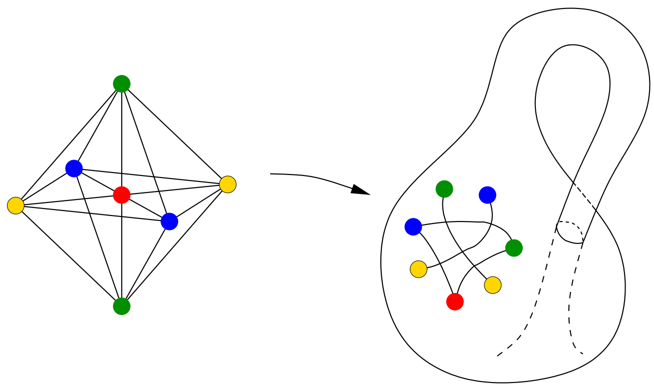
Equivariant topology methods in discrete geometry



A dissertation presented by

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Equivariant topology methods in discrete geometry

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Contents

Preface

This thesis splits into two independent chapters and an appendix. The first chapter is on the colored Tverberg problem, which is joint work with Pavle Blagojević and Günter Ziegler [BMZ09], [BMZ11a], [BMZ11b]. First we find and prove a new and tight colored version of Tverberg's theorem that implies the Bárány–Larman conjecture for primes minus one and asymptotically in general. We generalize it further to a transversal theorem and then to manifolds.

The second chapter contains new results on inscribing squares and rectangles into closed curves in the plane. These problems are old, beautiful, and very complicated. They show that we still don't completely understand continuous plane curves. The results in this chapter are disjoint from the ones in [Mat08, Chap. III]. They will appear in [Mat09] and [Mat11].

The appendix contains two independent results on polytopes that I don't want to omit. The first one is joint work with Francisco Santos and Christophe Weibel [MSW11] on d-spindles with large width, the second with Julian Pfeifle and Vincent Pilaud [MPP11] on productsimplicial-neighborly polytopes.

Background material The books [Mat03], [dL12] and the articles [Bjö95], [Živ96], [Živ97], and [Živ98] serve as very good introductions to topological combinatorics, especially on how to attach topological methods to problems in discrete geometry and combinatorics. For more background on equivariant topology I recommend [Bre72], [tD87], [Hsi75], [AM94], [Bro82], and [Lüc89].

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Notations

The following notation is used throughout the book. Here, X denotes always a topological space, K a simplicial complex, G a finite group which possibly acts on X and K, and the rest are positive or non-negative integers.

$\{*\}$	one-point space.
$[n]$	$= \{1, 2, \ldots, n\}$.
Δ_n	n-dimensional simplex.
$\Delta_{n,m}$	(n, m)-chessboard complex, $\Delta_{n,m} = [n]^{*m}_{\Delta(2)} = [m]^{*n}_{\Delta(2)}$. Its nm vertices correspond to an $n \times m$-grid, and a subset of the grid is a face if it takes from each row and each column at most one vertex.
Δ_{X^n}	thin diagonal of X^n, $\Delta_{X^n} = \{(x, \ldots, x) \in X^n \mid x \in X\}$.
$X^r_{\Delta(k)}$	r-fold k-wise deleted product of a space X, $X^r_{\Delta k} = \{(x_1, \ldots, x_r) \mid \text{the } x_i \text{ are } k\text{-wise disjoint}\}$. Here an n-tuple is k-wise disjoint if no k of them are equal.
X^{*r}	r-fold join of a space X.
$X^{*r}_{\Delta(k)}$	r-fold k-wise deleted join of a space X, $X^{*r}_{\Delta k} = \{\sum \lambda_i x_i \mid \text{if } \lambda_1 = \ldots = \lambda_r \text{ then the } x_i \text{ are } k\text{-wise disjoint}\}$.
$K^r_{\Delta(k)}$	r-fold k-wise deleted product of a simplicial complex K, $K^r_{\Delta k} = \{F_1 \times \ldots \times F_r \mid \text{the faces } F_i \text{ of } K \text{ have } k\text{-wise disjoint support}\}$.
K^{*r}	r-fold join of a simplicial complex K.
$K^{*r}_{\Delta(k)}$	r-fold k-wise deleted join of a simplicial complex K, $K^{*r}_{\Delta k} = \{F_1 \uplus \ldots \uplus F_r \mid \text{the faces } F_i \text{ of } K \text{ have } k\text{-wise disjoint support}\}$.
$t(d, r)$	see Colored Tverberg Problem 1.2, page 1.
$\widetilde{t}(d, r)$	see Topological Colored Tverberg Problem 1.5, page 2.
S_r	symmetric group on r elements.
$\mathbb{Z}_r$	$= \mathbb{Z}/r\mathbb{Z}$.

W_r	standard representation of S_r, or its restriction to $\mathbb{Z}_r$, $W_r = \{x \in X^r \mid \sum x_i = 0\}$.
EG	$= G * G * G * \dots$.
BG	classifying space, $BG = EG/G$.
$EG \times_G X$	Borel construction, $EG \times_G X = (EG \times X)/G$ (modding out the diagonal action).
$H^*(X)$	we usually mean Čech cohomology with coefficients in $\mathbb{F}_r$ for some prime r.
$H^*_G(X)$	equivariant cohomology, $H^*_G(X) = H^*(EG \times_G X)$.
$H^*(G)$	group cohomology, $H^*(G) = H^*_G(\{*\}) = H^*(BG)$.
$\mathrm{Index}^B_G(X)$	cohomological index of a G-space X with respect to a projection $p : X \to B$ to a trivial G-space B, $\mathrm{Index}^B_G(X) = \ker\big(H^*_G(B) \xrightarrow{p^*} H^*_G(X)\big) \subseteq H^*_G(B) \cong H^*(BG) \otimes H^*(B)$.
$\mathrm{Index}_G(X)$	$= \mathrm{Index}^{\{*\}}_G(X)$.
t, x, y	generators of $H^*(\mathbb{Z}_2; \mathbb{F}_2) = \mathbb{F}_2[t]$ and $H^*(\mathbb{Z}_r; \mathbb{F}_r) = \mathbb{F}_r[x, y]/y^2$ for r an odd prime; $\deg(t) = 1$, $\deg(x) = 2$, $\deg(y) = 1$, $x = \beta(y)$.
$\mathcal{N}_*(X)$	unoriented bordism group of X.
$\Omega_*(X)$	oriented bordism group of X.

Chapter 1

The colored Tverberg problem

1 A new colored Tverberg theorem

1.1 Introduction

More than 50 years ago, the Cambridge undergraduate Bryan Birch [Bir59] showed that any "$3r$ points in a plane" can be split into r triples that span triangles with a non-empty intersection. He also conjectured a sharp, higher-dimensional version of this, which was proved by Helge Tverberg [Tve66] in 1964 (freezing, in a hotel room in Manchester):

Theorem 1.1 (Tverberg)**.** *Any family of $N + 1 := (d + 1)(r - 1) + 1$ points in $\mathbb{R}^d$ can be partitioned into r sets whose convex hulls intersect.*

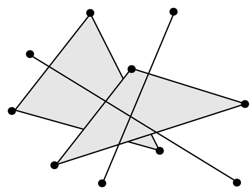


Figure 1.1: Example of Theorem 1.1 for $d = 2$, $r = 4$, $N + 1 = 10$.

Figure 1.1 shows an example. A look at the codimensions of intersections shows that the number $(d+1)(r-1)+1$ of points is minimal for this. The special case $r = 2$ is known as Radon's theorem [Rad21].

In their 1990 study of halving lines and halving planes, Bárány, Füredi & Lovász [BFL90] observed "we need a colored version of Tverberg's theorem" and provided a first case, for three triangles in the plane. Soon after that Bárány & Larman [BL92] in 1992 formulated the following general problem and proved it for the planar case.

Problem 1.2 (The colored Tverberg problem)**.** *Determine the smallest number $t = t(d, r)$ such that for every collection $\mathcal{C} = C_1 \uplus \ldots \uplus C_{d+1}$ of points in $\mathbb{R}^d$ with $|C_i| \geq t$, there are r disjoint subcollections $F_1, \ldots, F_r$ of $\mathcal{C}$ satisfying $|F_i \cap C_j| \leq 1$ for every $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, d + 1\}$, and $\operatorname{conv}(F_1) \cap \cdots \cap \operatorname{conv}(F_r) \neq \emptyset$.*

A family of such disjoint subcollections $F_1, \ldots, F_r$ that contain at most one point from each *color class* C_i is called a *rainbow r-partition.* (We do not require $F_1 \cup \cdots \cup F_r = \mathcal{C}$ for this.) Multiple points are allowed in these collections of points, but then the cardinalities have to account for these.

A trivial lower bound is $t(d,r) \geq r$: Collections $\mathcal{C}$ with only $(r-1)(d+1)$ points in general position do not admit an intersecting r-partition, again by codimension reasons.

Bárány and Larman showed that the trivial lower bound is tight in the cases $t(1,r) = r$ and $t(2,r) = r$, presented a proof by Lovász for $t(d,2) = 2$, and conjectured the following equality.

Conjecture 1.3 (The Bárány–Larman Conjecture)**.** *$t(d,r) = r$ for all $r \geq 2$ and $d \geq 1$.*

Still in 1992, Živaljević & Vrećica [ŽV92] established for r prime the upper bound $t(d,r) \leq 2r-1$. The same bound holds for prime powers according to Živaljević [Živ96]. The bound for primes also yields bounds for arbitrary r: For example, one gets $t(d,r) \leq 4r-3$, since there is a prime p (and certainly a prime power!) between r and $2r$.

Topological versions. As in the case of Tverberg's classical theorem, one can consider a topological version of the colored Tverberg problem.

Theorem 1.4 (The topological Tverberg theorem; [BSS81], [Öza87], [Mat03, Sect. 6.4])**.** *Let $r \geq 2$ be a prime power, $d \geq 1$, and $N = (d+1)(r-1)$. Then for every continuous map of an N-simplex Δ_N to $\mathbb{R}^d$ there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N whose images under f intersect in $\mathbb{R}^d$.*

Problem 1.5 (The topological colored Tverberg problem)**.** *Determine the smallest number $t = \widetilde{t}(d,r)$ such that for every simplex Δ with $(d+1)$-colored vertex set $\mathcal{C} = C_1 \uplus \ldots \uplus C_{d+1}$, $|C_i| \geq t$, and every continuous map $f : \Delta \to \mathbb{R}^d$ there are r disjoint faces $F_1, \ldots, F_r$ of Δ satisfying $|F_i \cap C_j| \leq 1$ for every $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, d+1\}$, and $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.*

The family of faces $F_1, \ldots, F_r$ is called a *topological rainbow partition*.

The argument from [ŽV92] and [Živ96] gives the same upper bound $\widetilde{t}(d,r) \leq 2r-1$ for r a prime power, and consequently the upper bound $\widetilde{t}(d,r) \leq 4r-3$ for arbitrary r. Notice that $t(d,r) \leq \widetilde{t}(d,r)$.

Conjecture 1.6 (The topological Bárány–Larman conjecture)**.** *$\widetilde{t}(d,r) = r$ for all $r \geq 2$ and $d \geq 1$.*

The Lovász proof for $t(d,2) = 2$ presented in [BL92] is topological and thus also valid for the topological Bárány–Larman conjecture. Therefore $\widetilde{t}(d,2) = 2$.

The general case of the topological Bárány–Larman conjecture would classically be approached via a study of the existence of an S_r-equivariant map

$$\Delta_{r,|C_1|} * \cdots * \Delta_{r,|C_{d+1}|} \quad \longrightarrow_{S_r} \quad S(W_r^{\oplus(d+1)}) \;\simeq\; S^{(r-1)(d+1)-1}, \tag{1.7}$$

where W_r is the standard $(r-1)$-dimensional real representation of S_r obtained by restricting the coordinate permutation action on $\mathbb{R}^r$ to $\{(\xi_1, \ldots, \xi_r) \in \mathbb{R}^r : \xi_1 + \cdots + \xi_r = 0\}$ and $\Delta_{r,n}$ denotes the $r \times n$ chessboard complex $([r])^{*n}_{\Delta(2)}$; cf. [Mat03, Remark after Thm. 6.8.2].

However, this approach fails when applied to the colored Tverberg problem directly, since an S_r-equivariant map (1.7) exists. This is because the square chessboard complexes $\Delta_{r,r}$ admit S_r-equivariant collapses that reduce the dimension.

The key in solving the topological Bárány–Larman conjecture (when $r+1$ is a prime) is first to reduce it geometrically to another, new, colored Tverberg problem that in turn *can* be attacked with topological methods, that is, for which the corresponding test map in question does not exist.

1.2 The main result

Our main result (Blagojević, M, Ziegler [BMZ09]) is the following strengthening of (the prime case of) the topological Tverberg theorem.

Theorem 1.8 (Main theorem)**.** *Let $r \geq 2$ be prime, $d \geq 1$, and $N := (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of the vertex set into "color classes"*

$$\mathcal{C} \quad = \quad C_1 \uplus \ldots \uplus C_m,$$

with $|C_i| \leq r-1$ for all i.

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint "rainbow" faces $F_1, \ldots, F_r$ of Δ_N whose images under f intersect, that is, $|F_i \cap C_j| \leq 1$ for every $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, m\}$, and $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.

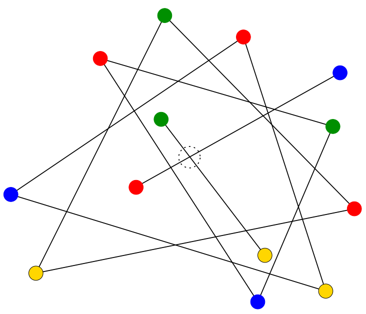


Figure 1.2: Example of Theorem 1.8 for $d = 2$, $r = 5$, $N + 1 = 13$.

The requirement $|C_i| \leq r-1$ forces that there are at least $d+2$ non-empty color classes. Theorem 1.8 is tight in the sense that there would exist counter-examples f if $|C_1| = r$ and $|C_2| = \ldots = |C_m| = 1$.

We will present a generalization for maps into manifolds in Section 3 and a "transversal" generalization in Section 2.

1.3 Applications

Either of our Theorems 1.8 and 1.18 immediately implies the topological Tverberg theorem for the case when r is a prime, as it holds for an *arbitrary* partition of the vertex set into color classes of the specified sizes. Thus it is a "constrained" Tverberg theorem as discussed recently by Hell [Hel08].

It remains to be explored how the constraints can be used to derive lower bounds for the number of Tverberg partitions; compare Vučić & Živaljević [VŽ93] [Mat03, Sect. 6.3].

More importantly, however, Theorem 1.8 implies the topological Bárány–Larman conjecture for the case when $r+1$ is a prime, as follows.

Corollary 1.9. *If $r+1$ is prime, then $t(d,r) = \widetilde{t}(d,r) = r$.*

Proof. We prove that if $r \geq 2$ is prime, then $\widetilde{t}(d,r-1) \leq r-1$. For this, let Δ_{N-1} be a simplex with vertex set $\mathcal{C} = C_1 \uplus \ldots \uplus C_{d+1}$, $|C_i| = r-1$, and let $f : \Delta_{N-1} \to \mathbb{R}^d$ be continuous. Extend this to a map $\Delta_N \to \mathbb{R}^d$, where Δ_N has an extra vertex v_N, and set $C_{d+2} := \{v_N\}$. Then Theorem 1.8 can be applied, and yields a topological colored Tverberg partition into r parts. Ignore the part that contains v_N. ☐

Using estimates on prime numbers one can derive from this tight bounds for the colored Tverberg problem also in the general case. The classical Bertrand's postulate ("For every r there is a prime p with $r+1 \leq p < 2r$") can be used here, but there are also much stronger estimates available, such as the existence of a prime p between r and $r + r^{6/11+\varepsilon}$ for arbitrary $\varepsilon > 0$ if r is large enough according to Lou & Yao [LY92].

Corollary 1.10. *For any $r \geq 2$ we have $r \leq t(d,r) \leq \widetilde{t}(d,r) \leq 2r-2$. Asymptotically, $r \leq t(d,r) \leq \widetilde{t}(d,r) \leq (1+o(1))\,r$ as $r \to \infty$.*

Proof. The first, explicit estimate is obtained from Bertrand's postulate: For any given r there is a prime p with $r+1 \leq p < 2r$. We use $|C_i| \geq 2r-2 \geq p-1$ to derive the existence of a colored Tverberg $(p-1)$-partition, which in particular yields an r-partition since $p-1 \geq r$. The second, asymptotic estimate uses the Lou & Yao bound instead. ☐

Remark 1.11. The colored Tverberg problem was originally posed by Bárány & Larman [BL92] in a different version than the one we have given above (we followed Bárány, Füredi & Lovász [BFL90] and Vrećica & Živaljević [ŽV92]): Bárány and Larman had asked for an upper bound $N(d,r)$ on the *cardinality of the union* $|\mathcal{C}|$ that together with $|C_i| \geq r$ would force the existence of a rainbow r-partition.

This original formulation has two major difficulties: Firstly, Vrećica's and Živaljević's result gives no estimate on $N(d,r)$, since their approach needs a lower bound on the color class sizes that is larger than r. Secondly, the prime case of it doesn't suffice to give any estimate on $N(d,r)$ when r is not a prime. That is, there is no induction on r: A bound on $N(d,r)$ for some r does not seem to give a bound on $N(d,r')$ for $r' \neq r$.

Our Corollary 1.9 also solves the original version for the case when $r+1$ is a prime.

The colored Tverberg problem originally arose as a tool to obtain complexity bounds in computational geometry. As a consequence, our new bounds can be applied to improve these bounds, as follows. Note that in some of these results $t(d,d+1)^d$ appears in the exponent, so even slightly improved estimates on $t(d,d+1)$ have considerable effect. For surveys see [Bár93], [Mat02, Sect. 9.2], and [Živ97, Sect. 11.4.2].

Let $S \subseteq \mathbb{R}^d$ be a set of size n in general position, that is, such that no $d+1$ points of S are on a hyperplane. Let $h_d(n)$ denote the number of hyperplanes that bisect the set S and are spanned by the elements of the set S. According to Bárány [Bár93, p. 239],

$$h_d(n) = O(n^{d-\varepsilon_d}) \qquad \text{with} \qquad \varepsilon_d = t(d,d+1)^{-(d+1)}.$$

Thus we obtain the following bound and equality.

Corollary 1.12. *If $d+2$ is a prime then*

$$h_d(n) = O(n^{d-\varepsilon_d}) \qquad \text{with} \qquad \varepsilon_d = (d+1)^{-(d+1)}.$$

For general d, we obtain e.g. $\varepsilon_d \geq (d+1)^{-(d+1)-O(\log d)}$.

Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a finite set. A *$\mathcal{C}$-simplex* is the convex hull of some collection of $d+1$ points of $\mathcal{C}$. The second selection lemma [Mat02, Thm. 9.2.1] claims that for any n-point set $\mathcal{C} \subseteq \mathbb{R}^d$ and a family $\mathcal{F}$ of $\alpha\binom{n}{d+1}$ $\mathcal{C}$-simplices with $\alpha \in (0,1]$ there exists a point contained in at least $c \cdot \alpha^{s_d}\binom{n}{d+1}$ $\mathcal{C}$-simplices of $\mathcal{F}$. Here $c = c(d) > 0$ and s_d are constants depending only on d. For dimensions $d > 2$, the presently known proof gives that $s_d \approx t(d, d+1)^{d+1}$. Again, Corollary 1.10 yields the following, much better bounds for the constant s_d.

Corollary 1.13. *If $d+2 > 4$ is a prime then the second selection lemma holds for $s_d = (d+1)^{d+1}$, and in general e.g. for $s_d = (2d+2)^{d+1}$.*

Let $X \subset \mathbb{R}^d$ be an n element set. A *k-facet* of the set X is an oriented $(d-1)$-simplex $\operatorname{conv}\{x_1, \ldots, x_d\}$ spanned by elements of X such that there are exactly k points of X on its strictly positive side. When $n-d$ is even $\frac{n-d}{2}$-facets of the set X are called *halving facets*. From [Mat02, Thm. 11.3.3] we have a new, better estimate for the number of halving facets.

Corollary 1.14. *For $d > 2$ and $n-d$ even, the number of halving facets of an n-set $X \subset \mathbb{R}^d$ is $O(n^{d-\frac{1}{(2d)^d}})$.*

1.4 The configuration space/test map scheme

Suppose we are given a continuous map

$$f : \Delta_N \longrightarrow \mathbb{R}^d$$

and a coloring of the vertex set $\operatorname{vert}(\Delta_N) = [N+1] := \{1, \ldots, N+1\} = C_1 \uplus \cdots \uplus C_m$ such that $|C_i| \leq r-1$. We want to find a colored Tverberg partition $F_1, \ldots, F_r$.

To measure those solutions we construct a test map F out of f. Let $f^{*r} : (\Delta_N)^{*r} \longrightarrow_{\mathbb{Z}_r} (\mathbb{R}^d)^{*r}$ be the r-fold join of f, which is equivariant with respect to the $\mathbb{Z}_r$-action that shifts the join constituents cyclically. Since we are interested in pairwise disjoint faces $F_1, \ldots, F_r$, we restrict the domain of f^{*r} to the r-fold 2-wise deleted join of Δ_N, $(\Delta_N)^{*r}_{\Delta(2)} = [r]^{*(N+1)}$. (See [Mat03] for an introduction to these notions.) Since we are interested in colorful F_js, we restrict the domain further to the subcomplex

$$K := (C_1 * \ldots * C_m)^{*r}_{\Delta(2)} = [r]^{*|C_1|}_{\Delta(2)} * \cdots * [r]^{*|C_m|}_{\Delta(2)}.$$

The space $[n]^{*m}_{\Delta(2)}$ is known as the *chessboard complex* $\Delta_{n,m}$. Hence K can be written as

$$K = \Delta_{r,|C_1|} * \cdots * \Delta_{r,|C_m|}. \tag{1.15}$$

Thus by restricting the domain of f^{*r} to K we get a $\mathbb{Z}_r$-equivariant map

$$F'' : K \longrightarrow_{\mathbb{Z}_r} (\mathbb{R}^d)^{*r}.$$

Let $\mathbb{R}[\mathbb{Z}_r] \cong \mathbb{R}^r$ be the regular representation of $\mathbb{Z}_r$ and $W_r \subseteq \mathbb{R}^r$ the orthogonal complement of the all-one vector $\mathbb{1} = e_1 + \dots + e_r$. We write W_r^{d+1} for $(W_r)^{\oplus(d+1)}$. The orthogonal projection

$$p : \mathbb{R}^r \longrightarrow_{\mathbb{Z}_r} W_r$$

yields a $\mathbb{Z}_r$-equivariant map

$$\begin{array}{lll} (\mathbb{R}^d)^{*r} & \longrightarrow_{\mathbb{Z}_r} & W_r^{d+1} \\ \sum_{j=1}^r \lambda_j x_j & \longmapsto & (p(\lambda_1, \dots, \lambda_r), p(\lambda_1 x_{1,1}, \dots, \lambda_r x_{r,1}), \dots, p(\lambda_1 x_{1,d}, \dots, \lambda_r x_{r,d}). \end{array}$$

The composition of this map with F'' gives us the test map F',

$$F' : K \longrightarrow_{\mathbb{Z}_r} W_r^{d+1}. \tag{1.16}$$

The preimages $(F')^{-1}(0)$ of zero correspond exactly to the colored Tverberg partitions. Hence the image of F' contains 0 if and only if the map f admits a colored Tverberg partition. Suppose that 0 is not in the image, then we get a map

$$F : K \longrightarrow_{\mathbb{Z}_r} S(W_r^{d+1}) \tag{1.17}$$

into the representation sphere by composing F' with the radial projection map. We will derive contradictions to the existence of such an equivariant map.

1.5 First proof of the main theorem

In this Section we give a degree proof of Theorem 1.8.

Geometric reduction lemma. Our first step towards the proof is a *reduction* of Theorem 1.8 to the following essential special case.

Theorem 1.18 (Essential special case)**.** *Let $r \geq 2$ be prime, $d \geq 1$, and $N := (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of the vertex set into $d+2$ parts*

$$\mathcal{C} \quad = \quad C_1 \uplus \dots \uplus C_{d+1} \uplus C_{d+2},$$

with $|C_i| = r-1$ for $1 \leq i \leq d+1$ and $|C_{d+2}| = 1$.

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint rainbow faces $F_1, \dots, F_r$ of Δ_N whose images under f intersect.

In fact, this is exactly the special case we need in order to prove Corollary 1.9, the prime+1 case of the Bárány–Larman conjecture.

Lemma 1.19 ([BMZ09])**.** *Theorem 1.18 implies Theorem 1.8.*

Proof. Suppose we are given such a map f and a coloring $C_1 \uplus \ldots \uplus C_m$ of the vertex set of Δ_N. Let $N' := (r-1)m$ and $C_{m+1} := \emptyset$. We enlarge the color classes C_i by $N' - N = (r-1)(m-(d+1))$ new vertices and obtain color classes $C'_1, \ldots, C'_{m+1}$, such that $C_i \subseteq C'_i$ for all i, and $|C'_1| = \ldots = |C'_m| = r-1$ and $|C'_{m+1}| = 1$. We construct from f a new map $f' : \Delta_{N'} \to \mathbb{R}^{d'}$, where $d' := m-1$, as follows: We regard $\mathbb{R}^d$ as the subspace of $\mathbb{R}^{d'}$ where the last $d'-d$ coordinates are zero. So we let f' be the same as f on the N-dimensional front face of $\Delta_{N'}$ that is spanned by the original vertices $C_1 \uplus \ldots \uplus C_m$. We assemble the further $N'-N$ vertices into $d'-d$ groups $V_1, \ldots, V_{d'-d}$ of $r-1$ vertices each. The vertices in V_i shall be mapped to e_{d+i}, the $(d+i)$st standard basis vector of $\mathbb{R}^{d'}$. We extend this map linearly to all of $\Delta_{N'}$ and we obtain f'. We apply Theorem 1.18 to f' and the coloring $C'_1, \ldots, C'_{m+1}$ and obtain disjoint faces $F'_1, \ldots, F'_r$ of $\Delta_{n'}$. Let $F_i := F'_i \cap \Delta_N$ be the intersection of F'_i with the N-dimensional front face of $\Delta_{N'}$. By construction of f', the intersection $f'(F'_1) \cap \cdots \cap f'(F'_r)$ lies in R^d. Therefore, already $F_1, \ldots, F_r$ is a colorful Tverberg partition for f', and hence it is for f: We have $f(F_1) \cap \cdots \cap f(F_r) = \emptyset$. □

Such a reduction previously appears in Sarkaria's proof for the prime power Tverberg theorem [Sar00, (2.7.3)]; see also Longueville's exposition [dL01, Prop. 2.5].

Degree proof. By the reduction, it suffices to consider the special case $K = K' * [r]$ where $K' = (\Delta_{r,r-1})^{*(d+1)}$. Let $M = F|_{K'} : K' \to S(W_r^{d+1})$ be the restriction of F to K'. The chessboard complex $\Delta_{r,r-1}$ for $r \geq 3$ is a connected orientable pseudo-manifold, hence K' is one as well. For $r = 2$, K' is the boundary of a $d+1$-dimensional crosspolytope, hence a d-sphere. The dimensions $\dim K' = N-1 = \dim S(W_r^{d+1})$ coincide. Thus we can talk about the degree $\deg(M) \in \mathbb{Z}$. Here we are not interested in the actual sign, hence we do not need to fix orientations. Since K' is a free $\mathbb{Z}_r$-space and S^{N-1} is $(N-2)$-connected, the degree $\deg(M)$ is uniquely determined modulo r: This is because M is unique up to $\mathbb{Z}_r$-homotopy on the codimension one skeleton of K', and changing M on top-dimensional cells of K' has to be done $\mathbb{Z}_r$-equivariantly, hence it affects $\deg(M)$ by a multiple of r.

To determine $\deg(M) \mod r$, we let f be the affine map that takes the vertices in C_1 to $-\mathbb{1} = -(e_1 + \cdots + e_d)$ and the vertices in C_i $(2 \leq i \leq d+1)$ to e_i, where e_i is the ith standard basis vector of $\mathbb{R}^d$. The position of the singleton C_{d+2} does not matter, we can choose it arbitrarily in $\mathbb{R}^d$. Let $P \in S(W_r^{d+1})$ be the normalization of the point $(p(1, \ldots, 1, 0), 0, \ldots, 0) \in W_r^{d+1}$.

The preimage $M^{-1}(P)$ is exactly the set of barycenters of the $(r-1)!^{d+1}$ top-dimensional faces of $K' \cap (\Delta_{r-1,r-1})^{*(d+1)}$. With $\Delta_{r-1,r-1}$ we mean the full subcomplex $[r-1]^{*(r-1)}_{\Delta(2)}$ of $\Delta_{r,r-1}$. One checks that all preimages of P have the same preimage orientation. This was essentially done in [BMZ09] when we calculated that $c_f(\Phi) = (r-1)!^d \zeta$. Hence

$$\deg(M) = \pm(r-1)!^{d+1} = \pm 1 \mod r. \tag{1.20}$$

Alternatively one can take any map $m : \Delta_{r,r-1} \longrightarrow_{\mathbb{Z}_r} S(W_r)$, show that its degree is ± 1 by a similar preimage argument in dimension $d = 1$, and deduce that

$$\deg(M) = \deg(m^{*(d+1)}) = \deg(m)^{d+1} = \pm 1 \mod r.$$

Degree proof of Theorem 1.8. Since $\deg(M) \neq 0$, M is not null-homotopic. Thus M does not extend to a map with domain $K' * [1] \subseteq K$. Therefore the test map F of (1.17) does not exist. □

Remark 1.21. The degree $\deg(M)$ is even uniquely determined modulo $r!$. To see this one uses the S_r-equivariance of M and the fact that M is given uniquely up to S_r-homotopy on the non-free part, which lies in the codimension one skeleton of K'. The latter can be shown with the modified test map F_0 from [BMZ09]. This might possibly be an ansatz for a proof of the affine version of Theorem 1.8 for non-primes r.

Matoušek, Wagner, and Tancer [MTW10] found a point configuration for the non-prime case $r = 4$ where the degree is 0. In their example however, the desired colored Tverberg partition does exist nevertheless.

During a visit in Prague, Marek Krčál, Martin Tancer and I found an easy point configuration for $r = 4$ where the degree is zero: Put on each vertex of a triangle in $\mathbb{R}^2$ three points, a red, a blue, and a green one. If one puts a further black point in the center of the triangle, this point configuration admits exactly two colored Tverberg partitions, both of which appear with the opposite sign in the degree. We also noted that for any point configuration for parameters (r, d) such that the degree is zero, there is another point configuration for $(r, d+1)$ also with degree zero. For this simply add $r-1$ new points of a new color above the given point configuration as in the proof of Lemma 1.19.

Other degree zero point configurations have been found by Kleist [Kle11].

Remark 1.22. The original proof of Theorem 1.18 was done via obstruction theory [BMZ09]. The degree proof is basically extracting the non-existence part from the obstruction theory proof. The obstruction theory proof has the advantage that it shows exactly when the test map in question exists. That is, the proof method fails when r is not a prime (except for $d = 0$ and $r = 4$). We omit this proof here, because it is completely due to my coauthors (I only helped slightly with the reverse direction, that is, when the test map in question does exist). After the obstruction theory proof was published, Vrećica and Živaljević [VŽ11a] derived from it the same degree proof, independently from us.

1.6 Problems

Approach for the affine main theorem for non-primes

Does Theorem 1.8 still hold for non-primes r, even in the special case when f is affine?

Originally Tverberg proved his Theorem 1.1 by a deformation argument [Tve66]: When we move one point of a given point configuration generically such that at a certain time a Tverberg partition becomes invalid then one can rearrange the partition such that it will serve as a Tverberg partition from that time on for some time.

Newer proofs [Tve81], [Sar00], use Bárány's colored Carathéodory theorem [Bár82]. These proofs can be written in terms of the test-map (1.16).

The affine version of Theorem 1.8 would now follow similarly from a corresponding "multicolored Carathéodory conjecture", if this exists. Similarly one can state a corresponding "multi-colored Helly conjecture". Both of them also seem to be interesting on their own.

Existence of the test-map for non-primes

What are the right conditions on the color class sizes such that the S_r-equivariant test-map (1.17) does not exist?

In [BMZ09] we proved that for $|C_0| = \ldots = |C_d| = r - 1$ and $|C_{d+1}| = 1$, (1.17) exists if and only if r divides $(r-1)!^d$. On the other hand, if all color classes are singletons then Özaydin [Öza87] proved that (1.17) does not exist if and only if r is a prime power.

One guess is that, given $r = p^k$, the test-map (1.17) does not exist if all color classes are of size at most $\varphi(r)$, where $\varphi(r) = (p-1)p^{k-1}$ is Euler's totient function, and that this upper bound is best possible.

2 A transversal generalization

2.1 Introduction

In their 1993 paper [TV93] H. Tverberg and S. Vrećica presented a conjectured common generalization of some Tverberg type theorems, some ham sandwich type theorems and many intermediate results. See [Živ99] for a further collection of implications.

Conjecture 2.1 (Tverberg–Vrećica Conjecture)**.** *Let $0 \leq k \leq d$ and let $\mathcal{C}^0, \ldots, \mathcal{C}^k$ be finite point sets in $\mathbb{R}^d$ of cardinality $|\mathcal{C}^\ell| = (r_\ell - 1)(d-k+1)+1$. Then one can partition each $\mathcal{C}^\ell$ into r_ℓ sets $F_1^\ell, \ldots, F_{r_\ell}^\ell$ such that there is a k-plane P in $\mathbb{R}^d$ that intersects all the convex hulls* $\mathrm{conv}(F_j^\ell)$, $0 \leq \ell \leq k$, $1 \leq j \leq r_\ell$.

The Tverberg–Vrećica Conjecture has been verified for the following special cases:

- $k = d$ (trivial),
- $k = 0$ (Tverberg's theorem [Tve66]),
- $k = d-1$ (Tverberg & Vrećica [TV93]),
- for $k = d-2$ a weakened version was shown in [TV93] (one requires two more points for each $\mathcal{C}^\ell$),
- k and d are odd, and $r_0 = \cdots = r_k$ is an odd prime (Živaljević [Živ99]),
- $r_0 = \cdots = r_k = 2$ (Vrećica [Vre03]), and
- $r_\ell = p^{a_\ell}$, $a_\ell \geq 0$, for some prime p, and $p(d-k)$ is even or $k = 0$ (Karasev [Kar07]).

In this section we consider the following colorful generalization of the Tverberg–Vrećica conjecture.

Conjecture 2.2. *Let $0 \leq k \leq d$, $r_\ell \geq 2$ $(\ell = 0, \ldots, k)$ and let $\mathcal{C}^\ell$ $(\ell = 0, \ldots, k)$ be subsets of $\mathbb{R}^d$ of cardinality $|\mathcal{C}^\ell| = (r_\ell - 1)(d-k+1)+1$. Let the $\mathcal{C}^\ell$ be colored,*

$$C^\ell \;=\; \biguplus C_i^\ell,$$

such that no color class is too large, $|C_i^\ell| \le r_\ell - 1$. Then we can partition each C^ℓ into sets $F_1^\ell, \dots, F_{r_\ell}^\ell$ that are colorful *(in the sense that $|C_i^\ell \cap F_j^\ell| \le 1$ for all i, j, ℓ) and find a k-plane P that intersects all the convex hulls* conv(F_j^ℓ)*.*

The Tverberg–Vrećica Conjecture 2.1 is the special case of the previous conjecture when all color classes are given by singletons. The main result of Section 2 is the following special case.

Theorem 2.3 (Transversal main theorem, [BMZ11a])**.** *Let r be prime and $0 \le k \le d$ such that $r(d-k)$ is even or $k = 0$. Let $\mathcal{C}^\ell$ $(\ell = 0, \dots, k)$ be subsets of $\mathbb{R}^d$ of cardinality $|\mathcal{C}^\ell| = (r-1)(d-k+1)+1$. Let the $\mathcal{C}^\ell$ be colored,*

$$\mathcal{C}^\ell = \biguplus C_i^\ell,$$

such that no color class is too large, $|C_i^\ell| \le r-1$. Then we can partition each $\mathcal{C}^\ell$ into colorful sets $F_1^\ell, \dots, F_r^\ell$ and find a k-plane P that intersects all the convex hulls conv(F_j^ℓ)*.*

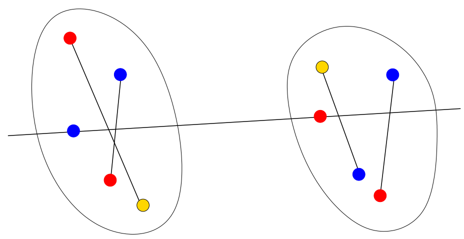


Figure 1.3: Example of Theorem 2.3 for $d = 2$, $k = 1$, $r = 3$, $|\mathcal{C}^\ell| = 5$.

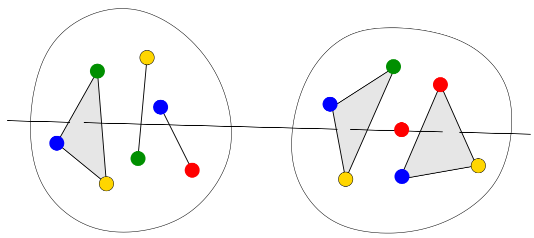


Figure 1.4: Example of Theorem 2.3 for $d = 3$, $k = 1$, $r = 3$, $|\mathcal{C}^\ell| = 7$.

Our earlier Main Theorem 1.8 is the special case when $k = 0$.

In Section 2.5 we will see that this theorem is quite tight in the sense that it becomes false if one single color class C_i^ℓ has r_ℓ elements and all the other ones are singletons.

Since we will prove Theorem 2.3 topologically it has a natural topological extension, Theorem 2.12.

In Section 2.2 we present an alternative proof for the case $k = 0$, based on the configuration space/test map scheme from Section 1.4. It puts the first one proof from Section 1.5 into the language of cohomological index theory. For this, we calculate the cohomological index of joins of chessboard complexes. This allows for a more direct proof of the case $k = 0$, which is the first of two keys for the Transversal Main Theorem 2.3.

The second key is a new Borsuk–Ulam type theorem for equivariant bundles. We establish it in Section 2.4, and prove the transversal main theorem in Section 2.5. The new Borsuk–Ulam type theorem can also be applied to obtain an alternative proof of Karasev's above-mentioned result from [Kar07]; see Section 2.5. Karasev has also obtained a colored version of the Tverberg–Vrećica conjecture, different from ours, even for prime powers, which can also alternatively be obtained from our new Borsuk–Ulam type theorem.

For another recently established parametrized version of the Borsuk–Ulam theorem we refer to [SSST11].

2.2 Second proof of the main theorem

In this section we present a new proof of Theorem 1.8. It puts the degree proof from Section 1.5 into the language of cohomological index theory, as developed by Fadell and Husseini [FH88]. Even though the second proof needs calculations and looks more difficult it actually allows for a more direct path, since it avoids the non-topological reduction of Lemma 1.19. This requires non-trivial index calculations, which however are very valuable since they provide a first key step towards our proof of the transversal main theorem 2.3 in Section 2.5.

Cohomological index theory

Let H^* denote Čech cohomology with $\mathbb{Z}_r$-coefficients, where r is prime. The *equivariant cohomology* of a G-space X is defined as

$$H^*_G(X) \;:=\; H^*(EG \times_G X),$$

where EG is a contractible free G-CW complex and $EG \times_G X := (EG \times X)/G$. The *classifying space* of G is $BG := EG/G$. If $p : X \to B$ is furthermore a projection to a trivial G-space B, we denote the *cohomological index* of X over B, also called the *Fadell–Husseini index* [FH87], [FH88], by

$$\mathrm{Index}^B_G(X) \;:=\; \ker\left(H^*_G(B) \xrightarrow{p^*} H^*_G(X)\right) \;\subseteq\; H^*_G(B) \cong H^*(BG) \otimes H^*(B).$$

If $B = \{*\}$ is a point then one also writes $H^*_G(\{*\}) = H^*(G)$ and $\mathrm{Index}_G(X) := \mathrm{Index}^{\{*\}}_G(X)$.

The cohomological index has the four properties

- Monotonicity: If there is a bundle map $X \longrightarrow_G Y$ then
$$\mathrm{Index}^B_G(X) \supseteq \mathrm{Index}^B_G(Y). \tag{2.4}$$
- Additivity: If $(X_1 \cup X_2, X_1, X_2)$ is excisive, then
$$\mathrm{Index}^B_G(X_1) \cdot \mathrm{Index}^B_G(X_2) \subseteq \mathrm{Index}^B_G(X_1 \cup X_2).$$
- Joins:
$$\mathrm{Index}^B_G(X) \cdot \mathrm{Index}^B_G(Y) \subseteq \mathrm{Index}^B_G(X * Y).$$

- Subbundles: If there is a is a bundle map $f : X \longrightarrow_G Y$ and a closed subbundle $Z \subseteq Y$ then
$$\mathrm{Index}_G^B(f^{-1}(Z)) \cdot \mathrm{Index}_G^B(Y) \subseteq \mathrm{Index}_G^B(X). \tag{2.5}$$

The first two properties imply the other two. The last one uses furthermore the continuity of Čech cohomology H^*. For more information about this index theory see [FH87] and [FH88].

If r is odd then the cohomology of $\mathbb{Z}_r$ as a $\mathbb{Z}_r$-algebra is

$$H^*(\mathbb{Z}_r) = H^*(B\mathbb{Z}_r) \cong \mathbb{Z}_r[x, y]/(y^2),$$

where $\deg(x) = 2$ and $\deg(y) = 1$. If r is even, then $r = 2$ and $H^*(\mathbb{Z}_2) \cong \mathbb{Z}_2[t]$, $\deg t = 1$.

Index calculations

Theorem 2.6. *Let r be a prime. Let K be an n-dimensional connected free $\mathbb{Z}_r$-CW complex and let S be an n-dimensional $(n-1)$-connected free $\mathbb{Z}_r$-CW complex. If there is a $\mathbb{Z}_r$-map $M : K \longrightarrow_{\mathbb{Z}_r} S$ that induces an isomorphism on H^n, then*

$$\mathrm{Index}_{\mathbb{Z}_r}^{\{*\}}(K) = H^{*\geq n+1}(B\mathbb{Z}_r).$$

Proof. The $\mathbb{Z}_r$-equivariant map $M : K \longrightarrow_{\mathbb{Z}_r} S$ induces a map of fibrations,

$$\begin{tikzcd}
K \arrow[r, "M"] \arrow[d] & S \arrow[d] \\
E\mathbb{Z}_r \times_{\mathbb{Z}_r} K \arrow[r, "(\mathrm{id}, M)"] \arrow[d] & E\mathbb{Z}_r \times_{\mathbb{Z}_r} S \arrow[d] \\
B\mathbb{Z}_r \arrow[r, "\mathrm{id}"] & B\mathbb{Z}_r.
\end{tikzcd}$$

Consequently, M induces a morphism $E_*^{*,*}(M)$ between associated Leray–Serre spectral sequences $E_*^{*,*}(K)$ and $E_*^{*,*}(S)$, see Figure 1.5. It has the property that $E_2^{*,0}(M) = id_{H^*(B\mathbb{Z}_r)}$. For background on Leray–Serre spectral sequences see [McC01, Chapters 5 and 6]. Moreover, the nth rows $E_2^{*,n}(K) = H^*(\mathbb{Z}_r; H^n(K))$ and $E_2^{*,n}(S) = H^*(\mathbb{Z}_r; H^n(S))$ at the E_2-pages are identified via $E_2^{*,n}(M)$.

At the E_∞-pages both spectral sequences have to satisfy $E_\infty^{p,q} = 0$ whenever the total degree $p + q \geq n + 1$. This is because K is free $\mathbb{Z}_r$-space, hence $H^*_{\mathbb{Z}_r}(K) \cong H^*(K/\mathbb{Z}_r)$, which is zero in degrees $* \geq n + 1$. The same holds for S. Therefore, the elements $E_*^{*\geq n+1,0}(S) = H^{*\geq n+1}(\mathbb{Z}_r)$ in the bottom row of the spectral sequence $E_*^{*,*}(S)$ must be hit by some differential. These differentials can come only from the nth row at the E_{n+1}-page (this argument even gives us the $H^*(\mathbb{Z}_r)$-module structure of the nth row). Hence there is a non-zero transgressive element $w \in E_2^{0,n}(S) = H^0(\mathbb{Z}_r; H^n(S)) = H^n(S)^{\mathbb{Z}_r}$, that is, $d_{n+1}(w) \neq 0$. Let $z := E_r^{0,n}(M)(w) \in E_2^{0,n}(K) = H^n(K)^{\mathbb{Z}_r}$. Then $d_i(z) = d_i(E_r^{0,n}(M)(w)) = E_r^{i,n-i+1}(M)(d_i(w))$, which is zero for $i \leq n$. Therefore z survives at least until E_{n+1}. Analogously, the whole nth row survives until E_{n+1}. We know that all

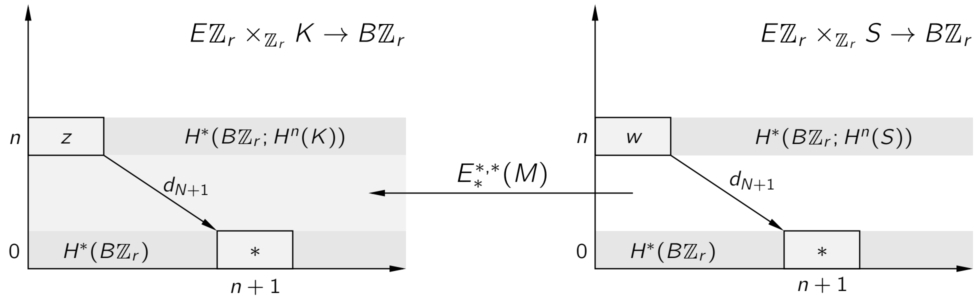


Figure 1.5: The morphism $E_*^{*,*}(M)$ between the spectral sequences $E_*^{*,*}(S)$ and $E_*^{*,*}(K)$.

elements in $E_{n+1}^{*\geq 1,n}(K)$ have to die eventually, so they do it exactly on page E_{n+1}. Thus these elements are exactly the elements whose differentials make the part $E_*^{*\geq n+2,0}$ of the bottom row vanish.

We claim that no non-zero differential can arrive at the bottom row on an earlier page of $E_*^{*,*}(K)$. Assume that $d_i(\alpha) = x^a y^b \in E_i^{*,0}$ for some α and $i \leq n$. This would imply that $d_i(x^k\alpha) = x^{a+k}y^b$ for all $k > 0$. But we already know that the elements in $E_*^{*\geq n+2,0}(K)$ survive until page E_{n+1}, which gives the desired contradiction.

Therefore at $E_\infty(K)$, the non-zero part of the bottom row is $H^{*\leq n}(\mathbb{Z}_r)$. The index defining map $H^*_{\mathbb{Z}_r}(\{*\}) \to H^*_{\mathbb{Z}_r}(K)$ is the edge homomorphism, which is the composition

$$H^*_{\mathbb{Z}_r}(\{*\}) \xrightarrow{\cong} E_2^{*,0}(K) \twoheadrightarrow E_\infty^{*,0} \hookrightarrow H^*_{\mathbb{Z}_r}(K).$$

Therefore the index of K is everything in the bottom row that got hit by a differential, that is,

$$\operatorname{Index}^{\{*\}}_{\mathbb{Z}_r}(K) = H^{*\geq n+1}(B\mathbb{Z}_r).$$

□

We apply this theorem to the above maps $M : K' \to S(W_r^{d+1})$ and $(M * \mathrm{id}) : K' * [r] \to S(W_r^{d+1}) * [r]$. By the degree calculation (1.20) (p. 7), M induces an isomorphism in H^{N-1}.

Corollary 2.7. *The $\mathbb{Z}_r$-index of $K' = (\Delta_{r,r-1})^{*(d+1)}$ is*

$$\operatorname{Index}^{\{*\}}_{\mathbb{Z}_r}(K') = H^{*\geq N}(B\mathbb{Z}_r)$$

and the $\mathbb{Z}_r$-index of $K' * [r]$ is

$$\operatorname{Index}^{\{*\}}_{\mathbb{Z}_r}(K' * [r]) = H^{*\geq N+1}(B\mathbb{Z}_r).$$

Using the first part of this corollary we can compute the index for more general joins of chessboard complexes.

Corollary 2.8. *Let $0 \leq c_0, \dots, c_m \leq r-1$ and let $s := \sum c_i$. Let $K := \Delta_{r,c_0} * \cdots * \Delta_{r,c_m}$. Then*

$$\operatorname{Index}^{\{*\}}_{\mathbb{Z}_r}(K) = H^{*\geq s}(B\mathbb{Z}_r).$$

Proof. Let $L := \Delta_{r,r-1-c_0} * \cdots * \Delta_{r,r-1-c_m}$ and $K' := (\Delta_{r,r-1})^{*(m+1)}$. Then $\dim K = s-1$ and $\dim K' = (r-1)(m+1)-1$. We calculate $\dim K'+1 = (\dim K+1)+(\dim L+1)$. There is an inclusion $K' \longrightarrow_{\mathbb{Z}_r} K * L$. This implies

$$\mathrm{Index}_{\mathbb{Z}_r}(K') \supseteq \mathrm{Index}_{\mathbb{Z}_r}(K * L) \supseteq \mathrm{Index}_{\mathbb{Z}_r}(K) \cdot \mathrm{Index}_{\mathbb{Z}_r}(L). \tag{2.9}$$

Since K is a free $\mathbb{Z}_r$-space, $H^*_{\mathbb{Z}_r}(K) = H^*(K/\mathbb{Z}_r)$, hence

$$\mathrm{Index}_{\mathbb{Z}_r}(K) \supseteq H^{*\geq \dim K+1}(B\mathbb{Z}_r), \tag{2.10}$$

and analogously

$$\mathrm{Index}_{\mathbb{Z}_r}(L) \supseteq H^{*\geq \dim L+1}(B\mathbb{Z}_r). \tag{2.11}$$

The dimension $a := \dim K'$ is odd if r is odd. Using Corollary 2.7, we find that $\mathrm{Index}_{\mathbb{Z}_r}(K') = H^{\geq a+1}(B\mathbb{Z}_r) = \langle x^{\frac{a+1}{2}} \rangle$ if r is odd, and $\mathrm{Index}_{\mathbb{Z}_r}(K') = \langle t^{a+1} \rangle$ if $r = 2$. Together with equation (2.9), the inclusions (2.10) and (2.11) have to hold with equality. □

It is interesting that the last argument of the proof would fail for odd r if $a+1$ was odd, due to the relation $y^2 = 0$ in $H^*(\mathbb{Z}_r)$.

Now we plug in the configuration space K from (1.15) and obtain the second proof of Theorem 1.8.

Second proof of Theorem 1.8. According to the monotonicity of the index, see (2.4), the existence of the test map $F : K \longrightarrow_{\mathbb{Z}_r} S(W_r^{d+1})$ of (1.17) would imply that

$$\mathrm{Index}^{\{*\}}_{\mathbb{Z}_r}(K) \;\supseteq\; \mathrm{Index}^{\{*\}}_{\mathbb{Z}_r}(S(W_r^{d+1})).$$

This is a contradiction since $\mathrm{Index}^{\{*\}}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r)$, whereas $\mathrm{Index}^{\{*\}}_{\mathbb{Z}_r}(S(W_r^{d+1})) = H^{*\geq N}(B\mathbb{Z}_r)$, since $S(W_r^{d+1})$ is an $(N-1)$-dimensional free $\mathbb{Z}_r$-sphere. □

On the surface this proof seems to be a more difficult reformulation of the first proof. However, its view point is essential for the transversal generalization, since we do not rely on the geometric tools of the Reduction Lemma 1.19 anymore, and such a reduction lemma does not seem to exist for the Tverberg–Vrećica type transversal theorem. Thus we need to use the more general configuration space $\Delta_{r,|C_0|} * \cdots * \Delta_{r,|C_m|}$ of (1.15) instead of $(\Delta_{r,r-1})^{*(d+1)} * [r]$.

2.3 The transversal configuration space/test map scheme

The proof of our transversal main theorem 2.3 is based on a configuration space/test map scheme for vector bundles. Such a proof scheme was already used in [Dol87], [Dol93], [Živ99], [Vre03] and [Kar07]. Our progress in this paper stems from the topological index calculations of Section 2.2 and from the Borsuk–Ulam type Theorem 2.15 in Section 2.4.

The proof gives actually the following more general topological version. Theorem 2.3 is the special case when all maps f_ℓ are affine.

Theorem 2.12 (Topological transversal main theorem)**.** *Let r be prime and $0 \leq k \leq d$ such that $r(d-k)$ is even or $k=0$. Let $\mathcal{C}^\ell$ ($\ell = 0, \dots, k$) be sets of cardinality $|\mathcal{C}^\ell| = (r-1)(d-k+1)+1$, which we identify with the vertex sets of simplices $\Delta_{|\mathcal{C}^\ell|-1}$. We color them*

$$\mathcal{C}^\ell = \biguplus C_i^\ell,$$

such that no color class is too large, $|C_i^\ell| \leq r-1$. Let

$$f_\ell : \Delta_{|\mathcal{C}^\ell|-1} \to \mathbb{R}^d$$

be continuous maps. Then we can find r disjoint rainbow faces $F_1^\ell, \dots, F_r^\ell$ in each simplex $\Delta_{|\mathcal{C}^\ell|-1}$ (that is, $|F_j^\ell \cap C_i^\ell| \leq 1$) and a k-plane $P \subseteq \mathbb{R}^d$ that intersects all the sets $f_\ell(F_j^\ell)$.

The proof scheme for our situation works as follows. Suppose we are given $\mathcal{C}^\ell$s, f_ℓs and r as in the assertion of Theorem 2.12 together with the colorings

$$\mathcal{C}^\ell = \biguplus_{i=0}^{m_\ell} C_i^\ell.$$

A collection of rainbow faces F_j^ℓ of the simplices $\Delta_{|\mathcal{C}^\ell|-1}$ admits a common k-plane P that intersects all images $f(F_j^\ell)$ if and only if one can project these images orthogonally to a $(d-k)$-dimensional subspace of $\mathbb{R}^d$ (namely the orthogonal complement of P) such that the convex hulls of the projected F_j^ℓs have a point in common (this point is the image of P under the projection).

Calculations turn out to be easier if we look first at the set of colored Tverberg points of all projections of one single fixed $\mathcal{C}^\ell$ and then show that the corresponding sets for all $\mathcal{C}^\ell$s have to intersect.

Fix an $\ell \in \{0, 1, \dots, k\}$. Let $B := G_{d,d-k}$ be the Grassmannian manifold of all $(d-k)$-dimensional subspaces of $\mathbb{R}^d$ and $\gamma \to B$ the tautological bundle over B. For definitions and context, see Chapter 5 of [MS74]. Let ε denote the trivial line bundle over B. Let $B \times W_r$ be the trivial bundle over B with fiber W_r, which was defined in Section 2.2. Let $E := (B \times W_r) \oplus \gamma^{\oplus r}$. The group $G := \mathbb{Z}_r$ acts on $[r]$ by left translations and on E by fiberwise shifting the coordinates cyclically. E is a G-bundle over the trivial G-space B whose fixed-point subbundle $\Delta := E^G = (B \times W_r)^G \oplus (\gamma^{\oplus r})^G \cong \gamma$ is the thin diagonal bundle.

The space

$$K := \Delta_{r,|C_0^\ell|} * \dots * \Delta_{r,|C_{m_\ell}^\ell|} = (C_0^\ell * \ldots * C_{m_\ell}^\ell)^{*r}_{\Delta(2)} \subseteq (\Delta_{|\mathcal{C}^\ell|-1})^{*r} \tag{2.13}$$

will again be the configuration space. For each $b \in B$, we can compose the map f_ℓ with the orthogonal projection to the $(d-k)$-space given by b, which can be identified with the fiber over b in γ. This is gives function

$$B \times \Delta_{|\mathcal{C}^\ell|-1} \to \gamma,$$

which is bundle map over B, $B \times \Delta_{|\mathcal{C}^\ell|-1}$ being the trivial bundle over B. Doing the analogous construction as in Section 1.4 for the testmap in the non-transversal situation, we get a $\mathbb{Z}_r$-equivariant bundle map

$$B \times K \xrightarrow{M} E,$$

where the r join coefficients in $K = (C_0^\ell * \ldots * C_{m_\ell}^\ell)^{*r}_{\Delta(2)}$ are mapped into the r trivial summands ε of E. Define $T^\ell := \mathrm{im}(M) \cap \Delta$, which is the set of colored Tverberg points of the respective projected sets $\mathrm{im}(F_\ell)$. Each point of $T^\ell \subseteq \Delta$, which lies in the fiber over say $b \in B$, lies in the intersection of the images $\mathrm{im}(F_j^\ell)$ of r disjoint rainbow faces projected to b.

Hence we need to show that

$$T^0 \cap \cdots \cap T^k \neq \emptyset. \tag{2.14}$$

We will apply our results on the index of the configuration space K, derived in Corollary 2.8, and tools from Section 2.4 to show that this is indeed the case. The proof of Theorem 2.12 continues in Section 2.5.

2.4 A new Borsuk–Ulam type theorem

In this section we prove the following Borsuk–Ulam type theorem. It is the second topological step towards the proof of Theorem 2.3. This theorem will be applied in combination with the subsequent intersection Lemma 2.24.

Theorem 2.15 (Borsuk–Ulam type)**.** *Let*

- *p be a prime,*
- *$G = (\mathbb{Z}_p)^m$ an elementary abelian group,*
- *K a G-CW-complex with index $\mathrm{Index}_G^{\{*\}}(K) \subseteq H^{*\geq n+1}(BG; \mathbb{Z}_p)$,*
- *B a connected, trivial G-space,*
- *$E \xrightarrow{\phi} B$ a G-vector bundle (all fibers carry the same G-representation),*
- *$\Delta := E^G \to B$ the fixed-point subbundle of $E \to B$,*
- *$C \to B$ its G-invariant orthogonal complement subbundle ($E = C \oplus \Delta$),*
- *F be the fiber of the sphere bundle $S(C) \to B$.*

Suppose that

- *$n = \mathrm{rank}(C)$,*
- *$\pi_1(B)$ acts trivially on $H^*(F; \mathbb{Z}_p)$ (that is, $C \to B$ is orientable if $p \neq 2$), and*
- *we are given a G-bundle map M,*

$$\begin{tikzcd} B \times K \arrow[rr, "M"] \arrow[dr, "pr_1"'] & & E = C \oplus \Delta \arrow[dl, "\phi"] \\ & B & \end{tikzcd} .$$

Then for $S := M^{-1}(\Delta)$ and $T := M(S) = \mathrm{im}(M) \cap \Delta$ the composition

$$H^*(B;\mathbb{Z}_p) \longrightarrow H^*_G(B;\mathbb{Z}_p) \xrightarrow{(pr_1|_S)^*} H^*_G(S;\mathbb{Z}_p) \tag{2.16}$$

and the map induced by projection

$$H^*(B;\mathbb{Z}_p) \xrightarrow{(\phi|_T)^*} H^*(T;\mathbb{Z}_p) \tag{2.17}$$

are both injective.

If K is free then (2.16) is the same as the map induced by projection $H^*(B;\mathbb{Z}_p) \to H^*(S/G;\mathbb{Z}_p)$.

Remark 2.18. The theorem generalizes

- a lemma of Volovikov [Vol96], which is the special case when $B = \{*\}$ and K is $(n-1)$-$\mathbb{Z}_p$-acyclic,
- and Theorem 4.2 of Živaljević [Živ99], from whose proof one can extract the special case when $m = 1$ and K is $(n-1)$-$\mathbb{Z}_p$-acyclic. and in particular
- the Borsuk–Ulam theorem, which is the special case when $G = \mathbb{Z}_2$, $B = \{*\}$, $K = S^n$, $E = \mathbb{R}^n$, E and K with antipodal action.

Proof of Theorem 2.15. We use Čech cohomology with $\mathbb{Z}_p$-coefficients.
(1.) Let $b \in B$ be the point over which F is the fiber in the sphere bundle $S(C) \to B$. We denote by $E^{*,*}_*(F)$ and $E^{*,*}_*(S(C))$ the Leray–Serre spectral sequences associated to the fibrations

$$F \hookrightarrow EG \times_G F \to BG \times b \tag{2.19}$$

and

$$F \hookrightarrow EG \times_G S(C) \to BG \times B, \tag{2.20}$$

respectively, see Figure 1.6. For details on the Leray–Serre spectral sequence, see [McC01, Chapters 5 and 6].

The E_2-page $E_2^{*,*}(F)$ has only two non-zero rows, the 0-row and the $(n-1)$-row. The local coefficients in $E_2^{p,q}(F) = H^p(BG, H^q(F))$ are given by the $\pi_1(BG)$-module structure on $H^q(F)$. Since $G = \pi_1(BG)$ is an elementary abelian group and F is a sphere, the $H^*(BG)$-module structure on $H^q(F)$ is trivial, for the G action on $H^q(F)$ is induced by homeomorphisms $F \to F$, and the degree of this homeomorphism has to be 1 if p is odd. Therefore

$$E_2^{p,q}(F) = H^p(BG, H^q(F)) = H^p(BG) \otimes H^q(F).$$

The differentials are $H^*(BG)$-homomorphism, and

$$E_n^{*,n-1}(F) = E_2^{*,n-1}(F) = H^*(BG) \otimes H^{n-1}(F)$$

is a $H^*(BG)$-module generated by

$$1 \in E_n^{0,n-1}(F) = H^0(BG) \otimes H^{n-1}(F),$$

where 1 is regarded as the generator of $H^{n-1}(F)$. Hence there is a non-vanishing differential in $E_*^{*,*}(F)$ if and only if the differential $d_n : E^{0,n-1}(F) \to E^{n,0}(F)$ is non-zero. Since F is fixed-point free, the edge homomorphism $H^*(BG) \to H_G^*(F)$ is not injective [tD87, Prop. 3.14, p. 196]. Thus there must be a non-vanishing differential. Therefore there is a non-zero element $\alpha = d_n(1) \in \mathrm{Index}_G^{\{*\}}(F)$ of degree n.

(2.) Now the inclusion $F \hookrightarrow S(C)$ gives a bundle map from (2.19) to (2.20),

$$\begin{tikzcd} EG \times_G F \arrow[r] \arrow[d] & EG \times_G S(C) \arrow[d] \\ BG \times b \arrow[r] & BG \times B \end{tikzcd} \tag{2.21}$$

which induces a morphism $E_*^{*,*}(S(C)) \to E_*^{*,*}(F)$ of associated Leray–Serre spectral sequences, see Figure 1.6.

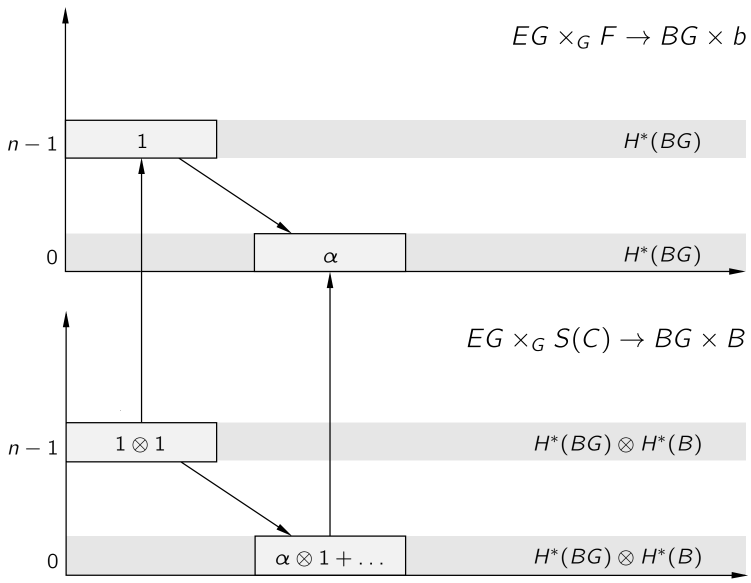


Figure 1.6: The morphism of spectral sequences induced by the bundle map (2.21).

The E_2-page of $E_*^{*,*}(S(C))$ is $E_2^{p,q}(S(C)) = H^p(BG \times B, H^q(F))$, where the local coefficients are given by the $\pi_1(BG \times B)$-module structure on $H^q(F)$. Since $H^*(F)$ is a trivial $G \times \pi_1(B)$-module, for $q = 0$ and $q = n-1$ we have

$$E_2^{p,q}(S) = H^p(BG \times B; H^q(F)) = \bigoplus_{i=0}^{p} H^i(BG) \otimes H^{p-i}(B).$$

The morphism of the spectral sequences $E_*^{*,*}(S(C)) \to E_*^{*,*}(F)$ on the 0-row and on the $(n-1)$-row of the E_2-page,

$$\bigoplus_{i=0}^{p} H^i(BG) \otimes H^{p-i}(B) \to H^p(BG),$$

is zero on $\bigoplus_{i=1}^{p} H^i(BG) \otimes H^{p-i}(B)$. On $H^p(BG) \otimes H^0(B) = H^p(BG)$ it is just the identity. The differential of the generator $1 \otimes 1 \in H^0(BG) \otimes H^0(B)$ of $E_n^{0,n-1}(S(C))$ hits an element $\gamma \in E_n^{n-1,0}(S(C))$ of the bottom row $\bigoplus_{i=0}^{n} H^i(BG) \otimes H^{n-i}(B)$. Since the differentials commute with morphisms of spectral sequences, γ is an element in $\mathrm{Index}_G^B(S(C)) \subseteq H^*(BG) \otimes H^*(B)$ that restricts to α under the map $\bigoplus_{i=0}^{n} H^i(BG) \otimes H^{n-i}(B) \to H^n(BG)$, hence $\gamma \neq 0$. Since α and γ are of degree n, γ has the form

$$\gamma = \alpha \otimes 1 + \sum_i \delta_i \otimes \varepsilon_i, \tag{2.22}$$

for some δ_i and ε_i with $\deg \delta_i + \deg \varepsilon_i = n$ and $\deg \delta_i \leq n-1$.

(3.) Formula (2.5) of Section 2.2 yields

$$\mathrm{Index}_G^B(S) \cdot \mathrm{Index}_G^B(S(C)) \subseteq \mathrm{Index}_G^B(B \times K) = \mathrm{Index}_G^{\{*\}}(K) \otimes H^*(B). \tag{2.23}$$

We know that $\mathrm{Index}_G^{\{*\}}(K) \subseteq H^{*\geq n+1}(BG)$, and in (2.22) we got an element $\gamma \in \mathrm{Index}_G^B(S(C))$. We claim that $\mathrm{Index}_G^B(S) \subseteq H^*(BG) \otimes H^*(B)$ does not contain any non-zero element of the form $1 \otimes \beta$, $\beta \in H^*(B)$. Indeed, if $1 \otimes \beta \in \mathrm{Index}_G^B(S) \backslash \{0\}$, then

$$(1 \otimes \beta) \cdot \gamma = \alpha \otimes \beta + \sum_i \delta_i \otimes (\beta \cdot \varepsilon_i) \in \mathrm{Index}_G^{\{*\}}(K) \otimes H^*(B).$$

Here we omitted signs for simplicity. Since $\deg(\delta_i) < \deg(\alpha) = n$, this implies that $\alpha \in \mathrm{Index}_G^{\{*\}}(K)$. This contradicts $\mathrm{Index}_G^{\{*\}}(K) \subseteq H^{*\geq n+1}(BG)$.

Hence the following composition is injective,

$$H^*(B) \xrightarrow{1 \otimes \mathrm{id}} H^*(BG) \otimes H^*(B) \to H_G^*(S),$$

where both maps are induced by projection. The following diagram is induced by the obvious maps

$$\begin{tikzcd}
H_G^*(S) & H_G^*(T) = H^*(BG) \otimes H^*(T) \arrow[l] & H^*(T) \arrow[l] \\
& H^*(B) \arrow[ul, "\text{inj.}"] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

which shows that $H^*(B) \to H^*(T)$ has to be injective as well. □

We will use Theorem 2.15 together with the following intersection lemma.

Lemma 2.24 (Intersection lemma)**.** *Let p be prime and $\Delta \xrightarrow{pr} B$ be a vector bundle over a $\mathbb{Z}_p$-orientable compact manifold B, whose mod-p Euler class $e := e(\Delta) \in H^*(B; \mathbb{Z}_p)$ satisfies $e^k \neq 0$. Let $T_0, \ldots, T_k \subseteq \Delta$ be closed sets such that $H^{\dim B}(B; \mathbb{Z}_p) \xrightarrow{(pr|_{T_i})^*} H^{\dim B}(T_i; \mathbb{Z}_p)$ is injective for all i. Then*

$$T_0 \cap \cdots \cap T_k \neq \emptyset.$$

A proof for the case where p is prime can be extracted from [Živ99]. For the convenience of the reader and for Remark 2.26 we repeat the argument.

Proof of Lemma 2.24. In the case $k = 0$ we need to show that $T_0 \neq \emptyset$. This is true since $(pr|_{T_0})^*$ is injective. So let us assume that $k \geq 1$.

Let $D(\Delta)$ and $S(\Delta)$ be the disc and sphere bundles of Δ. As a bundle, Δ is $\mathbb{Z}_p$-orientable since $e \neq 0$, and B is a $\mathbb{Z}_p$-orientable manifold. Hence $D(\Delta)$ is a $\mathbb{Z}_p$-orientable manifold as well. We may assume that the T_is lie in the interior of $D(\Delta)$ by rescaling the fibers. Let $b := \dim(B)$ and $r := \operatorname{rank}(\Delta)$. Let $\tau \in H^r(D(\Delta), S(\Delta))$ be the Thom class. We have $e = i^*(\tau)$, where $i : (B, \emptyset) \to (D(\Delta), S(\Delta))$ is the inclusion. Hence $\tau^{k+1} = e^k\tau \neq 0$, since $e^k \neq 0$ and multiplication by τ is the Thom isomorphism.

Suppose that $T_0 \cap \cdots \cap T_k = \emptyset$. Then there are open neighborhoods V_i of T_i such that $V_0 \cap \cdots \cap V_k = \emptyset$. We will derive a contradiction to that. Since the Čech-cohomology of T_i is the limit of the ordinary cohomology over open neighborhoods in E, we can make the neighborhoods V_i smaller such that the maps $H^b(B;\mathbb{Z}_p) \xrightarrow{(pr|_{V_i})^*} H^b(V_i;\mathbb{Z}_p)$ are injective for all i. Therefore the universal coefficient theorem for cohomology implies that $H_b(V_i;\mathbb{Z}_p) \to H_b(B;\mathbb{Z}_p)$ is surjective, because we use field coefficients.

In the following commutative diagram, the vertical arrows are Poincaré–Alexander–Lefschetz duality and the bottom map j_i is induced by inclusion.

$$\begin{tikzcd}
H_b(V_i) \arrow[r] \arrow[d, "\cong"'] & H_b(B) \arrow[r, "\cong"] & H_b(D(\Delta)) \arrow[d, "\cong"] \\
H^r(D(\Delta), D(\Delta)\backslash V_i) \arrow[rr, "j_i"] & & H^r(D(\Delta), S(\Delta))
\end{tikzcd}$$

Since τ is the Poincaré dual of the orientation class $[B] \in H^b(B) = H^b(D(\Delta))$ and the top map is surjective, we find an element $\alpha_i \in H^r(D(\Delta), D(\Delta)\backslash V_i)$ such that $j_i(\alpha_i) = \tau$. Hence

$$\tau^{k+1} = j_0(\alpha_0) \cdot \ldots \cdot j_k(\alpha_k) = j(\alpha_0 \cdot \ldots \cdot \alpha_k),$$

where j is the map induced by inclusion,

$$j : H^{r(k+1)}(D(\Delta), D(\Delta)\backslash(V_0 \cap \ldots \cap V_k)) \to H^{r(k+1)}(D(\Delta), S(\Delta)).$$

Since the image of the map j contains $\tau^{k+1} \neq 0$, we find that $V_0 \cap \ldots \cap V_k \neq 0$. $\square$

Possible extensions

Remark 2.25 (Extension of Theorem 2.15). Suppose that all the assumptions of the parameterized Borsuk–Ulam type theorem 2.15 are fulfilled and that the index of K satisfies the stronger bound

$$\operatorname{Index}_G^{\{*\}}(K) \subseteq H^{*\geq n+1+x}(BG;\mathbb{Z}_p),$$

for some $x \in \mathbb{N}$. Then the above proof yields a stronger implication: The composition

$$H^{*\leq x}(BG) \otimes H^*(B;\mathbb{Z}_p) \hookrightarrow H^*_G(B;\mathbb{Z}_p) \xrightarrow{(pr_1|_S)^*} H^*_G(S;\mathbb{Z}_p)$$

is injective. For this extension, only part (3.) of the proof needs adjustments. Sometimes one can get an even stronger implication if one directly uses inequality (2.23).

Remark 2.26 (Extension of Lemma 2.24). The intersection lemma can be extended to all positive integers p. This is possibly interesting for applications, however it won't be needed here. We have to change the argument only at the point where we need $H_b(V_i; \mathbb{Z}_p) \to H_b(V_i; \mathbb{Z}_p)$ to be surjective. The key is that $\mathbb{Z}_p$ is an injective $\mathbb{Z}_p$-module, which can be deduced from the criterion [Hun96, Lemma IV.3.8]. Hence $\hom(\_, \mathbb{Z}_p)$ is an exact functor. Therefore we have natural isomorphisms $H^b(B) \xrightarrow{\cong} \hom(H_b(B), \mathbb{Z}_p)$ and $H^b(V_i) \xrightarrow{\cong} \hom(H_b(V_i), \mathbb{Z}_p)$ and the injectivity of the map $H^b(B) \to H^b(V_i)$ implies that $H_b(V_i) \to H_b(B)$ is surjective.

2.5 Proof of the transversal main theorem

Now we have all the topological tools to prove the transversal main theorem. We continue from where we stopped at in Section 2.3. We need to prove (2.14), that is,

$$T_0 \cap \ldots \cap T_k \neq \emptyset.$$

Continuation of the proof of Theorem 2.12. First we assume that $p = 2$ or that d and k are odd. The remaining case, when p is odd and d and k are even, will be a consequence of an elementary reduction lemma at the end of the proof.

The configuration space K is of dimension $(r-1)(d-k)$. The ranks of E and Δ are $r(d-k+1)-1$ and $d-k$. We claim that the orthogonal complement bundle C of Δ in E is $\mathbb{Z}_r$-orientable. Since all vector bundles are $\mathbb{Z}_2$-orientable, we only need to deal with the case where r is odd. Then $r-1$ is even and C is stably isomorphic to γ^{r-1}, which is an even power of a bundle, hence orientable. Therefore we can apply the Borsuk–Ulam type Theorem 2.15 and get that $H^*(B) \to H^*(T_i)$ is injective. To apply the Intersection Lemma 2.24, we need that $e^k \neq 0$ for the mod-r Euler class $e \in H^{d-k}(B)$ of $\Delta \cong \gamma$.

If $r = 2$ then e is the top Stiefel–Whitney class w_{d-k}, whose k-th power is the mod-2 fundamental class of B (see e.g. [Hil80, Lemma 1.2]), which proves the theorem in this case. Now we come to the case where r is odd. If $\mathrm{rank}(\gamma) = d-k$ is odd then the mod-r Euler class is zero and our method yields no conclusion. If $d-k$ is even then we may assume that d and k are odd, otherwise we prove the theorem for $d' = d+1$ and $k' = k+1$ first and use the Reduction Lemma 2.27 below afterwards. Then $e^k \neq 0$ was proved in Proposition 4.9 of [Živ99], based on [FH88]. In fact, he even shows it for the tautological bundle over the *oriented* Grassmannian. Since this bundle is the pullback of γ we are done by naturality of the Euler class. Now the Intersection Lemma 2.24 gives that $T_0 \cap \cdots \cap T_k \neq \emptyset$. Hence by the preliminary work of Section 2.3 we are done.

Finally we prove the elementary Reduction Lemma 2.27 that also implies the case when p is odd and d and k are even.

Lemma 2.27 (Reduction Lemma). *If Conjecture 2.2 holds for parameters $(d, k, r_0, \ldots, r_k)$ then so it does for $(d', k', r_0, \ldots, r_{k-1})$ with $d' := d-1$ and $k' := \max(k-1, 0)$.*

Proof. We will prove only the case $k \geq 1$, since the case $k = 0$ is exactly the reduction that is used in the proof of Lemma 1.19 [BMZ09].

Assume that Conjecture 2.2 is true for parameters $(d, k, r_0, \ldots, r_k)$ and suppose we are given colored sets $\mathcal{C}^0, \ldots, \mathcal{C}^{k-1} \subseteq \mathbb{R}^{d-1}$ where we have to partition $\mathcal{C}^\ell$ into r_ℓ pieces such that some $(k-1)$-dimensional plane meets the convex hulls of all pieces. To do this, view $\mathbb{R}^{d-1}$ as the hyperplane in $\mathbb{R}^d$ where the last coordinate is zero, and define $\mathcal{C}^k \subset \mathbb{R}^d$ to be a set of $(r_k-1)(d-k+1)+1$ points all of which lie close to $(0,\dots,0,1)$. We color $\mathcal{C}^k$ in an arbitrary way. For example, we may give each point a different color. Since Conjecture 2.2 is true for $(d,k,r_0,\dots,r_k)$, we can partition the sets $\mathcal{C}^\ell$ appropriately and find a k-plane P meeting all of the convex hulls of the pieces. Since P goes through the convex hull of $\mathcal{C}^k$, it cannot be fully contained in $\mathbb{R}^{d-1}$. Therefore $P \cap \mathbb{R}^{d-1}$ is a $(k-1)$-plane intersecting the convex hulls of the pieces of the sets $\mathcal{C}^0,\dots,\mathcal{C}^{k-1}$. □

This finishes the proof of the transversal main theorem. □

Our proof of the transversal main theorem does not extend to prime powers $r_\ell = p^{a_\ell}$ over the same prime p. The basic reason is that the degree of M vanishes modulo r if and only if r divides $(r-1)!^d$ (see (1.20)). Therefore this proof can only work if r is a prime or if $r=4$ and $d=1$. For $k=0$, even using the full symmetry group S_r does not help since an S_r-equivariant test map exists if and only if r divides $(r-1)!^d$; see [BMZ09]. To see this one needs to take a closer look at the obstruction; the degree proof from Section 2.2 does not yield this.

A new proof for Karasev's result [Kar07]

We can extend the above proof to arbitrary powers of a fixed prime p if all color classes are singletons, or in other words, if we omit all the color constraints. In this case, the configuration space K of (2.13) becomes

$$K = [r_\ell]^{*(N+1)} = (\Delta_N)^{*r_\ell}_{\Delta(2)},$$

which is the r_ℓ-wise 2-fold deleted join of an N-simplex. It follows from the connectivity relation $\operatorname{conn}(A*B) \geq \operatorname{conn}(A)+\operatorname{conn}(B)+2$ for CW-complexes that K is $(N-1)$-connected. As symmetry group we take instead of $\mathbb{Z}_r$ a subgroup $G \cong (\mathbb{Z}_p)^{m_\ell}$ of S_{r_ℓ} that acts fixed-point freely on $[r_\ell]$. By the connectivity of K, $\operatorname{Index}_G^{\{*\}}(K) \subseteq H^{*\geq N+1}(BG)$, as we can directly deduce from the Leray–Serre spectral sequence of $K \hookrightarrow EG \times_G K \to BG$. We obtain Karasev's result.

Theorem 2.28 ([Kar07])**.** *The Tverberg–Vrećica Conjecture 2.1 holds in the special case $r_\ell = p^{a_\ell}$, where p is a prime such that $p(d-k)$ is even or $k=0$.*

Tightness of the Transversal Main Theorem 2.3

Observation 2.29. *For any $0 \leq k \leq d-1$, $0 \leq \ell^* \leq k$, $r_{\ell^*} \geq 2$, we can choose point sets $\mathcal{C}^\ell \subset \mathbb{R}^d$ of size $|\mathcal{C}^\ell| = (r_\ell - 1)(d-k+1)+1$ and make all the color classes singletons except for one single color class $C_0^{\ell^*}$ of size r_{ℓ^*} such that there are no colorful partitions of the $\mathcal{C}^\ell$s into r_ℓ pieces each that admit a common k-dimensional transversal.*

Proof. Let V^ℓ, $0 \leq \ell \leq k$, be pairwise parallel $(d-k)$-dimensional affine subspaces of $\mathbb{R}^d$ such that their projections to an orthogonal k-space are the $k+1$ vertices of a k-simplex.

On each V^ℓ we place a standard point configuration $\mathcal{C}^\ell$: Take a $(d-k)$-simplex Δ_ℓ in V^ℓ, let $\mathcal{C}^\ell$ have $r_\ell - 1$ points on each vertex of Δ_ℓ and put the last vertex of $\mathcal{C}^\ell$ into the center c^ℓ of Δ_ℓ.

The sets $\mathcal{C}^\ell$ admit only one Tverberg point, namely c^ℓ. Hence a potential common k-dimensional transversal P must intersect all c^ℓ. Since the V^ℓ have been chosen generically enough, P is uniquely determined and $P \cap V^\ell = \{c^\ell\}$.

Now we color the points of an arbitrary $\mathcal{C}^{\ell^*}$ at an arbitrary vertex of Δ_{ℓ^*} red, together with a further point at another vertex. Even if all other color classes in $\mathcal{C}^{\ell^*}$ are singletons there will be no colored Tverberg partition of $\mathcal{C}^{\ell^*}$. Together with $P \cap V^{\ell^*} = \{c^{\ell^*}\}$ this proves the observation. □

3 Colored Tverberg on manifolds

3.1 Introduction

In this section we present an extension of Theorem 1.8 that treats continuous maps $\Delta_N \to M$ from the N-simplex to an arbitrary d-dimensional manifold M in place of $\mathbb{R}^d$, [BMZ11b].

Suppose that the vertices of Δ_N are colored with color classes $C_1, \ldots, C_m$. Let $R = C_1 * \ldots * C_m$ denote the subcomplex of rainbow faces in Δ_N.

Theorem 3.1 (New colored Tverberg theorem for M, [BMZ11b]). *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$. Let R be the corresponding rainbow subcomplex of Δ_N.*

Then for every continuous map $f : R \to M$ to a d-dimensional manifold M, the simplex Δ_N has r disjoint rainbow faces whose images under f have a point in common.

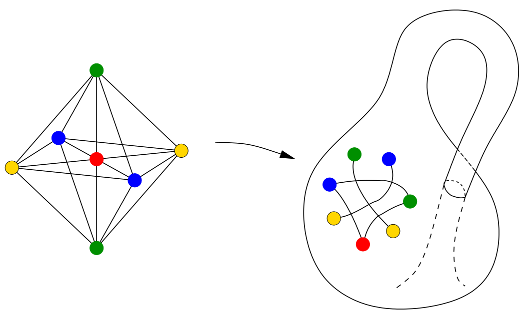


Figure 1.7: Example of Theorem 3.1 for $d = 2$, $r = 3$, $N+1 = 7$, and M is the Klein bottle.

Theorem 3.1 without color constraints (that is, when all color classes are of size 1, and thus all faces are rainbow faces) was previously obtained by Volovikov [Vol96], which also works for prime powers r. The prime power case for the colored version, Theorem 3.1, seems however out of reach at this point, since so it does in the case $M = \mathbb{R}^d$.

Finally we remark that it is not obvious that Theorem 3.1 holds when the domain of f is less than the whole N-simplex, even though any colored Tverberg partition will take only faces in $R \subset \Delta_N$ of dimension at most $N - r + 1$. Let us give an example to illustrate that. Let $d = r = 2$ and let M be the 2-dimensional sphere. Then $N = 3$ and we give the vertices of the tetrahedron Δ_N all different colors. Since the N-dimensional face of Δ_N is never part of a Tverberg partition, we might guess that the conclusion of Theorem 3.1 should hold true also for any map $f : \partial\Delta_3 \to M$. However this is wrong: any homeomorphism f gives a counter-example!

3.2 Proof

The configuration space/test map scheme

Suppose we are given a continuous map

$$f : R \longrightarrow M,$$

and a coloring of the vertex set $\operatorname{vert}(\Delta_N) = [N+1] = C_0 \uplus \cdots \uplus C_m$ such that the color classes C_i are of size $|C_i| \leq r - 1$.

This time we take the *deleted product* of R as our configuration space,

$$K^\times = R^r_{\Delta(2)} = (C_0 * \ldots * C_m)^r_{\Delta(2)},$$

instead of the deleted join $K = R^{*r}_{\Delta(2)}$ that we used in Section 1.4.

The index of $K = R^{*r}_{\Delta(2)}$ was computed in Corollary 2.8,

$$\operatorname{Index}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r).$$

We can use this together with a lemma of Karasev [Kar09a, Lemma 3.2], which was independently proved by Carsten Schultz in a different way (unpublished).

Lemma 3.2 (Karasev, Schultz)**.** *Let r be a prime and let R be a simplicial complex. If $r = 2$ then*

$$t \cdot \operatorname{Index}_G(R^r_{\Delta(2)}) \subseteq \operatorname{Index}_G(R^{*r}_{\Delta(2)}),$$

and if r is odd then

$$x^{(r-1)/2} \cdot \operatorname{Index}_G(R^r_{\Delta(2)}) \subseteq \operatorname{Index}_G(R^{*r}_{\Delta(2)}).$$

Hence the index of $K^\times$ is given by

$$\operatorname{Index}_{\mathbb{Z}_r}(K^\times) = H^{*\geq N-r+2}(B\mathbb{Z}_r). \tag{3.3}$$

Note that $N - r + 2 = d(r-1) + 1$. Now we only need to use a Borsuk–Ulam type theorem of Volovikov [Vol93, Theorem 1] to finish the proof of Theorem 3.1, as it is done by Volovikov in [Vol96] to prove the (uncolored) topological Tverberg problem for prime powers (which was first proved by Özaydin [Öza87] using equivariant obstruction theory).

Theorem 3.4 (Volovikov)**.** *Let r be a prime, let M be a compact d-dimensional topological manifold that is orientable if r is odd, and let $K^\times$ be a free paracompact $\mathbb{Z}_r$-space. Suppose that $h : K^\times \to M$ is a map that induces zero in reduced cohomology and that $\operatorname{Index}_{\mathbb{Z}_r}(K^\times) \subseteq H^{*\geq d(p-1)+1}\mathbb{Z}_r$. Then there exists an $x \in K^\times$ whose orbit $\mathbb{Z}_p \cdot x$ gets mapped to a point by h.*

We apply this theorem to the map $h : K^\times \to M$ that is given by $h(x_1, \ldots, x_r) := f(x_1)$. The assumption that $\widetilde{H}^*(M) \to \widetilde{H}^*(K^\times)$ is zero is indeed fulfilled: Since all color classes are of size less than r we have at least $d+2$ color classes. Hence $R = C_0 * \ldots * C_m$ is d-connected. Since h factors as $K^\times \xrightarrow{pr_1} R \xrightarrow{f} M$, h is contractible. If M is non-orientable then one can instead take its universal cover $\widetilde{M}$ and apply Theorem 3.4 to the lift $\widetilde{h} : K^\times \to \widetilde{M}$.

The conclusion of Theorem 3.4 yields r points on disjoint faces of R whose images under f are the same. □

Remark 3.5. Both, Lemma 3.2 and Theorem 3.4 have analogous versions also for prime powers $r = p^k$ and $\mathbb{Z}_p^k$-actions, see [Kar09a] and [Vol93].

Let us also remark that the index of $K^\times$ becomes larger with respect to inclusion than in (3.3) if just one color class C_i has more than $r-1$ elements. That is, in this case this proof of Theorem 3.1 does not work anymore. In fact, for any r and d there exist $N+1$ colored points in $\mathbb{R}^d$ such that one color class is of size r and all other color classes are singletons that admit no colored Tverberg partition. In our paper [BMZ11b] we gave another proof of Theorem 3.1 based on the deleted join scheme and the monotonicity of the index, however only in the case when the map f is a map from the full simplex $f : \Delta_N \to M$, or if the number of color classes is large enough. Hence this proof does not give the full result, but it is simpler than Volovikov's proof of Theorem 3.4.

3.3 Remarks

Theorem 3.1 strictly generalizes Theorem 1.8

One may ask whether Theorem 3.1 can be reduced to Theorem 1.8 by factorizing the given map $f : \Delta^N \to M$ over $\mathbb{R}^d$,

$$f : \Delta^N \xrightarrow{f'} \mathbb{R}^d \to M.$$

When ever this is possible Theorem 1.8 immediately implies Theorem 3.1. However this is not always possible.

Proposition 3.6. *Let f be the composed map $\Delta^3 \to S^3 \to S^2$ that first quotients out the boundary of Δ^3 and then sends S^3 to S^2 via the Hopf map. Then f does not factor over $\mathbb{R}^d$.*

The following proof is due to Elmar Vogt.

Proof. Suppose that f factors as $f : \Delta^N \xrightarrow{f'} \mathbb{R}^d \xrightarrow{g} M$. Let $h : S^3 \to S^2$ denote the Hopf map. Let $z := h^{-1}(n) \subset S^3$, where n is the north pole of S^2. We think of z being the closure of the z-axis in the stereographic projection of S^3 to $\mathbb{R}^3$ in the one-point compactification of $\mathbb{R}^3$. Let D be the halfspace $\{(x, y, z) \mid x > 0,\ y = 0\}$, which is a disc in S^3 whose boundary is z. Then all fibers $h^{-1}(x)$ other than z intersect D transversally. In particular, h maps D homeomorphically to $S^2 \backslash \{n\}$. Then f' also maps D (regarded as a disc in Δ^3) homeomorphically to a set $D' \subset \mathbb{R}^2$. Moreover, for every $x \neq n$, $f'(h^{-1}(x))$ is a singleton in D'. Further, D' is bounded, since Δ^3 is compact. Let $p \in z$ and let (p_i) be a sequence in D converging to p. Then the fibers $h^{-1}(h(p_i))$ contain sequences of points that come close to any other point of z. By continuity, $f'(z)$ must be a singleton in $\mathbb{R}^2$ as well. But $f'(z)$ must also be the boundary of D' which is bounded and homeomorphic to an open disc, which gives a contradiction. $\square$

Deleted joins versus deleted products

An interesting question is which test map scheme is more powerful, the one coming from the deleted-product construction or the one from the deleted-join construction?

In many applications we investigate the existence of a S_r-equivariant test map of deleted products

$$f^\times : X^r_{\Delta(\ell)} \longrightarrow_{S_r} Y^r_{\Delta(k)}, \tag{3.7}$$

where X is a simplicial complex and Y is a space. Here S_r stands for the group of permutations on r letters. The corresponding S_r-equivariant test map for deleted joins would be

$$f^* : X^{*r}_{\Delta(\ell)} \longrightarrow_{S_r} Y^{*r}_{\Delta(k)}. \tag{3.8}$$

In the case when $Y = \mathbb{R}^d$, for some d, the existence of the S_r-equivariant map $f^\times$ implies the existence of the S_r-equivariant map f^*. Indeed, if $f^\times : X^r_{\Delta(\ell)} \longrightarrow_{S_r} (\mathbb{R}^d)^r_{\Delta(k)}$ is given, then we can define

$$f^*(\lambda_1 x_1 + \ldots + \lambda_r x_r) := \sum_{i=1}^{r} \lambda_i y_i,$$

where

$$y_i := \left(\prod_{j=1}^{r} r\lambda_j\right) \cdot f_i^\times(x_1, \ldots, x_r) \in \mathbb{R}^d.$$

The constant factor r^r is included in the definition of y_i because $X^r_{\Delta(\ell)}$ can be seen as a subspace of $X^{*r}_{\Delta(\ell)}$ where all join coefficients are $\frac{1}{r}$. The cell $F_1 \times \ldots \times F_r$ of the deleted product complex can be identified with the subspace $\{\frac{1}{r} \cdot x_1 + \ldots + \frac{1}{r} \cdot x_r \mid x_i \in F_i\}$ of the simplex $F_1 * \ldots * F_r$ in the deleted join complex.

Therefore, in the case when $Y = \mathbb{R}^d$, the deleted-product scheme (3.7) is at least as strong as the deleted-join scheme (3.8).

Nevertheless, *proving* the non-existence of f^* might be easier than proving the non-existence of $f^\times$. For instance, if one only wants to argue with the high connectivity of the domain, then this is usually easier for f^*, see e.g. [Mat03, Sections 5.5–5.8].

Also the monotonicity of the Fadell–Husseini index sometimes puts a stronger condition on f^* than on $f^\times$. In particular, this affects Theorem 3.1. The range of $f^\times$ is $M^r_{\Delta(r)}$. If $M = \mathbb{R}^d$ then the corresponding index is

$$\mathrm{Index}_{\mathbb{Z}_r}((\mathbb{R}^d)^r_{\Delta(r)}) = H^{*\geq d(r-1)}(B\mathbb{Z}_r),$$

since $(\mathbb{R}^d)^r_{\Delta(r)}$ deformation retracts equivariantly to a fixed-point free sphere whose dimension is $d(r-1)-1$. Hence we can show the non-existence of $f^\times$ using the monotonicity of the index. However, for $M = S^d$ the index is smaller with respect to inclusion by the following proposition, so the monotonicity of the index alone is not enough to prove Theorem 3.1 in the deleted-product scheme.

Proposition 3.9. *If $d \geq 2$, then*

$$\mathrm{Index}_{\mathbb{Z}_r}((S^d)^r_{\Delta(r)}) \subseteq H^{*\geq d(r-1)+1}(B\mathbb{Z}_r).$$

Proof. We have to show that in the Leray–Serre spectral sequence associated to $EG \times_G (S^d)^r_{\Delta(r)} \to BG$, no non-zero differential hits the bottom row in filtration degree smaller or equal to $d(r-1)$. The E_2 entries are $H^*(BG, H^*((S^d)^r \backslash \Delta))$, where Δ is the thin diagonal in $(S^d)^r$. Now,

$$H^i((S^d)^r \backslash \Delta) \cong H_{dr-i}((S^d)^r, \Delta) \cong H^{dr-i}((S^d)^r, \Delta).$$

From the long exact sequence in cohomology of the pair $((S^d)^r, \Delta)$,

$$\ldots \to H^*((S^d)^r, \Delta) \to H^*((S^d)^r) \to H^*(\Delta) \to \ldots,$$

we see that $H^{dr}((S^d)^r, \Delta) = \mathbb{F}_r$, $H^d((S^d)^r, \Delta) \cong \mathbb{F}_r[\mathbb{Z}_r]/(1 + t + \ldots + t^r)\mathbb{F}_r$, and for $d < j < dr$ we have $H^j((S^d)^r, \Delta) \cong \mathbb{F}_r[\mathbb{Z}_r]^{\oplus \alpha_j}$, where $\alpha_j \geq 0$ depends on j. Therefore the first non-zero row (up to the 0-column entries) in the spectral sequence above the bottom row is the $d(r-1)$-row. Thus the first element in the bottom row that is hit by a differential has degree at least $d(r-1)+1$. □

On the other hand, the monotonicity of the Fadell–Husseini index proves the non-existence of f^* for $M = S^d$, since $(S^d)^{*r}_{\Delta(r)}$ deformation retracts equivariantly to an $(N-1)$-dimensional fixed-point free sphere, whose index is equal to $H^{*\geq N}(B\mathbb{Z}_r)$.

So, in this particular instance of proving the non-existence of the test-map for Theorem 3.1 using the Fadell–Husseini index, the deleted-join scheme is stronger than the deleted-product scheme.

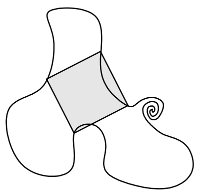
Chapter 2

On the square peg problem and some relatives

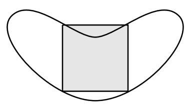
1 Introduction

The square peg problem was first posed by Otto Toeplitz in 1911:

Conjecture 1.1 (Square peg problem, Toeplitz [Toe11])**.** *Every continuous embedding* $\gamma : S^1 \to \mathbb{R}^2$ *contains four points that are the vertices of a square.*



The name square peg problem might be a bit misleading. We do not require the square to lie inside the curve, otherwise there are easy counter-examples:



Toeplitz' problem has been solved affirmatively for various restricted classes of curves such as convex curves and curves that are "smooth enough", by various authors; the strongest version so far was due to W. Stromquist [Str89, Thm. 3] who established the square peg problem for "locally monotone" curves. All known proofs are based on the fact that generically the number of squares on a curve is odd, which can be measured in various topological ways. See Section 2.1 for a short survey. For general embedded plane curves, the problem is still open.

We start our discussion with a proof idea due to Shnirel'man [Shn44] (in a modern version, in terms of a bordism argument), which establishes the square peg problem for the class of smooth curves.

In Section 2.4 we prove it for a new class of curves that is defined by a weaker smoothness criterion than Stromquist's. Hence it contains all previous known curves for which the square peg problem is proved, see Theorem 2.5. The first drawing above is an example that lies in this new class, but not in Stromquist's. The proof extends to curves in arbitrary metric spaces.

In Section 2.5 we present the first known *open* set of curves in the compact-open topology (equivalently, in $((\mathbb{R}^2)^{S^1}, ||.||_\infty)$) for which the square peg problem holds, see Theorem 2.9. It does neither require the curve to be smooth nor injective, and it finds a square whose size is bounded from below; see Section 2 for the rather simple proof and variations of the statement.

In Section 2.6 we deal with *immersed* planar curves and the parity of their inscribed squares. Cantarella [Can08] conjectured that this parity is an isotopy invariant and he stated a precise formula based on examples. We disprove Cantarella's conjecture and state in Theorem 2.13 how the parity can be computed from the angles at the intersection points. Theorem 2.14 gives a similar formula for the parity of inscribed rectangles of a fixed aspect ratio.

Section 3 deals with the existence of rectangles with a given aspect ratio on smooth curves.

Conjecture 1.2 (Rectangular peg problem)**.** *Every C^∞ embedding $\gamma : S^1 \to \mathbb{R}^2$ contains four points that are the vertices of a rectangle with a prescribed aspect ratio $r > 0$.*

We state this conjecture for smooth curves only, since already this seems to be a hard problem. It is equivalent to stating this conjecture for piecewise linear curves. So far it is only known to hold in the case $r = 1$, that is, for inscribed squares. The proof in Griffiths' paper [Gri90] unfortunately contains a fatal error in the calculation of intersection numbers; see [Mat08] for details. The difficulty comes from the fact that, counted with orientations, every smooth curve contains generically zero rectangles of a prescribed aspect ratio. E.g. an ellipse contains two rectangles with opposite orientations. This makes the configuration space-test map method fail. That is, more geometric arguments are needed to attack the problem.

In Section 3.1 we will present some intuition why those inscribed rectangles should exist for all r. Then in Section 3.2 we prove the a first non-square special case of the rectangular peg problem, namely for $r = \sqrt{3}$ and 60°-angular convex curves, see Theorem 3.4.

The final Section 4 treats a higher-dimensional analog. We ask for d-dimensional regular crosspolytopes on smoothly embedded $(d-1)$-spheres in $\mathbb{R}^d$. The square peg problem for smooth curves is the case $d = 2$. This crosspolytopal peg problem is proved for prime powers $d \geq 3$ [Mak03], [Kar09b], and is open otherwise. As one possible generalization one might ask whether any smoothly embedded $(d-1)$ in a Riemannian manifold contains a regular crosspolytope. We use Koschorke's obstruction theory [Kos81] to derive that for $d = 3$, a natural topological approach for a proof fails: The strong test map in question exists.

2 Squares on curves

2.1 Some short historic remarks

The square peg problem first appears in the literature in the conference report [Toe11] (1911). It states the problem and that Toeplitz was only able to find a solution for convex curves. Afterwards Emch [Emc13, Emc15] presents two proofs of the square peg problem for "smooth enough" convex curves. In [Emc15] he states that he had not been aware of Toeplitz' and his students' work and that the problem was suggested to him by Kempner.

In 1929 Schnirel'man proved it for class of curves that is slightly larger than C^2. An extended version [Shn44], which corrects also some minor errors, was published posthumously in 1944. Guggenheimer [Gug65] states that the extended version still contains errors which he claims to correct. However in my point of view Schnirel'man's proof is up to minor errors correct. His main idea is a bordism argument. On the other hand, Guggenheimer's main lemma admits counter-examples [Mat08, Section III.9]; he was not aware that squares can vanish pairwise when one deforms the curve.

Other proofs are due to Jerrard [Jer61] for analytic curves, Stromquist [Str89] for locally monotone curves, Vrećica & Živaljević [VŽ11b] for Stromquist's curves, Pak [Pak08, 2. proof] for piecewise linear curves (his first proof unfortunately contains an error), and Cantarella, Denne & McCleary [CDM] for curves with bounded total curvature. Stronquist's result was the so far strongest one: A curve $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ is **locally monotone** if every point of $x \in S^1$ admits a neighborhood U and a linear functional $\ell : \mathbb{R}^2 \to \mathbb{R}$ such that $\ell \circ \gamma|_U$ is monotone. Another interesting proof for convex curves is due to Fenn [Fen70]. He obtains it as a corollary of his table theorem. For more historical background and related problems, see [KW96, Problem 11], [Pak08], [Mat08].

2.2 Notations and the parameter space of polygons on curves

For an element x of the unit circle $S^1 \cong \mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}$ we define $x + t \in S^1$ as the result of a counter-clockwise rotation of x by the angle $2\pi t$ around 0. Let $\Delta_n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum t_i = 1\}$ be the standard n-simplex.

The natural **parameter space** of polygons is

$$P_n := S^1 \times \Delta_{n-1}.$$

It parameterizes polygons on S^1 or on some given curve $S^1 \to \mathbb{R}^\infty$ by their vertices in the following way

$$\varphi : \; P_n \to (S^1)^n : \; (x; t_0, \ldots, t_{n-1}) \mapsto (x, x + t_0, x + t_0 + t_1, \ldots, x + \sum_{i=0}^{n-2} t_i).$$

The so parameterized polygons are the ones that are lying counter-clockwise on S^1. The map φ is not injective, as all $(x; 0, \ldots, 0, 1, 0, \ldots, 0)$ are mapped to the same point $(x, \ldots, x)$; but it is injective on $P_n \backslash \varphi^{-1}(\Delta_{(S^1)^n})$, where $\varphi^{-1}(\Delta_{(S^1)^n}) = (S^1 \times \mathrm{vert}(\Delta_{n-1}))$, and on this set φ bijects to $(S^1)^n \backslash \Delta_{(S^1)^n}$. The map φ identifies the interior P_n° with the set of n-tuples of pairwise distinct points in counter-clockwise order on S^1. We define the boundary as $\partial P_n^\circ := P_n \backslash P_n^\circ$.

We let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} = \langle \varepsilon \rangle$ act on P_n by

$$\varepsilon \cdot (x; t_0, \ldots, t_{n-1}) = (x + t_0; t_1, \ldots, t_{n-1}, t_0).$$

This corresponds to a cyclic relabeling of the vertices of the parameterized polygon.

A substitution

The following coordinate transformation makes the $\mathbb{Z}_n$-action on P_n look nicer. We substitute $(x; t_0, \dots, t_{n-1}) \in P_n$ by $(x^*; t_0, \dots, t_{n-1})$, where $x^* := x + \sum_{k=1}^{n-1} \frac{n-k}{n} \cdot t_{k-1} \in S^1$. In terms of the new coordinates,

$$\varepsilon \cdot (x^*; t_0, \dots, t_{n-1}) = (x^* + \frac{1}{n}; t_1, \dots, t_{n-1}, t_0).$$

Further notations

When we talk about an *arc* on S^1 from a point x to y, we always mean the arc that goes counter-clockwise. For $x, y \in S^1$, we denote by $y - x$ the length of the arc from x to y, normalized with the factor $\frac{1}{2\pi}$. For an n-tuple $(x_1, \dots, x_n) \in \varphi(P_n) \subset (S^1)^n$ we write

$$[x_1, \dots, x_n] := (x_1; x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1}, 1 - \sum_{k=2}^{n}(x_k - x_{k-1})) \in P_n.$$

The function $[\dots] : \varphi(P_n) \to (S^1)^n$ is right-inverse to φ, but not continuous.

Smooth means C^∞ for us. An *ε-close square* is a quadrilateral whose ratios between the edges and diagonals are up to an ε-error the ones of a square. The precise definition will not matter. We will use "ε-closeness" with other polygons analogously.

2.3 Shnirel'man's proof for the smooth square peg problem

We start with L. G. Shnirel'man's proof [Shn44], since it is in my point of view the most beautiful one. The following presentation uses transversality and a bordism argument; in Shnirel'man's days, these notions had not been formalized and baptized yet, but his argument works like this.

Proof. Suppose that γ is smooth. P_4 parameterizes quadrilaterals on γ. Let $f : P_4 \to \mathbb{R}^6$ be the function that measures the four edges and the two diagonals of the quadrilaterals,

$$\begin{aligned} f : P_4 &\longrightarrow \mathbb{R}^4 \times \mathbb{R}^2 \\ [x_1, x_2, x_3, x_4] &\longmapsto (||\gamma(x_1) - \gamma(x_2)||, ||\gamma(x_2) - \gamma(x_3)||, ||\gamma(x_3) - \gamma(x_4)||, \\ &\qquad ||\gamma(x_4) - \gamma(x_1)||, ||\gamma(x_1) - \gamma(x_3)||, ||\gamma(x_2) - \gamma(x_4)||) \end{aligned} \tag{2.1}$$

We can compose f with the quotient map $\mathbb{R}^6 \to \mathbb{R}^6/_{\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2}} \cong \mathbb{R}^4$ and get $\overline{f} : P_4 \to \mathbb{R}^4$. $\overline{f}$ measures squares, since $Q := \overline{f}^{-1}(0) \setminus \phi^{-1}(\Delta_{(S^1)^4}) = \overline{f}^{-1}(0) \cap P_4^\circ$ is the set of all squares that lie counter-clockwise on γ. $\overline{f}$ is $\mathbb{Z}_4$-equivariant with respect to the natural $\mathbb{Z}_4$-actions. Here we removed the $\phi^{-1}(\Delta_{(S^1)^4})$, i.e. the set of squares that are degenerate to a point. Since γ is smooth there is a small neighborhood of ∂P_4 in P_4 that intersects $\overline{f}^{-1}(0)$ only in $\varphi^{-1}(\Delta_{(S_1)^4}) \subset \partial P_4$. We can deform $\overline{f}$ relative to a possibly smaller neighborhood of ∂P_4° equivariantly by a small ε-homotopy to make 0 a regular value of $\overline{f}|_{P_4^\circ}$. Note that Q lies in P_4°, which is free. So Q becomes a zero-dimensional $\mathbb{Z}_4$-manifold of ε-close squares. If we deform the curve smoothly to another curve (e.g. the ellipse) then Q changes by a $\mathbb{Z}_4$-bordism. We can do such a deformation between any two embedded curves for simplicity in

$\mathbb{R}^4$; since $\overline{f}$ depends only on distances. This bordism stays away from the boundary of P_4°, if the homotopy is chosen smoothly, since then no curve inscribes ε-close squares which have arbitrarily small edges (the angles get too close to π). Hence Q represents a unique class $[Q]$ in the zero-dimensional unoriented bordism group $\mathcal{N}_0(P_4^\circ/_{\mathbb{Z}_4}) \cong H_0(P_4^\circ/_{\mathbb{Z}_4}; \mathbb{Z}_2) \cong \mathbb{Z}_2$. If γ is an ellipse then 0 is a regular value of $\overline{f}$ and Q consists of one $\mathbb{Z}_4$-orbit. Hence $[Q]$ is the generator of $\mathbb{Z}_2$, so Q is non-empty for any smooth curve γ. Letting ε go to zero and taking a convergent subsequence of ε-close squares finishes the proof. □

If γ is only continuous one might try to approximate it with smooth curves and then take a convergent subsequence of the squares that we get on them. The problem is to guarantee that this subsequence does not converge to a square that degenerates to a point. Natural candidates for which this works are continuous curves with bounded total curvature without cusps, see Cantarella, Denne & McCleary [CDM]. So far, nobody managed to do this for all continuous curves.

Shnirel'man's proof can be refined to get a slightly stronger result.

Corollary 2.2 (of the proof)**.** *We may assume that γ goes counter-clockwise around its interior. Then one can find and order four vertices of a square on γ, such that they lie counter-clockwise on γ and also label the square counter-clockwise.*

Proof. This can be achieved by restricting Q in the above proof to the set of squares $[x_1, x_2, x_3, x_4]$ in P_4 that are labeled by $(\gamma(x_1), \dots, \gamma(x_4))$ in counter-clockwise order. Along a path in the bordism this cannot change (here we take a bordism that is induced by a deformation of the curve in the plane). If γ is an ellipse then it is clear that the restricted Q is equal to Q, so it represents the generator in $\mathcal{N}_0(P_4^\circ/_{\mathbb{Z}_4})$. □

Remark 2.3. The square peg problem is still open for general continuous curves, for no valid approximation argument is known. If we approximate a given continuous curve by smooth we get a sequence of squares on the approximating curves, which contains a subsequence that converges to a square on the given curve. However the limit square might be degenerate to a point.

Hence one would like to find "big" squares to make the limit argument work (this idea is old). It seems reasonable to conjecture that any smooth curve γ in the plane inscribes a square with side length at least $\sqrt{2}a$ if the interior of γ contains a ball of radius a (which would be a tight bound). This conjecture implies the same statement for all continuous curves using an outer approximation and hence the square peg problem. A very similar conjecture was independently posed by Tverberg [Tve11].

2.4 A weaker smoothness criterion

Let $\gamma : S^1 \to \mathbb{R}^2$ be a simple closed curve (that is, injective and continuous). We need some preparation. Let $f : P_4 \to \mathbb{R}^6$ be the corresponding test map that measure the four edges and two diagonals, which was defined in equation (2.1). For $y_1, y_4 \in S^1$, $y_1 \neq y_4$, let

$$P_4(y_1, y_4) := \{[y_1, x_2, x_3, y_4] \in P_4^\circ\}$$

the set of all quadrilaterals counter-clockwise on S^1 whose first and last vertices are given. For a path $y : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2}$, $y(t) = (y_1(t), y_4(t))$, we define

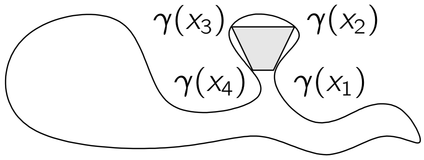
$$P_4(y) := \bigcup_{t \in S^1} P_4(y(t)) = \{[y_1(t), x_2, x_3, y_4(t)] \in P_4^\circ \mid t \in S^1\}.$$

Definition 2.4. We call a quadrilateral on γ given by $[x_1, x_2, x_3, x_4]$ **special** if

$$f([x_1, x_2, x_3, x_4]) = (a, a, a, b, e, e) \text{ with } a \geq b, \text{ for some reals } a, b, e.$$

Let S denote the set of all special quadrilaterals in P_4. The **size** of a special quadrilateral $[x_1, x_2, x_3, x_4]$ is the normalized arc length $x_4 - x_1$, measured in the domain of γ.

The following figure shows a special quadrilateral of small size on γ.



Theorem 2.5. *Suppose there is an $\varepsilon \in (0, 1)$, such that γ inscribes no (or generically an even number of) special quadrilateral of size ε. Then γ circumscribes a square.*

This theorem is probably most useful for very small $\varepsilon > 0$.

Proof. Use the following lemma with $y_1 := \mathrm{id}_{S^1}$ and $y_4 := \mathrm{id}_{S^1} + \varepsilon$. □

Lemma 2.6. *Suppose there is a path $y : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2} \cong P_2^\circ$, $y(t) = (y_1(t), y_4(t))$, that represents a generator in $\pi_1((S^1)^2 \backslash \Delta_{(S^1)^2}) \cong \pi_1(S^1) \cong \mathbb{Z}$. If γ does not inscribe a square then the mod-2 intersection number of $P_4(y)$ and S is 1.*

The mod-2 intersection number will be described in the proof. The proof is based on equivariant obstruction theory, which was first used in connection to the square peg problem by Vrećica and Živaljević [VŽ11b]. The second part of our proof will be very close to what they did. One can of course use different topological methods, but their way is quite straight-forward and beautiful. Another point of view will be sketched in Remarks 2.8

Proof. $P_4(y)$ can be parameterized by $g : S^1 \times \Delta_2 \to P_4(y)$, where S^1 parameterizes y and Δ_2 the three arc lengths between the points $y_1(t)$, x_2, x_3 and $y_4(t)$. The map g is injective if and only if y is.

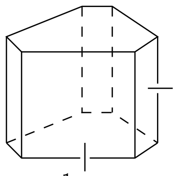
The mod-2 intersection number in the theorem is defined as the mod-2 intersection number of $f(g(S^1 \times \Delta_2))$ and $V := \{(a, a, a, b, e, e) \in \mathbb{R}^6 \mid a \geq b\}$ in $\mathbb{R}^6$. This is only well-defined if $f(g(S^1 \times \partial\Delta_2)) \cap V = \emptyset$ and $\mathrm{im}(f \circ g) \cap \partial V = \emptyset$. The former is trivially fulfilled, the latter if and only if no quadrilateral on γ given by $P_4(y)$ is a square (this is interesting if one deforms y; compare with Remark 1.) in 2.8). The map $f \circ g$ can now be deformed by a homotopy rel $S^1 \times \partial\Delta_2$, such that at no time it intersects the boundary of V, and such that it becomes transversal to V. The intersection number then counts the preimages of V under $f \circ g$ modulo 2.

Suppose that γ does not inscribe a square, but the described mod-2 intersection number is zero. We want to derive a contradiction.

For some $\varepsilon \in (0, \frac{1}{2})$ (later we might choose $\varepsilon = \frac{1}{3}$), let $T = T^\varepsilon \subset \Delta_3$ be a polytope obtained from a tetrahedron by cutting an open vertex figure of size ε from the vertices (we delete all points $(t_0, \ldots, t_3) \in \Delta_3$ that have an entry $> 1-\varepsilon$). The four vertices of Δ_3 are given by the standard basis vectors $e_0, \ldots, e_3$ of $\mathbb{R}^3$. The four corresponding triangular facets of T are denoted by $T_0, \ldots, T_3$, and their opposite hexagonal facets by $H_0, \ldots, H_3$.

$S^1 \times T_3 \subset P_4$ parameterizes the 4-tuples $(x_1, \ldots, x_4) \in (S^1)^4$ with $x_4 - x_1 = \varepsilon$.

Here is a sketch of T in one dimension smaller where we draw $S^1 \times T \subset P_4$ as a cylinder whose the bottom and top face are identified:

$S^1 \times T$: 

We will construct for some small $\delta > 0$ a $\mathbb{Z}_4$-equivariant map

$$h : S^1 \times T^\varepsilon \longrightarrow_{\mathbb{Z}_4} S^1 \times T^\delta$$

that satisfies the following conditions:

1. h maps $S^1 \times H_i$ to $S^1 \times H_i$, $0 \le i \le 3$,
2. h is prescribed on $S^1 \times T_3^\varepsilon \subset P_4$ as
$$h(t; t_0, t_1, t_2, t_3 = 1-\varepsilon) := (y_1(t); \lambda_t t_0, \lambda_t t_1, \lambda_t t_2, y_1(t) - y_4(t)),$$
where $\lambda_t > 0$ is chosen uniquely such that the last four entries sum up to one, that is, we want $h(t; \_, \_, \_, 1-\varepsilon) \in P_4(y_1(t), y_2(t))$.

The second condition prescribes h on all $S^1 \times T_i$, $i = 0, \ldots, 3$, since h is $\mathbb{Z}_4$-equivariant.

Now we construct h. If $y = (y_1, y_4)$ is given by $(\mathrm{id}_{S^1}, \mathrm{id}_{S^1} + \varepsilon)$, then we can choose $\delta = \varepsilon$ and $h = \mathrm{id}_{S^1 \times T^\varepsilon}$. Otherwise there is a homotopy $Y_s : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2}$, $s \in [0,1]$, from y to the previous one. For each time $s \in [0,1]$ we can now ask how to find an h_s as above for Y_s. If we only require condition (2) then this is a homotopy extension problem. Since $(S^1 \times T^\varepsilon, S^1 \times (T_0 \cup \ldots \cup T_3))$ is a pair of free $\mathbb{Z}_4$-CW-complexes, we can solve this. The standard proof for this, see e.g. [Bre93, Corollary VII.1.4], gives a solution that automatically satisfies condition (1) at each time, so especially for y. Therefore h exists.

Hence we get a test map

$$t := pr \circ f \circ h : \ S^1 \times T \xrightarrow{f \circ h}_{\mathbb{Z}_4} \mathbb{R}^6 \backslash (\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2}) \underset{\simeq\, \mathbb{Z}_4}{\xrightarrow{pr}} \mathbb{R}^4 \backslash \{0\}$$

which is avoiding $0 \in \mathbb{R}^4$ since we assumed that γ inscribes no square.

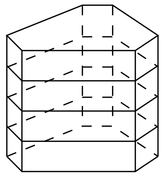
The range $\mathbb{R}^4 \backslash \{0\}$ of t is a product of the standard $\mathbb{Z}_4$-representation $W_4 := \mathbb{R}^4/_{\Delta_{\mathbb{R}^4}}$ and $U := \mathbb{R}^2/_{\Delta_{\mathbb{R}^2}}$ ($\varepsilon \cdot u = -u$, $u \in U$), with 0 deleted. The corresponding components of

t are t_W and t_U. The images $f_0, \ldots, f_3 \in W_4$ of the four standard basis vectors $e_0, \ldots, e_3$ of $\mathbb{R}^4$ span a tetrahedron which defines a fan with apex in 0 and with four facets, which we label by $F_0, \ldots, F_3$, such that $-f_i \in F_i$. $V \subset \mathbb{R}^6$ projects under pr in $\mathbb{R}^4 = W_4 \times U$ to $V' := \mathbb{R}_{\leq 0} \cdot (-f_3) \times \{0\}$.

We have enough information to disprove the existence of t using an obstruction argument. Assume that only the restriction of t to $\partial(S^1 \times T) = S^1 \times \partial T$ is given, we look whether we can extend it.

We are allowed to deform t by an arbitrary $\mathbb{Z}_4$-homotopy. First of all we make t transversal to V' on $S^1 \times T_3$ relative to its boundary (and extend this deformation $\mathbb{Z}_4$-equivariantly). Let $t^{-1}(V') \cap (S^1 \times T_3) = \{p_1, \ldots, p_{2k}\}$.

From now on we write $S^1 \times T \subset P_n$ in the coordinates that were introduces in Section 2.2. We see that it has a simple $\mathbb{Z}_4$-CW-complex structure with only one four-dimensional $\mathbb{Z}_4$-cell orbit:



One three-cell e shall be $* \times T$, $* \in S^1$. We may assume that $t(\partial(e) \cap T_3) \cap V' = \emptyset$ and analogously for the other T_i, since there are only finitely many points $* \in S^1$ which are forbidden in this way (namely the S^1-coordinates of the p_i and their $\mathbb{Z}_4$-translates).

Note that $t_W(S^1 \times H_i) \subset F_i \backslash \{0\}$. This is because on such points the t_i-coordinate is zero, hence the corresponding edge of the parameterized quadrilateral is zero and thus minimal among all edges. Therefore we can $\mathbb{Z}_4$-deform t_U on a sufficiently small neighborhood of $S^1 \times (H_0 \cup \ldots \cup H_3)$ such that t_U becomes zero on $S^1 \times (H_0 \cup \ldots \cup H_3)$ and such that during no time of this deformation change the new intersections of $t(S^1 \times T_3)$ and V'.

By the *degree* of a map $S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ we mean the degree of the normalized map to S^{n-1}, or the scaling factor of the induced map on homology $H_{n-1}(\_)$.

Since $t(\partial(e) \cap T_3) \cap V = \emptyset$, we can also deform t on a small neighborhood of $\partial(e) \cap T_3$ such that $t_W|_{\partial(e) \cap T_3}$ lies in $F_0 \cup F_1 \cup F_2$ and such that $t_U|_{\partial(e) \cap T_3}$ is zero, without changing the intersections of $t(S^1 \times T_3)$ and V. Suppose we have extended t on e such that t_U is positive on the interior of e. Then t_U is negative on the interior of $\varepsilon \cdot e$ ($\mathbb{Z}_4 = \langle \varepsilon \rangle$). Let E be the 4-cell of $S^1 \times T$ that has e and $\varepsilon \cdot e$ as boundary faces. The degree of t_W on ∂e is one.

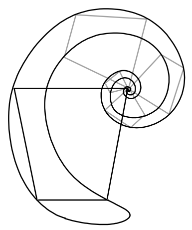
Recall $t^{-1}(V) \cap (S^1 \times T_3) = \{p_1, \ldots, p_{2k}\}$. If $2k = 0$, then one could also deform t on $\partial E \cap \partial(S^1 \times T)$ as we did with t on ∂e. In this case, $t|_{\partial E}$ is homotopic to the suspension of $t_W|_{\partial e}$, hence it was of degree 1. However for every $p_i \in \partial E$ the degree changes by one. This also happens at the other facets $\partial E \cap (S^1 \times T_i)$ of ∂E with the $\mathbb{Z}_4$-translates of $\{p_1, \ldots, p_{2k}\}$. In total there are $2k$ such points, hence the degree of $t|_{\partial E}$ is odd. If $t|_e$ was chosen differently, the degree of $t|_E$ would change twice $\pm$ the same number, once for e and once for $\varepsilon \cdot e$. Hence one cannot extend t to E, contradiction. □

Corollary 2.7. *Suppose there is a path* $y : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2} = P_2^\circ$, $y(t) = (y_1(t), y_4(t))$, *that represents a generator in $\pi_1((S^1)^2 \backslash \Delta_{(S^1)^2}) \cong \pi_1(S^1) \cong \mathbb{Z}$. If $P_4(y) \cap S = \emptyset$, then γ circumscribes a square.*

Proof. The mod-2 intersection number of Lemma 2.6 is here trivially zero. □

Remarks 2.8. 1.) An alternative view point to the proof of Lemma 2.6 is to look at S as a 1-dimensional manifold, after one made f transversal to V by a small ε-homotopy, at first on $P_4(y)$ and then on P_4. Here a technical trick is to choose ε not as a constant but as a function on P_4° that becomes arbitrarily small at the boundary, such that all technicalities work out. What Theorem 2.6 measures is the following.

$P_4(y)$ can be seen as a "membrane", which separates P_4 into two components if y is injective. If γ circumscribes no square then there is an *odd* number of paths in S that pass through $P_4(y)$ and approach the boundary at $S^1 \times e_3$, e_3 being the one vertex of Δ_3. These paths might look very chaotic close to the boundary. On the other side of the membrane $P_4(y)$, this odd number of paths cannot all end in each other. One of them has to end somewhere else. It might end suddenly in P_4°, which means that it found a square, or it might end somewhere else at ∂P_4°. My hope was that the latter is not possible, but it is:



The drawn path of special quadrilaterals starts in the middle of the spiral at $S^1 \times e_3$ with a quadrilateral that is degenerate to a point, and it stops when x_1 and x_4 moved together again, $x_4 - x_1 = 1$.

2.) Corollary 2.7 is sometimes good for proving the existence of a square, if the curve is piecewise C^1 but has cusps (points in which the tangent vector changes the direction). This however does not work in a large generality as the previous example shows.

3.) The whole Section 2.4 deals with the curve *intrinsically*, since the only datum of γ we used is the distances between points on γ. If we define a square in a metric space (X, d) to be a 4-tuple $(x_0, \dots, x_3) \in X^4$ such that $d(x_0, x_1) = d(x_1, x_2) = d(x_2, x_3) = d(x_3, x_0)$ and $d(x_0, x_2) = d(x_1, x_3)$, then the whole section also works for curves $\gamma : S^1 \to X$. More generally, X does not need to fulfill the triangle inequality. In other words, we do not need an embedded curve but a distance defining function $d : S^1 \times S^1 \to \mathbb{R}$ that is continuous, symmetric, non-negative, and zero exactly on the diagonal.

4.) The condition on curves γ of having no inscribed special trapezoid of size ϵ is an open condition in the following sense. Any embedding $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ without an inscribed special trapezoid of size ε has a neighborhood in $(\mathbb{R}^2)^{S^1}$ with respect to the compact-open topology with the same property. This shows that Theorem 2.5 is strictly stronger than Stromquist's criterion [Str89].

2.5 Squares on curves in an annulus

In this section we prove the first results on the square peg problem for an *open* set of continuous curves that inscribe a square whose *size is bounded from below*.

Theorem 2.9. *Let A denote the annulus $\{x \in \mathbb{R}^2 \mid 1 \leq ||x|| \leq 1+\sqrt{2}\}$. Suppose that $\gamma : S^1 \to A$ is a continuous closed curve in A that represents a generator of $\pi_1(A) = \mathbb{Z}$. Then γ inscribes a square of side length at least $\sqrt{2}$.*

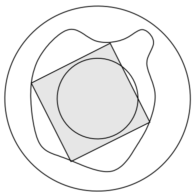


Figure 2.1: Example for Theorem 2.9.

Figure 2.1 shows an example. This theorem does *not* contain all previous known classes of curves for which the square peg problem is proved. It might be an Ansatz to prove the full square peg problem.

Here are two more versions, whose proofs are very similar; Figures 2.10 and 2.11 show examples.

Theorem 2.10. *Let S denote the set $\{x \in \mathbb{R}^2 \mid 1 \leq ||x||_\infty \leq 3\} = [0,3]^2 \backslash (1,2)^2$. Suppose that $\gamma : S^1 \to S$ is a continuous closed curve in S that represents a generator of $\pi_1(S) = \mathbb{Z}$. Then γ inscribes a square of side length at least $\sqrt{2}$.*

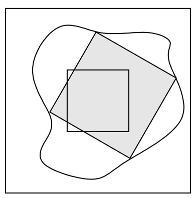


Figure 2.2: Example for Theorem 2.10.

Theorem 2.11. *Let Δ be an equilateral triangle in $\mathbb{R}^2$ whose center point is the origin. Let T be the closure of $((1+\sqrt{3}) \cdot \Delta) \backslash \Delta$. Suppose that $\gamma : S^1 \to T$ is a continuous closed curve in T that represents a generator of $\pi_1(T) = \mathbb{Z}$. Then γ inscribes a square of side length at least $2\sqrt{3} - 3$.*

It seems to be desirable to extend this method for much more general shapes in order to possibly prove the square peg problem for all curves. The proofs of Theorems 2.9, 2.10, and 2.11 follow from the following lemma.

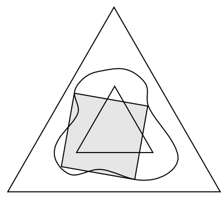


Figure 2.3: Example for Theorem 2.11.

Lemma 2.12. *Let A be a subset of $\mathbb{R}^2$. Let S_A be the set of 4-tuples $(P_1, \ldots, P_4) \in A^4$ that form the vertices of a possibly degenerate square in counter-clockwise order. Let C be a connected component of an ε-neighborhood of S_A that does not contain degenerate squares, that is, points of the form (P, P, P, P). Let $\widetilde{\gamma} : S^1 \to A$ be a generic curve that contains an odd number of squares in C. Then every continuous curve $\gamma : S^1 \to A$ that is homotopic to $\widetilde{\gamma}$ in A contains a square in C as well.*

Here, by a generic curve $\widetilde{\gamma}$ we mean a curve such that the corresponding test map that measures squares in C hits the test-space smoothly and transversally.

The proof of Lemma 2.12 is a simple bordism argument.

Proof of Theorem 2.9. We may assume that γ is actually a curve in the interior of A. The other cases follow by a limit argument, for which we use that on each approximating curve we can find a square of size at least $\sqrt{2}$. Some subsequence of this sequence of squares will then converge to a non-degenerate square of the given curve.

By compactness γ is a curve in $A' := \overline{U_\varepsilon(\overline{A})}$ for some $\varepsilon > 0$. Now we can apply Lemma 2.12, where we choose $\widetilde{\gamma}$ to being an ellipse in A'. $\square$

The proofs of Theorems 2.10, and 2.11 are analogous.

2.6 Squares and rectangles on immersed curves

Toeplitz' conjecture concerns inscribed squares on simple closed curves in the plane. There are plenty ways to generalize this problem. One possible way is to omit the requirement that γ has to be injective. Then there are several kinds of degenerate squares, which we have to deal with in that case. How should one define an inscribed square if the image of γ is a segment or a tree? To keep things simple, we will only consider smooth curves and transversal intersections, and we will not count degenerate squares.

In this section we will prove a simple mod-2 formula for the number of squares and rectangles that are inscribed in an immersed circle (or union of circles).

Squares on immersed curves. Let γ be a "generic" immersion of a finite union of circles in the plane.

There is a chequerboard coloring of the complement of γ, see Figure 2.4. That is, we color each component of $\mathbb{R}^2 \backslash \gamma$ black or white such that adjacent components get different colors. We may assume that the unbounded component is white. Let $b(\gamma)$ be the number of black components. We call a self-intersection of γ a crossing. We say that a crossing is

fat if the black angles at this crossing are larger than 90°. The fat crossings in Figure 2.4 are marked by a black dot. Let $f(\gamma)$ be the number of fat crossings.

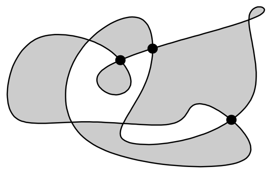


Figure 2.4: Chequerboard coloring associated to γ. Dots mark the fat crossings.

Theorem 2.13. *Suppose that γ is a generic immersion of finitely many circles in the plane. Then the number of non-degenerate squares inscribed in γ is congruent modulo 2 to $b(\gamma) + f(\gamma)$.*

Proof. By genericity of the curve, no inscribed square will have a vertex at a crossing. We smoothen the crossings of γ such that all white components become one big component. The number of inscribed squares increases by $f(\gamma)$ under this operation, see Figure 2.5. The new curve consists of $b(\gamma)$ separated simple closed curves. We can deform them by an ambient isotopy such that they become $b(\gamma)$ small ellipses and such that there is no inscribed square that touches more than one component. Therefore the resulting union of ellipses inscribes exactly $b(\gamma)$ squares. Using a bordism argument, the parity of the number of inscribes squares did not change during the isotopy. Since every ellipse inscribes exactly one square, this finishes the proof. ☐

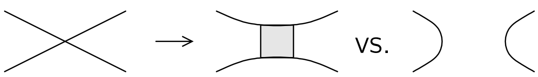


Figure 2.5: When we smoothen a crossing then a new square appears if and only if we opened the smaller angle.

Rectangles on immersed curves. The analogous theorem for rectangles of prescribed aspect ratio $0 < r < 1$ that are inscribed in immersed circles is slightly different. Let γ be again a generic immersion of a finite union of circles in the plane, and consider again the chequerboard coloring from above. Let $0 < \alpha(r) < \pi/2$ be the angle at the intersection of the two diagonals of a rectangle with aspect ratio r. We call a self-intersection of γ α-**orthogonal** if the angle at this crossing lies in the open interval $(\alpha, \pi - \alpha)$. Let $o(\gamma, r)$ denote the number of $\alpha(r)$-orthogonal crossings.

Theorem 2.14. *Let $0 < r < 1$. Suppose that γ is a generic immersion of finitely many circles in the plane. Then the number of non-degenerate rectangles with aspect ratio r inscribed in γ is congruent modulo 2 to $o(\gamma, r)$.*

Proof. The proof is very similar to the one of Theorem 2.13. Again, in a small neighborhood of a generic crossing we have no vertex of an inscribed rectangle with aspect ratio r. When we smoothen the crossing, 0, 1, or 2 new rectangles will appear, depending on whether the angle β that we smoothen satisfies $\beta < \alpha$, $\alpha < \beta < \pi - \alpha$, or $\pi - \alpha < \beta$; compare with Figure 2.6.

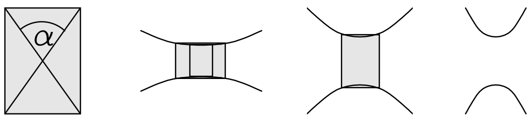
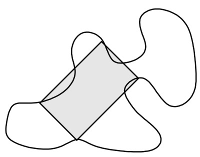


Figure 2.6: Smoothening a crossing changes the number of inscribed rectangles modulo 2 if and only if the crossing is α-orthogonal.

The rest is analogous to the previous proof. □

3 Rectangles on curves

In this section we deal with the smooth rectangular peg problem 1.2. It is very challenging and from the author's point of view the most beautiful open problem in this area of inscribing and circumscribing problems.



H. B. Griffiths [Gri90] gave a proof for Conjecture 1.2; unfortunately however his computations contain fatal errors concerning orientations. Hence the problem is still open for all cases $r \neq 1$, the case $r = 1$ being the smooth square peg problem.

The rectangular peg problem is much more difficult than the smooth square peg problem since the symmetry group of an oriented rectangle is $\mathbb{Z}_2$ instead of $\mathbb{Z}_4$. If one describes the set of rectangles R of a given fixed ratio r as the preimage of some $\mathbb{Z}_2$-equivariant test-map, it is easy to check that $\mathbb{Z}_2$ acts *orientation-preserving* on R. But it turns out the number of such rectangles, counted with sign, is zero, which can be deduced from an ellipse. Hence purely topological arguments will not work, any test-map in question will exist, even when one uses the boundary conditions that come from the smoothness of the curve. Still, topology gives some intuition, and here are two approaches.

3.1 Some intuition

Assuming that Conjecture 1.2 admitted a counter-example (γ, r), the following two lemmas derive conclusions that seem to be unintuitive, but more geometric ideas are needed to yield a contradiction.

Lemma 3.1. *Suppose there was a counter-example (γ, r). Then for all $\varepsilon > 0$, there is a $\mathbb{Z}_2$-invariant one-parameter family $S^1 \to P_4$ of ε-close parallelograms with aspect ratio in $[r-\varepsilon, r+\varepsilon]$ and with an odd winding number, such that during the whole one-parameter family one of the diagonals stays larger than the other one.*

Sketch of proof. For more details, we refer to [Mat09]. We define a test map,

$$\begin{aligned} g: P_4 &\longrightarrow_{\mathbb{Z}_2} \mathbb{R}^2 \times \mathbb{R} \\ [x_1, x_2, x_3, x_4] &\longmapsto \big((\gamma(x_1)+\gamma(x_3)) - (\gamma(x_2)+\gamma(x_4)), \\ &\qquad (||\gamma(x_1)-\gamma(x_2)|| + ||\gamma(x_3)-\gamma(x_4)||) - \\ &\qquad\quad r \cdot (||\gamma(x_2)-\gamma(x_3)|| + ||\gamma(x_4)-\gamma(x_1)||)\big). \end{aligned}$$

The preimage $P := (g|_{P_4^\circ})^{-1}(0)$ is exactly the set of parallelograms on γ of aspect ratio r. The action of $\mathbb{Z}_2$ preserves the orientation of P_4, $\mathbb{R}^2 \times \mathbb{R}$ and $\{0\}$. Hence $\mathbb{Z}_2$ preserves the preimage orientation of P as well. Since on smooth curves γ (and on smooth isotopies) the edge lengths of inscribed parallelograms of aspect ratio r are bounded from below, P does not intersect some small neighborhood of ∂P_4. Hence P defines an element in the 1-dimensional oriented bordism group $[P/\mathbb{Z}_2] \in \Omega_1(P_4/\mathbb{Z}_2) \cong \mathbb{Z}$. From a circle we deduce that this element is 1. $\square$

Remark 3.2. In Lemma 3.1, instead of looking at the set of parallelograms with aspect ratio r, we might look as well on the set of parallelograms whose diagonals intersect in an angle α, where α is the intersection angle of the diagonals in a rectangle of aspect ratio r. This gives an analogous lemma, which might be easier to deal with geometrically.

Lemma 3.3. *Suppose there was a counter-example (γ, r). Then for all $\varepsilon > 0$, there is a $\mathbb{Z}_4$-invariant one-parameter family $S^1 \to P_4$ of ε-close rectangles.*

Intuitively it seems that the conclusion of Lemma 3.3 can hold only for curves that are "close" to convex.

Proof. Let $f: P_4^\circ \longrightarrow_{\mathbb{Z}_4} \mathbb{R}^4 \times \mathbb{R}^2$ be the restricted map (2.1) from Section 2.3, measuring the edges and diagonals.

First of all we make f $\mathbb{Z}_4$-equivariantly transversal to $\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2}$ by a small δ-homotopy, and let $Q := f^{-1}(\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2})$ be the set of all squares (up to an δ-error, where δ is a function that decreases sufficiently fast near the boundary of P_4°). Then we make f $\mathbb{Z}_4$-equivariantly transversal to the $\mathbb{Z}_4$-invariant subspace $V := \{(a, b, a, b, e, e) \in \mathbb{R}^4 \times \mathbb{R}^2\}$ by a small δ-homotopy which leaves Q fixed, and let $R := f^{-1}(V)$ be the set of all rectangles on γ (up to an δ-error). If δ was chosen small enough, R consists only of ε-close rectangles.

Let R_Q be the set of all components of R that contain a square. We may assume that all these components are circles, otherwise a component would come arbitrary close to the boundary of P_4, so there would be an ε-close rectangle on it with aspect ratio r. If we could do this for all ε, then a limit argument would give us a proper rectangle of aspect ratio r. Hence if need be, we choose a smaller ε for which this does not happen.

R is a one-dimensional $\mathbb{Z}_4$-manifold, so $\mathbb{Z}_4$ acts on R_Q as well. We decompose $R_Q = R_1 \uplus R_2 \uplus R_4$, where R_1 is the set of components with isotropy group $\langle 0 \rangle$, R_2 with isotropy group $\mathbb{Z}_2 = \langle \varepsilon^2 \rangle \subset \mathbb{Z}_4$ and R_4 with $\mathbb{Z}_4$. Now we only need to count the number of squares on each R_i.

- $\sharp Q = 4 \mod 8$, since modulo $\mathbb{Z}_4$ it is odd (see Section 2.3).
- Every component $C \in \mathbb{R}_Q$ contains an even number of squares, since while passing a square the rectangle changes from fat to skinny or vice versa (this follows from the bijectivity of the differential $\mathrm{d}f$ at points in Q).
- 4 divides $\sharp R_1$, and every component in R_1 contains two squares. So the number of squares on components of R_1 is divisible by 8.
- 2 divides $\sharp R_2$, and if a component in R_2 contains a square S, then it contains also $\varepsilon^2 \cdot S$. When it goes through a square and changes from fat to skinny, then so it does at $\varepsilon^2 \cdot S$. Hence it has to go through $4k$ squares, $k \geq 1$. Thus the number of squares on components of R_2 is divisible by 8.
- If a component C of R_4 goes through a square S and changes from fat to skinny, then it also goes through $\varepsilon \cdot S$ and changes from skinny to fat. That is, in between it had to go through an even number of squares, all of which of course belong to a different $\mathbb{Z}_4$-orbit. Hence the number of square-orbits on C is odd, $\sharp(Q \cap C) = 4 \mod 8$.

Putting this modulo 8 together, we get $\sharp R_4 = 1 \mod 2$, which is even a bit stronger than what is stated in the lemma. □

3.2 Inscribed rectangles with aspect ratio $\sqrt{3}$

Let us now come to rectangles of aspect ratio $\sqrt{3}$. We call a smooth positively oriented plane curve $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ to have **angular convexity** at most α, if the signed curvature of γ restricted to any arc is at least $-\alpha$; see Figure 2.7.

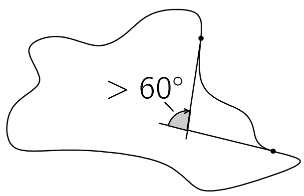


Figure 2.7: Example of a curve that is not 60°-angular convex.

Theorem 3.4. *Let $\gamma : S^1 \to \mathbb{R}^2$ be a C^∞ curve whose angular convexity is at most 60°. Then γ inscribes a rectangle with aspect ratio $\sqrt{3}$.*

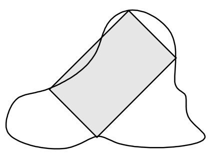


Figure 2.8: Example for Theorem 3.4.

The proof uses a hidden symmetry that appears for $r = \sqrt{3}$, which is a geometric piece of information.

Proof. Let us leave all technical details concerning transversality to the subsequent subsection below the proof. Suppose we are given a smooth curve $\gamma : S^1 \hookrightarrow \mathbb{R}^2$.

We define a map

$$\begin{aligned} f :(S^1)^4 &\longrightarrow_G \quad \mathbb{R}^2 \times S^1 \\ (x_1, x_2, y_1, y_2) &\longmapsto \quad (v, \alpha), \end{aligned}$$

where v is again the difference between the midpoints of the diagonals in the quadrilateral $(\gamma(x_1), \gamma(y_1), \gamma(x_2), \gamma(y_2))$ and α is the mod-180° angle between these diagonals (we measure angles always in counter-clockwise sense). If one diagonal is degenerate to a point we take the tangent of γ at this point to define α.

The map f is $G := \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\bar{0}_x, \bar{1}_x\} \times \{\bar{0}_y, \bar{1}_y\}$-invariant, where G acts on $(S^1)^4$ by $\bar{1}_x \cdot (x_1, x_2, y_1, y_2) = (x_2, x_1, y_1, y_2)$ and $\bar{1}_y \cdot (x_1, x_2, y_1, y_2) = (x_1, x_2, y_2, y_1)$.

Let $P := f^{-1}(0, 60°)$ be the set of parallelograms on γ having a 60°-angle modulo 180° between their diagonals. We call them 60°-parallelograms. We may assume that P is a union of connected 1-dimensional submanifolds K_i of $(S^1)^4$,

$$P = K_1 \cup \ldots \cup K_n, \;\; K_i \cong S^1,$$

where the union is disjoint except that points $(x_1, x_2, y_1, y_2) \in P$ of the form $x_1 = x_2$ or $y_1 = y_2$ occur exactly twice (for all technicalities, see Section 3.2). This is because P might contain parallelograms where one diagonal is degenerate to a point. These are exactly the points of P on that G does not act freely. However G acts freely on the disjoint union $K_1 \uplus \ldots \uplus K_n$. We denote $(S^1)^4/_G = M^2$ where $M := (S^1)^2/_{\mathbb{Z}_2}$ is the Möbius strip. The first factor M parameterizes x_1 and x_2 without their order and the second M parameterizes y_1 and y_2. Let $L_1 \uplus \ldots \uplus L_m \subset M^2$ be the quotient manifold $(\bigcup K_i)/_G$, which has corners at the points where it touches ∂M^2. Then L represents an element in the 1-dimensional unoriented bordism group $N_1(M^2) \cong N_1((S^1)^2) \cong (\mathbb{Z}_2)^2$, since all embedded circles γ are isotopic in the plane and G-homotopies of f change $K_1 \uplus \ldots \uplus K_n$ by a G-bordism.

If γ is the unit circle then we see that P is the disjoint union of four circles that all get identified by G. Their quotient L is one circle that represents $(\bar{1}, \bar{1}) \in N_1(M^2) \cong (\mathbb{Z}_2)^2$, where $\bar{1} \in \mathbb{Z}_2$ is the generator.

P does not contain parallelograms that have an edge that is degenerate to a point. Hence the x_1 and x_2-coordinates will always differ from the y_1 and y_2-coordinates at any point (x_1, x_2, y_1, y_2) in P. Therefore the circles L_i can only represent the elements $(\bar{0}, \bar{0})$ and $(\bar{1}, \bar{1})$ of $N_1(M^2) \cong (\mathbb{Z}_2)^2$.

Now we come to the "hidden symmetry", that is, the geometric piece of information that is the key in this proof. Let $W := \{(\alpha, \beta, \gamma) \in (S^1)^3 \mid \alpha + \beta + \gamma = 0° \mod 180°\}$. We define a map

$$\begin{aligned} F :(S^1)^6 &\longrightarrow \quad (\mathbb{R}^2)^3 \times W \\ (x_1, x_2, y_1, y_2, z_1, z_2) &\longmapsto \quad (m_x, m_y, m_z, \alpha_{xy}, \alpha_{yz}, \alpha_{zx}), \end{aligned}$$

where m_x is the mid-point of the segment $(\gamma(x_1), \gamma(x_2))$, $\alpha_x y$ is the mod-180°-angle between the segments $(\gamma(x_1), \gamma(x_2))$ and $(\gamma(y_1), \gamma(y_2))$, and analogously for the the other coordinates. F is equivariant with respect to the natural actions of the wreath product

$K := (\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_3$. Let

$$\widetilde{S} := F^{-1}(\Delta_{(\mathbb{R}^2)^3} \times \{(60°, 60°, 60°)\}).$$

We may assume that $\widetilde{S}$ is a 0-dimensional free K-manifold. We define $S := \widetilde{S}/_K$ to be the set of *stars*. See Figure 2.9 for an example. Every star $s \in S$ contains three 60°-parallelograms on γ, namely P_{xy}, P_{yz} and P_{zx}. Modulo G they lie in some components L_i, L_j and L_k (they are not necessarily pairwise distinct). We say that this star s relates L_i, L_j and L_k. Saying this is unique up to cyclic permutation of L_i, L_j and L_k. So we can draw a directed graph D whose nodes are the components of L, and we draw for each star a directed triangle $L_i \to L_j \to L_k \to L_i$.

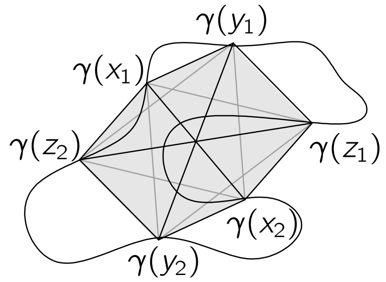


Figure 2.9: A star containing three 60°-parallelograms; $x_1z_1x_2z_2$ is skinny, the other two rectangles are fat.

Assume that γ does not contain a rectangle of aspect ratio $\sqrt{3}$. These are exactly the rectangles whose diagonals cross in a 60°-angle. Then all 60°-parallelograms on γ are *skinny* or *fat* in the sense that the x-diagonal is longer or shorter than the y-diagonal. By continuity this does not change along the components of L. Hence we can call the L_i's fat or skinny.

Recall that $[L_i] \in N_1(M^2)$ is $(\bar{0}, \bar{0})$ or $(\bar{1}, \bar{1})$. Correspondingly, we say that the *winding number* $w(L_i)$ of L_i is $\bar{0}$ (even) or $\bar{1}$ (odd), respectively.

Let $x, y : M^2 \to M$ be the projections to the first and to the second factor, respectively. An arc $L_i \to L_j$ in the graph D corresponds to an intersection of $y(L_i)$ and $x(L_j)$. The number of such intersections is

$$\sharp(y(L_i) \cap x(L_j)) = w(L_i) \cdot w(L_j) \mod 2. \tag{3.5}$$

We will derive a contradiction by double counting the number of stars $\sharp S$.

By (3.5), components of L with even winding number will have no influence on what follows. Let s be the number of skinny components of L with odd winding number, and let f be the number of fat components of L with odd winding number.

We know that $[L] = \sum_i [L_i] = (\bar{1}, \bar{1})$, thus

$$s + f = 1 \mod 2.$$

Note that no star relates three skinny or three fat 60°-parallelograms with each other. Hence every star gives exactly one arc from a skinny to a fat component of L. Modulo 2 and using (3.5), the number of these arcs is congruent $s \cdot f = 0 \mod 2$. Therefore,

$$\sharp S = 0 \mod 2.$$

On the other hand, every star relates three components, two of which are skinny or two of which are fat. So every star gives exactly one arc between two skinny components or between two fat components. Using (3.5), the number of arcs between skinny components modulo two is

$$s^2 = s \mod 2,$$

and the number of arcs between fat components modulo two is

$$f^2 = f \mod 2.$$

Together this gives,

$$\sharp S = s + f = 1 \mod 2.$$

This is a contradiction, which finishes the proof of Theorem 3.4. $\square$

Technical Details

In the previous section we assumed that the set of inscribed 60°-parallelograms P is a 1-dimensional manifold in the 4-manifold M^2. Also the set of stars should be finite. At the same time, when two parallelograms p_1 and p_2 have a common diagonal $y(p_1) = x(p_2)$ they form a star. Thus there should be another parallelogram p_3 such that $x(p_1) = y(p_3)$ and $y(p_2) = x(p_3)$. These triple intersection points come from the geometry, but they are in some sense *not generic*. That is, we need to be careful on how to make the test maps f and F simultaneously transversal in order to keep the geometric property of a star and without violating the equivariance. We solve this issue by perturbing the following two maps.

Let

$$m : (S^1)^2 \to \mathbb{R}^2$$

be the map that sends $(x_1, x_2) \in (S^1)^2$ to the mid-point $\frac{\gamma(x_1)+\gamma(x_2)}{2}$. Let

$$\alpha : (S^1)^2 \to S^1$$

be the map that sends $(x_1, x_2) \in (S^1)^2$ to the mod-180° angle of the line through $\gamma(x_1)$ and $\gamma(x_2)$ and some fixed line in the plane. The maps f and F can written in terms of m and α,

$$f(x_1, x_2, y_1, y_2) = (m(y_1, y_2) - m(x_1, x_2), \alpha(y_1, y_2) - \alpha(x_1, x_2))$$

and similarly F.

Let $\varphi_i : S^1 \to [0, 1]$, $i = 1 \dots k$, be a partition of unity of S^1 subordinate to a covering of S^1 with small ε-balls. We will perturb the maps m and α with two sets of parameters $S_m := ([-\varepsilon, +\varepsilon]^2)^{\binom{k+1}{2}}$ and $S_\alpha := [-\varepsilon, +\varepsilon]^{\binom{k+1}{2}}$ as follows:

$$\begin{aligned} m' : S_m \times (S^1)^2 &\longrightarrow \mathbb{R}^2 \\ (s_m, x_1, x_2) &\longmapsto m(x_1, x_2) + \sum_{i \le j} (\varphi_i(x_1)\varphi_j(x_2) + \varphi_i(x_2)\varphi_j(x_1)) \cdot (s_m)_{i,j}, \end{aligned}$$

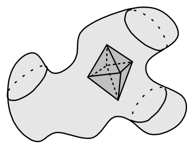
and

$$\begin{aligned} \alpha' : S_\alpha \times (S^1)^2 &\longrightarrow S^1 \\ (s_\alpha, x_1, x_2) &\longmapsto \alpha(x_1, x_2) + \sum_{i \le j} (\varphi_i(x_1)\varphi_j(x_2) + \varphi_i(x_2)\varphi_j(x_1)) \cdot (s_\alpha)_{i,j}, \end{aligned}$$

This defined analogous functions $f' : S_m \times S_\alpha \times (S^1)^4 \longrightarrow_G \mathbb{R}^2 \times S^1$ and $F' : S_m \times S_\alpha \times (S^1)^6 \longrightarrow_K (\mathbb{R}^2)^3 \times W$. Because of the additional parameter space f' and F' are transversal to the respective test-spaces $\{(0, 60°)\}$ and $\Delta_{(\mathbb{R}^2)^3} \times \{60°, 60°, 60°\}$. By the transversality theorem [GP74, p. 68], for almost all choices $s := (s_m, s_\alpha)$ (up to a zero set), the perturbations $f'_s := f'(s, \_)$ and $F'_s := F'(s, \_)$ are transversal to the test-spaces as well. Similarly one can show that for almost all s, $y(K_i)$ intersects $x(K_j)$ transversally for all i, j.

4 Crosspolytopes on spheres

In Klee & Wagon [KW96, Problem 11.5] is was asked whether every 3-dimensional convex body circumscribes the vertices of a regular octahedron. Makeev [Mak03] proved this for smooth convex bodies and Karasev [Kar09b] generalized the proof to smoothly embedded spheres in higher dimensions as follows.



Theorem 4.1 (Makeev, Karasev)**.** *Let d be an odd prime power. Then every smooth embedding $\Gamma : S^{d-1} \to \mathbb{R}^d$ contains the vertices of a regular d-dimensional crosspolytope.*

H. Guggenheimer [Gug65] gave already in 1965 a proof for all d, however there is unfortunately an error in his main lemma due to some connectivity arguments, which seems to invalidate the proof. Recently, Akopyan and Karasev [AK11] proved by a non-trivial approximation argument that for $d = 3$, the smooth embedding Γ can be replaced by the boundary of a simple polytope.

An interesting possible extension seems to be the following. Let M be a Riemann manifold. By a **crosspolytope** P on M we mean a set of $2d$ vertices v_i^ε, $\varepsilon \in \{+, -\}$, $i \in \{1, \dots, d\}$, on M. We call two vertices v_i^ε and v_j^δ **opposite** if $i = j$ and $\varepsilon = -\delta$. If any pair of non-opposite vertices of P have the same distance in M then we call P a **regular crosspolytope**.

Conjecture 4.2 ("Crosspolytopal peg problem for manifolds")**.** *Let d be a positive integer. Then every smooth embedding $\Gamma : S^{d-1} \to M$ into a Riemann manifold contains the vertices of a regular d-dimensional crosspolytope.*

The aim of this section is to show that the conjecture in general is probably very difficult.

The topological counter-example

A solution of the conjecture would involve deeper geometric reasoning, since there is the following "topological counter-example" for $d = 3$. Suppose we are given a smooth embedding $\Gamma : S^2 \to M$. Let $G \cong (\mathbb{Z}_2)^3 \rtimes S_3$ be the symmetry group of the regular octahedron and $G_{or} \subset G$ be the subgroup of orientation preserving symmetries. G acts on $(S^2)^6$ by permuting the coordinates in the same way as it permutes the vertices of the regular octahedron. Let G act on $\mathbb{R}^{12}$ by permuting the coordinates in the same way as it permutes the edges of the regular octahedron. The subrepresentation $(\Delta_{\mathbb{R}^{12}})^{\perp} \subset \mathbb{R}^{12}$ is denoted by Y. Let $\Delta^{fat}_{(S^2)^6}$ be the space of all 6-tuples in $(S^2)^6$ that contain at least two equal elements, that is, the fat diagonal. Let B be a small ε-neighborhood of $\Delta^{fat}_{(S^2)^6}$, where ε depends only on an isotopy of Γ to some nice embedding, that we will describe later. Then the complement $X := (S^2)^6 \backslash B$ is a free compact G-manifold with boundary and

$$X \simeq_G \{(x_1, \dots, x_6) \in (S^2)^6 \mid x_i \text{ are pairwise distinct}\} = (S^2)^6 \backslash \Delta^{fat}_{(S^2)^6}.$$

Then Γ gives us a test map

$$t : X \longrightarrow_G Y,$$

which measures the edges of the parameterized octahedra modulo $\mathbb{1} = (1, \dots, 1)$. This map depends only on the distance function $d : M \times M \to \mathbb{R}$ on M. Since ε was chosen small, $t|_{\partial X}$ is mapping uniquely up G-homotopy to $Y \backslash \{0\}$, if we change d by a homotopy relative to a small neighborhood of the diagonal Δ_{M^2} of M^2. We will use this fact later to assume that Γ is actually some nice embedding of S^{d-1} into $\mathbb{R}^d$. The solution set S of regular octahedra on Γ is $S := t^{-1}(0)$. The subset $S_{or} \subset S$ of positively oriented octahedra is a part of the preimage $t^{-1}(0)$, and t induces an isomorphism of G_{or}-vector bundles over S_{or},

$$TS_{or} \oplus (i_{S_{or}})^*(X \times Y) \cong (i_{S_{or}})^*(TX),$$

where $i_{S_{or}}$ denotes the inclusion $S_{or} \hookrightarrow X$. Thus S_{or} gives us together with this normal data an element $[S_{or}]$ in the equivariant normal bordism group (see Koschorke [Kos81, Chap. 2])

$$\Omega_1^{G_{or}}(X, X \times Y - TX) = \Omega_1(X/_{G_{or}}, X \times_{G_{or}} Y - T(X/_{G_{or}})),$$

which is well-defined, since $\mathbb{Z}_2$-homotopies of d relative to a small neighborhood of Δ_{M^2} change S only by a normal bordism that stays away from the ∂X if ε was chosen small enough, and components of octahedra of different orientation are always separated from each other. In Koschorke's notation, $[S_{or}]$ is the obstruction

$$\widetilde{\omega}_1(\underset{\sim}{\mathbb{R}}, X \times_{G_{or}} Y, (\mathrm{id}_{\partial X}, t|_{\partial X})/_{G_{or}}),$$

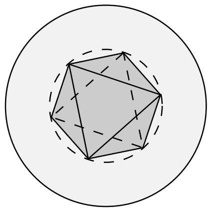
where $\underset{\sim}{\mathbb{R}}$ is the trivial line bundle.

Theorem 4.3. *The above defined $[S_{or}]$ is zero. Hence*

$$[S] \in \Omega_1^G(X, X \times Y - TX)$$

is zero as well. In particular, the test map t can be deformed G-equivariantly relative to ∂X to a map t', such that $0 \notin t'(X)$.

The existence of the test-map t' that fulfills the boundary conditions is what I call a topological counter-example.



Sketch of Proof. To construct a nice representative for $[S_{or}]$ we take the standard 2-sphere and scale it down linearly along the z-axis of $\mathbb{R}^3$. This is our Γ and we let t and S be the corresponding test map and solution set, respectively. S is a disjoint union of $16 = \frac{1}{3} \cdot \sharp G$ circles. One octahedron on the scaled sphere looks as follows (one looks along the z-axis):

If we rotate it around the z-axis then we get up to symmetry all octahedra on Γ. The G-bundles $X \times Y$ and TX are G-orientable. Therefore the relevant part of Koschorke's exact sequence [Kos81, Thm. 9.3] becomes

$$\begin{aligned} H_2(X/_{G_{or}};\mathbb{Z}) \to \mathbb{Z}_2 \to \Omega_1(X/_{G_{or}}, X \times_{G_{or}} Y - T(X/_{G_{or}})) \\ \to H_1(X/_{G_{or}};\mathbb{Z}) \to 0. \end{aligned}$$

It is not difficult to see that the image of $[S_{or}]$ in $H_1(X/_{G_{or}};\mathbb{Z}) = H_1(G_{or};\mathbb{Z})$ is zero. This is because the 120 degree rotation of a regular octahedron around an the line connecting the midpoints of two opposite triangles is an element of the commutator of G_{or}. It requires more visualization to see that $[S_{or}]$ is in fact the image of the generator of $\mathbb{Z}_2$. The hard part is to show that $\mathbb{Z}_2$ unfortunately lies in the image of $H_2(X/_{G_{or}};\mathbb{Z})$, which I could manage to do only with a very long computer program. It finds that $H_2(X/_{G_{or}};\mathbb{Z}) \cong \mathbb{Z}_4 \times (\mathbb{Z}_2)^3$, where one can choose the generators such that the first three map to zero and the last one to the generator of $\mathbb{Z}_2$.

The G_{or}-null-bordism of S_{or} can be extended to a G-null-bordism of S. By Theorem 3.1 of U. Koschorke [Kos81], we can extend the section as stated. □

Remarks to the algorithm

An economical S_6-CW-complex structure on $(S^2)^6$ is based on an S_6-cell decomposition of $\mathbb{R}^2$ of V. A. Vassiliev [Vas94], which has few high dimensional cells. $\Delta^{fat}_{(S^2)^6}$ is a subcomplex, so one can compute $H_2(X/_{G_{or}}) \cong H^{10}((S^2)^6/_{G_{or}}, (\Delta^{fat}_{(S^2)^6})/_{G_{or}})$. The Smith normal form is used to compute this cellular cohomology and the LLL-algorithm to choose economical generators. The image in $\mathbb{Z}_2$ is determined by computing second Stiefel-Whitney classes, which I implemented as obstruction classes.

Appendix A

Two classes of interesting polytopes

The following topics of my coauthor's and my work do not fit thematically to the previous chapters. This small chapter summarizes the main results.

A1 5-spindles and their width

Definition A1.1. A d-**spindle** is a d-dimensional polytope P with two distinguished vertices t and b such that each facet contains at least one of them. The **width** of P is the distance of t and b in the graph of P.

That is, the width counts the number of steps that one needs to walk from t to b along edges of P, where every edge has length one. 2-dimensional spindles are triangles and trapezoids, and their width is 2. Any 3-spindle has width at most 3 (exercise). Santos and independently Stephen & Thomas showed that also any 4-spindle has width at most 4 [SST11].

The recent interest in spindles with width larger than their dimension comes from the fact that they form the basis for the first *counter-example to the Hirsch conjecture*, which was found by Francisco Santos in 2010 [San10].

Theorem A1.2 (Santos 2010)**.** *If P is a d-spindle of with n facets and width w, then there is a $(d+1)$-spindle Q with $n+1$ facets and width $w+1$. Repeating this step $n-2d$ times, one arrives at a $(n-d)$-spindle with $2n-2d$ facets and width $n-2d+w$. If the starting d-spindle has width $w > d$ then the resulting polytope is a counter-example to the Hirsch conjecture.*

With Santos we constructed a family of 5-spindles whose width grow in the square root of the number of facets, or more precisely:

Theorem A1.3 (M, Santos, Weibel 2011)**.** *Let $d \geq 3$, $q \geq 2$ and $m := dq$. Then there exists a 5-prismatoid of width $4 + q/2$ with $m(m-d+2)$ facets in each base vertex.*

The same paper [MSW11] contains work of Christophe Weibel and Francisco Santos (without me) in which they find smaller counter-examples to the Hirsch conjecture, the smallest of which lives in dimension 20 and which has the advantages that it is has explicit coordinates and that it fits into a computer.

A2 Product-simplicial neighborly polytopes

Definition A2.1 (Product-simplicial neighborly polytopes)**.** Let $n_1, \dots, n_r \geq 0$, $r \geq 1$, and $k \geq 0$. A polytope P is called ***k*-product-simplicial neighborly** (or ***k*-psn**) with respect to the given data $(n_1, \dots, n_r)$ if its k-skeleton is combinatorially equivalent to the k-skeleton of the product of simplices $\Delta_{n_1} \times \dots \times \Delta_{n_r}$.

This definition interpolates between simplicial neighborly polytopes and cubically neighborly polytopes: A k-psn polytope with $r = 1$ is known as a simplicial $(k+1)$-neighborly polytope on $n_1 + 1$ vertices, and a k-psn polytope with $n_1 = \dots = n_r = 1$ is known as a cubically k-neighborly polytope on 2^r vertices.

As for simplicially and cubically neighborly polytopes it seems interesting to ask: What is the smallest dimension in which there exists k-psn polytopes to the given data $(n_1, \dots, n_r)$. The main result of my paper [MPP11] with Julian Pfeifle and Vincent Pilaud is the following theorem.

Theorem A2.2 (M–Pfeifle–Pilaud 2009)**.** *Let $r \geq 1$, $n_1, \dots, n_r \geq 1$, and $k \geq 0$. Then there is a k-psn polytope to this data already in dimension (at most) $2k + r + 1$.*

The proof is constructive up to finding a sufficiently small $\varepsilon > 0$. A very general construction method is to project a polytope that is combinatorially equivalent to $\Delta_{n_1} \times \dots \times \Delta_{n_r}$ to $\mathbb{R}^d$ in such a way that the k-skeleton survives. We found also new obstructions for this projection method using topological methods extending the one of Rörig and Sanyal [RS09].

It might be interesting to generalize the notion of product-simplicial neighborlyness to *embedded manifolds* by extending the notion of k-neighborly embedded manifolds from Perles [Per82] (see also Vassiliev [Vas98] and Kalai & Wigderson [KW08]) to the product-simplicial setting. We could call an embedding of a product of manifolds $e : M_1 \times \dots \times M_r \to \mathbb{R}^d$ k-product-simplicial neighborly, if for any chosen subsets $P_i \subset M_i$ whose cardinalities sum up to at most $k + r$ there exists a hyperplane in $\mathbb{R}^d$ that supports $e(M)$ at precisely the points $e(P_i \times \dots \times P_r)$. One can now ask: Given a product of manifolds $M = M_1 \times \dots \times M_r$, what is the smallest dimension d such that there exists a k-psn embedding of M into $\mathbb{R}^d$? The construction of Theorem A2.2 yields a k-psn embedding into $\mathbb{R}^{2k+r+1}$ if all M_i equal $\mathbb{R}$. First interesting special cases are $M_i = S^1$ and $M_i = \mathbb{R}^m$ for $m \geq 2$.

Appendix B

Summaries

B1 English summary

The first chapter of this thesis is on the *colored Tverberg problem*, which is joint work with Pavle Blagojević and Günter Ziegler [BMZ09], [BMZ11a], [BMZ11b]. First we present a new and tight colored version of Tverberg's theorem that implies the Bárány–Larman conjecture for primes minus one and asymptotically in general. This in turn improves the bounds in the second selection lemma, which is used in computational complexity for example to bound the number of halving sets of an n-set in $\mathbb{R}^d$. Then we generalize our theorem to a transversal version, a colored version of the Tverberg–Vrećica conjecture, which is a unifying theorem in the sense that it implies the ham sandwich theorem and the center transversal theorem. Finally we generalize our theorem to maps into manifolds. Two results of independent interest are a new parameterized Borsuk–Ulam-type theorem for equivariant vector bundles and the calculation of the Fadell–Husseini index of joins of chessboard complexes.

The second chapter is on *inscribing squares and rectangles* into closed curves in the plane. The results are disjoint from the ones in [Mat08, Chap. III], and they will appear in [Mat09] and [Mat11]. We present two new classes of Jordan curves that fulfill Toeplitz' still unsolved square peg problem from 1911, that is, these curves inscribe squares. One of them strictly contains all previously known classes; the other one is the first known *open* set of such curves. Then we disprove Cantarella's conjecture on the parity of inscribed squares for immersed plane curves and give the right answer, also for inscribed rectangles. We give another class of Jordan curves that inscribes rectangles of aspect ration $\sqrt{3}$, which is the first known partial result for an aspect ratios other than 1.

The appendix summarizes two papers on polytopes. The first one is joint work with Francisco Santos and Christophe Weibel [MSW11] on 5-*spindles with large width*, which are a building block for new counter-examples of the Hirsch conjecture. The second is joint work with Julian Pfeifle and Vincent Pilaud [MPP11] on *productsimplicial-neighborly polytopes*, where we construct polytopes that interpolate between being neighborly and cubically neighborly.

B2 Deutsche Zusammenfassung

Das erste Kapitel behandelt das *farbige Tverbergproblem* und ist in Zusammenarbeit mit Pavle Blagojević und Günter Ziegler [BMZ09], [BMZ11a], [BMZ11b] entstanden. Zuerst zeigen wir eine neue und optimale farbige Version des Tverbergsatzes, welches die Bárány–Larman-Vermutung für Primzahlen minus Eins und im Allgemeinen asymptotisch impliziert. Das wiederum verbessert die Schranken im Second-Selection-Lemma, was z.B. in der algorithmischen Komplexitätstheorie benutzt wird um die Anzahl der halbierenden Mengen in einer n-Menge im $\mathbb{R}^d$ nach oben abzuschätzen. Anschließend verallgemeinern wir unseren Satz zu einer transversalen Variante, einer farbigen Version der Tverberg–Vrećica-Vermutung, welche zudem das Ham-Sandwich-Theorem impliziert und allgemeiner das Center-Transversal-Theorem. Im dritten Abschnitt verallgemeinern wir unseren Satz für Abbildungen in beliebige Mannigfaltigkeiten. Zwei methodische Resultate sind ein neuer parametrisierter Borsuk–Ulam-Satz für äquivariante Vektorbündel und die Berechnung des Fadell–Husseini-Indexes von Joins von Schachbrettkomplexen.

Das zweite Kapitel beschäftigt sich mit *in Kurven einbeschriebenen Quadraten und Rechtecken*. Die Ergebnisse sind disjunkt von denen in [Mat08, Chap. III], und sie werden in [Mat09] und [Mat11] erscheinen. Wir zeigen für zwei neue Klassen von Jordankurven, dass sie die Toeplitz-Vermutung erfüllen, d.h. jede dieser Kurven enthält die vier Punkte eines Quadrats. Die erste Klasse enthält strikt alle bisher bekannten Klassen, und die andere ist die erste bekannte *offene* Menge solcher Kurven. Dann widerlegen wir eine Vermutung von Cantarella über die Parität der Anzahl von in immersierten planaren Kurven einbeschriebenen Quadraten, und geben die richtige Anzahl an, auch für einbeschriebene Rechtecke. Im zweiten Abschnitt geben wir eine Klasse von Jordankurven an, die Rechtecke des Seitenverhältnisses $\sqrt{3}$ einschreiben, welches das erste bekannte Teilergebnis für Seitenverhältnisse ungleich 1 ist.

Der Anhang fasst zwei Papers über Polytope zusammen. Das erste ist eine Zusammenarbeit mit Francisco Santos und Christophe Weibel [MSW11] über *breite* 5-*Spindeln*, von welchen neue Gegenbeispiele zur Hirschvermutung konstruiert werden können. Das zweite ist eine Zusammenarbeit mit Julian Pfeifle und Vincent Pilaud [MPP11] über *produktsimpliziale nachbarschaftliche Polytope*, in der wir unter anderem Polytope konstruieren, die zwischen nachbarschaftlichen und kubisch nachbarschaftlichen Polytopen interpolieren.

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