Estimating Growth at Risk with Skewed Stochastic Volatility Models

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1. Motivation

- Adrian et al. (2019) analyze the forecasting density of US GDP growth (gdp_{t+1}) based on current financial conditions ($nfci_t$) using a semi-parametric approach.
- **Findings:**
  (1) Lower quantiles of the conditional forecast distribution vary a lot over over time while the upper quantiles remain relatively stable.
  (2) A deterioration of national financial conditions coincides with increases in the interquartile range and decreases the mean.
  (3) Distributions are more symmetric in normal times and become left skewed in recessionary periods
- **Drawbacks of the semi-paramteric approach:**
  - Time variation of the distribution is not parametrically characterized
  - Does not allow for parameter inference
- **Use a fully parametric model to analyze the evolution of the conditional forecast distributions and conduct statistical inference on the estimated parameters.**

2. Skewed Stochastic Volatility Model

The Skewed Stochastic Volatility Model (SSV) is a non-linear, non-Gaussian state space model with measurement equation

$$gdp_{t+1} = \gamma_0 + \gamma_1 nfci_t + \varepsilon_{t+1} \quad \text{with} \quad \varepsilon_t \sim skew\,\mathcal{N}(0, \sigma_t, \alpha_t)$$

and state equations

$$\ln(\sigma_t) = \delta_{1,0} + \delta_{1,1} nfci_t + \delta_{1,2} \ln(\sigma_{t-1}) + \nu_{1,t}$$
$$\alpha_t = \delta_{2,0} + \delta_{2,1} nfci_t + \delta_{2,2} \alpha_{t-1} + \nu_{2,t}.$$

- $\nu_{1,t}$ and $\nu_{2,t}$ are assumed to be uncorrelated Gaussian White Noise innovations
- Errors in the measurement equation are distributed according to the skewed Normal distribution of Azzalini (2013).

Similar to the normal distribution it has has parameters for **location** (μ) and **scale** (σ) plus an additional parameter that determines its **shape** (α):

$$skew\,\mathcal{N}(y|\mu, \sigma, \alpha) = \frac{2}{\sqrt{(2\pi)}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \int_{-\infty}^{\alpha\frac{y-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

$\longrightarrow$ $nfci_t$ can affect all three moments.

$\longrightarrow$ Kurtosis evolves endogenously.

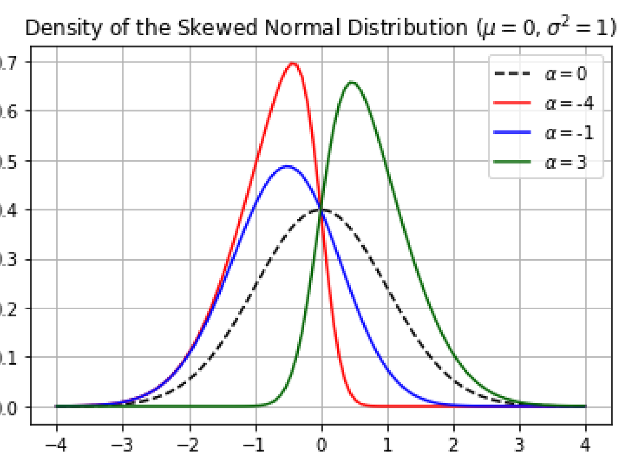

Figure: Skew-Normal Density

3. Estimation Method

Based on the work of Kim et al. (1998), the skewed stochastic volatility models is estimated using a Particle Metropolis Hastings algorithm:

- **Static Model Parameters** ($\theta = (\delta_0, \delta_1, \gamma_{1,0}, \gamma_{1,1}, \gamma_{1,2}, \gamma_{2,0}, \gamma_{2,1}, \gamma_{2,2}, \sigma_{\nu,1}, \sigma_{\nu,2})$) are estimated using a Metropolis Hastings sampler with stationary distribution

$$p(\theta|y_{1:T}, s_{1:T}) = \frac{p(y_{1:T}|s_{1:T}, \theta)p(s_{1:T}|\theta)p(\theta)}{p(y_{1:T})}$$

- **Time varying model parameters** ($s_t = (\ln \sigma_t, \alpha_t)$) are estimated using the tempered particle filter from Herbst and Schorfheide (2019) to approximate the filtering distribution

$$p(s_t|y_{1:t}) = \frac{p(y_t|s_t)p(s_t|y_{1:t-1})}{\int p(y_t|s_t)p(s_t|y_{1:t-1})ds_t}$$

  using sequential importance sampling.

3.1 Tempered Particle Filter

- The tempered particle filter uses annealed importance sampling to obtain a better proposal density using a sequence of N_ϕ bridge distributions.
- Building on the adaptive tempering schedule of Herbst and Schorfheide (2019), I use a tempering scheme that jointly tempers the variance and tilts the density towards the actual level while targeting a predetermined inefficiecy ratio r^*:

$$p_n(y_t|s_{t,i}) = skew\,\mathcal{N}(y_t|\mu_t, \sigma_{t,i}/\phi_n, \phi_n\alpha_{t,i}) \quad \text{with} \quad 0 < \phi_n < 1 \quad \text{and} \quad \lim_{n \to N_\phi} \phi_n = 1.$$

- This gives the unnormalized weights at the n^{th} tempering step as

$$\tilde{w}_{t,i}(\phi_n) = \left(\frac{\phi_n}{\phi_{n-1}}\right)^2 \exp\left(\frac{-(\phi_n - \phi_{n-1})(y_t - \mu_t)}{2\sigma_{t,i}}\right)^2 \tilde{\Lambda}_{t,i}(\phi_n)$$

  with

$$\tilde{\Lambda}_{t,i}(\phi_n) = \frac{\int_{-\infty}^{\alpha_{t,i}\phi_n^{2/3}\frac{(y_t-\mu_t)}{\sigma_{t,i}}} \exp\left(\frac{-t^2}{2}\right)dt}{\int_{-\infty}^{\alpha_{t,i}\phi_{n-1}^{2/3}\frac{(y_t-\mu_t)}{\sigma_{t,i}}} \exp\left(\frac{-t^2}{2}\right)dt}$$

- Additionally tempering the symmetry of the distribution reduces the number of tempering steps N_ϕ by about 25%.

3.2 Tuning

- The Tempered Particle Filter is tuned using $M = 40000$ particles with a targeted Inefficiency ratio $r^* = 1.2$ and 2 mutation steps in each tempering iteration.
- The model is estimated based on $N = 20000$ draws of the Particle Metropolis Hastings Algorithm using a standard Random Walk proposal with an estimate of the proposal variance $Var(\theta) = \hat{\Omega}$ based on a pre-run.
- Mixture of uninformative and informative priors on the static parameters.

4. Estimation Results

The model is estimated on the same data set as used by Adrian et al. (2019) with four Markov chains ran in parallel on the HPC-Cluster at the Freie Universität.

4.1 Static Parameters

Model Parameter	Mean	SD	q05	q95
γ_0	2.217	0.335	1.69	2.799
γ_1	**-0.695**	0.236	-1.125	-0.351
$\delta_{1,0}$	1.295	0.385	0.784	2.06
$\delta_{1,1}$	**0.292**	0.198	0.106	0.556
$\delta_{1,2}$	0.388	0.139	0.197	0.647
$\delta_{2,0}$	0.401	0.305	-0.904	0.649
$\delta_{2,1}$	**-0.429**	0.226	-0.81	-0.038
σ_{ν_1}	0.451	0.119	0.298	0.688
σ_{ν_2}	0.533	0.182	0.319	0.877

- $nfci_t$ has a negative impact on the mean and skewness and a positive impact on the volatility.
- Estimated parameter values for the effect of financial conditions on the variance and skewness of the conditional distributions are significant based on 90% credible sets.

Posteriors Densities:

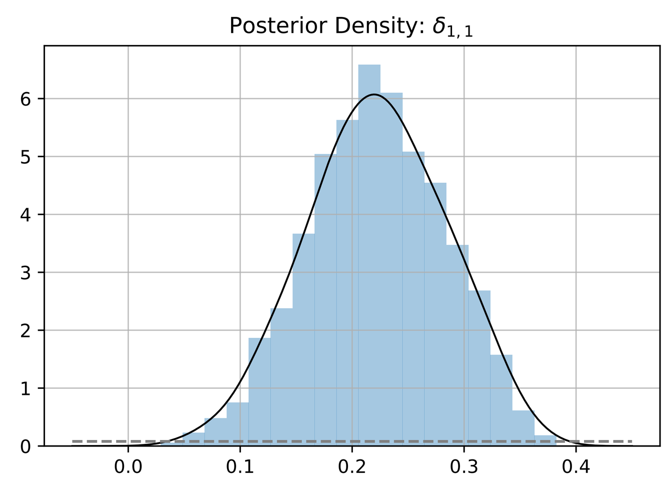

Figure: Impact of $nfci_t$ on the scale

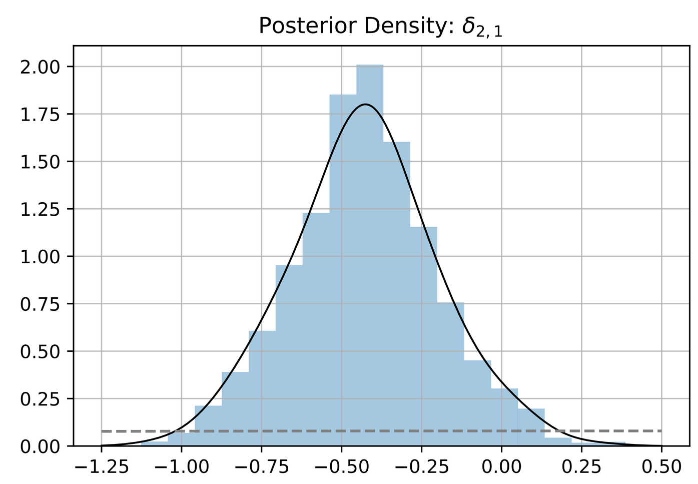

Figure: Impact of $nfci_t$ on the shape

4.2 Time-varying Parameters

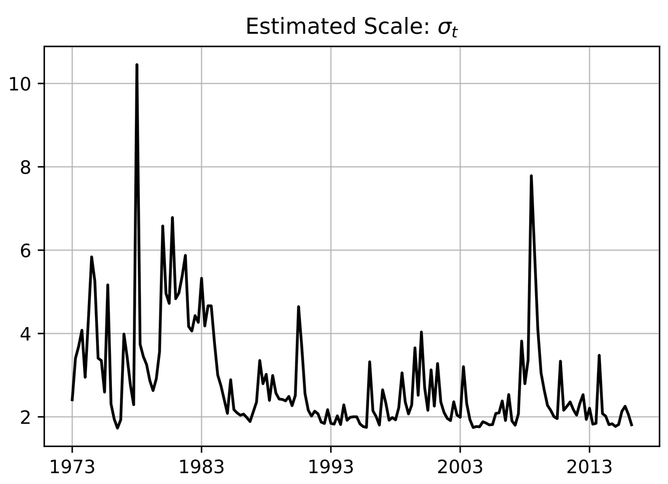

Figure: Volatility over time

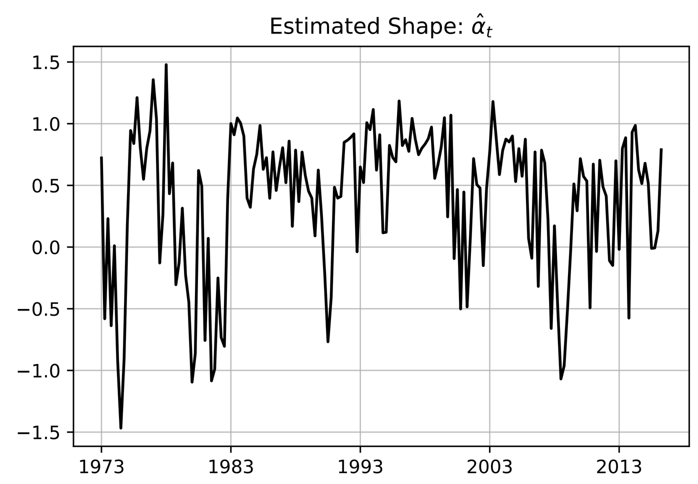

Figure: Skewness over time

- Volatility and downside risk increase in the 1980s and during the Great Recession
- Increases in volatility occur with an increase in downside risks ($\rho[\alpha_t, \sigma_t^2] = -0.41$)
- The estimated state of α_t even exhibits positive skewness in times of moderation similar to the findings of delle Monache et al. (2021)

4.3 Conditional Forecast Densities

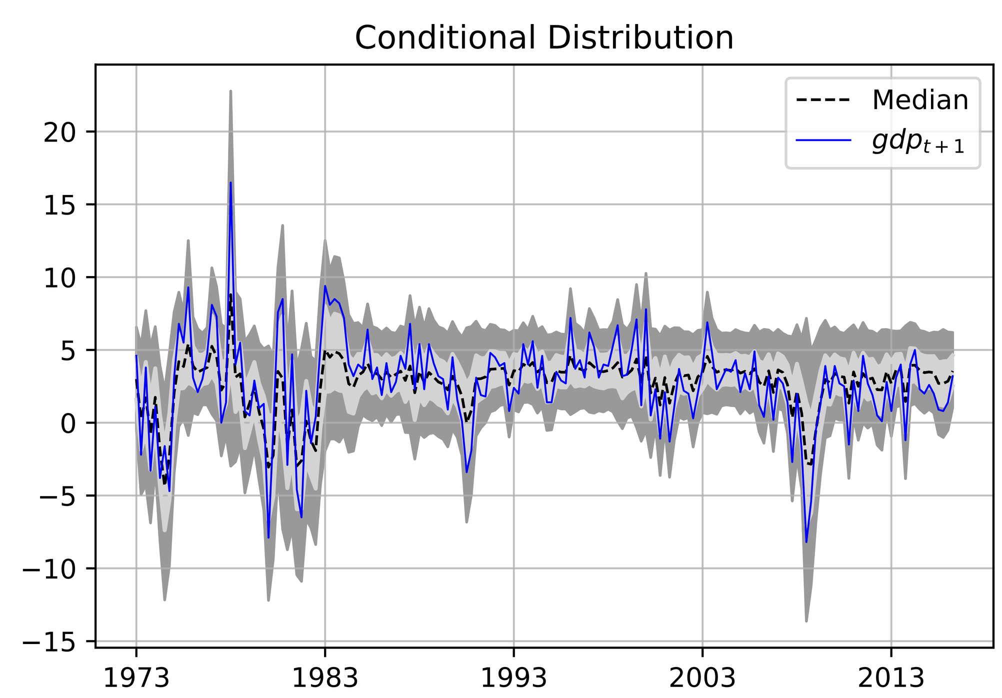

Figure: Estimated forecasting densities

- Lower and upper 5% and 25% percent quantiles of the one-period ahead forecasting density
- While the upper quantiles remain relatively stable, the lower quantiles vary strongly over time indicating increased downside risk to GDP growth in times of financial distress

5. Conclusion

- I propose a Skewed Stochastic Volatility model to analyze Growth at Risk and conduct statistical inference on the estimated parameters
- The model is estimated using Bayesian Methods. The tempering schedule of the tempered particle filter is adapted to asymmetric distributions.
- The estimated parameter values for the effect of financial conditions on the variance and skewness of the conditional distributions are significant and in line with other recent studies.

References

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[2] delle Monache, D., A. de Polis, and I. Petrella (2021). Modeling and forecasting macroeconomic downside risk. Working Papers 1324, Banca d'Italia
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