_{x_0}\rho_X(\gamma)v. \tag{3.22}$$

ii) ***Conjugacy:*** *For all $\gamma \colon j \to k$ and for all $v \in X_j$, the linearizations $\mathcal{L}_{x_0}$ and $\mathcal{L}_{\rho_X(\gamma)x_0}$ are conjugated as follows:*

$$\rho_Y(\gamma)\mathcal{L}_{x_0}v = \mathcal{L}_{\rho_X(\gamma)x_0}\rho_X(\gamma)v. \tag{3.23}$$

iii) ***Inheritance:*** *Let Y_j^ℓ be a vertex space of the map $\mathcal{L}_{x_0} \colon X \to Y$, i.e., let*

$$\rho_Y(\gamma)\mathcal{L}_{x_0}v = \mathcal{L}_{x_0}\rho_X(\gamma)v. \tag{3.24}$$

hold for $x_0 \in X_j$, $v \in X_\ell$, with $\gamma\colon \ell \to \ell$. Then for all k with $X_k \subseteq X_j$, Y_k^ℓ is a vertex space of $\mathcal{L}_{x_k}\colon X \to Y$ where $x_k \in X_k \subseteq X_j$.

Moreover, in the special case that $\{0\}$ is a vertex space of $\mathcal{F}$, the map $\mathcal{L}_0 = \mathrm{D}_x \mathcal{F}(0)\colon X \to Y$ is $(\Gamma \rightrightarrows I)$-equivaroid.

Proof. i) Isotropy: Let $x_0 \in X_0$, $\gamma \in K_0$. Then by equivariance,

$$\rho_Y(\gamma)\mathcal{F}(x_0) = \mathcal{F}(\rho_X(\gamma)x_0), \tag{3.25}$$

the Gateaux derivative at x_0 in the direction $v \in X_j$ combined with isotropy yields

$$\begin{aligned} \rho_Y(\gamma)\,\mathrm{D}_v\,\mathcal{F}(x_0)v &= \mathrm{D}_v\,\mathcal{F}(\rho_X(\gamma)x_0)\rho_X(\gamma)v \\ &= \mathrm{D}_v\,\mathcal{F}(x_0)\rho_X(\gamma)x. \end{aligned} \tag{3.26}$$

ii) Conjugacy: This is basically the same proof as for the isotropy. Linearizing (3.25) at x_0 in the direction $v \in X_j$, we obtain

$$\rho_Y(\gamma)\,\mathrm{D}_x\,\mathcal{F}(x_0)x = \mathrm{D}_x\,\mathcal{F}(\rho_X(\gamma)x_0)\rho_X(\gamma)x. \tag{3.27}$$

iii) Inheritance: We linearize on a smaller subspace X_k. Therefore the claim is less than the assumption. In spite of this result looking trivial, its explicit formulation is useful for bifurcation purposes.

Linearization at zero: Note that $\rho_X(\gamma)0 = 0$ for all $\gamma \in \Gamma$. Thus, combining the points on isotropy and conjugacy, we obtain

$$\rho_Y(\gamma)\mathcal{L}_0 x = \mathcal{L}_0\rho_X(\gamma)x. \tag{3.28}$$

This concludes the proof. $\square$

We are now ready to prove the following.

Lemma 3.3.2 (Invaroid kernel, range and eigenspaces)**.** *Let $\mathcal{L}\colon X \to Y$ be a $(\Gamma \rightrightarrows I)$-equivaroid closed linear operator.*

Then $\ker \mathcal{L}$ *and* $\operatorname{range} \mathcal{L}$ *as well as their complements* $(\ker \mathcal{L})^c$ *and* $(\operatorname{range} \mathcal{L})^c$ *are $(\Gamma \rightrightarrows I)$-invaroid subspaces of X and Y, respectively.*

Moreover, the (generalized) eigenspaces of $\mathcal{L}$ are also $(\Gamma \rightrightarrows I)$-invaroid.

Proof. We first show the invariance-property of the kernel: Let $x \in \ker \mathcal{L} \subseteq X$, and suppose additionally that x lies in a vertex space X_j. Let $\gamma\colon j \to k$, then

$$\mathcal{L}\rho_X(\gamma)x = \rho_Y(\gamma)\mathcal{L}x = \rho_Y(\gamma)0 = 0. \tag{3.29}$$

It follows that $\rho_X(\gamma)x \in \ker \mathcal{L}$, which proves the claim. Note that $\rho_X(\gamma)x$ might lie in a different vertex space of $\mathcal{L}$.

Next, let $x \in (\ker \mathcal{L})^c \subseteq X$, and again suppose additionally that x lies in a vertex space X_j. Let $\gamma\colon j \to k$, then

$$\mathcal{L}\rho_X(\gamma)x = \rho_Y(\gamma)\mathcal{L}x \neq 0. \tag{3.30}$$

It follows that $\rho_X(\gamma)x \in (\ker \mathcal{L})^c$, which proves the claim. As before, note that $\rho_X(\gamma)x$ might lie in a different vertex space of $\mathcal{L}$.

We proceed with the invariance property of the range: Let $y \in \operatorname{range} \mathcal{L} \subseteq Y$, and now suppose that y lies in a vertex space Y_j. Then there exists $x \in X_j$ such that

$$y = \mathcal{L}x. \tag{3.31}$$

Applying $\gamma\colon j \to k$ to both sides yields

$$\rho_Y(\gamma)y = \rho_Y(\gamma)\mathcal{L}x. \tag{3.32}$$

Since $\mathcal{L}$ is equivaroid, we find

$$\rho_Y(\gamma)y = \mathcal{L}\rho_X(\gamma)x. \tag{3.33}$$

It follows that $\rho_Y(\gamma)y \in \operatorname{range} \mathcal{L}$ which proves the claim. Again, note $\rho_Y(\gamma)Y$ might lie in a different vertex space of $\mathcal{L}$.

The claim on $(\operatorname{range}\mathcal{L})^c$ is proven via contradiction. Let $y \in (\operatorname{range}\mathcal{L})^c \subseteq Y$, and now suppose that y lies in a vertex space Y_j. Then there does not exist any $x \in X$ such that $y = \mathcal{L}x$. Let $\gamma\colon j \to k$ and assume that $\rho_Y(\gamma)y \in (\operatorname{range}\mathcal{L})^c$. Then there exists $x \in X_k$ such that

$$\rho_Y(\gamma)y = \mathcal{L}x. \tag{3.34}$$

Applying $\gamma^{-1} : k \to j$ to both sides implies

$$y = \rho_Y(\gamma^{-1})\mathcal{L}x. \tag{3.35}$$

By the source equivariance on the vertex space X_k, we find

$$y = \mathcal{L}\rho_X(\gamma^{-1})x, \tag{3.36}$$

which contradicts the assumption that $y \in (\operatorname{range}\mathcal{L})^c$.

The claims on the generalized eigenspaces follow analogously by considering the operator $\mathcal{L} - \lambda\,\mathrm{Id}$ instead, where λ is an eigenvalue of $\mathcal{L}$. $\square$

At the end of this section, we examine the linearization and its symmetry groupoid of an example in more detail.

Example 3.3.3 (Three coupled oscillators)**.** We consider the dynamical system of three coupled oscillators

$$\begin{aligned}
\dot{x}_1 &= f(x_1) + \kappa x_2\\
\dot{x}_2 &= f(x_2) + \kappa x_1\\
\dot{x}_3 &= f(x_3) + \kappa x_2,
\end{aligned} \tag{3.37}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1-function, and $\kappa \in \mathbb{R}\backslash\{0\}$. We additionally assume $f(0) = 0$ to allow for the trivial equilibrium. The vertex spaces and vertex symmetry groups of this system are listed in Table 3.7. There are six vertex spaces, but no conjugating symmetries.

Throughout, we denote the Jacobian at an equilibrium $(x_j, y_j, z_j) \in X_j$, $j = 1, \ldots, 6$, by $\mathcal{L}_j$.

j	Vertex spaces $X_j \subseteq \mathbb{R}^3$	Vertex symmetry groups H_j
1	$\mathbb{R}^3$	$\{\mathrm{Id}\}$
2	$\{(x_1, x_2, x_3) \mid x_1 = x_3\}$	$\left\{ \begin{pmatrix} a & 1 & -a \\ b & 0 & 1-b \\ c & 1 & -c \end{pmatrix} \middle\vert\, a \neq c \right\} \cup \left\{ \begin{pmatrix} a & 0 & 1-a \\ b & 1 & -b \\ c & 0 & 1-c \end{pmatrix} \middle\vert\, a \neq -c \right\}$
3	$\{(x_1, x_2, x_3) \mid x_1 = x_2\}$	$\left\{ \begin{pmatrix} a & 1-a & 0 \\ b & 1-b & 0 \\ c & -c & 1 \end{pmatrix} \middle\vert\, a \neq b \right\}$
4	$\{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$	$\left\{ \begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ e & f & 1-e-f \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$
5	$\{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$	$\left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix} \middle\vert\, ad - bc \neq 0 \right\}$
6	$\{(0, 0, 0)\}$	$\mathrm{GL}(3)$

Table 3.7.: Vertex spaces and vertex symmetry groups of the network (3.37). All parameters $a, b, c, d, e, f \in \mathbb{R}$.

Let us start with the three-dimensional vertex space $X_1 = \mathbb{R}^3$ as it will have the smallest symmetry groupoid of all linearizations. A linearization from this space is of the general form

$$\mathcal{L}_1 = \begin{pmatrix} F_1 & \kappa & 0 \\ \kappa & F_2 & 0 \\ 0 & \kappa & F_3 \end{pmatrix}, \tag{3.38}$$

where $F_1, F_2, F_3 \in \mathbb{R}$. The groupoid for this linearization is given in Table 3.8. Note that the symmetry of the space X_1 is trivial, but the symmetry groupoid of its linearization is far from trivial. We obtain eight (!) vertex spaces, three of which coincide with vertex spaces of the original symmetry groupoid. The vertex groups of the linearization contain more elements even for the same vertex space, since linear systems automatically allow for scaling symmetries.

Next, we consider the linearization $\mathcal{L}_2$ of the vertex space X_2 to illustrate the inheritance principle. The general form of the linearization is given by

$$\mathcal{L}_2 = \begin{pmatrix} F_1 & \kappa & 0 \\ \kappa & F_2 & 0 \\ 0 & \kappa & F_1 \end{pmatrix}. \tag{3.39}$$

The symmetry groupoid of this vertex space is quite large (remember that, by the inheritance principle, it will at least contain the symmetry groupoid of $\mathcal{L}_1$) and we will not do the full calculations here. Instead, let us focus on the isotropy, as X_2 is itself a vertex space with a nontrivial vertex group. In fact the vertex group of $Y_2^2 = X_2$ is given by

$$H_2^2 = \left\{ h = \begin{pmatrix} a & 0 & d-a \\ b & d & -b \\ c & 0 & d-c \end{pmatrix} \middle| \det h \neq 0 \right\}. \tag{3.40}$$

The vertex isotropy group of X_2, on the other hand, is given by

$$H_2 = \left\{ h = \begin{pmatrix} a & 0 & 1-a \\ b & 1 & -b \\ c & 0 & 1-c \end{pmatrix} \middle| \det h \neq 0 \right\}, \tag{3.41}$$

which is clearly a subgroup of H_2, as claimed in Lemma 3.3.1.

Vertex spaces $Y_1^\ell \subseteq \mathbb{R}^3$	Vertex symmetry groups H_1^ℓ
$Y_1^1 = X_1 = \mathbb{R}^3$	$\{a \operatorname{Id} \mid a \neq 0\}$
$Y_1^2 = \{(x_1, x_2, x_3) \mid x_1 = x_3 = 0)\}$	$\left\{ \left. \begin{pmatrix} a & 0 & 0 \\ b & a & -b \\ c & 0 & a-c \end{pmatrix} \right\vert a(a-c) \neq 0 \right\} \cup \left\{ \left. \begin{pmatrix} a & 0 & b \\ 0 & a+b & 0 \\ c & 0 & a+b-c \end{pmatrix} \right\vert (a+b)(a-c) \neq 0 \right\}$
$Y_1^3 = \{(x_1, x_2, x_3) \mid x_2 = 0\}$	$\left\{ \left. \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right\vert ab \neq 0 \right\}$
$Y_1^4 = \{(x_1, x_2, x_3) \mid x_2 = x_3 = 0\}$	$\left\{ \left. \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & d \end{pmatrix} \right\vert ad \neq 0 \right\}$
$Y_1^5 = X_5 = \{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$	$\left\{ \left. \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{pmatrix} \right\vert ad - bc \neq 0, g \neq 0 \right\}$
$Y_1^6 = \{(x_1, x_2, x_3) \mid x_1 = x_2, x_3 = 0\}$	$\left\{ \left. \begin{pmatrix} a & 0 & b \\ 0 & a+b & -b \\ 0 & 0 & a+b \end{pmatrix} \right\vert a(a+b) \neq 0 \right\}$
$Y_1^7 = \{(x_1, x_2, x_3) \mid x_1 = 0\}$	$\left\{ \left. \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & 0 & a-b \end{pmatrix} \right\vert a(a-b) \neq 0 \right\}$
$Y_1^8 = X_6 = \{(0, 0, 0)\}$	$\mathrm{GL}(3)$

Table 3.8.: Symmetry groupoid of the linearization $\mathcal{L}_1$. Far from being trivial, we find eight vertex spaces. Comparing with the symmetry groupoid from the full system (3.37), note in particular that $Y_1^1 = X_1, Y_1^5 = X_5, Y_1^8 = X_6$.

3.4. Application to infinite-dimensional dynamical systems

In this section we apply Definition 3.1.1 to generators of strongly continuous semigroups and prove that they yield again groupoid symmetries on the semigroup in the following sense.

Definition 3.4.1 (Groupoid symmetry for strongly continuous semigroups)**.** Consider a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space Y. Let $\{Y_j\}_{j\in I}$ be an indexed family of linear closed subspaces ("vertex spaces") of Y such that $T(t)Y_j \subseteq Y_j$ for all $t \geq 0$ and for all $j \in I$. Let $(\Gamma \rightrightarrows I)$ be a groupoid Γ over the base I.

We say that $\gamma\colon j \to k$, $\gamma \in \Gamma$, is a (Y_j, Y_k)-symmetry of the semigroup $(T(t))_{t\geq 0}$ if the following holds:

There exists a representation ρ_Y of the groupoid $(\Gamma \rightrightarrows I)$ on the space Y such that

$$\rho_Y(\gamma)\colon Y \to Y \text{ with } \rho_Y(\gamma)Y_j = Y_k \tag{3.42}$$

and

$$T(t)\rho_Y(\gamma)y_0 = \rho_Y(\gamma)T(t)y_0 \tag{3.43}$$

holds for all $y_0 \in Y_j$ and for all $t \geq 0$.

In this setting we adopt the terms vertex symmetry/isotropy/quotient groups, for bounded linear operators $\rho_Y(\gamma)$ on the space Y which possess an inverse. The same goes for (inherited) vertex conjugacy and conjugating symmetries.

Let us now assume that $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \to Y$ is the infinitesimal generator of the semigroup $T(t)$. Furthermore, assume that $\mathcal{A}$ is $(\Gamma \rightrightarrows I)$-equivaroid. Are the invariance of the generator $\mathcal{A}(\mathcal{D}(\mathcal{A})\cap Y_j) \subseteq Y_j$ and flow-invariance $T(t)Y_j \subseteq Y_j$ equivalent?

First, suppose that Y_j is a closed linear subspace of Y such that $T(t)Y_j \subseteq Y_j$ for all $t \geq 0$, in other words, Y_j is flow-invariant. Then the restrictions

$$T(t)_{|j} := T(t)|_{Y_j} \tag{3.44}$$

form a strongly continuous semigroup $(T(t)_{|j})_{t\geq 0}$ on the Banach space Y_j; see [19]. Its generator is constructed as follows.

Definition 3.4.2 (Part of a generator, [19])**.** The *part* of $\mathcal{A}$ in Y_j is the operator $\mathcal{A}_{|j}$ defined by

$$\mathcal{A}_{|j}y := \mathcal{A}y \tag{3.45}$$

with domain

$$\mathcal{D}(\mathcal{A}_{|j}) := \{y \in \mathcal{D}(\mathcal{A}) \cap Y_j \mid \mathcal{A}y \in Y_j\}. \tag{3.46}$$

Then the part $\mathcal{A}_{|j}$ coincides with the generator of the semigroup $(T(t)_{|j})_{t\geq 0}$ on the subspace Y_j and invariance of the generator follows.

Proposition 3.4.3 (Generator of the restricted semigroup, [19])**.** *Let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on Y. If the restricted semigroup $(T(t)_{|j})_{t\geq 0}$ is strongly continuous on some closed flow-invariant Banach space $Y_j \subseteq Y$, then the generator of $(T(t)_{|j})_{t\geq 0}$ is the part $(\mathcal{A}_{|j}, \mathcal{D}(\mathcal{A}_{|j}))$ of $\mathcal{A}$ in Y_j.*

Moreover, to see that groupoid symmetry implies source equivariance, let $T(t)y_0$ be the orbit of y_0 generated by $\mathcal{A}$ with $y_0 \in Y_j \subseteq Y$. Suppose that $T(t)y_0 \in Y_j$ for all $t \geq 0$, and let $\rho_Y(\gamma)$ be the action of a (j,k)-symmetry. Then for all $t \geq 0$, $\rho_Y(\gamma)T(t)y_0$ is in the vertex space Y_k. But $\rho_Y(\gamma)T(t)y_0$ is an orbit generated by $\mathcal{A}$ and therefore [48]:

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)\rho_Y(\gamma)y_0 = \mathcal{A}T(t)\rho_Y(\gamma)y_0. \tag{3.47}$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)\rho_Y(\gamma)y_0 = \rho_Y(\gamma)\frac{\mathrm{d}}{\mathrm{d}t}T(t)y_0 = \rho_Y(\gamma)\mathcal{A}T(t)y_0. \tag{3.48}$$

Since this holds for all $t \geq 0$ and in particular for $t = 0$ we conclude that

$$\mathcal{A}\,\rho_Y(\gamma)y_0 = \rho_Y(\gamma)\mathcal{A}\,y_0 \tag{3.49}$$

follows, as desired.

For the converse, let the part of $\mathcal{A}$ in Y_j be invariant, i.e., $\mathcal{A}(\mathcal{D}(\mathcal{A}) \cap Y_j) \subseteq Y_j$. Then the semigroup that it generates is indeed $(T(t)_{|j})_{t\geq 0}$, for which the flow-invariance of the space Y_j follows.

To see this in more detail, let

$$\mathcal{A}_j := \mathcal{A}|_{Y_j} \colon \mathcal{D}(\mathcal{A}) \cap Y_j \to Y_j. \tag{3.50}$$

Then $\mathcal{A}_j$ generates a C^0-semigroup

$$T_j(t) \colon Y_j \to Y_j. \tag{3.51}$$

Next, we fix $y_0 \in Y_j$ and define

$$f(s) := T(t-s)T_j(s)y_0. \tag{3.52}$$

Then $f(0) = T(t)y_0$, $f(t) = T_j(t)y_0$ and, by the product rule,

$$f'(s) = -T(t-s)\,\mathcal{A}\,T_j(s)y_0 + T(t-s)\,\mathcal{A}_j\,T_j(s)y_0. \tag{3.53}$$

But note that $T_j(s)y_0 \in Y_j$ and hence

$$\mathcal{A}\,T_j(s)x = \mathcal{A}_j\,T_j(s)y_0 \tag{3.54}$$

holds. This implies that $f \equiv 0$ and $f(0) = f(t)$ and thereby $T_j(t)y_0 = T(t)y_0$. Since $T_j(t)$ leaves Y_j invariant, so does $T(t)$.

Now suppose that $\mathcal{A}\,\rho_Y(\gamma)y_0 = \rho_Y(\gamma)\mathcal{A}\,y_0$ holds for $y_0 \in Y_j$ and $\gamma\colon j \to k$. We want to prove that this implies $T(t)\rho_Y(\gamma)y_0 = \rho_Y(\gamma)T(t)y_0$. To this end, we define

$$g(t) := T(t)\rho_Y(\gamma)y_0 - \rho_Y(\gamma)T(t)y_0. \tag{3.55}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) := \frac{\mathrm{d}}{\mathrm{d}t}T(t)\rho_Y(\gamma)y_0 - \rho_Y(\gamma)\frac{\mathrm{d}}{\mathrm{d}t}T(t)y_0, \tag{3.56}$$

and it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) = T(t)\mathcal{A}\,\rho_Y(\gamma)y_0 - \rho_Y(\gamma)T(t)\mathcal{A}\,y_0 \tag{3.57}$$
$$= T(t)\rho_Y(\gamma)\mathcal{A}\,y_0 - \rho_Y(\gamma)T(t)\mathcal{A}\,y_0 \tag{3.58}$$
$$= 0. \tag{3.59}$$

Since $g(0) = 0$, the claim follows.

In the following example, we apply the symmetries from a generator that we already know (see Example 3.1.3) to its semigroup.

Example 3.4.4 (Odd scalar-reaction-diffusion equations, continued from Example 3.1.3)**.** We consider the scalar reaction-diffusion equation of the form

$$u_t = u_{xx} + f(u) \tag{3.60}$$

$u \in \mathbb{R}$, $x \in [0, \pi]$, $t \geq 0$, with Dirichlet boundary conditions $u(t, 0) = u(t, \pi) = 0$ for all $t \geq 0$. We assume that f is an odd polynomial.

This defines a global C^0-semiflow, for example on the Sobolev space $Y = H_0^2([0, \pi], \mathbb{R})$; see [30]:

$$(t, u_0) \mapsto u(t) := T(t)u_0 \in Y. \tag{3.61}$$

The symmetries of the Chafee–Infante equation are acting by convolution, where $h\colon \mathbb{R} \to \mathbb{R}$ has finite L^1-norm:

$$(h * u)\,(x) := \frac{1}{\pi}\int_0^\pi h(x - y)u(y)\mathrm{d}y \tag{3.62}$$

By the theory above, we can recycle the vertex spaces and groupoid symmetries from Example 3.1.3, so we will not list them here.

Indeed, all equilibria of e.g., the Chafee–Infante equation; see [11], can be associated to one of the vertex spaces $Y_{j,2} \subseteq Y_{j,1}$ or Y_0. Moreover, note the somewhat hidden but well-known property that for every solution u^*, $-u^*$ is also a solution.

4. Steady-state bifurcation in equivaroid systems

So far, we have explored general properties of equivaroid systems. Now we want to prove the existence of equilibrium solutions with a given vertex isotropy group, which leads us to the generalization of the equivariant branching lemma to equivaroid systems. We first generalize equivariant Lyapunov–Schmidt reduction to equivaroid systems. This requires particular care, as the resulting bifurcation equation is not $(\Gamma \rightrightarrows I)$-equivaroid, but $(\Gamma \rightrightarrows I)$-subequivaroid with respect to the kernel and the range of the linearization. We then apply the theorem by Crandall and Rabinowitz on bifurcation from simple eigenvalues to the vertex space with said isotropy. In equivaroid systems, additional solutions might bifurcate within a vertex space if more than one eigenvalue crosses zero. We therefore additionally present the *iterated equivaroid branching lemma*, which provides the existence of solutions via an iterative process.

This chapter is organized as follows: In Section 4.1 we extend the classical equivariant Lyapunov–Schmidt reduction to equivaroid systems. In Section 4.2 we formulate and prove the *equivaroid branching lemma.* In addition to the steady-state solutions found with the equivaroid branching lemma, there might exist additional nontrivial branches of solutions, we will present a first result in this direction in Section 4.3. As always, the theory is accompanied by illustrative examples.

4.1. Equivaroid Lyapunov–Schmidt reduction

The Lyapunov–Schmidt reduction, which goes back to Lyapunov [42] and Schmidt [55], is a variant of the implicit function theorem: It decomposes and reduces a possibly infinite-dimensional equation to a finite-dimensional part which contains the solutions of the equation.

Let us start with the usual setting; see e.g., [12]. Throughout, let X, Y, Λ be real Banach spaces and let

$$\mathcal{F} \colon \Lambda \times X \to Y \tag{4.1}$$

be a C^k-map, $k \geq 1$. Suppose that $\mathcal{F}(0,0) = 0$. We are interested in bifurcation theory and hence we want to find solutions of

$$\mathcal{F}(\lambda, x) = 0 \tag{4.2}$$

near the origin $\lambda = 0$, $x = 0$. Here $\lambda \in \Lambda$ is the *bifurcation parameter*.

Let $\mathcal{L} := \mathrm{D}_x \mathcal{F}(0,0)$. Then there are two possibilities: Either $\mathcal{L}$ is invertible, in which case we can simply apply the implicit function theorem to find the solutions in a neighbourhood of the origin [66]. Or $\mathcal{L}$ is not invertible, in which case we will usually find a bifurcation. To this end, the standard Lyapunov–Schmidt reduction decomposes equation (4.2) into a possibly infinite-dimensional part where the implicit function theorem holds and into another, finite-dimensional part, which we call the *bifurcation equation*.

We now additionally assume that $\mathcal{F}$ is $(\Gamma \rightrightarrows I)$-equivaroid, as a generalization to the equivariant case as treated in [12]. The aim of this section is to extend the Lyapunov–Schmidt reduction to the equivaroid case. In particular, we want to find out how the groupoid symmetries are inherited by the bifurcation equation.

By Lemma 3.3.1, the linearization $\mathcal{L} := \mathrm{D}_x \mathcal{F}(0,0)$ at the origin is at least $(\Gamma \rightrightarrows I)$-equivaroid. Hence its kernel and range are $(\Gamma \rightrightarrows I)$-invaroid subspaces of X and Y, respectively; see Lemma 3.3.2. These facts allow us to construct an equivaroid Lyapunov–Schmidt reduction as follows.

Suppose that $\mathcal{L}$ is a Fredholm operator, i.e., $\mathcal{L}$ is closed, and both $\ker \mathcal{L}$ and the complement of $\operatorname{range} \mathcal{L}$ are finite-dimensional [66]. In parallel to the standard Lyapunov–Schmidt reduction we define continuous projections

$$P \colon X \to \ker \mathcal{L} =: U, \tag{4.3}$$
$$(\mathrm{Id} - P) \colon X \to (\ker \mathcal{L})^c =: V, \tag{4.4}$$
$$Q \colon Y \to \operatorname{range} \mathcal{L} =: W, \tag{4.5}$$
$$(\mathrm{Id} - Q) \colon Y \to (\operatorname{range} \mathcal{L})^c =: T, \tag{4.6}$$

and use the notation

$$u := Px, \; v := (\mathrm{Id} - P)x, \; w := Qy, \; t := (\mathrm{Id} - Q)y. \tag{4.7}$$

For now, let us assume that it is possible to choose the projections P and Q such that they induce $(\Gamma \rightrightarrows I)$-subequivaroid equations on the subspaces U and W. In other words, we request

$$\rho_U(\gamma)Px = P\rho_X x, \quad \rho_W(\gamma)Qy = Q\rho_Y(\gamma)y, \tag{4.8}$$

i.e., that the projections commute with the actions of the groupoid. Such a choice of projections is often possible, we will discuss the details at the end of this section.

Following [12], the original equation $\mathcal{F}(\lambda, x) = 0$ is equivalent to the system

$$Q\mathcal{F}(\lambda, u + v) = 0 \tag{4.9}$$
$$(\mathrm{Id} - Q)\mathcal{F}(\lambda, u + v) = 0. \tag{4.10}$$

To see this, note that $u + v = Px + (1 - P)x = x$ and $Q\mathcal{F}(\lambda, u + v) + (\mathrm{Id} - Q)\mathcal{F}(\lambda, u + v) = \mathcal{F}(\lambda, u + v)$.

Next, we write

$$\mathcal{F}(\lambda, x) = \mathcal{L}x + \mathcal{N}(\lambda, x), \tag{4.11}$$

where $\mathcal{N}$ denotes the nonlinear part of $\mathcal{F}$ with $D_x\mathcal{N}(0, 0) = 0$, which implies that $\mathcal{N}$ is $(\Gamma \rightrightarrows I)$-equivaroid.

We treat the invertible, possibly infinite-dimensional part first: Since $u = Px$ is in the kernel of $\mathcal{L}$, we can rewrite equation (4.9) as

$$Q\mathcal{L}v + Q\mathcal{N}(\lambda, u + v) = 0. \tag{4.12}$$

The operator $Q\mathcal{L}$ is invertible in $\operatorname{range}\mathcal{L}$ and its inverse is bounded. Therefore, the implicit function theorem can be applied at $u = v = 0$, $\lambda = 0$, and we obtain that for every (x, λ) in a neighborhood of $(0, 0)$ in $\Lambda \times X$, equation (4.9) has a unique solution $v(\lambda, u)$ such that $v(0, 0) = 0$ and v is also a C^k-function.

The important point is that $Q\mathcal{L}v + Q\mathcal{N}(\lambda, u + v)\colon \Lambda \times U \times V \to W$ is subequivaroid with respect to the spaces (U, V) and W. This implies that $\rho_V(\gamma)v(\lambda, u)$ is a solution of equation (4.12) if and only if $v(\lambda, \rho_U(\gamma)u)$ is a solution of equation (4.12) with $(u, v) \in (U_j, V_j) = X_j$ a vertex space:

$$\begin{aligned}
0 &= Q\mathcal{L}\rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u) \\
&\quad + Q\mathcal{N}(\lambda, u + \rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u)) \\
&= \rho_W(\gamma)Q\mathcal{L}\rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u) \\
&\quad + \rho_W(\gamma)Q\mathcal{N}(\lambda, u + \rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u)) \\
&= Q\rho_W(\gamma)\mathcal{L}\rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u) \\
&\quad + Q\rho_Y(\gamma)\mathcal{N}(\lambda, u + \rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u)) \\
&= Q\mathcal{L}\rho_V(\gamma)\rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u) \\
&\quad + Q\mathcal{N}(\lambda, \rho_U(\gamma)u + \rho_V(\gamma)\rho_V(\gamma)^{-1}v(\lambda, \rho_U(\gamma)u)) \\
&= Q\mathcal{L}v(\lambda, \rho_U(\gamma)u) + Q\mathcal{N}(\lambda, \rho_U(\gamma)u + v(\lambda, \rho_U(\gamma)u)).
\end{aligned} \tag{4.13}$$

By uniqueness of the solution from the implicit function theorem, we can thereby conclude that $\rho_V(\gamma)v(\lambda, u) = v(\lambda, \rho_U(\gamma)u)$, i.e., $v : U \to V$ is $(\Gamma \rightrightarrows I)$-(sub)equivaroid with respect to the spaces U and V.

It remains to study the second equation (4.10), which has a new form by the implicit function theorem, in the neighbourhood of $u = v = 0$, $\lambda = 0$:

$$(1 - Q)\mathcal{N}(\lambda, u + v(\lambda, u)) := g(\lambda, u) = 0, \tag{4.14}$$

with $g\colon \Lambda \times \ker \mathcal{L} = U \to (\operatorname{range} \mathcal{L})^c = T$; note that this is indeed a finite-dimensional problem. We need to show that equation (4.14) is $(\Gamma \rightrightarrows I)$-subequivaroid with respect to the spaces $U = \ker \mathcal{L}$, $T = (\operatorname{range} \mathcal{L})^c$.

Suppose that (u_j, v_j) is an element of some vertex space X_j and let $\gamma\colon j \to k$. Then subequivariance with respect to U and T follows from

$$\begin{aligned} \rho_T(\gamma) g(\lambda, u_j) &= \rho_T(\gamma)\big((\operatorname{Id} - Q)\mathcal{N}(\lambda, u_j + v_j(\lambda, u_j))\big) \\ &= (\operatorname{Id} - Q)\rho_Y(\gamma)\mathcal{N}(\lambda, u_j + v_j(\lambda, u_j)) \\ &= (\operatorname{Id} - Q)\mathcal{N}(\lambda, \rho_U(\gamma)u_j + \rho_V(\gamma)v_j(\lambda, u_j)) \\ &= (\operatorname{Id} - Q)\mathcal{N}(\lambda, \rho_U(\gamma)u_j + v_j(\lambda, \rho_U(\gamma)u_j)) \\ &= g(\lambda, \rho_U(\gamma)u_j). \end{aligned} \tag{4.15}$$

Note that, in contrast to the equivariant case, the morphism $\gamma\colon j \to k$ takes us out of the space X_j and into the space X_k.

In summary, we have proven the following theorem, as a direct generalization of the equivariant Lyapunov–Schmidt reduction [12] which now also allows for the equivaroid case.

Theorem 4.1.1 (Equivaroid Lyapunov–Schmidt reduction). *Let X, Y, Λ be real Banach spaces and let*

$$\mathcal{F}\colon \Lambda \times X \to Y \tag{4.16}$$

be a C^k-map, $k \geq 1$. Suppose that $\mathcal{F}(0,0) = 0$ and that the linearization $\mathcal{L} := \mathrm{D}_x \mathcal{F}(0,0)$ is a Fredholm operator. Moreover, suppose that $\mathcal{F}$ is $(\Gamma \rightrightarrows I)$-equivaroid, and that the projections $P\colon X \to \ker \mathcal{L} =: U$ and $Q\colon Y \to \operatorname{range} \mathcal{L} =: W$ fulfill

$$\rho_U(\gamma)Px = P\rho_X x, \quad \rho_W(\gamma)Qy = Q\rho_Y(\gamma)y. \tag{4.17}$$

Then the following holds:

i) The subspaces $\ker \mathcal{L}$ and $\operatorname{range} \mathcal{L}$ are $(\Gamma \rightrightarrows I)$-invaroid.

ii) the local solutions of the equation $\mathcal{F}(x, \lambda) = 0$ have the form $x(\lambda) = u(\lambda) + v(\lambda, u(\lambda))$. Here, $v\colon \Lambda \times U \to V$ is $(\Gamma \rightrightarrows I)$-subequivaroid with respect to the spaces U and V, i.e., $\rho_V(\gamma)v(\lambda, u) = v(\lambda, \rho_U(\gamma)u)$.

iii) the bifurcation equation $(1-Q)\mathcal{N}(\lambda, u+v(\lambda,u)) = 0$ *is* $(\Gamma \rightrightarrows I)$-*subequivaroid with respect to* U *and* T.

In words, the *equivaroid Lyapunov–Schmidt reduction* states that such a reduction can be performed in such a way that the reduced equation is subequivaroid. We illustrate this with the following example, which is modified from [12] to the equivaroid situation. Note that we consider the same equation but are able to describe a much finer structure in the framework of groupoid symmetries.

Example 4.1.2 (The pendulum equation)**.** We consider

$$\mathcal{F}(x) = \ddot{x} + \mu \sin(x) = 0, \tag{4.18}$$

with boundary conditions

$$\dot{x}(0) = \dot{x}(1) = 0, \tag{4.19}$$

and bifurcation parameter μ, which we redefine with $\lambda = \mu - \mu_k \in \Lambda = \mathbb{R}$, where $\mu_k = k^2\pi^2$ are the bifurcation points, $k \in \mathbb{N}$. We have

$$X = \{x \in C^2[0,1] \mid \dot{x}(0) = \dot{x}(1) = 0\}, \tag{4.20}$$

with the usual additive sup-norm on $x, \dot{x}$ and $\ddot{x}$, and $Y = C^0[0,1]$, with the standard sup-norm. Using the identities

$$\sin\left(\sum_{i=1}^{\infty} \theta_i\right) = \sum_{\text{odd } k \geq 1} (-1)^{\frac{k-1}{2}} \sum_{\substack{A \subseteq \mathbb{N} \\ |A| = k}} \left(\prod_{i \in A} \sin\theta_i \prod_{i \notin A} \cos\theta_i\right), \tag{4.21}$$

$$\sin(a_m \cos m\pi t) = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(a_m) \cos\left((2k+1)m\pi t\right), \tag{4.22}$$

$$\cos(a_m \cos m\pi t) = J_0(a_m) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(a_m) \cos(2km\pi t), \tag{4.23}$$

Vertex spaces $X_0 = Y_0, X_{j,i} = Y_{j,i}, j \in \mathbb{N}, i = 1, 2$
$X_0 \;\; = \left\{ x(t) = \sum\limits_{k=0}^{\infty} a_k \cos(k\pi t) \;\middle|\; \forall\, k \geq 1 : a_k = 0 \right\}$
$X_{j,1} = \left\{ x(t) = \sum\limits_{k=0}^{\infty} a_k \cos(k\pi t) \;\middle|\; \forall\, k \neq nj,\ n \in \mathbb{N}_0 : a_k = 0 \right\}$
$X_{j,2} = \left\{ x(t) = \sum\limits_{k=0}^{\infty} a_k \cos(k\pi t) \;\middle|\; \forall\, k \neq nj,\ n \in \mathbb{N} \text{ odd} : a_k = 0 \right\}$

Table 4.1.: The vertex spaces of the pendulum equation (4.18) (convergence conditions are ignored for simplicity).

where

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{z}{2}\right)^{2m+\alpha} \tag{4.24}$$

denotes the Bessel functions, as well as

$$\prod_{k=1}^{n} \cos\theta_k = \frac{1}{2^n} \sum_{e \in S} \cos(e_1\theta_1 + \cdots + e_n\theta_n) \tag{4.25}$$

where $e = (e_1, \ldots, e_n) \in S = \{1, -1\}^n$, all from [1], we find the vertex spaces of $\mathcal{F}\colon X \to Y$ as in Table 4.1. The corresponding vertex symmetry groups are listed in Table 4.2. The groupoid symmetries act by convolution, where $h\colon \mathbb{R} \to \mathbb{R}$ has finite L^1-norm:

$$(h * u)(t) := \int_0^1 h(t-s)u(s)\,\mathrm{d}s. \tag{4.26}$$

We write

$$\mathcal{F}(x, \lambda) = \ddot{x} + \mu \sin(x) = \mathcal{L}x + \mathcal{N}(\lambda, x), \tag{4.27}$$

where we define

$$\mathcal{L}_k x := \ddot{x} + \mu_k x \tag{4.28}$$

$$\mathcal{N}_k(\lambda, x) = \mu_k(\sin x - x) + \lambda \sin(x), \tag{4.29}$$

Vertex groups $H_0, H_{j,i}, j \in \mathbb{N}, i = 1, 2$
$H_0 \;\; = \left\{ h(t) = \pm \sum\limits_{k=0}^{\infty} b_k \cos(k\pi t) \,\middle|\, b_0 = 1 \right\}$
$H_{j,1} = \left\{ h(t) = \pm \sum\limits_{k=0}^{\infty} b_k \cos(k\pi t) \,\middle|\, \forall\, k = nj,\, n \in \mathbb{N}_0 : b_k = 1 \right\}$
$H_{j,2} = \left\{ h(t) = \pm \sum\limits_{k=0}^{\infty} b_k \cos(k\pi t) \,\middle|\, \forall\, k = nj,\; n \in \mathbb{N} \text{ odd} : b_k = 1 \right\}$

Table 4.2.: The vertex symmetry groups of the pendulum equation (4.18) (conditions on convergence and invertibility are ignored for simplicity).

for the linear and the nonlinear part of $\mathcal{F}$, with $\lambda = \mu - \mu_k$. At the bifurcation points $\mu_k = k^2\pi^2$, a nontrivial kernel exists:

$$\begin{aligned} \ker \mathcal{L}_k &= \{x(t) = a_k \cos(k\pi t) \mid a_k \in \mathbb{R}\} =: U, \\ \operatorname{range} \mathcal{L}_k &= Y \setminus \ker \mathcal{L}_k =: W. \end{aligned} \tag{4.30}$$

Hence, $\mathcal{L}_k$ is indeed a Fredholm operator at the bifurcation point μ_k. Next, we note that the subspace $\mathcal{L}_k$ is a subspace of $X_{k,1}$, $X_{k,2}$ and of no other vertex space. We define $U_{k,1} := PX_{k,1}$ and $U_{k,2} := PX_{k,2}$, as well as $V_{k,1} := (1-P)X_{k,1}$ and $V_{k,2} := (1-P)X_{k,2}$.

As the next step, we set $x(t) = a_k \cos(k\pi t) + v(t)$ and obtain from $\mathcal{L}_k(v) = -\mathcal{N}_k(\lambda, a_k \cos(k\pi t) + v)$:

$$\mathcal{L}_k v = -\lambda \sin(a_k \cos(k\pi t) + v) - \mu_k(\sin(a_k \cos(k\pi t) + v) - a_k \cos(k\pi t) - v). \tag{4.31}$$

Projecting this last equation to the space $W = \operatorname{range} \mathcal{L}_k$, we find that the **local solutions are subequivaroid** with respect to the spaces U and V as follows: Indeed $\rho_V(h)v(\lambda, u) \in V_{k,1}$ or $V_{k,2}$ with $u \in U_{k,1}$ or $U_{k,2}$ is a solution of equation (4.31) if and only if $v(\lambda, \rho_U(h)u) \in V_{k,1}$ or $V_{k,2}$ is a solution of equation (4.31) with h as in Table 4.2. Notice that this property does not hold for the entire spaces X and Y, but only on $(U_{k,1}, V_{k,1})$ and $(U_{k,2}, V_{k,2})$.

As the last step, we check that **the bifurcation equation is subequivaroid**: the bifurcation equation $(1-Q)\mathcal{N}_k(\lambda, u+v(\lambda,u)) = 0$ is $(\Gamma \rightrightarrows I)$-subequivaroid with respect to the spaces U and T. This is indeed the case, because the bifurcation equation is indeed odd with respect to the coefficient a_k (remember that v is also subequivaroid).

$$\begin{aligned}
&(1-Q)\mathcal{N}_k(\lambda, u+v(\lambda,u)) \\
=& (1-Q)\Big(\lambda \sin(a_k \cos(k\pi t) + v) \\
&+ \mu_k(\sin(a_k \cos(k\pi t) + v) + a_k \cos(k\pi t) + v)\Big).
\end{aligned} \tag{4.32}$$

As a conclusion of this example, we have seen how the groupoid symmetries are inherited by the Lyapunov–Schmidt reduction. It is indeed a much finer structure than shown in the previous treatment of this specific example in [12], where the only studied property was even/oddness of solutions due to the $\mathbb{Z}_2$-group equivariance.

We conclude this section by a discussion on a situation where the projections P and Q can indeed be chosen to commute with the action of the groupoid. In the following, suppose that the groupoid is *compact* [52] and possesses an *isometric representation*. In this context we adapt the arguments from [63] from groups to groupoids.

More precisely, we suppose that X, Y are real Banach spaces and that there exist continuous, symmetric and positive definite bilinear forms

$$\langle \cdot,\cdot \rangle_X \colon X \times X \to X, \quad \langle \cdot,\cdot \rangle_Y \colon Y \times Y \to Y \tag{4.33}$$

on the spaces X and Y, respectively. We require the representations ρ_X and ρ_Y to fulfill

$$\begin{aligned}
\langle \rho_X(\gamma)x_1, \rho_X(\gamma)x_2 \rangle_X &= \langle x_1, x_2 \rangle_X, \quad \text{for all } \gamma \colon j \to k,\ x_1, x_2 \in X_j \\
\langle \rho_Y(\gamma)y_1, \rho_Y(\gamma)y_2 \rangle_Y &= \langle y_1, y_2 \rangle_Y, \quad \text{for all } \gamma \colon j \to k,\ y_1, y_2 \in Y_j.
\end{aligned} \tag{4.34}$$

Recall once more that the spaces $U := \ker \mathcal{L} \subseteq X$ and $W := \operatorname{range} \mathcal{L} \subseteq Y$ as well as their complements are $(\Gamma \rightrightarrows I)$-invaroid. Moreover, since $\mathcal{L}$ is a Fredholm operator, U and T are finite-dimensional. Hence there exist bases $\{e_i^U \mid i = 1, \dots, n_U\}$ and $\{e_i^T \mid i = 1, \dots, n_T\}$ which are orthonormal with respect to the bilinear forms above. Here $n_U := \dim U$, $n_T := \dim T$. Then we define the specific projection

$$\begin{aligned} P\colon X &\to U \\ x &\mapsto \sum_{i=1}^{n_U} \langle x, e_i^U \rangle_X, \end{aligned} \tag{4.35}$$

as well as the specific projection

$$\begin{aligned} (\operatorname{Id} - Q)\colon Y &\to T \\ y &\mapsto \sum_{i=1}^{n_T} \langle x, e_i^T \rangle_Y. \end{aligned} \tag{4.36}$$

We then define

$$\begin{aligned} U^\perp &:= \{x \in X \mid \langle x, u \rangle_X = 0 \ \text{ for all } u \in U\}, \\ T^\perp &:= \{y \in Y \mid \langle y, t \rangle_Y = 0 \ \text{ for all } t \in T\}, \end{aligned} \tag{4.37}$$

and note that $Px = 0$ is equivalent to $x \in U^\perp$, as well as $(\operatorname{Id} - Q)y = 0$ is equivalent to $y \in T^\perp$.

Now U is $(\Gamma \rightrightarrows I)$-invaroid, that is, for all $j \in I$ such that $U \cap X_j \neq \emptyset$ and for all morphisms $\gamma\colon j \to k$,

$$\rho_X(\gamma)(U \cap X_j) \subseteq (U \cap X_k). \tag{4.38}$$

The same goes for the space T: For all $j \in I$ such that $T \cap X_j \neq \emptyset$ and for all morphisms $\gamma\colon j \to k$,

$$\rho_X(\gamma)(T \cap Y_j) \subseteq (T \cap Y_k). \tag{4.39}$$

Since both U and T are finite-dimensional and $\gamma\colon j \to k$ acts as an isomorphism, we conclude that equality holds in both (4.38) and (4.39).

Last, we show that also the spaces $U^\perp$ and $T^\perp$ are invaroid under the same groupoid actions as the spaces U and T (notice that this does not mean that $U^\perp$ and $T^\perp$ are $(\Gamma \rightrightarrows I)$-invaroid!): To see this, note that for all $\gamma\colon j \to k$, $u \in U \cap X_k$, $x \in U^\perp \cap X_j$ we have

$$\langle u, \rho_X(\gamma)x\rangle_X = \langle \rho^{-1}(\gamma)_X u, x\rangle_X = 0, \tag{4.40}$$

Analoguosly, we obtain for all $\gamma\colon j \to k$, $t \in T \cap Y_k$, $y \in T^\perp \cap Y_j$:

$$\langle t, \rho_Y(\gamma)y\rangle_Y = \langle \rho^{-1}(\gamma)_Y t, y\rangle_Y = 0. \tag{4.41}$$

We conclude

$$\rho_X(\gamma)x = \rho_X(\gamma)Px + \rho_X(\gamma)(\mathrm{Id} - P)x. \tag{4.42}$$

Applying the projection P to both sides, we obtain

$$P\rho_X(\gamma)x = \rho_X(\gamma)Px = \rho_U(\gamma)u, \tag{4.43}$$

as desired. For the projection $(\mathrm{Id} - Q)$, on the other hand, we conclude

$$\rho_Y(\gamma)x = \rho_Y(\gamma)(\mathrm{Id} - Q)x + \rho_X(\gamma)Qy, \tag{4.44}$$

implying

$$(\mathrm{Id} - Q)\rho_Y(\gamma)y = \rho_Y(\gamma)(\mathrm{Id} - Q)y = \rho_T(\gamma)t. \tag{4.45}$$

4.2. The equivaroid branching lemma

In this section we generalize the equivariant branching lemma by Vanderbauwhede [63] and Cicogna [13] to equivaroid equations. This is the first existence result of patterns generated by groupoid symmetries.

The setting of the equivaroid bifurcation theory is as follows: We consider C^k-maps, $k \geq 2$, $\mathcal{F}\colon \mathbb{R} \times X \to Y$, where X and Y are Banach spaces. Our goal is to solve

$$0 = \mathcal{F}(\lambda, x), \tag{4.46}$$

where $\mathcal{F}$ is $(\Gamma \rightrightarrows I)$-equivaroid. Again, we call $\lambda \in \mathbb{R}$ the bifurcation parameter. Suppose that there is a trivial solution $x = 0$ for all $\lambda \in \mathbb{R}$:

$$0 = \mathcal{F}(\lambda, 0). \tag{4.47}$$

We are interested in nontrivial solutions $x(\lambda)$ and their groupoid symmetries. Next to the equivaroid Lyapunov–Schmidt reduction, our main tool is the following theorem by Crandall & Rabinowitz.

Theorem 4.2.1 (Bifurcation from simple eigenvalues, cited from [15] without proof)**.** *Consider* $0 = \mathcal{F}(\lambda, x)$*,* $x \in X$*,* $\mathcal{F}\colon \mathbb{R} \times X \to Y$*,* X, Y *Banach spaces,* $\mathcal{F} \in C^k$*,* $k \geq 2$*. In addition, assume that* $\mathcal{L} := \mathrm{D}_x \mathcal{F}(0,0)$ *is a Fredholm operator of index* 0*,* $\mathcal{F}(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$ *and that*

i) $\dim \ker \mathcal{L} = 1$*;*

ii) $\mathrm{D}_\lambda \mathrm{D}_x \mathcal{F}(0,0)|_{\ker \mathcal{L}} \notin \operatorname{range} \mathcal{L}$ *("eigenvalue crossing condition").*

Then there exists a unique C^{k-1}*-branch* $(\lambda(s), x(s))$ *of nontrivial zeros of* $\mathcal{F}$ *in the neighbourhood of* $\lambda = 0$*,* $x = 0$*, and moreover,*

$$0 \neq \dot{x}(0) \in \ker \mathrm{D}_x \mathcal{F}. \tag{4.48}$$

While the condition $\dim \ker \mathcal{L} = 1$ holds generically, eigenvalues of equivaroid operators often have higher multiplicities; see some of the examples below. This is similar to the standard equivariant setting.

The main idea is to restrict the equation (4.46) to a vertex space X_j, which is flow-invariant by construction (either before or after reducing via Lyapunov–Schmidt). The theorem of Crandall and Rabinowitz then applies to the restricted equation, which implies that we can prove bifurcation from the trivial equilibrium into the vertex space X_j. Additionally, we then obtain the bifurcation of a conjugated solution in any vertex space X_k which is conjugated to X_j.

Theorem 4.2.2 (Equivaroid branching lemma)**.** *Consider* $0 = \mathcal{F}(\lambda, x)$*,* $x \in X$*,* $\mathcal{F}\colon \mathbb{R} \times X \to Y$*,* X, Y *Banach spaces,* $\mathcal{F} \in C^k$*,* $k \geq 2$ *and let* $\mathcal{F}$ *be* $(\Gamma \rightrightarrows I)$*-equivaroid for all* $\lambda \in \mathbb{R}$*.*

Let $X_j \subseteq X$ be a vertex space with vertex isotropy group K_j, such that $\mathcal{F}_j := \mathcal{F}|_{X_j} : \mathbb{R} \times X_j \to Y_j$, and suppose that $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

Assume that at $\lambda = 0$ the following conditions are fulfilled:

i) $\mathcal{L} := \mathrm{D}_x \mathcal{F}_j(0,0) \colon X \to Y$ *is a Fredholm operator of index 0;*

ii) $\dim \ker \mathcal{L}|_{X_j} = 1$*;*

iii) $\mathrm{D}_\lambda \mathrm{D}_x \mathcal{F}_j(0,0)|_{\ker \mathcal{L}} \notin \operatorname{range} \mathcal{L}|_{X_j}$*.*

Then the zero set of $\mathcal{F}$ near $(0,0)$ consists of the trivial branch and a nontrivial C^{k-1}-branch $(\lambda(s), x(s))$, through $\lambda(0) = 0$, $x(0) = 0$ and $\dot{x}(0) \in \ker \mathcal{L}|_{X_j}$. The branch is unique up to reparametrization.

Moreover, $x(s) \in X_j$, i.e., $x(s)$ has isotropy K_j.

Proof. Consider the restricted map $\mathcal{F}_j \colon \mathbb{R} \times X_j \to Y_j$. The conditions of the theorem by Crandall & Rabinowitz are fulfilled. The isotropy is clear since we have reduced to the vertex spaces X_j, Y_j. Alternatively, if suitable projections P and Q exist, we can perform the equivaroid Lyapunov–Schmidt reduction first. The key point is to remember that by Lemma 3.3.1, the linearization $\mathcal{L}$ at the origin is at least $(\Gamma \rightrightarrows I)$-equivaroid. Hence its kernel and range are $(\Gamma \rightrightarrows I)$-invaroid subspaces of X and Y, respectively; see Lemma 3.3.2. The Lyapunov–Schmidt reduction then gives us a subequivaroid bifurcation equations, on which we can restrict to the vertex space with index j, in this case U_j. The full isotropy of the solution $x(\lambda) = u(\lambda) + v(\lambda, u(\lambda))$ holds because v is subequivaroid. □

In the following, we will consider an example where standard and equivariant steady-state bifurcations fail, but the equivaroid branching lemma gives precise information on the bifurcating branches and their isotropy.

Example 4.2.3 (Two coupled cells II, continued from Example 2.1.9)**.** We consider again the dynamical system

$$\begin{aligned} \dot{x}_1 &= f(\lambda, x_1) + x_1 x_2 \\ \dot{x}_2 &= f(\lambda, x_2) + 2x_1 x_2, \end{aligned} \tag{4.49}$$

where $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(\lambda, 0) = 0$ for all $\lambda = 0$. We assume $\mathcal{L}_f := \mathrm{D}_x\, f(\lambda, 0) = \lambda$. Then the linearization at the trivial equilibrium $x_1 = x_2 = 0$,

$$\mathcal{L} := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{4.50}$$

has a double eigenvalue zero for $\lambda = 0$. Consequently, the standard bifurcation from simple eigenvalues fails. In addition, also the equivariant bifurcation theory fails because system (4.49) is not equivariant with respect to any nontrivial group.

However, we know that the system (4.49) is equivaroid, and we have calculated the vertex spaces as well as vertex symmetry groups in Example 2.1.9. The relevant vertex spaces are $X_2 = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 = 0\}$ and $X_3 = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 = 0\}$.

The important point is that the kernel of $\mathcal{L}|_{X_j}$, $j = 2, 3$, is one-dimensional as we restrict it to the vertex spaces X_2, X_3. This implies equivaroid steady-state bifurcation with the isotropies K_2 and K_3, respectively. Indeed, the system

$$\dot{x} = f(\lambda, x) \tag{4.51}$$

(as it looks both within X_2 and X_3) shows a bifurcation from a simple eigenvalue zero at $\lambda = 0$.

Moreover, the vertex spaces X_2 and X_3 are conjugated, and it therefore suffices to prove bifurcation in one of these spaces to find it in the other space as well.

The following example is $\mathbb{Z}_2$-equivariant in the standard group sense, but, as we remember from our previous discussion, it also possesses a symmetry groupoid which is much larger than the symmetry group $\mathbb{Z}_2$. This gives us an opportunity to compare the conclusions from the equivariant and the equivaroid branching lemmata.

Example 4.2.4 (Three coupled cells II, continued from Example 2.4.7)**.** We reconsider

$$\begin{aligned}\dot{x}_1 &= f(\lambda, x_1) + x_2 \\ \dot{x}_2 &= f(\lambda, x_2) + x_2 \\ \dot{x}_3 &= f(\lambda, x_3) + x_2,\end{aligned} \tag{4.52}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. We assume $\mathcal{L}_f := \mathrm{D}_x\, f(\lambda, 0) = \lambda$. The linearization at the trivial equilibrium $x_1 = x_2 = x_3$,

$$\mathcal{L} := \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda + 1 & 0 \\ 0 & 1 & \lambda \end{pmatrix} \tag{4.53}$$

has a double eigenvalue zero for $\lambda = 0$ and thus standard bifurcation from simple eigenvalues fails. There is also a bifurcation from a one-dimensional kernel at $\lambda = -1$ for which the Theorem 4.2.1 provides the existence of nontrivial solutions but no information on their isotropy.

What does the equivariant bifurcation theory tell us here? The system (4.52) is $\mathbb{Z}_2$-equivariant, where $\mathbb{Z}_2$ acts by permutation of the variables x_1 and x_3. The corresponding fixed-point subspace is given by

$$\Sigma = \{(x_1, x_2, x_3) \mid x_1 = x_3\}. \tag{4.54}$$

The kernel of $\mathcal{L}$ at $\lambda = 0$ is given by $\ker \mathcal{L} = \{(x_1, x_2, x_3) \mid x_2 = 0\}$. Hence $\dim \ker \mathcal{L}|_\Sigma = \dim\{(x_1, x_2, x_3) \mid x_2 = 0,\, x_1 = x_3\} = 1$ and the conditions for the equivariant branching lemma are fulfilled. We conclude the existence of nontrivial solutions within the fixed-point space Σ.

At $\lambda = -1$, the kernel is given by $\ker \mathcal{L} = \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$ and hence one-dimensional, as is its restriction to the fixed-point space Σ. Once more, we conclude the existence of nontrivial solutions within Σ.

Now let us turn to the equivaroid analysis. For the convenience of the reader, the vertex spaces are given in Tables 4.3 and 4.4. Conjugating symmetries exist between the spaces X_2 and X_4 as well as the vertex spaces X_7 and X_8; see Table 4.5.

j	Vertex spaces $X_j \subset \mathbb{R}^3$	Vertex symmetry groups H_j
1	$\mathbb{R}^3$	$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = \mathbb{Z}_2$
2	$\{(x_1, x_2, x_3) \,\vert\, x_1 = x_2\}$	$\left\{ \begin{pmatrix} a & 1-a & 0 \\ b & 1-b & 0 \\ c & -c & 1 \end{pmatrix} \middle\vert\, a \neq b \right\}$
3	$\{(x_1, x_2, x_3) \,\vert\, x_1 = x_3\}$	$\left\{ \begin{pmatrix} a & 0 & 1-a \\ b & 1 & -b \\ c & 0 & 1-c \end{pmatrix} \middle\vert\, a \neq c \right\}$
4	$\{(x_1, x_2, x_3) \,\vert\, x_2 = x_3\}$	$\left\{ \begin{pmatrix} 1 & a & -a \\ 0 & b & 1-b \\ 0 & c & 1-c \end{pmatrix} \middle\vert\, b \neq c \right\}$
5	$\{(x_1, x_2, x_3) \mid x_2 = 0\}$	$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 1 \end{pmatrix} \middle\vert\, b \neq 0 \right\}$
		$\cup \left\{ \begin{pmatrix} 0 & a & 1 \\ 0 & b & 0 \\ 1 & c & 0 \end{pmatrix} \middle\vert\, b \neq 0 \right\}$
6	$\{(x_1, x_2, x_3) \,\vert\, x_1 = x_2 = x_3\}$	$\left\{ \begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ e & f & 1 \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$

Table 4.3.: Vertex spaces and vertex symmetry group of the system (4.52), part 1. Here $a, b, c, d, e, f \in \mathbb{R}$.

j	Vertex spaces $X_j \subset \mathbb{R}^3$	Vertex symmetry groups H_j
7	$\{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$	$\left\{ \begin{pmatrix} a & 0 & b \\ c & 1 & d \\ e & 0 & f \end{pmatrix} \middle\vert\ \det \neq 0 \right\}$
8	$\{(x_1, x_2, x_3) \mid x_2 = x_3 = 0\}$	$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix} \middle\vert\ \det \neq 0 \right\}$
9	$\{(0, 0, 0)\}$	$\left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \middle\vert\ \det \neq 0 \right\}$

Table 4.4.: Vertex spaces and vertex symmetry group of the system (4.52), part 2. Here $a, b, c, d, e, f, g, h, i \in \mathbb{R}$.

The kernel $\ker \mathcal{L} = \{(x_1, x_2, x_3) \mid x_2 = 0\}$ of the linearization at $\lambda = 0$ is one-dimensional if restricted to the vertex spaces X_2, X_3, X_4, X_7, X_8. Since $X_7 \subseteq X_2$, $X_8 \subseteq X_4$, the equivaroid branching lemma guarantees us the existence of three nontrivial solutions into the vertex spaces X_7, X_8, X_3.

For the specific dynamics $f(\lambda, x) = \lambda x + x^2$, we get the explicit bifurcating solutions $(0, 0, -\lambda), (-\lambda, 0, 0), (-\lambda, 0, -\lambda)$, in accordance with the theoretical results on equivaroid branching.

The equivaroid bifurcation at $\lambda = -1$ leads to solutions within the vertex space X_6 and it could also have been treated by equivariant bifurcation theory. This would not have been as effective, though: The equivariant branching lemma guarantees a bifurcating solution into the fixed-point subspace Σ of the $\mathbb{Z}_2$-action, which is at the same time the vertex space X_3. Since $X_6 \subset X_3$, the equivaroid branching lemma is more precise, here.

Looking once more at the specific dynamics $f(\lambda, x) = \lambda x + x^2$, we find

(X_j, X_k)	Conjugating (X_j, X_k)-symmetries H_{jk}
(X_2, X_4)	$H_{24} = \left\{ \begin{pmatrix} a & -a & 1 \\ b & 1-b & 0 \\ c & 1-c & 0 \end{pmatrix} \middle\vert\, b \neq c \right\}$
(X_4, X_2)	$H_{42} = \left\{ \begin{pmatrix} 0 & a & 1-a \\ 0 & b & 1-b \\ 1 & c & -c \end{pmatrix} \middle\vert\, a \neq b \right\}$
(X_7, X_8)	$H_{78} = \left\{ \begin{pmatrix} a & b & 1 \\ c & d & 0 \\ e & f & 0 \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$
(X_8, X_7)	$H_{87} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 1 & e & f \end{pmatrix} \middle\vert\, \det \neq 0 \right\}$

Table 4.5.: Conjugating morphisms of the system (4.52) with $a, b, c, d, e, f \in \mathbb{R}$.

the explicit bifurcating solution $(-1-\lambda, -1-\lambda, -1-\lambda)$, in accordance with the theoretical results on equivaroid branching.

In the last example, we continue the analysis of the pendulum equation from the Lyapunov–Schmidt reduction.

Example 4.2.5 (The pendulum equation, continued from Example 4.1.2)**.** We reconsider

$$\mathcal{F}(x) = \ddot{x} + \mu \sin(x) = 0, \tag{4.55}$$

with boundary conditions

$$\dot{x}(0) = \dot{x}(1) = 0, \tag{4.56}$$

and bifurcation parameter μ where $\mu_k = k^2\pi^2$ is the bifurcation point, $k \in \mathbb{N}$. We find the vertex spaces of $\mathcal{F} \colon X \to Y$ as in Table 4.1. The corresponding vertex symmetry groups are listed in Table 4.2.

Using our previous analysis from Example 4.1.2, at each bifurcation point μ_k, $k \in \mathbb{N}$ we find bifurcation of a nontrivial solution into each of the subspaces $Y_{k,2}$. As a conclusion of this example, we have seen a much finer bifurcation structure than shown in the previous treatment in [12], where only even/oddness of solutions due to the $\mathbb{Z}_2$-group equivariance could be predicted. Now we can precisely tell by symmetry arguments alone which Fourier modes are present in the bifurcating solution.

4.3. The iterated equivaroid branching lemma

Equivaroid systems will often exhibit multiple eigenvalues crossing zero. Not all of these cases can be treated with the equivaroid branching lemma. This is because it essentially needs a one-dimensional eigenspace within the vertex space. In this section we will present a result on algebraically double or multiple eigenvalues crossing the imaginary axis where the dimensionality condition does not hold.

We will first state the relevant conditions and the new theorem. Then we demonstrate the way it can be used in two examples, and close with a short discussion.

Let X_j be the vertex space into which we want to prove the existence of a bifurcating steady-state solution. We can reduce to $\mathcal{F}(\lambda, x) = 0$, $\mathcal{F}\colon \mathbb{R} \times X_j \to Y_j$. To compensate for the missing condition of a one-dimensional kernel, we assume that within the vertex space X_j, there exist nonempty subspaces $M_j, N_j \subset X_j$ such that $M_j \oplus N_j = X_j$ and any solution $x \in X_j$ of $\mathcal{F}(\lambda, x) = 0$ can be written in the form $x = \mu + \nu(\mu)$ where $\mu \in M_j$ and $\nu \in N_j$. Importantly, we also suppose that μ solves $\mathcal{F}(\lambda, \mu) = 0$, $\mathcal{F}\colon \mathbb{R} \times M_j \to Y_j$, and can be written in the form $\mu(\lambda)$. It remains to solve the equation $\mathcal{F}(\lambda, \mu(\lambda) + \nu) = 0$, $\mathcal{F}\colon \mathbb{R} \times N_j \to Y_j$.

In fact there exist many dynamical systems which fulfill this requirement. It is for example often met in *feed-forward networks* [28], or in networks where a vertex space possesses a feed-forward structure (it is not necessarily for the full network to possess such a structure). Also we emphasize that we do *not* require flow-invariance of the subspaces M_j, N_j.

The main idea is to apply the standard equivariant branching lemma to the subsystem $\mathcal{F}(\lambda, \mu) = 0$, $\mathcal{F} : \mathbb{R} \times M_j \to Y_j$ and then apply it to the space N_j one more time. The condition is therefore a one-dimensional kernel on M_j first, and then in the next step, a one-dimensional kernel on N_j.

Theorem 4.3.1 (Iterated equivaroid branching lemma)**.** *Consider $\mathcal{F}(\lambda, x) = 0$, $x \in X$, $\mathcal{F}\colon \mathbb{R} \times X \to Y$, X, Y Banach spaces, $\mathcal{F} \in C^k$, $k \geq 2$ and let $\mathcal{F}$ be $(\Gamma \rightrightarrows I)$-equivaroid for all $\lambda \in \mathbb{R}$. Let $X_j \subseteq X$ be a vertex space with vertex isotropy group K_j, and suppose that $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.*

Suppose that $X_j = M_j \oplus N_j$ such that any solution $x \in X_j$ of $\mathcal{F}(\lambda, x) = 0$, $\mathcal{F}\colon \mathbb{R} \times X_j \to Y_j$ can be written in the form $x = \mu + \nu(\mu)$ where $\mu = \mu(\lambda)$ is a nonzero solution of $\mathcal{F}(\lambda, \mu) = 0$, $\mathcal{F} : \mathbb{R} \times M_j \to Y_j$ obtained by bifurcation from simple eigenvalues at $\lambda = 0$, and $\nu \in N_j$.

Assume that at $\lambda = 0$ the following conditions are fulfilled:

i) $\mathcal{L} := \mathrm{D}_x \mathcal{F}(\lambda, x)|_{\lambda=0,x=0} \colon X_j \to Y_j$ *is a Fredholm operator of index 0;*

ii) $\mathcal{L}_{N_j} := \mathrm{D}_\nu \mathcal{F}(\lambda, \mu(\lambda) + \nu)|_{\lambda=0,\nu=0} \colon N_j \to Y_j$, $\dim\ker \mathcal{L}_{N_j} = 1$, *where $\mu(\lambda)$ is a solution of $\mathcal{F}(\lambda, \mu) = 0$, $\mathcal{F} : \mathbb{R} \times M_j \to Y_j$ obtained by bifurcation from simple eigenvalues at $\lambda = 0$;*

iii) $\big(\mathrm{D}_\lambda \mathrm{D}_\nu \mathcal{F}(\lambda, \mu(\lambda) + \nu)|_{\lambda=0,\nu=0}\big)|_{\ker \mathcal{L}_{N_j}} \notin \operatorname{range} \mathcal{L}_{N_j}$.

Then the zero set of $\mathcal{F}$ near $(0, 0)$ consists of the trivial branch and a nontrivial C^{k-1}-branch $(\lambda(s), x(s)) = (\lambda(s), \mu(s) + \nu(s, \mu(s)))$, through $\lambda(0) = 0$, $x(0) = 0$ and $\dot{\mu}(0) \in \ker \mathcal{L}|_{M_j}$, $\dot{\nu}(0, 0) \in \mathcal{L}_{N_j}$.

Moreover, $x(s) \in X_j$, i.e., $x(s)$ possesses isotropy K_j.

Proof. By assumption, the theorem of bifurcation from simple eigenvalues theorem by Crandall & Rabinowitz gives a nontrivial solution $(\lambda(s), \mu(s))$ of $\mathcal{F}(\lambda, \mu) = 0$, $\mathcal{F} : \mathbb{R} \times M_j \to Y_j$ with $\lambda(0) = 0$, $\mu(0) = 0$ and $\dot{\mu}(0) \in \ker \mathcal{L}|_{M_j}$. The other conditions then assure another bifurcation of simple eigenvalues for the equation $\mathcal{F}(\lambda, \mu(\lambda) + \nu) = 0$ with $\mathcal{F} \colon \mathbb{R} \times N_j \to Y_j$ where $\nu(0, 0) = 0$, $\dot{\nu}(0, 0) \in \mathcal{L}_{N_j}$. Putting this together, any solution $x \in X_j$ of $\mathcal{F}(\lambda, x) = 0$, $\mathcal{F} \colon \mathbb{R} \times X_j \to Y_j$ can be written in the form $(\lambda(s), x(s)) = (\lambda(s), \mu(s) + \nu(s, \mu(s)))$, which proves the theorem. □

Remark 4.3.2. The iterated equivaroid branching lemma can be applied in situations with higher dimensional kernels as well. The procedure outlined in the proof then needs to be carried out multiple times. We have chosen to present only the two-dimensional case here in order to keep notation clean.

As we will see in the following two examples, the iterated equivaroid branching lemma allows us to prove the existence of nontrivial steady states where multiple eigenvalues cross zero within the same vertex space which cannot be divided into smaller vertex spaces.

Example 4.3.3 (A feed-forward network)**.** To see the iterated equivaroid branching lemma in action, let us consider the following two-cell network with feed-forward structure:

$$\begin{aligned} \dot{x}_1 &= f(\lambda, x_1) \\ \dot{x}_2 &= f(\lambda, x_2) + x_1 x_2, \end{aligned} \tag{4.57}$$

with $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(\lambda, 0) = 0$, $\mathrm{D}_x f(\lambda, 0) = \lambda$, where $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $x_1, x_2 \in \mathbb{R}$. Note that system (**??**) does not possess any group symmetries, but it exhibits a surprisingly rich set of groupoid symmetries from which we can deduce the bifurcation structure.

The vertex spaces of system (**??**) are given by

$$\begin{aligned} X_1 &= \mathbb{R}^2, \quad X_2 = \{(x_1, x_2) \,|\, x_1 = 0\}, \\ X_3 &= \{(x_1, x_2) \,|\, x_2 = 0\}, \quad X_4 = \{(0, 0)\}. \end{aligned} \tag{4.58}$$

The symmetry sets H_{jk} can be found in Table 4.6. The linearization at the trivial equilibrium is given by

$$\mathcal{L} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \tag{4.59}$$

which implies that there are two eigenvalues crossing zero at $\lambda = 0$. With the help of the equivaroid branching lemma, we see the existence of two branches of solutions into the subspaces X_2 and X_3, as the kernel of the linearization restricted to these spaces is one-dimensional and all other conditions are fulfilled.

In addition to those two branches, there is a third branch of solutions bifurcating from the trivial equilibrium at $\lambda = 0$. To see this, we divide $\mathbb{R}^2 = X = M \oplus N$, where $M = \{x_1 \in \mathbb{R}\}$ and $N = \{x_2 \in \mathbb{R}\}$.

In the space M, the steady-state solution $\mu(\lambda)$ of $\dot{\mu} = f(\lambda, \mu)$ can be found by bifurcation of simple eigenvalues. We therefore only need to solve for steady state solutions of

$$\dot{\nu} = f(\lambda, \nu) + \mu(\lambda)\nu. \tag{4.60}$$

Vertex pair (j,k)	Set of (j,k)-symmetries
$(1,1)$	$H_{11} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{\mathrm{Id}\}$
$(2,2)$	$H_{22} = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle\vert\ a \neq 0 \right\}$
$(3,3)$	$H_{33} = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \middle\vert\ b \neq 0 \right\}$
$(4,4)$	$H_{44} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle\vert\ ad - bc \neq 0 \right\}$
$(2,3)$	$H_{23} = \left\{ \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \middle\vert\ b \neq 0 \right\}$
$(3,2)$	$H_{32} = \left\{ \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \middle\vert\ a \neq 0 \right\}$
else	$H_{jk} = \emptyset$

Table 4.6.: The symmetry groupoid of system (4.57). Here $a, b, c, d \in \mathbb{R}$.

This equation has again a solution by the theorem by Crandall and Rabinowitz, which proves the existence of a solution $(\mu(\lambda), \nu(\lambda, \mu(\lambda))) \in \mathbb{R}^2$ of (4.57), outside of the vertex spaces X_2, X_3.

Note that this solution does not necessarily grow as $\sqrt{\lambda}$ on both coordinates. This unusual behaviour should be investigated more closely in the future. For networks of feed-forward structure it has been discussed under the name of *amplitude amplification*; see e.g., [28].

To see this phenomenon in an example, we consider the specific nonlinearity $f(\lambda, x) = \lambda x - x^3$. Then, apart from the trivial solution, we find the following nontrivial equilibrium solutions:

$$\begin{aligned}
(x_1^*, x_2^*) &= \left(0, \pm\sqrt{\lambda}\right), \\
(x_1^*, x_2^*) &= \left(\pm\sqrt{\lambda}, 0\right), \\
(x_1^*, x_2^*) &= \left(\sqrt{\lambda}, \pm\sqrt{\lambda + \sqrt{\lambda}}\right), \\
(x_1^*, x_2^*) &= \left(-\sqrt{\lambda}, \pm\sqrt{\lambda - \sqrt{\lambda}}\right).
\end{aligned} \tag{4.61}$$

Let us also note that the subspaces X_2 and X_3 behave differently in the subequivaroid bifurcation, although they are conjugated vertex spaces. Indeed, it is not important that X_2 is a vertex space (it just becomes somewhat easier to see the bifurcation).

Example 4.3.4 (Another feed-forward network)**.** Let us consider the following two-cell network with feed-forward structure:

$$\begin{aligned}
\dot{x}_1 &= f(\lambda, x_1) \\
\dot{x}_2 &= f(\lambda, x_2) + x_1,
\end{aligned} \tag{4.62}$$

with $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(\lambda, 0) = 0$, $\mathrm{D}_x f(\lambda, 0) = \lambda$, where $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $x_1, x_2 \in \mathbb{R}$.

The vertex spaces of system (4.62) are given by

$$X_1 = \mathbb{R}^2, \quad X_2 = \{(x_1, x_2) \mid x_1 = 0\}, \quad X_3 = \{(0, 0)\}. \tag{4.63}$$

The symmetry sets H_{jk} can be found in Table 4.7. The linearization at the trivial equilibrium solution is given by

$$\mathcal{L} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \tag{4.64}$$

which implies that there are two eigenvalues crossing zero at $\lambda = 0$. With the help of the equivaroid branching lemma, we see the existence of a nontrivial branch of solutions into the subspace X_2, as the kernel of the

Vertex pair (j,k)	Set of (j,k)-symmetries
$(1,1)$	$H_{11} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{\mathrm{Id}\}$
$(2,2)$	$H_{22} = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle\vert\, a \neq 0 \right\}$
$(3,3)$	$H_{33} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle\vert\, ad - bc \neq 0 \right\}$
else	$H_{jk} = \emptyset$

Table 4.7.: The symmetry groupoid of system (4.62). Here $a, b, c, d \in \mathbb{R}$.

linearization restricted to this space is one-dimensional and all other conditions are fulfilled.

Now note that the algebraic multiplicity of λ is two, but the geometric multiplicity is one. We find that every solution $(x_1^*, x_2^*) \in \mathbb{R}^2$ of (**??**) gives a solution x_1^* of $0 = f(\lambda, x_1^*)$. It remains to solve the equation

$$0 = f(\lambda, x_1). \tag{4.65}$$

This existence of the bifurcating solution $x_1^*(\lambda)$ is easily seen by the theorem of Crandall and Rabinowitz. Now, all we have to do is solve the equation

$$0 = f(\lambda, x_2) + x_1^*(\lambda). \tag{4.66}$$

The solution of this last equation can again be found by the theorem by Crandall and Rabinowitz. This proves the existence of a solution $(x_1^*, x_2^*) \in \mathbb{R}^2$ of (4.62), outside the vertex spaces X_2, X_3 but along the same eigenvector as the bifurcation into the subspace X_2.

Also in this case, the solution does not grow as $\sqrt{\lambda}$ on both coordinates; see Fig. 4.2 for an explicit example.

Example 4.3.5 (A circular network with nonlinear coupling)**.** It is not necessary to have a feed-forward structure to apply the iterated equivaroid branching lemma, as we will show with the following example:

$$\begin{aligned} \dot{x}_1 &= f(\lambda, x_1) + \, x_1 x_3 \\ \dot{x}_2 &= f(\lambda, x_2) + 2\, x_2 x_1 \\ \dot{x}_3 &= f(\lambda, x_3) + 3\, x_3 x_2, \end{aligned} \tag{4.67}$$

with $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(\lambda, 0) = 0$, $\mathrm{D}_x f(\lambda, 0) = \lambda$, where $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $x_1, x_2, x_3 \in \mathbb{R}$. Note that system (4.67) does not possess any group symmetries due to the nonidentical coupling, but it exhibits a surprisingly rich set of groupoid symmetries from which we can deduce the bifurcation structure.

Indeed, the vertex spaces of system (4.67) are given by

$$\begin{aligned} &X_1 = \mathbb{R}^3,\ X_2 = \{(x_1, x_2, x_3) \mid x_1 = 0\}, \\ &X_3 = \{(x_1, x_2, x_3) \mid x_2 = 0\},\ X_4 = \{(x_1, x_2, x_3) \mid x_3 = 0\}, \\ &X_5 = \{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\},\ X_6 = \{(x_1, x_2, x_3) \mid x_2 = x_3 = 0\}, \\ &X_7 = \{(x_1, x_2, x_3) \mid x_3 = x_1 = 0\},\ X_8 = \{(0, 0, 0)\}. \end{aligned} \tag{4.68}$$

The linearization at the trivial equilibrium is given by

$$\mathcal{L} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \tag{4.69}$$

which implies that there are three eigenvalues crossing zero at $\lambda = 0$. With the help of the equivaroid branching lemma, we can prove the existence of three branches of solutions into the subspaces X_5, X_6, and X_7, as the kernel of the linearization restricted to these vertex spaces is one-dimensional and all other conditions are fulfilled.

In this system, we find other branches as well, and we will now exemplarily treat iterated equivaroid bifurcation from the trivial equilibrium at $\lambda = 0$ into the two-dimensional X_4. To see this, we divide $X_4 = M_4 \oplus N_4$, where $M_4 = \{x_1 \in \mathbb{R}\}$ and $N_4 = \{x_2 \in \mathbb{R}\}$. Note that the third variable is identically zero in the vertex space X_4.

Then we can proceed as before: In the space M_4, the steady-state solution $\mu(\lambda)$ of $\dot{\mu} = f(\lambda, \mu)$ can be found by bifurcation of simple eigenvalues. We therefore only need to solve for steady state solutions of

$$\dot{\nu} = f(\lambda, \nu) + \mu(\lambda)\nu. \tag{4.70}$$

This equation has again a solution by the theorem by Crandall and Rabinowitz, which proves the existence of a solution $(\mu(\lambda), \nu(\lambda, \mu(\lambda)), 0) \in X_4$ of (**??**), within the vertex space X_4. By the same reasoning, or by noticing that X_2, X_3, X_4 are indeed conjugated vertex spaces, we can find conjugated solutions into the subspaces X_2 and X_3 as well.

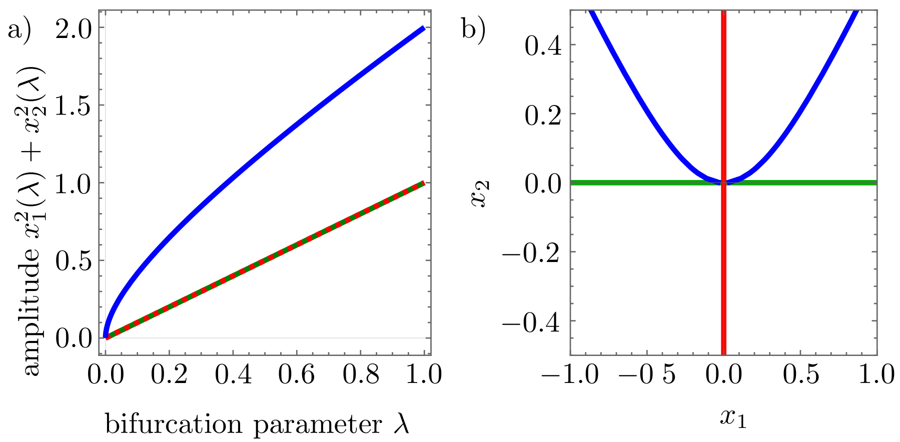

Figure 4.1.: Steady-state solutions of system (**??**) with $f(\lambda, x) = \lambda x - x^3$. a) Amplification of the amplitude growth depending on the bifurcation parameter $\lambda > 0$. The red and the green branches are obtained by the equivaroid branching lemma, and their quadratic growth is linear, as expected. The blue branch is obtained by the iterated equivaroid branching lemma, and its quadratic growth is superlinear. b) Steady-state solutions in the (x_1, x_2)-plane. The green branch lies in the vertex space X_3 and the red branch in the vertex space X_2 (both obtained by the equivaroid branching lemma), the blue branch (as obtained by the iterated equivaroid branching lemma) bifurcates tangentially to X_2, but into the full space $X_1 = \mathbb{R}^2$.

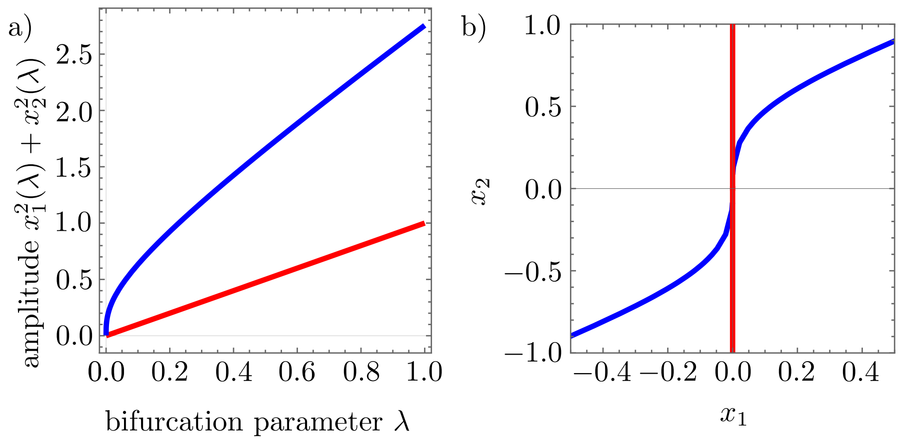

Figure 4.2.: Steady-state solutions of system (4.62) with $f(\lambda, x) = \lambda x - x^3$. a) Amplification of the amplitude growth depending on the bifurcation parameter $\lambda > 0$. The red branch is obtained by the equivaroid branching lemma, and its quadratic growth is linear, as expected. The blue branch is obtained by the iterated equivaroid branching lemma, and its quadratic growth is no longer linear. b) Steady-state solutions in the (x_1, x_2)-plane. The red branch (as obtained by the equivaroid branching lemma) lies in the vertex space X_2, and the blue branch (as obtained by the iterated equivaroid branching lemma) bifurcates tangentially, but into the full space $X_1 = \mathbb{R}^2$.

5. Spatio-temporal patterns and equivaroid Hopf bifurcation

The goal of this chapter is to prove (iterated) Hopf bifurcation in equivaroid systems. To this end, we generalize the notion of spatio-temporal patterns, introducing an action of the N-dimensional torus as a *componentwise time-shift*. This allows us to describe elaborate spatio-temporal patterns.

We proceed as follows. In Section 5.1 we recall the abstract functional analytic setting for Hopf bifurcation, following the construction of Vanderbauwhede [62, 63]. Readers familiar with these works can directly skip ahead to Section 5.2: Here we give a new definition of spatio-temporal patterns in the sense of groupoids and show its potential in applications with some examples. We then state and prove equivaroid Hopf bifurcation in Section 5.3. Last, as an extension, we prove iterated equivaroid Hopf bifurcation in Section 5.4, which allows existence proofs of periodic orbits with even more elaborate spatio-temporal patterns than the equivaroid Hopf bifurcation allows.

5.1. Functional analytic setting

In this section we first recall the general S^1-equivariant formulation of the Hopf bifurcation problem, following an argument by Vanderbauwhede [62, 63] and loosely following the presentation of [12].

To find p-periodic solutions $\tilde{x}(t+p) = \tilde{x}(t)$ of the ordinary differential equation

$$\dot{\tilde{x}} = f(\lambda, \tilde{x}(t)), \tag{5.1}$$

where $f\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a C^k-function, $k \geq 2$, we first rescale time such that the minimal period $p > 0$ becomes a fixed period 2π and then define

$$x(t) := \tilde{x}(\tfrac{p}{2\pi} t). \tag{5.2}$$

It follows that the periodic solutions $\tilde{x}(t)$ of (5.1) are in fact equilibrium solutions x of the equation

$$\mathcal{F}(\lambda, p, x) := \dot{x} - \frac{p}{2\pi} f(\lambda, x) = 0, \tag{5.3}$$

where $\mathcal{F}\colon \mathbb{R} \times \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$, with $\mathbb{X}$ and $\mathbb{Y}$ being Banach spaces of 2π-periodic functions, more precisely

$$\mathbb{Y} := \tilde{C}^0 := \left\{ x \in C^0(\mathbb{R}, \mathbb{R}^N) \;\middle|\; x(t + 2\pi) = x(t) \text{ for all } t \in \mathbb{R} \right\}, \tag{5.4}$$

$$\mathbb{X} := \tilde{C}^1 := \tilde{C}^0 \cap C^1. \tag{5.5}$$

Note that equation (5.3) is S^1-equivariant, where S^1 acts on $\mathbb{X}, \mathbb{Y}$ as a time-shift

$$(\rho(\vartheta)x)(t) := x(t + \vartheta), \tag{5.6}$$

with $\vartheta \in S^1$, $x \in \mathbb{X}$ or $\mathbb{Y}$, and $t \in \mathbb{R}$. Indeed,

$$\begin{aligned} \big(\rho(\vartheta)\mathcal{F}(\lambda, p, x)\big)(t) &= \mathcal{F}(\lambda, p, x)(t + \vartheta) \\ &= \dot{x}(t + \vartheta) - \frac{p}{2\pi} f(\lambda, p, x(t + \vartheta)) \\ &= \mathcal{F}(\lambda, p, \rho(\vartheta)x)(t). \end{aligned} \tag{5.7}$$

Thus, if $x(t)$ is a solution of equation (5.3), then also $x(t+\vartheta)$ is a solution for all $\vartheta \in S^1$.

Technically, we will treat Hopf bifurcation as a steady-state bifurcation in the setting (5.3). This is possible because the linearization on the trivial equilibrium,

$$\begin{aligned}\mathcal{L}(\lambda, p)y &:= \mathrm{D}_x\, \mathcal{F}(\lambda, p)y \\ &= \frac{d}{dt}y - \frac{p}{2\pi}\,\mathrm{D}_x\, f(\lambda, 0)y \\ &= \dot{y} - A(\lambda)y,\end{aligned} \tag{5.8}$$

with $A(\lambda) := \mathrm{D}_x\, f(\lambda, 0)$, is a Fredholm operator [62], which allows us to invoke Lyapunov–Schmidt reduction.

One possibility (which we will not pursue further in this thesis) is to treat the operator

$$\mathcal{F}(\lambda, p, x) := \dot{x} - \frac{p}{2\pi} f(\lambda, x) \tag{5.9}$$

as an equivaroid mapping of its own right, with vertex spaces, vertex symmetry and isotropy groups as well as conjugating morphisms. Then one could invoke the equivaroid branching lemma. While this procedure would yield rather general symmetries, it would also hide the interplay between space and time. We will therefore choose a different route, which we will outline in the following.

5.2. Spatio-temporal groupoid symmetries

With the setting of the previous section in mind, we now define *groupoid spatio-temporal patterns*. It turns out that these patterns are vastly more general than those discussed and predicted by equivariant bifurcation theory.

It is our goal to describe the *groupoid symmetries of a periodic orbit* (or more precisely: a set of periodic orbits). Contrary to the previous chapters, we will *not* describe the symmetries of the dynamical system.

To this end, let us introduce the following action of the N-dimensional torus $\mathbb{T}^N \cong (S^1)^N$ as a *componentwise time-shift* on $\mathbb{X}$ and $\mathbb{Y}$ via

$$(\rho(\vartheta)x)(t) := x(t+\vartheta) := (x_1(t+\vartheta_1), \ldots, x_N(t+\vartheta_N)), \tag{5.10}$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_N) \in \mathbb{T}^N$ and $x = (x_1, \ldots, x_N)$. Moreover, let $\gamma \in \mathrm{GL}(N)$ be a linear invertible isomorphism on $X = \mathbb{R}^N$ acting as a matrix. Then we define the groupoid symmetry of a periodic orbit as follows.

Definition 5.2.1 (Spatio-temporal groupoid symmetries)**.** Consider a solution $x \in \tilde{C}^1$ of

$$\mathcal{F}(\lambda, p, x) := \dot{x} - \frac{p}{2\pi} f(\lambda, x) = 0, \tag{5.11}$$

with $\mathcal{F} \colon \mathbb{R} \times \mathbb{R} \times \tilde{C}^1 \to \tilde{C}^0$.

We call a pair (γ, ϑ) a *spatio-temporal groupoid symmetry of the periodic orbit* $x(t)$ if $(\gamma, \vartheta)\, x \in \tilde{C}^1$ is also a solution of (5.11).

Then the set of spatio-temporal groupoid symmetries forms a groupoid, hence the name. This deserves some explanation:

First of all, the vertices of the groupoid are given by the periodic orbit themselves. In particular, we regard those orbits who are simple time-shifted versions of each other as individual vertices. This means that the groupoid vertices for periodic orbits of autonomous systems are never isolated, there is always at least a ring of vertices (unless of course, the periodic orbit is actually an equilibrium).

Next, the vertex group of the periodic orbit x is defined by

$$(H, \Theta)_x := \{(\gamma, \vartheta) \mid \gamma x(t+\vartheta) = x(t) \text{ for all } t \in \mathbb{R}\}, \tag{5.12}$$

that is, by all combinations of spatial transformations γ and componentwise time-shifts ϑ which leave the orbit unchanged. Why is the vertex group a group? Obviously, $(\mathrm{Id}, 0)$ acts as the identity, and $(\gamma^{-1}, -\vartheta)$ as the inverse element to (γ, ϑ).

An important subgroup of the vertex group is given by the *spatial vertex isotropy group* of the periodic orbit x,

$$K_x := \{(\gamma, 0) \mid \gamma x(t) = x(t) \text{ for all } t \in \mathbb{R}\}, \tag{5.13}$$

i.e., by those *spatial* actions $(\gamma, 0)$ which fix the the periodic orbit *point-wise*.

Another important subgroup of the vertex group is given by the *temporal vertex isotropy group*

$$J_x := \{(\mathrm{Id}, \vartheta) \mid x(t + \vartheta) = x(t) \text{ for all } t \in \mathbb{R}\}, \tag{5.14}$$

i.e., by those *temporal* actions (Id, ϑ) which fix the periodic orbit *point-wise*.

Moreover, the set of *conjugating morphisms* between different periodic orbits x_1, x_2 of the same equation is defined as follows:

$$(H, \Theta)_{x_1, x_2} := \{(\gamma, \vartheta) \mid \gamma x_1(t + \vartheta) = x_2(t) \text{ for all } t \in \mathbb{R}\}. \tag{5.15}$$

For example, as indicated above, all time-shifted versions of "the same" periodic orbit are conjugated, in this setting. As before, $(H, \Theta)_{x_1, x_2}$ is not a group. It does not contain an identity element. Its inverse elements are contained in the set $(H, \Theta)_{x_2, x_1}$.

We call the set of all spatio-temporal groupoid symmetries of all periodic orbits of a dynamical system the *twisted symmetry groupoid* $(\mathcal{G} \rightrightarrows \mathcal{I})$. This is in accordance with the previous literature on equivariant systems, compare the twisted symmetry group, e.g., in [20, 27].

Let us now describe the relation between the vertex group and the two different isotropy groups. We start with the spatial vertex isotropy group K_x. Here we define the group homomorphism to the N-dimensional torus,

$$\Omega : (H, \Theta)_x \to \mathbb{T}^N \tag{5.16}$$

$$(\gamma, \vartheta) \mapsto \vartheta. \tag{5.17}$$

The kernel of this homomorphism is given by the group K_x, i.e., $\ker \Omega = K_x$, and using the group homomorphism theorem [61], we find that

$$\operatorname{range} \Omega \cong (H, \Theta)_x / K_x. \tag{5.18}$$

To see that this is indeed a group homomorphism, note that $(\gamma_1, \vartheta_1) \circ (\gamma_2, \vartheta_2) = (\gamma_1\gamma_2, \vartheta_1 + \vartheta_2)$ and that we have

$$\begin{aligned} x(t + \Omega(\gamma_1\gamma_2, \vartheta_1 + \vartheta_2)) &= x(t + \vartheta_1 + \vartheta_2) \\ &= \gamma_2^{-1}\gamma_1^{-1} x(t) \\ &= \gamma_1^{-1} x(t + \Omega(\gamma_2, \vartheta_2)) \\ &= x(t + \Omega(\gamma_1, \vartheta_1) + \Omega(\gamma_2, \vartheta_2)), \end{aligned} \tag{5.19}$$

and therefore

$$\Omega(\gamma_1\gamma_2, \vartheta_1 + \vartheta_2) = \Omega(\gamma_1, \vartheta_1) + \Omega(\gamma_2, \vartheta_2) \mod 2\pi, \text{ componentwise.} \tag{5.20}$$

We now turn to the temporal vertex isotropy group. In this case we define the group homomorphism

$$\Xi : (H, \Theta)_x \to \mathrm{GL}(N) \tag{5.21}$$

$$(\gamma, \vartheta) \mapsto \gamma. \tag{5.22}$$

Its kernel is given by J_x, and therefore J_x is a normal subgroup of (H, Θ) and the range of Ξ satisfies

$$\operatorname{range} \Xi \cong (H, \Theta)_x / J_x. \tag{5.23}$$

Again, this is obviously a group homomorphism satisfying

$$\Xi(\gamma_1\gamma_2, \vartheta_1 + \vartheta_2) = \Xi(\gamma_1, \vartheta_1) \cdot \Xi(\gamma_2, \vartheta_2). \tag{5.24}$$

These considerations give us the opportunity to classify periodic orbits according to their symmetry groupoid. We start with a temporal classification, which revolves around the subgroups of a torus.

Definition 5.2.2 (Temporal classification of periodic orbits)**.** We call $x(t)$

i) a *rotating wave* if $\operatorname{range}\Omega \cong S^1$ [20];

ii) a *discrete wave* if there exists $n \in \mathbb{N}$ such that $\operatorname{range}\Omega \cong \mathbb{Z}_n$ [20];

iii) a *concentric wave* if $\operatorname{range}\Omega \cong \mathrm{Id}$ [20];

iv) a *multi-frequency wave* if there exist $n_1 \neq \ldots \neq n_k \in \mathbb{N}$ such that $\operatorname{range}\Omega \subseteq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$;

v) a *toroidal wave* if there exists $n \in \mathbb{N}$, $n \geq 1$, such that $\operatorname{range}\Omega = \mathbb{T}^n$;

vi) a *mixed wave* if there exist $n_1 \neq \ldots \neq n_k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $\operatorname{range}\Omega \subseteq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \times \mathbb{T}^n$;

The names *rotating, discrete and concentric wave* were chosen from [20] and they carry over to the groupoid context. However, since we have introduced the component-wise time-shifts, the theory now include more complex patterns as well, namely multi-frequency, toroidal, and mixed waves.

We proceed with a spatial classification.

Definition 5.2.3 (Spatial classification of periodic orbits)**.** Let $G \subseteq \mathrm{GL}(n)$. Then we say that a periodic orbit is *of G-type*, if $\operatorname{range}\Xi \cong G$.

Before we turn to some examples, let us ask the question question: Which spatio-temporal symmetries do we find if $\dot{x} = f(x)$ is $(\Gamma \rightrightarrows I)$-equivaroid? Then each periodic orbit x which lies in a vertex space X_j gives rise to a vertex of the symmetry groupoid and a subgroup $(\tilde{H}_j, \Theta_j)$ of the vertex group $(H, \Theta)_x$, where

$$\begin{aligned}(\tilde{H}_j, \Theta_j) := \big\{(\gamma, \vartheta) \mid \gamma \in H_j \text{ such that there exists } \vartheta \in S^1 \\ \text{with } \gamma x(t+\vartheta) = x(t) \text{ for all } t \in \mathbb{R}\big\}.\end{aligned} \tag{5.25}$$

Note that this does not completely describe the spatio-temporal symmetry of the periodic orbit. The group $(\tilde{H}_j, \Theta_j)$ will nevertheless be very useful

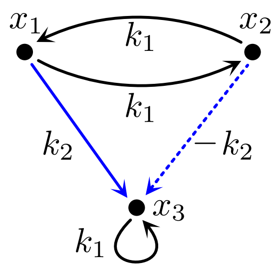

Figure 5.1.: Graphical depiction of the network (5.26), omitting the identical self coupling through f. The coupling k_1 is drawn black and solid, k_2 solid blue, and $-k_2$ dotted blue.

for describing the symmetries emerging at Hopf bifurcation, as we will do in the following section.

We will now proceed with several examples. In this way, we illustrate the vastness of possible spatio-temporal patterns. The simulations are performed with Mathematica using `NDSolve` and merely serve illustrational purposes.

Example 5.2.4 (A concentric wave of $\mathbb{Z}_2$-type with a 2-torus of conjugating spatio-temporal symmetries)**.** We choose the rather simple network

$$\begin{aligned} \dot{x}_1 &= f(x_1) + k_1\, g(x_2) \\ \dot{x}_2 &= f(x_2) + k_1\, g(x_1) \\ \dot{x}_3 &= f(x_3) + k_1\, g(x_3) + k_2\, g(x_1) - k_2\, g(x_2), \end{aligned} \tag{5.26}$$

with oscillators $x_1, x_2, x_3 \in \mathbb{R}^N$, $f, g\colon \mathbb{R}^N \to \mathbb{R}^N$, $k_1, k_2 \in \mathbb{R}^{N\times N}$; see Fig. 5.1. We find the invariant subspaces

$$\mathbb{X}_\theta = \{(x_1, x_2, x_3) \mid x_1(t) \equiv x_2(t), x_3(t-\theta) \equiv x_1(t)\} \subset \mathbb{X} \text{ for all } \theta \in S^1. \tag{5.27}$$

Note that these spaces are not vertex spaces of the original symmetry groupoid of the dynamical system, but they are still relatively easy to find and produce interesting patterns and symmetry groupoids.

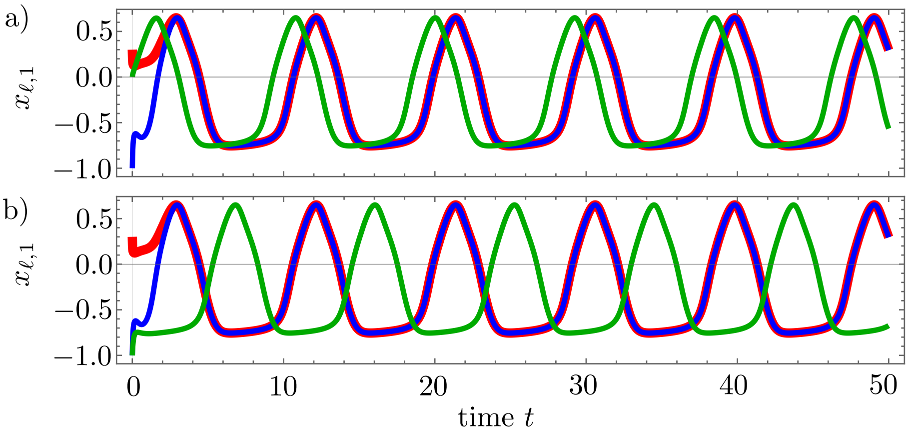

Figure 5.2.: Numerical simulations of (5.26). We use $N = 2$, that is, two-dimensional oscillators and depict the first component, respectively; red: $x_{1,1}$, blue: $x_{2,1}$, green: $x_{3,1}$) While x_1 and x_2 become indistinguishable after a short time, x_3 remains a shifted version, with the time-shift depending on the initial conditions. We approach a periodic orbits in $\mathbb{X}_\theta$. The two periodic orbits are conjugated through a component-wise time-shift in the space $\mathbb{X}_\theta$. The following explicit dynamics were used: $f = (f_1, f_2)$, $f_1(x_{i,1}, x_{i,2}) = 4x_{i,1} + x_{i,2} - 10\left(x_{i,1}^2 + x_{i,2}^2\right)x_{i,1} - 2\left(x_{i,1}^3 + x_{i,2}^3\right)x_{i,1}$, $f_2(x_{i,1}, x_{i,2}) = -x_{i,1} + x_{i,2} - 10\left(x_{i,1}^2 + x_{i,2}^2\right)x_{i,2} - 2\left(x_{i,1}^3 + x_{i,2}^3\right)x_{i,2}$, $g(x_i) = x_i$, $k_1 = 0.8$, $k_2 = -0.08$. Initial conditions: $x_{1,1}^0 = 0.3, x_{1,2}^0 = 2.0, x_{2,1}^0 = -1.0, x_{2,2}^0 = -1.0$, in the case a) $x_{3,1}^0 = 0.0, x_{3,2}^0 = 2.0$, in the case b) $x_{3,1}^0 = -1.0, x_{3,2}^0 = -0.4$).

On each periodic orbit (that is, vertex!) in $\mathbb{X}_\theta$, we find that the vertex symmetry group is equal to the spatial isotropy group $(H, \Theta)_x \cong K_x \cong \mathbb{Z}_2$, acting as a permutation of oscillators x_1 and x_2. The temporal isotropy group is given by $J_x^\Theta = \mathrm{Id}$. We therefore find range $\Xi = \mathbb{Z}_2$, which makes this a wave of $\mathbb{Z}_2$-type.

In this example, there exists a 2-torus of conjugated periodic solutions. Symmetry-wise, the periodic orbits in this example are connected via *conjugating morphisms* which act by component-wise delays as follows.

$$\begin{aligned} (\rho(\mathrm{Id}, \vartheta)x)(t) &:= \big(x_1(t), x_2(t), x_3(t+\vartheta)\big). \\ (\rho(\mathrm{Id}, \varphi)x)(t) &:= \big(x_1(t+\varphi), x_2(t+\varphi), x_3(t+\varphi)\big). \end{aligned} \tag{5.28}$$

Here ϑ represents the conjugation between the invariant subspaces $\mathbb{X}_\theta$ whereas φ represents the S^1-equivariance established by Vanderbauwhede. We therefore find a torus $\mathbb{T}^2$ of periodic solutions; see Fig. 5.2 for a specific example.

Example 5.2.5 (A rotating wave in a finite-dimensional subspace where the conjugating morphisms contain an extended singleton)**.** In some cases, it pays to look at *finite-dimensional* subspaces of $\mathbb{X}$. An example can be found in [53], by Röhm, Lüdge and this author, where the following system of coupled Stuart–Landau oscillators was considered:

$$\begin{aligned} \dot z_1 &= (\lambda + \mathrm{i}\omega + \gamma\,|z_1|^2)z_1 + \kappa e^{\mathrm{i}\phi}(z_2 - z_1) \\ \dot z_2 &= (\lambda + \mathrm{i}\omega + \gamma\,|z_2|^2)z_2 + \kappa e^{\mathrm{i}\phi}(z_1 - z_2), \end{aligned} \tag{5.29}$$

where $z_1, z_2 \in \mathbb{C}$, $\lambda, \omega, \kappa \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\phi \in S^1$. The system is obviously $\mathbb{Z}_2 \times S^1$-equivariant, which usually implies the search for in-phase and anti-phase rotating-wave solutions. But there is more to the story and we will show another spatio-temporal pattern which can be found.

The linear subspace

$$\mathbb{X}_1 := \{(z_1, z_2) \mid z_1 = a_1 e^{\mathrm{i}\omega t},\ z_2 = a_2 e^{\mathrm{i}\omega t},\ a_1, a_2 \in \mathbb{C}\} \subset \mathbb{X} \tag{5.30}$$

is invariant. On each such periodic orbit x, we find the vertex symmetry group

$$(H, \Theta)_x = (S^1, S^1), \tag{5.31}$$

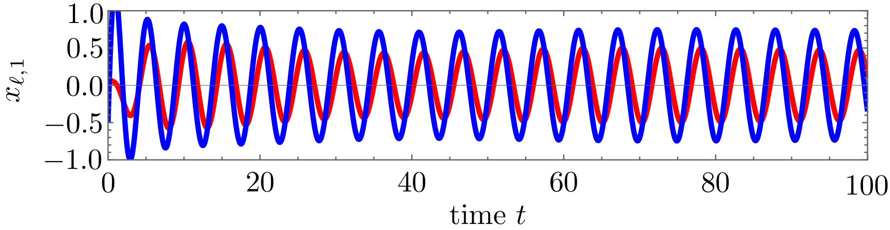

Figure 5.3.: Numerical simulations of (5.29) for $\lambda = 0.07$, $\gamma = -0.1 + 0.5\mathrm{i}$, $\kappa = 0.1$, $\phi = 0.2\pi$ (red: $x_{1,1}$, blue: $x_{2,1}$) with initial conditions $x^0_{1,2} = 2.0, x^0_{2,1} = -1.0, x^0_{2,2} = -1.0$. The two oscillators are both rotating waves with different (complex) amplitude.

which corresponds to the fact that the periodic solutions are rotating waves.

Both the spatial and the temporal vertex isotropy groups are trivial. Moreover, if $(a_1 e^{\omega \mathrm{i}t},\, a_2 e^{\omega \mathrm{i}t})$ is a periodic solution, then (due to the $\mathbb{Z}_2$-equivariance), also $(a_2 e^{\omega \mathrm{i}t},\, a_1 e^{\omega \mathrm{i}t})$ is a periodic solution. This gives rise to conjugating symmetries in the form of an *extended singleton*; see Example A.1.5. See also [53] for the detailed calculations and the explicit periodic solutions which arise in this four-dimensional invariant subspace (without the groupoid-theoretical explanation).

Example 5.2.6 (A (mixed) multi-frequency wave of $S^1 \times \mathbb{Z}_2$-type)**.** We consider the following network:

$$\begin{aligned}
\dot{x}_1(t) &= f(x_1) + k\, x_1 \\
\dot{x}_2(t) &= f(x_2) + k\,(x_{1,1}x_{2,1} + \mathrm{i}x_{1,2}x_{2,2}) \\
\dot{x}_3(t) &= f(x_3) + k\,(x_{2,1}x_{3,1} + \mathrm{i}x_{2,2}x_{3,2}),
\end{aligned} \tag{5.32}$$

$x_1 = x_{1,1} + \mathrm{i}x_{1,2},\ x_2 = x_{2,1} + \mathrm{i}x_{2,2},\ x_3 = x_{3,1} + \mathrm{i}x_{3,2},\ x_1, x_2, x_3 \in \mathbb{C},\ k \in \mathbb{C}$. For simplicity, we use an S^1-equivariant Stuart–Landau-nonlinearity $f\colon \mathbb{C} \to \mathbb{C}$. Let us look for periodic solutions in the invariant subspace

$$\mathbb{X}_{4,2,1} = \{(x_1, x_2, x_3) \in \mathbb{X} \mid x_1(t + p/4) = x_1,\ x_2(t + p/2) = x_2\}. \tag{5.33}$$

Note that, at this point, we only know that this subspace of the p-periodic solutions is invariant; we do not know whether it contains more than the trivial equilibrium. Existence of periodic orbits from such subspaces will be tackled in the next two subsections when we present the main results of this chapter — (iterated) equivaroid Hopf bifurcation. For now, we leave it at a numerical example.

Within the subspace $\mathbb{X}_{4,2,1}$, we find the following symmetries:

$$\begin{aligned}
x_1(t) &= e^{-i\phi\tau} x_1(t+\tau), \\
x_1(t) &= x_1(t+p/4), \\
x_1(t) &= -x_1(t+p/8), \\
x_2(t) &= x_2(t+p/2), \\
x_2(t) &= -x_2(t+p/4), \\
x_3(t) &= x_3(t+p), \\
x_3(t) &= -x_3(t+p/2).
\end{aligned} \tag{5.34}$$

The oscillator x_1 is of Stuart–Landau type and its periodic solution is therefore of the form $x_1(t) = ae^{\mathrm{i}\phi t}$. The spatial vertex isotropy group is trivial. Therefore, we find a mixed multi-frequency wave.

The temporal vertex isotropy group is generated by

$$\begin{aligned}
x_1(t) &= x_1(t+p/4), \\
x_2(t) &= x_2(t+p/2), \\
x_3(t) &= x_3(t+p).
\end{aligned} \tag{5.35}$$

This implies that we find waves of $S^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ type.

Example 5.2.7 (A mixed wave of $S^1 \times \mathbb{Z}_4$-type)**.** We consider the following network:

$$\begin{aligned}
\dot{x}_1 &= f(x_1) + k\, x_1 \\
\dot{x}_2 &= f(x_2) + k\, x_2\, (x_{1,2}^2 + \mathrm{i} x_{1,1}^2),
\end{aligned} \tag{5.36}$$

with $x_1 = x_{1,1} + \mathrm{i}x_{1,2},\ x_2 = x_{2,1} + \mathrm{i}x_{2,2},\ x_1, x_2 \in \mathbb{C},\ f\colon \mathbb{C} \to \mathbb{C}$ is S^1-equivariant, $k \in \mathbb{R}$. In standard equivariant bifurcation theory, this

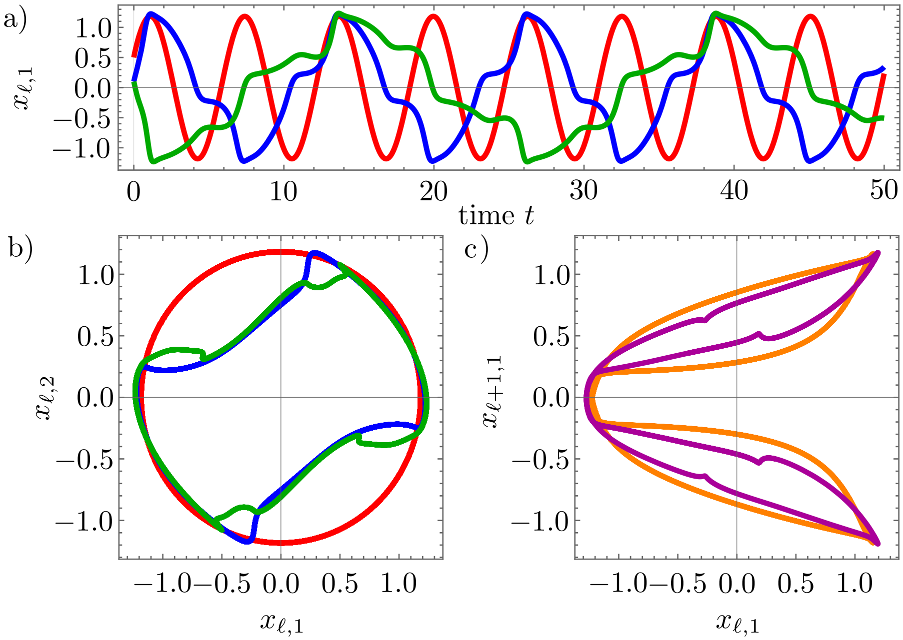

Figure 5.4.: Numerical simulations of (5.32) for $\lambda = 4$, $\gamma = -5$, $k = 3$ (red: $x_{1,1}$, blue: $x_{2,1}$, green: $x_{3,1}$) with initial conditions $x^0_{1,1} = 0.5, x^0_{1,2} = 1.0, x^0_{2,1} = 0.1, x^0_{2,2} = 1.0, x_{3,1} = 0.1, x_{3,2} = -1.0$ show a multi-frequency wave. Closed curves in the phase-space plots $(x_{\ell,1}, x_{\ell,2})$, $\ell = 1, 2, 3$, and $(x_{\ell,1}, x_{\ell+1,1})$, $\ell = 1, 2$ indicate a periodic orbit, the latter also shows the $1 : 2$ frequency relations (orange: x_1 vs. x_2, violet: x_2 vs. x_3).

system would be considered as an *odd* system, which would trigger us to look for odd periodic orbits of the form

$$(x_1, x_2)(t + p/2) = -(x_1, x_2)(t). \tag{5.37}$$

Let us see what more groupoid symmetries we can find.

Due to the S^1-equivariance of the nonlinearity f, we search for periodic orbits of (5.36) with $x_1(t) = ae^{\mathrm{i}\phi t}$. That is, the first oscillator x_1 describes a rotating wave,

$$x_1(t) = e^{-\mathrm{i}\phi\tau} x_1(t + \tau). \tag{5.38}$$

Moreover, switching real and imaginary parts, we also find that for each individual oscillator a rotation by $\pi/2$ corresponds to a phase-shift by 1/4, i.e.,

$$x_2(t) = \mathrm{i}x_2(t + p/4). \tag{5.39}$$

We therefore find the vertex group of the periodic orbit x to be

$$(H, \Theta)_x = \left(S^1 \times \mathbb{Z}_4, S^1 \times \mathbb{Z}_4\right). \tag{5.40}$$

Both the spatial and the temporal vertex isotropy groups are trivial.

Due to the form of the coupling, the input on x_2 has double the frequency of x_1, and we find the following conjugating morphisms:

$$\begin{aligned}
\big(\rho(\mathrm{Id}, (p/2, 0))x\big)(t) &:= \big(x_1(t + p/2), x_2(t)\big),\\
\big(\rho(\mathrm{Id}, (0, p/2))x\big)(t) &:= \big(x_1(t), x_2(t + p/2)\big),\\
\big(\rho((-1, 1), (0, 0)x\big)(t) &:= \big(-x_1(t), x_2(t)\big),\\
\big(\rho((1, -1), (0, 0)x\big)(t) &:= \big(x_1(t), -x_2(t)\big),\\
\big(\rho(\mathrm{Id}, \varphi)x\big)(t) &:= \big(x_1(t + \varphi), x_2(t + \varphi)\big).
\end{aligned} \tag{5.41}$$

We will discuss more examples of periodic orbits in Chapter 6.

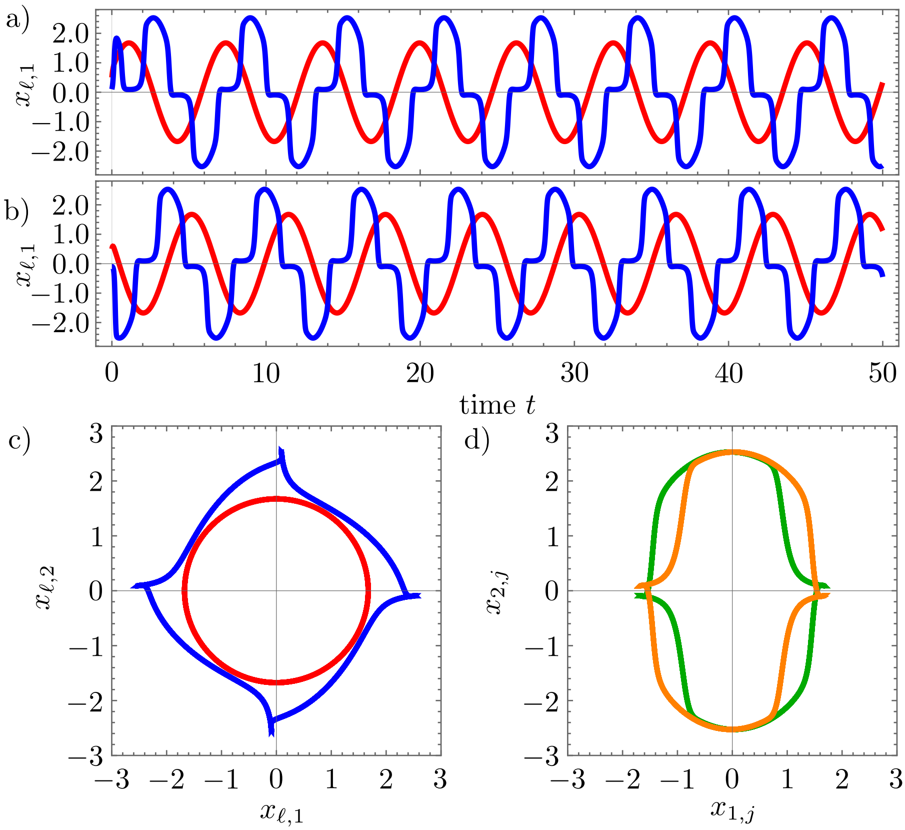

Figure 5.5.: Numerical simulations of (5.36) for $\lambda = 4$, $\gamma = -5$, $k = 10$ (red: $x_{1,1}$, blue: $x_{2,1}$) with initial conditions $x^0_{1,1} = 0.5, x^0_{1,2} = -1.0, x^0_{2,1} = -0.1, x^0_{2,2} = 1.0$ show a mixed wave of $S^1 \times \mathbb{Z}_4$-type. a) + b) Time plots of the two conjugated waves. c) Closed curves in the phase-space $(x_{\ell,1}, x_{\ell_2})$, $\ell = 1, 2$ indicate a periodic orbit of $S^1 \times \mathbb{Z}_4$-type. d) A different phase-plot shows that the conjugating morphisms also give rise to solutions of the same system.

5.3. Equivaroid Hopf bifurcation

In this section we will state and prove the main result of this chapter which is equivaroid Hopf bifurcation. Standard Hopf bifurcation goes back to Poincaré (1892) [51] and Andronov (1937) [5], as well Hopf (1942) [32]. We will use here the infinite-dimensional version due to Crandall and Rabinowitz (1977) [16], as well as the idea of proof going back to Vanderbauwhede (1980) [62].

Before proceeding to the main theorem, let us treat the linear part first, that is, for any system

$$\dot{x} = f(\lambda, x) = A(\lambda)x + N(\lambda, x), \tag{5.42}$$

where $A(\lambda) = \mathrm{D}_x\, f(\lambda, 0)$ and $N(\lambda, x) \in O(|x|^2)$, we now only look at

$$\dot{y} = A(0)y, \quad y \in \mathbb{R}^N,\ y(0) = y_0. \tag{5.43}$$

Then we know from Chapter 3 that the linearization at the trivial equilibrium of a $(\Gamma \rightrightarrows I)$-equivaroid system is at least $(\Gamma \rightrightarrows I)$-equivaroid. In particular, all vertex spaces X_j of (5.42) are also vertex spaces of (5.43) with vertex symmetry groups H_j.

Now assume that $A(0)$ possesses complex conjugated eigenvalues $\pm\mathrm{i}$ and a corresponding complex geometric eigenspace E. Then any $y(t) \in E$, $y(t) \neq 0$, possesses minimal period $p = 2\pi$. Similar to the equivariant case, the flow $\exp(A(0)t)$ itself defines an S^1-action on E:

$$\vartheta y_0 := e^{-2\pi i\vartheta} y_0 \tag{5.44}$$

Note that we will only treat the torus in the following section and stay with the S^1-action here. Now, for any $y_0 \in X_j$, where X_j is a vertex space of (5.42), and corresponding vertex symmetry group H_j, this defines an action of the twisted spatio-temporal symmetry group (H_j, Θ_j) on E, as follows. Let $(\gamma, \vartheta) \in (H_j, \Theta_j)$ and $y_0 \in X_j$, then

$$(\gamma, \vartheta)y_0 := \gamma e^{-2\pi i\vartheta} y_0 = \gamma y(-\vartheta). \tag{5.45}$$

In this way, we see that

$$(\tilde{H}_j, \Theta_j) := \{(\gamma, \theta) \mid \gamma y(t - \vartheta) = y(t) \text{ for all } t\} \tag{5.46}$$

is indeed a subgroup of the vertex isotropy group of $y(t)$ defined by (5.43), seen as an element of the space $\mathbb{X}$ of 2π-periodic functions.

We are now ready to formulate *equivaroid Hopf bifurcation*.

Theorem 5.3.1 (Equivaroid Hopf bifurcation)**.** *Let $f\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a C^k-vectorfield, $k \geq 2$, with $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Let f be $(\Gamma \rightrightarrows I)$-equivaroid for all $\lambda \in \mathbb{R}$. Let $X_j \subseteq X$ be a vertex space with vertex symmetry group H_j and corresponding twisted symmetry group $(\tilde{H}_j, \Theta_j)$.*

Let

$$\dot{x} = f(\lambda, x) = A(\lambda)x + N(\lambda, x), \tag{5.47}$$

where $A(\lambda) = \mathrm{D}_x f(\lambda, 0)$ and $N(\lambda, x) \in O(|x|^2)$. Assume

i) ***purely imaginary eigenvalues:*** $\pm \mathrm{i} \in \operatorname{spec} A(0)$ *are eigenvalues with equal algebraic and geometric multiplicity;*

ii) ***symmetry:*** $\dim_{\mathbb{R}} E_{(\tilde{H}_j, \Theta_j)} = 2$*;*

iii) ***nonresonance:*** $\pm n\mathrm{i} \notin \operatorname{spec} A(0)$ *for* $n \in \mathbb{Z} \backslash \{\pm 1\}$*;*

iv) ***transverse crossing:*** *the continuation* $\eta(\lambda) \in \operatorname{spec} A(\lambda)$ *of* $\eta(0) = \pm\mathrm{i}$ *crosses the imaginary axis transversely at $\lambda = 0$, i.e.,* $\operatorname{Re} \eta'(0) \neq 0$.

Then there exists a local C^{k-1}-branch, parametrized by s,

$$s \mapsto (\lambda(s), p(s), \tilde{x}(t, s)) \tag{5.48}$$

of periodic solutions $\tilde{x}(\cdot, s)$ of (5.47) with minimal period $p(s)$ for $s \neq 0$. At $s = 0$, bifurcation occurs from $(\lambda, p, x) = (0, 2\pi, 0)$, with $0 \neq \partial_s \tilde{x}(t, 0) \in E_{(\tilde{H}_j, \Theta_j)} \backslash \{0\}$. The branch is unique up to reparametrization of s and shifting time t. Moreover, the spatio-temporal symmetry of $\tilde{x}(s, \cdot) \in X_j$ is at least $(\tilde{H}_j, \Theta_j)$, for all $s \neq 0$.

Proof. We use the functional setting in Section 5.1. The equation $\mathcal{F}(\lambda, p, x) = \dot{x} - \frac{p}{2\pi} f(\lambda, x) = 0$ on the spaces $\mathbb{X}, \mathbb{Y}$ is then equivaroid with vertex spaces $\mathbb{X}_j, \mathbb{Y}_j$ and the vertex isotropy groups $(\tilde{H}_j, \Theta_j)$. We therefore first reduce to $\mathbb{X}_j$, $\mathbb{Y}_j$. Next, we apply standard Hopf bifurcation [25] to the restricted system, which provides us with a branch of nontrivial solutions which lie in X_j and whose spatio-temporal symmetry is at least given by $(\tilde{H}_j, \Theta_j)$, for all $s \neq 0$. □

The new equivaroid Hopf bifurcation theorem gives rise to those spatio-temporal patterns discussed in the previous section which can be described by an S^1-time shift. We will close this section with an example, before we move on to multi-frequency and toroidal waves in the following section.

Example 5.3.2 (Turing ring of four oscillators, inspired by [2], giving rise to a mixed wave of non-orthogonal type)**.** We consider

$$\begin{aligned} \dot{z}_1 &= f(\lambda, z_1) + k(z_4 + z_2) \\ \dot{z}_2 &= f(\lambda, z_2) + k(z_1 + z_3) \\ \dot{z}_3 &= f(\lambda, z_3) + k(z_2 + z_4) \\ \dot{z}_4 &= f(\lambda, z_4) + k(z_3 + z_1), \end{aligned} \tag{5.49}$$

where $f\colon \mathbb{C} \to \mathbb{C}$ is an odd function, $\mathrm{D}_x f(\lambda, 0) = \lambda + \mathrm{i}$, and $k \in \mathbb{C}$. Then we find the (flow-invariant) subspace

$$\mathbb{X}_0 = \{(z_1, z_2, z_3, z_4) \mid z_1 \equiv z_3 \equiv 0, z_2 \equiv -z_4\}. \tag{5.50}$$

Linearization of (5.49) at zero yields the system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} \lambda + \mathrm{i} & k & 0 & k \\ k & \lambda + \mathrm{i} & k & 0 \\ 0 & k & \lambda + \mathrm{i} & k \\ k & 0 & k & \lambda + \mathrm{i} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}. \tag{5.51}$$

At $\lambda = 0$, we find a geometrically and algebraically double eigenvalue $\pm\mathrm{i}$, and no other eigenvalues on the imaginary axis. The eigenspace is spanned by the eigenvectors $(0,-z,0,z)^T$ and $(-z,0,z,0)^T$. It has four dimensions. Restricted to the subspace $\mathbb{X}_0$, it is two-dimensional. As the eigenvalues cross the imaginary axis with non-zero speed, we find Hopf bifurcation into the subspace $\mathbb{X}_0$. Unfortunately, the bifurcating solution in this case is unstable, so we do not perform any simulation. See however Example 6.2.8 for the same pattern generated by a wider range of coupling matrices where this pattern is in effect stable.

5.4. Iterated equivaroid Hopf bifurcation

Similar to the case of steady-state bifurcation, the equivaroid Hopf bifurcation theorem does not always give us all periodic solutions near a bifurcation point. Some of these solutions can be found through *iterated equivaroid Hopf bifurcation.* This includes e.g., multi-frequency solutions but also other patterns with more complicated temporal structures.

Theorem 5.4.1 (Iterated equivaroid Hopf bifurcation)**.** *Let $f\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be a C^k-vectorfield, $k \geq 2$, with $f(\lambda,0)=0$ for all $\lambda\in\mathbb{R}$. Let f be $(\Gamma \rightrightarrows I)$-equivaroid for all $\lambda\in\mathbb{R}$. Let $X_j \subseteq \mathbb{R}^N$ be a vertex space with vertex symmetry group H_j and corresponding twisted symmetry group $(\tilde{H}_j,\Theta_j)$.*

Suppose that $X_j = M_j \oplus N_j$, such that any solution $x\in X_j$ of $\mathcal{F}(\lambda,x)=0$, $\mathcal{F}\colon \mathbb{R}\times X_j \to X_j$ can be written in the form $x=\mu+\nu(\mu)$ where $\mu=\mu(\lambda)$ is a solution of $\dot{\mu}=f(\lambda,\mu)$ with (not necessarily minimal) period 2π, $f:\mathbb{R}\times M_j \to X_j$, obtained by standard Hopf bifurcation of eigenvalues $\pm m_1\mathrm{i}$ at $\lambda=0$, and $\nu\in N_j$. Assume that $\dot{\nu}=f(\lambda,\mu+\nu(\mu))$ is S^1-equivariant in the sense of equation (5.6)*.*

Let

$$\dot{x} = f(\lambda,x) = A(\lambda)x + N(\lambda,x), \tag{5.52}$$

where $A(\lambda)=\mathrm{D}_x f(\lambda,0)$ and $N(\lambda,x)\in O(|x|^2)$. Assume *i)* ***purely imaginary eigenvalues:*** $\pm m_1\mathrm{i} \in \operatorname{spec} A(0)|_{M_j}$ *and* $\pm m_2\mathrm{i} \in \operatorname{spec} A(0)_{N_j}$ *are eigenvalues with algebraic multiplicity one in the restricted spaces,* $m_1, m_2 \in \mathbb{N}$ *and either* $m_1 = 1$ *or* $m_2 = 1$*. Here* $A(\lambda)_{N_j} := \mathrm{D}_\nu f(\lambda, \mu(\lambda) + \nu)|_{\lambda=0,\nu=0} \colon N_j \to X_j$*;*

ii) ***symmetry:*** $\dim_{\mathbb{R}} E_{N_j} = 2$*, where* E_{N_j} *denotes the complex geometric eigenspace to the eigenvalue* $\pm m_2\mathrm{i}$ *of* $A(\lambda)_{N_j}$*;*

iii) ***nonresonance:*** $\pm n\mathrm{i} \notin \operatorname{spec} A(0)$ *for* $n \in \mathbb{Z}\backslash\{\pm m_1, \pm m_2\}$*;*

iv) ***transverse crossing:*** *the continuation* $\eta(\lambda) \in \operatorname{spec} A(\lambda)_{N_j}$ *of* $\eta(0) = \pm m_2\mathrm{i}$ *crosses the imaginary axis transversely at* $\lambda = 0$*, i.e.,* $\operatorname{Re}\eta'(0) \neq 0$*.*

Then there exists a local C^{k-1}*-branch, parametrized by* s*,*

$$s \mapsto (\lambda(s), p(s), \tilde{x}(t,s)) \tag{5.53}$$

of periodic solutions $x(\cdot, s)$ *of* (5.52) *with minimal period* $p(s)$ *for* $s \neq 0$*. There is a 2-torus of solutions. At* $s = 0$*, bifurcation occurs from* $(\lambda, p, x) = (0, 2\pi, 0)$*. Moreover,* $x(t,s) \in X_j$*, and the minimal period on the components* μ, ν *is given by* $2\pi/m_1$*, and* $2\pi/m_2$*, respectively.*

Remark 5.4.2.

i) It seems that the condition on S^1-equivariance on the space N_j is not necessary, which means that the existence of many more spatio-temporal patterns could be proven with a similar construction. However, this is beyond the scope of this thesis and should be addressed in future research.

ii) As in the case of the iterated equivaroid branching lemma, we can apply iterated equivaroid Hopf bifurcation multiple times with multiple pairs of eigenvalues. We have chosen to present the version with a "double" Hopf bifurcation here to keep notation clean.

Proof. By assumption, the standard Hopf bifurcation Theorem by Crandall and Rabinowitz [16] gives a nontrivial solution $(\lambda(s), \mu(s))$ of $\dot{\mu} =$

$f(\lambda, \mu) = 0$, $f\colon \mathbb{R} \times M_j \to Y_j$ with $\lambda(0) = 0$, $\mu(0) = 0$. The other conditions then assure another standard Hopf bifurcation for the equation $\dot{\nu} = f(\lambda, \mu(\lambda) + \nu) = 0$ with $f\colon \mathbb{R} \times N_j \to Y_j$ where $\nu(0) = 0$. Putting this together, any solution $x \in X_j$ of $\dot{x} = f(\lambda, x) = 0$, $\mathcal{F}\colon \mathbb{R} \times X_j \to Y_j$ can be written in the form $(\lambda(s), x(s)) = (\lambda(s), \mu(s) + \nu(s, \mu(s)))$, which proves the theorem. $\square$

Let us end our discussion on iterated Hopf bifurcation with two examples. The first example gives rise to a multi-frequency wave.

Example 5.4.3 (Hopf-bifurcation of a multi-frequency wave)**.** We consider the following system

$$\begin{aligned} \dot{x}_1 &= (\lambda + 2\mathrm{i} + \gamma|x_1^2|)x_1 \\ \dot{x}_2 &= (\lambda + \ \ \mathrm{i} + \gamma|x_2^2|)x_2 + k\,(|x_1|^2 - |x_2|^2)x_2, \end{aligned} \tag{5.54}$$

where $x_1, x_2 \in \mathbb{C}$, $\lambda \in \mathbb{R}$ is the bifurcation parameter, $\gamma \in \mathbb{R}$ is the cubic coefficient, $\gamma \neq 0$, and $k \in \mathbb{R}$ denotes the coupling strength, $k \neq \gamma$. In this case the relevant vertex space is the full space $X = \mathbb{C}^2$, which we divide into $M = \{x_1 \in \mathbb{C}\}$ and $N = \{x_2 \in \mathbb{C}\}$. Linearization at the trivial zero equilibrium yields

$$\begin{aligned} \dot{z}_1 &= (\lambda + k + 2\mathrm{i})z_1 \\ \dot{z}_2 &= (\lambda + k + \mathrm{i})z_2. \end{aligned} \tag{5.55}$$

Consequentially, for $\lambda = 0$, we find the two purely imaginary pairs of eigenvalues $\pm\mathrm{i}, \pm 2\mathrm{i}$. On M we have the normal form of a Hopf bifurcation, which gives rise to periodic solutions with $|x_1(t)|^2 \equiv -\lambda/\gamma$. The equation on N is therefore given by

$$\begin{aligned} \dot{x}_2 &= (\lambda + \mathrm{i} + \gamma|x_2^2|)x_2 + k\left(-\tfrac{\lambda}{\gamma} - |x_2|^2\right)x_2 \\ &= \left(\lambda\left(1 - \tfrac{k}{\gamma}\right) + \mathrm{i} + (\gamma - k)|x_2^2|\right)x_2, \end{aligned} \tag{5.56}$$

which again corresponds to the normal form of a Hopf bifurcation. We therefore find a periodic solution with $|x_2(t)|^2 = -\lambda(1 - k/\gamma)/(\gamma - k) = -\lambda/\gamma$. We therefore find a Hopf bifurcation to a multi-frequency pattern. There is a torus of such solutions.

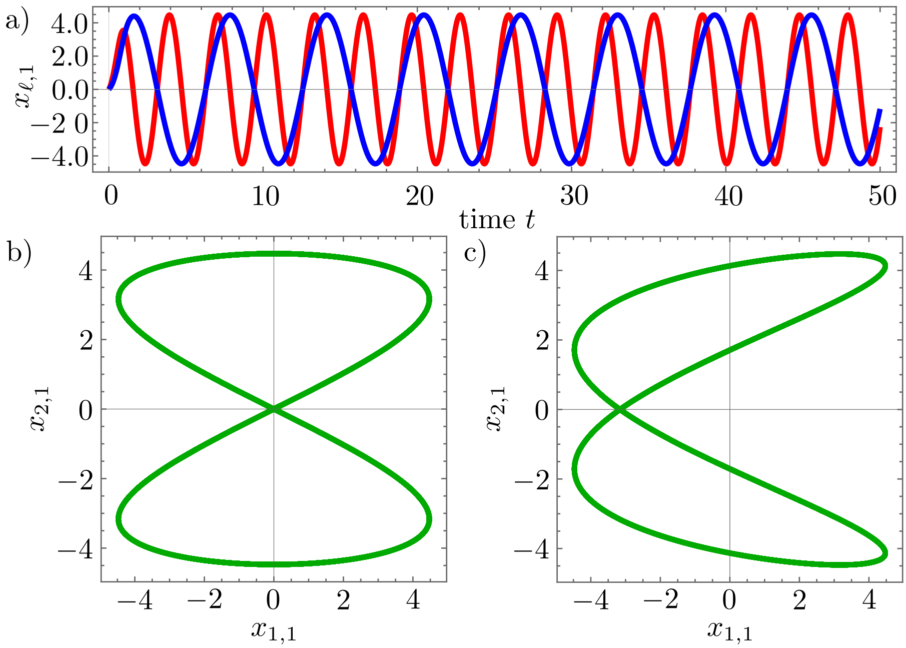

Figure 5.6.: Numerical simulations of (5.54) for $\lambda = -10$, $\gamma = -0.5$, $k = 10$ (red: $x_{1,1}$, blue: $x_{2,1}$) with initial conditions $x^0_{1,1} = 0.0, x^0_{1,2} = 1.0, x^0_{2,1} = 0.0, x^0_{2,2} = 2.0$ show a multi-frequency wave. a) Time-plot. b) Phase-plot: Closed curves in the phase-space $(x_{1,1}, x_{2,1})$ indicate a periodic orbit, the loop-like form also supports the 2 : 1-frequency relation. c) Phase-plot, but with different initial conditions, $x^0_{1,1} = -1.0, x^0_{1,2} = 1.0, x^0_{2,1} = 0.0, x^0_{2,2} = 2.0$. There is a torus of multi-frequency solutions.

In the following example we see that iterated equivaroid Hopf bifurcation can be useful also if both pairs of imaginary eigenvalues are $\pm\mathrm{i}$.

Example 5.4.4 (A concentric wave of $\mathbb{Z}_2$-type with a 2-torus of conjugating spatio-temporal symmetries, continued from Example 5.2.4)**.** We reconsider the network

$$\begin{aligned} \dot{x}_1 &= f(\lambda, x_1) + k_1\, g(x_2) \\ \dot{x}_2 &= f(\lambda, x_2) + k_1\, g(x_1) \\ \dot{x}_3 &= f(\lambda, x_3) + k_1\, g(x_3) + k_2\, g(x_1) - k_2\, g(x_2), \end{aligned} \tag{5.57}$$

with $x_1, x_2, x_3 \in \mathbb{R}^N$, $f\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $g\colon \mathbb{R}^N \to \mathbb{R}^N$, $k_1, k_2 \in \mathbb{R}^{N\times N}$, and $\lambda \in \mathbb{R}$ is the bifurcation parameter. We also suppose that $f(\lambda, 0) = 0$, $\mathrm{D}_x\, f(\lambda, 0) = \lambda + \mathrm{i}$, $g(0) = 0$, and $g'(0) = \alpha + \mathrm{i}\beta$. The Jacobi matrix $\mathcal{L}$ at the trivial equilibrium is then given by

$$\mathcal{L} = \begin{pmatrix} \lambda + \mathrm{i} & k_1\alpha + \mathrm{i}k_1\beta & 0 \\ k_1\alpha + \mathrm{i}k_1\beta & \lambda + \mathrm{i} & 0 \\ k_2\alpha + \mathrm{i}k_2\beta & -k_2\alpha - \mathrm{i}k_2\beta & \lambda + k_1\alpha + \mathrm{i} + \mathrm{i}k_1\beta \end{pmatrix}, \tag{5.58}$$

which means that there is a double pair of eigenvalues $\pm\mathrm{i}(1 + k_1\beta)$ at $\lambda = -k_1\alpha$. Each of these yields an equivaroid Hopf bifurcation into the vertex spaces $X_1 = \{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$ and $X_2 = \{(x_1, x_2, x_3) \mid x_3 = 0\}$. This is however not all that happens, and we can indeed observe iterated equivaroid Hopf bifurcation. To do this, we first need to find a suitable vertex space, and then a decomposition of that vertex space. The space that we are looking for is given by $X_1 = \{(x_1, x_2, x_3) \mid x_1 = x_2\}$. Identifying $x := x_1 = x_2$, the network reduces to

$$\begin{aligned} \dot{x} &= f(\lambda, x) + k_1\, g(x) \\ \dot{x}_3 &= f(\lambda, x_3) + k_1\, g(x_3). \end{aligned} \tag{5.59}$$

We can now split $X_1 = M_1 \oplus N_1$ where $M_1 = \{x \in \mathbb{R}\}$ and $N_1 = \{x_3 \in \mathbb{R}\}$ and we can perform standard Hopf bifurcation in either of these spaces, and then in the other, obtaining a 2-torus of bifurcating periodic solutions as discussed in Example 5.2.4. Note that in this specific example, there is no amplitude amplification because the system decouples completely.

6. Rational design of dynamical systems with groupoid symmetries

After introducing the notion of groupoid symmetries in the previous chapters, the next questions are: *Which dynamical systems allow for such groupoid symmetries? How can we design dynamical systems with prescribed symmetries?* Consequentially, in this chapter, we will address the theoretical background for the rational design of networks, moreover, we present explicit examples. These results also clearly illustrate the many different types of spatio-temporal patterns captured by the new groupoid approach.

In Section 6.1 a convenient notation of networks is introduced. In Section 6.2 we will design patterns defined purely by a vertex isotropy group. Section 6.3 deals with the design of networks where we prescribe not only the vertex isotropy group, but also the vertex quotient group.

6.1. Network Setting

In this chapter we deal exclusively with patterns in finite dimensions. This will help us concentrate on the patterns and avoid technical problems such as existence of uniqueness of solutions which could arise in infinite-dimensional systems. Note, however, that the general principle can and should be transferred to larger classes of dynamical systems.

Our focus lies on ordinary differential equations

$$\dot{x} = f(x), \tag{6.1}$$

with $x \in \mathbb{R}^{nN}$, $f\colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$, and we assume that equation (6.1) can be rewritten in network form, that is, x consists of n components of N-dimensional vectors,

$$x = (x_1, \ldots, x_n), \quad x_\ell \in \mathbb{R}^N. \tag{6.2}$$

Depending on the context, the individual components $x_\ell \in \mathbb{R}^N$ are called *oscillators*, *cells*, or *nodes* [26, 27]. Generalizations to oscillators with different dimensions are certainly possible but would lead us too far astray.

The individual oscillators x_i are *coupled* via M coupling matrices $\gamma^i \in \mathbb{R}^{nN\times nN}$, $i = 1, \ldots, M$, which are notated in block matrix form

$$\gamma^i = (\gamma^i_{\ell m})_{\ell,m=1,\ldots,n}, \tag{6.3}$$

with $\gamma^i_{\ell m} \in \mathbb{R}^{N\times N}$ for all $\ell, m = 1, \ldots, n$. We allow for nonlinear, but diagonal coupling functions

$$\begin{aligned} g^i\colon \mathbb{R}^{nN} &\to \mathbb{R}^{nN} \\ (x_1, \ldots, x_n) &\mapsto (g^i(x_1), \ldots, g^i(x_n)), \end{aligned} \tag{6.4}$$

which act on each oscillator x_ℓ, $\ell = 1, \ldots, n$, separately. In other words, in this notation we separate the nonlinear part g^i of the coupling from its nondiagonal part γ^i. In total, we consider networks of the form

$$\dot{x} = F\left(\gamma^1 g^1(x), \ldots, \gamma^M g^M(x)\right), \tag{6.5}$$

where $F\colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$. All involved functions are assumed to be sufficiently well-behaved to guarantee existence and uniqueness of solutions. This network class contains many examples studied in the literature, including e.g., [6, 9, 58, 60]. In particular, note that the oscillators are not required to possess the same internal dynamics.

Throughout the chapter, let $\tilde{X}$ be a linear subspace of $X = \mathbb{R}^{nN}$.

6.2. Designing patterns with a prescribed vertex isotropy group

In this section we prescribe a vertex space, or alternatively, its corresponding vertex isotropy group. It is our task to design a network of the form outlined in Section 6.1 which contains the desired vertex space and isotropy group.

It turns out that the coupling matrices γ^i of the network can be chosen out of the following monoid (a monoid fulfills all conditions of a group except invertibility).

Definition 6.2.1 (Full symmetry monoid of a vertex space $\tilde{X}$)**.** We call the set

$$\mathcal{M} := \{\gamma \in \mathbb{R}^{nN\times nN} \mid \gamma x \in \tilde{X} \text{ for all } x \in \tilde{X}\}, \tag{6.6}$$

together with matrix multiplication, the *full symmetry monoid* of $\tilde{X}$.

Why does the set $\mathcal{M}$ form a monoid? First, note that $\gamma = \mathrm{Id}$ is an element of $\mathcal{M}$. Second, for all $\gamma_1, \gamma_2 \in \mathcal{M}$, also $\gamma_1\gamma_2 \in \mathcal{M}$, and associativity follows by the standard associativity of matrix multiplication. On the other hand, we do not require invertibility of the elements $\gamma \in \mathcal{M}$, which implies that $\mathcal{M}$ does not form a group, but a monoid.

Remark 6.2.2. The set $\mathcal{M}$ is a linear subspace of $\mathbb{R}^{nN\times nN}$. Its crucial property is that its monoid action leaves the space $\tilde{X}$ invariant.

Together with the notation above, we obtain the following pattern design.

Theorem 6.2.3 (Designing patterns using the full symmetry monoid)**.** *Let the coupling matrices $\gamma^i \in \mathbb{R}^{nN\times nN}$, $i = 1, \ldots, M$, be elements of the full symmetry monoid $\mathcal{M}$ of $\tilde{X}$.*

Let a family of coupling functions $g^i \colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ with $g(\tilde{X}) \subseteq \tilde{X}$ be defined as in (6.4)*, and let $F(\tilde{X}, \ldots, \tilde{X}) \subseteq \tilde{X}$.*

Then $\tilde{X}$ is a vertex space of the following type of networks,

$$\dot{x} = F\left(\gamma^1 g^1(x), \ldots, \gamma^M g^M(x)\right). \tag{6.7}$$

Moreover, the vertex isotropy group of $\tilde{X}$ is given by

$$\tilde{K} = \{\gamma \in \mathbb{R}^{nN \times nN} \mid \gamma x = x \text{ for all } x \in \tilde{X}, \gamma \text{ invertible}\}. \tag{6.8}$$

Proof. The linear subspace $\tilde{X}$ is flow-invariant by construction: First, we require $g(\tilde{X}) \subseteq \tilde{X}$, then $\gamma^i \tilde{X} \subseteq \tilde{X}$ and lastly $F(\tilde{X}, \ldots, \tilde{X}) \subseteq \tilde{X}$. The vertex isotropy group follows immediately from Corollary 2.3.6. □

Remarks 6.2.4.

i) In particular, there always exist networks with the vertex space and vertex isotropy group. It is possible to construct any linear subspace, also prescribing seemingly exotic conditions like $x_1 = 3x_2$ or similar. The difficulty mostly lies in finding *nonlinear* F and g^i's which satisfy the flow-invariance conditions.

ii) We do not claim that all networks with a vertex space $\tilde{X}$ are of the form constructed in Theorem 6.2.3. This point should be investigated in future research.

iii) Note that such network constructions only yield information on the vertex isotropy groups, and none on the symmetry or quotient groups. In the next section we will discuss which $\gamma \in \mathcal{M}$ should be used to design networks with given vertex symmetry and quotient groups.

iv) Networks with more than one vertex space can be constructed by choosing coupling matrices which lie simultaneously in the full symmetry monoids of *all* the desired spaces. At this point, we do not know whether the intersection of two or more prescribed symmetry monoids contains more than the identity matrix. It remains to be studied which combinations of vertex spaces are readily realizable.

v) Conversely, constructions as in Theorem 6.2.3 will often yield more than one vertex space, since a chosen coupling matrix often belongs to the full symmetry monoid of another linear subspace of X as well.

vi) To obtain conjugating symmetries, it suffices to construct two distinct vertex spaces with the same dynamics on each vertex space.

Let us discuss a case which merits special attention: *full synchrony.* This corresponds to the subspace

$$X_S := \{(x_1, \ldots, x_n) \mid x_1 = \cdots = x_n\}. \tag{6.9}$$

We will consider here a relatively easy network type, with n identical oscillators x_ℓ, $\ell = 1, \ldots, n$, and simple additive coupling, for which we formulate a precise version of the folklore theorem: "Coupling matrices with constant row sums yield synchrony"; see e.g., [49].

Corollary 6.2.5 (Networks with full synchrony)**.** *Consider the synchrony subspace X_S. Choose a coupling matrix $\gamma \in \mathbb{R}^{nN \times nN}$ from its full symmetry monoid $\mathcal{M}_S$, which is given by*

$$\mathcal{M}_S = \left\{ \gamma \in \mathbb{R}^{nN \times nN} \,\middle|\, \sum_{m=1}^{n} \gamma_{1m} = \cdots = \sum_{m=1}^{n} \gamma_{nm} =: \hat{\gamma} \right\}. \tag{6.10}$$

Then X_S is a vertex space of the network

$$\dot{x}_\ell = f(x_\ell) + \sum_{m=1}^{n} \gamma_{\ell m}\, g(x_m), \qquad \ell = 1, \ldots, n, \tag{6.11}$$

with vertex isotropy group

$$K_S = \{\gamma \in \mathcal{M}_s \mid \gamma \text{ invertible, } \hat{\gamma} = \mathrm{Id}\}, \tag{6.12}$$

and the reduced dynamics on X_S is given by

$$\dot{x} = f(x) + \hat{\gamma}\, g(x). \tag{6.13}$$

Here, *reduced dynamics* means that all synchronized oscillators are identified.

Proof. The full symmetry monoid can be obtained by direct calculation. The remaining claims are an immediate corollary of Theorem 6.2.3. □

Let us discuss another special case, namely *partial synchrony*, or as it is also called, *balanced colourings* [26]. Partial synchrony is a partition P on the set of oscillators (more precisely: on its index set $\{1, \ldots, n\}$) such that the oscillators x_ℓ are identical on each cell in the partition; see also [26]. We define $\sim_P$ to be the equivalence relation with respect to the partition P, denote by $[\ell]$ the representative of the equivalence class containing ℓ, and its cell by

$$C^\ell := \{k \in \{1, \ldots .n\} \mid k \sim_P \ell\}. \tag{6.14}$$

We denote the *partial synchrony subspace* by

$$X_P = \{(x_1, \ldots, x_n) \mid \ell \sim_P m \text{ implies } x_\ell = x_m \text{ for all } \ell, m = 1, \ldots, n\}. \tag{6.15}$$

Corollary 6.2.6 (Networks with partial synchrony)**.** *Consider a partition P on the set of oscillators and its corresponding* partial synchrony subspace X_P*. Let the coupling matrix $\gamma \in \mathbb{R}^{nN\times nN}$ be an element of the full symmetry monoid $\mathcal{M}_P$ of the subspace X_P, which is given by*

$$\mathcal{M}_P = \left\{ \gamma \in \mathbb{R}^{nN\times nN} \,\middle|\, \sum_{\ell \in C^\ell} \gamma_{p\ell} = \sum_{\ell \in C^\ell} \gamma_{\tilde{p}\ell} =: \gamma_{[m][\ell]}, \right. \\ \left. p, \tilde{p} \in C^m, \text{for}\, m, \ell = 1, \ldots, n \right\}. \tag{6.16}$$

Then X_P is a vertex space of the network

$$\dot{x}_\ell = f(x_\ell) + \sum_{k=1}^{n} \gamma_{\ell k}\, g(x_k), \qquad \ell = 1, \ldots, n, \tag{6.17}$$

with vertex isotropy group

$$K_P = \left\{ \gamma \in \mathcal{M}_P \;\middle|\; \gamma \text{ invertible}, \; \sum_{\ell \in C^j} \gamma_{p\ell} = \mathrm{Id}, \; p \in C^j, \right. \\ \left. \sum_{\ell \in C^k} \gamma_{p\ell} = 0, \; p \in C^k, k \not\sim_P j, j, k = 1, \ldots, n \right\}. \tag{6.18}$$

The reduced network is given by

$$\dot{x}_{[m]} = f\big(x_{[m]}\big) + \sum_{\ell=1}^{M} \gamma_{[m][\ell]}\, g\big(x_{[m]}\big), \quad m = 1, \ldots, M, \tag{6.19}$$

where we choose one representative $[m]$ *of each equivalence class.*

Proof. The proof is completely analogous to the proof of Corollary 6.2.5. □

How robust are our results with respect to pertubations? This question concerns both the coupling matrices from the full symmetry monoid, as well as the invariance conditions on the nonlinearities. As a very first step towards an answer, each of the examples below includes also a perturbed version. We observe that the prescribed patterns are surprisingly robust. Further research needs to be conducted in this direction.

We present here three examples, which should illustrate the vastness of possible patterns even in very small networks. The simulations are performed with Mathematica using `NDSolve` and merely serve illustrational purposes.

Example 6.2.7 (Partial synchrony for three oscillators)**.** We prescribe the linear subspace $X_1 = \{(x_1, x_2, x_3) \mid x_1 = x_2\} \subset \mathbb{R}^{3N}$ where $x_\ell \in \mathbb{R}^N$ for $\ell = 1, 2, 3$ and want to construct a class of networks which possesses X_1 as a vertex space. As this is only a pattern of partial synchrony, there are no requirements on the nonlinearities f, g. In other words, the results are model-independent.

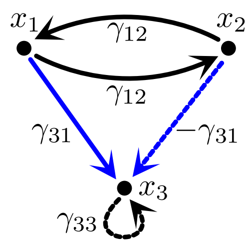

Figure 6.1.: Schematic depiction of the network (6.22). The coupling γ_{12} is drawn black and solid, γ_{31} solid blue, $-\gamma_{31}$ dotted blue, γ_{33} dashed black. The identical self coupling through f is omitted.

First, we determine the full symmetry monoid,

$$\mathcal{M}_1 = \left\{ \gamma \in \mathbb{R}^{3N\times 3N} \;\middle|\; \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{11}+\gamma_{12}-\gamma_{21} & \gamma_{13} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \right\}, \tag{6.20}$$

where $\gamma_{\ell m} \in \mathbb{R}^{N\times N}$. Choosing any coupling matrix out of the set $\mathcal{M}_1$, we *should* be able to observe two synchronous oscillators x_1 and x_2, and one non-synchronous oscillator x_3, according to the results in this section. *"Should"*, because we have not discussed stability yet. It turns out however, that it is easy to find a dynamical system with stable solutions within the subspace X_1, as we have many free parameters at our disposal: There are indeed seven matrices $\gamma_{\ell m}$ which can be chosen freely, and each matrix contains N^2 free parameters.

For the simulations, we choose the following set of coupling matrices,

$$\mathcal{H}_1 = \left\{ \gamma \in \mathbb{R}^{3N\times 3N} \;\middle|\; \gamma = \begin{pmatrix} 0 & \gamma_{12} & 0 \\ \gamma_{12} & 0 & 0 \\ \gamma_{31} & -\gamma_{31} & \gamma_{33} \end{pmatrix} \right\}, \tag{6.21}$$

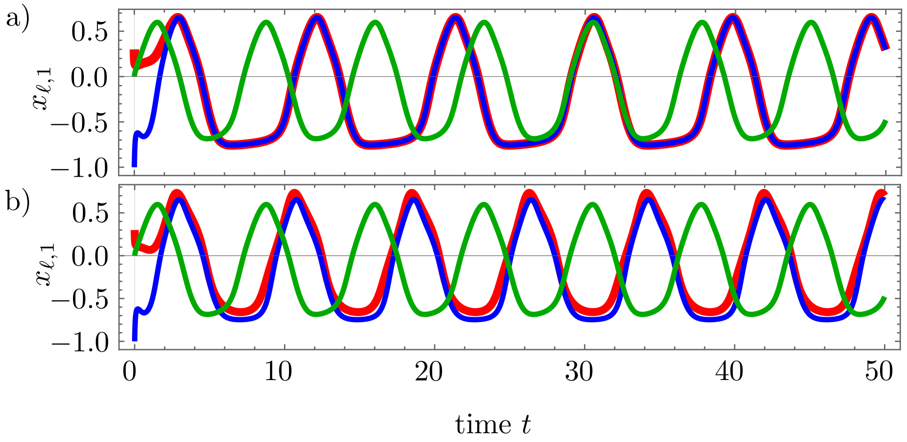

Figure 6.2.: a) Numerical simulation of the network (6.22) (red: $x_{1,1}$, blue: $x_{2,1}$, green: $x_{3,1}$). Here $x_{\ell,1}$, $x_{\ell,2}$ denote the first and the second component of x_ℓ, respectively. Initial conditions outside X_1 ($x^0_{1,1} = 0.3, , x^0_{1,2} = 2, x^0_{2,1} = -1.0, x^0_{2,2} = -1, x^0_{3,1} = 0, x^0_{3,2} = 2$) indicate a large region of stability for a periodic orbit in X_1. The oscillators x_1 and x_2 synchronize after a short transient while x_3 follows a different trajectory, as prescribed. The example uses no additional symmetries on f and g, specifically, the following dynamics were used: $f_1(x_{\ell,1}, x_{\ell,2}) = 4x_{\ell,1} + x_{\ell,2} - 10\left(x^2_{\ell,1} + x^2_{\ell,2}\right)x_{\ell,1} - 2\left(x^3_{\ell,1} + x^3_{\ell,2}\right)x_{\ell,1}$, $f_2(x_{\ell,1}, x_{\ell,2}) = -x_{\ell,1} + x_{\ell,2} - 10\left(x^2_{\ell,1} + x^2_{\ell,2}\right)x_{\ell,2} - 2\left(x^3_{\ell,1} + x^3_{\ell,2}\right)x_{\ell,2}$, $g(x_\ell) = x_\ell$, $\gamma_{12} = 0.8$, $\gamma_{13} = -0.08$, $\gamma_{33} = -0.02$. b) Perturbation of (6.22). Specifically, different $\gamma_{12} = 1.0$, $\gamma_{21} = 0.8$ for the coupling of the first and the second equation were used, and $f_1(x_{1,1}, x_{1,2}) = 4x_{1,1} + x_{1,2} - 10\left((x_{1,1} - 0.1)^2 + x^2_{1,2}\right)x_{1,1} - 2\left(x^3_{1,1} + x^3_{1,2}\right)x_{1,1}$. The pattern is still clearly visible.

which falls into the above category and yields the rather simple network

$$\begin{aligned}
\dot{x}_1 &= f(x_1) + \gamma_{12}\, g(x_2) \\
\dot{x}_2 &= f(x_2) + \gamma_{12}\, g(x_1) \\
\dot{x}_3 &= f(x_3) + \gamma_{31}\, g(x_1) - \gamma_{31}\, g(x_2) + \gamma_{33} g(x_3),
\end{aligned} \tag{6.22}$$

with $x_1, x_2, x_3 \in \mathbb{R}^2$, $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\gamma_{\ell m} \in \mathbb{R}^{2\times 2}$, $\ell, m = 1, 2, 3$. Graphically, the network is depicted in Fig. 6.1. The numerical simulations in Fig. 6.2 confirm a successful design of the prescribed pattern. The pattern is still clearly visible when a pertubation is included.

Last, the reduced network on the subspace X_1 is given by

$$\begin{aligned}
\dot{x}_{[1]} &= f(x_{[1]}) + \gamma_{12}\, g(x_{[1]}) \\
\dot{x}_{[2]} &= f(x_{[2]}) + \gamma_{31} g(x_{[2]}).
\end{aligned} \tag{6.23}$$

To finish this example, we find the vertex isotropy group as

$$K_1 = \left\{ \gamma \in \mathbb{R}^{3N\times 3N} \;\middle|\; \gamma = \begin{pmatrix} \gamma_{11} & \mathrm{Id} - \gamma_{11} & 0 \\ \gamma_{21} & \mathrm{Id} - \gamma_{21} & 0 \\ \gamma_{31} & -\gamma_{31} & \mathrm{Id} \end{pmatrix},\ \det \gamma \neq 0 \right\}, \tag{6.24}$$

where Id denotes the N-dimensional identity matrix, and 0 the N-dimensional zero matrix.

Example 6.2.8 (Partial amplitude death combined with algebraic conditions)**.** We prescribe the linear subspace

$$X_2 = \{(x_1, x_2, x_3, x_4) \mid x_1 = -x_3,\ x_2 = x_4 = 0\} \subset \mathbb{R}^{4N}, \tag{6.25}$$

where $x_\ell \in \mathbb{R}^N$ for $\ell = 1, 2, 3, 4$. We find the full symmetry monoid as

$$\mathcal{M}_2 = \left\{ \gamma \in \mathbb{R}^{4N\times 4N} \;\middle|\; \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{21} & \gamma_{24} \\ \gamma_{31} & \gamma_{32} & \gamma_{11} - \gamma_{13} + \gamma_{31} & \gamma_{34} \\ \gamma_{41} & \gamma_{42} & \gamma_{41} & \gamma_{44} \end{pmatrix} \right\}. \tag{6.26}$$

Next, to design X_2, we further suppose that f, g are odd functions. This ensures the invariance conditions, more precisely that $f(0) = 0$, $g(0) = 0$ and also allows for the condition $x_1 = -x_3$. We restrict ourselves to networks of the form

$$\dot{x}_\ell = f(x_\ell) + \sum_{m=1}^{n} \gamma_{\ell k}\, g(x_k), \qquad \ell = 1, \ldots, n. \tag{6.27}$$

For the simulations in Fig. 6.3, we choose the Stuart–Landau oscillator for the local dynamics and the coupling matrix

$$\gamma = 3 \begin{pmatrix} 0 & -10 & -1 & -1 \\ 5 & 2 & 5 & 1 \\ 3 & -1 & 4 & 3 \\ 2 & -6 & 2 & -10 \end{pmatrix}. \tag{6.28}$$

The reduced network on the subspace X_2 is given by

$$\begin{aligned} \dot{x}_{[1]} &= f\big(x_{[1]}\big) + 3\, g\big(x_{[1]}\big) \\ x_{[2]} &= 0. \end{aligned} \tag{6.29}$$

Note that the often used diffusive coupling of the form

$$\gamma = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \tag{6.30}$$

is an element of the full symmetry monoid as well, this explicit example and this specific subspace have been extensively discussed, since the invariant subspace does not correspond to a fixed-point subspace of the symmetry group; see e.g., in [2] about the Turing ring with four cells. Last, we find the vertex isotropy group as

$$K_2 = \left\{ \gamma \in \mathbb{R}^{4N\times 4N} \;\middle|\; \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{11} - \mathrm{Id} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{21} & \gamma_{24} \\ \gamma_{31} & \gamma_{32} & \gamma_{31} + \mathrm{Id} & \gamma_{34} \\ \gamma_{41} & \gamma_{42} & \gamma_{41} & \gamma_{44} \end{pmatrix},\ \det\gamma \neq 0 \right\}, \tag{6.31}$$

where $\gamma_{\ell m} \in \mathbb{R}^{N\times N}$. The desired dynamics are clearly visible, also for a randomly perturbed coupling matrix.

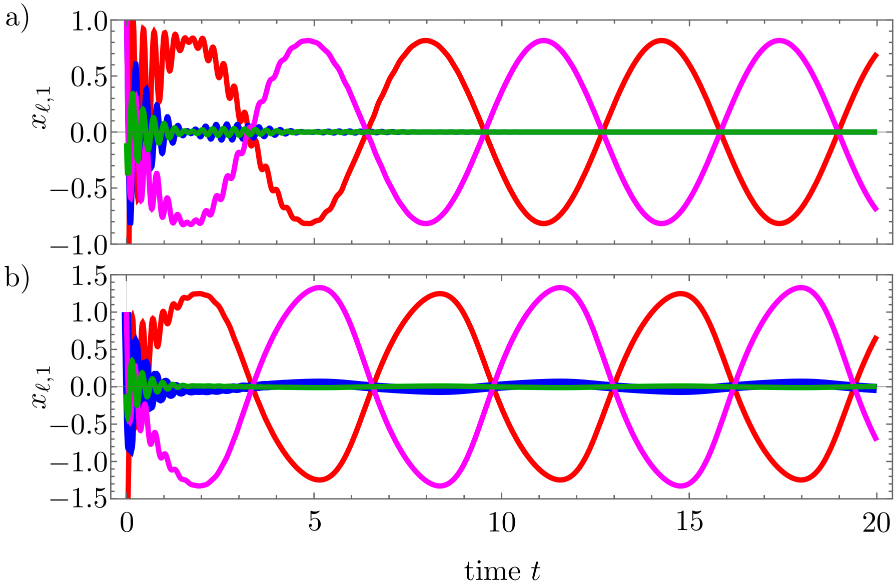

Figure 6.3.: a) Numerical simulation of (6.27) (red: $x_{1,1}$, blue: $x_{2,1}$, magenta: $x_{3,1}$, green: $x_{4,1}$). Here $x_{\ell,1}$, $x_{\ell,2}$ denote the first and the second component of x_ℓ, respectively. Initial conditions outside X_2 ($x^0_{1,1} = -2.0, x^0_{1,2} = 2.0, x^0_{2,1} = 1.0, x^0_{2,2} = -0.2, x^0_{3,1} = 1.0, x^0_{3,2} = -3.0, x^0_{4,1} = -0.1, x^0_{4,2} = 0.1$) indicate a large region of stability for an equilibrium in X_2. Specifically, the following dynamics were used: $f(x_{\ell,1}, x_{\ell,2}) = 4x_{\ell,1} + x_{i,2} - 1.5\left(x^2_{i,1} + x^2_{i,2}\right)x_{i,1}$, $f_2(x_{\ell,1}, x_{\ell,2}) = -x_{\ell,1} + 4x_{\ell,2} - 1.5\left(x^2_{\ell,1} + x^2_{\ell,2}\right)x_{\ell,2}$, $g(x_{\ell,1}, x_{\ell,2}) = (x_{\ell,1}, x_{\ell,2})$, γ as in equation (6.28). b) Pertubation of (6.27), random additions of $0, \pm 0.22$ to each element of the coupling matrix. The pattern is still visible.

Example 6.2.9 (Algebraic relations between oscillators)**.** As a final example, we now consider a two-oscillator network with $x_1, x_2 \in \mathbb{C}$ and the prescribed subspace

$$X_3 = \left\{ (x_1, x_2) \mid x_1 = e^{-\frac{2\pi \mathrm{i}}{3}} x_2 \right\}. \tag{6.32}$$

The full symmetry monoid of X_3 is given by

$$\mathcal{M}_3 = \left\{ \gamma \in \mathbb{C}^{2\times 2} \,\middle|\, \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{11} + \gamma_{12} e^{\frac{2\pi \mathrm{i}}{3}} - \gamma_{21} e^{-\frac{2\pi \mathrm{i}}{3}} \end{pmatrix} \right\}. \tag{6.33}$$

We consider the following explicit example,

$$\begin{aligned} \dot{x}_1 &= f(x_1) + \kappa\, e^{-\frac{2\pi \mathrm{i}}{3}} x_2 \\ \dot{x}_2 &= f(x_2) + \kappa\, e^{+\frac{2\pi \mathrm{i}}{3}} x_1, \end{aligned} \tag{6.34}$$

where $x_1, x_2 \in \mathbb{C}$, $\kappa \in \mathbb{C}$. Note that the coupling is an element of $\mathcal{M}_3$. To fulfill the invariance conditions, we additionally assume that f is $\mathbb{Z}_3$-equivariant, i.e.,

$$f\left(e^{-\frac{2\pi \mathrm{i}\ell}{3}} x\right) = e^{-\frac{2\pi \mathrm{i}\ell}{3}} f(x) \quad \text{for all } x \in \mathbb{C} \text{ and for } \ell = 0, 1, 2. \tag{6.35}$$

The reduced network on the subspace X_3 is given by

$$\dot{x}_{[1]} = f(x_{[1]}) + \kappa\, x_{[1]}. \tag{6.36}$$

Last, we determine the vertex isotropy group as

$$K_3 = \left\{ \gamma \in \mathbb{C}^{2\times 2} \,\middle|\, \gamma = \begin{pmatrix} \gamma_{11} & e^{-\frac{2\pi \mathrm{i}}{3}}(1 - \gamma_{11}) \\ \gamma_{21} & 1 - \gamma_{21} e^{-\frac{2\pi \mathrm{i}}{3}} \end{pmatrix},\ \det\gamma \neq 0 \right\}. \tag{6.37}$$

The pattern is clearly visible, also in the perturbed case; see Fig. 6.4.

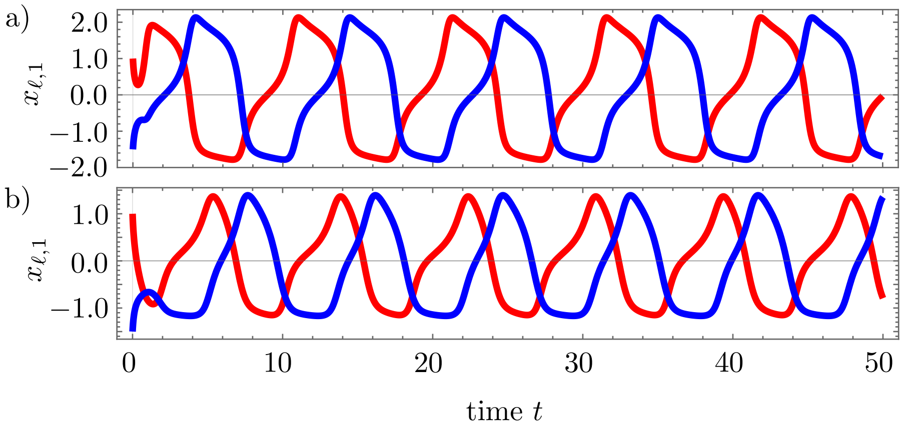

Figure 6.4.: a) Numerical simulation of (6.34) (red: $x_{1,1}$, blue: $x_{2,1}$, magenta: $x_{3,1}$, green: $x_{4,1}$). Here $x_{\ell,1}$, $x_{\ell,2}$ denote the first and the second component of x_ℓ, respectively. Initial conditions outside X_2 ($x^0_{1,1} = -2.0, x^0_{1,2} = 2.0, x^0_{2,1} = 1.0, x^0_{2,2} = -1.0, x^0_{3,1} = 1.0, x^0_{3,2} = -3.0, x^0_{4,1} = -0.5, x^0_{4,2} = 1$) indicate a large region of stability for an equilibrium in X_2. Specifically, the following dynamics were used: $f(x_{\ell,1}, x_{\ell,2}) = x_{\ell,1} - x_{\ell,2} - \left(x^2_{\ell,1} + x^2_{\ell,2}\right) x_{\ell,1} + 0.4 \left(x^2_{\ell,1} + x^2_{\ell,2}\right)$, $f_2(x_{\ell,1}, x_{\ell,2}) = x_{\ell,1} + x_{\ell,2} - \left(x^2_{\ell,1} + x^2_{\ell,2}\right) x_{\ell,1} - 0.8 \left(x^2_{\ell,1} + x^2_{\ell,2}\right)$, $\kappa = 1.4$. b) Pertubation of (6.34), f is no longer $\mathbb{Z}_3$-equivariant: $f(x_{\ell,1}, x_{\ell,2}) = x_{\ell,1} - x_{\ell,2} - 1.2 \left(x^2_{\ell,1} + x^2_{\ell,2}\right) x_{\ell,1} + 0.4 \left(x^2_{\ell,1} + x^2_{\ell,2}\right)$, $f_2(x_{\ell,1}, x_{\ell,2}) = x_{\ell,1} + x_{\ell,2} - 0.8 \left(x^2_{\ell,1} + x^2_{\ell,2}\right) x_{\ell,1} - 0.8 \left(x^2_{\ell,1} + x^2_{\ell,2}\right)$. The pattern is distorted yet still clearly visible.

6.3. Designing patterns with a nontrivial vertex quotient group

In this section we design networks with a prescribed vertex space as well as a prescribed nontrivial vertex quotient group. To this end, let us start with the following two definitions.

Definition 6.3.1 (Isotropy monoid of a linear subspace $\tilde{X} \subseteq X = \mathbb{R}^{nN}$)**.** We call the set

$$\mathcal{K} := \left\{ \gamma \in \mathbb{R}^{nN \times nN} \mid \gamma x = x \text{ for all } x \in \tilde{X} \right\}, \tag{6.38}$$

together with matrix multiplication, the *isotropy monoid* of $\tilde{X}$.

Note that the elements $\gamma \in \mathcal{K}$ need not be invertible, but that the vertex isotropy group $\tilde{K}$ is a subset of $\mathcal{K}$. We are dealing with a monoid since the identity is clearly an element of $\mathcal{K}$ and all elements can be composed.

Let us suppose that we want to prescribe a finite vertex quotient group $Q = \{q_1, \ldots, q_m\}$ on the space $\tilde{X}$. Then for each $x \in \tilde{X}$, the vertex quotient group reaches a finite number of points $q_j x \in \tilde{X}$, $j = 1, \ldots, m$.

Definition 6.3.2 (Symmetry monoid with a given quotient)**.** Let $\tilde{X} \subseteq X$ be a linear subspace of $X = \mathbb{R}^{nN}$. Let Q be a prescribed vertex quotient group of $\tilde{X}$, acting on the full space X. Let $\mathcal{K}$ be the isotropy monoid of the space $\tilde{X}$. We call the set

$$\begin{aligned} \mathcal{H} := \{\gamma \in \mathbb{R}^{nN \times nN} \mid \ & \text{for all } x \in \tilde{X} \text{ there exists a} \\ & j \in \{1, \ldots, m\} \text{ such that } \gamma x = q_j x\}, \end{aligned} \tag{6.39}$$

together with matrix multiplication, the *symmetry monoid of $\tilde{X}$ with quotient group Q*.

Let us reflect shortly on this definition. First, we are again dealing with a monoid: The identity is clearly an element of $\mathcal{H}$, and all elements may be composed. Second, note that $\mathcal{H}$ leaves the space $\tilde{X}$ invariant.

The symmetry monoid with a given quotient group can give rise to the prescribed pattern as described in the following theorem.

Theorem 6.3.3 (Patterns via the symmetry monoid with quotient group Q)**.** *Let the coupling matrices $\gamma^i \in \mathbb{R}^{nN\times nN}$, $i = 1,\dots,M$, be elements of the symmetry monoid $\mathcal{H}$ of $\tilde{X}$ with quotient group Q. Let $\tilde{H} = \{\gamma \in \mathcal{H} \mid \gamma \text{ invertible}\}$.*

Let a family of coupling functions $g^i\colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ with $g(\tilde{X}) \subseteq \tilde{X}$ be defined as above and let $F(\tilde{X},\dots,\tilde{X}) \subseteq \tilde{X}$. Additionally, for all $h \in \tilde{H}$ and for all $x, x_1, \dots, x_M \in \tilde{X}$, suppose that $hg(x) = g(hx)$, $h\gamma^i = \gamma^i h$ and $F(hx_1,\dots,hx_M) = hF(x_1,\dots,x_M)$.

Then $\tilde{X}$ is a vertex space of the following type of networks,

$$\dot{x} = F\left(\gamma^1 g^1(x), \dots, \gamma^M g^M(x)\right). \tag{6.40}$$

Moreover, the vertex isotropy group of $\tilde{X}$ is given by

$$\tilde{K} = \{\gamma \in \mathbb{R}^{nN\times nN} \mid \gamma x = x \text{ for all } x \in \tilde{X}, \gamma \text{ invertible}\}, \tag{6.41}$$

and its vertex symmetry group is given at least by $\tilde{H}$.

Proof. The linear subspace $\tilde{X}$ is flow-invariant by construction, as before. Moreover, F is $(\tilde{H} \rightrightarrows \tilde{X})$-equivaroid. $\square$

Remarks 6.3.4.

i) Again, we do not claim that all networks with a vertex space $\tilde{X}$ are of the above form and suggest this as a topic of future research.

ii) Compared to Theorem 6.2.3, the set of possible coupling matrices is notably smaller.

iii) The condition $F(hx_1,\dots,hx_M) = hF(x_1,\dots,x_M)$ is automatically fulfilled for additive coupling terms.

iv) To obtain inherited conjugating symmetries, it is useful to construct first the vertex space with the quotient group which should be inherited to smaller vertex spaces as conjugated morphisms.

In the following example, we will construct a class of networks which contains a specific vertex space, combined with a prescribed vertex quotient group. We will also comment on the spatio-temporal pattern that we would expect in this case and provide a simulation.

Example 6.3.5. We prescribe the linear subspace $X_3 = \{(x_1, x_2, x_3) \mid x_1 = x_3\} \subset \mathbb{R}^{3N}$ where $x_\ell \in \mathbb{R}^N$ for $\ell = 1, 2, 3, 4$. In this case, the isotropy monoid is given by

$$\mathcal{K}_3 = \left\{ \gamma \in \mathbb{R}^{3N\times 3N} \;\middle|\; \gamma = \begin{pmatrix} \gamma_{11} & 0 & \mathrm{Id} - \gamma_{11} \\ \gamma_{21} & \mathrm{Id} & -\gamma_{21} \\ \gamma_{31} & 0 & \mathrm{Id} - \gamma_{31} \end{pmatrix} \right\}, \tag{6.42}$$

where $\gamma_{\ell m} \in \mathbb{R}^{N\times N}$, $\ell, m = 1, 2, 3$.

Moreover, we prescribe the vertex quotient group, as it acts on $\mathbb{R}^{3N}$ as

$$Q_3 = \left\{ \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & -\mathrm{Id} \\ 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} \right\}. \tag{6.43}$$

Note that this group is isomorphic to $\mathbb{Z}^2$ and that it exchanges oscillators x_1 and x_3 with oscillator x_2.

From this, let us determine the symmetry monoid with the given quotient,

$$\mathcal{H}_3 = \left\{ \gamma \in \mathbb{R}^{3N\times 3N} \;\middle|\; \gamma = \begin{pmatrix} \gamma_{11} & \mathrm{Id} & -\gamma_{11} \\ \gamma_{21} & 0 & \mathrm{Id} - \gamma_{21} \\ \gamma_{31} & \mathrm{Id} & -\gamma_{31} \end{pmatrix}, \begin{pmatrix} \gamma_{11} & 0 & \mathrm{Id} - \gamma_{11} \\ \gamma_{21} & \mathrm{Id} & -\gamma_{21} \\ \gamma_{31} & 0 & \mathrm{Id} - \gamma_{31} \end{pmatrix} \right\}. \tag{6.44}$$

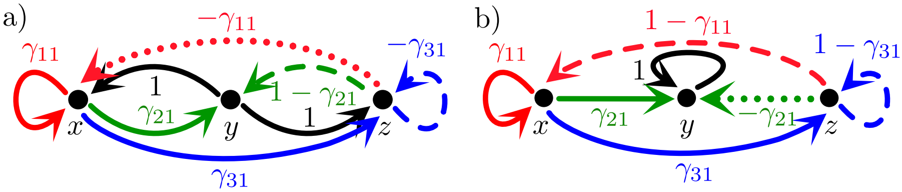

Figure 6.5.: The two different network types which allow for the vertex space X_3 in combination with the vertex quotient group Q_3.

As we choose additive coupling, there are no further conditions on commutativity. For the simulations, we now choose the following system:

$$\begin{aligned} \dot{x}_1 &= f(x_1) + \kappa g(x_2) \\ \dot{x}_2 &= f(x_2) + \kappa g(x_1) \\ \dot{x}_3 &= f(x_3) + \kappa g(x_2). \end{aligned} \tag{6.45}$$

Specifically, we choose $N = 2$ and $\kappa \in \mathbb{R}$. Its coupling matrix is given by

$$\gamma^* = \begin{pmatrix} 0 & \kappa \,\mathrm{Id} & 0 \\ \kappa \,\mathrm{Id} & 0 & 0 \\ 0 & \kappa \,\mathrm{Id} & 0 \end{pmatrix}, \tag{6.46}$$

which makes it an element of the symmetry monoid $\mathcal{H}_3$. Theorem 6.3.3 states that the network indeed possesses the prescribed vertex space, vertex isotropy, symmetry and quotient groups.

The vertex quotient group tell us that whenever there is a solution of (6.45) of the form (ξ_1, ξ_2, ξ_1), there is also a solution of the form (ξ_2, ξ_1, ξ_2), that is, the system restricted to the vertex space X_3 is $\mathbb{Z}_2$-equivariant. For periodic orbits we can therefore reasonably expect anti-phase patterns of the form

$$\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_1(t) \end{pmatrix} = \begin{pmatrix} \xi_2(t-p/2) \\ \xi_1(t-p/2) \\ \xi_2(t-p/2) \end{pmatrix}, \tag{6.47}$$

where $p > 0$ denotes the minimal period of said periodic orbit. For a specific example, such a periodic orbit can be found in Fig. 6.6 .

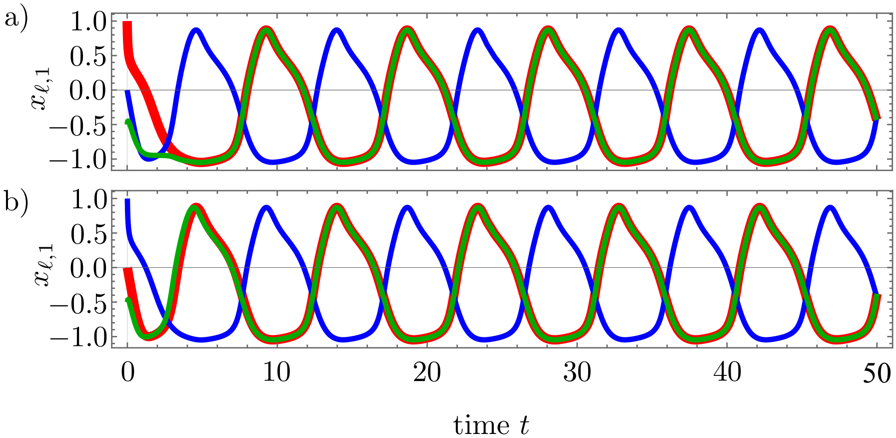

Figure 6.6.: a) and b) Numerical simulation of (6.45) for $N = 2$ (red: $x_{1,1}$, blue: $x_{2,1}$, green: $x_{3,1}$). Initial conditions far outside X_3 (for a) $x^0_{1,1} = 0.0, x^0_{1,2} = -1.0, x^0_{2,1} = 1.0, x^0_{2,2} = -2.0, x^0_{3,1} = -0.5, x^0_{3,2} = -2.0$, for b) $x^0_{1,1} = 1.0, x^0_{1,2} = -2.0, x^0_{2,1} = 0.0, x^0_{2,2} = -1.0, x^0_{3,1} = -0.5, x^0_{3,2} = -2.0$) indicate a large region of stability for a periodic orbit in X_3. While x_1 and x_3 become indistinguishable after a short time, x_2 is time-shifted by half the minimal period. We see both solutions of the form (ξ_1, ξ_2, ξ_1) and (ξ_2, ξ_1, ξ_2) (compare a) and b)). The example uses no additional symmetries on f and g, as this is not required. Specifically, the following dynamics were used: $f_1(x_{i,1}, x_{i,2}) = 4x_{i,1} + x_{i,2} - 5\left(x_{i,1}^2 + x_{i,2}^2\right)x_{i,1} - \left(x_{i,1}^3 + x_{i,2}^3\right)x_{i,1}, f_2(x_{i,1}, x_{i,2}) = -x_{i,1} + x_{i,2} - 5\left(x_{i,1}^2 + x_{i,2}^2\right)x_{i,2} - \left(x_{i,1}^3 + x_{i,2}^3\right)x_{i,2}$, $g(x_i) = 0.4x_i$.

7. Conclusion and discussion

To conclude this thesis on a new theory of symmetric dynamical systems which allows both a more general and a more refined investigation of spatio-temporal patterns, we comment on the symmetry groupoid and on equivaroid dynamical systems from a variety of perspectives. What have we achieved in this thesis? How does our work compare to other recent generalizations of symmetry groups? And which research questions will need to be answered in the future?

We present a short overview of our aims, methods, and results in Section 7.1. In Section 7.2 we discuss quiver symmetries and the groupoid formalism in comparison to the symmetry groupoid as presented in this thesis. In Section 7.3 we indicate open problems and give an outlook on further research.

7.1. Conclusion

It is the main goal of this dissertation to generalize the existing definition of symmetry and dynamical systems and create a refined but much more widely applicable theory of symmetric dynamical systems. In particular, we want to describe patterns in systems without group symmetry.

To this aim, we redefine symmetry on the basis of linear flow-invariant subspaces. Any linear isomorphism which maps a solution from one subspace to another solution, which might lie in the same subspace or in a different one, is defined to be a symmetry. In this way, we break down symmetry to the phenomenon that we want to describe: *Solutions are mapped to solutions.* In contrast to group symmetry, we do not require that all solutions are mapped to solutions, but only that all solutions in a given linear flow-invariant subspace are mapped to solutions. In view of their central importance, the linear flow-invariant subspaces are called *vertex spaces.*

It turns out that the algebraic structure of symmetries defined in this way is given by a *groupoid* and we define the set of all symmetries as the *symmetry groupoid.* This gives rise to notions such as *vertex spaces*, *vertex symmetry, isotropy and quotient groups* as well as *conjugating symmetries.*

We formulate the new symmetry definition in terms of a generalized equivariance — this provides us with *equivaroid maps* as well as a convenient condition for the generalization to the infinite-dimensional case and the bifurcation theorems.

On a more technical note, an equivaroid system restricted to a certain linear subspace is called *subequivaroid*, a term describing the particular way in which vertex spaces as well as symmetries are inherited to the restricted system. We also introduce *invaroid* subspaces, that is, special linear subspaces which are left invariant under the action of the full groupoid.

The equivaroid setting allows for a significant generalization of the well-known equivariant bifurcation theory. In particular, we prove an equivaroid version of the Lyapunov–Schmidt reduction and from this, two generalizations of steady-state bifurcation in equivaroid systems.

Moreover, the treatment of spatio-temporal patterns can be vastly generalized in the setting of equivariant systems: Going beyond discrete and rotating waves, we are now able to incorporate e.g. multi-frequency patterns into the theory. Existence of these patterns is established through the new (iterated) equivaroid Hopf bifurcation theorem.

Lastly, we have shown how to rationally design dynamical systems that allow for prescibed groupoid symmetries. In the context of networks, we find that the main algebraic objects in this context are the *symmetry and the isotropy monoids* of the prescribed vertex space. The corresponding results also illustrate the many different types of spatio-temporal patterns captured by the new groupoid approach.

7.2. Discussion of quiver symmetries and the groupoid formalism

The need to establish a theory of symmetry in dynamical systems which is more general than symmetry groups has been identified before. Most notably, the concept of *quiver symmetries* [46] and the *groupoid formalism* [26] have been developed over the past years. We will now shortly discuss both of these theories and compare them to the symmetry groupoid as developed in this thesis.

Let us start with the theory of quiver symmetries [46] by Nijholt, Rink and von der Gracht. Roughly speaking, *quivers* are algebraic structures similar to groupoids. They also consist of a set V of *objects or vertices* and of a set A of *morphisms* between those vertices with source and target maps.

As algebraic objects, quivers require less structure than groupoids: In particular, no inverse and no identity are required and there are no restrictions concerning any (partial) composition of morphisms. Therefore quivers are interpreted or even defined as directed multigraphs. For their use in the context of finite-dimensional dynamical systems, quivers are represented as follows: Each vertex v is represented by a finite-dimensional vector space E_v, and each morphism a is represented by a linear map $R_a : E_{s(a)} \to E_{t(a)}$ between those vector spaces, where $s(a)$ and $t(a)$ denote source and target of the morphism a.

Then a map F is called *quiver-equivariant* if it consists of a collection of maps $F_v : E_v \to E_v$, $v \in V$, such that

$$F_{t(a)} \circ R_a = R_a \circ F_{s(a)} \quad \text{for all } a \in A. \tag{7.1}$$

Comparing the symmetry quiver and the symmetry groupoid, we find that these objects share a subset of their morphisms, namely what we call the vertex quotient groups Q_j in this thesis. The other morphisms are either only found in the symmetry groupoid, e.g., the morphisms which form the vertex isotropy groups, or they are only found in the quiver, in this case the noninvertible morphisms between vertex spaces of different dimensions. This is because the definitions of symmetry and equivariance in the two contexts differ, and while the quiver requires less structure on the algebraic object, the groupoid symmetries require less structure on the dynamical system.

Both definitions of symmetry share that they are preserved by the Lyapunov–Schmidt reduction. Further similarities and differences will need to be examined in the future, as well as the possibility of an even more general theory which encompasses both definitions of symmetry.

Another theory which generalizes symmetry is the *groupoid formalism* by Golubitsky and Stewart [26]. It is explicitly intended to describe the symmetries in networks of coupled dynamical systems. As such, their symmetry groupoid only uses morphisms which can be described as *permutations* between different cells.

More precisely, in a directed network consisting of a finite set $\mathcal{C}$ of coupled cells, and a finite set $\mathcal{E}$ of arrows ("couplings") between them, the *input set* of $c \in \mathcal{C}$ is defined as

$$I(c) = \{e \in \mathcal{E} \mid e \text{ points towards } c\}. \tag{7.2}$$

Then an equivalence relation $\sim_I$ on C is defined by

$$c_1 \sim_I c_2 \iff \beta\colon I(c_1) \to I(c_2) \text{ is an arrow-preserving bijection,} \tag{7.3}$$

where we call β an *input isomorphism.* Let $B(c_1, c_2)$ denote the set of all input isomorphisms from cell c_1 to cell c_2. Then the *symmetry groupoid* *of a network* in the context of the groupoid formalism is defined as

$$\mathcal{B} = \bigcup_{c_j, c_k \in \mathcal{C}} B(c_j, c_k). \tag{7.4}$$

From this, the authors construct *polydiagonals*, that is, flow-invariant subspaces of partial synchrony. The advantage is that these subspaces can be found directly by simply “colouring” the cells (“balanced colouring”).

Comparing with the symmetry groupoid as defined in this thesis, we find that it consists only of a subset of the groupoid we discussed in this thesis, namely of those morphisms which act as permutations. The groupoid formalism by Golubitsky and Stewart is therefore not able to describe and predict patterns which go beyond partial synchrony and rigid rotating waves.

7.3. Open questions and work for the future

In this thesis, we have introduced a new theory of symmetric dynamical systems which allows both a more general and a more refined investigation of spatio-temporal patterns. The new theory leaves many open questions as well as prospective research topics, some of which we will discuss in the following. Here we present a selection of questions which arise directly from the results of this thesis.

We order the questions according to topic.

On groupoid symmetries in general

- *Is there a systematic way to find all vertex spaces and groupoid symmetries of a given dynamical system?*
  This is of course a central question as groupoid symmetries get used in applications and should be given a high priority, especially in the context of high- or infinite-dimensional dynamical systems, where the symmetries are far from obvious. In finite networks, the patterns of partial synchrony can be partly covered by the balanced colorings from [26, 36]. However, they cover only a very small

subset of the dynamical systems and patterns presented in this thesis.

- *Is there a feasible way to unify the theories of quiver and groupoid theories?*
  In this thesis, we have focused on invertible morphisms as symmetries; thus excluding quivers as the algebraic structure behind symmetries. As we discussed in the previous section, there is an overlap with the quiver symmetries. Comparing bifurcation scenarios would be highly interesting, however, to date, there is no general quiver-equivariant bifurcation theory.

- *How do the symmetries persist when a system is only approximately equivaroid?*
  This question is aimed at applications in the natural sciences, where perfectly equivaroid systems are an unrealistic scenario. First numerical examples indicate that the patterns are quite robust; see Chapter 6 for some examples, but this is of course no evidence and the question needs to be addressed in future research.

- *How can we find symmetries when the corresponding flow-invariant subspaces are not linear?*
  The choice of coordinates is an important question even for group-symmetric systems, and sometimes a better choice of coordinates can make more symmetries visible (such as in the Kepler problem or the hydrogen atom [7, 21, 45]). In equivaroid systems, this problem becomes even more evident as some of the vertex spaces and their corresponding symmetries can be "overlooked" due to an unfortunate choice of coordinates.

On equivaroid bifurcation theory

- *Is it possible to obtain global bifurcation results?*
  In the spirit of global bifurcation results such as in [3, 20] it will be a future challenge to extend the local results from this thesis to global bifurcation theorems.

- *Do certain bifurcations in their normal form imply more groupoid symmetries?*
  To finite order, Hopf bifurcations are S^1-equivariant. Can a similar phenomenon occur in different bifurcation scenarios, but then show groupoid symmetries instead? Such a scenario seems to be given in the situation of bifurcation without parameters [41], for example in the case of transcritical bifurcation with and without additional reflection symmetry.

- *Can we find different reversible Hopf bifurcation patterns in equivaroid systems?*
  In this thesis, we have focused on an equivaroid generalization of the equivariant Hopf bifurcation theorem. Consequently, we have neglected reversible Hopf bifurcation. It should however be very well possible to find spatio-temporal patterns which possess more than one reflection symmetry in time.

- *Which mode-interactions can occur in equivaroid systems?*
  This is a research topic which has been extensively discussed in the context of equivariant dynamical systems. Obviously, since all equivariant systems are also equivaroid, the known theory extends to our case as well. It should however be possible to find many more interesting examples due to the extended notion of patterns and isotropy. Mode-interactions from equivaroid systems are therefore a research topic which should be discussed in the future.

- *Which secondary bifurcations are possible in equivaroid systems?*
  So far, our results on bifurcation prove only bifurcation from the trivial equilibrium. Especially in the groupoid case, however, where the trivial equilibrium is seen as its own vertex space, we should be concerned with secondary bifurcation from nontrivial equilibria and periodic orbits. Our goal could be the construction of a *bifurcation tree*, similar to [18, 24].

On the rational design of equivaroid dynamical systems

- *Is it possible to generalize our results on the rational design of dynamical systems with prescribed vertex spaces to systems which are not of finite-network type?*
  The precise network form of the dynamical system that we have chosen in Chapter 6 is definitely not necessary for the construction, however, it simplifies matters and gives first insights. In the future, these results should be generalized.

- *How can we design spatio-temporal patterns not induced by a given vertex space?*
  To design spatio-temporal patterns not related to a given vertex quotient group of the symmetry group, we need to focus specifically on the nonlinearities. First examples of such constructions (although not yet in a systematic way) can be found in Chapter 5; see in particular Examples 5.2.6 and 5.2.7.

*****

In summary, we have set out to explore symmetries in dynamical systems form a new point of view. Based on their description as linear isomorphisms between linear flow-invariant subspaces and subsequently their algebraic groupoid structure, we have successfully developed an equivaroid bifurcation theory as well as a method to design systems with given symmetries. Many examples have shown an abundance of spatio-temporal patterns which can be studied with our new methods as well as a broad applicability in all the natural sciences.

A. A short introduction to groupoids

In this appendix we review the definition and some of the standard facts on groupoids without proofs. Moreover, we give a brief introduction to the representation theory of groupoids. As for prerequisites, the reader is assumed to be familiar with groups and their representation theory.

We proceed as follows: In Section A.1 we give the basic definition of a groupoid as well as several examples. In Section A.2 we proceed with the study the inner structure of a groupoid, focussing on vertex groups, conjugating morphisms and orbits. In Section A.3 we summarize without proof the relevant material on representation theory of groupoids.

A.1. Groupoid — Definition

The word *groupoid* consists of two parts: *group* and *-oid*. The suffix -oid comes from the greek εἶδος meaning *likeness, form* [35, 59].

Definition A.1.1 (Groupoid, [34, 38, 43, 47, 64], repetition of Def. 2.1.5)**.** Let B be a set. A *groupoid* is a set Γ of morphisms $\gamma : B \to B$, $\gamma \in \Gamma$, supplemented with the following maps:

- a surjective *source map* $s\colon \Gamma \to B$, $\gamma \mapsto s(\gamma)$,

- a surjective *target map* $t\colon \Gamma \to B$, $\gamma \mapsto t(\gamma)$,
- a *partial binary operation* defined on the set of composable morphisms $\Gamma \star \Gamma := \{(\gamma_2, \gamma_1) \in \Gamma \times \Gamma \,|\, t(\gamma_1) = s(\gamma_2)\}$:

$$\begin{aligned} \circ\colon \quad \Gamma \star \Gamma &\to \Gamma \\ (\gamma_2, \gamma_1) &\mapsto \gamma_2 \circ \gamma_1, \end{aligned} \tag{A.1}$$

- an injective *identity map* $e\colon B \to \Gamma$, $b \mapsto e(b) =: e_b$,

which satisfy the following properties:

i) the partial binary operation is associative, that is, for all (γ_3, γ_2), $(\gamma_2, \gamma_1) \in \Gamma \star \Gamma$, the identity $(\gamma_3 \circ \gamma_2) \circ \gamma_1 = \gamma_3 \circ (\gamma_2 \circ \gamma_1)$ holds;

ii) the identity map defines a family of identity morphisms in the following sense:

   a) for all $b \in B$: $s(e_b) = t(e_b) = b$,

   b) for all γ such that $s(\gamma) = b$: $\gamma \circ e_b = \gamma$,

   c) for all γ such that $t(\gamma) = b$: $e_b \circ \gamma = \gamma$;

iii) each morphism $\gamma \in \Gamma$ has a two-sided inverse $\gamma^{-1} \in \Gamma$ such that

$$s(\gamma) = t(\gamma^{-1}), \quad t(\gamma) = s(\gamma^{-1}), \quad \text{and} \tag{A.2}$$

$$\gamma^{-1} \circ \gamma = e_{s(\gamma)}, \quad \gamma \circ \gamma^{-1} = e_{t(\gamma)}. \tag{A.3}$$

We denote such a groupoid by $\Gamma \rightrightarrows B$. The set B is called the *base*, and its elements are called *objects*. Moreover, we call $s(\gamma)$ the *source* of the morphism γ, and $t(\gamma)$ its *target*.

Remarks A.1.2.

i) A groupoid is sometimes called a Brandt groupoid or a virtual group.

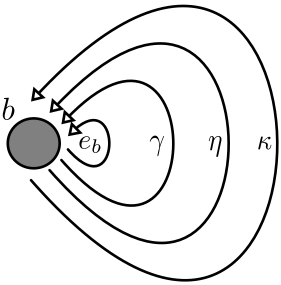

Figure A.1.: Graphical depiction of a group with four elements or morphisms $e_b, \gamma, \eta, \kappa$ from the groupoid viewpoint. It has one object b which is the source and target of all morphisms.

ii) In the language of category theory, a groupoid corresponds to a small category where every morphism is invertible [34]. This amounts to the same, since a category, by definition, consists of two classes, the objects and the morphisms, with two objects associated to every morphism, the source and the target.

Groupoids can come in surprisingly many forms, we will outline a few examples here.

Example A.1.3 (Groups, [34])**.** The simplest example of groupoids are *groups*. A group is a groupoid which only has one object, let us call it b. The morphisms of the groupoid are simply the elements of the group. They all have the same source and target object b. In this way all morphisms of the group under the groupoid viewpoint can be composed. Therefore, the source and target maps are usually omitted.

Example A.1.4 (Disjoint union of groups, [10])**.** Let G_k be groups, k in some index set K. Then any disjoint union $\Gamma = \bigcup_k G_k$ is a groupoid: In this case we assign to each group G_k a base point $k \in K$, the partial binary operation $g_2 \circ g_1$ of two morphisms g_1, g_2 is defined if and only if

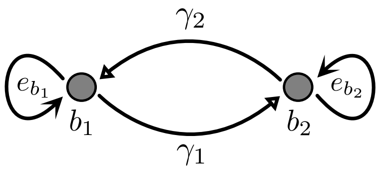

Figure A.2.: Graphical depiction of the extended singleton A_2 as an example of a groupoid which is not a group. It has two objects b_1 and b_2 as well as four morphisms $\gamma_1, \gamma_2, e_{b_1}, e_{b_2}$.

the morphisms belong to the same group G_k, in which case $g_2 \circ g_1$ is the product within the group G_k.

Example A.1.5 (The extended singleton, [34])**.** The extended singleton A_2 is the first example of a groupoid which is not a group (or multiple groups): It consists of two objects b_1, b_2 and four morphisms

$$e_{b_1} \colon b_1 \to b_1, \quad e_{b_2} \colon b_2 \to b_2, \quad \gamma_1 \colon b_1 \to b_2, \quad \gamma_2 \colon b_2 \to b_1, \tag{A.4}$$

where γ_1, γ_2 are invertible and

$$\gamma_2 \circ \gamma_1 = e_{b_1}, \quad \gamma_2 \circ \gamma_1 = e_{b_2}. \tag{A.5}$$

The extended singleton is not a group since $\gamma_1 \circ \gamma_1$ as well as $\gamma_2 \circ \gamma_2$ are not defined. Moreover, there exist two different identity elements, which is not permitted in a group.

Example A.1.6 (Groupoid of objects, [34])**.** On the other end of the spectrum, we can trivially associate a groupoid to any set B. Its objects are the elements $b \in B$ and its morphisms are the identity elements e_b. The sources and targets are given as follows:

$$s(e_b) = t(e_b) = b, \quad b \in B. \tag{A.6}$$

Example A.1.7 (Groupoid of ordered pairs, [34])**.** Let B be a set, and let $\Gamma := B \times B$. Define the objects as the individual elements of B, and its morphisms are given by the pairs $(b_2, b_1) \in B \times B$. The sources and targets are given by

$$s(b_2, b_1) = b_1, \quad t(b_2, b_1) = b_2. \tag{A.7}$$

Multiplication of these pairs is defined by

$$(b_3, b_2) \circ (b_2, b_1) = (b_3, b_1), \tag{A.8}$$

the left and right identity elements of (b_2, b_1) are given by (b_2, b_2) and (b_1, b_1), respectively:

$$(b_2, b_2) \circ (b_2, b_1) \circ (b_1, b_1) = (b_2, b_1), \tag{A.9}$$

inverses are given by $(b_2, b_1)^{-1} = (b_1, b_2)$.

Example A.1.8 (The action groupoid, [47, 34])**.** Let G be a group acting on a set B. Then we denote by $\Gamma(G, B) \rightrightarrows B$ the groupoid associated to the action of the group G on B. Its objects are the elements of B and whose morphisms are given by triples $(b_2, g, b_1) \in B \times G \times B$ such that $b_2 = gb_1$. The sources and targets are

$$s(b_2, g, b_1) = b_1, \ t(b_2, g, b_1) = b_2, \tag{A.10}$$

and the composition law is

$$(b_3, g', b_2) \circ (b_2, g, b_1) = (b_3, g'g, b_1). \tag{A.11}$$

Example A.1.9 (Equivalence relations, [64])**.** Let B be a set with an equivalence relation $\sim$ which we interpret as follows under the groupoid viewpoint: The objects of the groupoid are the elements of B, and for any two elements $b_1, b_2 \in B$, there is a (single) morphism from b_1 to b_2 if and only if $b_1 \sim b_2$.

Example A.1.10 (Lie groupoids, [38, 43, 44])**.** Building on Lie groups, which are groups that are also smooth manifolds, we define Lie groupoids [43, 44]: A groupoid $\Gamma \rightrightarrows B$ is called a Lie groupoid if Γ and B are smooth manifolds, the map e (identity), the inverse and the partial multiplication are all smooth and the maps s (source), and t (target) are subjective submersions.

Suppose B is a smooth manifold. Both the groupoid of objects on B and the pair groupoid are examples of Lie groupoids. Moreover, if G is a Lie group acting smoothly on B, then the corresponding action groupoid $\Gamma(G, B) \rightrightarrows B$ is a Lie groupoid.

Example A.1.11 (The hydrogen spectrum, [14, 29])**.** Last we consider a fascinating example from physics, the hydrogen spectrum. The lines of the hydrogen spectrum can be organized into a series of discrete wavelengths λ_{mn} indexed by m, n, each having the form

$$\frac{1}{\lambda_{mn}} = \frac{R}{m^2} - \frac{R}{n^2}, \tag{A.12}$$

where $m, n \in \mathbb{N}, n > m$, and R is the Rydberg constant. Let c be the wave speed. Now if we consider frequencies $\nu = c/\lambda$ instead of wavelengths, then the measured spectrum can by defined as a *set of differences of frequencies*. To this end, we introduce a set N of auxiliary frequencies $\nu_i = \frac{Rc}{i^2}$.

It follows that the spectrum of the hydrogen atom is the set of differences $\nu_{ij} = \nu_i - \nu_j$, where $\nu_i, \nu_j \in N, j > i$. To see the groupoid structure, we note that for two frequencies ν_{ij} and ν_{jk} in the spectrum, $\nu_{ik} = \nu_{ij} + \nu_{jk}$ is also in the spectrum. This is indeed a partially defined law of composition; in order to be combined, the frequencies must share the index j.

Formally, the groupoid is constructed as follows:

- base $B = N$,
- morphisms $\Gamma = \{(i, j) \in \mathbb{N} \times \mathbb{N} \,|\, j > i\}$,
- source $s(i, j) = \nu_j$,
- target $t(i, j) = \nu_i$,

- binary operation defined if $s(i,j) = t(j,k)$:

$$(i,j) \circ (j,k) = (i,k). \tag{A.13}$$

The connection of the abstract groupoid to the spectrum is given by $(i,j) \mapsto \nu_{ij}$. Note that this is a groupoid of pairs; see Example A.1.7.

A.2. Vertex groups, conjugating morphisms, and orbits

In this section we look more closely at the inner structure of a groupoid, focussing in particular on those notions are needed in order to develop the theory of symmetry of dynamical systems.

Definition A.2.1 (Vertex group [34, 43])**.** Let $\Gamma \rightrightarrows B$ be a groupoid and $b \in B$. The set of all morphisms $\gamma\colon b \to b$, with source and target b, is denoted by Γ_b and called the *vertex group* of b, i.e.,

$$\Gamma_b := \Gamma(b) := \Gamma(b,b) = \{\gamma \in \Gamma \mid s(\gamma) = b, t(\gamma) = b\}. \tag{A.14}$$

Remarks A.2.2.

i) In the literature, vertex groups are also called *isotropy groups*. We avoid this terminology because it is easily confused with "isotropy subgroups" from equivariant bifurcation theory.

ii) The vertex groups Γ_b, $b \in B$, of the groupoid $\Gamma \rightrightarrows B$ are indeed groups: To see this, note that every pair of morphisms in the set Γ_b is in the set of composable morphisms $\Gamma \star \Gamma$, since all elements of Γ_b have the same source and target.

The vertex group is a collection of morphisms from one base point to itself. A groupoid also allows for morphisms between different base points, which we call *conjugating morphisms*.

Definition A.2.3 ((b_j, b_k)-conjugating morphisms)**.** Let $b_j, b_k \in B$. We denote by $\Gamma(b_k, b_j)$ the set of morphisms between b_j to b_k, i.e.,

$$\Gamma(b_k, b_j) := \{\gamma \in \Gamma \,|\, s(\gamma) = b_j,\, t(\gamma) = b_k\}. \tag{A.15}$$

We call such morphisms *conjugating* if $b_j \neq b_k$.

Note that the vertex group of the base point b is simply given by $\Gamma(b, b)$.

It turns out that vertex groups are isomorphic if there exist conjugating morphisms between them, a fact that we will extensively use.

Proposition A.2.4 (Isomorphic vertex groups, [34])**.** *Let $b_j, b_k \in B$ and $b_j \neq b_k$. If there exist conjugating morphisms between two objects b_j, b_k, i.e., if $\Gamma(b_k, b_j) \neq \emptyset$, then the vertex groups Γ_{b_j} and Γ_{b_k} are isomorphic.*

Proof. Let $\gamma : b_j \to b_k$ belong to $\Gamma(b_k, b_j)$. Then

$$\Phi_\gamma \colon \Gamma_{b_j} \to \Gamma_{b_k} \tag{A.16}$$

$$\gamma_{b_j} \to \gamma_{b_k} = \gamma \circ \gamma_{b_j} \circ \gamma^{-1} \tag{A.17}$$

is a group homomorphism. □

An important notion for groupoids are *orbits*.

Definition A.2.5 (Groupoid orbit, [34])**.** Given an object $b_* \in B$, the *orbit* $\mathcal{O}_{b_*}$ of the groupoid $\Gamma \rightrightarrows B$ through b_* is the set of *objects* b_k corresponding to the target objects of morphisms starting at b_*. In other words, $b_k \in \mathcal{O}_{b_*}$ if and only if there exists a morphism $\gamma \in \Gamma$ such that $\gamma \colon b_* \to b_k$.

Note that the orbits $\mathcal{O}_b$ define a partition of the base B into disjoint sets.

Definition A.2.6 (Connected groupoids, [43])**.** We say that a groupoid is *connected* or *transitive* if it has just one orbit, i.e., if all objects are connected via morphisms.

We finish this section by revisiting the groupoid examples from above and determining the vertex groups, conjugating morphisms, and the orbits, each.

Example A.2.7 (Groups, continued from Example A.1.3)**.** A group G has only one base point b for the vertex group Γ_b equals G, i.e., *the* group. As a consequence, there are no conjugating morphisms, and only one orbit, consisting of the only object b. Thus a group is a connected groupoid.

Example A.2.8 (Disjoint union of groups, continued from Example A.1.4)**.** In a disjoint union $\Gamma = \bigcup_k G_k$ of groups G_k, k in some index set K, we allocate a base point b_k to each group. Then the vertex group of the base point b_k is given by the group $\Gamma_{b_k} = G_k$. There are no conjugating morphisms between the groups. The groupoid orbits are given by the individual objects $\mathcal{O}(b_k) = \{b_k\}$, and the groupoid is therefore not connected (unless of course, the disjoint union of groups consists of exactly one group; see Example A.2.7 above).

Example A.2.9 (The extended singleton, continued from Example A.1.5)**.** The extended singleton has two base points whose vertex groups are as follows:

$$\Gamma_{b_1} = \{e_{b_1}\}\,, \quad \Gamma_{b_2} = \{e_{b_2}\}\,. \tag{A.18}$$

The conjugating morphisms of the groupoids are of two types, from b_1 to b_2 and vice versa:

$$\eta\colon b_1 \to b_2, \quad \gamma\colon b_2 \to b_1. \tag{A.19}$$

There is only one orbit: $\mathcal{O}_{b_1} = \mathcal{O}_{b_2} = \{b_1, b_2\}$. Hence, the extended singleton is connected.

Example A.2.10 (Groupoid of objects, continued from Example A.1.6)**.** In this groupoid, where objects are the base points $b \in B$ and the only morphisms are the identity elements e_b, the vertex groups are $\Gamma_b = \{e_b\}$. There are no conjugating morphisms. Consequentially, the groupoid of orbits is totally disconnected, that is, $\mathcal{O}_b = \{b\}$ holds for all objects $b \in B$.

Example A.2.11 (Groupoid of pairs, continued from Example A.1.7)**.** Recall that the groupoid of pairs is given by $\Gamma = B \times B$ on a set B. The vertex group corresponding to a base point $b \in B$ is given by $\Gamma_b = \{(b, b)\}$. Moreover, the conjugating morphisms between two distinct objects b_1, b_2 are given by (b_2, b_1) as well as its inverse (b_1, b_2). As all objects are connected via morphisms, it follows that $\mathcal{O}_b = B$ for any $b \in B$, implying that the groupoid of pairs is connected.

Example A.2.12 (The action groupoid, continued from Example A.1.8)**.** The groupoid $\Gamma(G, B) \rightrightarrows B$ associated to the action of the group G on B explains the double use of "isotropy groups" (which we call vertex groups in the groupoid context to avoid confusion; see Remark A.2.2). Indeed, the vertex groups $\Gamma(G, B)_b$ of the action groupoid coincide with the isotropy subgroups G_b of the group G acting on the set B. There is one more analogy that we should note: There is a one-to-one correspondence between the orbit $\mathcal{O}_b$ through b of the action groupoid and the orbit Gb of the group action. As a consequence, the action groupoid is connected if and only if the group action is transitive.

Example A.2.13 (Equivalence relations, continued from Example A.1.9)**.** Let B be a set with an equivalence relation. Then B with this equivalence relation forms a subgroupoid of the groupoid of pairs. It is therefore straightforward to conclude the following: The set B with an equivalence relation $\sim$ has the vertex groups $\Gamma_b = \{(b, b)\}$ for each element b. The conjugating morphisms between two disjoint objects b_1, b_2 is given by (b_2, b_1) as well as its inverse (b_1, b_2) if and only if $b_1 \sim b_2$. The interesting assertion is that the orbits of the equivalence relation groupoid correspond directly to the equivalence classes. The groupoid is therefore connected if and only if there is only a single equivalence class, and disconnected otherwise.

Example A.2.14 (Lie groupoids, continued from Example A.1.10)**.** The theory of Lie groupoids has been developed extensively [38, 43, 44]. In particular, the following holds: The vertex groups Γ_b of a Lie groupoid are Lie groups. The set of composable morphisms $\Gamma \star \Gamma$ is a closed submanifold of $\Gamma \times \Gamma$. For each $b \in B$, the fibers $s^{-1}(b)$ and $t^{-1}(b)$ are submanifolds of Γ. Moreover, the orbits $\mathcal{O}_b$ are immersed submanifolds of the set B.

A.3. Groupoid representation theory

For our purposes, it will be useful to represent the base points of a groupoid by vector spaces and the morphisms of a groupoid by linear operators between these vector spaces. In this way we can connect groupoids and dynamical systems. Let us start off with the definition of a linear representation of a groupoid.

Definition A.3.1 (Linear groupoid representation, [34, 38]). A linear representation ρ of a groupoid $\Gamma \rightrightarrows B$ consists of a family $\{X_b\}_{b\in B}$ of vector spaces assigned to the base B, i.e., $X_b := \rho(b)$, and a collection $\{\rho(\gamma)\}_{\gamma\in\Gamma}$ of invertible linear maps $\rho(\gamma)\colon \rho(b_1) \to \rho(b_2)$ assigned to each morphism $\gamma\colon b_1 \to b_2$ such that

i) for all base points $b \in B$, the identity elements are represented by the identity operation on the space X_b, i.e., $\rho(e_b) = \mathrm{Id}_{X_b}$,

ii) the representation preserves the partial binary operation of the groupoid, i.e., $\forall(\gamma, \eta) \in \Gamma \star \Gamma$, $\rho(\eta \circ \gamma) = \rho(\eta)\rho(\gamma)$ holds,

iii) the representation of the inverse element is the inverse of the representation, i.e., for all $\gamma \in \Gamma$, $\rho(\gamma^{-1}) = \rho(\gamma)^{-1}$ holds.

A particularly useful property of groupoid representations is given in the following proposition.

Proposition A.3.2 (Connected components have isomorphic representation spaces, [34]). *Let ρ be a linear representation of the groupoid $\Gamma \rightrightarrows B$. Then, on each connected component of the groupoid, the linear spaces X_b are isomorphic. In particular, if the groupoid is connected, all linear spaces X_b are isomorphic.*

As we have seen in the previous section, the vertex groups form the "building blocks" of the groupoid. We therefore introduce the notion of the vertex representation, which is the restriction of the groupoid representation to a single vertex group. Here $\mathrm{GL}(X)$ denotes the general linear group on the space X.

Definition A.3.3 (Vertex representation, [34])**.** Let ρ be a linear representation of the groupoid $\Gamma \rightrightarrows B$. By the *vertex representation* ρ_b of ρ of the vertex group Γ_b we denote the subset $\rho_b := \{\rho \colon \Gamma_b \to \mathrm{GL}(X_b)\}$, and we define $\rho_b(\gamma) := \rho(\gamma) \colon X_b \to X_b$ for all $\gamma \in \Gamma_b$.

Note in particular that the vertex representation is simply a group representation. Inspired by Proposition A.3.2, we can therefore ask whether the vertex group representations are equivalent (in the standard group sense). In fact, this is the case in connected components:

Proposition A.3.4 (Equivalence of vertex representations in connected components, [34])**.** *Let R be a linear representation of the groupoid $\Gamma \rightrightarrows B$ and let b_1, b_2 be two objects in the same connected component. Then the vertex representations ρ_{b_1}, ρ_{b_2} of the vertex groups $\Gamma_{b_1}, \Gamma_{b_2}$ are equivalent.*

For our purposes, the concept of *irreducibility* of a representation is of high importance, as it is needed in equivariant bifurcation theory, which we strive to generalize.

Definition A.3.5 (Internally irreducible groupoid representations, [34])**.** A representation ρ of a groupoid $\Gamma \rightrightarrows \rho$ is called *internally irreducible*, if all its vertex representations are irreducible.

Example A.3.6 (Groups and disjoint union of groups, continued from Examples A.1.3, A.1.4, A.2.7, A.2.8)**.** In this case, the linear representation ρ of the groupoid becomes the standard linear representation of the group(s).

Example A.3.7 (The extended singleton, continued from Example A.1.5, A.2.9)**.** Let us find a simple representation here: The extended singleton has two base points b_1, b_2 to which we associate the spaces $X_{b_1} = X_{b_2} = \mathbb{R}$, each. Then the representation of the identities is the unit in $\mathbb{R}$,

$$\rho(e_{b_1}) = \rho(e_{b_2}) = 1 \in \mathbb{R}. \tag{A.20}$$

The conjugating morphism $\gamma_1 \colon b_1 \to b_2$, can be represented by any number $c \in \mathbb{R}\backslash\{0\}$, and then its inverse $\gamma_2 \colon b_2 \to b_1$ must be represented by c^{-1}:

$$\rho(\eta) = c \in \mathbb{R}\backslash\{0\} \quad \rho(\gamma_2) = \rho(\gamma_1^{-1}) = \rho(\gamma_1)^{-1} = c^{-1} \in \mathbb{R}\backslash\{0\}. \tag{A.21}$$

Note that indeed, for the representation of this connected groupoid found here, the spaces X_{b_1}, X_{b_2} are isomorphic and the representations of the vertex groups are equivalent.

Example A.3.8 (Groupoid of objects, continued from Example A.1.6, A.2.10)**.** In this groupoid, we assign an arbitrary vector space X_b to each of the base points $b \in B$ and the identity on the respective space to the morphisms, $\rho(e_b) = \mathrm{Id}_{X_b}$.

Example A.3.9 (Groupoid of pairs and equivalence relations, continued from Examples A.1.7, A.1.9, A.2.11, A.2.13)**.** A simple, albeit maybe not so useful, representation of the groupoid of pairs can be constructed as follows: To each base point $b \in B$, we assign the vector space $X_b := X$ (since by Proposition A.3.2, all spaces X_b must be isomorphic, we may as well choose them identical). Then to any pair (b_j, b_k), we assign the identity operator. The same construction can be used on every equivalence class for the representation of an equivalence relation as a groupoid.

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Selbständigkeitserklärung

Erklärung gemäß §4 der Habilitationsordnung des Fachbereiches Mathematik und Informatik der Freien Universität Berlin Amtsblatt der Freien Universität Berlin, 4.10.1999. Ausgabe Nr. 2311999

Ich erkläre, dass ich die vorliegende schriftliche Habilitationsleistung mit dem Thema

Symmetry Groupoids in Dynamical Systems

Spatio-temporal patterns and a generalized equivariant bifurcation theory

in Form einer Monografie selbstständig angefertigt habe und nur die benannten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

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