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How the Higgs potential got its shape

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Abstract

String-localized quantum field theory allows renormalizable couplings involving massive vector bosons, without invoking negative-norm states and compensating ghosts. We analyze the most general coupling of a massive vector boson to a scalar field, and find that the scalar field necessarily comes with a quartic potential which has the precise shape of the shifted Higgs potential. In other words: the shape of the Higgs potential has not to be assumed, but arises as a consistency condition among fundamental principles of QFT: Hilbert space, causality, and covariance. The consistency can be achieved by relaxing the localization properties of auxiliary quantities, including interacting charged fields, while observable fields and the S-matrix are not affected. This is an instance of the "L-V formalism" – a novel model-independent scheme that can be used as a tool to "renormalize the non-renormalizable" by adding a total derivative to the interaction.

1. Introduction

The present study is part of a long-term program [20,26,30,31] whose aim is to build renormalizable perturbation theory for the Standard Model on quantum principles (notably Hilbert space and causality), and detach it from formal quantization based on classical field theories which requires to sacrifice the Hilbert space as soon as the spin or helicity exceeds $\frac{1}{2}$. Well-established quantitative predictions are unaltered, but the theoretical reasoning changes from

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recipes to principles. The latter are powerful enough to determine the structure of interactions without invoking classical gauge symmetry [17,19]. The program also includes the construction of "off-shell" interacting quantum fields. This is a great advantage over the BRST method: Recall that the BRST variation of interacting charged fields is non-zero (typically a ghost-valued gauge transformation). Such fields are therefore not defined on the positive quotient Hilbert space [6,22]. In contrast, with the new method, all interacting fields are defined on the same Hilbert space. The charged ones, however, will have a weaker localization due to the interaction. The interaction thus "selects" the observables of the theory, namely those interacting fields which remain well localized as required by causality.

The weaker localization properties of "off-shell" (i.e., beyond the S-matrix) charged fields already allowed to re-address and solve salient infrared problems of QED [24,25], including the well-known conflict between locality and the Gauss Law [12], and the infrared superselection structure [13]. We expect that it may also shed new light on confinement in QCD.

The focus in the present paper is on *massive* vector bosons. Massive vector bosons play a central role in the Standard Model, exhibiting self-couplings and minimal couplings to fermions. The immediate problem with these couplings is that the free massive vector field on a Hilbert space (the Proca field with spin 1) has more singular correlation functions than scalar fields. Its "short-distance dimension 2" means that the field causes stronger ultraviolet vacuum polarizations, which in turn makes ultraviolet divergences in loop diagrams stronger than with scalar fields. In technical parlance: the interaction density coupling the massive vector bosons to itself and to Fermi fields (the weak interaction) is power-counting non-renormalizable.

It has become common practice to make the interaction renormalizable by using vector potentials on a Krein space, which means that one admits states of negative norm square. In order to get rid of the latter, one needs gauge invariance. Gauge invariance not only requires indefinite metric, it also does not permit a mass term. The Higgs mechanism is invoked to make the massless gauge bosons behave "as if" they were massive particles [27,32,34].

The construction to be presented here is an alternative way to secure renormalizability that goes without states of negative norm square, ghosts, and spontaneously broken gauge symmetry. The same effect of taming the vacuum polarizations can be achieved by "allowing more room in space". This means, one replaces the Proca fields in the interaction by fields that are localized on "strings" (rays extending to infinity), see Sect. 2.1. They live on the Hilbert space of the massive Proca fields, but have a better UV behaviour. We show that they can have self-interactions only in the presence of a scalar field, and the perturbative renormalizability of such couplings requires the scalar field to have a potential in the familiar shape of the shifted Higgs potential

$$V(H) = \frac{m_H^2}{2}\big(H^2 + \frac{g}{m}H^3 + \frac{g^2}{4m^2}H^4\big) = \frac{g^2 m_H^2}{8m^2}H^2\big(H + \frac{2m}{g}\big)^2 \tag{1.1}$$

with its two degenerate minima.

String-localized quantum field theory (SQFT) offers a variety of tools to prevent that an interaction involving string-localized fields makes the entire theory non-local. These tools implement what is called the "Principle of String Independence" (PSI). The first purpose of this article is to elaborate the more general "L-V formalism", of which the PSI is a prominent instance. We then apply it to the Abelian Higgs Model with only one massive vector boson. One realizes in first order of perturbation theory, that the vector boson of mass m must have a unique cubic coupling to a scalar field called H of (arbitrary) mass m_H, which in turn may have a cubic self-coupling H^3. In second order, the PSI admits a quartic self-coupling H^4. In third order, the PSI uniquely fixes the cubic and quartic coefficients. Together with the mass term, the outcome is (1.1), which is the "shifted" version of the symmetric Higgs potential

$$V(\Psi) = \kappa\Big(\Psi^*\Psi - \frac{v^2}{2}\Big)^2. \tag{1.2}$$

(1.2) is usually invoked to trigger the Higgs mechanism, where the complex scalar field Ψ is minimally coupled to a *massless* gauge field. The symmetry of (1.2) is assumed to be broken spontaneously, and the resulting vacuum expectation value of the complex field makes the massless gauge bosons behave as if they were massive particles. The physical real Higgs field H describes the fluctuations of the complex field around its vacuum expectation value. Expressed in terms of H, (1.2) becomes (1.1) with $m = gv$ and $m_H^2 = 2\kappa v^2$.

To emphasize that (1.1) arises as a *prediction* of SQFT, rather than an *input* to define the model, is the second main purpose of our article. It retrospectively justifies the name "Higgs field" for the scalar field H. But the mass of the vector boson is not generated by spontaneous symmetry breaking. It is there from the start.

Popular as it is (and successful as far as the S-matrix is concerned), we think that the Higgs mechanism suffers from conceptual weaknesses: To which extent can the degenerate classical minima be regarded as different ground states of a quantum algebra (which would justify the term "spontaneous symmetry breaking")? The very (perturbative) construction of such an algebra already picks one of the classical minima to expand around.[1] Moreover, the algebra cannot be constructed on a Hilbert space. The latter has to be recovered with the help of compensating ghost fields and the principle of BRST invariance. The interacting Higgs and other fields of interest are not BRST invariant and hence are not defined on the BRST Hilbert space.

These detriments can be avoided with SQFT, without compromising on the fundamental principles of quantum theory. Remarkably, the *outcome* of the SQFT approach to the Abelian Higgs Model is equivalent to the *input* of the Higgs mechanism: the presence of a neutral Higgs particle with the potential (1.1) with its two degenerate minima. Gauge symmetry is not assumed in the SQFT approach (and not spontaneously broken).

The Abelian Higgs Model was previously treated in [18] and [7, Sect. 5.1], see also [29, Chap. 4.1], in the causal BRST setting, in order to equally emphasize the latter fact: that gauge symmetry needs not to be assumed. In their setting, the shape of the Higgs potential was inferred from the consistency of the BRST method in higher orders of perturbation theory. Parts of our analysis are in fact quite similar to theirs, with the PSI in the place of BRST invariance. But SQFT goes a step further by working in a Hilbert space and with only physical degrees of freedom from the outset.

As said before, this is made possible by admitting the vector potential for the massive particle to be string-localized (see Sect. 2.1). One can then write down a renormalizable interaction density and establish that the resulting theory is equivalent (in a sense to be explained in Sect. 2.3) to the theory with the non-renormalizable interaction density. We shall show this for the coupling to the Higgs field in Sect. 3, and point out that the coupling to Dirac fields can be added without difficulties, see Sect. 3.5.

The Abelian Higgs Model is only a toy model for the weak interaction. The actual weak interaction (with four vector bosons, electrons and neutrinos, and Higgs with their experimentally given masses) was treated without Higgs mechanism in the BRST setting [1,2,10,29], and in

[1] The Feynman rules that give the vector boson mass derive from the "shifted" Lagrangian.

SQFT [17,19]. In these papers, it was found by way of necessary consistency conditions, that the coefficients of cubic self-interaction among several vector bosons *must be* the structure constants of a Lie algebra of compact type, and in the same way the quartic self-couplings are found. Moreover, the coupling of the massive vector bosons to the electrons (neutrinos) *must be* chiral (completely chiral), and Yukawa couplings of the Higgs can only involve scalar Fermi currents.

The decisive mechanism in the BRST setting (in [1,2,10,18,29]) is the consistency of BRST invariance of the S-matrix in higher orders of perturbation theory, as the necessary tool to recover Hilbert space positivity, while locality is manifest. The decisive mechanism in SQFT (in [17,19] and the present paper) is the consistent implementation of the PSI in higher orders of perturbation theory, as the necessary tool to control locality, while positivity is manifest. Neither uses gauge symmetry. The universality of results (Yang-Mills type of self-couplings, chirality of couplings to Fermi currents, and the shape of the Higgs potential) in several variants of BRST and SQFT (see Sect. 4) rather signals an intrinsic consistency between the fundamental principles of Hilbert space positivity and locality, which shows up in many different guises.

Plan of the paper. In Sect. 2, we briefly recall the definition and properties of string-localized quantum fields and the idea how to use them to improve the renormalizability of perturbative quantum field theories that are power-counting non-renormalizable. The S-matrix and interacting quantum fields are constructed perturbatively along the lines of "causal perturbation theory" [6,11], which best permits to control the locality of interacting fields.

We then formulate (in a model-independent way) the PSI to ensure that the resulting theory does not depend on the auxiliary string variables, and develop in quite some detail the recursive scheme that generically induces higher order interaction terms. This "induction" mechanism is what in the model of our interest eventually produces the potential (1.1).

There are actually two variants of the PSI, referred to as "L-V" and "L-Q" formalism, respectively (Sect. 2.3). While SQFT deploys its full power in the former, the latter is much easier, and therefore best suited to familiarize oneself with the calculus (cancellation of "obstructions"). For this reason, we shall in the application to the model (Sect. 3.3) present the L-Q computations in more detail, mostly referring for the L-V variant to "straightforward computations" in Sect. 3.4. It is worth mentioning that the L-V formalism is also useful outside SQFT, whenever one wants to assess the effect of total derivatives in the interaction, see Sect. 2.2, and Sect. 4.5 for an example.

The core section is Sect. 3. We apply the method to the Abelian Higgs Model, whose only input is its free field content: a string-localized massive vector field (constructed from the Proca field), and a canonical scalar field (and no ghosts or Stückelberg fields). The PSI in first order determines a unique cubic coupling among these fields plus a cubic self-coupling of the scalar field with an undetermined coefficient. In second order, a quartic self-coupling is induced, and string independence in third order fixes the cubic and quartic coefficients. The outcome is the Higgs potential. We conclude Sect. 3 with a discussion of interacting fields and local observables in SQFT in general, and in the Abelian Higgs model in particular.

In Sect. 4, we contrast the L-Q and L-V variants of the string-localized approach on the physical Hilbert space with various alternative approaches, beginning with the standard spontaneous symmetry breaking (Sect. 4.1). They also include the BRST approach with ghost and Stückelberg fields (Sect. 4.2) and a point-localized approach without auxiliary fields in Krein space (Sect. 4.5). In both of them, the restoration of Hilbert space positivity is the principle that fixes higher orders of the interaction. The latter, however, turns out to be inconsistent in third order. We also present another string-localized ghost-free approach on the Krein space. All these alternative approaches need unphysical field degrees of freedom.

While the main result of Sect. 3 was the equivalence between a renormalizable string-dependent interaction and a non-renormalizable point-localized interaction (which gives rise to a prescription to renormalize the latter), we present in Sect. 4.7 strong evidence that the latter is also equivalent to the (equally non-renormalizable) interaction in the unitary gauge of the gauge-theoretic approach. See more on the comparison of approaches in Sect. 4.7.

A crucial message is that the Higgs potential (1.1) is *the same in all consistent approaches* – whether they assume a spontaneously broken classical gauge symmetry, or whether they impose quantum principles. We take this as evidence that the observation of the Higgs *particle* and its couplings should not be misconceived as a proof of the Higgs *mechanism* as a physical process.

2. The *L*-*V* formalism

2.1. String-localized quantum fields

String-localization is the mildest form of relaxing the localization, bringing substantial benefits. In [24,25], we have shown how the string-localized *massless* vector potential of QED

$$A_\mu(x,e) = \int_0^\infty ds\, F_{\mu\nu}(x+se)e^\nu, \tag{2.1}$$

where $F_{\mu\nu}$ is the Maxwell tensor and $e \in \mathbb{R}^4$ is a suitable spacelike string direction, can be employed for a new understanding of the singular infrared structure and the "photon clouds" of QED, with the usual gauge redundancy turned into a rich superselection structure. Other advantages have been discussed in [23].

In the present paper, the focus is instead on the improved ultraviolet behaviour, i.e., the renormalization of interactions that are non-renormalizable in Hilbert space formulations of point-local perturbation theory.

In the Abelian Higgs Model, the string-localized field is given by the same formula (2.1) with $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, the field strength of the *massive* Proca field B_μ, in the place of $F_{\mu\nu}$. It is defined on the Wigner Fock space of the Proca field and it creates the same physical particle states as the latter. It only differs by an "operator-valued gauge transformation":

$$A_\mu(x,e) = B_\mu(x) + \partial_\mu \phi(x,e), \tag{2.2}$$

where the massive "escort field" $\phi(x,e)$ is given by

$$\phi(x,e) = \int_0^\infty ds\, B_\mu(x+se)e^\mu. \tag{2.3}$$

In particular, if smeared with $c(e)$ of total weight 1, $A_\mu(c)$ is another potential for the field strength:

$$G_{\mu\nu} = \partial_\mu A_\nu(c) - \partial_\nu A_\mu(c).$$

The crucial feature is that, thanks to the integration along the string, its short-distance dimension is 1, while that of B_μ is 2. Thus, while the Proca couplings $B_\mu B^\mu H$ and $B_\mu j^\mu$ to the scalar Higgs field H and to the Dirac current are non-renormalizable, the couplings $A_\mu(c)B^\mu H$ and $A_\mu(c)j^\mu$ are renormalizable. See [15] why "power counting" is the appropriate criterium for renormalizability also with string-localized fields like (2.1).

2.2. *L-V pairs and L-Q pairs*

An "L-V **pair**" is a relation

$$L = L' + \partial_\mu V^\mu$$

between two interaction densities with specific properties, depending on the case. Adding a total derivative to the interaction may be beneficial.

E.g., L may be defined on a Hilbert subspace of a Krein space where L' is defined. This happens in BRST, where the variation of the interaction density L' is a total derivative because $s(A^K_\mu) = -\partial_\mu u$. To make it zero by adding a total derivative to L', one may use string-localized free fields. E.g., the escort field of QED on the Krein space satisfies $s(\phi(c)) = u$, so that $A(c) = A^K + \partial\phi(c)$ and $L(c) = A(c)j$ are BRST invariant. For another example, see Sect. 4.4.

Or L may be power-counting renormalizable while L' is not. This is the situation in the present paper, where both L and L' live on the physical Hilbert space of the Abelian Higgs model. Renormalizability of L is achieved by string-localization.

In classical field theory, total derivatives in the Lagrangian are ineffective for the equations of motion because the total action is the same. In contrast, in quantum field theory, the S-matrix

$$S = Te^{i\int d^4x\, L(x)}$$

is the time-ordered exponential of the action. Because the time-ordering does not commute with derivatives, adding a derivative will in general change the S-matrix.

The general formalism to be developed below allows to add derivative terms *without altering* the S-matrix. It rather *provides an equivalent reformulation of a theory* in which complementary principles are manifestly satisfied, see (2.14). The equivalence then shows that all principles hold simultaneously. E.g., while a non-renormalizable interaction has no autonomous interpretation, it is provided by a renormalizable (string-localized) reformulation, see Remark 2.3.

The general idea is quite flexible, and can also be used outside SQFT, whenever derivative terms play a role. In the main body of the paper, both L and L' live on the physical Hilbert space of the Abelian Higgs model. This is not possible for QED because of the vanishing photon mass and the IR problem: instead, an L-V pair reformulating the standard indefinite (Feynman gauge) Krein space interaction as an (embedded) string-localized Hilbert space interaction has proven to be a powerful tool for the understanding of infrared features [25]. In Sect. 4.5 of this paper, we consider an L-V pair for the Abelian Higgs model between two point-localized formulations, one on the Hilbert space and the other on the Krein space.

In Sect. 2.3, we develop the L-V formalism specifically for a pair of a point-localized non-renormalizable and a string-localized renormalizable interaction. In the general case, only the specific properties of the admitted interaction terms have to be changed.

When one adds a string-localized derivative term in order to render a point-localized interaction renormalizable, one has to *impose* the **Principle of String Independence** that the S-matrix in higher orders (and the local observables of the theory) are independent of the auxiliary string direction.

The implementation of this principle may require just standard Ward identities, as in QED. In general, as in the Abelian Higgs Model, it requires the addition of "induced" interactions. Namely, while the bulk terms of perturbation theory are integrals with delocalized propagators, the obstruction terms are integrals with δ-functions, which makes it possible to cancel them with interactions of higher-order. Moreover, the cancellation may fix values of parameters that were free parameters in lower orders. This is how the Lie-algebra structure of self-interactions of vector bosons (Yang-Mills), the necessity of a Higgs coupling in the massive case, and the chirality of the coupling of massive vector bosons to Fermi fields (weak interaction) were shown [17,19]. This is also how the correct coefficients in (1.1) will be fixed in Sect. 3 of the present paper.

String-localized interactions always depart from an L-V pair, or, slightly more flexible, an L-Q pair. The former consists of a renormalizable string-localized interaction density $L_1(c)$ and a Lorentz vector of Wick polynomials $V_1^\mu(c)$ such that

$$L_1(c) = L_1^{\text{pt}} + \partial_\mu V_1^\mu(c), \tag{2.4}$$

where L_1^{pt} is a (typically non-renormalizable) point-localized interaction density. Here c stands for the dependence on a string direction e that may be smeared with a smearing function $c(e)$. The simplest L-V pairs are of the form

$$A_\mu(c) j^\mu = B_\mu j^\mu + \partial_\mu(\phi(c) j^\mu), \tag{2.5}$$

where j is a conserved current, B_μ is a point-localized vector potential, and $A_\mu(c) = B_\mu + \partial_\mu\phi(c)$ is an associated string-localized vector potential of the form (2.2) (smeared with $c(e)$). In QED, B_μ is the Feynman gauge vector potential defined on a Krein Fock space.

A more flexible version is the L-Q pair formalism. It starts from the weaker condition

$$\delta_c L_1(c) = \partial_\mu Q_1^\mu(c) \tag{2.6}$$

which only states that the *variation* of the string-localized interaction density w.r.t. the string direction or its smearing function is a total derivative. (When $L_1(c)$ belongs to an L-V pair, then (2.6) is trivially fulfilled with $Q_1^\mu = \delta_c V_1^\mu$.)

Its main advantage is that it does not assume the existence of $V_1^\mu(c)$ such that $Q_1^\mu(c) = \delta_c V_1^\mu(c)$, nor of an associated point-localized density L_1^{pt} such that (2.4) holds. It therefore does not reformulate a given non-renormalizable interaction, but rather allows to establish string-independence of a renormalizable theory whose interaction is string-localized from the outset. In the Abelian Higgs Model, already the L-Q-pair approach allows to fix the Higgs potential. But strictly speaking only the L-V-pair approach allows to identify this potential with the Higgs potential of other approaches, because it contains the point-localized interaction to compare with, and thus is closer to conventional model building.

Some interactions can be formulated as an L-Q pair on the Wigner Hilbert space of free fields, but not as an L-V pair. An example is massless Yang-Mills on the Hilbert space of the free field strengths $F^a_{\mu\nu}$ [16,17]. Thus, one will have to resort to the L-Q formalism for QCD, if one wants to preserve positivity. See also Sect. 5.

In a way, the L-Q equation (2.10) describes only the infinitesimal departure (in first order of $\partial\chi$) from the adiabatic limit of the L-V identity (2.14). This results in drastic computational simplifications. For this reason we begin the next subsection, after some preparations, with the exposition of the former.

2.3. Implementing the Principle of String Independence

Notations. In the sequel, we consider fields X as Wick polynomials in a basis of free fields labelled by φ. In order not to overburden the notation, we shall frequently write $X^{(\prime)}$ for fields $X(x^{(\prime)})$. Wick ordering is always understood but not written, except in some lemmas and their proofs where otherwise there may be ambiguities. The time-ordering symbol T is meant to apply to all fields to its right (brackets omitted). $\langle\cdot\rangle$ is the free vacuum expectation value. $\delta_{xx'}$ stands for $\delta(x-x')$ and $\delta_{xx'x''}\equiv\delta_{xx'}\delta_{x'x''}$ for the total δ-function; and $\mathfrak{S}_n$ for $\frac{1}{n!}$ times the sum over all permutations of n points $x, x', \ldots$.

Obstructions. Because we are interested in *necessary* conditions for string-independence (see Remark 2.3), we shall investigate the SI condition only at tree level.

String-independence possibly fails because time-ordering does not commute with derivatives. It turns out that the PSI at tree level can be formulated in terms of "obstructions" of the form

$$O_Y(X') := [T,\partial_\mu]Y^\mu X'\Big|^{\text{tree}} \equiv T\partial_\mu Y^\mu(x)X(x')\Big|^{\text{tree}} - \partial_\mu TY^\mu(x)X(x')\Big|^{\text{tree}}, \tag{2.7}$$

where Y^μ ($=V^\mu$ or Q^μ, respectively) are vector-valued fields. The quantities $O_Y(X')$ can be expanded in terms of the numerical "two-point obstructions" among the basis fields

$$O_\mu(\varphi;\varphi') \equiv [T,\partial_\mu]\varphi(x)\varphi'(x') \equiv \langle T\partial_\mu\varphi(x)\varphi'(x')\rangle - \partial_\mu\langle T\varphi(x)\varphi'(x')\rangle. \tag{2.8}$$

The latter are δ-functions, or derivatives or string-integrals of δ-functions to be determined in each model, see App. B. For the Abelian Higgs Model, they are displayed in (3.10)–(3.12).

Lemma 2.1. *It holds*

$$O_Y(X') = \sum_{\varphi,\varphi'} O_\mu(\varphi;\varphi')\cdot :\frac{\partial Y^\mu}{\partial\varphi}\frac{\partial X'}{\partial\varphi'}: . \tag{2.9}$$

Corollary 2.2. *The maps $X'\mapsto O_Y(X')$ are derivations on Wick polynomials, i.e., one has the Leibniz rule*

$$O_Y(:XY:') = :O_Y(X')Y': + :X'O_Y(Y'): .$$

Proof of the lemma. By the Wick expansion, because there is only one contraction at tree level, it holds

$$TY^\mu X'\Big|^{\text{tree}} = \sum_{\varphi,\varphi'}\langle T\varphi(x)\varphi(x')\rangle :\frac{\partial Y^\mu}{\partial\varphi}(x)\frac{\partial X}{\partial\varphi'}(x'): .$$

Apply the same expansion to $T(\partial_\mu Y^\mu)X'\Big|^{\text{tree}}$, where $\partial_\mu Y^\mu = \sum_\varphi :\frac{\partial Y^\mu}{\partial\varphi}\partial_\mu\varphi:$, and subtract the expansion of $\partial_\mu TY^\mu X'\Big|^{\text{tree}}$. The result (2.9) is obtained, because all terms cancel in which the derivative hits the uncontracted factors $:\frac{\partial Y^\mu}{\partial\varphi}(x)\frac{\partial X}{\partial\varphi'}(x'):$. □

Proof of the corollary. Obvious, because the maps $X'\mapsto\frac{\partial X'}{\partial\varphi'}$ are derivations. □

Thus, once the two-point obstructions have been determined in a model, the computation of $O_Y(X')$ is straightforward. We have automatized it in many higher-order cases involving iterations of maps O_Y, as in (2.20) or (3.23).

L-Q-pair formalism. The L-Q-pair approach implements the string independence of a model with a renormalizable string-localized interaction density in the adiabatic limit when the spacetime cutoff function χ for the coupling constant g goes to 1.

We want to find conditions on a string-dependent renormalizable interaction such that the variation of the tree-level S-matrix with respect to the string smearing function c vanishes in the adiabatic limit:

$$\lim_{\chi\to 1} \delta_c T e^{iL[\chi,c]}\big|^{\text{tree}} \stackrel{!}{=} 0. \tag{2.10}$$

Here $L[\chi, c]$ is a series of the form

$$L[\chi,c] \equiv \int dx\, \Big(g\,\chi(x)L_1(x,c) + \frac{g^2}{2}\chi^2(x)L_2(x,c) + \dots\Big) \tag{2.11}$$

with a sequence $L_1(x,c), L_2(x,c), \dots$ of properly adjusted *power-counting renormalizable string-localized* interaction densities.

In first order (no time-ordering needed), the condition (2.10) amounts to the statement that $\int dx\, \delta_c L_1(x,c) = 0$. Thus, $\delta_c L_1(x,c)$ must be a total derivative of the form (2.6), which is therefore always the starting point of the recursion.

Imposing (2.10) at tree level order by order, results in a recursive scheme determining the higher-order interactions L_n: in order n, the sum of contributions from all L_m with $m < n$ may not vanish, and this "obstruction" has to be cancelled by L_n. Whether this is possible, depends on the model, i.e., on the "initial data" $L_1(c)$, $Q_1(c)$. We refer to the fulfillability of (2.10) in each order as the condition of string-independence (**SI condition**). The SI condition may induce higher-order interactions, and at the same time also fix parameters from lower orders.

After expanding (2.10) to second order:

$$\frac{i^2}{2}\int dx\,dx'\,\chi(x)\chi(x')\big(\delta_c T L_1(c)L_1'(c) - i\delta_{x,x'}\delta_c L_2(c)\big)\big|^{\text{tree}} \stackrel{!}{=} 0,$$

one may insert $\delta_c L_1 = \partial Q_1$, and replace $T\partial Q_1 L_1'$ by $[T,\partial]Q_1 L_1'$ because the subtracted term $\partial T Q_1 L_1'$ vanishes in the adiabatic limit. The resulting second-order obstruction $O^{(2)}_{LQ}$ must be cancelled by $\delta_c L_2$ up to another total derivative. This is the second order SI condition:

$$\begin{aligned} O^{(2)}_{LQ}(x,x') &:= [T,\partial]Q_1 L_1' + [T,\partial']Q_1' L_1 = O_{Q_1}(L_1') + O_{Q_1'}(L_1) \\ &\stackrel{!}{=} i\delta_{xx'}\cdot(\delta_c L_2(x) - \partial Q_2(x)). \end{aligned} \tag{2.12}$$

The condition (2.12) determines L_2 and Q_2 (possibly with some free parameters). If L_1 is cubic in the fields, then $O^{(2)}_{LQ}$ and the second-order densities L_2, Q_2 are quartic.

After expanding (2.10) to third order, one may insert $\delta_c L_1 = \partial Q_1$ and (using (2.12)) $\delta_c L_2 = \partial Q_2 - i\int dx''\, O^{(2)}_{LQ}(x,x'')$. One obtains

$$\begin{aligned} &\frac{i^3}{6}\int dx\,dx'\,dx''\,\chi(x)\chi(x')\chi(x'')\cdot \\ &\cdot\Big(3T\partial Q_1 L_1' L_1'' - 3i\delta_{xx''}\big(T\partial Q_1 L_2' + T\partial Q_2 L_1'\big) - 3T\,O^{(2)}_{LQ}(x,x'')L_1' - \delta_{xx'x''}\delta_c L_3\Big) \stackrel{!}{=} 0. \end{aligned}$$

One may again replace $T\partial Q_m \dots$ by $[T,\partial]Q_m\dots$ wherever it occurs. The resulting term $3\mathfrak{S}_3([T,\partial]Q_1 L_1' L_1'')$ can be expanded at tree level, using Lemma A.1:

$$3\mathfrak{S}_3\big([T,\partial]Q_1 L_1' L_1''\big|^{\text{tree}}\big) = 3\mathfrak{S}_3\big(2T\,O_{Q_1}(L_1')L_1''\big|^{\text{tree}}\big),$$

and cancels the term $\mathfrak{S}_3(T\,O^{(2)}_{LQ}(x,x'')L_1'\big|^{\text{tree}})$. The terms that are left define the third-order obstruction $O^{(3)}_{LQ}$ and should be cancelled by $\delta_c L_3$ up to another total derivative:

$$O^{(3)}_{LQ}(x,x',x'') := -3i\mathfrak{S}_3\Big(\delta_{x'x''}\big(O_{Q_1}(L'_2)+O_{Q_2}(L'_1)\big)\Big) \stackrel{!}{=} \delta_{xx'x''}\cdot(\delta_c L_3 - \partial Q_3). \quad (2.13)$$

The condition (2.13) determines L_3 and Q_3. If L_1 and Q_1 are cubic in free fields, then L_n and Q_n are of polynomial order $n+2$. The recursion must stop with $L_3 = 0$, because renormalizable L_n of polynomial order > 4 do not exist.

L-V-pair formalism. The more ambitious L-V formalism not only allows to construct *some* string-independent S-matrix. It also establishes the equivalence with a possibly non-renormalizable point-localized interaction *before* the adiabatic limit is taken, see Remark 2.3.

We want to establish the identity

$$Te^{i(L[\chi;c]+V\circ\partial[\chi;c])} = Te^{iL^{\rm pt}[\chi]} \quad (2.14)$$

to hold at tree level for arbitrary cutoff functions χ, where the term $V\circ\partial[\chi]$ vanishes[2] in the adiabatic limit $\chi\to 1$. More precisely, on the right-hand side

$$L^{\rm pt}[\chi] \equiv \int dx\,\Big(g\,\chi(x)L^{\rm pt}_1(x) + \frac{g^2}{2}\chi(x)^2L^{\rm pt}_2(x)+\dots\Big) \quad (2.15)$$

is a series of *possibly power-counting non-renormalizable point-localized* interaction densities. Similarly, $L[\chi,c]$ on the left-hand side is again given by (2.11) with a series of *power-counting renormalizable string-localized* interaction densities. Finally,

$$V\circ\partial[\chi] = \int dx\,\Big(g\,\partial_\mu\chi(x)V^\mu_1(x,c) + \frac{g^2}{2}\big[\partial_\mu\chi(x)^2V^\mu_2(x,c) + \partial_\mu\chi(x)\partial_\nu\chi(x)W^{\mu\nu}_2(x,c)\big]+\dots\Big) \quad (2.16)$$

with a series of string-localized, possibly non-renormalizable tensor densities $V^\mu_n,\dots,W^{\mu_1\dots\mu_n}_n$.

Remark 2.3. The virtue of the formula (2.14) is that the left-hand side is renormalizable in the adiabatic limit, where the term $V\circ\partial[\chi]$ vanishes. It thus serves to "renormalize the non-renormalizable right-hand side", which is manifestly string-independent and point-localized, by fixing infinitely many renormalization constants appearing in loop diagrams in terms of finitely many constants on the left-hand side. For this prescription to work, (2.14) must be an identity at tree level. We therefore shall restrict the analysis to tree level.

Again, we refer to the fulfillability of (2.14) as the **SI condition**. In each order $O(g^n)$, it is an equality between operator-valued distributions evaluated on $\chi^{\otimes n}$. It constitutes a recursive system, that has to be solved for $L_n(c)$, $L^{\rm pt}_n$, $V_n(c)$, $W_n(c)$ with the specifications as given above.

In first order, the SI condition simply reads

$$\int dx\,\chi(x)\big(L_1(x,c) - \partial_\mu V^\mu_1(x,c) - L^{\rm pt}_1(x)\big) \stackrel{!}{=} 0. \quad (2.17)$$

Its validity for all χ is equivalent to the L-V-pair condition (2.4), which is therefore always the starting point of the recursion.

[2] The notation "$V\circ\partial$" is only suggestive. The main parts of $V\circ\partial[\chi]$ are $V^\mu_n(\partial_\mu(\chi^n))$, see (2.16). The need to add such a term in order to get an identity *before* the adiabatic limit is taken, was first noticed by Duch [5].

In n-th order, one collects all terms involving $L_m^{\rm pt}, L_m, V_m, W_m$ with $m < n$ in (2.14), and writes them with the help of integrations by parts as

$$\frac{i^n}{n!}\int dx_1 \dots dx_n\, \chi(x_1)\dots\chi(x_n)\, O^{(n)}(x_1,\dots,x_n).$$

The "n-th order obstruction" $O^{(n)}$ must then be cancelled by the linear contribution from $L_n^{\rm pt}, L_n, V_n, W_n$. This condition determines the latter, possibly with free parameters.

Proposition 2.4. *The tree-level obstruction in second order is*

$$O^{(2)}(x,x') = \mathfrak{S}_2\Big(2O_{V_1}(L_1^{\rm pt\prime}) + O_{V_1}(\partial' V_1') - \partial' O_{V_1}(V_1')\Big). \tag{2.18}$$

For the proof, see App. A. Because χ is arbitrary, the SI condition requires the cancellation

$$O^{(2)}(x,x') \stackrel{!}{=} i\delta_{xx'}\cdot\big(L_2(x,c) - L_2^{\rm pt}(x) - \partial_\mu V_2^\mu(x,c)\big) + \partial_\mu\partial_\nu'\big[i\delta_{xx'}\cdot W_2^{\mu\nu}(x,c)\big]. \tag{2.19}$$

Proposition 2.5. *After cancellation of the second-order obstruction, the tree-level obstruction in third order is*

$$\begin{aligned} O^{(3)}(x,x',x'') = \mathfrak{S}_3\Big(& O_{V_1}\big(O_{V_1'}(L_1^{\rm pt\prime\prime} + 2L_1'')\big) - 2\partial'' O_{V_1}\big(O_{V_1'}(V_1'')\big) \\ &+ 3O_{O_{V_1}(V_1')}(L_1^{\rm pt\prime\prime}) - 3i\delta_{x'x''}\big(O_{V_1}(L_2') - \partial' O_{V_1}(V_2') + O_{V_2'}(L_1^{\rm pt})\big) \\ &+ 3i\partial_\nu''\delta_{x'x''}\cdot O_{W_2'^\nu}(L_1^{\rm pt}) - 3i\partial'\partial''\big[\delta_{x'x''}O_{V_1}(W_2')\big]\Big) \end{aligned} \tag{2.20}$$

with obvious contractions of Lorentz indices (and $O_{W_2^\nu}(L_1^{\rm pt\prime\prime}) \equiv [T,\partial_\mu]W_2^{\mu\nu}L_1^{\rm pt\prime\prime}$).

For the proof, see App. A. It is interesting to notice, that all terms in (2.20) are various iterations of expressions of the form $O_Y(X')$ as in (2.7). The ensuing SI condition is

$$\begin{aligned} O^{(3)}(x,x',x'') \stackrel{!}{=} & \delta_{xx'x''}\cdot\big(L_3(x) - L_3^{\rm pt}(x) - \partial_\mu V_3^\mu(x)\big) \\ &+ \mathfrak{S}_3\big(\partial_\mu\partial_\nu'\big[\delta_{xx'x''}\cdot W_3^{\mu\nu}(x)\big]\big). \end{aligned} \tag{2.21}$$

If the initial L-V pair is cubic in free fields, then the higher-order densities are of polynomial order $n+2$. Renormalizability requires that $L_3 = 0$.

3. The Abelian Higgs Model

The basic field content of the Abelian Higgs Model in the SQFT formulation is given by the fields $A_\mu(c)$ and $\phi(c) = -m^{-2}\partial_\mu A^\mu(c)$ of mass $m > 0$, and the scalar Higgs field H of mass $m_H > 0$. In this spirit, B_μ (the Proca field) is rather a short-hand notation for the string-independent combination $A_\mu(c) - \partial_\mu\phi(c)$, see Sect. 2.1. Notice that $A_\mu(c)$ and $\phi(c)$ are defined as in (2.2) and (2.3) smeared with $c(e)$ of total weight 1, so that (2.2) still holds.

However, for the purpose of the computation of obstructions in the subsequent analysis, it is more convenient to work in the basis

$$B_\mu, A_\mu(c), \phi(c), H, \partial_\mu H. \tag{3.1}$$

We list $\partial_\mu H$ as an independent field because time-ordering does not respect differential relations among fields. In contrast, $\partial_\mu\phi(c) = A_\mu(c) - B_\mu$ can be expressed in terms of the basis fields.

We denote the string variation of the escort field by

$$w := \delta_c \phi(c). \tag{3.2}$$

Its precise form is not relevant here; see, e.g., [23]. Then,

$$\delta_c \partial_\mu \phi(c) = \delta_c A_\mu(c) = \partial_\mu w, \qquad \delta_c H = \delta_c \partial_\mu H = 0. \tag{3.3}$$

3.1. First order

We determine the initial L-Q and L-V pairs that define the model. We shall see that self-interactions of the massive vector field are only possible with the intervention of the scalar Higgs field.

Because $A_\mu(c)$, $\phi(c)$ and H have short-distance dimension 1, the only renormalizable couplings are cubic or quartic Wick polynomials, involving at most one derivative in the cubic case. The most general candidate for $L_1(c)$ is

$$\begin{aligned} L_1 = {} & a_1\, A^\mu A^\nu \partial_\mu A_\nu + a_2\, A^\mu A_\mu \phi + a_3\, A^\mu \phi \partial_\mu \phi + a_4\, \phi^3 + \\ & + b_1\, A^\mu A_\mu H + b_2\, A^\mu \partial_\mu \phi H + b_3\, A^\mu \phi \partial_\mu H + b_4\, \phi^2 H + \\ & + c_1\, A^\mu H \partial_\mu H + c_2\, \phi H^2 + d H^3 + e_1 (A^\mu A_\mu)^2 + e_2\, A^\mu A_\mu \phi^2 + e_3\, \phi^4 + \\ & + f_1\, A^\mu A_\mu \phi H + f_2\, \phi^3 H + g_1\, A^\mu A_\mu H^2 + g_2\, \phi^2 H^2 + h\, \phi H^3 + j\, H^4. \end{aligned} \tag{3.4}$$

Here, we suppress the string-dependence of the fields.

Proposition 3.1. *The interaction density $L_1(c)$ is part of an L-Q pair if, and only if, it is (up to a global factor to be absorbed in the coupling constant g) of the form*

$$L_1 = m\Big(A^\mu B_\mu H + A^\mu \phi \partial_\mu H - \frac{m_H^2}{2}\phi^2 H + a\, H^3\Big) + a'\, H^4 + \partial_\mu \sum\nolimits_i \alpha_i U_i^\mu, \tag{3.5}$$

where $U_1^\mu = A^\mu A^\nu A_\nu$, $U_2^\mu = A^\mu \phi^2$, $U_3^\mu = A^\mu \phi H$, $U_4^\mu = A^\mu H^2$. At this point, a, a' and α_i ($i = 1, 2, 3, 4$) are free real parameters. It holds $\delta_c L_1(c) = \partial_\mu Q_1^\mu(c)$ and $Q_1^\mu(c) = \delta_c V_1^\mu(c)$ with

$$Q_1^\mu = m\big(B^\mu w H + \phi w \partial^\mu H\big) + \sum\nolimits_i \alpha_i \delta_c U_i^\mu, \tag{3.6}$$

$$V_1^\mu = m\Big(B^\mu \phi H + \frac{1}{2}\phi^2 \partial_\mu H\Big) + \sum\nolimits_i \alpha_i U_i^\mu, \tag{3.7}$$

hence $L_1(c)$ is also part of an L-V pair $L_1(c) - \partial_\mu V_1(c) = L_1^{\mathrm{pt}}$ with

$$L_1^{\mathrm{pt}} = m\big(B^\mu B_\mu H + a\, H^3\big) + a'\, H^4. \tag{3.8}$$

Remark 3.2. We shall show in Lemma 3.3 that the SI condition in second order requires $\alpha_i = 0$ ($i = 1, 2, 3, 4$). The term aH^3 in (3.5), (3.8) will become the cubic part of the potential

$$V(H) = \frac{1}{2} m_H^2 H^2 - g \cdot m a H^3 - \frac{g^2}{2} \cdot b H^4. \tag{3.9}$$

The quartic part will arise in second order in L_2 (whereas a' as part of L_1 must vanish), and the coefficients a and b will be fixed in third order, see below. That the SI condition uniquely fixes the Higgs potential as in (1.1), is the result referred to in the title of this paper.

Proof of Proposition 3.1. We must fix the coefficients in (3.4) such that the string-variation $\delta_c L_1(x,c)$ is a total x-derivative. The terms $dH^3 + jH^4$ in (3.4) are string-independent. The four combinations $\partial_\mu U_i^\mu$ trivially satisfy this condition via $\delta_c \partial_\mu U_i^\mu = \partial_\mu(\delta_c U_i^\mu)$. We also have (using (3.2), (3.3), and $\Box H = -m_H^2 H$ and $\partial A = \Box\phi = -m^2\phi$)

$$\begin{aligned}\delta_c(A^2H + A\phi\overset{\leftrightarrow}{\partial} H - \frac{1}{2}m_H^2\phi^2 H) &= (A\partial H - m_H^2\phi H)w + (AH + \phi\overset{\leftrightarrow}{\partial} H)\partial w \\ &= \partial[(AH - \phi\overset{\leftrightarrow}{\partial} H)w].\end{aligned}$$

Since $A - \partial\phi = B$, these are the solutions displayed in (3.5) and (3.6), with a relabelling $d \to ma$, $j \to a'$ of the coefficients. If the coupling constant g is dimensionless, then b_1 is dimensionful. The choice $b_1 = m$ in (3.5) is a matter of convenience.

To prove that there are no further solutions, we may use the given solutions to freely adjust the coefficients a_1, a_3, b_1, b_3, c_1. For an independent solution we may thus assume $a_1 = a_3 = b_1 = b_3 = c_1 = 0$. By homogeneity in the Higgs fields and in the vector boson fields, the conditions on the remaining coefficients decouple from each other for the terms with coefficients labelled by different letters. E.g., the remaining a-terms give

$$\delta_c(a_2A^2\phi + a_4\phi^3) = a_2(A^2w + 2(A\partial w)\phi) + 3a_4\phi^2 w,$$

which cannot be a total derivative unless $a_2 = a_4 = 0$. The remaining b-terms give

$$\delta_c(b_2A\partial\phi H + b_4\phi^2 H) = b_2(A + \partial\phi)\partial w H + 2b_4\phi w H,$$

which cannot be a total derivative unless $b_2 = b_4 = 0$. Similar for the terms with coefficients e_i, f_i, g_i, h, which do not admit combinations whose string-derivatives are a total x-derivative.

The asserted equalities $\delta_c V_1 = Q_1$ and $L_1 - \partial V_1 = L_1^{\mathrm{pt}}$ are verified by direct computation. □

3.2. Two-point obstructions

In order to compute the higher obstructions $O_{LQ}^{(n)}$ of the S-matrix, one needs the two-point obstructions involving the fields (3.1) (and w in the L-Q-pair approach) of the Abelian Higgs Model. By (2.7), the latter are directly obtained from the propagators as determined in App. B, in combination with the field equations. The given scaling degrees of the propagators allow two free renormalization parameters c_H, c_B in the Higgs and Proca sector, respectively. The relevant two-point obstructions for the Higgs field are

$$\begin{aligned}O_\mu(\partial^\mu H; \partial'_\nu H') &= -i(1 + c_H)\partial_\nu\delta(x - x'), \\ O_\mu(\partial^\mu H; H') &= i\delta(x - x'), \\ O_\mu(H; \partial'_\nu H') &= ic_H\eta_{\mu\nu}\delta(x - x'), \\ O_\mu(H; H') &= 0.\end{aligned} \tag{3.10}$$

Those for the fields B, A, ϕ are

$$\begin{aligned}O_\mu(B^\mu; B'_\nu) &= -i(1 + c_B)\cdot m^{-2}\partial_\nu\delta(x - x'), \\ O_\mu(B^\mu; \phi') &= -im^{-2}\delta(x - x'), \\ O_\mu(\phi; B'_\nu) &= -ic_B\cdot m^{-2}\eta_{\mu\nu}\delta(x - x'), \\ O_\mu(\phi; \phi') &= 0,\end{aligned} \tag{3.11}$$

as well as

$$\begin{aligned} O_\mu(A^\mu; A'_\nu) &= -i \cdot \big(e_\nu I_e - (ee')I_e I_{-e'}\partial\big)\delta(x-x'), \\ O_\mu(A^\mu; \phi') &= -i \cdot (ee')I_e I_{-e'}\delta(x-x'), \\ O_\mu(A^\mu; B'_\nu) &= -i \cdot e_\nu I_e \delta(x-x'), \\ O_\mu(\phi; A'_\nu) &= 0, \\ O_\mu(B^\mu; A'_\nu) &= 0. \end{aligned} \tag{3.12}$$

In the L-Q-approach, one also needs the two-point obstructions $O_\mu(w; X')$ for $X = A, B, \phi$. These are all found to be zero.

The two-point obstructions $O(A, X')$ are string-localized. They could spoil the SI conditions, because obstructions of (2.10) or (2.14) involving string-integrated δ-functions cannot be cancelled by higher-order densities. Fortunately, this does not happen, as we shall see.

Lemma 3.3. *The SI condition (both in the L-Q and L-V-pair approach) requires in second order that in Proposition 3.1,*

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.$$

Proof. Q_1^μ and V_1^μ contain the field A only in the terms $\delta_c U_i$ and U_i. By (2.9), obstructions with string-integrated δ-functions can occur in second order only through these terms in (3.6) and (3.7). Because their coefficients according to (2.9) are linearly independent, there can be no cancellations. These terms must therefore be excluded altogether: $\alpha_i = 0$. □

We proceed with the L-Q-pair approach. The L-V-pair approach will be treated in Sect. 3.4.

3.3. The L-Q-pair approach

The initial L-Q-pair (2.6) of the Abelian Higgs Model is specified by Proposition 3.1 and Lemma 3.3:

$$\begin{aligned} L_1 &= m\Big(ABH + A\phi\partial H - \frac{m_H^2}{2}\phi^2 H + aH^3\Big) + a'H^4, \\ Q_1 &= m\big(BH + \phi\partial H\big)w. \end{aligned} \tag{3.13}$$

Second order. The string-localized two-point obstructions $O_\mu(A^\mu; X')$ in (3.12) do not contribute to the second-order obstruction (2.12) of the S-matrix, because the field A (or $\partial\phi = A - B$) does not occur in Q^μ. This feature of the model distinguishes the choice of the "kinematical" propagators for the string-localized fields, as discussed in App. B.3, leaving only the parameters c_H in (3.10) and c_B in (3.11) free.

Proposition 3.4. *The SI condition in second order (2.12) requires that the parameter $a' = 0$ in (3.13). Then the condition is solved by*

$$\begin{aligned} L_2 &= m^2\Big(\big(3a + \frac{m_H^2}{m^2}\big)\phi^2 H^2 - \frac{m_H^2}{4}\phi^4 + (1+c_H)\cdot A^2\phi^2\Big) + (1+c_B)\cdot A^2H^2 + bH^4, \\ Q_2 &= m^2(1+c_H)\cdot A\phi^2 w + (1+c_B)\cdot AwH^2. \end{aligned} \tag{3.14}$$

The quartic term of the Higgs potential appears with the coefficient b undetermined.

Proof. The task is to compute the obstruction $O^{(2)}_{LQ}$ in (2.12). The straightforward computation using (2.9) with (3.10), (3.11) yields

$$\begin{aligned}
O_{Q_1}(L_1') = m^2\Big(&\phi w\cdot\delta_{xx'}\cdot\big(AB - \frac{m_H^2}{2}\phi^2 + 3aH^2 + \frac{4a'}{m}H^3\big)\\
&+ c_H Bw\cdot\delta_{xx'}\cdot A\phi - (1+c_H)\phi w\cdot\partial\delta_{xx'}\cdot A'\phi'\Big)\\
&- wH\cdot\delta_{xx'}\cdot(A\partial H - m_H^2\phi H) - c_B w\partial H\cdot\delta_{xx'}\cdot AH\\
&- (1+c_B)wH\cdot\partial\delta_{xx'}\cdot A'H'.
\end{aligned}$$

Adding $O_{Q_1'}(L_1)$ and using (C.1), one obtains

$$\begin{aligned}
O^{(2)}_{LQ} &= i\delta_{xx'}\cdot m^2\Big[2\big(3a+\frac{m_H^2}{m^2}\big)\phi wH^2 - m_H^2\phi^3 w + (1+c_H)\big(2AB\phi w + (A\overleftrightarrow{\partial} w)\phi^2\big)\Big]+\\
&\quad + i\delta_{xx'}\cdot(1+c_B)\big(-2AwH\partial H + (A\overleftrightarrow{\partial} w)H^2\big) + i\delta_{xx'}\cdot 8ma'\phi wH^3 =\\
&= i\delta_{xx'}\cdot m^2\Big[\delta_c\Big(\big(3a+\frac{m_H^2}{m^2}\big)\phi^2H^2 - \frac{m_H^2}{4}\phi^4\Big) + (1+c_H)\Big(\delta_c(A^2\phi^2) - \partial(A\phi^2 w)\Big)\Big]+\\
&\quad + i\delta_{xx'}\cdot(1+c_B)\big(\delta_c(A^2H^2) - \partial(AwH^2)\big) + i\delta_{xx'}\cdot ma'\delta_c(\phi^2H^3).
\end{aligned} \tag{3.15}$$

The term ϕ^2H^3 has dimension 5 and is not admissible in L_2, hence we must have $a' = 0$. Then, L_2 and Q_2 in (2.12) are read off (3.15). □

Third order. Q_2 consists of two terms involving the field A with coefficients $1 + c_H$ and $1 + c_B$. By (3.18) and (3.12), these would contribute string-integrated δ-functions in $O^{(3)}_{LQ}$, which cannot be cancelled. This forces us to fix the renormalization parameters as

$$c_H = -1, \qquad c_B = -1. \tag{3.16}$$

In particular, $Q_2 = 0$ with this choice.

Proposition 3.5. *The SI condition in third order (2.13) requires that the parameters a in (3.13) and b in (3.14) take the values*

$$a = -\frac{1}{2}\frac{m_H^2}{m^2}, \qquad b = -\frac{1}{4}\frac{m_H^2}{m^2}. \tag{3.17}$$

Then the condition is solved by $L_3 = 0$ and $Q_3 = 0$.

Corollary 3.6. *The values a and b determined by Proposition 3.5 yield the precise form of the Higgs potential (1.1).*

Proof of the proposition. The task is to compute $O^{(3)}_{LQ}$ as in (2.13). The straightforward computation yields[3]

[3] We do not know the significance of the following observation. If one works with general c_B and c_H, then one gets – apart from the string-localized obstructions, that do not cancel – four additional point-localized contributions to (3.18)

$$+2\big((1+c_H) + c_B(1+c_H) - (1+c_B) - c_H(1+c_B)\big)A^2\phi wH = 0,$$

$$O^{(3)}_{LQ} = 3\delta_{xx'x''} \cdot \Big[m^3\Big(2\big(3a + \frac{m_H^2}{m^2}\big) + \frac{m_H^2}{m^2}\Big)\phi^3 wH + \\ + m\Big(4b - 2\big(3a + \frac{m_H^2}{m^2}\big)\Big)\phi w H^3\Big]. \tag{3.18}$$

Since the fields $\phi^3 wH$ and $\phi w H^3$ cannot be written as $\delta_c L_3 - \partial Q_3$ with renormalizable L_3, their coefficients must vanish. This fixes the parameters a and b, and $\delta_c L_3 - \partial Q_3 = 0$. □

Proof of the corollary. (3.9) with a and b as in (3.17) is (1.1). □

3.4. L-V-pair approach

The initial L-V-pair (2.4) of the Abelian Higgs Model, as specified by Proposition 3.1 and Lemma 3.3, is

$$L_1^{\rm pt} = m\big(B^\mu B_\mu H + a\,H^3\big) + a'\,H^4, \tag{3.19}$$
$$L_1 = m\Big(A^\mu B_\mu H + A^\mu \phi\partial_\mu H - \frac{m_H^2}{2}\phi^2 H + aH^3\Big) + a' H^4,$$
$$V_1^\mu = m\Big(B^\mu \phi H + \frac{1}{2}\phi^2\partial^\mu H\Big).$$

Second order. The string-localized two-point obstructions $O_\mu(A^\mu; X')$ in (3.12) do not contribute to the second-order obstruction (2.18) of the S-matrix, because the field A does not occur in V^μ.

Proposition 3.7. *The SI condition in second order (2.19) requires that the parameter $a' = 0$ in (3.19). Then the condition is solved by*

$$L_2^{\rm pt} = -3B^2H^2 + (1+c_B)\cdot 4B^2H^2 + bH^4, \tag{3.20}$$
$$L_2 = m^2\Big((3a + \frac{m_H^2}{m^2})\phi^2H^2 - \frac{m_H^2}{4}\phi^4 + (1+c_H)\cdot A^2\phi^2\Big) + (1+c_B)\cdot A^2H^2 + bH^4,$$
$$V_2^\mu = -B^\mu\phi H^2 + \frac{m^2}{6}B^\mu\phi^3 + (1+c_H)\cdot\frac{m^2}{2}A^\mu\phi^3 + (1+c_B)\cdot A^\mu\phi H^2,$$
$$W_2^{\mu\nu} = \Big((1+c_H)\cdot\frac{m^2}{4}\phi^4 + (1+c_B)\cdot\phi^2H^2\Big)\eta^{\mu\nu},$$

with one new free parameter b.

Proof. We have to compute (2.18), using (2.7). We begin with the choice (3.16) for c_H, c_B. With this choice, the obstructions $O_V(X')$ contain no derivatives or string-integrals of $\delta(x-x')$, and it is convenient to write

$$O_V(X') =: i\delta(x-x')\cdot\Omega_V(X). \tag{3.21}$$

In particular, one has $\Omega_{V_1}(V_1) = 0$. This simplifies (2.18) and (2.19) to

which identically cancel each other. This is a remarkable independence of the renormalization parameters. In particular, the values of the Higgs potential parameters a and b are not affected.

$$O^{(2)}(x,x') = i\delta(x-x')\cdot\Omega_{V_1}(L_1^{\mathrm{pt}}+L_1), \qquad \Omega_{V_1}(L_1^{\mathrm{pt}}+L_1) \overset{!}{=} L_2 - L_2^{\mathrm{pt}} - \partial V_2. \quad (3.22)$$

The straightforward computation yields

$$\Omega_{V_1}(L_1^{\mathrm{pt}}+L_1) = ABH^2 + 2B^2H^2 + 2B\phi H\partial H + \\ + m^2\Big(\big(3a+\frac{m_H^2}{m^2}\big)\phi^2H^2 - \frac{1}{2}(A-B)B\phi^2 - \frac{m_H^2}{4}\phi^4\Big) + 4ma'\phi^2H^3,$$

which can be re-written as

$$= 3B^2H^2 + m^2\Big(\big(3a+\frac{m_H^2}{m^2}\big)\phi^2H^2 - \frac{m_H^2}{4}\phi^4\Big) + \partial\big(B\phi H^2 - \frac{m^2}{6}B\phi^3\big) + 4ma'\phi^2H^3.$$

The first term is point-localized, the second term is renormalizable, and the third term is a derivative. The last term is not compatible with the required form of the cancelling second-order densities in (3.22). Thus $a'=0$. Comparing the other terms with (2.19), one reads off (3.20) for the special values $c_H = c_B = -1$.

The additional contributions to $O^{(2)}(x,x')$ due to different values of c_H, c_B involve derivatives of $\delta(x-x')$. The formulae in Lemma C.1 in App. C nicely deal with the symmetrization in $x \leftrightarrow x'$, and reduce the result to the additional terms

$$\cdots + (1+c_H)\cdot im^2\Big(\delta(x-x')\big(A^2\phi^2 - \frac{1}{2}\partial(A\phi^3)\big) + \frac{1}{4}\partial\partial'\big[\delta(x-x')\phi^4\big]\Big) + \\ + (1+c_B)\cdot i\Big(\delta(x-x')\big(A^2H^2 - 4B^2H^2 - \partial(A\phi H^2)\big) + \partial\partial'\big[\delta(x-x')\phi^2H^2\big]\Big).$$

This yields the additional terms displayed in (3.20). □

L_2 in (3.20) coincides with L_2 in (3.14) in the L-Q-pair approach. Notice that, while $\delta_c V_1 = Q_1$, the general setup does not imply that $\delta_c V_2$ should equal Q_1.

Third order. V_2 contains two terms involving the field A with coefficients $1+c_H$ and $1+c_B$. By (2.20) and (3.12), these would contribute string-integrated δ-functions in $O^{(3)}$, which cannot be cancelled. This forces us again to fix the renormalization parameters as in (3.16). In this case, the two-point obstructions (3.10) and (3.11) do not contain derivatives of δ-functions and can be cancelled in each order by induced triples $L_n, V_n, L_n^{\mathrm{pt}}$ (i.e., all $W_n = 0$). In particular, one has (3.21) with $\Omega_{V_1}(V_1)=0$, so that $W_2 = 0$ and (2.20) simplifies considerably:

$$O^{(3)}(x,x',x'') = \delta_{xx'x''}\big(-\Omega_{V_1}^2(L_1^{\mathrm{pt}}+2L_1) + 3\Omega_{V_1}(L_2) + 3\Omega_{V_2}(L_1^{\mathrm{pt}}) - 2\partial\Omega_{V_1}(V_2)\big). \quad (3.23)$$

Proposition 3.8. *The SI condition in third order (2.21) requires that the parameters a in (3.19) and b in (3.20) take the values (3.17). In this case, it is solved by*

$$L_3^{\mathrm{pt}} = \frac{12}{m}B^2H^3, \quad (3.24)$$
$$L_3 = 0,$$
$$V_3 = \frac{2}{m}B\phi H^3 - \frac{1}{m}\phi^2H^2\partial H - \frac{5m}{3}B\phi^3H + \frac{m}{12}\phi^4\partial H,$$
$$W_3 = 0.$$

Proof. The computation of (3.23) is straightforward. The result, inserted in the third-order SI condition (2.21), can be written as

$$L_3 - \partial V_3 - L_3^{\mathrm{pt}} \stackrel{!}{=} -\frac{12}{m}B^2H^3 - \partial\Big(\frac{2}{m}B\phi H^3 - \frac{1}{m}\phi^2H^2\partial H - \frac{5m}{3}B\phi^3H + \frac{m}{12}\phi^4\partial H\Big) + \\ +m\Big(6b - 9a - 3\frac{m_H^2}{m^2}\Big)\phi^2H^3 + m^3\Big(\frac{9a}{2} + \frac{9}{4}\frac{m_H^2}{m^2}\Big)\phi^4H, \tag{3.25}$$

where L_3 must vanish because renormalizable fifth-order Wick polynomials do not exist. The last two terms in (3.25) are not compatible with the required form of the left-hand side. Their vanishing requires the values given in (3.17), and hence implies the precise form of the Higgs potential (1.1), as in Corollary 3.6.

The first term is a total derivative and should be identified with $-\partial V_3$. The second term is string-independent, and should be identified with $-L_3^{\mathrm{pt}}$. These are the third-order terms (3.24). □

The renormalizable string-localized interaction density $L[\chi;c]$ in (2.11) terminates with the quartic terms L_2, because renormalizable candidates of higher polynomial order do not exist. Higher-order terms appear only in the form of L_n^{pt} and $V_n(c)$. Recall that the purpose of the (non-renormalizable) derivative terms $V_n^\mu(\partial_\mu\chi^n)$ is to dispose of the non-renormalizable contributions of the point-localized interaction $L_n^{\mathrm{pt}}(\chi^n)$ in (2.14), and that they vanish in the adiabatic limit.

3.5. Coupling to Dirac fields

The Abelian Higgs Model serves as a simplified model for the self-coupling of massive vector bosons in the weak interaction, when a cubic self-interaction of a single massive vector field is not viable, see the discussion in Sect. 1 and Proposition 3.1. The model can easily be extended to include also the coupling to a fermionic current $j^\mu = \overline{\psi}\gamma^\mu\psi$. Namely, the interaction (2.5) is another L-V pair that can be added to the L-V pair (3.19) of the Abelian Higgs Model.

In order to compute the effect of the extension on the SI condition, one proceeds as before, using that the obstruction $O_\mu(j^\mu; j'^\nu) = -\partial_\mu Tj^\mu(x)j^\nu(x')$ vanishes by the usual Ward identity. With

$$\Delta L_1 = Aj, \qquad \Delta L_1^{\mathrm{pt}} = Bj, \qquad \Delta V_1 = \phi j,$$

one finds that the second and third order SI conditions are satisfied with

$$\Delta L_2 = 0, \qquad \Delta L_2^{\mathrm{pt}} = -4m^{-1}\cdot BHj - m^{-2}\cdot j^2, \qquad \Delta V_2 = -m^{-1}\cdot \phi Hj,$$
$$\Delta L_3 = 0, \qquad \Delta L_3^{\mathrm{pt}} = 18m^{-2}\cdot BH^2j + 6m^{-3}\cdot Hj^2, \qquad \Delta V_3 = 2m^{-2}\cdot\phi H^2 j - \frac{2}{3}\cdot\phi^3 j.$$

In particular, the Higgs potential is not affected by the extension. This was expected, because the parameters a and b are fixed at tree level, whereas diagrams involving Dirac fields, that could possibly contribute to the coefficients of ϕ^3H^2 and ϕH^4, must necessarily contain Dirac loops.

3.6. Local observables

The "off-shell" interacting quantum fields are of prime interest for the perturbative construction of an actual QFT, beyond the S-matrix needed for predictions of experiments. Not all of them are local observables; e.g., in BRST the observables are by definition those fields that commute with the interacting BRST operator. They are, however, usually not computed in the literature [2,29].

Interacting fields are computed in causal perturbation theory by the variation of "relative S-matrices" with respect to a source function

$$\Phi\big|_{L(\chi)}(x) := -i\frac{\delta}{\delta f(x)} S(\chi,0)^* S(\chi,f)\big|_{f=0} \tag{3.26}$$

where $S(\chi, f) = Te^{i(L(\chi)+\Phi(f))}$. By axiomatizing properties of relative S-matrices, this is the way to give a precise meaning (as the adiabatic limit $\chi \to 1$ of (3.26)) to Bogoliubov's formula

$$\Phi\big|_L(x) := \big(Te^{i\int dx\, L(x)}\big)^* T\big(\Phi(x)e^{i\int dx\, L(x)}\big).$$

In the L-V approach at hand, the formula has to be qualified: We shall first establish the identity

$$T\Big(\Phi(x)e^{i(L[\chi;c]+V\circ\partial[\chi;c])}\Big) = T\Big(\Phi_{[g\chi]}(x)e^{iL^{\rm pt}[\chi]}\Big). \tag{3.27}$$

(3.27) is (2.14) with the insertion of a local free field $\Phi(x)$ on the left-hand side, and of

$$\Phi_{[g\chi]}(x) = \Phi(x) + g\chi(x)\Phi_{[1]}(x) + \frac{g^2}{2}\chi(x)^2\Phi_{[2]}(x) + \ldots \tag{3.28}$$

on the right-hand side. The corrections $\Phi_{[n]}(x)$ are free Wick polynomials that are recursively determined by (3.27), see below.

The combination of (2.14) and (3.27) yields

$$\Big(Te^{i(L[\chi;c]+V\circ\partial[\chi;c])}\Big)^* T\Big(\Phi(x)e^{i(L[\chi;c]+V\circ\partial[\chi;c])}\Big) = \Big(Te^{iL^{\rm pt}[\chi]}\Big)^* T\Big(\Phi_{[g\chi]}(x)e^{iL^{\rm pt}[\chi]}\Big). \tag{3.29}$$

Either of these two expressions (to be taken in the adiabatic limit $\chi \to 1$) defines the interacting field $\Phi\big|_L$, where the left-hand side is renormalizable, and the right-hand side is local *if and only if $\Phi_{[g\chi]}(x)$ is point-localized*. As in Remark 2.3, infinitely many renormalization constants on the right-hand side are fixed as functions of finitely many constants of the left-hand side.

Definition 3.9. A free Wick polynomial Φ such that $\Phi_{[g\chi]}$ is point-localized (i.e., Φ and all its corrections $\Phi_{[n]}$ are point-localized), is called the "seed" of the local interacting field $\Phi|_L$ given by (3.29). The resulting interacting fields are the local observables of the theory.

In other words: *The perturbation theory selects the local observables of the theory*. The condition (vanishing of $\Phi_{[n]}$) can be decided at the level of the free field.

(3.29) generalizes the construction of the interacting Dirac field of QED as a point-localized perturbation of the string-localized "dressed Dirac field" [25, Eq. (2.14)]. In QED, the SI condition is fulfilled without any higher order interactions added to the massless L-V pair (2.5). In the case at hand, the role of the dressing transformation is taken by the map $\Phi \mapsto \Phi_{[g\chi]}$ when $\chi \to 1$.

We now turn to the determination of the correction terms $\Phi_{[n]}(x)$ in (3.28). The strategy is the same as for the S-matrix, cf. App. A. We sketch it here again in a model-independent way, but with the simplifying assumption that the two-point obstructions involve no derivatives of δ-functions (otherwise, one would have to admit terms with derivatives of χ in (3.28)), and $\Omega_{V_1}(V_1) = 0$. Recall that in the Abelian Higgs model, the SI condition forces us to choose the renormalizations (3.16) such that the simplifying assumptions hold, and as a consequence $W_2 = W_3 = 0$.

Expanding both sides of (3.27) to first order in g, we get

$$i \int dy \big(\chi(y) T L_1(y)\Phi(x) + \partial\chi(y) T V_1(y)\Phi(x)\big)$$
$$= i \int dy \big(\chi(y) T L_1^{\rm pt}(y)\Phi(x) + \chi(x)\Phi_{[1]}(x)\big).$$

Inserting $L_1 = L_1^{\rm pt} + \partial V_1$, we get

$$i \int dy\, \chi(y)[T, \partial^y] V_1(y)\Phi(x) = \chi(x)\Phi_{[1]}(x) \quad \overset{(3.21)}{\Rightarrow} \quad \Phi_{[1]} = -\Omega_{V_1}(\Phi). \tag{3.30}$$

Expanding (3.27) in second order, we insert $L_1 = L_1^{\rm pt} + \partial V_1$. This cancels the terms involving the cubic time-ordered products $T L_1^{\rm pt}(y) L_1^{\rm pt}(y')\Phi(x)$. The remaining cubic terms are of the form $O_Y(X', \Phi(x))$ as in (A.2) and can be evaluated using Lemma A.1. This produces terms $2T L_1^{\rm pt}\Omega_{V_1}(\Phi(x)) + T\Phi(x)\Omega_{V_1}(2L_1^{\rm pt} + \partial V_1)$, which cancel the quadratic contributions $T L_1^{\rm pt}\Phi_{[1]}$ and $T(L_2 - \partial V_2 - L_2^{\rm pt})\Phi(x)$. The remaining terms give the surprisingly simple result:

$$\Phi_{[2]} = \Omega_{V_1} \circ \Omega_{V_1}(\Phi) - \Omega_{V_2}(\Phi). \tag{3.31}$$

If the simplifying assumptions as above are fulfilled, then we conjecture for higher orders:

Conjecture 3.10. *All corrections $\Phi_{[n]}$ are linear combinations of iterated obstructions $\Omega_{V_{n_1}} \circ \cdots \circ \Omega_{V_{n_k}}(\Phi)$ with $\sum_k n_k = n$.*

That Ω_{V_n} should come in $\Phi_{[n]}$ with the coefficient -1, can be seen rather easily. Specifically, for reasons to be explained below and after (3.34), we guess[4]

$$\Phi_{[3]} = \big(-\Omega_{V_1} \circ \Omega_{V_1} \circ \Omega_{V_1} + 2\Omega_{V_1} \circ \Omega_{V_2} + \Omega_{V_2} \circ \Omega_{V_1} - \Omega_{V_3}\big)(\Phi). \tag{3.32}$$

This guess, together with (3.30) and (3.31) and the derivation property of Ω_{V_n}, implies

$$(XY)_{[2]} = X_{[2]}Y + 2X_{[1]}Y_{[1]} + XY_{[2]}, \quad (XY)_{[3]} = X_{[3]}Y + 3X_{[2]}Y_{[1]} + 3X_{[1]}Y_{[2]} + XY_{[3]}.$$

This structure in turn entails that the point-locality of the corrections of two fields passes to the corrections of their Wick product, hence it warrants that the seeds Φ of local interacting fields in Definition 3.9 form an *algebra*.[5] However, this feature only constrains, but does not fix the coefficients in (3.32).

The actual computations in the Abelian Higgs model are again straightforward, using (2.9) and the known two-point obstructions. The first corrections

$$B^\mu_{[1]} = -\frac{1}{m}(B^\mu H + \phi\partial^\mu H), \qquad H_{[1]} = -\frac{m}{2}\phi^2, \qquad (\partial_\mu H)_{[1]} = m B_\mu \phi,$$

of the Proca and the Higgs fields are string-localized. Thus neither the interacting Higgs field nor the interacting Proca field are local observables of the model.

4 The effort required for the recursive analysis of (3.27) in third order is comparable to (2.14) in fourth order.

5 This is a desirable feature, but not an axiom because the map $\Phi \mapsto \Phi_{\rm int}$ in Definition 3.9 must not be expected to be an algebra homomorphism.

For the massive Proca field strength tensor $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, one needs also two-point obstructions involving $G_{\mu\nu}$. The kinematical propagators displayed in App. B.3 yield

$$O_\kappa(B^\kappa, G'_{\mu\nu}) = O_\kappa(\phi, G'_{\mu\nu}) = 0, \qquad O_\kappa(A^\kappa, G'_{\mu\nu}) = -iI_e e_{[\mu}\partial_{\nu]}\delta(x-x'). \tag{3.33}$$

Thus, the first two (three, if the above structural conjecture is correct) corrections of $G_{\mu\nu}$ are zero for the simple reason that the field A does not occur in V_1 and V_2 (and V_3).

For a Wick polynomial Φ in B, H, and ∂H to be the seed of a local interacting field, $\Phi_{[1]} = -\Omega_{V_1}(\Phi)$ must be independent of ϕ. This condition is a differential equation for Φ, which implies that Φ must be a polynomial in the composite field

$$Z := m^2B^2 + (\partial H)^2.$$

Indeed, also the next two corrections of Z are point-localized:

$$Z_{[1]} = -2mB^2H, \qquad Z_{[2]} = 6B^2H^2, \qquad Z_{[3]} = -\frac{24}{m}B^2H^3. \tag{3.34}$$

$Z_{[3]}$ was computed with (3.32), which is the unique (up to a factor) combination with a point-localized outcome. We take this as a strong support for the guess (3.32). Clearly, a better understanding of the higher-order corrections is strongly desired.

With these evidences, we conjecture

Conjecture 3.11. *The interacting fields $G_{\mu\nu}$ and $Z = m^2B^2 + (\partial H)^2$ are local observables of the Higgs model.*

Other fields, like the interacting Higgs field, may still be regarded as part of the theory, e.g., in order to create Higgs particle states. But, just as the interacting dressed Dirac field of QED [25], they cannot be local fields in the sense of the usual QFT axiomatics.

When a Dirac field is added to the Abelian Higgs model (see Sect. 3.5), one expects the current to be a local observable. Indeed, all corrections $j_{[n]}$ vanish because $O_\mu(j^\mu; j'^\nu) = 0$. The corrections to the Dirac field are computed with the help of

$$O_\mu(j_\mu; \psi') = -\partial_\mu Tj^\mu(x)j^\nu(x') = \delta_{xx'}\psi(x),$$

giving

$$\psi_{[1]} = i\phi\psi, \qquad \psi_{[2]} = -\phi^2\psi, \qquad \psi_{[3]} = -i\phi^3\psi.$$

This is in perfect agreement with the perturbative expansion of the "dressed Dirac field" $\psi_{qc} = e^{iq\phi(c)} \cdot \psi$ as discussed in [25] for the QED coupling to massless photons, where a version of (3.29) is used to construct the interacting fields. In contrast to the massless case where ψ_{qc} can be defined as a string-localized field, the massive Wick exponential $e^{ig\phi(c)}$ is a "Jaffe field" of very poor localization properties [21].

4. Comparison with other approaches

We compile here the various approaches to the Abelian Higgs model, not least because the same symbols tend to stand for different objects in different settings. See for this Sect. 4.7.

4.1. Spontaneous symmetry breaking in unitary gauge

In the textbook narrative of the Higgs mechanism, one starts classically with the minimal coupling of a charged scalar Higgs field Ψ with the potential $U(\Psi^*\Psi)$ to a massless vector field B with field strength $G = \partial \wedge B$:

$$L = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + (D_\mu\Psi)^* D^\mu\Psi - U(\Psi^*\Psi), \qquad U(\Psi^*\Psi) = \kappa\Big(\Psi^*\Psi - \frac{v^2}{2}\Big)^2. \tag{4.1}$$

Parameterizing

$$\Psi(x) = \frac{1}{\sqrt{2}}(v + H(x))e^{i\chi(x)/v},$$

one can “gauge away” the Goldstone mode $\chi(x)$. One then writes the resulting Lagrangian in terms of B and H. After suitable identification of the parameters ($m^2 := g^2v^2$ where g is the gauge coupling constant, and $m_H^2 := 2\kappa v^2$), one arrives at

$$L = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{m^2}{2}B_\mu B^\mu + \frac{1}{2}\partial_\mu H\partial^\mu H - \frac{m_H^2}{2}H^2 + L^{\rm Pr}, \tag{4.2}$$

which contains the free massive Proca and Higgs Lagrangians along with the interaction density

$$L^{\rm Pr} = mg\Big(B^2H - \frac{m_H^2}{2m^2}H^3\Big) + \frac{g^2}{2}\Big(B^2H^2 - \frac{m_H^2}{4m^2}H^4\Big). \tag{4.3}$$

The Higgs mass term in (4.2) and the cubic and quartic terms in (4.3) together constitute the Higgs potential (1.1). As a quantum interaction, (4.3) is non-renormalizable.

By regarding the unitary gauge as a limiting case at tree level[6] of the renormalizable R_ξ gauges in indefinite metric, and exploiting the unbroken gauge invariance to establish the necessary Ward identities, it is concluded that the theory is renormalizable and unitary.

4.2. BRST approach

In [2] and [29, Chap. 4.1], the Abelian Higgs model is constructed without spontaneous symmetry breaking. The method of securing BRST invariance of the S-matrix is similar to our L-Q-pair method: one recursively fixes induced interaction terms to cancel obstructions in each order.

One starts on the indefinite Fock space (Krein space) of the massive vector potential $A^{\rm K}$ in the Feynman gauge, the Higgs field H, the ghost fields u, $\widetilde{u}$, and the independent positive-definite scalar Stückelberg field Φ of the same mass as $A^{\rm K}$ [28]. The cubic interaction is given by the power-counting renormalizable interaction density[7]

$$L^{\rm BRST} = mg\Big(A^{\rm K}(A^{\rm K}H + \frac{1}{m}\Phi\overleftrightarrow{\partial}H) + u\widetilde{u}H - \frac{m_H^2}{2m^2}\Phi^2H + aH^3\Big). \tag{4.4}$$

[6] This is not true for loop corrections [37].

[7] In [2,7,29], the Higgs field H is denoted by ϕ or φ. We reserve ϕ for the escort field.

The nilpotent BRST transformation s is implemented by the graded commutator with a nilpotent free BRST operator Q, whose cohomology $\mathcal{H} = \mathrm{Ker}(Q)/\mathrm{Ran}(Q)$ is the positive-definite physical Hilbert space of the free theory. The BRST variations $s(X) = i[Q, X]_\pm$ are

$$s(A^{\mathrm{K}}_\mu) = -\partial_\mu u, \quad s(\Phi) = -mu, \quad s(u) = 0, \quad s(\widetilde{u}) = \partial_\mu A^{\mathrm{K}\mu} + m\Phi, \quad s(H) = 0.$$

The interaction (4.4) is distinguished by the property that its BRST variation is a total derivative:

$$s\big(L^{\mathrm{BRST}}\big) = \partial_\mu\big((mA^{\mathrm{K}\mu} H + \Phi \overset{\leftrightarrow}{\partial}{}^\mu H)u\big).$$

The nontrivial condition to secure Hilbert space positivity of the interacting theory is that this feature must persist in higher orders of perturbation theory for the S-matrix. This condition in second order requires to add quartic terms

$$\frac{g^2}{2}\Big(A^{\mathrm{K}2} H^2 + A^{\mathrm{K}2}\Phi^2 - \frac{m_H^2}{4m^2}\Phi^4 + \big(3a + \frac{m_H^2}{m^2}\big)\Phi^2 H^2 + bH^4\Big)$$

to the interaction density, in order to cancel obstructions. The cubic and quartic coefficients of the Higgs potential are fixed in third order. The result is the values (3.17).

The field

$$B_\mu := A^{\mathrm{K}}_\mu - m^{-1}\partial_\mu\Phi \tag{4.5}$$

is BRST-invariant. Its two-point function coincides with the positive-definite Proca two-point function (B.2). However, $\partial_\mu B^\mu = \partial_\mu A^\mu + m\Phi \neq 0$ on the Fock space. But this quantity is a null field in the range of s, hence it vanishes on the BRST Hilbert space $\mathcal{H}$. Thus, (4.5) on $\mathcal{H}$ is the Proca field. Moreover, on $\mathcal{H}$ also u vanishes, and the interaction density (4.4) coincides with the cubic part of (4.3) up to a total derivative.

4.3. String-localized L-Q-pair approach (on the Hilbert space)

The L-Q-pair approach was exhibited at length in Sect. 3.3. Apart from the values of the parameters a and b, its result $L[\chi; c]$ cannot be directly compared to point-localized approaches. But it can be asserted that the initial L-Q-pair (3.13) defines a string-independent S-matrix iff the parameters a and b in L_1 and L_2 take the values (3.17).

4.4. String-localized L-V-pair approach (on the Hilbert space)

The L-V-pair approach was exhibited at length in Sect. 3.4. Here, the assertion is that for the initial L-V pair (3.19), the S-matrix in the left-hand side of (2.14) coincides with (and actually defines in the adiabatic limit) the string-independent S-matrix on the right-hand side.

Collecting the pieces L_n^{pt} in (2.15) as computed in Sect. 3.4 with the choice (3.16) of renormalization parameters, one gets

$$L^{\mathrm{pt}} = mg \cdot \big(B^2 H - \frac{m_H^2}{2m^2}H^3\big) + \frac{g^2}{2}\cdot\big(-3B^2H^2 - \frac{m_H^2}{4m^2}H^4\big) + \frac{g^3}{6}\cdot\frac{12}{m}B^2H^3 + \dots. \tag{4.6}$$

It contains the interaction part of the Higgs potential (3.9) whose coefficients are the same as in all other approaches; plus possibly further coupling terms like $g^n \cdot B^2 H^n$ $(n > 3)$. The latter will be discussed in Sect. 4.7.

The string-localized interaction density $L[c]$ is also defined on the Krein space of Sect. 4.2, where B is given by (4.5) and $A(c)$ and $\phi(c)$ are defined by (2.2) and (2.3) (smeared with $c(e)$). It then holds, for any c

$$s(\phi(c)) = u. \tag{4.7}$$

The cubic SQFT interaction (3.5) (with $a' = \alpha_i = 0$) differs from (4.4) by

$$L_1(c) = L_1^{\rm BRST} - s\big((m\phi(c) - \Phi)\widetilde{u}H\big) + \partial_\mu\Big(B^\mu(m\phi(c) - \Phi)H + \frac{1}{2m}(m^2\phi(c)^2 - \Phi^2)\partial^\mu H\Big).$$

Up to the term in $\mathrm{Ran}(s)$ which can be included in $L^{\rm BRST}$ at no expense, this is another instance of the example mentioned in the beginning of Sect. 2.2.

4.5. Krein space L-V-pair approach (point-localized)

One may also work with a point-localized L-V pair in the Krein space of the massive vector potential $A^{\rm K}$ and the Higgs field, without the ghost and Stückelberg fields. The Proca field B is embedded into the Krein space as

$$B_\mu := m^{-2}\partial^\nu G^{\rm K}_{\mu\nu} = A^{\rm K}_\mu - \partial_\mu\phi^{\rm K} \tag{4.8}$$

where $G^{\rm K} := \partial \wedge A^{\rm K}$ is the field strength, and $\phi^{\rm K} := -m^{-2}(\partial A^{\rm K})$. We refer to the subspace generated from the Krein vacuum by B_μ and H as the "embedded Hilbert space".

Then one has an L-V pair

$$L_1^{\rm K} = L_1^{\rm Pr} + \partial V_1^{\rm K} \tag{4.9}$$

with

$$L_1^{\rm Pr} = m\cdot\big(B^2H + aH^3\big), \tag{4.10}$$
$$L_1^{\rm K} = m\cdot\Big(A^{\rm K}BH + A^{\rm K}\phi^{\rm K}\partial H - \frac{m_H^2}{2}\phi^{\rm K2}H + aH^3\Big),$$
$$V_1^{\rm K} = m\cdot\Big(B\phi^{\rm K}H + \frac{1}{2}\phi^{\rm K2}\partial H\Big).$$

We want to use this pair as the starting point of a recursion as in Sect. 2.3 to reformulate a non-renormalizable point-localized interaction of the Proca and Higgs fields on the embedded Hilbert space, as a renormalizable point-localized interaction on the Krein space. The PSI in this case is replaced by the principle of Hilbert space positivity, i.e., the right-hand side of the analogue

$$Te^{i(L^{\rm K}[\chi;c]+V^{\rm K}[\chi;c])} = Te^{iL^{\rm Pr}[\chi]}$$

of the identity (2.14) should be defined on the embedded Hilbert space.

We then proceed as in Sect. 2.3 and recursively determine the higher-order densities with the specification that $L_n^{\rm K}$ are renormalizable are $L_n^{\rm Pr}$ are defined on the embedded Hilbert space. This would secure a positive-definite renormalizable theory in the adiabatic limit.

The triple $L_1^{\rm Pr}$, $L_1^{\rm K}$, $V_1^{\rm K}$ is identical with the triple $L_1^{\rm pt}$, $L_1(c)$, $V_1(c)$ in Sect. 3.4 with $A(c)$ replaced by $A^{\rm K}$ and $\phi(c)$ replaced by $\phi^{\rm K}$. However, the two-point obstructions are different, due to the different scaling degrees of the two-point functions and different linear relations among the fields and their derivatives, see in App. B.2. The two-point obstructions (3.10) in the Higgs sector are unchanged, those in the vector boson sector are

$$\begin{aligned}
O_\mu(A^{\mathrm{K}\mu}; B'_\nu) = O_\mu(B^\mu; A^{\mathrm{K}\prime}_\nu) = O_\mu(B^\mu; B'_\nu) &= -i(1+c_B)\cdot m^{-2}\partial_\nu\delta(x-x'),\\
O_\mu(A^{\mathrm{K}\mu}; \phi^{\mathrm{K}\prime}) &= -im^{-2}\delta(x-x'),\\
O_\mu(\phi^{\mathrm{K}}; A^{\mathrm{K}\prime}_\nu) &= -ic_B\cdot m^{-2}\eta_{\mu\nu}\delta(x-x'),\\
O_\mu(A^{\mathrm{K}\mu}; A^{\mathrm{K}\prime}_\nu) = O_\mu(\phi^{\mathrm{K}}; \phi^{\mathrm{K}\prime}) &= 0,\\
O_\mu(B^\mu; \phi^{\mathrm{K}\prime}) = O_\mu(\phi^{\mathrm{K}}; B'_\nu) &= 0.
\end{aligned} \tag{4.11}$$

With these, one computes the second-order obstruction (2.18) of the L-V pair $L_1^{\mathrm{K}} = L_1^{\mathrm{Pr}} + \partial V_1^{\mathrm{K}}$. One finds that it can be cancelled with

$$\begin{aligned}
L_2^{\mathrm{Pr}} =& (1+4c_B)\cdot B^2H^2 + bH^4, \qquad (4.12)\\
L_2^{\mathrm{K}} =& (A^{\mathrm{K}2} + 3c_BA^{\mathrm{K}}B)H^2\\
&+ m^2\Big((3a + \frac{m_H^2}{m^2} + c_B)\phi^{\mathrm{K}2}H^2 - \frac{m_H^2}{4}\phi^{\mathrm{K}4} + (1+c_H)A^{\mathrm{K}2}\phi^{\mathrm{K}2}\Big) + bH^4,\\
V_2^{\mathrm{K}} =& \big(A^{\mathrm{K}} - (1-c_B)\cdot B\big)\phi^{\mathrm{K}}H^2 + \frac{m^2}{6}B\phi^{\mathrm{K}3} + (1+c_H)\cdot\frac{m^2}{2}A^{\mathrm{K}}\phi^{\mathrm{K}3} + c_B\cdot\phi^{\mathrm{K}2}H\partial H,\\
W_2^{\mathrm{K}} =& (1+c_H)\cdot\frac{m^2}{4}\phi^{\mathrm{K}4} + (1+c_B)\cdot\phi^{\mathrm{K}2}H^2
\end{aligned}$$

with a free coefficient b of the quartic part of $V(H)$. However, the term $A^{\mathrm{K}}BH^2$ in L_2^{K} has dimension 5 and is not renormalizable. One therefore has to choose $c_B = 0$. Quite amazingly, precisely with this choice the expressions (4.12) are identical with (3.20) (with the replacement of string-localized fields by Krein fields).

With $c_B = 0$, the complete computation of the third-order obstruction as in (2.20) is more contrived because of the derivatives of δ-functions. We have used (A.6) to compute it up to derivatives as in (2.21). It turns out that the third-order obstruction cannot be cancelled by third-order densities with L_3^{Pr} positive-definite and L_3^{K} renormalizable, for any value of c_H. Thus, this approach fails in third order.

4.6. String-localized L-V-pair approach in Krein space

For the sake of completeness, we report yet another L-V pair, which reformulates the point-localized Krein space interaction as in Sect. 4.5 as the renormalizable string-localized Hilbert space interaction as in Sect. 3.4. Unlike in Sect. 4.5, non-renormalizable higher-order terms L_n^{K} are admitted in the Krein space interaction. We thus want to establish the identity

$$Te^{i(\widetilde{L}[\chi;c]+\widetilde{V}\circ\partial[\chi;c])} \stackrel{!}{=} Te^{i\widetilde{L}^{\mathrm{K}}[\chi]} \tag{4.13}$$

with the initial L-V pair

$$\widetilde{L}_1 = \widetilde{L}_1^{\mathrm{K}} + \partial\widetilde{V}_1,$$

where $\widetilde{L}_1 = L_1$ as in (3.19) and $\widetilde{L}_1^{\mathrm{K}} = L_1^{\mathrm{K}}$ as in (4.10), thus (because L_1^{Pr} is identical with L_1^{pt} embedded into the Krein space)

$$\widetilde{V}_1 = V_1 - V_1^{\mathrm{K}}.$$

Along with the Proca field (4.8), also the string-localized fields are embedded into the Krein space via (2.2) and (2.3), and it holds

$$B = A - \partial\phi = A^{\mathrm{K}} - \partial\phi^{\mathrm{K}}. \tag{4.14}$$

The two-point obstructions among the Hilbert space fields and among the Krein space fields are as before. One also needs mixed two-point obstructions, which turn out to be

$$\begin{aligned}
&O_\mu(\phi;\phi^{\mathrm{K}\prime}) = O_\mu(A^\mu;\phi^{\mathrm{K}\prime}) = O_\mu(\phi^{\mathrm{K}};\phi') = O_\mu(\phi^{\mathrm{K}};A'_\nu) = 0,\\
&O_\mu(\phi;A^{\mathrm{K}\prime}_\nu) = O_\mu(\phi;B'_\nu) = -ic_B\cdot m^{-2}\eta_{\mu\nu}\delta_{xx'},\\
&O_\mu(A^{\mathrm{K}\mu};\phi') = O_\mu(B^\mu;\phi') = -im^{-2}\delta_{xx'},\\
&O_\mu(A^\mu;A^{\mathrm{K}\prime}_\nu) = O_\mu(A^\mu;B'_\nu) = -ie_\nu I_e\delta_{xx'}, \quad O_\mu(A^{\mathrm{K}\mu};A'_\nu) = O_\mu(B^\mu;A'_\nu) = 0.
\end{aligned}$$

Notice that $O_\mu(A^\mu;A^{\mathrm{K}\prime}_\nu)$ is string-localized.

Because in this approach, the right-hand side of (4.13) is not required to be renormalizable, terms like $A^{\mathrm{K}}BH^2$ are admitted in $\widetilde{L}^{\mathrm{K}}_2$ (in contrast to L^{K}_2 in Sect. 4.5). The second-order obstruction can be cancelled by $\widetilde{L}_2 - \partial\widetilde{V}_2 - \widetilde{L}^{\mathrm{K}}_2$ for arbitrary values of c_B and c_H, but $\widetilde{V}_2$ contains terms involving the string-localized field A with coefficients $(1+c_B)$ or $(1+c_H)$. As in Sect. 3.4, such terms would produce string-localized δ-functions in the third-order obstruction, which cannot be cancelled. Therefore, we have to choose again $c_B = c_H = -1$. With this choice,

$$\begin{aligned}
\widetilde{L}_2 &= m^2\Big((3a + \frac{m_H^2}{m^2})\phi^2H^2 - \frac{m_H^2}{4}\phi^4\Big), \qquad (4.15)\\
\widetilde{L}^{\mathrm{K}}_2 &= A^{\mathrm{K}}(A^{\mathrm{K}} - 3B)H^2 + m^2\Big(\frac{1}{2}(B - A^{\mathrm{K}})B\phi^{\mathrm{K}2} + (3a + \frac{m_H^2}{m^2} - 1)\phi^{\mathrm{K}2}H^2 - \frac{m_H^2}{4}\phi^{\mathrm{K}4}\Big),\\
\widetilde{V}_2 &= (B - A^{\mathrm{K}})\phi^{\mathrm{K}}H^2 - B\phi H^2 + \frac{m^2}{6}B\phi^3 + \frac{m^2}{2}B(\phi^{\mathrm{K}} - \phi)\phi\phi^{\mathrm{K}} + (\phi^{\mathrm{K}} - \phi)\phi^{\mathrm{K}}H\partial H.
\end{aligned}$$

We have then computed the third-order obstruction using (3.23).[8] All its "mixed terms" (products of string-localized and Krein fields) and string-localized terms can be cancelled by derivatives $\partial\widetilde{V}_3$, except precisely the same last two terms as in (3.25). Because $\widetilde{L}_3$ must vanish by renormalizability of the left-hand side of (4.13), this means that the third-order obstruction can be cancelled if and only if the parameters a and b take the values (3.17) of the Higgs potential.

4.7. Synopsis

In all approaches Sect. 4.2–Sect. 4.4, one has the renormalizable interaction density of the form

$$L_1 = m\Big(ABH + A\phi\partial H - \frac{1}{2}m_H^2\phi^2H + aH^3\Big)$$

[8] It contains 39 terms. Although the explicit expressions are of little interest, we just report the final $\widetilde{L}^{\mathrm{K}}_3$:

$$\begin{aligned}
L^{\mathrm{K}}_3 = {}& m^{-1}(2A^{\mathrm{K}2} - 11A^KB)H^3 + m\big((3A^{\mathrm{K}2} + \frac{3}{2}A^{\mathrm{K}}B + 3B^2)\phi^{\mathrm{K}2}H + (2 - 3\frac{m_H^2}{m^2})\phi^{\mathrm{K}2}H^3\big)\\
&+ m^3(1 - \frac{m_H^2}{4m^2})\phi^{\mathrm{K}4}H.
\end{aligned}$$

(plus ghost terms in the BRST setting), and a relation of the form

$$B_\mu = A_\mu - \partial_\mu \phi.$$

However, not only the meanings of the symbols A_μ and ϕ are very different, but also their correlations, hence propagators and obstructions in perturbation theory. This explains the different induced quartic and higher interaction densities found in the various approaches.

In the string-localized approach of the main body of the paper (Sect. 3.4), $A = A(c)$ is a string-localized potential and $\phi = \phi(c)$ its string-localized escort field (an integral over the Proca field), both defined on the physical Hilbert space of the Proca field. In BRST (Sect. 4.2), $A = A^{\mathrm{K}}$ is the Feynman gauge vector potential and ϕ is (up to the factor m) the independent positive-definite Stückelberg field Φ with $\langle A^{\mathrm{K}}\Phi\rangle = 0$. In the Krein space L-V-pair approach (Sect. 4.5), $A = A^{\mathrm{K}}$ as in BRST, but $\phi = \phi^{\mathrm{K}} = -m^{-2}(\partial A^{\mathrm{K}})$ is a derivative of the former and negative-definite. Finally, in Sect. 4.6, we have two sets of fields $A(c), \phi(c)$ and $A^{\mathrm{K}}, \phi^{\mathrm{K}}$, related to each other by the two representations (4.14) of the Proca field.

In the BRST approach, $\partial A + m\Phi$ is in the range of the BRST transformation, hence it vanishes on the physical Hilbert space. Thus, the positive-definite Stückelberg field is "identified" with the negative-definite $m\phi^{\mathrm{K}} = -m^{-1}(\partial A^{\mathrm{K}})$ of the Krein space approach. This is of course only possible because their difference is a null field, which is zero in the BRST quotient space.

It is remarkable that, although the obstructions appearing in perturbation theory are different, they can be cancelled in the BRST and the string-localized approaches, and the fixing of the coefficients a, b of the Higgs potential in third order gives the same values in all of them. The fact that the cancellation is not possible in the point-localized Krein space approach (without a Stückelberg field) shows that it is by no means automatic that identities like (2.14) can be recursively fulfilled. Rather, there must be some hidden features of the model whose general nature is not transparent to us. Apparently, string-localization provides the necessary flexibility that in the BRST and the string-localized Krein space approaches is provided by the blowing-up of the field content.

The L-Q-pair approach does not allow to compute L_n^{pt}. For L_n, it gives compatible results with the L-V-pair approach.

Having established, by virtue of the identity (2.14), the equivalence between the renormalizable string-localized interaction $L(e)$ with the non-renormalizable point-localized interaction L^{pt}, we should ask whether the latter is equivalent to the interaction L^{Pr} as in Sect. 4.1. L^{pt} in (4.6) and L^{Pr} in (4.3) differ by the coefficient of the second-order term B^2H^2 and by a new third-order term B^2H^3 (and possibly higher-order terms B^2H^n).[9] On the other hand, the former uses the renormalization of the Proca propagator with $c_B = -1$, that was necessary in order to eliminate string-localized obstructions in third order. The latter uses the kinematical choice $c_B = 0$. We shall now give evidence for the presumed equivalence.

This apparent discrepancy is a variation of the familiar observation in scalar QED, that the renormalization (by adding a multiple of the δ-function) of a propagator connecting two cubic vertices, just amounts to another quartic vertex. That the renormalization of the propagator $\langle T\partial_\mu\varphi\partial_\nu\varphi\rangle$ can be traded for the coefficient of the quartic vertex $A^2\varphi^*\varphi$ of scalar QED, has been proven in all orders in various settings [8,9,33]. Requiring gauge invariance of the total Lagrangian would select the kinematical propagator. But causal perturbation theory does not need

[9] Also the BRST approach in Sect. 4.2 produces a different quartic interaction, except for the Higgs potential.

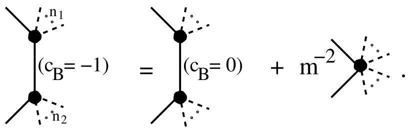
gauge invariance and can be done with an arbitrary renormalization. The result is equivalent up to a renormalization group transformation interpolating between both values.

The case at hand, with many vertices B^2H^n and the Higgs self-couplings, is more contrived than scalar QED. Yet, for the tree-level scattering amplitudes for processes

2 vector bosons $\to$ n Higgs,

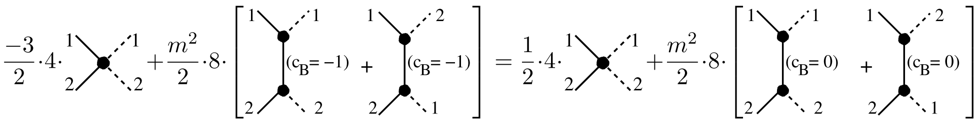
when computed with (4.6) in the L-V-pair approach ($c_B = -1$) and with (4.3) in the unitary gauge ($c_B = 0$), we have verified the match for $n = 2$ and for $n = 3$, as follows.

For the comparison of different values of the renormalization parameter, (B.3) can be written graphically (solid lines $= B$, $n_1 + n_2$ broken lines $= H$) as



(4.16)

This immediately implies the match between (4.6) and (4.3) for $n = 2$ in order g^2:



(where the factors 4 and 8 are the counting factors for equivalent contractions). Note that the coupling constants for the B^2H^2-term differ by the factor of -3. The difference is made up by the contributions from the second diagram on the left-hand side to the first diagram on the right-hand side, due to (4.16). By the same method, we have verified the match also in the case $n = 3$ (seven diagrams with permutations, of which two diagrams need not be considered because they do not contain differing coupling constants or renormalized propagators), – thus justifying the presence of the term B^2H^3 in L_3^{pt}, and confirming the value of its coefficient.

In view of this evidence for the equivalence between L^{Pr} in (4.3) with $c_B = 0$ and L^{pt} in (4.6) with $c_B = -1$, it would be most rewarding to find again a renormalization group transformation interpolating between them.

All approaches discussed here may be regarded as attempts to define a renormalization of a power-counting non-renormalizable interaction. They do not differ in their physical predictions, but in the way how (and whether) fundamental principles of quantum field theory are implemented. What stands out is the universality of the Higgs potential, that is the same in all consistent approaches. Its universal shape is recognized to be an intrinsic consistency condition, rather than an input to trigger a spontaneous breaking of gauge symmetry.

5. Discussion

The ubiquitous clashes between Hilbert space, causality, and renormalizability are worrying us since the early days of QFT. The L-V-pair formalism developed in this paper allows to establish equivalences between formulations of QFT models, in which complementary subsets of these fundamental principles are fulfilled, such that, by the very equivalence, all of them hold simultaneously – but possibly not in any single formulation. Even more, it allows to fix physical parameters (coupling constants) as consistency conditions for the equivalences to hold.

In particular, by providing the necessary L-V pairs, string-localized QFT can be employed with various benefits. In the present paper, we have considered an instance where it can be used to "renormalize the unrenormalizable", provided certain parameters are appropriately fixed to secure consistency of the method. The physical manifestation of these parameters is the Higgs potential. It owes its universal shape to the fundamental principles of Hilbert space positivity and locality, rather than an aesthetic but positivity-violating gauge principle.

In the same way, SQFT has been used earlier to explain the "gauge theory pattern" of massive vector boson self-couplings [17], and the chirality of the weak interaction [19].

In a very different way, it has been used to explore the infrared structure of QED. Here, the logarithmic infrared divergence of the string integration defining the escort field becomes instrumental for a new understanding of the superselection structure of QED [25], in which the string smearing function describes the "shape of the photon cloud" of charged states [24]. SQFT allows to construct string-dependent charged fields: the Principle of String Independence holds for the S-matrix in the neutral sector and for observable fields, but string dependence of charged states and unobservable fields becomes a physical feature (the "shape of photon clouds").

QED is in fact the prototype of an SQFT, which has no second-order obstructions and hence no higher-order interactions L_n $(n > 1)$. This property distinguishes QED from the model treated in this work, and allows a non-perturbative construction leading "halfways" to QED [25].

It is natural to consider an SQFT treatment of QCD. Massless Yang-Mills theory is an instance where an L-Q pair

$$\begin{aligned} L_1(c) &= f_{abc}A^a_\mu(c)A^b_\nu(c)F^{c\mu\nu}, \\ Q^\mu_1 &= 2f_{abc}w^a A^b_\nu(c)F^{c\mu\nu}, \end{aligned} \tag{5.1}$$

with f_{abc} completely antisymmetric and $w^a = \delta_c A^a(c)$, exists on the Wigner Hilbert space of the free field strengths $F^a_{\mu\nu}$, but no L-V pair [16]. (Notice that the same expression (5.1) on the Krein space, as in Sect. 4.5, is not even an L-Q pair, because $\partial_\mu A^{a\mathrm{K}\mu} \neq 0$.)

The case of QCD still remains to be worked out. The L-Q pair (5.1) has second-order obstructions, so that there is no immediate analogue of the non-perturbative construction that gives rise to the infrared superselection structure of QED. Instead, it is expected that no colour-charged states can be constructed at all, which would be a new mechanism to explain confinement.

The use of string-localized quantum fields in the interaction gives us occasion to comment on the fact (underlying causal perturbation theory also in the point-localized case): *Interaction does not need a free Lagrangian.* This is advantageous, because "canonical quantization" based on free Lagrangians is beset with difficulties. The zero-component of the Maxwell four-potential has no canonically conjugate momentum: one needs a "gauge-fixing term" to cure this problem, and one needs another cure (the Gupta-Bleuler condition) to make the first cure ineffective for the dynamics. Why is the classically purely auxiliary four-potential treated as fundamental in the first place, and not the observable Maxwell field tensor? For massive tensor fields of higher spin, "free Lagrangians" need a host of auxiliary fields to implement constraints [14]. For spinor fields, *anti*-commutation relations have no a priori "canonical" justification: they are needed to reconcile covariance with Hilbert space positivity after the quantization has been performed at the one-particle level, and Dirac's theory to deal with first-class constraints is needed to save the idea of canonical quantization with a free Lagrangian that is linear in the momenta.

Weinberg [34] has shown how one can bypass all these pains. Given a unitary one-particle representation of the Poincaré group as classified by Wigner [35], one directly constructs free fields on the Fock space (i.e., a Hilbert space) over the one-particle space. Their interaction is described by the *interaction* density $L \equiv L_{\rm int}$ alone. There is no reference to an interacting equation of motion (which in the literal sense does not exist in QFT). The interacting quantum field is constructed perturbatively by "causal perturbation theory" due to Glaser and Epstein [11], who turned Bogoliubov's somewhat heuristic formula [4] into a rigorous working scheme.

A side-message of Weinberg's construction is that quantum fields associated with a given particle are by no means unique: the intertwiner condition on the coefficients of creation and annihilation operators has many solutions. The resulting fields are all defined on the same Fock space and create the same particle states. This is trivially true for derivatives of a given field, and derivatives are the only operations that respect causal (anti-)commutativity. But if one is willing to relax localization (e.g., in order to tame the UV singularity of the propagator), then more flexibility is gained with string-localized fields. Even the NoGo result against local fields for infinite-spin particles [36] can be overcome [26] without the need to sacrifice the Hilbert space.

To conclude: There exist several different but equivalent ways to set up the perturbation theory of the same QFT model. The setups may be competitive in which fundamental principles they respect manifestly, and which ones have to be concluded indirectly. In the case of the Abelian Higgs model, SQFT seems to be closest to the "best of all worlds" in which Hilbert space positivity, covariance, locality and renormalizability are all satisfied at the same time and at every step. (String-localization is a very mild relaxation of locality for non-observable fields.) By avoiding unphysical field degrees of freedom, it is also the most economic one. In addition, we have stressed that the precise shape of the Higgs potential is determined by internal consistency with fundamental principles, without invoking the usual gauge theoretical arguments.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Second and third-order SI conditions

For the proofs of Proposition 2.4 and Proposition 2.5 it is immaterial that the densities $L_n(x,c)$ are string-localized and $L_n^{\rm pt}(x)$ point-localized. We write more generally just L_n and K_n instead, so as to cover also the L-V pairs to be discussed in Sect. 4.5 and Sect. 4.6.

Proof of Proposition 2.4. The expansion of (2.14) in second order reads

$$\frac{i^2}{2}\int dx\,dx'\,\big(\chi\chi'\cdot T[L_1L_1'] + \partial_\mu\chi\chi'\cdot T[V_1^\mu L_1']$$
$$+\chi\partial_\nu'\chi'\cdot T[L_1V_1'^\nu] + \partial_\mu\chi\partial_\nu'\chi'\cdot T[V_1^\mu V_1'^\nu]\big) -$$

$$+\frac{i}{2}\int dx\,(\chi^2\cdot L_2+\partial_\mu\chi^2\cdot V_2^\mu+\partial_\mu\chi\partial_\nu\chi\cdot W_2^{\mu\nu})$$
$$\stackrel{!}{=}\frac{i^2}{2}\int dx\,dx'\,\chi\chi'\cdot T[K_1K_1']+\frac{i}{2}\int dx\,\chi\cdot K_2.$$

We insert the initial L-V pair relation $L_1=K_1+\partial_\mu V_1^\mu$, and integrate by parts. After the obvious cancellations, this becomes the determining condition for L_2, K_2, V_2, W_2

$$\int dx\,dx'\,\chi\chi'\cdot\Big[(O^{(2)}(x,x')-i\delta(x-x')(L_2-K_2-\partial_\mu V_2^\mu)-\partial_\mu\partial'_\nu\big(i\delta(x-x')W_2^{\mu\nu}\big)\Big]\stackrel{!}{=}0,$$

where

$$O^{(2)}(x,x')=[T,\partial_\mu]V_1^\mu K_1{}'+[T,\partial'_\mu]K_1V_1'^\mu+[[T,\partial_\mu],\partial'_\nu]V_1^\mu V_1'^\nu.$$

With the notation (2.7), this is (2.18). $\square$

Proof of Proposition 2.5. Expanding (2.14) in third order, eliminating L_1 by the first-order condition, and integrating by parts, we get

$$\int dx\,dx'\,dx''\,\chi\chi'\chi''\big[O^{(3)}(x,x',x'')-\delta_{xx'x''}(L_3-K_3-\partial V_3)-\mathfrak{S}_3\big(\partial\partial'[\delta_{xx'x''}W_3]\big)\big]\stackrel{!}{=}0,$$

where

$$\begin{aligned}O^{(3)}(x,x',x'')=\mathfrak{S}_3\Big(&3[T,\partial]V_1K_1'K_1''+3[[T,\partial],\partial']V_1V_1'K_1''\\&+[[[T,\partial],\partial'],\partial'']V_1V_1'V_1''-\\&-3i\delta_{x'x''}\big([T,\partial]V_1L_2'+[T,\partial']V_2'K_1-\partial'[T,\partial]V_1V_2'\big)-\\&-3i\delta_{x'x''}TK_1(L_2'-K_2'-\partial'V_2')\\&-3i\partial'\partial''\big(\delta_{x'x''}([T,\partial]V_1W_2'+TK_1W_2')\big)\Big).\end{aligned}\tag{A.1}$$

The subsequent Lemma A.1 is the tree-level version of the "Master Ward Identity" of [3, Sect. 2.4], which the authors postulate to hold as a natural renormalization condition in all loop orders. It will allow substantial systematic cancellations in (A.1).

Lemma A.1. *For Wick polynomials Y and X_i, let*

$$O_\mu(Y;X_1,\dots,X_n):=[T,\partial_\mu]Y(x)X_1(x_1),\dots,X_n(x_n)\big|^{\rm tree}.\tag{A.2}$$

It holds

$$O_Y(:X_1:,\dots,:X_n:)=\sum_{i=1}^n T\big(:O_Y(X_i):\ :X_1:\ \dots:\cancel{X_i}:\ \dots:X_n:\big)\big|^{\rm tree}.\tag{A.3}$$

Proof. We insert $\partial Y=\sum_\varphi\frac{\partial Y}{\partial\varphi}\partial\varphi$ (as Wick polynomials) in $T\big(\partial YX_1\dots X_n\big)$. In the Wick expansion, the terms in which $\partial\varphi$ is not contracted, cancel against the corresponding terms in the Wick expansion of $\partial T\big(YX_1\dots X_n\big)$ in which the derivative hits a noncontracted factor of Y.

The terms in $T\big(\partial Y X_1 \dots X_n\big)$ in which $\partial\varphi$ is contracted with one of the fields X_i, can be written as

$$\sum_{\varphi,\chi_i} \langle T\partial_\mu\varphi\, \chi_i\rangle \cdot T\big(:\frac{\partial Y}{\partial\varphi}\frac{\partial X_i}{\partial\chi_i}:\ :X_1:\ \dots :\cancel{X_i}:\ \dots :X_n:\big)\big|^{\text{tree}}.$$

Note that at tree level, there are no further contractions between $\frac{\partial Y}{\partial\varphi}$ and $\frac{\partial X_i}{\partial\chi}$, so the latter appear in a single Wick product. These terms can be paired with the corresponding terms

$$\sum_{\varphi,\chi_i} \partial_\mu\langle T\varphi\, \chi_i\rangle \cdot T\big(:\frac{\partial Y}{\partial\varphi}\frac{\partial X_i}{\partial\chi_i}:\ :X_1:\ \dots :\cancel{X_i}:\ \dots :X_n:\big)\big|^{\text{tree}}$$

arising in the Wick expansion of $\partial T\big(Y X_1 \dots X_n\big)$, in which the derivative hits a contracted factor of Y. Thus, by (2.8), we have

$$\begin{aligned}&[T,\partial_\mu]\big(:Y:\ :X_1:\ \dots :X_n:\big)\\ &= \sum_i\sum_{\varphi,\chi_i} O_\mu(\varphi;\chi_i)\cdot T\Big(:\frac{\partial Y}{\partial\varphi}\frac{\partial X_i}{\partial\chi}:\ :X_1:\ \dots :\cancel{X_i}:\ \dots :X_n:\Big)\Big|^{\text{tree}}.\end{aligned}$$

By (2.9), this proves the claim. □

Proof of Proposition 2.5 (cont'd). We need the case $n=2$ of Lemma A.1, where $Y^\mu = V_1^\mu$ is a vector field. With notation $O_{Y_1}(X',Z'') \equiv O_\mu(Y_1^\mu; X, Z)$, the first line of (A.1) can be written as

$$\begin{aligned}&\mathfrak{S}_3\big(3O_{V_1}(K_1',K_1'') + 3O_{V_1}(\partial'V_1',K_1'') - 3\partial'O_{V_1}(V_1',K_1'') +\\ &+ O_{V_1}(\partial'V_1',\partial''V_1'') - 2\partial'O_{V_1}(V_1',\partial''V_1'') + \partial'\partial''O_{V_1}(V_1',V_1'')\big).\end{aligned} \tag{A.4}$$

By Lemma A.1, and with some rearrangements, this becomes

$$\begin{aligned}&\mathfrak{S}_3\Big(3T\,O^{(2)}(x,x')K_1(x'') + 3O_{O_{V_1}(V_1')}(K_1'')\\ &\qquad + O_{V_1}\big(O_{V_1'}(3K_1''+2\partial''V_1'')\big) - 2\partial''O_{V_1}\big(O_{V_1'}(V_1'')\big)\Big)\end{aligned}$$

with $O^{(2)}(x,x')$ as in (2.18). Expressing the latter by (2.19) in terms of the second-order fields L_2, K_2, V_2, W_2, one can cancel the term involving $\delta_{x'x''}\cdot TK_1(L_2' - K_2' - \partial'V_2')$ in the last line of (A.1), and rewrite all the remaining terms as (2.20). □

If one is interested only in the third-order densities K_3 and L_3, it suffices to compute the integral over (2.21) and demand:

$$\int dx\,dx'\,dx''\,O^{(3)}(x,x',x'') \stackrel{!}{=} \int dx\,(L_3(x) - K_3(x)). \tag{A.5}$$

Because all total derivatives drop out, the integral over (2.20) reduces to

$$\begin{aligned}\int dx\,dx'\,dx''\,O^{(3)}(x,x',x'') = \int dx\,\big(&\Omega_{V_1}(3L_2 - \Omega_{V_1}(K_1+2L_1))\\ &+ 3\Omega_{V_2-\Omega_{V_1}(V_1)}(K_1)\big)(x),\end{aligned} \tag{A.6}$$

where for vector fields Y^μ,

$$\Omega_Y(X) := -i\int dx'\,O_Y(X')$$

coincides with (3.21) in the case when there are no derivatives of δ-functions. The form (A.6) is easy to evaluate, even when the two-point obstructions involve derivatives of δ-functions. It then suffices to equate the integrands of (A.5) and (A.6).

Appendix B. Propagators and two-point obstructions

We denote by $W_m(x-x')$ and $T_m(x-x')$ the canonical scalar two-point function and time-ordered two-point function of mass m, such that iT_m is the Feynman propagator, and

$$(\Box + m^2)W_m(x-x') = 0, \qquad (\Box + m^2)T_m(x-x') = -i\delta(x-x').$$

B.1. Propagators and two-point obstructions for the Higgs field

The two-point function of the Higgs field is the canonical scalar two-point function of mass m_H:

$$\langle H(x)H(x')\rangle = W_{m_H}(x-x').$$

The two-point functions of derivatives of H are derivatives of W_{m_H}. We define the "kinematical" propagators among the fields H and ∂H by the same differential operators acting on the massive Feynman propagator $iT_{m_H}(x-x')$. However, the time-ordering prescription fixes the propagators only outside the point $x-x'=0$. The freedom to add an arbitrary derivative of $\delta(x-x')$ is constrained, apart from Lorentz covariance, by the scaling degree that must not exceed the scaling degree of the kinematical propagators. T_{m_H} has the canonical scaling degree 2. Every derivative increases the scaling degree by 1. The δ-function has scaling degree 4. Therefore, $\langle T\partial H\partial' H'\rangle$ has a freedom of renormalization:

$$\langle T\partial_\mu H(x)\partial'_\nu H(x')\rangle = -\partial_\mu\partial_\nu T_{m_H}(x-x') + ic_H\eta_{\mu\nu}\delta(x-x') \tag{B.1}$$

with an arbitrary real constant c_H.

Using $\Box H = -m_H^2 H$ and the definition (2.8), one computes the two-point obstructions $O_\mu(\varphi;\varphi')$ among the fields H and ∂H, as displayed in (3.10):

$$\begin{aligned}
O_\mu(H;\partial'_\nu H') &= \langle T\partial_\mu H\partial'_\nu H'\rangle - \partial_\mu\langle TH\partial'_\nu H'\rangle = ic_H\eta_{\mu\nu}\delta(x-x'),\\
O_\mu(\partial^\mu H;H') &= \langle T\Box HH'\rangle - \partial_\mu\langle T\partial^\mu HH'\rangle = -(m_H^2+\Box)T_{m_H}(x-x') = i\delta(x-x'),\\
O_\mu(\partial^\mu H;\partial'_\nu H') &= \langle T\Box H\partial'_\nu H'\rangle - \partial_\mu\langle T\partial^\mu H\partial'_\nu H'\rangle = -i(c_H+1)\partial_\nu\delta(x-x').
\end{aligned}$$

B.2. Propagators and two-point obstructions for Krein space fields

The Feynman gauge two-point function of the Krein potential A^{K} of mass m is

$$\langle A^{\mathrm{K}}_\mu(x)A^{\mathrm{K}}_\nu(x')\rangle = -\eta_{\mu\nu}W_m(x-x').$$

By the definitions $\phi^{\mathrm{K}} := -m^{-2}(\partial A^{\mathrm{K}})$ and $B := A^{\mathrm{K}} - \partial\phi^{\mathrm{K}}$ as in Sect. 4.5, one computes the two-point functions (with Lorentz indices and arguments suppressed in an obvious way)

$$\begin{aligned}
&\langle\phi^{\mathrm{K}}\phi^{\mathrm{K}\prime}\rangle = -m^{-2}W_m,\\
&\langle\phi^{\mathrm{K}}\partial\phi^{\mathrm{K}\prime}\rangle = -\langle\partial\phi^{\mathrm{K}}\phi^{\mathrm{K}\prime}\rangle = \langle\phi^{\mathrm{K}}A^{\mathrm{K}\prime}\rangle = -\langle A^{\mathrm{K}}\phi^{\mathrm{K}\prime}\rangle = m^{-2}\partial W_m,\\
&\langle\partial\phi^{\mathrm{K}}\partial\phi^{\mathrm{K}\prime}\rangle = \langle\partial\phi^{\mathrm{K}}A^{\mathrm{K}\prime}\rangle = \langle A^{\mathrm{K}}\partial\phi^{\mathrm{K}\prime}\rangle = m^{-2}\partial\partial W_m,
\end{aligned}$$

$$\langle BB'\rangle = \langle BA^{\mathrm{K}\prime}\rangle = \langle A^{\mathrm{K}}B'\rangle = -(\eta + m^{-2}\partial\partial)W_m,$$
$$\langle B\phi^{\mathrm{K}\prime}\rangle = \langle \phi^{\mathrm{K}}B'\rangle = \langle B\partial\phi^{\mathrm{K}\prime}\rangle = \langle \partial\phi^{\mathrm{K}}B'\rangle = 0.$$

We define the kinematical propagators by the same differential operators acting on the massive Feynman propagator iT_m. Only those propagators involving $\partial_\mu\partial_\nu T_m$ have scaling degree 4 and admit a renormalization proportional to $\eta_{\mu\nu}\delta(x-x')$. By linearity and $A^{\mathrm{K}}_\mu - B_\mu = \partial_\mu\phi^{\mathrm{K}}$, there is only one independent parameter c_B:

$$\langle TBB'\rangle = \langle TA^{\mathrm{K}}B'\rangle = \langle TBA^{\mathrm{K}\prime}\rangle = -(\eta + m^{-2}\partial\partial)T_m(x-x') + ic_B\cdot m^{-2}\eta\delta(x-x'),$$
$$\langle T\partial\phi^{\mathrm{K}}\partial\phi^{\mathrm{K}\prime}\rangle = \langle TA^{\mathrm{K}}\partial\phi^{\mathrm{K}\prime}\rangle = \langle T\partial\phi^{\mathrm{K}}A^{\mathrm{K}\prime}\rangle = m^{-2}\partial\partial T_m(x-x') - ic_B\cdot m^{-2}\eta\delta(x-x').$$

With $\partial B = 0$, $\partial A^{\mathrm{K}} = -m^2\phi^{\mathrm{K}}$, and $\Box\phi^{\mathrm{K}} = -m^2\phi^{\mathrm{K}}$, one computes the relevant two-point obstructions among the fields A^{K}, B, ϕ^{K}. The results are (4.11).

B.3. Propagators and two-point obstructions for string-localized fields

The positive-definite two-point function of the Proca field B of mass m is

$$\langle B_\mu(x)B_\nu(x')\rangle = -(\eta_{\mu\nu} + m^{-2}\partial_\mu\partial_\nu)W_m(x-x'). \tag{B.2}$$

We use the short-hand notation $(I_eX)(x) := \int_0^\infty ds\, X(x+se)$. By the definitions $\phi(e) := I_e(eB)$, $A(e) := B + \partial\phi(e) = B + I_e\partial(eB)$, one computes the two-point functions

$$\langle A_\mu B'_\nu\rangle = -(\eta_{\mu\nu} + e_\nu I_e\partial_\mu)W_m, \qquad \langle B_\mu A'_\nu\rangle = -(\eta_{\mu\nu} - e'_\mu I_{-e'}\partial_\nu)W_m,$$
$$\langle A_\mu\phi'\rangle = -(e'_\mu I_{-e'} + (ee')I_eI_{-e'}\partial_\mu)W_m, \qquad \langle \phi A'_\nu\rangle = -(e_\nu I_e - (ee')I_eI_{-e'}\partial_\nu)W_m,$$
$$\langle A_\mu A'_\nu\rangle = -(\eta_{\mu\nu} + e_\nu I_e\partial_\mu - e'_\mu I_{-e'}\partial_\nu - (ee')I_eI_{-e'}\partial_\mu\partial_\nu)W_m,$$
$$\langle B_\mu\phi'\rangle = -(e'_\mu I_{-e'} + m^{-2}\partial_\mu)W_m, \qquad \langle \phi B'_\nu\rangle = -(e_\nu I_e - m^{-2}\partial_\nu)W_m,$$
$$\langle \phi\phi'\rangle = -((ee')I_eI_{-e'} - m^{-2})W_m,$$

where it was used repeatedly that $\Box W_m = -m^2W_m$ and $(e\partial)(I_eX)(x) = -X(x)$.

We define the kinematical propagators by the same differential and integral operators as in the two-point functions, acting on $iT_m(x-x')$. The propagator $i\langle TBB\rangle$ has scaling degree 4 and admits the renormalization (the same as in the Krein space approach App. B.2)[10]:

$$\langle TB_\mu B'_\nu\rangle = -(\eta_{\mu\nu} + m^{-2}\partial_\mu\partial_\nu)T_m(x-x') + ic_B\cdot m^{-2}\eta_{\mu\nu}\delta(x-x'). \tag{B.3}$$

By inspection of the scaling degree that is lowered by 1 by a string integration, one observes that all other propagators admit only renormalizations involving string-integrals over δ-functions.

The kinematical propagators produce string-localized two-point obstructions $O_\mu(A^\mu; X')$, as displayed in (3.12). We have made a careful analysis of possible string-localized renormalizations of the propagators. Their precise structure is dictated by the scaling degree, Lorentz invariance, the number of string integrations, homogeneity in e and e', and the identity $-(e\partial)I_e = 1$ which

[10] The freedom of propagator renormalization in causal perturbation theory should not be confused with propagators in different gauges, like R_ξ gauges. The latter would violate causality because of their momentum space denominators $k^2 - \xi m^2$.

implies the axiality property $e^\mu A_\mu = 0$. It turns out that it is impossible to make all string-localized two-point obstructions vanish – one would rather produce more of them.[11] There is thus the risk that the second-order obstructions (2.12) and (2.18) of the S-matrix become string-localized. In this case, they cannot be cancelled by admissible higher-order densities, as outlined in Sect. 2.3. In the Abelian Higgs model, they do not occur, thanks to a characteristic feature of the model (namely, the string-localized field A does not appear within V_n) that prefers the kinematical choice of propagators for the string-localized fields, see Sect. 3.

We take therefore all relevant propagators in the Proca sector except (B.3) to be the kinematical ones. One can then directly compute the relevant two-point obstructions. One obtains (3.11) and (3.12).

For the L-Q-approach in Sect. 3.3, we also need obstructions $O_\mu(w; X')$ involving the field $w = \delta_c\phi(c)$. These obstructions vanish because $O_\mu(\phi; X')$ in (3.11) and (3.12) are string-independent, and $O_\mu(w; X') = \delta_c O_\mu(\phi; X')$.

In Sect. 3.6, we also need obstructions of the field strength $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The obstructions (3.33) follow from the unique propagators

$$\langle TB^\kappa G'_{\mu\nu}\rangle = (\delta^\kappa_\nu\partial_\mu - \delta^\kappa_\mu\partial_\nu)T_m, \quad \langle T\phi(e)G'_{\mu\nu}\rangle = (e_\nu\partial_\mu - e_\mu\partial_\nu)I_eT_m \tag{B.4}$$

and the kinematical propagator

$$\langle TA^\kappa(e)G'_{\mu\nu}\rangle = \big((\delta^\kappa_\nu\partial_\mu - \delta^\kappa_\mu\partial_\nu) + (e_\nu\partial_\mu - e_\mu\partial_\nu)\partial^\kappa I_e\big)T_m. \tag{B.5}$$

Appendix C. Useful identities

The following structures appear in the computation of $O^{(2)}$ as in (2.18). Let throughout $X^{(\prime)} \equiv X(x^{(\prime)})$, $\partial^{(\prime)} \equiv \partial_{x^{(\prime)}}$, and $\delta_{xx'} \equiv \delta(x-x')$ and $\delta_{xx'x''} \equiv \delta_{xx'}\delta_{x'x''}$.

Lemma C.1. *It holds*

$$\begin{aligned} X\cdot\delta_{xx'}\cdot Y' + X'\cdot\delta_{xx'}\cdot Y &= \delta_{xx'}\cdot 2XY, \\ X\cdot\partial_\alpha\delta_{xx'}\cdot Y' + X'\cdot\partial'_\alpha\delta_{xx'}\cdot Y &= \delta_{xx'}\cdot X\overset{\leftrightarrow}{\partial}_\alpha Y, \\ \partial'_\alpha(X\cdot\delta_{xx'}\cdot Y') + \partial_\alpha(X'\cdot\delta_{xx'}\cdot Y) &= \delta_{xx'}\cdot\partial_\alpha(XY), \\ \partial'_\alpha(X\cdot\partial_\beta\delta_{xx'}\cdot Y') + \partial_\alpha(X'\cdot\partial'_\beta\delta_{xx'}\cdot Y) &= (\partial'_\alpha\partial_\beta + \partial_\alpha\partial'_\beta)(\delta_{xx'}\cdot XY) - \delta_{xx'}\cdot\partial_\alpha(Y\partial_\beta X). \end{aligned} \tag{C.1}$$

Proof. The proof is elementary, using identities of the form $X\cdot\partial\delta_{xx'}\cdot Y' = \partial(X\cdot\delta_{xx'}\cdot Y') - \partial X\cdot\delta_{xx'}\cdot Y'$, as well as $(\partial+\partial')\delta_{xx'} = 0$. □

The following structures appear in the computation of $O^{(3)}$ as in (3.23).

Lemma C.2. *It holds*

$$\begin{aligned} 3\mathfrak{S}_3\big(\delta_{x'x''}\cdot\partial'\delta_{xx'}\cdot X\big) &= 2\delta_{xx'x''}\cdot\partial X, \\ 3\mathfrak{S}_3\big(\delta_{x'x''}\cdot\partial'\delta_{xx'}\cdot XY'\big) &= \delta_{xx'x''}\cdot(2Y\partial X - X\partial Y). \end{aligned} \tag{C.2}$$

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[11] E.g., contributions to the obstruction $O_\mu(A^\mu; A'_\nu)$ in (3.12) could come from renormalizations of $\langle TA^\mu A'_\nu\rangle$ or $\langle T\phi A'_\nu\rangle$ (via $\partial_\mu A^\mu = -m^2\phi$). But the latter (scaling degree 1) admits no renormalization at all, and the renormalization of the former (scaling degree 2) by $((ee')\eta_{\mu\nu} - e'_\mu e_\nu)I_eI_{-e'}\delta(x-x')$ would (via $\partial_\mu\phi = A_\mu - B_\mu$) produce a non-zero obstruction $O_\mu(\phi; A'_\nu)$.

Proof. For the first identity, write

$$3\delta_{x'x''}\cdot\partial'\delta_{xx'}\cdot X=-3\delta_{x'x''}\cdot\partial\delta_{xx'}\cdot X=3\delta_{xx'x''}\cdot\partial X-3\partial(\delta_{xx'x''}\cdot X).$$

In the second term, $3\delta_{xx'x''}\cdot X=\delta_{xx'x''}\cdot(X+X'+X'')$ is separately symmetric. Apply the symmetrization:

$$3\mathfrak{S}_3\big(\delta_{x'x''}\cdot\partial'\delta_{xx'}\cdot X\big)=3\delta_{xx'x''}\cdot\partial X-\frac{1}{3}(\partial+\partial'+\partial'')\big(\delta_{xx'x''}\cdot(X+X'+X'')\big),$$

and use that $(\partial+\partial'+\partial'')\delta_{xx'x''}=0$ while $\delta_{xx'x''}\cdot(\partial+\partial'+\partial'')(X+X'+X'')=3\delta_{xx'x''}\cdot\partial X$. This proves the first identity. For the second identity write

$$\partial'\delta_{xx'}\cdot XY'=\partial'(\delta_{xx'}\cdot XY')-\delta_{xx'}\cdot X\partial'Y'=\partial'\delta_{xx'}\cdot XY-\delta_{xx'}\cdot X\partial Y,$$

and apply the first identity. □

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