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ORIGINAL PAPER

On the relationship between school mathematics and university mathematics: a comparison of three approaches

Thorsten Scheiner[1,2] · **Marianna Bosch**[3]

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Abstract
This paper examines how different approaches in mathematics education conceptualise the relationship between school mathematics and university mathematics. The approaches considered here include: (a) Klein's elementary mathematics from a higher standpoint; (b) Shulman's transformation of disciplinary subject matter into subject matter for teaching; and (c) Chevallard's didactic transposition of scholarly knowledge into knowledge to be taught. Similarities and contrasts between these three approaches are discussed in terms of how they frame the relationship between the academic discipline and the school subject, and to what extent they problematise the reliance and bias towards the academic discipline. The institutional position implicit in the three approaches is then examined in order to open up new ways of thinking about the relationship between school mathematics and university mathematics.

Keywords Didactic transposition · Elementarisation · School mathematics · Transformation of disciplinary subject matter · University mathematics

1 Introduction

By way of introduction, it can be said that the relationship between school mathematics and university mathematics can be approached and conceptualised in various ways across different traditions, shaped culturally and historically and influenced by socio-political interests of the time. At one extreme, school mathematics and university mathematics can be considered independent fields of knowledge, where one does not influence the other and should therefore be considered as separate entities. At the other extreme, school mathematics and university mathematics are identified as a single entity, 'mathematics', with different elements selected as school content or university content. Take, for example, the secondary-tertiary transition (i.e., the transition from secondary school to university), where consideration of the relationship between school mathematics and university mathematics is often a focal point (see e.g., Di Martino et al., 2023; Gueudet, 2008; Kaiser & Buchholtz, 2014; Pinto & Koichu, 2023; Rach & Heinze, 2017). The notion of transition signifies a process of change, suggesting that the passage from school mathematics to university mathematics tends to be discontinuous (see e.g., Gueudet et al., 2016; Hochmuth et al., 2021).

Arguably, school subjects and academic disciplines have quite different goals. The primary objective of academic disciplines is to address and solve open problems that are relevant to society and to the discipline itself. In contrast, the objectives of school subjects are more diverse and include educational goals related to values (e.g., justice, democracy) and competencies (e.g., critical thinking), for example. According to Dewey (1972), school subjects are primarily concerned with "the subject as a special mode of personal experience for children, rather than the discipline as a body of wrought-out facts and scientifically tested principles" (p. 169).

This paper aims to examine the relationship between school mathematics and university mathematics, looking at three concrete approaches that have been pioneering or influential in many ways. These approaches include: (a) Klein's

✉ Thorsten Scheiner
thorsten.scheiner@acu.edu.au

Marianna Bosch
marianna.bosch@ub.edu

1 Institute for Learning Sciences and Teacher Education, Australian Catholic University, Brisbane, Australia

2 Freie Universität Berlin, Berlin, Germany

3 Universitat de Barcelona, Barcelona, Spain

(2016a) consideration of elementary mathematics from a higher standpoint; (b) Shulman's (1986, 1987) approach of transforming disciplinary subject matter into subject matter for teaching; and (c) Chevallard's (1985) didactic transposition of scholarly knowledge into knowledge to be taught.

The selection of Klein, Shulman, and Chevallard as the three approaches for examination in this study is based on several reasons. Firstly, each approach has had a significant impact on the field of mathematics education and offers unique perspectives on the relationship between the academic discipline and the school subject. By considering these diverse perspectives, this paper aims to develop a more nuanced understanding of the relationship between school mathematics and university mathematics, contributing to the ongoing discourse on the topic.

Secondly, these three approaches span different historical periods and cultural contexts, and their authors have distinct research profiles. Klein was a mathematician, Shulman a general educator, and Chevallard a mathematics educator. Despite these differences, the three scholars were interested in similar issues related to teacher preparation and have influenced national and international reform movements in the professionalisation of teaching (see e.g., De Bock, 2023; Gispert & Schubring, 2011).

Thirdly, these approaches are particularly relevant to discussions around teacher education and the preparation of secondary mathematics teachers. They are frequently cited in writings on teacher education and continue to inspire educators and researchers today.[1] However, while these approaches have been examined and compared concerning issues of preparing mathematics for teaching (Scheiner et al., 2022) and conceptualising teacher knowledge (Scheiner & Buchholtz, 2022), they have not been studied regarding how they position the relationship between school mathematics and university mathematics. This is especially relevant in university mathematics education research (Winsløw & Rasmussen, 2020), particularly in relation to the transitions (epistemological, cognitive, institutional) that may occur in secondary mathematics teacher education (Artigue, 2022).

Although the three approaches examined in this paper do not encompass the entire spectrum of possibilities, they were selected to demonstrate similarities and differences in the way they position the relationship between school mathematics and university mathematics. The selection was made to provide a nuanced understanding of how different perspectives shape the relationship between school mathematics and university mathematics, highlighting cultural and historical differences in thinking about this topic.[2] This selection was made in consideration of the theme of this Special Issue of *ZDM–Mathematics Education*, "Exploring and strengthening university mathematics courses for secondary teacher preparation" (Buchbinder et al., 2023), where the interactions between university mathematics and school mathematics take on specific importance.

The paper is structured in three parts. The first part provides an overview of the three approaches (Klein, Shulman and Chevallard) and how they conceptualise the relationship between the school subject and the academic discipline. The second part of the paper explores the directionality of the relationship between school mathematics and university mathematics assumed in the three approaches and the extent to which this relationship has been problematised. Finally, the third part addresses the differences between the three approaches by examining the institutional position implicitly assumed between the school and the academic environments to open up new ways of thinking about the relationship between school mathematics and university mathematics.

2 Approaches in considering the relationship between school mathematics and university mathematics

In this section, the focus is on how the approaches of Klein, Shulman and Chevallard represent the relationship between school mathematics and university mathematics. While these approaches addressed various issues that were critical in their cultural contexts at the time, they remain relevant in many parts of the world today (see e.g., Bass, 2005; Heinze et al., 2016; Jahnke et al., 2022; Winsløw & Grønbæk, 2014) and, we assert, offer valuable insights into reflecting on the relationship between school mathematics and university mathematics. However, it should be noted that the three scholars under consideration have written about this relationship rather unsystematically and often implicitly. As a result, interpretations must be made carefully. This is all the more important because the intellectual climate of the time and the region strongly influenced the way Klein, Shulman and Chevallard expressed themselves.

[1] For works that take Klein's approach to teacher preparation, see, for example, Allmendinger et al. (2023), Buchholtz et al. (2013) and Winsløw and Grønbæk (2014); for writings on Shulman's approach, see, for example, Depaepe et al. (2013) and Scheiner (2022); and for works on Chevallard's approach, see, for example, Barquero et al. (2019), Chevallard (2022), and Winsløw (2015).

[2] It is important to note that the focus on the three approaches (Klein, Shulman and Chevallard) represents a limited range of perspectives for examining the relationship between school mathematics and university mathematics. These approaches were chosen because they consider this relationship from the viewpoint of the distinction between the school subject and the academic discipline and imply, either explicitly or implicitly, a certain orientation of one field of knowledge towards the other.

2.1 Felix Klein's elementary mathematics from a higher standpoint

The German mathematician Felix Klein (1849–1925) had a profound impact on the mathematical preparation of prospective teachers for teaching secondary mathematics, particularly in Germany (see Weigand et al., 2019). He presented his ideas for university mathematics lectures for prospective teachers in three books entitled '*Elementary Mathematics from a Higher Standpoint*' (Klein, 2016a, 2016b, 2016c). These lectures aimed to provide prospective teachers with an epistemological approach to mathematics by explaining connections within mathematics as a whole and showing the links to specific mathematical topics and questions within school mathematics.

Klein's lectures were especially important because they addressed a central issue in the preparation of mathematics teachers: the discontinuity between the school subject and the academic discipline. By connecting problems in the main disciplines of mathematics with problems in school mathematics, Klein aimed to bridge this discontinuity and provide a more comprehensive understanding of mathematics for prospective teachers.

> For a long time […], university [departments] were practising exclusively research of optimal quality, without giving a thought to the needs of the schools, without even caring to establish a connection with school mathematics. What is the result of this practice? The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fell in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching. (Klein, 2016a, p. 1)

Klein's work aimed to overcome this *double discontinuity* in the transition from school to university and in the return to school by bringing school mathematics into a productive relationship with the processes of the main disciplines of mathematics. Despite recognising the discontinuity between the two, Klein held a view of mathematics in which there was not necessarily such kind of discontinuity.[3] Rather, he viewed mathematics holistically, as constantly evolving and reforming through a process of *elementarisation* (for a discussion, see Schubring, 2016).

In this context, elementarisation refers to a conceptual exposition of the epistemological essence of a mathematical domain or idea, rather than a simplification of the subject matter. This process can lead to a restructuring of parts of different disciplines of mathematics that may have grown independently of each other instead of building up genetically.

Klein saw the historical evolution of mathematics as an ongoing process of elementarisation, where fundamental elements of a mathematical field or idea are unraveled to create a new architecture of mathematics.

> The normal course of development […] in science is that higher and more complicated parts gradually become more elementary through the increasing clarification of concepts and through simplified representation. (Klein & Schummack, 1907, p. 90; our translation)

This method makes it possible to identify the *elementary* of a mathematical discipline. The term 'elementary' in this context is not meant to imply simplicity or basicness in the everyday sense, nor is it used as a direct contrast to 'scientific' or 'academic'. Instead, it refers to the outcome of elementarisation of complex developments in mathematics. The elementary can then be understood as the fundamental concepts of mathematics, which are related to the discipline as a whole and correspond to its newly structured architecture (see Schubring, 2019).

> 'Elementary' in a domain is what, due to its relative simplicity, is suitable for a natural introduction to the subject. Elementary mathematics will then comprise those parts of mathematics that, according to the present state of science, are accessible to a human mind of average ability without further specialised study. School mathematics, however, will again have to select from this elementary mathematics what best corresponds to the aim of secondary schools, namely: to provide a general basis for understanding our present-day culture. (Klein & Schummack, 1907, p. 111; our translation)

[3] Recently, Liang et al. (2023), for example, reported that prospective teachers' experiences of transitioning from school to university were somewhat coherent (as opposed to discontinuous). In particular, the 'double discontinuity' (in Klein's sense) became a crucial resource that shaped prospective teachers' thinking about mathematics teaching and from which they capitalised as they transitioned into their new roles.

The practice of elementarisation and the focus on the elementary have a rich tradition in German-language didactics of mathematics (see Scheiner et al., 2022). Elementarisation does not merely mean a reduction of mathematics to its basic components, but rather a concretisation or an embodiment of the essential meaning inherent in the mathematics in question (see Kirsch, 1977). Through this process, the mathematics may become even more complex as it is concentrated, intensified or abstracted to which is fundamental to the development of a deep understanding of, and insight into, the mathematical topic, subject area or working method in question.

The practice of elementarisation and the identification of the elementary have fostered the development of important didactic constructs, such as *fundamental ideas* ('fundamentale Ideen') and *basic ideas* ('Grundvorstellungen'). Fundamental ideas describe the underlying principles or essence of a subject area, such as the ideas of algorithm, approximation, or symmetry (Schweiger, 1992; Vohns, 2016). Basic ideas are more localised than fundamental ideas and provide adequate interpretations of mathematical concepts that give them meaning, such as the idea of 'equal sharing' for dividing natural numbers (Greefrath et al., 2016; vom Hofe & Blum, 2016).

2.2 Lee S. Shulman's transformation of disciplinary subject matter into subject matter for teaching

The American educational psychologist Lee S. Shulman (1938–) has been a major driving force in advancing the teaching profession, particularly in the Anglo-American educational research literature, by claiming that teachers have a specialised kind of content knowledge that goes beyond subject matter knowledge per se and encompasses subject matter knowledge for teaching (Shulman, 1986, 1987). Shulman called this specialised kind of content knowledge *pedagogical content knowledge* and defined it as "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organised, represented, and adapted to diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). In this way, Shulman created a way to overcome the previously mutually exclusive categories of subject matter knowledge and pedagogical knowledge in thinking about teacher education and the teaching profession.[4]

The notion of pedagogical content knowledge arose from the *Knowledge Growth in Teaching* research programme of Shulman and his colleagues (e.g., Grossman et al., 1989; Shulman, 1986, 1987; Wilson et al., 1987), which examined the interaction of content knowledge and pedagogical development in novice schoolteachers in a variety of subjects (including English, mathematics, science, and social studies). The focus of this research programme was on how novice teachers *transform* disciplinary subject matter acquired in college or university into forms that are suitable for teaching. For Shulman, the central intellectual task of teaching was to transform the disciplinary subject matter content a teacher possessed into pedagogical forms that were accessible to students. This pedagogical transformation is what Shulman (1987) considered to be the defining principle for a knowledge base of teaching:

> […] the key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students. (p. 15)

Shulman asserted that the subject matter content taught in school is a pedagogical and personal revision of a teacher's disciplinary content knowledge (for a discussion, see Scheiner, 2022). The teacher's orientation towards the structure of the discipline and the structure of students' minds, including their prior knowledge and dispositions, provides the basis for transforming content knowledge into pedagogical forms that are appropriate for students and specific to the task of teaching.

> […] excellent teachers transform their own content knowledge into pedagogical representations that connect with prior knowledge and dispositions of learners. The effectiveness of these representations depends on their fidelity to the essential feature of the subject matter and to the prior knowledge of the learners. The capacity to teach […] is highly dependent on […] how well one understands ways of transforming the subject matter into pedagogically powerful representations. (Shulman & Quinlan, 1996, p. 409)

2.3 Yves Chevallard's didactic transposition of scholarly knowledge into knowledge to be taught

The French didactician Yves Chevallard (1946–) has become widely recognised for his influential work on *didactic*

[4] It is important to note that Shulman has been an educator and reformer in teaching and teacher education, not specifically in mathematics education. Therefore, his perspective on the relationship between school mathematics and university mathematics can only be conceived through his view on the relationship between the academic discipline and the school subject.

transposition ('transposition didactique'), first published in French (Chevallard, 1985), in which he demonstrated and analysed the profound changes that knowledge undergoes when it is transposed from one institution to another (for overviews, see Bosch & Gascón, 2006; Chevallard & Bosch, 2020b).

Knowledge, for Chevallard, is a changing reality that adapts to its institutional habitat, in which it occupies a more or less narrow niche.

> A given piece of knowledge K is found in various types of institutions I, which are, in terms of the ecology of knowledge, so many different habitats for it. Now, when we consider these habitats, we immediately see that the knowledge in question regularly occupies very distinct niches. Or, to put it another way, the institutional relationship of I to K, $R_I(K)$, which we will also call the problematic of I in relation to K, can be very different. Correlatively, the way in which the agents of the institution will 'manipulate' this knowledge will also vary. (Chevallard, 1991, p. 210; our translation)

Knowledge is, therefore, always relative to the institution in which it 'lives'. This institutional relativity of knowledge means there is and always will be an inevitable difference or gap between the knowledge that is created or implemented and used in one institution (e.g., academic mathematics in universities) and the knowledge that is created or implemented and used in another institution (e.g., school mathematics in schools). With this view, Chevallard provided a different perspective on the then-growing gap between academic mathematics and school mathematics and the more or less obvious failure of earlier attempts to close this gap by reconstructing school mathematics based on academic mathematics.

> It is in the confrontation of these two terms [scholarly knowledge and taught knowledge], in the distance that separates them, beyond what brings them together and imposes to confront them, that we can best grasp the specificity of the didactic treatment of knowledge. (Chevallard, 1991, p. 20; our translation).

Chevallard used the notion of didactic transposition to refer to the transition from scholarly knowledge ('savoir savant'), produced by the scientific community and legitimised by the academic institution, to taught knowledge ('savoir enseigné'), which results from a teaching process in a particular educational institution. Further, he introduced the intermediated entity 'knowledge to be taught' ('savoir à enseigner') and distinguished between external didactic transposition and internal didactic transposition. External didactic transposition takes place from scholarly knowledge to knowledge to be taught, and is undertaken by the so-called 'noosphere', the sphere of those who think about education, including curriculum developers, textbook designers, and policymakers—an intermediary institution between the education system and the society (see Chevallard & Bosch, 2020b). Internal didactic transposition refers to the processes that lead from knowledge to be taught to the knowledge actually taught in the classroom and is carried out by the teacher and the students.

Since different actors (disciplinary experts or 'scholars', noosphere agents and teachers) are always subject to various constraints in the transposition process, and knowledge is institutionally embedded, the transposition from scholarly knowledge (e.g., academic mathematics) to taught knowledge (e.g., school mathematics) can hardly be taken for granted or predetermined. Moreover, the construction of the knowledge to be taught is a historical and collective endeavour, that typically begins in the scholarly institution and is continuously modified by various actors in the noosphere. The culmination of this process is the production of curriculum guidelines, textbooks, treatises and exercise books, which are essential in comprehending the (occasionally challenging) relationships between school mathematics and university mathematics.

The New Math reform appeared as a paradigmatic example of the tensions that can arise between scholarly mathematics and mathematics to be taught, with the latter being considered outdated by the scholarly institution (Chevallard, 1985). The historical evolution of the didactic transposition process is also essential to understand the changes operated in the knowledge to be taught, as well as the invariants that seem difficult to modify. Many reminiscences of the New Math reform, such as the construction of numbers as autonomous entities and the secondary role of quantities in such a construction (Chambris, 2018; Chambris & Visnovska, 2022), still remain in today's school mathematics and appear to be strongly conditioned by their status in scholarly mathematics (Chevallard & Bosch, 2000).

Another more subtle tension arises from the legitimation processes associated with the teaching of disciplines in schools. The theory of didactic transposition posits that school mathematics cannot be identical to scholarly mathematics because it is situated in different institutions; however, it cannot be drastically divergent either. What is taught in school should be recognised by society as 'mathematics', and scholarly institutions play a pivotal role in such recognition. This is why the didactic transposition process includes its denial by its participants: what is taught at schools has to be presented as 'authentic' mathematics rather than just a transposed variant of it.

Didactic transposition processes take place in all institutions as far as teaching and learning processes occur. This includes the university itself because transformations are also needed to elaborate the ‘university mathematics knowledge to be taught’. In this case, even if the positions of ‘scholar’, ‘noospherian’ and ‘teacher’ differ, the persons that occupy them can coincide, for instance, when a researcher in topology—a ‘scholar’—teaches this subject in an undergraduate course and even participates in the design of the subject syllabus and its weight and position in the degree program, thus acting as a ‘noospherian’. The analysis of university didactic transposition is taken up by research in undergraduate mathematics education, with the corresponding questioning of the transformations produced on the scholarly organisation of knowledge (see e.g., Bosch et al., 2018). However, to avoid complexity, in what follows ‘university mathematics’ will be mainly used as a synonym of ‘scholarly mathematics’.

3 On the reliance of the academic discipline and the directionality from university mathematics to school mathematics

The school subject is not a mere simplification or reduction of the academic discipline. The approaches by Klein, Shulman and Chevallard acknowledge the necessity of modifying academic knowledge for teaching purposes, attributing specific statuses to this modification process through processes such as elementarisation (Klein, 2016a), pedagogical transformation (Shulman, 1987) and didactic transposition (Chevallard, 1991). While these processes have been scrutinised elsewhere (Scheiner, 2022; Scheiner et al., 2022), this paper delves into the ways they position the relationship between school mathematics and university mathematics.

It is important to note that the three approaches differ in the extent to which they are tied to the subject of mathematics. While Klein’s approach is firmly rooted in the context of mathematics, the notion of elementarisation has also garnered considerable interest in general didactic considerations across various subjects (e.g., Duit et al., 2012; Klafki, 1954; Krüger, 2008). Shulman’s approach, on the other hand, is subject-independent and has been applied to diverse subjects, including English, history, social studies, and mathematics (e.g., Gudmundsdottir & Shulman, 1987; Marks, 1990). Notably, Shulman was not a mathematics educator or mathematician, but rather a general educationalist. Chevallard’s approach, initially situated within the realm of mathematics education, has also demonstrated its applicability in other subjects, such as physics, chemistry, biology, language, physical education, and music, among others.

The three approaches by Klein, Shulman and Chevallard share the idea that school mathematics and university mathematics are related yet distinct entities. They also hold the merit of recognising and legitimising the specific position of teachers in relation to both. The key differences among these approaches lie in their assumptions regarding the directionality of the relationship between school mathematics and university mathematics and the extent to which it is questioned. It is towards understanding this relationship and its implications that this section is devoted.

All three approaches posit that school mathematics is derived from university mathematics, with the latter representing the actual knowledge, and the former serving as its transformed, transposed or elementarised version for pedagogical or educational use. However, the extent to which the academic discipline is considered as the exclusive starting point varies among these approaches, as does the ‘status’ attributed to it.

Klein’s approach suggests that by revealing the nature of school subjects and academic disciplines as mutually determining moments in the process of elementarisation, the underlying ideas of the academic discipline and the intellectual orientation of the school subject can be discerned. In this approach, school mathematics appears as an epistemic imitation of the underlying discipline, reflecting its basic ideas and fundamental working methods.

In contrast, Shulman (1986, 1987) explicitly asserted that the academic discipline precedes the school subject, starting from the premise that teachers need to transform their subject matter knowledge acquired at college or university (including academic mathematical knowledge) into subject matter knowledge for teaching (including school mathematical knowledge). The transition from the disciplinary content teachers learned at university to the content they teach at school is seen as the ‘central problem’ facing teachers, particularly novice ones. According to Shulman, the teacher’s orientation to the academic discipline forms the basis for restructuring content knowledge for pedagogical purposes. The school subject is a pedagogical revision of the logical knowledge of the academic discipline in relation to the cognitive development of the learner but aimed at a high degree of fidelity to the structures of the discipline.

Another version of the academic discipline preceding the school subject is presented in Chevallard’s (1991) theory of didactic transposition. This theory suggests that the knowledge taught in school is derived from a pole of scholarly knowledge and transposed to a seemingly subordinate pole in the classroom. This asymmetry is rooted in the social legitimacy attributed to the disciplines taught in school. While what is taught in school as ‘mathematics’ cannot be entirely different from what exists in society as ‘mathematics’, the scholarly institution primarily defines this subject.

Pointing out this asymmetry is part of the contribution of the theory of didactic transposition.

In each of the three approaches, there is a more or less explicit reliance on the academic discipline. Generally, these approaches leave unscrutinised the status of the academic discipline as the 'authoritative' source from which the school subject is derived, and thus in control of the substantive possibilities for the school subject. However, the approaches differ in the extent to which they question or problematise this reliance on the academic discipline as the source of school mathematics.

Shulman's approach upholds the reliance on the academic discipline as the basis for what is taught in school. This approach aligns with the 'structure of the discipline' movement of the 1960s (see Bruner, 1960; Schwab, 1964), which regarded knowledge as propositional and foundational.[5] This viewpoint recognises the value of seeking and confirming knowledge through academic disciplines as reservoirs of knowledge. It generally prioritises facts over values, perceiving knowledge as the outcome of objective inquiry. This is possibly more reflective of the positivist tradition in the philosophy of knowledge prevalent in Western societies, rather than the intrinsic nature of disciplines like mathematics.

Chevallard, in contrast, brings out the negotiable issues of curriculum purpose, content and practice. The introduction of an intermediate institution, the *noosphere*, points at the agents who intervene, the advanced reasons and the practices that are carried out to select the bodies of knowledge (and other entities like competencies, skills or attitudes) and elaborate the knowledge to be taught as specific organisations of discourses and practices structured sequentially. It also underscores the transposition work undertaken to produce the instructional resources that support the transformation of the 'knowledge to be taught' into real teaching and learning practices to produce the 'actually taught knowledge'. While scholarly (and university) mathematicians play a crucial role in this transposition work—especially in its first historical steps—they are by no means the only ones involved.

Klein's approach, on the other hand, problematises the elementary as a result of the elementarisation of mathematics. The directionality between university mathematics and school mathematics is expressed in a way that differs from Shulman and Chevallard. Klein proposed to overcome the gap between university mathematics and school mathematics by further developing university mathematics and by proposing new ways of elementarisation.

Overall, the three approaches construe the subject matter of a school subject in terms of the basic ideas, methods, and ways of thinking and knowing embedded in the academic discipline. However, they are not located in the same institutional position concerning both school mathematics and academic mathematics, as will be discussed in the next section.

4 Locating the positions of Klein, Shulman and Chevallard at different levels

In Klein's approach, both school mathematics and university mathematics are problematised by the development of an overarching mathematical framework. Shulman's approach, on the other hand, problematises teacher effectiveness in complementing subject matter content with pedagogical forms appropriate for students. It also revindicates the existence—and legitimacy—of a new entity of knowledge that exists in school institutions and is different from university knowledge, even though it is closely related to it. Chevallard's approach takes a more external perspective, problematising the institutional constraints and mutual determinations that influence the subject matter content in the transposition processes originating from various institutions: the school itself, its noosphere, and the scholarly institutions (university, research centres, professional associations, etc.). Other institutions also play a role, even if they are not always explicitly mentioned, such as those responsible for teacher education and professional development.

This section takes the frame of didactic transposition to highlight what appears as a central contrast in terms of the more or less implicitly presumed institutional position among the three approaches. We will thus take up Chevallard's approach to explore the other two approaches, which can be seen as the starting point of a dialogue between the three approaches (Bikner-Ahsbahs & Prediger, 2014; Bosch et al., 2017; Scheiner, 2020) that will remain in progress.[6]

Let us consider the process of didactic transposition with its different entities and institutions: the 'scholarly knowledge' corresponding to the academic discipline or university mathematics, the 'knowledge to be taught' as delineated and elaborated by the noosphere, and the 'taught knowledge' implemented in schools (see Fig. 1).

[5] Bruner (1960) and Schwab (1964) had different understandings of structure, but they defined the subject matter to be taught in school in terms of concepts, principles, methods and habits of thought derived from the academic discipline. Bruner (1960) characterised structure in terms of fundamental disciplinary ideas, concepts and relationships. Schwab (1964) characterised structure in terms of the organisation of a discipline, and the substantive and syntactic structure of a discipline.

[6] In this regard, it is worth mentioning Travers and Westbury (1989), who compared and contrasted mathematics curricula in twenty countries and educational systems, focusing particularly on what mathematics is to be taught, what mathematics is actually taught, how this mathematics is taught and what mathematics is learned in these different countries.

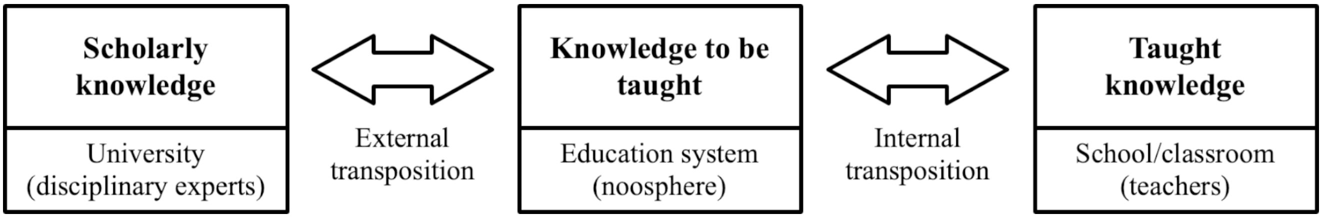

Fig. 1 The process of didactic transposition

In Chevallard's approach, the object of study is the didactic transposition process itself, the transposition work done to elaborate the 'knowledge to be taught' and the constraints it places on the elaboration of the knowledge taught by teachers and learned by students (see Fig. 2). Tools are needed to describe and problematise the different forms of knowledge that occur in the didactic transposition process. For this reason, the theory of didactic transposition has been further developed into what is now known as the 'anthropological theory of the didactic' (see Bosch et al., 2020; Bosch & Gascón, 2014; Chevallard et al., 2022). Unlike other approaches, the anthropological theory of the didactic provides a general epistemological model to describe human activities taking place in social institutions and the knowledge they produce and activate. This model is mainly based on the notion of *praxeology*, the inseparable union of a *praxis* or know-how and a *logos* or discourse to describe, explain, organise and justify the praxis (for an overview, see Chevallard & Bosch, 2020a). The analysis in terms of praxeologies is then used to describe the different elements of the didactic transposition process: the mathematical knowledge taught in school, the mathematical knowledge to be taught and the related scholarly knowledge.

Klein's approach, on the other hand, can be located within the scholarly institution or between the scholarly institution and the noosphere (see Fig. 2). Elementarisation provides connections between school mathematics and university mathematics in the preparation of secondary mathematics teachers. It is part of the transposition work undergone to overarching university mathematics and school mathematics in the elaboration of the knowledge to be taught. This work is done by scholars or disciplinary experts, not by the (prospective) teachers themselves, who are expected to learn elementary mathematics from a higher standpoint. In this process, and in contrast to the other two approaches, the relationship between scholarly knowledge and school knowledge is a reciprocal one: both school knowledge and scholarly knowledge are problematised, questioned and reconstructed.

Shulman's approach highlights a singular transpositive work carried out by individual teachers, in which they transform disciplinary subject matter into subject matter for teaching. This underscores the critical role teachers play in making disciplinary knowledge accessible to students. Unlike other approaches, Shulman's approach does not posit an intermediary entity in this transformation process, which spans from (university) disciplinary knowledge to the knowledge taught in the classroom (see Fig. 2). Instead, with the introduction of pedagogical content knowledge, Shulman pointed at the existence of a unique kind of subject matter knowledge, distinct from the subject matter knowledge per se, namely "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9). This form of knowledge is held by individual teachers and actualised within the school or classroom context.

5 Conclusion

The selection, organisation and transformation of knowledge into the subject matter of a school subject is always a form of social, cultural and economic reproduction (e.g., Gispert & Schubring, 2011), and any hegemony of disciplinarity is always closely intertwined with issues of ideology, privilege and power (Bourdieu, 1992). Thus, the recourse to university (or scholarly) mathematics as the basis for school mathematics can reinforce the 'disciplinary power' (Foucault, 1972) of university mathematicians and educators, with university mathematics setting the boundaries for school mathematics. However, this view risks reinforcing established content knowledge without critical examination, potentially leaving both university mathematics and school mathematics unchallenged.

The approaches by Klein, Shulman and Chevallard acknowledge the central role our societies ascribe to the academic discipline in determining school content but also identify forms of knowledge that are specific to educational institutions and teaching processes, attributing a particular status and recognition to them. Shulman's concept of pedagogical content knowledge highlights the distinct professional knowledge necessary for effective teaching, which involves transforming disciplinary knowledge into pedagogical forms that are accessible to students. Klein's approach focuses on developing the transposition work that links scholarly knowledge to the knowledge to be taught

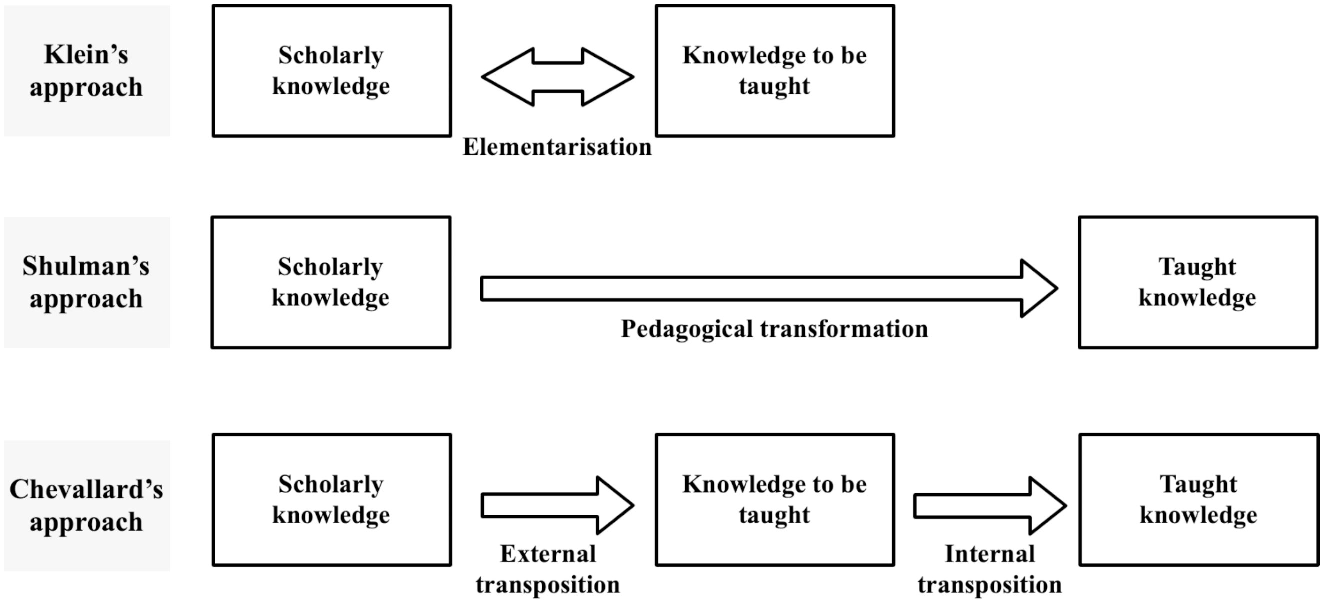

Fig. 2 Positioning Klein, Shulman and Chevallard along different levels

through processes of elementarisation. It is considered part of the external didactic transposition process (Bosch & Winsløw, 2020), with a focus on meeting the needs of teachers. Finally, Chevallard's approach provides a framework for understanding the different statuses of knowledge that emerge in the transposition process, as well as the roles played by various institutions and their hierarchies. It also illuminates how the specific forms of knowledge to be taught resulting from the transposition process condition teaching and learning processes. To reorganise or challenge these forms of knowledge, collective efforts and institutional recognition are necessary. Taken together, these approaches offer valuable insights into the complex relationship between university mathematics and school mathematics, and into the specific needs of teachers who have to deal with it, thus contributing to a more nuanced understanding of how to approach this critical issue.

A central challenge and opportunity in university mathematics teacher education is to expand beyond the academic discipline as the sole epistemic foundation for the school subject (for a discussion, see Stengel, 1997). It is important for researchers, educators and teachers to move beyond the scholarly view of mathematics and consider what mathematics is important to be taught in school and why. Although many have used the approaches of Klein, Shulman or Chevallard to think about the academic discipline and the school subject and their relationship, there is an opportunity to re-examine these approaches to recognise that what counts as subject matter in school is not only a logical and epistemic question but also a normative, ethical and socio-political one.

Indeed, the integrity of the academic discipline should not overshadow what and how students should learn. While scholarly mathematics plays a crucial role in legitimising the knowledge to be taught, it should not dictate how school mathematics is defined. A broader, reconstructive educational analysis is needed to investigate how the different social institutions—including scholarly mathematics—intervene in the didactic transposition process to shape school mathematics and determine what mathematics is done at school and why. Therefore, any approach, including those of Klein, Shulman and Chevallard, should be scrutinised to ensure that they provide the necessary epistemological and pedagogical tools to question all forms of knowledge and practices that define educational projects. University mathematics courses for secondary teacher preparation are a crucial arena for this endeavour (for additional insights, see Wasserman et al., 2023).

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