Enharmonic motion:

Towards the global dynamics of negative delayed feedback

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Abstract

In this thesis, we establish a new method for describing the qualitative dynamics of the so-called Hopf–Smale attractors in scalar delay differential equations with symmetric negative delayed feedback.

The dynamics of Hopf–Smale attractors are robust under regular perturbations. Qualitatively, the attractor consists of an equilibrium, periodic orbits, and connections between them. We describe the mechanism that produces the periodic orbits and show how their formation creates new connecting orbits via sequences of Hopf bifurcations. As a result, we obtain an enumeration of all the phase diagrams, that is, the directed graphs encoding the equilibrium and periodic orbits as vertices and the connections as edges.

In particular, we have obtained a prototype, the so-called enharmonic oscillator, that realizes all Hopf–Smale phase diagrams. Besides describing the Hopf–Smale attractors, our method also sheds insight into the formation process of certain global attractors with positive delayed feedback.

Contents

Chapter 1

Introduction

1.1 Self-regulation

Delayed negative feedback is a widespread method for **self-regulation** in real-world processes. It involves a control mechanism that counteracts the deviation of a dynamical variable from a baseline value or reference state. Within this setting, the system often overcompensates due to a time lag in the control mechanism, resulting in sustained oscillations.

Intuitively, picture a jet pilot steering a jet fighter. **Pilot-induced oscillations** arise from the interplay between the quick response of the plane and the comparatively slow reaction time of a human pilot who tries to keep the plane aligned with the runway; see [McR95]. In a different timescale, the same phenomenon emerges in a regular car whose driver has a slower reaction time than a well-trained pilot. The reader may experiment with this give-and-take between system dynamics and delayed control risk-free and first-person. It is why a longer stick is easier to balance on the tip of their finger.

In this work, we discuss mathematical models of systems with delayed self-regulation that satisfy three assumptions:

(A1) The system is scalar and possesses one discrete delay only. In other words, we model the change in time of a single real-valued variable denoted by $x(t)$, and we assume that the derivative $\dot{x}(t)$ is a differentiable function of $x(t)$ and $x(t-1)$, only.

(A2) The system has **negative delayed feedback**, i.e., the control mechanism counteracts the deviation of $x(t-1)$ from the baseline value. We model this by assuming that $\dot{x}(t)$ is a monotonically decreasing function of the delayed state $x(t-1)$ of the system.

(A3) The feedback strength is **symmetric**. In other words, the strength of the control term depends on the size of the deviation from the baseline, $(|x(t)|, |x(t-1)|)$, only.

In the pilot-induced oscillations example above, the quantity $x(t)$ in (A1) represents the angle by which the plane's trajectory deviates from following a straight line. The negative feedback (A2) corresponds to the pilot turning more sharply as the deviation increases. Finally, (A3) applies to ambidextrous pilots in perfectly symmetric environments that cannot distinguish between left and right.

Mathematically, we model any quantity $x(t)$ satisfying (A1)–(A2) as a solution of a delay differential equation (abbr., DDE) of the form

$$\dot{x}(t) = f(x(t), x(t-1)). \tag{1.1}$$

Here the nonlinearity $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function whose derivative in the second component satisfies $\partial_2 f(\xi, \eta) < 0$ for all $(\xi, \eta) \in \mathbb{R}^2$. Furthermore, we enforce (A3) by assuming the even-odd symmetry $f(\xi, \eta) = f(-\xi, \eta) = -f(\xi, -\eta)$ for all $(\xi, \eta) \in \mathbb{R}^2$.

The assumptions (A1)–(A2) lie at the core of this pilot-vehicle interaction and often appear in applied sciences. Their use is especially prominent in biology, where delays often appear as response times in the self-regulation of a biological or chemical species presenting self-inhibition. Hutchinson [Hut48] already pointed out that oscillations appear due to delay effects in the equation

$$\dot{x}(t) = \lambda x(t)(1 - x(t-1)), \quad \lambda > 0. \tag{1.2}$$

Here (1.2) satisfies (A1)–(A2) for all biologically relevant solutions. Originally, (1.2) was proposed as a model for population dynamics where the time delay and negative feedback appear as a result of the competitive advantage of experienced individuals over younger ones in the intraspecies fighting of certain rodent populations.

Later, Lasota and Wazewska [WL76] interpreted the delays in erythrocyte production as a result of the maturation time during the formation process in the bone marrow. They proposed the DDE

$$\dot{x}(t) = -\lambda_1 x(t) + \lambda_2 \mathrm{e}^{-x(t-1)}, \quad \lambda_j > 0,$$

modeling the concentration of red blood cells, where negative feedback appears due to the observable self-regulation process by which a healthy human body stimulates erythrocyte production after a sudden drop in population. Simultaneously, Mackey and Glass [MG77] considered the class

$$\dot{x}(t) = -\lambda_1 x(t) + \frac{\lambda_2}{\lambda_3 + (x(t-1))^n}, \quad \lambda_j > 0, \tag{1.3}$$

satisfying (A1)–(A2) for all positive solutions. In their model, $x(t)$ represents the concentration of a population of blood cells, and the delays appear due to cellular maturation times. Furthermore, they postulated that the oscillations arising as the delay increases were a potential indicator of hematopoietic disease processes.

More recently, Lewis [Lew03] has proposed autoinhibitory delayed effects as a fundamental agent in the vertebrate somitogenesis oscillator. In mammals [YKMN$^+$20], delayed negative feedback arises in two different ways. At an intracellular level, delays appear due to mRNA transcription and are fundamental for self-regulation in the periodic expression of the Hes7 gene. At an extracellular level, delays appear through lags in the Notch signaling pathways used for cell-to-cell synchronization in the Hes7 expression. In either case, a nonlinearity of Mackey–Glass type (1.3) governs the quantity of functional Hes7 protein. Takashima et al. [TOG$^+$11] have tested the influence of delays in the segmentation clock empirically. They have shown that an effective reduction in the intracellular RNA transcription times via intron removal within the Hes7 gene produces a faster ticking of the segmentation clock. The result is the development of individuals presenting too many vertebrae.

Further applications of delayed negative feedback to cellular dynamics include the linear and solvable DDE

$$\dot{x}(t) = \lambda(x(t) - x(t-1)), \quad \lambda > 0. \tag{1.4}$$

The dynamics of (1.4) resemble the mechanism regulating the concentration of calcium ion levels within the sarcoplasmic reticulum in a cardiomyocyte; see [Tho13]. Here the delay appears due to the recovery time of the ryanodine receptors.

The linear equation (1.4) also plays a vital role in modeling epidemics; see [Del20]. Particularly, a delay in the recovery time of an infected patient in a compartmental SIR epidemic model allows for replacing a three-dimensional nonlinear ODE with the infinite-dimensional but solvable DDE (1.4).

In environmental sciences, Suárez and Schopf [SS88] have proposed the delayed action oscillator

$$\dot{x}(t) = \lambda_1 \left(x(t) - (x(t))^3 \right) - \lambda_2 x(t-1), \quad \lambda_j > 0,$$

as a model for the El Niño Southern Oscillation phenomenon. In particular, they suggested that a negative delayed feedback loop drives the temperature $x(t)$ in the Central Pacific region. Such interaction appears as the West-East oceanic Kelvin waves weaken the East-West atmospheric currents by raising the air temperature of the Central Pacific region. Time delays are a result of the traveling times of the Kelvin waves, usually in the order of months.

In economics, price fluctuations for commodities emerge as a result of the negative delayed feedback in

$$\dot{x}(t) = \lambda \left(x(t) - x(t-1) \right) - |x(t)| x(t), \quad \lambda > 0.$$

A delay arises as the market responds to changes in the supply, and a negative feedback loop appears as the higher price stimulates production. In turn, the increase in the supply makes the price sink; see [BEW04].

Finally, the DDE (1.1) arises in analyzing pure mathematical problems. At around the same time as Hutchinson, Wright [Wri55] used the Hutchinson equation (1.2) to discuss the asymptotic sparsity of the prime number distribution.

The solutions of some ordinary differential equations (abbr. ODEs) having spatiotemporal symmetries also solve (1.1). For instance, the traveling wave solutions of the one-directional lattice ODE

$$\dot{x}_j = f(x_j, x_{j+1}), \quad j \in \mathbb{Z}, \tag{1.5}$$

satisfy the DDE (1.1); see [MP99]. The linear chain trick [Smi11] uses a lattice equation of the type (1.5) to represent the dynamics of a single DDE with distributed delay given by a gamma-distribution type kernel. Loos et al. [LHK21] applied this approach successfully to model a single stochastic colloidal particle under the influence of a control kernel with memory. They concluded that a linear DDE

$$\dot{x}(t) = \lambda_1 x(t) - \lambda_2 x(t-1)$$

characterizes the equilibrium stability at infinite memory capacity.

In contrast to assumptions (A1)–(A2), (A3) rarely appears in the literature. The role of (A1)–(A2) in the discussion of the DDE (1.1) is connected to the structural stability of the dynamics, that is, the resistance to perturbations. However, (A3) enables the most powerful tools we use throughout this work to accurately describe such robust dynamics. Moreover, although (A3) is not explicitly connected to previous academic work, it often appears after taking regular limits for the equations considered above. For example, (A3) holds for the Mackey–Glass equation (1.3) with $\lambda_1 = \lambda_3 = 0$ and $n = 3$. Thus our framework is broader than it seems thanks to the structural stability inherited from (A1)–(A2).

1.2 Long-term dynamics

We are interested in describing the long-term behavior of the solutions to the DDE (1.1). Following [HVL93], we regard (1.1) as the infinitesimal recipe for constructing a one-parameter family of transformations $S_f(t)$ with $t \geq 0$ called the **semiflow**. The semiflow $S_f(t)$ transports bits of information, or **initial conditions**, ϕ living on a yet-to-be-specified Hilbert space H. Any differential equation generating such a notion of semiflow, which mimics the action of time, is called an **evolution equation**; see [Lad91]. Further examples include ODEs, for which $H = \mathbb{R}^n$, and parabolic partial differential equations (abbr. PDEs), with a suitable choice of a Sobolev space H. Specifically for DDEs, the phase space H often goes by the name of **history space**.

We assume that the semiflow is **dissipative**, that is, for all initial conditions $\phi \in H$, the evolution $S_f(t)\phi$ enters a uniformly bounded ball in the history space H as $t \to \infty$. In this setting, there exists a maximal compact semiflow-invariant subset $\mathcal{A}(f) \subset H$ known as the **global attractor** of (1.1); see [HMO02, Lad91]. The most important property of $\mathcal{A}(f)$ is that it *attracts* all initial conditions $\phi \in H$ as $t \to \infty$. Thus, $\mathcal{A}(f)$ contains all of the *observable states* of the real-world processes governed by the negative delayed feedback DDE (1.1). Still, not all the invariant subsets of $\mathcal{A}(f)$ are attracting; some elements are often repelling and not observable in practice. Nevertheless, the repelling sets are essential because they play the role of transition states, thereby determining the basins of stability within which the stable sets attract.

The existence of a global attractor $\mathcal{A}(f)$ is a widespread phenomenon in dissipative evolution equations. However, its mathematical construction is abstract and conveys no detail about the complexity of the dynamics on $\mathcal{A}(f)$. In absence of (A1)–(A2) complicated dynamics arise quickly and $\mathcal{A}(f)$ can contain chaotic dynamics even in the simple-looking equation

$$\dot{x}(t) = -\lambda \sin(x(t-1)), \quad \lambda > 0, \quad x(t) \in \mathbb{R}/2\pi\mathbb{Z},$$

modeling the phase difference of two identical Kuramoto oscillators with symmetric coupling; see [LWS02].

However, (A1)–(A2) provide additional structure and vastly simplify the dynamics within the attractor. Thanks to a **Poincaré–Bendixson theorem** [MPS96a, KW01], there exists a graph $\Gamma(f)$ called **phase diagram** that describes the dynamics on the global attractor $\mathcal{A}(f)$ of the DDE (1.1) satisfying (A1)–(A2). Typically, the vertices of $\Gamma(f)$ correspond to **critical elements**, i.e., the closed $S_f(t)$-orbits in H. Denoting the set of critical elements by $\mathrm{Crit}(f)$, $\Gamma(f)$ contains the directed edge $(\gamma^\dagger, \gamma^*)$ for $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$ if and only if there exists an $S_f(t)$-orbit connecting $\gamma^\dagger$ to γ^*. In other words, if there exists $\phi \in H$ satisfying

$$\gamma^\dagger \xleftarrow{t\to-\infty} S_f(t)\phi \xrightarrow{t\to\infty} \gamma^*.$$

Most examples in the literature describe the global attractor of (1.1) under the **positive delayed feedback** assumption $\partial_2 f(\xi, \eta) > 0$ for all $(\xi, \eta) \in \mathbb{R}^2$. However, such systems share the Poincaré–Bendixson theorem of negative delayed feedback systems; see [MPS96a, KW01]. In particular, $\mathcal{A}(f)$ also admits a phase diagram description $\Gamma(f)$ in the positive delayed feedback setting. Two examples have heavily influenced this work:

1. **Spindle attractors**, that is, topological spheres on which the dynamics consist of a single *center* equilibrium from which solutions emanate

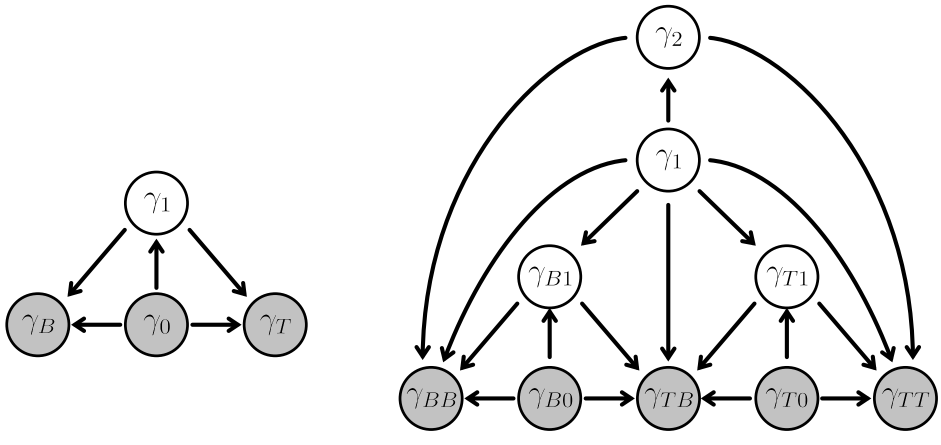


Figure 1.1: (Left) Phase diagram of a three-dimensional spindle possessing three equilibria (grey) and one periodic orbit (white). (Right) The phase diagram of the Vas attractor consists of five equilibria (grey) and four periodic orbits (white).

and connect to a family of nontrivial periodic solutions lying on an equatorial region and two *sink* equilibria at the poles each one of which attracts all solutions lying on the North and South hemisphere, respectively; see [KWW99b, KWW99a] and Figure 1.1 (Left).

2. The **Vas attractor**, first discussed in Vas' doctoral work [Vas11], that consists of two three-dimensional spindles as in Figure 1.1 (Left) glued by the tips and enveloped by a superstructure formed by two large-amplitude periodic solutions γ_1 and γ_2 oscillating around the equator; see also [KV11] and Figure 1.1 (Right).

The relative simplicity of the phase diagram representation is a consequence of a discrete Lyapunov function or **zero number** developed by Mallet-Paret, Sell, and others; see [MPS96b, Cao90, MP88, Mys55]. Analogous versions of the zero number exist for both negative and positive delayed feedback systems. Essentially, they provide an abstract quantized entropy function for the difference of any two solutions to (1.1). The zero number only takes integer values and yields three critical properties:

(P1) A **Poincaré–Bendixson theorem** as mentioned above; see [MPS96a, Theorem 2.1]. This guarantees that solutions $x(t)$ accumulate to planar sets in the long time limit $t \to \infty$. We highlight that only the limiting behavior of the solutions is planar; in general, the dynamics on $\mathcal{A}(f)$ have a higher dimensionality, as showcased both by spindles and the

Vas attractor above.

(P2) An eigenvalue structure resembling the **Sturm–Liouville eigenvalue structure** in linear second-order ODEs; see [MPN13, MPS96b].

(P3) A characterization of the **structural stability** of $\mathcal{A}(f)$ in terms of local stability properties of the critical elements $\mathrm{Crit}(f)$. This is a consequence of the appearance of exponential dichotomies due to (P1)–(P2); see [LN17].

The zero number is not exclusive to monotone delayed feedback systems. Analogous objects, also known as **nodal properties**, appear in completely unrelated settings. We highlight the work of Mallet-Paret and Smith [MPS90] in **monotone cyclic feedback systems**, Fusco and Oliva [FO87] in **Jacobi systems**, as well as Angenent and Matano [Ang88, Mat82] in **scalar reaction-diffusion** PDEs with compact one-dimensional domains

$$\partial_t u(t,\xi) = \partial_\xi^2 u(t,\xi) + h(\xi, u(t,\xi), \partial_\xi u(t,\xi)), \quad t > 0,\ \xi \in [0,L]. \tag{1.6}$$

Thus properties (P1)–(P3) are not specific to DDEs; instead, they seem to be a mere consequence of the existence of an abstract notion of zero number; see [Ter94, Sá09].

Akin versions of the Poincaré–Bendixson theorem (P1) exist for monotone cyclic feedback systems [MPS90] and reaction-diffusion on circular topologies [FMP89]. The asymptotic dynamics become even lower-dimensional when considering Jacobi systems [FO87] for which the only possible accumulation points of the dynamics are equilibria. The same holds for reaction-diffusion PDEs with separate boundary conditions, for which Matano and Zelenyak [Mat88, Zel68] showed the existence of real-valued Lyapunov functions. Recently, Lappicy and Fiedler [LF19] have extended Matano's gradient structure to fully nonlinear parabolic PDEs.

On the other hand, the eigenvalue structure (P2) results from a generalized Krein–Rutman theory distinctive of any evolution process possessing a zero number; see [FO91, PT93, MPN13]. Having (P2) is vital in proving the structural stability property (P3), a defining feature of the so-called **Morse–Smale systems**; see [Pal69, PdM82, Oli00]. In their independent pioneering work, Henry [Hen85] and Angenent [Ang86] proved the Morse–Smale property for parabolic PDEs (1.6) under separate boundary conditions. Later, Hale et al. [CCH92] extended the proof to time-periodic PDEs with separate boundary conditions. More recently, Czaja and Rocha [CR08] proved that the result still holds for scalar reaction-diffusion systems on circular topologies. Furthermore, the Morse–Smale property is *typical* for the PDE (1.6) in a topological sense. More precisely, Joly and Raugel [JR10a, JR10b] showed that the attractor $\mathcal{A}^{\mathrm{PDE}}(h)$ of (1.6) possesses the Morse–Smale property for a **generic** family of nonlinearities h in the Baire category sense. For comparison, we highlight that the set of Morse–Smale ODEs is not generic; see [Sma66]. The same ideas as in the PDE setting have been employed successfully to show structural stability properties for the ODE examples above [FO87, FO90, WZ17] and similar techniques yield the DDE variant [LN17].

This similarity between systems possessing zero numbers hints at a deep connection relating monotone delayed feedback DDEs (1.1) to parabolic PDEs (1.6). However, a grand theory encompassing all the known examples remains undiscovered. The first step in this direction is to understand specific cases better. In that sense, the better-understood class of equations is reaction-diffusion PDEs (1.6) under separate boundary conditions, that is, in the gradient regime of Matano and Zelenyak [Mat88, Zel68]. In this setting, all nonstationary solutions on the global attractor $\mathcal{A}^{\mathrm{PDE}}(h)$ correspond to connections between different equilibria $\partial_t u(t,\xi) \equiv 0$. The equilibria form an ordered set with nonstationary solutions only flowing from top to bottom.

Under Neumann boundary conditions, the equilibria in (1.6) arise by solving the ODE boundary value problem

$$(1.7)\quad \partial_\xi^2 u(0,\xi) = -h(\xi, u(0,\xi), \partial_\xi u(0,\xi)), \quad \partial_\xi u(0,0) = \partial_\xi u(0,L) = 0.$$

Surprisingly, Fusco and Rocha [FR91] showed that the complete connectivity web between the equilibria in (1.6) is determined by the shooting curve resulting from solving (1.7) for *time* $\xi = L$. Thus the shooting curve obtained in this way determines a **meander permutation**, which encodes a sequence of local bifurcations. The phase diagram $\Gamma^{\mathrm{PDE}}(h)$ of $\mathcal{A}^{\mathrm{PDE}}(h)$ can then be recovered thanks to the Morse–Smale property (P3) above [FR96]. The result is a complete, well-organized classification of the **Sturm attractors** $\mathcal{A}^{\mathrm{PDE}}(h)$ up to topological orbit conjugacy [FR00]. In this way, Sturm attractors form a mathematical phylogenetic tree whose complexity increases quickly as the number of equilibria grows. The meander permutations obtained this way do not yield unique phase diagrams. However, Fiedler and Rocha [FR18a, FR18b, FR18c] have since pushed the study of Sturm attractors well beyond the graph structure $\Gamma^{\mathrm{PDE}}(h)$ by showing that the meander permutations are an enumeration of a finer structure called **signed hemisphere decomposition**.

Finally, we point out that the theory of Neumann-type Sturm attractors above has been extended to more complex settings. Nontrivially, under S^1-equivariance assumptions, it is possible to *freeze* all the rotating waves appearing for periodic boundary conditions and reduce the description of the global attractor to the case with separate boundary conditions; see [FRW04, FRW12b, FRW12a]. Further generalizations, following related techniques, have been made to describe unbounded global attractors in the slowly nondissipative regime [Pim16, PR16], and quasilinear [LP18, Lap20] and fully nonlinear [Lap22] cases.

1.3 Goal and method

The purpose of this thesis is the development of a systematic method to design global attractors $\mathcal{A}(f)$ of the DDE (1.1) satisfying the symmetric negative delayed feedback assumptions (A1)–(A3). We achieve this by applying the methods developed over the last decades for Sturm attractors in reaction-diffusion PDEs (1.6) and applying them to negative delayed feedback DDEs (1.1). Our main achievement is developing a new constructive method to obtain an infinite family of DDE phase diagrams $\Gamma(f)$, the vast majority of which were previously unknown.

The vertex set $\mathrm{Crit}(f)$ of a phase diagram $\Gamma(f)$ obtained by our method consists of a single equilibrium, denoted γ_0, and a family of periodic orbits $\gamma_1, \ldots, \gamma_N$. We construct the edges in $\Gamma(f)$ through a homotopy, i.e., a smooth deformation f_λ of nonlinearities satisfying (A1)–(A3) for all $\lambda \in [0,1]$ and such that:

(i) The attractor of f_0 satisfies $\mathcal{A}(f_0) = \gamma_0$, and $f_1 = f$.

(ii) The periodic orbits $\gamma_1, \ldots, \gamma_N$ appear through a finite number of nondegenerate **Hopf bifurcations**, only.

By the Morse–Smale property (P3), the structure of $\Gamma(f)$ can only change at a finite number of parameter values or **Hopf points** at which the equilibrium γ_0 of (1.1) either *absorbs* or *emits* a single **bifurcating periodic orbit**. This emission-absorption mechanism happens in two distinct ways:

Subcritically: if there exists an obit connecting the bifurcating periodic solution to γ_0.

Supercritically: if there exists an orbit connecting γ_0 to the bifurcating periodic orbit.

In this way, a combinatorial structure arises. More precisely, the admissible sequences of Hopf bifurcations producing N periodic orbits enumerate the phase diagrams that can be obtained by this method. The Morse–Smale attractors $\mathcal{A}(f)$ that can be built by this process consisting exclusively of Hopf bifurcation are called **Hopf–Smale attractors**.

Although our results apply to Hopf–Smale attractors in general, the subclass of **enharmonic oscillators** realizes all possible Hopf–Smale phase diagrams. We call enharmonic oscillators to the DDEs satisfying assumptions (A1)–(A3) that have the special form

$$\dot{x}(t) = -\frac{\pi}{2}\Omega\left(\sqrt{(x(t))^2 + (x(t-1))^2}\right)x(t-1). \tag{1.8}$$

Here $\Omega > 0$ is a nonlinear **frequency function** depending on the amplitude $\sqrt{(x(t))^2 + (x(t-1))^2}$, only. We will show that all of the periodic solutions $x^*(t)$ of (1.8) are of harmonic type, that is, they satisfy

$$x^*(t) = a\sin\left(\frac{\pi}{2}\Omega(a)t + t^*\right), \quad a,\, t^* > 0,\ \Omega(a) = 4n-3,$$

for some $n \in \mathbb{N}$. In other words, all periodic solutions are *harmonic* with amplitude a and integer frequency of the form $4n-3$. Hence the choice of the term **enharmonic** borrowed from music theory. A sound is said to possess an enharmonic equivalent whenever it admits more than one spelling on the score. Just like in a keyboard E sharp and F produce the same sound, both the DDE (1.8) and the standard harmonic oscillator described by Hooke's law

$$\ddot{\xi} = \lambda\xi, \tag{1.9}$$

possess harmonic solutions. However, the periodic solutions (1.8) are the offspring of self-inhibition, while (1.9) is a conservative system and loses any periodic behavior in the presence of friction. We have coined the name enharmonic oscillator to highlight this analogous property of two completely disconnected evolution equations. We think of (1.8) as a figurative enharmonic equivalent of (1.9).

This thesis discusses three topics to prove the method above:

1. **Transverse intersections of invariant manifolds**

   Given a critical element γ^* of the DDE (1.1), the set of initial conditions $\phi \in H$ such that $S_f(t)\phi$ approaches γ^* in positive time direction is a $S_f(t)$-invariant manifold called **stable manifold** $W^{\mathrm{s}}(\gamma^*)$. Likewise, the **unstable manifold** is the set $W^{\mathrm{u}}(\gamma^*)$ containing all initial conditions that approach γ^* as $t \to -\infty$.

   Given two different $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$, it was shown in [LN17] that if both $\gamma^\dagger$ and γ^* are **hyperbolic**, their invariant manifolds intersect **transversely**. In other words, either the intersection $W^{\mathrm{u}}(\gamma^\dagger) \cap W^{\mathrm{s}}(\gamma^*)$ is empty, or the sum of the tangent spaces $T_\phi W^{\mathrm{u}}(\gamma^\dagger) + T_\phi W^{\mathrm{s}}(\gamma^*)$ spans the phase space H at any point of intersection. The method of proof follows Hale et al. [CCH92] and uses an **Oseledets-type filtration** [Mañ83] together with the zero number [MPS96b] to construct complementary subspaces within $T_\phi W^{\mathrm{u}}(\gamma^\dagger)$ and $T_\phi W^{\mathrm{s}}(\gamma^*)$.

   We refine and complete the results in [LN17] by showing:

   - The transverse intersection property still holds at Hopf points, i.e., if one of the critical elements involved is γ_0 and it is nonhyperbolic.
   - If all the equilibria and periodic orbits of the DDE (1.1) are hyperbolic, then (1.1) is a Morse–Smale system. Hence the global attractor $\mathcal{A}(f)$ is orbitally structurally stable.

   Proving that (1.1) is Morse–Smale requires a full characterization of the **nonwandering set**, i.e., the set of recurrent dynamics of (1.1). We do this by using a zero number argument that goes back to Mallet-Paret [MP88]. More recently, Shen et al. [SWZ21] have used a similar argument to describe the nonwandering set of reaction-diffusion systems that are not known to be gradient.

2. **Periodic solutions and local stability**

   So far, a main obstacle in discussing $\mathcal{A}(f)$ was the lack of a method to describe the formation process of periodic solutions in (1.1). A particularly cumbersome problem was the discussion of multiple periodic orbits oscillating with the same frequency.

   We have solved this issue by showing that if (1.1) satisfies (A1)–(A3), then the curve $(\xi(t), \eta(t)) := (x^*(t), x^*(t-1))$ solves the planar ODE

$$\begin{aligned} \dot{\xi} &= f(\xi, \eta), \\ \dot{\eta} &= -f(\eta, \xi), \end{aligned} \tag{1.10}$$

for all periodic solutions $x^*(t)$ of the DDE (1.1). Thus, in the flavor of [Sch90], the periodic solutions of (1.1) can be identified with certain level sets of the **period map** p_f taking the amplitude of the periodic solutions to their minimal period in the **reference ODE** (1.10). Our results are the converse to those of Kaplan and Yorke [KY74, KY75], showing that the periodic solutions of (1.10) translate into DDE periodic solutions via the existence of spatiotemporal symmetry. Furthermore, we also complete the results of Nussbaum [Nus79], and Cao [Cao96], who showed that all slowly oscillating periodic solutions of the DDE (1.1) satisfy the ODE (1.10) under further convexity assumptions. Our fundamental tool is a planar projection developed by Mallet-Paret and Sell [MPS96a] that maps DDE orbits onto $\mathbb{R}^2$ homeomorphically. Since both negative and positive delayed feedback DDEs share this property, the discussion in this thesis is a specialized version of the results in [LN20].

Additionally, we show that the local stability properties of the periodic orbits γ^* are encoded in p'_f. This characterization of the periodic orbits and their local stability in terms of a time map is surprisingly reminiscent of the situation in reaction-diffusion systems on the circle; see [FRW04].

The enharmonic oscillator (1.8) has the convenient feature that the period map is explicitly given by

$$p_f(a) = \frac{4}{\Omega(a)}.$$

This gives us complete control over the appearance of periodic orbits in (1.8), enabling our analysis.

3. **Hopf bifurcation analysis**

   The phase diagram $\Gamma(f)$ changes at Hopf points only thanks to the Morse–Smale property (P3) above. Furthermore, it is possible to track the changes that local Hopf bifurcations have on the global structure of the attractor. The two key tools are:

   - Center manifold theory to guarantee a connection between the equilibrium γ_0 and the bifurcating periodic orbit.
   - A principle of transitivity of connections to show that a connection between two critical elements can only vanish if it proxies through γ_0 at a Hopf point.

The conclusion will be that a $\{-1, 1\}$-sequence denoted $\chi(f)$ records all the changes in $\Gamma(f)$. Roughly, $\chi(f)$ encodes the *shape* of the period map p_f. We call $\chi(f)$ the **signature** of f because it is analogous, but not equal, to the **Rocha signature** used by Rocha [Roc07] to classify period maps of planar Hamiltonian systems. In particular, counting the number of Hopf–Smale diagrams reduces to counting all admissible signatures.

1.4 Outline

The previous sections already constitute a rough sketch of our main achievements and their relation to the current literature. Casual readers are recommended to skip ahead to Chapter 7. There we show which signatures produce Hopf–Smale attractors and discuss how to reconstruct the phase diagram from the signature. More curious readers are welcome to read the intermediate chapters.

Chapters 2 and 3 contain the technical fundamentals used throughout the rest of the thesis. In contrast, Chapters 4 through 6 have the proofs of the main results. Finally, Appendices A through C discuss generalities on differential geometry and dynamical systems. We include them for completeness.

In Chapter 2, we introduce the basic mathematical setting for treating DDEs. Our approach deviates from the canon by introducing a slightly unconventional phase space. We present the global attractor and define the zero number, a distinctive property of scalar equations with negative delayed feedback. Towards the end, we derive some essential properties of periodic solutions and present the Sturm–Liouville eigenvalue structure.

Chapter 3 characterizes the invariant manifolds of the critical elements γ^*. The most crucial point is that we characterize the tangent spaces in terms of asymptotic convergence rates along the variational flow. Our approach uses standard invariant manifold theory for compact maps; see [HPS77].

Chapters 4, 5, and 6 deal respectively with the Morse–Smale structure generated by the semiflow, the two-dimensional reduction for finding critical elements, and the appearance of connections through Hopf bifurcations.

Chapter 7 concludes the thesis with a summary of our methods that includes a collection of examples. Furthermore, we discuss extending our results to positive delayed feedback systems and present a selection of open problems.

Chapter 2

Technical fundamentals

This chapter contains the basic framework for all our results. All of the content that we present here is either well-known in the literature or is a rearrangement in such a way that it is useful for the remainder of the thesis.

In Section 2.1, we define the semiflow $S_f(t)$ mentioned in Chapter 1 and set the phase space H. Our approach uses a Sobolev-type Hilbert space H rather than the traditional space of continuous functions equipped with the uniform norm $C := (C^0([-1,0],\mathbb{R}), \|\cdot\|_{C^0})$ used by Hale and Lunel [HVL93] and Diekmann et al. [DvGVLW95].

In Section 2.2, we define the global attractor $\mathcal{A}(f)$ as the set that contains all the eternal solutions of the DDE (1.1). Our approach follows the general dynamical systems theory for dissipative compact semiflows by Ladizhenskaya [Lad91] and Hale et al. [HMO84, HMO02].

In Section 2.3, we present the theory developed by Mallet-Paret and Sell [MPS96b, MPS96a] for cyclic monotone delayed feedback systems. We focus on the so-called zero number and a Poincaré–Bendixson theorem that vastly simplifies the asymptotic dynamics of the DDE (1.1). This motivates the existence of the phase diagram $\Gamma(f)$ that we announced in Chapter 1 and endows the periodic solutions of (1.1) with unique properties that we will use extensively in Chapter 5.

Finally, in Section 2.4, we discuss a Sturm–Liouville eigenvalue structure for the linearization of the semiflow at periodic orbits and equilibria. More precisely, we highlight the most vital consequences of the general results by Mallet-Paret and Nussbaum [MPN13] in our simpler setting under assumptions (A1)–(A3).

2.1 Solution semiflow

We consider twice continuously differentiable nonlinearities f with finite C^2-norm and denote this by $f \in BC^2(\mathbb{R}^2, \mathbb{R})$. A solution of the DDE (1.1) is a continuous curve $x(t)$ defined for all $t \geq -1$, such that $x(t)$ is continuously differentiable for all $t > 0$ and the derivative satisfies the differential equality (1.1). In agreement with the assumptions (A1)–(A3) in Chapter 1, we assume that f satisfies the **negative delayed feedback** assumption

$$\partial_2 f(\xi, \eta) < 0 \quad \text{for all } (\xi, \eta) \in \mathbb{R}^2, \tag{2.1}$$

and the **even-odd symmetry**

$$f(\xi, \eta) = f(-\xi, \eta) = -f(\xi, -\eta) \quad \text{for all } (\xi, \eta) \in \mathbb{R}^2. \tag{2.2}$$

Hence, from this point, $\mathfrak{X}^-$ denotes the cone of functions in $BC^2(\mathbb{R}^2, \mathbb{R})$ satisfying both assumptions (2.1) and (2.2).

We describe the solutions of (1.1) by considering the initial value problem

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-1)), \quad t \geq 0, \\ x(\theta) &= \phi(\theta), \quad \theta \in [-1, 0], \end{aligned} \tag{2.3}$$

where $\phi \in H$ is a continuous function acting as an **initial condition**. Our approach differs slightly from the traditional choice of function space in DDE literature. We only consider initial conditions ϕ in a subset $H \subset C$, where H denotes the compact embedding into C of the standard **Sobolev space** $H^1([-1, 0], \mathbb{R})$. To be accurate, H is the vector space containing all continuous functions $\phi \in C$ with a square-integrable weak derivative $\phi' : [-1, 0] \to \mathbb{R}$, i.e., ϕ' such that

$$\int_{-1}^{0} (\phi'(\theta))^2 \, \mathrm{d}\theta < \infty \quad \text{and} \quad \int_{-1}^{0} \phi(\theta)\dot{\varphi}(\theta) \, \mathrm{d}\theta = -\int_{-1}^{0} \phi'(\theta)\varphi(\theta) \, \mathrm{d}\theta,$$

for all test functions $\varphi \in C^\infty((-1, 0), \mathbb{R})$ with compact support. It is well known that H is a separable Hilbert space when equipped with the norm

$\|\phi\|_H := \sqrt{\langle \phi, \phi \rangle}$ induced by the **Sobolev inner product**

$$\langle \phi, \tilde{\phi} \rangle := \int_{-1}^{0} \phi(\theta)\tilde{\phi}(\theta)\, \mathrm{d}\theta + \int_{-1}^{0} \phi'(\theta)\tilde{\phi}'(\theta)\, \mathrm{d}\theta.$$

We denote $\|\cdot\| := \|\cdot\|_H$ and adopt the Sobolev topology on H unless specified otherwise.

Since the initial value problem (2.3) has just one discrete delay $x(t-1)$, the existence and uniqueness of solutions follows by the Picard–Lindelöf theorem using the so-called method of steps, i.e., step-by-step integration of the nonautonomous ODEs

$$\begin{aligned} \dot{x}_n(t) =& f_n(t, x_n(t)) \\ :=& f(x_n(t), x_{n-1}(t)), \quad t \in [-1, 0], \\ x_n(-1) =& x_{n-1}(0). \end{aligned} \tag{2.4}$$

Indeed, any solution $x(t)$, $t \in [-1, \infty)$, of the initial value problem (2.3) connects to (2.4) by means of the notation

$$x_n(\theta) := x(n+\theta), \quad x_0(\theta) := \phi(\theta), \quad \theta \in [-1, 0].$$

Furthermore, it follows from the C^2-boundedness of $f \in \mathfrak{X}^-$ that the solutions of the initial value problem (2.3) are well defined for all $t \geq 0$.

However, (2.3) differs considerably from standard ODEs since the solutions $x(t)$ are not necessarily well defined for values $t < -1$. Specifically, the DDE (1.1) defines an irreversible process. Indeed, solving by the method of steps (2.4) shows that continuous initial data $x_0 = \phi \in H$ regularizes in time so that $x(t)$ is continuously differentiable for all $t > 0$. In particular, after one iteration, we have $x_1 \in C^1([-1, 0], \mathbb{R}) \subset H$, which highlights a time-direction asymmetry innate to DDE; see [HVL93, Section 2.5] for a detailed discussion.

By the remarks above, the **semiflow** announced in Chapter 1 is the one-parameter family of operators $S_f(t) : H \to H$ with $t \geq 0$ defined by setting $S_f(t)x_0 = x_t$, where we used the notation $x_t(\theta) := x(t+\theta)$, $\theta \in [-1, 0]$. The trick of regarding the solutions of the DDE (1.1) as curves of functions $x_t \in H$, rather than curves of real points $x(t) \in \mathbb{R}$ motivates calling H a **history space**. On the other hand, the term semiflow follows directly from the fact that S_f defines, by construction, an action of the positive real line $(\mathbb{R}_+, +)$ on H. In other words, it satisfies the algebraic relations

$$S_f(0) = \mathrm{Id} \quad \text{and} \quad S_f(t) \circ S_f(s) = S_f(t+s), \quad \text{for all } t, s \geq 0.$$

Driver and Melvin [Dri65, Mel72] studied neutral DDEs with a Sobolev-type history space like H for their better regularity properties compared to the traditional choice of C. Nishiguchi [Nis19] gives a detailed discussion of how this choice affects the regularity of (1.1) under changes in the delay. Rather than proving the regularity of $S_f(t)$ now, we postpone it to Appendix A. Nevertheless, the general rule to remember is that all the relevant regularity properties of the traditional semiflow defined on all of C are inherited by the restriction to the smaller Hilbert space H.

More precisely, for all $t \geq 0$, the map $S_f(t)$ is twice continuously differentiable in ϕ and f. Moreover, for all $t \geq 2$, $S_f(t)$ is **compact**, that is, it maps bounded sets to precompact sets in H; see Propositions A.2 and A.3. Since $f \in \mathfrak{X}^-$ is C^2-uniformly bounded, [HMO02, Proposition A.0.5] guarantees that the semiflow $S_f(t)$ is **dissipative** in the sense of Chapter 1, i.e., there exists a constant $K > 0$ such that for any initial condition $\phi \in H$ there exists an *entry time* $t^* = t^*(\phi)$ so that $\|S_f(t)\phi\| < K$ for all $t > t^*$.

2.2 Global attractors

As mentioned above, the processes modeled by the DDE (1.1) are irreversible, i.e., switching the time direction $t \mapsto -t$ yields an ill-defined initial value problem. However, **eternal solutions**, i.e., the uniformly bounded differentiable curves $x(t)$ solving (1.1) for all $t \in \mathbb{R}$ are essential in studying the long-time limit $t \to \infty$ for (1.1).

Let us consider the subset of H formed by all the initial conditions yielding eternal solutions of (1.1) and denote it

$$\mathcal{A}(f) := \left\{ \phi \in H : \sup_{t \in \mathbb{R}} \|S_f(t)\phi\| < \infty \right\}.$$

The monotone feedback condition (2.2) makes $S_f(t)$ injective for all $t \geq 0$; see Lemma A.4. Therefore, the semiflow S_f restricted to $\mathcal{A}(f)$ defines a group action, also known as **flow**, on $\mathcal{A}(f)$. In particular, $\mathcal{A}(f)$ is $S_f(t)$**-invariant** by construction, that is,

$$S_f(t)\mathcal{A}(f) = \mathcal{A}(f) \quad \text{for all } t \in \mathbb{R}. \tag{2.5}$$

For this reason, from this point on, the notation $\tilde{\phi} = S_f(t)\phi$ with $t < 0$ is shorthand for *there exists a* $\tilde{\phi} \in H$ *such that* $\phi = S_f(-t)\tilde{\phi}$.

Given a compact dissipative semiflow, such as $S_f(t)$, the set $\mathcal{A}(f)$ is nonempty, compact, and attracts all bounded sets in H; see [HMO02,Lad91]. In other words, for all bounded $U \subset H$ we have

$$\lim_{t\to\infty} \left(\sup_{\phi\in U} \operatorname{dist}(S_f(t)\phi, \mathcal{A}(f)) \right) = 0. \tag{2.6}$$

Thus, under these conditions, we call $\mathcal{A}(f)$:

- **Global attractor**, since it attracts any bounded sets.
- Eternal core, since it contains all the eternal initial conditions.
- Maximal compact invariant set, since it contains all other compact invariant subsets by the attractivity property (2.6).

Notice that, by the regularizing property of the semiflow above, the invariance (2.5) guarantees that $\mathcal{A}(f) \subset C^1([-1,0],\mathbb{R})$. Moreover, the identity map

$$\begin{aligned} \text{Id} : (\mathcal{A}(f), \|\cdot\|_H) \ &\to \ (\mathcal{A}(f), \|\cdot\|_{C^0}) \\ \phi \ &\mapsto \ \phi, \end{aligned} \tag{2.7}$$

is a continuous bijection because H induces a stronger relative topology on $\mathcal{A}(f)$ than C does. Thus, by the compactness of the global attractor $(\mathcal{A}(f), \|\cdot\|_H)$, (2.7) is a homeomorphism. In particular, our results for the global attractor on the Sobolev space H apply to the *standard* global attractor on C.

It follows from the maximality of $\mathcal{A}(f)$ that given any $\phi \in \mathcal{A}(f)$, the closure of the **orbit**

$$\gamma := \{S_f(t)\phi : t \in \mathbb{R}\},$$

belongs to $\mathcal{A}(f)$. Less trivially, given $\phi \in H$ the set of forward accumulation points of the trajectory $S_f(t)\phi$, also known as the **ω-limit set** $\omega(\phi)$, belongs to $\mathcal{A}(f)$. Indeed, since the semiflow $S_f(t)$ is compact and dissipative, the set

$$\omega(\phi) := \bigcap_{s\geq 0} \operatorname{clos}\left(\{S_f(t)\phi : t \geq s\}\right) \tag{2.8}$$

is a nested intersection of nonempty, connected, and compact sets. Thus $\omega(\phi)$ is nonempty, connected, compact, and $S_f(t)$-invariant by construction. In particular, $\omega(\phi) \subset \mathcal{A}(f)$ for all $\phi \in H$.

The analogous definition in the backward time direction is the **α-limit set**, $\alpha(\phi)$, which we can only define for the eternal initial conditions $\phi \in \mathcal{A}(f)$. Imitating (2.8), we define

$$\alpha(\phi) := \bigcap_{s \le 0} \operatorname{clos}\left(\{S_f(t)\phi : t \le s\}\right),$$

the same arguments above show that $\alpha(\phi) \subset \mathcal{A}(f)$ is nonempty, connected, and compact for all $\phi \in \mathcal{A}(f)$.

Thus the initial conditions $\phi \in \mathcal{A}(f)$ whose orbits coincide with their ω-limit set, i.e., those such that

$$\gamma = \omega(\phi) = \alpha(\phi), \tag{2.9}$$

play a special role. The orbits satisfying (2.9) are the equilibria and periodic orbits of (1.1).

We say that an orbit γ is an **equilibrium** of (1.1) if it is a singleton, i.e., if $\gamma = \{\phi\}$ for $\phi \in H$. In particular, equilibria correspond to initial conditions $\phi(\theta) \equiv x^* \in \mathbb{R}$ where x^* is a zero of $f(x^*, x^*)$. In our setting, the assumption $f \in \mathfrak{X}^-$ ensures that the DDE (1.1) possesses only one equilibrium denoted $\gamma_0 := \{0\}$ where 0 stands for the constant zero function in H. On the other hand, **periodic orbits** are often harder to spot and correspond to the nonequilibrium orbits γ through $\phi \in H$ for which there exists a **period** $p > 0$ such that $S_f(t + p)\phi = \phi$. The smallest such $p > 0$ is called the **minimal period** of $\gamma(\phi)$. Due to (2.9), we shall refer to both the equilibria and periodic orbits of (1.1) as **critical elements** and denote them

$$\operatorname{Crit}(f) := \{\gamma \subset \mathcal{A}(f) : \gamma = \omega(\phi) = \alpha(\phi)\}.$$

The initial conditions $\phi \in H$ whose orbit is a critical element $\gamma \in \operatorname{Crit}(f)$ are called **periodic points**. We denote the set of periodic points by $\operatorname{Per}(f)$. Notice that the constant zero function in H is a periodic point of the DDE (1.1). However, the corresponding orbit γ_0 is not periodic.

In addition, we define the set of **connecting points** consisting of all the initial conditions $\phi \in \mathcal{A}(f)$ whose orbit $\gamma \subset \mathcal{A}(f)$ satisfies $\alpha(\phi) \in \operatorname{Crit}(f)$ and $\omega(\phi) \in \operatorname{Crit}(f)$. If ϕ is a connecting point we say that the orbit γ is **homoclinic** if $\alpha(\phi) = \omega(\phi)$, otherwise we call γ a **heteroclinic orbit**.

Last, we define the **nonwandering set** $\operatorname{Nw}(f) \subset \mathcal{A}(f)$ containing all of the initial conditions $\phi \in H$ such that the nearby dynamics are *recurrent*. In other words, we require that for each $\delta > 0$ and $t^* > 0$ we can find $\tilde{\phi} \in H$ and $t > t^*$ such that

$$\|\phi - \tilde{\phi}\| < \delta \quad \text{and} \quad \|\phi - S_f(t)\tilde{\phi}\| < \delta.$$

Given the standing assumptions, $\mathrm{Nw}(f)$ is nonempty, compact, and invariant under the semiflow; see [HMO02, Proposition 2.0.2]. Moreover, by definition, any initial condition $\phi \in H$ such that $\phi \in \omega(\phi)$ satisfies $\phi \in \mathrm{Nw}(f)$. In particular, we obtain that $\mathrm{Per}(f) \subset \mathrm{Nw}(f)$. Nevertheless, the main advantage of considering the bigger set $\mathrm{Nw}(f)$ over $\mathrm{Per}(f)$ is that $\mathrm{Nw}(f)$ is closed by construction. On compact manifolds, Pugh's closing lemma [Pug67] characterizes $\mathrm{Nw}(f)$ as the set of points that belong to $\mathrm{Per}(\tilde{f})$ after an ε-small perturbation of the nonlinearity $\|f - \tilde{f}\|_{C^2} < \varepsilon$. Thus $\mathrm{Nw}(f) \setminus \mathrm{Per}(f)$ represents the boundary of existence of critical elements.

2.3 Zero number

An essential feature of the DDE (1.1) possessing the negative delayed feedback condition (2.1) is the existence of a specialized comparison principle. In the following, we consider linear DDEs of the form

$$\begin{aligned} \dot{y}(t) &= c_1(t)y(t) + c_2(t)y(t-1), \quad t \geq s \\ y_s(\theta) &= \psi(\theta), \quad \theta \in [-1, 0] \end{aligned} \tag{2.10}$$

with coefficients $c_1(t), c_2(t) \in BC^1([s, \infty), \mathbb{R})$ and such that $c_2(t)$ satisfies the negative feedback condition

$$c_2(t) < 0 \quad \text{for all } t \in \mathbb{R}. \tag{2.11}$$

Notice that (2.10) arises naturally in studying the original DDE (1.1) with $f \in \mathfrak{X}^-$. Indeed, let $x^*(t), x^\dagger(t)$, $t \geq 0$, be two different solutions of (1.1). Then $\dot{x}^*(t)$ solves (2.10) with the coefficients

$$c_1(t) = \partial_1 f(x^*(t), x^*(t-1)), \quad c_2(t) = \partial_2 f(x^*(t), x^*(t-1)). \tag{2.12}$$

Furthermore, the difference $x^*(t) - x^\dagger(t)$ also solves (2.10) by choosing

$$\begin{aligned} c_1(t) &= \int_0^1 \partial_1 f(\theta x^*(t) + (1-\theta)x^\dagger(t), x^*(t-1))\, \mathrm{d}\theta, \\ c_2(t) &= \int_0^1 \partial_2 f(x^\dagger(t), \theta x^*(t-1) + (1-\theta)x^\dagger(t-1))\, \mathrm{d}\theta. \end{aligned} \tag{2.13}$$

In either (2.12) or (2.13), the inequality (2.11) is induced by the negative feedback assumption (2.1). The method of steps used in solving (2.4) allows us to produce an **evolution** $S_c(t,s) : H \to H$ via the relation $S_c(t,s)y_s = y_t$, that is, a two-parameter family of linear operators solving the nonautonomous DDE (2.10) for all $t \geq s$.

We show the regularity properties of $S_c(t,s)$ in Proposition A.2. The most relevant is the continuity with respect to perturbations of the coefficients $c_j(t)$ and the initial condition $\psi \in H$.

The negative feedback (2.11) becomes particularly important after we define the **sign changes of** of a function $\psi \in H \setminus \{0\}$. In other words, the possibly infinite quantity

$$\mathrm{sc}(\psi) := \sup \{k \in \mathbb{N} : \psi(\theta_i)\psi(\theta_{i+1}) < 0,\ -1 < \theta_1 < \cdots < \theta_{k+1} < 0\}\,. \tag{2.14}$$

For all $\psi \in H \setminus \{0\}$, the **zero number** is the value

$$z(\psi) := \begin{cases} \mathrm{sc}(\psi) & \text{if } \mathrm{sc}(\psi) < \infty \text{ is odd,} \\ \mathrm{sc}(\psi) + 1 & \text{if } \mathrm{sc}(\psi) < \infty \text{ is even,} \\ \infty & \text{otherwise.} \end{cases} \tag{2.15}$$

The zero number is, by construction, lower semicontinuous. However, it is continuous at C^1-functions $\psi^* \in H$ possessing solely simple interior zeros, provided that the convergence occurs in the uniform C^1-norm. In turn, C^1-convergence can be enforced by solving the DDE (2.10) for a small amount of time. The following lemma covers a common situation throughout our work.

Lemma 2.1. *The zero number* (2.15) *is lower semicontinuous, i.e., given a sequence* $\psi^{(n)} \xrightarrow{n\to\infty} \psi^*$ *in the* H*-norm, then*

$$z(\psi^*) \leq \liminf_{n\to\infty} z\left(\psi^{(n)}\right). \tag{2.16}$$

Moreover, let $S_c(t,0)$ *be the solution operator of the linear equation* (2.10). *We denote by* $S_c^{(n)}(t,0)$ *the solution operator of*

$$\dot{y}^{(n)}(t) = c_1^{(n)}(t)y^{(n)}(t) + c_2^{(n)}(t)y^{(n)}(t-1),$$

with scalar coefficients in $BC^1(\mathbb{R}_+,\mathbb{R})$ *satisfying the pointwise limit*

$$c_j^{(n)}(t) \xrightarrow{n\to\infty} c_j(t), \quad j = 1, 2.$$

If there exists a $t^* > 1$ *such that all the zeros of* $\tilde{\psi}^* := S_c(t^*, 0)\psi^*$ *are simple and are contained in the open interval* $(-1, 0)$*, then there exists an* $n_0 \in \mathbb{N}$ *such that*

$$z\left(S_c^{(n)}(t^*, 0)\psi_0^{(n)}\right) = z\left(\tilde{\psi}^*\right) < \infty, \quad \textit{for all } n > n_0. \tag{2.17}$$

Proof. The lower semicontinuity (2.16) follows by definition; see [MPS96a]. The continuity (2.17) is a corollary to [MPS96a, Theorem 4.4]. Indeed, by continuous dependence on the linear coefficients $c_j^{(n)}$, $j = 1, 2$, we have the pointwise limit

$$|\dot{y}^{(n)}(t^* + \theta) - \dot{y}(t^* + \theta)| \quad \text{for all } \theta \in [-1, 0].$$

Thus the limit

$$S_c^{(n)}(t^*, 0)\psi^{(n)} \xrightarrow{n\to\infty} \tilde{\psi}^*,$$

happens in the uniform C^1-norm $\|\cdot\|_{C^1}$. Following [MPS96a, Theorem 4.4], the implicit function theorem shows that the set of C^1-functions with simple zeros inside $(-1, 0)$ is open in the C^1-topology. Thus there exists a $\delta > 0$ such that $z(\tilde{\psi}) = z(\tilde{\psi}^*) < \infty$ for all $\tilde{\psi} \in C^1([-1, 0], \mathbb{R})$ satisfying $\|\tilde{\psi} - \tilde{\psi}^*\|_{C^1} < \delta$. In particular $z(S_c(t^*, 0)\psi^{(n)}) = z(\tilde{\psi}^*) < \infty$ for all $n \in \mathbb{N}$ sufficiently large, which shows (2.17) and completes the proof. □

Provided that it is finite, the zero number (2.15) acts as a *discrete valued entropy* or *discrete Lyapunov function* for the linear DDE (2.10). More precisely, the zero number evaluated along solutions $y_t := S_c(t, s)y_s$ of (2.10) is a monotonically nonincreasing function, dropping strictly near multiple zeros $y(t^*) = \dot{y}(t^*) = 0$, as shown in the following proposition.

Proposition 2.2 ([MPS96b, Theorems 2.1–2.2])**.** *Let* $y(t)$*,* $t \in [s - 1, \infty)$ *solve the linear nonautonomous equation* (2.10) *for* $t \geq s$*. Then*

$$z(y_{t_2}) \leq z(y_{t_1}) \quad \textit{for all } s \leq t_1 \leq t_2.$$

Moreover, if $y(t^*) = \dot{y}(t^*) = 0$ *for some* $t^* \in [s+3, \infty)$*, then the zero number must have dropped strictly, i.e.,*

$$\textit{either } z(y_{t^*}) < z(y_{t^*-3}) \quad \textit{or} \quad z(y_{t^*}) = \infty.$$

Proposition 2.2 only provides valuable information if the zero number is finite. Nevertheless, this is always the case for solutions of the original DDE (1.1) lying on the global attractor.

Lemma 2.3. *Let $f \in \mathfrak{X}^-$. If $\phi, \tilde{\phi} \in \mathcal{A}(f)$ and $\phi \neq \tilde{\phi}$, then*

$$z(\phi - \tilde{\phi}) < \infty.$$

Furthermore, if ϕ is nonconstant, then we also have

$$z(\dot{\phi}) < \infty.$$

Proof. Indeed, since we chose different solutions, $z(S_f(t)\phi - S_f(t)\tilde{\phi})$ is defined for all $t \in \mathbb{R}$. Denoting by $x^*(t)$ and $x^\dagger(t)$ the solutions with initial condition ϕ and $\tilde{\phi}$, respectively, the difference $x^*(t) - x^\dagger(t)$ is uniformly bounded for $t \in \mathbb{R}$ and solves (2.10) with the choice of uniformly bounded coefficients (2.13). Applying [MPS96b, Theorem 2.4] we obtain $z(S_f(t)\phi - S_f(t)\tilde{\phi}) < \infty$ for all $t \in \mathbb{R}$. The proof is completely analogous for $\dot{\phi}$ if we consider the coefficients (2.12) instead. □

A significant consequence is that the **planar projection**

$$\begin{aligned} P : H &\to \mathbb{R}^2 \\ \phi &\mapsto (\phi(0), \phi(-1)), \end{aligned} \tag{2.18}$$

defines a homeomorphism $\omega(\phi) \cong P\omega(\phi) \subset \mathbb{R}^2$ for all $\phi \in H$; see [MPS96a, Theorem 2.1]. In particular, this planar homeomorphism property yields a Poincaré–Bendixson theorem.

Proposition 2.4. *If $f \in \mathfrak{X}^-$, then for all $\phi \in H$ either $\omega(\phi) = \gamma_0$ or $\omega(\phi)$ is a single periodic orbit.*

Proof. By the even-odd symmetry of $f \in \mathfrak{X}^-$, the zero function 0 is the only equilibrium in $\mathrm{Crit}(f)$. Thus the result follows from [MPS96a, Theorem 2.1]. □

The **Poincaré–Bendixson theorem** 2.4 highlights the importance of the set $\mathrm{Crit}(f)$ as it contains the orbits to which trajectories of the semiflow $S_f(t)$ can accumulate in the forward time direction. Moreover, an analog of Proposition 2.4 holds replacing ω-limit sets by α-limit sets [KW01, Proposition 4.2]. Thus we obtain a finer decomposition of the attractor.

Proposition 2.5. *If $f \in \mathfrak{X}^-$, then $\mathcal{A}(f)$ admits the decomposition*

$$\mathcal{A}(f) = \mathrm{Per}(f) \cup \mathrm{Conn}(f).$$

Proof. All that is needed is an analog of Proposition 2.4 in the backward time direction. For positive delayed feedback systems satisfying $\partial_2 f(\xi, \eta) > 0$ for all $(\xi, \eta) \in \mathbb{R}^2$, this is true by [KW01, Proposition 4.2]. However, the main argument is that the methods used to prove the original Poincaré–Bendixson theorem [MPS96a, Theorem 2.1] can be used after reversing time on the global attractor $\mathcal{A}(f)$. This procedure can also be performed in our setting. □

The main implication of Proposition 2.5 is that the hub of recurrent dynamics in the attractor $\mathcal{A}(f)$ consists solely of periodic orbits and equilibria, and the rest of the dynamics are transient connections between critical elements.

Provided that $\mathrm{Crit}(f)$ contains $N + 1$ elements, Proposition 2.5 allows us to define the **phase diagram** $\Gamma(f)$, that is, a directed graph describing the global attractor. The vertices of $\Gamma(f)$ are the critical elements $\{\gamma_0, \gamma_1, \ldots, \gamma_N\} = \mathrm{Crit}(f)$ and the ordered pair $(\gamma^\dagger, \gamma^*)$ is an edge of $\Gamma(f)$ if and only if there exists a $\phi \in \mathrm{Conn}(f)$ such that $\alpha(\phi) = \gamma^\dagger$ and $\omega(\phi) = \gamma^*$.

More importantly, the projection (2.18) also endows all periodic solutions with unique properties that we summarize as follows.

Lemma 2.6. *Let $x^*(t)$ and $x^\dagger(t)$ be two periodic solutions of the DDE* (1.1) *with $f \in \mathfrak{X}^-$ and let us denote their orbits by γ^* and $\gamma^\dagger$ and their minimal periods by p and $p^\dagger$, respectively. Then, the following statements hold:*

(i) *The planar projection $P\gamma^*$ is an embedding of γ^*. In particular, $P\gamma^*$ is a simple closed C^1-curve in $\mathbb{R}^2$.*

(ii) *The planar projections of two different orbits γ^* and $\gamma^\dagger$ do not intersect, i.e.,*

$$P\gamma^* \cap P\gamma^\dagger \neq \emptyset \quad \textit{if and only if} \quad \gamma^* = \gamma^\dagger.$$

(iii) *$x^*(t)$ is sinusoidal, i.e., it moves monotonically in between its positive maximum and negative minimum values, and it reaches them exactly once over every minimal period.*

(iv) $x^*(t)$ *has the* ***odd symmetry***

$$x^*(t) = -x^*\left(t - \frac{p}{2}\right). \tag{2.19}$$

(v) $P\gamma^*$ *contains* $P\gamma_0 = (0,0)$ *in its interior region.*

(vi) $z(x_t^*) = z(\dot{x}_t^*)$ *and* $z(x_t^* - x_{t+t^*}^\dagger)$ *are independent of* $t, t^* \in \mathbb{R}$. *Moreover, if* $x_t^* \neq x_{t+t^*}^*$, *then* $z(\dot{x}_t^*) = z(x_t^* - x_{t+t^*}^*)$.

Proof. Part (i) follows from [MPS96a, Theorem 2.1]. Part (ii) is [MPS96a, Lemma 5.7] and (iii) is [MPS96a, Theorem 7.1]. Part (iv) follows from [MPS96a, Theorem 7.2] and Part (v) is [MPS96a, Corollary 7.4] for the special case in which $x \equiv 0$ is the only equilibrium of the DDE (1.1). To show Part (vi), $z(\dot{x}_t^*)$ is constant in t by [MPS96a, Lemma 5.1], since $\dot{x}^*(t)$ solves (2.21) with coefficients (2.12). Likewise, $z(x_t^*)$, $z(x_t^* - x_{t+t^*}^\dagger)$, and $z(x_t^* - x_{t+t^*}^*)$ are constant because of [MPS96a, Lemma 5.1]. The equality, $z(x_t^*) = z(\dot{x}_t^*)$ follows from Part (iii) above. Finally, to see $z(\dot{x}_t^*) = z(x_t^* - x_{t+t^*}^*)$, notice that

$$z(x_t^* - x_{t+t^*}^*) = z\left(\frac{x_t^* - x_{t+t^*}^*}{t^*}\right) \tag{2.20}$$

is independent of the choice of an arbitrarily small $t^* > 0$ and the right-hand side of (2.20) approximates $\dot{x}_t^*$ as $t^* \to 0$. Shifting t, if necessary, so that all the sign changes of $\dot{x}_t^*$ take place in the interior of $[-1, 0]$, the continuity of the zero number (2.17) in Lemma 2.1 guarantees that for t^* small enough

$$z\left(\frac{x_t^* - x_{t+t^*}^*}{t^*}\right) = z(\dot{x}_t^*),$$

thereby completing the proof. □

2.4 Sturm–Liouville eigenvalue structure

We now focus our attention on the behavior of the solutions of the DDE (1.1), close to a critical element γ^*. We denote $\mathcal{S} := S_f(p)$ where $p > 0$ is the minimal period of $\gamma^* \in \mathrm{Crit}(f)$ in case γ^* is a periodic orbit, and $p = 1$, in case $\gamma^* = \gamma_0$. The standard **Floquet theory** [HVL93] ensures

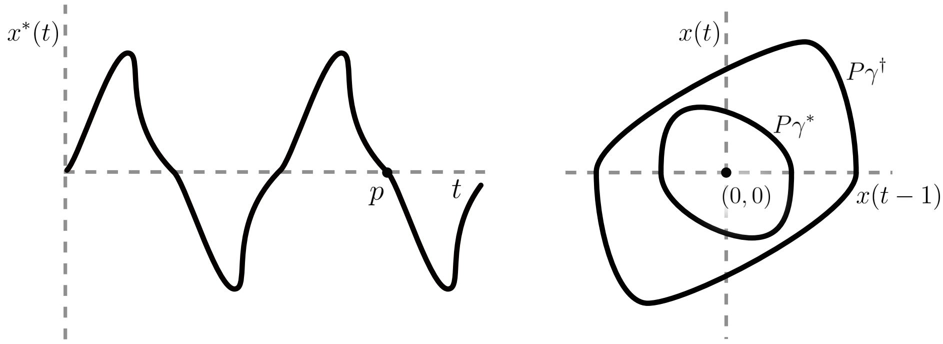


Figure 2.1: (Left) Schematic picture for a sinusoidal periodic solution of the DDE (1.1). (Right) Disjoint planar projections of two different periodic orbits.

that the spectrum of the Fréchet derivative in H of $\mathcal{S}$ at any point $x_0^* \in \gamma^*$ characterizes the local orbital stability of γ^*. We call **monodromy operator** to such derivative and denote it by $M := D\mathcal{S}(x_0^*)$. Here we omit x_0^* in the notation of the monodromy operator because it is known that the spectrum $\mathrm{Spec}(M)$ is independent of the choice of x_0^*.

We denote by $x^*(t)$ the solution of (1.1) with the initial condition $x_0^* \in \gamma^*$. By Proposition A.2, M coincides with the time-p evolution $S_c(p,0)$ of the initial value problem (2.10) with the linear coefficients given by (2.12). Since it plays a special role throughout the rest of the work, we denote the linearized equation at a critical element via

$$\dot{y}(t) = A(t)y(t) + B(t)y(t-1) \tag{2.21}$$

with the linear coefficients

$$A(t) := \partial_1 f(x^*(t), x^*(t-1)) \quad \text{and} \quad B(t) := \partial_2 f(x^*(t), x^*(t-1)).$$

Additionally, the even-odd symmetry of the nonlinearity (2.2), together with the odd symmetry of the solutions (2.19) imply that the coefficients in (2.10) have period $p/2$, i.e., $A(t+p/2) = A(t)$ and $B(t+p/2) = B(t)$. Because of this $p/2$-periodicity, we consider N given by the time-$p/2$ solution operator of (2.10). Due to the trivial relation $M = N^2$, we call N the **half-monodromy operator** of γ^*.

Moreover, by Proposition A.3, there exists an integer $n \geq 1$ such that N^{2n} is a compact operator on H. Hence the spectrum $\mathrm{Spec}(N)$ consists of 0 and countably many eigenvalues accumulating to 0. Since the spectrum of N is independent of the choice of a base point $x_0^* \in \gamma^*$, we call the eigenvalues of N **half-multipliers** of the critical element γ^*. In analogy, we call **characteristic multipliers** of γ^* to the eigenvalues of M.

A remarkable consequence of the negative delayed feedback assumption (2.2) is that the eigenfunctions of N associated with eigenvalues of smaller size have a larger zero number (2.15).

Proposition 2.7 ([MPN13, Theorem 5.1])**.** *Consider $f \in \mathfrak{X}^-$ and let N (resp., M) be the half-monodromy operator (resp., monodromy operator) of $\gamma^* \in \mathrm{Crit}(f)$. Then the spectrum of N (resp., M) consists of countably many pairs of eigenvalues $\{\mu_{2j-1}, \mu_{2j}\}$. The eigenvalues admit the ordering*

$$|\mu_1| \geq |\mu_2| > |\mu_3| \geq |\mu_4| > \cdots > |\mu_{2j-1}| \geq |\mu_{2j}| > \ldots \xrightarrow{j\to\infty} 0,$$

counting algebraic multiplicities. For each $j \in \mathbb{N}$, the corresponding pair of eigenvalues satisfies $\mu_{2j-1}\mu_{2j} > 0$ and is associated to a real generalized eigenspace

$$E^j := \mathrm{Re}\left(\bigcup_{k=1}^{\infty} \ker(\mu_{2j-1}\,\mathrm{Id} - N)^k \oplus \bigcup_{k=1}^{\infty} \ker(\mu_{2j}\,\mathrm{Id} - N)^k\right), \tag{2.22}$$

such that $\dim(E^j) = 2$. Furthermore, all the generalized eigenfunctions $\psi_j \in E^j$ satisfy

$$z(\psi_j) = 2j - 1.$$

We refer to Proposition 2.7 as the **Sturm–Liouville eigenvalue structure** to highlight the qualitative resemblance to the well-known boundary value problem in linear second-order ODEs; see [CL55]. We also point out that originally Mallet-Paret and Nussbaum [MPN13, Theorem 5.1] showed their results for the analog of N (resp., M) defined on the larger space C of continuous functions. However, Proposition 2.7 holds on H because all the eigenfunctions of the operator N (resp., M) belong at least to $C^1([-1,0],\mathbb{R}) \subset H$.

Notice that, since $M = N^2$, $\mathrm{Spec}(M)$ consists of the squares of the eigenvalues $\mathrm{Spec}(N)$. Moereover, M and N share the same generalized eigenspaces E^j in (2.22).

The quantity $i(\gamma^*)$ is called the **Morse index** or **unstable dimension** of the periodic orbit γ^*. By $i(\gamma^*)$, we denote the number of characteristic multipliers of γ^* with an absolute value bigger than 1, counting algebraic multiplicities. We say that a critical element γ^* is **hyperbolic** if γ^* has at most one characteristic multiplier on the unit circle, counting multiplicities.

If γ^* is a periodic orbit, differentiating the original DDE (1.1) shows that $\dot{x}^*(t)$ is a periodic solution of the linearized equation (2.21). Furthermore, the odd symmetry in Lemma 2.6 (iv) yields that $\dot{x}_0^*$ is an eigenvector of the half-monodromy operator N, associated with the half-multiplier $-1 \in \mathrm{Spec}(N)$. Within our setting, we can distinguish the following three situations for a periodic orbit γ^*.

Corollary 2.8. *In the setting of Proposition 2.7, let γ^* be a periodic orbit. Then there exists a real half-multiplier $\mu_c < 0$ of γ^* associated with $\Psi \in \ker(\mu_c \mathrm{I} - N)^2$, a real generalized eigenfunction satisfying $z(\Psi) = z(\dot{x}_0^*)$ and $\Psi \notin \mathrm{Span}_{\mathbb{R}}\{\dot{x}_0^*\}$. Moreover, exactly one of the following statements holds:*

(i) $\mu_c < -1$, *γ^* is hyperbolic,* $\dim \mathrm{Re}(\ker(\mu_c \mathrm{I} - N)) = 1$, *and* $i(\gamma^*) = z(\dot{x}_0^*)$.

(ii) $\mu_c > -1$, *γ^* is hyperbolic,* $\dim \mathrm{Re}(\ker(\mu_c \mathrm{I} - N)) = 1$, *and* $i(\gamma^*) = z(\dot{x}_0^*) - 1$.

(iii) $\mu_c = -1$, *γ^* is not hyperbolic,* $\dim \mathrm{Re}(\ker(-\mathrm{I} - N)^2) = 2$, *and* $i(\gamma^*) = z(\dot{x}_0^*) - 1$.

In either case, the estimate $z(\dot{x}_0^*) \geq i(\gamma^*)$ holds.

Proof. Let $z(\dot{x}_0^*) = 2k - 1$ for some $k \in \mathbb{N}$. Recall that for a periodic orbit we always have $-1 \in \mathrm{Spec}(N)$. Via Proposition 2.7 we have that $\mu_{2k-1}\mu_{2k} > 0$ and (i)–(iii) in Corollary 2.8 correspond to the cases (i) $\mu_c = \mu_{2k-1} < -1$ and $\mu_{2k} = -1$, (ii) $\mu_{2k-1} = -1$ and $\mu_c = \mu_{2k} > -1$, and (iii) $\mu_c = \mu_{2k-1} = \mu_{2k} = -1$. In either of the three cases, μ_c is real. Thus we can pick a real generalized eigenfunction $\Psi \in \bigcup_{k=1}^{\infty} \ker(\mu_c \,\mathrm{Id} - N)^k$ so that $E^k = \mathrm{Span}_{\mathbb{R}}\{\dot{x}_0^*, \Psi\}$, which completes the proof. □

Since μ_c is the real pair of the trivial half-multiplier -1, in the sense of Proposition 2.7, it completely characterizes the unstable dimension and the hyperbolicity of the periodic orbit γ^*. Thus we say that μ_c is the **critical half-multiplier** of γ^* with **critical eigenfunction** Ψ. However, the critical half-multiplier is ill-defined when linearizing at the origin γ_0. At γ_0, the pairs of eigenvalues in Proposition 2.2 are complex conjugates. As a result, we obtain a finer description of the characteristic multipliers that we show in the next lemma.

Lemma 2.9. *Let* $f \in \mathfrak{X}^-$ *and denote* $B := \partial_2 f(0,0)$. *Then*

$$i(\gamma_0) = \begin{cases} 0 & \text{if } B \in \left[-\frac{\pi}{2}, 0\right), \\ 2n & \text{if } B \in \left[-\frac{(4n+1)\pi}{2}, -\frac{(4n-3)\pi}{2}\right). \end{cases} \tag{2.23}$$

In other words, $B_n = (4n-3)\pi/2$ *are Hopf points at which* M *possesses a simple pair of complex conjugate eigenvalues* $\{\mu_{i(\gamma_0)}, \mu_{i(\gamma_0)+1}\}$ *such that* $|\mu_{i(\gamma_0)}| = |\mu_{i(\gamma_0)+1}| = 1$.

Proof. Since the nonlinearity $f \in \mathfrak{X}^-$ in (1.1) is even in the first component, we have that $\partial_1 f(0,0) = 0$. Therefore, N is the time-1/2 evolution $S_c(1/2, 0)$ solving the autonomous equation

$$\dot{y} = By(t-1),\ t \geq 0. \tag{2.24}$$

Following [HVL93], it is known that the relation $\mu_k = e^{\nu_k/2}$ connects the half-multipliers μ_k of γ_0 with the roots ν_k of the characteristic equation

$$\nu = Be^{-\nu}. \tag{2.25}$$

Notice that (2.25) possesses purely imaginary roots if and only if

$$\begin{aligned} 0 &= B\cos(\nu), \\ \nu &= -B\sin(\nu). \end{aligned}$$

Thus (2.25) has imaginary solutions at $B = -(4n-3)\pi/2$, $n \in \mathbb{N}$, only. These values correspond to Hopf bifurcation points at which $\sin(-Bt)$ and $\cos(-Bt)$ solve (2.24). Choosing eigenfunctions $\psi_n(\theta) := \sin((4n-3)\pi\theta/2)$, $\theta \in [-1, 0]$, we have that $z(\psi_n) = 2n-1$. Hence Proposition 2.7 yields the values (2.23) for the Morse index, completing the proof. □

Chapter 3

Invariant manifolds

Given a critical element $\gamma^* \in \mathrm{Crit}(f)$, the neighboring initial conditions $\phi \in H$ that converge towards γ^* in the backward time direction lie on the **local unstable manifold** $W^{\mathrm{u}}_{\mathrm{loc}}(\gamma^*)$. Likewise, in the forward time direction, we obtain the so-called **local stable manifold** denoted $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*)$. Although the manifold structure of $W^{\mathrm{u}}_{\mathrm{loc}}(\gamma^*)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*)$ is well known for a hyperbolic critical element γ^* [HVL93, Chapter 10], this chapter settles three aspects:

1. We construct global versions of $W^{\mathrm{u}}_{\mathrm{loc}}(\gamma^*)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*)$. Indeed, the initial conditions $\phi \in \mathrm{Conn}(f)$ are those such that $S_f(t)\phi$ belongs to a local invariant manifold for large $|t|$. However, ϕ itself need not belong to any local invariant manifold. To circumvent this issue, we study the global sets

$$W^{\mathrm{u}}(\gamma^*) := \left\{ \phi \in H : \lim_{t \to -\infty} \mathrm{dist}(S_f(t)\phi, \gamma^*) = 0 \right\}, \tag{3.1}$$

$$W^{\mathrm{s}}(\gamma^*) := \left\{ \phi \in H : \lim_{t \to \infty} \mathrm{dist}(S_f(t)\phi, \gamma^*) = 0 \right\}. \tag{3.2}$$

2. We need an accurate description of the tangent spaces of the invariant manifolds above. This level of detail will be fundamental in the upcoming discussion of transverse intersections in Chapter 4.

3. If γ^* is nonhyperbolic, the invariant manifolds above may develop boundaries. We discuss the center manifold dynamics in case the equilibrium at the origin $\gamma^* = \gamma_0$ is nonhyperbolic and show that the sets (3.1)–(3.2) are indeed differentiable manifolds.

Notice that the sets (3.1)–(3.2) are $S_f(t)$-invariant by construction. Moreover, they satisfy

$$W^{\mathrm{u}}(\gamma^*) := \{\phi \in H : \alpha(\phi) = \gamma^*\}\,,$$
$$W^{\mathrm{s}}(\gamma^*) := \{\phi \in H : \omega(\phi) = \gamma^*\}\,.$$

Thus the connecting points $\mathrm{Conn}(f)$ correspond to intersections

$$W^{\mathrm{u}}(\gamma^\dagger) \cap W^{\mathrm{s}}(\gamma^*),$$

for $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$. Nevertheless, it is not apparent that (3.1)–(3.2) should carry a manifold structure. We prove this by subscribing to the viewpoint of Hale et al. [CCH92], who regard γ^* as a set consisting of fixed points under a time map $\mathcal{S} := S_f(p)$, where p is the minimal period if γ^* is a periodic orbit and $p = 1$ if $\gamma^* = \gamma_0$. Thus, our approach to invariant manifold theory follows the perspective of discrete map iterations going back to the work of Perron and Hadamard; see [HPS77] and the references therein. This viewpoint uses smooth cutoff functions extensively, however, the problem of finding smooth cutoff functions is rooted in functional analysis and they do not exist in C. It is for this reason that we chose H as a history space. Our method of proof relies on three steps that we exemplify using (3.2):

Step 1: Construction of a local manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*)$ for the discrete map $\mathcal{S}$ near a $\mathcal{S}$-fixed point $x_0^* \in \gamma^*$. In particular, we choose a local manifold consisting of a single chart $(W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*), P^{\mathrm{s}})$, where P^{s} is the canonical projection onto a suitable subspace of H. We summarize the mathematical framework required for this construction in Appendix C.

Step 2: Showing that there exists $t^* > 0$ such that

$$W^{\mathrm{s}}(\gamma^*) = \bigcup_{m \geq 0} S_f(mt^*)^{-1}\left(W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*)\right), \tag{3.3}$$

holds for the global stable set in (3.2). We emphasize that (3.3) does not imply that all initial conditions on the local stable manifold produce eternal solutions. Rather, it shows that $W^{\mathrm{s}}(\gamma^*)$ is the maximal backward extension of $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*)$, as a set.

Step 3: Definition of an atlas on the global extension (3.3) by using $S_f(t^*)$ to build new charts $(S_f(t^*)^{-1}W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*), P^{\mathrm{s}} \circ S_f(t^*))$ via translation of the local chart $(W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*), P^{\mathrm{s}})$. The union of manifolds obtained in this way remains a manifold because $S_f(t^*)$ is injective and the Fréchet derivative $D(S_f(t^*))(\phi)$ has a dense range, by Lemmata A.4 and A.5.

We highlight that Step 2 produces injectively immersed manifolds rather than embedded ones. In particular, $W^{\mathrm{s}}(\gamma^*)$ constructed in this way is allowed to self-intersect and accumulate to itself. Hence $W^{\mathrm{s}}(\gamma^*)$ is not a manifold in the relative subset topology of the history space H, but rather when equipped with the base inherited from translating the relative subset topology of the fundamental domain $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma^*)$.

The remaining sections of the chapter are dedicated to the study of a hyperbolic equilibrium, a nonhyperbolic equilibrium and a hyperbolic periodic orbit, respectively. We use the notation that $\mathcal{S}^{-1}\phi$ represents the unique preimage of ϕ under $\mathcal{S}$ and extend it to the Fréchet derivative. Thus, from this point, $D\mathcal{S}^{-1}(\phi)\psi$ represents the unique preimage of ψ under $D\mathcal{S}(\mathcal{S}^{-1}\phi)$, provided that it exists.

3.1 Hyperbolic equilibrium

Recall from Proposition 2.7 and Lemma 2.9 that the characteristic multipliers at the equilibrium γ_0 are the eigenvalues μ_j of the monodromy operator $M := D\mathcal{S}(0)$ and can be ordered with repetitions so that

$$|\mu_1| \geq |\mu_2| > \cdots \geq |\mu_{i(\gamma_0)}| > 1 \geq |\mu_{i(\gamma_0)+1}| \geq \cdots \to 0.$$

This way, we obtain an M-invariant splitting denoted by

$$H = H^{\mathrm{u}} \oplus H^{\mathrm{s}},$$

into an **unstable eigenspace** H^{u} and a **stable eigenspace** H^{s}. Recall that $i(\gamma_0)$ is always even by Lemma 2.9, thus we choose

$$H^{\mathrm{u}} := \bigoplus_{j \leq i(\gamma_0)/2} E^j \quad \text{and} \quad H^{\mathrm{s}} := (H^{\mathrm{u}})^{\perp}.$$

Here E^j denotes the real generalized eigenspaces as in Proposition 2.7 and $\perp$ denotes the closed complement in H. In particular, $\dim H^{\mathrm{u}} = i(\gamma_0) =$

codim H^{s} is the Morse index of the origin. Moreover, since γ_0 is hyperbolic, we have that

$$
\begin{aligned}
\inf\{|\mu| : \mu \in \operatorname{Spec}(M|_{H^{\mathrm{u}}})\} > 1 \\
\sup\{|\mu| : \mu \in \operatorname{Spec}(M|_{H^{\mathrm{s}}})\} < 1,
\end{aligned}
\tag{3.4}
$$

thus meeting the conditions of Theorem C.3.

Theorem 3.1. *Let γ_0 be a hyperbolic equilibrium of the DDE* (1.1) *with $f \in \mathfrak{X}^-$. Then the $S_f(t)$-invariant sets* (3.1)–(3.2) *are injectively immersed C^2-manifolds. Moreover,* $\dim W^{\mathrm{u}}(\gamma_0) = \operatorname{codim} W^{\mathrm{s}}(\gamma_0) = i(\gamma_0)$ *and $W^{\mathrm{u}}(\gamma_0)$ satisfies*

$$
\begin{aligned}
W^{\mathrm{u}}(\gamma_0) &= \left\{\phi \in H : \lim_{n\to-\infty} \mathcal{S}^n\phi = 0\right\}, \\
T_\phi W^{\mathrm{u}}(\gamma_0) &= \left\{\psi \in H : \lim_{n\to-\infty} D\mathcal{S}^n(\phi)\psi = 0\right\}.
\end{aligned}
\tag{3.5}
$$

Likewise, for $W^{\mathrm{s}}(\gamma_0)$ we have the characterization

$$
\begin{aligned}
W^{\mathrm{s}}(\gamma_0) &= \left\{\phi \in H : \lim_{n\to\infty} \mathcal{S}^n\phi = 0\right\}, \\
T_\phi W^{\mathrm{s}}(\gamma_0) &= \left\{\psi \in H : \lim_{n\to\infty} D\mathcal{S}^n(\phi)\psi = 0\right\}.
\end{aligned}
\tag{3.6}
$$

Proof. We show (3.6) and (3.5) follow along the same steps in the backward time direction. We use $\kappa = 1$ in the splitting of Theorem C.3 and construct a local 1-stable manifold $W^{1,\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$ close to γ_0. Moreover, since $1 \notin \operatorname{Spec}(M)$, Corollary C.2 shows that $W^{\mathrm{ss}}_{\mathrm{loc}}(\gamma_0) = W^{1\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$ and by Theorem C.6 $W^{1,\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$ is locally unique. In particular, we can choose a neighborhood U of γ_0 such that

$$
\begin{aligned}
W^{1\mathrm{s}}_{\mathrm{loc}}(\gamma_0) &= \left\{\phi \in U : \begin{array}{c}\{\mathcal{S}^n\phi\}_{n\geq 0} \subset U \text{ and} \\ \lim_{n\to\infty} \mathcal{S}^n\phi = 0\end{array}\right\}, \\
T_\phi W^{1\mathrm{s}}_{\mathrm{loc}}(\gamma_0) &= \left\{\psi \in H : \lim_{n\to\infty} D\mathcal{S}^n(\phi)\psi = 0\right\}.
\end{aligned}
\tag{3.7}
$$

Furthermore, $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0) := W^{1\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$ is C^2-diffeomorphic to an open subset of H^{s} via the canonical projection $P^{\mathrm{s}} : H \to H^{\mathrm{s}}$. Notice that for all $t \in [n, n+1]$ we can write $S_f(t)\phi = S_f(t-n)\mathcal{S}^n\phi$, where $t - n \in [0, 1]$. For every $\phi \in W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$, we have $\mathcal{S}^n\phi \to 0$ and $t - n$ is contained in a compact set. Hence we conclude that $\lim_{t\to\infty} S_f(t)\phi = 0$ and obtain the inclusion

$W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0) \subset W^{\mathrm{s}}(\gamma_0)$, by definition (3.2). Conversely for all $\phi \in W^{\mathrm{s}}(\gamma_0)$ we can find an $n_0 \in \mathbb{N}$ such that $\mathcal{S}^n\phi \in W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$ for all $n \geq n_0$. This is immediate since (3.2) implies $\lim_{n\to\infty} \mathcal{S}^n\phi = 0$ and $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$ is locally uniquely defined as we discussed above. Thus we just showed that

$$W^{\mathrm{s}}(\gamma_0) = \bigcup_{m\leq 0} \mathcal{S}^m W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0), \tag{3.8}$$

which in turn implies

$$W^{\mathrm{s}}(\gamma_0) = \left\{\phi \in H : \lim_{n\to\infty} \mathcal{S}^n\phi = 0\right\}.$$

It is left to show that the representation (3.8) admits a manifold structure. We know from Lemmata A.4 and A.5 that $\mathcal{S}$ is injective and $D\mathcal{S}(\phi)$ has a dense range for all $\phi \in H$. Thus we proceed as in Step 3 above and use [Hen81, Theorem 6.1.9] to construct charts near all $\phi \in W^{\mathrm{s}}(\gamma_0)$ via translation of the local chart $(W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0), P^{\mathrm{s}})$. The stable manifold equipped with this atlas is an injectively immersed C^2-manifold of codimension codim $W^{\mathrm{s}}(\gamma_0) =$ codim $H^{\mathrm{s}} = i(\gamma_0)$. To characterize the tangent spaces, we plug the translation by $\mathcal{S}$ into the characterization (3.7). In particular, by (3.8), given $\tilde{\phi} \in W^{\mathrm{s}}(\gamma_0)$ we choose an $m \geq 0$ such that $\mathcal{S}^m\tilde{\phi} \in W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0)$ and, by the chain rule, the tangent space satisfies

$$\begin{aligned} T_{\tilde{\phi}}W^{\mathrm{s}}(\gamma_0) &= \left\{\psi \in H : D\mathcal{S}^m(\tilde{\phi})\psi \in T_{\mathcal{S}^m\tilde{\phi}}W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0)\right\} \\ &= \left\{\psi \in H : \lim_{n\to\infty} DS^n(\mathcal{S}^m\tilde{\phi})D\mathcal{S}^m(\tilde{\phi})\psi = 0\right\} \\ &= \left\{\psi \in H : \lim_{n\to\infty} D\mathcal{S}^n(\tilde{\phi})\psi = 0\right\}. \end{aligned} \tag{3.9}$$

□

3.2 Nonhyperbolic equilibrium

In the setting of Section 3.1 let us assume that γ_0 is nonhyperbolic. Taking into account Lemma 2.9, the ordered sequence of eigenvalues of M is now

$$|\mu_1| \geq |\mu_2| > \cdots > |\mu_{i(\gamma_0)+1}| = |\mu_{i(\gamma_0)+2}| = 1 > |\mu_{i(\gamma_0)+3}| \geq \cdots \to 0.$$

Therefore, we obtain an M-invariant splitting of H possessing a **center eigenspace** H^{c}, that is,

$$H = H^{\mathrm{u}} \oplus H^{\mathrm{c}} \oplus H^{\mathrm{s}},$$

with

$$H^{\mathrm{u}} := \bigoplus_{j \leq i(\gamma_0)/2} E^j, \quad H^{\mathrm{c}} := E^{i(\gamma_0)/2+1}, \quad \text{and} \quad H^{\mathrm{s}} := (H^{\mathrm{u}} \oplus H^{\mathrm{c}})^{\perp}.$$

Notice that this splitting satisfies $|\mu| = 1$ for all $\mu \in \mathrm{Spec}(M|_{H^{\mathrm{c}}})$ in addition to (3.4). Recall that, by Lemma 2.9, the origin is nonhyperbolic, exactly, at the Hopf points, i.e., the values $\partial_2 f(0,0) = -(4n-3)\pi/2$ for some $n \in \mathbb{N}$. Furthermore, the center dimension always satisfies $\dim H^{\mathrm{c}} = 2$, which guarantees $\mathrm{codim}\, H^{\mathrm{s}} = i(\gamma_0) + 2$. We also consider the further splitting into the **center-unstable**, and **center-stable eigenspaces** of the form

$$H := H^{\mathrm{cu}} \oplus H^{\mathrm{s}} \quad \text{and} \quad H^{\mathrm{u}} \oplus H^{\mathrm{cs}}.$$

If γ_0 is an isolated critical element, a center manifold reduction as in Appendix C produces the following dichotomy.

Theorem 3.2. *Let γ_0 be an isolated nonhyperbolic critical element of* (1.1). *The $S_f(t)$-invariant sets* (3.1)–(3.2) *are injectively immersed C^2-manifolds. Moreover, exactly one of the two following situations occurs:*

Case 1: *The dynamics near γ_0 are repelling in the center direction, moreover,* $\dim W^{\mathrm{u}}(\gamma_0) = \mathrm{codim}\, W^{\mathrm{s}}(\gamma_0) = i(\gamma_0)+2$, *and $W^{\mathrm{u}}(\gamma_0)$ in* (3.1) *satisfies*

$$\begin{aligned} W^{\mathrm{u}}(\gamma_0) &= \left\{ \phi \in H : \lim_{n\to-\infty} \frac{\mathcal{S}^n \phi}{\kappa^n} = 0 \text{ for all } \kappa < 1 \right\}, \\ T_\phi W^{\mathrm{u}}(\gamma_0) &= \left\{ \psi \in H : \lim_{n\to-\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \text{ for all } \kappa < 1 \right\}. \end{aligned} \tag{3.10}$$

In contrast, the convergence to γ_0 along the stable set $W^{\mathrm{s}}(\gamma_0)$ in (3.2) *happens exponentially fast and satisfies*

$$\begin{aligned} W^{\mathrm{s}}(\gamma_0) &= \left\{ \phi \in H : \begin{array}{l} \text{there exists } \kappa < 1 \\ \text{such that } \lim_{n\to\infty} \frac{\mathcal{S}^n \phi}{\kappa^n} = 0 \end{array} \right\}, \\ T_\phi W^{\mathrm{s}}(\gamma_0) &= \left\{ \psi \in H : \begin{array}{l} \text{there exists } \kappa < 1 \\ \text{such that } \lim_{n\to\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\}. \end{aligned} \tag{3.11}$$

Additionally, the ***global fast unstable manifold*** $W^{\mathrm{uu}}(\gamma_0)$ *containing all the solutions repelled by γ_0 at an exponential rate is* *an injectively immersed C^2-manifold of dimension* $\dim W^{\mathrm{uu}}(\gamma_0) = i(\gamma_0)$ *characterized by*

(3.12)
$$W^{\mathrm{uu}}(\gamma_0) := \left\{ \phi \in H : \begin{array}{l} \textit{there exists } \kappa > 1 \\ \textit{such that } \lim\limits_{n\to-\infty} \dfrac{\mathcal{S}^n\phi}{\kappa^n} = 0 \end{array} \right\},$$
$$T_\phi W^{\mathrm{uu}}(\gamma_0) = \left\{ \psi \in H : \begin{array}{l} \textit{there exists } \kappa > 1 \\ \textit{such that } \lim\limits_{n\to-\infty} \dfrac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\}.$$

Case 2: γ_0 *is attracting in the center dimension, furthermore* $\dim W^{\mathrm{u}}(\gamma_0) = \operatorname{codim} W^{\mathrm{s}}(\gamma_0) = i(\gamma_0)$ *and* $W^{\mathrm{u}}(\gamma_0)$ *in* (3.1) *satisfies*

$$W^{\mathrm{u}}(\gamma_0) = \left\{ \phi \in H : \begin{array}{l} \textit{there exists } \kappa > 1 \\ \textit{such that } \lim\limits_{n\to-\infty} \dfrac{\mathcal{S}^n\phi}{\kappa^n} = 0 \end{array} \right\},$$
$$T_\phi W^{\mathrm{u}}(\gamma_0) = \left\{ \psi \in H : \begin{array}{l} \textit{there exists } \kappa > 1 \\ \textit{such that } \lim\limits_{n\to-\infty} \dfrac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\}.$$

Analogously to $W^{\mathrm{u}}(\gamma_0)$ *in* (3.10) *above, the convergence to* γ_0 *along* $W^{\mathrm{s}}(\gamma_0)$ *may only happen at an algebraic rate. Thus,* $W^{\mathrm{s}}(\gamma_0)$ *satisfies*

$$W^{\mathrm{s}}(\gamma_0) = \left\{ \phi \in H : \lim_{n\to\infty} \frac{\mathcal{S}^n\phi}{\kappa^n} = 0 \textit{ for all } \kappa > 1 \right\},$$
$$T_\phi W^{\mathrm{s}}(\gamma_0) = \left\{ \psi \in H : \lim_{n\to\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \textit{ for all } \kappa > 1 \right\}.$$

Additionally, the ***global fast stable manifold*** $W^{\mathrm{ss}}(\gamma_0)$ *is the injectively immersed C^2-manifold of codimension* $\operatorname{codim} W^{\mathrm{ss}}(\gamma_0) = i(\gamma_0) + 2$ *satisfying*

$$W^{\mathrm{ss}}(\gamma_0) := \left\{ \phi \in H : \begin{array}{l} \textit{there exists } \kappa < 1 \\ \textit{such that } \lim\limits_{n\to\infty} \dfrac{\mathcal{S}^n\phi}{\kappa^n} = 0 \end{array} \right\},$$
$$T_\phi W^{\mathrm{ss}}(\gamma_0) = \left\{ \psi \in H : \begin{array}{l} \textit{there exists } \kappa < 1 \\ \textit{such that } \lim\limits_{n\to\infty} \dfrac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\}.$$

Proof. First, we use Theorem C.11 to see that Cases 1 and 2 are the only possibilities. Hence we construct a nonunique local center manifold $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0)$

meeting the assumptions of Corollary C.10. We must however be careful to obtain a reduced two-dimensional ODE on $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0)$ from the DDE (1.1). Indeed, the $\mathcal{S}$-invariant center manifold produced by Theorem C.9 may not be invariant under the semiflow $S_f(t)$; see [Kri05]. To bypass this issue we apply Theorem C.1 to a modified map $\mathcal{S}_\delta$ which coincides with $\mathcal{S}$ within a δ-ball U_δ around γ_0. We ensure that $\mathcal{S}_\delta$ is compatible with $S_f(t)$ at the DDE level by choosing $\mathcal{S}_\delta := S_{\bar{f}}(1)$, solving the modified DDE

$$\begin{aligned}\dot{\bar{x}} &= \bar{f}(\bar{\mathrm{x}}_t) \\ &:= B\,\bar{\mathrm{x}}(t-1) + \sigma\left(\frac{\|\,\bar{\mathrm{x}}_t\,\|}{\delta}\right)\left(f(\bar{\mathrm{x}}(t), \bar{\mathrm{x}}(t-1)) - B\,\bar{\mathrm{x}}(t-1)\right).\end{aligned}$$

Here, σ is a smooth cutoff function that equals 1 within the unit ball in H and has bounded support. Indeed, $\mathcal{S}_\delta$ agrees with $\mathcal{S}$ within U_δ by construction and, by the variation-of-constants formula; see [HVL93]. Moreover, we can choose $\delta > 0$ so small that $\sup_{\phi\in H}\|D\mathcal{S}_\delta(\phi) - M\|$ is arbitrarily small. Performing the flattening trick (C.7) in (C.8)–(C.9), we obtain that $\mathcal{S}_\delta$ admits a global center manifold $\overline{W}^{\mathrm{c}}(\gamma_0)$. Moreover, $\mathcal{S}_\delta$ commutes with the semiflow $S_{\bar{f}}(t)$ and agrees with $S_f(t)$ sufficiently close to γ_0. As a result, the characterization in terms of asymptotics in Theorem C.1 guarantees that the $\mathcal{S}_\delta$-invariant center manifold $\overline{W}^{\mathrm{c}}(\gamma_0)$ is also $S_{\bar{f}}(t)$-invariant, i.e., $S_{\bar{f}}(t)\overline{W}^{\mathrm{c}}(\gamma_0) = \overline{W}^{\mathrm{c}}(\gamma_0)$ for all $t \in \mathbb{R}$. This in turn implies that for all $\phi \in W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0) := \overline{W}^{\mathrm{c}}(\gamma_0) \cap U_\delta$ we can find $t^* > 0$ such that for all $t \in (-t^*, t^*)$ we have $S_f(t)\phi \in W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0)$.

Let us denote $x_t := S_f(t)\phi$ and choose coordinates so that

$$x(t+\theta) = \xi(t)\Psi^{(1)}(\theta) + \eta(t)\Psi^{(2)}(\theta) + h^{\mathrm{c}}\left(\xi(t)\Psi^{(1)} + \eta(t)\Psi^{(2)}\right)(\theta),$$

where $H^{\mathrm{c}} = \mathrm{Span}_{\mathbb{R}}\{\Psi^{(1)}, \Psi^{(2)}\}$ and $h^{\mathrm{c}} : H^{\mathrm{c}} \to H^{\mathrm{u}} \oplus H^{\mathrm{s}}$ is the C^2-graph representing the center manifold as in Lemma C.7. Recall from the proof of Lemma 2.9 that, if $B := \partial_2 f(0,0) < 0$, then we can choose $\Psi^{(1)}(\theta) = \cos(B\theta)$ and $\Psi^{(2)}(\theta) = -\sin(B\theta)$, thereby obtaining

$$\partial_\theta \Psi^{(1)} = -B\Psi^{(2)} \quad \text{and} \quad \partial_\theta \Psi^{(2)} = B\Psi^{(1)}.$$

Differentiating $x(t+\theta)$, we obtain that

$$\partial_t x(t+\theta) = \partial_\theta x(t+\theta),$$

and projecting onto the eigenfunctions $\Psi^{(1)}$ and $\Psi^{(2)}$, we obtain ODEs in $\mathbb{R}^2$ of the form

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + o(\|(\xi,\eta)\|_2). \tag{3.13}$$

Here we used the Landau little-o notation. Therefore, the linearization at the origin of the ODE (3.13) corresponds to a center in the traditional classification of planar equilibria. Thus, following [CL55, Chapter 15, Theorem 4.1], we face three possible scenarios for the reduced flow on the center manifold:

Case 1: The origin is unstable.

Case 2: The origin is asymptotically stable.

Case 3: The origin is an accumulation point of periodic initial conditions.

Since we prevented Case 3 by assuming that γ_0 is an isolated critical element, we find ourselves in either Case 1 or 2. In particular, for all $\delta > 0$ we can find a neighborhood U^{c} of γ_0 in $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0)$ such that $\mathrm{clos}(\mathcal{S}^{-1}U^{\mathrm{c}}) \subset \mathring{U}^{\mathrm{c}}$ in Case 1 and $\mathrm{clos}(\mathcal{S}U^{\mathrm{c}}) \subset \mathring{U}^{\mathrm{c}}$ in Case 2.

From this point, we shall discuss Case 1 only and point out how to adapt the proof to Case 2 towards the end. Using Theorem C.1 with $\kappa = 1 + \varepsilon$ and $0 < \varepsilon \ll 1$, by Theorem C.11, there exists $\delta > 0$ and an open δ-ball $U_\delta \subset H$ containing an open neighborhood U of γ_0 such that we can construct a center-unstable manifold that satisfies

$$W^{\mathrm{cu}}_{\mathrm{loc}}(\gamma_0) = \left\{ \phi \in U : \begin{array}{c} \{\mathcal{S}^n\phi\}_{n\leq 0} \subset U \text{ and} \\ \lim\limits_{n\to-\infty} \mathcal{S}^n\phi = 0 \end{array} \right\}.$$

Furthermore, the tangent space satisfies

$$T_\phi W^{\mathrm{cu}}_{\mathrm{loc}}(\gamma_0) = \left\{ \psi \in H : \lim_{n\to-\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \text{ for all } \kappa < 1 \right\},$$

and $W^{\mathrm{cu}}_{\mathrm{loc}}(\gamma_0)$ is C^2-diffeomorphic to an open ball in H^{cu} via the canonical projection $P^{\mathrm{cu}} : H \to H^{\mathrm{cu}}$. In addition, choosing $\kappa = 1 + \varepsilon$ and $\kappa = 1 - \varepsilon$ in Theorem C.1, we obtain a local fast unstable manifold $W^{\mathrm{uu}}_{\mathrm{loc}}(\gamma_0)$ and a local fast stable manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(\gamma_0)$, respectively. Choosing possibly different open neighborhoods U of γ_0, we may define them so that

$$W^{\mathrm{uu}}_{\mathrm{loc}}(\gamma_0) = \left\{ \phi \in U : \begin{array}{c} \{\mathcal{S}^n\phi\}_{n\leq 0} \subset U \text{ and there exists } \kappa > 1 \\ \text{such that } \lim\limits_{n\to-\infty} \dfrac{\mathcal{S}^n\phi}{\kappa^n} = 0 \end{array} \right\},$$

$$T_\phi W^{\mathrm{uu}}_{\mathrm{loc}}(\gamma_0) = \left\{ \psi \in H : \begin{array}{c} \text{there exists } \kappa > 1 \\ \text{such that } \lim\limits_{n\to-\infty} \dfrac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\},$$

and

$$W^{\text{ss}}_{\text{loc}}(\gamma_0) = \left\{ \phi \in U : \begin{array}{l} \{\mathcal{S}^n\phi\}_{n\geq 0} \subset U \text{ and there exists } \kappa < 1 \\ \text{such that } \lim\limits_{n\to\infty} \dfrac{\mathcal{S}^n\phi}{\kappa^n} = 0 \end{array} \right\},$$

$$T_\phi W^{\text{ss}}_{\text{loc}}(\gamma_0) = \left\{ \psi \in H : \begin{array}{l} \text{there exists } \kappa < 1 \\ \text{such that } \lim\limits_{n\to\infty} \dfrac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\}$$

Moreover, $W^{\text{uu}}_{\text{loc}}(\gamma_0)$ and $W^{\text{ss}}_{\text{loc}}(\gamma_0)$ are C^2-diffeomorphic to H^{u} and H^{s} via the canonical projections P^{u} and P^{s}, respectively. Hence, at this point, we want to show

$$\begin{aligned} W^{\text{u}}(\gamma_0) &= \bigcup_{m\geq 0} \mathcal{S}^m W^{\text{cu}}_{\text{loc}}(\gamma_0), \\ W^{\text{uu}}(\gamma_0) &= \bigcup_{m\geq 0} \mathcal{S}^m W^{\text{uu}}_{\text{loc}}(\gamma_0), \\ W^{\text{s}}(\gamma_0) &= \bigcup_{m\leq 0} \mathcal{S}^m W^{\text{ss}}_{\text{loc}}(\gamma_0), \end{aligned} \tag{3.14}$$

for the invariant sets (3.1)–(3.2). Once we have proved (3.14), the characterizations (3.10), (3.11), and (3.12) follow immediately.

First, notice that (3.14) follows for $W^{\text{uu}}(\gamma_0)$ because of the definition (3.12).

Second, consider $W^{\text{u}}(\gamma_0)$ in (3.14) and notice that $W^{\text{cu}}_{\text{loc}}(\gamma_0) \subset W^{\text{u}}(\gamma_0)$ follows automatically by the same argument as in Theorem 3.1, using the semigroup property $S_f(t)\phi = S_f(t-n)\mathcal{S}^n\phi$. Conversely, for all $\phi \in W^{\text{u}}(\gamma_0)$, the definition (3.1) ensures that there exists an $n_0 \leq 0$ such that $\mathcal{S}^n\phi \in U$ for all $n \leq n_0$. This shows that $\mathcal{S}^n\phi \in W^{\text{cu}}_{\text{loc}}(\gamma_0)$ and proves (3.14) for $W^{\text{u}}(\gamma_0)$.

Third, we show (3.14) for $W^{\text{s}}(\gamma_0)$. As above, the arguments used in Theorem 3.1 to show $W^{\text{s}}_{\text{loc}}(\gamma_0) \subset W^{\text{s}}(\gamma_0)$ also prove $W^{\text{ss}}_{\text{loc}}(\gamma_0) \subset W^{\text{s}}(\gamma_0)$. To see the reverse inclusion, consider $\phi \in W^{\text{s}}(\gamma_0)$, by the characterization (3.2) there exists an $n_0 \geq 0$ such that $\mathcal{S}^n\phi \in U$ for all $n \geq n_0$. Hence, the exponential attractivity in Lemma C.8 for the local center manifold $W^{\text{c}}_{\text{loc}}(\gamma_0)$ constructed above guarantees that $\mathcal{S}^n\phi$ approaches $W^{\text{c}}_{\text{loc}}(\gamma_0)$ at an exponential rate. However, since we are in Case 1 above, the only orbit in $W^{\text{c}}_{\text{loc}}(\gamma_0)$ that converges to γ_0 under the forward iteration of $\mathcal{S}$ is γ_0 itself. Therefore, we can find a $\kappa < 1$ such that $\mathcal{S}^n\phi \in W^{\kappa\text{s}}_{\text{loc}}(\gamma_0) \subset W^{\text{ss}}_{\text{loc}}(\gamma_0)$, where $W^{\kappa\text{s}}_{\text{loc}}(\gamma_0)$ is a local κ-stable manifold as obtained by Theorem C.3. As a result, we obtain $W^{\mathrm{s}}(\gamma_0) = \bigcup_{m\leq 0} \mathcal{S}^m W^{\mathrm{ss}}_{\mathrm{loc}}(\gamma_0)$.

Finally, from Lemmata A.4 and A.5, recall that $\mathcal{S}$ is injective and $D\mathcal{S}(\phi)$ is injective and has a dense range for all $\phi \in H$. Since we have the $\mathcal{S}$-invariance properties

$$\begin{aligned} W^{\mathrm{cu}}_{\mathrm{loc}}(\gamma_0) \cap \mathcal{S}U_\delta &\subset \mathcal{S}W^{\mathrm{cu}}_{\mathrm{loc}}(\gamma_0), \\ W^{\mathrm{uu}}_{\mathrm{loc}}(\gamma_0) \cap \mathcal{S}U_\delta &\subset \mathcal{S}W^{\mathrm{uu}}_{\mathrm{loc}}(\gamma_0), \\ \mathcal{S}W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0) \cap U_\delta &\subset W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_0), \end{aligned}$$

we apply [Hen81, Theorem 6.1.9] and obtain that the invariant sets in (3.14) are indeed injectively immersed C^2-manifolds of the claimed dimensionality. After applying the chain rule like in (3.9), we have characterized the tangent spaces and concluded the proof for Case 1.

The method to show Case 2 is completely analogous except that we construct a local center-stable manifold $W^{\mathrm{cs}}_{\mathrm{loc}}(\gamma_0)$ instead. As a result, the identity (3.14) transforms into

$$\begin{aligned} W^{\mathrm{s}}(\gamma_0) &= \bigcup_{m\leq 0} \mathcal{S}^m W^{\mathrm{cs}}_{\mathrm{loc}}(\gamma_0), \\ W^{\mathrm{ss}}(\gamma_0) &= \bigcup_{m\leq 0} \mathcal{S}^m W^{\mathrm{ss}}_{\mathrm{loc}}(\gamma_0), \\ W^{\mathrm{uu}}(\gamma_0) &= \bigcup_{m\geq 0} \mathcal{S}^m W^{\mathrm{uu}}_{\mathrm{loc}}(\gamma_0). \end{aligned}$$

Then we carry out the arguments above in the opposite time direction, completing the proof. □

3.3 Hyperbolic periodic orbits

Let us now discuss the case in which γ^* is a hyperbolic periodic orbit with minimal period p. Analogously to the case of the equilibrium at the origin discussed in Sections 3.1 and 3.2, we aim to discuss the geometric structure of the sets (3.1)–(3.2). We denote by $x^*(t)$ the periodic solution of (1.1) generating the orbit γ^* and, denoting $\mathcal{S} := S_f(p)$, we consider the monodromy operator $M = D\mathcal{S}(x^*_0)$. By the discussion in Section 2.4, M always possesses a trivial eigenvalue 1 associated with the derivative $\dot{x}^*_0$. Thus, with the ordering of Proposition 2.7, the characteristic multipliers satisfy

$$|\mu_1| \geq |\mu_2| > \cdots \geq |\mu_{i(\gamma^*)}| > 1 > |\mu_{i(\gamma^*)+2}| \geq \cdots \to 0.$$

Here, the parity of the unstable dimension $i(\gamma^*)$ depends on the size of the critical half-multiplier μ_{c} in Corollary 2.8. In particular, we have an M-invariant splitting of H possessing a trivial center direction, that is,

$$H = H^{\mathrm{u}} \oplus H^{\mathrm{c}} \oplus H^{\mathrm{s}}.$$

Denoting by Ψ the critical eigenfunction in Corollary 2.8 we have two different configurations depending on μ_{c}:

(C1) If $\mu_{\mathrm{c}} < -1$, then $i(\gamma^*)$ is odd and $\mathrm{Span}_{\mathbb{R}}\{\Psi\} \leq H^{\mathrm{u}}$.

(C2) If $\mu_{\mathrm{c}} > -1$, then $i(\gamma^*)$ is even and $\mathrm{Span}_{\mathbb{R}}\{\Psi\} \leq H^{\mathrm{s}}$.

We start by constructing our manifold structure with respect to a Poincaré first return map. Consider first the affine section $\bar{H} := x_0^* + \mathrm{Span}_{\mathbb{R}}\{\dot{x}_0^*\}^{\perp}$. Here, $\perp$ stands for the closed complement in H. Hence, by the Hahn–Banach theorem, there exists a linear functional $\ell : H \to \mathbb{R}$ satisfying $\ell(\dot{x}_0^*) = 1$ and $\mathrm{Span}_{\mathbb{R}}\{\dot{x}_0^*\}^{\perp} := \ker(\ell) \leq H$. Since $\dot{x}_0^* \neq 0$, the section $\bar{H}$ is transverse to the semiflow $S_f(t)$ at x_0^* by construction, i.e., $T_{x_0^*}\bar{H} + \mathrm{Span}_{\mathbb{R}}\{\dot{x}_0^*\} = H$. Recall that

$$\ell(\mathcal{S}x_0^* - x_0^*) = 0,$$

and $\partial_t \ell(S_f(t)x_0^*)|_{t=p} = \ell(\dot{x}_0^*) = 1$. Therefore, there exists an open neighborhood $x_0^* \in \mathcal{U} \subset H$ on which we can define a C^2-map $\bar{\mathcal{T}} : \mathcal{U} \to \mathbb{R}$ solving

$$\bar{\mathcal{T}}(x_0^*) = p \quad \text{and} \quad \ell(S_f(\bar{\mathcal{T}}(\phi))\phi - x_0^*) = 0 \quad \text{for all } \phi \in \mathcal{U}. \tag{3.15}$$

The map $\bar{\mathcal{T}}$ is the first return map giving the time it takes the semiflow $S_f(t)$ to return an initial condition $\phi \in \mathcal{U}$ to the section $\bar{H}$. Furthermore, by Lemmata A.4 and A.5, the solution semiflow $S_f(t)$ is injective and the Fréchet derivative $D(S_f(t))(\phi)$ possesses a dense range for all $t \geq 0$ and $\phi \in H$. Thus, the level sets $\mathcal{U}_\tau := \bar{\mathcal{T}}^{-1}(\tau) \cap \mathcal{U}$ yield a foliation of $\mathcal{U}$ by codimension one manifolds. In particular, we can consider the first return time from the leaf $\mathcal{U}_\tau$ to itself by using the implicit function theorem to solve

$$\mathcal{T}(x_0^*) = p \quad \text{and} \quad \bar{\mathcal{T}}(S_f(\mathcal{T}(\phi))\phi) - \bar{\mathcal{T}}(\phi) = 0 \quad \text{for all } \phi \in \mathcal{U}. \tag{3.16}$$

Indeed, denoting by $M := D\mathcal{S}(x_0^*)$ the monodromy operator of γ^*, we obtain by differentiating (3.15) that $\ell(\dot{x}_0^* D\bar{\mathcal{T}}(x_0^*)\psi + M\psi) = 0$ for all $\psi \in H$. Hence, by the hyperbolicity of γ^*, $D\bar{\mathcal{T}}(x_0^*)\psi = -\ell(M\psi)$ and $D\bar{\mathcal{T}}(x_0^*)\dot{x}_0^* = -1$, which allows us to obtain the C^2-**Poincaré time** $\mathcal{T} : \mathcal{U} \to \mathbb{R}$. The composition

$$(3.17)\qquad \begin{array}{rcl} \mathcal{P} : \mathcal{U} & \to & H \\ \phi & \mapsto & S_f(\mathcal{T}(\phi))\phi, \end{array}$$

is called the **Poincaré map** at x_0^*. We summarize the main properties of the Poincaré map and the $\bar{\mathcal{T}}$-foliation of $\mathcal{U}$ as follows.

Lemma 3.3. *In the notation above, both the Poincaré map $\mathcal{P}$ and the semi-flow $S_f(t)$ preserve the leaves $\mathcal{U}_\tau$ of the foliation $\mathcal{U} = \bigcup_{\tau \in (-\varepsilon,\varepsilon)} \mathcal{U}_\tau$, i.e., $\mathcal{P}(\mathcal{U}_\tau) \cap \mathcal{U} \subset \mathcal{U}_\tau$, $\mathcal{U}_\tau \cap \mathcal{P}(\mathcal{U}) \subset \mathcal{P}(\mathcal{U}_\tau)$, and $S_f(t)\mathcal{U}_\tau \cap \mathcal{U} \subset \mathcal{U}_{t+\tau}$ for all $\tau, t+\tau \in (-\varepsilon, \varepsilon)$. Moreover, we have that $D\mathcal{P}(x_0^*) = M$ and the Poincaré time satisfies $D\mathcal{T}(x_0^*) = 0$.*

Proof. Indeed, the invariance properties follow immediately from the construction. To see $D\mathcal{T}(x_0^*) = 0$ and $D\mathcal{P}(x_0^*) = M$, we differentiate (3.16) at x_0^* obtaining

$$D\bar{\mathcal{T}}(x_0^*)\left(\dot{x}_0^* D\mathcal{T}(x_0^*)\psi + M\psi\right) - D\bar{\mathcal{T}}(x_0^*)\psi = 0 \quad \text{for all } \psi \in H.$$

We showed above that $D\bar{\mathcal{T}}(x_0^*)\psi = -\ell(M\psi)$; thus we obtain $D\mathcal{T}(x_0^*) = 0$ and differentiating (3.17) we get

$$\begin{aligned} D\mathcal{P}(x_0^*)\psi &= \dot{x}_0^* D\mathcal{T}(x_0^*)\psi + M\psi \\ &= M\psi, \end{aligned}$$

which completes the proof. □

After these preliminaries, we prove the section's main theorem by characterizing the manifolds (3.1)–(3.2) via exponential in-phase convergence. More precisely, the stable and unstable manifolds of a hyperbolic periodic orbit admit a foliation by a one-parameter family of fast $\mathcal{S}$-invariant manifolds along which the convergence rate is exponential.

Theorem 3.4. *Let γ^* be a hyperbolic periodic orbit of the DDE* (1.1). *Then the sets* (3.1)–(3.2) *are injectively immersed C^2-manifolds whose dimensions* *satisfy* $\dim W^{\mathrm{u}}(\gamma^*) = i(\gamma^*) + 1$ *and* $\operatorname{codim} W^{\mathrm{s}}(\gamma^*) = i(\gamma^*)$. *Moreover, the unstable manifold* (3.1) *satisfies*

$$
\begin{aligned}
W^{\mathrm{u}}(\gamma^*) &= \bigcup_{\tau\in[0,p)} \left\{ \phi \in H : \begin{array}{c} \textit{there exists } \kappa > 1 \\ \textit{such that } \displaystyle\lim_{n\to-\infty} \frac{\mathcal{S}^n\phi - x_\tau^*}{\kappa^n} = 0 \end{array} \right\}, \\
T_\phi W^{\mathrm{u}}(\gamma^*) &= \left\{ \psi \in H : \lim_{n\to\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \textit{ for all } \kappa < 1 \right\}.
\end{aligned} \tag{3.18}
$$

Likewise, the stable manifold (3.2) *is given by*

$$
\begin{aligned}
W^{\mathrm{s}}(\gamma^*) &= \bigcup_{\tau\in[0,p)} \left\{ \phi \in H : \begin{array}{c} \textit{there exists } \kappa < 1 \\ \textit{such that } \displaystyle\lim_{n\to\infty} \frac{\mathcal{S}^n\phi - x_\tau^*}{\kappa^n} = 0 \end{array} \right\}, \\
T_\phi W^{\mathrm{s}}(\gamma^*) &= \left\{ \psi \in H : \lim_{n\to\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \textit{ for all } \kappa > 1 \right\}.
\end{aligned} \tag{3.19}
$$

Proof. We show (3.19) and (3.18) is completely symmetric with respect to a change in the time direction. First, consider the Poincaré map $\mathcal{P}$ defined in (3.17) in a neighborhood $\mathcal{U}$ of $x_0^* \in \gamma^*$. We prove the identity

$$W^{\mathrm{s}}(\gamma^*) = \bigcup_{m\le 0} S_f(mt^*) W_{\mathrm{loc}}^{\mathrm{cs}}(x_0^*). \tag{3.20}$$

where $t^* > 0$ is chosen so that $\gamma^* \subset \bigcup_{m\le 0} S_f(mt^*)\mathcal{U}$ and $W_{\mathrm{loc}}^{\mathrm{cs}}(\gamma^*)$ denotes a local center-stable manifold built using Theorem C.11 for $\mathcal{P}$ in a δ-ball $\in U_\delta \subset H$ around x_0^*. Choosing $\mathcal{U}$ small enough, we have that

$$W_{\mathrm{loc}}^{\mathrm{cs}}(x_0^*) = \left\{ \phi \in \mathcal{U} : \begin{array}{c} \{\mathcal{P}^n(\phi)\}_{n\ge 0} \subset \mathcal{U} \text{ and} \\ \displaystyle\lim_{n\to\infty} \frac{\mathcal{P}^n(\phi)}{\kappa^n} = 0 \text{ for all } \kappa > 1 \end{array} \right\}.$$

Consider first $\phi \in W^{\mathrm{s}}(\gamma^*)$, by construction of the Poincaré map $\mathcal{P}$, we can find an $n_0 \ge 0$ such for all $n \ge 0$ we have $\mathcal{P}^n(S_f(n_0t^*)\phi) \in \mathcal{U}$ and $\mathcal{P}^n(S_f(n_0t^*)\phi) \to \gamma^*$. Hence $S_f(n_0t^*)\phi \in W_{\mathrm{loc}}^{\mathrm{cs}}(x_0^*)$ and we have proved $W^{\mathrm{s}}(\gamma^*) \subset \bigcup_{m\le 0} S_f(mt^*) W_{\mathrm{loc}}^{\mathrm{cs}}(x_0^*)$.

To see the reverse inclusion, let $\phi \in W_{\mathrm{loc}}^{\mathrm{cs}}(x_0^*)$. By Lemma 3.3, if $\phi \in \mathcal{U}_\tau$, then $\mathcal{P}(\phi) \in \mathcal{U}_\tau$ stays on the same leaf as long as ϕ is chosen sufficiently close to γ^*. In particular, it follows that $\mathcal{U}_\tau \cap W_{\mathrm{loc}}^{\mathrm{cs}}(x_0^*)$ coincides with a local fast stable manifold for the hyperbolic map resulting from restricting $\mathcal{P}$ to the invariant leaf $\mathcal{U}_\tau$. Therefore, by Theorem C.6, we may assume without loss of generality that $\mathcal{P}^n(\phi) \in \mathcal{U}_\tau$ for all $n \geq 0$ and $\lim_{n\to\infty} \mathcal{P}^n(\phi) = x^*_{-\tau} \in \gamma^*$.

Recall that $\mathcal{T}$, the Poincaré time defined via (3.16), satisfies $\mathcal{T}(x_0^*) = p$. Thus, for any sequence $t_k \xrightarrow{k\to\infty} \infty$ we can find sequences $n_k \xrightarrow{k\to\infty} \infty$ and $\tau_k \in [0, 2p]$ such that

$$S_f(t_k)x_0 = S_f(\tau_k)\mathcal{P}^{n_k}(x_0).$$

Choosing a convergent subsequence, if needed, and denoting the limit $\tau_k \to \tau^* \in [0, 2p]$, we conclude that

$$\begin{aligned}\lim_{k\to\infty} S_f(t_k)\phi &= \lim_{k\to\infty} S_f(\tau_k)\mathcal{P}^{n_k}(\phi) \\ &= S_f(\tau^*)x^*_\tau \in \gamma^*.\end{aligned}$$

Hence, $\lim_{t\to\infty} \operatorname{dist}(S_f(t)\phi, \gamma^*) = 0$, which shows $W^{\mathrm{cs}}(x_0^*) \subset W^{\mathrm{s}}(\gamma^*)$ and proves the identity (3.20) by $S_f(t)$-invariance of $W^{\mathrm{s}}(\gamma^*)$.

In order to see that $W^{\mathrm{s}}(\gamma^*)$ is indeed a differentiable manifold of the claimed codimension, notice that $W^{\mathrm{cs}}_{\mathrm{loc}}(x_0^*)$ is C^2-diffeomorphic to an open ball in the center-stable eigenspace $H^{\mathrm{cs}} = (H^{\mathrm{u}})^\perp$ with the diffeomorphism given by the canonical projection P^{cs} onto H^{cs}. Recall that $S_f(\tau)$ is injective, the derivative $D(S_f(\tau))(\phi)$ has a dense range, and the local invariance

$$S_f(t^*)W^{\mathrm{cs}}_{\mathrm{loc}}(x_0^*) \cap U_\delta \subset W^{\mathrm{cs}}_{\mathrm{loc}}(x_0^*)$$

holds by Lemma 3.3. Thus [Hen81, Theorem 6.1.9] allows us to extend the local chart $(W^{\mathrm{cs}}_{\mathrm{loc}}(x_0^*), P^{\mathrm{cs}})$ to an atlas on all of $W^{\mathrm{s}}(\gamma^*)$ via the translation by the map $S_f(t^*)$.

In particular, we have shown that γ^* is a hyperbolic periodic orbit with unstable dimension equal to zero for the reduced semiflow resulting from restricting $S_f(t)$ to $W^{\mathrm{s}}(\gamma^*)$. Under these conditions, the standard ODE proof for in-phase convergence to the periodic orbit γ^* holds; see [Ama83, Theorem 23.10]. This guarantees that for all $\phi \in W^{\mathrm{s}}(\gamma^*)$ there exist $x^*_r \in \gamma^*$ and $\kappa < 1$ such that

$$\lim_{n\to\infty} \frac{\mathcal{S}^n\phi - x^*_r}{\kappa^n} = 0. \tag{3.21}$$

Thus showing the first line in (3.19). To see the second line of (3.19), notice that by (3.21) we can find a neighborhood $U \subset H$ of x_0^* such that $W^{\mathrm{s}}(\gamma^*) \cap$

$U \subset \widetilde{W}^{\mathrm{cs}}_{\mathrm{loc}}(x_0^*)$ for a local center-stable manifold $\widetilde{W}^{\mathrm{cs}}_{\mathrm{loc}}(x_0^*)$ built with respect to $\mathcal{S}$ rather than $\mathcal{P}$. In particular, by Theorem C.3, the tangent space satisfies

$$T_\phi \widetilde{W}^{\mathrm{cs}}_{\mathrm{loc}}(\gamma^*) = \left\{ \psi \in H : \lim_{n\to\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \text{ for all } \kappa > 1 \right\}.$$

Furthermore, locally near x_0^*, both $W^{\mathrm{s}}(\gamma^*)$ and $\widetilde{W}^{\mathrm{cs}}_{\mathrm{loc}}(x_0^*)$ are graphs over H^{cs}. Hence, the description (3.19) holds for all $\phi \in W^{\mathrm{s}}(\gamma^*)$ sufficiently close to x_0^*, using the chain rule shows (3.19) and completes the proof. □

Chapter 4

Morse–Smale property

In this chapter, we develop sufficient conditions to characterize the structural stability of a global attractor $\mathcal{A}(f)$ in terms of the local stability properties of the individual critical elements $\gamma^* \in \mathrm{Crit}(f)$. More precisely, the solution semiflow $S_f(t)$ solving the DDE (1.1) with $f \in \mathfrak{X}^-$ is called **Morse–Smale**, if and only if:

(MS1) $\mathrm{Crit}(f)$ consists of finitely many hyperbolic orbits.

(MS2) The nonwandering set coincides with the set of periodic points.

(MS3) For all $\gamma^*, \gamma^\dagger \in \mathrm{Crit}(f)$, the invariant manifolds $W^{\mathrm{u}}(\gamma^\dagger)$ and $W^{\mathrm{s}}(\gamma^*)$ defined in Chapter 3 intersect **transversely**, that is,

$$T_\phi W^{\mathrm{u}}(\gamma^\dagger) + T_\phi W^{\mathrm{s}}(\gamma^*) = H \quad \text{for all} \quad \phi \in W^{\mathrm{u}}(\gamma^\dagger) \cap W^{\mathrm{s}}(\gamma^*).$$

If an intersection is transverse, we denote it by

$$W^{\mathrm{u}}(\gamma^\dagger) \pitchfork W^{\mathrm{s}}(\gamma^*).$$

We generalize and call the attractor $\mathcal{A}(f)$ **Morse–Smale** if the semiflow $S_f(t)$ is Morse–Smale. Morse–Smale attractors have the quality that regular perturbations of $f \in \mathfrak{X}^-$ leave the global attractor $\mathcal{A}(f)$ orbitally topologically unchanged.

Theorem 4.1 ([Oli00, Theorem 3.2]). *Let $f \in \mathfrak{X}^-$. If $\mathcal{A}(f)$ is Morse–Smale, then there exists $\delta > 0$ such that for all $\tilde{f} \in \mathfrak{X}^-$ satisfying $\|f - \tilde{f}\|_{BC^2} < \delta$*

there is a homeomorphism $\Xi : \mathcal{A}(f) \to \mathcal{A}(\tilde{f})$ mapping orbits of $S_f(t)$ to orbits of $S_{\tilde{f}}(t)$ and preserving the time direction.

Those $\mathcal{A}(f)$ satisfying the conclusions of Theorem 4.1 are called ***A*-stable**. Furthermore, we call any two attractors $\mathcal{A}(f)$ and $\mathcal{A}(\tilde{f})$ related by a homeomorphism Ξ as above **orbit equivalent**. In particular, the phase diagrams $\Gamma(f)$ and $\Gamma(\tilde{f})$ of two orbit equivalent attractors are isomorphic as graphs. Indeed, the orbit homeomorphism Ξ maps $\mathrm{Crit}(f)$ to $\mathrm{Crit}(\tilde{f})$ while preserving the connections, thereby mapping the vertex set of $\Gamma(f)$ to the vertex set of $\Gamma(\tilde{f})$ and keeping the edges. Hence Ξ induces a trivial graph isomorphism via vertex identification.

The main goal of this section is to characterize the Morse–Smale attractors $\mathcal{A}(f)$ with $f \in \mathfrak{X}^-$. Within this setting, our method of proof of the Morse–Smale property for two hyperbolic $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$ is as follows:

Step 1: We relate the zero number evaluated on the invariant manifolds $W^{\mathrm{u}}(\gamma^\dagger)$ and $W^{\mathrm{s}}(\gamma^*)$ to the unstable dimension $i(\gamma^*)$. This allows us to derive a gradient structure whereby $(W^{\mathrm{s}}(\gamma^\dagger)\backslash\gamma^\dagger) \cap (W^{\mathrm{s}}(\gamma^*)\backslash\gamma^*) \neq \emptyset$ implies $i(\gamma^\dagger) > i(\gamma^*)$.

Step 2: As a consequence of Step 1, we show that if all the critical elements $\gamma^* \in \mathrm{Crit}(f)$ are hyperbolic, then the set of nonwandering points $\mathrm{Nw}(f)$ consists of periodic points, only.

Step 3: For all $\phi \in W^{\mathrm{u}}(\gamma^\dagger) \cap W^{\mathrm{s}}(\gamma^*)$, we follow [CCH92] and construct **Oseledets subspaces** $H_j^{\mathrm{u}}(\phi) \leq T_\phi W^{\mathrm{u}}(\gamma^\dagger)$ and $H_j^{\mathrm{s}}(\phi) \leq T_\phi W^{\mathrm{s}}(\gamma^*)$. Thanks to Step 1, the choice can be made so that $\dim H_j^{\mathrm{u}}(\phi) = \operatorname{codim} H_j^{\mathrm{s}}(\phi) = 2j$. Furthermore, the subspaces satisfy $z(\psi) \geq 2j+1$ for all $\psi \in H_j^{\mathrm{s}}(\phi)$ and $z(\psi) \leq 2j-1$ for all $\psi \in H_j^{\mathrm{u}}(\phi)$, which ensures

$$H_j^{\mathrm{u}}(\phi) \oplus H_j^{\mathrm{s}}(\phi) = H.$$

Thus, in practice, checking (MS1) ensures that (MS2)–(MS3) hold automatically. Nevertheless, requiring hyperbolicity is not essential for the existence of transverse intersections. The fast invariant manifolds of a nonhyperbolic equilibrium γ_0 seen in Theorem 3.2 also possess transversality properties that play a vital role in the future discussion of Hopf bifurcations.

In Section 4.1 we introduce the Lyapunov numbers. Specialized technical tools that we use to prove the results in the remainder of the section.

Section 4.2 uses the Lyapunov numbers to establish a relation between the unstable dimension of a critical element and the zero number of the initial conditions lying on its invariant manifolds. In particular, this induces a gradient structure on the global attractor that Mallet-Paret used to define Morse decompositions [MP88].

In Section 4.3, we used ideas inherited from the Morse decomposition in [MP88] and show that if (MS1) holds then (MS2) follows automatically.

In parallel to Section 4.3, Section 4.4 uses the gradient structure above to recover transversality results from [LN17]. Following [CCH92], we extend the ideas in [LN17] and apply them to the case of a nonhyperbolic equilibrium γ_0.

Finally, in Section 4.5, we combine the results above to obtain the chapter's main result: that in our scenario (MS1) is a sufficient condition for checking A-stability.

4.1 Lyapunov numbers

Given a critical element $\gamma^* \in \mathrm{Crit}(f)$, we define the map $\mathcal{S} := S_f(p)$ with $p = 1$ if $\gamma^* = \gamma_0$ and, if γ^* is a periodic orbit, we choose p to be the minimal period. For $x_0^* \in \gamma^*$, recall that the monodromy operator is the Fréchet derivative $M = D\mathcal{S}(x_0^*)$ in H-direction. We consider linear iterations of the form

$$\psi^{(n+1)} = M_n \psi^{(n)}, \quad \psi^{(0)} = \psi \in H, \tag{4.1}$$

where M_n are a family of compact, bounded linear operators on H that converge to M in the operator norm.

Recalling (2.10)–(2.13), the recursion (4.1) enters the DDE (1.1) naturally when considering initial conditions $\phi \in H$ such that $\mathcal{S}^n \phi \to x_0^* \in \gamma^*$. Indeed, denoting by $x(t)$ the solution of (1.1) through $\phi \in H$, we set $M_n := S_c^{(n)}(p, 0)$

to be the time-p solution operator of the initial value problems

$$\begin{aligned} \dot{y}^{(n)}(t) &= c_1^{(n)}(t)y^{(n)}(t) + c_2^{(n)}(t)y^{(n)}(t-1), \quad t \geq 0 \\ y_0^{(n)} &= \psi^{(n)}. \end{aligned} \tag{4.2}$$

If we choose the coefficients $c_j^{(n)}(t)$, $j = 1, 2$ to be given by the linearization along $x(t)$, i.e.,

$$c_j^{(n)}(t) := \partial_j f(x(t+np), x(t+np-1)), \quad j = 1, 2, \tag{4.3}$$

then $\psi^{(n)} = \dot{x}_n$ satisfy (4.1). Moreover, if instead we choose the coefficients

$$\begin{aligned} c_1^{(n)}(t) &= \int_0^1 \partial_1 f(\theta x(t+np) + (1-\theta)x^*(t), x(t+np-1))\mathrm{d}\theta, \\ c_2^{(n)}(t) &= \int_0^1 \partial_2 f(x^*(t), \theta x(t+np-1) + (1-\theta)x^*(t-1))\mathrm{d}\theta, \end{aligned} \tag{4.4}$$

the difference of solutions $\psi^{(n)} = x_n - x_0^*$ solves (4.1). In (4.3)–(4.4) the pointwise limit $c_j^{(n)}(t) \to \partial_j f(x^*(t), x^*(t-1))$ for $j = 1, 2$ implies the limit $M_n \xrightarrow{n\to\infty} M$ in operator norm by the continuity of the evolution in Proposition A.2.

Our main tool for the rest of the chapter will be the asymptotic growth and contraction rates of the sequence $\{\psi^{(n)}\}$ as $n \to \pm\infty$. This can be regarded as a generalization for the characteristic multipliers at a critical element presented in Section 2.4.

Lemma 4.2. *Let $M_n \xrightarrow{n\to\infty} M$ (resp., $M_n \xrightarrow{n\to-\infty} M$) be as above and assume that the sequence $\{\psi^{(n)}\}_{n\geq 0}$ (resp., $\{\psi^{(n)}\}_{n\leq 0}$) satisfies* (4.1)*. Then the quantity*

$$\Lambda^+(\psi) := \inf\left\{\kappa > 0 : \lim_{n\to\infty} \frac{\psi^{(n)}}{\kappa^n} = 0\right\}$$

$$(\textit{resp.},\ \Lambda^-(\psi) := \sup\left\{\kappa > 0 : \lim_{n\to-\infty} \frac{\psi^{(n)}}{\kappa^n} = 0\right\})$$

is well-defined and there exists $\mu \in \mathrm{Spec}(M)$ such that $\Lambda^+(\psi) = |\mu|$ (resp., $\Lambda^-(\psi) = |\mu|$).

Proof. Indeed, by Proposition 2.7, the iteration (4.1) and the monodromy operator M satisfy the assumptions of [CCH92, Corollary B.3 and Theorem B.9]. □

Under the assumptions of Lemma (4.2), the quantity $\Lambda^+(\psi)$ is called **Lyapunov number** of ψ with respect to (4.1). If we consider the limit $n \to -\infty$ instead, we call $\Lambda^-(\psi)$ the **backward Lyapunov number** of ψ with respect to (4.1). If the iteration (4.1) is generated by solving (4.2)–(4.3), the Lyapunov numbers give a measure of the asymptotic distortion rate of the unit sphere in H under the action of the linearized semiflow $D(S_f(t))(\phi)$. Moreover, typically the sequences $\{\psi^{(n)}\}$ define by the iteration (4.1) align with the eigenfunctions of M in the following sense.

Lemma 4.3. *Under the assumptions of Lemma 4.2, let μ_j denote the eigenvalues of M with the ordering in Proposition 2.7. If $\Lambda^+(\psi) \in \{|\mu_{2j-1}|, |\mu_{2j}|\}$ (resp., $\Lambda^-(\psi) \in \{|\mu_{2j-1}|, |\mu_{2j}|\}$), then there exists $\psi^* \in E^j$ and a subsequence $\psi^{(n_k)}$ such that*

$$\boldsymbol{\psi}^{(n_k)} := \frac{\psi^{(n_k)}}{\|\psi^{(n_k)}\|} \xrightarrow{n_k \to \infty} \psi^* \quad \left(resp.,\ \boldsymbol{\psi}^{(n_k)} \xrightarrow{n_k \to -\infty} \psi^*\right).$$

Here E^j denotes the two-dimensional real generalized eigenspace associated with the pair of characteristic multipliers $\{\mu_{2j-1}, \mu_{2j}\}$, as defined in Proposition 2.7.

Proof. Indeed, by [CCH92, Theorem B.4], we have that $\{\boldsymbol{\psi}^{(n)}\}_{n\geq 0}$ is precompact and the compact ω-limit set of the sequence given by

$$\omega\left(\left\{\boldsymbol{\psi}^{(n)}\right\}_{n\geq 0}\right) := \bigcap_{n\geq 0} \operatorname{clos}\left(\left\{\boldsymbol{\psi}^{(m)} : m \geq n\right\}\right),$$

is contained in the unit ball of E^j. Since $\dim E^j = 2$ by Proposition 2.7, the closed unit ball is compact, and the existence of the accumulation point $\psi^* \in E^j$ is guaranteed.

For the case $n \to -\infty$, we apply [CCH92, Theorem B.9], hence the α-limit set of the sequence

$$\alpha\left(\left\{\boldsymbol{\psi}^{(n)}\right\}_{n\leq 0}\right) := \bigcap_{n\leq 0} \operatorname{clos}\left(\left\{\boldsymbol{\psi}^{(m)} : m \leq n\right\}\right)$$

is contained in E^j, which is two-dimensional. In particular, the accumulation point ψ^* exists. □

4.2 Invariant manifolds and the zero number

A posteriori, Lemma 4.2 allows us to reformulate the invariant manifold theory in Chapter 3 in the language of Lyapunov numbers. In particular, Lemma 4.3 and the Sturm–Liouville eigenvalue structure in Proposition 2.7 relate the zero number to the unstable dimension $i(\gamma^*)$ of a critical element $\gamma^* \in \mathrm{Crit}(f)$. To be precise, we have the following.

Lemma 4.4. *Let $\gamma^* \in \mathrm{Crit}(f)$ with $f \in \mathfrak{X}^-$ be hyperbolic. For all $\phi \in W^{\mathrm{u}}(\gamma^*)\setminus\gamma^*$ (resp., $\phi \in W^{\mathrm{s}}(\gamma^*)\setminus\gamma^*$) there exists $x_0^* \in \gamma^*$ such that $z(\phi - x_0^*) \leq i(\gamma^*)$ (resp., $z(\phi - x_0^*) > i(\gamma^*)$).*

Proof. We give the proof for $\phi \in W^{\mathrm{s}}(\gamma^*) \setminus \gamma^*$, the case $\phi \in W^{\mathrm{u}}(\gamma^*) \setminus \gamma^*$ follows along the same arguments after exchanging time directions and is slightly simpler due to the finite dimensionality of $W^{\mathrm{u}}(\gamma^*)$ and the fact that the inequality is not strict. First, we consider a hyperbolic periodic orbit γ^* with minimal period $p > 0$ and later give the details for the hyperbolic equilibrium γ_0.

Given any $\phi \in W^{\mathrm{s}}(\gamma^*) \setminus \gamma^*$ and denoting the solution through ϕ by $x(t)$, we know from Theorem 3.4 that there exists $x_0^* \in \gamma^*$ such that $x_{np} \xrightarrow{n\to\infty} x_0^*$. Moreover, we have discussed in Section 4.1 that $\psi^{(n)} := (x_{np} - x_0^*)$ satisfy the iteration (4.1) for the choice $M_n = S_c^{(n)}(p, 0)$. Here $S_c^{(n)}(p, 0)$ is the time-p evolution solving (4.2) with the coefficients (4.4). Applying Lemma 4.2, there exist a Lyapunov number $\Lambda^+(\psi)$ with respect to the sequence $\{\psi^{(n)}\}_{n\geq 0}$ and $\mu \in \mathrm{Spec}(M)$ such that $\Lambda^+(\psi) = |\mu|$. We distinguish two situations.

First, if $\Lambda^+(\psi) = 0$, then the difference of solutions $(x(t) - x^*(t))$ converges to zero at a superexponential rate, i.e.,

$$\lim_{t\to\infty} \mathrm{e}^{\beta t}|x(t) - x^*(t)| = 0 \quad \text{for all } \beta \in \mathbb{R}.$$

Applying [Cao90, Theorem 2.8], we conclude that $z(\phi - x_0^*) = \infty$ and our claims hold.

Second, if $\Lambda^+(\psi) > 0$, notice that the exponential convergence rate (3.19) in Theorem 3.4 guarantees that $\Lambda^+(\psi) < 1$. Denoting by μ_k the eigenvalues of M ordered as in Proposition 2.7, there exists an eigenvalue pair such that $\Lambda^+(\psi) \in \{|\mu_{2j-1}|, |\mu_{2j}|\}$. Therefore, Lemma 4.3 shows that the normalized sequence

$$\boldsymbol{\psi}^{(n)} := \frac{\psi^{(n)}}{\|\psi^{(n)}\|},$$

accumulates to an eigenfunction $\psi^* \in E^j$. Recalling from Proposition 2.7 that $z(\psi_j) = 2j-1$ for all $\psi_j \in E^j$, we show that $\lim_{n\to\infty} z(\boldsymbol{\psi}^{(n)}) = z(\psi^*)$. To this end, we consider a small modification and shift the iteration by defining

$$\widehat{\boldsymbol{\psi}}^{(n)} := \frac{S_f(np+t^*)\phi - x^*_{t^*}}{\|S_f(np+t^*)\phi - x^*_{t^*}\|} \xrightarrow{n\to\infty} \frac{D(S_f(t^*))(x^*_0)\psi}{\|D(S_f(t^*))(x^*_0)\psi\|} =: \widehat{\psi}^*. \tag{4.5}$$

Recalling that ψ^* is an eigenfunction of M, Proposition 2.2 shows that $\widehat{\psi}^*$ possesses only simple zeros and satisfies $z(\widehat{\psi}^*) = z(\psi^*)$. Furthermore, choosing $t^* > 1$ in such a way that $(\widehat{\psi}^*(0), \widehat{\psi}^*(-1)) \neq 0$, we apply the continuity of the zero number (2.17) in Lemma 2.1 to obtain $z(\widehat{\boldsymbol{\psi}}^{(n)}) = z(\widehat{\psi}^*) = 2j-1$ for $n \in \mathbb{N}$ sufficiently large.

The monotonicity of the zero number in Proposition 2.2, ensures that $z(\phi - x^*_0) \geq z(\widehat{\boldsymbol{\psi}}^{(n)}) \geq 2j-1$. It is left to compare $2j-1$ to $i(\gamma^*)$. For this, recall that $\dot{x}^*_0$ is an eigenfunction of M associated with the trivial multiplier 1. In particular, $1 > \Lambda^+(\psi)$ guarantees that $z(\psi^*) \geq z(\dot{x}^*_0)$ with the equality happening only for the spectral configuration (C2) in Section 3.3. That is, if $1 = |\mu_{2j-1}|$, $\Lambda^+(\psi) = |\mu_{2j}|$, and $2j-1 = i(\gamma^*)+1$. Since $z(\dot{x}^*_0) \geq i(\gamma^*)$, by Lemma 2.6 (vi), we have that $z(\psi^*) > i(\gamma^*)$ in either case.

For a hyperbolic equilibrium $\gamma^* = \gamma_0$, a completely analogous study takes $p = 1$ and $x^*_0 = 0$. The main difference is that the eigenvalues of M come in complex conjugate pairs; see Lemma 2.9. Hence we obtain that $\Lambda^+(\psi) = |\mu_{2j-1}| = |\mu_{2j}| < 1$. However, we still have the bound $i(\gamma^*) > 2j-1$, ensuring the lemma holds. □

Notice that the exponential in-phase convergence towards a hyperbolic periodic orbit in Theorem 3.4 is vital in showing the strict inequality $z(\phi - x^*_0) > i(\gamma^*)$ for $\phi \in W^{\mathrm{s}}(\gamma^*) \setminus \gamma^*$ in Lemma 4.4. Nevertheless, if phase convergence is removed, a weak version of Lemma 4.4 still holds by applying the same methods.

Lemma 4.5. *Under the assumptions of Lemma 4.4, the zero number satisfies* $z(\phi - x_0^*) \leq i(\gamma^*)$ *(resp.,* $z(\phi - x_0^*) \geq i(\gamma^*)$*) for all* $\phi \in W^{\mathrm{u}}(\gamma^*) \setminus \gamma^*$ *(resp.,* $\phi \in W^{\mathrm{s}}(\gamma^*) \setminus \gamma^*$*) and* $x_0^* \in \gamma^*$*.*

Proof. Indeed, $\mathcal{S}^n\phi$ may converge to $x_{t^*}^* \in \gamma^*$ such that $x_{t^*}^* \neq x_0^*$. Taking limits and applying Proposition 2.2 together with Lemma 2.6 (vi) we obtain that

$$\begin{aligned} z(\phi - x_0^*) &\geq z(x_{t^*}^* - x_0^*) \\ &= z(\dot{x}_0^*) \\ &\geq i(\gamma^*). \end{aligned}$$

□

Furthermore, the proof also applies if we consider an isolated nonhyperbolic equilibrium $\gamma_0 \in \mathrm{Crit}(f)$.

Lemma 4.6. *Let* $\gamma_0 \in \mathrm{Crit}(f)$ *with* $f \in \mathfrak{X}^-$ *be isolated and nonhyperbolic. If* γ_0 *satisfies Theorem 3.2 Case 1 (resp., Case 2), then* $z(\phi) < i(\gamma_0)$ *(resp.,* $z(\phi) > i(\gamma_0)$*) for all* $\phi \in W^{\mathrm{uu}}(\gamma_0) \setminus \gamma_0$ *(resp.,* $\phi \in W^{\mathrm{ss}}(\gamma_0) \setminus \gamma_0$*).*

Proof. Indeed, the proof of Lemma 4.4 relies upon showing the strict inequalities $\Lambda^+(\psi) < 1$ and $\Lambda^-(\psi) < 1$ for the Lyapunov numbers with respect to the iteration (4.1) obtained by choosing $\psi := \phi - 0$ and $\psi^{(n)} := S_f(n)\phi$. In this case, the Lyapunov number inequalities are enforced by considering initial conditions on the fast invariant manifolds. The proof is then completed by mimicking Lemma 4.4. We point out that, unlike in Lemma 4.7, in this case the inequality is strict on the fast unstable dimension. □

As a result, Lemma 4.4 induces a gradient structure on the set $\mathrm{Crit}(f)$. More precisely, following an argument of Czaja and Rocha [CR08], we show that the orbits connecting critical elements are only allowed to go in the direction decreasing the Morse index.

Corollary 4.7. *Let* $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$ *with* $f \in \mathfrak{X}^-$ *be hyperbolic and assume* $(W^{\mathrm{u}}(\gamma^\dagger) \setminus \gamma^\dagger) \cap (W^{\mathrm{s}}(\gamma^*) \setminus \gamma^*) \neq \emptyset$*. Then* $i(\gamma^\dagger) > i(\gamma^*)$*. In particular, there exists no homoclinic connection to* γ^**, i.e.,* $(W^{\mathrm{u}}(\gamma^*) \setminus \gamma^*) \cap (W^{\mathrm{s}}(\gamma^*) \setminus \gamma^*) = \emptyset$*.*

Proof. Indeed, take $\phi \in (W^{\mathrm{u}}(\gamma^{\dagger}) \setminus \gamma^{\dagger}) \cap (W^{\mathrm{s}}(\gamma^{*}) \setminus \gamma^{*})$ and let $x(t)$ with $t \in \mathbb{R}$ solve (1.1) with initial condition $x_0 = \phi$. Choosing $x_0^{\dagger}$ and x_0^{*} as in Lemma 4.4, recall that the difference of solutions $(x(t) - x^{\iota}(t))$ solves (2.10) with the coefficients given by (2.13) for $\iota = \dagger, *$. Hence, by Proposition 2.2 and Lemma 4.4, we have that

$$\begin{aligned} z(x_0 - x_0^{\dagger}) &\leq z(x_t - x_t^{\dagger}) \\ &\leq i(\gamma^{\dagger}) \quad \text{for all } t \leq 0, \end{aligned}$$

and

$$\begin{aligned} z(x_0 - x_0^{*}) &\geq z(x_t - x_t^{*}) \\ &> i(\gamma^{*}) \quad \text{for all } t \geq 0. \end{aligned}$$

By Lemma 2.6 (vi) and Proposition 2.2, all the zeros of $(x_t^{\dagger} - x_t^{*})$ are simple, moreover, shifting t if necessary, we may assume that $(x_t^{\dagger}(0) - x_t^{*}(0)) \neq 0$ and $(x_t^{\dagger}(-1) - x_t^{*}(-1)) \neq 0$. By the continuity of the zero number (2.17) in Lemma 2.1, there exists $\delta > 0$ such that if $\|x_{t-1} - x_{t-1}^{\dagger}\| < \delta$, then we have that $z(x_t - x_t^{\dagger} + x_t^{\dagger} - x_t^{*}) = z(x_t^{\dagger} - x_t^{*})$. Since $\phi \in W^{\mathrm{u}}(\gamma^{\dagger})$, we may always choose $t^{\dagger} < 0$ so small that

$$\begin{aligned} i(\gamma^{*}) &< z(x_0 - x_0^{*}) \\ &\leq z(x_{t^{\dagger}} - x_{t^{\dagger}}^{\dagger} + x_{t^{\dagger}}^{\dagger} - x_{t^{\dagger}}^{*}) \\ &= z(x_{t^{\dagger}}^{\dagger} - x_{t^{\dagger}}^{*}). \end{aligned} \tag{4.6}$$

Furthermore, since $\phi \in W^{\mathrm{s}}(\gamma^{*}) \setminus \gamma^{*}$, we may argue similarly for a sufficiently large $t^{*} > 0$ to obtain that

$$\begin{aligned} z(x_{t^{*}}^{\dagger} - x_{t^{*}}^{*}) &= z(x_{t^{*}}^{\dagger} + x_{t^{*}}^{*} - x_{t^{*}} - x_{t^{*}}^{*}) \\ &= z(x_{t^{*}}^{\dagger} - x_{t^{*}}) \\ &\leq z(x_0 - x_0^{\dagger}) \leq i(\gamma^{\dagger}). \end{aligned} \tag{4.7}$$

Finally, by Lemma 2.6 (vi) we have that $z(x_{t^{\dagger}}^{\dagger} - x_{t^{\dagger}}^{*}) = z(x_{t^{*}}^{\dagger} - x_{t^{*}}^{*})$, hence we combine (4.6)–(4.7) to obtain $i(\gamma^{*}) < i(\gamma^{\dagger})$, as claimed. □

4.3 Nonwandering set

In this section, we characterize the nonwandering set $\mathrm{Nw}(f)$, provided all $\gamma^{*} \in \mathrm{Crit}(f)$ are hyperbolic. To this end, notice that there is a minimal distance between any two hyperbolic critical elements.

Lemma 4.8. *Let γ^* be a hyperbolic critical element of the DDE* (1.1)*, then γ^* is isolated in* $\mathrm{Crit}(f)$*, i.e., there exists $\delta > 0$ such that* $\mathrm{dist}(\gamma^*, \gamma^\dagger) > \delta$ *for all $\gamma^* \neq \gamma^\dagger \in \mathrm{Crit}(f)$. In particular, if all $\gamma^* \in \mathrm{Crit}(f)$ are hyperbolic, then* $\mathrm{Crit}(f)$ *is finite.*

Proof. Recall from Chapter 3 that any point $x_0^* \in \gamma^*$ is fixed under a C^1-map, given by

- the time-1 map $\mathcal{S} := S_f(1)$ acting on H, if γ^* is an equilibrium;
- the Poincaré map $\mathcal{P}$ defined in (3.17) and acting on a section $\mathcal{U}_0$ transverse to γ^* at x_0^*.

Thus, by Lemma 3.3, assuming that γ^* is hyperbolic is sufficient to guarantee that $\mathcal{S}(\phi) - \phi$ and $\mathcal{P}(\phi) - \phi$ are invertible close to x_0^*. In particular, x_0^* is the only fixed point in a δ-ball around x_0^*. Recalling that $\mathrm{Crit}(f) \subset \mathcal{A}(f)$ and $\mathcal{A}(f)$ is compact we obtain that the cardinality of $\mathrm{Crit}(f)$ is finite. □

Combining Lemma 4.8 with the gradient structure in Corollary 4.7, we obtain that hyperbolic critical elements are not only isolated within $\mathrm{Crit}(f)$, but also as $S_f(t)$-invariant sets.

Corollary 4.9. *Condider $f \in \mathfrak{X}^-$ and let $\gamma^* \in \mathrm{Crit}(f)$ be hyperbolic. Then there exists $\delta > 0$ such that γ^* is the maximal $S_f(t)$-invariant set contained in*

$$U_\delta(\gamma^*) := \{\phi \in H : \mathrm{dist}(\phi, \gamma^*) < \delta\}.$$

That is, γ^* is an isolated $S_f(t)$-invariant set.

Proof. Indeed, suppose that for all $n \in \mathbb{N}$ there exists a $S_f(t)$-invariant set $V_n \subset U_{1/n}(\gamma^*) \setminus \gamma^*$. Lemma 4.8 together with the Poincaré–Bendixson theorem in Proposition 2.4 guarantees that for n sufficiently large $\omega(\phi) = \gamma^*$ for all $\phi \in V_n$. Furthermore, applying Proposition 2.4 to $\alpha(\phi)$, we obtain that $\gamma^* \subset \alpha(\phi)$ for all $\phi \in V_n$. It follows that, $(W^{\mathrm{u}}(\gamma^*) \setminus \gamma^*) \cap (W^{\mathrm{s}}(\gamma^*) \setminus \gamma^*) \neq \emptyset$, in contradiction to Corollary 4.7. □

Next, we introduce a general auxiliary lemma that captures the ideas that Mallet-Paret used to show a Morse decomposition for $\mathcal{A}(f)$; see [MP88].

More recently, Sheng et al. have used a similar approach to characterize the nonwandering set in reaction-diffusion PDEs (1.6); see [SWZ21].

Lemma 4.10. *Consider $\phi \in \mathrm{Nw}(f)$ with $f \in \mathfrak{X}^-$ and assume that $\phi \notin \omega(\phi)$ and $\omega(\phi)$ is an isolated $S_f(t)$-invariant set. Then there exist sequences $\{\tilde{\phi}^{(n)}\}_{n\geq 0} \subset H$, $\tau_n \to \infty$ and a point $\tilde{\phi} \in H \setminus \omega(\phi)$ such that $\alpha(\tilde{\phi}) \subset \omega(\phi)$, $\tilde{\phi}^{(n)} \to \tilde{\phi}$, and $S_f(\tau_n)\tilde{\phi}^{(n)} \to \phi$.*

Proof. Consider an open $U \subset H$ such that $\phi \notin U$ and $\omega(\phi)$ is the only $S_f(t)$-invariant in U. By definition, we can find a sequence $t_n \to \infty$ such that for all $n \in \mathbb{N}$ we have $\mathrm{dist}(S_f(t_n)\phi, \omega(\phi)) < 1/n$. Moreover, since $\phi \in \mathrm{Nw}(f)$, we can also find sequences $\phi^{(n)} \to \phi$ and $t_n^* \to \infty$ such that $t_n^* > t_n$, $S_f(t_n^*)\phi^{(n)} \to \phi$ and $\mathrm{dist}(S_f(t_n)\phi^{(n)}, \omega(\phi)) < 1/n$. Recalling that $\phi \notin U$, for n large enough, there exists a sequence of first exit times from U after t_n, i.e., $t_n < \tilde{t}_n < t_n^*$ such that $S_f(\tilde{t}_n)\phi^{(n)} \in \partial U$.

The sequence $\{S_f(\tilde{t}_n)\phi^{(n)}\}_{n\geq 0}$ is precompact and possesses an accumulation point because the semiflow $S_f(t)$ is compact by Proposition A.2. Taking a subsequence if needed, we denote $\tilde{\phi} := S_f(\tilde{t}_n)\phi^{(n)} \in \partial U$. Furthermore, since $\tilde{t}_n - t_n \to \infty$, a diagonal argument allows us to choose nested subsequences so that the prehistories $S_f(-m)\tilde{\phi} := \lim_{n\to\infty} S_f(\tilde{t}_n - m)\phi^{(n)}$ are defined and satisfy $S_f(-m)\tilde{\phi} \in \mathring{U}$ for all $m \in \mathbb{N}$. Since $\omega(\phi)$ is the only compact invariant set inside $\mathring{U}$, we conclude that the sequence $\{S_f(-m)\tilde{\phi}\}_{m\geq 0} \subset \mathring{U}$ accumulates to $\omega(\phi)$, which guarantees $\alpha(\tilde{\phi}) \subset \omega(\phi)$. Setting $\tau_n := t_n^* - \tilde{t}_n$ and $\tilde{\phi}^{(n)} := S_f(\tilde{t}_n)\phi^{(n)}$ concludes the proof. □

We have formulated Lemma 4.10 with $f \in \mathfrak{X}^-$ for convenience. However, the key ingredient is that the semiflow $S_f(t)$ acts compactly on a metric space. Thus Lemma 4.10 admits more general reformulations and does not require the even-odd symmetry (2.2). Under these conditions, we can combine the zero number estimates in Lemma 4.4 with Lemma 4.10 to obtain the following.

Proposition 4.11. *Consider an initial condition $\phi \in \mathrm{Nw}(f)$ with $f \in \mathfrak{X}^-$. If $\omega(\phi) \in \mathrm{Crit}(f)$ is hyperbolic, then $\phi \in \omega(\phi)$. In other words, $\phi \in \mathrm{Per}(f)$.*

Proof. By Corollary 4.9, $\gamma^* := \omega(\phi)$ is an isolated invariant set. We proceed by contradiction and suppose that $\phi \notin \omega(\phi)$. By Lemma 4.10, there exists an element $\tilde{\phi} \in H$ such that $\alpha(\tilde{\phi}) \subset \omega(\phi)$ together sequences $\{\tilde{\phi}^{(n)}\}_{n\geq 0} \subset H$ and $\tau_n \to \infty$ such that $\tilde{\phi}^{(n)} \to \tilde{\phi}$ and $S_f(\tau_n)\tilde{\phi}^{(n)} \to \phi$.

Thus, we have that $\phi \in W^{\mathrm{s}}(\gamma^*)\setminus\gamma^*$ and $\tilde{\phi} \in W^{\mathrm{u}}(\gamma^*)\setminus\gamma^*$. Furthermore, by Lemma 4.4, we can find $x_0^* \in \gamma^*$ such that $z(\tilde{\phi}-x_0^*) < i(\gamma^*)$. In particular, we choose $t^* > 0$ such that $S_f(t^*)\tilde{\phi}-x^*_{t^*}$ possesses solely simple zeros, all of which are contained in the open interval $(-1, 0)$. By the continuity (2.17) in Lemma 2.1, for n sufficiently large we have that $z(S_f(t^*)\tilde{\phi}^{(n)}-x^*_{t^*}) = z(S_f(t^*)\tilde{\phi}-x^*_{t^*})$. Furthermore, using the lower semicontinuity (2.16) of the zero number in Lemma 2.1, we can choose subsequences such that $z(S_f(\tau_n)\tilde{\phi}^{(n)} - x^*_{\tau_n}) \geq z(\phi - x^*_{\tau_n})$. Since $\phi \in W^{\mathrm{u}}(\gamma^*)\setminus\gamma^*$, Lemma 4.5 guarantees $z(\phi - x^*_{\tau_n}) \geq i(\gamma^*)$. Thus the monotonicity in Proposition 2.2 ensures

$$\begin{aligned} i(\gamma^*) &> z(S_f(t^*)\tilde{\phi} - x^*_{t^*}) \\ &= z(S_f(t^*)\tilde{\phi}^{(n)} - x^*_{t^*}) \\ &\geq z(S_f(\tau_n)\tilde{\phi}^{(n)} - x^*_{\tau_n}) \\ &\geq z(\phi - x^*_{\tau_n}) \\ &\geq i(\gamma^*). \end{aligned}$$

Hence we reached a contradiction and $\phi \in \omega(\phi)$. □

4.4 Transversality

The goal of this section is to prove the transversality results announced Chapter 1. A key element of the proof are the so-called **Oseledets subspaces** associated with the asymptotic expansion and contraction rates for the variational semiflow $D(S_f(t))(\phi)$ for $\phi \in H$.

Lemma 4.12. *Let $\gamma^* \in \mathrm{Crit}(f)$ with $f \in \mathfrak{X}^-$ be either a hyperbolic periodic orbit or, in case $\gamma^* = \gamma_0$, assume it is isolated. We denote by μ_k the characteristic multipliers of γ^* ordered with repetitions as in Proposition 2.2 and consider $\kappa > 0$ so that*

$$|\mu_{2j}| > \kappa > |\mu_{2j+1}|.$$

Then for all $\phi \in W^{\mathrm{u}}(\gamma^*)$, the set

$$H_j^{\mathrm{u}}(\phi) := \left\{ \psi \in H : \lim_{n\to-\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \right\}$$

is a closed $2j$*-dimensional subspace of* H *and* $z(\psi) \leq 2j-1$ *for all* $\psi \in H^{\mathrm{u}}_j(\phi)$. *Analogously, for all* $\phi \in W^{\mathrm{s}}(\gamma^*)$*, the set*

$$H^{\mathrm{s}}_j(\phi) := \left\{ \psi \in H : \lim_{n\to\infty} \frac{D\mathcal{S}^n(\phi)\psi}{\kappa^n} = 0 \right\}$$

is a closed codimension-$2j$ *subspace of* H *and* $z(\psi) \geq 2j+1$ *for all* $\psi \in H^{\mathrm{s}}_j(\phi)$.

Proof. We show the claims for $H^{\mathrm{s}}_j(\phi)$, and the proof is analogous for $H^{\mathrm{u}}_j(\phi)$ by exchanging the time direction and considering backward Lyapunov exponents instead. Since $\mathcal{S}^n\phi \xrightarrow{n\to\infty} \gamma^*$, we prove the lemma for $H^{\mathrm{s}}_j(\mathcal{S}^n\phi)$ with large $n \geq 0$ first and then show that we can recover our conclusions for the preimage $H^{\mathrm{s}}_j(\phi) = (D\mathcal{S}^n(\phi))^{-1}(H^{\mathrm{s}}_j(\mathcal{S}^n\phi))$.

Recall from Theorems 3.1–3.4 that there exists $x^*_0 \in \gamma^*$ acting as the limit $\mathcal{S}^n\phi \xrightarrow{n\to\infty} x^*_0$. Thus, denoting by $x(t)$ the solution through $\phi \in H$, the remarks in Section 4.1 show that $M_n := S^{(n)}_c(p, 0)$ generated by (4.2) with coefficients (4.4) yields an iteration of the type (4.1). Moreover, from Proposition A.2, we have that $M_n = D\mathcal{S}(\mathcal{S}^n\phi)$ and the chain rule yields that $D\mathcal{S}^n(\phi) = M_n \circ \cdots \circ M_0$.

Since $M_n \xrightarrow{n\to\infty} M$, we use a Lipschitz invariant manifold theorem for uniformly bounded sequences $\{\psi^{(n)}\}_{n\geq 0}$ satisfying (4.1). More precisely, using [CCH92, Proposition B.8], there exists $n_0 \geq 0$ such that for all $n \geq n_0$ the set $H^{\mathrm{s}}_j(\mathcal{S}^n\phi)$ is isomorphic via the canonical projection to $(\oplus_{k\geq j} E^j)^\perp$. Here E^j are the real generalized eigenspaces defined in Proposition 2.2, and $\perp$ denotes the closed complement in H. In particular, $H^{\mathrm{s}}_j(\mathcal{S}^n\phi) \leq H$ is a closed subspace of the claimed codimension.

To see the bound on the zero number, consider $\psi \in H^{\mathrm{s}}_j(\phi)$. The Lyapunov number lemma 4.2 with respect to the iteration (4.1) with M_n as above shows that there exists $\mu \in \mathrm{Spec}(M)$ such that $\Lambda^+(\psi) = |\mu|$. If $\Lambda^+(\psi) = 0$, then $\psi^{(n)}$ approach 0 superexponentially and, using [Cao90, Theorem 2.8] and the monotonicity of the zero number in Proposition 2.2, we obtain $z(\psi) \geq z(\psi^{(n)}) = \infty$. Otherwise, $\Lambda^+(\psi) > 0$ and, by assumption, $\Lambda^+(\psi) \leq \kappa < |\mu_{2j}|$. Therefore, by Lemma 4.3, there exists an eigenfunction $\psi^* \in E^l$, $l > j$ such that $z(\psi^*) = 2l - 1 \geq 2j + 1$ and

$$\boldsymbol{\psi}^{(n)} := \frac{\psi^{(n)}}{\|\psi^{(n)}\|} \to \psi^*.$$

In analogy to (4.5), consider the shifted sequence

$$\widehat{\boldsymbol{\psi}}^{(n)} := \frac{D(S_f(t^*))(\mathcal{S}^n\phi)\psi^{(n)}}{\|D(S_f(t^*))(\mathcal{S}^n\phi)\psi^{(n)}\|} \xrightarrow{n\to\infty} \frac{D(S_f(t^*))(x_0^*)\psi^*}{\|D(S_f(t^*))(x_0^*)\psi^*\|} =: \widehat{\psi}^*.$$

Recall that ψ^* is an eigenfunction of the monodromy operator M, therefore, by the strict zero dropping in Proposition 2.2, $\widehat{\psi}^*$ possesses finitely many simple zeros and we can always choose $t^* \geq 1$ so that $\widehat{\psi}^*(0)\widehat{\psi}^*(-1) \neq 0$. Using Lemma 2.1, there exists a $n_0 > 0$ sufficiently large so that

$$\begin{aligned} z\left(\psi^{(n)}\right) &= z(\psi^*) \\ &\geq 2j+1 \end{aligned} \tag{4.8}$$

for all $n \geq n_0$ and the proof of Lemma 4.12 is complete for $H_j^{\mathrm{s}}(\mathcal{S}^n\phi)$.

To show the claims for $H_j^{\mathrm{s}}(\phi)$, notice that $H_j^{\mathrm{s}}(\mathcal{S}^n\phi) = \ker(\ell)$ for a bounded linear operator $\ell : H \to \mathbb{R}^{2j}$ with total range. Recalling that $D\mathcal{S}^n(\phi)$ has dense range by Lemma A.5, we obtain that $\ell \circ D\mathcal{S}^n(\phi)$ has total range and $H_j^{\mathrm{s}}(\phi) = \ker(\ell \circ D\mathcal{S}^n(\phi))$ is a codimension-$2j$ closed subspace of H. Furthermore, for all $\psi \in H_j^{\mathrm{s}}(\phi)$ Proposition 2.2 and (4.8) yield $z(\psi) \geq z(\psi^{(n)}) \geq 2j+1$ for all $\psi^{(n)} \in H_j^{\mathrm{s}}(\mathcal{S}^n\phi)$, which completes the proof. □

We highlight that the subspaces produced in Lemma 4.12 depend on j, only, rather than depending on the choice of κ. The underlying reason is that the Lyapunov numbers in Lemma 4.2 only take a discrete set of values determined by the characteristic multipliers $\mathrm{Spec}(M)$. Choosing $1 > |\mu_{2j+1}|$, the subspaces $H_j^{\mathrm{s}}(\phi)$ yield the **Oseledets filtration**

$$T_\phi W^{\mathrm{s}}(\gamma^*) \geq H_j^{\mathrm{s}}(\phi) \geq H_{j+1}^{\mathrm{s}}(\phi) \geq \ldots,$$

where $H_j^{\mathrm{s}}(\phi) \leq T_\phi W^{\mathrm{s}}(\gamma^*)$ follows from the characterization of the tangent spaces that we gave in Theorems 3.1 and 3.4. This insight is the main tool used in proving the following result, which can also be found in [LN17].

Theorem 4.13. *Let $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$ with $f \in \mathfrak{X}^-$ be hyperbolic. Then*

$$W^{\mathrm{u}}(\gamma^\dagger) \pitchfork W^{\mathrm{s}}(\gamma^*).$$

Proof. In case $W^{\mathrm{u}}(\gamma^\dagger) \cap W^{\mathrm{s}}(\gamma^*) = \emptyset$, the proof is finished. Otherwise, we consider an initial condition $\phi \in W^{\mathrm{u}}(\gamma^\dagger) \cap W^{\mathrm{s}}(\gamma^*)$ and, by Corollary 4.7, we have that $i(\gamma^\dagger) > i(\gamma^*)$. We denote the characteristic multipliers of $\gamma^\dagger$ and

γ^* by $\mu_k^\dagger$ and μ_k^*, respectively, and adopt the ordering with repetitions from Proposition 2.7. Let us define

$$j := \begin{cases} i(\gamma^*)/2, & \text{if } i(\gamma^*) \text{ is even,} \\ (i(\gamma^*)+1)/2, & \text{otherwise,} \end{cases}$$

and emphasize that, by assumption, we have $i(\gamma^\dagger) \geq 2j \geq i(\gamma^*)$. Therefore, we always have that

$$|\mu_{2j}^\dagger| \geq 1 \quad \text{and} \quad 1 \geq |\mu_{2j+1}^*|, \tag{4.9}$$

moreover, by the choice of j, one of the two inequalities in (4.9) is always strict. Hence we choose an arbitrary κ such that $|\mu_{2j}^\dagger| > \kappa > |\mu_{2j+1}^*|$ and apply Lemma 4.12. In this way, we obtain closed subspaces such that $\dim H_j^{\mathrm{u}}(\phi) = \operatorname{codim} H_j^{\mathrm{s}}(\phi) = 2j$, also for all $\psi \in H_j^{\mathrm{u}}(\phi) \cap H_j^{\mathrm{s}}(\phi)$ we have that $2j+1 \leq z(\psi) \leq 2j-1$. In particular, we have $H_j^{\mathrm{u}}(\phi) \cap H_j^{\mathrm{s}}(\phi) = \emptyset$, thereby yielding that $H_j^{\mathrm{u}}(\phi) + H_j^{\mathrm{s}}(\phi) = H$. Furthermore, by the characterization of the tangent spaces in Theorems 3.1 and 3.4, we have that $H_j^{\mathrm{u}}(\phi) \leq T_\phi W^{\mathrm{u}}(\gamma^\dagger)$ and $H_j^{\mathrm{s}}(\phi) \leq T_\phi W^{\mathrm{s}}(\gamma^*)$, obtaining $W^{\mathrm{u}}(\gamma^\dagger) \pitchfork W^{\mathrm{s}}(\gamma^*)$. □

Theorem 4.13 showcases that the most important aspect of our transverse intersections is the existence of a sufficiently large difference in unstable dimensions between $\gamma^\dagger$ and γ^*. Indeed, transverse intersections still happen for invariant manifolds involving nonhyperbolic equilibria.

Theorem 4.14. *Consider $\gamma_0, \gamma^* \in \mathrm{Crit}(f)$ with $f \in \mathfrak{X}^-$. We assume that γ_0 is isolated and nonhyperbolic and γ^* is a hyperbolic periodic orbit. Then*

$$W^{\mathrm{u}}(\gamma_0) \pitchfork W^{\mathrm{s}}(\gamma^*) \quad \text{and} \quad W^{\mathrm{u}}(\gamma^*) \pitchfork W^{\mathrm{s}}(\gamma_0), \tag{4.10}$$

furthermore, we distinguish the following two situations:

Case 1: *If γ_0 satisfies Theorem 3.2 Case 1, then $W^{\mathrm{uu}}(\gamma_0) \pitchfork W^{\mathrm{s}}(\gamma^*)$.*

Case 2: *If γ_0 satisfies Theorem 3.2 Case 2, then $W^{\mathrm{u}}(\gamma^*) \pitchfork W^{\mathrm{ss}}(\gamma_0)$.*

Proof. The proof is the same as in Theorem (4.13). We show (4.10), and Cases 1 and 2 follow as a consequence.

Let first $\phi \in W^{\mathrm{u}}(\gamma_0) \cap W^{\mathrm{s}}(\gamma^*) \neq \emptyset$, then Lemma 4.5 and the arguments in Corollary 4.7 show that $i(\gamma_0) \geq i(\gamma^*)$. Moreover, by Lemma 2.9, $i(\gamma_0)$ is even and we choose

$$j := i(\gamma_0)/2$$

which yields $i(\gamma_0) = 2j \geq i(\gamma^*)$. We use the order from Proposition 2.2 for the characteristic multipliers μ_k^0 and μ_k^* of γ_0 and γ^*, respectively. Thus we have that $|\mu_{2j}^0| > 1 \geq |\mu_{2j+1}^*|$ and, by Theorems 3.2 and 3.4, the subspaces $H_j^{\mathrm{u}}(\phi)$ and $H_j^{\mathrm{s}}(\phi)$ obtained by Lemma 4.12 satisfy $H_j^{\mathrm{u}}(\phi) \leq T_\phi W^{\mathrm{u}}(\gamma_0)$ and $H_j^{\mathrm{s}}(\phi) \leq T_\phi W^{\mathrm{s}}(\gamma^*)$. Furthermore, $\dim H_j^{\mathrm{u}}(\phi) = \operatorname{codim} H_j^{\mathrm{s}}(\phi) = 2j$ and for all $\psi \in H_j^{\mathrm{u}}(\phi) \cap H_j^{\mathrm{s}}(\phi)$ we have $2j+1 \leq z(\psi) \leq 2j-1$. Thus $H = H_j^{\mathrm{u}}(\phi) \oplus H_j^{\mathrm{s}}(\phi)$ and, as we claimed, $W^{\mathrm{u}}(\gamma_0) \pitchfork W^{\mathrm{s}}(\gamma^*)$. Notice that if we are in Case 1 above, we can take $\phi \in W^{\mathrm{uu}}(\gamma_0) \cap W^{\mathrm{s}}(\gamma^*)$ and the same choice of subspaces satisfies $H_j^{\mathrm{u}}(\phi) \leq T_\phi W^{\mathrm{uu}}(\gamma_0)$ and $H_j^{\mathrm{s}}(\phi) \leq T_\phi W^{\mathrm{s}}(\gamma^*)$. Hence we actually showed $W^{\mathrm{uu}}(\gamma_0) \pitchfork W^{\mathrm{s}}(\gamma^*)$ and proved Case 1.

Assume next that $\phi \in W^{\mathrm{u}}(\gamma^*) \cap W^{\mathrm{s}}(\gamma_0) \neq \emptyset$, then we prove (4.10). As above, we have that $i(\gamma^*) \geq i(\gamma_0)$. However, since $i(\gamma_0)$ is even, the inequality along $W^{\mathrm{s}}(\gamma_0) \setminus \gamma_0$ in Lemma 2.9 is strict, and yields the strict inequality $i(\gamma^*) = i(\gamma_0)$. Denoting $j = i(\gamma_0)/2$, then $i(\gamma^*) \geq 2j+1$. Using the same convention as above for the characteristic multipliers, we obtain the spectral splitting $|\mu_{2j+2}^*| \geq 1 > |\mu_{2j+3}^0|$. Thus Theorems 3.2 and 3.4 ensure that the subspaces $H_{j+1}^{\mathrm{u}}(\phi)$ and $H_{j+1}^{\mathrm{s}}(\phi)$ in Lemma 4.12 satisfy $H_{j+1}^{\mathrm{u}}(\phi) \leq T_\phi W^{\mathrm{u}}(\gamma^*)$ and $H_{j+1}^{\mathrm{s}}(\phi) \leq T_\phi W^{\mathrm{s}}(\gamma_0)$. Arguing as above, we obtain that $H = H_{j+1}^{\mathrm{u}}(\phi) \oplus H_{j+1}^{\mathrm{s}}(\phi)$ and $W^{\mathrm{u}}(\gamma^*) \pitchfork W^{\mathrm{s}}(\gamma_0)$. Finally, notice that if we are in Case 2, then for any $\phi \in W^{\mathrm{u}}(\gamma^*) \cap W^{\mathrm{ss}}(\gamma_0)$ the choice of subspaces above satisfies $H^{\mathrm{u}}(\phi) \leq T_\phi W^{\mathrm{u}}(\gamma^*)$ and $H_{j+1}^{\mathrm{s}}(\phi) \leq T_\phi W^{\mathrm{ss}}(\gamma_0)$. This proves Case 2. $\square$

4.5 Structural stability

Combining the results of Section 4.3 and Section 4.4, we conclude that all we have to do to show that a global attractor $\mathcal{A}(f)$ is A-stable is to check that all critical elements $\mathrm{Crit}(f)$ are hyperbolic, i.e., condition (MS1) above. As a result, the global structural stability of $\mathcal{A}(f)$ follows from the local linearization at an equilibrium or periodic orbit.

Theorem 4.15. *Given the DDE* (1.1) *with* $f \in \mathfrak{X}^-$. *If all critical elements* $\gamma^* \in \mathrm{Crit}(f)$ *are hyperbolic, then* $\mathcal{A}(f)$ *is Morse–Smale and therefore*

A-stable. In other words (MS1) implies (MS2)–(MS3).

Proof. Indeed, by Proposition 4.11, we obtain that $\mathrm{Nw}(f) = \mathrm{Per}(f)$ and Theorem 4.13 shows that the invariant manifolds of the critical elements intersect transversely. □

Abstract Morse–Smale systems possess several well-known properties. One of them is the **Morse–Smale order** on $\mathrm{Crit}(f)$, that is, the partial order on $\mathrm{Crit}(f)$ defined by

$$\gamma^\dagger \succ \gamma^* \quad \text{if and only if} \quad W^{\mathrm{u}}(\gamma^\dagger) \cap W^{\mathrm{s}}(\gamma^*) \neq \emptyset. \tag{4.11}$$

Proposition 4.16 ([Oli00, Proposition 3.4])**.** *Let $\mathcal{A}(f)$ be Morse–Smale. Then the relation* (4.11) *is a partial order on* $\mathrm{Crit}(f)$*. In particular, given $\gamma^\dagger, \gamma^\diamond, \gamma^* \in \mathrm{Crit}(f)$ such that $\gamma^\dagger \succ \gamma^\diamond$ and $\gamma^\diamond \succ \gamma^*$, we have that $\gamma^\dagger \succ \gamma^*$.*

As a consequence, the phase diagram $\Gamma(f)$ of any Morse–Smale attractor $\mathcal{A}(f)$ is a **directed acyclic graph**, that is, a digraph with no directed cycles. Indeed, we have that $(\gamma^\dagger, \gamma^*)$ is an edge of a Morse–Smale phase diagram $\Gamma(f)$ if and only if $\gamma^\dagger \succ \gamma^*$. Hence the transitivity in Proposition 4.16 shows that $\Gamma(f)$ coincides with its transitive closure via the partial order $\succ$, i.e., all edges in $\Gamma(f)$ are transitive.

Chapter 5

Critical elements

This chapter introduces the period map, a real function connected to a two-dimensional boundary value problem that we use to control the formation process of periodic orbits in the global attractor $\mathcal{A}(f)$. We pointed out in Chapter 1 that our method for constructing global attractors of the DDE (1.1) requires all periodic orbits to appear by a sequence of Hopf bifurcations. The problem of controlling the periodic orbits is also central in the description of spindles by Krisztin et al. [KW01, KWW99b, KWW99a] and in the Vas attractor [Vas11, KV11] presented in Chapter 1.

Our results follow an idea of Kaplan and Yorke [KY74, KY75] that uses spatiotemporal symmetry in a two-dimensional ODE to construct periodic solutions of the DDE (1.1). They study the periodic solutions with period p of the Hamiltonian ODE

$$\begin{aligned} \dot{\xi} &= g(\eta), \\ \dot{\eta} &= -g(\xi). \end{aligned} \tag{5.1}$$

Here g is an odd real function satisfying $g(\eta)\eta < 0$. Thus, for $\eta \neq 0$, they obtain the spatiotemporal relation $\eta(t) = \xi(t - p/4)$ and $\xi(t)$ is a periodic solution of the DDE

$$\dot{x}(t) = g(x(t - p/4)).$$

In this thesis, the main difference with (5.1) is that we consider nonlinearities $f \in \mathfrak{X}^-$. Thus our case becomes slightly more general because f

possesses a ξ-component, but also more limited because we require the negative delayed feedback (2.1). We use (2.1) to prove a converse to the results in [KY74, KY75]. More precisely, we give sufficient conditions such that all the periodic solutions of the DDE (1.1) appear as solutions of a so-called **reference ODE**

$$\begin{aligned} \dot{\xi} &= f(\xi, \eta), \\ \dot{\eta} &= -f(\eta, \xi). \end{aligned} \tag{5.2}$$

We will see that the dynamics in (5.2) consist of a single equilibrium $(0, 0)$ surrounded by a continuum of periodic orbits. We call **amplitude** to the maximum in the ξ-component of $(\xi(t), \eta(t))$ solving (5.2). Thus we parametrize the periodic solutions of (5.2) in amplitude and define the **period map** p_f to be the function taking the amplitude of a periodic solution to its minimal period.

Garab and Krisztin [GK11] took a similar approach to ours and described the branches $\gamma^*(\lambda)$ of periodic orbits in a one-parameter family of DDEs

$$\dot{x}(t) = -\lambda f(x(t), x(t-1)), \quad \lambda > 0,$$

with f satisfying the delayed feedback assumption (2.1). Although their approach does not require the symmetric feedback (2.2), it has the disadvantage that their period map singularizes at **saddle-node bifurcations** with respect to the parameter λ. In particular, an unexpected consequence of our analysis is that if $f \in \mathfrak{X}^-$, then the branches of periodic orbits in (1.1) possess no turns when parametrized in amplitude.

Section 5.1 shows how the reference ODE (5.2) produces DDE solutions via spatiotemporal symmetry. Our analysis is entirely analogous to [KY74] except that a global reversible Hopf bifurcation at the origin replaces the Hamiltonian of the ξ-independent setting. We define the period map and derive its most relevant properties.

In Section 5.2, we find conditions such that all the periodic solutions of the DDE (1.1) solve the reference ODE (5.2). First, we consider the planar projection $P\gamma^*$ defined via (2.18) of a DDE periodic orbit γ^*. We show that the set $P\gamma^*$ is an orbit of (5.2) if and only if $P\gamma^*$ intersects both vertical and horizontal axes orthogonally. After characterizing the *shape* of the projection $P\gamma^*$ we assume that the period map p_f possesses no *plateaus* and show that all projections $P\gamma^*$ have the desired shape, thanks to Lemma 2.6 (ii).

In Section 5.3, we discuss the local stability properties of the DDE periodic solutions obtained by the reduction process in Sections 5.1 and 5.2. Our main argument is that the two-dimensional nature of the periodic solutions obtained in this way produces a one-parameter family of linear two-dimensional ODEs. Here the parameter corresponds to the half-multipliers seen in Chapter 2. We fully characterize the unstable dimension of the periodic orbits by comparing the winding velocities of the solutions of the linear system above.

Finally, Section 5.4 gives a visualization of our results.

5.1 From ODE to DDE

In this section, we define the period map p_f for the reference ODE (5.2) with the nonlinearity $f \in \mathfrak{X}^-$. We also show how p_f yields periodic solutions of the DDE (1.1).

We denote the planar vector field in (5.2) by $F(\xi, \eta) := (f(\xi, \eta), -f(\eta, \xi))$ and point out two symmetries:

- A **reversibility** under reflections

$$\varrho_1(\xi, \eta) := (\eta, \xi) \quad \text{such that} \quad \varrho_1 \circ F \circ \varrho_1 = -F. \tag{5.3}$$

- An **equivariance** under ninety-degree rotations

$$\varrho_2(\xi, \eta) = (\eta, -\xi) \quad \text{such that} \quad \varrho_2 \circ F \circ \varrho_2^{-1} = F. \tag{5.4}$$

By composition, the linear maps ϱ_1 and ϱ_2 generate a representation of the symmetry group of the square Dih_4 in $\mathbb{R}^2$. We denote by $(\xi(a,t), \eta(a,t))$ the solution of (5.2) with initial condition $(\xi(a,0), \eta(a,0)) = (a, 0)$. A characteristic property of (5.2) is that it possesses a single equilibrium at $(0,0)$ and that, thanks to the reversibility (5.3), the solutions $(\xi(a,t), \eta(a,t))$ with initial condition $(\xi(a,0), \eta(a,0)) := (a, 0)$ are periodic in t for all $a > 0$. We summarize this in the following lemma.

Lemma 5.1. *If $f \in \mathfrak{X}^-$, then the solutions $(\xi(a,t), \eta(a,t))$ of the reference ODE (5.2) are periodic for all $a > 0$ with minimal period $p_f(a) < \infty$.*

Moreover, the map $p_f(a)$ is C^2 at any $a > 0$ and admits a C^1-extension $p_f : [0,\infty) \to (0,\infty)$ satisfying

$$p_f(0) = \frac{-2\pi}{\partial_2 f(0,0)} \quad \textit{and} \quad p_f'(0) = 0.$$

Proof. Indeed, linearize (5.2) at $(0,0)$ we observe that

$$\mathrm{Spec}(DF(0,0)) = \{Bi, -Bi\} \quad \text{with} \quad B := \partial_2 f(0,0).$$

Next, recalling the reversibility (5.3), we apply a reversible Hopf bifurcation theorem at $(0,0)$; see [Van82, Theorem 7.5.4.]. This shows that there exists an $\varepsilon > 0$ such that for all $a \in (0,\varepsilon)$ the solutions $(\xi(a,t), \eta(a,t))$ are periodic. Denoting

$$\bar{\varepsilon} := \sup\{a > 0 : (\xi(a,t), \eta(a,t)) \text{ is periodic}\},$$

we prove by contradiction that $\bar{\varepsilon} = \infty$. Suppose that $\bar{\varepsilon} < \infty$, hence the orbit $O_{\bar{\varepsilon}} := \{(\xi(\bar{\varepsilon},t), \eta(\bar{\varepsilon},t)) : t \in \mathbb{R}\}$ belongs to the nonwandering set of the ODE (5.2). However, since we are considering planar ODEs and $(0,0)$ is the only equilibrium of (5.2), the standard Poincaré–Bendixson theorem in $\mathbb{R}^2$ shows that $O_{\bar{\varepsilon}}$ is periodic. Moreover, $O_{\bar{\varepsilon}}$ must either attract or repel the initial conditions in a small neighborhood of its exterior region, in contradiction to the reversibility (5.3), thus $\bar{\varepsilon} = \infty$.

In particular, the map p_f that assigns the minimal period $p_f(a)$ to the periodic solution $(\xi(a,t), \eta(a,t))$ is well defined for all $a > 0$ and solves the equation

$$\eta(a, p_f(a)) - \eta(a, 0) = 0.$$

Since $\dot{\eta}(a, p_f(a)) = -f(0,a) \neq 0$ for $a > 0$, the implicit function theorem ensures that p_f inherits its regularity from f on the open interval $(0,\infty)$.

It is only left to show $p_f(0) = -2\pi/B$ and $p_f'(0)$. To this end, recall that near the origin p_f comes from a Hopf bifurcation in the ODE (5.2). In particular, p_f arises via bifurcation from simple eigenvalues in an O(2)-symmetric problem, see [Van82, Theorem 7.2.7.]. Thus p_f is differentiable at 0 with derivative $p_f'(0) = 0$. □

The C^1-map $p_f : [0,\infty) \to (0,\infty)$ in Lemma 5.1 is called the **period map** of f. As seen above, the existence of p_f is, at least locally, a consequence of the reversibility (5.3). On the other hand, the equivariance (5.4) allows us to connect solutions of the ODE (5.2) to solutions of the DDE (1.1) using spatiotemporal symmetries.

Lemma 5.2. *In the setting of Lemma 5.1, the periodic solutions for $a > 0$ satisfy*

$$(\xi(a,t),\eta(a,t)) = \left(-\eta\left(a, t-\frac{p_f(a)}{4}\right), \xi\left(a, t-\frac{p_f(a)}{4}\right)\right). \tag{5.5}$$

In particular, the components go into antiphase after a half-period, i.e.,

$$\xi\left(a, t-\frac{p_f(a)}{2}\right) = -\xi(a,t) \quad \text{and} \quad \eta\left(a, t-\frac{p_f(a)}{2}\right) = -\eta(a,t). \tag{5.6}$$

Proof. By Lemma 5.1 the orbits $O_a := \{(\xi(a,t),\eta(a,t)) : t \in \mathbb{R}\} \subset \mathbb{R}^2$ are simple curves surrounding $(0,0)$. Moreover, the symmetry under the rotation ϱ_2 in (5.4) ensures that O_a are ϱ_2-invariant, i.e., $\varrho_2 O_a = O_a$.

Since $\varrho_2(\xi(a,t),\eta(a,t)) = (\eta(a,t), -\xi(a,t))$ solves (5.2) and shares orbits with $(\xi(a,t),\eta(a,t))$, we have that $\eta(a,t) = \xi(a,t-\tau)$ for a suitable $\tau \in [0, p_f(a)]$. Recalling that $\varrho_2^4 = \mathrm{Id}$, we obtain that 4τ is a multiple of the period. Therefore, τ can only take the values $p_f(a)/4$ and $3p_f(a)/4$. Direct examination of (5.2) shows that $(\xi(a,t),\eta(a,t))$ winds in counterclockwise direction around the origin $(0,0)$ as t grows. Thus $\eta(a,t)$ increases monotonically until it reaches its maximum at a and then decreases until it reaches its minimum $-a$. From this we conclude that $\xi(a,0) = \eta(a, p_f(a)/4) = a$ and $\tau = p_f(a)/4$. This proves (5.5), applying (5.5) twice we obtain (5.6). □

5.2 From DDE to ODE

In this section, we give sufficient conditions such that the periodic solutions of the DDE (1.1) with $f \in \mathfrak{X}^-$ solve the reference ODE (5.2).

We establish a relation between the period p of a periodic solution $x^*(t)$ solving (1.1) and the shape of the planar projection (2.18) of the corresponding orbit γ^*. First, we point out that $p = 2$ is never a period of γ^*.

Lemma 5.3 ([CMP78, Lemma 4.1]). *Let $x^*(t)$ be a periodic solution of the DDE* (1.1), *then $x^*(t+2) \neq x^*(t)$.*

Proof. Indeed, otherwise, the ODE

$$\begin{aligned} \dot{\xi} &= f(\xi, \eta), \\ \dot{\eta} &= f(\eta, \xi), \end{aligned} \tag{5.7}$$

would possess a periodic solution $(x^*(t), x^*(t-1))$. However, this is impossible since $(0,0)$ is the only equilibrium and the diagonal $\{(\xi,\xi) : \xi \in \mathbb{R}\}$ is invariant under the dynamics of (5.7). ☐

In particular, no number of the form $2/(2m-1)$ for $m \in \mathbb{N}$ is a period of γ^*. However, because of Lemma (5.3), the periodic orbits γ^* with period 4 possess a distinctive planar projection $P\gamma^*$.

Lemma 5.4. *Let $x^*(t)$ be a periodic solution of the DDE* (1.1) *with $f \in \mathfrak{X}^-$. We denote the orbit of $x^*(t)$ by γ^*. Then $x^*(t)$ has period 4 if and only if the planar projection $P\gamma^*$ given by* (2.18) *intersects the vertical axis orthogonally.*

Proof. Suppose that $x^*(t)$ has minimal period p and $x^*(t+4) = x(t)$ for all $t \in \mathbb{R}$. Since 2 is not a period of $x^*(t)$ by Lemma 5.3, we have that $(2m-1)p = 4$ for some $m \in \mathbb{N}$. Then $x^*(t-2) = x^*(t-mp+mp/2) = -x^*(t)$ by the odd symmetry (2.19) and $(x^*(t), x^*(t-1))$ is a solution of the planar ODE (5.2). In particular, $P\gamma^*$ is an orbit of (5.2). Since $f \in \mathfrak{X}^-$, we have that $f(\xi, 0) = 0$ for all $\xi \in \mathbb{R}$ and the intersections of $P\gamma^*$ with the horizontal axis happen orthogonally at the maximum and minimum in the first component, denoted $(\bar{x}, 0)$ and $(\underline{x}, 0)$, respectively. By the equivariance of (5.2) under the rotation ϱ_2 defined in (5.4), we have that $P\gamma^* = \varrho_2(P\gamma^*)$ and $P\gamma^*$ intersects the vertical axis orthogonally.

To see the converse implication, assume that the simple curve $P\gamma^*$ intersects the vertical axis orthogonally. We set without loss of generality $x^*(0) = \bar{x} := \max_{t\in\mathbb{R}} x^*(t)$ and claim that $x^*(-2) = -x^*(0) = -\bar{x}$. Indeed, $x^*(0) = \bar{x}$ is a maximum, and thus

$$\dot{x}^*(0) = f(x^*(0), x^*(-1)) = 0. \tag{5.8}$$

For $f \in \mathfrak{X}^-$, (5.8) implies that $x^*(-1) = 0$. Lemma 2.6 (iii) states that $x^*(t)$ moves monotonically between extrema. Therefore, the point

$$(x^*(-1), x^*(-2)) = (0, x^*(-2)),$$

corresponds to one of the two intersections that $P\gamma^*$ has with the vertical axis. As a result,

$$\dot{x}^*(-2) = f(x^*(-2), x^*(-3)) = 0,$$

and $f \in \mathfrak{X}^-$ yields $x^*(-3) = 0$. By Lemma 2.6 (iii) and the odd symmetry (2.19), $P\gamma^*$ intersects the horizontal axis at $(\bar{x}, 0)$ and $(-\bar{x}, 0)$, only. Since 2 is not a period for $x^*(t)$ by Lemma 5.3, the only possibility is $x^*(-2) = -\bar{x} = -x^*(0)$, as claimed. Repeating the procedure shows that $x^*(-4) = x^*(0)$. Moreover, since $x^*(t)$ is sinusoidal, it only attains the maximum $x^*(0) = x^*(-4) = \bar{x}$ once over a minimal period. Hence $x^*(t) = x^*(t-4)$ for all $t \in \mathbb{R}$ and $x^*(t)$ has period 4. $\square$

In particular, Lemma 5.4 indicates that if a periodic solution $x^*(t)$ of (1.1) satisfies $x^*(t) \neq x^*(t+4)$, then the planar projection $P\gamma^*$ must intersect the vertical axis at a slanted angle. Together with the foliation of $\mathbb{R}^2$ by periodic solutions with spatiotemporal symmetry of the reference ODE (5.2), this imposes heavy restrictions on the regions of the plane $\mathbb{R}^2$ where the projections $P\gamma^*$ such that $x^*(t) \neq x^*(t+4)$ can lie.

Proposition 5.5. *In the setting of Lemma 5.4, assume that the periodic solution $x^*(t)$ satisfies $x^*(t+4) \neq x^*(t)$ and denote $\bar{x} := \max_{t\in\mathbb{R}} x^*(t)$. Consider the orbit $O_a := \{(\xi(a,t), \eta(a,t)) : t \in \mathbb{R}\}$ associated to the solution of the ODE* (5.2) *with initial condition $(a, 0)$, $a > 0$, and define*

$$\begin{aligned} \bar{a} &:= \inf\{a \geq \bar{x} > 0 : O_a \cap P\gamma^* = \emptyset\}, \\ \underline{a} &:= \sup\{\bar{x} \geq a > 0 : O_a \cap P\gamma^* = \emptyset\}. \end{aligned} \tag{5.9}$$

Then the period map p_f satisfies $p_f'(a) = 0$ for all $a \in (\underline{a}, \bar{a})$.

Proof. Let $x^*(t)$ have period p. Notice that $x^*(t)$ solves the family of DDEs $\dot{x}^*(t) = f(x^*(t), x^*(t-1-np))$ for all $n \in \mathbb{N}$. Rescaling time, we have that $x^{(n)}(t) := x^*((1+np)t)$ solves the DDE

$$\dot{x}(t) = (1+np)f(x(t), x(t-1)).$$

Denoting the DDE orbit of $x_0^{(n)}(t)$ by $\gamma^{(n)}$, for all $n \in \mathbb{N}$ the planar projections $P\gamma^{(n)}$, given by (2.18), satisfy

$$\begin{aligned} P\gamma^{(n)} &= \{(x^{(n)}(t), x^{(n)}(t-1)) : t \in \mathbb{R}\} \\ &= \{(x^*(t), x^*(t-1-np)) : t \in \mathbb{R}\} \\ &= \{(x^*(t), x^*(t-1) : t \in \mathbb{R}\} \\ &= P\gamma^*. \end{aligned} \tag{5.10}$$

By Lemma 5.4, $x^*(t+4) \neq x^*(t)$ implies that any intersection of $P\gamma^*$ with the vertical axis is not orthogonal. We denote the top intersection point by

$$(0, \hat{x}) := P\gamma^* \cap \{(0, \eta) : \eta \in \mathbb{R}^+\}.$$

Since $f \in \mathfrak{X}^-$, the ODE orbit O_a, $a > 0$, intersects both the horizontal and the vertical axes orthogonally, thus $\bar{a} - \underline{a}$ defined via (5.9) is strictly positive. Furthermore, by Lemma 5.2 we know that $\xi(a,t)$ satisfies the DDE

$$\dot{\xi}(a,t) = f\left(\xi(a,t), \xi\left(a, t - \frac{p_f(a)}{4}\right)\right),$$

and, therefore, the functions

$$\xi^{(a,m)}(t) := \xi\left(a, \left(\frac{p_f(a)}{4} + m p_f(a)\right) t\right),$$

have period 4 for all $m \in \mathbb{N}$ and solve the DDEs

$$\dot{x}(t) = \lambda f(x, x(t-1)), \tag{5.11}$$

for parameter value choices

$$\lambda = \frac{p_f(a)}{4} + m p_f(a).$$

Denoting by $\gamma^{(a,m)}$ the DDE orbit of $\xi^{(a,m)}(t)$, we proceed as in (5.10) to show that

$$\begin{aligned} P\gamma^{(a,m)} &= \{(\xi^{(a,m)}(t), \xi^{(a,m)}(t-1)) : t \in \mathbb{R}\} \\ &= \left\{\left(\xi\left(a,t\right), \xi\left(a, t - \frac{p_f(a)}{4} - m p_f(a)\right)\right) : t \in \mathbb{R}\right\} \\ &= O_a, \end{aligned} \tag{5.12}$$

for all $m \in \mathbb{N}$.

By contradiction, suppose that there exists $\hat{a} \in (\underline{a}, \bar{a})$ such that $p_f'(\hat{a}) \neq 0$. Hence we can assume without loss of generality, by making a new choice of $\underline{a}$ and $\bar{a}$ if necessary, that $p_f(\bar{a}) \neq p_f(\hat{a})$ and $p_f(\underline{a}) \neq p_f(\hat{a})$. Therefore, for all $\varepsilon > 0$ we can find $n^*, m^* \in \mathbb{N}$ such that

$$\left(m^* + \frac{1}{4}\right)(p_f(\hat{a}) + \varepsilon) = n^* p + 1.$$

Choosing $\varepsilon < \min\{|p_f(\bar{a}) - p_f(\hat{a})|, |p_f(\hat{a}) - p_f(\underline{a})|\}$, the intermediate value theorem yields the existence of $a^* \in (\underline{a}, \bar{a})$ such that $p_f(a^*) = p_f(\hat{a}) + \varepsilon$. Rescaling time, we have that $x^{(n^*)}(t)$ and $\xi^{(a^*,m^*)}(t)$ are both solutions of (5.11) with parameter $\lambda = 1 + n^*p$. By the identities (5.10) and (5.12), we have that $P\gamma^{(n^*)} \cap P\gamma^{(a^*,m^*)} = P\gamma^* \cap O_{a^*} \neq \emptyset$, in contradiction to Lemma 2.6 (ii). Therefore,

$$\begin{aligned} x^*((1+n^*p)t+4) &= x^{(n^*)}(t+4) \\ &= \xi^{(a^*,m^*)}(t+4) \\ &= \xi^{(a^*,m^*)}(t) \\ &= x^*((1+n^*p)t), \end{aligned}$$

which contradicts our assumptions. □

Thus Proposition 5.5 shows that all the periodic orbits of the DDE (1.1) have period 4, provided that we can prevent p_f from having *plateaus*. We say that p_f is called **locally nonconstant** if it presents no plateaus at all, i.e., if for all $a \geq 0$ we can find a sequence $a_n \to a$ such that $p_f(a_n) \neq p_f(a)$ for all $n \in \mathbb{N}$.

Theorem 5.6. *Let $x^*(t)$ be a nontrivial periodic solution of the DDE* (1.1) *with $f \in \mathfrak{X}^-$ and assume that the period map p_f is locally nonconstant. Denoting $\bar{x} := \max_{t\in\mathbb{R}} x^*(t)$, we have that*

$$p_f(\bar{x}) = \frac{4}{4n-3} \quad \textit{for some } n \in \mathbb{N}. \tag{5.13}$$

Moreover, the planar curve $(x^*(t), x^*(t-1))$ solves the two-dimensional reference ODE (5.2).

Conversely, let $(\xi(a,t), \eta(a,t))$ be the solution of (5.2) *with initial condition $(a, 0)$, $a > 0$. If $p_f(a) = 4/(4n-3)$ for some $n \in \mathbb{N}$, then $\xi(a,t)$ solves the DDE* (1.1).

Proof of Theorem 5.6. Indeed, let p_f be locally nonconstant. Thus, by Proposition 5.5, any periodic solution $x^*(t)$ of the DDE (1.1) satisfies $x^*(t) = x^*(t+4)$. From Lemma 2.6 (ii) and Lemma 5.3, we know that $x^*(t+2) = -x^*(t)$ and we obtain that $(x^*(t), x^*(t-1))$ solves the ODE (5.2). The converse follows immediately from Lemma 5.2. □

In virtue of Theorem 5.6, the periodic orbits of the DDE (1.1) with a locally nonconstant period map are in one-to-one correspondence with the periodic solutions of the reference ODE (5.2) such that the minimal period takes the form (5.13). We rephrase this statement more concisely as follows.

Corollary 5.7. *Under the assumptions of Theorem 5.6, a differentiable function $x^*(t)$ is a periodic solution of the DDE* (1.1) *if and only if $(x^*(t), x^*(t-1))$ solves the ODE boundary value problem*

$$\begin{aligned} \dot{\xi} &= f(\xi, \eta), \\ \dot{\eta} &= -f(\eta, \xi), \end{aligned} \qquad \text{and} \quad (\xi(t+2), \eta(t+2)) = -(\xi(t), \eta(t)). \tag{5.14}$$

Proof. In the setting of Theorem 5.6, let $x^*(t)$ with $\bar{x} = \max_{t\in\mathbb{R}} x^*(t)$ be a periodic solution of the DDE (1.1). By (5.13), we have that $x^*(t+2) = x^*(t + (4n-3)p_f(\bar{x})/2) = x^*(t + p_f(\bar{x})/2)$. Using the odd symmetry (2.19), we have that $x^*(t+2) = -x^*(t)$ and $(x^*(t), x^*(t-1))$ solves (5.14). The converse follows analogously by Lemma 5.2. □

5.3 Local stability

Finally, we show how the minimal period p_f of the periodic solutions obtained in Theorem 5.6 for the DDE (1.1) connects to the unstable dimension of their orbits. To be precise, we consider a periodic solution $x^*(t)$ with amplitude $\bar{x} := \max\{x^*(t) : t \in \mathbb{R}\}$ of (1.1). Our goal is to show that if $x^*(t)$ has minimal period $p_f(\bar{x}) = 4/(4n-3)$ for some $n \in \mathbb{N}$, then the unstable dimension, $i(\gamma^*)$, of the corresponding orbit is fully determined by the value of n and the sign of the derivative $p_f'(\bar{x})$.

In particular, the results obtained throughout this chapter characterize all the bifurcation phenomena near critical elements $\mathrm{Crit}(f)$, provided that p_f is locally nonconstant. We begin by proving a lemma that connects the minimal period of the periodic solutions in Theorem 5.6 to their corresponding zero number.

Lemma 5.8. *Consider a periodic solution $x^*(t)$ of the DDE* (1.1) *with $f \in \mathfrak{X}^-$. If $x^*(t)$ has minimal period $p := 4/(4n-3)$ for some $n \in \mathbb{N}$, then $z(\dot{x}_0^*) = 2n-1$.*

Proof. We know from Lemma 2.6 (iii) that $x^*(t)$ acquires its maximum $\bar{x}$ (resp., its minimum $\underline{x}$) once over every minimal period and moves monotonically between the maximum and the minimum. By the odd symmetry (2.19), half a minimal period separates any two neighboring sign changes of $\dot{x}^*(t)$. Therefore, due to $p = 4/(4n-3)$, we obtain the bounds

$$2n-2 = \left\lfloor \frac{2}{p} \right\rfloor \leq z(\dot{x}_0^*) \leq \left\lceil \frac{2}{p} \right\rceil = 2n-1.$$

Here we used the usual notation $\lfloor \cdot \rfloor$ (resp., $\lceil \cdot \rceil$) for the standard floor (resp., ceiling) function. Since the zero number is always odd, we have that $z(\dot{x}_0^*) = 2n-1$. □

When combined with the spectral characterization in Proposition 2.9, Lemma 5.8 yields the following three situations only.

Corollary 5.9. *In the setting of Lemma 5.8, let γ^* be the orbit of $x^*(t)$ and denote by $\mu_\mathrm{c} < 0$ the critical half-multiplier of γ^* defined in Corollary 2.8. Then exactly one of the following statements holds:*

(i) $\mu_\mathrm{c} < -1$, *γ^* is hyperbolic, and* $i(\gamma^*) = 2n-1$.

(ii) $\mu_\mathrm{c} > -1$, *γ^* is hyperbolic, and* $i(\gamma^*) = 2n-2$.

(iii) $\mu_\mathrm{c} = -1$, *γ^* is not hyperbolic, and* $i(\gamma^*) = 2n-2$.

Proof. The result follows immediately from plugging Lemma 5.8 into the three cases we distinguished in Corollary 2.8. □

Before we characterize the unstable dimension, we study Corollary 5.9 (iii) in detail. Our conclusion, summarized in the following lemma, is that γ^* loses its hyperbolicity at critical points $\bar{x}$ such that $p_f'(\bar{x}) = 0$.

Lemma 5.10. *In the setting of Lemma 5.8, let $\bar{x} := \max_{t\in\mathbb{R}} x^*(t)$ and denote by γ^* the orbit of $x^*(t)$. Then γ^* is hyperbolic if and only if $p_f'(\bar{x}) \neq 0$.*

Proof. Denoting $p := p_f(\bar{x})$ and $p' := p_f'(\bar{x})$, we first suppose that $p' = 0$. Following Theorem 5.6, we let $x^*(a,t)$ be the solution of the DDE

$$\dot{x}(t) = f\left(x(t), x\left(t - \frac{(4n-3)p_f(a)}{4}\right)\right),$$

with minimal period $p_f(a)$ and amplitude $a = \max_{t\in\mathbb{R}} x^*(a,t)$. Where we normalize the initial condition so that

$$x^*(a,0) = a. \tag{5.15}$$

We may assume, shifting time if necessary, that $x^*(t) = x^*(\bar{x},t)$. Then, the amplitude derivative $y^*(a,t) := \partial_a x^*(a,t)$ solves the linear inhomogeneous nonautonomous DDE

$$\begin{aligned}
\dot{y}(t) = \partial_1 f\left(x^*(a,t), x^*\left(a, t - \frac{(4n-3)p_f(a)}{4}\right)\right) y(t) \\
+ \partial_2 f\left(x^*(a,t), x^*\left(a, t - \frac{(4n-3)p_f(a)}{4}\right)\right) y\left(t - \frac{(4n-3)p_f(a)}{4}\right) \\
- \dot{x}^*\left(a, t - \frac{(4n-3)p_f(a)}{4}\right) \frac{(4n-3)p_f'(a)}{4}.
\end{aligned}$$

Since $p' = 0$, $y^*(t) := y^*(\bar{x},t)$ satisfies the linearized equation (2.21) around $x^*(t)$. In addition, $y^*(t)$ is periodic by

$$\begin{aligned}
y^*(t+p) &= \partial_a x^*(\bar{x}, t+p) + p'\partial_t x^*(\bar{x}, t+p) \\
&= \partial_a x^*(\bar{x}, t+p) \\
&= y^*(t).
\end{aligned}$$

Notice that $y^*(t)$ satisfies $y^*(0) = 1$, in contrast to the trivial solution $\dot{x}^*(t)$ for which $\dot{x}^*(0) = 0$ due to the normalization (5.15). Therefore, $\dot{x}_0^*$ and y_0^* are linearly independent. In other words, the characteristic multiplier 1 of γ^* has geometric multiplicity 2. Thus γ^* is not hyperbolic.

We show the converse by supposing that γ^* is not hyperbolic. By Proposition 2.9, the half-multiplier $-1 \in \mathrm{Spec}(N)$ has algebraic multiplicity two. Here N half-monodromy operator given by the time-$p/2$ evolution of the linearized equation (2.21).

By the Floquet theory in [HVL93], (2.21) possesses a solution $y^*(t)$ of the form

$$y^*(t) = v^*(t) + \beta t \dot{x}^*(t),$$

where $v^*(t)$ has period p and satisfies $v^*(t - p/2) = -v^*(t)$ for some $\beta \in \mathbb{R}$. Note that $(4n-3)p/2 = 2$ and therefore $v^*(t-2) = -v^*(t)$. In the following we denote

$$A(t) := \partial_1 f(x^*(t), x^*(t-1)) \quad \text{and} \quad B(t) := \partial_2 f(x^*(t), x^*(t-1)), \tag{5.16}$$

moreover, we set $\bar{A}(t) := A(t-1)$ and $\bar{B}(t) := B(t-1)$, we obtain that $(v^*(t), v^*(t-1))$ solves the inhomogeneous ODE

$$\begin{aligned} \dot{v}(t) &= A(t)v(t) + B(t)w(t) - \beta B(t)\dot{x}^*(t-1) - \beta\dot{x}^*(t), \\ \dot{w}(t) &= -\bar{B}(t)v(t) + \bar{A}(t)w(t) + \beta\bar{B}(t)\dot{x}^*(t) - \beta\dot{x}^*(t-1). \end{aligned} \tag{5.17}$$

By a Fredholm alternative argument in [Hal69, Section IV.1, Lemma 1.1], the inhomogeneous ODE (5.17) have solutions with period p if and only if

$$\beta\int_0^p (\hat{v}^*(t), \hat{w}^*(t)) \begin{pmatrix} -B(t)\dot{x}^*(t-1) - \dot{x}^*(t) \\ \bar{B}(t)\dot{x}^*(t) - \dot{x}^*(t-1) \end{pmatrix} \mathrm{d}t = 0, \tag{5.18}$$

for all $(\hat{v}^*(t), \hat{w}^*(t))$ with period p solving the homogeneous adjoint equation

$$\begin{aligned} \dot{\hat{v}}(t) &= -A(t)\hat{v}(t) + \bar{B}(t)\hat{w}(t), \\ \dot{\hat{w}}(t) &= -B(t)\hat{v}(t) - \bar{A}(t)\hat{w}(t). \end{aligned} \tag{5.19}$$

The choice

$$(\hat{v}^*(t), \hat{w}^*(t)) := \mathrm{e}^{-\int_{t-1}^t A(s)\mathrm{d}s} \left(-\dot{x}^*(t-1), \dot{x}^*(t)\right), \tag{5.20}$$

provides a p-periodic solution of (5.19). Plugging (5.20) into the condition (5.18), we obtain that

$$\beta\int_0^p \mathrm{e}^{-\int_{t-1}^t A(s)\mathrm{d}s} \left(B(t)|\dot{x}^*(t-1)|^2 + \bar{B}(t)|\dot{x}^*(t)|^2\right) \mathrm{d}t = 0.$$

By the monotone feedback (2.1), we have that $B(t) \neq 0$ for all $t \in \mathbb{R}$. Thus the integral term never vanishes and $\beta = 0$.

As a result, the generalized eigenfunction $y^*(t) = v^*(t)$ is a periodic solution of the linearized equation (2.21) and y_0^* is linearly independent of $\dot{x}_0^*$, by construction. Moreover, $y^*(t-2) = -y^*(t)$, which implies that the linear ODE

$$\begin{aligned} \dot{v}(t) &= A(t)v(t) + B(t)w(t), \\ \dot{w}(t) &= -\bar{B}(t)v(t) + \bar{A}(t)w(t), \end{aligned} \tag{5.21}$$

has two linearly independent solutions, $(\dot{x}^*(t), \dot{x}^*(t-1))$ and $(y^*(t), y^*(t-1))$, both with period 4.

For $a \geq 0$, we denote by $(\xi(a,t), \eta(a,t))$ the solution of the reference ODE (5.2) with the initial condition $(a, 0)$, i.e.,

$$\begin{aligned} \dot{\xi}(a,t) &= f(\xi(a,t), \eta(a,t)), \\ \dot{\eta}(a,t) &= -f(\eta(a,t), \xi(a,t)), \end{aligned}$$

satisfying the boundary condition

$$(a,0) = (\xi(a,0), \eta(a,0)) = (\xi(a, p_f(a)), \eta(a, p_f(a))).$$

By oddness of the periodic solution (2.19) and even-odd symmetry of $f \in \mathfrak{X}^-$, we have that the amplitude derivative $(\bar{v}(t), \bar{w}(t)) := \partial_a(\xi(\bar{x},t), \eta(\bar{x},t))$ solves the linear ODE (5.21). Moreover, it takes the boundary values

$$\begin{aligned}(1,0) &= (\bar{v}(0), \bar{w}(0)) \\ &= (\bar{v}(4), \bar{w}(4)) + (4n-3)p'(\dot{x}^*(0), \dot{x}^*(-1)).\end{aligned} \tag{5.22}$$

However, $(\bar{v}(t), \bar{w}(t))$ can be written as a linear combination of the eigenfunctions $(\dot{x}^*(t), \dot{x}^*(t-1))$ and $(y^*(t), y^*(t-1))$ of the linear ODE (5.21), both of which are p-periodic and also 4-periodic. Since $\dot{x}^*(-1) \neq 0$, by the normalized form (5.15) it follows that $p' = 0$ so that (5.22) is satisfied. □

Theorem 5.11. *In the setting of Lemma 5.10, the unstable dimension of the periodic orbit γ^* with minimal period $p_f(\bar{x}) = 4/(4n-3)$, $n \in \mathbb{N}$ comes given by*

$$i(\gamma^*) = \begin{cases} 2n-2, & \text{if } p_f'(\bar{x}) \geq 0, \text{ and} \\ 2n-1, & \text{otherwise.} \end{cases}$$

Proof. We use the notation $p := p_f(\bar{x})$ and $p' := p_f'(\bar{x})$. We know from Theorem 5.6 that $(x^*(t), x^*(t-1))$ solves the ODE (5.2). Without loss of generality we normalize $x^*(t)$ so that $x^*(0) = \bar{x}$.

By Lemma 5.10, if $p' = 0$, then the critical half-multiplier μ_c of γ^* satisfies $\mu_c = -1$ and the half-monodromy operator N solving the linearized equation (2.21) has a geometrically double eigenvalue -1. By Corollary 5.9 (iii), the Morse index of the periodic orbit γ^* is

$$i(\gamma^*) = 2n-2.$$

If $p' \neq 0$, we know by Lemma 5.10 that γ^* is hyperbolic and Corollary 2.8 (i)–(ii) imply that $\mu_c < 0$ has geometric multiplicity one. Furthermore, the associated critical eigenfunction Ψ satisfies $z(\Psi) = z(\dot{x}_0^*)$.

In virtue of Corollary 5.9 we have that $i(\gamma^*) = 2n-1$ (resp., $i(\gamma^*) = 2n-2$) if $\mu_c < -1$ (resp., $\mu_c > -1$). We now show by a comparison argument that if $p' > 0$, then $\mu_c > -1$.

Let $y^*(t)$ denote the solution of the linearized equation (2.21) whose initial condition satisfies $y_0^* = \Psi$. Thus it satisfies

$$y^*(t) = \mu_{\mathrm{c}} y^* \left(t - \frac{p}{2} \right), \tag{5.23}$$

recalling that $(4n-3)p/2 = 2$, we define $\tilde{\mu} := \mu_{\mathrm{c}}^{4n-3}$. This choice yields $y^*(t) = \tilde{\mu} y^*(t-2)$ and

$$\begin{aligned} \mu_{\mathrm{c}} < -1 \quad &\text{if and only if} \quad \tilde{\mu} < -1, \\ \mu_{\mathrm{c}} > -1 \quad &\text{if and only if} \quad \tilde{\mu} > -1. \end{aligned}$$

In particular, using the notation (5.16), $(v(t), w(t)) = (y^*(t), y^*(t-1))$ solves the ODE

$$\begin{aligned} \dot{v}(t) &= A(t)v(t) + B(t)w(t), \\ \dot{w}(t) &= \frac{1}{\tilde{\beta}} \bar{B}(t)v(t) + \bar{A}(t)w(t), \end{aligned} \tag{5.24}$$

for the parameter value $\tilde{\beta} = \tilde{\mu}$.

For $a \geq 0$, we denote by $(\xi(a,t), \eta(a,t))$ the solution of the ODE (5.2) with the initial condition $(a, 0)$. Let us consider the amplitude derivative

$$(\bar{v}(t), \bar{w}(t)) := \partial_a (\xi(\bar{x}, t), \eta(\bar{x}, t)),$$

which solves the linear equation (5.24) with the parameter $\tilde{\beta} = -1$ and takes the boundary values

$$\begin{aligned} (1, 0) &= (\bar{v}(0), \bar{w}(0)) \\ &= (\bar{v}(p), \bar{w}(p)) + p'(0, \dot{x}^*(-1)). \end{aligned} \tag{5.25}$$

Notice that $y^*(t)$, $\dot{x}^*(t)$, and $\bar{v}(t)$ are all solutions of the second-order ODE

$$\begin{aligned} \ddot{v}(t) = &\left(A(t) + \frac{\dot{B}(t)}{B(t)} + \bar{A}(t) \right) \dot{v}(t) \\ &+ \left(\dot{A}(t) - \left(\frac{\dot{B}(t)}{B(t)} + \bar{A}(t) \right) A(t) + \frac{1}{\tilde{\beta}} B(t)\bar{B}(t) \right) v(t), \end{aligned} \tag{5.26}$$

for parameter values $\tilde{\beta} = \tilde{\mu}$ in the case of $y^*(t)$, and $\tilde{\beta} = -1$ for $\dot{x}^*(t)$ and $\bar{v}(t)$. Let $v^1(t)$ and $v^2(t)$ be two nonzero solutions of (5.26) for parameter values $\tilde{\beta}_1$ and $\tilde{\beta}_2$, respectively. Since $B(t) \neq 0$, the only stationary solution of (5.26) is 0 and we can define the angle variables

$$\varphi^j(t) := \arctan\left(\frac{v^j(t)}{\dot{v}^j(t)}\right) \quad \text{for } j = 1, 2.$$

A comparison theorem [CL55, Chapter 8, Theorem 1.2] guarantees that for parameter values $|\tilde{\beta}_1| > |\tilde{\beta}_2|$ and initial angles $\varphi^1(0) \leq \varphi^2(0)$, the angle variables satisfy

$$\varphi^1(t) < \varphi^2(t) \quad \text{for all } t > 0. \tag{5.27}$$

We first prove that the initial condition $(y^*(0), \dot{y}^*(0))$ for the second-order ODE (5.26) satisfies $y^*(0) \neq 0$. By contradiction, suppose $y^*(0) = 0$ and compare the angles

$$\varphi^\Psi(t) := \arctan\left(\frac{y^*(t)}{\dot{y}^*(t)}\right) \quad \text{and} \quad \varphi^*(t) := \arctan\left(\frac{\dot{x}^*(t)}{\ddot{x}^*(t)}\right).$$

By assumption, we can set $\varphi^\Psi(0) = \varphi^*(0) = \pi/2$. Since we are in the hyperbolic setting, we have $\tilde{\mu} \neq -1$ and the comparison principle (5.27) yields

$$\text{either } \varphi^\Psi(t) > \varphi^*(t) \text{ or } \varphi^\Psi(t) < \varphi^*(t) \text{ for all } t > 0. \tag{5.28}$$

Now we recall the property $z(y^*) = z(\dot{x}_0^*)$ from Lemma 5.8 and the identity (5.23). As a result, the normalized curves

$$\boldsymbol{y}^*(t) := \frac{(y^*(t), \dot{y}^*(t))}{\|(y^*(t), \dot{y}^*(t))\|} \quad \text{and} \quad \dot{\boldsymbol{x}}^*(t) := \frac{(\dot{x}^*(t), \ddot{x}^*(t))}{\|(\dot{x}^*(t), \ddot{x}^*(t))\|},$$

are both periodic with minimal period p. However, the comparison (5.28) and the fact that $\boldsymbol{y}^*(t)$ and $\dot{\boldsymbol{x}}^*(t)$ wind counterclockwise, due to the negative feedback (2.1), implies that $5\pi/2 = \varphi^*(p) \neq \varphi^\Psi(p) = 5\pi/2$. Hence we have reached a contradiction and $y^*(0) \neq 0$.

Multiplying $y^*(t)$ by a real scalar if necessary, we assume without loss of generality $(y^*(0), \dot{y}^*(0)) = (1, r)$. Now we compare the angle variable $\varphi^\Psi(t)$ to $\varphi^\zeta(t)$ given by

$$\varphi^\zeta(t) := \arctan\left(\frac{\zeta(t)}{\dot{\zeta}(t)}\right),$$

where $\zeta(t) := \bar{v}(t) + r\dot{x}^*(t)/\ddot{x}^*(0)$ solves the second-order ODE (5.26) for $\tilde{\beta} = -1$ and has initial the condition $(\zeta(0), \dot{\zeta}(0)) = (y^*(0), r)$. By construction we have $\varphi^\Psi(0) = \varphi^\zeta(0)$.

Again we proceed by contradiction. Let $p' > 0$ and suppose that $\mu_c < -1$. Then by the inequality (5.27) it follows that

$$\varphi^\zeta(t) > \varphi^\Psi(t) \quad \text{for all } t > 0.$$

The normalized curve $\boldsymbol{y^*}(t)$ winds clockwise around $(0,0)$ once in time p. By (5.25) we have that $\zeta(t)$ satisfies the boundary condition

$$(\zeta(p), \dot{\zeta}(p)) = (\zeta(0), \dot{\zeta}(0)) + p'(0, -B(0)\dot{x}^*(-1)). \tag{5.29}$$

Now we assemble a series of facts. First, both normalized curves $\boldsymbol{y^*}(t)$ and

$$\boldsymbol{\zeta}(t) := \frac{(\zeta(t), \dot{\zeta}(t))}{\|(\zeta(t), \dot{\zeta}(t))\|},$$

wind around zero counterclockwise. Therefore, the comparison principle (5.27) implies that $\boldsymbol{\zeta}(t)$ winds around zero *faster* than $\boldsymbol{y^*}(t)$. At the same time, we compare $\boldsymbol{\zeta}(p)$ and $\boldsymbol{y^*}(p)$ by (5.29). Since we have negative feedback (2.1) and $p' > 0$, it follows in (5.29) that $-B(0)p'\dot{x}^*(-1) > 0$. Therefore, $\zeta(t)$ changes signs at least twice more than $y^*(t)$ over the time interval $[0, p]$. However, $y^*(t)$ changes signs exactly twice in that interval, and therefore $\zeta(t)$ changes signs at least four times for $t \in [0, p]$. By a comparison argument [CL55, Chapter 8, Theorem 1.1], there is a sign change of $\dot{x}^*(t)$ inserted between every two zeros of $\zeta(t)$. Thus $\dot{x}^*(t)$ changes signs at least four times for $t \in [0, p]$, which is a contradiction to p being the minimal period of $\dot{x}^*(t)$, by Lemma 2.6 (iii). In this way we have proved that if $p' > 0$, then $\mu_c > -1$ and thereby $i(\gamma^*) = 2n - 2$.

To prove that $p' < 0$ implies $\mu_c < -1$, we follow a completely analogous argument by contradiction. However, $\zeta(t)$ has two fewer sign changes than y^* over the time interval $[0, p]$. Again, this is a contradiction when we consider the sign changes of the trivial eigenfunction $\dot{x}^*(t)$. Therefore, $p' < 0$ implies that $\mu_c > -1$ and the unstable dimension is $i(\gamma^*) = 2n - 1$, completing the proof. □

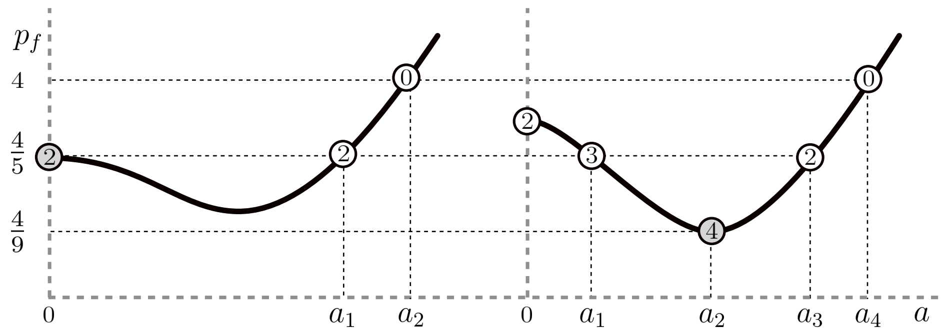


Figure 5.1: Schematic plots of two different period maps producing periodic orbits in the DDE (1.1). The unstable dimensions are indicated inside the circles. (Left) The equilibrium at the origin (grey) is at a Hopf point. (Right) The periodic orbit with amplitude a_2 (grey) is nonhyperbolic and is at a saddle-node point.

5.4 Interpretation

Although Theorems 5.6 and 5.11 are cumbersome at first glance, they are simple to represent graphically. On the one hand, Theorem 5.6 states that the periodic orbits of the DDE (1.1) with $f \in \mathfrak{X}^-$ correspond to the intersections of the period map p_f with the level sets $4n-3$ for $n \in \mathbb{N}$; see Figure 5.1. On the other hand, Theorem 5.11 ensures that a periodic orbit γ^* with amplitude a^* is hyperbolic if and only the graph of p_f intersects the level set $p_f(a^*)$ transversely. This is equivalent to requiring $p_f'(a^*) \neq 0$, moreover, the direction in which p_f intersects the level set $p_f(a^*)$ gives the unstable dimension $i(\gamma^*)$; see Figure 5.1.

The result is that there are two mechanisms behind the periodic solutions in (1.1):

1. **Hopf bifurcations** happening when $p_f(0) = 4n - 3$ for some $n \in \mathbb{N}$. Hence a periodic orbit is either emitted or absorbed by the equilibrium γ_0; see Figure 5.1 (Left).
2. **Saddle-node bifurcations** at which two hyperbolic periodic orbits collide, forming a nonhyperbolic periodic orbit whose amplitude a^* corresponds to a critical point $p_f'(a^*) = 0$. Immediately after, the periodic orbit vanishes; see Figure 5.1 (Right).

In particular, Theorems 5.6 and 5.11 complement the structural stability results in Chapter 4. The reason is that in Theorem 4.15 we saw that checking the hyperbolicity condition (MS1) ensures that the global attractor $\mathcal{A}(f)$ is Morse–Smale. Thus Theorem 5.11 provides a criterion for checking the structural stability of $\mathcal{A}(f)$.

Chapter 6

Connecting orbits

This chapter shows how the phase diagram $\Gamma(f)$ changes as the global attractor $\mathcal{A}(f)$ undergoes a Hopf bifurcation. Our analysis is a Hopf analog of the study of pitchfork bifurcations carried out by Fusco and Rocha [FR91] in scalar reaction-diffusion PDE (1.6) with Neumann boundary conditions.

We consider families of nonlinearities $f_\lambda \in \mathfrak{X}^-$ with C^1-dependence on a parameter $\lambda \in \mathbb{R}$. Under these conditions, the A-stability in Theorem 4.13 guarantees that the phase diagram $\Gamma(f_\lambda)$ remains unchanged as long as all $\gamma^* \in \mathrm{Crit}(f_\lambda)$ stay hyperbolic. Our goal is to discuss the changes happening at **isolated Hopf points**, i.e., parameter values λ_* so that:

(H1) There exist $\varepsilon > 0$ and a neighborhood $J := (\lambda_* - \varepsilon, \lambda_* + \varepsilon)$ such that critical elements $\gamma^*(\lambda) \in \mathrm{Crit}(f_\lambda)$ are hyperbolic for all $\lambda \in J \setminus \{\lambda_*\}$.

(H2) At $\lambda = \lambda_*$, the only equilibrium γ_0 becomes nonhyperbolic, while all other critical elements remain hyperbolic.

(H3) Denoting $J_- := (\lambda_* - \varepsilon, \lambda_*)$ and $J_+ := (\lambda_*, \lambda_* + \varepsilon)$, for $\lambda \in J_+$, there exists a single **bifurcating branch of hyperbolic periodic orbits** $\gamma^{\mathrm{c}}(\lambda)$ such that, in the standard notation for one-sided limits we have

$$\lim_{\lambda \to \lambda_*^+} \gamma^{\mathrm{c}}(\lambda) = 0.$$

Fully understanding the process of Hopf bifurcation is vital because it is the keystone in our new constructive method to design an infinitely large class of Morse–Smale global attractors $\mathcal{A}(f)$. More precisely, it allows us to produce a complete classification of the **Hopf–Smale attractors**, that is, those Morse–Smale attractors $\mathcal{A}(f)$ for which there exists a smooth deformation f_λ of nonlinearities such that $f_\lambda \in \mathfrak{X}^-$ for all $\lambda \in [0,1]$ and:

- The attractor of f_0 satisfies $\mathcal{A}(f_0) = \gamma_0$. In other words, $\mathcal{A}(f_0)$ consists of a single equilibrium and $f_1 = f$.
- The number of elements in $\mathrm{Crit}(f_\lambda)$ changes solely at a finite number of isolated Hopf bifurcation points in the sense of (H1)–(H3) above.

We describe the Hopf bifurcation process in three steps:

Step 1: Characterize the type of Hopf bifurcation, be it sub- or supercritical, in terms of the period map p_f in Theorem 5.6. In particular, each bifurcation type will correspond to the center manifold dynamics discussed in Theorem 3.2 Cases 1 and 2.

Step 2: Determine necessary conditions for the vanishing of connections between $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f_\lambda)$. More precisely, thanks to a refinement of the transitivity in Proposition 4.16, we show that if $\gamma^\dagger(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in J_-$ and $\gamma^\dagger(\lambda_*) \not\succ \gamma^*(\lambda_*)$ at a Hopf bifurcation point λ_*, then $\gamma^\dagger(\lambda) \succ \gamma_0(\lambda)$ and $\gamma_0(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in J_-$.

Step 3: We show that the connections broken by the method in Step 2 are inherited by the branch of bifurcating periodic solutions $\gamma^{\mathrm{c}}(\lambda)$ for $\lambda \in J_+$.

In Section 6.1, we show that the types of isolated Hopf points in the DDE (1.1) are completely determined by the period map from Chapter 1.

Section 6.2 is an extension to Hopf bifurcation of the transitivity property for Morse–Smale systems in Proposition 4.16.

Sections 6.2 to 6.4 introduce **blocking**, **adjacency**, and **liberalism**. Three terms imported from the classification of reaction-diffusion PDE attractors; see [FRW04]. Furthermore, we shall see that liberalism is an invariant of the Hopf bifurcations in our system. Thus completely determining the phase diagram of the Hopf–Smale attractors.

Lastly, Sections 6.5 to 6.7 show how to decode the notions of blocking and liberalism from the **signature**, a binary sequence encoding the shape of p_f. We present the first example of Hopf–Smale attractors with the Chafee–Infante class and explore the realizability of the signatures by using the enharmonic oscillator (1.8) from Chapter (1).

6.1 Types of Hopf bifurcation

A key aspect in studying the connectivity changes in a Hopf bifurcation is first to discuss what happens to the invariant manifolds involved in the process. Indeed, proceeding as in Lemma 4.8, a hyperbolic periodic orbit $\gamma^*(\lambda_*) \in \mathrm{Crit}(f_{\lambda_*})$ admits a unique local continuation $\gamma^*(\lambda)$ for all $\lambda \in J$. Denoting by $M(\lambda)$ the monodromy operator along the continuation, the discussion in Chapter 3 shows that the history space H admits a continuous $M(\lambda)$-invariant splitting into the unstable, center, and stable eigenspaces

$$H = H^{\mathrm{u}}(\lambda) \oplus H^{\mathrm{c}}(\lambda) \oplus H^{\mathrm{s}}(\lambda). \tag{6.1}$$

Furthermore, by hyperbolicity, they satisfy

$$\begin{aligned}
|\mu| > 1 \quad & \text{for all } \mu \in \mathrm{Spec}\left(M(\lambda)|_{H^{\mathrm{u}}(\lambda)}\right), \\
|\mu| = 1 \quad & \text{for all } \mu \in \mathrm{Spec}\left(M(\lambda)|_{H^{\mathrm{c}}(\lambda)}\right), \\
|\mu| < 1 \quad & \text{for all } \mu \in \mathrm{Spec}\left(M(\lambda)|_{H^{\mathrm{s}}(\lambda)}\right).
\end{aligned}$$

Recall that the construction of the global stable manifolds $W^{\mathrm{s}}(\gamma^*)$ was based on Theorem C.3. We constructed a single local chart $(W^{\mathrm{s}}_{\mathrm{loc}}(x_0^*), P^{\mathrm{s}})$ near points $x_0^* \in \gamma^*$ in such a way that the fundamental domain $W^{\mathrm{s}}_{\mathrm{loc}}(x_0^*)$ is C^1-uniformly continuous with respect to f_λ. However, the global extension $W^{\mathrm{s}}(\gamma^*(\lambda))$ required translations of the fundamental domain by the semiflow $S_f(t)$; see Theorems 3.1, 3.2, and 3.4, thereby we obtained an injectively immersed manifold rather than an embedded one. In particular, we lost the uniform continuity of the whole manifold in favor of uniform convergence on individual charts. Therefore, the expression

$$\lim_{\lambda \to \lambda_*} W^{\mathrm{s}}(\gamma^*(\lambda)) = W^{\mathrm{s}}(\gamma^*(\lambda_*)), \tag{6.2}$$

means that the convergence is considered uniformly on the charts above. The same argument holds for the unstable manifold $W^{\mathrm{u}}(\gamma^*)$. Nevertheless, the situation becomes more complicated when the splitting (6.1) experiences a discontinuity at an isolated Hopf point. By Theorem 5.11, the period map $p_{f_{\lambda_*}}$ satisfies $p_{f_{\lambda_*}}(0) = 4/(4n-3)$ for some $n \in \mathbb{N}$, and we distinguish two situations corresponding to Cases 1 and 2 in Theorem 3.2:

Case 1: $p_{f_{\lambda_*}}$ has a strict maximum at 0. Since we have standardized the bifurcation branch to appear for $\lambda \in J_+$, Lemma 2.9 and Theorem 5.11 guarantee that $i(\gamma_0(\lambda_*)) = 2n-2$ and $2n = i(\gamma^c(\lambda)) > i(\gamma_0(\lambda)) = 2n-2$ for all $\lambda \in J_+$. Thus, by Theorem 5.11, through the bifurcation the origin becomes more stable and emanates an unstable period orbit. Letting $M(\lambda)$ denote the monodromy operator at $\gamma_0(\lambda)$, we obtain that the $M(\lambda)$-invariant splitting (6.1) for $\lambda \in J_-$ satisfies

$$\lim_{\lambda\to\lambda_*^-} H^{\mathrm{u}}(\lambda) = H^{\mathrm{u}}(\lambda_*) + H^{\mathrm{c}}(\lambda_*) \quad \text{and} \quad \lim_{\lambda\to\lambda_*^-} H^{\mathrm{s}}(\lambda) = H^{\mathrm{s}}(\lambda_*).$$

Case 2: $p_{f_{\lambda_*}}$ possesses a strict minimum at 0, $i(\gamma_0(\lambda_*)) = 2n-2$, and $2n = i(\gamma_0(\lambda)) > i(\gamma^c(\lambda)) = 2n-2$ for all $\lambda \in J_+$. In particular, by Theorem 5.11 the origin destabilizes through the emission of a more stable period orbit. In the notation above, the $M(\lambda)$-invariant splitting at the origin (6.1) satisfies

$$\lim_{\lambda\to\lambda_*^-} H^{\mathrm{u}}(\lambda) = H^{\mathrm{u}}(\lambda_*) \quad \text{and} \quad \lim_{\lambda\to\lambda_*^-} H^{\mathrm{s}}(\lambda) = H^{\mathrm{s}}(\lambda_*) + H^{\mathrm{c}}(\lambda_*).$$

The discussion above allows us to distinguish two types of Hopf bifurcations.

Proposition 6.1. *Consider a smooth family of nonlinearities $f_\lambda \in \mathfrak{X}^-$ and let λ_* be an isolated Hopf point. Then using the standard one-sided limit convention, the following hold:*

$$\begin{aligned} \lim_{\lambda\to\lambda_*^-} W^{\mathrm{u}}(\gamma_0(\lambda)) &= W^{\mathrm{u}}(\gamma_0(\lambda_*)) = \lim_{\lambda\to\lambda_*^+} W^{\mathrm{u}}(\gamma^{\mathrm{c}}(\lambda)),\\ \lim_{\lambda\to\lambda_*^-} W^{\mathrm{s}}(\gamma_0(\lambda)) &= W^{\mathrm{s}}(\gamma_0(\lambda_*)) = \lim_{\lambda\to\lambda_*^+} W^{\mathrm{s}}(\gamma^{\mathrm{c}}(\lambda)). \end{aligned} \tag{6.3}$$

Furthermore, based on Cases 1 and 2 above, we can distinguish two scenarios:

Case 1: *If $p_{f_{\lambda_*}}$ has a strict local maximum at 0, then $i(\gamma^c(\lambda)) = i(\gamma_0(\lambda)) + 1 = z(\gamma^c(\lambda))$ and $\gamma^c(\lambda) \succ \gamma_0(\lambda)$ for all $\lambda \in J_+$. Additionally,*

$$\lim_{\lambda\to\lambda_*^+} W^{\mathrm{u}}(\gamma_0(\lambda)) = W^{\mathrm{uu}}(\gamma_0(\lambda_*)). \tag{6.4}$$

Case 2: *If $p_{f_{\lambda_*}}$ has a strict local minimum at* 0*, then* $i(\gamma^{\mathrm{c}}(\lambda)) = i(\gamma_0(\lambda)) - 2 = z(\gamma^{\mathrm{c}}) - 1$ *and* $\gamma_0(\lambda) \succ \gamma^{\mathrm{c}}(\lambda)$ *for all* $\lambda \in J_+$*. Moreover,*

$$\lim_{\lambda \to \lambda_*^+} W^{\mathrm{s}}(\gamma_0(\lambda)) = W^{\mathrm{ss}}(\gamma_0(\lambda_*)).$$

Proof. The proof essentially requires us to discuss the center manifold dynamics. We consider first Case 1 and Case 2 is completely analogous with the dynamics on the center manifold reversed. Indeed, we have that $i(0) > i(\gamma^{\mathrm{c}}(\lambda))$ for all $\lambda \in J_+$. Sufficiently close to λ_*, we use Theorem C.7 and Lemma B.2 to construct a λ-continuation $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$ of the two-dimensional local center manifold $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda_*))$ for the time-1 map $S_{f_\lambda}(1)$ in an open neighborhood around the origin. By construction, both $\gamma_0(\lambda)$ and $\gamma^{\mathrm{c}}(\lambda)$ belong to $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$ for $\lambda \in J_+$ sufficiently close to λ_*. Furthermore, $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$ is $S_{f_\lambda}(1)$-invariant, and $\gamma^{\mathrm{c}}(\lambda)$ defines a $S_{f_\lambda}(1)$-invariant circle on $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$. Since the restriction of $S_{f_\lambda}(1)$ to $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$ is orientation preserving, γ^* is a separatrix for the dynamics on $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$. As a result, all initial conditions $\phi \in W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$ possess both α- and ω-limit sets contained in the closure of the interior region determined by $\gamma^{\mathrm{c}}(\lambda)$. Recalling that $i(\gamma^{\mathrm{c}}(\lambda)) > i(\gamma_0(\lambda))$, Proposition 2.5 and Corollary 4.7 show that $\gamma^{\mathrm{c}}(\lambda) \succ \gamma_0(\lambda)$ for all λ sufficiently close to λ_* and, by Theorem 4.13, also for all $\lambda \in J_+$.

Moreover, recall that $\gamma^{\mathrm{c}}(\lambda)$ is hyperbolic for $\lambda \in J_+$. Thus, by the discussion above, any solutions in $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$ lying outside of $\gamma^{\mathrm{c}}(\lambda)$ are repelled by $\gamma^{\mathrm{c}}(\lambda)$. Using continuity, we shrink $\gamma^{\mathrm{c}}(\lambda)$ to zero as $\lambda \to \lambda_*$; by Theorem C.7 and the instability of γ^{c} in $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda))$, we observe that $\gamma_0(\lambda_*)$ is unstable within the center manifold $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda_*))$, as in Theorem 3.2 Case 1. Therefore, we can use the same continuity argument for the local charts that we used in the hyperbolic case (6.2) to conclude (6.3) and (6.4). ☐

Proposition (6.1) finally connects Cases 1 and 2 in Theorem (3.2) with the two bifurcation types described in Chapter 1. Furthermore, the characterization can be made in terms of the period map p_f. Thus, given an isolated Hopf point λ_*, we say that the Hopf bifurcation is **subcritical** if the assumptions of Case 1 in Proposition 6.1 are met. Likewise, we say that λ_* is **supercritical** if the assumptions of Case 2 hold.

6.2 Transitivity

In this section, we give a weak version of the transitivity property in Proposition 4.16. This guaranteed that given three different hyperbolic critical elements $\gamma^\dagger, \gamma^\diamond, \gamma^* \in \mathrm{Crit}(f)$, the connections $\gamma^\dagger \succ \gamma^\diamond$ and $\gamma^\diamond \succ \gamma^*$ can be concatenated to obtain $\gamma^\dagger \succ \gamma^*$. In the following, we show that only the intermediate critical element $\gamma^\diamond$ has to remain hyperbolic for transitivity to hold.

Lemma 6.2. *Let $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$ with $f \in \mathfrak{X}^-$ be isolated critical elements such that if any of them is nonhyperbolic, then it is the equilibrium γ_0. If there exists a hyperbolic $\gamma^\diamond \in \mathrm{Crit}(f)$ such that $\gamma^\dagger \succ \gamma^\diamond$ and $\gamma^\diamond \succ \gamma^*$, then $\gamma^\dagger \succ \gamma^*$.*

Proof. The result is a consequence of the λ-lemma [Pal69, Lemma 1.1], which also possesses a version for compact injective semiflows $S_f(t)$ with injective Fréchet derivative; see [HMO02, Proposition 6.2.3]. The case $\gamma^\diamond = \gamma_0$ reduces automatically to the standard transitivity [HMO02, Proposition 6.2.4] since all the elements involved are hyperbolic.

Thus, we consider the case in which $\gamma^\diamond$ is a periodic orbit and Theorem 4.13 guarantees that

$$W^{\mathrm{u}}(\gamma^\dagger) \pitchfork W^{\mathrm{s}}(\gamma^\diamond) \quad \text{and} \quad W^{\mathrm{u}}(\gamma^\diamond) \pitchfork W^{\mathrm{s}}(\gamma^*).$$

Consider $x_0^\diamond \in \gamma^\diamond$ and the dynamical system induced by the Poincaré map $\mathcal{P}$ defined in (3.17) for an invariant leaf $x_0^\diamond \in \mathcal{U}_0^\diamond$. Denoting the unstable manifold of the Poincaré map by

$$W^{\mathrm{u}}_{\mathrm{loc}}(x_0^\diamond) := W^{\mathrm{u}}(\gamma^\diamond) \cap \mathcal{U}_0^\diamond,$$

we take an open disk $U^\diamond \subset W^{\mathrm{u}}(x_0^\diamond)$ centered at $x_0^\diamond$. The λ-lemma for maps [HMO02, Proposition 6.1.12] shows that for all $\delta > 0$, we can find a submanifold $U_\delta^\dagger \subset W^{\mathrm{u}}(\gamma^\dagger) \cap \mathcal{U}_0$ of dimension $\dim U_\delta^\dagger = \dim W^{\mathrm{u}}_{\mathrm{loc}}(x_0^\diamond)$ such that $U_\delta^\dagger$ and $U^\diamond$ are δ-close in the C^1-uniform norm. In other words, we can find C^1-coordinate charts for $U_\delta^\dagger$ and $U^\diamond$ that are δ-close in the C^1-topology.

Let us choose a point $\phi \in U^\diamond \pitchfork (W^{\mathrm{s}}(\gamma^*) \cap \mathcal{U}_0^\diamond) \subset W^{\mathrm{u}}_{\mathrm{loc}}(x_0^\diamond) \pitchfork (W^{\mathrm{s}}(\gamma^*) \cap \mathcal{U}_0^\diamond)$, where transversality now takes place on the leaf $\mathcal{U}_0^\diamond$, which is a manifold of codimension one in H. By Theorem 4.13, we can find an open disk $U^* \in$

$(W^s(\gamma^*) \cap \mathcal{U}_0^\diamond)$, centered at x_0, such that U^* is transverse to $U^\diamond$ in $\mathcal{U}_0^\diamond$. Thus, choosing δ sufficiently small, we guarantee that $U_\delta^\dagger \cap U^* \neq \emptyset$, which shows $W^u(\gamma^\dagger) \cap W^s(\gamma^*) \neq \emptyset$. □

An immediate consequence is that to break a connection $\gamma^\dagger \succ \gamma^*$ we need it to proxy through a nonhyperbolic critical element $\gamma^\diamond$.

Lemma 6.3. *Consider an isolated Hopf point λ_* for a smooth family of nonlinearities $f_\lambda \in \mathfrak{X}^-$. Let $\gamma^\dagger(\lambda), \gamma^*(\lambda) \in \mathrm{Crit}(f_\lambda)$ denote two different branches of critical elements that remain hyperbolic for all $\lambda \in J$. If $\gamma^\dagger(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in J_-$ (resp., $\lambda \in J_+$) and $\gamma^\dagger(\lambda) \not\succ \gamma^*(\lambda)$ for all $\lambda \in J_+$ (resp., $\lambda \in J_-$), then $\gamma^\dagger(\lambda_*) \succ \gamma_0(\lambda_*)$ and $\gamma_0(\lambda_*) \succ \gamma^*(\lambda_*)$.*

Proof. Since the argument relies solely on the compactness of the solution operator $S_{f_\lambda}(t)$, for t sufficiently large, the proof is the same as the one of [Hen85, Theorem 9] for parabolic PDEs. We reproduce it for completeness.

The core idea is that, since $\gamma^\dagger(\lambda) \succ \gamma^*(\lambda)$ for $\lambda < \lambda_*$, in the limit $\lambda \to \lambda_*$ we can construct a finite chain of critical elements $\gamma^{(j)} \in \mathrm{Crit}(f_{\lambda_*})$ such that

$$\gamma^\dagger(\lambda_*) \succ \gamma^{(n)} \succ \cdots \succ \gamma^{(0)} \succ \gamma^*(\lambda_*).$$

At this point, by assumptions (H1)–(H3), we obtain a dichotomy:

(i) Either all $\gamma^{(j)}$ in the chain are hyperbolic periodic orbits, or

(ii) there exists some $j^* \in \{0, \dots, n\}$ for which $\gamma^{(j^*)} = 0$.

In case (i), we apply the transitivity in Lemma 6.2 to obtain that $\gamma^\dagger(\lambda_*) \succ \gamma^*(\lambda_*)$, thus Theorem 4.13 guarantees that $\gamma^\dagger(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in \{\lambda_*\} \cup J_+$.

In case (ii), however, there exist connections from $\gamma^\dagger(\lambda_*)$ to $\gamma^*(\lambda_*)$ that proxy through the single nonhyperbolic critical element γ_0. Therefore, we have two chains such that

$$\begin{aligned} &\gamma^\dagger(\lambda_*) \succ \gamma^{(n)} \succ \cdots \succ \gamma^{(j^*+1)} \succ \gamma_0(\lambda_*), \\ &\gamma_0(\lambda_*) \succ \gamma^{(j^*-1)} \succ \dots \gamma^{(0)} \succ \gamma^*(\lambda_*). \end{aligned}$$

Applying Lemma 6.2, we obtain that $\gamma^\dagger(\lambda_*) \succ \gamma_0(\lambda_*)$ and $\gamma_0(\lambda_*) \succ \gamma^*(\lambda_*)$. □

6.3 Blocking

In the following, we denote the existence of an undirected connection between $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$ by $\gamma^\dagger \sim \gamma^*$, that is, if either $\gamma^\dagger \succ \gamma^*$ or $\gamma^* \succ \gamma^\dagger$. Likewise, we denote the absence of a connection by $\gamma^\dagger \not\sim \gamma^*$.

Recall that, by Lemma 2.6 (vi), the **relative zero number** is well defined for formal differences of different critical elements via

$$z(\gamma^\dagger - \gamma^*) := z(x_0^\dagger - x_0^*), \quad \text{for any } x_0^\dagger \in \gamma^\dagger, x_0^* \in \gamma^*.$$

We can show that the relative zero number is uniquely determined by the relative **amplitude** $a(\gamma^*) := \max_{\phi \in \gamma^*} \|\phi\|_{C^0}$.

Lemma 6.4. *Consider two critical elements $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$ with $f \in \mathfrak{X}^-$ such that $a(\gamma^*) > a(\gamma^\dagger)$. If the period map p_f is locally nonconstant, then the relative zero number satisfies $z(\gamma^* - \gamma^\dagger) = z(\gamma^* - \gamma_0) = z(\gamma^*)$.*

Proof. Indeed, the result is equivalent to [BF88, Lemma 4.2] in the setting of reaction-diffusion PDEs. The main reason is that, analogously to equilibria in reaction-diffusion, the periodic solutions of the DDE (1.1) solve a second-order ODE. Still, we adapt the proof for completeness.

The result is immediate if $\gamma^\dagger = \gamma_0$. Otherwise, we denote by $x^*(t)$ and $x^\dagger(t)$ the solutions of the DDE (1.1) with orbits γ^* and $\gamma^\dagger$ respectively. We shall compare the sign changes of $x^*(t)$, $\dot{x}^*(t)$, and $x^*(t) - x^\dagger(t)$. Since all these functions solve linear systems of the form (2.10), they meet the assumptions of Proposition 2.2 and possess only simple zeros. Furthermore, by Lemma 2.6 (ii) and (iii), the projection $P\gamma^*$ is nested inside $P\gamma^\dagger$, implying that if $x^*(t) = x^\dagger(t)$, then we have the identity

$$\mathrm{sign}(x^*(t-1) - x^\dagger(t-1)) = \mathrm{sign}(x^*(t-1)).$$

Thus we obtain that $x^*(t) - x^\dagger(t)$ changes signs at most once in between any two zeros of $x^*(t)$. We will now show that

$$z(\gamma^* - \gamma^\dagger) \leq z(\gamma^*). \tag{6.5}$$

Indeed, the remarks above show that for all $t \in \mathbb{R}$ we have

$$\mathrm{sc}(x_t^* - x_t^\dagger) \leq \mathrm{sc}(x_t^*) + 1,$$

where sc stands for the sign change function defined in (2.14). In particular, if $\mathrm{sc}(x^*_t)$ is even, then (6.5) follows from the definition of the zero number (2.15). If $n := \mathrm{sc}(x^*_t)$ is odd, however, we have to discard the case $\mathrm{sc}(x^*_t - x^\dagger_t) = n+1$ since it does not satisfy (6.5). By contradiction, we denote the zeros of $x^*_t - x^\dagger_t$ by $-1 < \theta_1 < \theta_2 < \dots \theta_{n+1} < 0$, likewise, the zeros of x^*_t are the θ^*_j satisfying

$$-1 < \theta_1 < \theta^*_1 < \theta_2 < \cdots < \theta^*_n < \theta_{n+1} < 0.$$

Next, we shift time forward by $t^* > 0$ so that the zeros of $x^*_{t+t^*} - x^\dagger_{t+t^*}$ and $x^*_{t+t^*}$ satisfy

$$\theta_1 - t^* < -1 < \theta^*_1 - t^* < \theta_2 - t^* < \cdots < \theta^*_n - t^* < \theta_{n+1} - t^* < 0.$$

Since n was assumed to be odd, we have that $z(\gamma^*) = n$. Thus $x^*_{t+t^*}$ possesses no sign changes in the interval $(\theta^*_n + t^*, 0)$ and neither does $x^*_{t+t^*} - x^\dagger_{t+t^*}$ because $\theta_{n+1} - t^*$ is its only zero in the interval $(\theta^*_n + t^*, 0)$. Hence we have shown that $\mathrm{sc}(x^*_{t+t^*} - x^\dagger_{t+t^*}) = n$, so that

$$z(x^*_t - x^\dagger_t) = n + 2 \quad \text{and} \quad x^*_{t+t^*} - x^\dagger_{t+t^*} = n,$$

in contradiction to Lemma (2.10) (vi). This completes the proof of (6.5). To see the reverse inequality, notice that the nestedness of the projections yields

$$\mathrm{sign}(x^*(t) - x^\dagger(t)) = \mathrm{sign}(x^*(t)),$$

at $\dot{x}^*(t) = 0$. Since $x^*(t)$ is sinusoidal, this implies that $x^*(t) - x^\dagger(t)$ has at least one zero for every sign change of $\dot{x}^*(t)$. Hence

$$z(x^*_t - x^\dagger_t) + 1 \geq z(\dot{x}^*_t)$$

for all $t \in \mathbb{R}$. Notice that the argument used to prove (6.5) is still valid if we replace $x^*_t - x^\dagger_t$ by $\dot{x}_t$ and x^*_t by $x^*_t - x^\dagger_t$. Hence, we obtain

$$z(\gamma^* - \gamma^\dagger) \geq z(\dot{x}^*_t),$$

for all $t \in \mathbb{R}$. Recalling from Lemma 2.1 (vi) that $z(\dot{x}^*_t) = z(x^*_t) = z(\gamma^*)$, we have that

$$z(\gamma^* - \gamma^\dagger) \geq z(\gamma^*),$$

which completes the proof. □

Therefore, the relative zero number between two critical elements is fully determined by the element that possesses the largest amplitude. In particular, following the same method as in parabolic PDEs [FRW04] allows us to define a simple condition preventing the existence of connections between two critical elements.

Lemma 6.5. *Consider three different critical elements $\gamma^\dagger, \gamma^\diamond, \gamma^* \in \mathrm{Crit}(f)$ with $f \in \mathfrak{X}^-$ such that p_f is locally nonconstant. Let $a(\gamma^\dagger) < a(\gamma^\diamond) < a(\gamma^*)$, if $z(\gamma^\diamond) = z(\gamma^*)$, then there exist no connections between $\gamma^\dagger$ and γ^*. i.e., $\gamma^\dagger \not\sim \gamma^*$.*

Proof. We proceed by contradiction. Suppose that $\gamma^\dagger \succ \gamma^*$, i.e., that there exists an initial condition $\phi \in \mathcal{A}(f)$ such that $\alpha(\phi) = \gamma^\dagger$ and $\omega(\phi) = \gamma^*$. Let us denote by $x(t)$ the solution through ϕ. Using Lemma 6.4, for any $x_0^\diamond \in \gamma^\diamond$ we have that the zero number satisfies $\lim_{t\to\infty} z(x_t - x_t^\diamond) = z(\gamma^* - \gamma^\diamond) = z(\gamma^*)$ and $\lim_{t\to-\infty} z(x_t - x_t^\diamond) = z(\gamma^\dagger - \gamma^\diamond) = z(\gamma^\diamond)$.

By Lemma 2.6 (ii), we have that the planar projections of the periodic orbits are nested, i.e., $\mathrm{Int}\left(P\gamma^\dagger\right) \subset \mathrm{Int}\left(P\gamma^\diamond\right) \subset \mathrm{Int}\left(P\gamma^*\right)$ where Int refers to the interior region defined by the corresponding Jordan curve. Moreover, by continuity of the planar projection P, there exists a t^* such that $Px_{t^*} \in P\gamma^\diamond \neq \emptyset$. Since $(x(t) - x^\diamond(t))$ solves the linear equation (2.10) with the coefficients (2.13), by the monotonicity of the zero number in Proposition 2.2 we obtain

$$\begin{aligned} z(\gamma^*) &= z\left(\gamma^\diamond\right) \\ &= z\left(\gamma^\dagger - \gamma^\diamond\right) \\ &> z\left(x_{t^*} - x_{t^*}^\diamond\right) \\ &\geq z\left(\gamma^* - \gamma^\diamond\right) \\ &= z\left(\gamma^*\right). \end{aligned}$$

Thus, reaching a contradiction and obtaining $\gamma^\dagger \not\succ \gamma^*$. The argument is symmetric under the exchange of $\gamma^\dagger$ and γ^*; thus, we have also proved $\gamma^* \not\succ \gamma^\dagger$, which yields $\gamma^\dagger \not\sim \gamma^*$. □

In the setting of Lemma 6.5, we say that $\gamma^\diamond$ **blocks the connections** between $\gamma^\dagger$ and γ^*. We say that $\gamma^\dagger$ and γ^* are **adjacent** if there is no critical element $\gamma^\diamond$ blocking the connections between $\gamma^\dagger$ and γ^*.

6.4 Liberalism

We say that a global attractor $\mathcal{A}(f)$ satisfies **liberalism** if all adjacent critical elements are connected. The goal of this section is to determine that if $\mathcal{A}(f)$

is Hopf–Smale, then liberalism holds. To this end, we first show the following proposition.

Proposition 6.6. *Let $\lambda = \lambda_*$ be an isolated Hopf point for a smooth family of nonlinearities $f_\lambda \in \mathfrak{X}^-$ and let $\gamma^\dagger(\lambda), \gamma^*(\lambda) \in \mathrm{Crit}(f_\lambda)$ be two branches of periodic orbits that remain hyperbolic for all $\lambda \in J$. Denoting by $\gamma^{\mathrm{c}}(\lambda)$ the branch of bifurcating periodic orbits, we have that:*

(i) $\gamma^\dagger(\lambda) \sim \gamma^*(\lambda)$ *for* $\lambda \in J_-$ *if and only if* $\gamma^\dagger(\lambda) \sim \gamma^*(\lambda)$ *for* $\lambda \in J_+$.

(ii) $\gamma_0(\lambda) \sim \gamma^*(\lambda)$ *for* $\lambda \in J_-$ *if and only if* $\gamma^{\mathrm{c}}(\lambda) \sim \gamma^*(\lambda)$ *for* $\lambda \in J_+$.

(iii) $\gamma_0(\lambda) \sim \gamma^*(\lambda)$ *for* $\lambda \in J_-$ *and* $|i(\gamma_0(\lambda_*)) + 1 - z(\gamma^*(\lambda_*))| \geq 2$ *if and only if* $\gamma_0(\lambda) \sim \gamma^*(\lambda)$ *for* $\lambda \in J_+$.

Proof. To see (i), we proceed by contradiction. Indeed, let us suppose that $\gamma^\dagger(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in J_-$ and that connection between them vanishes for $\lambda \in J_+$. Thus, Lemma 6.3 shows that the only way the connection can be broken for $\lambda \in J_+$ is if we have $\gamma^\dagger(\lambda_*) \succ \gamma_0(\lambda_*)$ and $\gamma_0(\lambda_*) \succ \gamma^*(\lambda_*)$. Furthermore, the C^1-limits (6.3) ensure that

$$W^{\mathrm{u}}(\gamma^\dagger(\lambda_*)) \cap W^{\mathrm{s}}(\gamma^{\mathrm{c}}(\lambda_*)) = W^{\mathrm{u}}(\gamma^\dagger(\lambda_*)) \cap W^{\mathrm{s}}(\gamma_0(\lambda_*)) \neq \emptyset$$

and

$$W^{\mathrm{u}}(\gamma^{\mathrm{c}}(\lambda_*)) \cap W^{\mathrm{s}}(\gamma^*(\lambda_*)) = W^{\mathrm{u}}(\gamma_0(\lambda_*)) \cap W^{\mathrm{s}}(\gamma^*(\lambda_*)) \neq \emptyset.$$

Here $W^{\mathrm{s}}(\gamma^{\mathrm{c}}(\lambda_*))$ and $W^{\mathrm{u}}(\gamma^{\mathrm{c}}(\lambda_*))$ refer to the limits of the corresponding invariant manifolds as $\lambda \to \lambda_*$. As a result, Theorem 4.14 shows that

$$W^{\mathrm{u}}(\gamma^\dagger(\lambda_*)) \pitchfork W^{\mathrm{s}}(\gamma^{\mathrm{c}}(\lambda_*)) \neq \emptyset \quad \text{and} \quad W^{\mathrm{u}}(\gamma^{\mathrm{c}}(\lambda_*)) \pitchfork W^{\mathrm{s}}(\gamma^*(\lambda_*)) \neq \emptyset,$$

which allows us to use Corollary B.2 and continue the intersection to $\lambda \in J_+$, yielding $\gamma^\dagger(\lambda) \succ \gamma^{\mathrm{c}}(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in J_+$. Finally, by the transitivity Lemma 6.2, we obtain that $\gamma^\dagger(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in J_+$, yielding a contradiction. Reversing the direction of bifurcation shows the converse implication by an analogous argument.

To see (ii), we consider the limits (6.3) together with Theorem 4.14. Indeed, assume that $\gamma^*(\lambda) \succ \gamma_0(\lambda)$ for all $\lambda \in J_-$. Then we can continue the nontrivial intersections

$$W^{\mathrm{u}}(\gamma^*(\lambda_*)) \pitchfork W^{\mathrm{s}}(\gamma_0(\lambda_*)) = W^{\mathrm{u}}(\gamma^*(\lambda_*)) \pitchfork W^{\mathrm{s}}(\gamma^{\mathrm{c}}(\lambda_*)) \neq \emptyset$$

so that $W^{\mathrm{u}}(\gamma^*(\lambda)) \pitchfork W^{\mathrm{s}}(\gamma^{\mathrm{c}}(\lambda)) \neq \emptyset$ for all $\lambda \in J_+$. The argument is completely symmetric under an exchange of the direction of bifurcation, proving the converse. If we assume $\gamma_0(\lambda) \succ \gamma^*(\lambda)$ instead, then the proof follows analogously by swapping the roles of γ_0 and γ^*.

To see (iii), we show first the only if part. Indeed, assume that $\gamma_0(\lambda) \sim \gamma^*(\lambda)$ for all $\lambda \in J_+$. In particular, $\gamma^{\mathrm{c}}(\lambda)$ does not block the connections between $\gamma_0(\lambda)$ and $\gamma^*(\lambda)$. Hence $z(\gamma^{\mathrm{c}}(\lambda_*)) \neq z(\gamma^*(\lambda_*))$ and, by Proposition 6.1, we have that $z(\gamma^{\mathrm{c}}(\lambda)) = i(\gamma_0(\lambda_*)) + 1$ and obtain the bound $|i(\gamma_0(\lambda_*)) + 1 - z(\gamma^*(\lambda_*))| \geq 2$. By Lemma 6.3, a necessary condition for breaking the connection $\gamma_0(\lambda_*) \sim \gamma^*(\lambda_*)$ for $\lambda \in J_-$ is that $\gamma_0(\lambda_*) \sim \gamma^*(\lambda_*)$. Hence the connection $\gamma_0(\lambda_*) \sim \gamma^*(\lambda_*)$ can be continued to $\lambda \in J_-$ thanks to Theorem 4.14.

Next we show the if part in (iii) at a subcritical Hopf point, i.e., Case 1 in Proposition 6.1. The supercritical case is analogous. Assume that $\gamma_0(\lambda) \sim \gamma^*(\lambda)$ for all $\lambda \in J_-$ and $|i(\gamma_0(\lambda_*)) + 1 - z(\gamma^*(\lambda_*))| \geq 2$. If $\gamma^*(\lambda) \succ \gamma_0(\lambda)$ for all $\lambda \in J_-$, then Proposition 6.1 and Part (ii) above show that $\gamma^{\mathrm{c}}(\lambda) \succ \gamma_0(\lambda)$ and $\gamma^*(\lambda) \succ \gamma^{\mathrm{c}}(\lambda)$ for all $\lambda \in J_+$. Thus, by the transitivity in Lemma 6.2, we obtain $\gamma^*(\lambda) \succ \gamma_0(\lambda)$ for all $\lambda \in J_+$.

Finally, we show what happens if the bifurcation is subcritical and $\gamma_0(\lambda) \succ \gamma^*(\lambda)$ for all $\lambda \in J_-$. By Lemma 6.3, the connection $\gamma_0(\lambda) \succ \gamma^*(\lambda)$ for $\lambda \in J_+$ is only broken if $\gamma_0(\lambda_*) \succ \gamma^*(\lambda_*)$. Hence Theorem 4.14 and Lemma B.1 show that

$$\dim(W^{\mathrm{u}}(\gamma_0(\lambda_*)) \pitchfork W^{\mathrm{s}}(\gamma^*(\lambda_*))) = i(\gamma_0(\lambda_*)) + 2 - i(\gamma^*(\lambda_*)).$$

Moreover, the inequality $|i(\gamma_0(\lambda_*))+1-z(\gamma^*(\lambda_*))| \geq 2$ ensures that $i(\gamma_0(\lambda_*)) > i(\gamma^*(\lambda_*))$, obtainiting $\dim(W^{\mathrm{u}}(\gamma_0(\lambda_*)) \pitchfork W^{\mathrm{s}}(\gamma^*(\lambda_*))) \geq 3$. This proves that $W^{\mathrm{uu}}(\gamma_0(\lambda_*)) \pitchfork W^{\mathrm{s}}(\gamma^*(\lambda_*))) \neq \emptyset$ because otherwise all the connections from $\gamma_0(\lambda_*)$ to $\gamma^*(\lambda_*)$ would lie on the two-dimensional center manifold $W^{\mathrm{c}}_{\mathrm{loc}}(\gamma_0(\lambda_*))$. It follows from Theorem 4.14 and the limit (6.4) that the connection $\gamma_0(\lambda) \succ \gamma^*(\lambda)$ can be continued to all $\lambda \in J_+$, completing the proof. □

The connectivity rules (i)–(iii) in Proposition 6.6 above can be summarized in one by the following corollary.

Corollary 6.7. *Under the assumptions of Proposition 6.6, the global attractor $\mathcal{A}(f_\lambda)$ satisfies liberalism for all $\lambda \in J_-$ if and only if $\mathcal{A}(f_\lambda)$ satisfies* *liberalism for all $\lambda \in J_+$.*

Proof. Indeed, if liberalism holds for all $\lambda \in J_-$, then (i) preserves liberalism between the orbits that are not directly involved in the Hopf bifurcation. Furthermore, (ii) shows that γ^c inherits all the connections that γ_0 had before the bifurcation took place, and we know from Proposition 6.1 that $\gamma_0 \sim \gamma^c$ for all $\lambda \in J_+$. Therefore, liberalism holds for the connections between γ^c and any other critical element for $\lambda \in J_+$. Finally, (iii) tells us that all the connections with γ_0 that are not blocked by γ^c survive for all $\lambda \in J_+$. Thus, liberalism holds for all the connections involving γ_0, and we have that $\mathcal{A}(f_\lambda)$ satisfies liberalism for all $\lambda \in J_+$.

Conversely, let liberalism hold for all $\lambda \in J_+$. Liberalism for $\lambda \in J_-$ holds between any two orbits that are not involved directly in the Hopf bifurcation due to the symmetry of (i). However, (ii) shows that for $\lambda \in J_-$, γ_0 absorbs any connections that were blocked by γ^c for $\lambda \in J_+$. Furthermore, any connections that were not blocked by γ^c, are preserved by (iii), which completes the proof. □

Ultimately, this leads to a law of connection for all possible Hopf–Smale attractors.

Corollary 6.8. *Let $\mathcal{A}(f)$ with $f \in \mathfrak{X}^-$ be a Hopf–Smale attractor. Then it satisfies liberalism.*

Proof. Trivially, the attractor $\mathcal{A}(f_0) = \gamma_0$ at the beginning of the Hopf homotopy satisfies liberalism. By Corollary 6.7, the result follows from the definition of Hopf–Smale attractor. □

6.5 Signatures

In particular, Corollary 6.8 reduces the description of the Hopf–Smale global attractors $\mathcal{A}(f)$ to the study of the possible configurations of relative zero numbers and unstable dimensions for the critical elements $\mathrm{Crit}(f)$. Given a Hopf–Smale attractor $\mathcal{A}(f)$ with $f \in \mathfrak{X}^-$, Lemma 4.8 ensures that $\mathrm{Crit}(f) = \{\gamma_0, \ldots, \gamma_N\}$. Here the critical elements are ordered by increasing amplitudes

$a_j := a(\gamma_j) < a(\gamma_{j+1})$ for $j = 0, \dots, N-1$. The binary $\{-1, 1\}$-sequence given by

$$\chi(f) := \big(1, \mathrm{sign}(p_f'(a_1)), \dots, \mathrm{sign}(p_f'(a_{N-1})), 1\big)\,, \tag{6.6}$$

is called **signature** of f. In this section, we show that $\chi(f)$ contains enough information to reconstruct the phase diagram $\Gamma(f)$.

Lemma 6.9. *Let $f \in \mathfrak{X}^-$ be such that $\mathcal{A}(f)$ is Morse–Smale and, as above, let us denote the critical element with the largest amplitude by γ_N. Then $p_f(a) > 4$ for all $a > a(\gamma_N)$, $i(\gamma_N) = 0$, and $z(\gamma_N) = 1$.*

Proof. Indeed, by the compactness of the global attractor $\mathcal{A}(f)$, we can always find a $K > 0$ so large that the constant function $\phi \equiv K$ satisfies $z(\phi - \tilde{\phi}) = 1$ for all $\tilde{\phi} \in \mathcal{A}(f)$. Denote by $x(t)$ the solution with the initial condition ϕ and let $x^*(t)$ be a periodic solution with orbit γ_N. We recall that $(x(t) - x^*(t))$ solves the linear DDE (2.10) with coefficients (2.13). Thus, by Proposition 2.2, we have that $z(x_t - x_t^*) \geq z(\phi - x_0^*) = 1$ for all $t \geq 1$. Moreover, since the zero number can not take negative values, we have that $Px_t \cap P\gamma_N = \emptyset$ for all $t \geq 0$, i.e., the planar projection $\{(x(t), x(t-1)) : t \geq 0\}$ is confined to the exterior connected component of $\mathbb{R}^2$ defined by the Jordan curve $P\gamma_N$. However, the global attractor being the maximal compact invariant set implies $\omega(\phi) \subset \mathcal{A}(f)$. In particular, we obtain that $\omega(\phi) = \gamma_N$ and it follows from Theorem 3.4 that $\phi \in W^{\mathrm{s}}(\gamma_N)$; however, if $K > 0$ is large enough, the same argument is valid for all $\tilde{\phi} \in H$ sufficiently close to ϕ. Thus $W^{\mathrm{s}}(\gamma_N)$ contains an open ball in H, yielding $i(\gamma_N) = 0$. By assumption γ_N is hyperbolic and Theorem 5.11 implies that $p_f(a(\gamma_N)) = 4$ and $\mathrm{sign}(p_f'(a(\gamma_N))) = 1$. Since γ_N is the periodic solution with the largest amplitude, the proof is complete by Theorem 5.6. □

In particular, all the global attractors discussed in this work possess a stable outermost periodic solution. As a result of Lemma 6.9, the signature (6.6) encodes the shape of the period map p_f for any Morse–Smale nonlinearity $f \in \mathfrak{X}^-$. Thus our signature is just a special case of the **Rocha signature** used by Rocha to describe the period maps of planar Hamiltonian ODEs with Morse potentials; see [Roc07]. This property allows us to reconstruct the relative zero numbers and Morse indices of all periodic orbits from the signature.

Lemma 6.10. *In the setting of Lemma 6.9, let us denote*

$$(s_0, \dots, s_N) := \chi(f)$$

and construct the sequences $(z_0, \dots, z_N)$ *and* $(i_0, \dots, i_N)$ *via*

$$\begin{aligned} z_j &:= z_{j+1} + s_{j+1} + s_j, \quad j = 0, \dots, N-1, \quad z_N := 1, \\ i_j &:= z_j - \frac{s_j + 1}{2}, \qquad\quad j = 0, \dots, N. \end{aligned} \tag{6.7}$$

Then $z_j = z(\gamma_j)$ *for* $j = 1, \dots, N$, *and* $i_j = i(\gamma_j)$ *for* $j = 0, \dots, N$.

Proof. The claims are immediate if $a_N = 0$. Otherwise, Lemma 6.9 shows that $i(\gamma^{(N)}) = 0$ and $z(\gamma^{(N)}) = 1$, as required. We now proceed by induction and suppose that our claims hold for $j + 1$. Certainly, if $s_j = s_{j+1}$, we have that $p_f(a_j)$ and $p_f(a_{j+1})$ are consecutive realizable periods, i.e.,

$$p_f(a_j) = \frac{4}{4n - 3} \quad \text{and} \quad p_f(a_{j+1}) = \frac{4}{4m - 3} \quad \text{with } |n - m| = 1.$$

By Lemma 5.8, we have that $|z(\gamma^{(j)}) - z(\gamma^{(j+1)})| = 2$ with the sign of the difference being given by s_j, hence $z(\gamma^{(j)}) = z(\gamma^{(j+1)}) + s_j + s_{j+1}$. If $s_j \neq s_{j+1}$, the same argument yields $p_f(a_j) = p_f(a_{j+1})$ and $z(\gamma^{(j)}) = z(\gamma^{(j+1)}) + s_j + s_{j+1}$, this completes the proof for z_j.

To prove the claims for i_j, $j = 1, \dots, N$, recall from Theorem 5.11 that

$$i(\gamma^{(j)}) = \begin{cases} z(\gamma^{(j)}) - 1, & \text{if } s_j = 1, \\ z(\gamma^{(j)}), & \text{if } s_j = -1. \end{cases}$$

Then the result follows immediately. However, for i_0 we proceed slightly differently and notice that by Lemma 2.9 and Theorem 5.6 we have

$$\begin{aligned} i(\gamma_0) &= z_1 + \frac{s_1 + 1}{2} \\ &= z_1 + s_1 + s_0 - \frac{s_0 + 1}{2} \\ &= z_0 - \frac{s_0 + 1}{2} \\ &= i_0. \end{aligned}$$

□

Combining Lemma 6.10 with Corollary 6.8, we conclude that the phase diagram $\Gamma(f)$ of any Hopf–Smale attractor $\mathcal{A}(f)$ can be decoded from the signature $\chi(f)$ by using the sequences (6.7). Indeed, notice that (6.7) contain all the information necessary to determine if two critical elements are adjacent. Thus we obtain our main result.

Theorem 6.11. *Let $f \in \mathfrak{X}^-$ be such that $\mathcal{A}(f)$ is Hopf–Smale and define the sequences $(z_0, \ldots, z_N)$ and $(i_0, \ldots, i_N)$ as in Lemma 6.10. Then there exists a connection $\gamma_k \succ \gamma_l$ if and only if*

$$i_k > i_l, \quad \textit{and} \quad z_j \neq z_{\max\{k,l\}} \quad \textit{for all } \min\{k,l\} < j < \max\{k,l\}. \tag{6.8}$$

In particular, the condition (6.8) *determines the phase diagram* $\Gamma(f)$ *completely.*

Proof. By Corollary 6.8, $\mathcal{A}(f)$ satisfies liberalism. In particular, by Lemma 6.10, the condition (6.8) is equivalent to preventing blocking in the sense of Lemma 6.5, which is the same as guaranteeing liberalism. □

Furthermore, $\chi(f)$ can be recovered if we know $\Gamma(f)$ and the labels of the vertices.

Lemma 6.12. *In the setting of Theorem 6.11, the signature $\chi(f)$ is the unique sequence $(s_0, \ldots, s_N)$ satisfying $s_0 = s_N = 1$ and*

$$s_{j+1} = \begin{cases} 1, & \textit{if } \gamma_j \succ \gamma_{j+1}, \\ -1, & \textit{if } \gamma_{j+1} \succ \gamma_j, \end{cases} \qquad j = 0, \ldots, N-1. \tag{6.9}$$

Proof. Theorem 6.11 shows that $\gamma_j \sim \gamma_{j+1}$ for $j = 0, \ldots, N-1$. Therefore, the direction of the connection indicates the sign of $i(\gamma_{j+1}) - i(\gamma_j)$. Furthermore, by (6.7), we have that

$$i(\gamma_{j+1}) - i(\gamma_j) = \frac{-3s_{j+1} - s_j}{2};$$

hence $\operatorname{sign}(i(\gamma_{j+1}) - i(\gamma_j)) = -s_{j+1}$, from which (6.9) follows. □

In particular, we conclude that the method in Theorem 6.11 produces unique phase diagrams.

Proposition 6.13. *In the setting of Theorem 6.11, the signature* $\chi(f)$ *determines a unique phase diagram* $\Gamma(f)$ *up to a graph isomorphism. That is, if there exists a Hopf–Smale nonlinearity* $\tilde{f} \in \mathfrak{X}^-$ *such that* $\Gamma(f) \cong \Gamma(\tilde{f})$*, then* $\chi(f) = \chi(\tilde{f})$*.*

Proof. By contradiction, let $\Gamma(f) \cong \Gamma(\tilde{f})$, that is, for $\mathrm{Crit}(\tilde{f}) := \{\tilde{\gamma}_0, \dots, \tilde{\gamma}_N\}$ there exists a bijection $\Xi : \mathrm{Crit}(f) \to \mathrm{Crit}(\tilde{f})$ mapping edges of $\Gamma(f)$ to edges of $\Gamma(\tilde{f})$ and suppose that $\chi(f) \neq \chi(\tilde{f})$.

By Theorem 6.11, the relabeling Ξ maps nearest neighbors $\gamma_k, \gamma_{k+1} \in \mathrm{Crit}(f)$ to nearest neighbors $\gamma_l, \gamma_{l+1} \in \mathrm{Crit}(\tilde{f})$. Therefore, the only two possibilities are either $\Xi(\gamma_j) = \tilde{\gamma}_j$, or $\Xi(\gamma_j) = \tilde{\gamma}_{N-j}$ for $j = 0, \dots, N$. Lemma 6.12 shows that $\chi(f)$ can be recovered from the connections between neighboring equilibria, yielding $\chi(f) = \chi(\tilde{f})$ and completing the proof if the isomorphism of graphs is of type $\Xi(\gamma_j) = \tilde{\gamma}_j$.

Otherwise, the isomorphism is of type $\Xi(\gamma_j) = \tilde{\gamma}_{N-j}$ and it reverses the partial order defined by $\succ$ on $\Gamma(f)$. Lemma 6.12 together with Lemma 6.9 guarantee that the coefficients $(\tilde{s}_0, \dots, \tilde{s}_N) := \chi(\tilde{f})$ satisfy $\tilde{s}_0 = \tilde{s}_N = 1$ and $\tilde{s}_j = -s_{j-N}$ for $j = 1, \dots, N-1$. Moreover, since $\gamma_{N-1} \succ \gamma_N$ and $\tilde{\gamma}_{N-1} \succ \tilde{\gamma}_N$ always holds by Lemma 6.9, a necessary condition for $\Gamma(f) \cong \Gamma(\tilde{f})$ is that $s_1 = -1$. Under these conditions, (6.7) applied to $\chi(\tilde{f})$ yield the sequence

$$\tilde{z}_j = z_{N-j} - z_0 + 1. \tag{6.10}$$

We claim that if there exists $k \geq 1$ such that $s_k s_{k+1} = -1$, then $z(\gamma_k) = z(\gamma_{k+1}) = 1$. By contradiction suppose that $z(\gamma_k) > 1$, then Lemma 6.10 yields $z(\gamma_k) = z(\gamma_{k+1}) > 1$. In particular, since Lemma 6.9 shows $z(\gamma_N) = 1$ and the zero number leaps at most by two between neighboring periodic orbits, there exists a γ_l with $k + 1 < l \leq N$ such that $z(\gamma_l) < z(\gamma_{k+1})$ and γ_l is adjacent to γ_k. By Theorem 6.11, we conclude that $\gamma_k \succ \gamma_l$ and $\gamma_{k+1} \succ \gamma_l$. However, after reversing the indices via $\Xi(\gamma_j) = \tilde{\gamma}_{N-j}$, we obtain from (6.10) that the relative zero numbers of the critical elements have only been shifted by a constant. As a result, $z(\tilde{\gamma}_{N-k-1}) = z(\tilde{\gamma}_{N-k}) > z(\tilde{\gamma}_{N-l})$, but Lemma 6.5 shows that $\tilde{\gamma}_{N-k-1}$ blocks the connections between $\tilde{\gamma}_{N-k}$ and $\tilde{\gamma}_{N-l}$ contradicting that $\Gamma(f) \cong \Gamma(\tilde{f})$. With this, we just showed that $\max_j z_j = 1$ or, equivalently, that $s_j s_{j+1} = -1$ for $j = 1, \dots, N-1$ and $s_1 = -1$. Therefore, $\Gamma(f) \cong \Gamma(\tilde{f})$ via a graph isomorphism reversing the ordering of the labeling if and only if $\chi(f) = (1, -1, 1, \dots, 1, -1, 1)$, which yields $\chi(\tilde{f}) = \chi(f)$ and completes the proof. ∎

6.6 Example: The Chafee–Infante class

As an example, let us apply Theorem 6.11 to **soft-spring nonlinearities**, i.e., those $f(\xi,\eta) = g(\eta) \in \mathfrak{X}^-$ that satisfy the sublinear growth condition

$$\partial_\eta \left(\frac{g(\eta)}{\eta} \right) < 0 \quad \text{for all } \eta < 0.$$

Proposition 6.14. *Let $f \in \mathfrak{X}^-$ be a soft-spring nonlinearity as above, and assume that $\mathcal{A}(f)$ is Morse–Smale with N periodic orbits $\gamma_1, \dots, \gamma_N$ ordered by amplitudes. Then the signature $\chi(f) = (s_0, \dots, s_N)$ satisfies $s_j = 1$ for $j = 1, \dots, N$, $\mathcal{A}(f)$ is Hopf–Smale, and $\gamma_k \succ \gamma_l$ if and only if $k < l$.*

Proof. Indeed, [Nus79, Theorem 1.3] shows that the period map p_f is strictly increasing. Hence, the signature $\chi(f)$ consists of positive entries, only. Furthermore, the parametric family λf for $\lambda > 0$ has period map $p_{\lambda f}(a) = p_f(a)/\lambda$. Thus, taking $\lambda_0 > 0$ sufficiently small Theorem 5.6 ensures that $\mathcal{A}(\lambda_0 f) = \gamma_0$. Furthermore, as λ increases to 1, the period map possesses no critical points for $a > 0$. Therefore, $\mathcal{A}(\lambda f)$ undergoes solely isolated supercritical Hopf bifurcations for $\lambda \in (\lambda_0, 1)$. Using Lemma 6.10, we obtain the sequences

$$z_j = 1 + 2(N - j), \quad \text{and} \quad i_j = 2(N - j),$$

for $j = 0, \dots, N$. Thus, by Theorem 6.11, we obtain $\gamma_k \succ \gamma_l$ if and only if $k < l$, as claimed. □

We call any Hopf–Smale attractors $\mathcal{A}(f)$ with the signature of Proposition 6.14 **Chafee–Infante** because the formation process via a sequence of supercritical Hopf bifurcations resembles that of Chafee–Infante attractors in the reaction-diffusion PDE (1.6); see [CI74, Hen81]. By Proposition 6.14, the only possible phase diagrams for soft-spring nonlinearities are Chafee–Infante. However, since the set of soft-spring nonlinearities is convex, we have the following stronger result.

Theorem 6.15. *Let $f, \tilde{f} \in \mathfrak{X}^-$ be soft-spring nonlinearities and assume that there exists $n \in \mathbb{N}$ such that*

$$\partial_2 f(0,0), \partial_2 \tilde{f}(0,0) \in \left(-\frac{(4n+1)\pi}{2}, -\frac{(4n-3)\pi}{2} \right). \tag{6.11}$$

Then $\mathcal{A}(f)$ and $\mathcal{A}(\tilde{f})$ are More–Smale and orbit equivalent.

Proof. Since the set of soft-spring nonlinearities is convex, the standard homotopy $f_\lambda := (1-\lambda)f + \lambda\tilde{f}$ is soft-spring for all $\lambda \in [0,1]$. Thus, by [Nus79, Theorem 1.3], the period map p_{f_λ} is strictly increasing with a minimum at 0. Moreover, by (6.11), we have that $\partial_2 f_\lambda(0,0) \in \left(\frac{(4n+1)\pi}{2}, \frac{(4n-3)\pi}{2}\right)$ and Lemma 2.9 shows that there are no Hopf points along the homotopy. Thus the critical elements $\gamma_j(\lambda)$ remain hyperbolic through the whole process and $\mathcal{A}(f) \cong \mathcal{A}(\tilde{f})$ by Theorem 4.13. □

In particular, this shows that the Chafee–Infante class is the only possibility for soft-spring nonlinearities up to orbit equivalence.

6.7 Realizability: Enharmonic oscillators

We conclude the chapter by discussing the realizability of the Hopf–Smale dynamics discussed in Section 6.2. Lemma 6.10 and Theorem 6.11 prompt the question of finding all the $\{-1,1\}$-sequences $(s_0,\ldots,s_N)$ such that there exists a Hopf–Smale nonlinearity $f \in \mathfrak{X}^-$ whose signature satisfies $\chi(f) = (s_0,\ldots,s_N)$. We call a sequence $(s_0,\ldots,s_N)$ **enharmonic** if $s_N = 1$ and it satisfies $\sum_{j\geq k} s_j \geq 0$ for all $k = 1,\ldots,N$. We will show that all enharmonic sequences are indeed realizable. To this end, we recover the enharmonic oscillators introduced in Chapter 1, i.e., the DDEs

$$\dot{x}(t) = -\frac{\pi}{2}\Omega\left(\sqrt{(x(t))^2 + (x(t-1))^2}\right)x(t-1), \tag{6.12}$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a C^2-**frequency function** such that

$$f_\Omega(\xi,\eta) := -\frac{\pi}{2}\Omega\left(\sqrt{\xi^2+\eta^2}\right)\eta \in \mathfrak{X}^-.$$

Notice that f_Ω is even-odd in the sense of (2.2), by construction. Furthermore, by direct differentiation, the negative delayed feedback assumption (2.1) is equivalent to the lower growth bound

$$\frac{\Omega'(a)}{\Omega(a)} > -\frac{1}{a} \quad \text{for all } a > 0. \tag{6.13}$$

By assumption, the left-hand side of (6.13) is negative. Furthermore, (6.13) singularizes at amplitude $a = 0$, making the lower growth bound essentially void sufficiently close to the origin. More generally, integrating both sides of (6.13) shows that for all $a^* > 0$ there exists a constant $C > 0$ such that

$$\Omega(a) > \frac{C}{a}, \quad \text{for all } a > a^*.$$

Hence (6.13) allows Ω to decay, but only at a sublinear rate.

The main advantage of considering the enharmonic (6.12) is that the period map p_{f_Ω} is explicitly known. More precisely, notice that the associated ODEs from Lemma 5.1 are

$$\begin{aligned} \dot{\xi} &= -\frac{\pi}{2}\Omega\left(\sqrt{\xi^2+\eta^2}\right)\eta, \\ \dot{\eta} &= \frac{\pi}{2}\Omega\left(\sqrt{\xi^2+\eta^2}\right)\xi. \end{aligned} \tag{6.14}$$

The quantity $\xi^2 + \eta^2$ is a first integral of motion of (6.14). Therefore, Ω acts as a constant time rescaling on the orbits of a standard harmonic oscillator (1.9), and all the solutions of (6.12) are harmonic, i.e., they take the form

$$(\xi(a,t), \eta(a,t)) = a\left(\cos\left(\frac{\pi}{2}\Omega(a)t + t^*\right), \sin\left(\frac{\pi}{2}\Omega(a)t + t^*\right)\right),$$

for values $a, t^* \geq 0$. As a result, the period map is given by

$$p_{f_\Omega}(a) = \frac{4}{\Omega(a)},$$

unlike in Theorem 5.6, the special form of the enharmonic oscillator (6.12) allows us to remove the nondegeneracy assumption on the period map.

Proposition 6.16. *A differentiable curve $x^*(t)$ is a nontrivial periodic solution of the enharmonic oscillator* (6.12) *if and only if $x^*(t)$ is of the form*

$$x^*(t) = a\cos\left(\frac{\pi}{2}\Omega(a)t + t^*\right) \quad \textit{for some } a > 0 \textit{ and } t^* \in \mathbb{R}. \tag{6.15}$$

Here $\Omega(a) = (4n-3)$ for some $n \in \mathbb{N}$. Moreover, the associated orbit γ^* is hyperbolic if and only if $\Omega'(a) \neq 0$.

Proof. At first sight, the proposition seems to be a combination of Theorem 5.6, Lemma 5.8, and Theorem 5.11. Such is the case if p_{f_Ω} is locally nonconstant. However, we do not require this in general. We proceed by contradiction and assume that $x^*(t)$ is not of the form (6.15). Thus Proposition 5.5 shows that the range of $(x^*(t))^2 + (x^*(t-1))^2$ is contained in a region where the period map p_{f_Ω} experiences a plateau $p_{f_\Omega} \equiv 2\pi/B$. Hence $x^*(t)$ solves the linear autonomous DDE

$$\dot{x}(t) = -Bx(t-1). \tag{6.16}$$

Recalling the detailed description of the spectrum of the solution operator of (6.16) given in Lemma 2.9, we conclude that $x^*(t)$ is harmonic, independently of the nondegeneracy of p_{f_Ω}. Finally, we apply Lemma 5.8 and Theorem 5.11 to complete the proof. □

Combining Proposition 6.16 with Theorem 5.11 we obtain a complete characterization of the Morse–Smale enharmonic oscillators as follows.

Corollary 6.17. *The enharmonic oscillator* (6.12) *is Morse–Smale if and only if there exist finitely many $a_j > 0$, $j = 1, \dots, N$ such that $\Omega(a_j) = 4n-3$ for some $n \in \mathbb{N}$, $\Omega(0) \neq 4n-3$ for all $n \in \mathbb{N}$, and $\Omega'(a_j) \neq 0$ for all $j = 1, \dots, N$.*

Proof. Recalling from Lemma 4.8 that Morse–Smale attractors possess only finitely many critical elements, the result follows immediately from applying Theorem 4.15 to Proposition 6.16. □

However, Corollary 6.17 does not guarantee that the enharmonic nonlinearity f_Ω can be obtained by a sequence of isolated Hopf points.

Proposition 6.18. *Consider $f_\Omega \in \mathfrak{X}^-$. If $\mathcal{A}(f_\Omega)$ is Morse–Smale, then it is also Hopf–Smale.*

Proof. Let $\Omega_0 := \Omega$ denote the frequency of f_Ω. We show that we can remove the periodic orbit $\gamma_1 \in \mathrm{Crit}(f_\Omega)$ with the smallest amplitude $a_1 := a(\gamma_1)$ via an isolated Hopf point produced through a standard homotopy of frequencies

$$\Omega_\lambda := (1-\lambda)\Omega_0 + \lambda\Omega_1. \tag{6.17}$$

Then, an induction argument completes the proof.

Indeed, notice that the lower growth bound (6.13) is preserved along the homotopy (6.17) provided that the ends Ω_0 and Ω_1 satisfy it. In the following, we describe the construction of the frequency Ω_1.

For the sake of simplicity, let us assume that there exists a small $\varepsilon > 0$ such that $\Omega_0'(a) < 0$ for all $a \in (0, a_1 + \varepsilon)$ and $\Omega_0(a) > \Omega_0(a_1)$ for all $a \in [0, a_1)$. Later we shall prove that this can be assumed without loss of generality provided that $\Omega_0(0) > \Omega_0(a_1)$. Next, we construct a new enharmonic C^2-frequency function Ω_1 such that $\Omega_1(a) = \Omega(a)$ for all $a \geq a_1 + \varepsilon$. Close to the origin, we choose Ω_1 so that $\Omega_1'(a) < 0$ for all $a \in (0, a_1 + \varepsilon)$ and $\Omega_1(a) < \Omega_0(a_1)$ for all $a \in [0, a_1 + \varepsilon)$. Notice that, since Ω_0 was assumed to be decreasing in $(0, a_1 + \varepsilon)$, we have that $\Omega_0(a_1 + \varepsilon) < \Omega_0(a_1)$ for $\varepsilon > 0$ small enough. Thus to check if such an Ω_1 is constructible, we only have to check the inequality (6.13). For $a \geq a_1 + \varepsilon$, this is immediate because Ω_0 satisfies (6.13) by assumption. In case $a \in [0, a_1 + \varepsilon]$, the question is equivalent to finding a smooth decreasing function Ω_1 whose range satisfies $\Omega_1(a) \in [\Omega_0(a_1), \Omega_0(a_1 + \varepsilon)]$ for all $a \in [0, a_1 + \varepsilon)$. However, (6.13) is a condition on the derivative $\Omega_1'(a)$, which can be taken arbitrarily close to zero, but still negative, for $a \in [0, a_1 + \varepsilon)$. Hence if ε above is chosen so small that $a_1 + \varepsilon < a(\gamma_2)$ where γ_2 is the second periodic orbit of $\mathcal{A}(f_{\Omega_0})$, we conclude from Corollary 6.17 that $\mathcal{A}(f_{\Omega_1})$ possesses exactly one fewer periodic orbit. Furthermore, the homotoped frequency obtained via (6.17) satisfies $\Omega_\lambda(a) = \Omega_0(a)$ for all $a \geq a_1 + \varepsilon$ and $\Omega_\lambda'(a) < 0$ for all $a \in (0, a_1 + \varepsilon)$. This ensures that the only nonhyperbolic critical element along the homotopy is γ_0 when a subcritical Hopf bifurcation occurs and removes the periodic orbit γ_1.

A completely analogous treatment can be given if $\Omega_0(0) < \Omega_0(a_1)$. In this case, we can assume without loss of generality that there exists a small $\varepsilon > 0$ such that $\Omega_0'(a) > 0$ for all $a \in (0, a_1 + \varepsilon)$ and $\Omega_0(a) < \Omega_0(a_1)$ for all $a \in [0, a_1)$. Then we construct a new frequency Ω_1 such that $\Omega_1(a) = \Omega_0(a)$ for all $a \geq a_1 + \varepsilon$, $\Omega_1'(a) > 0$ for all $a \in (0, a_1 + \varepsilon)$, and $\Omega_1(a) > \Omega_0(a_1)$ for all $a \in [0, a_1 + \varepsilon)$. The construction is, however, simpler because (6.13) is not a constraint when Ω_1' is positive. The standard homotopy (6.17) is analogous to the case above, with the difference that the Hopf bifurcation taking place is supercritical.

We complete the proof by showing that if $\Omega_0(0) > \Omega_0(a_1)$, we can always assume the existence of $\varepsilon > 0$ such that $\Omega_0'(a) < 0$ for all $a \in (0, a_1 + \varepsilon)$ and

$\Omega_0(a) > \Omega_0(a_1)$ for all $a \in [0, a_1)$. This is done via a preparation homotopy

$$\widehat{\Omega}_\lambda := (1-\lambda)\Omega_0 + \lambda\widehat{\Omega}_1 \tag{6.18}$$

to a new C^2-frequency $\widehat{\Omega}_1$ that satisfies the claims. Furthermore, we choose $\widehat{\Omega}_1$ so that no bifurcations occur along (6.18). Indeed, $\Omega_0(a) > \Omega_0(a_1)$ for all $a \in [0, a_1)$ follows immediately by Corollary 6.17 from the fact that γ_1 is the smallest amplitude periodic orbit of $\mathcal{A}(f_{\Omega_0})$. Furthermore, since $\mathcal{A}(f_{\Omega_0})$ is Morse–Smale and $\Omega_0(0) > \Omega_0(a_1)$, Corollary 6.17 also ensures that $\Omega_0'(a_1) < 0$. Hence the existence of $\varepsilon > 0$ such that $\Omega_0'(a) < 0$ for all $a \in (a_1-\varepsilon, a_1+\varepsilon)$. We shall choose the new frequency $\widehat{\Omega}_1$ in such a way that $\widehat{\Omega}_1(a) = \Omega_0(a)$ for all $a \geq a_1 - \varepsilon$, $\widehat{\Omega}_0'(a) < 0$ for all $a \in (0, a_1-\varepsilon]$, and so that the graph satisfies $\widehat{\Omega}_1(a) \in (\Omega_0(a_1-\varepsilon), \Omega_0(0))$. For $\widehat{\Omega}_1$ to be a frequency, it must satisfy the growth inequality (6.13). However, we can force $\widehat{\Omega}_1$ to become arbitrarily flat near the origin by choosing the initial value $\widehat{\Omega}_0(0)$ arbitrarily close to $\widehat{\Omega}_0(a_1-\varepsilon)$. For $a \geq a_1 - \varepsilon$ (6.13) is automatically enforced since Ω_0 satisfies it. Finally, if $\varepsilon > 0$ is chosen so small that the global attractor of $\widehat{\Omega}_1$ possesses the same number of periodic orbits as $\mathcal{A}(f_{\Omega_0})$, then the homotopy (6.18) leaves the attractor unchanged. Indeed, the attractor for $\widehat{\Omega}_1$ is Morse–Smale by construction and Corollary (6.17). Moreover, the homotoped frequency (6.18) satisfies $\widehat{\Omega}_\lambda = \Omega_0(a)$ for all $a \geq a_1 - \varepsilon$, $\widehat{\Omega}_\lambda(a) \in (\Omega_0(a_1), \Omega_0(0))$ for all $a \in [0, a_1)$. Thus the attractor associated to $\widehat{\Omega}_\lambda$ stays Morse–Smale along the homotopy and we may assume without loss of generality that Ω_0 has the claimed properties.

The case $\Omega_0(0) < \Omega_0(a_1)$ admits the construction of a completely analogous preparation homotopy, which completes the proof. □

The following is the section's main result: all enharmonic sequences produce realizable phase diagrams in the class of Hopf–Smale attractors.

Theorem 6.19. *Let $(s_0, \ldots, s_N)$ be an enharmonic sequence. Then there exists a Hopf–Smale nonlinearity $f \in \mathfrak{X}^-$ such that $\chi(f) = (s_0, \ldots, s_N)$.*

Proof. Indeed, the lower growth condition (6.13) is essentially void for small amplitudes $a > 0$. Thus ensuring that we can construct a frequency function Ω satisfying the assumptions of Corollary 6.17 and such that $(s_0, s_1, \ldots, s_N) = \chi(f_\Omega)$. Moreover, by Proposition 6.18, any such f_Ω is Hopf–Smale, which completes the proof. □

Moreover, within the class of enharmonic oscillators, we can improve on Proposition 6.13 by showing that same signatures imply orbit equivalence in the sense of Theorem 4.1.

Proposition 6.20. *Consider two different Morse–Smale enharmonic oscillators* (6.12) *with* $f, \tilde{f} \in \mathfrak{X}^-$. *Then the signatures satisfy* $\chi(f) = \chi(\tilde{f})$ *if and only if* $\mathcal{A}(f)$ *and* $\mathcal{A}(\tilde{f})$ *are orbit equivalent.*

Proof. By Theorem 4.1, we have to show that there exists a smooth homotopy of Morse–Smale enharmonic oscillators $f_\lambda \in \mathfrak{X}^-$ such that $f = f_0$ and $\tilde{f} = f_1$. First, let $a_1 < \cdots < a_N$ be the sequence of amplitudes of the periodic orbits in $\mathrm{Crit}(f)$. By Corollary 6.17 and the lower bound (6.13), given any $a > a_1$, we are always allowed to smoothly deform the frequency Ω of f to a new frequency $\widehat{\Omega}$ while preserving the Morse–Smale property along the homotopy and so that the new amplitudes of the periodic solutions satisfy $\hat{a}_k = a$, $\hat{a}_j = a_j$ for $j < k$, and $\hat{a}_j \geq a_j$ for $j > k$. Moreover, by Theorem (4.1), the attractors $\mathcal{A}(f)$ and $\mathcal{A}(\hat{f})$ are orbit equivalent in the sense above.

Suppose now that the smallest amplitude $\tilde{a}_1$ in $\mathrm{Crit}(\tilde{f})$ satisfies $\tilde{a}_1 > a_1$, we can use the argument above to produce a new Morse–Smale enharmonic oscillator such that the amplitudes $\hat{a}_1$ and $\tilde{a}_1$ line up. Iterating this process, we may assume without loss of generality that the periodic orbits of f and $\tilde{f}$ appear at the same amplitudes $a_j = \tilde{a}_j$ for $j = 1, \ldots, N$. Hence we can use the standard homotopy $f_\lambda := (1-\lambda)f + \lambda\tilde{f}$ which, by Corollary 6.17, is guaranteed to produce a family of Morse–Smale enharmonic oscillators. As a result, the signature determines the orbital equivalence class.

To see that the equivalence class determines the signature, recall that the orbit homeomorphism $\Xi : \mathcal{A}(f) \to \mathcal{A}(\tilde{f})$ induces a graph isomorphism $\Gamma(f) \cong \Gamma(\tilde{f})$ via its action on the vertices. Thus, we apply Proposition 6.13 to conclude that $\chi(f) = \chi(\tilde{f})$ for any two orbit equivalent Hopf–Smale attractors $\mathcal{A}(f)$ and $\mathcal{A}(\tilde{f})$. □

In particular, the enharmonic sequences enumerate the orbit equivalence classes of the enharmonic oscillators. The precise number is as follows.

Corollary 6.21. *Up to orbit equivalence, there are exactly*

$$\binom{N}{\lfloor N/2 \rfloor} \tag{6.19}$$

different Morse–Smale enharmonic oscillators with N periodic orbits.

Proof. By Proposition 6.20 and Corollary 6.19, we have to count the number of enharmonic sequences of length $N + 1$. Given that the signature (6.6) encodes the intersections of a continuous curve with certain levels, the problem admits a restatement as an enumeration of lattice walks. In this case, a closed formula is known and [BKK$^+$19, Theorem 4.10] shows that the number of Morse–Smale enharmonic oscillators with N periodic orbits is given by (6.19). □

Chapter 7

Conclusion and examples

This chapter concludes the lengthy method for designing global attractors devised over Chapters 4 through 6. The main conclusion is a rule to recover the phase diagram $\Gamma(f)$ of a Hopf–Smale attractor $\mathcal{A}(f)$ from the signature $\chi(f)$, a binary sequence that we introduced in Section 6.5. Two big drawbacks of our analysis are that it is difficult to determine whether $\mathcal{A}(f)$ is Hopf–Smale and that the signature $\chi(f)$ is in general not known. In Proposition 6.14 we solved the problem for **soft-spring nonlinearities** $f(\xi,\eta) = g(\eta) \in \mathfrak{X}^-$ such that

$$\partial_\eta \left(\frac{g(\eta)}{\eta} \right) < 0 \quad \text{for all } \eta < 0.$$

Moreover, in Theorem 6.15 we showed that if two Morse–Smale attractors $\mathcal{A}(f)$ and $\mathcal{A}(\tilde{f})$ with soft-spring nonlinearities $f, \tilde{f} \in \mathfrak{X}^-$ possess the same number of periodic orbits, then the global attractors $\mathcal{A}(f)$ and $\mathcal{A}(\tilde{f})$ are orbit equivalent in the sense of Theorem 4.1. In other words, there exists a homeomorphism of attractors mapping orbits to orbits and preserving the time direction. All the soft-spring attractors arise via sequences of supercritical Hopf bifurcations. Thus they resemble the sequence of supercritical **pitchfork bifurcations** producing the Chafee–Infante attractors in reaction-diffusion PDE with Neumann boundary conditions; see [CI74, Hen81].

The problem of characterizing the Hopf–Smale class within $\mathfrak{X}^-$ remains open. Hence we have focused on solving an inverse problem. **This thesis characterizes the phase diagrams $\Gamma(f)$ of the Hopf–Smale attractors $\mathcal{A}(f)$ up to a graph isomorphism.**

Our method is constructive. We summarize it as follows:

Step 1: Prescribe an **enharmonic sequence**, that is, a finite sequence $(s_0, s_1, \dots, s_N)$ such that $s_j \in \{-1, 1\}$, $s_0 = s_N = 1$, and

$$\sum_{j \geq k} s_j \geq 0 \quad \text{for all } j = 1, \dots, N.$$

Step 2: Use Lemma 6.10 to derive the *relative zero numbers* $(z_0, \dots, z_N)$ and the *Morse indices* $(i_0, \dots, i_N)$ via

$$\begin{aligned} z_j &:= z_{j+1} + s_{j+1} + s_j, \quad j = 0, \dots, N-1, \quad z_N := 1, \\ i_j &:= z_j - \frac{s_j + 1}{2}, \qquad\quad j = 0, \dots, N. \end{aligned}$$

Step 3: By Corollary 6.19 and Theorem 6.11, there exists a Hopf–Smale attractor $\mathcal{A}(f)$ whose signature satisfies $\chi(f) = (s_0, \dots, s_N)$. Moreover, the phase diagram $\Gamma(f)$ has the vertex set $\{\gamma_0, \dots, \gamma_N\}$ and contains the edge (γ_k, γ_l) if and only if $i_k > i_l$ and

$$z_j \neq z_{\max\{k,l\}}, \quad \text{for all } \min\{k, l\} < j < \max\{k, l\}.$$

Restricted to enharmonic oscillators (1.8), Proposition 6.20, shows that each enharmonic sequence corresponds to a Morse–Smale enharmonic oscillator up to topological orbit equivalence in the sense of Theorem 4.1. Moreover, by Corollary 6.21, the number of orbit equivalence classes of enharmonic oscillators grows like

$$\binom{N}{\lfloor N/2 \rfloor} \sim \sqrt{\frac{2}{\pi}} \frac{2^N}{\sqrt{N}}.$$

7.1 Examples

Next, we show some new examples from applying the algorithm in Steps 1 to 3 above to the first twenty-three enharmonic sequences. A general feature of all phase diagrams is that they possess a single vertex γ_0 representing the equilibrium at the origin surrounded by N periodic orbits, the outermost one of which, denoted γ_N, is always stable as seen in Lemma 6.9. Tables

7.1 and 7.2 are the result of applying the connectivity rules above to all the enharmonic sequences with up to six entries. In particular, they correspond to the orbit equivalence classes of all possible enharmonic oscillators (1.8) possessing up to five periodic orbits. We point out that, for the sake of simplicity and thanks to the transitivity in Lemma 6.2, Tables 7.1 and 7.2 only represent the transitive reduction of the phase diagram $\Gamma(f)$. Two patterns stand out in Tables 7.1 and 7.2.

First, the attractors having signature $s_j = (-1)^N$ for $j = 1, \ldots, N-1$ are the Hopf–Smale **planar attractors** described by Walther [Wal95]. They appear by an alternating sequence of super- and subcritical Hopf bifurcations in the sense of Proposition 6.1. By Lemma 6.10, they are two-dimensional and the methods of [MPS96a] show that the planar projection

$$P\mathcal{A}(f) := \{(\phi(0), \phi(-1)) : \phi \in \mathcal{A}(f)\} \subset \mathbb{R}^2$$

is homeomorphic to $\mathcal{A}(f)$. Moreover, all neighboring periodic orbits are connected, and the unstable periodic orbits act as separatrices between the basins of attraction of the stable critical elements. The simplest possible example corresponds to the signature $(s_0, s_1) = (1, 1)$ in Table 7.1. However, our method spans an infinite family; the enharmonic sequences $(1, -1, 1)$, $(1, 1, -1, 1)$, $(1, -1, 1, -1, 1)$, and $(1, 1, -1, 1, -1, 1)$ in Tables 7.1 and 7.2 yield planar attractors.

The attractors associated with the enharmonic sequence $s_j = 1$ for $j = 0, \ldots, N$ are the Chafee–Infante attractors above. They arise via a sequence of exclusively supercritical Hopf bifurcations that destabilize the origin, producing N periodic orbits so that $\dim W^{\mathrm{u}}(\gamma_0) = 2N$.

7.2 Discussion: Positive delayed feedback

We highlight that the examples presented here can be extended to delayed positive feedback systems. However, we must first apply a perturbation argument. Notice that, if we define the set of nonlinearities with symmetric positive feedback to be $\mathfrak{X}^+ := -\mathfrak{X}^-$, then the dissipativity assumptions required for the existence of a global attractor $\mathcal{A}(f)$ are immediately violated. Indeed, consider the equation

$$\dot{x}(t) = \lambda \arctan(x(t-1)), \quad \lambda > 0, \tag{7.1}$$

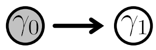
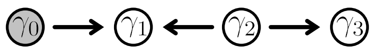
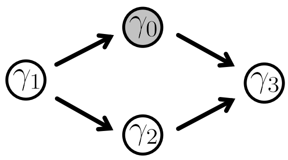
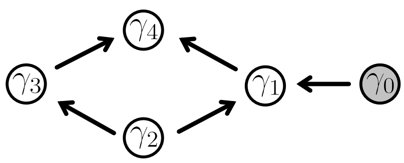
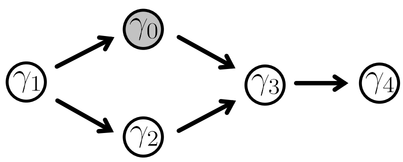
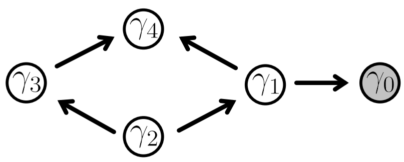
$\Gamma(f)$	$\chi(f)$
	(1)
	$(1, 1)$
	$(1, 1, 1)$
	$(1, -1, 1)$
	$(1, 1, 1, 1)$
	$(1, 1, -1, 1)$
	$(1, -1, 1, 1)$
	$(1, 1, 1, 1, 1)$
	$(1, 1, 1, -1, 1)$
	$(1, -1, 1, -1, 1)$
	$(1, 1, -1, 1, 1)$
	$(1, -1, 1, 1, 1)$
	$(1, -1, -1, 1, 1)$

Table 7.1: Hopf–Smale phase diagrams containing up to four periodic orbits, all edges are transitive.

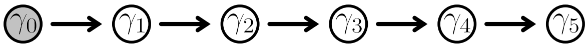
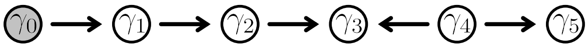
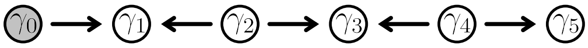
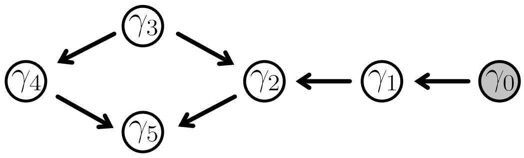
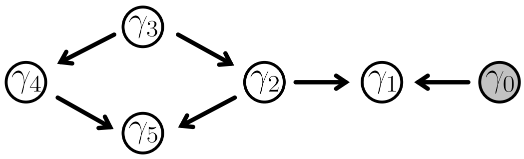
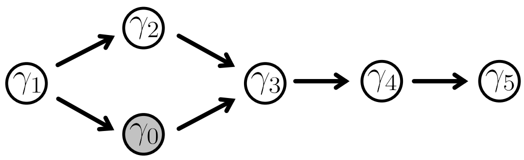
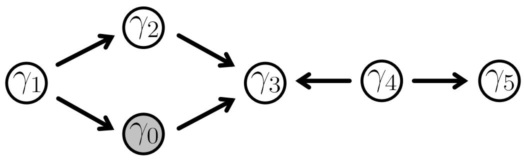
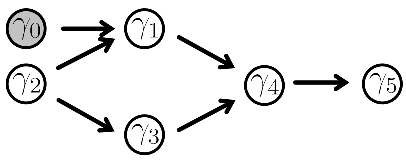
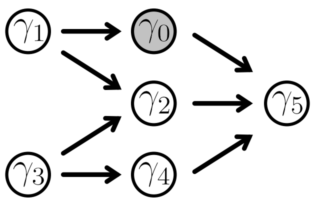
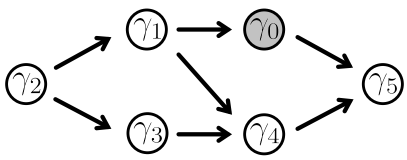
$\Gamma(f)$	$\chi(f)$
	$(1, 1, 1, 1, 1, 1)$
	$(1, 1, 1, 1, -1, 1)$
	$(1, 1, -1, 1, -1, 1)$
	$(1, 1, 1, -1, 1, 1)$
	$(1, 1, -1, -1, 1, 1)$
	$(1, -1, 1, 1, 1, 1)$
	$(1, -1, 1, 1, -1, 1)$
	$(1, 1, -1, 1, 1, 1)$
	$(1, -1, 1, -1, 1, 1)$
	$(1, -1, -1, 1, 1, 1)$

Table 7.2: Hopf–Smale phase diagrams containing exactly five periodic orbits, all edges are transitive.

with an initial condition $\phi \in H$ such that $\phi(\theta) \neq 0$ for all $\theta \in [-1, 0]$. It is known from [KWW99a] that such solutions drift to plus or minus infinity, and there exists a codimension-one manifold acting as a separatrix or equator of the dynamics. We claim that the methods used in this thesis can be used in the description of the set of uniformly bounded solutions

$$\mathcal{B}(f) := \left\{ \phi \in H : \sup_{t \in \mathbb{R}} \|S_f(t)\phi\| < \infty \right\}.$$

More precisely, following [LN17], the transversality properties in Chapter 4 also hold in the positive feedback setting. Moreover, the extensions of Chapter 5 in [LN20] show that the period map of the reference ODE (5.2) in Chapter 5 also characterizes the critical elements in positive delayed feedback systems $f \in \mathfrak{X}^+$. The connection results in Chapter 6 admit an adaption, verbatim, to the positive feedback setting, thus yielding a phase diagram for the set $\mathcal{B}(f)$. In the special case (7.1), $f(\xi, \eta) = \arctan(\eta)$ is a soft-spring nonlinearity. Therefore, adapting in Corollary 6.14 to positive feedback systems yields a connection graph for $\mathcal{B}(f)$ of Chafee–Infante type.

To produce a global attractor, we introduce a small friction coefficient and consider the modified DDE

$$\begin{aligned} \dot{x}(t) &= \tilde{f}(x(t), x(t-1)) \\ &:= -\varepsilon x(t) + \lambda \arctan(x(t-1)), \end{aligned}$$

which is dissipative for all $\varepsilon > 0$; see [KWW99a]. However, this is done at the cost of leaving the symmetry class, and $\tilde{f} \notin \mathfrak{X}^+$. Even worse, $\tilde{f}$ is not a uniformly small perturbation of $f \in \mathfrak{X}^+$ and generates two new equilibria γ_T and γ_B close to infinity. However, the global attractor $\mathcal{A}(\tilde{f})$ consists of the two new equilibria together with connections to an equatorial invariant set $\mathcal{B}(\tilde{f})$ whose dynamics are orbit equivalent to $\mathcal{B}(f)$. In this way, we can produce new attractors for positive delayed feedback systems by designing the equatorial dynamics on $\mathcal{B}(\tilde{f})$ with the Steps 1 to 3 presented above and then adding connections from $\mathcal{B}(\tilde{f})$ to both top and bottom equilibria γ_T and γ_B, respectively.

From this perspective, the spindle in Figure 1.1 (Left) is the unstable suspension of a planar Chafee–Infante attractor in the equator with signature $(1, 1)$. Furthermore, we can construct more complex three-dimensional spindle attractors by prescribing any planar signature for the equatorial dynamics. Nevertheless, our perturbation argument does not reproduce the complexity of the Vas attractor in Figure 1.1 (Right), whose explanation requires a better understanding of the critical elements in systems with delayed monotone feedback.

7.3 Selection of open problems

As is often the case in mathematics, the work presented here poses more new questions than it solves. For this reason, we have included a selection of open problems into which the current work provides some insight. The order of the problems is by what we consider increasing difficulty:

1. Our characterization in the periodic orbits in Theorem 5.6 includes a nondegeneracy condition, requiring that the period map p_f is locally nonconstant. However, such a condition is merely a limitation of the technique used throughout the proof, and it is safe to conjecture that it is possible to remove it. Removing it, however, goes beyond the scope of the current work.
2. Based on the techniques inherited from the study of reaction-diffusion systems, it seems apparent that the main roadblock in developing a similar program for equations with monotone delayed feedback is the absence of a sufficiently precise method to describe the appearance of periodic solutions. Although a first attempt at this problem has been made by Vas [Vas17] by realizing orbit configurations, studying the connecting orbits between them requires a higher degree of precision. This thesis solves the issue by considering the additional symmetry assumption (A3). However, the general case is understood poorly.
3. In the negative feedback regime $f \in \mathfrak{X}^-$, the transversality property $W^{\mathrm{u}}(\gamma^\dagger) \pitchfork W^{\mathrm{s}}(\gamma^*)$ holds for any two hyperbolic $\gamma^\dagger, \gamma^* \in \mathrm{Crit}(f)$, unless both $\gamma^\dagger$ and γ^* are so-called saddle equilibria, i.e., equilibria having two real characteristic multipliers of different signs. The connections between saddle equilibria often consist of homoclinic orbits and heteroclinic loops which usually sign the endpoints of branches of periodic orbits. Thus, determining the possible configurations for saddle-saddle connections is fundamental to understanding the branches of periodic solutions.

4. This work exploits Hopf bifurcations to analyze changes in the phase diagram $\Gamma(f)$. However, a description of the changes in connectivity through saddle-node bifurcations of periodic orbits is unknown. The main obstacle is that the branch of periodic solutions vanishes to one side of the bifurcation diagram, complicating how connections are inherited. To our knowledge, the analogous version of this problem in reaction-diffusion systems remains open.

5. We have discussed in Chapter 1 that DDEs with monotone delayed feedback share multiple traits with reaction-diffusion PDEs and ODEs possessing specific feedback structures. This degree of similarity has been pushed in this work, showing that it is possible to use dynamical systems techniques native to PDEs to describe the global structure of DDEs. Thus, it is tempting to conjecture that a common framework connects all the examples above. Efforts in this direction have already been carried out by considering PDE discretizations [FR00], but the connection to DDEs remains unexplored.

Appendix A

Solution operator in Sobolev space

In Chapter 2, we chose H, i.e., the compact embedding of the standard Sobolev space $H^1([-1,0],\mathbb{R})$ into the space of continuous functions $C := (C^0([-1,0],\mathbb{R}), \|\cdot\|_{C^0})$, as a phase space for the DDE (1.1), thereby deviating from the standard DDE literature; see [DvGVLW95, HVL93]. However, H has the advantage of being a Hilbert space. This appendix shows that the restriction of the standard solution semiflow of (1.1) defined on C to the Sobolev-type space H preserves the regularity properties. We prove our claims for the general setting of a nonautonomous DDE

$$\begin{aligned} \dot{x}(t) &= g(t, x_t), \quad t \geq s \\ x_s(\theta) &= \phi(\theta), \quad \theta \in [-1,0]. \end{aligned} \tag{A.1}$$

Here $g \in C^0(\mathbb{R} \times C, \mathbb{R})$ is bounded and globally Lipschitz, and we used the notation $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1,0]$. The assumption that g is globally Lipschitz is sufficient for this work because, in the main body, we focus on dissipative nonlinearities. Nevertheless, it is possible to produce a local more general theory.

The classical DDE theory, e.g. [HVL93, Chapter 2, Theorem 2.1], ensures the existence of a solution operator

$$\begin{aligned} S(g,t,s,\cdot) : C &\to C \\ x_s &\mapsto x_t \end{aligned}$$

for a small $t \geq s$ due to a fixed point theorem. However, under our Lipschitz continuity assumption, $S(g, t, s, \phi)$ is defined for all $t \geq s$ and is continuous in (g, ϕ); see [HVL93, Chapter 2, Theorem 2.2].

In addition, $S(g, t, s, \phi)$ is a C^k-map in (g, ϕ) for all $k \geq 1$ provided that $g \in C^k(\mathbb{R} \times C, \mathbb{R})$; see [HVL93, Chapter 2, Theorem 4.1], and is compact for all $t \geq 1$. In other words, $S(g, t, s, U)$ is a precompact set for all bounded sets $U \subset C$ and $t \geq s + 1$ [HVL93, Chapter 3, Corollary 6.2].

Next, we consider the restricted map $\bar{S}(g, t, s, \cdot) := S(g, t, s, \cdot)|_H$. Notice that, since the solution $x(t)$ of the DDE (A.1) is continuously differentiable for all $t > s$, we have that $\bar{S}(g, t, s, \cdot)$ maps H-functions back into H. Our concern is whether the restricted solution map $\bar{S}$ inherits the regularity properties of the original map S defined on all C. We claim that this is the case based on two observations:

(i) The Sobolev H-norm defines a stronger topology on H than the supremum norm does, i.e., there exists a constant $K > 0$ such that

$$\|\phi\|_{C^0} \leq K\|\phi\|_H \quad \text{for all } \phi \in H. \tag{A.2}$$

(ii) Given two solutions $x_t := S(g, t, s, x_s)$ and $x_t^* := S(g^*, t, s, x_s^*)$, the derivatives satisfy

$$|\dot{x}(t) - \dot{x}^*(t)| = |g(t, x_t^*) - g^*(t, x_t^*)| \quad \text{for all } t > s. \tag{A.3}$$

Proposition A.1. *The restricted solution operator $\bar{S}(g, t, s, \phi)$, $\phi \in H$ defined above is continuous in (g, s, ϕ).*

Proof. Indeed, by (A.2), we have that $\bar{S}(g, t, s, \phi)$ is continuous in H equipped with the C^0-norm. Thus we only need to prove that the weak derivative $\bar{S}(g^*, t, s, \phi^*)'$ converges in L^2-norm to the weak derivative $\bar{S}(g, t, s, \phi)'$ as $(g^*\phi^*) \to (g, \phi)$ with the C-norm in ϕ. Recall that for $t > s$, we have the pointwise limit (A.3). Denoting $J := [-1, 0] \cap (t - s, \infty)$, by dominated Lebesgue convergence we have that

$$\lim_{(g^*, \phi^*) \to (g, \phi)} \int_J |g(t + \theta, x_{t+\theta}^*) - g^*(t + \theta, x_{t+\theta}^*)|^2 = 0,$$

which implies H-convergence on the interval J, i.e., $\|(x_t^* - x_t)|_J\|_H \to 0$. Finally, we consider two cases if $t \in [s, s+1]$. For $\theta \in [-1, s-t]$, the solution operator corresponds to the translation

$$|\bar{S}(g,t,s,\phi)(\theta) - \bar{S}(g^*,t,s,\phi^*)(\theta)| = |\phi(t+\theta) - \phi^*(t+\theta)|$$

Therefore, we immediately obtain H-convergence $\|(x_t^* - x_t)|_{[-1,t-s]}\|_H \to 0$, which completes the proof. □

Proposition A.2. *Let g as above be C^k, $k \geq 1$. Then $\bar{S}(g,t,s,\phi)$ is C^1-differentiable with respect to (g,ϕ). Moreover, denoting $x_t := S(g,t,s,\phi)$, we have that the H-derivative $\partial_4 \bar{S}(g,t,s,\phi)\psi$ is y_t such that*

$$\begin{aligned} \dot{y}(t) &= \partial_2 g(t,x_t) y_t, \\ y_s(\theta) &= \psi(\theta), \quad \theta \in [-1,0]. \end{aligned} \tag{A.4}$$

Also, for all $\bar{g} \in C^k(\mathbb{R} \times C, \mathbb{R})$, $\partial_1 \bar{S}(g,t,s,\phi)\bar{g}$ is $\bar{y}_t$ such that

$$\begin{aligned} \dot{\bar{y}}(t) &= \partial_2 g(t,x_t)\bar{y}_t + \bar{g}(t,x_t), \\ \bar{y}_s(\theta) &= 0, \quad \theta \in [-1,0]. \end{aligned} \tag{A.5}$$

Proof. By [HVL93, Chapter 2, Theorem 4.1], our claims are true if we equip H with the supremum C-norm, we show (A.4) and (A.5) follows analogously. To clear up the notation, we drop the dependence in (g,t,s) throughout the proof. Indeed, we have that

$$\lim_{\|\psi\|_{C^0} \to 0} \frac{\|\bar{S}(\phi+\psi) - \bar{S}(\phi) - D\bar{S}(\phi)\psi\|_{C^0}}{\|\psi\|_{C^0}} = 0,$$

in particular, by (A.2), this implies that

$$\lim_{\|\psi\|_H \to 0} \frac{\|\bar{S}(\phi+\psi) - \bar{S}(\phi) - D\bar{S}(\phi)\psi\|_{L^2}}{\|\psi\|_H} = 0, \tag{A.6}$$

Hence we only have to show (A.6) for the weak derivatives, i.e.,

$$\lim_{\|\psi\|_H \to 0} \frac{\|\bar{S}(\phi+\psi)' - \bar{S}(\phi)' - \left(D\bar{S}(\phi)\psi\right)'\|_{L^2}}{\|\psi\|_H} = 0. \tag{A.7}$$

To see that (A.7) holds, we proceed as in the proof of Proposition A.1. Indeed, as long as $t \in [s, s+1]$, the function $\bar{S}(\phi+\psi)'(\theta) - \bar{S}(\phi)'(\theta) - \left(D\bar{S}(\phi)\psi\right)'(\theta)$ with $\theta \in [-1, s-t]$ is zero almost everywhere, therefore

$$\int_{[-1,s-t]} |\bar{S}(\phi+\psi)'(\theta) - \bar{S}(\phi)'(\theta) - \left(D\bar{S}(\phi)\psi\right)'(\theta)|^2 \mathrm{d}\theta = 0,$$

and (A.7) holds restricted to $[-1, s-t]$. Given $t > s+1$, we denote $x_t^{\psi} := S(\phi+\psi)$, hence the function inside the norm of the numerator in (A.7) is given by

$$g(t+\theta, x_{t+\theta}^{\psi}) - g(t+\theta, x_{t+\theta}) - \partial_2 g(t+\theta, x_{t+\theta}) y_{t+\theta},$$

for all $\theta \in J := [-1,0] \cap (s-t, \infty)$. Next, we apply Taylor expansion in the C-norm to obtain the pointwise limit

$$\lim_{\|\psi\|_{C^0} \to 0} \frac{|g(t+\theta, x_{t+\theta}^{\psi}) - g(t+\theta, x_{t+\theta}) - \partial_2 g(t+\theta, x_{t+\theta}) y_{t+\theta}|}{\|\psi\|_{C^0}} = 0,$$

for all $\theta \in J$. Hence dominated Lebesgue convergence guarantees

$$\lim_{\|\psi\|_H \to 0} \frac{\int_J |g(t+\theta, x_{t+\theta}^{\psi}) - g(t+\theta, x_{t+\theta}) - \partial_2 g(t+\theta, x_{t+\theta}) y_{t+\theta}|^2 \mathrm{d}\theta}{\|\psi\|_H} = 0.$$

Thus showing

$$\lim_{\|\psi\|_H \to 0} \frac{\|\bar{S}(\phi+\psi) - \bar{S}(\phi) - D\bar{S}(\phi)\psi\|_H}{\|\psi\|_H} = 0,$$

which completes the proof. □

Nishiguchi [Nis19] has shown that the solution semiflow on H is even differentiable in the time delay in (A.1), a regularity improvement compared to the standard phase space C. This property suggests that H is a better choice for analyzing DDEs that include state-dependent delays.

Proposition A.3. *If $t \geq s+2$, then $\bar{S}(g,t,s,\cdot)$ is a compact map, i.e., $\bar{S}(g,t,s,U)$ is precompact in H for all bounded $U \subset H$.*

Proof. It is enough to show that $\bar{S}(g,t,s,U)$ is sequentially compact for any bounded $U \subset H$, i.e., that for any sequence $\{x_t^{(n)}\}_{n\in\mathbb{N}} \subset \bar{S}(g,t,s,U)$ with initial conditions $\{x_s^{(n)}\}_{n\in\mathbb{N}} \subset U$, there exists a convergent subsequence $\{x_t^{(n_k)}\}_{k\in\mathbb{N}}$. Recall that, by [HVL93, Chapter 3, Corollary 6.2], $\bar{S}(g, s+1, s, U)$ is a precompact subset of C in the C-norm. In other words, $\{x_{s+1}^{(n)}\}_{n\in\mathbb{N}}$ possesses a subsequence $\{x_{s+1}^{n_k}\}_{k\in\mathbb{N}}$ for which there exists a point $\phi^* \in C$ such that $\|x_{s+1}^{n_k} - \psi^*\|_{C^0} \to 0$. Since $t \geq s+2$, we can guarantee that $\phi^* := S(g,t,s+1,\psi^*) \in C^1([-1,0],\mathbb{R}) \subset H$. We claim that

$$\|x_t^{n_k} - \phi^*\|_H = 0. \tag{A.8}$$

First, by C-continuity of $\bar{S}(g,t,s+1,\cdot)$, we have that $\|x_t^{n_k} - \phi^*\|_{L^2} \to 0$, automatically. Second, from (A.3) recall the pointwise limit

$$\lim_{k\to\infty} |\dot{x}_t^{n_k}(\theta) - \dot{\phi}^*(\theta)| = \lim_{k\to\infty} |g(t+\theta, x_{t+\theta}^{n_k}) - g(t+\theta, S(g,t+\theta,s,\psi^*)|,$$

for all $\theta \in [-1,0]$. Thus, by dominated Lebesgue convergence, we obtain that

$$\lim_{k\to\infty} \int_{[-1,0]} |\dot{x}_t^{n_k}(\theta) - \dot{\phi}^*(\theta)|^2 \mathrm{d}\theta = 0,$$

which shows (A.8) and completes the proof. □

Finally, we restrict ourselves to the special nonlinearity considered throughout the thesis, i.e., the DDE

$$\dot{x}(t) = f(x(t), x(t-1)), \tag{A.9}$$

where $f \in BC^2(\mathbb{R}^2, \mathbb{R})$ has negative delayed feedback, i.e., $\partial_2 f(\xi,\eta) < 0$ for all $(\xi,\eta) \in \mathbb{R}^2$.

Lemma A.4. *The solution operator* $\bar{S}(f,t,s,\cdot)$ *of* (A.9) *is injective.*

Proof. Since there is no explicit t-dependence in (A.9), we adopt the notation of the main sections $S_f(t-s)\phi := \bar{S}(f,t,s,\phi)$. Indeed, for all $x_t \in S_f(t)$ we can reconstruct $x(s)$ uniquely for all $s \in [t-1,t]$, via the formula

$$x(s) = f^{-1}(x(s+1), \dot{x}(s+1)).$$

Here f^{-1} is the unique function solving $f^{-1}(\xi, f(\xi,\eta)) = \eta$. Finally, the evolution property

$$S_f(t_2) = S_f(t_2 - t_1) \circ S_f(t_1),$$

for all $t_2 \geq t_1 \geq 0$ and an induction argument show that $S_f(t)$ is injective for all $t \geq 0$. □

Lemma A.5. *The H-derivative of* $\partial_4 \bar{S}(f,t,s,\phi)$ *solving* (A.9) *is injective and its range is dense in H for all $t \geq s$ and $\phi \in H$.*

Proof. Since we fixed f, we drop the f-dependence of the solution operator for simplicity of notation. By Proposition A.2, denoting $x_t := \bar{S}(t, s, \phi)$ we have that $y_t =: \partial_3 \bar{S}(t, s, \psi)$ solves the initial value problem

$$\begin{aligned} \dot{y}(t) &= \partial_1 f(x(t), x(t-1))y(t) + \partial_2 f(x(t), x(t-1))y(t-1), \\ y_s(\theta) &= \psi(\theta), \quad \theta \in [-1, 0]. \end{aligned}$$

Denoting $c_j(t) := \partial_j f(x(t), x(t-1))$, $j = 1, 2$, the injectivity of $\partial_3 \bar{S}(t, s, \phi)$ follows from the formula

$$y(s) = \frac{\dot{y}(s+1) - c_1(s+1)y(s)}{c_2(s+1)}, \tag{A.10}$$

which recovers $y(s)$, uniquely for all $s \in [t-1, t]$. We use induction to extend the argument to all $s < t-1$ by using the evolution property

$$\partial_3 \bar{S}(t, s, \phi) = \partial_3 \bar{S}(t, t-1, x_{t-1}) \partial_3 \bar{S}(t-1, s, \phi), \tag{A.11}$$

for all $t - 1 \geq s$.

To see that the range of $\partial_3 \bar{S}(t, s, \phi)$ is dense in H, we assume first that $s \in [t-1, t]$. Then, the prehistory formula (A.10) shows the characterization

$$\partial_3 \bar{S}(t, s, \phi)H = \left\{ \psi \in H : \begin{array}{c} \psi|_{[s-t,0]} \in C^1([s-t, 0], \mathbb{R}) \text{ and} \\ \dot{\psi}(0) = c_1(t)\psi(0) + c_2(t)\psi(-1) \end{array} \right\},$$

which is a dense subset of H. Recall that $\partial_3 \bar{S}(t, s, \phi)$ has a dense range if and only if the adjoint operator $\partial_3 \bar{S}(t, s, \phi)^*$ is injective. Here $\partial_3 \bar{S}(t, s, \phi)^*$ is defined on the dual space $H^* \cong H$ via the pairing with the H-inner product

$$\langle \partial_3 \bar{S}(t, s, \phi)^* \psi^*, \psi \rangle = \langle \psi^*, \partial_3 \bar{S}(t, s, \phi)\psi \rangle \quad \text{for all } \psi^*, \psi \in H.$$

Thus, to extend the result to $s \in [t-2, t-1)$, we use the evolution property (A.11) to split the interval $[s, t]$. Since (A.11) shows that $\partial_3 \bar{S}(t, s, \phi)^*$ is the composition of the corresponding injective adjoint operators $\partial_3 \bar{S}(t, t-1, x_{t-1})^*$ and $\partial_3 \bar{S}(t-1, s, \phi)^*$, we conclude that $\partial_3 \bar{S}(t, s, \phi)^*$ is injective and $\partial_3 \bar{S}(t, s, \phi)$ has dense range. Proceeding by induction, we obtain the result for any $t \geq s$ and complete the proof. □

Appendix B

Transversality

In this appendix, we discuss extending the intuitive three-dimensional notion of transversality to infinite dimensions. We do this by using the implicit function theorem. Given a Banach space $(X, \|\cdot\|)$, we say that two C^1-manifolds W^{u} of dimension $m \in \mathbb{N}$ and W^{s} if codimension $n \in \mathbb{N}$ intersect **transversely** if

$$T_\phi W^{\mathrm{u}} + T_\phi W^{\mathrm{s}} = X \quad \text{for all } \phi \in W^{\mathrm{u}} \cap W^{\mathrm{s}}.$$

If two manifolds intersect transversely, we denote it by $W^{\mathrm{u}} \pitchfork W^{\mathrm{s}}$. Notice that, by definition, two manifolds intersect transversely if their intersection is empty.

The main property of transverse intersections of manifolds is that $W^{\mathrm{u}} \pitchfork W^{\mathrm{s}}$ is again a manifold. We point out that the standard literature contains the definition we use here. Abraham, Robins, and Zeidler [AR67, Zei85] include topological splitting conditions on X. However, such splittings follow immediately if W^{u} has a finite dimension and W^{s} has a finite codimension.

Lemma B.1. *Let W^{u} and W^{s} be two C^1-manifolds as above such that $W^{\mathrm{u}} \pitchfork W^{\mathrm{s}} \neq \emptyset$. Then $W^{\mathrm{c}} = W^{\mathrm{u}} \pitchfork W^{\mathrm{s}}$ is a C^1-manifold of dimension*

$$\dim W^{\mathrm{c}} = m - n.$$

Moreover, in a neighborhood of W^{c} is locally C^1-diffeomorphic to $T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}}$. More precisely, locally $W^{\mathrm{c}} - \phi$ is the graph of a C^1-function $h^{\mathrm{c}} : T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}} \to (T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}})^\perp$ satisfying $h^{\mathrm{c}}(0) = 0$, $Dh^{\mathrm{c}}(0) = 0$. Here $(T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}})^\perp$ denotes the closed complement of $T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}}$ in X.

Proof. Indeed, given $\phi \in W^{\mathrm{c}}$, there exists $\delta > 0$ such that W^{u} is represented locally as $W^{\mathrm{u}} = \{\phi+\phi^{\mathrm{u}}+h^{\mathrm{u}}(\phi^{\mathrm{u}}) : \|\phi^{\mathrm{u}}\| < \delta\}$. Where $h^{\mathrm{u}} : T_\phi W^{\mathrm{u}} \to (T_\phi W^{\mathrm{u}})^\perp$ is a C^1-function such that $h^{\mathrm{u}}(0) = 0$ and $Dh(0) = 0$. Likewise, W^{s} is given locally by the zero set of a C^1-map $\bar{h}^{\mathrm{s}} : H \to (T_\phi W^{\mathrm{s}})^\perp$ with $\bar{h}^{\mathrm{s}}(\phi) = 0$ and $\ker D\bar{h}^{\mathrm{s}}(0) = T_\phi W^{\mathrm{s}}$. Thus, for any $\psi \in W^{\mathrm{c}}$ we can write, near ϕ $\psi = \phi + \psi^{\mathrm{uu}} + \psi^{\mathrm{c}} + h^{\mathrm{u}}(\psi^{\mathrm{uu}} + \psi^{\mathrm{c}})$ for $\psi^{\mathrm{c}} \in T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}}$ and ψ^{uu} in the closed complement of $T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}}$ in $T_\phi W^{\mathrm{u}}$ and, by the transversality assumption, coincides with $(T_\phi W^{\mathrm{s}})^\perp$. Hence, $\psi \in W^{\mathrm{c}}$ if and only if $\hat{h}^{\mathrm{c}}(\psi^{\mathrm{uu}}, \psi^{\mathrm{c}}) := \bar{h}^{\mathrm{s}}(\phi + \psi^{\mathrm{uu}} + \psi^{\mathrm{c}} + h^{\mathrm{u}}(\psi^{\mathrm{uu}} + \psi^{\mathrm{c}})) = 0$. However, we see by direct computation that $\partial_1 \hat{h}^{\mathrm{c}} = \mathrm{Id}|_{(T_\phi W^{\mathrm{s}})^\perp}$, therefore, there exists $\tilde{h}^{\mathrm{c}}$ such that $\hat{h}^{\mathrm{c}}(\phi + \psi^{\mathrm{c}} + \tilde{h}^{\mathrm{c}}(\psi^{\mathrm{c}}) + h^{\mathrm{u}}(\psi^{\mathrm{c}} + \tilde{h}^{\mathrm{c}}(\psi^{\mathrm{c}}))) = 0$ for all $\psi^{\mathrm{c}} \in T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}}$, sufficiently close to 0. Finally, we obtain $h^{\mathrm{c}} : T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}} \to (T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}})^\perp$ by setting $h^{\mathrm{c}}(\psi^{\mathrm{c}}) := \tilde{h}^{\mathrm{c}}(\psi^{\mathrm{c}}) + h^{\mathrm{u}}(\psi^{\mathrm{c}} + \tilde{h}^{\mathrm{c}}(\psi^{\mathrm{c}}))$. □

As a result, it follows that transverse intersections of manifolds resist small perturbations, i.e., they are structurally stable.

Corollary B.2. *Let W^{u}_λ and W^{s}_λ be two families of C^1-manifolds in X of dimension m and codimension n, respectively, and let the dependence of the parameter $\lambda \in \mathbb{R}$ be C^1. If W^{u}_0 and W^{s}_0 intersect transversely at $\phi \in X$, then there exists $\varepsilon > 0$ such that $W^{\mathrm{u}}_\lambda \cap W^{\mathrm{s}}_\lambda \neq \emptyset$ for all $\lambda \in (-\varepsilon, \varepsilon)$.*

Proof. Indeed, W^{u}_λ and W^{s}_λ are C^1-manifolds over the extended space $\mathcal{X} := X \times \mathbb{R}$. Moreover, by assumption, we have that $T_\phi W^{\mathrm{u}}_0 + T_\phi W^{\mathrm{s}}_0 = X$ for all $\phi \in W^{\mathrm{u}}_\lambda \cap W^{\mathrm{s}}_\lambda$. Thus, we obtain $T_{(\phi,0)} W^{\mathrm{u}}_\lambda + T_{(\phi,0)} W^{\mathrm{s}}_\lambda = (T_\phi W^{\mathrm{u}}_0 \times \mathbb{R}) + (T_\phi W^{\mathrm{s}}_0 \times \mathbb{R}) = \mathcal{X}$ and $W^{\mathrm{u}}_\lambda \pitchfork W^{\mathrm{s}}_\lambda$ with the intersection being transverse in $\mathcal{X}$. In particular, Lemma B.1 yields that $W^{\mathrm{u}}_\lambda \cap W^{\mathrm{s}}_\lambda$ is a C^1-manifold of dimension $m + 1 - n$ in $\mathcal{X}$ and at $(\phi, 0)$ the tangent space is $T_\phi W^{\mathrm{u}} \cap T_\phi W^{\mathrm{s}} \times \mathbb{R}$, this completes the proof. □

Appendix C

Invariant manifolds for maps

C.1 Local invariant manifolds

Given a nonlinear map $\mathcal{F} \in C^2(X, X)$ in a Hilbert space $(X, \|\cdot\|)$ such that $\mathcal{F}(0) = 0$. We denote $L \coloneqq D\mathcal{F}(0)$ and assume the existence of $\kappa > 0$ such that X admits a L-invariant splitting into closed subspaces

$$X = X^1 \oplus X^2, \tag{C.1}$$

where the restrictions $L_j \coloneqq L|_{X^j}$, $j = 1, 2$ satisfy

$$\inf\{|\mu| : \mu \in \operatorname{Spec}(L_1)\} > \kappa, \quad \text{and} \qquad \sup\{|\mu| : \mu \in \operatorname{Spec}(L_2)\} < \kappa.$$

Moreover, we assume that $\mathcal{F}$ and $D\mathcal{F}(\phi)$ are injective for all $\phi \in H$. Thus, in the following $\mathcal{F}^{-1}(\phi)$ is shorthand for *there exists a* $\tilde{\phi} \in X$ *such that* $\phi = \mathcal{F}(\tilde{\phi})$. Analogously, $D\mathcal{F}^{-1}(\phi)\psi$ refers to the unique preimage of ψ by $D\mathcal{F}(\mathcal{F}^{-1}(\phi))$. In this setting, we formulate the **global κ-invariant manifold** theorem.

Theorem C.1. *In the notation above, there exists* $\varepsilon > 0$ *such that if*

$$\max\left\{\sup_{\phi\in X} \|D\mathcal{F}(\phi) - L\|,\ \sup_{\phi\in X} \|D^2\mathcal{F}(\phi)\|\right\} < \varepsilon, \tag{C.2}$$

the κ-unstable and κ-stable sets

$$W^{\kappa\mathrm{u}}(0) = \left\{ \phi \in X : \lim_{n\to-\infty} \frac{\mathcal{F}^n(\phi)}{\kappa^n} = 0 \right\}, \tag{C.3}$$
$$W^{\kappa\mathrm{s}}(0) = \left\{ \phi \in X : \lim_{n\to\infty} \frac{\mathcal{F}^n(\phi)}{\kappa^n} = 0 \right\},$$

are, respectively, the graphs of C^2-functions

$$h_1 : X^1 \to X^2 \quad \text{and} \quad h_2 : X^2 \to X^1,$$

satisfying $h_j(0) = 0$ and $Dh_j(0) = 0$, $j = 1, 2$. Moreover, their tangent spaces are given by

$$T_\phi W^{\kappa\mathrm{u}}(0) = \left\{ \psi \in X : \lim_{n\to-\infty} \frac{D\mathcal{F}^n(\phi)\psi}{\kappa^n} = 0 \right\}, \tag{C.4}$$

$$T_\phi W^{\kappa\mathrm{s}}(0) = \left\{ \psi \in X : \lim_{n\to\infty} \frac{D\mathcal{F}^n(\phi)\psi}{\kappa^n} = 0 \right\}. \tag{C.5}$$

The manifolds $W^{\kappa\mathrm{u}}(0)$ and $W^{\kappa\mathrm{s}}(0)$ depend C^1-continuously on the map $\mathcal{F}$.

Proof. [CCH92, Appendix C] provides local versions of the theorem for the case $\kappa = 1$. All of our claims except for the results on differentiability and the description of the tangent spaces (C.4)–(C.5) follow immediately from [HPS77, Theorem 5.1]; we show (C.5) and (C.4) follows similarly.

Indeed, [HPS77, Theorem 5.1] constructs $W^{\kappa\mathrm{s}}(0)$ via a Banach contraction argument as the maximal backward invariant set within the cone

$$K := \{\phi^1 + \phi^2 \in X^1 + X^2 : \|\phi^2\| \geq \|\phi^1\|\}.$$

In other words,

$$W^{\kappa\mathrm{s}}(0) = \bigcap_{n\leq 0} \mathcal{F}^n(K),$$

moreover, by construction $W^{\kappa\mathrm{s}}(0)$ is the graph of a C^1-map $h_2 : X^2 \to X^1$ with global Lipschitz constant $\mathrm{Lip}(h_2) < 1$. Recall that $\mathcal{F}$ is C^2, therefore it induces a C^1-map $\boldsymbol{\mathcal{F}}$ on the Cartesian product $X \times X$ with the induced Euclidean norm $\|\cdot\|_{X\times X}$ via

$$\begin{aligned} \boldsymbol{\mathcal{F}} : X \times X &\to X \times X \\ (\phi, \psi) &\mapsto (\mathcal{F}\phi, D\mathcal{F}(\phi)\psi). \end{aligned}$$

Here, the first derivative is given by

$$D\boldsymbol{\mathcal{F}}(\phi,\psi)(u,v) = (D\mathcal{F}(\phi)u, D\mathcal{F}(\phi)v + D^2\mathcal{F}(\phi)(u,v)).$$

Setting $\boldsymbol{L}(u,v) := D\boldsymbol{\mathcal{F}}(0,0)(u,v) = (Lu, Lv)$, we obtain an $\boldsymbol{L}$-invariant splitting $X \times X = (X^1 \times X^1) \oplus (X^2 \times X^2)$. Hence we apply [HPS77, Theorem 5.1] to $\boldsymbol{\mathcal{F}}$ with the fixed point $(0,0)$ and obtain a C^1-manifold $\boldsymbol{W}^{\kappa \mathrm{s}}(0,0)$ of codimension $2\dim X^1$. Furthermore, $\boldsymbol{W}^{\kappa \mathrm{s}}(0,0)$ satisfies

$$\boldsymbol{W}^{\kappa \mathrm{s}}(0,0) := \bigcap_{n\leq 1} \boldsymbol{\mathcal{F}}^n(\boldsymbol{K}),$$

where $\boldsymbol{K}$ denotes the cone in extended space

$$\boldsymbol{K} := \left\{\Phi^1 + \Phi^2 \in (X^1 \times X^1) + (X^2 \times X^2) : \|\Phi^2\|_{X\times X} \geq \|\Phi^1\|_{X\times X}\right\},$$

and we know that $\boldsymbol{W}^{\kappa \mathrm{s}}(0,0)$ is characterized by

$$\boldsymbol{W}^{\kappa \mathrm{u}}(0,0) = \left\{(\phi,\psi) \in X \times X : \lim_{n\to\infty} \frac{\boldsymbol{\mathcal{F}}^n(\phi,\psi)}{\kappa^n} = 0\right\}$$

Recall that the tangent bundle $TW^{\kappa \mathrm{s}}(0)$ is the graph of a C^0-map

$$\begin{aligned} \boldsymbol{h}_2 : X^2 \times X^2 &\to X^1 \times X^1 \\ (\phi^2, \psi^2) &\mapsto (h_2(\phi^2), Dh_2(\phi^2)\psi^2)\,. \end{aligned}$$

Hence it is a C^0-manifold of codimension

$$\mathrm{codim}\,(TW^{\kappa \mathrm{s}}(0)) = 2\dim\left(X^1\right) = \mathrm{codim}\,(\boldsymbol{W}^{\kappa \mathrm{s}}(0,0))\,.$$

Since the map h_2 above satisfies $\mathrm{Lip}(h_2) < 1$, we have that $TW^{\kappa \mathrm{s}}(0)$ is a $\boldsymbol{\mathcal{F}}^{-1}$-invariant set satisfying $TW^{\kappa \mathrm{s}}(0) \subset K \times K \subset \boldsymbol{K}$. In particular, it follows that $TW^{\kappa \mathrm{s}}(0) \subset \boldsymbol{W}^{\kappa \mathrm{s}}(0,0)$. Since both sets are graphs over $X^2 \times X^2$, we conclude that

$$\boldsymbol{W}^{\kappa \mathrm{s}}(0,0) = TW^{\kappa \mathrm{s}}(0),$$

which finishes the proof of (C.5) and yields the claimed differentiability of h_2. □

We point out that, in general, we are interested in maps fixing any point $x^* \in X$. In such case, we will consider the iteration of the shifted map $\widetilde{\mathcal{F}}(\phi) := \mathcal{F}(\phi + x^*) - x^*$ and define

$$W^{\kappa \mathrm{u}}(x^*) := x^* + \widetilde{W}^{\kappa \mathrm{u}}(0) := x^* + \left\{\phi \in X : \begin{array}{l} \widetilde{\mathcal{F}}^n(\phi) \text{ exists for all } n \leq 0, \\ \text{and } \displaystyle\lim_{n\to -\infty} \frac{\widetilde{\mathcal{F}}^n(\phi)}{\kappa^n} = 0 \end{array}\right\}.$$

Using an analogous definition for $W^{\kappa\mathrm{s}}(x^*) := x^* + \widetilde{W}^{\kappa\mathrm{s}}(0)$, we are free to apply the results of this chapter to any fixed point. Moreover, the manifolds obtained by Theorem C.1 are determined by gaps in the spectrum of L rather than the specific choice of κ.

Corollary C.2. *In the setting of Theorem C.1, if $\tilde{\kappa} > \kappa$ is chosen so that $[\kappa, \tilde{\kappa}] \cap \{|\mu| : \mu \in \mathrm{Spec}(L)\} = \emptyset$, then we have that*

$$W^{\kappa\mathrm{u}}(0) = W^{\tilde{\kappa}\mathrm{u}}(0) \quad \text{and} \quad W^{\kappa\mathrm{s}}(0) = W^{\tilde{\kappa}\mathrm{s}}(0).$$

Proof. By the construction in [HPS77, Theorem 5.1], the choice of $\varepsilon > 0$ in (C.2) is the same for κ and $\tilde{\kappa}$. Thus, Theorem C.1 and the characterization (C.3) yield $W^{\kappa\mathrm{s}}(0) \subset W^{\tilde{\kappa}\mathrm{s}}(0)$. Since both manifolds are graphs over X^2, they must be equal. The proof works analogously for the unstable manifold, since $W^{\tilde{\kappa}\mathrm{u}}(0) \subset W^{\kappa\mathrm{u}}(0)$. $\square$

Verifying the vague smallness assumption (C.2) in Theorem C.1 is generally unrealistic. We need to ensure that the nonlinear map $\mathcal{F}$ is uniformly C^2-close to the Fréchet derivative L. We achieve this by considering a modified map $\mathcal{F}_\delta$ satisfying the assumptions of Theorem C.1 and such that $\mathcal{F}_\delta(\phi) = \mathcal{F}(\phi)$ for all ϕ within the open δ-ball around 0, which we denote U_δ. Due to the differentiability of the inner product in X, we can introduce a smooth cutoff function $\sigma : X \to X$ satisfying

$$\sigma(\phi) = \begin{cases} 1, & \text{if } \|\phi\| < 1, \\ 0, & \text{if } \|\phi\| > 2, \end{cases} \tag{C.6}$$

and such that, for $\delta > 0$ small enough, the modified map

$$\mathcal{F}_\delta(\phi) \coloneqq L\phi + (\mathcal{F}(\phi) - L\phi)\,\sigma\left(\frac{\phi}{\delta}\right)$$

satisfies that $\sup_{\phi\in X} \|D\mathcal{F}_\delta(\phi) - L\|$ in (C.2) is arbitrarily small. Moreover, for all $\delta > 0$, we have that $\sup_{\phi\in X} \|D^2_\phi \mathcal{F}_\delta\| < \infty$. Rather than shrinking $\sup_{\phi\in X} \|D^2_\phi \mathcal{F}_\delta\|$, we build the invariant manifolds for the *flattened* map

$$\bar{\mathcal{F}}_\delta(\phi) := \mathcal{F}_\delta(\bar{\delta}\phi)/\bar{\delta}, \tag{C.7}$$

instead. For $\bar{\delta} > 0$ small enough, $\bar{\mathcal{F}}_\delta$ satisfies the assumptions of Theorem C.1, fixes 0, and its derivative at 0 is L. Denoting the $\bar{\mathcal{F}}_\delta$-invariant manifolds obtained by these means by

$$\overline{W}^{\kappa\mathrm{u}}(0) := \left\{ \phi \in X : \lim_{n\to-\infty} \frac{\bar{\mathcal{F}}_{\delta}^{n}(\bar{\delta}\phi)}{\kappa^{n}} = 0 \right\}, \tag{C.8}$$

$$\overline{W}^{\kappa\mathrm{s}}(0) := \left\{ \phi \in X : \lim_{n\to\infty} \frac{\bar{\mathcal{F}}_{\delta}^{n}(\bar{\delta}\phi)}{\kappa^{n}} = 0 \right\}, \tag{C.9}$$

we call **local κ-unstable manifold** and **κ-stable manifold** to the sets

$$W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0) := \left\{ \phi \in X : \frac{\phi}{\bar{\delta}} \in \overline{W}^{\kappa\mathrm{u}}(0) \right\} \cap U_{\delta}, \tag{C.10}$$

$$W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0) := \left\{ \phi \in X : \frac{\phi}{\bar{\delta}} \in \overline{W}^{\kappa\mathrm{s}}(0) \right\} \cap U_{\delta}, \tag{C.11}$$

respectively. Both $W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ and $W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$ preserve a certain degree of invariance in the sense that they satisfy

$$\begin{aligned} &W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0) \cap \mathcal{F}(U_{\delta}) \subset \mathcal{F}(W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)), \\ &\mathcal{F}(W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)) \cap U_{\delta} \subset W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0). \end{aligned} \tag{C.12}$$

We summarize the comments above in the following theorem.

Theorem C.3. *Let $\mathcal{F}$ satisfy the assumptions of Theorem C.1 except for the smallness estimate* (C.2)*. Then there exists $\delta > 0$ such that for all open sets $U \subset U_{\delta}$ there exists a subset $W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ that is locally C^1-diffeomorphic to X^1 and contains all sequences $\{\mathcal{F}^n(\phi)\}_{n\leq 0} \subset U$ such that*

$$\lim_{n\to-\infty} \frac{\mathcal{F}^{n}(\phi)}{\kappa^{n}} = 0. \tag{C.13}$$

Furthermore, if $\{\mathcal{F}^n(\phi)\}_{n\leq 0} \subset U$ and (C.13) *holds, then*

$$T_{\phi}W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0) = \left\{ \psi \in X : \lim_{n\to-\infty} \frac{D\mathcal{F}^{n}(\phi)\psi}{\kappa^{n}} = 0 \right\}. \tag{C.14}$$

Analogously, we can find a subset $W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$ that is locally C^1-diffeomorphic to X^2 and contains all sequences $\{\mathcal{F}^n(\phi)\}_{n\leq 0} \subset U$ such that

$$\lim_{n\to\infty} \frac{\mathcal{F}^{n}(\phi)}{\kappa^{n}} = 0. \tag{C.15}$$

If $\{\mathcal{F}^n(\phi)\}_{n\geq 0} \subset U$ and (C.13) *holds, then*

$$T_{\phi}W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0) = \left\{ \psi \in X \lim_{n\to\infty} \frac{D\mathcal{F}^{n}(\phi)\psi}{\kappa^{n}} = 0 \right\}. \tag{C.16}$$

Moreover, the manifolds $W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ and $W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$ depend C^1-continuously on the restriction $\mathcal{F}|_U$.

Proof. Indeed, we proceed as above and choose a modified map $\mathcal{F}_\delta$ that we use to construct global κ-invariant manifolds. Then the convergence properties (C.13)–(C.16) follow by Theorem C.1. $\square$

In general, the local invariant manifolds in Theorem C.3 depend not only on the choice of a neighborhood U but also on the choice of a cutoff function (C.6) for their construction; see [Kel67]. In other words, two local invariant manifolds may a priori differ in arbitrarily small neighborhoods of 0. We say that an invariant manifold $W^{\kappa\iota}_{\mathrm{loc}}(0)$ is **locally unique** if given another local invariant manifold $\widetilde{W}^{\kappa\iota}_{\mathrm{loc}}(0)$ there exists a neighborhood U of the origin on which we have $W^{\kappa\iota}_{\mathrm{loc}}(0) \cap U = \widetilde{W}^{\kappa\iota}_{\mathrm{loc}}(0) \cap U$, $\iota = \mathrm{u}, \mathrm{s}$. The characterizations in terms of asymptotics (C.13) and (C.15) show that the existence of invariant neighborhoods on the local invariant manifolds (C.8)–(C.9) plays a fundamental role in proving the uniqueness of their local structure. If $\kappa > 1$, then we show that sufficiently close to 0, the map $\mathcal{F}$ expands any local unstable manifold $W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$, uniformly. In particular, it follows that independently of the chosen cutoff function (C.6), the fixed point 0 possesses arbitrarily small backward $\mathcal{F}$-invariant neighborhoods in $W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$. Conversely, if $\kappa < 1$, then $\mathcal{F}$ contracts $W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$ and there exist arbitrarily small $\mathcal{F}$-invariant neighborhoods of 0 in $W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$.

Lemma C.4. *Let $\kappa > 1$ (resp. $\kappa < 1$). Then there exist $\delta > 0$, $\beta < 1$, and an equivalent norm $|||\cdot|||$ in X such that if $\phi \in W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ (resp., $\phi \in W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$) and $\|\phi\| < \delta$, then*

$$|||\phi||| \le \beta |||\mathcal{F}(\phi)||| \quad (\textit{resp., } |||\mathcal{F}(\phi)||| \le \beta |||\phi|||).$$

In particular, for all $\delta > 0$ there exists a neighborhood of the origin $U^{\mathrm{u}} \subset W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ (resp. $U^{\mathrm{s}} \subset W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$) such that $U^{\mathrm{u}} \subset U_\delta$ and $U^{\mathrm{u}} \subset \mathcal{F}(U^{\mathrm{u}})$ (resp., $U^{\mathrm{s}} \subset U_\delta$ and $\mathcal{F}(U^{\mathrm{s}}) \subset U^{\mathrm{s}}$).

Proof. Indeed, we can always choose an equivalent norm $|||\cdot|||$ in X such that, using the notation of (C.1), the restriction L_1 satisfies $\inf_{\psi \in X_1} |||L_1\psi|||/|||\psi||| > \kappa$. Thus, the restricted map $\mathcal{F}_1 := \mathcal{F}|_{W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)}$ has linearization $D\mathcal{F}_1(0) = L_1$ and expanding at 0 we obtain

$$\begin{aligned} |||\mathcal{F}(\phi)||| &= |||\mathcal{F}_1(\phi)||| \\ &= |||L_1\phi + o(|||\phi|||)||| \\ &\ge (\kappa - \varepsilon)|||\phi|||, \end{aligned}$$

for all $\phi \in W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$. Here we used the Landau little-o notation, and $\varepsilon > 0$ is made arbitrarily small for a sufficiently small choice of $\delta > 0$. The result follows by taking $1/(\kappa - \varepsilon) < \beta < 1$. The proof is completely analogous for $\phi \in W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$, where the restricted map $\mathcal{F}|_{W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)}$ defines a uniform contraction near 0. □

Corollary C.5. *Under the assumptions of Lemma C.4 the local invariant manifold* $W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ *(resp.,* $W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$*) is locally unique.*

Proof. Indeed, let $\widetilde{W}^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ be a local invariant manifold constructed using a different cutoff function (C.6) and choose $\delta > 0$ so small that the maps $\mathcal{F}_\delta$ and $\widetilde{\mathcal{F}}_\delta$ used to define $W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ and $\widetilde{W}^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ coincide on U_δ. By Lemma C.13, there exists a neighborhood of the origin $U^{\mathrm{u}} \subset U_\delta \cap W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$ such that $U^{\mathrm{u}} \subset \mathcal{F}(U^{\mathrm{u}})$. It is then clear from the characterization (C.13) that $U^{\mathrm{u}} \cap W^{\kappa\mathrm{u}}_{\mathrm{loc}}(0) \subset \widetilde{W}^{\kappa\mathrm{u}}_{\mathrm{loc}}(0)$. The implication for $W^{\kappa\mathrm{s}}_{\mathrm{loc}}(0)$ follows analogously by considering (C.15) instead. □

C.2 Unstable, center, and stable dynamics

In the setting of Section C.1, let us assume that we can consider the special L-invariant splitting

$$X = X^{\mathrm{u}} \oplus X^{\mathrm{c}} \oplus X^{\mathrm{s}} \tag{C.17}$$

where the restrictions $L_\iota := L|_{X^\iota}$, $\iota = \mathrm{u}, \mathrm{c}, \mathrm{s}$, satisfy

- $\inf\{|\mu| : \mu \in \mathrm{Spec}(L_{\mathrm{u}})\} > \kappa_+ > 1$,
- $|\mu| = 1$, for all $\mu \in \mathrm{Spec}(L_{\mathrm{c}})$, and
- $\sup\{|\mu| : \mu \in \mathrm{Spec}(L_{\mathrm{s}})\} < \kappa_- < 1$.

Under these circumstances, the subspaces X^{u}, X^{c}, and X^{s} are called **unstable**, **center** and **stable** , respectively. Additionally, we consider two further L-invariant splittings

$$X = X^{\mathrm{cu}} \oplus X^{\mathrm{s}}, \quad \text{and} \quad X = X^{\mathrm{u}} \oplus X^{\mathrm{cs}}, \tag{C.18}$$

where $X^{\mathrm{cu}} := X^{\mathrm{u}} \oplus X^{\mathrm{c}}$ and $X^{\mathrm{cs}} := X^{\mathrm{s}} \oplus X^{\mathrm{s}}$ are the **center-unstable** and **center-stable eigenspaces**, respectively. We consider the local invariant manifolds (C.10)–(C.11) obtained the cutoff method Section C.2 with respect to a δ-ball $U_\delta \subset X$:

$$
\begin{aligned}
W^{\mathrm{uu}}_{\mathrm{loc}}(0) &:= W^{\kappa+\mathrm{u}}_{\mathrm{loc}}(0), \quad W^{\mathrm{cu}}_{\mathrm{loc}}(0) := W^{\kappa-\mathrm{u}}_{\mathrm{loc}}(0),\\
W^{\mathrm{ss}}_{\mathrm{loc}}(0) &:= W^{\kappa-\mathrm{s}}_{\mathrm{loc}}(0), \quad W^{\mathrm{cs}}_{\mathrm{loc}}(0) := W^{\kappa+\mathrm{s}}_{\mathrm{loc}}(0).
\end{aligned}
\tag{C.19}
$$

We refer to them **local fast unstable**, **center-unstable**, **fast stable**, and **center-stable manifold**, as indicated by the respective superscripts.

As discussed in Corollary C.5, the fast unstable and fast stable manifolds play an essential role in studying the dynamics close to 0. In particular, $W^{\mathrm{uu}}_{\mathrm{loc}}(0)$ and $W^{\mathrm{ss}}_{\mathrm{loc}}(0)$ are locally uniquely defined as the sets of initial conditions that converge to 0 at an at least exponential rate under the iteration of $\mathcal{F}$. We have the following characterization.

Theorem C.6. *There exists an open neighborhood $U \subset U_\delta$ and a locl fast unstable manifold $W^{\mathrm{uu}}_{\mathrm{loc}}(0)$ that satisfies*

$$
W^{\mathrm{uu}}_{\mathrm{loc}}(0) \cap U = \left\{ \phi \in U : \begin{array}{l} \{\mathcal{F}^n(\phi)\}_{n\le 0} \subset U \text{ and there exists } \kappa > 1 \\ \text{such that } \lim\limits_{n\to-\infty} \dfrac{\mathcal{F}^n(\phi)}{\kappa^n} = 0 \end{array} \right\}.
$$

Moreover, for all $\phi \in W^{\mathrm{uu}}_{\mathrm{loc}}(0) \cap U$ the tangent space satisfies

$$
T_\phi W^{\mathrm{uu}}_{\mathrm{loc}}(0) = \left\{ \psi \in X : \begin{array}{l} \text{there exists } \kappa > 1 \\ \text{such that } \lim\limits_{n\to-\infty} \dfrac{D\mathcal{F}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\}.
$$

Likewise, there exists an open neighborhood $U \subset U_\delta$ and a fast stable manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(0)$ that satisfies

$$
W^{\mathrm{ss}}_{\mathrm{loc}}(0) \cap U = \left\{ \phi \in U : \begin{array}{l} \{\mathcal{F}^n(\phi)\}_{n\ge 0} \subset U \text{ and there exists } \kappa < 1 \\ \text{such that } \lim\limits_{n\to\infty} \dfrac{\mathcal{F}^n(\phi)}{\kappa^n} = 0 \end{array} \right\},
$$

and for all $\phi \in W^{\mathrm{ss}}_{\mathrm{loc}}(0) \cap U$ we have

$$
T_\phi W^{\mathrm{ss}}_{\mathrm{loc}}(0) = \left\{ \psi \in X : \begin{array}{l} \text{there exists } \kappa < 1 \\ \text{such that } \lim\limits_{n\to\infty} \dfrac{D\mathcal{F}^n(\phi)\psi}{\kappa^n} = 0 \end{array} \right\}.
$$

Proof. The inclusion

$$W^{\mathrm{uu}}_{\mathrm{loc}}(0) \cap U \supset \left\{ \phi \in U : \begin{array}{c} \{\mathcal{F}^n(\phi)\}_{n\leq 0} \subset U \text{ and there exists } \kappa > 1 \\ \text{such that } \lim\limits_{n\to-\infty} \dfrac{\mathcal{F}^n(\phi)}{\kappa^n} = 0 \end{array} \right\},$$

follows from Theorem C.3 and Corollary C.2. The converse inclusion follows by Lemma C.4, hence we can find and open $U \subset U_\delta$ such that $U^{\mathrm{u}} := W^{\mathrm{uu}}_{\mathrm{loc}}(0) \cap U$ satisfies $U^{\mathrm{u}} \subset \mathcal{F}(U^{\mathrm{u}})$. The characterization of $T_\phi W^{\mathrm{uu}}_{\mathrm{loc}}(0)$ follows immediately from (C.14). The implications for $W^{\mathrm{ss}}_{\mathrm{loc}}(0)$ follow analogously by considering the characterizations (C.15)–(C.16) instead. □

In contrast, the center manifolds $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ and $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$ in (C.19) are not uniquely defined in general; see [Kel67]. However, from this point on, we fix a choice of local center-unstable and center-stable manifolds and define the **local center manifold** to be the intersection

$$W^{\mathrm{c}}_{\mathrm{loc}}(0) := W^{\mathrm{cu}}_{\mathrm{loc}}(0) \cap W^{\mathrm{cs}}_{\mathrm{loc}}(0).$$

If $\dim X^{\mathrm{cu}} < \infty$, then $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ is indeed a manifold of dimension $\dim X^{\mathrm{c}}$.

Theorem C.7. *In the setting above, let* $\dim X^{\mathrm{cu}} < \infty$*. Then there exists* $\delta > 0$ *such that for all open sets* $U \subset U_\delta$ *there exists a subset* $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ *that is locally* C^1*-diffeomorphic to* X^{c}*. More precisely,* $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ *is the graph of a* C^1*-function* $h^{\mathrm{c}} : X^{\mathrm{c}} \to X^{\mathrm{u}} \oplus X^{\mathrm{s}}$ *satisfying* $h^{\mathrm{c}}(0) = 0$ *and* $Dh^{\mathrm{c}}(0) = 0$*.* $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ *contains all sequences such taht* $\{\mathcal{F}^n(\phi)\}_{n\in\mathbb{Z}} \subset U$*.*

Proof. The claims follow by Lemma B.1. Indeed, we have that $T_0 W^{\mathrm{cu}}_{\mathrm{loc}}(0) = X^{\mathrm{cu}}$ and $T_0 W^{\mathrm{cs}}_{\mathrm{loc}}(0) = X^{\mathrm{cs}}$. Hence 0 is a point of transverse intersection for the manifolds $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ and $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$, which yields a function h^{c} as above. Indeed $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ contains all sequences $\{\mathcal{F}^n(\phi)\}_{n\in\mathbb{Z}} \subset U$ by Theorem C.3. □

By construction, $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ inherits both the forward and backward invariance properties (C.12) from $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ and $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$. Moreover, by Theorem C.7, $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ contains any $\mathcal{F}$-fixed or $\mathcal{F}$-periodic points sufficiently close to 0.

Although the local center manifold is most often nonunique by construction, for diffeomorphisms $\mathcal{F}$ acting on finite-dimensional phase spaces X, it is known that the dynamics within two different local center manifolds are topologically conjugate; see [Tak71]. However, since we are in an a priori infinite-dimensional setting, we proceed differently. For $\kappa = 1$, the uniform contraction rates towards the origin in Lemma C.4 survive in the sense that $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ attracts any iterations within $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ and $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$, at a uniform rate. Thereby, any sequence $\{\mathcal{F}^n(\phi)\}_{n\geq 0} \subset W^{\mathrm{cs}}_{\mathrm{loc}}(0)$ shall approach the associated center manifold $W^{\mathrm{c}}_{\mathrm{loc}}(0) \subset W^{\mathrm{cs}}_{\mathrm{loc}}(0)$ exponentially fast until $\mathcal{F}^n(\phi)$ becomes virtually indistinguishable from an orbit in $W^{\mathrm{c}}_{\mathrm{loc}}(0)$. This phenomenon, which reduces the dynamics on both $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ and $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$ to studying the lower-dimensional center manifold is known as **center manifold reduction**. The remainder of the section presents sufficient conditions on the local center manifold dynamics to justify a center manifold reduction. In the following, P^ι denotes the **canonical projections** associated to the corresponding splittings (C.17)–(C.18) for $\iota = \mathrm{u}, \mathrm{cu}, \mathrm{c}, \mathrm{cs}, \mathrm{s}$.

Lemma C.8. *In the setting of Theorem C.7, there exist $\delta > 0$, $\beta < 1$, and an equivalent norm $|||\cdot|||$ on X, such that if $\phi \in W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ (resp., $\phi \in W^{\mathrm{cs}}_{\mathrm{loc}}(0)$) and $\|\phi\| < \delta$, then*

$$|||\phi - P^{\mathrm{c}}\phi - h^{\mathrm{c}}(P^{\mathrm{c}}\phi)||| \leq \beta |||\mathcal{F}(\phi) - P^{\mathrm{c}}\mathcal{F}(\phi) - h^{\mathrm{c}}(P^{\mathrm{c}}\mathcal{F}(\phi))|||$$
$$(\textit{resp., } |||\mathcal{F}(\phi) - P^{\mathrm{c}}\mathcal{F}(\phi) - h^{\mathrm{c}}(P^{\mathrm{c}}\mathcal{F}(\phi))||| \leq \beta |||\phi - P^{\mathrm{c}}\phi - h^{\mathrm{c}}(P^{\mathrm{c}}\phi)|||).$$

Proof. Indeed, choosing $\delta > 0$ small enough, we can guarantee that $\mathcal{F}^{-1}(\phi) \in W^{\mathrm{cu}}_{\mathrm{loc}}(0)$. Let $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ be the graph of $h^{\mathrm{cu}} : X^{\mathrm{cu}} \to X^{\mathrm{s}}$, then $\mathcal{F}$ induces a map on X^{cu} via $\widetilde{\mathcal{F}}^{\mathrm{cu}}(\phi^{\mathrm{cu}}) := P^{\mathrm{cu}}\mathcal{F}(\phi^{\mathrm{cu}} + h^{\mathrm{cu}}(\phi^{\mathrm{cu}}))$. Moreover, since $\mathcal{F}$ is injective, the inverse is a well-defined C^2-map. Thus, $(\widetilde{\mathcal{F}}^{\mathrm{cu}})^{-1}$ fulfills the assumptions of [MM76, Theorem 2.1 (b)], whose proof consists in showing that $\psi := P^{\mathrm{c}}\widetilde{\mathcal{F}}^{-1}(\phi) + h^{\mathrm{c}}(P^{\mathrm{c}}(\widetilde{\mathcal{F}}^{\mathrm{cu}})^{-1}(\phi))$ satisfies

$$\begin{aligned} \left|\left|\left|\mathcal{F}^{-1}(\phi) - \psi\right|\right|\right| &= \left|\left|\left|(\widetilde{\mathcal{F}}^{\mathrm{cu}})^{-1}(\phi) - \psi\right|\right|\right| \\ &\leq \beta |||\phi - P^{\mathrm{c}}\phi - h^{\mathrm{c}}(P^{\mathrm{c}}\phi)|||. \end{aligned}$$

Here $|||\cdot|||$ is an equivalent norm such that the inverse of the unstable linear part L_{u} is a uniform contraction. The proof of the first case is complete by lifting the result to $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ and choosing a new $\beta < 1$, if necessary.

The result for $\phi \in W^{\mathrm{cs}}_{\mathrm{loc}}(0)$ follows along the same lines, by considering instead $\widetilde{\mathcal{F}}^{\mathrm{cs}}(\phi^{\mathrm{cs}}) := P^{\mathrm{cs}}\mathcal{F}(\phi^{\mathrm{cs}} + h^{\mathrm{cs}}(\phi^{\mathrm{cs}}))$, the C^2-map that $\mathcal{F}$ induces on X^{cs}. ☐

A consequence of this is that, provided that 0 possesses a $\mathcal{F}$-invariant neighborhood within $W^{\mathrm{c}}_{\mathrm{loc}}(0)$, we can construct invariant neighborhoods within $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ and $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$.

Corollary C.9. *In the setting of Lemma C.8 assume that there exists* $U^{\mathrm{c}} \subset W^{\mathrm{c}}_{\mathrm{loc}}(0) \cap U_\delta$ *such that* U^{c} *is open in the relative topology and* $\mathrm{clos}(\mathcal{F}^{-1}(U^{\mathrm{c}})) \subset \mathrm{int}(U^{\mathrm{c}})$ *(resp.,* $\mathrm{clos}(\mathcal{F}(U^{\mathrm{c}})) \subset \mathrm{int}(U^{\mathrm{c}})$*). Then there exists* $U^{\mathrm{cu}} \subset W^{\mathrm{cu}}_{\mathrm{loc}}(0)$ *(resp.,* $U^{\mathrm{cs}} \subset W^{\mathrm{cs}}_{\mathrm{loc}}(0)$*) open in the relative topology such that* $U^{\mathrm{cu}} \subset \mathcal{F}(U^{\mathrm{cu}})$ *(resp.,* $\mathcal{F}(U^{\mathrm{cs}}) \subset U^{\mathrm{cs}}$*).*

Proof. Let us choose a $\mathcal{F}^{-1}$-invariant neighborhood $0 \in U^{\mathrm{c}} \subset W^{\mathrm{c}}_{\mathrm{loc}}(0)$ as above. We claim that for $\varepsilon > 0$ sufficiently small, the set

$$U^{\mathrm{cu}}_\varepsilon := \{\phi \in W^{\mathrm{cu}}_{\mathrm{loc}}(0) : \|\phi - P^{\mathrm{c}}\phi - h^{\mathrm{c}}(P^{\mathrm{c}}\phi)\| < \varepsilon\}$$

is $\mathcal{F}^{-1}$-invariant. We proceed by contradiction, suppose that this is not the case. Thus we can always find a sequence of points $\phi^{(n)} \notin U^{\mathrm{cu}}_{1/n}$ and such that $\mathcal{F}(\phi^{(n)}) \in U^{\mathrm{cu}}_{1/n}$. By the attractivity of $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ in Lemma C.8, this is only possible if $P^{\mathrm{c}}\phi^{(n)} + h^{\mathrm{c}}(P^{\mathrm{c}}\phi^{(n)}) \notin U^{\mathrm{c}}$. By construction, we have that $\mathcal{F}(\phi^{(n)}) \to \mathrm{clos}(U^{\mathrm{c}})$ and, recalling that $\mathrm{clos}(\mathcal{F}^{-1}(U^{\mathrm{c}})) \subset \mathrm{int}(U^{\mathrm{c}})$, there exists $\varepsilon > 0$ such that $\mathrm{dist}(\phi^{(n)}, \mathcal{F}^{-1}(U^{\mathrm{c}})) > \varepsilon$. Thus, we reached a contradiction to the continuity of $\mathcal{F}$ and completed the proof for $W^{\mathrm{cu}}_{\mathrm{loc}}(0)$. The result for $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$ follows analogously by exchanging directions of iteration. □

Finally, if the trajectories within $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ drift away from the origin under the iteration of $\mathcal{F}$, the same will hold for the center-unstable manifold. Likewise, if the $\mathcal{F}$-iterates in $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ decay to 0, the same holds on the center-stable manifold.

Corollary C.10. *Assume that for all* $\delta > 0$ *we can find* U^{c} *and* U^{cu} *(resp.,* U^{cs}*) as in Corollary C.9. If* $\lim_{n\to-\infty} \mathcal{F}^n(\phi) = 0$ *for all* $\phi \in U^{\mathrm{c}}$ *with (resp.,* $\lim_{n\to\infty} \mathcal{F}^n(\phi) = 0$ *for all* $\phi \in U^{\mathrm{c}}$*), then* $\lim_{n\to-\infty} \mathcal{F}^n(\psi) = 0$ *for all* $\psi \in U^{\mathrm{cu}}$ *(resp.,* $\lim_{n\to\infty} \mathcal{F}^n(\psi) = 0$ *for all* $\psi \in U^{\mathrm{cs}}$*).*

Proof. By Lemma C.8, every sequence $\{\mathcal{F}^n(\psi)\}_{n\le 0}$ within the backward invariant set U^{cu} approaches U^{c} at an exponential rate. We claim that, for $n \le 0$ sufficiently small, the iteration becomes indistinguishable from a trajectory on $W^{\mathrm{c}}_{\mathrm{loc}}(0)$, which converges to 0. Indeed, suppose otherwise, then we can find $\psi \in U^{\mathrm{cu}}$ and a subsequence $n_k \to -\infty$ such that $\mathcal{F}^{n_k}(\psi) \to \psi^* \in U^{\mathrm{c}} \setminus \{0\}$. However, this implies the existence of a compact α-limit set

$\alpha(\psi) \subset U^{\mathrm{c}} \setminus \{0\}$ containing $\psi^* \neq 0$ and such that $\mathcal{F}(\alpha(\psi)) = \alpha(\psi)$, which is a clear contradiction to $\lim_{n\to-\infty} \mathcal{F}^n(\phi) = 0$ for all $\phi \in U^{\mathrm{c}}$. The claim for $W^{\mathrm{cs}}_{\mathrm{loc}}(0)$ is analogous, iterating in the opposite direction. $\square$

Corollary C.10 allows us to produce locally unique center manifolds via the methods used in Theorem C.6. More precisely, we have the following.

Theorem C.11. *In the setting of Corollary C.9, if* $\lim_{n\to-\infty} \mathcal{F}^n(\phi) = 0$ *for all* $\phi \in U^{\mathrm{c}}$, *then there exists an open subset* $U \subset U_\delta$ *such that*

$$W^{\mathrm{cu}}_{\mathrm{loc}}(0) \cap U = \left\{ \phi \in U : \begin{array}{l} \{\mathcal{F}^n(\phi)\}_{n\leq 0} \subset U \text{ and} \\ \text{such that } \lim\limits_{n\to-\infty} \mathcal{F}^n(\phi) = 0 \end{array} \right\}.$$

Moreover, for all $\phi \in W^{\mathrm{cu}}_{\mathrm{loc}}(0) \cap U$ *we have that*

$$T_\phi W^{\mathrm{cu}}_{\mathrm{loc}}(0) = \left\{ \psi \in X : \lim_{n\to-\infty} \frac{D\mathcal{F}^n(\phi)\psi}{\kappa^n} = 0 \text{ for all } \kappa < 1 \right\}.$$

Analogously, if $\lim_{n\to-\infty} \mathcal{F}^n(\phi) = 0$ *for all* $\phi \in U^{\mathrm{c}}$, *then we can find an open subset* $U \subset U_\delta$ *such that*

$$W^{\mathrm{cs}}_{\mathrm{loc}}(0) \cap U = \left\{ \phi \in U : \begin{array}{l} \{\mathcal{F}^n(\phi)\}_{n\geq 0} \subset U \text{ and} \\ \text{such that } \lim\limits_{n\to\infty} \mathcal{F}^n(\phi) = 0 \end{array} \right\}.$$

For all $\phi \in W^{\mathrm{cs}}_{\mathrm{loc}}(0) \cap U$ *we have*

$$T_\phi W^{\mathrm{cs}}_{\mathrm{loc}}(0) = \left\{ \psi \in X : \lim_{n\to\infty} \frac{D\mathcal{F}^n(\phi)\psi}{\kappa^n} = 0 \text{ for all } \kappa > 1 \right\}.$$

Proof. The proof is analogous to that of Theorem C.6. Thus it follows from Theorem C.3 and the existence of invariant neighborhood in Corollary C.10. $\square$

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Index of definitions

Deutsche Zusammenfassung

In dieser Arbeit wird eine neue Methode zur Beschreibung der qualitativen Dynamik der sogenannten Hopf–Smale-Attraktoren in skalaren retardierten Differentialgleichung mit symmetrischer negativer verzögerter Rückkopplung entwickelt.

Die Dynamik von Hopf–Smale-Attraktoren ist robust gegenüber regelmäßigen Störungen. Qualitativ besteht der Attraktor aus einem Gleichgewicht, periodischen Orbits und Orbits zwischen diesen. Wir beschreiben den Mechanismus, der die periodischen Orbits erzeugt und zeigen, wie dieser neue verbindende Orbits über Sequenzen von Hopf-Bifurkationen erzeugt. Als Ergebnis erhalten wir eine Aufzählung aller Phasendiagramme, d.h. der gerichteten Graphen, die die Gleichgewichts- und periodischen Bahnen als Knoten und die Verbindungen als Kanten kodieren.

Insbesondere haben wir einen Prototyp, den sogenannten enharmonischen Oszillator, gefunden, der alle Hopf–Smale-Phasendiagramme verwirklicht. Neben der Beschreibung der Hopf–Smale-Attraktoren gibt unsere Methode auch Aufschluss über den Entstehungsprozess bestimmter globaler Attraktoren mit positiver verzögerter Rückkopplung.

Selbstständigkeitserklärung

Hiermit bestätige ich, López Nieto, Alejandro, dass ich die vorliegende Dissertation mit dem Thema

Enharmonic motion:
Towards the global dynamics of negative delayed feedback

selbstständig angefertigt und nur die genannten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

Berlin, den 23. März 2023.

Curriculum Vitae

Der Lebenslauf ist in der Online-Version aus Gründen des Datenschutzes nicht erhalten.