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# The Daugavet Property and Translation-Invariant Subspaces

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# Contents

# Introduction

A Banach space $X$ has the *Daugavet property* if every rank-one operator $T : X \to X$ fulfills the *Daugavet equation*

$$\|\mathrm{Id} + T\| = 1 + \|T\| .$$

V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner introduced this notion, inspired by the result that all weakly compact operators on a Banach space $X$ fulfill the Daugavet equation if all operators on $X$ of rank one do. Classical examples of spaces with the Daugavet property are $C(K)$-spaces if $K$ has no isolated points, and $L^1(\Omega, \Sigma, \mu)$-spaces if $(\Omega, \Sigma, \mu)$ is a non-atomic probability space. The Daugavet property depends crucially on the norm of the space but it has isomorphic consequences too. If $X$ has the Daugavet property, then $X$ fails the Radon-Nikodým property, contains a copy of $\ell^1$, and does not embed into a space with an unconditional basis. So Banach spaces with the Daugavet property can be considered as "big" and it is an interesting question which closed subspaces and which quotients of a space with the Daugavet property inherit this property. But one can even ask a little bit more. Which closed subspaces $Y$ are *rich* subspaces of $X$, i.e., satisfy that every closed subspace $Z$ of $X$ with $Y \subset Z$ has the Daugavet property, and which closed subspaces $Y$ are *poor* subspaces of $X$, i.e., satisfy that $X/Z$ has the Daugavet property for every closed subspace $Z \subset Y$?

If $G$ is an infinite compact abelian group with Borel $\sigma$-algebra $\mathscr{B}(G)$ and Haar measure $m$, then $G$ has no isolated points and $(G, \mathscr{B}(G), m)$ is non-atomic. Hence $C(G)$ and $L^1(G, \mathscr{B}(G), m)$ have the Daugavet property. Since $G$ is a group, we can translate every function that is defined on $G$ and have a special class of subspaces of $C(G)$ or $L^1(G)$, the *translation-invariant* ones. The purpose of this work is to study which closed translation-invariant subspaces have the Daugavet property, are rich or poor subspaces, and which quotients with respect to a closed translation-invariant subspace inherit the Daugavet property.

In Chapter I, we present the parts of the theory of the Daugavet property that will be needed later on. This includes the characterization of the Daugavet property via slices of the unit sphere and the concept of *narrow* operators which is necessary to characterize rich subspaces and poor subspaces. Since we want to study subspaces of $C(K)$- or $L^1(\Omega, \Sigma, \mu)$-spaces, we focus our attention on narrow operators on these spaces. The only new result builds a link to a property that was considered by G. Godefroy, N. J. Kalton, and D. Li.

**Proposition I.4.10** *Let $(\Omega, \Sigma, \mu)$ be a non-atomic probability space. A closed subspace $X$ of $L^1(\Omega)$ is poor if for no $A \in \Sigma$ with $\mu(A) > 0$ the operator $f \mapsto \chi_A f$ maps the space $X$ onto $\{f \in L^1(\Omega) : \chi_A f = f\}$.*

We conclude the chapter by presenting a weaker version of the Daugavet property, the so-called *almost Daugavet property*, which can be characterized for separable spaces $X$ via the *thickness* $T(X)$ of $X$.

In Chapter II, we deal with the question which closed subspaces of a space with the almost Daugavet property $X$ inherit this property. We first consider closed subspaces $Y$ of $X$ such that the quotient space $X/Y$ is "small" and obtain the following results.

**Theorem II.1.3** *Let $X$ be a Banach space with $T(X) = 2$. If $Y$ is a closed subspace of $X$ such that the quotient space $X/Y$ contains no copy of $\ell^1$, then $T(Y) = 2$.*

**Corollary II.1.4** *Let $X$ be a separable Banach space with the almost Daugavet property. If $Y$ is a closed subspace of $X$ such that the quotient space $X/Y$ contains no copy of $\ell^1$, then $Y$ has the almost Daugavet property as well.*

After that, we turn our attention to a special class of Banach spaces. A Banach space $X$ is said to be *$L$-embedded* if there exists a closed subspace $X_s$ of the bidual of $X$ such that $X^{**} = X \oplus_1 X_s$. Using the principle of local reflexivity, it is easy to check that every non-reflexive $L$-embedded space has thickness two. Since $L$-embedded spaces are weakly sequentially complete, every non-reflexive closed subspace of an $L$-embedded space contains a sequence $(e_n)_{n\in\mathbb{N}}$ which is equivalent to the canonical basis of $\ell^1$. Using the $L$-decomposition of $X^{**}$, we can construct an element in $X_s$ which is "close" to a weak* accumulation point of $(e_n)_{n\in\mathbb{N}}$. This allows us to prove the following.

**Theorem II.2.12** *Let $X$ be an $L$-embedded space and let $Y$ be a closed subspace of $X$ which is not reflexive. Then $T(Y) = 2$.*

**Corollary II.2.13** *Let $X$ be an $L$-embedded space and let $Y$ be a separable, closed subspace of $X$. If $Y$ is not reflexive, then $Y$ has the almost Daugavet property.*

In Chapter III, we give an overview of the basic concepts of abstract harmonic analysis that are needed to deal with translation-invariant subspaces. We define for example the *convolution*, the *dual group* $\Gamma$, the space of *trigonometric polynomials*, the *Fourier transform*, and the *Fourier-Stieltjes transform*. We then show that for every closed translation-invariant subspace $X$ of $C(G)$ or $L^1(G)$ there is a subset $\Lambda$ of the dual group of $G$ such that $X$ consists exactly of those elements of $C(G)$ or $L^1(G)$ whose spectrum is contained in $\Lambda$. We will denote subspaces of this form by $C_\Lambda(G)$ or $L^1_\Lambda(G)$. Afterwards, we present various classes of subsets of a dual group that will be important later on. It is furthermore possible to transfer arguments that Y. Meyer used during his studies of *Riesz sets* to the class of *semi-Riesz sets*, which plays a crucial role in the study of rich subspaces of $C(G)$.

**Proposition III.4.32** *Let $\tau$ be the topology of pointwise convergence on $\Gamma$. If for every $\gamma \in \Gamma$ there exists a $\tau$-open neighborhood $V$ of $\gamma$ such that $\Lambda \cap V$ is a semi-Riesz set, then $\Lambda$ is a semi-Riesz set.*

**Proposition III.4.34** *Let $\tau$ be the topology of pointwise convergence on $\Gamma$. If $\Lambda_1$ is a semi-Riesz set and $\Lambda_2$ is a $\tau$-closed semi-Riesz set, then $\Lambda_1 \cup \Lambda_2$ is a semi-Riesz set.*

In Chapter IV, we study which translation-invariant subspaces or which quotients with respect to a translation-invariant subspace inherit the Daugavet property. D. Werner proved that $C_\Lambda(G)$ has the Daugavet property if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set. This can be extended, because in order to prove that a closed translation-invariant subspace $Y$ of $C(G)$ or $L^1(G)$ is rich, we do not have to consider all closed subspaces of $C(G)$ or $L^1(G)$ containing $Y$ but only the translation-invariant ones.

**Corollary IV.2.4** *If $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set, then $C_\Lambda(G)$ is a rich subspace of $C(G)$.*

The converse implication is valid too.

**Theorem IV.2.6** *If $C_\Lambda(G)$ is a rich subspace of $C(G)$, then $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set.*

Applying the result from V. M. Kadets, R. V. Shvidkoy, and D. Werner that a closed subspace $Y$ of a Banach space $X$ with the Daugavet property is rich if $(X/Y)^*$ has the Radon-Nikodým property, we deduce that $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$ if $\Gamma \setminus \Lambda^{-1}$ is a *Rosenthal set*. We can furthermore prove a necessary but not sufficient condition. If $\mu \in M_{\Gamma\setminus\Lambda^{-1}}(G)$ is a non-diffuse measure, then there exists a Borel set $E$ with $m(E) > 0$ such that the restriction of the convolution operator $f \mapsto \mu * f$ to the subspace $\{f \in L^1(G) : \chi_E f = f\}$ is an isomorphism onto its image. As $L^1_\Lambda(G)$ is contained in the kernel of this convolution operator, it cannot be a rich subspace of $L^1(G)$. So we get the following result.

**Theorem IV.2.16** *If $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$, then $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set.*

After that, we turn our attention to the product of two infinite compact abelian groups $G_1$ and $G_2$ and the link between rich subspaces of $C(G_1 \oplus G_2)$ (or $L^1(G_1 \oplus G_2)$) and rich subspaces of $C(G_1)$ and $C(G_2)$ (or $L^1(G_1)$ and $L^1(G_2)$). Using these results, we can construct peculiar examples of translation-invariant subspaces of $C(G)$ and $L^1(G)$ that have the Daugavet property but are not rich. Furthermore, we find a set $\Lambda$ such that $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$ but $\Gamma \setminus \Lambda^{-1}$ is not a Rosenthal set.

Considering quotients with respect to translation-invariant subspaces, we find an interesting relation between rich subspaces of $C(G)$ and quotients of $L^1(G)$ and vice versa. The key ingredient is the observation that the unit ball of $C_\Lambda(G)$ is weak* dense in the unit ball of $L^\infty_\Lambda(G)$ and that the unit ball of $L^1_\Lambda(G)$ is weak* dense in the unit ball of $M_\Lambda(G)$.

**Theorem IV.4.4** *If $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$, then $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property.*

**Theorem IV.4.2** *If $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$, then $C(G)/C_\Lambda(G)$ has the Daugavet property.*

Applying results by G. Godefroy, N. J. Kalton, and D. Li, we can deduce for a metrizable group $G$, that $L^1_\Lambda(G)$ is a poor subspace of $L^1(G)$ if $\Lambda$ is a nicely placed semi-Riesz set. This can partially be extended to the general case.

**Theorem IV.5.3** *If $\Lambda$ is a nicely placed Riesz set, then $L^1_\Lambda(G)$ is a poor subspace of $L^1(G)$.*

In Chapter V, we study which translation-invariant subspaces of $C(G)$ or $L^1(G)$ have the almost Daugavet property. Applying our results about subspaces of $L$-embedded spaces, we get the following corollaries.

**Corollary V.1.1** *The space $L^1_\Lambda(G)$ has thickness two if and only if $\Lambda$ is not a $\Lambda(1)$ set.*

**Corollary V.1.2** *Let $G$ be a metrizable compact abelian group. The space $L^1_\Lambda(G)$ has the almost Daugavet property if and only if $\Lambda$ is not a $\Lambda(1)$ set.*

The translation-invariant subspaces of $C(G)$ that have thickness two can also be fully characterized. If we consider the circle group $\mathbb{T}$ and have an infinite set $\Lambda$ of integers, we can find functions in $C_\Lambda(\mathbb{T})$ that oscillate arbitrarily fast. So $C_\Lambda(\mathbb{T})$ has thickness two. If $G$ is the direct product $\prod_{n=1}^\infty G_n$ of infinitely many compact abelian groups, we observe that the evaluation of a trigonometric polynomial on $G$ just depends on finitely many coordinates. Using this, we can show that $C_\Lambda(G)$ has thickness two if $G$ is a direct product of finite groups and $\Lambda$ is an infinite set. The general case can be treated by applying the result that in every abelian group there is an exhausting sequence of subgroups that are direct products of cyclic groups.

**Theorem V.2.8** *If $\Lambda$ is an infinite subset of $\Gamma$, then $T(C_\Lambda(G)) = 2$.*

**Corollary V.2.9** *Let $G$ be a metrizable compact abelian group. The space $C_\Lambda(G)$ has the almost Daugavet property if and only if $\Lambda$ contains infinitely many elements.*

In Chapter VI, we conclude our work by stating some problems that could not be solved during our studies.

Various results of this work are published in the following articles:

S. Lücking, *Subspaces of almost Daugavet spaces*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2777–2782.

S. Lücking, *The almost Daugavet property and translation-invariant subspaces*, Colloq. Math. **134** (2014), no. 2, 151–163.

S. Lücking, *The Daugavet property and translation-invariant subspaces*, Stud. Math. **221** (2014), no. 3, 269–291.

The first one contains the result that a closed subspace $Y$ of a separable Banach space $X$ with the almost Daugavet property inherits this property if the quotient space $X/Y$ does not contain a copy of $\ell^1$. The second one contains the results concerning the almost Daugavet property of subspaces of $L$-embedded spaces and of translation-invariant subspaces. The third one contains the results of Chapter IV.

Finally, I want to express my sincere gratitude to my supervisor Dirk Werner who got me enthusiastic about Banach space theory in the first place and supported me with helpful hints and advice during my studies. Furthermore, I am grateful to Miguel Martín Suárez and the Departamento de Análisis Matemático de la Universidad de Granada for their hospitality during my stay in Granada. I also wish to thank the Land Berlin for granting an Elsa-Neumann-Stipendium.

# Notation

We will only work with complex vector spaces. For $z \in \mathbb{C}$, we denote by $\operatorname{Re} z$ its real part, by $\operatorname{Im} z$ its imaginary part, by $\overline{z}$ its complex conjugate, and by $|z|$ its absolute value. The multiplicative group of all complex numbers of absolute value one, the so-called circle group, is denoted by $\mathbb{T}$.

Let $X$ be a Banach space. If $A$ is a subset of $X$, we write $\operatorname{lin} A$ for its linear span, $\operatorname{conv} A$ for its convex hull, and $\operatorname{diam} A$ for its diameter.

In the case that $M_1$ is a set, $f : M_1 \to \mathbb{C}$ is a complex-valued function, and $x$ is an element of $X$, we write $f \otimes x$ for the map $t \mapsto f(t)x$ from $M_1$ to $X$. If $x$ is also a complex-valued function on a set $M_2$, we sometimes identify $f \otimes x$ with the map $(s,t) \mapsto f(s)x(t)$ from $M_1 \times M_2$ to $\mathbb{C}$.

We denote the unit ball of $X$ by $B_X$ and its unit sphere by $S_X$. If $Y$ is another Banach space, $L(X,Y)$ stands for the space of all bounded, linear operators from $X$ into $Y$ which is equipped with the usual norm

$$\|T\| = \sup\{\|T(x)\| : x \in B_X\}.$$

A special case is the dual space of $X$ that we denote by $X^*$. If $T \in L(X,Y)$, we write $\ker(T)$ for its kernel, $\operatorname{ran}(T)$ for its range, and $T^*$ for its adjoint operator.

If $X$ and $Y$ are Banach spaces, we denote by $X \oplus_1 Y$ the direct sum of $X$ and $Y$ equipped with the norm

$$\|(x,y)\| = \|x\| + \|y\| \quad (x \in X, y \in Y),$$

and by $X \oplus_\infty Y$ the direct sum of $X$ and $Y$ equipped with the norm

$$\|(x,y)\| = \max\{\|x\|, \|y\|\} \quad (x \in X, y \in Y).$$

We write $c_0$ for the space of complex sequences that converge to zero, $c_{00}$ for the space of complex sequences with finite support, $\ell^1$ for the space of absolutely summable complex sequences equipped with the norm

$$\|(x_n)_{n\in\mathbb{N}}\|_1 = \sum_{n=1}^{\infty} |x_n|,$$

and $\ell^\infty$ for the space of bounded complex sequences equipped with the norm

$$\|(x_n)_{n\in\mathbb{N}}\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}.$$

For an arbitrary index set $J$, we denote by $\ell^1(J)$ the space

$$\left\{ f : I \to \mathbb{C} : \sum_{j \in J} |f(j)| < \infty \right\}$$

equipped with the norm

$$\|f\|_1 = \sum_{j \in J} |f(j)| \,.$$

Let $K$ be a locally compact space. A continuous function $f : K \to \mathbb{C}$ vanishes at infinity if for every $\varepsilon > 0$ there exists a compact set $A \subset K$ such that $|f(x)| < \varepsilon$ for all $x \in K \setminus A$. We denote by $C_0(K)$ the space of continuous, complex-valued functions on $K$ that vanish at infinity. It is equipped with the uniform norm $\|\cdot\|_\infty$ and its dual space can be identified with $M(K)$, the space of all regular Borel measures on $K$ of bounded variation. For $f \in C_0(K)$, we define its support by $\operatorname{supp}(f) = \overline{\{f \neq 0\}}$. If $K$ is a compact space and $X$ a Banach space, then $C(K)$ stands for the space of continuous, complex-valued functions on $K$ and $C(K, X)$ for the space of continuous, $X$-valued functions on $K$.

Let $(\Omega, \Sigma, \mu)$ be a probability space. For $0 < p < \infty$, we denote by $L^p(\Omega, \Sigma, \mu)$ (or $L^p(\Omega)$ for short) the Lebesgue space of measurable, complex-valued functions $f$ such that $|f|^p$ is integrable and by $L^\infty(\Omega, \Sigma, \mu)$ (or $L^\infty(\Omega)$ for short) the Lebesgue space of measurable, essentially bounded, complex-valued functions. As usual, we identify functions that coincide almost everywhere. The map

$$\|f\|_p = \left( \int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}$$

is a quasi-norm for $p < 1$ and a norm for $p \geq 1$. The space $L^\infty(\Omega)$ is equipped with the essential supremum norm $\|\cdot\|_\infty$. The corresponding spaces of Bochner-measurable, $X$-valued functions are denoted by $L^p(\Omega, X)$ and $L^\infty(\Omega, X)$.

Let $J$ be a directed set, let $\mathscr{U}$ a ultrafilter on $J$, and let $X$ be a Hausdorff space. We say that a net $(x_j)_{j \in J}$ in $X$ converges along $\mathscr{U}$ to $x$ and write $\lim_{j,\mathscr{U}} x_j = x$ if for every neighborhood $V$ of $x$ the set $\{j \in J : x_j \in V\}$ belongs to $\mathscr{U}$. In the case that $\mathscr{U}$ contains the filter base

$$\{\{j \geq j_0\} : j_0 \in J\}$$

and $(x_j)_{j \in J}$ converges along $\mathscr{U}$ to $x$, we can find a subnet of $(x_j)_{j \in J}$ that converges in the usual sense to $x$. Especially, if $(x_j)_{j \in J}$ converges in the usual sense to $x$, then $(x_j)_{j \in J}$ converges along $\mathscr{U}$ to $x$.

# I The Daugavet Property

## I.1 The Daugavet equation

I. K. Daugavet [10] proved in 1963 that all compact operators $T$ on $C[0,1]$ fulfill the norm identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|\,,$$

which has become known as the *Daugavet equation*. C. Foiaş and I. Singer [15] extended this result to all weakly compact operators on $C[0,1]$ and A. Pełczyński [15, p. 446] observed that their argument can also be used for weakly compact operators on $C(K)$ provided that $K$ is a compact space without isolated points. Shortly afterwards, G. Ya. Lozanovskiĭ [43] showed that the Daugavet equation holds for all compact operators on $L^1[0,1]$ and J. R. Holub [29] extended this result to all weakly compact operators on $L^1(\Omega, \Sigma, \mu)$ where $\mu$ is a $\sigma$-finite non-atomic measure.

Other classes of spaces on which all weakly compact operators fulfill the Daugavet equation were constructed by Yu. A. Abramovich [1], who considered spaces of the form $L^\infty(\Omega, \Sigma, \mu) \oplus_1 L^\infty(\Omega, \Sigma, \nu)$ and $L^1(\Omega, \Sigma, \mu) \oplus_\infty L^1(\Omega, \Sigma, \nu)$ where $\mu$ and $\nu$ are non-atomic probability measures. This approach was extended by P. Wojtaszczyk [64], who showed that if all weakly compact operators on $X_1$ and $X_2$ fulfill the Daugavet equation, then all weakly compact operators on $X_1 \oplus_1 X_2$ and $X_1 \oplus_\infty X_2$ fulfill the Daugavet equation as well.

## I.2 The Daugavet property

V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner [35] proved that the validity of the Daugavet equation for weakly compact operators already follows from the corresponding statement for operators of rank one. This led to the following definition.

**Definition I.2.1** Let $X$ be a Banach space. We call $X$ a *Daugavet space* or say that $X$ has the *Daugavet property* if every operator $T : X \to X$ of rank one satisfies the Daugavet equation.

If $X$ has the Daugavet property, then not only all weakly compact operators on $X$ satisfy the Daugavet equation but also all strong Radon-Nikodým operators [35, Theorem 2.3], meaning operators $T$ for which $\overline{T[B_X]}$ is a Radon-Nikodým set, and operators not fixing a copy of $\ell^1$ [56, Theorem 4].

**Examples I.2.2**

1. I. K. Daugavet's result shows that $C[0,1]$ is a Daugavet space [10].
2. Let $K$ be a compact space and let $E$ be a Banach space. The space $C(K,E)$ of all continuous, $E$-valued functions on $K$ has the Daugavet property if $K$ has no isolated points [30, Theorem 4.4] or if $E$ is a Daugavet space [46, Remark 6].
3. G. Ya. Lozanovskiĭ's result shows that $L^1[0,1]$ is a Daugavet space [43].
4. Let $(\Omega, \Sigma, \mu)$ be a probability space and let $E$ be a Banach space. The Bochner space $L^1(\Omega, E)$ has the Daugavet property if $\mu$ is a non-atomic measure [35, Example after Theorem 2.3] or if $E$ is a Daugavet space [46, Remark 9].
5. A *uniform algebra* $A$ on a compact space $K$ is a closed subalgebra of the space of continuous, complex-valued functions $C(K)$ that separates the points of $K$ and contains the constant functions. Its *Shilov boundary* is the smallest closed subset of $K$ on which every $f \in A$ attains its maximum. A uniform algebra $A$ has the Daugavet property if its Shilov boundary has no isolated points [64, Theorem 2]. The disk algebra $A(\mathbb{D})$ has therefore the Daugavet property because its Shilov boundary is $\{z \in \mathbb{C} : |z| = 1\}$ [17, Example V.1.4].

The Daugavet property can be characterized in terms of slices of the unit ball or the dual unit ball. If $X$ is a Banach space, we denote by

$$S(x^*, \varepsilon) = \{x \in B_X : \operatorname{Re} x^*(x) \geq 1 - \varepsilon\}$$

the slice of $B_X$ determined by $x^* \in S_{X^*}$ and $\varepsilon > 0$ and by

$$S(x, \varepsilon) = \{x^* \in B_{X^*} : \operatorname{Re} x^*(x) \geq 1 - \varepsilon\}$$

the weak* slice of $B_{X^*}$ determined by $x \in S_X \subset S_{X^{**}}$ and $\varepsilon > 0$.

**Lemma I.2.3** [2, Lemma 2.1] *Let $x$ and $y$ be elements of a normed space $X$. If $\|x+y\| = \|x\| + \|y\|$, then $\|\alpha x + \beta y\| = \alpha \|x\| + \beta \|y\|$ for all $\alpha, \beta \geq 0$.*

*Proof.* We may assume that $\alpha \geq \beta \geq 0$. Then

$$\begin{aligned} \|\alpha x + \beta y\| &= \|\alpha(x+y) - (\alpha - \beta)y\| \geq \alpha \|x+y\| - (\alpha - \beta)\|y\| \\ &= \alpha(\|x\| + \|y\|) - (\alpha - \beta)\|y\| = \alpha \|x\| + \beta \|y\| \end{aligned}$$

and the desired equality follows. □

**Lemma I.2.4** [35, Lemma 2.2] *Let $X$ be a Banach space. The following assertions are equivalent:*

(i) *$X$ has the Daugavet property.*

(ii) *For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$ there exists $y \in S(x^*, \varepsilon)$ such that $\|x+y\| \geq 2 - \varepsilon$.*

(iii) *For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$ there exists $y^* \in S(x, \varepsilon)$ such that $\|x^* + y^*\| \geq 2 - \varepsilon$.*

*Proof.* (i) $\Rightarrow$ (ii): Fix $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$. As the operator $x^* \otimes x$ satisfies the Daugavet equation, we can choose $y \in S_X$ with $\|y + x^*(y)x\| \geq 2 - \frac{\varepsilon}{2}$ and $x^*(y) \geq 0$. So $x^*(y) \geq 1 - \frac{\varepsilon}{2}$ and $y$ belongs to $S(x^*, \varepsilon)$. Furthermore,

$$\|x + y\| \geq \|y + x^*(y)x\| - |x^*(y) - 1| \, \|x\| \geq \left(2 - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} = 2 - \varepsilon.$$

(i) $\Rightarrow$ (iii): This implication works quite similarly. Fix $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$. As the operator $T = x^* \otimes x$ satisfies the Daugavet equation, its adjoint operator $T^*$ satisfies the Daugavet equation as well. Pick $y^* \in S_{X^*}$ with $\|y^* + T^*(y^*)\| \geq 2 - \frac{\varepsilon}{2}$ and $y^*(x) \geq 0$. Using the definition of $T^*$, we get $\|y^* + y^*(x)x^*\| \geq 2 - \frac{\varepsilon}{2}$. So $y^*(x) \geq 1 - \frac{\varepsilon}{2}$ and $y^*$ belongs to $S(x, \varepsilon)$. Furthermore,

$$\|x^* + y^*\| \geq \|y^* + y^*(x)x^*\| - |y^*(x) - 1| \, \|x^*\| \geq \left(2 - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} = 2 - \varepsilon.$$

(ii) $\Rightarrow$ (i): Let $T : X \to X$ be an operator of rank one and $\varepsilon > 0$. We may suppose by Lemma I.2.3 that $\|T\| = 1$. So there exist $x \in S_X$ and $x^* \in S_{X^*}$ with $T = x^* \otimes x$. By assumption, we can pick $y \in S(x^*, \varepsilon)$ with $\|x + y\| \geq 2 - \varepsilon$. Then

$$\|\mathrm{Id} + x^* \otimes x\| \geq \|y + x^*(y)x\| \geq \|x + y\| - |x^*(y) - 1| \, \|x\| \geq (2 - \varepsilon) - (\varepsilon + \sqrt{2\varepsilon}).$$

This implies that $\|\mathrm{Id} + T\| = 2$ because $\varepsilon$ was chosen arbitrarily.

(iii) $\Rightarrow$ (ii): This implication is again similar to the previous one. Let $T : X \to X$ be an operator of rank one and $\varepsilon > 0$. We may suppose by Lemma I.2.3 that $\|T\| = 1$. So there exist $x \in S_X$ and $x^* \in S_{X^*}$ with $T = x^* \otimes x$. By assumption, we can pick $y^* \in S(x, \varepsilon)$ with $\|x^* + y^*\| \geq 2 - \varepsilon$. Then

$$\begin{aligned} \|\mathrm{Id}_X + T\| = \|\mathrm{Id}_{X^*} + T^*\| &\geq \|y^* + T^*(y^*)\| = \|y^* + y^*(x)x^*\| \\ &\geq \|x^* + y^*\| - |y^*(x) - 1| \, \|x^*\| \geq (2 - \varepsilon) - (\varepsilon + \sqrt{2\varepsilon}). \end{aligned}$$

This implies that $\|\mathrm{Id} + T\| = 2$ because $\varepsilon$ was chosen arbitrarily. □

The Daugavet property depends crucially on the norm of the space and can easily be spoiled by arbitrarily small perturbations of the norm [64, Corollary 2]. But the Daugavet property does have isomorphic consequences too. If $X$ has the Daugavet property, then every slice of $B_X$ has diameter 2 as a consequence of Lemma I.2.4 and $X$ fails the Radon-Nikodým property [64, Corollary 1]. Furthermore, $X$ contains a copy of $\ell^1$ [35, Theorem 2.9], does not have an unconditional basis [30, Corollary 2.3], and does not even embed into a space with an unconditional basis [35, Corollary 2.7].

## I.3 Narrow operators and rich subspaces

Daugavet spaces are in a certain sense "big". It is therefore an interesting question which subspaces of a space $X$ with the Daugavet property inherit this property. One approach is to look at closed subspaces $Y$ such that the quotient space $X/Y$ is "small". For this purpose, V. M. Kadets and M. M. Popov [34] used the class of *narrow* operators that is a generalization of the class of compact operators and that was introduced by M. M. Popov and A. M. Plichko [50] for operators on $L^1[0,1]$. V. M. Kadets and M. M. Popov extended the notion of narrow operators to operators on $C[0,1]$ and used narrow operators to find closed subspaces of $C[0,1]$ or $L^1[0,1]$ that inherit the Daugavet property. This concept was transferred to Daugavet spaces by V. M. Kadets, R. V. Shvidkoy, and D. Werner [36]. Note that it is still unknown if the following definition of narrow operators on a Daugavet space coincides on $L^1[0,1]$ with the definition of narrow operators due to M. M. Popov and A. M. Plichko.

**Definition I.3.1** Let $X$ be a Banach space with the Daugavet property and let $E$ be an arbitrary Banach space. An operator $T \in L(X,E)$ is called *narrow* if for every two elements $x, y \in S_X$, for every $x^* \in X^*$, and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|T(y-z)\| + |x^*(y-z)| \leq \varepsilon$ and $\|x+z\| \geq 2-\varepsilon$. A closed subspace $Y$ of $X$ is said to be *rich* if the quotient map $\pi : X \to X/Y$ is narrow.

**Examples I.3.2** Let $X$ be a Banach space with the Daugavet property.

1. All strong Radon-Nikodým operators on $X$ are narrow [36, Theorem 3.13]. Consequently, $Y$ is a rich subspace of $X$ if the quotient space $X/Y$ has the Radon-Nikodým property.
2. All operators on $X$ which do not fix $\ell^1$ are narrow [36, Theorem 4.13]. This implies that $Y$ is a rich subspace of $X$ if the quotient space $X/Y$ contains no copy of $\ell^1$ or if $(X/Y)^*$ has the Radon-Nikodým property [36, Proposition 5.3].

**Lemma I.3.3** *Let $X$ be a Banach space with the Daugavet property, let $E$ be an arbitrary Banach space, and let $T : X \to E$ be a narrow operator. Then for every $x, y \in X$, for every $x^* \in X^*$, and for every $\varepsilon > 0$ there is an element $z \in \|y\| S_X$ with $\|T(y-z)\| + |x^*(y-z)| \leq \varepsilon$ and $\|x+z\| \geq \|x\| + \|z\| - \varepsilon$.*

*Proof.* If $y = 0$, set $z = 0$, and if $x = 0$, set $z = y$. Let us now assume that $x \neq 0$ and $y \neq 0$. Since $T$ is narrow, there exists $z_0 \in S_X$ with

$$\left\| T\left(\frac{y}{\|y\|} - z_0\right)\right\| + \left| x^*\left(\frac{y}{\|y\|} - z_0\right)\right| \leq \frac{\varepsilon}{\|y\|}$$

and

$$\left\| \frac{x}{\|x\|} + z_0 \right\| \geq 2 - \min\left\{ \frac{\varepsilon}{\|x\|}, \frac{\varepsilon}{\|y\|} \right\}.$$

Set $z = \|y\| z_0$. Then it is obvious that $\|T(y-z)\| + |x^*(y-z)| \leq \varepsilon$. Furthermore, if

$\|x\| \geq \|z\|$, then

$$\begin{aligned}\|x+z\| &= \left\| \|x\| \left( \frac{x}{\|x\|} + z_0 \right) - (\|x\| - \|z\|) z_0 \right\| \\ &\geq \left( 2 - \frac{\varepsilon}{\|x\|} \right) \|x\| - \|x\| + \|z\| \\ &= \|x\| + \|z\| - \varepsilon.\end{aligned}$$

The case $\|x\| < \|z\|$ can be treated similarly. □

A rich subspace inherits the Daugavet property. But even a little bit more is true.

**Proposition I.3.4** [36, Theorem 5.2] *Let $X$ be a Daugavet space and let $Y$ be a rich subspace of $X$. Then for every $x \in S_X$, $y^* \in S_{Y^*}$, and $\varepsilon > 0$ there exists $y \in S_Y$ with $\operatorname{Re} y^*(y) \geq 1 - \varepsilon$ and $\|x + y\| \geq 2 - \varepsilon$.*

*Proof.* Fix $x \in S_X$, $y^* \in S_{Y^*}$, and $\varepsilon > 0$. Choose $\delta > 0$ with $\frac{1-3\delta}{1+\delta} \geq 1 - \varepsilon$ and $z \in S_Y$ with $\operatorname{Re} y^*(z) \geq 1 - \delta$. Since $Y$ is a rich subspace of $X$, there exists $x_0 \in S_X$ with $d(x_0, Y) = d(z - x_0, Y) < \delta$, $|y^*(z - x_0)| \leq \delta$ and $\|x + x_0\| \geq 2 - \delta$. Fix $y_0 \in Y$ with $\|x_0 - y_0\| \leq \delta$ and set $y = \frac{y_0}{\|y_0\|}$. Then

$$\operatorname{Re} y^*(y_0) \geq \operatorname{Re} y^*(z) - |y^*(z - x_0)| - \|x_0 - y_0\| \geq 1 - 3\delta$$

and

$$\|x_0 - y\| \leq \|x_0 - y_0\| + \|y_0 - y\| \leq 2\delta.$$

So we get by our choice of $\delta$ that $\operatorname{Re} y^*(y) \geq 1 - \varepsilon$ and $\|x + y\| \geq 2 - \varepsilon$. □

If $Y$ is a rich subspace of $X$, then all closed subspaces of $X$ which contain $Y$ are rich subspaces as well and inherit the Daugavet property. This property actually characterizes rich subspaces.

**Theorem I.3.5** [36, Theorem 5.12] *Let $X$ be a Daugavet space and let $Y$ be a closed subspace of $X$. The following assertions are equivalent:*

(i) *$Y$ is a rich subspace of $X$.*

(ii) *For every $x, y \in X$, the linear span of $Y$, $x$ and $y$ has the Daugavet property.*

(iii) *If $Z$ is a closed subspace of $X$ with $Y \subset Z$, then $Z$ has the Daugavet property.*

The narrow operators on $C(K)$-spaces and $L^1(\Omega)$-spaces can be characterized in a more convenient form. In the sequel, $K$ denotes a compact space, $(\Omega, \Sigma, \mu)$ a non-atomic probability space, and $D$ and $E$ arbitrary Banach spaces.

**Definition I.3.6** An operator $T \in L(C(K), E)$ is called *$C$-narrow* if for every non-empty open set $O$ and every $\varepsilon > 0$ there is a function $f \in S_{C(K)}$ with $f|_{K \setminus O} = 0$ and $\|T(f)\| \leq \varepsilon$. A closed subspace $Y$ of $C(K)$ is said to be *$C$-rich* if the quotient map $\pi : C(K) \to C(K)/Y$ is $C$-narrow.

**Lemma I.3.7** *If $T \in L(C(K), E)$ is a $C$-narrow operator, then for every non-empty open set $O$ and every $\varepsilon > 0$ there is a* real-valued *and* non-negative *function $f \in S_{C(K)}$ with $f|_{K\setminus O} = 0$ and $\|T(f)\| \le \varepsilon$.*

*Proof.* This proof is essentially the same as the proof of [34, Lemma 1.4] but with some minor modifications to cover the complex case as well.

Fix a non-empty open set $O$ and $\varepsilon > 0$. Set $O_0 = O$ and pick $\delta \in (0,1)$ and $n \in \mathbb{N}$ with $\frac{1}{1-\delta}\left(\delta + \delta\,\|T\| + \frac{2}{n}\,\|T\|\right) < \varepsilon$. As $T$ is $C$-narrow, there is a function $f_1 \in S_{C(K)}$ with $f|_{K\setminus O_0} = 0$ and $\|T(f_1)\| \le \delta$. We may assume that

$$\max_{x\in O_0} |f_1(x)| = \max_{x\in O_0} \operatorname{Re} f_1(x) = 1$$

because otherwise we multiply by an adequate scalar of modulus one. Let $O_1 \subset O_0$ be the non-empty open set where $\operatorname{Re} f_1 > 1-\delta$ and $|\operatorname{Im} f_1| < \delta$. In an analogous manner, we choose for $k = 2, \dots, n$ functions $f_k \in S_{C(K)}$ and non-empty open sets $O_k$ so that $O_1 \supset O_2 \supset \cdots \supset O_n$, $O_k = \{\operatorname{Re} f_k > 1-\delta\} \cap \{|\operatorname{Im} f_k| < \delta\}$, $f_k|_{K\setminus O_{k-1}} = 0$, and $\|T(f_k)\| \le \delta$. If we set $g = \frac{1}{n}\sum_{k=1}^n f_k$, we see that $\|g\|_\infty \le 1$, $g|_{K\setminus O} = 0$, and $\|Tg\| \le \delta$. In addition, $\operatorname{Re} g(x) > 1-\delta$ and $|\operatorname{Im} g(x)| < \delta$ for $x \in O_n$. Hence $1-\delta \le \|\operatorname{Re} g\|_\infty \le 1$.

Let us prove that $\operatorname{Re} g \ge -\frac{1}{n}$ and $|\operatorname{Im} g| \le \delta + \frac{1}{n}$. On $O_n$, the required estimate is already known. Furthermore, $g$ vanishes outside $O$. Fix $k \in \{0, \dots, n-1\}$. For $x \in O_k \setminus O_{k+1}$ we have

$$\operatorname{Re} g(x) = \frac{1}{n}\left(\sum_{l=1}^{k} \operatorname{Re} f_l(x) + \operatorname{Re} f_{k+1}(x)\right) \ge \frac{1}{n}(k(1-\delta) - 1) \ge -\frac{1}{n}$$

and

$$|\operatorname{Im} g(x)| \le \frac{1}{n}\left(\sum_{l=1}^{k} |\operatorname{Im} f_l(x)| + |\operatorname{Im} f_{k+1}(x)|\right) \le \frac{1}{n}(k\delta + 1) \le \delta + \frac{1}{n}.$$

This proves the claim for all $x$ because $O \setminus O_n = \bigcup_{k=0}^{n-1} (O_k \setminus O_{k+1})$.

Now we can define the required function $f$ by

$$f = \frac{g^+}{\|g^+\|_\infty}, \quad \text{where } g^+ = \frac{\operatorname{Re} g + |\operatorname{Re} g|}{2}.$$

Since $\operatorname{Re} g(x) \ge 1-\delta$ for $x \in O_n$, we have $1-\delta \le \|g^+\|_\infty \le 1$. For the distance between $g$ and $g^+$ we get

$$\|g - g^+\|_\infty \le \|\operatorname{Re} g - g^+\|_\infty + \|\operatorname{Im} g\|_\infty \le \frac{1}{n} + \left(\delta + \frac{1}{n}\right) = \frac{2}{n} + \delta.$$

Therefore,

$$\begin{aligned}\|T(f)\| = \frac{\|T(g^+)\|}{\|g^+\|_\infty} &\le \frac{1}{1-\delta}\left(\|T(g)\| + \|T\|\,\|g - g^+\|_\infty\right) \\ &\le \frac{1}{1-\delta}\left(\delta + \|T\|\left(\frac{2}{n} + \delta\right)\right) \le \varepsilon.\end{aligned}$$

By construction, $f$ is real-valued and non-negative, $f \in S_{C(K)}$, and $f|_{K\setminus O} = 0$. □

**Proposition I.3.8** [36, Theorem 3.5] *Let $K$ have no isolated points. If $T \in L(C(K), E)$ is narrow, then it is $C$-narrow.*

*Proof.* Fix a non-empty open set $O$ and $\varepsilon > 0$. We have to find $f \in S_{C(K)}$ with $f|_{K\setminus O} = 0$ and $\|T(f)\| \le \varepsilon$.

We may assume that $O \neq K$. Then $A = K \setminus O$ is a non-empty closed set. Since $K$ is normal, we can choose an open neighborhood $V$ of $A$ and a non-empty open set $W$ with $V \cap W = \emptyset$. We will construct inductively four sequences of functions.

Fix $\delta \in (0, \frac{1}{2})$ with $\frac{1}{1-3\delta}(\delta + 2\,\|T\|\,\delta) \le \varepsilon$. Let $p_1 \in S_{C(K)}$ be a real-valued, non-negative function supported on $W$ and let $q_1 \in S_{C(K)}$ be supported on $V$. As $T$ is narrow, we can select $g_1 \in S_{C(K)}$ with $\|T(q_1 - g_1)\| \le \delta$ and $\|p_1 + g_1\|_\infty \ge 2 - \delta$. Set $h_1 = g_1 - q_1$. Let $p_2 \in S_{C(K)}$ be a real-valued, non-negative function supported on the set $\{x \in W : \operatorname{Re} h_1(x) > \sup_{y\in W} \operatorname{Re} h_1(y) - \delta\}$, i.e., on the subset of $W$ where $\operatorname{Re} h_1$ attains its supremum on $W$ up to $\delta$. Using Tietze's extension theorem [63, Theorem 15.8], we can pick $q_2 \in S_{C(K)}$ such that $q_2$ is supported on $V$ and $\|\,h_1|_A\|_\infty\, q_2$ coincides on $A$ with $h_1$. Now we use again that $T$ is narrow, select $g_2 \in S_{C(K)}$ with $\|T(q_2 - g_2)\| \le \delta$ and $\|p_2 + g_2\|_\infty \ge 2 - \delta$, and set $h_2 = (g_1 - q_1) + (g_2 - q_2)$. Going on like this, we get four sequences of functions satisfying the following properties (with $h_0 = 0$):

- Every $p_n$ is a real-valued, non-negative function that belongs to $S_{C(K)}$ and is supported on $\{x \in W : \operatorname{Re} h_{n-1}(x) > \sup_{y\in W} \operatorname{Re} h_{n-1}(y) - \delta\}$.
- Every $q_n$ belongs to $S_{C(K)}$, is supported on $V$, and $\|\,h_{n-1}|_A\|_\infty\, q_n$ coincides on $A$ with $h_{n-1}$.
- Every $g_n$ belongs to $S_{C(K)}$ and satisfies $\|T(q_n - g_n)\| \le \delta$ and $\|p_n + g_n\|_\infty \ge 2 - \delta$.
- $h_n = \sum_{k=1}^{n} (g_k - q_k)$.

We first claim that $\|\,h_n|_A\|_\infty \le 3$ for all $n \in \mathbb{N}$. This is certainly true for $n = 1$ because

$$\|h_1\|_\infty = \|g_1 - q_1\|_\infty \le 2.$$

Now induction yields for $x \in A$

$$\begin{aligned}
|h_{n+1}(x)| &= |h_n(x) + g_{n+1}(x) - q_{n+1}(x)| \\
&= |\|\,h_n|_A\|_\infty\, q_{n+1}(x) + g_{n+1}(x) - q_{n+1}(x)| \\
&\le |g_{n+1}(x)| + |\|\,h_n|_A\|_\infty - 1|\,|q_{n+1}(x)| \\
&\le 1 + 2 = 3.
\end{aligned}$$

Our second claim is that

$$\sup_{x\in W} \operatorname{Re} h_n(x) \ge n(1 - 3\delta) \quad (n \in \mathbb{N}).$$

Let us start with $n = 1$. We can pick $x_1 \in W$ with $|p_1(x_1) + g_1(x_1)| \ge 2 - \delta$ because $p_1$ is supported on $W$ and $\|p_1 + g_1\|_\infty \ge 2 - \delta$. Since $p_1$ is real-valued and non-negative, it is easy to check that $\operatorname{Re} g_1(x_1) \ge 1 - 2\delta$. The functions $p_1$ and $q_1$ are disjointly supported and therefore $q_1(x_1) = 0$. Consequently, $\operatorname{Re} h_1(x_1) = \operatorname{Re} g_1(x_1) \ge 1 - 2\delta$. To perform the induction step, we use the same argument to find a point $x_{n+1}$ in the support of $p_{n+1}$

at which $\operatorname{Re} g_{n+1}(x_{n+1}) \geq 1 - 2\delta$ and $q_{n+1}(x_{n+1}) = 0$. At this point $x_{n+1}$ the function $\operatorname{Re} h_n$ attains its supremum on $W$ up to $\delta$. So

$$\begin{aligned}\sup_{x\in W} \operatorname{Re} h_{n+1}(x) &\geq \operatorname{Re} h_{n+1}(x_{n+1}) = \operatorname{Re} h_n(x_{n+1}) + \operatorname{Re} g_{n+1}(x_{n+1}) \\ &\geq \left(\sup_{x\in W} \operatorname{Re} h_n(x) - \delta\right) + (1-2\delta) \\ &\geq n(1-3\delta) + 1 - 3\delta = (n+1)(1-3\delta).\end{aligned}$$

Our second claim implies that $\|h_n\|_\infty \geq n(1-3\delta)$ for all $n \in \mathbb{N}$. On the other hand, we have for all $n \in \mathbb{N}$

$$\|T(h_n)\| \leq \sum_{k=1}^{n} \|T(q_k - g_k)\| \leq n\delta.$$

Fix $n_0 \in \mathbb{N}$ with $\frac{3}{n_0(1-3\delta)} \leq \delta$ and set

$$g = \frac{g_{n_0}}{\|g_{n_0}\|_\infty}.$$

Then $\|g|_A\|_\infty \leq \delta$ and $\|T(g)\| \leq \frac{\delta}{1-3\delta}$. The set $B = \{|g| \geq 2\delta\}$ is non-empty and closed and $A \cap B = \emptyset$. Using Urysohn's lemma [63, Lemma 15.6], we can pick a continuous function $\varphi : K \to [0,1]$ with $\varphi|_A = 0$ and $\varphi|_B = 1$. We now set $f = g\varphi$. Then $f \in S_{C(K)}$, $f|_{K\setminus O} = f|_A = 0$, and $\|f - g\|_\infty \leq 2\delta$. Finally,

$$\|T(f)\| \leq \|T(g)\| + \|T\| \, \|f-g\|_\infty \leq \frac{\delta}{1-3\delta} + \|T\| \, 2\delta \leq \varepsilon. \qquad \square$$

**Proposition I.3.9** [4, Proposition 4.3; 36, Theorem 3.7] *Let $K$ have no isolated points and let $T : C(K,D) \to E$ be a bounded operator. Suppose that there exists for every non-empty open set $O$, every $d \in D$, and every $\varepsilon > 0$ a real-valued and non-negative function $f \in S_{C(K)}$ with $f|_{K\setminus O} = 0$ and $\|T(f \otimes d)\| \leq \varepsilon$. Then $T$ is narrow.*

*Proof.* Fix $f, g \in S_{C(K,D)}$, $x^* \in C(K,D)^*$, and $\varepsilon > 0$. We have to find $h \in S_{C(K,D)}$ with $\|T(g-h)\| + |x^*(g-h)| \leq \varepsilon$ and $\|f+h\|_\infty \geq 2 - \varepsilon$.

Let $\mu$ be the $D^*$-valued, regular Borel measure of bounded variation which generates $x^*$. Fix $\delta > 0$ with $(3 + \|\mu\| + 2\|T\|)\delta \leq \varepsilon$ and consider the non-empty open set $O = \{\|f\|_D > 1 - \delta\}$. The space $K$ has no isolated points, thus the open set $O$ contains infinitely many elements and we can select $x_0 \in O$ with $|\mu|(\{x_0\}) < \delta$. As $|\mu|$ is a regular measure and $f$ and $g$ are continuous, we can choose an open neighborhood $V$ of $x_0$ with $V \subset O$, $|\mu|(V) \leq \delta$ and

$$\|g(x) - g(x_0) + f(x) - f(x_0)\|_D \leq \delta \quad (x \in V).$$

By assumption, there exists a real-valued and non-negative $\varphi \in S_{C(K)}$ with $\varphi|_{K\setminus V} = 0$ and $\|T(\varphi \otimes g(x_0) - \varphi \otimes f(x_0))\| \leq \delta$. Set

$$h_0 = \varphi f + (1-\varphi) g \quad \text{and} \quad h = \frac{h_0}{\|h_0\|_\infty}.$$

Then $1-\delta \le \|h_0\|_\infty \le 1$ and

$$\|f+h\|_\infty \ge \|f+h_0\|_\infty - \delta \ge \sup_{x\in V} \|f(x)+h_0(x)\|_D - \delta \ge 2-3\delta \ge 2-\varepsilon.$$

Furthermore,

$$\begin{aligned}
|x^*(g-h)| &= \left|\int_K (g-h)\,d\mu\right| \le \left|\int_K (g-h_0)\,d\mu\right| + \|\mu\|\,\delta \\
&= \left|\int_K \varphi(g-f)\,d\mu\right| + \|\mu\|\,\delta \\
&\le |\mu|\,(V)\,\|\varphi\|_\infty\,\|g-f\|_\infty + \|\mu\|\,\delta \\
&\le (2+\|\mu\|)\delta
\end{aligned}$$

and

$$\begin{aligned}
\|T(g-h)\| &\le \|T(g-h_0)\| + \|T\|\,\delta = \|T(\varphi(g-f))\| + \|T\|\,\delta \\
&\le \|T\|\,\|\varphi(g-g(x_0)-f+f(x_0))\|_\infty + \|T(\varphi\otimes g(x_0)-\varphi\otimes f(x_0))\| + \|T\|\,\delta \\
&\le \|T\|\,\delta + \delta + \|T\|\,\delta = (1+2\,\|T\|)\delta.
\end{aligned}$$

Combining these inequalities, we get

$$\|T(g-h)\| + |x^*(g-h)| \le (3+\|\mu\|+2\,\|T\|)\delta \le \varepsilon.$$

So $T$ is a narrow operator. □

**Corollary I.3.10** *Let $K$ have no isolated points. An operator $T \in L(C(K),E)$ is narrow if and only if it is $C$-narrow.*

**Definition I.3.11** Let $A$ be an element of $\Sigma$ and let $\varepsilon$ be a positive number. A real-valued function $f \in L^1(\Omega)$ is said to be a *balanced $\varepsilon$-peak* on $A$ if $f \ge -1$, $\chi_A f = f$, $\int_\Omega f\,d\mu = 0$, and $\mu(\{f=-1\}) \ge \mu(A)-\varepsilon$.

**Lemma I.3.12** *Let $A$ be an element of $\Sigma$ with $\mu(A)>0$ and let $\varepsilon$ be a positive number. If $f \in S_{L^1(\Omega)}$ satisfies $\left|1-\int_A f\,d\mu\right| \le \varepsilon$, then $\|f-\chi_B \operatorname{Re} f\|_1 \le \varepsilon+\sqrt{2\varepsilon}$ for $B = A\cap\{\operatorname{Re} f \ge 0\}$.*

*Proof.* We first note that

$$1-\int_A \operatorname{Re} f\,d\mu \le \left|1-\int_A f\,d\mu\right| \le \varepsilon.$$

Consequently, $\int_A \operatorname{Re} f\,d\mu \ge 1-\varepsilon$. Using Hölder's inequality, we get

$$\begin{aligned}
\left(\int_\Omega |\operatorname{Im} f|\,d\mu\right)^2 &= \left(\int_\Omega \frac{|\operatorname{Im} f|}{|f|}\,|f|\,d\mu\right)^2 \le \int_\Omega \frac{(\operatorname{Im} f)^2}{|f|^2}\,|f|\,d\mu \\
&= 1-\int_\Omega \frac{(\operatorname{Re} f)^2}{|f|^2}\,|f|\,d\mu \\
&\le 1-\left(\int_\Omega |\operatorname{Re} f|\,d\mu\right)^2 \\
&\le 1-(1-\varepsilon)^2 \le 2\varepsilon.
\end{aligned}$$

Furthermore,

$$\int_{\Omega\setminus B} |\operatorname{Re} f|\, d\mu = \int_{\Omega} |\operatorname{Re} f|\, d\mu - \int_B \operatorname{Re} f\, d\mu \le 1 - \int_A \operatorname{Re} f\, d\mu \le \varepsilon.$$

Combining these two estimates, we get

$$\|f - \chi_B \operatorname{Re} f\|_1 \le \int_{\Omega\setminus B} |\operatorname{Re} f|\, d\mu + \int_{\Omega} |\operatorname{Im} f|\, d\mu \le \varepsilon + \sqrt{2\varepsilon}. \qquad \square$$

**Proposition I.3.13** [31, Theorem 2.1; 36, Theorem 6.1] *If $T \in L(L^1(\Omega), E)$ is narrow, then there exists for every $A \in \Sigma$ and every $\delta, \varepsilon > 0$ a balanced $\varepsilon$-peak $f$ on $A$ with $\|T(f)\| \le \delta$.*

*Proof.* Fix $A \in \Sigma$ with $\mu(A) > 0$ and $\delta, \varepsilon > 0$. Pick $\eta > 0$ with $\|T\| \frac{2\eta}{1-\eta} + \eta \le \min\{\delta, \varepsilon\}$.

Since $T$ is narrow, there exists $g \in S_{L^1(\Omega)}$ with

$$\left\| T\left(\frac{\chi_A}{\mu(A)} - g\right)\right\| + \left|1 - \int_A g\, d\mu\right| \le \eta$$

and

$$\left\| g - \frac{\chi_A}{\mu(A)}\right\|_1 \ge 2 - \eta.$$

We may assume by Lemma I.3.12 that $g$ is a real-valued and non-negative with $\chi_A g = g$. Denote by $B$ the set $\{g > \frac{1}{\mu(A)}\}$. Then

$$\begin{aligned} 2 - \eta \le \left\| g - \frac{\chi_A}{\mu(A)}\right\|_1 &= \int_B \left(g - \frac{\chi_A}{\mu(A)}\right) d\mu + \int_{A\setminus B} \left(\frac{\chi_A}{\mu(A)} - g\right) d\mu \\ &\le \left(1 - \frac{\mu(B)}{\mu(A)}\right) + \left(1 - \int_{A\setminus B} g\, d\mu\right) \\ &= 2 - \frac{\mu(B)}{\mu(A)} - \int_{A\setminus B} g\, d\mu. \end{aligned}$$

This implies that

$$\mu(B) \le \eta\mu(A)$$

and

$$\|g - \chi_B g\|_1 = \|\chi_{A\setminus B} g\|_1 \le \eta.$$

Set

$$f = \frac{\mu(A)}{\alpha}\chi_B g - \chi_A$$

with $\alpha = \int_B g\, d\mu$ so that $\int_\Omega f\, d\mu = 0$. Then $f$ is real-valued, $f \ge -1$, and

$$\mu(\{f = -1\}) \ge \mu(A) - \mu(B) \ge \mu(A) - \eta \ge \mu(A) - \varepsilon.$$

So $f$ is a balanced $\varepsilon$-peak on $A$. Since $\int_A g\,d\mu = 1$ and $\|g - \chi_B g\|_1 \leq \eta$, we have that $\alpha \geq 1 - \eta$. So

$$\left\|\frac{1}{\alpha}\chi_B g - g\right\|_1 \leq \frac{1}{\alpha}\|\chi_B g - g\|_1 + \left\|\frac{g}{\alpha} - g\right\|_1 \leq \frac{\eta}{1-\eta} + \left(\frac{1}{\alpha} - 1\right) \leq \frac{2\eta}{1-\eta}.$$

Using this inequality, we conclude that

$$\begin{aligned}\|T(f)\| &= \mu(A)\left\|T\left(\frac{1}{\alpha}\chi_B g - \frac{\chi(A)}{\mu(A)}\right)\right\| \\ &\leq \mu(A)\|T\|\left\|\frac{1}{\alpha}\chi_B g - g\right\|_1 + \mu(A)\left\|T\left(\frac{\chi_A}{\mu(A)} - g\right)\right\| \\ &\leq \mu(A)\left(\|T\|\frac{2\eta}{1-\eta} + \eta\right) \leq \delta.\end{aligned}$$

□

**Proposition I.3.14** [8, Theorem 2.4; 36, Theorem 6.1] *Let $T : L^1(\Omega, D) \to E$ be a bounded operator. If there exists for every $A \in \Sigma$, every $d \in D$, and every $\delta, \varepsilon > 0$ a balanced $\varepsilon$-peak $f$ on $A$ with $\|T(f \otimes d)\| \leq \delta$, then $T$ is narrow.*

*Proof.* Fix $f, g \in S_{L^1(\Omega,D)}$, $x^* \in L^1(\Omega, D)^*$, and $\varepsilon > 0$. We have to find $h \in S_{L^1(\Omega,D)}$ with $\|T(g-h)\| + |x^*(g-h)| \leq \varepsilon$ and $\|f + h\|_1 \geq 2 - \varepsilon$.

By density arguments, we may assume without loss of generality that $f$ and $g$ are step functions taking values in a finite-dimensional subspace $F \subset D$. We can represent $x^*|_{L^1(\Omega,F)}$ by a function $r \in L^\infty(\Omega, F^*)$ because $L^1(\Omega, F)^* \cong L^\infty(\Omega, F^*)$ [12, Theorem IV.1]. This function $r$ can be approximated by step functions as $F$ is finite-dimensional. So we may assume that there is a partition $A_1, \ldots, A_n$ of $\Omega$ such that

$$f = \sum_{k=1}^n \chi_{A_k} \otimes f_k, \quad g = \sum_{k=1}^n \chi_{A_k} \otimes g_k, \quad \text{and} \quad r = \sum_{k=1}^n \chi_{A_k} \otimes r_k^*$$

where $f_k, g_k \in F$ and $r_k^* \in F^*$ for $k = 1, \ldots, n$.

Choose $\delta > 0$ with $\max\{n\delta, \sum_{k=1}^n 2\delta\|f_k\|_D\} \leq \varepsilon$. By assumption, we can pick for every $k \in \{1, \ldots, n\}$ a balanced $\delta$-peak $p_k$ on $A_k$ with $\|T(p_k \otimes g_k)\| \leq \delta$. Set

$$h = \sum_{k=1}^n (\chi_{A_k} + p_k) \otimes g_k.$$

As every $p_k$ is a balanced $\delta$-peak, we get that

$$\|h\|_1 = \sum_{k=1}^n \int_{A_k} (1 + p_k)\,d\mu\,\|g_k\|_D = \sum_{k=1}^n \mu(A_k)\|g_k\|_D = \|g\|_1 = 1$$

and

$$|x^*(g-h)| = \left|\sum_{k=1}^n x^*(p_k \otimes g_k)\right| = \left|\sum_{k=1}^n \int_{A_k} r_k^*(g_k)p_k\,d\mu\right| = 0.$$

Furthermore,

$$\|T(g-h)\| \leq \sum_{k=1}^{n} \|T(p_k \otimes g_k)\| \leq n\delta \leq \varepsilon.$$

Denote for $k = 1, \ldots, n$ the set $\{p_k = -1\}$ by $B_k$. Using the fact that $\mu(B_k) \geq \mu(A_k) - \delta$ for $k = 1, \ldots, n$, we deduce that

$$\begin{aligned}
\|f+h\|_1 &= \sum_{k=1}^{n} \|\chi_{A_k} \otimes f_k + (\chi_{A_k} + p_k) \otimes g_k\|_1 \\
&\geq \sum_{k=1}^{n} \left(\|\chi_{B_k} \otimes f_k + (\chi_{A_k} + p_k) \otimes g_k\|_1 - \|\chi_{A_k \setminus B_k} \otimes f_k\|_1\right) \\
&= \sum_{k=1}^{n} (\mu(B_k)\, \|f_k\|_D + \mu(A_k)\, \|g_k\|_D - \mu(A_k \setminus B_k)\, \|f_k\|_D) \\
&\geq \sum_{k=1}^{n} (\mu(A_k)\, \|f_k\|_D + \mu(A_k)\, \|g_k\|_D) - \sum_{k=1}^{n} 2\delta\, \|f_k\|_D \\
&\geq 2 - \varepsilon.
\end{aligned}$$

So $T$ is a narrow operator. □

**Corollary I.3.15** *An operator $T \in L(L^1(\Omega), E)$ is narrow if and only if there exists for every $A \in \Sigma$ and $\delta, \varepsilon > 0$ a balanced $\varepsilon$-peak $f$ on $A$ with $\|T(f)\| \leq \delta$.*

We have seen in Example I.2.2.4, that $L^1(\Omega, X)$ has the Daugavet property if $X$ does. The analogous result for rich subspaces is valid too.

**Proposition I.3.16** [5, Corollary 4.2; 31, Lemma 2.8] *Let $(\Omega, \Sigma, \mu)$ be an arbitrary probability space and let $X$ be a Banach space with the Daugavet property. If $Y$ is a rich subspace of $X$, then $L^1(\Omega, Y)$ is a rich subspace of $L^1(\Omega, X)$.*

*Proof.* Fix $f, g \in S_{L^1(\Omega,X)}$, $x^* \in L^1(\Omega, X)^*$, and $\varepsilon > 0$. We have to find $h \in S_{L^1(\Omega,X)}$ with $d(g-h, L^1(\Omega, Y)) + |x^*(g-h)| \leq \varepsilon$ and $\|f+h\|_1 \geq 2 - \varepsilon$.

By density arguments, we may assume that there is a partition $A_1, \ldots, A_n$ of $\Omega$ such that

$$f = \sum_{k=1}^{n} \chi_{A_k} \otimes f_k, \quad \text{and} \quad g = \sum_{k=1}^{n} \chi_{A_k} \otimes g_k$$

where $f_k, g_k \in X$ for $k = 1, \ldots, n$. If we set

$$Z = \left\{ \sum_{k=1}^{n} \chi_{A_k} \otimes x_k : x_k \in X \text{ for } k = 1, \ldots, n \right\},$$

then $Z$ is a closed subspace of $L^1(\Omega, X)$ and is isometrically isomorphic to $X \oplus_1 \cdots \oplus_1 X$. The functional $x^*|_Z$ can now be written as $\sum_{k=1}^{n} \chi_{A_k} \oplus x_k^*$ where $x_k^* \in X^*$ for $k = 1, \ldots, n$.

As $Y$ is a rich subspace of $X$, we can pick by Lemma I.3.3 for every $k \in \{1, \dots, n\}$ an element $h_k \in \|g_k\|_X S_X$ with

$$d(g_k - h_k, Y) + |x_k^*(g_k - h_k)| \leq \varepsilon$$

and

$$\|f_k + h_k\|_X \geq \|f_k\|_X + \|h_k\|_X - \varepsilon.$$

Considering $h = \sum_{k=1}^n \chi_{A_k} \oplus h_k$, we first note that

$$\|h\|_1 = \sum_{k=1}^n \|h_k\| \, \mu(A_k) = \sum_{k=1}^n \|g_k\| \, \mu(A_k) = \|g\|_1 = 1.$$

Furthermore,

$$d(g - h, L^1(\Omega, Y)) + |x^*(g - h)| \leq \sum_{k=1}^n (d(g_k - h_k, Y) + |x_k^*(g_k - h_k)|)\mu(A_k) \leq \varepsilon$$

and

$$\|f + h\|_1 = \sum_{k=1}^n \|f_k + h_k\|_X \, \mu(A_k) \geq \sum_{k=1}^n (\|f_k\|_X + \|h_k\|_X)\mu(A_k) - \varepsilon = 2 - \varepsilon. \qquad \square$$

## I.4 Poor subspaces

Recall that a closed subspace $Y$ of a Daugavet space $X$ is rich if and only if every closed subspace $Z$ of $X$ with $Y \subset Z$ has the Daugavet property. The corresponding notion for quotients of $X$ was introduced by V. M. Kadets, V. Shepelska, and D. Werner [32].

**Definition I.4.1** Let $X$ be a Banach space with the Daugavet property. A closed subspace $Y$ of $X$ is called *poor* if $X/Z$ has the Daugavet property for every closed subspace $Z \subset Y$.

**Example I.4.2** Let $Y$ be a reflexive subspace of a Banach space $X$ with the Daugavet property. Then $X/Y$ has the Daugavet property [56, Theorem 6]. Consequently, $Y$ is a poor subspace of $X$ because every closed subspace of $Y$ is reflexive as well.

To study poor subspaces, the following generalizations of the Daugavet property, of narrowness and of richness were considered [32].

**Definition I.4.3** Let $X$ be a Banach space and let $U$ be a norming subspace of $X^*$, i.e., $\sup_{u^* \in S_U} |u^*(x)| = \|x\|$ for all $x \in X$. We say that $X$ has the *Daugavet property with respect to* $U$ if the Daugavet equation holds true for every rank-one operator $T : X \to X$ of the form $T = u^* \otimes x$ where $x \in X$ and $u^* \in U$.

This property was introduced during the studies of ultraproducts of Daugavet spaces. It was motivated by the fact that the ultraproduct of Banach spaces with the Daugavet property has the Daugavet property with respect to the ultraproduct of the dual spaces [5, Lemma 2.6].

**Definition I.4.4** Let $X$ be a Banach space that has the Daugavet property with respect to some norming subspace $U$ and let $E$ be an arbitrary Banach space. An operator $T \in L(X, E)$ is called *narrow with respect to* $U$ (or $U$*-narrow* for short) if for every two elements $x, y \in S_X$, for every $u^* \in U$, and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|T(y-z)\| + |u^*(y-z)| \leq \varepsilon$ and $\|x+z\| \geq 2-\varepsilon$. A closed subspace $Y$ of $X$ is said to be *rich with respect to* $U$ (or $U$*-rich* for short) if the quotient map $\pi : X \to X/Y$ is $U$-narrow.

If $Y$ is a $U$-rich subspace of $X$, then $Y$ has the Daugavet property with respect to $U|_Y = \{u^*|_Y : u^* \in U\}$. This remains true for every closed subspace of $X$ which contains $Y$. As in the case of rich subspaces, this property characterizes $U$-rich subspaces: A closed subspace $Y$ of $X$ is $U$-rich if and only if every closed subspace $Z$ of $X$ with $Y \subset Z$ has the Daugavet property with respect to $U|_Z$ [32, Theorem 5.5].

A Banach space $X$ has the Daugavet property if and only if $X^*$ has the Daugavet property with respect to $X \subset X^{**}$. Consequently, a closed subspace $Y$ of a Daugavet space $X$ is poor if and only if for every closed subspace $Z \subset Y$ the space $(X/Z)^* \cong Z^\perp$ has the Daugavet property with respect to $X/Z \cong X|_{Z^\perp}$. This leads to the following characterization of poor subspaces.

**Theorem I.4.5** [32, Theorem 5.8] *Let $X$ be a Banach space with the Daugavet property. A closed subspace $Y$ of $X$ is poor if and only if $Y^\perp$ is an $X$-rich subspace of $X^*$.*

If we want to describe the poor subspaces of $L^1(\Omega)$, we have to study operators $T : L^\infty(\Omega) \to E$ that are narrow with respect to $L^1(\Omega)$. These can be characterized in a convenient way. In the sequel, $(\Omega, \Sigma, \mu)$ denotes a non-atomic probability space. For every $A \in \Sigma$, we write $L^1(A)$ for the subspace $\{f \in L^1(\Omega) : \chi_A f = f\}$ and $P_A$ for the projection from $L^1(\Omega)$ onto $L^1(A)$ defined by $P_A(f) = \chi_A f$.

**Proposition I.4.6** [32, Theorem 6.5] *Let $E$ be an arbitrary Banach space. An operator $T \in L(L^\infty(\Omega), E)$ is narrow with respect to $L^1(\Omega)$ if and only if for every $A \in \Sigma$ with $\mu(A) > 0$ and every $\varepsilon > 0$ there exists $f \in S_{L^\infty(\Omega)}$ with $\chi_A f = f$ and $\|T(f)\| \leq \varepsilon$.*

*Proof.* Arguing as in the proof of Proposition I.3.8, we can show that the condition is necessary.

Let us now prove that the condition is sufficient. Fix $f, g \in S_{L^\infty(\Omega)}$, $h \in L^1(\Omega)$, and $\varepsilon > 0$. We have to find $r \in S_{L^\infty(\Omega)}$ with $\|T(g-r)\| + \left|\int_\Omega (g-r)h\,d\mu\right| \leq \varepsilon$ and $\|f+r\|_\infty \geq 2-\varepsilon$.

By density arguments, we may assume without loss of generality that $f, g$ and $h$ are step functions and that there is a partition $A_1, \ldots, A_n$ of $\Omega$ such that

$$f = \sum_{k=1}^n f_k \chi_{A_k}, \quad g = \sum_{k=1}^n g_k \chi_{A_k}, \quad \text{and} \quad h = \sum_{k=1}^n h_k \chi_{A_k}$$

where $f_k, g_k, h_k \in \mathbb{C}$ for $k = 1, \ldots, n$. Since $\|f\|_\infty = 1$, there exists $k_0 \in \{1, \ldots, n\}$ with $|f_{k_0}| = 1$ and $\mu(A_{k_0}) > 0$. Fix $\delta > 0$ with $(2 + 2\,|h_{k_0}|)\delta \leq \varepsilon$ and let $B \in \Sigma$ be a subset of $A_{k_0}$ with $0 < \mu(B) \leq \delta$. By assumption, there exists $p \in S_{L^\infty(\Omega)}$ with $\chi_B p = p$

and $\|T(p)\| \leq \delta$. Applying the same reasoning as in the proof of Lemma I.3.7, we may assume that $p$ is real-valued and non-negative. Set

$$r = pf + (1-p)g = g + (f_{k_0} - g_{k_0})p.$$

Then $\|r\|_\infty = 1$. Observing that for all $\eta > 0$

$$\begin{aligned}\mu(\{|f+r| \geq 2-2\eta\}) &\geq \mu(\{|f_{k_0}\chi_B + g_{k_0}\chi_B + (f_{k_0} - g_{k_0})p| \geq 2-2\eta\}) \\ &= \mu(\{|(\chi_B + p)f_{k_0} + (\chi_B - p)g_{k_0}| \geq 2-2\eta\}) \\ &\geq \mu(\{p \geq 1-\eta\}) > 0,\end{aligned}$$

we conclude that $\|f+r\|_\infty = 2$. Furthermore,

$$\|T(g-r)\| = \|T((f_{k_0} - g_{k_0})p)\| \leq 2\|T(p)\| \leq 2\delta$$

and

$$\left|\int_\Omega (g-r)h\,d\mu\right| = \left|\int_\Omega (f_{k_0} - g_{k_0})ph\,d\mu\right| \leq 2\int_B |h|\,d\mu \leq 2|h_{k_0}|\,\mu(B) \leq 2|h_{k_0}|\,\delta.$$

Combining these estimates, we get

$$\|T(g-r)\| + \left|\int_\Omega (g-r)h\,d\mu\right| \leq (2 + 2|h_{k_0}|)\delta \leq \varepsilon.$$

So $T$ is narrow with respect to $L^1(\Omega)$. □

**Corollary I.4.7** [32, Corollary 6.6] *A closed subspace $X$ of $L^1(\Omega)$ is poor if and only if for every $A \in \Sigma$ of positive measure and every $\varepsilon > 0$ there exists $f \in S_{L^\infty(\Omega)}$ with $\chi_A f = f$ and $\|f|_X\| \leq \varepsilon$ where we interpret $f$ as a functional on $L^1(\Omega)$.*

Using this characterization, we can build a link to a property that was studied by G. Godefroy, N. J. Kalton, and D. Li [20].

**Definition I.4.8** A closed subspace $X$ of $L^1(\Omega)$ is said to be *small* if there is no $A \in \Sigma$ of positive measure such that $P_A$ maps $X$ onto $L^1(A)$.

**Proposition I.4.9** [32, Corollary 6.7] *If $X$ is a poor subspace of $L^1(\Omega)$, then it is small.*

*Proof.* Fix $A \in \Sigma$ with $\mu(A) > 0$. We have to show that $P_A$ does not map $X$ onto $L^1(A)$.

By the open mapping theorem, $P_A$ does not map $X$ onto $L^1(A)$ if and only if there is no $M > 0$ with $B_{L^1(A)} \subset MP_A[B_X]$. Fix $M > 0$. By Corollary I.4.7, there exists $f \in S_{L^\infty(\Omega)}$ with $\chi_A f = f$ and $\|f|_X\| \leq \frac{1}{2M}$ where we interpret $f$ as a functional on $L^1(\Omega)$. Then

$$\sup\left\{\left|\int_A fg\,d\mu\right| : g \in B_{L^1(A)}\right\} = 1$$

and

$$\sup\left\{\left|\int_A fg\,d\mu\right| : g \in MP_A[B_X]\right\} \leq \frac{1}{2}.$$

Consequently, $B_{L^1(A)} \not\subset MP_A[B_X]$ and $X$ is small. □

**Proposition I.4.10** *If $X$ is a small subspace of $L^1(\Omega)$, then it is poor.*

*Proof.* Fix $A \in \Sigma$ with $\mu(A) > 0$ and $\varepsilon > 0$. By Corollary I.4.7, we have to find $f \in S_{L^\infty(\Omega)}$ with $\chi_A f = f$ and $\| f|_X \| \leq \varepsilon$.

Since $X$ is small, the projection $P_A : L^1(\Omega) \to L^1(A)$ does not map $X$ onto $L^1(A)$. By the (proof of the) open mapping theorem [14, Theorem II.2.1], the set $P_A[\varepsilon^{-1}B_X]$ is nowhere dense in $L^1(A)$. Pick $g \in B_{L^1(A)}$ with $g \notin \overline{P_A[\varepsilon^{-1}B_X]}$. As $\overline{P_A[\varepsilon^{-1}B_X]}$ is absolutely convex, there exists by the Hahn-Banach theorem a function $f \in S_{L^\infty(\Omega)}$ with $\chi_A f = f$ and

$$\sup\left\{\left|\int_A fh\,d\mu\right| : h \in \frac{1}{\varepsilon}B_X\right\} \leq \operatorname{Re}\int_A fg\,d\mu.$$

Using this inequality, we get

$$\| f|_X \| = \sup\left\{\left|\int_A fh\,d\mu\right| : h \in B_X\right\} \leq \varepsilon \operatorname{Re}\int_A fg\,d\mu \leq \varepsilon. \qquad \square$$

**Corollary I.4.11** *A closed subspace $X$ of $L^1(\Omega)$ is poor if and only if it is small.*

## I.5 The almost Daugavet property

**Definition I.5.1** A Banach space $X$ is called an *almost Daugavet space* or a space with the *almost Daugavet property* if it has the Daugavet property with respect to some norming subspace $U \subset X^*$.

**Examples I.5.2**

1. If $X$ has the Daugavet property, then $X^*$ has the Daugavet property with respect to $X \subset X^{**}$. So every dual of a Daugavet space is an almost Daugavet space.
2. The sequence space $\ell^1$ has the almost Daugavet property but fails the Daugavet property [33, Proposition 2.6].

V. M. Kadets, V. Shepelska, and D. Werner coined this term [33] and were able to characterize separable Banach spaces with the almost Daugavet property using the following parameter that was introduced by R. Whitley [62].

**Definition I.5.3** Let $X$ be a Banach space. We call a set $A$ an *inner $\varepsilon$-net* for $S_X$ if $A \subset S_X$ and for every $x \in S_X$ there exists $y \in A$ with $\|x - y\| \leq \varepsilon$. Then the *thickness* $T(X)$ of $X$ is defined by

$$T(X) = \inf\{\varepsilon > 0 : \text{there exists a finite inner } \varepsilon\text{-net for } S_X\}.$$

The thickness is essentially an inner measure of non-compactness of the unit sphere $S_X$.

**Examples I.5.4**

1. Let $X$ be a Banach space. If $X$ is finite-dimensional, then $T(X) = 0$, and if $X$ is infinite-dimensional, then $1 \leq T(X) \leq 2$ [62, Lemma 2].

2. Let $K$ be a compact space which contains an infinite number of points. Then $T(C(K)) = 1$ if $K$ contains an isolated point, and $T(C(K)) = 2$ otherwise [62, Lemma 3].
3. $T(\ell^p) = 2^{1/p}$ for $1 \leq p < \infty$ [62, Lemma 4].

**Theorem I.5.5** [33, Theorem 1.1] *A separable Banach space $X$ has the almost Daugavet property if and only if $T(X) = 2$.*

# II Subspaces of Almost Daugavet Spaces

As in the case of Daugavet spaces, almost Daugavet spaces are in a certain sense "big". They have thickness two and contain a copy of $\ell^1$ [33, Corollary 3.3]. So it is again an interesting question which subspaces of a space with the almost Daugavet property inherit this property.

## II.1 "Big" subspaces of almost Daugavet spaces

Studying subspaces of Daugavet spaces, it was proved that a closed subspace $Y$ of a Daugavet space $X$ inherits the Daugavet property if the quotient space $X/Y$ is in a certain sense "small". We will use a similar approach for subspaces of almost Daugavet spaces.

**Lemma II.1.1** *Let $X$ be a Banach space with $T(X) = 2$ and let $Y$ be a finite-dimensional subspace of $X$. For every $\varepsilon > 0$ there exists $x \in S_X$ with*

$$\|y + \alpha x\| \geq (1 - \varepsilon)(\|y\| + |\alpha|) \quad (y \in Y, \alpha \in \mathbb{C}).$$

*Proof.* Fix $\varepsilon > 0$ and let $\{y_1, \ldots, y_n\}$ be an inner $\frac{\varepsilon}{2}$-net for $S_Y$. Since $T(X) = 2$, we can choose $x \in S_X$ with $\|y_k + x\| \geq 2 - \frac{\varepsilon}{2}$ for $k = 1, \ldots, n$. To prove the desired inequality, it suffices to consider the case that $y \in Y$ and $\alpha = 1$. Furthermore, let us assume that $\|y\| \geq 1$. The argumentation in the case $\|y\| < 1$ is essentially the same. Pick $y_{k_0} \in \{y_1, \ldots, y_n\}$ with $\|y/\|y\| - y_{k_0}\| \leq \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
\|y + x\| &= \left\| \|y\| \left( \frac{y}{\|y\|} + x \right) - (\|y\| - 1)x \right\| \\
&\geq \|y\| \left\| \frac{y}{\|y\|} + x \right\| - (\|y\| - 1)\|x\| \\
&\geq \|y\| \left( \|y_{k_0} + x\| - \frac{\varepsilon}{2} \right) - \|y\| + 1 \\
&\geq \|y\| (2 - \varepsilon) - \|y\| + 1 \\
&\geq (1 - \varepsilon)(\|y\| + 1).
\end{aligned}$$

□

**Corollary II.1.2** *If $X$ is a Banach space with $T(X) = 2$, then $X$ contains a copy of $\ell^1$.*

**Theorem II.1.3** *Let $X$ be a Banach space with $T(X) = 2$. If $Y$ is a closed subspace of $X$ such that the quotient space $X/Y$ contains no copy of $\ell^1$, then $T(Y) = 2$.*

*Proof.* Fix $y_1, \dots, y_n \in S_Y$ and $\varepsilon \in (0,1)$. We have to find $z \in S_Y$ with $\|y_k + z\| \geq 2 - \varepsilon$ for $k = 1, \dots, n$.

Let $(\delta_l)_{l \in \mathbb{N}}$ be a sequence of positive numbers such that $\prod_{l=1}^{\infty}(1-\delta_l) \geq 1 - \frac{\varepsilon}{3}$. Using Lemma II.1.1, we find an element $e_1 \in S_X$ with

$$\|x + \alpha e_1\| \geq (1-\delta_1)(\|x\| + |\alpha|) \quad (x \in \operatorname{lin}\{y_1, \dots, y_n\}, \alpha \in \mathbb{C}).$$

Going on like this, we can inductively construct a normalized sequence $(e_l)_{l \in \mathbb{N}}$ such that

$$\|x + \alpha e_l\| \geq (1-\delta_l)(\|x\| + |\alpha|) \quad (x \in \operatorname{lin}\{y_1, \dots, y_n, e_1, \dots, e_{l-1}\}, \alpha \in \mathbb{C})$$

for all $l \geq 2$. We then have for every $l \in \mathbb{N}$, $x \in \operatorname{lin}\{y_1, \dots, y_n, e_1, \dots, e_l\}$, and every linear combination $\sum_{j=l+1}^{N} \alpha_j e_j$ that

$$\begin{aligned}
\left\| x + \sum_{j=l+1}^{N} \alpha_j e_j \right\| &\geq (1-\delta_N) \left\| x + \sum_{j=l+1}^{N-1} \alpha_j e_j \right\| + (1-\delta_N)\,|a_N| \\
&\geq \cdots \geq \prod_{j=l+1}^{N} (1-\delta_j)\,\|x\| + \sum_{j=l+1}^{N} (1-\delta_j)\,|\alpha_j| \\
&\geq \left(1 - \frac{\varepsilon}{3}\right) \left( \|x\| + \sum_{j=l+1}^{N} |\alpha_j| \right). \qquad (1.1)
\end{aligned}$$

So the sequence $(e_l)_{l \in \mathbb{N}}$ is equivalent to the canonical basis of $\ell^1$. Since $X/Y$ does not contain a copy of $\ell^1$, the quotient map $\pi : X \to X/Y$ fails to be bounded below on $\operatorname{lin}\{e_l : l \in \mathbb{N}\}$. So we can choose a normalized linear combination $\sum_{j=1}^{N} \beta_j e_j$ and $z \in S_Y$ with $\|\sum_{j=1}^{N} \beta_j e_j - z\| \leq \frac{\varepsilon}{3}$. Fix $k \in \{1, \dots, n\}$. Using (1.1), we get

$$\begin{aligned}
\|y_k + z\| &\geq \left\| y_k + \sum_{j=1}^{N} \beta_j e_j \right\| - \frac{\varepsilon}{3} \geq \left(1 - \frac{\varepsilon}{3}\right) \left( \|y_k\| + \sum_{j=1}^{N} |\beta_j| \right) - \frac{\varepsilon}{3} \\
&\geq 2\left(1 - \frac{\varepsilon}{3}\right) - \frac{\varepsilon}{3} = 2 - \varepsilon.
\end{aligned}$$

□

**Corollary II.1.4** *Let $X$ be a separable almost Daugavet space. If $Y$ is a closed subspace of $X$ such that the quotient space $X/Y$ contains no copy of $\ell^1$, then $Y$ has the almost Daugavet property as well.*

## II.2 The almost Daugavet property and $L$-embedded spaces

Let us consider a special class of Banach spaces whose subspaces of thickness two can be fully characterized.

**Definition II.2.1** Let $X$ be a Banach space. A linear projection $P : X \to X$ is called an *$L$-projection* if

$$\|x\| = \|P(x)\| + \|x - P(x)\| \quad (x \in X).$$

A closed subspace of $X$ is called an *$L$-summand* if it is the range of an $L$-projection.

**Examples II.2.2**

1. Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. For every $A \in \Sigma$, the projection $f \mapsto \chi_A f$ is an $L$-projection. Furthermore, every $L$-projection on $L^1(\Omega)$ is of this type [24, Example I.1.6(a)].
2. Let $M[0,1]$ be the space of all regular Borel measures on $[0,1]$ and let $\lambda \in M[0,1]$ be a probability measure. By Lebesgue's decomposition theorem, there exist for every measure $\mu \in M[0,1]$ two measures $\mu_{\mathrm{ac}}$, $\mu_{\mathrm{sing}} \in M[0,1]$ such that $\mu_{\mathrm{ac}}$ is absolutely continuous with respect to $\lambda$, $\mu_{\mathrm{sing}}$ and $\lambda$ are singular, and $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$. The map $\mu \mapsto \mu_{\mathrm{ac}}$ is an $L$-projection on $M[0,1]$ [24, Example I.1.6(b)].

**Definition II.2.3** A Banach space $X$ is called *$L$-embedded* if $X$ is an $L$-summand in its bidual $X^{**}$ where we identify $X$ with its image under the canonical embedding $i_X : X \to X^{**}$.

**Examples II.2.4**

1. Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. Then $L^1(\Omega)$ is an $L$-embedded space [24, Example IV.1.1(a)].
2. Let $H$ be a Hilbert space. A *von Neumann algebra* $M$ is a unitial, selfadjoint subalgebra of $L(H)$ which is closed with respect to the weak operator topology. The space

   $$M_* = \{f \in M^* : f|_{B_M} \text{ is continuous with respect to the weak operator topology}\}$$

   is the only predual of $M$ [59, Theorem II.2.6 and Corollary III.3.9] and $L$-embedded [24, Example IV.1.1(b)].
3. The Hardy space $H^1$ is $L$-embedded [24, Example IV.1.1(d)].

**Proposition II.2.5** *If an $L$-embedded Banach space $X$ is not reflexive, then $T(X) = 2$.*

*Proof.* Our proof is similar to the argumentation in [24, Remark IV.2.4].

Fix $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$. We have to find $y \in S_X$ with $\|x_k + y\| \geq 2 - \varepsilon$ for $k = 1, \ldots, n$.

Let $X_s$ be the $L$-summand in $X^{**}$ which is complementary to $X$, i.e., $X^{**} = X \oplus_1 X_s$. Since $X$ is not reflexive, we can choose $x_s^{**} \in S_{X_s}$. Set $Y = \operatorname{lin}\{x_1, \ldots, x_n, x_s^{**}\}$ and pick $\delta > 0$ with $\frac{1}{1+\delta} - 2\delta \geq 1 - \varepsilon$. By the principle of local reflexivity [3, Theorem 11.2.4], there exists an operator $T : Y \to X$ with

$$(1 - \delta)\,\|x^{**}\| \leq \|T(x^{**})\| \leq (1 + \delta)\,\|x^{**}\| \quad (x^{**} \in Y)$$

and

$$T(x) = x \quad (x \in Y \cap X).$$

Set $y = \frac{T(x_s^{**})}{\|T(x_s^{**})\|}$ and note that $\|T(x_s^{**})\| \leq 1 + \delta$. We then get for every $k \in \{1, \ldots, n\}$

$$\begin{aligned}
\|x_k + y\| &= \left\| T(x_k) + \frac{T(x_s^{**})}{\|T(x_s^{**})\|} \right\| \geq (1-\delta) \left\| x_k + \frac{x_s^{**}}{\|T(x_s^{**})\|} \right\| \\
&= (1-\delta) \left( \|x_k\| + \frac{\|x_s^{**}\|}{\|T(x_s^{**})\|} \right) \geq (1-\delta) \left( 1 + \frac{1}{1+\delta} \right) \\
&\geq 1 + \frac{1}{1+\delta} - 2\delta \geq 2 - \varepsilon.
\end{aligned}$$

□

**Corollary II.2.6** *Let $X$ be a separable $L$-embedded space. If $X$ is not reflexive, then $X$ has the almost Daugavet property.*

**Proposition II.2.7** [19, Lemme 4] *Every $L$-embedded space is weakly sequentially complete.*

*Proof.* Let $X$ be a Banach space with $X^{**} = X \oplus_1 X_s$. Let $(y_n)_{n\in\mathbb{N}}$ be a weak Cauchy sequence in $X$ and denote by $y^{**}$ its weak* limit. We may assume that $y^{**} \in X_s$ because otherwise we decompose $y^{**}$ into $y + y_s^{**}$ with $y \in X$ and $y_s^{**} \in X_s$ and pass to the sequence $(y_n - y)_{n\in\mathbb{N}}$. So we have to show that $y^{**} = 0$.

Fix $z \in X$ with $\|z\| = \|y^{**}\|$ and $\varepsilon \in \{\pm 1\}$. We then have for every $x \in X$ and $\alpha \in \mathbb{C}$ that

$$\|x\| \leq \|x - \varepsilon\alpha z\| + \|\alpha z\| = \|x - \varepsilon\alpha z\| + \|\alpha y^{**}\| = \|x - \alpha(y^{**} + \varepsilon z)\|.$$

If we interpret $x \in X$ as a functional on $X^*$, this implies that

$$\|x|_{\ker(y^{**}+\varepsilon z)}\| = d(x, \operatorname{lin}\{y^{**} + \varepsilon z\}) = \|x\|.$$

By the Hahn-Banach theorem, we can therefore deduce that $\ker(y^{**}+\varepsilon z) \cap B_{X^*}$ is weak* dense in $B_{X^*}$. Observe that

$$\ker(y^{**} + \varepsilon z) = \bigcap_{k,l=1}^{\infty} \left\{ x^* \in X^* : \text{there exists } n \geq l \text{ such that } |x^*(y_n + \varepsilon z)| < \frac{1}{k} \right\}.$$

So $\ker(y^{**} + \varepsilon z) \cap B_{X^*}$ is a weak* $G_\delta$ set in $B_{X^*}$. By Baire's category theorem, the set $\ker(y^{**} + z) \cap \ker(y^{**} - z) \cap B_{X^*}$ is therefore weak* dense in $B_{X^*}$. Note that this set is contained in $\ker z \cap B_{X^*}$. Since $z$ is weak* continuous, we have that $z = 0$ and $y^{**} = 0$. □

**Corollary II.2.8** [24, Corollary IV.2.3] *Every closed, non-reflexive subspace of an $L$-embedded space $X$ contains a copy of $\ell^1$.*

*Proof.* Let $Y$ be a closed, non-reflexive subspace of $X$. Its unit ball $B_Y$ is not weakly compact and by the Eberlein-Šmulian theorem [3, Theorem 1.6.3] not weakly sequentially compact. So there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $B_Y$ which does not have a weakly convergent subsequence. By Proposition II.2.7, $X$ is weakly sequentially complete and therefore $(y_n)_{n\in\mathbb{N}}$ does not have weakly Cauchy subsequences either. Now Rosenthal's $\ell^1$ theorem [3, Theorem 10.2.1] yields that a subsequence of $(y_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of $\ell^1$. □

**Lemma II.2.9** [24, Lemma IV.1.4] *Let $X$ be an $L$-embedded space with $L$-projection $P$ from $X^{**}$ onto $X$. Suppose that the closed subspace $Y \subset X$ is an almost $L$-summand in its bidual in the sense that there is $\varepsilon \in (0, \frac{1}{4})$ and a closed subspace $Z \subset X^{**}$ such that $Y^{\perp\perp} = Y \oplus Z$ and*

$$\|y + z^{**}\| \geq (1-\varepsilon)(\|y\| + \|z^{**}\|) \quad (y \in Y, z^{**} \in Z).$$

*(Note that we identify $Y^{**}$ and $Y^{\perp\perp}$.) Then $\|P|_{Y^{\perp\perp}} - Q\| \leq 3\sqrt{\varepsilon}$ where $Q$ denotes the projection from $Y^{\perp\perp}$ onto $Y$.*

*Proof.* Since for all $y \in Y$ and $z^{**} \in Z$

$$\|P(y + z^{**}) - Q(y - z^{**})\| = \|P(z^{**})\|$$

and

$$(\sqrt{\varepsilon} + 2\varepsilon)\,\|z^{**}\| \leq \frac{\sqrt{\varepsilon} + 2\varepsilon}{1-\varepsilon}\,\|y + z^{**}\| \leq 3\sqrt{\varepsilon}\,\|y + z^{**}\|\,,$$

it suffices to show that $\|P(z^{**})\| \leq (\sqrt{\varepsilon} + 2\varepsilon)\,\|z^{**}\|$ for every $z^{**} \in Z$.

Let $X_s$ be the $L$-summand in $X^{**}$ which is complementary to $X$, i.e., $X^{**} = X \oplus_1 X_s$. Fix $z^{**} \in Z$ and decompose it into the sum $x + x_s^{**}$ with $x \in X$ and $x_s^{**} \in X_s$. If $\|x\| = \|P(z^{**})\| \leq \sqrt{\varepsilon}\,\|z^{**}\|$, there is nothing to show. So we assume $\|x\| > \sqrt{\varepsilon}\,\|z^{**}\|$ from now on. For every $y \in Y$, we get

$$\begin{aligned}
\|y + x\| &= \|y + z^{**}\| - \|x_s^{**}\| \\
&\geq (1-\varepsilon)(\|y\| + \|z^{**}\|) - \|x_s^{**}\| \\
&= (1-\varepsilon)(\|y\| + \|x\| + \|x_s^{**}\|) - \|x_s^{**}\| \\
&= (1-\varepsilon)(\|y\| + \|x\|) - \varepsilon\,\|x_s^{**}\| \\
&\geq (1-\varepsilon)(\|y\| + \|x\|) - \varepsilon\,\|z^{**}\| \\
&\geq (1-\varepsilon)(\|y\| + \|x\|) - \sqrt{\varepsilon}\,\|x\| \\
&\geq (1-2\sqrt{\varepsilon})(\|y\| + \|x\|). \qquad (2.1)
\end{aligned}$$

Let us show that this inequality extends to all $y^{\perp\perp} \in Y^{\perp\perp}$. First note that by (2.1)

$$d(x, Y) \geq (1 - 2\sqrt{\varepsilon})\,\|x\| > 0$$

and therefore $x \notin Y$. Thus it makes sense to consider the direct sum $Y \oplus \operatorname{lin}\{x\} \subset X$. Denoting by $T$ the identity from $Y \oplus \operatorname{lin}\{x\}$ onto $Y \oplus_1 \operatorname{lin}\{x\}$, we conclude from (2.1)

that $\|T\| \leq (1-2\sqrt{\varepsilon})^{-1}$. Since $(Y \oplus_1 \operatorname{lin}\{x\})^{**} \cong Y^{\perp\perp} \oplus_1 \operatorname{lin}\{x\}$, we get for every $y^{\perp\perp} \in Y^{\perp\perp}$

$$\|y^{\perp\perp} + x\| \geq (1-2\sqrt{\varepsilon})\,\|T^{**}(y^{\perp\perp} + x)\| = (1-2\sqrt{\varepsilon})(\|y^{\perp\perp}\| + \|x\|).$$

Using this last inequality for $-z^{**}$, we obtain

$$\|x_s^{**}\| = \|z^{**} - x\| \geq (1-2\sqrt{\varepsilon})(\|z^{**}\| + \|x\|) \geq (1-2\sqrt{\varepsilon})(\|z^{**}\| + \sqrt{\varepsilon}\,\|z^{**}\|)$$

and finally

$$\begin{aligned}\|P(z^{**})\| = \|x\| &= \|z^{**}\| - \|x_s^{**}\| \\ &\leq \|z^{**}\| - (1-2\sqrt{\varepsilon})(1+\sqrt{\varepsilon})\,\|z^{**}\| \\ &= (\sqrt{\varepsilon} + 2\varepsilon)\,\|z^{**}\|\,.\end{aligned}$$

□

**Proposition II.2.10** [61, Aufgabe III.6.6] *The dual of $\ell^\infty$ can be written as $\ell^1 \oplus_1 c_0^\perp$.*

*Proof.* It is clear that $\ell^1$ and $c_0^\perp$ are closed subspaces of $(\ell^\infty)^*$ with $\ell^1 \cap c_0^\perp = \{0\}$.

Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis of $\ell^1$ and fix $x^* \in (\ell^\infty)^*$. Set $x_n = x^*(e_n)$ for every $n \in \mathbb{N}$ and note that $(x_n)_{n\in\mathbb{N}} \in \ell^1$. Let $x_1^*$ be the functional on $\ell^\infty$ that is generated by $(x_n)_{n\in\mathbb{N}}$ and set $x_2^* = x^* - x_1^*$. Then $x_2^* \in c_0^\perp$ and $x^* = x_1^* + x_2^*$. Therefore, $(\ell^\infty)^* = \ell^1 \oplus c_0^\perp$.

Fix $x_1^* \in \ell^1$ and $x_2^* \in c_0^\perp$. It remains to show that $\|x_1^* + x_2^*\| = \|x_1^*\| + \|x_2^*\|$. Fix $\varepsilon > 0$ and choose $(x_n)_{n\in\mathbb{N}} \in c_{00}$ and $(y_n)_{n\in\mathbb{N}} \in \ell^\infty$ with $\|(x_n)\|_\infty = \|(y_n)\|_\infty = 1$ such that $\operatorname{Re} x_1^*((x_n)) \geq \|x_1^*\| - \varepsilon$ and $\operatorname{Re} x_2^*((y_n)) \geq \|x_2^*\| - \varepsilon$. Since $(x_n)_{n\in\mathbb{N}} \in c_{00}$, there exists $n_0 \in \mathbb{N}$ with $x_n = 0$ for all $n > n_0$. Set $z_n = x_n$ for $n = 1, \ldots, n_0$ and $z_n = y_n$ for $n > n_0$. Then $\|(z_n)\|_\infty = 1$ and

$$\|x_1^* + x_2^*\| \geq |x_1^*((z_n)) + x_2^*((z_n))| = |x_1^*((x_n)) + x_2^*((y_n))| \geq \|x_1^*\| + \|x_2^*\| - 2\varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this finishes the proof. □

**Lemma II.2.11** [24, claim in the proof of Theorem IV.2.7] *Let $X$ be an $L$-embedded space with $X^{**} = X \oplus_1 X_s$. Suppose that there exist $\varepsilon \in (0, \frac{1}{4})$ and a sequence $(y_l)_{l\in\mathbb{N}}$ in $X$ such that*

$$(1-\varepsilon)\sum_{l=1}^{\infty} |\alpha_l| \leq \left\|\sum_{l=1}^{\infty} \alpha_l y_l\right\| \leq \sum_{l=1}^{\infty} |\alpha_l|$$

*for any sequence of scalars $(\alpha_l)_{l\in\mathbb{N}}$ with finite support. Then there exists $x_s^{**} \in X_s$ such that*

$$1 - 4\sqrt{\varepsilon} \leq \|x_s^{**}\| \leq 1$$

*and for all $\delta > 0$, all $x_1^*, \ldots, x_n^* \in X^*$, and all $l_0 \in \mathbb{N}$ there is $l \geq l_0$ with*

$$|x_s^{**}(x_k^*) - x_k^*(y_l)| \leq 3\sqrt{\varepsilon}\,\|x_k^*\| + \delta \quad (k = 1, \ldots, n).$$

*In other words, there is $x_s^{**} \in X_s$ which is "close" to a weak* accumulation point of $(y_l)_{l\in\mathbb{N}}$.*

*Proof.* By Proposition II.2.10, we can write the bidual of $\ell^1$ as $\ell^1 \oplus_1 c_0^\perp$. Denote by $P_{\ell^1}$ the $L$-projection from $(\ell^1)^{**}$ onto $\ell^1$. Let $e^{**}$ be a weak* accumulation point of the canonical basis of $\ell^1$. Then $\|e^{**}\| = 1$ and $e^{**} \in c_0^\perp$. We will map $e^{**}$ into $X_s$.

Set $Y = \overline{\operatorname{lin}}\{y_l : l \in \mathbb{N}\}$. By assumption, the canonical isomorphism $T : Y \to \ell^1$ satisfies

$$\|y\| \le \|T(y)\| \le \frac{1}{1-\varepsilon}\,\|y\| \quad (y \in Y).$$

Identifying $Y^{**}$ and $Y^{\perp\perp}$, this can be extended to

$$\|y^{\perp\perp}\| \le \|T^{**}(y^{\perp\perp})\| \le \frac{1}{1-\varepsilon}\,\|y^{\perp\perp}\| \quad (y^{\perp\perp} \in Y^{\perp\perp}).$$

In particular, $1-\varepsilon \le \|z_s^{**}\| \le 1$ for $z_s^{**} = (T^{**})^{-1}(e^{**})$. Denote by $Q$ the canonical projection from $Y^{\perp\perp}$ onto $Y$, i.e., $Q = (T^{**})^{-1}P_{\ell^1}T^{**}$, and set $Y_s = \ker Q$. Then $z_s^{**} \in Y_s$ because $e^{**} \in c_0^\perp = \ker P_{\ell^1}$. Furthermore, $z_s^{**}$ is a weak* accumulation point of $(y_l)_{l\in\mathbb{N}}$ as $(T^{**})^{-1}$ is weak*-to-weak* continuous. Finally, put $x_s^{**} = (\mathrm{Id}_{X^{**}} - P)(z_s^{**}) \in X_s$ where $P$ denotes the $L$-projection from $X^{**}$ onto $X$. Note that $\|x_s^{**}\| \le 1$ since $\|\mathrm{Id}_{X^{**}} - P\| = 1$.

If we decompose $y^{\perp\perp} \in Y^{\perp\perp}$ into $y + y_s^{**}$ with $y \in Y$ and $y_s^{**} \in Y_s$, then

$$\begin{aligned}\|y + y_s^{**}\| &\ge (1-\varepsilon)\,\|T^{**}(y) + T^{**}(y_s^{**})\| \\ &= (1-\varepsilon)(\|T^{**}(y)\| + \|T^{**}(y_s^{**})\|) \\ &\ge (1-\varepsilon)(\|y\| + \|y_s^{**}\|).\end{aligned}$$

Since $\varepsilon < \frac{1}{4}$, the assumptions of Lemma II.2.9 are satisfied and $\|P|_{Y^{\perp\perp}} - Q\| \le 3\sqrt{\varepsilon}$. This implies that

$$\|x_s^{**} - z_s^{**}\| = \|P(z_s^{**})\| = \|P(z_s^{**}) - Q(z_s^{**})\| \le 3\sqrt{\varepsilon}\,\|z_s^{**}\|\,. \tag{2.2}$$

Hence

$$\|x_s^{**}\| = \|z_s^{**} - P(z_s^{**})\| \ge \|z_s^{**}\| - \|P(z_s^{**})\| \ge (1 - 3\sqrt{\varepsilon})\,\|z_s^{**}\| \ge 1 - 4\sqrt{\varepsilon}.$$

Fix now $\delta > 0$, $x_1^*, \dots, x_n^* \in X^*$, and $l_0 \in \mathbb{N}$. Since $z_s^{**}$ is a weak* accumulation point of $(y_l)_{l\in\mathbb{N}}$, there exists $l \ge l_0$ such that

$$|z_s^{**}(x_k^*) - x_k^*(y_l)| \le \delta \quad (k = 1, \dots, n).$$

Combining these inequalities with (2.2), we get for every $k \in \{1, \dots, n\}$

$$\begin{aligned}|x_s^{**}(x_k^*) - x_k^*(y_l)| &\le |x_s^{**}(x_k^*) - z_s^{**}(x_k^*)| + |z_s^{**}(x_k^*) - x_k^*(y_l)| \\ &\le 3\sqrt{\varepsilon}\,\|x_k^*\| + \delta.\end{aligned}$$

□

**Theorem II.2.12** *Let $X$ be an $L$-embedded space and let $Y$ be a closed subspace of $X$ which is not reflexive. Then $T(Y) = 2$.*

*Proof.* Fix $x_1, \dots, x_n \in S_Y$ and $\varepsilon > 0$. We have to find $y \in S_Y$ with $\|x_k + y\| \geq 2 - \varepsilon$ for $k = 1, \dots, n$.

Choose $\delta > 0$ with $7\sqrt{\delta} + 2\delta \leq \varepsilon$. By Corollary II.2.8 and James's $\ell^1$ distortion theorem [3, Theorem 10.3.1], there is a sequence $(y_l)_{l\in\mathbb{N}}$ in $Y$ with

$$(1-\delta)\sum_{l=1}^{\infty}|\alpha_l| \leq \left\|\sum_{l=1}^{\infty}\alpha_l y_l\right\| \leq \sum_{l=1}^{\infty}|\alpha_l|$$

for any sequence of scalars $(\alpha_l)_{l\in\mathbb{N}}$ with finite support. Let $X_s$ be the $L$-summand in $X^{**}$ which is complementary to $X$ and let $x_s^{**} \in X_s$ be "close" to a weak* accumulation point of $(y_l)_{l\in\mathbb{N}}$ as in Lemma II.2.11. Since $X^{**} = X \oplus_1 X_s$, we have for $k = 1, \dots, n$

$$\|x_k + x_s^{**}\| = \|x_k\| + \|x_s^{**}\| \geq 2 - 4\sqrt{\delta}.$$

Thus there exist functionals $x_1^*, \dots, x_n^* \in S_{X^*}$ with

$$|x_k^*(x_k) + x_s^{**}(x_k^*)| \geq 2 - 4\sqrt{\delta} - \delta$$

and $l \in \mathbb{N}$ with

$$|x_s^{**}(x_k^*) - x_k^*(y_l)| \leq 3\sqrt{\delta} + \delta$$

for $k = 1, \dots, n$.

Fix $k \in \{1, \dots, n\}$. Using the last two inequalities leads to

$$\begin{aligned}\|x_k + y_l\| &\geq |x_k^*(x_k) + x_k^*(y_l)| \\ &\geq |x_k^*(x_k) + x_s^{**}(x_k^*)| - |x_s^{**}(x_k^*) - x_k^*(y_l)| \\ &\geq (2 - 4\sqrt{\delta} - \delta) - (3\sqrt{\delta} + \delta) \\ &\geq 2 - \varepsilon.\end{aligned}$$

□

**Corollary II.2.13** *Let $X$ be an $L$-embedded space and let $Y$ be a separable, closed subspace of $X$. If $Y$ is not reflexive, then $Y$ has the almost Daugavet property.*

We say that a Banach space $X$ has the *fixed point property* if given any non-empty, closed, bounded and convex subset $C$ of $X$, every non-expansive mapping $T : C \to C$ has a fixed point. Here $T$ is non-expansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. By considering

$$C = \{(x_n)_{n\in\mathbb{N}} \in S_{\ell^1} : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$$

and the right shift operator, it can be shown that $\ell^1$ does not have the fixed point property [13, Theorem 1.2]. This counterexample can be transferred to all Banach spaces that contain an *asymptotically isometric copy of* $\ell^1$. A Banach space $X$ is said to contain an asymptotically isometric copy of $\ell^1$ if there is a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, 1)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ such that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|\alpha_n| \leq \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\| \leq \sum_{n=1}^{\infty}|\alpha_n|$$

for any sequence of scalars $(\alpha_n)_{n\in\mathbb{N}}$ with finite support. Using Lemma II.1.1, it can be shown that every Banach space $X$ with $T(X) = 2$ contains an asymptotically isometric copy of $\ell^1$. So Theorem II.2.12 gives another proof of the fact that every non-reflexive subspace of $L^1[0,1]$ or more generally every non-reflexive subspace of an $L$-embedded space fails the fixed point property (cf. [13, Theorem 1.4; 49, Corollary 4]).

# III Translation-Invariant Subspaces

Let $\mathbb{T}$ be the circle group, i.e., the multiplicative group of all complex numbers of absolute value one. If we endow $\mathbb{T}$ with the canonical topology, it becomes a compact topological group without isolated points. Considering the Borel $\sigma$-algebra $\mathscr{B}(\mathbb{T})$ and the normalized Lebesgue measure $\frac{1}{2\pi}\lambda$ on $\mathbb{T}$, we observe that $(\mathbb{T}, \mathscr{B}(\mathbb{T}), \frac{1}{2\pi}\lambda)$ is a probability space without atoms. So $C(\mathbb{T})$ and $L^1(\mathbb{T})$ have the Daugavet property.

Since $\mathbb{T}$ has a group structure, we can translate every function $f : \mathbb{T} \to \mathbb{C}$ and define for $t \in \mathbb{T}$ the function $f_t$ by

$$f_t(u) = f(ut^{-1}) \quad (u \in \mathbb{T}).$$

For every $t \in \mathbb{T}$, the operator $f \mapsto f_t$ is an isometry from $C(\mathbb{T})$ onto $C(\mathbb{T})$ and from $L^1(\mathbb{T})$ onto $L^1(\mathbb{T})$. A natural class of subspaces of $C(\mathbb{T})$ or $L^1(\mathbb{T})$ are the translation-invariant subspaces, i.e., subspaces that contain with a function $f$ all possible translates $f_t$. Now the questions arise which closed translation-invariant subspaces of $C(\mathbb{T})$ or $L^1(\mathbb{T})$ inherit the Daugavet property and which quotients with respect to a closed translation-invariant subspace are Daugavet spaces.

But there is no need to restrict our studies to $\mathbb{T}$.

## III.1 Basic concepts of abstract harmonic analysis

Let $G$ be a locally compact abelian group, with multiplication as group operation and $e_G$ as identity element, and denote by $\mathscr{B}(G)$ its Borel $\sigma$-algebra. Then there exists a measure $m$ on $\mathscr{B}(G)$ with the following properties [25, Theorem IV.15.5 and Remarks IV.15.8]:

- $m$ is locally finite but not identically zero.
- $m$ is outer regular, i.e., $m(A) = \inf\{m(O) : O \text{ is open and } A \subset O\}$ for all $A \in \mathscr{B}(G)$.
- For every open set $O$, we have $m(O) = \sup\{m(K) : K \text{ is compact and } K \subset O\}$.
- $m$ is translation-invariant, i.e., $m(Ax) = m(A)$ for all $x \in G$ and $A \in \mathscr{B}(G)$.

This measure is unique up to a positive multiplicative constant and is called the *Haar measure* of $G$. If $G$ is a compact group, then $m$ is actually a regular measure and it is customary to normalize $m$ so that $m(G) = 1$. If $G$ is discrete and infinite, we choose $m$ to be the counting measure. We can now consider the space $L^p(G, \mathscr{B}(G), m)$ for $1 \le p \le \infty$ and will write $L^p(G)$ instead of $L^p(G, \mathscr{B}(G), m)$.

Using the group structure of $G$, we can define the convolution of two functions.

**Definition III.1.1** Let $f, g : G \to \mathbb{C}$ be measurable functions. We define their convolution $f * g$ by the formula

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\, dm(y),$$

provided that

$$\int_G \left| f(xy^{-1})g(y) \right| \, dm(y) < \infty.$$

If $f \in L^1(G)$ and $g \in L^\infty(G)$, then $f * g$ is continuous [53, Theorem 1.1.6(b)], and if $f, g \in L^1(G)$, then $f * g$ is defined almost everywhere and $f * g \in L^1(G)$. If multiplication is defined by convolution, $L^1(G)$ becomes a commutative Banach algebra [53, Theorem 1.1.7].

The multiplicative functionals of $L^1(G)$ can be described via the so-called characters of $G$.

**Definition III.1.2** A (group) homomorphism from $G$ to the circle group $\mathbb{T}$ is called a *character* of $G$. The set of all continuous characters forms a group $\Gamma$, the *dual group* of $G$, if multiplication is defined by

$$(\gamma_1\gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad (x \in G; \gamma_1, \gamma_2 \in \Gamma).$$

We set $T(G) = \operatorname{lin} \Gamma$ and call every element of $T(G)$ a *trigonometric polynomial.*

Note that $\gamma^{-1} = \overline{\gamma}$ for every $\gamma \in \Gamma$ and that the identity element of $\Gamma$ coincides with the function identically equal to one that will be denoted by $\mathbf{1}_G$.

**Definition III.1.3** If $f \in L^1(G)$, the function $\hat{f}$ defined on $\Gamma$ by

$$\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)} \, dm(x) \quad (\gamma \in \Gamma),$$

is called the *Fourier transform* of $f$.

If $\gamma \in \Gamma$, the map $f \mapsto \hat{f}(\gamma)$ is a complex (algebra) homomorphism of $L^1(G)$ and is not identically zero. Conversely, every complex (algebra) homomorphism of $L^1(G)$ is obtained in this way and distinct characters induce distinct homomorphisms [53, Theorem 1.2.2]. So $\Gamma$ can be identified with the maximal ideal space of $L^1(G)$ and $\hat{f}$ is precisely the Gelfand transform of $f$. If we endow $\Gamma$ with the weak* topology, it becomes a locally compact abelian group [53, Theorem 1.2.6] and $\hat{f} \in C_0(\Gamma)$ for every $f \in L^1(G)$.

The "classical" groups that are studied in Fourier analysis are the real line $\mathbb{R}$ and the circle group $\mathbb{T}$.

$\mathbb{R}$ together with its canonical topology is a locally compact abelian group and its Haar measure is given by an adjusted Lebesgue measure. The function $x \mapsto \mathrm{e}^{\mathrm{i}xy}$ is a continuous character for every $y \in \mathbb{R}$. Via this correspondence, the dual group of $\mathbb{R}$ can be identified with $\mathbb{R}$ itself [53, Examples 1.2.7] and the Fourier transform takes the form

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}xy} \, dx \quad (y \in \mathbb{R}).$$

The circle group $\mathbb{T}$ together with its canonical topology is a compact abelian group and its Haar measure is given by the normalized Lebesgue measure. If we associate

an integer $n$ with the function $t \mapsto t^n$, we can identify the dual group of $\mathbb{T}$ with $\mathbb{Z}$ [53, Examples 1.2.7] and the Fourier transform takes the form

$$\hat{f}(n) = \int_{\mathbb{T}} f(t)t^{-n}\, dm(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\vartheta})\mathrm{e}^{-\mathrm{i}n\vartheta}\, d\vartheta \quad (n \in \mathbb{Z}).$$

We observe that the dual group of the compact group $\mathbb{T}$ is the discrete group $\mathbb{Z}$. Generally, the dual of a compact group is always a discrete group and the dual of a discrete group is always a compact group [53, Theorem 1.2.5].

Since $\Gamma$ is again a locally compact abelian group, we can consider the dual group of $\Gamma$. For every $x \in G$, a character on $\Gamma$ is given by evaluation at $x$. These characters are continuous [53, Theorem 1.2.6(a)] and by the Pontryagin duality theorem every continuous character on $\Gamma$ is of this form [53, Theorem 1.7.2]. So $G$ can be identified with its bidual group.

We can not only define the convolution of elements of $L^1(G)$, but also the convolution of measures.

**Definition III.1.4** [53, 1.3.1] Let $\lambda$ and $\mu$ be members of $M(G)$, let $\lambda \times \mu$ be their product measure on the space $G^2$, and associate to every Borel set $A$ in $G$ the set

$$A_{(2)} = \{(x, y) \in G^2 : xy \in A\}.$$

Then $A_{(2)}$ is a Borel set in $G^2$ and we define $\lambda * \mu$ by

$$(\lambda * \mu)(A) = (\lambda \times \mu)(A_{(2)}).$$

$M(G)$ is a commutative Banach algebra with unit if multiplication is defined by convolution [53, Theorem 1.3.2]. $L^1(G)$ can canonically be regarded as a subset of $M(G)$ and is actually a closed ideal of $M(G)$ [53, Theorem 1.3.5]. The Fourier transform can now be extended to $M(G)$.

**Definition III.1.5** If $\mu \in M(G)$, the function $\hat{\mu}$ defined on $\Gamma$ by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma(x)}\, d\mu(x) \quad (\gamma \in \Gamma)$$

is called the *Fourier-Stieltjes transform* of $\mu$.

For every $\gamma \in \Gamma$, the map $\mu \mapsto \hat{\mu}(\gamma)$ is a complex (algebra) homomorphism [53, Theorem 1.3.3]. Furthermore, the map $\mu \mapsto \hat{\mu}$ is injective, i.e., if $\hat{\mu} = 0$, then $\mu = 0$ [53, 1.7.3(b)].

The Banach algebra $L^1(G)$ does not have a unit, unless $G$ is discrete [53, 1.7.3(d)]. But approximate units are always available.

**Proposition III.1.6** [53, Theorem 1.1.8] *Given $f \in L^1(G)$ and $\varepsilon > 0$, there exists an open neighborhood $O$ of $e_G$ with the following property: if $v \in S_{L^1(G)}$ is a real-valued, non-negative function with $\chi_O v = v$, then $\|f - f * v\|_1 \leq \varepsilon$.*

We will furthermore need the following technical result.

**Proposition III.1.7** [53, Theorem 2.6.2] *If $O$ is an open set in $\Gamma$ which contains a compact set $K$, then there exists $v \in L^1(G)$ such that $\hat{v} = 1$ on $K$ and $\hat{v} = 0$ outside $O$.*

If $G$ is compact, various things become easier. Its dual group separates points (which is also true for locally compact abelian groups [53, 1.5.2]) and so the space of trigonometric polynomials $T(G)$ is dense in $C(G)$, by the Stone-Weierstrass theorem. Every continuous function on $G$ is integrable and especially $\Gamma \subset L^1(G)$. If $f \in C(G)$ and $\mu \in M(G)$, then $f * \mu \in C(G)$ and $C(G)$ is an ideal in $M(G)$. Furthermore, there exists always a net of trigonometric polynomials that is an approximate unit of $L^1(G)$ and $C(G)$.

**Proposition III.1.8** [26, Theorem VIII.33.12 and Remark VIII.32.33(a)] *Let $G$ be a compact abelian group. There is a net $(v_j)_{j \in J}$ in $L^1(G)$ with the following properties:*

(i) $\|f - f * v_j\|_1 \longrightarrow 0$ *for every* $f \in L^1(G)$*;*

(ii) $\|f - f * v_j\|_\infty \longrightarrow 0$ *for every* $f \in C(G)$*;*

(iii) $v_j \geq 0$, $v_j \in T(G)$ *and* $\hat{v}_j \geq 0$ *for every* $j \in J$*;*

(iv) $\|v_j\|_1 = 1$ *for every* $j \in J$*;*

(v) $\hat{v}_j(\gamma) \longrightarrow 1$ *for every* $\gamma \in \Gamma$*;*

**Example III.1.9** [65, Section III.3] The classical approximate unit of $L^1(\mathbb{T})$ that fulfills all properties mentioned in Proposition III.1.8 is the sequence of *Fejér kernels*. The Fejér kernel of order $n$ is defined by

$$K_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) t^k \quad (t \in \mathbb{T}).$$

## III.2 Subgroups, quotient groups and direct products

**Definition III.2.1**

(a) Let $H$ be a closed subgroup of $G$. The *annihilator* of $H$ is defined by

$$H^{\perp} = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in H\}$$

and is therefore a closed subgroup of $\Gamma$.

(b) If $(G_j)_{j\in J}$ is a family of locally compact abelian groups, we define their *direct product* (or their *complete direct sum*) by

$$\prod_{j\in J} G_j = \left\{ f : J \to \bigcup_{j\in J} G_j : f(j) \in G_j \text{ for every } j \in J \right\}$$

and define the group operation coordinatewise. Their *direct sum* is the subgroup

$$\bigoplus_{j\in J} G_j = \left\{ f \in \prod_{j\in J} G_j : f(j) = e_{G_j} \text{ for all but finitely many } j \in J \right\}.$$

**Proposition III.2.2** [53, Theorem 2.1.2] *Let $H$ be a closed subgroup of $G$. The dual group of $H$ can be identified with $\Gamma/H^{\perp}$ and the dual group of $G/H$ can be identified with $H^{\perp}$.*

**Proposition III.2.3** [53, Theorem 2.2.3] *Let $(G_j)_{j\in J}$ be a family of compact abelian groups. The dual group of $\prod_{j\in J} G_j$ can be identified with $\bigoplus_{j\in J} \Gamma_j$ if we interpret every $(\gamma_j)_{j\in J} \in \bigoplus_{j\in J} \Gamma_j$ as the function*

$$(x_j)_{j\in J} \mapsto \prod_{j\in J} \gamma_j(x_j).$$

## III.3 Translation-invariant subspaces

Using the group structure of $G$, we can translate functions and consider translation-invariant subspaces of $L^1(G)$ or $C(G)$.

**Definition III.3.1** Let $f : G \to \mathbb{C}$ be a function and let $x$ be an element of $G$. The *translate* $f_x$ of $f$ is defined by

$$f_x(y) = f(yx^{-1}) \quad (y \in G).$$

A subspace $X$ of $L^1(G)$ or $C(G)$ is called *translation-invariant* if $X$ contains with a function $f$ all possible translates $f_x$.

**Proposition III.3.2** [53, Theorem 7.1.2] *Let $G$ be a compact abelian group. A closed subspace $X$ of $L^1(G)$ is translation-invariant if and only if $X$ is an ideal of $L^1(G)$. Analogously, a closed subspace $X$ of $C(G)$ is translation-invariant if and only if $X$ is an ideal of $C(G)$.*

*Proof.* We will only prove the result for subspaces of $L^1(G)$. The proof for subspaces of $C(G)$ works the same way.

We start with the following observation. Let $X$ be a translation-invariant subspace of $L^1(G)$ and suppose that $\varphi \in L^\infty(G)$ annihilates $X$. That means

$$\int_G f(y^{-1})\varphi(y)\,dm(y) = 0 \quad (f \in X).$$

(We identify here an element $\varphi \in L^\infty(G)$ with the functional $f \mapsto \int_G f(y^{-1})\varphi(y)\,dm(y)$ on $L^1(G)$. This gives us an isometry between $L^1(G)^*$ and $L^\infty(G)$ as well.) Since $X$ contains every translate of $f$, if $f \in X$, we also have

$$\int_G f(xy^{-1})\varphi(y)\,dm(y) = 0 \quad (f \in X, x \in G).$$

Hence, to say that $\varphi \in L^\infty(G)$ annihilates $X$ is the same as to say that $f * \varphi = 0$ for all $f \in X$.

For $f, g \in L^1(G)$ and $\varphi \in L^\infty(G)$, we have

$$\int_G (f * g)(x^{-1})\varphi(x)\,dm(x) = \int_G g(y^{-1})(f * \varphi)(y)\,dm(y), \tag{3.1}$$

since each of these expressions is $(f * g * \varphi)(e_G)$.

Suppose now that $X$ is closed and translation-invariant, $\varphi \in L^\infty(G)$ annihilates $X$, $f \in X$, and $g \in L^1(G)$. Then $f * \varphi = 0$, the right-hand side of (3.1) is zero, and $\varphi$ annihilates $f * g$. Since this is true for every $\varphi$ that annihilates $X$, the Hahn-Banach theorem implies that $f * g \in X$, and $X$ is an ideal.

Conversely, suppose that $X$ is a closed ideal, $\varphi \in L^\infty(G)$ annihilates $X$, $f \in X$ and $g \in L^1(G)$. Then $f * g \in X$ and the left-hand side of (3.1) is zero. Hence $f * \varphi$ annihilates every $g \in L^1(G)$ and so $f * \varphi = 0$. This means that $\varphi$ annihilates every translate of $f$, and if we apply the Hahn-Banach theorem once more, we see that $X$ contains every translate of $f$. □

If $G$ is a compact abelian group, all closed ideals of $L^1(G)$ or $C(G)$ have a special form. Namely, they consist of all functions whose spectrum is contained in a subset $\Lambda$ of $\Gamma$.

**Definition III.3.3** For every $\mu \in M(G)$, we define its *spectrum* by

$$\operatorname{spec}(\mu) = \{\gamma \in \Gamma : \hat{\mu}(\gamma) \neq 0\}.$$

**Definition III.3.4** Let $G$ be a compact abelian group. For $\Lambda \subset \Gamma$, we set

$$L^1_\Lambda(G) = \{f \in L^1(G) : \operatorname{spec}(f) \subset \Lambda\}.$$

Analogously, we define $C_\Lambda(G)$, $L^\infty_\Lambda(G)$, $M_\Lambda(G)$, and $T_\Lambda(G)$.

**Proposition III.3.5** *Let $G$ be a compact abelian group and let $\Lambda$ be a subset of $\Gamma$. Then $T_\Lambda(G)$ is $\|\cdot\|_1$-dense in $L^1_\Lambda(G)$ and $\|\cdot\|_\infty$-dense in $C_\Lambda(G)$.*

*Proof.* This is an immediate consequence of the existence of trigonometric polynomials that are an approximate unit in $L^1(G)$ and $C(G)$ (see Proposition III.1.8). □

**Corollary III.3.6** *Let $G$ be a compact abelian group and let $\Lambda$ be a subset of $\Gamma$. Then $C_\Lambda(G)^\perp = M_{\Gamma\setminus\Lambda^{-1}}(G)$ and $L^1_\Lambda(G)^\perp = L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$.*

**Proposition III.3.7** [53, Theorem 7.1.5] *Let $G$ be a compact abelian group. A closed subspace $X$ of $L^1(G)$ or $C(G)$ is an ideal if and only if there exists $\Lambda \subset \Gamma$ such that $X = L^1_\Lambda(G)$ or $X = C_\Lambda(G)$.*

*Proof.* We consider just the case that $X$ is a subspace of $L^1(G)$. The other case can be proved similarly.

Since the map $f \mapsto \hat{f}(\gamma)$ is multiplicative and continuous for every $\gamma \in \Gamma$, it is clear that every subspace of the form $L^1_\Lambda(G)$ is a closed ideal.

Assume now that $X$ is a closed ideal of $L^1(G)$. Set

$$\Lambda = \bigcup_{f\in X} \operatorname{spec}(f).$$

If $\gamma \in \Lambda$, there exists $g \in X$ with $\hat{g}(\gamma) = 1$, and hence $g * \gamma = \gamma$, regarding $\gamma$ as a member of $L^1(G)$. Since $X$ is an ideal, $g * \gamma \in X$, and so $\gamma \in X$. It follows that $X$ contains $T_\Lambda(G)$. Since $T_\Lambda(G)$ is dense in $L^1_\Lambda(G)$ (see Proposition III.3.5) and $X$ is closed, we conclude that $L^1_\Lambda(G) = X$. □

## III.4 Special subsets of $\Gamma$

If not stated otherwise, $G$ denotes in the sequel a compact abelian group, $m$ its normalized Haar measure, $\Gamma$ its discrete dual group, and $\Lambda$ a subset of $\Gamma$. To measure how thin $\Lambda$ is, various Banach space properties of the spaces $C_\Lambda(G)$, $L^1_\Lambda(G)$, $L^\infty_\Lambda(G)$, and $M_\Lambda(G)$ are considered.

### III.4.1 Sidon sets

**Definition III.4.1** A set $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ of natural numbers which for some $q$ satisfies the inequalities

$$\frac{\lambda_{n+1}}{\lambda_n} > q > 1 \quad (n \in \mathbb{N})$$

is called a *Hadamard set*, in view of the Ostrowski-Hadamard gap theorem concerning the natural boundaries of power series of the form $\sum_{n=1}^\infty \alpha_{\lambda_n} z^{\lambda_n}$.

S. Sidon proved in 1927 the following result:

**Proposition III.4.2** [58] *Let $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ be a Hadamard set. If $f \in L^\infty_\Lambda(\mathbb{T})$, then*

$$\sum_{n=1}^\infty |\hat{f}(\lambda_n)| < \infty.$$

This result inspired in the 1950s the following definition:

**Definition III.4.3** We say that $\Lambda$ is a *Sidon set* if there is a constant $C$ (depending on $\Lambda$) such that

$$\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| \le C \|f\|_\infty \quad (f \in T_\Lambda(G)).$$

In other words, if the Fourier transform is continuous from $T_\Lambda(G)$ into $\ell^1(\Gamma)$.

**Proposition III.4.4** [42, Theorem 1.3] *The following assertions are equivalent:*

(i) *$\Lambda$ is a Sidon set.*

(ii) *To every bounded function $\varphi$ on $\Lambda$ there corresponds a measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in \Lambda$.*

(iii) *To every function $\varphi : \Lambda \to \{\pm 1\}$ there corresponds a measure $\mu \in M(G)$ such that*

$$\sup_{\gamma\in\Lambda} |\hat{\mu}(\gamma) - \varphi(\gamma)| < 1.$$

(iv) *If $f \in L^\infty_\Lambda(G)$, then $\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| < \infty$.*

(v) *If $f \in C_\Lambda(G)$, then $\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| < \infty$.*

*Proof.* It is obvious that (ii) implies (iii) and that (iv) implies (v).

(i) $\Rightarrow$ (ii): Suppose $\Lambda$ is a Sidon set with constant $C$ and let $\varphi$ be a bounded function on $\Lambda$. Hence

$$T(f) = \sum_{\gamma\in\Lambda} \hat{f}(\gamma)\varphi(\gamma)$$

defines a linear functional $T$ on the subspace $T_\Lambda(G)$ of $C(G)$. It is bounded since

$$|T(f)| \le \sum_{\gamma\in\Lambda} |\hat{f}(\gamma)||\varphi(\gamma)| \le C \|f\|_\infty \|\varphi\|_\infty .$$

We can extend $T$ by the Hahn-Banach theorem to a bounded functional on $C(G)$ and by the Riesz representation theorem there is a measure $\mu \in M(G)$ such that $\|\mu\| \le C \|\varphi\|_\infty$ and

$$T(f) = \int_G \overline{f(x)}\, d\mu(x) \quad (f \in T_\Lambda(G)).$$

Putting $f = \gamma \in \Lambda$, we obtain $\hat{\mu}(\gamma) = \varphi(\gamma)$.

(iii) $\Rightarrow$ (iv): For $f \in L^1(G)$, define $\tilde{f}$ by

$$\tilde{f}(x) = \overline{f(x^{-1})} \quad (x \in G).$$

The Fourier transform of $\tilde{f}$ is the complex conjugate of $\hat{f}$. If we define analogously for $\mu \in M(G)$ the measure $\tilde{\mu}$ by

$$\tilde{\mu}(A) = \overline{\mu(A^{-1})} \quad (A \in \mathscr{B}(G)),$$

then the Fourier-Stieltjes transform of $\tilde{\mu}$ is the complex conjugate of $\hat{\mu}$. For $f \in L^\infty_\Lambda(G)$, we can write $f = f_1 + \mathrm{i}f_2$ where $f_1 = \frac{1}{2}(f + \tilde{f})$ and $\mathrm{i}f_2 = \frac{1}{2}(f - \tilde{f})$. Then $\hat{f}_1$ and $\hat{f}_2$ are real-valued and $f_1, f_2 \in L^\infty_\Lambda(G)$. Since it suffices to show that (iv) is valid for $f_1$ and $f_2$, we can suppose without loss of generality that $\hat{f}$ is real-valued. Now define $\varphi : \Lambda \to \{\pm 1\}$ so that $\varphi\hat{f} = |\hat{f}|$. If $\Lambda$ has property (iii), then there exists a measure $\mu \in M(G)$ and $\delta > 0$ such that

$$|\hat{\mu}(\gamma) - \varphi(\gamma)| \leq 1 - \delta \quad (\gamma \in \Lambda). \tag{4.1}$$

If $\nu = \frac{1}{2}(\mu + \tilde{\mu})$, then $\hat{\nu}$ is the real part of $\hat{\mu}$ and $\hat{\nu}$ also satisfies (4.1). The function $g = \nu * f$ belongs to $L^\infty_\Lambda(G)$ and

$$\begin{aligned} |\hat{g}(\gamma) - |\hat{f}(\gamma)|| &= |\hat{\nu}(\gamma)\hat{f}(\gamma) - \varphi(\gamma)\hat{f}(\gamma)| \\ &= |\hat{\nu}(\gamma) - \varphi(\gamma)||\hat{f}(\gamma)| \\ &\leq (1-\delta)|\hat{f}(\gamma)| \end{aligned}$$

for all $\gamma \in \Lambda$. It follows that

$$\hat{g}(\gamma) \geq \delta|\hat{f}(\gamma)| \quad (\gamma \in \Lambda).$$

Let $(v_j)_{j\in J}$ be an approximate unit of $L^1(G)$ with $v_j \in T(G)$, $\hat{v}_j \geq 0$ and $\|v_j\|_1 = 1$ for all $j \in J$ (see Proposition III.1.8). Then we have for finite sets $\Delta \subset \Gamma$

$$\begin{aligned} \delta \sum_{\gamma\in\Delta} |\hat{f}(\gamma)|\hat{v}_j(\gamma) \leq \sum_{\gamma\in\Delta} \hat{g}(\gamma)\hat{v}_j(\gamma) &\leq \sum_{\gamma\in\Gamma} \hat{g}(\gamma)\hat{v}_j(\gamma) \\ &= (g * v_j)(e_G) \leq \|g * v_j\|_\infty \\ &\leq \|g\|_\infty \|v_j\|_1 = \|g\|_\infty . \end{aligned}$$

Since $\hat{v}_j(\gamma) \longrightarrow 1$ for each $\gamma \in \Gamma$, we conclude that

$$\delta \sum_{\gamma\in\Delta} |\hat{f}(\gamma)| \leq \|g\|_\infty .$$

Since $\Delta$ is an arbitrary finite subset of $\Gamma$, we have that $\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| < \infty$.

(v) $\Rightarrow$ (i): $C_\Lambda(G)$ is a closed subspace of $C(G)$, and if $\Lambda$ has the property (v), then the map $f \mapsto \hat{f}$ is well-defined and bijective from $C_\Lambda(G)$ onto $\ell^1(\Lambda)$. It is bounded since $\|f\|_\infty \leq \|\hat{f}\|_{\ell^1}$ for every $f \in C_\Lambda(G)$. Hence it is an isomorphism by the open mapping theorem and $\Lambda$ is a Sidon set. □

**Proposition III.4.5** [53, Theorem 5.7.5] *Let $\Lambda = \{\gamma_1, \gamma_2, \dots\}$ be countable and for any $\gamma \in \Gamma$ and any $k \in \mathbb{N}$ let $R_k(\Lambda, \gamma)$ be the number of representations of $\gamma$ in the form*

$$\gamma = \gamma_{n_1}^{\pm 1}\gamma_{n_2}^{\pm 1} \cdots \gamma_{n_k}^{\pm 1} \quad (n_1 < n_2 < \cdots < n_k). \tag{4.2}$$

*Suppose that $\Lambda$ satisfies the following conditions:*

(i) *If* $\gamma \in \Lambda$ *and* $\gamma \neq \gamma^{-1}$, *then* $\gamma^{-1} \notin \Lambda$.

(ii) *There is a constant* $C$ *such that*

$$R_k(\Lambda, \gamma) \leq C^k \quad (k \in \mathbb{N})$$

*for all* $\gamma \in \Lambda$ *and for* $\gamma = e_\Gamma$.

*Then* $\Lambda$ *is a Sidon set.*

*Proof.* We may assume without loss of generality that $e_\Gamma \notin \Lambda$ and will show that $\Lambda$ satisfies condition (iii) of Proposition III.4.4. Set $\beta = \frac{1}{3C^2}$ and let $\varphi : \Lambda \to \{\pm\beta\}$ be an arbitrary function. Define for all $l \in \mathbb{N}$ and $x \in G$

$$f_l(x) = \begin{cases} 1 + \varphi(\gamma_l)\gamma_l(x) + \varphi(\gamma_l)\gamma_l^{-1}(x) & \text{if } \gamma_l \neq \gamma_l^{-1} \\ 1 + \varphi(\gamma_l)\gamma_l(x) & \text{if } \gamma_l = \gamma_l^{-1} \end{cases}$$

and consider the Riesz products

$$P_n(x) = \prod_{l=1}^{n} f_l(x) \quad (x \in G, n \in \mathbb{N}).$$

Multiplying out, we see that

$$P_n(x) = 1 + \sum_{l=1}^{n} \varphi(\gamma_l)\gamma_l(x) + \sum_{\substack{l=1 \\ \gamma_l \neq \gamma_l^{-1}}}^{n} \varphi(\gamma_l)\gamma_l^{-1}(x) + \sum_{\gamma \in \Gamma} c_n(\gamma)\gamma(x), \tag{4.3}$$

where

$$|c_n(\gamma)| \leq \sum_{k=2}^{n} \sum |\varphi(\gamma_{n_1}) \cdots \varphi(\gamma_{n_k})|;$$

the inner sum extends over all $\gamma_{n_1}, \ldots, \gamma_{n_k}$ which satisfy (4.2) and hence has at most $C^k$ terms if $\gamma \in \Lambda$ or if $\gamma = e_\Gamma$. So

$$|c_n(\gamma)| \leq \sum_{k=2}^{\infty} C^k \beta^k = \frac{C^2\beta^2}{1 - C\beta} \leq \frac{1}{6C^2} \quad (\gamma \in \Lambda, \gamma = e_\Gamma). \tag{4.4}$$

Every $f_l$ is real-valued by construction and non-negative since $\beta < \frac{1}{2}$. So $P_n(x) \geq 0$ and by (4.4)

$$\|P_n\|_1 = 1 + c_n(e_\Gamma) \leq 1 + \frac{1}{6C^2} \quad (n \in \mathbb{N}).$$

In particular, $(P_n)_{n\in\mathbb{N}}$ is a bounded sequence in $M(G)$ and has therefore a weak* accumulation point $\mu \in M(G)$. (4.3) and (4.4) imply that

$$|\hat{\mu}(\gamma) - \varphi(\gamma)| \leq \frac{1}{6C^2} = \frac{\beta}{2} \quad (\gamma \in \Lambda).$$

Hence $\Lambda$ satisfies the condition (iii) of Proposition III.4.4. □

**Corollary III.4.6** [53, Example 5.7.6.(a)] *Every infinite subset of $\Gamma$ contains an infinite Sidon set.*

*Proof.* Let $\Lambda$ be an infinite subset of $\Gamma$. Fix $\gamma_1$ and $\gamma_2 \in \Lambda$ with $\gamma_2 \neq \gamma_1^{\pm 1}$. Set

$$S_3 = \{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \gamma_1^{\pm 1}\gamma_2^{\pm 1}\}.$$

Then $S_3$ is finite and we can find $\gamma_3 \in \Lambda \setminus S_3$. If $\gamma_1, \dots, \gamma_n$ are chosen, let $S_n$ be the set of all $\gamma \in \Gamma$ of the form

$$\gamma = \gamma_{k_1}^{\pm 1} \cdots \gamma_{k_l}^{\pm 1} \quad (k_1 < \cdots < k_l; 1 \leq l \leq n).$$

Then $S_n$ is finite and we can pick $\gamma_{n+1} \in \Lambda$ outside $S_n$. The infinite set $\{\gamma_n : n \in \mathbb{N}\}$ is contained in $\Lambda$ and satisfies the hypotheses of Proposition III.4.5. □

**Examples III.4.7**

1. $\{3^n : n \in \mathbb{N}\}$ is a Hadamard set and by Proposition III.4.2 together with Proposition III.4.4 a Sidon set.
2. The set
$$\Lambda = \{3^{2^{n+2}} + 3^{2^n + k} : k = 0, \dots, 2^n - 1; n \in \mathbb{N}_0\}$$
fulfills the hypotheses of Proposition III.4.5 with $C = 1$ and is therefore a Sidon set. But it cannot be written as a finite union of Hadamard sets [28, Examples 5.6].

## III.4.2 Rosenthal sets

Let us look back at condition (iv) of Proposition III.4.4. If $\Lambda$ is a Sidon set and $f \in L^\infty_\Lambda(G)$, then $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| < \infty$. Hence $\sum_{\gamma \in \Gamma} \hat{f}(\gamma)\gamma$ converges uniformly to a continuous function $g$. Since $\hat{f}(\gamma) = \hat{g}(\gamma)$ for all $\gamma \in \Gamma$, the functions $f$ and $g$ coincide almost everywhere. Summarized, we have that $L^\infty_\Lambda(G) = C_\Lambda(G)$. H. P. Rosenthal observed that this condition does not characterize Sidon sets and gave the following counterexample.

**Proposition III.4.8** [51, Corollary 4] *Set $\Lambda_n = \{1, \dots, n\}$ for $n \in \mathbb{N}$ and*

$$\Lambda = \bigcup_{n=1}^{\infty} (2n)!\Lambda_{2n}.$$

*Then $L^\infty_\Lambda(\mathbb{T}) = C_\Lambda(\mathbb{T})$, but $\Lambda$ is not a Sidon set.*

**Definition III.4.9** We say that $\Lambda$ is a *Rosenthal set* if $L^\infty_\Lambda(G) = C_\Lambda(G)$.

Let us consider various properties of Rosenthal sets.

**Lemma III.4.10** *If $X$ is a separable closed subspace of $C_\Lambda(G)$, then there exists a countable subset $\Lambda'$ of $\Lambda$ such that $X$ is contained in $C_{\Lambda'}(G)$.*

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be dense in $X$. By Proposition III.3.5, $T_\Lambda(G)$ is dense in $C_\Lambda(G)$ and so we can find sequences $(v_{n,k})_{k\in\mathbb{N}}$ in $T_\Lambda(G)$ with $v_{n,k} \longrightarrow x_n$ $(k \to \infty)$ for every $n \in \mathbb{N}$. If

$$\Lambda' = \bigcup_{n,k=1}^{\infty} \operatorname{spec}(v_{n,k}),$$

then $\Lambda'$ is countable and $X \subset C_{\Lambda'}(G)$. □

The proof of the following proposition is part of the proof of [44, Theorem 3].

**Proposition III.4.11** *If $\Lambda$ is a Rosenthal set, then $C_\Lambda(G)$ does not contain $c_0$.*

*Proof.* Suppose that $C_\Lambda(G)$ contains $c_0$. By Lemma III.4.10, we may assume that $\Lambda$ is countable. $L^\infty_\Lambda(G)$ is a dual space (namely, the dual of $L^1(G)/L^1_{\Gamma\setminus\Lambda^{-1}}(G)$, see Corollary III.3.6) and contains $c_0$ as well. A theorem due to C. Bessaga and A. Pełczyński states that if a dual space contains $c_0$, then it contains $\ell^\infty$ as well [41, Proposition I.2.e.8]. Hence $L^\infty_\Lambda(G)$ cannot be the same space as the separable space $C_\Lambda(G)$ and $\Lambda$ is not a Rosenthal set. □

**Theorem III.4.12** [24, Theorem IV.4.7; 45, Théorème 1] *The following assertions are equivalent:*

(i) *$\Lambda$ is a Rosenthal set.*

(ii) *$C_\Lambda(G)$ has the Radon-Nikodým property.*

(iii) *$C_{\Lambda'}(G)$ is a separable dual space for all countable $\Lambda' \subset \Lambda$.*

*Proof.* (i) ⇒ (iii): It is clear that $C_{\Lambda'}(G)$ is separable if $\Lambda'$ is countable. Every subset $\Lambda'$ of $\Lambda$ is a Rosenthal set and by Corollary III.3.6 we have

$$C_{\Lambda'}(G) = L^\infty_{\Lambda'}(G) \cong (L^1(G)/L^1_{\Gamma\setminus(\Lambda')^{-1}}(G))^*.$$

(iii) ⇒ (ii): It suffices to show that every separable closed subspace $X$ of $C_\Lambda(G)$ has the Radon-Nikodým property [12, Theorem III.3.2]. Using Lemma III.4.10, we can construct for such a space a countable $\Lambda' \subset \Lambda$ such that $X$ is contained in $C_{\Lambda'}(G)$. But the space $C_{\Lambda'}(G)$ has the Radon-Nikodým property as a separable dual space [12, Theorem III.3.1] and so $X$ has the Radon-Nikodým property as well [12, Theorem III.3.2].

(ii) ⇒ (i): We will split this proof into four parts.

First part: Fix $h \in L^\infty_\Lambda(G)$. We have to show that $h$ coincides almost everywhere with a continuous function $f$. Since the Fourier transform is multiplicative and the convolution of two bounded functions is continuous, the following map on $\mathscr{B}(G)$ is well-defined:

$$\begin{aligned} F : \mathscr{B}(G) &\longrightarrow C_\Lambda(G) \\ A &\longmapsto \chi_A * h. \end{aligned}$$

Since the convolution is a linear operator, it is clear that $F$ is finitely additive. We also have that for every Borel set $A$

$$\|F(A)\|_\infty = \|\chi_A * h\|_\infty \leq \|\chi_A\|_1 \|h\|_\infty = \|h\|_\infty m(A).$$

This implies that $F$ is $\sigma$-additive, absolutely continuous with respect to the Haar measure $m$, and of bounded variation. By the very definition of the Radon-Nikodým property, there is a Bochner integrable function $g : G \to C_\Lambda(G)$ such that

$$F(A) = \int_A g(x)\, dm(x) \quad (A \in \mathscr{B}(G)).$$

Second part: We are going to show that there is a null set $N$ such that

$$g(y)_{y^{-1}} = g(x)_{x^{-1}}$$

for almost all $y \in G$ if $x \notin N$. (Recall that $f_x$ denotes the translate of $f$ by $x$, i.e. $f_x(y) = f(yx^{-1})$; $T_x : f \mapsto f_x$ is the corresponding operator on $C(G)$.) If we fix $z \in G$, we obtain for every Borel set $A$

$$\begin{aligned}
\int_A g(x)\, dm(x) = F(A) = \chi_A * h = (\chi_{Az^{-1}} * h)_z &= \left(\int_{Az^{-1}} g(x)\, dm(x)\right)_z \\
&= T_z\left(\int_G \chi_{Az^{-1}}(x) g(x)\, dm(x)\right) = \int_G T_z(\chi_{Az^{-1}}(x) g(x))\, dm(x) \\
&= \int_G \chi_{Az^{-1}}(x) g(x)_z\, dm(x) = \int_G \chi_A(xz) g(x)_z\, dm(x) \\
&= \int_G \chi_A(x) g(xz^{-1})_z\, dm(x) = \int_A g(xz^{-1})_z\, dm(x).
\end{aligned}$$

The density function of a vector measure is uniquely determined [12, Corollary II.2.5] and so there exists a null set $N_z$ with $g(x) = g(xz^{-1})_z$ for $x \notin N_z$. Since $(x, z) \mapsto g(xz^{-1})_z$ is Bochner integrable, we deduce by using Fubini's theorem that

$$\int_G \int_G \left\|g(x) - g(xz^{-1})_z\right\|_\infty dm(z) dm(x) = \int_G \int_G \left\|g(x) - g(xz^{-1})_z\right\|_\infty dm(x) dm(z) = 0.$$

So there is a null set $N$ with

$$\int_G \left\|g(x) - g(xz^{-1})_z\right\|_\infty dm(z) = 0 \quad (x \notin N).$$

This proves that for almost every $y \in G$

$$g(y)_{y^{-1}} = g(x)_{x^{-1}},$$

if $x \notin N$.

Third part: Let us fix $x_0 \notin N$ and let us define

$$f = g(x_0)_{x_0^{-1}} \in C_\Lambda(G).$$

Consequently, $g(x) = f_x$ for almost every $x$. We will now prove that

$$\chi_A * h = \chi_A * f \quad (A \in \mathscr{B}(G)). \tag{4.5}$$

For an arbitrary $\mu \in M(G) = C(G)^*$, we get by Fubini's theorem the following:

$$\begin{aligned}
\langle \chi_A * f, \mu \rangle &= \int_G (\chi_A * f)(y)\, d\mu(y) = \int_G \int_G \chi_A(x) f(yx^{-1})\, dm(x) d\mu(y) \\
&= \int_G \int_A f(yx^{-1})\, dm(x) d\mu(y) = \int_A \int_G f(yx^{-1})\, d\mu(y) dm(x) \\
&= \int_A \langle f_x, \mu \rangle\, dm(x) = \left\langle \int_A f_x\, dm(x), \mu \right\rangle \\
&= \left\langle \int_A g(x)\, dm(x), \mu \right\rangle = \langle F(A), \mu \rangle \\
&= \langle \chi_A * h, \mu \rangle .
\end{aligned}$$

This implies (4.5).

Fourth part: If (4.5) holds, we have $\varphi * h = \varphi * f$ for all simple functions $\varphi$ and by continuity of the convolution in $L^1(G)$ for all $\varphi \in L^1(G)$. Especially, $\gamma * h = \gamma * f$ for all $\gamma \in \Gamma$. For fixed $\gamma \in \Gamma$, we obtain

$$\hat{h}(\gamma)\gamma = \gamma * h = \gamma * f = \hat{f}(\gamma)\gamma.$$

This means that $\hat{h}(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \Gamma$ and that $h$ and $f$ must coincide almost everywhere. □

### III.4.3 $\Lambda(p)$ sets

S. Sidon proved another interesting result during his studies of lacunary subsets of $\mathbb{Z}$.

**Proposition III.4.13** [57, Satz I] *Let $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ be a Hadamard set. Then there exists a constant $C$ such that*

$$\|f\|_2 \le C \|f\|_1 \quad (f \in T_\Lambda(\mathbb{T})).$$

This led to the following definitions:

**Definition III.4.14**

(i) Suppose $0 < r < p < \infty$. We say that $\Lambda$ *is of type* $(r, p)$ if there is a constant $C$ (depending on $\Lambda, r, p$) such that

$$\|f\|_p \le C \|f\|_r \quad (f \in T_\Lambda(G)).$$

In other words, if $\|\cdot\|_r$ and $\|\cdot\|_p$ are equivalent on $T_\Lambda(G)$.

(ii) Suppose $1 \le p < \infty$. We say that $\Lambda$ is a $\Lambda(p)$ *set* if $\Lambda$ is of type $(r, p)$ for some $r < p$.

An application of Hölder's inequality shows that if $\Lambda$ is of type $(r, p)$, then it is of type $(s, p)$ for all $s < p$ [52, Theorem 1.4]. Moreover, if $\Lambda$ is a $\Lambda(p)$ set and $r < p$, then $\Lambda$ is a $\Lambda(r)$ set.

Using this terminology, S. Sidon's result states that every Hadamard set is a $\Lambda(2)$ set. But more is true: Every Sidon set is a $\Lambda(p)$ set for every $p \geq 1$ [53, Theorem 5.7.7]. The converse is not valid, because there exist sets $\Lambda \subset \mathbb{Z}$ which are $\Lambda(p)$ sets for all $p \geq 1$, but which are not Rosenthal and especially not Sidon sets [39]. The Rosenthal set constructed in Proposition III.4.8 is not a $\Lambda(p)$ set due to the following result, which reinforces the impression that $\Lambda(p)$ sets are very small.

**Definition III.4.15**

(i) If $a$ and $b$ are integers with $b \neq 0$ and $n$ is a natural number, the set

$$\{a, a+b, a+2b, \ldots, a+(n-1)b\}$$

is called *arithmetic progression* with first term $a$, common difference $b$ and of length $n$.

(ii) We say that a subset $P$ of $\Gamma$ is a *parallelepiped of dimension* $n$ if $|P| = 2^n$ and if there exist $\chi_k, \psi_k \in \Gamma$ for $k = 1, \ldots, n$ with

$$P = \left\{ \prod_{k=1}^{n} \gamma_k : \gamma_k \in \{\chi_k, \psi_k\} \right\}.$$

Parallelepipeds are generalizations of arithmetic progressions. Any arithmetic progression of length $2^n$ is a parallelepiped of dimension $n$.

**Proposition III.4.16** [22, Theorem 1.2] *If $\Lambda$ is a $\Lambda(p)$ set for $p \geq 1$, then $\Lambda$ does not contain parallelepipeds of arbitrarily large dimension.*

We can characterize the translation-invariant subspaces of $L^1(G)$ that are reflexive using the notion of $\Lambda(p)$ sets.

**Proposition III.4.17** [23, Corollary] *$L^1_\Lambda(G)$ is reflexive if and only if $\Lambda$ is a $\Lambda(1)$ set.*

### III.4.4 Riesz sets

Let us recall the classical theorem of F. and M. Riesz that appears in the study of Hardy spaces.

**Theorem III.4.18** [54, Theorem 17.13] *Every $\mu \in M_{\mathbb{N}}(\mathbb{T})$ is absolutely continuous with respect to the Lebesgue measure.*

**Definition III.4.19** We call $\Lambda$ a *Riesz set* if every $\mu \in M_\Lambda(G)$ is absolutely continuous with respect to the Haar measure of $G$.

With this terminology, the F. and M. Riesz theorem states that $\mathbb{N}$ is a Riesz set of $\mathbb{Z}$. All Rosenthal sets are Riesz sets [45, Théorème 3], but for example $\mathbb{N}$ is not a

Rosenthal set since $L^\infty_{\mathbb{N}}(\mathbb{T})$ can be identified with the Hardy space $H^\infty$ which is not separable [17, Example V.1.5]. The natural numbers contain arbitrarily long arithmetic progressions and so $\mathbb{N}$ is no $\Lambda(p)$ set either, but conversely, every $\Lambda(p)$ set is a Riesz set [42, Theorem 5.3] (for $p = 1$ see also [23, Theorem]).

We saw in Theorem III.4.12 that $\Lambda$ is a Rosenthal set if and only if $C_\Lambda(G)$ has the Radon-Nikodým property. With essentially the same proof, one can show the following analogous result.

**Theorem III.4.20** [24, Theorem IV.4.7; 45, Théorème 2] *$\Lambda$ is a Riesz set if and only if $L^1_\Lambda(G)$ has the Radon-Nikodým property.*

### III.4.5 Shapiro sets

Let us introduce a special class of Riesz sets.

**Definition III.4.21** Let $(\Omega, \Sigma, \mu)$ be a probability space and let $X$ be a closed subspace of $L^1(\Omega)$. We say that $X$ is *nicely placed* if the unit ball of $X$ is closed with respect to convergence in measure.

A subset $\Lambda$ of $\Gamma$ is *nicely placed* if $L^1_\Lambda(G)$ is a nicely placed subspace of $L^1(G)$.

**Proposition III.4.22** [24, Theorem IV.3.5] *For a closed subspace $X$ of $L^1(\Omega)$ the following statements are equivalent:*

(i) *$B_X$ is closed with respect to convergence in measure.*

(ii) *$X$ is $L$-embedded, i.e., there is a closed subspace $X_s$ of $X^{**}$ with $X^{**} = X \oplus_1 X_s$ (where we identify $X$ with its image under the canonical embedding $i_X : X \to X^{**}$).*

**Definition III.4.23** We say that $\Lambda$ is a *Shapiro set* if all subsets of $\Lambda$ are nicely placed.

G. Godefroy coined the notion of "nicely placed subspaces" [19] and of "Shapiro sets" [18]. The second one was motivated by the work of J. H. Shapiro [55].

If $\Lambda$ is a $\Lambda(p)$ set and $\Lambda'$ a subset of $\Lambda$, then $L^1_{\Lambda'}(G)$ is reflexive (see Proposition III.4.17) and so trivially $L$-embedded. Hence every $\Lambda(p)$ set is a Shapiro set. Rosenthal sets need not be Shapiro sets [18, Proposition 3.8.1]. $\mathbb{N}$ is a Shapiro set [24, Example IV.4.11] but we have already seen in Section III.4.4 that $\mathbb{N}$ is neither a $\Lambda(p)$ set nor a Rosenthal set.

**Lemma III.4.24** [6, Théorème II; 55, Lemma 1.1] *Let $\mathscr{S}$ be the net of symmetric, open neighborhoods of $e_G$. For $V \in \mathscr{S}$, set $u_V = m(V)^{-1}\chi_V$. If $\mu \in M(G)$ is singular with respect to the Haar measure, then $(u_V * \mu)_{V\in\mathscr{S}}$ converges in Haar measure to zero.*

*Proof.* Since $|u_V * \mu| \le u_V * |\mu|$, we may assume without loss of generality that $\mu$ is a positive measure. Fix $\varepsilon, \delta > 0$. We have to find $V_0 \in \mathscr{S}$ such that

$$m(\{u_V * \mu \ge \varepsilon\}) \le \delta$$

for every $V \subset V_0$. By the regularity and singularity of $\mu$, there exist sets $K \subset O \subset G$ with $K$ compact, $O$ open, and

$$\begin{aligned}
&\mu(O) = \mu(G) = \|\mu\|\,, \\
&\mu(O \setminus K) \le \frac{\delta\varepsilon}{2}, \\
&m(O) \le \frac{\delta}{2}.
\end{aligned}$$

If we define $\lambda$ by $\lambda(A) = \mu(A \cap K)$ for every Borel set $A$ of $G$, then $\mu = \lambda + \vartheta$ where $\lambda$ is concentrated on $K$ and $\vartheta(G) \le \frac{\delta\varepsilon}{2}$.

Choose $V_0 \in \mathscr{S}$ such that $KV_0 \subset O$. The symmetry of $V_0$ implies that $(V_0 x) \cap K = \emptyset$ if $x \notin O$. Since

$$(u_V * \lambda)(x) = \frac{\lambda(Vx)}{m(V)} = \frac{\mu((Vx) \cap K)}{m(V)},$$

$u_V * \lambda$ vanishes off $O$ whenever $V \subset V_0$. So we have $(u_V * \mu)(x) = (u_V * \vartheta)(x)$ if $V \subset V_0$ and $x \notin O$. Hence

$$\int_{G \setminus O} u_V * \mu \, dm = \int_{G \setminus O} u_V * \vartheta \, dm \le \|u_V\|_1 \, \|\vartheta\| \le \frac{\delta\varepsilon}{2}.$$

Using Chebyshev's inequality, we get that

$$m(\{u_V * \mu \ge \varepsilon\} \cap (G \setminus O)) \le \frac{\delta}{2}.$$

Consequently, we have for every $V \subset V_0$ that

$$m(\{u_V * \mu \ge \varepsilon\}) \le \frac{\delta}{2} + m(O) \le \delta. \qquad \square$$

**Lemma III.4.25** [24, Lemma IV.4.3] *A subset $\Lambda$ of $\Gamma$ is a Riesz set if it has the following property: If $\Lambda' \subset \Lambda$ and $\mu \in M_{\Lambda'}(G)$, then $\mu_s \in M_{\Lambda'}(G)$ where $\mu_s$ denotes the part of $\mu$ that is singular with respect to the Haar measure.*

*Proof.* Fix $\mu \in M_\Lambda(G)$. We must show that $\mu_s = 0$. We have $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \notin \Lambda$, since $\mu_s \in M_\Lambda(G)$. Take now $\gamma \in \Lambda$ and consider $\Lambda' = \Lambda \setminus \{\gamma\}$ and $\nu = \mu - \hat{\mu}(\gamma)\gamma$. Then $\nu \in M_{\Lambda'}(G)$ and by assumption $\nu_s \in M_{\Lambda'}(G)$. But $\nu_s = \mu_s$, so $\hat{\mu}_s(\gamma) = \hat{\nu}_s(\gamma) = 0$. Therefore, $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in \Gamma$ and $\mu_s = 0$. $\square$

**Proposition III.4.26** [24, Proposition IV.4.5] *Every Shapiro set is a Riesz set.*

*Proof.* Suppose $\Lambda$ is a Shapiro set. We are going to show that $\Lambda$ meets the hypotheses of Lemma III.4.25. Fix $\Lambda' \subset \Lambda$, $\mu \in M_{\Lambda'}(G)$, and let $\mu = \mu_s + f\,dm$ be the Lebesgue decomposition of $\mu$ with respect to the Haar measure. We have to show that $\mu_s \in M_{\Lambda'}(G)$ or equivalently that $f \in L^1_{\Lambda'}(G)$. Since the net $(u_V)_{V \in \mathscr{S}}$ considered in Lemma III.4.24 is an approximate unit of $L^1(G)$ (see Proposition III.1.6), we get that

$\|f - u_V * f\|_1 \longrightarrow 0$. Thus $(u_V * f)_{V \in \mathscr{S}}$ converges in Haar measure to $f$. Lemma III.4.24 yields that $(u_V * \mu_s)_{V \in \mathscr{S}}$ converges in Haar measure to zero. So

$$u_V * \mu = u_V * f + u_V * \mu_s \xrightarrow{m} f.$$

Because $L^1(G)$ and $M_{\Lambda'}(G)$ are ideals, $(u_V * \mu)_{V \in \mathscr{S}}$ is a bounded net in $L^1_{\Lambda'}(G)$. Since $\Lambda$ is a Shapiro set, every closed ball in $L^1_{\Lambda'}(G)$ is closed with respect to convergence in Haar measure and so $f \in L^1_{\Lambda'}(G)$. □

The class of Shapiro sets is a proper subclass of the Riesz sets. The set

$$\Lambda = \bigcup_{n \in \mathbb{N}_0} \{k2^n : |k| \leq 2^n\}$$

is an example of a nicely placed Riesz set which is not a Shapiro set [24, Example IV.4.12].

### III.4.6 Semi-Riesz sets

**Definition III.4.27** Let $K$ be a compact space. A measure $\mu \in M(K)$ is said to be *diffuse* or *non-atomic* if $\mu(A) = 0$ for all countable sets $A \subset K$. We denote by $M_{\text{diff}}(K)$ the space of all diffuse members of $M(K)$.

If $G$ is an infinite compact abelian group, then the Haar measure on $G$ is diffuse and therefore every measure which is absolutely continuous with respect to the Haar measure as well. Hence every $\mu \in M_\Lambda(G)$ is diffuse if $\Lambda$ is a Riesz set. But this property is weaker than the property of being a Riesz set and gives rise to the following definition.

**Definition III.4.28** We call $\Lambda$ a *semi-Riesz set* if every $\mu \in M_\Lambda(G)$ is diffuse.

N. Wiener's theorem states that

$$\lim_{n \to \infty} \frac{1}{2n+1} \sum_{k=-n}^{n} |\hat{\mu}(k)|^2 = \sum_{t \in \mathbb{T}} |\mu(\{t\})|^2$$

for every $\mu \in M(\mathbb{T})$ [21, Theorem A.2.1]. Hence subsets $\Lambda$ of $\mathbb{Z}$ with density zero, that is

$$\lim_{n \to \infty} \frac{|\Lambda \cap \{-n, \ldots, n\}|}{2n+1} = 0,$$

are semi-Riesz sets. Let us construct a proper semi-Riesz set that is furthermore nicely placed.

**Example III.4.29** [20, p. 265] Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers with

$$\lambda_{n+1} > 4(\lambda_1 + \cdots + \lambda_n) \quad (n \in \mathbb{N})$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_{n+1}} < \infty.$$

Consider the Riesz products

$$P_n(t) = \prod_{k=1}^{n} \left(1 + \frac{1}{2}t^{\lambda_k} + \frac{1}{2}t^{-\lambda_k}\right) \quad (t \in \mathbb{T}, n \in \mathbb{N}).$$

Multiplying out, we see that

$$P_n(t) = \sum_{\varepsilon_1 \in \{-1,0,1\}} \cdots \sum_{\varepsilon_n \in \{-1,0,1\}} 2^{-(|\varepsilon_1|+\cdots+|\varepsilon_n|)} t^{\varepsilon_1\lambda_1+\cdots+\varepsilon_n\lambda_n}$$

and that

$$\Lambda_n = \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k : \varepsilon_k \in \{-1, 0, 1\} \right\}$$

is the spectrum of $P_n$. Every element of $\Lambda_n$ has a unique representation in the form $\sum_{k=1}^{n} \varepsilon_k \lambda_k$, since the sequence $(\lambda_n)_{n\in\mathbb{N}}$ grows very fast. Thus

$$\widehat{P}_n(l) = \begin{cases} 0 & l \notin \Lambda_n \\ 2^{-(|\varepsilon_1|+\cdots+|\varepsilon_n|)} & l = \sum_{k=1}^{n} \varepsilon_k \lambda_k \in \Lambda_n. \end{cases}$$

By construction, every $P_n$ is real-valued and non-negative. Therefore

$$\|P_n\|_1 = \int_{\mathbb{T}} P_n \, dm = \widehat{P}_n(0) = 1.$$

The set $\{P_n : n \in \mathbb{N}\}$ is consequently a bounded subset of $M(\mathbb{T})$ and we can find a weak* accumulation point $\mu \in M(\mathbb{T})$. If we set $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$, we get that

$$\hat{\mu}(l) = \begin{cases} 0 & l \notin \Lambda \\ 2^{-(|\varepsilon_1|+\cdots+|\varepsilon_n|)} & l = \sum_{k=1}^{n} \varepsilon_k \lambda_k \in \Lambda_n. \end{cases}$$

Hence $\mu$ is not only a weak* accumulation point but the weak* limit of $(P_n)_{n\in\mathbb{N}}$. The measure $\mu$ cannot belong to $L^1(\mathbb{T})$ because $\hat{\mu}(\lambda_n) = \frac{1}{2}$ for every $n \in \mathbb{N}$ and thus $\hat{\mu}(l) \not\to 0$ for $|l| \to \infty$. So the set $\Lambda$ is not a Riesz set, but has density zero and is therefore a semi-Riesz set.

Using [47, Théorème 6] and [18, Corollary 2.6], we can deduce that $\Lambda$ is nicely placed.

The idea to construct measures that are not absolutely continuous with respect to the Haar measure by considering appropriate Riesz products first appeared in a work of E. Hewitt and H. S. Zuckerman [27]. This idea was combined with N. Wiener's theorem by R. W. Chaney [9] in order to construct proper semi-Riesz sets.

### III.4.7 Localizable families

If $G$ is a compact abelian group, then $\Gamma$ is discrete. But there is another useful topology on $\Gamma$. If we write $G_d$ for $G$ equipped with the discrete topology and denote by $b\Gamma$ the dual group of $G_d$, then $b\Gamma$ is a compact abelian group that contains $\Gamma$ as a dense subgroup [53, Theorem 1.8.2]. We call $b\Gamma$ the *Bohr compactification* of $\Gamma$. Let us denote by $\tau$ the topology induced on $\Gamma$ by the Bohr compactification $b\Gamma$. It coincides with the topology of pointwise convergence and the sets

$$U(\gamma_0, x_1, \dots, x_n, \varepsilon) = \{\gamma \in \Gamma : |\gamma_0(x_k) - \gamma(x_k)| < \varepsilon \text{ for } k = 1, \dots, n\}$$

form a basis of $\tau$ [53, Theorem 1.2.6].

Y. Meyer used this topology to characterize Riesz sets [47] and his ideas led to the following definition.

**Definition III.4.30** Let $\mathscr{C}$ be a family of subsets of $\Gamma$. We say that $\mathscr{C}$ is *localizable* if the following holds: a subset $\Lambda$ of $\Gamma$ belongs to $\mathscr{C}$ if and only if for every $\gamma \in \Gamma$ there exists a $\tau$-open neighborhood $V$ of $\gamma$ such that $\Lambda \cap V \in \mathscr{C}$.

If $\mathscr{C}$ is a localizable family of subsets of $\Gamma$ which contains all finite sets, then $\mathscr{C}$ has to contain sets $\Lambda$ such that $\Lambda$ contains parallelepipeds of arbitrarily large dimension [38, Remark IV.2]. This implies that the class of Hadamard sets, the class of Sidon sets and the class of $\Lambda(p)$ sets is not localizable (see Proposition III.4.16). The class of Rosenthal sets is not localizable either [44, Theorem 3]. Y. Meyer showed that the class of Riesz sets is localizable [47, Théorème 1] and that the union of a Riesz set and a $\tau$-closed Riesz set is again a Riesz set [47, Théorème 2]. G. Godefroy extended these results and showed that the class of nicely placed sets and the class of Shapiro sets are localizable as well [18, Theorem 2.3].

Mimicking the proofs of Y. Meyer [47, Théorème 1 and Théorème 2], we get analogous results for the class of semi-Riesz sets.

We can decompose every element of $M(G)$ into its discrete and its diffuse part. Therefore

$$M(G) = \ell^1(G) \oplus_1 M_{\text{diff}}(G).$$

Denote by $P$ the projection from $M(G)$ onto $\ell^1(G)$.

**Lemma III.4.31** *If $\mu \in M(G)$ and $\sigma \in \ell^1(G)$, then*

$$P(\mu * \sigma) = P(\mu) * \sigma.$$

*Proof.* If $\nu \in M(G)$, $x \in G$, and $A \in \mathscr{B}(G)$, then

$$(\nu * \delta_x)(A) = \int_G \nu(Ay^{-1})\, d\delta_x(y) = \nu(Ax^{-1}).$$

Hence $\nu \in \ell^1(G)$ implies $\nu * \delta_x \in \ell^1(G)$ and $\nu \in M_{\text{diff}}(G)$ implies $\nu * \delta_x \in M_{\text{diff}}(G)$. Since $\ell^1(G)$ and $M_{\text{diff}}(G)$ are closed subspaces of $M(G)$ and every discrete measure is a limit

of measures with finite support, we conclude that $\ell^1(G)$ and $M_{\text{diff}}(G)$ are closed with respect to convolution with an element of $\ell^1(G)$.

If $\mu \in M(G)$ and $\sigma \in \ell^1(G)$, then

$$\mu * \sigma = P(\mu) * \sigma + (\text{Id} - P)(\mu) * \sigma.$$

Since $P(\mu)*\sigma \in \ell^1(G)$ and $(\text{Id}-P)(\mu)*\sigma \in M_{\text{diff}}(G)$, we have that $P(\mu*\sigma) = P(\mu)*\sigma$. □

**Proposition III.4.32** *Let $\Lambda$ be a subset of $\Gamma$. If there exists for every $\gamma \in \Gamma$ a $\tau$-open neighborhood $V$ of $\gamma$ such that $\Lambda \cap V$ is a semi-Riesz set, then $\Lambda$ is a semi-Riesz set.*

*Proof.* Fix $\mu \in M_\Lambda(G)$. We have to show that $P(\mu) = 0$ or equivalently that $\widehat{P(\mu)}(\gamma) = 0$ for all $\gamma \in \Gamma$. Fix $\gamma \in \Gamma$ and let $V$ be a $\tau$-open neighborhood of $\gamma$ such that $\Lambda \cap V$ is a semi-Riesz set. Identifying $\ell^1(G)$ and $L^1(G_d)$, Proposition III.1.7 implies that there exists $\sigma \in \ell^1(G)$ with $\hat{\sigma}(\gamma) = 1$ and $\hat{\sigma} = 0$ outside $V$. The spectrum of $\mu * \sigma$ is contained in $\Lambda \cap V$ and since $\Lambda \cap V$ is a semi-Riesz set we get by Lemma III.4.31 that

$$0 = P(\mu * \sigma) = P(\mu) * \sigma.$$

Consequently,

$$0 = (\widehat{P(\mu) * \sigma})(\gamma) = \widehat{P(\mu)}(\gamma)\hat{\sigma}(\gamma) = \widehat{P(\mu)}(\gamma).$$ □

**Corollary III.4.33** *The class of semi-Riesz sets is localizable.*

**Proposition III.4.34** *Let $\Lambda_1$ be a semi-Riesz set and let $\Lambda_2$ be a $\tau$-closed semi-Riesz set. Then $\Lambda_1 \cup \Lambda_2$ is a semi-Riesz set.*

*Proof.* Fix $\mu \in M_{\Lambda_1 \cup \Lambda_2}(G)$. We have to show that $P(\mu) = 0$. Since $\Lambda_2$ is a semi-Riesz set, it suffices to deduce that $\text{spec}(P(\mu)) \subset \Lambda_2$. Fix $\gamma \in \Gamma \setminus \Lambda_2$. Identifying $\ell^1(G)$ and $L^1(G_d)$, Proposition III.1.7 implies that there exists $\sigma \in \ell^1(G)$ with $\hat{\sigma}(\gamma) = 1$ and $\hat{\sigma} = 0$ on $\Lambda_2$. The spectrum of $\mu * \sigma$ is therefore contained in the semi-Riesz set $\Lambda_1$. Thus $P(\mu * \sigma) = 0$. Using Lemma III.4.31, we get

$$0 = \widehat{P(\mu * \sigma)}(\gamma) = (\widehat{P(\mu) * \sigma})(\gamma) = \widehat{P(\mu)}(\gamma)\hat{\sigma}(\gamma) = \widehat{P(\mu)}(\gamma).$$ □

**Examples III.4.35**

1. The proper semi-Riesz set that we constructed in Section III.4.6 is $\tau$-closed [47, Théorème 6].
2. Let $\mathbb{P}$ be the set of all prime numbers. Then $\{-1, 1\} \cup \mathbb{P}$ is a $\tau$-closed Riesz set [47, Proposition 3].
3. The set $\{n^2 : n \in \mathbb{Z}\}$ of all square numbers is a $\tau$-closed Riesz set [47, Proposition 4].

### III.4.8 Uniformly distributed sets

**Definition III.4.36** Let $\Lambda$ be a subset of $\mathbb{Z}$ and let $\lambda_1, \lambda_2, \dots$ be an enumeration of $\Lambda$ with $|\lambda_1| \leq |\lambda_2| \leq \cdots$. We say that $\Lambda$ is *uniformly distributed* if

$$\frac{1}{n}\sum_{k=1}^{n} t^{\lambda_k} \longrightarrow 0 \quad (t \in \mathbb{T}, t \neq 1).$$

The name comes from H. Weyl's classical criterion for the equidistribution of a real sequence mod $2\pi$. Note that J. Bourgain [7] used the notion of *ergodic sequence*.

Using the geometric summation formula, it is easy to show that $\mathbb{N}$ and $\mathbb{Z}$ are uniformly distributed. By a probabilistic approach, it is possible to prove the existence of uniformly distributed sets that are $\Lambda(p)$ sets for all $p \geq 1$ [40, Theorem II.2]. So uniformly distributed sets are in one sense rather large but can be quite thin, since they do not have to contain arbitrarily long arithmetic progressions.

# IV The Daugavet Property and Translation-Invariant Subspaces

We have now provided all the necessary terminology and all the necessary results in order to study the question which subspaces of the form $C_\Lambda(G)$ or $L^1_\Lambda(G)$ and which quotients of the form $C(G)/C_\Lambda(G)$ or $L^1/L^1_\Lambda(G)$ have the Daugavet property. If not stated otherwise, $G$ denotes in the sequel an infinite compact abelian group, $m$ its normalized Haar measure, $\Gamma$ its discrete dual group, and $\Lambda$ a subset of $\Gamma$.

## IV.1 Structure-preserving isometries

The Daugavet property depends crucially on the norm of a space and is preserved under isometries but in general not under isomorphisms. Considering translation-invariant subspaces of $C(G)$ and $L^1(G)$, it would be useful to know isometries that map translation-invariant subspaces onto translation-invariant subspaces.

**Definition IV.1.1** Let $G_1$ and $G_2$ be locally compact abelian groups with dual groups $\Gamma_1$ and $\Gamma_2$. Let $H : G_1 \to G_2$ be a continuous homomorphism. The *adjoint homomorphism* $H^* : \Gamma_2 \to \Gamma_1$ is defined by

$$H^*(\gamma) = \gamma \circ H \quad (\gamma \in \Gamma_2).$$

The adjoint homomorphism $H^*$ is continuous [25, Theorem VI.24.38], $H^{**} = H$ [25, VI.24.41.(a)], and $H^*[\Gamma_2]$ is dense in $\Gamma_1$ if and only if $H$ is one-to-one [25, VI.24.41.(b)].

**Lemma IV.1.2** *Let $H : G \to G$ be a continuous and surjective homomorphism. Then $H$ is measure-preserving, i.e., each Borel set $A$ of $G$ satisfies $m(H^{-1}[A]) = m(A)$.*

*Proof.* Denote by $\mu$ the push-forward of $m$ under $H$. It is easy to see that $\mu$ is regular and $\mu(G) = 1$. Since the Haar measure is uniquely determined, it suffices to show that $\mu$ is translation-invariant.

Fix $A \in \mathscr{B}(G)$ and $x \in G$. The mapping $H$ is surjective and thus there is $y \in G$ with $H(y) = x$. It is not difficult to check that $H^{-1}[AH(y)] = H^{-1}[A]y$. Using this equality, we get

$$\mu(Ax) = m(H^{-1}[AH(y)]) = m(H^{-1}[A]y) = m(H^{-1}[A]) = \mu(A). \qquad \square$$

**Proposition IV.1.3** *Suppose $H : \Gamma \to \Gamma$ is a one-to-one homomorphism. Then $C_\Lambda(G) \cong C_{H[\Lambda]}(G)$ and $L^1_\Lambda(G) \cong L^1_{H[\Lambda]}(G)$.*

*Proof.* If we define $T : C(G) \to C(G)$ by

$$T(f) = f \circ H^* \quad (f \in C(G)),$$

then $T$ is well-defined and an isometry because $H^*$ is continuous and surjective. (Note that $H^*[G]$ is dense since $H$ is one-to-one and that $H^*[G]$ is compact since $H^*$ is continuous.) For every trigonometric polynomial $f = \sum_{k=1}^n \alpha_k \gamma_k$ and every $x \in G$ we get

$$T(f)(x) = \sum_{k=1}^n \alpha_k \gamma_k(H^*(x)) = \sum_{k=1}^n \alpha_k H(\gamma_k)(x).$$

Hence $T$ maps for every $\Lambda \subset \Gamma$ the space $T_\Lambda(G)$ onto $T_{H[\Lambda]}(G)$ and by density the space $C_\Lambda(G)$ onto $C_{H[\Lambda]}(G)$.

Let us look at the same $T$ but now as an operator from $L^1(G)$ into itself. It is again an isometry because $H^*$ is measure-preserving by Lemma IV.1.2. It still maps for every $\Lambda \subset \Gamma$ the space $T_\Lambda(G)$ onto $T_{H[\Lambda]}(G)$ and so by density $L^1_\Lambda(G)$ onto $L^1_{H[\Lambda]}(G)$. □

**Corollary IV.1.4** *Let $H : \Gamma \to \Gamma$ be a one-to-one homomorphism. If $C_\Lambda(G)$ has the Daugavet property, then $C_{H[\Lambda]}(G)$ has the Daugavet property as well. Analogously, if $L^1_\Lambda(G)$ has the Daugavet property, then $L^1_{H[\Lambda]}(G)$ has the Daugavet property as well.*

**Example IV.1.5** Every one-to-one homomorphism on $\mathbb{Z}$ is of the form $k \mapsto nk$ where $n \neq 0$ is a fixed integer. So $C_\Lambda(\mathbb{T}) \cong C_{n\Lambda}(\mathbb{T})$ and $L^1_\Lambda(\mathbb{T}) \cong L^1_{n\Lambda}(\mathbb{T})$ for every integer $n \neq 0$.

## IV.2 Rich subspaces

We have seen in Theorem I.3.5 that a closed subspace $Y$ of a Daugavet space $X$ is rich if and only if every closed subspace $Z$ of $X$ with $Y \subset Z$ has the Daugavet property. In order to prove that a closed translation-invariant subspace $Y$ of $C(G)$ or $L^1(G)$ is rich, we do not have to consider all closed subspaces of $C(G)$ or $L^1(G)$ containing $Y$ but only the translation-invariant ones.

**Lemma IV.2.1** *Let $X$ be a Banach space. Suppose that for every $\varepsilon > 0$ there is a Daugavet space $Y$ and a surjective operator $T : X \to Y$ with*

$$(1-\varepsilon)\|x\| \leq \|T(x)\| \leq (1+\varepsilon)\|x\| \quad (x \in X). \tag{2.1}$$

*Then $X$ has the Daugavet property.*

*Proof.* Let $S : X \to X$ be an operator of rank one. We have to show that $S$ fulfills the equation $\|\mathrm{Id}_X + S\| = 1 + \|S\|$.

Fix $\varepsilon > 0$. By assumption, there exists a Banach space $Y$ with the Daugavet property and a surjective operator $T : X \to Y$ satisfying (2.1). It is easy to check that for every continuous operator $R : X \to X$ the norm of $TRT^{-1}$ can be estimated by

$$\frac{1-\varepsilon}{1+\varepsilon}\|R\| \leq \|TRT^{-1}\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|R\|.$$

Using this estimation and the fact that $Y$ has the Daugavet property, we get

$$\begin{aligned}\|\mathrm{Id}_X + S\| &\geq \frac{1-\varepsilon}{1+\varepsilon}\,\|\mathrm{Id}_Y + TST^{-1}\| \\ &= \frac{1-\varepsilon}{1+\varepsilon}\,(1 + \|TST^{-1}\|) \\ &\geq \frac{1-\varepsilon}{1+\varepsilon}\left(1 + \frac{1-\varepsilon}{1+\varepsilon}\,\|S\|\right).\end{aligned}$$

This finishes the proof because $\varepsilon > 0$ was chosen arbitrarily. □

**Proposition IV.2.2** *Suppose that $\Lambda$ is a subset of $\Gamma$ such that $C_\Theta(G)$ has the Daugavet property for all $\Lambda \subset \Theta \subset \Gamma$. Then $C_\Lambda(G)$ is a rich subspace of $C(G)$. The analogous statement is valid for subspaces of $L^1(G)$.*

*Proof.* We will only prove the result for subspaces of $C(G)$. The proof for subspaces of $L^1(G)$ works the same way.

By Theorem I.3.5, it suffices to show that for arbitrary $f_1, f_2 \in S_{C(G)}$ the linear span of $C_\Lambda(G)$, $f_1$ and $f_2$ has the Daugavet property. In order to do this, we are going to prove that $X = \mathrm{lin}\{C_\Lambda(G) \cup \{f_1, f_2\}\}$ meets the assumptions of Lemma IV.2.1.

Fix $\varepsilon > 0$ and let us suppose that $f_1$ does not belong to $C_\Lambda(G)$ and that $f_2$ does not belong to $\mathrm{lin}\{C_\Lambda(G) \cup \{f_1\}\}$; the other cases can be treated similarly. Then $X$ is isomorphic to $C_\Lambda(G) \oplus_1 \mathrm{lin}\{f_1\} \oplus_1 \mathrm{lin}\{f_2\}$ and there exists $M > 0$ with

$$M(\|h\|_\infty + |\alpha| + |\beta|) \leq \|h + \alpha f_1 + \beta f_2\|_\infty \quad (h \in C_\Lambda(G), \alpha, \beta \in \mathbb{C}).$$

Using the density of $T(G)$ in $C(G)$, we can choose $g_1, g_2 \in S_{T(G)}$ with $\|f_k - g_k\|_\infty \leq M\varepsilon$ for $k = 1, 2$. If we define $T : X \to \mathrm{lin}\{C_\Lambda(G) \cup \{g_1, g_2\}\}$ by

$$T(h + \alpha f_1 + \beta f_2) = h + \alpha g_1 + \beta g_2 \quad (h \in C_\Lambda(G), \alpha, \beta \in \mathbb{C}),$$

then $T$ is surjective and meets the assumption of Lemma IV.2.1 since

$$\begin{aligned}\|T(h + \alpha f_1 + \beta f_2) - (h + \alpha f_1 + \beta f_2)\|_\infty &\leq M\varepsilon(|\alpha| + |\beta|) \\ &\leq \varepsilon\,\|h + \alpha f_1 + \beta f_2\|_\infty\end{aligned}$$

for $h \in C_\Lambda(G)$ and $\alpha, \beta \in \mathbb{C}$.

To complete the proof, we have to show that $Y = \mathrm{lin}\{C_\Lambda(G) \cup \{g_1, g_2\}\}$ has the Daugavet property. Set $\Delta = \mathrm{spec}(g_1) \cup \mathrm{spec}(g_2)$. Since $g_1$ and $g_2$ are trigonometric polynomials, the set $\Delta$ is finite. By assumption, $C_{\Lambda\cup\Delta}(G)$ has the Daugavet property. The space $Y$ is a finite-codimensional subspace of $C_{\Lambda\cup\Delta}(G)$ and has therefore the Daugavet property as well (see Examples I.3.2). □

Not all translation-invariant subspaces of $C(G)$ or $L^1(G)$ with the Daugavet property must be rich. The space $C_{2\mathbb{Z}}(\mathbb{T})$ has the Daugavet property because $C(\mathbb{T}) \cong C_{2\mathbb{Z}}(\mathbb{T})$ by Corollary IV.1.4. But every $f \in C_{2\mathbb{Z}}(\mathbb{T})$ satisfies

$$f(t) = f(-t) \quad (t \in \mathbb{T})$$

and therefore $C_{2\mathbb{Z}}(\mathbb{T})$ cannot be a rich subspace of $C(\mathbb{T})$. Similarly, $L^1_{2\mathbb{Z}}(\mathbb{T})$ has the Daugavet property but is not a rich subspace of $L^1(\mathbb{T})$.

### IV.2.1 Rich subspaces of $C(G)$

D. Werner showed that $C_{\Gamma\setminus\Lambda^{-1}}(G)$ has the Daugavet property if $\Lambda$ is a semi-Riesz set [60, Theorem 3.7]. Let us present his proof in a different form using the characterization of the Daugavet property by weak* slices of the dual unit ball.

**Theorem IV.2.3** *If $\Lambda$ is a semi-Riesz set, then $C_{\Gamma\setminus\Lambda^{-1}}(G)$ has the Daugavet property.*

*Proof.* We first use Corollary III.3.6 and observe that

$$\begin{aligned} C_{\Gamma\setminus\Lambda^{-1}}(G)^* &\cong M(G)/M_\Lambda(G) \cong \left(\ell^1(G) \oplus_1 M_{\text{diff}}(G)\right)/M_\Lambda(G) \\ &\cong \ell^1(G) \oplus_1 M_{\text{diff}}(G)/M_\Lambda(G). \end{aligned}$$

This means that every $x^* \in C_{\Gamma\setminus\Lambda^{-1}}(G)^*$ can be identified with a pair $(\sum_{n=1}^\infty \alpha_n\delta_{x_n}, [\mu])$ where $[\mu] \in M_{\text{diff}}(G)/M_\Lambda(G)$ and $\sum_{n=1}^\infty \alpha_n\delta_{x_n}$ is an absolutely convergent sum of Dirac measures. Furthermore, $\|x^*\| = \|(\sum_{n=1}^\infty \alpha_n\delta_{x_n}, [\mu])\| = \sum_{n=1}^\infty |\alpha_n| + \|[\mu]\|$.

Fix $f \in C_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|f\|_\infty = 1$, $x^* \in C_{\Gamma\setminus\Lambda^{-1}}(G)^*$ with $\|x^*\| = 1$, and $\varepsilon > 0$. By Lemma I.2.4, it suffices to find $y^* \in C_{\Gamma\setminus\Lambda^{-1}}(G)^*$ with $\|y^*\| = 1$, $\operatorname{Re} y^*(f) \geq 1-\varepsilon$ and $\|x^* + y^*\| \geq 2-\varepsilon$. The open set $O = \{|f| > 1-\varepsilon\}$ is non-empty and contains infinitely many elements because $G$ has no isolated points. Let us identify $x^*$ with $(\sum_{n=1}^\infty \alpha_n\delta_{x_n}, [\mu])$. Since $(\alpha_n)_{n\in\mathbb{N}} \in \ell^1$, we can choose $y \in O$ satisfying

$$\sum_{n=1}^\infty |\alpha_n|\delta_{x_n}(\{y\}) \leq \varepsilon.$$

If we set $\lambda = \frac{|f(y)|}{f(y)}$ and identify $(\lambda\delta_y, 0)$ with $y^* \in C_{\Gamma\setminus\Lambda^{-1}}(G)^*$, we get

$$\operatorname{Re} y^*(f) = \operatorname{Re} \lambda f(y) = |f(y)| \geq 1-\varepsilon$$

and

$$\begin{aligned} \|x^* + y^*\| &= \left\|\left(\sum_{n=1}^\infty \alpha_n\delta_{x_n} + \lambda\delta_y, [\mu]\right)\right\| = \left\|\sum_{n=1}^\infty \alpha_n\delta_{x_n} + \lambda\delta_y\right\| + \|[\mu]\| \\ &\geq \sum_{n=1}^\infty |\alpha_n| + |\lambda| - \sum_{n=1}^\infty |\alpha_n|\delta_{x_n}(\{y\}) + \|[\mu]\| \\ &\geq \|x^*\| + 1 - \varepsilon = 2-\varepsilon. \end{aligned}$$

□

Combining this result with the fact that every subset of a semi-Riesz set is again a semi-Riesz set, we get by Proposition IV.2.2 the following corollary.

**Corollary IV.2.4** *If $\Lambda$ is a semi-Riesz set, then $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$.*

The converse implication is also valid.

**Lemma IV.2.5** *If $C_\Lambda(G)$ is a rich subspace of $C(G)$, then there exists for every $x \in G$, every open neighborhood $O$ of $e_G$, and every $\varepsilon > 0$ a real-valued and non-negative $f \in S_{C(G)}$ with $f(x) = 1$, $f|_{G\setminus(xO)} = 0$, and $d(f, C_\Lambda(G)) \leq \varepsilon$.*

*Proof.* Let $V$ be an open neighborhood of $e_G$ with $VV^{-1} \subset O$. Since $C_\Lambda(G)$ is a rich subspace of $C(G)$, we can pick by Corollary I.3.10 and Lemma I.3.7 a real-valued, non-negative $g \in S_{C(G)}$ with $g|_{G\setminus V} = 0$ and $d(g, C_\Lambda(G)) \leq \varepsilon$. Fix $x_0 \in V$ with $g(x_0) = 1$ and set $f = g_{xx_0^{-1}}$. This function is still at a distance of at most $\varepsilon$ from $C_\Lambda(G)$ because $C_\Lambda(G)$ is translation-invariant. Furthermore, $f(x) = 1$ and $f|_{G\setminus(xO)} = 0$ by our choice of $V$. In fact, if we pick $y \in G$ with $f(y) \neq 0$, we get that

$$g(yx_0x^{-1}) = f(y) \neq 0.$$

Consequently, $yx_0x^{-1} \in V$ and

$$y \in xx_0^{-1}V \subset xVV^{-1} \subset xO. \qquad \square$$

**Theorem IV.2.6** *If $C_\Lambda(G)$ is a rich subspace of $C(G)$, then $C_\Lambda(G)^\perp$ consists of diffuse measures.*

*Proof.* Let $[\mu]$ denote the equivalence class of $\mu$ in $M(G)/C_\Lambda(G)^\perp$. It suffices to show the following: For every $x \in G$, every $\alpha \in \mathbb{C}$, and every $\mu \in M(G)$ with $\mu(\{x\}) = 0$ we have $\|[\alpha\delta_x] + [\mu]\| = |\alpha| + \|[\mu]\|$. Indeed, if the preceding statement is true, we get for every $\mu \in C_\Lambda(G)^\perp$ and every $x \in G$ that

$$0 = \|[\mu]\| = \|[\mu(\{x\})\delta_x] + [\mu - \mu(\{x\})\delta_x]\| = |\mu(\{x\})| + \|[\mu - \mu(\{x\})\delta_x]\|.$$

Hence $|\mu(\{x\})| = 0$ and $\mu$ is a diffuse measure.

Fix $x \in G$, $\alpha \in \mathbb{C} \setminus \{0\}$, $\mu \in M(G)$ with $\mu(\{x\}) = 0$, and $\varepsilon > 0$. Choose $f \in S_{C_\Lambda(G)}$ with $\operatorname{Re} \int_G f\, d\mu \geq \|[\mu]\| - \varepsilon$. Since $|\mu|$ is a regular Borel measure and $f$ is a continuous function, there is an open neighborhood $O$ of $e_G$ with $|\mu|(xO) < \varepsilon$ and $|f(x) - f(xy)| < \varepsilon$ for all $y \in O$. As $C_\Lambda(G)$ is a rich subspace of $C(G)$, we can pick by Lemma IV.2.5 a real-valued, non-negative $g_0 \in S_{C(G)}$ with $g_0(x) = 1$, $g_0|_{G\setminus(xO)} = 0$ and $d(g_0, C_\Lambda(G)) < \varepsilon$. Let $g$ be an element of $C_\Lambda(G)$ with $\|g - g_0\|_\infty \leq \varepsilon$.

If we set

$$h_0 = f + \left(\frac{|\alpha|}{\alpha} - f(x)\right) g_0 \quad \text{and} \quad h = f + \left(\frac{|\alpha|}{\alpha} - f(x)\right) g,$$

then $h \in C_\Lambda(G)$ and $\|h - h_0\|_\infty \leq 2\varepsilon$. Furthermore,

$$\alpha h_0(x) = |\alpha| \tag{2.2}$$

and

$$\begin{aligned} \operatorname{Re} \int_G h_0\, d\mu &= \operatorname{Re} \int_G \left(f + \left(\frac{|\alpha|}{\alpha} - f(x)\right) g_0\right) d\mu \\ &\geq \|[\mu]\| - \varepsilon - 2\int_G g_0\, d|\mu| \\ &= \|[\mu]\| - \varepsilon - 2\int_{xO} g_0\, d|\mu| \\ &\geq \|[\mu]\| - \varepsilon - 2|\mu|(xO) \\ &\geq \|[\mu]\| - 3\varepsilon. \end{aligned} \tag{2.3}$$

Let us estimate the norm of $h$. We get for $y \in G \setminus (xO)$

$$\begin{aligned} |h(y)| &= \left| f(y) + \left( \frac{|\alpha|}{\alpha} - f(x) \right) g(y) \right| \\ &\leq \|f\|_\infty + 2 \, \|g|_{G\setminus(xO)}\|_\infty \leq 1 + 2\varepsilon \end{aligned}$$

and for $y \in xO$

$$\begin{aligned} |h(y)| &= \left| f(y) + \left( \frac{|\alpha|}{\alpha} - f(x) \right) g(y) \right| \\ &\leq |f(y) - f(x) g_0(y)| + g_0(y) + 2 \, \|g - g_0\|_\infty \\ &\leq |f(y) - f(x)| + |f(x)|(1 - g_0(y)) + g_0(y) + 2\varepsilon \\ &\leq \varepsilon + (1 - g_0(y)) + g_0(y) + 2\varepsilon = 1 + 3\varepsilon. \end{aligned}$$

Hence $\|h\|_\infty \leq 1 + 3\varepsilon$. Combining this estimate with (2.2) and (2.3), we get

$$\begin{aligned} (1 + 3\varepsilon) \, \|[\alpha\delta_x] + [\mu]\| &\geq \left| \int_G h \, d(\alpha\delta_x + \mu) \right| \\ &\geq \left| \int_G h_0 \, d(\alpha\delta_x + \mu) \right| - 2\varepsilon \, \|\alpha\delta_x + \mu\| \\ &\geq |\alpha| + \|[\mu]\| - 3\varepsilon - 2\varepsilon \, \|\alpha\delta_x + \mu\| \, . \end{aligned}$$

We can choose $\varepsilon > 0$ arbitrarily small and so $\|[\alpha\delta_x] + [\mu]\| = |\alpha| + \|[\mu]\|$. □

**Corollary IV.2.7** *The space $C_\Lambda(G)$ is a rich subspace of $C(G)$ if and only if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set.*

In the proof of Theorem IV.2.6 we showed that every Dirac measure $\delta_x$ still has norm one and still spans an $L$-summand if we consider it as an element of $C_\Lambda(G)^*$. Such subspaces are called *nicely embedded* and were studied by D. Werner [60]. His proof of the fact that $C_\Lambda(G)$ has the Daugavet property if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set, is as well based on the observation that then $C_\Lambda(G)$ is nicely embedded.

Let us present an alternative proof of Corollary IV.2.7 for the case that $G$ is metrizable. It is based on results of V. M. Kadets and M. M. Popov [34].

**Definition IV.2.8** Let $K$ be a metrizable, compact space and let $E$ be a Banach space. We say that an operator $T \in L(C(K), E)$ *vanishes* at a point $x \in K$ if there exist a sequence $(O_n)_{n\in\mathbb{N}}$ of open neighborhoods of $x$ with $\operatorname{diam} O_n \longrightarrow 0$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of real-valued and non-negative functions satisfying that $f_n \in S_{C(K)}$, $f_n|_{K\setminus O_n} = 0$, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $\chi_{\{x\}}$, and $\|T(f_n)\| \longrightarrow 0$. We denote by $\operatorname{van}(T)$ the set of all vanishing points of $T$.

**Lemma IV.2.9** [34, Lemma 1.6] *Let $K$ be a metrizable, compact space without isolated points and let $E$ be a Banach space. An operator $T \in L(C(K), E)$ is narrow if and only if* $\operatorname{van}(T)$ *is dense in $K$.*

*Proof.* Recall first that $T$ is narrow if and only if $T$ is $C$-narrow (see Corollary I.3.10). It is clear that $T$ is $C$-narrow if $\operatorname{van}(T)$ is dense in $K$.

Suppose now that $T$ is $C$-narrow. We define for every $n \in \mathbb{N}$ the sets

$$A_n = \left\{ f \in S_{C(K)} : f \geq 0,\ \operatorname{diam}\operatorname{supp}(f) \leq \frac{1}{n},\ \|T(f)\| \leq \frac{1}{n} \right\}$$

and

$$B_n = \bigcup_{f \in A_n} \left\{ f > 1 - \frac{1}{n} \right\}.$$

The sets $B_n$ are open and dense in $K$ because $T$ is $C$-narrow. By Baire's category theorem, $B = \bigcap_{n=1}^{\infty} B_n$ is dense in $K$.

Let us prove that $\operatorname{van}(T) = B$. Since $\operatorname{van}(T) \subset B_n$ for every $n \in \mathbb{N}$, the set $\operatorname{van}(T)$ is contained in $B$. Fix now $x \in B$. For every $n \in \mathbb{N}$, the point $x$ belongs to $B_n$ and there exists a function $f_n \in A_n$ such that $f_n(x) \geq 1 - \frac{1}{n}$. The sequence $(f_n)_{n\in\mathbb{N}}$ meets all necessary requirements of Definition IV.2.8. So $x \in \operatorname{van}(T)$. □

**Lemma IV.2.10** [34, Lemma 1.7] *Let $K$ be a metrizable, compact space and let $E$ be a Banach space. For $T \in L(C(K), E)$ and $x \in K$ the following assertions are equivalent:*

(i) $x \in \operatorname{van}(T)$*;*

(ii) *For every $e^* \in E^*$, the point $x$ is not an atom of the measure corresponding to $T^*(e^*)$.*

*Proof.* (i) $\Rightarrow$ (ii): Fix $e^* \in E^*$ and denote by $\mu$ the measure corresponding to $T^*(e^*)$. Since $x \in \operatorname{van}(T)$, there exists a sequence of real-valued and non-negative functions $(f_n)_{n\in\mathbb{N}}$ in $S_{C(K)}$ which converges pointwise to $\chi_{\{x\}}$ and satisfies that $\|T(f_n)\| \longrightarrow 0$. Then

$$\begin{aligned} |\mu(\{x\})| &= \left| \int_K \chi_{\{x\}}\, d\mu \right| = \left| \lim_{n\to\infty} \int_K f_n\, d\mu \right| \\ &= \left| \lim_{n\to\infty} e^*(T(f_n)) \right| \leq \|e^*\| \lim_{n\to\infty} \|T(f_n)\| = 0 \end{aligned}$$

and $x$ is not an atom of $\mu$.

(ii) $\Rightarrow$ (i): Let $(O_n)_{n\in\mathbb{N}}$ be a sequence of open neighborhoods of $x$ with $\operatorname{diam} O_n \longrightarrow 0$ and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of real-valued and non-negative functions such that $g_n \in S_{C(K)}$, $g_n|_{K\setminus O_n} = 0$, and $g(x) = 1$ for every $n \in \mathbb{N}$. By assumption, $(T(g_n))_{n\in\mathbb{N}}$ converges weakly to zero. By Mazur's lemma, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in \operatorname{conv}\{g_k : k \geq n\}$ for every $n \in \mathbb{N}$ and $\|T(f_n)\| \longrightarrow 0$. This sequence meets all necessary requirements of Definition IV.2.8. Hence $x \in \operatorname{van}(T)$. □

**Theorem IV.2.11** *Let $G$ be a metrizable, infinite compact abelian group. The space $C_\Lambda(G)$ is a rich subspace of $C(G)$ if and only if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set.*

*Proof.* Let $\pi : C(G) \to C(G)/C_\Lambda(G)$ be the canonical quotient map and note that

$$\operatorname{ran}(\pi^*) = C_\Lambda(G)^\perp = M_{\Gamma\setminus\Lambda^{-1}}(G).$$

If $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set, then every element of $M_{\Gamma\setminus\Lambda^{-1}}(G)$ is a diffuse measure. By Lemma IV.2.10, we therefore have $\operatorname{van}(\pi) = G$ and $\pi$ is a narrow operator.

Conversely, if $\pi$ is narrow, it is an easy consequence of Lemma IV.2.5 that $\operatorname{van}(\pi) = G$. By Lemma IV.2.10, $M_{\Gamma\setminus\Lambda^{-1}}(G)$ must consist of diffuse measures and $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set. □

R. Demazeux studied uniformly distributed sets of $\mathbb{Z}$ and stated the question if there is a connection between uniformly distributed sets and semi-Riesz sets. Using one of his results, we can give a partial answer.

**Theorem IV.2.12** [11, Théorème I.1.7] *If $\Lambda \subset \mathbb{Z}$ is uniformly distributed, then $C_\Lambda(\mathbb{T})$ is a rich subspace of $C(\mathbb{T})$.*

*Proof.* Fix an open subset $O$ of $\mathbb{T}$ with $O \neq \mathbb{T}$ and $\varepsilon \in (0,1)$. By Corollary I.3.10, we have to find $f \in S_{C(\mathbb{T})}$ with $f|_{\mathbb{T}\setminus O} = 0$ and $d(f, C_\Lambda(\mathbb{T})) \leq \varepsilon$.

Since $C_\Lambda(\mathbb{T})$ is translation-invariant, we may assume that $1 \in O$. Set $A = \mathbb{T} \setminus O$. Let $\lambda_1, \lambda_2, \dots$ be an enumeration of $\Lambda$ with $|\lambda_1| \leq |\lambda_2| \leq \cdots$ and consider for every $n \in \mathbb{N}$ the function

$$g_n(t) = \frac{1}{n}\sum_{k=1}^{n} t^{\lambda_k} \quad (t \in \mathbb{T}).$$

Every $g_n$ belongs to $S_{C_\Lambda(\mathbb{T})}$ and satisfies that $g_n(1) = 1$. As $\Lambda$ is uniformly distributed, the sequence $(g_n|_A)_{n\in\mathbb{N}}$ converges pointwise to zero. If we interpret every $g_n|_A$ as an element of $C(A)$, then $(g_n|_A)_{n\in\mathbb{N}}$ converges weakly to zero and by Mazur's lemma there exists $g \in \operatorname{conv}\{g_n : n \in \mathbb{N}\}$ with $\|g|_A\|_\infty \leq \frac{\varepsilon}{2}$. Note that $g(1) = 1$ and $g \in S_{C_\Lambda(\mathbb{T})}$. The set $B = \{|g| \geq \varepsilon\}$ is non-empty and closed and $A \cap B = \emptyset$. Using Urysohn's lemma [63, Lemma 15.6], we can pick a continuous function $\varphi : \mathbb{T} \to [0,1]$ with $\varphi|_A = 0$ and $\varphi|_B = 1$. We now set $f = g\varphi$. Then $f \in S_{C(\mathbb{T})}$, $f|_{\mathbb{T}\setminus O} = f|_A = 0$, and $d(f, C_\Lambda(\mathbb{T})) \leq \|f - g\|_\infty \leq \varepsilon$. □

**Corollary IV.2.13** *If $\Lambda \subset \mathbb{Z}$ is uniformly distributed, then $\mathbb{Z} \setminus (-\Lambda)$ is a semi-Riesz set.*

### IV.2.2 Rich subspaces of $L^1(G)$

We have mentioned in Examples I.3.2 that a closed subspace $Y$ of a Daugavet space $X$ is rich if $(X/Y)^*$ has the Radon-Nikodým property. Let us apply this result to translation-invariant subspaces of $L^1(G)$.

**Proposition IV.2.14** *If $\Lambda$ is a Rosenthal set, then $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$.*

*Proof.* Suppose that $\Lambda$ is a Rosenthal set. By Corollary III.3.6, $L^\infty_\Lambda(G)$ can be identified with the dual space of $L^1(G)/L^1_{\Gamma\setminus\Lambda^{-1}}(G)$. Since $\Lambda$ is a Rosenthal set, $L^\infty_\Lambda(G)$ has the Radon-Nikodým property (see Theorem III.4.12) and $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$. □

In Section IV.3, we will give an example of a non-Rosenthal set $\Lambda$ such that $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$.

If we apply the same reasoning to the case of translation-invariant subspaces of $C(G)$, we can conclude that $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$ if $\Lambda$ is a Riesz set.

**Lemma IV.2.15** *If $O$ is an open neighborhood of $e_G$, then there exists a covering of $G$ by disjoint Borel sets $B_1, \dots, B_n$ with $B_k B_k^{-1} \subset O$ for $k = 1, \dots, n$.*

*Proof.* Let $V$ be an open neighborhood of $e_G$ with $VV^{-1} \subset O$. Since $G$ is compact, we can choose $x_1, \dots, x_n \in G$ with $G = \bigcup_{k=1}^n (x_k V)$. Set $B_1 = x_1 V$ and

$$B_k = (x_k V) \setminus \bigcup_{l=1}^{k-1} B_l \quad (k = 2, \dots, n).$$

Then $B_1, \dots, B_n$ is a covering of $G$ by disjoint Borel sets and for every $k \in \{1, \dots, n\}$

$$B_k B_k^{-1} \subset (x_k V)(x_k V)^{-1} \subset VV^{-1} \subset O.$$ □

**Theorem IV.2.16** *If $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$, then $\Lambda$ is a semi-Riesz set.*

*Proof.* The following proof is based on arguments used by G. Godefroy, N. J. Kalton and D. Li [20, Proposition III.10] and N. J. Kalton [37, Theorem 5.4].

Suppose that $\Lambda$ is not a semi-Riesz set. We will show that $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is not a rich subspace of $L^1(G)$.

Let $\mu \in M_\Lambda(G)$ be a non-diffuse measure and assume that $\mu(\{e_G\}) = 1$, i.e., $\mu = \delta_{e_G} + \nu$ with $\nu(\{e_G\}) = 0$. (If $\mu$ is not of this form, fix $x \in G$ with $\mu(\{x\}) \neq 0$ and consider the measure $\mu(\{x\})^{-1}(\mu * \delta_{x^{-1}}) \in M_\Lambda(G)$.) Let $R, S, T : L^1(G) \to L^1(G)$ be the convolution operators defined by $R(f) = \mu * f$, $S(f) = \nu * f$, and $T(f) = |\nu| * f$. Observe that $R = \mathrm{Id} + S$. Note that for every $\lambda \in M(G)$ and $f \in L^1(G)$ we have

$$(\lambda * f)(x) = \int_G f(xy^{-1})\, d\lambda(y)$$

for $m$-almost all $x \in G$ [25, Theorem V.20.12]. Therefore,

$$\|S(\chi_A)\|_1 \leq \|T(\chi_A)\|_1 \quad (A \in \mathscr{B}(G)).$$

We will first show that there exists $E \in \mathscr{B}(G)$ with $m(E) > 0$ such that $R|_{L^1(E)}$ is an isomorphism onto its image. (We write $L^1(E)$ for the subspace $\{f \in L^1(G) : \chi_E f = f\}$.) Since $\nu(\{e_G\}) = 0$, we can choose a sequence $(O_n)_{n\in\mathbb{N}}$ of open neighborhoods of $e_G$ with

$|\nu|(O_n) \longrightarrow 0$. For each $n \in \mathbb{N}$, use Lemma IV.2.15 to find a covering of $G$ by disjoint Borel sets $B_{n,1}, \ldots, B_{n,N_n}$ with $B_{n,k}B_{n,k}^{-1} \subset O_n$ for $k = 1, \ldots, N_n$. Set for every $n \in \mathbb{N}$

$$R_n = \sum_{k=1}^{N_n} P_{B_{n,k}} R P_{B_{n,k}}, \quad S_n = \sum_{k=1}^{N_n} P_{B_{n,k}} S P_{B_{n,k}}, \quad \text{and} \quad T_n = \sum_{k=1}^{N_n} P_{B_{n,k}} T P_{B_{n,k}}$$

where $P_A$ denotes for every $A \in \mathscr{B}(G)$ the projection from $L^1(G)$ onto $L^1(A)$ defined by $P_A(f) = \chi_A f$. Let for every $n \in \mathbb{N}$ the map $\rho_n$ be defined by

$$\rho_n(A) = \|T_n(\chi_A)\|_1 \quad (A \in \mathscr{B}(G)).$$

Since $T_n$ is continuous and maps positive functions to positive functions, it is a consequence of the monotone convergence theorem that $\rho_n$ is a positive Borel measure on $G$. Every $\rho_n$ is absolutely continuous with respect to the Haar measure $m$ and we denote by $\omega_n$ its Radon-Nikodým derivative with respect to $m$. For each $n \in \mathbb{N}$, we get

$$\begin{aligned}
\rho_n(G) = \|T_n(\chi_G)\|_1 &= \sum_{k=1}^{N_n} \int_{B_{n,k}} T(\chi_{B_{n,k}})(x)\, dm(x) \\
&= \sum_{k=1}^{N_n} \int_{B_{n,k}} \int_G \chi_{B_{n,k}}(xy^{-1})\, d|\nu|(y) dm(x) \\
&= \sum_{k=1}^{N_n} \int_{B_{n,k}} |\nu|(xB_{n,k}^{-1})\, dm(x) \\
&\leq \sum_{k=1}^{N_n} \int_{B_{n,k}} |\nu|(B_{n,k}B_{n,k}^{-1})\, dm(x) \leq |\nu|(O_n).
\end{aligned}$$

Therefore, $\rho_n(G) \longrightarrow 0$ and in particular $(\omega_n)_{n\in\mathbb{N}}$ converges in Haar measure to zero. So there exists a Borel set $D$ of $G$ with $m(D) > 0$ and $n_0 \in \mathbb{N}$ satisfying

$$\omega_{n_0}(x) \leq \frac{1}{2} \quad (x \in D).$$

Consequently,

$$\|S_{n_0}(\chi_A)\|_1 \leq \|T_{n_0}(\chi_A)\|_1 \leq \frac{1}{2} m(A)$$

for all Borel sets $A \subset D$ and $\| S_{n_0}|_{L^1(D)}\| \leq \frac{1}{2}$. Thus $(\mathrm{Id} + S_{n_0})|_{L^1(D)} = R_{n_0}|_{L^1(D)}$ is an isomorphism onto its image. Fix $k_0 \in \{1, \ldots, N_{n_0}\}$ with $m(D \cap B_{n_0,k_0}) > 0$ and set $E = D \cap B_{n_0,k_0}$. Then $R|_{L^1(E)}$ is an isomorphism onto its image because $\|R_{n_0}(f)\|_1 \leq \|R(f)\|_1$ for all $f \in L^1(E)$.

We will now finish the proof by showing that $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is not a rich subspace of $L^1(G)$. Let $\pi : L^1(G) \to L^1(G)/\ker(R)$ be the canonical quotient map and let $\widetilde{R} : L^1(G)/\ker(R) \to L^1(G)$ be a bounded operator with $R = \widetilde{R} \circ \pi$. Since $R|_{L^1(E)}$ is an isomorphism, $\pi|_{L^1(E)}$ is bounded below. By Corollary I.3.15, the operator $\pi$ cannot be narrow. As the space $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is contained in $\ker(R)$, it is not a rich subspace of $L^1(G)$. $\square$

**Corollary IV.2.17** *If $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$, then $C_\Lambda(G)$ is a rich subspace of $C(G)$.*

The sets $\mathbb{N}$ and $\mathbb{Z} \setminus \mathbb{N}$ are Riesz sets. Using Corollary IV.2.7, we deduce that $C_\mathbb{N}(\mathbb{T})$ is a rich subspace of $C(\mathbb{T})$. But by Theorem III.4.20, the space $L^1_\mathbb{N}(\mathbb{T})$ has the Radon-Nikodým property and therefore not the Daugavet property. So the converse of Corollary IV.2.17 is not true. Let us consider a more extreme example. There exists $\Lambda \subset \mathbb{Z}$ such that $\Lambda$ is uniformly distributed and a $\Lambda(p)$ set for all $p \geq 1$ [40, Theorem II.2]. Then $C_\Lambda(\mathbb{T})$ is a rich subspace of $C(\mathbb{T})$ by Theorem IV.2.12 and $L^1_\Lambda(\mathbb{T})$ is reflexive by Proposition III.4.17.

## IV.3 Products of compact abelian groups

If $G_1$ and $G_2$ are compact abelian groups, then $G_1 \oplus G_2$ is again a compact abelian group. Let us study the connection between rich subspaces of $C(G_1 \oplus G_2)$ or $L^1(G_1 \oplus G_2)$ and rich subspaces of $C(G_1)$ and $C(G_2)$ or $L^1(G_1)$ and $L^1(G_2)$.

**Proposition IV.3.1** *Let $G_1$ be an infinite compact abelian group, let $G_2$ be an arbitrary compact abelian group, let $\Lambda_1$ be a subset of $\Gamma_1$, and let $\Lambda_2$ be a subset of $\Gamma_2$.*

(a) *Suppose that $C_{\Lambda_1}(G_1)$ is a rich subspace of $C(G_1)$ and that $C_{\Lambda_2}(G_2)$ is a rich subspace of $C(G_2)$ (or, if $G_2$ is finite, that $\Lambda_2 = \Gamma_2$). Then $C_{\Lambda_1 \times \Lambda_2}(G_1 \oplus G_2)$ is a rich subspace of $C(G_1 \oplus G_2)$.*

(b) *Suppose that $C_{\Lambda_1}(G_1)$ is a rich subspace of $C(G_1)$ and that $\Lambda_2$ is non-empty. Then $C_{\Lambda_1 \times \Lambda_2}(G_1 \oplus G_2)$ has the Daugavet property.*

*Proof.* Set $G = G_1 \oplus G_2$ and $\Lambda = \Lambda_1 \times \Lambda_2$.

We start with part (a). Let $O$ be a non-empty open set of $G$ and let $\varepsilon$ be a positive number. By Corollary I.3.10, we have to find $f \in S_{C(G)}$ with $f|_{G \setminus O} = 0$ and $d(f, C_\Lambda(G)) \leq \varepsilon$.

Pick non-empty open sets $O_1 \subset G_1$ and $O_2 \subset G_2$ with $O_1 \times O_2 \subset O$ and $\delta > 0$ with $2\delta + \delta^2 \leq \varepsilon$. By assumption, there exist $f_k \in S_{C(G_k)}$ and $g_k \in T_{\Lambda_k}(G_k)$ with $f_k|_{G_k \setminus O_k} = 0$ and $\|f_k - g_k\|_\infty \leq \delta$ for $k = 1, 2$. If we set $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$, then $f \in S_{C(G)}$, $g \in T_\Lambda(G)$, and $f|_{G \setminus O} = 0$. Furthermore,

$$\begin{aligned} d(f, C_\Lambda(G)) &\leq \|f - g\|_\infty \\ &\leq \|f_1\|_\infty \|f_2 - g_2\|_\infty + \|g_2\|_\infty \|f_1 - g_1\|_\infty \\ &\leq \delta + (1 + \delta)\delta \leq \varepsilon. \end{aligned}$$

Let us now consider part (b). The space $C_{\Gamma_1 \times \Lambda_2}(G)$ can canonically be identified with $C(G_1, C_{\Lambda_2}(G_2))$, the space of all continuous functions from $G_1$ into $C_{\Lambda_2}(G_2)$, and has therefore the Daugavet property (see Examples I.2.2). We will prove that $C_\Lambda(G)$ is a rich subspace of $C_{\Gamma_1 \times \Lambda_2}(G)$. To do this, we will use Proposition I.3.9. So it is sufficient to show that for every non-empty open set $O$ of $G_1$, every $g \in T_{\Lambda_2}(G_2)$ with $\|g\|_\infty = 1$,

and every $\varepsilon > 0$ there exists $f \in S_{C(G_1)}$ with $f|_{G_1 \setminus O} = 0$ and $d(f \otimes g, C_\Lambda(G)) \leq \varepsilon$. Since $C_{\Lambda_1}(G_1)$ is a rich subspace of $C(G_1)$, there exist $f \in S_{C(G_1)}$ and $h \in T_{\Lambda_1}(G_1)$ with $f|_{G_1 \setminus O} = 0$ and $\|f - h\|_\infty \leq \varepsilon$. Then $h \otimes g \in T_\Lambda(G)$ and

$$d(f \otimes g, C_\Lambda(G)) \leq \|f \otimes g - h \otimes g\|_\infty \leq \|f - h\|_\infty \|g\|_\infty \leq \varepsilon. \qquad \square$$

**Proposition IV.3.2** *Let $G$ be the product of two compact abelian groups $G_1$ and $G_2$ and denote by $p$ the projection from $\Gamma = \Gamma_1 \oplus \Gamma_2$ onto $\Gamma_1$. If $C_\Lambda(G)$ is a rich subspace of $C(G)$, then $C_{p[\Lambda]}(G_1)$ is a rich subspace of $C(G_1)$ (or $p[\Lambda] = \Gamma_1$ if $G_1$ is finite).*

*Proof.* Let $O$ be a non-empty open set of $G_1$ and let $\varepsilon > 0$ be a positive number. By Corollary I.3.10, we have to find $f \in S_{C(G_1)}$ with $f|_{G_1 \setminus O} = 0$ and $d(f, C_{p[\Lambda]}(G_1)) \leq \varepsilon$. (Note that this is sufficient in the case of finite $G_1$ as well.)

Since $C_\Lambda(G)$ is a rich subspace of $C(G)$, there exist $f_0 \in S_{C(G)}$ and $g_0 \in T_\Lambda(G)$ with $f_0|_{G \setminus (O \times G_2)} = 0$ and $\|f_0 - g_0\|_\infty \leq \varepsilon$. Fix $(x_0, y_0) \in G$ with $|f_0(x_0, y_0)| = 1$. Setting $f = f_0(\,\cdot\,, y_0)$ and $g = g_0(\,\cdot\,, y_0)$, we get that $f \in S_{C(G_1)}$, $g \in T_{p[\Lambda]}(G_1)$, and $f|_{G_1 \setminus O} = 0$. Finally,

$$d(f, C_{p[\Lambda]}(G_1)) \leq \|f - g\|_\infty \leq \|f_0 - g_0\|_\infty \leq \varepsilon. \qquad \square$$

**Proposition IV.3.3** *Let $G_1$ and $G_2$ be infinite compact abelian groups, let $\Lambda_1$ be a subset of $\Gamma_1$, and let $\Lambda_2$ be a subset of $\Gamma_2$.*

(a) *If $L^1_{\Lambda_2}(G_2)$ is a rich subspace of $L^1(G_2)$, then $L^1_{\Gamma_1 \times \Lambda_2}(G_1 \oplus G_2)$ is a rich subspace of $L^1(G_1 \oplus G_2)$.*

(b) *Suppose that $L^1_{\Lambda_1}(G_1)$ is a rich subspace of $L^1(G_1)$ and that $\Lambda_2$ is non-empty. Then $L^1_{\Lambda_1 \times \Lambda_2}(G_1 \oplus G_2)$ has the Daugavet property.*

*Proof.* Set $G = G_1 \oplus G_2$ and $\Lambda = \Lambda_1 \times \Lambda_2$.

We start with part (a). The space $L^1(G)$ can canonically be identified with the Bochner space $L^1(G_1, L^1(G_2))$ and $L^1_{\Gamma_1 \times \Lambda_2}(G)$ with the subspace $L^1(G_1, L^1_{\Lambda_2}(G_2))$. Since $L^1_{\Lambda_2}(G_2)$ is a rich subspace of $L^1(G_2)$, the space $L^1(G_1, L^1_{\Lambda_2}(G_2))$ is rich in $L^1(G_1, L^1(G_2))$ by Proposition I.3.16.

Let us now consider part (b). Identifying again $L^1_{\Gamma_1 \times \Lambda_2}(G)$ with the Bochner space $L^1(G_1, L^1_{\Lambda_2}(G_2))$, we see that $L^1_{\Gamma_1 \times \Lambda_2}(G)$ has the Daugavet property (see Examples I.2.2). We will show that $L^1_\Lambda(G)$ is a rich subspace of $L^1_{\Gamma_1 \times \Lambda_2}(G)$. By Proposition I.3.14, it is sufficient to find for every Borel set $A$ of $G_1$, every $g \in T_{\Lambda_2}(G_2)$ with $\|g\|_1 = 1$, and every $\delta, \varepsilon > 0$ a balanced $\varepsilon$-peak $f$ on $A$ with $d(f \otimes g, L^1_\Lambda(G)) \leq \delta$.

Since $L^1_{\Lambda_1}(G_1)$ is a rich subspace of $L^1(G_1)$, there exist a balanced $\varepsilon$-peak $f$ on $A$ and $h \in T_{\Lambda_1}(G_1)$ with $\|f - h\|_1 \leq \delta$. Then $h \otimes g \in T_\Lambda(G)$ and

$$d(f \otimes g, L^1_\Lambda(G) \leq \|f \otimes g - h \otimes g\|_1 = \|f - h\|_1 \|g\|_1 \leq \delta. \qquad \square$$

**Proposition IV.3.4** *Let $G$ be the product of two compact abelian groups $G_1$ and $G_2$ and denote by $p$ the projection from $\Gamma = \Gamma_1 \oplus \Gamma_2$ onto $\Gamma_1$. If $L^1_\Lambda(G)$ is a rich subspace of $L(G)$, then $L^1_{p[\Lambda]}(G_1)$ is a rich subspace of $L^1(G_1)$ (or $p[\Lambda] = \Gamma_1$ if $G_1$ is finite).*

*Proof.* If $p[\Lambda] = \Gamma_1$, we have nothing to show. So let us assume that there exists $\gamma \in \Gamma_1 \setminus p[\Lambda]$. Set $\vartheta = \overline{\gamma} \otimes \mathbf{1}_{G_2}$ and $\Theta = \vartheta\Lambda$. The map $f \mapsto \vartheta f$ is an isometry from $L^1(G)$ onto $L^1(G)$ and maps $L^1_\Lambda(G)$ onto $L^1_\Theta(G)$. Analogously, the map $f \mapsto \overline{\gamma} f$ is an isometry from $L^1(G_1)$ onto $L^1(G_1)$ and maps $L^1_{p[\Lambda]}(G_1)$ onto $L^1_{\overline{\gamma}p[\Lambda]}(G_1)$. Note that

$$\overline{\gamma}p[\Lambda] = p[(\overline{\gamma}, \mathbf{1}_{G_2})\Lambda] = p[\Theta]$$

and that $\mathbf{1}_{G_1} \notin \overline{\gamma}p[\Lambda]$. Taking into account that $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$ if and only if $L^1_\Theta(G)$ is a rich subspace of $L^1(G)$ and that $L^1_{p[\Lambda]}(G_1)$ is a rich subspace of $L^1(G_1)$ if and only if $L^1_{p[\Theta]}(G_1)$ is a rich subspace of $L^1(G_1)$, we may assume that $\mathbf{1}_{G_1} \notin p[\Lambda]$.

Fix a Borel subset $A$ of $G_1$ and $\delta, \varepsilon > 0$. By Corollary I.3.15, we have to find a balanced $\varepsilon$-peak $f$ on $A$ with $d(f, L^1_{p[\Lambda]}(G_1)) \leq \delta$. By assumption, $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$ and therefore there are a balanced $\frac{\varepsilon}{3}$-peak $f_0$ on $A \times G_2$ and $g \in T_\Lambda(G)$ with $\|f_0 - g\|_1 \leq \frac{\delta}{6}$. Set

$$B = \{y \in G_2 : m_1(\{f_0(\,\cdot\,, y) = -1\}) > m_1(A) - \varepsilon\}$$

and

$$C = \left\{y \in G_2 : \|f_0(\,\cdot\,, y) - g(\,\cdot\,, y)\|_1 \leq \frac{\delta}{2}\right\}.$$

Note that the Haar measure on $G$ coincides with the product measure $m_1 \times m_2$ [25, Example IV.15.17.(i)] and that we may assume that $f_0(\,\cdot\,, y) \in L^1(G_1)$ for all $y \in G_2$ and that $B$ and $C$ are measurable [25, Theorem III.13.8]. We then get

$$\begin{aligned} m_1(A) - \frac{\varepsilon}{3} \leq m(\{f_0 = -1\}) &= \int_{G_2}\int_{G_1} \chi_{\{f_0=-1\}}(x,y)\, dm_1(x) dm_2(y) \\ &= \int_{G_2} m_1(\{f_0(\,\cdot\,, y) = -1\})\, dm_2(y) \\ &\leq m_2(B)m_1(A) + (1 - m_2(B))(m_1(A) - \varepsilon) \\ &= m_1(A) + m_2(B)\varepsilon - \varepsilon \end{aligned}$$

and

$$\begin{aligned} \frac{\delta}{6} \geq \|f_0 - g\|_1 &= \int_{G_2} \|f_0(\,\cdot\,, y) - g(\,\cdot\,, y)\|_1\, dm_2(y) \\ &\geq \frac{\delta}{2}(1 - m_2(C)). \end{aligned}$$

Hence $m_2(B) \geq \frac{2}{3}$ and $m_2(C) \geq \frac{2}{3}$. Therefore $B \cap C \neq \emptyset$ and we can choose $y_0 \in B \cap C$.

Let us gather the properties of $f_0(\,\cdot\,, y_0) \in L^1(G_1)$. It is clear that $f_0(\,\cdot\,, y_0)$ is real-valued, $f_0(\,\cdot\,, y_0) \geq -1$, and $\chi_A f_0(\,\cdot\,, y_0) = f_0(\,\cdot\,, y_0)$. As $y_0$ belongs to $B$ and $C$, we have

$$m_1(\{f_0(\,\cdot\,, y_0) = -1\}) > m_1(A) - \varepsilon$$

and

$$\|f_0(\,\cdot\,, y_0) - g(\,\cdot\,, y_0)\|_1 \leq \frac{\delta}{2}.$$

The function $g(\,\cdot\,, y_0)$ belongs to $T_{p[\Lambda]}(G)$ and $\mathbf{1}_{G_1} \notin p[\Lambda]$. So $\int_{G_1} g(x, y_0)\, dm_1(x) = 0$ and $|\int_{G_1} f_0(x, y_0)\, dm_1(x)| \leq \frac{\delta}{2}$. Modifying $f_0(\,\cdot\,, y_0)$ a little bit, we get a balanced $\varepsilon$-peak $f$ on $A$ with $\|f - g(\,\cdot\,, y_0)\|_1 \leq \delta$. □

Set $\Lambda = \mathbb{Z} \times \{0\}$. Then $\Lambda$ is not a Rosenthal set because $C_\Lambda(\mathbb{T}^2) \cong C(\mathbb{T})$ contains a copy of $c_0$ (see Proposition III.4.11). But $\mathbb{Z}^2 \backslash (-\Lambda) = \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ and $L^1_{\mathbb{Z}\times(\mathbb{Z}\backslash\{0\})}(\mathbb{T}^2)$ is a rich subspace of $L^1(\mathbb{T}^2)$ by Proposition IV.3.3(a). So the converse of Proposition IV.2.14 is not true.

Let us come back to examples of translation-invariant subspaces that have the Daugavet property but are not rich. The examples mentioned in Section IV.2 are of the following type: We take a one-to-one homomorphism $H : \Gamma \to \Gamma$ that is not onto. Then $C_{H[\Gamma]}(G)$ and $L^1_{H[\Gamma]}(G)$ have the Daugavet property but are not rich subspaces of $C(G)$ or $L^1(G)$. In this case, $\bigcap_{\gamma \in H[\Gamma]} \ker(\gamma)$ contains $\ker(H^*) \neq \{e_G\}$. Set $\Lambda = \mathbb{Z} \times \{1\}$. Using Proposition IV.3.1.(b) and IV.3.3.(b), we see that $C_\Lambda(\mathbb{T}^2)$ and $L^1_\Lambda(\mathbb{T}^2)$ have the Daugavet property. But they are not rich subspaces of $C(\mathbb{T}^2)$ or $L^1(\mathbb{T}^2)$ by Proposition IV.3.2 and IV.3.4. Furthermore, $\bigcap_{\gamma \in \Lambda} \ker(\gamma) = \{(1, 1)\}$.

## IV.4 Quotients with respect to translation-invariant subspaces

We will now study quotients of the form $C(G)/C_\Lambda(G)$ and $L^1(G)/L^1_\Lambda(G)$. The following lemma is the key ingredient for all results of this section.

**Lemma IV.4.1** *If we interpret $f \in C(G)$ as a functional on $M(G)$, we have*

$$\|\, f|_{L^1_\Lambda(G)}\| = \|\, f|_{M_\Lambda(G)}\|\,.$$

*Analogously, if we interpret $g \in L^1(G)$ as a functional on $L^\infty(G)$, we have*

$$\|\, g|_{C_\Lambda(G)}\| = \|\, g|_{L^\infty_\Lambda(G)}\|\,.$$

*Proof.* We will just show the first statement. The proof of the second statement works the same way.

It is clear that $\|\, f|_{L^1_\Lambda(G)}\| \leq \|\, f|_{M_\Lambda(G)}\|$ because $L^1_\Lambda(G) \subset M_\Lambda(G)$. In order to prove the reverse inequality, we may assume without loss of generality that $\|\, f|_{M_\Lambda(G)}\| = 1$. Fix $\varepsilon > 0$ and an approximate unit $(v_j)_{j\in J}$ of $L^1(G)$ that fulfills the properties listed in Proposition III.1.8. Pick $\mu \in M_\Lambda(G)$ with $\|\mu\| = 1$ and $|\int_G f\, d\mu| \geq 1 - \frac{\varepsilon}{2}$. Let $g = \sum_{k=1}^n \alpha_k \gamma_k$ be a trigonometric polynomial. Using that $\hat{v}_j(\gamma) \longrightarrow 1$ for every $\gamma \in \Gamma$, we can deduce that

$$\int_G g\, d(\mu * v_j) = \sum_{k=1}^n \alpha_k \hat{\mu}(\overline{\gamma_k}) \hat{v}_j(\overline{\gamma_k}) \longrightarrow \sum_{k=1}^n \alpha_k \hat{\mu}(\overline{\gamma_k}) = \int_G g\, d\mu.$$

So $\mu$ is the weak* limit of $(\mu * v_j)_{j\in J}$ because $T(G)$ is dense in $C(G)$. Fix $j_0 \in J$ with $|\int_G f\, d(\mu * v_{j_0})| \geq 1 - \varepsilon$. Since $\mu * v_{j_0} \in L^1_\Lambda(G)$ and $\|\mu * v_{j_0}\|_1 \leq 1$, we have that $\|\, f|_{L^1_\Lambda(G)}\| \geq 1 - \varepsilon$. As $\varepsilon > 0$ was chosen arbitrarily, this finishes the proof. □

**Theorem IV.4.2** *If $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$, then $C(G)/C_\Lambda(G)$ has the Daugavet property.*

*Proof.* Recall that $C_\Lambda(G)^\perp = M_{\Gamma\setminus\Lambda^{-1}}(G)$ because $T_\Lambda(G)$ is dense in $C_\Lambda(G)$ (see Proposition III.3.5). We can therefore identify the dual space of $C(G)/C_\Lambda(G)$ with $M_{\Gamma\setminus\Lambda^{-1}}(G)$.

Fix $[f] \in C(G)/C_\Lambda(G)$ with $\|[f]\| = 1$, $\mu \in M_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|\mu\| = 1$, and $\varepsilon > 0$. By Lemma I.2.4, we have to find $\nu \in M_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|\nu\| = 1$, $\operatorname{Re}\int_G f\,d\nu \geq 1-\varepsilon$, and $\|\mu+\nu\| \geq 2-\varepsilon$. Let $\mu = \mu_s + g\,dm$ be the Lebesgue decomposition of $\mu$ where $\mu_s$ and $m$ are singular and $g \in L^1(G)$.

If we interpret $f$ as a functional on $M(G)$, we have by Lemma IV.4.1 that

$$\|f|_{L^1_{\Gamma\setminus\Lambda^{-1}}(G)}\| = \|f|_{M_{\Gamma\setminus\Lambda^{-1}}(G)}\| = 1.$$

By assumption, $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$ and so there exists by Proposition I.3.4 a function $h \in L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|h\|_1 = 1$, $\operatorname{Re}\int_G fh\,dm \geq 1-\varepsilon$, and $\|g/\|g\|_1 + h\|_1 \geq 2-\varepsilon$. Setting $\nu = h\,dm$, we therefore get

$$\begin{aligned}
\|\mu+\nu\| = \|\mu_s\| + \|g+h\|_1 &= \|\mu_s\| + \left\|\frac{g}{\|g\|_1} + h - (1-\|g\|_1)\frac{g}{\|g\|_1}\right\|_1 \\
&\geq \|\mu_s\| + \left\|\frac{g}{\|g\|_1} + h\right\|_1 - (1-\|g\|_1) \\
&\geq \|\mu_s\| + (2-\varepsilon) - (1-\|g\|_1) \\
&= \|\mu\| + 1 - \varepsilon = 2-\varepsilon.
\end{aligned}$$

□

**Corollary IV.4.3** *If $\Lambda$ is a Rosenthal set, then $C(G)/C_\Lambda(G)$ has the Daugavet property.*

**Theorem IV.4.4** *If $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$, then $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property.*

*Proof.* Let us begin as in the proof of Theorem IV.4.2. By Corollary III.3.6, we can identify the dual space of $L^1(G)/L^1_\Lambda(G)$ with $L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$.

Fix $[f] \in L^1(G)/L^1_\Lambda(G)$ with $\|[f]\| = 1$, $g \in L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|g\|_\infty = 1$, and $\varepsilon > 0$. By Lemma I.2.4, we have to find $h \in L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|h\|_\infty = 1$, $\operatorname{Re}\int_G fh\,dm \geq 1-\varepsilon$, and $\|g+h\|_\infty \geq 2-\varepsilon$.

Choose $\delta \in (0,1)$ with $\frac{1-5\|f\|_1\delta}{1+3\delta} \geq 1-\frac{\varepsilon}{2}$, $\eta > 0$ such that $\int_A |f|\,dm \leq \delta$ for all $A \in \mathscr{B}(G)$ with $m(A) \leq \eta$, and $t \in \mathbb{T}$ with

$$m\left(\left\{\operatorname{Re} t^{-1}g \geq 1 - \frac{\varepsilon}{2}\right\}\right) > 0.$$

If we interpret $f$ as a functional on $L^\infty(G)$, we have by Lemma IV.4.1 that

$$\|f|_{C_{\Gamma\setminus\Lambda^{-1}}(G)}\| = \|f|_{L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)}\| = 1.$$

Pick $h_0 \in C_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|h_0\|_\infty = 1$ and $\operatorname{Re}\int_G fh_0\,dm \geq 1-\delta$. Since $h_0$ is uniformly continuous, there exists an open neighborhood $O$ of $e_G$ with

$$|h_0(x) - h_0(y)| \leq \delta \quad (xy^{-1} \in O)$$

and $m(O) \leq \eta$. By assumption, $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$ and so there exist by Lemma IV.2.5 a real-valued, non-negative $p_0 \in S_{C(G)}$ with $p_0|_{G\setminus O} = 0$ and $p_0(e_G) = 1$ and $p \in C_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|p_0 - p\|_\infty \leq \delta$. Then $V = \{p_0 > 1 - \delta\}$ is an open neighborhood of $e_G$ and $V \subset O$. An easy compactness argument shows that there exists $x_0 \in G$ with

$$m\left(\left\{x \in x_0V : \operatorname{Re} t^{-1}g(x) \geq 1 - \frac{\varepsilon}{2}\right\}\right) > 0.$$

If we set

$$h_1 = h_0 + (t - h_0(x_0))p_{x_0} \quad \text{and} \quad h = \frac{h_1}{\|h_1\|_\infty},$$

then $h$ is normalized and belongs by construction to $C_{\Gamma\setminus\Lambda^{-1}}(G)$. Let us estimate the norm of $h_1$. We get for $x \in G \setminus (x_0O)$

$$|h_1(x)| = |h_0(x) + (t - h_0(x_0))p(xx_0^{-1})| \leq \|h_0\|_\infty + 2\|p|_{G\setminus O}\|_\infty \leq 1 + 2\delta$$

and for $x \in x_0O$

$$\begin{aligned}
|h_1(x)| &= |h_0(x) + (t - h_0(x_0))p(xx_0^{-1})| \\
&\leq |h_0(x) - h_0(x_0)p_0(xx_0^{-1})| + p_0(xx_0^{-1}) + 2\,\|p - p_0\|_\infty \\
&\leq |h_0(x) - h_0(x_0)| + |h_0(x_0)|(1 - p_0(xx_0^{-1})) + p_0(xx_0^{-1}) + 2\delta \\
&\leq \delta + (1 - p_0(xx_0^{-1})) + p_0(xx_0^{-1}) + 2\delta \\
&= 1 + 3\delta.
\end{aligned}$$

Consequently, $\|h_1\|_\infty \leq 1 + 3\delta$. Let us check that $h$ is as desired. We first observe that

$$\begin{aligned}
\operatorname{Re}\int_G fh_1\,dm &\geq \operatorname{Re}\int_G fh_0\,dm - 2\int_G |fp_{x_0}|\,dm \\
&\geq (1 - \delta) - 2\int_{x_0O} |f|\,dm - 2\,\|f\|_1\,\|p_0 - p\|_\infty \\
&\geq (1 - \delta) - 2\delta - 2\,\|f\|_1\,\delta = 1 - (3 + 2\,\|f\|_1)\delta.
\end{aligned}$$

Therefore, $\operatorname{Re}\int_G fh\,dm \geq 1 - \varepsilon$ by our choice of $\delta$. If $x \in x_0V$, we get

$$\begin{aligned}
\operatorname{Re} t^{-1}h_1(x) &\geq \operatorname{Re} t^{-1}h_0(x) + \operatorname{Re}(1 - t^{-1}h_0(x_0))p_0(xx_0^{-1}) - 2\,\|p_0 - p\|_\infty \\
&\geq \operatorname{Re} t^{-1}h_0(x) + \operatorname{Re}(1 - t^{-1}h_0(x_0))(1 - \delta) - 2\delta \\
&\geq 1 - 3\delta - |h_0(x) - h_0(x_0)| \geq 1 - 4\delta
\end{aligned}$$

and hence $\operatorname{Re} t^{-1}h(x) \geq 1 - \frac{\varepsilon}{2}$ by our choice of $\delta$. Thus

$$\begin{aligned}
m(\{|g + h| \geq 2 - \varepsilon\}) &\geq m(\{\operatorname{Re} t^{-1}(g + h) \geq 2 - \varepsilon\}) \\
&\geq m\left(\left\{\operatorname{Re} t^{-1}g \geq 1 - \frac{\varepsilon}{2}\right\} \cap \left\{\operatorname{Re} t^{-1}h \geq 1 - \frac{\varepsilon}{2}\right\}\right) \\
&\geq m\left(\left\{\operatorname{Re} t^{-1}g \geq 1 - \frac{\varepsilon}{2}\right\} \cap (x_0V)\right) > 0
\end{aligned}$$

and $\|g + h\|_\infty \geq 2 - \varepsilon$. □

**Corollary IV.4.5** *If $\Lambda$ is a semi-Riesz set, then $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property.*

## IV.5 Poor subspaces of $L^1(G)$

In Section IV.4, we have seen some cases in which the quotient space $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property. It is now a natural question which subspaces of $L^1(G)$ are poor. Recall that a closed subspace $Y$ of a Daugavet space $X$ is called poor if $X/Z$ has the Daugavet property for every closed subspace $Z \subset Y$. Let us start with an example that is due to D. Werner.

**Example IV.5.1** Identify the Hardy space $H^1_0$ with $L^1_{\mathbb{N}}(\mathbb{T})$. Since $\mathbb{N}$ is a Riesz set, we have by Corollary IV.4.5 that $L^1(\mathbb{T})/H^1_0$ has the Daugavet property, a result that was already observed by P. Wojtaszczyk [64, p. 1051]. But $H^1_0$ is even a poor subspace of $L^1(\mathbb{T})$. Let us sketch a proof. The dual of $L^1(\mathbb{T})/H^1_0$ is $L^\infty_{\mathbb{N}_0}(\mathbb{T})$ and can be identified with $H^\infty$. Let $X$ be the maximal ideal space of $L^\infty(\mathbb{T})$. Under the Gelfand transform, $L^\infty(\mathbb{T})$ is isometric to $C(X)$ and $H^\infty$ is isometric to a closed subalgebra of $C(X)$. The Shilov boundary of $H^\infty$ (the smallest closed subset of $X$ on which every $f \in H^\infty$ attains its maximum) coincides with $X$ [17, Theorem V.1.7]. Let $O$ be a non-empty open subset of $X$ with $O \neq X$ and let $\varepsilon$ be a positive number. Since $A = X \setminus O$ is a proper closed subset of $X$, we can find a function $g \in S_{H^\infty}$ such that $\|g|_A\|_\infty < 1$. If $n \in \mathbb{N}$ is large enough, $\|g^n|_A\|_\infty \leq \frac{\varepsilon}{2}$. So we can construct a function $f \in S_{C(K)}$ with $f|_A = 0$ and $d(f, H^\infty) \leq \varepsilon$. Hence $H^\infty$ is a rich subspace of $L^\infty(\mathbb{T})$ by Corollary I.3.10 and $L^1_{\mathbb{N}}(\mathbb{T})$ is a poor subspace of $L^1(\mathbb{T})$ by Theorem I.4.5.

This example can now be extended. The key observation is that $\mathbb{N}$ is not only a Riesz set but also nicely placed, i.e., the unit ball of $L^1_{\mathbb{N}}(\mathbb{T})$ is closed with respect to convergence in measure.

In the sequel, we denote for $A \in \mathscr{B}(G)$ by $L^1(A)$ the space $\{f \in L^1(G) : \chi_A f = f\}$ and by $P_A$ the projection from $L^1(G)$ onto $L^1(A)$ defined by $P_A(f) = \chi_A f$.

**Lemma IV.5.2** *Let $X$ be a nicely placed subspace of $L^1(G)$ and suppose that there exists $A \in \mathscr{B}(G)$ with $m(A) > 0$ such that $P_A$ maps $X$ onto $L^1(A)$, i.e., suppose that $X$ is not small. Then there exists a continuous operator $T : L^1(A) \to X$ with $j_A = P_A T$ where $j_A : L^1(A) \to L^1(G)$ is the natural injection.*

*Proof.* This proof is a modification of a proof by G. Godefroy, N. J. Kalton, and D. Li [20, Lemma III.5]. We identify $X^{**}$ with $X^{\perp\perp} \subset L^1(G)^{**}$ and denote by $P$ the $L$-projection from $L^1(G)^{**}$ onto $L^1(G)$. Recall that A. V. Buhvalov and G. Ya. Lozanovskiĭ showed that $P[B_{X^{\perp\perp}}] = B_X$ if $X$ is nicely placed in $L^1(G)$ [24, Theorem IV.3.4].

Denote by $\mathscr{N}$ the directed set of open neighborhoods of $e_G$. (We turn $\mathscr{N}$ into a directed set by setting $V \leq W$ if and only if $V$ contains $W$.) Let $\mathscr{U}$ be an ultrafilter on $\mathscr{N}$ which contains the filter base

$$\{\{W \in \mathscr{N} : V \leq W\} : V \in \mathscr{N}\}.$$

$P_A$ is an open map by the open mapping theorem. So we can fix $M > 0$ with $B_{L^1(A)} \subset M P_A[B_X]$. For every $V \in \mathscr{N}$, use Lemma IV.2.15 and choose disjoint Borel sets $B_{V,1}, \ldots, B_{V,N_V}$ with $A = \bigcup_{k=1}^{N_V} B_{V,k}$ and $B_{V,k} B_{V,k}^{-1} \subset V$ for $k = 1, \ldots, N_V$.

Picking $f_{V,k} \in MB_X$ with $P_A(f_{V,k}) = m(B_{V,k})^{-1}\chi_{B_{V,k}}$ for $k = 1, \ldots, N_V$, we define $S_V : L^1(A) \to X$ by

$$S_V(f) = \sum_{k=1}^{N_V} \left( \int_{B_{V,k}} f \, dm \right) f_{V,k} \quad (f \in L^1(A)).$$

As the norm of every $S_V$ is bounded by $M$, we can define $S : L^1(A) \to X^{\perp\perp}$ by

$$S(f) = w^*\text{-}\lim_{V,\mathscr{U}} S_V(f) \quad (f \in L^1(A))$$

and set $T = PS$.

Let us check that $j_A = P_A T$. Fix $f \in L^1(A)$. Since $C(G)$ is dense in $L^1(G)$, we may assume that $f$ is the restriction to $A$ of a continuous function. Let $(S_{\varphi(j)}(f))_{j\in J}$ be a subnet of $(S_V(f))_{V\in\mathscr{N}}$ with $S(f) = w^*\text{-}\lim_j S_{\varphi(j)}(f)$. Since $f$ is uniformly continuous, it is easy to construct an increasing sequence $(j_n)_{n\in\mathbb{N}}$ in $J$ with

$$\sup \left\{ \left\| f - P_A S_{\varphi(j)}(f) \right\|_\infty : j \geq j_n \right\} \longrightarrow 0. \tag{5.1}$$

Furthermore, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $L^1(G)$ that converges $m$-almost everywhere to $PS(f)$ with $g_n \in \operatorname{conv}\{S_{\varphi(j)}(f) : j \geq j_n\}$ for all $n \in \mathbb{N}$ [24, Lemma IV.3.1]. Hence we have by (5.1) that for $m$-almost all $x \in A$

$$T(f)(x) = PS(f)(x) = \lim_n g_n(x) = f(x)$$

and therefore $j_A = P_A T$. □

**Theorem IV.5.3** *If $\Lambda$ is a nicely placed Riesz set, then $L^1_\Lambda(G)$ is a small subspace of $L^1(G)$.*

*Proof.* Assume that $\Lambda$ is a nicely placed Riesz set such that $L^1_\Lambda(G)$ is not a small subspace of $L^1(G)$.

Since $L^1_\Lambda(G)$ is not small, there exists a Borel set $A$ of positive measure such that $P_A$ maps $L^1_\Lambda(G)$ onto $L^1(A)$. Using Lemma IV.5.2, we find $T : L^1(A) \to L^1_\Lambda(G)$ with $j_A = P_A T$. This operator is an isomorphism onto its image and $L^1_\Lambda(G)$ contains a copy of $L^1(A)$. So $L^1_\Lambda(G)$ fails the Radon-Nikodým property. But this contradicts our assumption because $\Lambda$ is a Riesz set if and only if $L^1_\Lambda(G)$ has the Radon-Nikodým property (see Theorem III.4.20). □

**Corollary IV.5.4** *If $\Lambda$ is a Shapiro set, then $L^1_\Lambda(G)$ is a poor subspace of $L^1(G)$.*

Theorem IV.5.3 can be strengthened if $G$ is metrizable. Let $\Lambda$ be nicely placed. Then $L^1_\Lambda(G)$ is a poor subspace of $L^1(G)$ if and only if $\Lambda$ is a semi-Riesz set [20, Proposition III.10]

# V The Almost Daugavet Property and Translation-Invariant Subspaces

If $G$ is an infinite compact abelian group, then $C(G)$ and $L^1(G)$ have the Daugavet property and a fortiori the almost Daugavet property. But which translation-invariant subspaces inherit this property? We will characterize the translation-invariant subspaces of thickness two what leads in the case of metrizable $G$ to characterizations of the translation-invariant subspaces with the almost Daugavet property.

## V.1 Translation-invariant subspaces of $L^1(G)$

To deal with translation-invariant subspaces of $L^1(G)$, we will use the results of Section II.2.

**Corollary V.1.1** *The space $L^1_\Lambda(G)$ has thickness two if and only if $\Lambda$ is not a $\Lambda(1)$ set.*

*Proof.* Recall that the space $L^1_\Lambda(G)$ is reflexive if and only if $\Lambda$ is a $\Lambda(1)$ set (see Proposition III.4.17).

Suppose first that $T(L^1_\Lambda(G)) = 2$. By Corollary II.1.2, the space $L^1_\Lambda(G)$ contains a copy of $\ell^1$ and is not reflexive. So $\Lambda$ is not a $\Lambda(1)$ set.

Suppose now that $\Lambda$ is not a $\Lambda(1)$ set. Then $L^1_\Lambda(G)$ is a non-reflexive subspace of the $L$-embedded space $L^1(G)$. Hence $T(L^1_\Lambda(G)) = 2$ by Theorem II.2.12. □

**Corollary V.1.2** *Let $G$ be a metrizable compact abelian group. The space $L^1_\Lambda(G)$ has the almost Daugavet property if and only if $\Lambda$ is not a $\Lambda(1)$ set.*

*Proof.* If $G$ is a metrizable compact abelian group, then $\Gamma$ is countable [53, Theorem 2.2.6] and $L^1(G)$ is separable. Since a separable space has the almost Daugavet property if and only if it has thickness two (see Theorem I.5.5), it is a consequence of Corollary V.1.1 that $L^1_\Lambda(G)$ has the almost Daugavet property if and only if $\Lambda$ is not a $\Lambda(1)$ set. □

The almost Daugavet property is strictly weaker than the Daugavet property for translation-invariant subspaces of $L^1(G)$. Since $\mathbb{N}$ is a Riesz set, the space $L^1_{\mathbb{N}}(\mathbb{T})$ has the Radon-Nikodým property by Theorem III.4.20 and fails the Daugavet property. But $\mathbb{N}$ contains arbitrarily large arithmetic progressions and is therefore not a $\Lambda(1)$ set (see Proposition III.4.16). So $L^1_{\mathbb{N}}(\mathbb{T})$ has the almost Daugavet property.

## V.2 Translation-invariant subspaces of $C(G)$

We will show that $T(C_\Lambda(G)) = 2$ if and only if $\Lambda$ is an infinite set. We will split the proof into various cases that depend on the structure of $G$.

Recall that for a family of abelian groups $(G_j)_{j\in J}$ we denote by $\prod_{j\in J} G_j$ their direct product and by $\bigoplus_{j\in J} G_j$ their direct sum. If all $G_j$ coincide with the group $G$, we write $G^J$ or $G^{(J)}$ for the direct product or the direct sum. We denote by $p_{G_j}$ the projection from $\prod_{j\in J} G_j$ onto $G_j$. If we consider products of the form $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{Z}^n$, we denote by $p_1, p_2, \dots$ the corresponding projections onto $\mathbb{Z}$.

**Proposition V.2.1** *Let $A$ be a compact abelian group, set $G = \mathbb{T} \oplus A$, and let $\Lambda$ be a subset of $\Gamma = \mathbb{Z} \oplus \Gamma_A$. If $p_{\mathbb{Z}}[\Lambda]$ is infinite, then $T(C_\Lambda(G)) = 2$.*

*Proof.* Fix $f_1, \dots, f_n \in S_{C_\Lambda(G)}$ and $\varepsilon > 0$. We have to find $g \in S_{C_\Lambda(G)}$ satisfying $\|f_k + g\|_\infty \geq 2 - \varepsilon$ for $k = 1, \dots, n$.

Every $f_k$ is uniformly continuous and therefore there exists $\delta > 0$ such that for $k = 1, \dots, n$ and all $a \in A$

$$|\varphi - \vartheta| \leq \delta \Longrightarrow \left| f_k(\mathrm{e}^{\mathrm{i}\varphi}, a) - f_k(\mathrm{e}^{\mathrm{i}\vartheta}, a) \right| \leq \varepsilon.$$

Since $p_{\mathbb{Z}}[\Lambda]$ contains infinitely many elements, we can pick $s \in p_{\mathbb{Z}}[\Lambda]$ with $|s|2\delta \geq 2\pi$. By our choice of $s$, we get for all $\vartheta \in [0, 2\pi]$

$$\left\{\mathrm{e}^{\mathrm{i}s\varphi} : |\varphi - \vartheta| \leq \delta\right\} = \left\{\mathrm{e}^{\mathrm{i}\varphi} : |\varphi - \vartheta| \leq |s|\delta\right\} = \mathbb{T}. \tag{2.1}$$

Choose $g \in \Lambda$ with $p_{\mathbb{Z}}(g) = s$ and fix $k \in \{1, \dots, n\}$. Since $f_k \in S_{C_\Lambda(G)}$, there exists $(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)}) \in G$ with

$$\left| f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)}) \right| = 1.$$

By (2.1), we can pick $\varphi^{(k)} \in \mathbb{R}$ with

$$\left| \varphi^{(k)} - \vartheta^{(k)} \right| \leq \delta$$

and

$$\mathrm{e}^{\mathrm{i}s\varphi^{(k)}} = \frac{f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)})}{g(1, a^{(k)})}.$$

Note that the right-hand side of the last equation has absolute value one because $g$ is a character of $G$. Consequently,

$$g(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)}) = g(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, e_A) g(1, a^{(k)}) = e^{\mathrm{i}s\varphi^{(k)}} g(1, a^{(k)}) = f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)}).$$

Finally,

$$\begin{aligned} \|f_k + g\|_\infty &\geq \left| f_k(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)}) + g(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)}) \right| \\ &\geq 2 \left| f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)}) \right| - \left| f_k(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)}) - f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)}) \right| \\ &\geq 2 - \varepsilon. \end{aligned}$$

□

**Proposition V.2.2** *Let $A$ be a compact abelian group, set $G = \mathbb{T}^{\mathbb{N}} \oplus A$, and let $\Lambda$ be a subset of $\Gamma = \mathbb{Z}^{(\mathbb{N})} \oplus \Gamma_A$. If we find arbitrarily large $l \in \mathbb{N}$ with $p_l[\Lambda] \neq \{0\}$, then $T(C_\Lambda(G)) = 2$.*

*Proof.* Fix $f_1, \dots, f_n \in S_{C_\Lambda(G)}$. Since $T_\Lambda(G)$ is dense in $C_\Lambda(G)$, we may assume without loss of generality that $f_1, \dots, f_n$ are trigonometric polynomials. We are going to find $g \in S_{C_\Lambda(G)}$ with $\|f_k + g\|_\infty = 2$ for $k = 1, \dots, n$.

Setting $\Delta = \bigcup_{k=1}^n \operatorname{spec}(f_k)$, we get a finite subset of $\Lambda$ because every $f_k$ is a trigonometric polynomial and therefore has a finite spectrum. Consequently, there exists $l_0 \in \mathbb{N}$ with $p_l[\Delta] = \{0\}$ for all $l > l_0$ and the evaluation of $f_1, \dots, f_n$ at a point $(t_1, t_2, \dots, a) \in G$ just depends on the coordinates $t_1, \dots, t_{l_0}$ and $a$.

By assumption, we can find $l > l_0$ and $g \in \Lambda$ with $s = p_l(g) \neq 0$. Fix $k \in \{1, \dots, n\}$. Since $f_k \in S_{C(G)}$, there exists $x^{(k)} = (t_1^{(k)}, t_2^{(k)}, \dots, a^{(k)}) \in G$ with $|f_k(x^{(k)})| = 1$. Pick $u^{(k)} \in \mathbb{T}$ with

$$(u^{(k)})^s = \frac{f_k(x^{(k)})}{g(t_1^{(k)}, \dots, t_{l-1}^{(k)}, 1, t_{l+1}^{(k)}, t_{l+2}^{(k)}, \dots, a^{(k)})}.$$

Note that the right-hand side of the last equation has absolute value one because $g$ is a character of $G$. With the same reasoning as at the end of the proof of Proposition V.2.1 we get that

$$g(t_1^{(k)}, \dots, t_{l-1}^{(k)}, u^{(k)}, t_{l+1}^{(k)}, t_{l+2}^{(k)}, \dots, a^{(k)}) = f_k(x^{(k)}).$$

Finally,

$$\begin{aligned} \|f_k + g\|_\infty &\geq \left|(f_k + g)(t_1^{(k)}, \dots, t_{l-1}^{(k)}, u^{(k)}, t_{l+1}^{(k)}, t_{l+2}^{(k)}, \dots, a^{(k)})\right| \\ &= 2\left|f_k(x^{(k)})\right| = 2. \end{aligned}$$

□

**Lemma V.2.3** *Let $\varepsilon$ be a positive number. If $z_1, \dots, z_n \in \{z \in \mathbb{C} : |z| \leq 1\}$ satisfy*

$$\left|\sum_{k=1}^n z_k\right| \geq n(1 - \varepsilon),$$

*then*

$$|z_k| \geq 1 - n\varepsilon \quad \text{and} \quad |z_k - z_l| \leq 2n\sqrt{\varepsilon}$$

*for $k, l = 1, \dots, n$.*

*Proof.* The first assertion is an easy consequence of the triangle inequality.

For fixed $k, l \in \{1, \dots, n\}$ we have

$$\begin{aligned} \operatorname{Re} z_k \overline{z_l} &= \operatorname{Re} \sum_{s,t=1}^n z_s \overline{z_t} - \operatorname{Re} \sum_{\substack{s,t=1 \\ (s,t) \neq (k,l)}}^n z_s \overline{z_t} = \left|\sum_{k=1}^n z_k\right|^2 - \operatorname{Re} \sum_{\substack{s,t=1 \\ (s,t) \neq (k,l)}}^n z_s \overline{z_t} \\ &\geq n^2(1 - \varepsilon)^2 - (n^2 - 1) = 1 - 2n^2\varepsilon + n^2\varepsilon^2 \\ &\geq 1 - 2n^2\varepsilon. \end{aligned}$$

Using this inequality, we get

$$\begin{aligned}|z_k - z_l|^2 &= |z_k|^2 + |z_l|^2 - 2\operatorname{Re} z_k\overline{z_l}\\ &\le 2 - 2(1 - 2n^2\varepsilon) = 4n^2\varepsilon.\end{aligned}$$

□

The following lemma shows that if we are given $n$ subsets of the unit circle that do not meet a circular sector with central angle bigger than $\frac{2\pi}{n}$, then we can rotate these $n$ subsets such that their intersection becomes empty.

**Lemma V.2.4** *Let $W_1, \dots, W_n$ be subsets of $\{z \in \mathbb{C} : |z| \le 1\}$. Suppose that for every $k \in \{1, \dots, n\}$ there exist $\varphi_k \in [0, 2\pi]$ and $\vartheta_k \in [\frac{2\pi}{n}, 2\pi]$ with*

$$W_k \cap \{r\mathrm{e}^{\mathrm{i}\alpha} : r \in [0,1], \alpha \in [\varphi_k, \varphi_k + \vartheta_k]\} = \emptyset.$$

*Then there exist $t_1, \dots, t_n \in \mathbb{T}$ with*

$$\bigcap_{k=1}^{n} t_k W_k = \emptyset.$$

*Proof.* Setting for $k = 1, \dots, n$ (with $\vartheta_0 = 0$)

$$t_k = \mathrm{e}^{\mathrm{i}\sum_{l=0}^{k-1}\vartheta_l}\mathrm{e}^{-\mathrm{i}\varphi_k},$$

we get

$$t_k W_k \cap \left\{ r\mathrm{e}^{\mathrm{i}\alpha} : r \in [0,1], \alpha \in \left[\sum_{l=0}^{k-1}\vartheta_l, \sum_{l=0}^{k}\vartheta_l\right]\right\} = \emptyset.$$

Fix $\alpha \in [0, 2\pi]$ and $r \in [0,1]$. Since $\sum_{k=1}^{n}\vartheta_k \ge 2\pi$, there is $k \in \{1, \dots, n\}$ with

$$\alpha \in \left[\sum_{l=0}^{k-1}\vartheta_l, \sum_{l=0}^{k}\vartheta_l\right].$$

Consequently, $r\mathrm{e}^{\mathrm{i}\alpha}$ does not belong to $t_k W_k$ and $\bigcap_{k=1}^{n} t_k W_k = \emptyset$. □

**Lemma V.2.5** *Let $\varepsilon, \delta > 0$, let $W$ be a subset of $\{z \in \mathbb{C} : 1 - \delta \le |z| \le 1\}$, and set $W_\varepsilon = \{z \in \mathbb{C} :$ there exists $w \in W$ with $|w - z| \le \varepsilon\}$. Suppose that there exists $\vartheta \in [0, 2\pi]$ such that for every $\varphi \in [0, 2\pi]$*

$$W_\varepsilon \cap \{r\mathrm{e}^{\mathrm{i}\alpha} : r \in [0,1], \alpha \in [\varphi, \varphi + \vartheta]\} \ne \emptyset.$$

*Then $W$ is a $(2\varepsilon + \delta + \vartheta)$-net for $\mathbb{T}$.*

*Proof.* Fix $\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{T}$. We have to find $w \in W$ with $|w - e^{\mathrm{i}\varphi}| \le 2\varepsilon + \delta + \vartheta$.

By assumption, there exist $s\mathrm{e}^{\mathrm{i}\beta} \in W_\varepsilon \cap \{r\mathrm{e}^{\mathrm{i}\alpha} : r \in [0,1], \alpha \in [\varphi, \varphi + \vartheta]\}$ and $w \in W$ with $|w - s\mathrm{e}^{\mathrm{i}\beta}| \le \varepsilon$. It is easy to see that $s \ge 1 - \delta - \varepsilon$. Finally,

$$\begin{aligned}|w - \mathrm{e}^{\mathrm{i}\varphi}| &\le |w - s\mathrm{e}^{\mathrm{i}\beta}| + |s\mathrm{e}^{\mathrm{i}\beta} - s\mathrm{e}^{\mathrm{i}\varphi}| + |s\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{\mathrm{i}\varphi}|\\ &\le \varepsilon + \vartheta + (\delta + \varepsilon) = 2\varepsilon + \delta + \vartheta.\end{aligned}$$

□

**Proposition V.2.6** *Let $A$ be a compact abelian group, let $(G_l)_{l\in\mathbb{N}}$ be a sequence of finite abelian groups, set $G = \prod_{l=1}^{\infty} G_l \oplus A$, and let $\Lambda$ be an infinite subset of $\Gamma = \bigoplus_{l=1}^{\infty} \Gamma_l \oplus \Gamma_A$. If $p_{\Gamma_A}[\Lambda]$ is a finite set, then $T(C_\Lambda(G)) = 2$.*

*Proof.* The beginning is almost like in the proof of Proposition V.2.2.

Fix $f_1, \dots, f_n \in S_{C_\Lambda(G)}$ and $\varepsilon > 0$. Since $T_\Lambda(G)$ is dense in $C_\Lambda(G)$, we may assume without loss of generality that $f_1, \dots, f_n$ are trigonometric polynomials. We have to find $g \in S_{C_\Lambda(G)}$ with $\|f_k + g\|_\infty \geq 2 - \varepsilon$ for $k = 1, \dots, n$.

Setting $\Delta = \bigcup_{k=1}^{n} \operatorname{spec}(f_k)$, we get a finite subset of $\Lambda$ because every $f_k$ is a trigonometric polynomial and therefore has a finite spectrum. Consequently, there exists $l_0 \in \mathbb{N}$ with $p_{\Gamma_l}[\Delta] = \{\mathbf{1}_{G_l}\}$ for all $l > l_0$ and the evaluation of $f_1, \dots, f_n$ at a point $(x_1, x_2, \dots, a) \in G$ just depends on the coordinates $x_1, \dots, x_{l_0}$ and $a$.

Since $\Gamma_1, \dots, \Gamma_{l_0}$ and $p_{\Gamma_A}[\Lambda]$ are finite sets and $\Lambda$ is an infinite set, there exist an infinite subset $\Lambda_0$ of $\Lambda$ and elements $\gamma_1 \in \Gamma_1, \dots, \gamma_{l_0} \in \Gamma_{l_0}, \gamma_A \in \Gamma_A$ with $p_{\Gamma_l}[\Lambda_0] = \{\gamma_l\}$ for $l = 1, \dots, l_0$ and $p_{\Gamma_A}[\Lambda_0] = \{\gamma_A\}$. In other words, all elements of $\Lambda_0$ coincide in the first $l_0$ coordinates of $\bigoplus_{l=1}^{\infty} \Gamma_l$ and in the coordinate that corresponds to $\Gamma_A$. We can also assume that $\Lambda_0$ is a Sidon set because every infinite subset of $\Gamma$ contains an infinite Sidon set (see Corollary III.4.6). So if $\{\lambda_1, \lambda_2, \dots\}$ is an enumeration of $\Lambda_0$, then $(\lambda_s)_{s\in\mathbb{N}}$ is equivalent to the canonical basis of $\ell^1$.

Set $\gamma = (\overline{\gamma_1}, \dots, \overline{\gamma_{l_0}}, \mathbf{1}_{G_{l_0+1}}, \mathbf{1}_{G_{l_0+2}}, \dots, \overline{\gamma_A}) \in \Gamma$. The sequence $(\gamma\lambda_s)_{s\in\mathbb{N}}$ is still equivalent to the canonical basis of $\ell^1$ and we have for every character $\gamma\lambda_s$ that $p_{\Gamma_A}(\gamma\lambda_s) = \mathbf{1}_A$ and $p_{\Gamma_l}(\gamma\lambda_s) = \mathbf{1}_{G_l}$ for $l = 1, \dots, l_0$. Thus the evaluation of $\gamma\lambda_1, \gamma\lambda_2, \dots$ at a point $(x_1, x_2, \dots, a) \in G$ does not depend on the coordinates $x_1, \dots, x_{l_0}$ and $a$.

Choose $n_0 \in \mathbb{N}$ with $\frac{2\pi}{n_0} \leq \frac{\varepsilon}{3}$ and $\delta \in (0,1)$ with $4n_0\sqrt{\delta} \leq \frac{\varepsilon}{3}$. By James's $\ell^1$ distortion theorem [3, Theorem 10.3.1], there is a normalized block basis sequence $(g_s)_{s\in\mathbb{N}}$ of $(\gamma\lambda_s)_{s\in\mathbb{N}}$ with

$$(1-\delta)\sum_{s=1}^{\infty} |z_s| \leq \left\| \sum_{s=1}^{\infty} z_s g_s \right\|_\infty \leq \sum_{s=1}^{\infty} |z_s|$$

for any sequence of complex numbers $(z_s)_{s\in\mathbb{N}}$ with finite support. It follows that for every $n_0$-tuple $(z_1, \dots, z_{n_0}) \in \mathbb{T}^{n_0}$ there is $x \in G$ with

$$\left| \sum_{s=1}^{n_0} z_s g_s(x) \right| \geq n_0(1-\delta).$$

Using Lemma V.2.3, we have for $s, t = 1, \dots, n_0$

$$|g_s(x)| \geq 1 - n_0\delta \quad \text{and} \quad |z_s g_s(x) - z_t g_t(x)| \leq 2n_0\sqrt{\delta}.$$

Setting for $s = 1, \dots, n_0$

$$W_s = g_s[G] \cap \{z \in \mathbb{C} : |z| \geq 1 - n_0\delta\}$$

and

$$\widetilde{W_s} = \{z \in \mathbb{C} : \text{there exists } w \in W_s \text{ with } |w - z| \leq 2n_0\sqrt{\delta}\},$$

we conclude that for every tuple $(z_1, \dots, z_{n_0}) \in \mathbb{T}^{n_0}$

$$\bigcap_{s=1}^{n_0} z_s \widetilde{W_s} \neq \emptyset.$$

By Lemma V.2.4, there is $s_0 \in \{1, \dots, n_0\}$ such that for any $\varphi \in [0, 2\pi]$

$$\widetilde{W_{s_0}} \cap \left\{ r\mathrm{e}^{\mathrm{i}\alpha} : r \in [0,1], \alpha \in \left[\varphi, \varphi + \frac{2\pi}{n_0}\right] \right\} \neq \emptyset.$$

It follows from Lemma V.2.5 and our choice of $n_0$ and $\delta$ that $W_{s_0}$ is an $\varepsilon$-net for $\mathbb{T}$.

The function $g = \overline{\gamma} g_{s_0}$ is by construction a normalized trigonometric polynomial with spectrum contained in $\Lambda$. Fix $k \in \{1, \dots, n\}$. Since $f_k \in S_{C_\Lambda(G)}$, there exists $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, a^{(k)}) \in G$ with $\left|f_k(x^{(k)})\right| = 1$, and by our choice of $g_{s_0}$, we can find a point $y^{(k)} = (y_1^{(k)}, y_2^{(k)}, \dots, b^{(k)}) \in G$ with

$$\left|\gamma(x^{(k)}) f_k(x^{(k)}) - g_{s_0}(y^{(k)})\right| \leq \varepsilon.$$

Note that $\gamma(x^{(k)}) f_k(x^{(k)}) \in \mathbb{T}$ since $\gamma$ is a character. We therefore get

$$\begin{aligned}
\|f_k + g\|_\infty &= \|\gamma f_k + g_{s_0}\|_\infty \\
&\geq \left|(\gamma f_k + g_{s_0})(x_1^{(k)}, \dots, x_{l_0}^{(k)}, y_{l_0+1}^{(k)}, y_{l_0+2}^{(k)}, \dots, a^{(k)})\right| \\
&= \left|\gamma(x^{(k)}) f_k(x^{(k)}) + g_{s_0}(y^{(k)})\right| \\
&\geq 2\left|\gamma(x^{(k)}) f_k(x^{(k)})\right| - \left|\gamma(x^{(k)}) f_k(x^{(k)}) - g_{s_0}(y^{(k)})\right| \\
&\geq 2 - \varepsilon.
\end{aligned}$$

□

Recall that the *order* $o(\gamma)$ of an element $\gamma \in \Gamma$ is the smallest positive integer $m$ such that $\gamma^m = e_\Gamma$. If no such $m$ exists, $\gamma$ is said to have infinite order.

**Lemma V.2.7** *Let $G$ be a compact abelian group and let $\gamma$ be an element of $\Gamma$.*

(a) *If $o(\gamma) = m$, then $\gamma[G] = \{\mathrm{e}^{2\pi\mathrm{i}\frac{k}{m}} : k = 0, \dots, m-1\}$, i.e., the image of $G$ under $\gamma$ is the set of the $m$th roots of unity.*

(b) *If $o(\gamma) = \infty$, then $\gamma[G] = \mathbb{T}$.*

*Proof.* If $o(\gamma) = m$, we have $\gamma(x)^m = 1$ for every $x \in G$. Thus every element of $\gamma[G]$ is an $m$th root of unity. Setting $n = |\gamma[G]|$, it follows from Lagrange's theorem that $\gamma(x)^n = 1$ for every $x \in G$. Therefore $n \geq m$ and $\gamma[G]$ has to coincide with $\{\mathrm{e}^{2\pi\mathrm{i}\frac{k}{m}} : k = 0, \dots, m-1\}$.

The set $\gamma[G]$ is a compact and therefore closed subgroup of $\mathbb{T}$. Since all proper closed subgroups of $\mathbb{T}$ are finite [48, Corollary 2.3], we have $\gamma[G] = \mathbb{T}$ if $o(\gamma) = \infty$. □

**Theorem V.2.8** *If $\Lambda$ is an infinite subset of $\Gamma$, then $T(C_\Lambda(G)) = 2$.*

*Proof.* We start like in the proofs of Proposition V.2.2 and V.2.6.

Fix $f_1, \dots, f_n \in S_{C_\Lambda(G)}$ and $\varepsilon > 0$. Since $T_\Lambda(G)$ is dense in $C_\Lambda(G)$, we may assume without loss of generality that $f_1, \dots, f_n$ are trigonometric polynomials. We have to find $g \in S_{C_\Lambda(G)}$ with $\|f_k + g\|_\infty \geq 2 - \varepsilon$ for $k = 1, \dots, n$.

Setting $\Delta = \bigcup_{k=1}^n \operatorname{spec}(f_k)$, we get a finite subset of $\Lambda$ because every $f_k$ is a trigonometric polynomial and therefore has a finite spectrum.

We can assume, by passing to a countably infinite subset if necessary, that $\Lambda$ is countable. Hence $\langle \Lambda \rangle$, the group generated by $\Lambda$, is a countable subgroup of $\Gamma$.

Let $M$ be a maximal independent subset of $\langle \Lambda \rangle$ and let $\Gamma_1 = \langle M \rangle$ be the subgroup of $\Gamma$ that is generated by $M$. Recall that a subset $M$ of $\Gamma$ is *independent* if $\gamma_1^{k_1} \cdots \gamma_n^{k_n} = e_\Gamma$ implies $\gamma_1^{k_1} = \cdots = \gamma_n^{k_n} = e_\Gamma$ for every choice of distinct elements $\gamma_1, \dots, \gamma_n \in M$ and integers $k_1, \dots, k_n$. Defining inductively

$$\Gamma_l = \{\gamma \in \langle \Lambda \rangle : \gamma^l \in \Gamma_{l-1}\}$$

for $l = 2, 3, \dots$, we get an increasing sequence $(\Gamma_l)_{l \in \mathbb{N}}$ of subgroups of $\langle \Lambda \rangle$. Since $M$ is a maximal independent subset of $\langle \Lambda \rangle$, we have that $\bigcup_{l=1}^\infty \Gamma_l = \langle \Lambda \rangle$. Furthermore, every $\Gamma_l$ is a direct sum of cyclic groups [16, Corollary 18.4]. We distinguish two cases depending on whether or not there exists $\Gamma_l$ that contains $\Delta$ and infinitely many elements of $\Lambda$.

*First case:* Suppose that there exists $l_0 \in \mathbb{N}$ such that $\Delta \subset \Gamma_{l_0}$ and $\Lambda_0 = \Lambda \cap \Gamma_{l_0}$ is an infinite set.

By our choice of $\Gamma_{l_0}$, the functions $f_1, \dots, f_n$ and all characters $\gamma \in \Lambda_0$ are constant on the cosets of $G/(\Gamma_{l_0})^\perp$ and can therefore be considered as functions and characters on $G_0 = G/(\Gamma_{l_0})^\perp$. (To simplify notation, we continue to write $f_1, \dots, f_n$.) Note that $\Gamma_{l_0}$ is the dual group of $G_0$. Since $\Gamma_{l_0}$ is a direct sum of cyclic groups, there exists a sequence $(\Phi_s)_{s \in \mathbb{N}}$ of finite abelian groups such that $\Gamma_{l_0} = \mathbb{Z}^{(\mathbb{N})} \oplus \bigoplus_{s=1}^\infty \Phi_s$ or $\Gamma_{l_0} = \mathbb{Z}^{n_0} \oplus \bigoplus_{s=1}^\infty \Phi_s$ for adequate $n_0 \in \mathbb{N}$. Hence $G_0 = \mathbb{T}^\mathbb{N} \oplus \prod_{s=1}^\infty F_s$ or $G_0 = \mathbb{T}^{n_0} \oplus \prod_{s=1}^\infty F_s$ where the dual group of $F_s$ is $\Phi_s$. Let $p_1, p_2, \dots$ be the projections from $\Gamma_{l_0}$ onto $\mathbb{Z}$.

If there exists $s_0 \in \mathbb{N}$ such that $p_{s_0}[\Lambda_0]$ contains infinitely many elements or if there exist arbitrarily large $s \in \mathbb{N}$ with $p_s[\Lambda_0] \neq \{0\}$, then $T(C_{\Lambda_0}(G_0)) = 2$ by Proposition V.2.1 or V.2.2. Otherwise $p_{\mathbb{Z}^{(\mathbb{N})}}[\Lambda_0]$ (or $p_{\mathbb{Z}^{n_0}}[\Lambda_0]$) is a finite set and $T(C_{\Lambda_0}(G_0)) = 2$ by Proposition V.2.6. So we can find $\tilde{g} \in S_{C_{\Lambda_0}(G_0)}$ with $\|f_k + \tilde{g}\|_\infty \geq 2 - \varepsilon$ for $k = 1, \dots, n$. Setting $g = \tilde{g} \circ \pi$ where $\pi$ is the canonical map from $G$ onto $G_0 = G/(\Gamma_0)^\perp$, we get $\|f_k + g\|_\infty \geq 2 - \varepsilon$ for $k = 1, \dots, n$.

*Second case:* Suppose that there exist arbitrarily large $l \in \mathbb{N}$ with $\Lambda \cap (\Gamma_l \setminus \Gamma_{l-1}) \neq \emptyset$.

Fix $l_0 \in \mathbb{N}$ with $\Delta \subset \Gamma_{l_0}$ and choose $l_1 \in \mathbb{N}$ with $l_1 > l_0^2$, $\frac{2\pi}{l_1} \leq \varepsilon$ and $(\Gamma_{l_1} \setminus \Gamma_{l_1 - 1}) \cap \Lambda \neq \emptyset$. By our choice of $\Gamma_{l_0}$, the functions $f_1, \dots, f_n$ are constant on the cosets of $G/(\Gamma_{l_0})^\perp$ and therefore

$$f_k(xy) = f_k(x) \quad (x \in G, y \in (\Gamma_{l_0})^\perp) \tag{2.2}$$

for $k = 1, \dots, n$. Pick $g \in (\Gamma_l \setminus \Gamma_{l-1}) \cap \Lambda$ and denote by $\tilde{g}$ the restriction of $g$ to $(\Gamma_{l_0})^\perp$. What can we say about the order of $\tilde{g}$? Since $(\Gamma_{l_0})^{\perp\perp} = \Gamma_{l_0}$, we have for every $m \in \mathbb{N}$ that $\tilde{g}^m = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ if and only if $g^m \in \Gamma_{l_0}$.

Suppose that $\tilde{g}^m = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ for some $2 \leq m \leq l_0$. Then $\tilde{g}^{ml_0} = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ as well and $g^{ml_0} \in \Gamma_{l_0}$. Consequently, $g \in \Gamma_{ml_0}$ because $g^{ml_0} \in \Gamma_{l_0} \subset \Gamma_{ml_0 - 1}$. But this contradicts

our choice of $g$ and $l_1$ because $l_1 > ml_0$. Assuming that $\tilde{g}^m = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ for some $l_0 < m < l_1$ leads to the same contradiction. The order of $\tilde{g}$ is therefore at least $l_1$. By our choice of $l_1$ and by Lemma V.2.7, we get that $\tilde{g}[(\Gamma_{l_0})^\perp]$ is an $\varepsilon$-net for $\mathbb{T}$.

Fix now $k \in \{1, \dots, n\}$ and choose $x^{(k)} \in G$ with $|f_k(x^{(k)})| = 1$ and $y^{(k)} \in (\Gamma_{l_0})^\perp$ with

$$\left| f_k(x^{(k)}) - g(x^{(k)})\tilde{g}(y^{(k)}) \right| \leq \varepsilon. \tag{2.3}$$

Note that $g$ is a character and hence $g(x^{(k)}) \in \mathbb{T}$. Using (2.2) and (2.3), we get

$$\begin{aligned} \|f_k + g\|_\infty &\geq \left| f_k(x^{(k)}y^{(k)}) + g(x^{(k)}y^{(k)}) \right| \\ &= \left| f_k(x^{(k)}) + g(x^{(k)})\tilde{g}(y^{(k)}) \right| \\ &\geq 2\left| f_k(x^{(k)}) \right| - \left| f_k(x^{(k)}) - g(x^{(k)})\tilde{g}(y^{(k)}) \right| \\ &\geq 2 - \varepsilon. \end{aligned}$$

□

**Corollary V.2.9** *Let $G$ be a metrizable compact abelian group. The space $C_\Lambda(G)$ has the almost Daugavet property if and only if $\Lambda$ contains infinitely many elements.*

*Proof.* Every almost Daugavet space is infinite-dimensional and so the condition is necessary.

If $G$ is a metrizable compact abelian group, then $\Gamma$ is countable [53, Theorem 2.2.6] and $C(G)$ is separable. Since a separable space has the almost Daugavet property if and only if it has thickness two (see Theorem I.5.5), it is a consequence of Theorem V.2.8 that $C_\Lambda(G)$ has the almost Daugavet property. □

The almost Daugavet property is strictly weaker than the Daugavet property for translation-invariant subspaces of $C(G)$. If we set $\Lambda = \{3^n : n \in \mathbb{N}\}$, then $\Lambda$ is a Sidon set by Proposition III.4.5. So $C_\Lambda(\mathbb{T})$ is isomorphic to $\ell^1$, has the Radon-Nikodým property and therefore not the Daugavet property. But $\Lambda$ is an infinite set and $C_\Lambda(\mathbb{T})$ has the almost Daugavet property.

# VI Open Problems

1. By Corollary IV.2.7, the space $C_\Lambda(G)$ is a rich subspace of $C(G)$ if and only if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set. Can we characterize the rich subspaces of $L^1(G)$ in a similar way?

2. If $\Lambda \subset \mathbb{Z}$ is uniformly distributed, then $\mathbb{Z} \setminus (-\Lambda)$ is a semi-Riesz set (see Corollary IV.2.13). Is there a semi-Riesz set $\Lambda \subset \mathbb{Z}$ such that $\mathbb{Z} \setminus (-\Lambda)$ is not uniformly distributed?

3. Considering products of infinite, compact abelian groups, we have shown in Proposition IV.3.3 and in Proposition IV.3.4 that $L^1_{\Gamma_1 \times \Lambda_2}(G_1 \oplus G_2)$ is a rich subspace of $L^1(G_1 \oplus G_2)$ if and only if $L^1_{\Lambda_2}$ is a rich subspace of $L^1(G_2)$. Does it hold that $L^1_{\Lambda_1 \times \Lambda_2}(G_1 \oplus G_2)$ is a rich subspace of $L^1(G_1 \oplus G_2)$ if $L^1_{\Lambda_1}(G_1)$ is a rich subspace of $L^1(G_1)$ and $L^1_{\Lambda_2}(G_2)$ is a rich subspace of $L^1(G_2)$? We only know that then $L^1_{\Gamma_1 \times \Lambda_2}(G_1 \oplus G_2)$ is a rich subspace of $L^1(G_1 \oplus G_2)$ and $L^1_{\Lambda_1 \times \Lambda_2}(G_1 \oplus G_2)$ is a rich subspace of $L^1_{\Gamma_1 \times \Lambda_2}(G_1 \oplus G_2)$. So $L^1_{\Lambda_1 \times \Lambda_2}(G_1 \oplus G_2)$ is a rich subspace of a rich subspace of $L^1(G_1 \oplus G_2)$.

4. Set $\Lambda = \mathbb{Z} \times \{1\}$. In Section IV.3, we have shown that $C_\Lambda(\mathbb{T}^2)$ has the Daugavet property but is not a rich subspace of $C(\mathbb{T}^2)$. Furthermore, $\bigcap_{\gamma \in \Lambda} \ker(\gamma) = \{(1,1)\}$. Is it possible to construct such an example in $\mathbb{Z}$? In other words, is there a subset $\Lambda \subset \mathbb{Z}$ such that $C_\Lambda(\mathbb{T})$ has the Daugavet property, $C_\Lambda(\mathbb{T})$ is not a rich subspace of $C(\mathbb{T})$, and $\bigcap_{\gamma \in \Lambda} \ker(\gamma) = \{1\}$? Naturally, the analogous question can also be asked for translation-invariant subspaces of $L^1(\mathbb{T})$.

5. We have proved in Proposition IV.2.14 that $L^1_{\Gamma \setminus \Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$ if $\Lambda$ is a Rosenthal set. In Section IV.3, we have given an example of a non-Rosenthal set $\Lambda \subset \mathbb{Z}^2$ such that $L^1_{\mathbb{Z}^2 \setminus (-\Lambda)}(\mathbb{T}^2)$ is a rich subspace of $L^1(\mathbb{T}^2)$. Is it possible to construct such an example in $\mathbb{Z}$? In other words, is there a subset $\Lambda \subset \mathbb{Z}$ such that $L^1_{\mathbb{Z} \setminus (-\Lambda)}(\mathbb{T})$ is a rich subspace of $L^1(\mathbb{T})$ and $\Lambda$ is not a Rosenthal set?

6. If $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$, then $C_\Lambda(G)$ is a rich subspace of $C(G)$ (see Corollary IV.2.17). Does $C_\Lambda(G)$ have the Daugavet property if $L^1_\Lambda(G)$ does?

7. In Section IV.5, we have given some results concerning poor subspaces of $L^1(G)$. Can anything be said about translation-invariant subspaces of $C(G)$ that are poor? For example, is $C_\Lambda(G)$ poor if $\Lambda$ is a Sidon set or a Rosenthal set?

8. A separable Banach space $X$ has the almost Daugavet property if and only if $T(X) = 2$ (see Theorem I.5.5). Is this also true for non-separable Banach spaces? Looking back at the proofs of Section V.2, we can even prove the following result: Let $\Lambda$ be an

infinite subset of $\Gamma$, let $A$ be a subset of $S_{C_\Lambda(G)}$ whose cardinality is strictly smaller than the cardinality of $\Lambda$, and let $\varepsilon$ be a positive number. Then there exists $g \in S_{C_\Lambda(G)}$ with $\|f+g\|_\infty \geq 2-\varepsilon$ for all $f \in A$. Can this result be used to show that $C_\Lambda(G)$ has the almost Daugavet property if $\Lambda$ is not countably infinite?

# List of Symbols

| | | |
|---|---|---|
| $\operatorname{Re} z$ | real part of $z$ | 11 |
| $\operatorname{Im} z$ | imaginary part of $z$ | 11 |
| $\overline{z}$ | complex conjugate of $z$ | 11 |
| $\lvert z \rvert$ | absolute value of $z$ | 11 |
| $\mathbb{T}$ | group of all complex numbers of absolute value one | 11 |
| | | |
| $\operatorname{lin} A$ | linear span of $A$ | 11 |
| $\operatorname{conv} A$ | convex hull of $A$ | 11 |
| $\operatorname{diam} A$ | diameter of $A$ | 11 |
| $f \otimes x$ | function defined by $t \mapsto f(t)x$ or by $(s,t) \mapsto f(s)x(t)$ | 11 |
| $B_X$ | unit ball of $X$ | 11 |
| $S_X$ | unit sphere of $X$ | 11 |
| $T(X)$ | thickness of $X$ | 28 |
| $L(X,Y)$ | space of all bounded, linear operators from $X$ into $Y$ | 11 |
| $X^*$ | dual space of $X$ | 11 |
| $\ker(T)$ | kernel of $T$ | 11 |
| $\operatorname{ran}(T)$ | range of $T$ | 11 |
| $T^*$ | adjoint operator of $T$ | 11 |
| $S(x^*, \varepsilon)$ | slice of $B_X$ determined by $x^* \in S_{X^*}$ and $\varepsilon > 0$ | 14 |
| $S(x, \varepsilon)$ | weak* slice of $B_{X^*}$ determined by $x \in S_X$ and $\varepsilon > 0$ | 14 |
| $X \oplus_1 Y$ | $\ell^1$-sum of $X$ and $Y$ | 11 |
| $X \oplus_\infty Y$ | $\ell^\infty$-sum of $X$ and $Y$ | 11 |
| | | |
| $c_0$ | space of complex sequences which converge to zero | 11 |
| $c_{00}$ | space of complex sequences with finite support | 11 |
| $\ell^1$ | space of absolutely summable, complex sequences | 11 |
| $\ell^1(J)$ | $\ell^1$-space over $J$ | 12 |
| $\ell^\infty$ | space of bounded, complex sequences | 11 |
| | | |
| $C_0(K)$ | space of continuous, complex-valued functions on $K$ that vanish at infinity | 12 |
| $C(K)$ | space of continuous, complex-valued functions on $K$ | 12 |
| $C(K,X)$ | space of continuous, $X$-valued functions on $K$ | 12 |
| $M(K)$ | space of regular Borel measures on $K$ of bounded variation | 12 |
| $M_{\text{diff}}(K)$ | space of diffuse members of $M(K)$ | 58 |
| $\operatorname{supp}(f)$ | support of $f$ | 12 |
| $\operatorname{van}(T)$ | set of vanishing points of $T$ | 68 |

| | | |
|---|---|---|
| $\lim_{j,\mathscr{U}} x_j$ | limit of $(x_j)_{j\in J}$ along $\mathscr{U}$ | 12 |
| $L^p(\Omega, \Sigma, \mu)$ | space of measurable, complex-valued functions $f$ on $\Omega$ such that $\lvert f\rvert^p$ is integrable | 12 |
| $L^\infty(\Omega, \Sigma, \mu)$ | space of measurable, essentially bounded, complex-valued functions on $\Omega$ | 12 |
| $L^p(\Omega, X)$ | space of Bochner-measurable, $X$-valued functions $f$ on $\Omega$ such that $\lVert f\rVert_X^p$ is integrable | 12 |
| $L^\infty(\Omega, X)$ | space of Bochner-measurable, essentially bounded, $X$-valued functions on $\Omega$ | 12 |
| $L^1(A)$ | space of all $f \in L^1(\Omega)$ with $\chi_A f = f$ | 26 |
| $P_A$ | projection defined by $P_A(f) = \chi_A f$ | 26 |
| $e_G$ | identity element of $G$ | 41 |
| $o(\gamma)$ | order of $\gamma$ | 86 |
| $\langle \Lambda \rangle$ | subgroup generated by $\Lambda$ | 87 |
| $\mathscr{B}(G)$ | Borel $\sigma$-algebra of $G$ | 41 |
| $m$ | Haar measure of $G$ | 41 |
| $G_d$ | $G$ equipped with the discrete topology | 60 |
| $f_x$ | translate of $f$ | 45 |
| $\lambda * \mu$ | convolution of $\lambda$ and $\mu$ | 43 |
| $\mathbf{1}_G$ | function on $G$ that is identically equal to one | 42 |
| $\Gamma$ | dual group of $G$ | 42 |
| $\tau$ | topology of pointwise convergence on $\Gamma$ | 60 |
| $\Lambda$ | subset of $\Gamma$ | 46 |
| $H^*$ | adjoint homomorphism of $H$ | 63 |
| $H^\perp$ | annihilator of $H$ | 45 |
| $\prod_{j\in J} G_j$ | direct product or complete direct sum of $(G_j)_{j\in J}$ | 45 |
| $G^J$ | direct product of copies of $G$ | 82 |
| $\bigoplus_{j\in J} G_j$ | direct sum of $(G_j)_{j\in J}$ | 45 |
| $G^{(J)}$ | direct sum of of copies of $G$ | 82 |
| $p_{G_j}$ | projection from $\prod_{j\in J} G_j$ onto $G_j$ | 82 |
| $p_1$ | projection from $\mathbb{Z}^n$ onto $\mathbb{Z}$ | 82 |
| $\hat{f}$ | Fourier transform of $f$ | 42 |
| $\hat{\mu}$ | Fourier-Stieltjes transform of $\mu$ | 43 |
| $\mathrm{spec}(\mu)$ | spectrum of $\mu$ | 46 |
| $T(G)$ | space of trigonometric polynomials of $G$ | 42 |
| $T_\Lambda(G)$ | space of all $f \in T(G)$ whose spectrum is contained in $\Lambda$ | 46 |
| $L^1_\Lambda(G)$ | space of all $f \in L^1(G)$ whose spectrum is contained in $\Lambda$ | 46 |
| $C_\Lambda(G)$ | space of all $f \in C(G)$ whose spectrum is contained in $\Lambda$ | 46 |
| $L^\infty_\Lambda(G)$ | space of all $f \in L^\infty(G)$ whose spectrum is contained in $\Lambda$ | 46 |
| $M_\Lambda(G)$ | space of all $\mu \in M(G)$ whose spectrum is contained in $\Lambda$ | 46 |

# Glossary

**almost Daugavet property**
A Banach space $X$ has the almost Daugavet property if there exists a norming subspace $U$ of $X^*$ such that $X$ has the Daugavet property with respect to $U$.

**balanced $\varepsilon$-peak**
Let $(\Omega, \Sigma, \mu)$ be a probability space, let $A$ be an element of $\Sigma$ and let $\varepsilon$ be a positive number. A real-valued function $f \in L^1(\Omega)$ is said to be a balanced $\varepsilon$-peak on $A$ if $f \geq -1$, $\chi_A f = f$, $\int_\Omega f \, d\mu = 0$, and $\mu(\{f = -1\}) \geq \mu(A) - \varepsilon$.

**$C$-narrow operator**
Let $K$ be a compact space and let $E$ be an arbitrary Banach space. An operator $T \in L(C(K), E)$ is called $C$-narrow if for every non-empty open set $O$ and every $\varepsilon > 0$ there is a function $f \in S_{C(K)}$ with $f|_{K \setminus O} = 0$ and $\|T(f)\| \leq \varepsilon$.

**Daugavet equation**
Let $X$ be a Banach space. An bounded operator $T : X \to X$ satisfies the Daugavet equation if $\|\mathrm{Id} + T\| = 1 + \|T\|$.

**Daugavet property**
A Banach space $X$ has the Daugavet property if every bounded operator $T : X \to X$ of rank one satisfies the Daugavet equation.

**Daugavet property with respect to $U$**
Let $X$ be a Banach space and let $U$ be a norming subspace of $X^*$. We say that $X$ has the Daugavet property with respect to $U$ if the Daugavet equation holds true for every rank-one operator $T : X \to X$ of the form $T = u^* \otimes x$ where $x \in X$ and $u^* \in U$.

**Hadamard set**
A set $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ of natural numbers which for some $q$ satisfies the inequalities

$$\frac{\lambda_{n+1}}{\lambda_n} > q > 1 \quad (n \in \mathbb{N})$$

is called a Hadamard set.

**inner $\varepsilon$-net**
Let $X$ be a Banach space and let $A$ be a subset of $X$. We call a set $B$ an inner $\varepsilon$-net for $A$ if $B \subset A$ and for every $x \in A$ there exists $y \in B$ with $\|x - y\| \leq \varepsilon$.

**$L$-embedded space**
A Banach space $X$ is called $L$-embedded if $X$ is an $L$-summand in its bidual $X^{**}$ where we identify $X$ with its image under the canonical embedding $i_X : X \to X^{**}$.

**$L$-projection**
Let $X$ be a Banach space. A linear projection $P : X \to X$ is called an $L$-projection if

$$\|x\| = \|P(x)\| + \|x - P(x)\| \quad (x \in X).$$

**$L$-summand**
Let $X$ be a Banach space. A closed subspace $Y$ of $X$ is an $L$-summand of $X$ if it is the range of an $L$-projection.

**$\Lambda(p)$ set**
Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. A subset $\Lambda$ of $\Gamma$ is called a $\Lambda(p)$ set for $p \geq 1$ if there exist $r < p$ and a constant $C$ such that

$$\|f\|_p \leq C \|f\|_r \quad (f \in T_\Lambda(G)).$$

**localizable family**
Let $G$ be a compact abelian group, let $\Gamma$ be its dual group, let $\tau$ be the topology of pointwise convergence on $\Gamma$, and let $\mathscr{C}$ be a family of subsets of $\Gamma$. We say that $\mathscr{C}$ is localizable if the following holds: a subset $\Lambda$ of $\Gamma$ belongs to $\mathscr{C}$ if and only if for every $\gamma \in \Gamma$ there exists a $\tau$-open neighborhood $V$ of $\gamma$ such that $\Lambda \cap V \in \mathscr{C}$.

**nicely placed set**
Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. A subset $\Lambda$ of $\Gamma$ is called nicely placed if $L^1_\Lambda(G)$ is a nicely placed subspace of $L^1(G)$.

**nicely placed subspace**
Let $(\Omega, \Sigma, \mu)$ be a probability space. A closed subspace $X$ of $L^1(\Omega)$ is said to be nicely placed if the unit ball of $X$ is closed with respect to convergence in measure.

**narrow operator**
Let $X$ be a Banach space with the Daugavet property and let $E$ be an arbitrary Banach space. An operator $T \in L(X, E)$ is called narrow if for every two elements $x, y \in S_X$, for every $x^* \in X^*$, and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|T(y - z)\| + |x^*(y - z)| \leq \varepsilon$ and $\|x + z\| \geq 2 - \varepsilon$.

**narrow operator with respect to $U$**
Let $X$ be a Banach space that has the Daugavet property with respect to some norming subspace $U$ and let $E$ be an arbitrary Banach space. An operator $T \in L(X, E)$ is called narrow with respect to $U$ if for every two elements $x, y \in S_X$, for every $u^* \in U$, and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|T(y - z)\| + |u^*(y - z)| \leq \varepsilon$ and $\|x + z\| \geq 2 - \varepsilon$.

**poor subspace**
Let $X$ be a Banach space with the Daugavet property. A closed subspace $Y$ of $X$ is called poor if $X/Z$ has the Daugavet property for every closed subspace $Z \subset Y$.

**rich subspace**
Let $X$ be a Banach space with the Daugavet property. A closed subspace $Y$ of $X$ is called rich if the quotient map $\pi : X \to X/Y$ is narrow.

**rich subspace with respect to $U$**
Let $X$ be a Banach space that has the Daugavet property with respect to some norming subspace $U$. A closed subspace $Y$ of $X$ is said to be rich with respect to $U$ if the quotient map $\pi : X \to X/Y$ is narrow with respect to $U$.

**Riesz set**
Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. A subset $\Lambda$ of $\Gamma$ is called a Riesz set if every $\mu \in M_\Lambda(G)$ is absolutely continuous with respect to the Haar measure, i.e., if $L^1_\Lambda(G) = M_\Lambda(G)$.

**Rosenthal set**
Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. A subset $\Lambda$ of $\Gamma$ is called a Rosenthal set if every equivalence class of $L^\infty_\Lambda(G)$ contains a continuous member, i.e., if $C_\Lambda(G) = L^\infty_\Lambda(G)$.

**semi-Riesz set**
Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. A subset $\Lambda$ of $\Gamma$ is called a semi-Riesz set if every $\mu \in M_\Lambda(G)$ is a diffuse measure, i.e., if $M_\Lambda(G) \subset M_{\text{diff}}(G)$.

**Shapiro set**
Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. A subset $\Lambda$ of $\Gamma$ is called a Shapiro set if all subsets of $\Lambda$ are nicely placed.

**Sidon set**
Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. A subset $\Lambda$ of $\Gamma$ is called a Sidon set if there is a constant $C$ such that

$$\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| \leq C \|f\|_\infty \quad (f \in T_\Lambda(G)).$$

**small subspace**
Let $(\Omega, \Sigma, \mu)$ be a probability space. A closed subspace $X$ of $L^1(\Omega)$ is said to be small if there is no $A \in \Sigma$ of positive measure such that the operator $f \mapsto \chi_A f$ maps $X$ onto $\{f \in L^1(\Omega) : \chi_A f = f\}$.

**thickness**
Let $X$ be a Banach space. The thickness $T(X)$ of $X$ is defined by

$$T(X) = \inf\{\varepsilon > 0 : \text{there exists a finite inner } \varepsilon\text{-net for } S_X\}.$$

**uniformly distributed set**
Let $\Lambda$ be a subset of $\mathbb{Z}$ and let $\lambda_1, \lambda_2, \ldots$ be an enumeration of $\Lambda$ with $|\lambda_1| \leq |\lambda_2| \leq \cdots$. We say that $\Lambda$ is uniformly distributed if

$$\frac{1}{n}\sum_{k=1}^{n} t^{\lambda_k} \longrightarrow 0 \quad (t \in \mathbb{T}, t \neq 1).$$

**vanishing point**
Let $K$ be a metrizable, compact space and let $E$ be an arbitrary Banach space. We say that an operator $T \in L(C(K), E)$ vanishes at a point $x \in K$ if there exist a sequence $(O_n)_{n\in\mathbb{N}}$ of open neighborhoods of $x$ with $\operatorname{diam} O_n \longrightarrow 0$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of real-valued and non-negative functions satisfying that $f_n \in S_{C(K)}$, $f_n|_{K\setminus O_n} = 0$, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $\chi_{\{x\}}$, and $\|T(f_n)\| \longrightarrow 0$.

# Bibliography


[1] Yu. Abramovich, *New classes of spaces on which compact operators satisfy the Daugavet equation*, J. Operator Theory **25** (1991), no. 2, 331–345.

[2] Yu. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw, *The Daugavet equation in uniformly convex Banach spaces*, J. Funct. Anal. **97** (1991), no. 1, 215–230.

[3] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Graduate Texts in Mathematics, vol. 233, Springer-Verlag, New York, 2006.

[4] D. Bilik, V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner, *Narrow operators on vector-valued sup-normed spaces*, Illinois J. Math. **46** (2002), no. 2, 421–441.

[5] D. Bilik, V. Kadets, R. Shvidkoy, and D. Werner, *Narrow operators and the Daugavet property for ultraproducts*, Positivity **9** (2005), no. 1, 45–62.

[6] J. Boclé, *Sur la théorie ergodique*, Ann. Inst. Fourier **10** (1960), 1–45.

[7] J. Bourgain, *On the maximal ergodic theorem for certain subsets of the integers*, Israel J. Math. **61** (1988), no. 1, 39–72.

[8] K. Boyko, V. Kadets, and D. Werner, *Narrow operators on Bochner $L_1$-spaces*, Zh. Mat. Fiz. Anal. Geom. **2** (2006), no. 4, 358–371.

[9] R. W. Chaney, *Note on Fourier-Stieltjes transforms of continuous and absolutely continuous measures*, Comment. Math. Prace Mat. **15** (1971), 147–149.

[10] I. K. Daugavet, *A property of completely continuous operators in the space $C$*, Uspekhi Mat. Nauk **18** (1963), no. 5 (113), 157–158.

[11] R. Demazeux, *Centres de Daugavet et opérateurs de composition à poids*, Thèse, Université d'Artois, 2011.

[12] J. Diestel and J. J. Uhl Jr., *Vector Measures*, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.

[13] P. N. Dowling and C. J. Lennard, *Every nonreflexive subspace of $L_1[0,1]$ fails the fixed point property*, Proc. Amer. Math. Soc. **125** (1997), no. 2, 443–446.

[14] N. Dunford and J. T. Schwartz, *Linear Operators. Part I: General Theory*, Pure and Applied Mathematics, vol. 7, Interscience Publishers, Inc., New York, 1958.

[15] C. Foiaş and I. Singer, *Points of diffusion of linear operators and almost diffuse operators in spaces of continuous functions*, Math. Z. **87** (1965), 434–450.

[16] L. Fuchs, *Infinite Abelian Groups. Vol. I*, Pure and Applied Mathematics, vol. 36-I, Academic Press, New York, 1970.

[17] J. B. Garnett, *Bounded Analytic Functions*, Revised first edition, Graduate Texts in Mathematics, vol. 236, Springer-Verlag, New York, 2007.

[18] G. Godefroy, *On Riesz subsets of abelian discrete groups*, Israel J. Math. **61** (1988), no. 3, 301–331.

[19] G. Godefroy, *Sous-espaces bien disposés de $L^1$-applications*, Trans. Amer. Math. Soc. **286** (1984), no. 1, 227–249.

[20] G. Godefroy, N. J. Kalton, and D. Li, *Operators between subspaces and quotients of $L^1$*, Indiana Univ. Math. J. **49** (2000), no. 1, 245–286.

[21] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Die Grundlehren der Mathematischen Wissenschaften, vol. 238, Springer-Verlag, New York, 1979.

[22] K. E. Hare, *Arithmetic properties of thin sets*, Pacific J. Math. **131** (1988), no. 1, 143–155.

[23] K. E. Hare, *An elementary proof of a result on $\Lambda(p)$ sets*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 829–834.

[24] P. Harmand, D. Werner, and W. Werner, *$M$-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Mathematics, vol. 1547, Springer-Verlag, Berlin, 1993.

[25] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I*, Second edition, Die Grundlehren der mathematischen Wissenschaften, vol. 115, Springer-Verlag, New York, 1979.

[26] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. II*, Die Grundlehren der mathematischen Wissenschaften, vol. 152, Springer-Verlag, New York, 1970.

[27] E. Hewitt and H. S. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proc. Cambridge Philos. Soc. **62** (1966), 399–420.

[28] E. Hewitt and H. S. Zuckerman, *Some theorems on lacunary Fourier series, with extensions to compact groups*, Trans. Amer. Math. Soc. **93** (1959), 1–19.

[29] J. R. Holub, *Daugavet's equation and operators on $L^1(\mu)$*, Proc. Amer. Math. Soc. **100** (1987), no. 2, 295–300.

[30] V. M. Kadets, *Some remarks concerning the Daugavet equation*, Quaestiones Math. **19** (1996), no. 1-2, 225–235.

[31] V. Kadets, N. Kalton, and D. Werner, *Remarks on rich subspaces of Banach spaces*, Studia Math. **159** (2003), no. 2, 195–206.

[32] V. Kadets, V. Shepelska, and D. Werner, *Quotients of Banach spaces with the Daugavet property*, Bull. Pol. Acad. Sci. Math. **56** (2008), no. 2, 131–147.

[33] V. Kadets, V. Shepelska, and D. Werner, *Thickness of the unit sphere, $\ell_1$-types, and the almost Daugavet property*, Houston J. Math. **37** (2011), no. 3, 867–878.

[34] V. M. Kadets and M. M. Popov, *The Daugavet property for narrow operators in rich subspaces of the spaces $C[0,1]$ and $L_1[0,1]$*, Algebra i Analiz **8** (1996), no. 4, 43–62; English transl., St. Petersburg Math. J. **8** (1997), no. 4, 571–584.

[35] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 855–873.

[36] V. M. Kadets, R. V. Shvidkoy, and D. Werner, *Narrow operators and rich subspaces of Banach spaces with the Daugavet property*, Studia Math. **147** (2001), no. 3, 269–298.

[37] N. J. Kalton, *The endomorphisms of $L_p$ $(0 \leq p \leq 1)$*, Indiana Univ. Math. J. **27** (1978), no. 3, 353–381.

[38] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, *Some translation-invariant Banach function spaces which contain $c_0$*, Studia Math. **163** (2004), no. 2, 137–155.

[39] D. Li, *A remark about $\Lambda(p)$-sets and Rosenthal sets*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3329–3333.

[40] D. Li, H. Queffélec, and L. Rodríguez-Piazza, *Some new thin sets of integers in harmonic analysis*, J. Anal. Math. **86** (2002), 105–138.

[41] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer-Verlag, Berlin, 1977.

[42] J. M. López and K. A. Ross, *Sidon Sets*, Lecture Notes in Pure and Applied Mathematics, vol. 13, Marcel Dekker Inc., New York, 1975.

[43] G. Ya. Lozanovskiĭ, *On almost integral operators in $KB$-spaces*, Vestnik Leningrad. Univ. **21** (1966), no. 7, 35–44.

[44] F. Lust-Piquard, *Bohr local properties of $C_\Lambda(T)$*, Colloq. Math. **58** (1989), no. 1, 29–38.

[45] F. Lust, *Ensembles de Rosenthal et ensembles de Riesz*, C. R. Acad. Sci. Paris Sér. A **282** (1976), no. 16, 833–835.

[46] M. Martín and R. Payá, *Numerical index of vector-valued function spaces*, Studia Math. **142** (2000), no. 3, 269–280.

[47] Y. Meyer, *Spectres des mesures et mesures absolument continues*, Studia Math. **30** (1968), 87–99.

[48] S. A. Morris, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*, London Mathematical Society Lecture Note Series, vol. 29, Cambridge University Press, Cambridge, 1977.

[49] H. Pfitzner, *A note on asymptotically isometric copies of $l^1$ and $c_0$*, Proc. Amer. Math. Soc. **129** (2001), no. 5, 1367–1373.

[50] A. M. Plichko and M. M. Popov, *Symmetric function spaces on atomless probability spaces*, Dissertationes Math. (Rozprawy Mat.) **306** (1990).

[51] H. P. Rosenthal, *On trigonometric series associated with weak$^*$ closed subspaces of continuous functions*, J. Math. Mech. **17** (1967), 485–490.

[52] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227.

[53] W. Rudin, *Fourier Analysis on Groups*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1990. Reprint of the 1962 original.

[54] W. Rudin, *Real and Complex Analysis*, Third edition, McGraw-Hill Book Co., New York, 1987.

[55] J. H. Shapiro, *Subspaces of $L^p(G)$ spanned by characters:* $0 < p < 1$, Israel J. Math. **29** (1978), no. 2-3, 248–264.

[56] R. V. Shvydkoy, *Geometric aspects of the Daugavet property*, J. Funct. Anal. **176** (2000), no. 2, 198–212.

[57] S. Sidon, *Ein Satz über trigonometrische Polynome mit Lücken und seine Anwendung in der Theorie der Fourier-Reihen*, J. Reine Angew. Math. **163** (1930), 251–252.

[58] S. Sidon, *Verallgemeinerung eines Satzes über die absolute Konvergenz von Fourierreihen mit Lücken*, Math. Ann. **97** (1927), no. 1, 675–676.

[59] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York, 1979.

[60] D. Werner, *The Daugavet equation for operators on function spaces*, J. Funct. Anal. **143** (1997), no. 1, 117–128.

[61] D. Werner, *Funktionalanalysis*, Sechste, korrigierte Auflage, Springer-Lehrbuch, Springer-Verlag, Berlin, 2007.

[62] R. Whitley, *The size of the unit sphere*, Canadian J. Math **20** (1968), 450–455.

[63] S. Willard, *General Topology*, Addison-Wesley Publishing Co., Reading, Mass., 1970.

[64] P. Wojtaszczyk, *Some remarks on the Daugavet equation*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 1047–1052.

[65] A. Zygmund, *Trigonometric Series. Volumes I & II combined*, Third edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.

# Zusammenfassung

Ein Banachraum $X$ hat die *Daugavet-Eigenschaft*, wenn jeder stetige und lineare Operator $T : X \to X$ mit eindimensionalem Bild die sogenannte Daugavet-Gleichung

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

erfüllt. Ein abgeschlossener Unterraum $Y$ von $X$ heißt *reichhaltig*, wenn jeder abgeschlossene Unterraum von $X$, der $Y$ enthält, die Daugavet-Eigenschaft hat, und er heißt *spärlich*, wenn der Quotient $X/Z$ die Daugavet-Eigenschaft hat für jeden abgeschlossenen Unterraum $Z$ von $Y$.

Ist $G$ eine unendliche, kompakte, abelsche Gruppe versehen mit dem Haarschen Maß, so haben $C(G)$ und $L^1(G)$ die Daugavet-Eigenschaft. Da auf $G$ eine Gruppenstruktur existiert, können wir jede Funktion auf $G$ um ein beliebiges $x \in G$ verschieben. Ein abgeschlossener Unterraum $X$ von $C(G)$ oder $L^1(G)$ heißt nun *translationsinvariant*, falls $X$ mit einer Funktion auch beliebige Verschiebungen von ihr enthält. Zu jedem solchen Unterraum $X$ existiert eine Teilmenge $\Lambda$ der dualen Gruppe von $G$, so daß $X$ genau diejenigen Elemente aus $C(G)$ oder $L^1(G)$ enthält, deren Spektrum in $\Lambda$ liegt. Solche Räume bezeichnen wir mit $C_\Lambda(G)$ oder $L^1_\Lambda(G)$. Ziel dieser Arbeit ist es zu untersuchen, welche translationsinvarianten Unterräume die Daugavet-Eigenschaft haben, welche reichhaltig oder spärlich sind, und welche Quotienten bezüglich translationsinvarianter Unterräume die Daugavet-Eigenschaft haben.

Wir erweitern ein Resultat von D. Werner und zeigen, daß $C_\Lambda(G)$ genau dann ein reichhaltiger Unterraum von $C(G)$ ist, wenn $\Gamma \setminus \Lambda^{-1}$ eine semi-Riesz-Menge ist. Außerdem wird gezeigt, daß $\Gamma \setminus \Lambda^{-1}$ eine semi-Riesz-Menge ist, wenn $L^1_\Lambda(G)$ ein reichhaltiger Unterraum von $L^1(G)$ ist. Somit ist $C_\Lambda(G)$ ein reichhaltiger Unterraum von $C(G)$, wenn $L^1_\Lambda(G)$ ein reichhaltiger Unterraum von $L^1(G)$ ist.

Beim Studium von Quotienten von $C(G)$ oder $L^1(G)$ beweisen wir eine interessante Verbindung zwischen reichhaltigen Unterräumen von $C(G)$ und Quotienten von $L^1(G)$ und umgekehrt. So hat $L^1(G)/L^1_\Lambda(G)$ die Daugavet-Eigenschaft, wenn $C_{\Gamma\setminus\Lambda^{-1}}(G)$ ein reichhaltiger Unterraum von $C(G)$ ist, und $C(G)/C_\Lambda(G)$ hat die Daugavet-Eigenschaft, wenn $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ ein reichhaltiger Unterraum von $L^1(G)$ ist.

Betrachtet man spärliche Unterräume von $L^1(G)$, dann kann eine Brücke zu Ergebnissen von G. Godefroy, N. J. Kalton, und D. Li geschlagen werden. Somit erhält man, daß $L^1_\Lambda(G)$ ein spärlicher Unterraum von $L^1(G)$ ist, wenn $\Lambda$ eine Riesz-Menge ist und die Einheitskugel von $L^1_\Lambda(G)$ abgeschlossen ist bezüglich Konvergenz dem Maße nach.

Wir untersuchen außerdem eine Abschwächung der Daugavet-Eigenschaft, die sogenannte *fast-Daugavet-Eigenschaft*. Wir zeigen, daß ein abgeschlossener Unterraum $Y$ eines separablen Raumes $X$ mit der fast-Daugavet-Eigenschaft diese Eigenschaft erbt, wenn der Quotient $X/Y$ keine Kopie von $\ell^1$ enthält. Ist $X$ ein $L$-eingebetteter Raum,

so hat ein separabler, abgeschlossener Unterraum von $X$ die fast-Daugavet-Eigenschaft, wenn er nicht reflexiv ist. Dies führt im Falle einer metrischen, kompakten, abelschen Gruppe dazu, daß $L^1_\Lambda(G)$ genau dann die fast-Daugavet-Eigenschaft hat, wenn $\Lambda$ keine $\Lambda(1)$-Menge ist. Betrachtet man auf einer metrischen, kompakten, abelschen Gruppe die stetigen Funktionen, so hat der Raum $C_\Lambda(G)$ genau dann die fast-Daugavet-Eigenschaft, wenn $\Lambda$ aus unendlich vielen Elementen besteht.

# Lebenslauf

Der Lebenslauf ist in der Online-Version aus Gründen des Datenschutzes nicht enthalten.