Freie Universität Berlin

Orientation Games and Minimal Ramsey Graphs

DISSERTATION
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

Vorgelegt von
Anita LIEBENAU

Betreut durch
Prof. Tibor SZABÓ, PhD

Eingereicht am

FACHBEREICH MATHEMATIK UND INFORMATIK
DER FREIEN UNIVERSITÄT BERLIN

Im August 2013

Erstgutachter und Betreuer: Prof. Tibor Szabó

Zweitgutachter: Prof. Michael Krivelevich

Tag der Disputation: 2. Dezember 2013

Contents

Preface **iii**

Acknowledgements **vii**

1 Introduction **1**

1.1 Maker–Breaker games with orientations 1

1.1.1 On the random graph intuition for the tournament game 1

1.1.2 Orientation games 4

1.2 Ramsey-minimal graphs 7

1.2.1 What is Ramsey-eqivalent to the clique? 7

1.2.2 Ramsey-minimal graphs for r colours 9

2 Tournament games **13**

2.1 The probabilistic analysis for the tournament game 13

2.2 A strategy for Maker in the tournament game 19

2.3 A strategy for OBreaker in the orientation tournament game 24

2.4 Concluding remarks 25

3 On the threshold bias in the oriented-cycle game **29**

3.1 α-structures 31

3.2 OBreaker's strategy for the monotone rules 34

3.3 OBreaker's strategy for the strict rules 38

3.4 Concluding remarks 48

4 What is Ramsey-equivalent to the clique? **51**

4.1 Hanging edges 51

4.2 Clique with some disjoint smaller cliques 56

4.3 Concluding remarks 62

5 On minimal r-Ramsey graphs for the clique **65**

5.1 Lower bounds on $s_r(K_k)$ 65

5.2 Upper bounds on $s_r(K_k)$ 71

5.3 Packing (n, r, k)-critical graphs 76

5.4 Signal senders and BEL-gadgets 83

5.5 Concluding remarks 92

Zusammenfassung **95**

Eidesstattliche Erklärung **97**

References **99**

Curriculum Vitae **101**

Preface

In this thesis, we consider problems of two different directions in extremal graph theory. In the first part, we study orientation games, which are a variation of positional games. Two players, referred to as OMaker and OBreaker, alternately direct edges of K_n, the complete graph on n vertices. OMaker wins the game if the final complete digraph (a tournament) has some predefined property $\mathcal{P}$. Otherwise, OBreaker wins. Orientation games were studied by several researchers, including Aigner, Alon, Beck, Ben-Eliezer, Bollobás, Krivelevich, Sudakov, Szabó, Tuza and many others. For a given tournament T_k on k vertices, we consider the orientation-tournament game $Or(T_k)$ in which the property $\mathcal{P}$ OMaker tries to achieve is that the final tournament contains a copy of T_k. We show that OMaker can win this game whenever $k \leqslant (2-o(1))\log_2 n$, whereas OBreaker has a winning strategy when k is roughly of order $4\log_2 n$. For the lower bound, we work in a setting studied earlier by Beck and Gebauer where OMaker wins if and only if the digraph consisting merely of her directed edges contains the given tournament T_k. We improve the best known constant factor. Moreover, our lower bound is tight for this setting, as is implied by the criterion of Erdős and Selfridge. The second orientation game we consider is the oriented-cycle game, where OMaker wins if the final tournament contains a directed cycle. As was recently shown by Ben-Eliezer, Krivelevich and Sudakov, OMaker has a winning strategy in this game even when OBreaker is allowed to direct up to roughly $n/2$ edges in each round. Let b be the number of edges OBreaker is allowed to direct in one round. As was observed by Bollobás and Szabó, OBreaker can win if $b \geqslant n-2$. We improve this trivial upper bound and show that OBreaker has a winning strategy when $b \geqslant 5n/6 + 2$. We adjust our strategy to the case when OBreaker is required to direct exactly b edges and thus refute a conjecture by Bollobás and Szabó.

In the second part, we study minimal Ramsey graphs. A graph G is r-Ramsey for a graph H, denoted by $G \to (H)_r$ if every r-colouring of the edges of G contains a monochromatic copy of H. A graph G is r-Ramsey minimal for a graph H if it is r-Ramsey for H, but no proper subgraph of G is r-Ramsey for H. Let $s_r(H)$ be the smallest minimum degree an r-Ramsey graph of H can have. The study of $s_2(H)$ was introduced by Burr, Erdős, and Lovász, who showed that $s_2(K_k) = (k-1)^2$. In this thesis, we settle a question by Szabó, Zumstein, and Zürcher and prove that $s_2(K_k \cdot K_2) = k-1$, where $K_k \cdot K_2$ is the graph on $k+1$ vertices consisting of K_k with a pendant edge. This has the following interesting consequence. Two graphs H and H' are Ramsey-equivalent if every graph G is 2-Ramsey for H if and only if it is 2-Ramsey for H'. A famous theorem of Nešetřil and Rödl implies that any graph H which is Ramsey-equivalent to K_k must contain K_k. Our result implies therefore that any graph H which is Ramsey-equivalent to K_k must be the disjoint union of K_k and a graph without a K_k. Let $\mu(k,t)$ be the maximum m such that the graph $H = K_k + mK_t$, consisting of K_k and m disjoint copies of K_t, is Ramsey-equivalent to K_k. Szabó et al. gave a lower bound on $\mu(k,t)$. We prove an upper bound on $\mu(k,t)$ which is roughly within a factor 2 of the lower bound. Furthermore, we study the dependency of $s_r(K_k)$ on r and show that, under the condition that k is constant, $s_r(K_k)$ has order of magnitude $r^{2+o(1)}$. We also give an upper bound on $s_r(K_k)$ which is polynomial in both r and k, and we determine $s_r(K_3)$ up to a factor of $\log r$.

Organisation.
In Chapter 1, we introduce all necessary concepts for both parts and state our results precisely. In Chapter 2, we study the orientation-tournament game and its aforementioned variant, the tournament game. In Chapter 3, we give a winning strategy for OBreaker in the oriented-cycle game. Chapter 4 and Chapter 5 are concerned with minimal Ramsey graphs. In Chapter 4, we only consider the case when the number of colours is two. We show that K_k and $K_k \cdot K_2$ are not Ramsey-equivalent, and we prove an upper bound on $\mu(k,t)$. Finally, in Chapter 5, we consider how the parameter $s_r(K_k)$ changes when the number of colours r tends to infinity.

General notation.
For a natural number n, we write $[n] := \{1,\ldots,n\}$ for the first n integers. The expression $(n)_k$ denotes the falling factorial, that is $(n)_k = n(n-1)\cdots(n-k+1)$. The logarithm $\log x$ is always in base 2, and $\ln n$ denotes the natural logarithm.

We use standard graph-theoretic notation and follow mainly the notation used in [3]. In this thesis, a graph is a pair $G = (V,E)$, where V is a finite set, and $E \subseteq \binom{V}{2}$ is a subset of the pairs of elements of V. In particular, our graphs are always simple and do not have loops. The elements in V are called vertices, and elements in E are called edges. We sometimes identify a graph G with its edge set for convenience. We write $v(G) := |V|$ for the number of vertices, and $e(G) := |E|$ for the number of edges. If $E = \binom{V}{2}$ we call G a *complete graph* on

$n := v(G)$ vertices (or n-*clique*), and denote it by K_n. We say two vertices v, w are *adjacent* if $vw \in E$, and we write $N(v) = N_G(v)$ for the set of all vertices adjacent to V, and denote by $\deg(v) = \deg_G(v) = |N(v)|$ the *degree of* v *in* G. Whenever the base graph G is clear from the context we omit the subscript. By $\delta(G)$ we denote the minimum degree of G, the smallest degree a vertex in G has. Further standard graph parameters we use are the independence number $\alpha(G)$, the maximum size of a subset of the vertices without edges; the clique number $\omega(G)$, the maximum size of a clique in G; and the chromatic number $\chi(G)$, the smallest number k such that the vertices can be coloured with k colours so that no two vertices of the same colour are adjacent.

More generally, we call $\mathcal{H}$ a hypergraph if $\mathcal{H} \subseteq 2^V$ is a subset of the powerset of some finite set V. The underlying set V of vertices is usually clear from the context. Elements in $\mathcal{H}$ are then called *hyperedges*. The hypergraph $\mathcal{H}$ is called t-*uniform*, if all hyperedges have cardinality t.

A directed graph D is a subset $D \subseteq V \times V$ for some finite set V, called the set of vertices again. Elements in D are called *arcs*. We provide more notation about directed graphs at the beginning of Chapter 3.

Our results are mostly of asymptotic nature and we use standard asymptotic notation, as for example in [3]. That is, for two functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f = O(g)$ if $f(n) \leqslant cg(n)$ for sufficiently large values of n, where $c > 0$ is an absolute constant. We write $f = \Omega(g)$ if $g \in O(f)$; and $f = \Theta(g)$ if $f \in O(g)$ and $f \in \Omega(g)$. If the ratio $f(n)/g(n)$ tends to zero as n tends to infinity we write $f = o(g)$.

Acknowledgements

I am grateful to Tibor Szabó for his invaluable advice and support, for proposing challenging mathematical problems, for ensuing interesting discussions, for his helpful critique, and for the coffee in the morning. I would like to thank Andrey, Dennis, Heidi, Jacob, Tibor and Yury, for the enriching experience of working together on problems that contributed to the thesis. I want to thank the team of our research group for providing a stimulating, cooperative and joyous working environment. I am thankful to Yury and Dennis who carefully read parts of the thesis and provided helpful advice and comments.

I am thankful to Jacob Fox for inviting me to work together at MIT and, together with his group of PhD students, to welcome me so warmly and to make my time in Cambridge so fruitful and special. I thank Asaf Ferber and Michael Krivelevich for waking my enthusiasm for positional games. I want to thank Emo Welzl and his research group; it was at the 10^{th} Gremo's Workshop on Open Problems that I first got in touch with the oriented-cycle game. Andrey, Asaf, Dennis and Guillem: thank you for the time together in front of the blackboard.

The past years were particularly carefree due to the support of the Berlin Mathematical School, which was not only of financial kind. In an unbureaucratic way the BMS helped me among others to pursue my trips to several conferences, my stay at MIT, to participate in seminars, and feel backed-up in the BMS-family. Thank you. I also want to thank the graduate school Methods for Discrete Structures that funded many workshops and conferences I attended. Margrit Barrett and Beate Pierchalla helped me tremendously to conquer the challenges of the real world, may it be filling out forms or providing a cooling footbath on a hot summer day. I want to thank Heike Siebert, Carsten Lange and Elmar Vogt, you were wonderful mentors.

Finally, I want to thank Martin for being Martin; Berthold for encouraging me to take the math path and persistently fighting my doubts; Franzi for the odd round around the lake when my head was overloaded; Deb for being my second mum; Ian, Rachael and Siena for making me feel at home far away from home, and my family for supporting me wherever possible.

1

Introduction

1.1 Maker–Breaker games with orientations

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the classical Maker–Breaker game $(X, \mathcal{F})$, two players, called Maker and Breaker, alternately claim elements of X, with Maker going first. X is usually called the *board*, and $\mathcal{F}$ is referred to as the *family of winning sets*. Maker wins the game if she claims all elements of some winning set; otherwise Breaker wins. A well-studied class of Maker–Breaker games are *graph games*, where the board is the edge set of a complete graph K_n, and Maker's goal is to create a graph which possesses some fixed (usually monotone) property P. A widely investigated example of such a game is the k*-clique game* (sometimes abbreviated by *clique game*) where Maker wins if and only if, by the end of the game, her graph contains a clique of size at least k. In [30], Erdős and Selfridge considered the largest value $k_{cl} = k_{cl}(n)$ such that Maker has a winning strategy in the k_{cl}-clique game. By applying their well-known Erdős-Selfridge criterion, they obtained that $k_{cl} \leqslant (2 - o(1)) \log n$ (all logarithms are in base 2, unless stated otherwise). Later, Beck [8] introduced the method of *self-improving potentials* and used his technique to determine k_{cl} exactly, namely

$$k_{cl} = \lfloor 2 \log n - 2 \log \log n + 2 \log e - 3 + o(1) \rfloor , \tag{1.1}$$

see Theorem 6.4 in [8].

1.1.1 On the random graph intuition for the tournament game

There is an interesting relation between k_{cl} and the corresponding extremal value k^*_{cl} for a game where Maker and Breaker are replaced with "random players" which select their edge in each round uniformly at random from all previously unclaimed edges: In this game, *RandomMaker* creates a random graph $G(n, m)$ with $m = \lceil \frac{1}{2}\binom{n}{2} \rceil$ edges chosen uniformly at random from all $\binom{n}{2}$ edges. It is well-known that the size of the largest clique of $G(n, m)$ is $(2 - o(1)) \log n$ asymptotically almost surely, so the threshold where the random k-clique game turns from a RandomMaker's win to a RandomBreaker's win is around $(2 - o(1)) \log n$; just like in the deterministic game, as shown by Theorem 1.1 .

For quite a few other games it has been found that the outcome of the random game is essentially the same as the outcome of the deterministic game (see, e.g., [6, 10, 42, 44, 54]). This phenomenon, known as the *random graph intuition* or the *Erdős paradigm*, was first pointed out by Chvátal and Erdős [19], and later investigated further in many papers of Beck [4, 5, 6, 7] and Bednarska and Łuczak [10].

A particular example that does not support the random graph intuition is the *biased* $(1 : b)$ non-planarity game. In that game, Maker claims one edge in each round and Breaker claims b edges in each round. Maker wins if her final graph has no planar embedding. Note that non-planarity is a monotone increasing property (every supergraph of a non-planar graph is non-planar), so that if Maker wins the $(1 : b)$ game for some Breaker bias b then Maker also wins the $(1 : b - 1)$ game. Therefore, the game has a natural *threshold bias* b_{np} where the game turns from a Maker win to a Breaker win. Hefetz, Krivelevich, Stojaković and Szabó [44] showed that in the deterministic game, the threshold bias b_{np} is asymptotically $n/2$. Note that in the random analogue, when both players choose their edges uniformly at random from all remaining edges, Makers final graph has the same distribution as $G(n, m')$, where $m' = \left\lceil \frac{1}{b+1}\binom{n}{2} \right\rceil$. But known results for random graphs imply that $G(n, m')$ is planar a.a.s. for $m' \leqslant n/2 + o(n)$; and $G(n, m')$ is non-planar a.a.s. for $m' \geqslant n/2 + o(n)$. That is, the threshold b^*_{np} from a RandomMaker win to a RandomBreaker win is asymptotically n, twice the threshold as for the deterministic game. It is largely open which suitable criteria guarantee for a given game that the random graph intuition holds.

We want to consider a variant of the k-clique game and investigate whether that variant supports the random graph intuition. A *tournament* is a directed graph where every pair of vertices is connected by a single arc (directed edge). The *k-tournament game* $\mathcal{T}(k, n)$ is played on K_n. At the beginning of the game Breaker fixes an arbitrary goal tournament T_k on k vertices. In each round, Maker and Breaker then alternately claim one unclaimed edge (as in classical graph games), and – additionally – select one of the two possible orientations for their chosen edge. If, at the end of the game, Maker's digraph contains a copy of the goal tournament T_k, she wins; otherwise, Breaker is the winner. Note that for the outcome of this particular game, the orientations of Breaker's edges are irrelevant. In the light of general *orientation games*, which we shall introduce shortly, they become meaningful though.

Let $k_t = k_t(n)$ denote the largest k such that Maker has a winning strategy in the game $\mathcal{T}(k,n)$. To get an indication for the value of k_t, Beck analyzed the *random tournament game* in which RandomMaker and RandomBreaker each choose their edge and the corresponding orientation uniformly at random. He remarks in [8] that the threshold where the random game turns from a RandomMaker's win to a RandomBreaker's win is around $(1-o(1))\log n$. We verify this observation and show in Section 2.1 that for $k \leqslant \log n - 2\log\log n$ RandomMaker wins the random tournament game a.a.s., and for $k \geqslant \log n + 1$ RandomBreaker wins the random tournament game a.a.s. Motivated by the question whether the tournament game supports the random graph intuition, Beck [8] asked to determine k_t. Since a winning strategy for Breaker in the k-clique game allows him to prevent Maker from achieving any tournament on k vertices, we have

$$k_t \leqslant k_{cl} = (2-o(1))\log n. \tag{1.2}$$

The second equation follows from (1.1). From the other side, Beck [8, p. 457] derived that

$$k_t \geqslant \left(\frac{1}{2} - o(1)\right)\log n.$$

In fact, he proved the stronger statement that for $k = (1/2 - o(1))\log n$, Maker has a strategy to occupy a graph containing a copy of *every* tournament on k vertices. The lower bound on k_t was improved by Gebauer in [41] to $k_t \geqslant (1-o(1))\log n$. We show that the upper bound is tight. This means that k_t is twice as large as the random graph intuition suggests.

Theorem 1.1.1 ([21])**.** *$k_t \geqslant 2\log n - 2\log\log n - 12 = (2-o(1))\log n$.*

This Theorem is joint work with Dennis Clemens and Heidi Gebauer, and we prove it in Chapter 2. As a direct consequence of (1.1), (1.2) and Theorem 1.1.1, the asymptotics of k_t are determined:

$$k_t = 2\log n - 2\log\log n + \Theta(1) = (2-o(1))\log n.$$

Remarkably, the upper bound in the clique-game and the lower bound in the tournament game differ only by an additive constant of 12 (for n large enough). Thus, the additional constraint that the edges have to be oriented in a particular way makes it only a little harder for Maker.

Our result seemingly refutes the random graph intuition described above. However, as a first step towards the proof of Theorem 1.1.1 we will define a suitable, *classical* graph game

$\mathcal{G}$ (with no edge-orientations involved), which has the property that every winning strategy for Maker in $\mathcal{G}$ directly gives her a winning strategy for the tournament game. The idea is as follows. Let T_k be the tournament on k vertices that Breaker chooses at the beginning, with $V(T_k) = \{u_1, \ldots, u_k\}$. First, Maker partitions the vertex set into k equally sized parts: $V(K_n) = V_1 \dot{\cup} \ldots \dot{\cup} V_k$. Then she identifies the class V_i with the vertex u_i: Whenever Maker claims an edge between V_i and V_j, she chooses the direction according to the direction of $u_i u_j$ in T_k. Therefore, her goal reduces to gaining a copy of a clique K_k, containing one vertex from each class V_i. Let us consider the corresponding random game on this reduced board, where in every round, each player claims a random unclaimed edge of the complete k-partite graph with vertex classes $V_1, \ldots, V_k$, each V_i having size rougly n/k. We shortly sketch why the threshold where this game turns from a RandomMaker's win to a RandomBreaker's win is around $(2 - o(1)) \log n$: By standard techniques it can be shown that the expected number of k-cliques in RandomMaker's graph is

$$(1 + o(1)) \left(\frac{n}{k}\right)^k 2^{-\binom{k}{2}},$$

which jumps from below one to above one at $k = 2 \log n - 2 \log \log n - 1 + o(1)$. Analogously to the proof of the concentration result for the largest clique size in the random graph $G(n, m)$ [14], it can be shown that if $\left\{ \begin{array}{l} k \leqslant (2 - o(1)) \log n \\ k \geqslant (2 - o(1)) \log n \end{array} \right\}$ then RandomMaker's graph $\left\{ \begin{array}{l} \text{contains} \\ \text{does not contain} \end{array} \right\}$ a k-clique a.a.s. Thus, from a more subtle point of view, the random graph intuition can be considered valid.

1.1.2 Orientation games

The tournament game discussed previously constitutes a bridge between classical Maker–Breaker games and orientation games. In the tournament game, the players do not merely claim edges, but also orient them. Still, for the outcome of the game, only Maker's arcs are relevant. If we go one step further and take Breaker's arcs into account for the outcome of the game, we find ourselves in the realm of orientation games.

Orientation games were studied among others by Ben-Eliezer, Krivelevich and Sudakov in [11], and we follow their notation. In orientation games, the board consists of the edges of the complete graph K_n. In the $(p : q)$ orientation game, the two players called OMaker and OBreaker, orient previously undirected edges alternately. OMaker starts, and in each round, OMaker directs between one and p edges, and then OBreaker directs between one and

q edges. At the end of the game, the final graph is a tournament on n vertices. OMaker wins the game if this tournament has some predefined property $\mathcal{P}$. Otherwise, OBreaker wins.

We only consider the case when $p = 1$. We refer to the $(1 : 1)$ game as the *unbiased* orientation game, and the $(1 : b)$-game as the *b-biased orientation game* when $b > 1$. Increasing b can only help OBreaker, so the game is *bias monotone*. Therefore, any such game has a threshold $t(n, \mathcal{P})$ such that OMaker wins the b-biased game when $b \leqslant t(n, \mathcal{P})$ and OBreaker wins the game when $b > t(n, \mathcal{P})$.

In a variant, OBreaker is required to direct exactly b edges. We refer to this variant as the *strict* b-biased orientation game, where *the strict rules apply*. Accordingly, we say the *monotone rules apply* when OBreaker is free to direct between one and b edges. Playing the exact bias in every round may be disadvantageous for OBreaker, so the existence of a threshold as for the monotone rules is not guaranteed in general. We therefore define $t^+(n, \mathcal{P})$ to be the largest value b such that OMaker has a strategy to win the strict b-biased orientation game, and $t^-(n, \mathcal{P})$ to be the largest integer such that for *every* $b \leqslant t^-(n, \mathcal{P})$, OMaker has a strategy to win the strict b-biased orientation game. Trivially, $t(n, \mathcal{P}) \leqslant t^-(n, \mathcal{P}) \leqslant t^+(n, \mathcal{P})$. The threshold bias $t(n, \mathcal{P})$ was investigated in [11] for several orientation games. However, the relation between all three parameters in question is still widely open. It is not even clear whether $t^-(n, \mathcal{P})$ and $t^+(n, \mathcal{P})$ need to be distinct values.

We consider the following orientation-version of the tournament game. Let T_k be a given tournament on k vertices. By $Or(T_k) = Or(T_k, n)$ we denote the unbiased orientation game in which OMaker aims to achieve that the final digraph contains a copy of T_k. In the spirit of the k-clique game and the k-tournament game, it is quite natural to ask for the largest integer $k_o = k_o(n)$ such that OMaker has a winning strategy for the game $Or(T_k)$ for every tournament T_k on k_o vertices.

Trivially, k_o is at least as large as the corresponding extremal number k_t for the ordinary tournament game. We show that, asymptotically, k_o is at most twice as large as k_t.

Theorem 1.1.2 ([21]). *Let n be large enough, let $k \geqslant 4 \log n + 2$ be an integer and let T_k be a tournament on k vertices. Then OBreaker has a strategy to win the game $Or(T_k, n)$.*

This is joint work with Dennis Clemens and Heidi Gebauer and we prove it in Chapter 2. Together with Theorem 1.1.1, since $k_t \leqslant k_o$ as mentioned, we therefore get

$$2 \log n(1 - o(1)) \leqslant k_o \leqslant 4 \log n(1 + o(1)).$$

The third game we consider here is the oriented-cycle game, which is an orientation game in which OMaker wins if the final tournament contains a directed cycle. Equivalently, OBreaker wins if the final tournament is a transitive tournament. Let $\mathcal{P}$ be the property of containing a directed cycle. It is easy to see that in the unbiased orientation game, OMaker wins on K_n as soon as $n \geqslant 4$ (and OBreaker wins for $n \leqslant 3$). Therefore, it is natural to provide OBreaker with more power, i.e. to introduce the aforementioned bias. The strict version of this game was studied by Alon (unpublished result) , and later by Bollobás and Szabó in [16]. They show that $t^+(n, \mathcal{P}) \geqslant \lfloor (2 - \sqrt{3})n \rfloor$. Moreover, they remark that the proof also works for the monotone rules, which implies that $t(n, \mathcal{P}) \geqslant \lfloor (2 - \sqrt{3})n \rfloor$. In [11], Ben-Eliezer, Krivelevich and Sudakov improve the lower bound and show that for $b \leqslant n/2 - 2$, OMaker has a strategy guaranteeing a cycle in the b-biased orientation game, i.e. $t(n, \mathcal{P}) \geqslant n/2 - 2$. For an upper bound, it is rather simple to see that OBreaker wins the b-biased oriented-cycle game for $b \geqslant n - 2$. A short argument is given in [11] when the monotone rules apply, which easily extends to the strict rules: During the game, let D denote the digraph of already directed edges, and let $A \subseteq [n]$ be the set of active vertices, i.e. the vertex set spanned by the complement of D. Then, no matter which edge (v, w) OMaker directs, OBreaker can always maintain the situation that $D[A]$ forms a directed star (he reduces A by at least one vertex in each round). We refer to this as the *trivial strategy in* K_{b+2}. Therefore, $t^+(n, \mathcal{P}) \leqslant n - 3$. Bollobás and Szabó conjectured that this upper bound is tight for the *strict* b-biased oriented-cycle game. We give a strategy for OBreaker when $b \geqslant 5n/6 + 2$ in the *monotone* b-biased oriented-cycle game.

Theorem 1.1.3 ([22]). *For $b \geqslant 5n/6 + 2$, OBreaker has a strategy to prevent OMaker from closing a directed cycle in the b-biased orientation game played on K_n. In particular, $t(n, \mathcal{P}) \leqslant 5n/6 + 1$.*

Furthermore, we adjust our strategy to the strict rules and show the following.

Theorem 1.1.4 ([22]). *Let $0 < c < 1$ be a constant. Then there exists $n_0 = n_0(c)$ such that for all $n \geqslant n_0$ and $b \geqslant n - c\sqrt{n}$, OBreaker has a strategy to prevent OMaker from closing a directed cycle in the strict b-biased orientation game played on K_n. In particular, $t^+(n, \mathcal{P}) \leqslant n - c\sqrt{n} - 1$.*

These two theorems are joint work with Dennis Clemens and we prove them in Chapter 3. We want to remark that in the latter theorem, the only requirement needed for $n_0(c)$ is that $\lfloor \sqrt{b} \rfloor \geqslant n - b + 1$. So for $c = \frac{1}{2}$ for example, this is true for all $n \geqslant 10$. Hence, Theorem 1.1.4 refutes the above conjecture of Bollobás and Szabó as soon as $n - \frac{1}{2}\sqrt{n} \leqslant n - 3$, which holds for $n \geqslant 36$.

1.2 Ramsey-minimal graphs

A graph G is *Ramsey* for a graph H (or H-*Ramsey*), denoted by $G \to H$, if any two-colouring of the edges of G contains a monochromatic copy of H. The fact that for every graph H there is a graph G such that G is H-Ramsey was first proved by Ramsey [49] in 1930 and rediscovered independently by Erdős and Szekeres a few years later [34]. Ramsey theory is currently one of the most active areas of combinatorics with connections to number theory, geometry, analysis, logic, and computer science.

1.2.1 What is Ramsey-eqivalent to the clique?

A fundamental problem in graph Ramsey theory is to understand the graphs G for which G is K_k-Ramsey, where K_k denotes the complete graph on k vertices. The Ramsey number $r(H)$ is the minimum number of vertices of a graph G which is H-Ramsey. The most famous question in this area is that of estimating the Ramsey number $R(k) := r(K_k)$. Classical results of Erdős [31] and Erdős and Szekeres [34] show that $2^{k/2} \leqslant R(k) \leqslant \binom{2k-2}{k-1} \leqslant 2^{2k}$. There have been several improvements on these bounds. A now standard application of the Lovász Local Lemma (proven by Spencer [53]) gives the best known lower bound of $\frac{\sqrt{2}}{e} k 2^{k/2}(1 + o(1)) \leqslant R(k)$. From the other side, Conlon [23] showed that for every $s > 0$, there is a constant C_s such that $R(k) \leqslant \frac{C_s}{k^s}\binom{2k-2}{k-1}$. Despite much attention, the constant factors in the above exponents remain the same though. Given these difficulties, the field has naturally stretched in different directions. Many foundational results were proved in the 1970s which showed the depth and breadth of graph Ramsey theory. For instance, the size-Ramsey number $\hat{r}(H)$, which is the minimum number of edges of a graph G which is H-Ramsey, was introduced by Erdős, Faudree, Rousseau and Schelp [32], and studied by many others extensively (see [35] for a survey). Another famous theorem of Nešetřil and Rödl [47] states that for every graph H there is a graph G with the same clique number as H such that $G \to H$.

Szabó, Zumstein, and Zürcher [55] defined two graphs H and H' to be *Ramsey-equivalent* if for every graph G, G is H-Ramsey if and only if G is H'-Ramsey. The result of Nešetřil and Rödl [47] above implies that any graph H which is Ramsey-equivalent to the clique K_k must contain a copy of K_k. We are interested in the problem of determining which graphs are Ramsey-equivalent to K_k. In other words, knowing that G is Ramsey for K_k, what additional monochromatic subgraphs must occur in any two-colouring of the edges of G?

In [55] it was conjectured that for large enough k, the clique K_k is Ramsey-equivalent to $K_k \cdot K_2$, the graph on $k+1$ vertices consisting of K_k with a pendant edge. We settle this conjecture in the negative, showing that, for all k, the graphs K_k and $K_k \cdot K_2$ are not Ramsey-equivalent. Together with the above discussion, this implies the following theorem.

Theorem 1.2.1 ([39]). *Any graph which is Ramsey-equivalent to the clique K_k must be the disjoint union of K_k and a graph of smaller clique number.*

It is therefore natural to study the following function. Let $\mu(k,t)$ be the maximum m such that K_k and $K_k + m \cdot K_t$ are Ramsey-equivalent, where $K_k + m \cdot K_t$ denotes the disjoint union of a K_k and m copies of K_t. It is easy to see [55] that $\mu(k,k) = 0$ and $\mu(k,1) = R(k) - k$. For $t \leqslant k-2$ Szabó et al. [55] proved the lower bound

$$\mu(k,t) \geqslant \frac{R(k, k-t+1) - 2(k-t)}{2t}, \tag{1.3}$$

where $R(k,s)$ is the *asymmetric Ramsey number* denoting the minimum n such that every red-blue edge-colouring of K_n contains a monochromatic red copy of K_k or a monochromatic blue copy of K_s. We prove the following theorem which, together with (1.3), determines $\mu(k,t)$ up to roughly a factor 2.

Theorem 1.2.2 ([39]). *For $k > t \geqslant 3$,*

$$\mu(k,t) \leqslant \frac{R(k, k-t+1) - 1}{t}.$$

Our Theorem misses the case when $t = 2$. However, it is easy to see that $\mu(k,2) \leqslant \frac{1}{2}(R(k) - k)$: If $m > \frac{1}{2}(R(k) - k)$, consider K_n where $n = R(k)$. By definition, $K_n \to K_k$, however K_n has too few vertices to accommodate a $K_k + m \cdot K_2$. Note that $R(k) \leqslant 2R(k, k-1)$, a consequence of the result of Erdős and Szekeres in [34]. So this trivial upper bound is roughly a factor two apart from the upper bound in Theorem 1.2.2; and roughly a factor of four away from the lower bound.

A graph G is *Ramsey-minimal with respect to H* (or *H-minimal)* if G is H-Ramsey but no proper subgraph of G is H-Ramsey. We denote the class of all H-minimal graphs by $\mathcal{M}(H)$. Note that G is H-Ramsey if and only if G contains an H-minimal graph, so determining the H-Ramsey graphs reduces to determining the H-minimal graphs. Also, two graphs H and H' are Ramsey-equivalent if and only if $\mathcal{M}(H) = \mathcal{M}(H')$.

Understanding the the class $\mathcal{M}(H)$ and properties of the graphs therein is an important problem in graph Ramsey theory. For example, the minimum number of vertices of a graph in

$\mathcal{M}(H)$ is precisely the Ramsey number $r(H)$, and the minimum number of edges of a graph in $\mathcal{M}(H)$ is precisely the size-Ramsey number $\hat{r}(H)$. In [18], Burr, Erdős and Lovász showed that $|\mathcal{M}(K_k)|$ is infinite. This was strengthened in [17] by Burr, Nešetřil and Rödl who showed that there are at least $c^{n \log n}$ graphs on n vertices which are Ramsey-minimal for K_k, where $c = c(k) > 1$ is a constant and n is large enough. Finally, Rödl and Siggers [50] showed that the latter statement is true if we replace $c^{n \log n}$ by c^{n^2}. Note that the order of magnitude of the exponent is best possible since there are at most 2^{n^2} graphs on n vertices.

Another parameter of interest is $s(H)$, the smallest minimum degree of an H-minimal graph. That is,

$$s(H) := \min_{G \in \mathcal{M}(H)} \delta(G),$$

where $\delta(G)$ is the minimum degree of G. It is a simple exercise to show [40] that for every graph H, we have $2\delta(H) - 1 \leqslant s(H) \leqslant r(H) - 1$. Rather surprisingly, the upper bound is far from optimal, at least for cliques. Indeed, Burr, Erdős, and Lovász [18] proved that $s(K_k) = (k-1)^2$. This is quite notable, as the simple upper bound mentioned above is exponential in k.

Szabó, Zumstein, and Zürcher [55] proved that the lower bound is tight for a large class of bipartite graphs, including paths, even cycles and trees. On the other hand, the authors proved in the same paper that $s(K_k \cdot K_2) \geqslant k - 1$. We prove the following theorem, showing that this lower bound is sharp.

Theorem 1.2.3 ([39])**.** *For all $k \geqslant 2$, $s(K_k \cdot K_2) = k - 1$.*

Note that Theorem 1.2.3 implies that K_k and $K_k \cdot K_2$ are not Ramsey-equivalent. Indeed, for $k = 2$ this is trivial, and for $k \geqslant 3$ we have $(k-1)^2 = s(K_k) > s(K_k \cdot K_2) = k - 1$. Hence, Theorem 1.2.1 is a corollary of Theorem 1.2.3.

1.2.2 Ramsey-minimal graphs for r colours

An interesting direction of research in graph Ramsey theory is to generalize the above notions to more than two colours and to obtain estimates on various parameters for the corresponding graphs. A graph G is *r-Ramsey for a graph H*, denoted by $G \to (H)_r$, if any r-colouring of the edges of G contains a monochromatic copy of H in some colour. It follows recursively from the result for two colours that for every graph H there exists a graph G that is r-Ramsey for H. Similarly as above, we denote by $R_r(k)$ the r-Ramsey number of the clique K_k on k vertices, the smallest natural number n such that any edge-colouring of the edges of K_n

contains a monochromatic copy of K_k. If the known bounds for the case when $r = 2$ have been already unsatisfactorily far apart, all we can hope for $r \geqslant 3$ is fishing in murky waters – or excitement of how much depth there is left to discover. We list some of the developments here and refer the reader to an updated survey by Radziszowski [48]. Even for $k = 3$, the triangle case, the best known upper bound, proven by Wan [56], is of order $\Theta(r!) = \exp[\Theta(r \ln r)]$, whereas from the other side, $R_r(3) \geqslant c^r$, for some fixed constant $3 < c < 3.2$, is the best known lower bound, see Xiadong et. al [58]. A general upper bound follows from a recursive formula given by Greenwood and Gleason [43], who show that $R_r(k) \leqslant 2 + rR(k, \ldots, k, k-1)$, where $R(k, \ldots, k, k-1)$ denotes the smallest natural number n such that in any r-colouring of the edges of K_n there exists either a monochromatic K_k in one of the colours $1, \ldots, r-1$ or a monochromatic K_{k-1} in colour r. The probabilistic method that Erdős used to prove the lower bound $2^{k/2} \leqslant R(k)$ generalizes to more than two colours and yields $R_r(k) \geqslant r^{k/2}$, as was observed by Chvátal and Harary in [20]. If k is constant the bound $R_r(k) \geqslant c(k)(2k-3)^r$ by Abbott and Hanson [1] gives a better estimate.

Again, we can study the graphs which are minimal with respect to being r-Ramsey. We call a graph G *r-Ramsey-minimal for H* (or *r-minimal for H*), if $G \to (H)_r$, but $G' \nrightarrow (H)_r$ for any proper subgraph $G' \subsetneq G$. Let then $\mathcal{M}_r(H)$ denote the class of all graphs G that are r-Ramsey-minimal with respect to H. As noted above, the set $\mathcal{M}_r(H)$ is nonempty for any H. Moreover, Rödl and Siggers [50] showed that for all $k \geqslant 3$ and $r \geqslant 2$ there exists a constant $c = c(r, k) > 1$ such that for n large enough there are at least c^{n^2} graphs G on at most n vertices that are r-Ramsey for the clique K_k. In particular, $|\mathcal{M}_r(K_k)|$ is infinite.

The following notion is a natural generalization of the smallest minimum degree of Ramsey-minimal graphs $s(H)$. Define $s_r(H)$ by

$$s_r(H) := \min_{G \in \mathcal{M}_r(H)} \delta(G),$$

the minimal minimum degree of r-Ramsey-minimal graphs.

We are interested in this quantity when H is a clique of fixed size, and we study how the behaviour of $s_r(K_k)$ changes as r grows. When H is the triangle, we determine the asymptotic behaviour of $s_r(K_3)$ up to a factor of roughly $\log r$.

Theorem 1.2.4 ([38]). *There exist constants $c, C > 0$ and $r_0 \in \mathbb{N}$ such that for all $r \geqslant r_0$, $cr^2 \ln r \leqslant s_r(K_3) \leqslant Cr^2 \ln^2 r$.*

It is a fairly simple observation that $s_r(K_k) > s_{r-1}(K_k)$, and it will follow from a stronger statement (cf. Lemma 5.1.3). On the other hand, fixing r, it is not clear that $s_r(K_k)$ is increasing in k as well: Removing a vertex of minimum degree of some r-minimal graph G leaves a graph G' which is not r-Ramsey for K_k; however, $G' \to (K_{k-1})_r$ and moreover, it is far from being r-minimal for K_{k-1}. Therefore, the lower bound on $s_r(K_3)$ does not necessarily imply that also $s_r(K_k) = \Omega\left(r^2 \log r\right)$ is true. We prove a lower bound on r which is super-quadratic and show that, under the condition of k being a constant, $s_r(K_k)$ is at most a polylog(r)-factor above the lower bound.

Theorem 1.2.5 ([38])**.** *For all $k \geqslant 4$, there exist constants $c = c(k), C = C(k) > 0$ and $r_0 \in \mathbb{N}$ such that for all $r \geqslant r_0$,*

$$c\,r^2 \sqrt{\frac{\ln r}{\ln \ln r}} \leqslant s_r(K_k) \leqslant C\,r^2\,(2\ln r)^{8(k-1)^2}.$$

The proof of the upper bounds in Theorem 1.2.4 and Theorem 1.2.5 are of asymptotic nature and require r to be rather large. Moreover, the exponent of the $(\log r)$-factor in the latter upper bound depends on the size of the clique. Therefore, we also prove a seemingly weaker upper bound of $s_r(K_k)$ which is polynomial both in r and in k, but which is applicable for small values of r and k.

Theorem 1.2.6 ([38])**.** *For $k, r \geqslant 3$, $s_r(K_k) \leqslant 8(k-1)^6 r^3$.*

The results of this Section are joint work with Jacob Fox, Andrey Grinshpun, Yury Person and Tibor Szabó, we prove them in Chapter 5.

2

Tournament games

In this chapter, we analyse the tournament game and the orientation tournament game introduced in Section 1.1.1 and 1.1.2. Recall that both games, $\mathcal{T}(k,n)$ and $Or(T_k,n)$ are played on the edge set of K_n. In the tournament game $\mathcal{T}(k,n)$, Breaker fixes a goal tournament T_k on k vertices at the beginning. In the orientation tournament game, the goal tournament T_k is given. Then, in both games, the two players alternately claim and direct one undirected edge. In $\mathcal{T}(k,n)$, Maker wins if her digraph contains a copy of T_k at the end of the game. Otherwise, Breaker wins. In $Or(T_k,n)$, OMaker wins if the tournament on n vertices (including OBreaker's arcs) contains a copy of T_k. Otherwise, OBreaker wins.

First, we study the random tournament game in which both players choose and direct their edge in each round uniformly at random from all remaining edges. We show that the threshold when the game turns from a RandomMaker win to a RandomBreaker win is asymptotically equal to $\log n$. In Section 2.2, we prove Theorem 1.1.1, and in Section 2.3, we prove Theorem 1.1.2. We close this chapter with a discussion on open problems related to both games.

2.1 The probabilistic analysis for the tournament game

In this section, we investigate the tournament game when the two intelligent players Maker and Breaker are replaced by two random players RandomMaker and RandomBreaker. At the beginning of the game, RandomBreaker chooses a goal tournament T_k on vertex set $[k]$ (the actual choice of T_k is irrelevant). Then, RandomMaker and RandomBreaker alternately each choose their edge among the unclaimed edges on K_n and the corresponding orientation uniformly at random. At the end of the game, RandomMaker's graph is a random digraph on $m = \lceil \frac{1}{2}\binom{n}{2} \rceil$ edges, each having one of the two possible directions with probablitiy 1/2. Let us denote by $\mathcal{D}(n,m)$ the following random digraph model: A digraph D is drawn from $\mathcal{D}(n,m)$, denoted by $D \sim \mathcal{D}(n,m)$, if D is a digraph on vertex set $[n]$ with m (undirected) edges chosen uniformly at random from all $\binom{n}{2}$ pairs, each edge given one of its two orientations with probability $1/2$. Intuitively, if we restrict our view to one particular edge, it belongs to Maker's graph with probability roughly $1/2$, and with probability $1/2$ it is directed "in the right" direction. In the binomial random graph $G(n,1/4)$, in which every possible edge occurs with probability $1/4$, the size of the largest clique is roughly $2\log_4 n = \log n$. This vague idea is made precise in the following theorem.

Theorem 2.1.1. *For $k \in \mathbb{N}$, let T_k be a given tournament on k vertices. Let $m = \lceil \frac{1}{2}\binom{n}{2} \rceil$ and let $D \sim \mathcal{D}(n,m)$.*

(i) If $k \leqslant (1-o(1))\log n$, then D contains a (labelled) copy of T_k a.a.s.

(ii) If $k \geqslant \log n + 2$, then D contains no copy of T_k a.a.s.

The proof of this theorem is a standard application of the first and second moment method.

Let $n \in \mathbb{N}$, and let $1 \leqslant k \leqslant n$. Further, let T_k be the tournament on vertex set $[k]$, and let $D \sim \mathcal{D}(n,m)$. A *labelled copy of T_k in D* is an injective function $\varphi : [k] \to [n]$ such that for all $i, j \in [k]$, $i \neq j$,

$$(i,j) \in E(T_k) \text{ if and only if } (\varphi(i), \varphi(j)) \in E(D).$$

We count the number of labelled copies of T_k in D. For an injection $\varphi : [k] \to [n]$, let X_φ denote the indicator random variable of the event that φ is a copy of T_k in D, and let $X = \sum_\varphi X_\varphi$ be the number of (labelled) copies of T_k in D, where the sum runs over all injective maps $[k] \to [n]$. Let $p := \mathbb{P}(X_\varphi)$ for some injection $\varphi : [k] \to [n]$.

Claim 2.1.2. *For $k = o(n)$,*

$$\left(1 - O\left(\frac{k^4}{n^2}\right)\right) 4^{-\binom{k}{2}} \leqslant p \leqslant 4^{-\binom{k}{2}}$$

Proof. Recall that $m = \lceil \frac{1}{2}\binom{n}{2} \rceil$. We assume that $\binom{n}{2}$ is even and omit ceiling signs from now on for clarity of presentation. The case when $\binom{n}{2}$ is odd is analogous. Let $\varphi = (n_1, \ldots, n_k)$, where $n_i = \varphi(i)$. Then p is the probability that D induces an ordered copy of T_k on vertices $n_1, \ldots, n_k$. For D to induce an ordered copy of T_k, all the (unordered) edges $n_i n_j$ need to be chosen to belong to D, and all pairs need to be given the correct orientation, according to the orientation of the edge ij in T_k. Therefore,

$$p = 2^{-\binom{k}{2}} \cdot \frac{\binom{\binom{n}{2}-\binom{k}{2}}{m-\binom{k}{2}}}{\binom{\binom{n}{2}}{m}} = 2^{-\binom{k}{2}} \cdot \frac{(m)_{\binom{k}{2}}}{\binom{n}{2}_{\binom{k}{2}}},$$

where $(a)_b$ denotes the falling factorial. Now,

$$\begin{aligned}(m)_{\binom{k}{2}} &= \left(\frac{1}{2}\binom{n}{2}\right)_{\binom{k}{2}} \\ &= \frac{1}{2}\binom{n}{2} \cdot \left(\frac{1}{2}\binom{n}{2} - 1\right) \cdot \ldots \cdot \left(\frac{1}{2}\binom{n}{2} - \binom{k}{2} + 1\right) \\ &= 2^{-\binom{k}{2}} \cdot \binom{n}{2} \cdot \left(\binom{n}{2} - 2\right) \cdot \ldots \cdot \left(\binom{n}{2} - 2\binom{k}{2} + 2\right).\end{aligned}$$

For the upper bound, note that therefore

$$p = 4^{-\binom{k}{2}} \cdot \frac{\binom{n}{2} \cdot \left(\binom{n}{2} - 2\right) \cdot \ldots \cdot \left(\binom{n}{2} - 2\binom{k}{2} + 2\right)}{\binom{n}{2} \cdot \left(\binom{n}{2} - 1\right) \cdot \ldots \cdot \left(\binom{n}{2} - \binom{k}{2} + 1\right)} \leqslant 4^{-\binom{k}{2}}.$$

For the lower bound, we estimate

$$\begin{aligned}p &= 4^{-\binom{k}{2}} \cdot \frac{\binom{n}{2} \cdot \left(\binom{n}{2} - 2\right) \cdot \ldots \cdot \left(\binom{n}{2} - 2\binom{k}{2} + 2\right)}{\binom{n}{2} \cdot \left(\binom{n}{2} - 1\right) \cdot \ldots \cdot \left(\binom{n}{2} - \binom{k}{2} + 1\right)} \\ &\geqslant 4^{-\binom{k}{2}} \left(\frac{\binom{n}{2} - 2\binom{k}{2} + 2}{\binom{n}{2} - \binom{k}{2} + 1}\right)^{\binom{k}{2}} \\ &= 4^{-\binom{k}{2}} \left(1 - \frac{\binom{k}{2} - 1}{\binom{n}{2} - \binom{k}{2} + 1}\right)^{\binom{k}{2}}.\end{aligned}$$

Now, for $k = o(n)$, $x := \frac{\binom{k}{2}-1}{\binom{n}{2}-\binom{k}{2}+1} \leqslant 1$ for n large enough, and so by Bernoulli's inequality $(1-x)^{\binom{k}{2}} \geqslant 1 - x\binom{k}{2}$, i.e.

$$\begin{aligned}p &\geqslant 4^{-\binom{k}{2}} \left(1 - \frac{\binom{k}{2} - 1}{\binom{n}{2} - \binom{k}{2} + 1}\right)^{\binom{k}{2}} \\ &\geqslant 4^{-\binom{k}{2}} \left(1 - \frac{\binom{k}{2}^2}{\binom{n}{2} - \binom{k}{2} + 1}\right) \\ &= 4^{-\binom{k}{2}} \left(1 - O\left(\frac{k^4}{n^2}\right)\right)\end{aligned}$$

□

The following is immediate.

Corollary 2.1.3. *For $k^2 = o(n)$, $\mathbb{E}(X) = p \cdot (n)_k = 4^{-\binom{k}{2}} \cdot (n)_k \cdot (1 - o(1))$.*

This observation enables us to prove part (*ii*) of the main theorem of the section.

Proof of Theorem 2.1.1 (ii). For $k = \log n + 2$, by the usual first moment argument,

$$\begin{aligned}&\mathbb{P}(D \text{ contains a (labelled) copy of } T_k)\\ &\leqslant \mathbb{E}(X) \leqslant n^k 4^{-\binom{k}{2}} = 2^{k(\log n - (k-1))} = 2^{-k} \to 0,\end{aligned}$$

for any predefined tournament T_k. That is, a.a.s. D does not contain T_k when $k = \log n + 2$, and therefore, D does not contain any given tournament on more than k vertices almost surely, which proves the claim. □

Proof of Theorem 2.1.1 (i). For part (*i*) of the theorem we want to apply Chebyshev's Inequality. So we need to consider the variance

$$\mathbb{V}(X) = \sum_{\varphi,\psi} \mathbb{E}(X_\varphi X_\psi) - \mathbb{E}(X_\varphi)\mathbb{E}(X_\psi), \tag{2.1}$$

where the sum runs over all injective maps $\varphi, \psi : [k] \to [n]$. To bound the sum, fix $0 \leqslant i \leqslant k$, and fix two injections $\varphi, \psi : [k] \to [n]$ such that their images intersect in i vertices, that is $|\varphi([k]) \cap \psi([k])| = i$. Since X_φ and X_ψ are indicator random variables,

$$\mathbb{E}(X_\varphi X_\psi) - \mathbb{E}(X_\varphi)\mathbb{E}(X_\psi) = \mathbb{P}(X_\varphi \wedge X_\psi) - \mathbb{P}(X_\varphi)\mathbb{P}(X_\psi). \tag{2.2}$$

We bound the terms $\mathbb{P}(X_\varphi \wedge X_\psi)$ depending on the size of the intersection $\varphi([k]) \cap \psi([k])$ of the two potential copies in the following two auxiliary claims.

Claim 2.1.4. *With the notation as above we have for $|\varphi([k]) \cap \psi([k])| \leqslant 1$ that $\mathbb{P}(X_\varphi \wedge X_\psi) \leqslant 2^{-4\binom{k}{2}}$.*

Proof. Let φ and ψ be such that $|\varphi([k]) \cap \psi([k])| \leqslant 1$. Then the two vertex sets $\varphi([k])$ and $\psi([k])$ do not share a common edge. The events of inducing a copy of T_k each are not independent though. However, they are "close enough" to being independent. In order for both injections to induce a copy of T_k each, all of the $2\binom{k}{2}$ edges need to be in D, and need to be given the correct orientation according to the orientation in T_k. We calculate

$$\mathbb{P}(X_\varphi \wedge X_\psi) = 2^{-2\binom{k}{2}} \frac{\binom{\binom{n}{2} - 2\binom{k}{2}}{m - 2\binom{k}{2}}}{\binom{\binom{n}{2}}{m}} = 2^{-2\binom{k}{2}} \frac{(m)_{2\binom{k}{2}}}{\binom{n}{2}_{2\binom{k}{2}}} \leqslant 2^{-4\binom{k}{2}},$$

since, similar as in the proof of Claim 2.1.2,

$$\begin{aligned}(m)_{2\binom{k}{2}} &= \left(\frac{1}{2}\binom{n}{2}\right)_{2\binom{k}{2}} \\ &= \frac{1}{2}\binom{n}{2} \cdot \left(\frac{1}{2}\binom{n}{2} - 1\right) \cdot \ldots \cdot \left(\frac{1}{2}\binom{n}{2} - 2\binom{k}{2} + 1\right) \\ &= 2^{-2\binom{k}{2}} \cdot \binom{n}{2} \cdot \left(\binom{n}{2} - 2\right) \cdot \ldots \cdot \left(\binom{n}{2} - 4\binom{k}{2} + 2\right) \\ &\leqslant 2^{-2\binom{k}{2}} \cdot \binom{n}{2}_{2\binom{k}{2}}. \end{aligned}$$ □

Claim 2.1.5. *With the notation as above, for $2 \leqslant i \leqslant k$ and for $|\varphi([k]) \cap \psi([k])| = i$, we have that $\mathbb{P}(X_\varphi \wedge X_\psi) \leqslant 2^{-4\binom{k}{2}+2\binom{i}{2}}$.*

Proof. When $|\varphi([k]) \cap \psi([k])| \geqslant 2$, then the orientations given by T_k do not need to agree on $\varphi([k]) \cap \psi([k])$. In this case, $\mathbb{P}(X_\varphi \wedge X_\psi) = 0$ and the claim holds trivially. If the orientations in the intersection agree, then

$$\begin{aligned}\mathbb{P}(X_\varphi \wedge X_\psi) &= 2^{-\left(2\binom{k}{2}-\binom{i}{2}\right)} \frac{\binom{\binom{n}{2} - 2\binom{k}{2} + \binom{i}{2}}{m - 2\binom{k}{2} + \binom{i}{2}}}{\binom{\binom{n}{2}}{m}} \\ &= 2^{-\left(2\binom{k}{2}-\binom{i}{2}\right)} \frac{(m)_{2\binom{k}{2}-\binom{i}{2}}}{\binom{n}{2}_{2\binom{k}{2}-\binom{i}{2}}} \\ &\leqslant 2^{-4\binom{k}{2}+2\binom{i}{2}}, \end{aligned}$$

similar to the proof of the previous claim. □

Having Claim 2.1.4 and Claim 2.1.5 in the back of our mind, we set

$$q_0 = q_1 := 2^{-4\binom{k}{2}} \quad \text{and} \quad q_i := 2^{-4\binom{k}{2}+2\binom{i}{2}} \text{ for } 2 \leqslant i \leqslant k.$$

Turning back to the expression of the variance (2.1), we need to count the number of injections $\varphi, \psi : [k] \to [n]$. Fix an injection $\varphi : [k] \to [n]$, and fix $0 \leqslant i \leqslant k$. Suppose the (image of the) second injection ψ meets the (image of the) first injection φ in i vertices, i.e. $|\varphi([k]) \cap \psi([k])| = i$. Then there are $\binom{k}{i}$ possibilities to choose the i intersection vertices in $\varphi([k])$. Call this set of i vertices S. Now, there are $(k)_i$ possibilities to choose the preimage $S^{-1} := \psi^{-1}(S) \subseteq [k]$. Finally, there are $(n-k)_{k-i}$ possibilities to choose the vertices $\psi([k]) \setminus S^{-1}$ in the subset

$[n] \setminus \varphi([k])$. Therefore, by (2.1), (2.2), Claim 2.1.4 and Claim 2.1.5,

$$\begin{aligned} \mathbb{V}(X) &= \sum_{\varphi,\psi} \mathbb{E}(X_\varphi X_\psi) - \mathbb{E}(X_\varphi)\mathbb{E}(X_\psi) \\ &\leqslant (n)_k \sum_{i=0}^{k} \binom{k}{i} (k)_i (n-k)_{k-i}(q_i - p^2), \end{aligned} \tag{2.3}$$

where $p = \mathbb{P}(X_\varphi)$ as before is independent of the actual choice of φ, and q_i was defined above. In order to apply Chebyshev's Inequality, we need to show that $\mathbb{V}(X) = o(\mathbb{E}(X)^2)$. Recall that by Corollary 2.1.3, $\mathbb{E}(X) = (n)_k 4^{-\binom{k}{2}}(1-o(1))$. In the above sum, we first bound the terms for $i = 0, 1$. By definition of $q_0 = q_1$ and by Claim 2.1.2,

$$q_0 - p^2 \leqslant 2^{-4\binom{k}{2}} \left(1 - \left(1 - O\left(\frac{k^4}{n^2}\right)\right)^2\right) \leqslant 2^{-4\binom{k}{2}} \cdot O\left(\frac{k^4}{n^2}\right).$$

Therefore,

$$\frac{(n)_k\big((n-k)_k + k^2(n-k)_{k-1}\big)(q_0 - p^2)}{\mathbb{E}(X)^2} \leqslant k^2 \cdot O\left(\frac{k^4}{n^2}\right) = o(1),$$

whenever $k^3 = o(n)$. That is, by (2.3), for $k^3 = o(n)$,

$$\begin{aligned} \frac{\mathbb{V}(X)}{\mathbb{E}(X)^2} &\leqslant \frac{(n)_k}{\mathbb{E}(X)^2} \sum_{i=0}^{k} \binom{k}{i} (k)_i (n-k)_{k-i}(q_i - p^2), \\ &\leqslant o(1) + \frac{(n)_k}{\mathbb{E}(X)^2} \sum_{i=2}^{k} \binom{k}{i} (k)_i (n-k)_{k-i} q_i \\ &= o(1) + (1+o(1)) \sum_{i=2}^{k} \frac{\binom{k}{i}(k)_i(n-k)_{k-i}}{(n)_k} 2^{2\binom{i}{2}}, \end{aligned} \tag{2.4}$$

by definition of q_i for $2 \leqslant i \leqslant k$ and by Corollary 2.1.3. For $2 \leqslant i \leqslant k$, set

$$g(i) := \frac{\binom{k}{i}(k)_i(n-k)_{k-i}}{(n)_k} 4^{\binom{i}{2}}.$$

We want to bound $\sum_{i=2}^{k} g(i)$ by the boundary values of g, and therefore consider

$$f(i) := \frac{g(i+1)}{g(i)} = \frac{(k-i)^2 4^i}{(i+1)(n-2k+i+1)}$$

for $2 \leqslant i \leqslant k-1$. Considering the derivative $f'(x)$ of the function f with domain $[2, k-1]$, we see that f is strictly increasing on $[2, k-2]$:

$$f'(x) = \frac{4^x(k-x)}{(x+1)(n-2k+x+1)} \underbrace{\left(\log 4 - \frac{2}{k-x} - \frac{1}{x+1} - \frac{1}{n-2k+x+1}\right)}_{\leqslant 0 \text{ for } 2 \leqslant x \leqslant k-2 \text{ and } n \text{ large enough}} < 0.$$

This implies that there is some $2 \leqslant i_0 \leqslant k-2$ such that

$$\text{for all } 2 \leqslant i \leqslant i_0 : \; g(i+1) < g(i)$$
$$\text{and for all } i_0 < i \leqslant k-2 : \; g(i+1) > g(i).$$

It follows that $\sum_{i=2}^{k} g(i) \leqslant k\big(g(2) + g(k-1)\big) + g(k)$. To finish the proof of Theorem 2.1.1 (i), we claim that for $k \leqslant \log n - 2\log\log n$, the upper bound in the previous line converges to zero. We calculate

$$\begin{aligned}
k\,g(2) &= \frac{4k\binom{k}{2}(k)_2(n-k)_{k-2}}{(n)_k} = O\left(\frac{k^5}{n^2}\right) = o(1) \\
&\qquad\qquad\text{for } k^5 = o(n^2), \\
g(k) &= \frac{k!}{(n)_k}\,4^{\binom{k}{2}} \leqslant 4^{\binom{k}{2}}\left(\frac{k}{n}\right)^k = o(1) \\
&\qquad\qquad\text{for } k \leqslant \log n - 2\log\log n, \\
k\,g(k-1) &= \frac{k^2 k!(n-k)}{(n)_k}\,4^{\binom{k-1}{2}} \leqslant \frac{2k^k}{(n-k)^{k-1}}\,4^{\binom{k-1}{2}} = o(1) \\
&\qquad\qquad\text{for } k \leqslant \log n - 2\log\log n.
\end{aligned}$$

We just proved that for $k \leqslant \log n - 2\log\log n$, $\mathbb{V}(X) = o\left(\mathbb{E}(X)^2\right)$. It follows with Chebyshev's Inequality (see e.g. Theorem 4.3.1 in [3]) that $X > 0$ almost surely, i.e. there exists a (labelled) copy of T_k in D with probability tending to one as n tends to infinity. □

2.2 A strategy for Maker in the tournament game

In this section, we prove Theorem 1.1.1, i.e. we show that for n large enough and $k \leqslant 2\log n - 2\log\log n - 12$, Maker has a strategy to win the tournament game $\mathcal{T}(k, n)$.

Proof of Theorem 1.1.1. Let $n \in \mathbb{N}$ be large enough, and let k be the largest integer such that $n \geqslant k2^{(k+9)/2}$. Note that by definition, $n < (k+1)2^{(k+10)/2}$, so $k \geqslant 2\log n - 2\log\log n - 12$. For clarity of presentation, we assume from now on that $n = k2^{(k+9)/2}$.

Let T_k be the tournament on k vertices that Breaker chooses at the beginning, with $V(T_k) = \{u_1, \ldots, u_k\}$. First, Maker partitions the vertex set into k equally sized parts: $V(K_n) = V_1 \dot\cup \ldots \dot\cup V_k$. Then she identifies the class V_i with the vertex u_i: Whenever Maker claims an edge between V_i and V_j, she chooses the direction according to the direction of $\{u_i, u_j\}$ in T_k. Therefore, her goal reduces to gaining a copy of a clique K_k, containing one vertex from each class V_i. Hence, she plays on the reduced board

$$X := \Big\{\{v_i, v_j\} : v_i \in V_i, v_j \in V_j, i \neq j\Big\}.$$

Our goal is to prove that she wins the classical Maker–Breaker game $(X, \mathcal{F})$ where $\mathcal{F}$ consists of all edge sets of k-cliques in the reduced k-partite graph:

$$\mathcal{F} := \left\{ \binom{S}{2} : S \subseteq V_1 \,\dot\cup \ldots \dot\cup\, V_k \text{ such that } |S \cap V_i| = 1, \text{ for every } 1 \leqslant i \leqslant k \right\}.$$

To this end, we will use a general criterion for Maker's win from [8]. Let us introduce the necessary notation first. For $p \in \mathbb{N}$, we define the set of *p-clusters* of $\mathcal{F}$ as

$$\mathcal{F}_2^p := \left\{ \bigcup_{1 \leqslant i \leqslant p} E_i \, : \{E_1, \ldots, E_p\} \in \binom{\mathcal{F}}{p}, \left| \bigcap_{1 \leqslant i \leqslant p} E_i \right| \geqslant 2 \right\}.$$

That is, $\mathcal{F}_2^p$ is the family consisting of all those subsets of X which can be represented as the union of p distinct winning sets sharing at least two elements of X. Furthermore, for any family $\mathcal{H}$ of finite sets, we consider the well-known potential function used in the Erdős-Selfridge criterion

$$T(\mathcal{H}) := \sum_{H \in \mathcal{H}} 2^{-|H|}.$$

According to Beck [8], we have the following sufficient condition for Maker's win.

Theorem 2.2.1 (Advanced Weak Win Criterion, [8])**.** *Maker has a winning strategy for the Maker–Breaker game $(X, \mathcal{F})$, if there exists an integer $p \geqslant 2$ such that*

$$\frac{T(\mathcal{F})}{|X|} > p + 4p\Big(T(\mathcal{F}_2^p)\Big)^{1/p}. \tag{2.5}$$

In the remainder of the proof we will show that our choice of $(X, \mathcal{F})$ satisfies (2.5) for $p = 4$ and n large enough. First, we note that

$$T(\mathcal{F}) = \sum_{F \in \mathcal{F}} 2^{-|F|} = \left(\frac{n}{k}\right)^k \cdot 2^{-\binom{k}{2}} = 2^{5k}$$

$$\text{and} \quad \frac{|X|}{T(\mathcal{F})} = \frac{\binom{k}{2}\left(\frac{n}{k}\right)^2}{2^{5k}} \leqslant \frac{k^2\, 2^{k+9}}{2^{5k}} = o(1). \tag{2.6}$$

As a first step towards the application of the Advanced Weak Win Criterion, we give an estimate on $T(\mathcal{F}_2^4)$. By definition,

$$\mathcal{F}_2^4 = \left\{ \bigcup_{1 \leqslant i \leqslant 4} E_i \, : \{E_1, \ldots, E_4\} \in \binom{\mathcal{F}}{4}, \left| \bigcap_{1 \leqslant i \leqslant 4} E_i \right| \geqslant 2 \right\}.$$

Note that any collection of cliques meets in two edges if and only if it meets in a triangle. Recall that the elements of $\mathcal{F}_2^4$ are referred to as *clusters*. Following the standard notation, we call a cluster a *sunflower* if there is a triangle such that any two of the four cliques meet

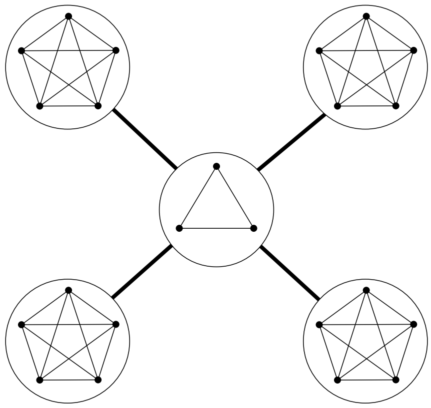


Figure 2.1: An example of a sunflower for $k = 8$. A thick line indicates that the vertices of the corresponding sets are pairwise connected.

in exactly this triangle. Figure 2.1 shows an illustration. We denote the subset of sunflowers of $\mathcal{F}_2^4$ by $\mathcal{S}_2^4$. By definition, a sunflower $F \in \mathcal{S}_2^4$ has exactly $4\binom{k}{2} - 9$ edges. In $\mathcal{F}_2^4$, there are at most $\binom{k}{3}\left(\frac{n}{k}\right)^3 \cdot \left(\frac{n}{k}\right)^{4(k-3)}$ sunflowers. Therefore,

$$T(\mathcal{S}_2^4) \leqslant \binom{k}{3}\left(\frac{n}{k}\right)^{4k-9} 2^{-4\binom{k}{2}+9} =: f(n,k).$$

It will turn out that $f(n,k)$ dominates the sum $T(\mathcal{F}_2^4)$.

For every $E \in \mathcal{F}$, we let $V(E)$ denote the set of vertices corresponding to E. Note that $|V(E)| = k$ for every $E \in \mathcal{F}$. As a first step of our analysis we use the technique of Beck [8] to assign to each cluster $F = \bigcup_{1 \leqslant i \leqslant 4} E_i$ *some* sequence $S(F) := (m_1, m_2, m_3)$ such that

$$\begin{aligned} m_1 &= |V(E_1) \cap V(E_2)|, \\ m_2 &= |(V(E_1) \cup V(E_2)) \cap V(E_3)|, \\ m_3 &= |(V(E_1) \cup V(E_2) \cup V(E_3)) \cap V(E_4)|. \end{aligned}$$

Note that for a given cluster F we may have several choices to select $S(F)$ (depending on the considered order of the E_i). Furthermore, we let $\mathcal{F}_2^4(m_1, m_2, m_3)$ denote the subset of clusters of $\mathcal{F}_2^4$ to which we assigned the sequence (m_1, m_2, m_3). Then obviously,

$$T(\mathcal{F}_2^4) \leqslant \sum_{m_1=3}^{k} \sum_{m_2=3}^{k} \sum_{m_3=3}^{k} T(\mathcal{F}_2^4(m_1, m_2, m_3)). \tag{2.7}$$

We now bound the cardinality of $\mathcal{F}_2^4(m_1, m_2, m_3)$.

Proposition 2.2.2. *For fixed* $3 \leqslant m_1, m_2, m_3 \leqslant k$, *we have that*

$$\left|\mathcal{F}_2^4(m_1, m_2, m_3)\right| \leqslant \binom{k}{3}\left(\frac{n}{k}\right)^{4k} \cdot \prod_{j=1}^{3} \binom{jk}{m_j - 3}\left(\frac{k}{n}\right)^{m_j}.$$

Furthermore, for any cluster $F \in \mathcal{F}_2^4(m_1, m_2, m_3)$ *we have* $|F| \geqslant 4\binom{k}{2} - \binom{m_1}{2} - \binom{m_2}{2} - \binom{m_3}{2}$.

Proof. We fix any m_1, m_2, m_3 with $3 \leqslant m_1, m_2, m_3 \leqslant k$, and we also fix any triple v_1, v_2, v_3 of vertices from distinct classes. We now derive an upper bound on the number of those clusters in $\mathcal{F}_2^4(m_1, m_2, m_3)$ where all four cliques contain v_1, v_2, and v_3. To this end we consider the number of possibilities to select $V(E_1)\setminus\{v_1, v_2, v_3\}$, $V(E_2)\setminus\{v_1, v_2, v_3\}$, $V(E_3)\setminus\{v_1, v_2, v_3\}$, $V(E_4)\setminus\{v_1, v_2, v_3\}$. Note that we have $\left(\frac{n}{k}\right)^{k-3}$ possibilities to choose the $k-3$ vertices of $V(E_1)\setminus\{v_1, v_2, v_3\}$.

Suppose that for some $1 \leqslant i \leqslant 3$ we have already determined the sets $V(E_1)\setminus\{v_1, v_2, v_3\}$, ... ,$V(E_i)\setminus\{v_1, v_2, v_3\}$. Then $V(E_1), \dots, V(E_i)$ cover at most $3 + i(k-3)$ vertices. Therefore, we have at most $\binom{3+i(k-3)}{m_i-3} \leqslant \binom{ik}{m_i-3}$ choices for those vertices of $\big(V(E_1)\cup\ldots\cup V(E_i)\big)\cap V(E_{i+1})$ which are different from v_1, v_2, v_3. Finally, there are at most $\left(\frac{n}{k}\right)^{k-m_i}$ possibilities to select $V(E_{i+1}) \setminus \big(V(E_1)\cup\ldots\cup V(E_i)\big)$.

Therefore, for any given m_1, m_2, m_3, every triple v_1, v_2, v_3 of vertices contributes at most

$$\left(\frac{n}{k}\right)^{k-3} \cdot \prod_{i=1}^{3} \binom{ik}{m_i - 3} \left(\frac{n}{k}\right)^{k-m_i}$$

to the number of clusters in $\mathcal{F}_2^4(m_1, m_2, m_3)$. Hence,

$$\begin{aligned} \left|\mathcal{F}_2^4(m_1, m_2, m_3)\right| &\leqslant \binom{k}{3} \left(\frac{n}{k}\right)^3 \left(\frac{n}{k}\right)^{k-3} \cdot \prod_{i=1}^{3} \binom{ik}{m_i - 3} \left(\frac{n}{k}\right)^{k-m_i} \\ &= \binom{k}{3} \left(\frac{n}{k}\right)^{4k} \cdot \prod_{i=1}^{3} \binom{ik}{m_i - 3} \left(\frac{k}{n}\right)^{m_i}, \end{aligned}$$

as claimed. For the second part of the proposition, note that $|E_1| = \binom{k}{2}$ and that every E_{i+1} contributes at least $\binom{k}{2} - \binom{m_i}{2}$ new edges to the cluster. □

We now show that $f(n,k)$ dominates the sum $T(\mathcal{F}_2^4)$.

Lemma 2.2.3. $T(\mathcal{F}_2^4) < k^3 f(n,k)$, *provided* k *is large enough.*

Proof. By definition of $T(\cdot)$ and Proposition 2.2.2 we have that

$$\begin{aligned} T\big(\mathcal{F}_2^4(m_1, m_2, m_3)\big) &\leqslant \binom{k}{3} \left(\frac{n}{k}\right)^{4k} \cdot \prod_{j=1}^{3} \left(\binom{jk}{m_j - 3} \left(\frac{k}{n}\right)^{m_j}\right) \\ &\qquad \times 2^{-4\binom{k}{2} + \binom{m_1}{2} + \binom{m_2}{2} + \binom{m_3}{2}} \\ &= \binom{k}{3} \left(\frac{n}{k}\right)^{4k} 2^{-4\binom{k}{2}} \cdot \prod_{j=1}^{3} \binom{jk}{m_j - 3} \left(\frac{k}{n}\right)^{m_j} 2^{\binom{m_j}{2}} \\ &= f(n,k) \cdot \prod_{j=1}^{3} \binom{jk}{m_j - 3} \left(\frac{k}{n}\right)^{m_j - 3} 2^{\binom{m_j}{2} - 3}. \end{aligned} \tag{2.8}$$

We set $g_j(m) := \binom{jk}{m-3}\left(\frac{k}{n}\right)^{m-3} 2^{\binom{m}{2}-3}$. We will show that $g_j(m) \leqslant 1$ for all $j \in \{1,2,3\}$ and $3 \leqslant m \leqslant k$, provided k is large enough. Indeed, for $3 \leqslant m \leqslant \frac{15k}{16}$ and k large enough, we have

$$\begin{aligned} g_j(m) &\leqslant (jk)^{m-3} \cdot 2^{-\frac{k+9}{2}(m-3)} \cdot 2^{\frac{(m+2)(m-3)}{2}} \\ &= \left(jk \cdot 2^{-\frac{k+9}{2}+\frac{m+2}{2}}\right)^{m-3} \leqslant \left(jk \cdot 2^{-\frac{k}{32}-\frac{7}{2}}\right)^{m-3} \leqslant 1. \end{aligned} \tag{2.9}$$

For $\frac{15k}{16} \leqslant m \leqslant k$ and k large enough, we obtain

$$\begin{aligned} g_j(m) &\leqslant 2^{jk}\left(2^{-\frac{k+9}{2}+\frac{m+2}{2}}\right)^{m-3} \\ &\leqslant 2^{3k}\left(2^{-\frac{7}{2}}\right)^{m-3} \leqslant 2^{3k}\left(2^{-\frac{7}{2}}\right)^{\frac{15k}{16}-3} \leqslant 1. \end{aligned} \tag{2.10}$$

Now, (2.8), (2.9) and (2.10) imply that $T(\mathcal{F}_2^4(m_1,m_2,m_3)) \leqslant f(n,k)$ for any sequence (m_1,m_2,m_3), provided k is large enough. Due to (2.7), we conclude that $T(\mathcal{F}_2^4) \leqslant k^3 f(n,k)$. □

Finally, we show that the Advanced Weak Win Criterion (Theorem 2.2.1) applies with $p = 4$.

Corollary 2.2.4. *For n large enough,* $T(\mathcal{F}) > 16|X|\left(\left(T(\mathcal{F}_2^4)\right)^{1/4} + \frac{1}{4}\right)$.

Proof. By Lemma 2.2.3, the definition of $f(n,k)$ and the fact that $|X| \leqslant n^2$ we get that

$$\begin{aligned} \frac{16\,|X|\,(T(\mathcal{F}_2^4))^{1/4}}{T(\mathcal{F})} &\leqslant \frac{16\,|X|\,(k^3 f(n,k))^{1/4}}{T(\mathcal{F})} \leqslant \frac{16\,n^2\,k^{\frac{3}{4}}\left(k^3\left(\frac{n}{k}\right)^{4k-9} 2^{-4\binom{k}{2}+9}\right)^{\frac{1}{4}}}{T(\mathcal{F})} \\ &\leqslant \frac{16\cdot 2^{\frac{9}{4}}\,n^2\,k^{\frac{6}{4}}\left(\frac{n}{k}\right)^{k-\frac{9}{4}} 2^{-\binom{k}{2}}}{\left(\frac{n}{k}\right)^k \cdot 2^{-\binom{k}{2}}} \leqslant \frac{100 n^2\,k^{\frac{6}{4}}}{\left(\frac{n}{k}\right)^{\frac{9}{4}}} \\ &= \frac{100 k^{\frac{15}{4}}}{n^{\frac{1}{4}}} < 100\cdot 2^{-\frac{k}{8}+\frac{14}{4}\log(k)} = o(1). \end{aligned}$$

By (2.6),

$$\frac{\frac{1}{4}\cdot 16|X|}{T(\mathcal{F})} = o(1),$$

and the claim follows. □

We have shown that the Advanced Weak Win Criterion applies for $p = 4$. Therefore, Maker has a winning strategy $\mathcal{S}$ in the Maker–Breaker game $(X, \mathcal{F})$, where X is the complete k-partite graph with vertex partition $V_1 \cup \cdots \cup V_k$ and $\mathcal{F}$ is the family of all k-cliques in that graph. In the tournament game, Maker now uses this winning strategy $\mathcal{S}$: Whenever $\mathcal{S}$ tells Maker to claim an edge $\{v_i, v_j\}$ for $v_i \in V_i$ and $v_j \in V_j$, $i \neq j$, she chooses the direction of $\{v_i, v_j\}$ according to the direction of $\{u_i, u_j\}$ in T_k. Clearly, since $\mathcal{S}$ guarantees her a copy of a k-clique on the k-partite graph, this strategy yields a copy of T_k at the end of the game. □

2.3 A strategy for OBreaker in the orientation tournament game

In this section we prove Theorem 1.1.2. We first generalize the notation of biased Maker–Breaker games, that we briefly touched in the introduction. In an $(a : b)$ Maker–Breaker game $(X, \mathcal{F})$ Maker claims a elements and Breaker claims b elements in each round. A game is then called *biased* if $a \neq 1$ or $b \neq 1$.

In order to provide OBreaker with a winning strategy for the game $Or(T_k)$ we associate with $Or(T_k)$ an auxiliary biased Maker–Breaker game. In the first step of the proof we show that Breaker has a strategy to win the auxiliary game, and in the second step we prove that this strategy directly gives him a winning strategy for $Or(T_k)$.

We will make use of the generalized Erdős-Selfridge-Criterion proven by Beck [8].

Theorem 2.3.1 (Generalized Erdős-Selfridge-Criterion)**.** *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$. If*

$$\sum_{F \in \mathcal{F}} (1+b)^{-|F|/a} < \frac{1}{1+b},$$

then Breaker has a winning strategy in the $(a : b)$ Maker–Breaker game $(X, \mathcal{F})$.

Let T_k be some tournament on k vertices. Consider the $(2 : 1)$ Maker–Breaker game $\mathcal{H}(T_k) = \mathcal{H}(T_k, n) = (X, \mathcal{F}(T_k))$ where

$$X := \{(u, v) : u, v \in V(K_n),\, u \neq v\}$$

is the board of the game consisting of $|X| = n(n-1)$ elements, and

$$\mathcal{F}(T_k) := \{S \subseteq X : S \text{ is a copy of } T_k\}$$

is the family of winning sets.

Claim 2.3.2. *For large enough n and $k \geqslant 4 \log n + 2$, Breaker has a winning strategy in $\mathcal{H}(T_k)$, for any tournament T_k on k vertices.*

Proof. We check that Theorem 2.3.1 applies. By definition, $|\mathcal{F}(T_k)| < n^k$, and $|F| = \binom{k}{2}$ for every $F \in \mathcal{F}(T_k)$. Therefore, and since $k \geqslant 4 \log n + 2$,

$$\sum_{F \in \mathcal{F}} 2^{-|F|/2} < n^k \cdot 2^{-k(k-1)/4} \leqslant \frac{1}{2}.$$ □

We conclude the proof of Theorem 1.1.2 with the following lemma.

Lemma 2.3.3. *Let T_k be a tournament on k vertices. Suppose that Breaker has a strategy to win the game $\mathcal{H}(T_k)$. Then there is also a winning strategy for OBreaker in the game $Or(T_k)$.*

Proof. We first need some notation. By *directing an edge* (u, v) we mean that we direct the edge spanned by u and v from u to v. We note that in each round of $\mathcal{H}(T_k)$, Maker is allowed to choose either two, one, or zero elements. (Otherwise she can just claim additional, arbitrary elements, and then follow her strategy. If this strategy calls for something she occupied before, she takes an arbitrary element; no extra element is disadvantageous for her.)

Suppose, for a contradiction, that OMaker has a winning strategy $\mathcal{S}$ for $Or(T_k)$. We now describe a strategy $\mathcal{S}'$ for Maker in $\mathcal{H}(T_k)$. During the play, Maker simulates (in parallel) a play of the game $Or(T_k)$, and maintains the invariant that after each of her moves in $\mathcal{H}(T_k)$, every pair $u, v \in V$ has the property that

(*i*) in $\mathcal{H}(T_k)$, if Breaker owns the element (u, v) then Maker owns (v, u), and

(*ii*) in $Or(T_k)$, there is a directed edge from u to v if and only if Maker has claimed the element (u, v) in $\mathcal{H}(T_k)$.

Let (a, b) be the edge $\mathcal{S}$ tells OMaker to direct in her first move. Then Maker claims the element (a, b) in the actual game $\mathcal{H}(T_k)$ (at this point she does not make use of the possibility to occupy two elements), and directs (a, b) in the parallel game $Or(T_k)$ as OMaker.

Suppose that i rounds have been played, and let $(u, v) \in X$ denote the element Breaker chose in his ith move. If Maker has already claimed (v, u) in a previous round then she does not claim a single element. Otherwise, as her $(i+1)$st move, she first occupies the element (v, u). In the parallel game $Or(T_k)$, she directs as OBreaker the edge (v, u). Then she identifies the edge (x, y) the strategy $\mathcal{S}$ tells OMaker to direct. Finally, she directs (x, y) as OMaker in $Or(T_k)$, and claims the element (x, y) in $\mathcal{H}(T_k)$.

We note that the invariants (i) and (ii) remain satisfied after Maker's $(i+1)$st move. So, by following $\mathcal{S}'$, Maker can guarantee that at the end of the game, these invariants still hold. Since by assumption, $\mathcal{S}$ is a winning strategy, the final digraph in $Or(T_k)$ contains a copy of T_k. Together with invariant (ii) this yields that Maker possesses all elements of some winning set in $\mathcal{H}(T_k)$. This contradicts the assumption that Breaker has a winning strategy for $\mathcal{H}(T_k)$. □

2.4 Concluding remarks

The strategy for Maker in Theorem 1.1.1 is independent of the actual tournament T_k Breaker chooses. On the other hand, the upper bound in (1.2) is the upper bound from the k-clique game. That means that for $k > 2 \log n - 2 \log \log n + o(1)$, Breaker has a strategy to prevent

Maker from building *any* given tournament on k vertices. So both, the lower and the upper bound do not depend on the tournament chosen at the beginning. On the other hand, there is a tournament on k_{cl} vertices which Maker can build: Consider the transitive tournament on the vertex set $\{u_1, \ldots, u_k\}$, where, say, for all indices $1 \leqslant i < j \leqslant k$ the edge between u_i and u_j is directed from u_i to u_j. Now, let $\{v_1, \ldots, v_n\}$ be an arbitrary enumeration of the vertices in K_n. Then Maker can just follow the strategy provided by the k-clique game. Whenever this strategy tells her to claim the edge $\{v_i, v_j\}$ for $i < j$, she chooses the direction (v_i, v_j). It would be interesting to determine whether all tournaments are "equally hard" for Maker. Therefore, we pose the following question.

Question 2.4.1. *Does there exist a tournament T_k on $k \leqslant k_{cl}$ vertices such that in the T_k-building game, Breaker has a strategy to prevent Maker from building T_k?*

Note that a negative answer to this question, together with Theorem 1.1, would give us the exact value of k_t. But even if there exists a tournament on at most k_{cl} vertices which Breaker can prevent, it is of particular interest to get rid of the gap in the constant term.

Problem 2.4.2. *Determine the exact value of k_t.*

One might wonder whether a more technical approach, applying the Advanced Weak Win Criterion with some p tending to infinity (as done in the proof of (1.1)), would help to get closer to this problem. Unfortunately, we were not even able to verify completely the original argument. In Section 25 of [8], when bounding the sizes of clusters (and showing that they are dominated by the sunflower terms), the collection of all p-clusters is divided into three classes. However, the applied case distinction does not seem to cover all possible clusters, and the set of uncovered clusters seems to be rather large. We were not able to fix this problem, but of course it is quite possible that we overlook something.

A similar discussion arises for the orientation-tournament game. To find the strategy for OBreaker in Theorem 1.1.2, we defined an auxiliary Maker–Breaker game $\mathcal{G}$ with the property that a winning stratey for Breaker in $\mathcal{G}$ yields a winning strategy for OBreaker in the orientation tournament game. Our strategy for the upper bound does not use the actual structure of the chosen tournament T_k. That is, OBreaker wins the game $Or(T_k)$ for *any* fixed tournament T_k on at least $4 \log n(1 + o(1))$ vertices. Similarly, for the lower bound strategy given by Theorem 1.1.1, OMaker wins $Or(T_k)$ for *any* tournament on at most $2 \log n(1 - o(1))$ vertices. Analogously to the ordinary tournament game we therefore pose the following question.

Question 2.4.3. *For $k_1 \in \mathbb{N}$, do there exist two (non-isomorphic) tournaments T and T' on k_1 vertices such that OMaker has a winning strategy in the orientation game $Or(T)$, but OBreaker has a winning strategy in the game $Or(T')$?*

Furthermore, we determined $k_o(n)$ only up to a factor of 2. The probabilistic analysis would suggest the breakpoint to be around $2 \log n$.

Problem 2.4.4. *Determine the constant $2 \leqslant c \leqslant 4$ such that $k_o(n) = (c + o(1)) \log n$.*

We finish this chapter with a short discussion about the *universal tournament game.* As noted in Subsection 1.1.1, Maker has a strategy to occupy a copy of *every* tournament on k vertices for $k \leqslant (1/2 - o(1)) \log n$. Beck conjectured that this result is not best possible and posed the following problem.

Problem 2.4.5 ([8], p. 457)**.** *Determine the largest k_u such that Maker has a strategy such that at the end of the game, her digraph contains* every *tournament on k_u vertices.*

For the upper bound, we can show that $k_u \leqslant (1 + o(1)) \log n$: We note that the number of non-isomorphic tournaments on k vertices is at least $c(k) := 2^{\binom{k}{2}}/k! > 2^{\binom{k}{2} - k \log k}$. By definition, Maker has a strategy to occupy a copy of every tournament on k_u vertices. Hence, at the end of the game, the underlying graph of Maker's graph contains $c(k_u)$ (not necessarily edge-disjoint) distinct cliques. However, a result of Bednarska and Łuczak (see Lemma 5 in [9]) asserts that there is some $k = (1+o(1)) \log n$ such that in the ordinary graph game (where no edge-orientations are involved), Breaker has a strategy to prevent Maker from claiming more than $c(k)$ distinct k-cliques. Thus, $k_u \leqslant (1 + o(1)) \log n$. To the best of our knowledge, nothing better is known for the universal tournament game.

3

On the threshold bias in the oriented-cycle game

In this chapter, we examine the b-biased oriented-cycle game for both the monotone and the strict rules. We will first introduce some terminology which is useful for dealing with directed graphs and for this particular game. In Section 3.1, we introduce a certain structure of digraphs which is helpful to control OMaker's edges *locally*. We then prove Theorem 1.1.3 in Section 3.2; and Theorem 1.1.4 in Section Section 3.3.

Notation

Let $V = [n]$ and let $D \subseteq V \times V$ be a digraph. We call elements $(v, w) \in D$ *arcs* and the underlying set $\{v, w\}$ *a pair* or *an edge*. An arc (v, v) is called a *loop* and (v, w) is called the *reverse arc* for (w, v). In this work we are only concerned with simple digraphs, without loops and reverse arcs. For an arc $e \in D$, we write e^+ for its tail and e^- for its head, i.e. $e = (e^+, e^-)$. For a subdigraph $S \subseteq D$, we denote by S^+ the set of all tails e^+ for $e \in S$, and by S^- the set of all heads e^- for $e \in S$. It will be convenient to denote by $\overleftarrow{D}$ the set of all reverse arcs, that is $\overleftarrow{D} := \{(v, w) \in V \times V : (w, v) \in D\}$. Moreover, the set $\mathcal{A}(D) := (V \times V) \setminus (D \cup \overleftarrow{D} \cup \mathcal{L})$ denotes the set of all *available arcs*, where $\mathcal{L} = \{(v, v) : v \in V\}$ is the set of all loops. Note that $\mathcal{A}(D)$ is symmetric, i.e. if $(v, w) \in \mathcal{A}(D)$ then also $(w, v) \in \mathcal{A}(D)$. Note also that for D without loops and reverse arcs it holds that $D \cap (\overleftarrow{D} \cup \mathcal{L}) = \emptyset$. We generalize the notation of an arc and say the k-tuple $(v_1, \ldots, v_k)$ *induces a transitive tournament in* D, denoted by $(v_1, \ldots, v_k) \in D$, if for all $1 \leqslant i < j \leqslant k$ we have that $(v_i, v_j) \in D$. Similarly, for two disjoint sets $A, B \subseteq V$ we write $(A, B) \in D$ if for all $v \in A$, $w \in B$ we have that $(v, w) \in D$. We then call the pair (A, B) a *uniformly directed biclique*, or short a UDB. We say the sequence

$P = (e_1, \ldots, e_k)$ is a *directed path* (or simply a *path*) in D if all $e_i \in D$ and for all $1 \leqslant i < k$ we have that $e_i^- = e_{i+1}^+$. In this case we also say that P is an e_1^+-e_k^--path. In our proofs we are concerned how D behaves on certain subsets of the vertices. Following standard graph theoretic notation, for a subset $A \subseteq V$ we denote by $D[A]$ the directed subgraph of D of arcs spanned by A.

Recall that the oriented-cycle game is played on the edge set of K_n where we may assume that $V = V(K_n) = [n]$. As in the orientation tournament game, we say a player *directs (or orients) the edge* (v, w) if (s)he directs the pair $\{v, w\}$ from v to w. That is, the player chooses the arc (v, w) to belong to the final digraph, and dismisses the arc (w, v) from the board. At a certain point in the game, we shall refer to $D \subseteq V \times V$ as the sub-digraph of already directed edges (arcs) by either player. We say a player *closes a cycle in* D *(by directing some edge* (v, w)*)* if there exists a w-v-path in D. Note that if a player can close a cycle in D, then (s)he can close a triangle (consider the shortest cycle a player can close, and consider any cord).

There are two essential concepts to our proof, the aforementioned UDB's and α-*structures.* A UDB is a complete bipartite digraph where all the edges are oriented in the same direction (i.e. from A to B). Our goal is to create a UDB (A, B) such that both parts fulfil $|A|, |B| \leqslant b$ and $A \cup B = V$. Suppose the following situation would be given to us for free. There is a partition $A \dot\cup B = V$ such that the pair (A, B) forms a UDB in D, both parts fulfil $|A|, |B| \leqslant b$ and both sets A and B are empty (i.e. $D[A] = D[B] = \emptyset$). OBreaker could then follow the "trivial strategy" inside A and B respectively (as OBreaker wins on K_{b+2}), even when the strict rules apply. However, while building such a UDB, OMaker will direct edges inside these sets, and OBreaker needs to control those. Moreover, to optimise the bias, OBreaker should be able to control those edges inside A and B with as few edges as possible.

To handle this obstacle, we introduce certain structures which we call α-*structures* and a procedure α to incorporate new (i.e. OMaker's) edges into an existing α-structure.

Before we move on to study these special structures let us mention that the idea of building a big UDB quickly will come up again in the proof of Theorem 1.1.4. However, the requirement of directing exactly b edges in each move puts some serious restrictions on the power of α-structures, so for the strict rules, we will consider only special α-structures, namely tournaments.

3.1 α-structures

The definition of an α-structure looks quite technical at first sight. So let us motivate the idea behind it.

Suppose OMaker's strategy is to build a long path first. (This indeed is the strategy for OMaker in the so far best-known lower-bound proof in [11].) Let $P = (e_1, \ldots, e_k)$ be a directed path of length k in D with arcs $e_i = (v_i, v_{i+1})$, and suppose OMaker enlengthens P by directing an edge (v_{k+1}, w) for some $w \in V$. Then all the pairs $\{w, v_i\}$ for $1 \leqslant i \leqslant k$ constitute potential threats as directing any (w, v_i) would close a cycle. So OBreaker better directs all edges (v_i, w) in his next move. This way, OBreaker fills up the missing arcs of an evolving transitive tournament with *spine* $e_1, \ldots, e_k$. Then formally, OBreaker sets $v_{k+2} := w$, $e_{k+1} := (v_{k+1}, w)$ and directs all edges (e_i^+, w) for $i \leqslant k$. Clearly, as long as there are isolated vertices, OMaker could follow this strategy and increase the number of threats that OBreaker *has to close immediately* by one in each move.

By defining the α-structure we show that this is best possible in the following sense: No matter how OMaker plays, OBreaker has a strategy such that in round k, he has to direct at most k edges to close immediate threats that would close cycles.

Let V be a set of vertices, and let $D \subset (V \times V) \setminus \mathcal{L}$ be a digraph without loops and reverse arcs (formally, $D \cap \overleftarrow{D} = \emptyset$). Let $S \subseteq D$ be a subdigraph. Then (D, S) is called an *α-structure of rank k* if $|S| = k$ and there exists a labelling $\{e_1, \ldots, e_k\} = S$ of the arcs in S such that

(α_1) for every $1 \leqslant i < j \leqslant k$: $(e_i^+, e_j^-) \in D$;

(α_2) and no other arcs are present in D.

Later, in our strategy, the arcs $e_1, \ldots, e_k$ will be the arcs that were directed by OMaker (though not necessarily in that order), and the arcs of "type" (α_1) are the ones directed by OBreaker. We will refer to $S = \{e_1, \ldots, e_k\}$ as the *special* arcs of an α-structure. All other edges in $D \setminus S$ are then *arcs of type* (α_1). Let us capture some immediate facts about α-structures.

Observation 3.1.1. *Let (D, S) be an α-structure of rank k on vertex set V. Then the following holds.*

(i) *For every directed path $P = (e_{i_1}, \ldots, e_{i_\ell})$ with $e_{i_j} \in S$: $i_1 < \ldots < i_\ell$.*

(ii) *For any subset $V' \subseteq V$, we have that $(D[V'], S[V'])$ is an α-structure of rank $k' \leqslant k$.*

(iii) The arcs of type (α_2) *are uniquely determined once* S *and a labelling of the arcs in* S *have been fixed. Moreover,* $D^+ = S^+$ *and* $D^- = S^-$.

Furthermore, it is easy to verify that (α_1)-(α_2) imply the following property. Recall that $\mathcal{A}(D) = (V \times V) \setminus (D \cup \overleftarrow{D} \cup \mathcal{L})$ denotes the set of available arcs.

Proposition 3.1.2 (Property (α_3))**.** *If* (D, S) *is an* α*-structure, then for every available* $e \in \mathcal{A}(D)$ *we have that* $D \cup e$ *is acyclic.*

Proof. Suppose there is a path $P = (f_1, \ldots, f_\ell)$ in D and an edge $e \in D \cup \mathcal{A}(D)$ such that $P \cup e$ forms a directed cycle. Moreover, let P be a shortest path with that property. Notice first that we may assume that $\ell = 2$. For if $\ell \geqslant 3$, consider the pair $\{f_1^+, f_\ell^+\}$. Either $(f_1^+, f_\ell^+) \in D$, or $(f_\ell^+, f_1^+) \in D$, or both pairs are available. In all cases, there is a shorter path P' with the property that one could close it to a cycle. So let $P = (f_1, f_2)$ be a path of length two in D, and let $f_1 = (v_1, v_2)$ and $f_2 = (v_2, v_3)$. Then by (α_2), both edges f_1 and f_2 must be of type (α_1) or belong to S. Hence, there exist (not necessarily distinct) edges $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4} \in S$ such that $v_1 = e_{i_1}^+$, $v_2 = e_{i_2}^- = e_{i_3}^+$, $v_3 = e_{i_4}^-$ and $i_1 \leqslant i_2$ and $i_3 \leqslant i_4$. Furthermore, (e_{i_2}, e_{i_3}) is a directed path of special edges, so by Observation 3.1.1 (i), $i_2 < i_3$. It follows that $i_1 < i_4$, so by (α_1), $(e_{i_1}^+, e_{i_4}^-) = (v_1, v_3) \in D$, a contradiction. □

In the light of our orientation game, we pin down the following important implication.

Corollary 3.1.3. *For some subset* $V' \subseteq V$*, suppose that in the oriented-cycle game, OBreaker maintains that* $(D[V'], S)$ *is an* α*-structure (of some rank* k *and for some* $S \subseteq D[V']$*). Then there is no cycle in* $D[V']$ *and OMaker cannot close a cycle inside* V' *in her next move.*

In order for OBreaker to maintain an α-structure on some subset of the vertices we need to know how to incorporate OMaker's edge into such a structure. The following is one of the key lemmas in OBreaker's strategy.

Lemma 3.1.4. *Let* (D, S) *be an* α*-structure of rank* k *on vertex set* V*, and let* $e \in \mathcal{A}(D)$ *be an available arc. Then there exist at most* $\min\{k, |V|\}$ *available arcs* $\{f_1, \ldots, f_t\} \subseteq \mathcal{A}(D)$ *such that* (D', S') *is an* α*-structure of rank* $k+1$*, where* $S' = S \cup \{e\}$ *and* $D' = D \cup \{e, f_1, \ldots, f_t\}$.

By *adding* e *to the* α*-structure* (D, S) we mean a strategy for OBreaker to direct the edges $\{f_1, \ldots, f_t\}$ given by the previous lemma. Before we prove the lemma, we need one more definition. Let (D, S) be an α-structure with special arcs $S = \{e_1, \ldots, e_k\}$, and let $x \in V$. We set

$$In(x) := \Big\{ e_i \,:\, \text{there exists a path } P = (e_i, e_{j_1}, \ldots, e_{j_m}) \text{ s.t. } x = e_{j_m}^- \Big\},$$
$$Out(x) := \Big\{ e_i \,:\, \text{there exists a path } P = (e_{j_1}, \ldots, e_{j_m}, e_i) \text{ s.t. } x = e_{j_1}^+ \Big\},$$

The following observation is rather simple.

Proposition 3.1.5. *Let (D, S) be an α-structure on vertex set V with special arcs $S = \{e_1, \ldots, e_k\}$. Further, let $x, y \in V$ be distinct vertices such that $(x, y) \in \mathcal{A}(D)$. Then*

(i) for all $e_i \in In(x)$, $e_j \in Out(x)$: $i < j$, and

(ii) for all $e_i \in In(x)$, $e_j \in Out(y)$: $i < j$.

In particular, $In(x) \cap Out(x) = \emptyset$ and $In(x) \cap Out(y) = \emptyset$.

Proof. For (i), let $e_i \in In(x)$, $e_j \in Out(x)$ and let P_i be the corresponding e_i^+-x-path starting with e_i, and let P_j be the corresponding x-e_j^--path ending with e_j. Since there are no directed cycles inside the structure (see property (α_3)), the concatenation of P_i and P_j is a directed path and (i) follows from Observation 3.1.1 (i). For (ii), let $e_i \in In(x)$, $e_j \in Out(y)$, let P_i be the corresponding e_i^+-x-path starting with e_i, let P_j be the corresponding y-e_j^--path ending with e_j, and assume $i \geqslant j$. When $i = j$ then $e_i = (y, x)$, a contradiction to $(x, y) \notin \overleftarrow{D}$. If $i > j$, then by property (α_1), the edge (e_j^+, e_i^-) is an arc in D. But then the concatenation of $(P_j - e_j)$, (e_j^+, e_i^-), $(P_i - e_i)$ and (x, y) would be a directed cycle, again a contradiction to property (α_3). $\square$

We are now ready to prove the above lemma.

Proof of Lemma 3.1.4. Let (D, S) be the α-structure of rank k on vertex set V, let $S = \{e_1, \ldots, e_k\}$ be the enumeration of the special edges, and let $e = (v, w) \in \mathcal{A}(D)$ be an available arc. Set $\ell := \min\{i : e_i \in Out(w)\}$ if $Out(w) \neq \emptyset$, and $\ell := k + 1$ otherwise. For all $i < \ell$, set $f_i := (e_i^+, w)$, and for all $i \geqslant \ell$, set $f_i := (v, e_i^-)$. We claim that for all $1 \leqslant i \leqslant k$, either $f_i \in D$ or $f_i \in \mathcal{A}(D)$.

First, let $i < \ell$ and suppose for a contradiction that $f := (w, e_i^+) \in D$. If $f \in S$, then by definition $e_i \in Out(w)$, so $i \geqslant \ell$, a contradiction. So we may assume that $f \notin S$. But then by property (α_2), f must be of type (α_1), that is by property (α_1) there exist $e_{j_1}, e_{j_2} \in S$ such that $f = (e_{j_1}^+, e_{j_2}^-)$ and $j_1 < j_2$. But then $P := e_{j_2} e_i$ is a path P consisting of special edges, so by Observation 3.1.1 (i), $j_2 < i$. Thirdly, since $e_{j_1}^+ = w$, by definition $e_{j_1} \in Out(w)$, so $\ell \leqslant j_1$. But this implies $\ell < i$, again a contradiction.
Now let $i \geqslant \ell$. The only additional observation we need to make here is that by Proposition 3.1.5, $\ell > j$ for all $e_j \in In(v)$. The rest is completely analogous to the first case. So we can assume that either $(v, e_i^-) \in \mathcal{A}(D)$, or $(v, e_i^-) \in D$.

We now check that the resulting structure (D', S') is an α-structure of rank $k + 1$, where

$S' = \{e'_1, \dots, e'_{k+1}\}$ with

$$e'_i = \begin{cases} e_i & \text{if } i < \ell \\ e & \text{if } i = \ell \\ e_{i-1} & \text{if } i > \ell, \end{cases}$$

and $D' = D \cup \{e, f_1, \dots, f_k\}$. Now, (α_1) is obvious since (D, S) is an α-structure and we preserved the relative order of S, and since the f_i we added are exactly the arcs of type (α_1) that are missing in (D', S'). Also, (α_2) follows since previously there were no other arcs, and all arcs f_i are of type (α_1) in (D', S').

Finally, for every existing arc $e_i \in S$, we added at most one new arc f_i. But also, for every vertex $z \in V$ at most one of the f_i contains z. So $|\{f_1, \dots, f_k\}| \leqslant \min\{k, |V|\}$. □

3.2 OBreaker's strategy for the monotone rules

Proof of Theorem 1.1.3. Recall that OMaker and OBreaker alternately direct edges of K_n, where OMaker directs exactly one edge in each round, and OBreaker directs at least one and at most b edges in each round, where $b \geqslant 5n/6 + 2$. OMaker's goal is to close a directed cycle, whereas OBreaker's goal is to prevent this. First, we provide OBreaker with a strategy, then we prove that he can follow that strategy and that it constitutes a winning strategy. At any point during the game let D denote the digraph of already directed edges. By the rules of the game, D has no loops and no reverse arcs. Let $e_1 = (v_1, w_1)$ be the very first edge OMaker directs. Then OBreaker picks two disjoint subsets $A, B \subseteq V$ such that $v_1 \in A$, $w_1 \in B$ and $|A| = |B| = 2 \leqslant \sqrt{b}$, and directs all edges (x, y) for $x \in A$, $y \in B$. The rest of the strategy is divided into three stages.

In **Stage I**, OBreaker maintains a UDB (A, B) such that after each of his moves

($S1.1$) $(D[V \setminus B], S_A)$ is an α-structure of rank k, for some $S_A \subseteq A \times A$,
($S1.2$) $(D[V \setminus A], S_B)$ is an α-structure of rank ℓ, for some $S_B \subseteq B \times B$,
($S1.3$) $k + \ell$ increases by one in each round, and
($S1.4$) $|A| - k$ and $|B| - \ell$ increase by one in each round.

($S1.3$) and ($S1.4$) imply that $k + \ell = \#$ rounds $= |A| - k = |B| - \ell$. OBreaker proceeds to Stage II after round $\lceil \frac{n}{6} \rceil$, that is as soon as $|A| - k = |B| - \ell \geqslant \frac{n}{6}$. We want to remark at this point that one could optimize OBreaker's bias by requiring "at least one" in ($S1.3$) and "at most one" in ($S1.4$). We comment on threshold optimization in the final section of this chapter, but for clarity of presentation keep this simple requirement.

Now let $e = (v, w)$ be the arc OMaker directs in a particular round of Stage I. Since (A, B) is a UDB, either $\{v, w\} \subseteq V \setminus A$ or $\{v, w\} \subseteq V \setminus B$. Assume first that $\{v, w\} \subseteq V \setminus B$. Then OBreaker adds e to the α-structure in $V \setminus B$ by directing the edges $\{f_1, \dots, f_t\} \in \mathcal{A}(D[V \setminus B])$

given by Lemma 3.1.4. If $v \in V \setminus (A \cup B)$ then he directs all edges (v, y) for $y \in B$ and sets $A := A \cup \{v\}$. Otherwise $v \in A$ already, so OBreaker picks an arbitrary new vertex $v' \in V \setminus (A \cup B)$, directs all edges (v', y) for $y \in B$ and sets $A := A \cup \{v'\}$. Similarly, if $w \in V \setminus (A \cup B)$ then he directs all edges (w, y) for $y \in B$ and sets $A := A \cup \{w\}$. Otherwise $w \in A$ already, so OBreaker picks an arbitrary new vertex $w' \in V \setminus (A \cup B)$, directs all edges (w', y) for $y \in B$ and sets $A := A \cup \{w'\}$. Furthermore, he picks an arbitrary element $y' \in V \setminus (A \cup B)$, directs all edges (x, y') for $x \in A$, and sets $B := B \cup \{y'\}$.
If $\{v, w\} \subseteq V \setminus A$, he similarly adds (v, w) to the α-structure in $V \setminus A$, and adds two vertices to B and one to A, depending whether $v, w \in B$ or not.
As soon as $|A| - k, |B| - \ell \geqslant n/6$, OBreaker proceeds to Stage II.

In **Stage II**, OBreaker stops increasing the values $|A| - k$ and $|B| - \ell$. He now maintains a UDB (A, B) such that after each of his moves

- $(S2.1)$ $(D[V \setminus B], S_A)$ is an α-structure of rank k, for some $S_A \subseteq D[V \setminus B]$, such that $S_A^+ \subseteq A$,
- $(S2.2)$ $(D[V \setminus A], S_B)$ is an α-structure of rank ℓ, for some $S_B \subseteq B \times B$,
- $(S2.3)$ $|A| - k$ and $|B| - \ell$ do not decrease, i.e. $|A| - k,\ |B| - \ell \geqslant \frac{n}{6} \geqslant n - b$.

Again, let $e = (v, w)$ be the arc OMaker directed in her previous move and assume first that $\{v, w\} \subseteq V \setminus B$. Then OBreaker adds e to the α-structure in $V \setminus B$ using Lemma 3.1.4. If $v \in V \setminus (A \cup B)$ then he directs all edges (v, y) for $y \in B$ and sets $A := A \cup \{v\}$. Otherwise $v \in A$ already, so OBreaker picks an arbitrary new vertex $v' \in V \setminus (A \cup B)$, directs all edges (v', y) for $y \in B$ and sets $A := A \cup \{v'\}$.

Assume now that $\{v, w\} \not\subseteq V \setminus B$. Then, since (A, B) is a UDB, $\{v, w\} \subseteq V \setminus A$ and at least one of the two vertices v, w is in B already. Assume $v \in B$ (the case $w \in B$ is analogous). Then OBreaker adds e to the α-structure in $V \setminus A$ by procedure α. If $w \in V \setminus (A \cup B)$, then he directs all edges (x, w) for $x \in A$ (unless already there) and sets $B := B \cup \{w\}$. Otherwise $w \in A$ already, so OBreaker picks an arbitrary new vertex $w' \in V \setminus (A \cup B)$, directs all edges (x, w') for $x \in A$ (unless already there) and sets $B := B \cup \{w'\}$.
Stage II ends as soon as $A \cup B = V$.

In **Stage III**, OBreaker maintains a UDB (A, B) with $A \cup B = V$ such that

- $(S3.1)$ $(D[A], S_A)$ forms an α-structure on A,
- $(S3.2)$ $(D[B], S_B)$ forms an α-structure on B, and
- $(S3.3)$ $|A|, |B| \leqslant b$.

Let again $e = (v, w)$ be the arc OMaker directed in her previous move. Either $\{v, w\} \subseteq A$ or $\{v, w\} \subseteq B$. In the first case, OBreaker adds e to the α-structure in A using Lemma 3.1.4, in the second case, OBreaker adds e to the α-structure in B, again by using Lemma 3.1.4.

Let us first remark that if OBreaker can follow the proposed strategy and reestablish the properties of the certain stage in each move, then OMaker can never close a cycle. Indeed, throughout the whole game, OBreaker maintains a UDB (A, B) such that D forms an α-structure on each, $V \setminus A$ and $V \setminus B$ (cf. $(S*.1)$ and $(S*.2)$ of each stage). Moreover, also by $(S*.1)$ and $(S*.2)$ of each stage, at any point during the game we have for any $(v, w) \in D$ that either $v \in A$ or $w \in B$ (or both). Suppose at some point, OMaker could close a cycle C by directing an edge $e = (v, w)$. Since (A, B) is a UDB and by the previous comment, all edges of C must lie either completely in $V \setminus A$ or completely in $V \setminus B$. However, $(D[V \setminus B], S_A)$ is an α-structure on $V \setminus B$ (and $(D[V \setminus A], S_B)$ on $V \setminus A$), so by Corollary 3.1.3, OMaker cannot close a cycle in $V \setminus A$ (or $V \setminus B$ respectively).

It remains to prove that OBreaker can follow the proposed strategy, that in each round he has to direct at most b edges, and that the properties of each stage are reestablished.

For the first move, it is clear that OBreaker can follow the strategy, and that it takes him at most $|A| \cdot |B| \leqslant b$ edges to direct. It is also clear that this first move establishes properties $(S1.1) - (S1.4)$ for $k = \ell = 0$.

Suppose now for **Stage I** that properties $(S1.1)-(S1.4)$ hold. If $|A|-k, |B|-\ell \geqslant n/6$, then OBreaker proceeds to Stage II, so we can assume $|A| - k, |B| - \ell < n/6$. As said previously, since (A, B) is a UDB, all the arcs (x, y) with $x \in A$ and $y \in B$ are present in D already, so OMaker's arc is either completely in $V \setminus A$ or completely in $V \setminus B$. Assume first that for OMaker's arc $e = (v, w)$ it holds that $\{v, w\} \subseteq V \setminus B$ before this round. Since $(D[V \setminus B], S_A)$ forms an α-structure by $(S1.1)$, and by Lemma 3.1.4, OBreaker can add e to that α-structure. By $(S1.2)$ we have $D[V \setminus A] \subseteq B \times B$. So for all $z \in V \setminus (A \cup B)$ all $y \in B$ none of the pairs $\{z, y\}$ has been directed so far. Similarly, by $(S1.1)$ we have $D[V \setminus B] \subseteq A \times A$. So for all $z \in V \setminus (A \cup B)$ all $x \in A$ none of the pairs $\{z, x\}$ has been directed so far. So OBreaker can claim all edges (v, y), (w, y) (or (v', y) and (w', y) respectively) for $y \in B$, and all edges (x, y') for $x \in A$ as requested by the strategy.
By Lemma 3.1.4, adding e to the α-structure in $V \setminus B$ takes OBreaker at most k edges to direct. Furthermore, since $|A| - k, |B| - \ell$, and $k + \ell$ are bounded by $n/6$ (by assumption and $(S1.4)$), the strategy asks OBreaker to direct at most

$$k + 2|B| + |A| + 2 = 2(|B| - \ell) + 2(k + \ell) + (|A| - k) + 2 \leqslant 5 \left\lfloor \frac{n}{6} \right\rfloor + 2 \leqslant b$$

edges in one round of Stage I. We need to show that the properties are restored. For ease of notation, let us assume that v and w were in $V \setminus (A \cup B)$, so OBreaker added those to A. Let $f_1, \ldots, f_t$ be the arcs OBreaker directed. Let $S'_A = S_A \cup \{e\}$ and $D' = D \cup \{e, f_1, \ldots, f_t\}$ be the new digraph after OBreaker's move. It is obvious from the strategy description that the pair (A', B') forms a UDB again, where $A' = A \cup \{v, w\}$ and $B' = B \cup \{y'\}$. Since both v and w were added to A, and by Lemma 3.1.4, $(D'[V \setminus B'], S'_A)$ is an α-structure of rank $k + 1$

in $V \setminus B'$, and $S'_A \subseteq A' \times A'$. So $(S1.1)$ holds again. Since OBreaker's edges either belong to the UDB (A', B') or live in $A' \times A'$, $(S1.2)$ still holds trivially. For $(S1.4)$ note that k, the rank of the α-structure in $V \setminus B$, increased by one. But OBreaker added two vertices to A, so $|A| - k$ increased by one. Also ℓ, the rank of the α-structure in $V \setminus A$, did not change, while we increased $|B|$ by one. Finally, for $(S1.3)$ note that $k + \ell$ increases by exactly one in each round.
The case $\{v, w\} \subseteq V \setminus A$ is analogous due to the symmetry of the properties.

For **Stage II**, it is clear that $(S1.1)$ and $(S1.2)$ imply $(S2.1)$ and $(S2.2)$. $S(2.3)$ follows by assumption of entering Stage II. So assume, the three properties hold before OMaker's move in this stage. Assume first that for OMaker's arc $e = (v, w)$ it holds that $\{v, w\} \subseteq V \setminus B$. As in Stage I, OBreaker can add e to the α-structure in $V \setminus B$. Similarly as in Stage I, by $(S2.2)$, for all edges $z \in V \setminus (A \cup B)$ all $y \in B$ none of the pairs $\{z, y\}$ has been directed so far. So OBreaker can direct all edges (v, y), or (v', y) respectively for $y \in B$. Furthermore, the strategy asks him to direct at most

$$|B| + k = |V| - (|A| - k) \leqslant b$$

edges, by property $(S2.3)$. Finally, the properties are restored. By Lemma 3.1.4, $(D[V \setminus B], S_A)$ is an α-structure in $V \setminus B$ again. Also, $e = (v, w)$ is added to the α-structure as an edge of type (α_1). Since v is added to A by directing all edges (v, b) for $b \in B$, $(S2.1)$ follows. There is nothing to prove for $(S2.2)$. For $(S2.3)$, note that only k increased, and OBreaker added exactly one new vertex to A (v or v'), so $|A| - k$ did not decrease and the claim follows.
Now assume that for OMaker's arc $e = (v, w)$ it holds that $\{v, w\} \not\subseteq V \setminus B$. As mentioned, at least one of the two vertices must lie in B then, and we may assume without loss of generality that $v \in B$, and also that $w \in V \setminus (A \cup B)$. As usual, by Lemma 3.1.4, OBreaker can add e to the α-structure in $V \setminus A$. By $(S2.1)$, for all $x \in A$, either the pair $\{x, w\}$ is not directed yet, or $(x, w) \in D$ (in this case, $w = e_i^-$ for some e_i being a special edge of the α-structure in $V \setminus B$). So OBreaker can follow the proposed strategy. Similar to the first case, this takes him at most $|A| + \ell = |V| - (|B| - \ell) \leqslant b$ edges. It is also easy to see now that the properties are restored. For (2.1), note that we may delete a vertex from the α-structure in $V \setminus B$ (if $w = e_i^-$ for some e_i), and all incident arcs. But as we observed in 3.1.1 (i), deleting a vertex and all incident arcs does not harm the α-structure. Property $(S2.2)$ follows again by Lemma 3.1.4, and since OBreaker added w to B. $(S2.3)$ follows as before for $|B| - \ell$ ($|A| - k$ might have increased though if we deleted one or more special edges from the α-structure $(D[V \setminus B], S_A)$.

Finally, it is straight-forward that OBreaker can follow the strategy proposed in **Stage III**. Since OBreaker plays in Stage II until $A \cup B = V$ (and the sets indeed enlarge in each round), and by $(S2.3)$ it follows that $|A|, |B| \leqslant b$. He then plays either inside A or B according to the strategy given by Lemma 3.1.4 until all edges are claimed. Therefore, in one round, OBreaker needs to direct at most $|A| \leqslant b$ or $|B| \leqslant b$ edges.

This finishes the proof of Theorem 1.1.3. □

3.3 OBreaker's strategy for the strict rules

In the proof of Theorem 1.1.3 in the previous section, OBreaker's strategy was to build a UDB (A, B) such that both parts have size at least $n - b$. Then, depending which edge OMaker directs, OBreaker plays either inside $V \setminus A$ or $V \setminus B$, both of size at most b. Inside these sets, $V \setminus A$ or $V \setminus B$, OBreaker then plays a variant of the trivial strategy. This particular variant used the notion of α-structures, which are powerful to maintain certain properties by directing only few edges. When OBreaker is asked to direct exactly b edges in every round, he might have to direct edges which could harm him. We will again use the idea of building a UDB of size at least $n-b$, so that OBreaker can play either inside $V \setminus A$ or $V \setminus B$. However, we abandon the idea of using α-structures.

Proof of Theorem 1.1.4. Let $b \geqslant n - c\sqrt{n}$ for some constant $c < 1$. We first provide OBreaker with a strategy, then we prove that he can follow that strategy, and that it constitutes a winning strategy.

Let $e_1 = (v_1, w_1)$ be the very first edge OMaker directs. Then OBreaker picks two disjoint subsets $A, B \subseteq V$ such that $v_1 \in A$, $w_1 \in B$ and $|A| = |B| = \lfloor \sqrt{b} \rfloor$. Note that for $n = n(c)$ large enough,

$$\lfloor \sqrt{b} \rfloor \geqslant \sqrt{b} - 1 \geqslant n - b + 1.$$

He then directs all edges (x, y) for $x \in A$ and $y \in B$. Let $t = b - \lfloor \sqrt{b} \rfloor^2 - 1$ be the number of edges OBreaker still needs to direct. He picks a vertex $a \in A$ and t arbitrary vertices $v_1, \dots, v_t \in V \setminus (A \cup B)$ and directs all edges (a, v_i) for $1 \leqslant i \leqslant t$.

After his first move, OBreaker maintains a UDB (A, B) and partitions $A = A_D \dot\cup A_{AD} \dot\cup A_S \dot\cup A_0$ and $B = B_D \dot\cup B_{AD} \dot\cup B_S \dot\cup B_0$, where A_D, A_{AD}, etc. may be empty at any point during the game, and A_S and B_S are sets of at most one vertex. The subscripts stand for *Dead*, *Almost Dead* and *Star*. In a particular round of the game, before OMaker's move, let $k = |A_{AD}|$ and $\ell = |B_{AD}|$. If $A_S \neq \emptyset$ we refer to the distinct element in A_S as v_{k+1}. Similarly, w_1 is the distinct element in B_S, if it exists. Furthermore, the following properties hold immediately after OBreaker's move.

(*a*) *Dead vertices:* The pairs $(A_D, V \setminus A_D)$ and $(V \setminus B_D, B_D)$ form UDB's. Furthermore, $D[A_D]$ and $D[B_D]$ form transitive tournaments each.

(b) *Almost-dead vertices:* The pairs $(A_{AD}, V \setminus A)$ and $(V \setminus B, B_{AD})$ form UDB's.

(c) *Structure of $A_{AD} \cup A_S$ and $B_{AD} \cup B_S$:* There are enumerations $A_{AD} = \{v_1, \ldots, v_k\}$, $B_{AD} = \{w_2, \ldots, w_{\ell+1}\}$ such that

  (i) $(v_1, \ldots, v_k, v_{k+1})$ and $(w_1, w_\ell, \ldots, w_{\ell+1})$ induce transitive tournaments in D.

  (ii) For all $1 \leqslant i < j \leqslant k+1$, all $w \in A_0$: If $(v_j, w) \in D$ then $(v_i, w) \in D$.

  (iii) For all $1 \leqslant i < j \leqslant \ell+1$, all $v \in B_0$: If $(v, w_i) \in D$ then $(v, w_j) \in D$.

(d) *Stars:* If $A_S \neq \emptyset$ then there exists at most one $w \in A_0$ such that $(v_{k+1}, w) \in D$. Similarly, if $B_S \neq \emptyset$ then there exists at most one $v \in B_0$ such that $(v, w_1) \in D$.

(e) For all edges $(v, w) \in D$, either $v \in A_D \cup A_{AD} \cup A_S$ or $w \in B_D \cup B_{AD} \cup B_S$; or $(v, w) \in A \times B$.

(f) *Sufficient sizes:* $|A_D \dot\cup A_0| \geqslant n - b$ and $|B_0 \dot\cup B_D| \geqslant n - b$.

The local and the global structure that is maintained is illustrated in Figure 3.1 and 3.2. Note that once the partitions are declared, the properties (a)-(f) determine exactly which arcs are in D and which are not, except for the ones starting in A_S, those ending in B_S, (i.e. arcs of the form (v_{k+1}, w) and (v, w_1) for $v, w \in V \setminus (A \cup B)$) and except for possible arcs from $A_{AD} \cup A_S$ to A_0 and arcs from B_0 to $B_{AD} \cup B_S$. Before we give the explicit strategy how to maintain the above properties, consider the *dual* structure $\overleftarrow{D}$ of all reverse arcs, with sets $A' := B$, $B' := A$ and partitions

$$A' = A'_D \dot\cup A'_{AD} \dot\cup A'_S \dot\cup A'_0 \text{ and } B' = B'_D \dot\cup B'_{AD} \dot\cup B'_S \dot\cup B'_0,$$
$$\text{where } A'_* = B_* \text{ and } B'_* = A_*. \tag{3.1}$$

Then (A', B') is a UDB in $\overleftarrow{D}$, and properties (a)-(f) hold for the given partition (in (c), reverse the order of $v_1, \ldots, v_{k+1}$ and of $w_1, \ldots, w_{\ell+1}$). This observation shortens our case distinction by a significant amount. Finally, note also that (f) implies that

$$|A| \geqslant |A| - |A_{AD}| \geqslant |A_D \dot\cup A_0| \geqslant n - b \quad \text{and}$$
$$|B| \geqslant |B| - |B_{AD}| \geqslant |B_0 \dot\cup B_D| \geqslant n - b.$$

Before a move of OMaker, let a UDB (A, B) be given with partitions $A = A_D \dot\cup A_{AD} \dot\cup A_S \dot\cup A_0$ and $B = B_D \dot\cup B_{AD} \dot\cup B_S \dot\cup B_0$ such that all properties (a)-(f) hold. Let (v, w) be OMakers next edge. We first provide OBreaker with a *base strategy* to direct *at most* b edges to include (v, w) into the structure and to restore properties (a)-(f). Let t be the number of edges OBreaker directed in the base strategy. We then provide OBreaker with an *add-edges strategy* to direct $b - t$ further edges (and suitably update the sets) such that the properties hold again.

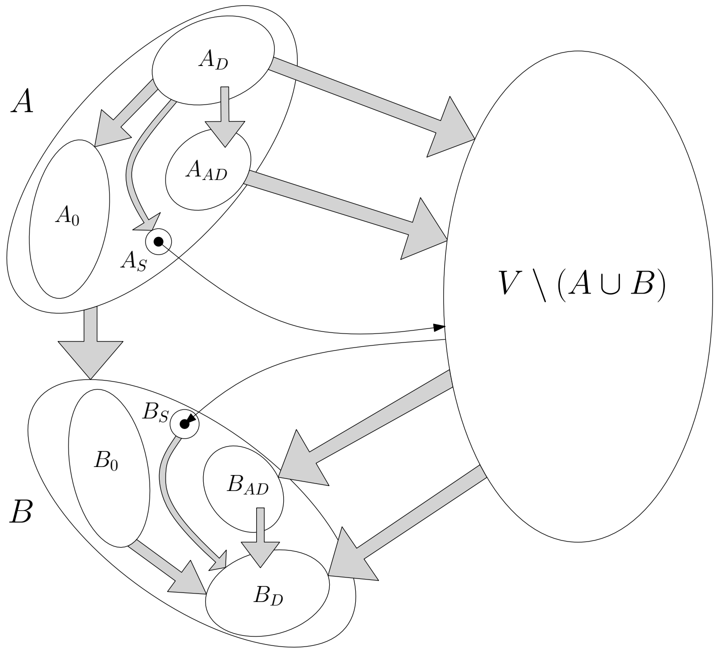


Figure 3.1: Global structure. A thick arrow indicates a UDB. A thin arrow indicates that arcs in that direction could be there, but not necessarily.

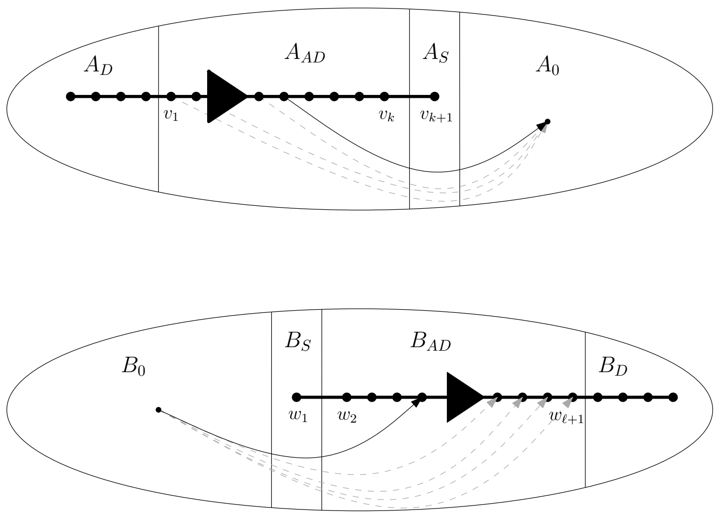


Figure 3.2: Local structure. The thick arrow indicates a tournament. The thin arrow indicates that arcs in that direction could be there, but not necessarily.

Base strategy:
Let (v, w) be OMakers arc. Assume first that $\{v, w\} \in V \setminus B$. We need to divide the strategy into cases depending on whether $v \in V \setminus (A \cup B)$ (whence we want to add it to A) or $v \in A$. The division into subcases is necessary for precision and notation, though we want to stress that the general philosophy is similar in all subcases in Case 1, and all subcases in Case 2. Whenever the strategy asks OBreaker to direct an edge (x, y) such that $(x, y) \in D$ already, he ignores that command and continues. When the strategy asks OBreaker to direct an edge (x, y) such that $(y, x) \in D$, then he forfeits the game.
Case 1: $v \in V \setminus (A \cup B)$. Then by Property (a) and (b), $w \notin A_D \cup A_{AD}$.

1.1 If $A_S = \emptyset$. Then OBreaker directs all edges (v, w') for $w' \in V \setminus A$; and all edges (v_i, w) for $1 \leqslant i \leqslant k$. He updates $A_{AD} := A_{AD} \cup \{v\}$.

1.2 If $A_S = \{v_{k+1}\} \neq \emptyset$ and $w \neq v_{k+1}$. Note that then $w \in A_0 \cup (V \setminus (A \cup B))$. Then, for all $w' \in V \setminus (A \cup B)$, OBreaker directs all (v_{k+1}, w'). Furthermore, he directs all (v, y) for $y \in B$, and for all $1 \leqslant i \leqslant k+1$, he directs (v_i, w). He updates $A_{AD} := A_{AD} \cup A_S$ and $A_S := \{v\}$.

1.3 If $w = v_{k+1}$ is the unique element in A_S. Then OBreaker directs all edges (v, w') for $w' \in V \setminus A$. Furthermore, for all edges (w, w'), he directs (v, w'). He updates $A_{AD} := A_{AD} \cup \{v\}$.

Case 2: $v \in A$. By (a), $v \notin A_D$.

2.1 If $v = v_i \in A_{AD}$. Then by (a), (b) and $(c)(i)$, $w \in A_0$. Then OBreaker directs all (v_j, w) for $1 \leqslant j < i$.

2.2 If $v \in A_S$. Then by (a), and $(c)(i)$, $w \in A_0 \cup (V \setminus (A \cup B))$. Then for all $w' \in V \setminus (A \cup B)$, OBreaker directs the edges (v, w'). And for all $1 \leqslant i \leqslant k$, he directs the edge (v_i, w). He then updates $A_{AD} := A_{AD} \cup A_S$ and $A_S := \emptyset$.

2.3 If $v \in A_0$. For a technical reason, we need to divide whether v is an endpoint of the tournament in $A_{AD} \cup A_S$, or not. Let u be the last vertex in the tournament in $A_{AD} \cup A_S$. That is, if $A_S = \emptyset$ then $u = v_k$, and otherwise $u = v_{k+1} \in A_S$.

2.3.1 If $A_{AD} \cup A_S \neq \emptyset$ and $(u, v) \in D$. Then by property $(c)(ii)$ for all $1 \leqslant j \leqslant k+1$: $(v_j, v) \in D$. Then OBreaker directs all edges (u, w') for $w' \in V \setminus (A \cup B)$. Furthermore, he directs all (v_j, w) for all $1 \leqslant j \leqslant k+1$. Finally, he directs all (v_1, a) for $a \in A_0$. Note that since $A_{AD} \cup A_S \neq \emptyset$, v_1 exists. He then updates $A_D := A_D \cup \{v_1\}$, $A_0 := A_0 \setminus \{v\}$, $A_{AD} := (A_{AD} \cup A_S) \setminus \{v_1\}$ and $A_S := \{v\}$.

2.3.2 If $A_{AD} \cup A_S = \emptyset$ or $(u, v) \notin D$. Set $i := 1$ if $A_{AD} \cup A_S = \emptyset$ and otherwise set $i := \min\{j : (v_j, v) \notin D\}$. That is, by property $(c)(ii)$, for all $1 \leqslant j < i$: $(v_j, v) \in D$, and for all $i \leqslant j \leqslant k+1$: $(v_j, v) \notin D$. Then OBreaker directs all edges (v, w') for $w' \in V \setminus (A \cup B)$. Furthermore, he directs all (v_j, w) for all $j < i$, and all (v, v_j) for $j \geqslant i$. For $a \in A_0$, if $(v_i, a) \in D$, then OBreaker directs the edge (v, a). He now updates $v_j := v_{j-1}$ for all $j > i$ and $v_i := v$. He then directs all (v_1, a) for $a \in A_0$. Finally, he updates $A_D := A_D \cup \{v_1\}$, $A_0 := A_0 \setminus \{v_i\}$, and $A_{AD} := (A_{AD} \cup \{v_i\}) \setminus \{v_1\}$.

Assume now that $\{v, w\} \not\subseteq V \setminus B$. Since (A, B) is a UDB we therefore have that $\{v, w\} \subseteq V \setminus A$. Consider now the dual $\overleftarrow{D}$ with partitions given by (3.1), where (A', B') is the main UDB. As mentioned, the properties (a)-(f) are fulfilled for that choice of sets and partitions. Furthermore, $\{v, w\} \subseteq V \setminus B'$. Let $\mathcal{S}$ be the strategy given above for inserting the reverse arc (w, v) into the dual structure $\overleftarrow{D}$. That is, $\mathcal{S}$ gives OBreaker a set of arcs $\{f_1, \ldots, f_t\} \subseteq \mathcal{A}(\overleftarrow{D}) = \mathcal{A}(D)$ to direct and some update rules. OBreaker now directs the reverse arcs $\overleftarrow{f_1}, \ldots, \overleftarrow{f_t}$, and updates the sets and partitions according to the dualization.

Add-edges strategy:

In a particular round, let t be the number of edges OBreaker directed to follow the base strategy. He then directs $b - t$ further edges by repeatedly applying the following proposition.

Proposition 3.3.1. *Let there be sets A, B with partitions*

$$A = A_D \dot{\cup} A_{AD} \dot{\cup} A_S \dot{\cup} A_0 \text{ and } B = B_D \dot{\cup} B_{AD} \dot{\cup} B_S \dot{\cup} B_0$$

with $|A|, |B| \geqslant n - b + 1$ such that (A, B) forms a UDB and such that the properties (a)-(f) hold. Then, unless D is a transitive tournament on K_n, there exists an available arc $(x, y) \in \mathcal{A}(D)$ such that for $D \cup (x, y)$ and suitably updated sets A and B, the properties (a)-(f) hold again. Moreover, the update does not decrease $|A|$ or $|B|$.

Note that after OBreaker's first move, $|A|, |B| \geqslant n - b + 1$, and that the base strategy described above never decreases $|A|$ or $|B|$. Hence, under the assumption that the base strategy yields properties (a)-(f) again, OBreaker can indeed apply this "add-edges" proposition. We now need to check that

(1) OBreaker can follow the strategy in his first move and the properties (a)-(f) hold after OBreakers first move;

(2) if the properties (a)-(f) hold, then there is no cycle in D, nor can OMaker close a cycle in her next move;

(3) OBreaker can follow the proposed base strategy, and he needs to direct at most b edges;

(4) after OBreaker's move of the base strategy, the properties (a)-(f) are restored with the updated sets;

(5) Proposition 3.3.1 is true.

For (1), it is clear that OBreaker can build a UDB (A, B) such that both sets have size $\lfloor b \rfloor \geqslant n - b + 1$. Furthermore, $|V \setminus (A \cup B)| = n - 2\lfloor b \rfloor \geqslant b - \lfloor b \rfloor^2 - 1$, so OBreaker can claim the remaining edges (a, v_i) for $v_i \in V \setminus (A \cup B)$, as requested. After the first move of OBreaker, properties (a)-(f) hold with $A_D = A_{AD} = B_D = B_{AD} = B_S = \emptyset$ and $A_S = \{a\}$.
For (2), suppose properties (a)-(f) hold, and let P be a path in D and $e \in D \cup \mathcal{A}(D)$ such that $P \cup e$ is a directed cycle. Since (A, B) is a UDB in D, the critical edge e lies either in $V \setminus A$ or in $V \setminus B$. Now, by property (e), no edge enters A from outside A, and no edge leaves B to the outside of B, so we have that $V(P) \subseteq A \cup B$ (where $V(P)$ are the vertices contained in P). Since $P \cup e$ is a cycle, we therefore have that either $V(P \cup e) \subseteq A$ or $V(P \cup e) \subseteq B$. Suppose first that $V(P \cup e) \subseteq A$. Since $P \subseteq D$ and by property (e) again, we have that $P^+ \subseteq A_D \cup A_{AD} \cup A_S$. But $A_D \cup A_{AD} \cup A_S$ induces a transitive tournament by $(c)(i)$. Then the only possibility to form a cycle is when $e^+ \in A_0$ and $e^- \in A_D \cup A_{AD} \cup A_S$. Let v_i be the last vertex in the order of that tournament in P, and let $v_j = e^-$. Then $j < i$ by choice of v_i, and $(v_i, e^+) \in P \subseteq D$. So by property $(c)(ii)$, $(e^-, e^+) = (v_j, e^+) \in D$, a contradiction to $e \in D \cup \mathcal{A}(D)$. The case when $V(P \cup e) \subseteq B$ is analogous.

Point (3) and (4), we prove case by case. Let $e = (v, w)$ be the arc directed by OMaker in her previous move. If $\{v, w\} \not\subseteq V \setminus B$, then we considered the dual $\overleftarrow{D}$ with UDB (A', B'), where $A' = B$ and $B' = A$, and inserted the reverse arc (w, v) into the dual structure, where $\{v, w\} \subseteq V \setminus B'$. Therefore, we only need to consider the case when $\{v, w\} \subseteq V \setminus B$. For the sake of completeness, we list all (sub)cases here. We want to mention though for the impatient reader that all cases follow the same pattern. If $v \in V \setminus (A \cup B)$, the strategy adds it to either A_{AD} or A_S. If $v \in A$, then some vertex (namely the top vertex in the tournament of A_{AD}) is added to A_D, and therefore *dead*. We consider Case 2.3.1 to be the not-so-trivial and maybe most interesting case.
Case 1: $v \in V \setminus (A \cup B)$.

1.1 By property (e), all arcs of the form (w', v) have $w' \in A$. Also, all arcs of the form (w, w') have $w' \in B$. So, OBreaker can follow the proposed strategy. Furthermore, the strategy asks him to direct at most $|V \setminus A| + |A_{AD}| = |V| - (|A| - |A_{AD}|) \leqslant b$ edges by property (f). Finally, the properties are restored. Since A_D, A_S, A_0 and B are unchanged, there is nothing to prove for (a), (d) and (f). (b) and (e) follow immediately from the strategy description. For (c), set $v_{k+1} := v$ and note that since $v \in V \setminus (A \cup B)$ before OBreaker's move, all arcs (v_i, v) are already edges in D, so (i) follows. Further, for any $(v, w') \in D$, either $w' \in B$ (whence the edges (v_i, w') are already present by (b))

or $w' = w$ in which case the edges (v_i, w) are added by the strategy. So, (ii) follows. (iii) concerns the structure in B and is irrelevant in this case.

1.2 By property (e), all edges of the form (w', v_{k+1}) or (w', v) have $w' \in A$. Also, all edges of the form (w, w') have $w' \in B$. So, OBreaker can follow the proposed strategy. Furthermore, the strategy asks him to direct at most $|V \setminus (A \cup B)| + |B| + |A_{AD} \cup A_S| = |V| - |A_D \cup A_0| \leqslant b$ edges by property (f). Finally, the properties are restored. Since A_D, A_0 and B are unchanged, there is nothing to prove for (a) and (f). (b) and (e) follow immediately from the strategy description. (A, B) is a UDB again since only v was moved to A and was connected to all of B. (c) follows similarly as above: set $v_{k+2} := v$. The arcs (v_i, v_{k+2}) are all present in D for $1 \leqslant i \leqslant k$ by (b) and since $v \in V \setminus (A \cup B)$ before the update. The edge (v_{k+1}, v_{k+2}) is added by the strategy unless it was there already. So $(c)(i)$ follows. For $(c)(ii)$, note that the only edges added of the form (v_{k+1}, w') have $w' \in V \setminus (A \cup B)$, so each (v_i, w') is in D already. Also, the only edges of the form (v_{k+2}, w') have $w \in B \cup \{w\}$, so the arcs (v_i, w') are either already in D, or added by the strategy. (d) follows because the only arcs of the form (v_{k+2}, w') have $w' \in B \cup \{w\}$.

1.3 By property (e), all arcs of the form (w', v) have $w' \in A_D \cup A_{AD}$. Also, all arcs of the form $(w, w') = (v_{k+1}, w')$ have $w' \in V \setminus (A_D \cup A_{AD})$ by (a) and $(c)(i)$. So, OBreaker can follow the proposed strategy. Moreover, by property (d), there is at most one $w' \in A_0$ such that $(v_{k+1}, w') \in D$. Therefore, the strategy asks him to direct at most $|V \setminus A| + 1 = |V \setminus A| + |A_S| \leqslant |V| - |A_D \cup A_0| \leqslant b$ edges by property (f). Finally, the properties are restored. (a), (b), (d), (e) and (f) are straight-forward as in Case 1.1. For (c), since v is added to A_{AD}, set $v_{k+2} := v_{k+1}$ and then $v_{k+1} := v$. We need to show that $(v_1, \ldots, v_k, v_{k+1}, v_{k+2})$ induces a transitive tournament. All the arcs (v_i, v_{k+1}), $1 \leqslant i \leqslant k$, are already present in D by (b). The arc (v_{k+1}, v_{k+2}) was directed by OMaker. Since $(v_1, \ldots, v_k, v_{k+2})$ was a transitive tournament already, $(c)(i)$ follows. We need to show $(c)(ii)$ only for $j = k+1$ and $j = k+2$. Any arc of the form (v_{k+1}, w') was either directed by OMaker or OBreaker in the very last move. Therefore, either $w' \in V \setminus A$ or $w' = v_{k+2}$, or $(v_{k+2}, w') \in D$ as well. In each case, all arcs (v_i, w') are present in D since (b), $(c)(i)$ and $(c)(ii)$ hold before OMaker's move by assumption. Finally, for any arc of the form (v_{k+2}, w'), the arcs (v_i, w') are present for $1 \leqslant i \leqslant k$ by assumption and $(c)(ii)$, and for $i = k+1$ by the strategy description.

Case 2: $v \in A$.

2.1 The only property that might be invalid after this move of OMaker is Property $(c)(ii)$. But this is exactly restored by the strategy description. By property (b), OBreaker can direct all those edges (v_j, w), and he needs to direct at most $k \leqslant |A| \leqslant |V \setminus B| \leqslant b$ edges.

2.2 This case is also straight forward. The strategy is exactly property restoring; and it takes at most $|V \setminus (A \cup B)| + |A_{AD}| \leqslant b$ edges as before.

2.3.1 Recall that we set $u = v_k$ if $A_S = \emptyset$ and $u = v_{k+1}$ otherwise. Note that if A_S is empty, the label $k+1$ does not have a host vertex, the following argument is still valid though. By property (e), all arcs of the form (w', u) have $w' \in A$, so for $w' \in V \setminus (A \cup B)$, either the arc (u, w') is already present in D, or OBreaker can direct it. Furthermore, all (v_j, w) for $1 \leqslant j \leqslant k+1$ are either in D already or in $\mathcal{A}(D)$. So OBreaker can direct (v_j, w). Finally, all arcs of the form (a, v_1) have $a \in A_D$, so for $a \in A_0$, OBreaker can direct (v_1, a). In total, the strategy asks OBreaker to direct at most $|V \setminus (A \cup B)| + k + 1 + |A_0| \leqslant |V \setminus B| \leqslant b$ edges. Now, we show that all the properties are restored. For (a), note that v_1 is added to A_D. Since $D[A_D]$ was a transitive tournament before, and since $(A_D, V \setminus A_D)$ was a UDB (both by (a)), adding v_1 yields that $A_D \cup \{v_1\}$ also induces a transitive tournament. The arcs (v_1, w') for $w' \in V \setminus A$ are present in D since either $v_1 \in A_{AD}$ before OMaker's move and by property (b); or $v_1 \in A_S$ whence the edges (v_1, w') for $w' \in V \setminus (A \cup B)$ were added by the strategy. The arcs (v_1, w') for $w' \in A_{AD} \cup A_S$ were present in D by property $(c)(i)$. Finally, the arcs (v_1, w') for $w' \in A_0$ were either present before or they were added by OBreaker in this round. So, $(A_D \cup \{v_1\}, V \setminus (A_D \cup \{v_1\}))$ is a UDB again, as claimed. For (b), we have to note that v_1 was shifted from A_{AD} to A_D (and thus does not harm property (b)) and u was shifted to A_{AD} (unless $A_S = \emptyset$ in which case there is nothing to prove for (b)). However, by the strategy description, OBreaker directed all edges (u, w') for $w' \in V \setminus (A \cup B)$; and all arcs (u, w') for $w' \in B$ were already in D. Hence, (b) holds again. For $(c)(i)$, we need to check that $(v_2, \ldots, u, v)$ induces a tournament. Clearly, $(v_2, \ldots, u)$ induces a tournament, since $(c)(i)$ was true before OMaker's move. By assumption of this case, $(u, v) \in D$ and $v \in A_0$ before the update, so by $(c)(ii)$ all the arcs (v_j, v) were in D before for $1 \leqslant j \leqslant k+1$. So $(c)(i)$ follows. Now, for $(c)(ii)$ set $v_{k+2} := v$. We need to check $(c)(ii)$ for $2 \leqslant i < j \leqslant k+2$. Clearly, the property holds for $2 \leqslant i < j \leqslant k+1$, since this was true before. Now, the only arcs of the form (v_{k+2}, w') have $w' \in B \cup \{w\}$ since $v_{k+2} = v \in A_0$ before the update and by property (e). But all arcs (v_i, w') for $2 \leqslant i < j \leqslant k+1$ and $w' \in B$ were present in D before, and all arcs (v_i, w) were either present already or added by OBreaker. So, $(c)(ii)$ follows. For $(c)(iii)$, there is nothing to prove. For property (d), after the update we have $A_S = \{v\}$. But the only edges of the form (v, w') have $w' \in B \cup \{w\}$ by (e), so (d) holds again. Now, (e) holds since we added v to A_S and OBreaker only directed edges (v', w') with

$v' \in A_{AD} \cup A_S$. Finally, (f) follows since we deleted one vertex (namely v) from A_0 and added one vertex (namely v_1) to A_D.

2.3.2 Recall that we set $i := 1$ if $A_{AD} \cup A_S = \emptyset$ and $i := \min\{j : (v_j, v) \notin G\} \leqslant k+1$ otherwise. Again, the label $k+1$ may have no host vertex. By property (e), all arcs of the form (w', v) have $w' \in A$, so for $w' \in V \setminus (A \cup B)$, either the edge (v, w') is already present in D (which only happens if $w' = w \in V \setminus (A \cup B)$), or OBreaker can direct it. Furthermore, by definition of i and by property $(c)(ii)$, for all $1 \leqslant j < i$, $(v_j, v) \in G$ already, and for all $i \leqslant j \leqslant k+1$, the arc (v, v_j) is available before OMaker's move ($w = v_j$ of OMaker's edge (v, w) is possible, in which case one of the pairs has direction (v, v_j)). Therefore, OBreaker can direct all arcs (v, v_j) for $i \leqslant j \leqslant k+1$, as requested by the strategy. Moreover, either $w = v_\ell$ for some $\ell \geqslant i$, or $w \in A_0 \cup V \setminus (A \cup B)$. In both cases, for all $1 \leqslant j < i$, the arc (v_j, w) is either already present in D or undirected, by properties $(c)(i)$ and (e). Hence, OBreaker can claim (v_j, w) for $1 \leqslant j < i$ as well. Finally, by property (e), all arcs (v, a) for $a \in A_0$ are available (unless $a = w$) since $v \in A_0$ and by property (e). Therefore, OBreaker can direct all these edges (v, a). Similarly, he can direct all edges (v_1, a) for $a \in A_0$.
To see that OBreaker directs at most b edges, note that for each $a \in A_0$, either $(v_i, a) \in D$, or the arc is available. In the second case, the strategy asks OBreaker to direct (v_1, a). In the first case, the strategy asks OBreaker to direct (v, a) *and* (v_1, a). However, by property $(c)(ii)$, the edge (v_1, a) is already present in D. So, for all $a \in A_0$, OBreaker directs at most one edge. Furthermore, for all $1 \leqslant j \leqslant k+1$, OBreaker directs at most one edge (either (v_j, w) or (v, v_j)). Hence, in total, OBreaker directs at most $|V \setminus (A \cup B)| + |A_{AD} \cup A_S \cup A_0| \leqslant n - |B| \leqslant b$ edges by (f).
Now, we show that all the properties are restored. For (a), assume first that $A_{AD} \neq \emptyset$. Then the vertex v_1 which we move to A_D was in A_{AD} before, and thus, similar to the previous case, all arcs of the form (v_1, w') for $w' \in A_{AD} \cup A_S \cup (V \setminus B)$ were already present, and arcs (v_1, w') with $w' \in A_0$ were added by OBreaker in this round. So assume now for (a) that $A_{AD} = \emptyset$. Then by assumption $(u, v) \notin D$ (or $A_S = \emptyset$ as well) and thus $i = 1$. Hence, $v = v_1$ (after the update) is the vertex moved to A_D. We check that all arcs (v, w') for $w' \in V \setminus A_D$ are there. For $w' \in B$ this is certainly true since $v \in A_0$ before the update and (A, B) was a UDB. Now, for $w' \in V \setminus (A \cup B)$ the edges (v, w') are added by OBreaker in this round. If $A_S = \{u\} \neq \emptyset$, then (v, u) is directed by OBreaker. Furthermore, all edges (v, w') for $w' \in A_0$ are directed by OBreaker as well. Property (a) follows. Property (b) follows similarly as in the previous case: v is (potentially) added to A_{AD}, and all necessary edges (v, w') for $w' \in V \setminus (A \cup B)$ are directed by OBreaker or are already there. For (c), consider the following ordering

$(v'_1, \dots, v'_{k+2})$ where

$$v'_j := \begin{cases} v_j & \text{for } 1 \leqslant j < i \\ v & \text{for } j = i \\ v_{j-1} & \text{for } i < j \leqslant k+2. \end{cases}$$

Now, $(v_1, \dots, u)$ induces a transitive tournament in D because property $(c)(i)$ holds before OMakers move. We claim that adding v between position $i-1$ and i forms again a transitive tournament after OBreakers move. For all $j < i$, the edges $(v'_j, v'_i) = (v_j, v)$ are already present in D as mentioned above. For all $j > i$, the pairs $\{v'_i, v'_j\} = \{v, v_{j-1}\}$ are undirected before OBreakers move, as mentioned above, and OBreaker directs all edges (v'_i, v'_j) by the strategy description. The claim follows. Now, $(c)(ii)$ follows for the same ordering exactly by the strategy description.
Since A_S is unchanged, there is nothing to prove for (d). Property (e) holds, since the only new start point of an edge v is moved to $A_D \cup A_{AD} \cup A_S$. Finally, v is removed from A_0, but v_1 is added to A_D, so (f) follows.

It remains to show (5).

Proof of Proposition 3.3.1. Assume first that $A_{AD} \neq \emptyset$. Then we may assume that there exists a $y \in A_0$ such that the pair $\{v_1, y\}$ is not directed, where v_1 is the top vertex in the tournament on A_{AD} as before. Otherwise, we set $A_D := A_D \cup \{v_1\}$ and $A_{AD} := A_{AD} \setminus \{v_1\}$ and reapply the Proposition. Then OBreaker directs (v_1, y) and it is obvious that the properties (a)-(f) hold again.
So assume from now on that $A_{AD} = \emptyset$. If $A_S \neq \emptyset$ let v_1 be the unique element in A_S (note that since $k = |A_{AD}| = 0$ this is consistent with our notation of v_{k+1} above). We similarly may assume that there exists a $y \in V \setminus (A \cup B)$ such that the pair $\{v_1, y\}$ is not directed, for otherwise we reapply the proposition with $A_{AD} := A_S$ and $A_S := \emptyset$. OBreaker then directs (v_1, y) and it is obvious that the properties (a)-(f) hold again.
So assume from now on that $A_{AD} = A_S = \emptyset$. If $A_0 \neq \emptyset$ and $V \setminus (A \cup B) \neq \emptyset$, then pick some $x \in A_0$, $y \in V \setminus (A \cup B)$, direct (x, y) and set $A_S := \{x\}$ and $A_0 := A_0 \setminus \{x\}$. It is easy to see that all the structural properties (a)-(e) hold again. For (f), since by assumption $|A| \geqslant n - b + 1$ and $A = A_D \cup A_0$, it follows that after moving x to A_S (f) still holds.
If $|A_0| \geqslant 2$ and $V \setminus (A \cup B) = \emptyset$, then pick $x, y \in A_0$, direct (x, y) and set $A_{AD} := \{x\}$ and $A_0 := A_0 \setminus \{x\}$. The properties follow as in the previous case. If $A_0 = \{x\}$ is a singleton and $V \setminus (A \cup B) = \emptyset$, set $A_D := A_D \cup A_0$ and $A_0 = \emptyset$ and reapply the lemma.
So we may assume that $A_{AD} = A_S = A_0 = \emptyset$, that is, $A = A_D$ and thus $D[A]$ is a transitive tournament and $(A, V \setminus A)$ is a UDB.
Now, by a similar analysis, we can either direct an edge in $V \setminus A$ or deduce that $B = B_D$,

that is, $D[B]$ forms a transitive tournament and $(V \setminus B, B)$ is a UDB. By assumption, D is not a transitive tournament on K_n, hence there must be an undirected pair $\{x, y\}$. Since $A = A_D$ and $B = B_D$, both x, y must lie in $V \setminus (A \cup B)$. Then OBreaker directs (x, y) and sets $A_S := \{x\}$ and updates $A := A \cup \{x\}$. Note that since $(V \setminus B, B)$ is a UDB, all the edges (x, z) for $z \in B$ are present. It is easy to see that (a)-(f) hold for the updated sets. □

This finishes the proof of Theorem 1.1.4 □

3.4 Concluding remarks

Before we comment on how good the upper bounds in Theorem 1.1.3 and 1.1.4 are let us first take a look at the strategy for OMaker given in [11] which gives the so far best known lower bound. Here, the idea is that OMaker first builds a long directed path (of length $n-1$) in at most $n-1$ rounds. When $b \leqslant n/2 - 2$, OBreaker cannot have directed all "backward" edges of this path, so OMaker can direct one of those and close a cycle.

From the perspective of OBreaker, it is indeed most harmful if OMaker builds a long path, as we have seen at the beginning of Section 3.1. Indeed, if OMaker builds a long path throughout the game, then in the k^{th} round she can create potentially k immediate threats. On the other hand, using the procedure α, OBreaker can ensure that he needs to orient at most k edges to answer every threat, even if OMaker plays another strategy than building a long path (cf. Lemma 3.1.4).

Even though the use of α-structures (and Lemma 3.1.4) suggests that building a long path is essentially the best OMaker can do, the lower bound of roughly $n/2$ and our upper bound of roughly $5n/6$ do not match at all. In the following, let us describe briefly why $n/\sqrt{2}$ is a lower bound to our strategy.

Suppose OBreaker managed to build a UDB (A, B) of size $n - b - 1$, and observe that then at least $(n-b-1)^2/(b+1)$ rounds (in Stage I) were played. Suppose that we are in the case, that OMaker always claimed edges in $V \setminus B$ such that the size k of the α-structure increased in each round. Assume further that OBreaker only wants to increase one of the values $|A| - k$ and $|B| - \ell$ (in order to decrease the number of edges to direct in one particular round). Without loss of generality let this be $|A| - k$. Then, in order to follow procedure α and to increase $|A| - k$ (by adding two vertices to A), OBreaker needs to direct at least $k + 2|B| \geqslant (n-b-1)^2/(b+1) + 2(n-b-1)$ edges, which is only possible if $b \geqslant n/\sqrt{2} + o(n)$.

Optimizing the constant of $5n/6$ in that direction seems to be rather technical, and since the best we could hope for is $n/\sqrt{2}$ which is still far from the lower bound of $n/2$, we rather kept the proof simple. We conjecture that the correct threshold is asymptotically at least $n/\sqrt{2}$.

Conjecture 3.4.1. *For n large enough and $b \leqslant n/\sqrt{2} - o(n)$, OMaker has a strategy to close a directed cycle in the monotone b-biased orientation game.*

Concerning the strict rules, OBreaker has to be a lot more careful where to direct remaining edges, since any additional arc can be used by OMaker to her advantage. So far, we have proven that for every $0 < c < 1$ and every large enough n, $t^+(n, \mathcal{P}) \leqslant n - c\sqrt{n} - 1$, where $\mathcal{P}$ is the property of containing an oriented cycle. The bound essentially comes from the first round in which OBreaker claims a UDB (A, B) with both parts having size at least $n - b$. With a smaller bias, OBreaker needs more rounds to build such a UDB (that was Stage I in the monotone-rules strategy). Building up a UDB *and* maintaining certain invariant properties in the strict rules seem possible, though very technical. We conjecture that there is a constant $\varepsilon > 0$ such that $t^+(n, \mathcal{P}) \leqslant (1 - \varepsilon)n$, i.e. that for $b \geqslant (1 - \varepsilon)n$ and n large enough, OBreaker has a strategy to prevent directed cycles in the strict b-biased orientation game. Moreover, we wonder whether $t^+(n, \mathcal{P})$ and $t(n, \mathcal{P})$ are (asymptically) equal.

4

What is Ramsey-equivalent to the clique?

In this chapter we study the questions which graphs are Ramsey-equivalent to the clique, and prove the results of Subsection 1.2.1. Recall that we say two graphs H and H' are Ramsey-equivalent if for every graph G, G is Ramsey for H if and only if it is Ramsey for H'. Throughout this chapter, all colourings are *red-blue*-colourings, and we omit the subscript $r = 2$ in our notation. As mentioned in the introduction, any graph H which is Ramsey-equivalent to the clique K_k must contain a copy of K_k. Here, we are concerned with the question of how much "bigger" than K_k such a graph H can be. In the first section of this chapter we study the clique with a hanging edge, denoted by $K_k \cdot K_2$. We show that for any $k \geqslant 3$, there exists a graph G which is Ramsey for K_k, but not Ramsey for $K_k \cdot K_2$. We will extend this result and prove a stronger statement about the minimum degree of minimal Ramsey graphs for $K_k \cdot K_2$, namely $s(K_k \cdot K_2) \leqslant k - 1$. It follows that if some graph H is Ramsey-equivalent to K_k it needs to be the disjoint union of a K_k and some graphs on fewer vertices. In Section 4.2, we will then study the question of how many disjoint (smaller) cliques we can add to K_k so that the resulting graph is still Ramsey-equivalent to K_k.

4.1 Hanging edges

In this section, we study the minimum degrees of graphs that are $K_k \cdot K_2$-minimal. Our plan is to construct a graph G that contains a vertex v of degree $k - 1$ which is "crucial" for G to be Ramsey for $K_k \cdot K_2$. That is, $G \to K_k \cdot K_2$, but $G - v \nrightarrow K_k \cdot K_2$. This implies that any minimal $K_k \cdot K_2$-Ramsey subgraph $G' \subseteq G$ (and certainly there is one!) has to contain v

and hence has to have minimum degree at most $k-1$. We therefore obtain the missing upper bound in Theorem 1.2.3.

We now proceed to develop tools useful for proving Theorem 1.2.3. The following theorem of Nešetřil and Rödl [47] states that there is a K_k-free graph F so that any two-colouring of the edges of F has a monochromatic K_{k-1}.

Theorem 4.1.1. *For every $k \geqslant 2$ there is some graph F so that F is K_k-free and $F \to K_{k-1}$.*

By a *circuit of length s* in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ we mean a sequence $e_1, v_1, e_2, v_2, \ldots, e_s, v_s$ of distinct edges $e_1, \ldots, e_s \in \mathcal{E}$ and distinct vertices $v_1, \ldots, v_s \in V$ such that $v_j \in e_j \cap e_{j+1}$ for all $1 \leqslant j < s$, and $v_s \in e_s \cap e_1$. In particular, if two distinct hyperedges intersect in two or more vertices, we consider this as a circuit of length 2. By the *girth* of a hypergraph $\mathcal{H}$ we denote the length of the shortest circuit in $\mathcal{H}$. The following lemma is proved in [33] by a now standard application of the probabilistic method [3].

Lemma 4.1.2. *For all integers $k, m \geqslant 2$ and every $\epsilon > 0$ there is a k-uniform hypergraph of girth at least m and independence number at most ϵn, where n is the number of vertices in the hypergraph.*

We will need a strengthening of Theorem 4.1.1 which states that there is a K_k-free graph F so that any two-colouring of the edges of F has a monochromatic K_{k-1} inside of every ε-fraction of the vertices.

Definition 4.1.3. *We write $F \xrightarrow{\varepsilon} K_{k-1}$ to mean that for every $S \subseteq V(F)$, $|S| \geqslant \varepsilon v(F)$ implies $F[S] \to K_{k-1}$.*

Lemma 4.1.4. *For every $\varepsilon > 0$ and $k \geqslant 2$ there exists a graph F which is K_k-free and $F \xrightarrow{\varepsilon} K_{k-1}$.*

Proof. The case where $k = 2$ is trivial, so we will assume that $k \geqslant 3$. Take F_0 to be as in Theorem 4.1.1. By Lemma 4.1.2 there is some $v(F_0)$-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ of girth at least 4 and independence number less than $\varepsilon |V|$. We construct a graph F on vertex set V. The edges of F are created by placing a copy of F_0 inside of each hyperedge in $\mathcal{E}$.

Since $\mathcal{H}$ has girth at least 4, any triangle of F must be contained in a single hyperedge of $\mathcal{H}$. Therefore, the vertex set of any copy of K_k in F must be contained in a single hyperedge of $\mathcal{H}$ as well. However, a single hyperedge forms just a copy of F_0 in F and F_0 has no copy of K_k, so F has no copy of K_k.

Since $\mathcal{H}$ has independence number less than $\varepsilon |V|$, any set S of at least $\varepsilon |V|$ vertices must contain some hyperedge. Hence, $F[S]$ contains a copy of F_0. As $F_0 \to K_{k-1}$, we also have $F[S] \to K_{k-1}$. □

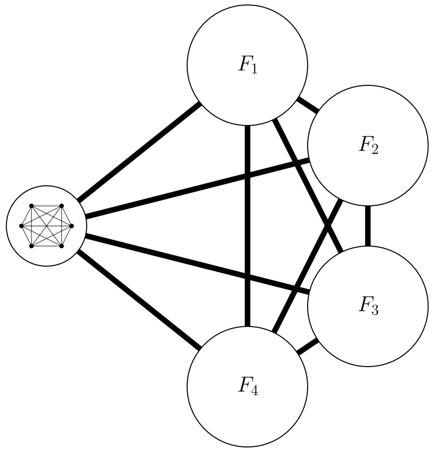


Figure 4.1: The gadget graph G_0 in Lemma 4.1.5 for $k = 6$. A thick line indicates that the vertices of the corresponding sets are pairwise connected.

From this F we construct a gadget graph G_0 with a useful property, namely that a particular copy of K_k is forced to be monochromatic.

Lemma 4.1.5. *There exists a graph G_0 with a subgraph H isomorphic to K_k contained in G_0 such that*

1. *there is a colouring of G_0 without a red $K_k \cdot K_2$ and without a blue K_k,*

2. *and every colouring of G_0 without a monochromatic copy of $K_k \cdot K_2$ results in H being monochromatic.*

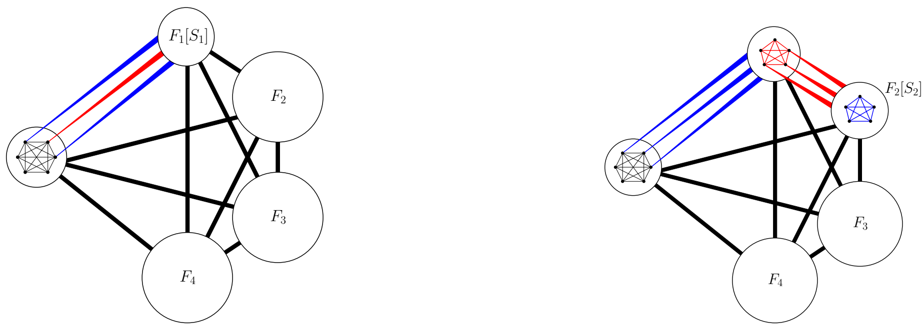
Proof. If $k = 2$ then taking G_0 to be a single edge suffices. We will henceforth assume $k \geqslant 3$. Take $\varepsilon = 2^{-k^2}$ and let $F_1, \ldots, F_{k-2}$ be copies of the graph F from Lemma 4.1.4. Add complete bipartite graphs between any two of these copies. Add a copy H of K_k and connect it to every vertex in every F_i. The resulting graph is G_0 (see Figure 4.1). To show $G_0 \nrightarrow K_k \cdot K_2$, colour all edges inside every F_i and inside H red, and all the remaining edges blue. The largest red clique is H, with only blue edges leaving H. The F_i are K_k-free, and any edge leaving F_i is blue as well. Since the graph of blue edges is $(k-1)$-chromatic ($F_1, \ldots, F_{k-2}, H$ is a partition into independent sets), the largest blue clique has order $k - 1$. This verifies (1).

For (2), assume χ is a red-blue colouring of the edges of G_0 without a monochromatic $K_k \cdot K_2$. We show that this forces H to be monochromatic. Define for a vertex v in G_0 and for a subset $S \subseteq N_{G_0}(v)$ the *colour pattern* $\mathbf{c}_v$ *of* v *with respect to* S to be the function with domain S that maps a vertex $w \in S$ to the colour of the edge $\{v, w\}$. We will now use a procedure of pruning vertices by their colour patterns; we refer to it as colour focusing, and will use it again later.

For a vertex $v \in V(F_1)$, consider the colour pattern $\mathbf{c}_v$ with respect to $V(H)$. There are at most 2^k such patterns, so at least a 2^{-k}-fraction of the vertices of F_1 has the same colour pattern. Fix an arbitrary subset $S_1 \subseteq V(F_1)$ such that $|S_1| \geqslant 2^{-k}v(F)$ and $\mathbf{c}_{v_1} = \mathbf{c}_{v_2}$ for every $v_1, v_2 \in S_1$ (see Figure 4.2a). Then $|S_1| > \varepsilon v(F)$, hence $F_1[S_1] \to K_{k-1}$. Fix a monochromatic copy H_1 of K_{k-1} contained in S_1, and assume without loss of generality that H_1 is red. We claim that all edges between $V(H)$ and S_1 (and in particular to $V(H_1)$) are blue. Indeed, since all vertices $v \in S_1$ have the same colour pattern to $V(H)$, say $\mathbf{c}$, all the edges $\{i, v\}$ with $v \in S_1$ have the same colour for any fixed vertex $i \in V(H)$, namely $\mathbf{c}(i)$. If one vertex i of H were red to S_1, then i along with H_1 and one (arbitrary) other vertex v of S_1 would contain a red copy of $K_k \cdot K_2$, a contradiction to our assumption on the colouring χ.

We now iterate this argument. Assume we have found red cliques $H_1, \dots, H_{t-1}$ in $F_1, \dots, F_{t-1}$ with vertex sets $V_1, \dots, V_{t-1}$, respectively, and that all the edges between these cliques as well as to H are blue. In F_t there is some, at least 2^{-tk}-fraction large subset $S_t \subseteq V(F_t)$ of the vertices, which all have the same colour pattern with respect to $V(H) \cup V_1 \cup V_2 \cup \dots \cup V_{t-1}$. Since $|S_t| > \varepsilon v(F_t)$, we get $F_t[S_t] \to K_{k-1}$. We find a monochromatic copy of K_{k-1} in S_t and call it H_t. Assume for contradiction that H_t is blue. In this case as before, all the edges between H_t and H as well as between H_t and $H_1, \dots, H_{t-1}$ would have to be red, otherwise there would be a blue $K_k \cdot K_2$. But if all these edges are red, any two vertices of H_t together with H_1 form a red $K_k \cdot K_2$ (see Figure 4.2b). Hence, H_t must be red, and as before all edges between H_t and H as well as between H_t and $H_1, \dots, H_{t-1}$ must be blue.

After applying this argument to F_{k-2}, we have a collection $H_1, \dots, H_{k-2}$ of red $(k-1)$-cliques and complete bipartite blue graphs between any two of $H, H_1, \dots, H_{k-2}$. Now, if some edge in H were blue, this edge along with one vertex from each of $H_1, \dots, H_{k-2}$ and any (arbitrary) other vertex from H_1 would create a blue $K_k \cdot K_2$. Therefore, every edge of H must be red, as desired. □



(a) Colour-focusing between $H = K_6$ and F_1. (b) There cannot be a blue K_5 in $F_2[S_2]$.

Figure 4.2: Illustrating the proof of Lemma 4.1.5.

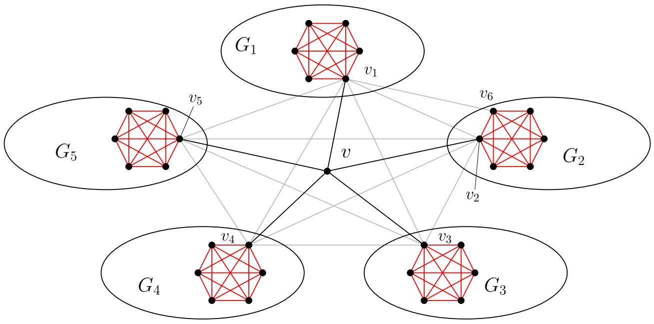


Figure 4.3: An example of the graph G in Lemma 4.1.6 for $k = 6$.

The following lemma completes the proof of Theorem 1.2.3.

Lemma 4.1.6. *For every $k \geqslant 3$ there is a graph G which contains a vertex v of degree $k-1$ so that $G \to K_k \cdot K_2$ but $G - v \nrightarrow K_k \cdot K_2$.*

Proof. Take $k-1$ copies $G_1, \dots, G_{k-1}$ of the gadget graph G_0 from Lemma 4.1.5, and let $H_1, \dots, H_{k-1}$ be the copies of K_k guaranteed to be monochromatic in any colouring without a monochromatic $K_k \cdot K_2$. Pick one vertex v_i in each H_i, and insert all edges between the v_i's. That is, we pick a vertex from each K_k and connect them to form a K_{k-1}. In addition, pick an arbitrary vertex $v_k \neq v_2$ from $V(H_2)$ and insert an edge between it and v_1. Finally, add a vertex v to the graph, and connect it to $v_1, \dots, v_{k-1}$. This completes the construction of G (see Figure 4.3). Clearly, $\deg(v) = k-1$. To see that $G - v \nrightarrow K_k \cdot K_2$, colour each G_i so it has no red $K_k \cdot K_2$ and no blue K_k. By property (2) of the gadget G_0 this also means that every H_i is monochromatic red. Colour the edges between $\{v_1, \dots, v_{k-1}\}$ and the additional edge $\{v_1, v_k\}$ blue. Since none of the G_i had a red $K_k \cdot K_2$ and we did not add any red edges, this colouring has no red $K_k \cdot K_2$. The G_i have no blue K_k, and for $i = 1, \dots, k-1$ the vertex v_i has no blue edges leaving G_i except those to the other v_j. But the edge $\{v_2, v_k\}$ is red, therefore there is no blue K_k and in particular no blue $K_k \cdot K_2$.

Finally, we show that $G \to K_k \cdot K_2$. Let any colouring of G be given, and suppose none of the copies of G_0 contains a monochromatic copy of $K_k \cdot K_2$. Then all of $H_1, \dots, H_{k-1}$ are monochromatic. We claim they have the same colour. Indeed, if H_i and H_j had different colours, then the edge $v_i v_j$ would induce a monochromatic $K_k \cdot K_2$ with whichever copy of K_k had the same colour as its own.

So all of the H_i have the same colour; without loss of generality, let this colour be red. If any of the edges $v_i v_j$, for $1 \leqslant i \leqslant j \leqslant k-1$ or for $i = 1$, $j = k$ were red, then along with H_i it would form a red $K_k \cdot K_2$. Similarly, if any of the edges $v v_i$ were red, then along with H_i it would induce a red $K_k \cdot K_2$. Otherwise, all these edges are blue and then $v, v_1, \dots, v_{k-1}$ and v_k form a blue $K_k \cdot K_2$, as desired. □

4.2 Clique with some disjoint smaller cliques

Recall that $K_k + m \cdot K_t$ denotes the disjoint union of a K_k and m copies of K_t. Also, $\mu(k,t)$ is the largest number m so that K_k and $K_k + m \cdot K_t$ are Ramsey-equivalent. In this section, we prove Theorem 1.2.2, which gives an upper bound on $\mu(k,t)$ and determines it up to roughly a factor of 2.

Proof of Theorem 1.2.2. Let $m = \left\lfloor \frac{R(k,k-t+1)-1}{t} \right\rfloor + 1$. We will construct a graph G with the following two properties.

$(G1)$ $G \to K_k$ and

$(G2)$ $G \nrightarrow K_k + m \cdot K_t$.

Construction of G.
Set $h := R(k, k-t+1) + k - 1$ and $\varepsilon_0 := 2^{-h-1}$. Let G_0 be a graph given by Lemma 4.1.4 such that

$$K_{k-1} \nsubseteq G_0 \quad \text{and} \quad G_0 \xrightarrow{\varepsilon_0} K_{k-2}. \tag{4.1}$$

Now, set $n_0 := v(G_0)$ and assume without loss of generality that $V(G_0) = [n_0]$. For every $1 \leqslant j \leqslant n_0$, we define the building blocks F_j of our graph G iteratively. First, let $\varepsilon_1 := 2^{-(h+n_0)}$ and let F_1 be a graph given by Lemma 4.1.4 such that $K_t \nsubseteq F_1$ and $F_1 \xrightarrow{\varepsilon_1} K_{t-1}$. For $2 \leqslant j \leqslant n_0$, assume we have defined $\varepsilon_1, \dots, \varepsilon_{j-1}$ and $F_1, \dots, F_{j-1}$. We then set

$$\varepsilon_j := 2^{-\left(h+n_0-j+\sum_{i=1}^{j-1} v(F_i)\right)} \tag{4.2}$$

and let F_j be a graph given by Lemma 4.1.4 such that

$$K_t \nsubseteq F_j \quad \text{and} \quad F_j \xrightarrow{\varepsilon_j} K_{t-1}. \tag{4.3}$$

We are now ready to define the graph $G = G(V,E)$. Define pairwise disjoint sets V_H and V_j, $1 \leqslant j \leqslant n_0$, such that $V_H = \{v_1, \dots, v_h\}$ and $|V_j| = |V(F_j)|$. Now set $V := V_H \cup \bigcup_{j=1}^{n_0} V_j$. The edge set E is defined as follows.

- $H := G[V_H] \cong K_h$,
- $G[V_j] \cong F_j$ for all $1 \leqslant j \leqslant n_0$,
- $v_i w \in E(G)$ for all $1 \leqslant i \leqslant h$, and all $w \in \bigcup_{j=1}^{n_0} V_j$,

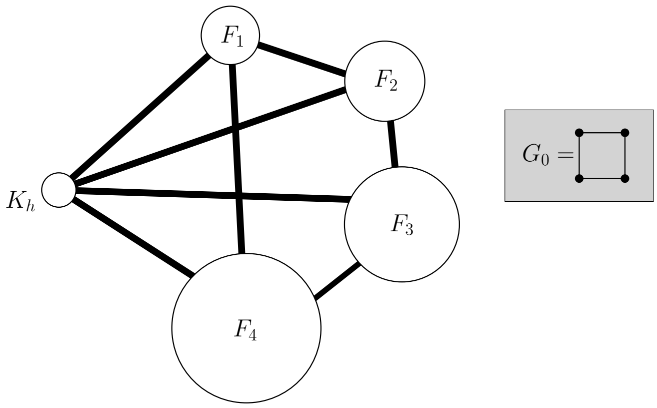


Figure 4.4: An illustration of the gadget graph G when $G_0 = C_4$. A thick line indicates that the vertices of the corresponding sets are pairwise connected.

- for all $u \in V_i$, $w \in V_j$, $uw \in E(G)$ if and only if $ij \in E(G_0)$

That is, our gadget graph consists of one copy of each F_j together with a copy H of a complete graph on $h = R(k, k-t+1)+k-1$ vertices. Furthermore, we place a complete bipartite graph between F_i and F_j whenever ij is an edge in G_0, and a complete bipartite graph between H and $\bigcup_{j=1}^{n_0} F_j$ (see Figure 4.4). We now show that the graph G fulfills the two conditions $(G1)$ and $(G2)$ above.

The graph G has property $(G2)$.
To see that $G \nrightarrow K_k + m \cdot K_t$, colour all edges inside H and inside the copy of each F_j red, and all edges between H and any F_j, and between any F_i and F_j $(i \neq j)$ blue. Then the largest blue clique has size $k-1$ (since G_0 is K_{k-1}-free). So any monochromatic copy of $K_k + m \cdot K_t$ would need to be red. Since all the F_j's are K_t-free, the red copy of $K_k + m \cdot K_t$ needs to lie inside H. However, $v(K_k + m \cdot K_t) = k + mt \geqslant R(k, k-t+1) + k > v(H)$. So H cannot host a copy of $K_k + m \cdot K_t$.

The graph G has property $(G1)$.
Let $\chi : E \to \{\text{red, blue}\}$ be a 2-colouring of G. We apply a similar "colour-focusing" procedure as in the proof of Lemma 4.1.5. This technique is used to obtain Lemma 4.2.1, which shows that there is a vertex subset for which the colouring is highly structured. From this lemma, it is not difficult to prove that there must be a monochromatic K_k.

Lemma 4.2.1. *There exist a subset $J \subseteq [n_0]$ and subsets $W_j \subseteq V_j$ for each $j \in J$ such that the following holds.*

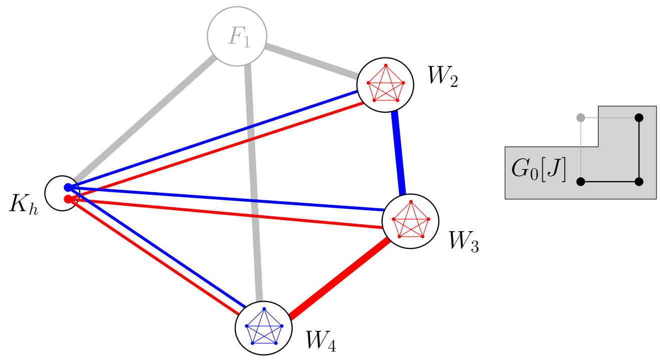


Figure 4.5: The colour patterns we find with Lemma 4.2.1.

(a) $|J| \geqslant n_0/2^h = 2\varepsilon_0 n_0$,

(b) *for all* $j \in J$, W_j *is the vertex set of a monochromatic* K_{t-1} *under* χ,

(c) *for all* $i, j \in J$, *if* $ij \in E(G_0)$ *then there exists* $c_{ij} \in \{$*red, blue*$\}$ *such that for all* $u \in W_i$, $w \in W_j$, $\chi(uw) = c_{ij}$.

(d) *for all* $v_i \in V_H$, *there exists* $c_i \in \{$*red, blue*$\}$ *such that for all* $u \in \bigcup_{j \in J} W_j$, $\chi(v_i u) = c_i$.

The structure of the sets J and W_j in Lemma 4.2.1 is depicted in Figure 4.5. Before proving the lemma, we first show how it implies that there is a monochromatic K_k in G, which implies $(G1)$.

Proof of $(G1)$ *assuming Lemma 4.2.1.* Let $J' \subseteq J$ with $|J'| \geqslant |J|/2$ be such that all W_j with $j \in J'$ are monochromatic of the *same* colour. Consider the induced subgraph $G_0' := G_0[J']$ of G_0. Let χ' be the edge-colouring of G_0' where each edge $ij \in E(G_0')$ has colour $\chi'(ij) := c_{ij}$. Since $|J'| \geqslant |J|/2 \geqslant \varepsilon_0 n_0$ by property (a), and since $G_0 \xrightarrow{\varepsilon_0} K_{k-2}$ by definition of G_0, there exists a monochromatic copy of K_{k-2} in G_0' under χ'. Let $I \subseteq J'$ denote the vertex set of this monochromatic copy, and assume without loss of generality that it is blue. Then for all $i, j \in I$, $i \neq j$, the sets W_i and W_j are connected by complete bipartite graphs, all edges being blue under χ. The monochromatic W_j with $j \in I$ are all the same colour, and we may assume that they are all red. Indeed, otherwise the union of the W_j with $j \in I$ form a monochromatic blue clique of order $(k-2)(t-1) \geqslant k$ (since $t \geqslant 3$), and thus there is a monochromatic K_k. So we may assume from now on that each W_j, $j \in I$, is a red K_{t-1}.

Consider now the vertices in V_H. Any such vertex has either only red edges or only blue edges to $\bigcup_{j \in J'} W_j$, by property (d). We call $v_i \in V_H$ *red* if $c_i =$ red, and *blue* otherwise. Suppose there exist two vertices, $v_i, v_j \in V_H$ which are both *blue*, such that $\chi(v_i v_j) =$ blue.

Then they form a blue K_k with one vertex from each $W_j, j \in I$. So we can assume that for two *blue* vertices $v_i, v_j \in V_H$ we have $\chi(v_i v_j) = \text{red}$. But then we can also assume that there are at most $k-1$ *blue* vertices inside H, since otherwise they form a red K_k inside H. So, there are at least $v(H) - (k-1) = R(k, k-t+1)$ *red* vertices $V_{red} \subseteq V_H$ in H. By definition of $R(k, k-t+1)$, V_{red} contains either a red K_{k-t+1} or a blue K_k. In the second case, we are done. In the first case, the vertex set $V_{red} \cup W_j$ contains a red K_k for any $j \in I$, so we are done as well. □

To finish the proof of Theorem 1.2.2 it remains to prove Lemma 4.2.1.

Proof of Lemma 4.2.1. We prove the lemma in two steps. First, we "colour-focus" each $v_i \in V_H$ ensuring property (d). We then restrict further down to sets inside $V(F_j)$ ensuring property (c) of monochromatic bipartite graphs between the vertex sets. These two steps are made precise in Claim 4.2.2 and Claim 4.2.3, and are illustrated in Figure 4.6 and 4.7.

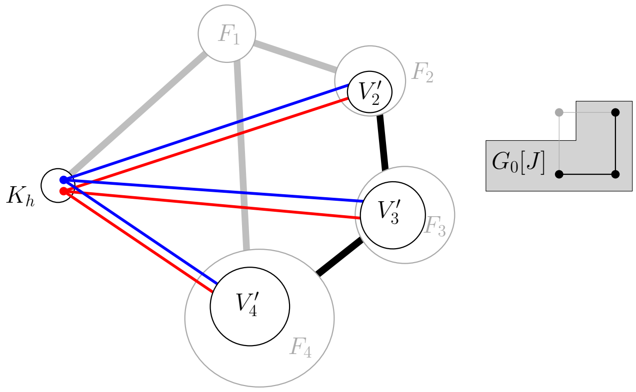


Figure 4.6: Applying Claim 4.2.2 to G and χ: Every $v \in V_H$ has only red or only blue edges going to the active sets V_2', V_3' and V_4'.

Claim 4.2.2. *For every 2-colouring $\chi : E \to \{red, blue\}$ of the edge set of G, there exists an index set $J \subseteq [n_0]$ and subsets $V_j' \subseteq V_j$ for each $j \in J$ such that the following properties hold.*

(a) $|J| \geqslant n_0/2^h$,

(b′) *for all $j \in J$, $|V_j'| \geqslant v(F_j)/2^h$, and*

(d′) *for all $v_i \in V_H$, there exists $c_i \in \{red\ blue\}$ such that for all $u \in \bigcup_{j \in J} V_j'$, $\chi(v_i u) = c_i$.*

Claim 4.2.3. *Let $\chi : E \to \{red, blue\}$ be a 2-colouring of the edge set of G, and let subsets J and V_j' (for $j \in J$) be given by Claim 4.2.2. Then there exist subsets $V_j'' \subseteq V_j'$ for each $j \in J$ such that the following holds.*

(b'') *For all* $j \in J$, $|V_j''| \geqslant v(F_j) \cdot 2^{-\left(h+n_0-j+\sum_{i<j} v(F_i)\right)} = \varepsilon_j \cdot v(F_j)$, *and*

(c') *for all* $i, j \in J$, *if* $ij \in E(G_0)$ *then there exists* $c_{ij} \in \{$*red, blue*$\}$ *such that for all* $u \in V_i''$, $w \in V_j''$, $\chi(uw) = c_{ij}$.

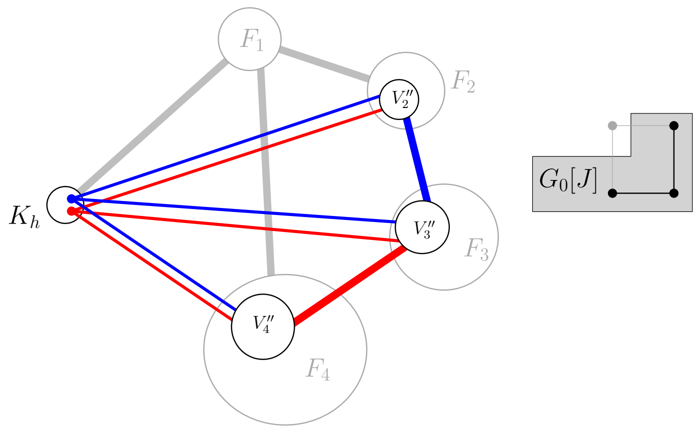


Figure 4.7: Applying Claim 4.2.3 to G and χ: The complete bipartite graph between any two sets V_i'' and V_j'' is monochromatic if $ij \in E(G_0)$.

It is now straightforward to see that Lemma 4.2.1 follows: since each $F_j \xrightarrow{\varepsilon_j} K_{t-1}$ and by property (b''), (b) follows, where the $W_j \subseteq V_j''$ are the host vertices for the monochromatic K_{t-1}. Now, since $W_j \subseteq V_i''$, (c) and (d) follow trivially from (c') and (d').

Proof of Claim 4.2.2. For each $j \in [n_0]$, of the 2^h colour patterns to V_H, there is a most common colour pattern, from the vertices in F_j to V_H. Call this colour pattern $\mathbf{c_j}$. Take $\mathbf{c}$ to be the colour pattern that occurs most frequently among the $\mathbf{c_j}$'s. Let J be the set of j for which $\mathbf{c_j} = \mathbf{c}$. For all $v_i \in V_H$, we have c_i is the colour of v_i in colour pattern $\mathbf{c}$.

By the pigeonhole principle, the set V_j' of vertices in V_j with colour pattern $\mathbf{c_j}$ to V_H has $|V_j'| \geqslant |V_j|/2^h$. Again by the pigeonhole principle, at least a 2^{-h} fraction of the $j \in [n_0]$ are in J. □

Proof of Claim 4.2.3. Let $J \subseteq [n_0]$ and $V_j' \subset V_j$ for $j \in J$ be the sets found in Claim 4.2.2 that fulfill the properties (a), (b') and (d'). We want to find subsets $V_j'' \subseteq V_j'$ inside the active sets $\{V_j : j \in J\}$ such that the induced bipartite graphs $G[V_i'', V_j'']$ are monochromatic (if $ij \in E(G_0)$) and the V_j'' are not too small, i.e. $|V_j''| \geqslant \varepsilon_j v(F_j)$. For ease of notation we will assume without loss of generality that $J = [\ell]$. Now, for every $1 \leqslant j \leqslant \ell$, we will define a sequence of subsets

$$V_j' \supseteq V_j^{(j)} \supseteq V_j^{(j+1)} \supseteq \ldots \supseteq V_j^{(\ell)} =: V_j''$$

such that the collection $\left\{V_j'' : 1 \leqslant j \leqslant \ell\right\}$ fulfills (b'') and (c').
For $1 \leqslant k \leqslant \ell$, we will iteratively maintain the following two properties.

$(C_1(k))$ For all $1 \leqslant j \leqslant k$, we have that $\left|V_j^{(k)}\right| \geqslant v(F_j) \cdot 2^{-\left(h+k-j+\sum_{i<j} v(F_i)\right)}$.

$(C_2(k))$ For all $1 \leqslant i < j \leqslant k$, if $ij \in E(G_0)$ then the induced bipartite graph $G[V_i^{(k)}, V_j^{(k)}]$ is monochromatic, i.e. there exists c_{ij} such that for all $u \in V_i^{(k)}$, all $w \in V_j^{(k)}$ it holds that $\chi(uw) = c_{ij}$.

First, note that for $V_1^{(1)} := V_1'$, $(C_1(1))$ holds by (b'), and that $(C_2(1))$ is trivially satisfied. Note also, that $(C_1(\ell))$ and $(C_2(\ell))$ imply (b'') and (c').
Suppose now that for $1 \leqslant k < \ell$ the properties $(C_1(k))$ and $(C_2(k))$ hold. We will colour-focus backwards from $V_{k+1}^{(k)} := V_{k+1}'$ and define subsets $V_i^{(k+1)} \subseteq V_i^{(k)}$ for all $1 \leqslant i \leqslant k+1$ such that $(C_1(k+1))$ and $(C_2(k+1))$ are satisfied.
Consider the vertices $w \in V_{k+1}^{(k)}$ and let $\mathbf{c}(w)$ be the colour pattern of w with respect to $V_1^{(k)} \cup \ldots \cup V_k^{(k)}$
There are at most

$$2^{\left|V_1^{(k)}\right|+\ldots+\left|V_k^{(k)}\right|} \leqslant 2^{\sum_{i\leqslant k} v(F_i)}$$

distinct colour patterns, so at least

$$\frac{\left|V_{k+1}^{(k)}\right|}{2^{\sum_{i\leqslant k} v(F_i)}} \geqslant \frac{v(F_{k+1})}{2^{h+\sum_{i\leqslant k} v(F_i)}}$$

vertices in $V_{k+1}^{(k)}$ must have the same colour pattern, say $\mathbf{c_{k+1}}$. Set

$$V_{k+1}^{(k+1)} := \left\{w \in V_{k+1}^{(k)} : \mathbf{c}(w) = \mathbf{c_{k+1}}\right\}.$$

Then by definition

$$\left|V_{k+1}^{(k+1)}\right| \geqslant \frac{v(F_{k+1})}{2^{h+\sum_{i\leqslant k} v(F_i)}}. \tag{4.4}$$

Now, for all $u \in V_1^{(k)} \cup \ldots \cup V_k^{(k)}$ and all $w_1, w_2 \in V_{k+1}^{(k+1)}$ we have

$$\chi(uw_1) = (\mathbf{c}(w_1))_u = (\mathbf{c_{k+1}})_u = (\mathbf{c}(w_2))_u = \chi(uw_2).$$

That is, all $u \in V_1^{(k)} \cup \ldots \cup V_k^{(k)}$ see the vertex set $V_{k+1}^{(k+1)}$ either only via red edges or only via blue edges. For $1 \leqslant i \leqslant k$, let $c_{i,k+1}$ be the colour which is more common between $V_i^{(k)}$ and $V_{k+1}^{(k+1)}$, i.e.

$$c_{i,k+1} = \begin{cases} \text{red} & \text{if } \left|\{u \in V_i^{(k)} : (\mathbf{c_{k+1}})_u = \text{red}\}\right| \geqslant |V_i^{(k)}|/2 \\ \text{blue} & \text{otherwise.} \end{cases}$$

We then restrict our attention to those vertices $u \in V_i^{(k)}$ which "see" $V_{k+1}^{(k+1)}$ in the more common colour and thus set

$$V_i^{(k+1)} := \left\{ u \in V_i^{(k)} : \chi(uw) = c_{i,k+1} \text{ for some } w \in V_{k+1}^{(k+1)} \right\}.$$

Then it follows immediately and by $(C_1(k))$ that

$$\left| V_i^{(k+1)} \right| \geqslant \frac{|V_i^{(k)}|}{2} \geqslant \frac{v(F_i)}{2^{h+\sum_{i'<i} v(F_{i'})+(k+1)-i}}.$$

Together with (4.4), this implies $(C_1(k+1))$ holds. Also, by construction of $V_i^{(k+1)}$ for $1 \leqslant i \leqslant k+1$ and by $(C_2(k))$ it follows that $(C_2(k+1))$ holds.
We have shown that the iterative colour focusing ensures $(C_1(\ell))$ and $(C_2(\ell))$. Set now $V_j'' := V_j^{(\ell)}$ for all $j \in J$. Then $(C_1(\ell))$ and $(C_2(\ell)$ translate to (b'') and (c'), which finishes the proof. □

As noted earlier, Claim 4.2.2 and 4.2.3 imply Lemma 4.2.1. This finishes the proof of Theorem 1.2.2. □

4.3 Concluding remarks

Recall that $\mu(k,t)$ is the maximum m such that K_k and $K_k + m \cdot K_t$ are Ramsey-equivalent. We determined $\mu(k,t)$ up to roughly a factor 2 for $k-1 > t > 2$. It would be of interest to close the gap between the lower and upper bounds.

Problem 4.3.1. *Determine* $\mu(k,t)$*.*

A special case of this problem already asked in [55] is the following. Note that we have shown that $\mu(k,k-1) \leqslant 1$. That is, if K_k and $K_k + K_{k-1}$ are Ramsey-equivalent, then $\mu(k,k-1) = 1$ and otherwise $\mu(k,k-1) = 0$.

Question 4.3.2. *Are* K_k *and* $K_k + K_{k-1}$ *Ramsey-equivalent?*

We proved that every graph (other than K_k) that is Ramsey-equivalent to K_k is not connected. This naturally leads to the following question.

Question 4.3.3. *Is there a pair of non-isomorphic connected graphs* H_1, H_2 *that are Ramsey-equivalent?*

We say H' is formed by adding a pendant edge to H if H' has a vertex v of degree 1 so that $H' - v$ is isomorphic to H. A related question is the following.

Question 4.3.4. *Is there a connected graph H and some graph H' formed by adding a pendant edge to H so that H and H' are Ramsey-equivalent?*

We have recently shown [37] that $K_{t,t}$ and $K_{t,t} \cdot K_2$, the graph formed by adding a pendant edge to $K_{t,t}$, are not Ramsey-equivalent. Furthermore, we proved $s(K_{t,t} \cdot K_2) = 1$ while it was shown in [40] that $s(K_{t,t}) = 2t - 1$.

We do not have a good understanding of how large a connected subgraph can be added to K_k and such that the resulting graph is Ramsey-equivalent to K_k. For example, we have the following problem.

Problem 4.3.5. *Let $g(k)$ be the maximum g such that K_k is Ramsey-equivalent to $K_k + K_{1,g}$, the disjoint union of K_k and the star $K_{1,g}$ with g leaves. Determine $g(k)$.*

We know that $g(k)$ is at most exponential in k.

5

On minimal r-Ramsey graphs for the clique

In this chapter, we generalize the notion of being Ramsey, denoted by $G \to H$, to an arbitrary number of colours. Recall that we say a graph G is r-Ramsey for a graph H, denoted by $G \to (H)_r$, if in any r-colouring of the edges of G there exists a monochromatic copy of H. Such a graph G is minimal r-Ramsey for H if no proper subgraph of G has this property. Here, we want to investigate the minimal minimum degree of minimal r-Ramsey graphs for the clique, denoted by $s_r(K_k)$. It was shown by Burr, Erdős and Lovász [18] that $s_2(K_k) = (k-1)^2$. We are interested how the quantity of $s_r(K_k)$ changes when $r \to \infty$, and mostly assume that k is constant. We will prove lower bounds on $s_r(K_k)$ as stated in Theorems 1.2.5 and 1.2.4. We shall do that in the next section. To prove upper bounds on $s_r(K_k)$, we need to introduce two special graph classes. We do so in Section 5.2 and show how the existence of these graphs implies upper bounds in Theorem 1.2.5, Theorem 1.2.6 and Theorem 1.2.4. In Section 5.3 and 5.4 we prove the existence of those special graphs. We close this chapter with some concluding remarks.

5.1 Lower bounds on $s_r(K_k)$

Our approach is to use induction on r to show lower bounds of order $\Theta^*(r^2)$, that is of order r^2 up to a factor that is polynomial in $\log r$. The induction step though works only for large values of r. Therefore, we first give a simple lower bound which is linear in r and which will be helpful in the induction step.

Lemma 5.1.1. *For all $r \geqslant 2,\ k \geqslant 3$ and all graphs H, we have that $s_r(H) > r\,(\delta(H) - 1)$.*

In particular, $s_r(K_k) \geqslant r(k-2)$. In the special case when $r=2$, the proof of this lemma is included in [40].

Proof. Assume $s_r(H) \leqslant r\,(\delta(H)-1)$. Then there exists a graph $G \in \mathcal{M}_r(H)$ such that $\delta(G) \leqslant r\,(\delta(H)-1)$. Let v be a vertex in G of degree at most $r(\delta(H)-1)$. By minimality, there is an r-colouring χ of $G-v$ without a monochromatic copy of H. Now extend χ to the incident edges of v by colouring at most $\delta(H)-1$ edges in colour i, for each colour i. This extended colouring also does not contain a monochromatic copy of H, since any monochromatic copy of H would contain v by assumption on χ, but in any colour, v has too little degree to be contained in such a copy of H. This contradicts the fact that G is r-Ramsey for H. □

Next, we pin down a simple observation which we use frequently for both, lower and upper bounds.

Observation 5.1.2. *Let $r \geqslant 2$, let H and G be graphs, and let v be a vertex in G. Further, assume that $G \to (H)_r$ and $G - v \nrightarrow (H)_r$. Then $s_r(H) \leqslant \deg(v)$.*

Proof. By assumption, any subgraph G' of G that is r-Ramsey-minimal for H (which exists) needs to contain v. Therefore, $s_r(H) \leqslant \delta(G') \leqslant \deg_{G'}(v) \leqslant \deg_G(v)$. □

The next lemma captures the main idea for our quadratic lower bounds. For a graph F, let $\alpha_k(F)$ denote the k-independence number of the graph F, that is, the largest cardinality of a subset $I \subseteq V(F)$ without a copy of K_k in F. For $k=2$, this is the usual independence number $\alpha(F)$.

Lemma 5.1.3. *Let $k, r \geqslant 3$ and let G be a graph such that $G \to (K_k)_r$. Let v be a vertex in G such that $G - v \nrightarrow (K_k)_r$. Let χ be an r-colouring of $G - v$ without a monochromatic copy of K_k, and let $G_1, \dots, G_r$ be the colour classes of χ in $N(v)$. Then $\deg_G(v) \geqslant s_{r-1}(K_k) + \max_i \alpha_{k-1}(G_i)$. In particular, $s_r(K_k) \geqslant s_{r-1}(K_k) + \max_i \alpha_{k-1}(G_i)$.*

An illustration of the proof idea can be found in Figure 5.1.

Proof. Fix a colour $i \in [r]$, and let $I_i \subseteq N(v)$ be a $(k-1)$-independent set of size $\alpha_{k-1}(G_i)$ in the graph G_i. Our plan is to remove a K_k-free subgraph F from G. Let F be the set of all edges of colour i under χ in $G-v$, and all edges from v to the set I_i. Now, F is K_k-free. Indeed, v cannot be contained in a copy of K_k in F, since I_i contains no copy of K_{k-1} in colour i by assumption. Moreover, χ does not contain a monochromatic copy of K_k in colour i, hence $F - v$ is also K_k-free. Call the resulting graph $G' = G \setminus F$. Then since F is K_k-free and $G \to (K_k)_r$ by assumption, we have that $G' \to (K_k)_{r-1}$ (we could use colour r

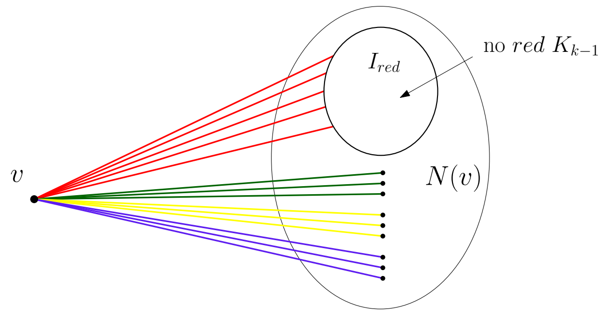


Figure 5.1: An illustration of Lemma 5.1.3 for $r = 4$.

for all edges in F without creating a monochromatic copy of K_k). Moreover, χ restricted to $G' - v = (G \setminus F) - v$ yields an $(r-1)$-colouring of the edges of $G' - v$ without a monochromatic copy of K_k. Therefore, by Observation 5.1.2 applied to $r-1$, K_k and G',

$$s_{r-1}(K_k) \leqslant \deg_{G'}(v) = \deg_G(v) - \deg_F(v) = \deg_G(v) - \alpha_{k-1}(G_i).$$

Since colour i was arbitrary, the first claim follows. Now, if G attains the minimum of $s_r(K_k)$ and if v is a vertex of minimum degree in G, then the second claim follows. □

Lemma 5.1.3 reveals the main idea of our proofs for lower bounds: We want to use induction on r and find a large $(k-1)$-independent set in at least one of the graphs G_i. Since the graphs G_i are colour classes in a colouring χ without a monochromatic copy of K_k, the G_i are naturally K_k-free graphs on $n = \deg_G(v)$ vertices. Consider the function $f_{s,k}(n)$ which is defined to be the minimum of $\alpha_s(F)$, where the minimum is taken over all K_k-free graphs F on n vertices. This function was first studied by Erdős and Gallai [28], and then formally defined by Erdős and Rogers [29]. In the literature, it is nowadays called the *Erdős-Rogers function*, see for example [27] for a recent survey. Then *all* our graphs G_i on n vertices satisfy $\alpha_{k-1}(G_i) \geqslant f_{k-1,k}(n)$. The following is therefore an immediate consequence of the previous lemma.

Corollary 5.1.4. *For all $r, k \geqslant 3$ we have that $s_r(K_k)$ satisfies the following recursion: $s_r(K_k) \geqslant s_{r-1}(K_k) + f_{k-1,k}(s_r(K_k))$.*

Therefore, we are interested in good lower bounds on the Erdős-Rogers function $f_{s,k}(n)$ in the special case when $s = k-1$. In the case when $k = 3$, it is easy to see that every triangle-free graph F on n vertices contains an independent set of size at least $\lfloor \sqrt{n} \rfloor$: If $\Delta := \Delta(F) \geqslant \lfloor \sqrt{n} \rfloor$ then there exists a vertex v of degree at least $\lfloor \sqrt{n} \rfloor$, and $N(v)$ is an independent set. Otherwise, $\Delta \leqslant \lfloor \sqrt{n} \rfloor - 1$ and we use the well known fact that $\alpha(F) \geqslant n/(\Delta(F)+1)$ (cf. [3]) to deduce that $\alpha(F) \geqslant \lfloor\sqrt{n}\rfloor$. Therefore, $f_{2,3}(n) = \min\{\alpha_2(F) : |V(F)| = n \text{ and } K_3 \nsubseteq F\} \geqslant \lfloor\sqrt{n}\rfloor$. A result of Shearer ([51] and Theorem 5.1.6 below) implies that $f_{2,3}(n) \geqslant (1/2 - o(1))\sqrt{n \ln n}$, which is the best known lower bound on $f_{2,3}(n)$ until this day. In [15], Bollobás and Hind proved that $f_{3,4}(n) \geqslant \sqrt{2n}$ and for more general $k \geqslant 5$, $f_{k-1,k}(n) \geqslant \sqrt{n}$. This lower bound was subsequently improved by Krivelevich [46] for $k \geqslant 4$. Recently, Dudek and Mubayi [24] noted that the result in [46] can be strengthened to $f_{k-1,k}(n) = \Omega\left(\sqrt{\frac{n \log n}{\log \log n}}\right)$, using a result of Shearer [52]. Using standard analytic tools it is easy to check that for $x > e^e$, the function $\frac{x \ln x}{\ln \ln x}$ is increasing. Therefore, the previous lower bound implies the following.

Theorem 5.1.5 ([24],[46] and [52])**.** *For every $k \geqslant 4$, there exists a constant $c_f = c_f(k) > 0$ such that for $n \geqslant 27$ we have that*

$$f_{k-1,k}(n) \geqslant c_f \cdot \sqrt{\frac{n \ln n}{\ln \ln n}}.$$

We are ready to prove our lower bound on $s_r(K_k)$ for $k \geqslant 4$.

Proof of lower bound in Theorem 1.2.5. Let $k \geqslant 4$ be given, and let $c_f = c_f(k)$ be the constant from Theorem 5.1.5. Now fix a constant $c = c(k) > 0$ such that

(1) $s_{28}(K_k) \geqslant c \cdot (28)^2 \sqrt{\frac{\ln 28}{\ln \ln 28}}$ and

(2) $100c < (c_f)^2$.

Since $s_{28}(K_k) > 0$, such a constant certainly exists. We show that

$$s_r(K_k) \geqslant cr^2 \sqrt{\frac{\ln r}{\ln \ln r}} \tag{5.1}$$

by induction on r. For brevity, set $s_r := s_r(K_k)$. For $r_0 := 28$, the claim follows from condition (1) on the constant c. Now for $r > r_0$, note that by the simple lower bound in Lemma 5.1.1, $s_r \geqslant r \geqslant 27$, so we can apply Theorem 5.1.5. By Corollary 5.1.4 it follows that $s_r \geqslant s_{r-1} + f_{k-1,k}(s_r)$, and with Theorem 5.1.5 we therefore get that,

$$s_r \geqslant s_{r-1} + c_f \cdot \sqrt{\frac{s_r \ln s_r}{\ln \ln s_r}}. \tag{5.2}$$

To bound the terms under the root, we use again that $s_r \geqslant r \geqslant 27$ and that $\frac{\ln x}{\ln \ln x}$ is increasing for $x \geqslant 27$. Also note that $s_r \geqslant s_{r-1} + f_{k-1,k}(s_r)$ trivially implies that $s_r \geqslant s_{r-1}$. Therefore, using the induction hypothesis on s_{r-1} we get that $s_r \geqslant c \cdot (r-1)^2 \geqslant \frac{c}{2} r^2$ since $r \geqslant 4$. Thus, from (5.2) we obtain, using the induction hypothesis again,

$$\begin{aligned}
s_r &\geqslant s_{r-1} + c_f \cdot \sqrt{\frac{s_r \ln s_r}{\ln\ln s_r}} \\
&\geqslant c \cdot (r-1)^2 \sqrt{\frac{\ln(r-1)}{\ln\ln(r-1)}} + c_f \cdot \sqrt{\frac{cr^2 \ln(r-1)}{2\ln\ln(r-1)}} \\
&\geqslant c\,r^2 \sqrt{\frac{\ln r}{\ln\ln r}} - c\,r^2 \left(\sqrt{\frac{\ln r}{\ln\ln r}} - \sqrt{\frac{\ln(r-1)}{\ln\ln(r-1)}}\right) \\
&\quad + \left(c_f r\sqrt{\frac{c}{2}} - 2cr\right)\sqrt{\frac{\ln(r-1)}{\ln\ln(r-1)}}.
\end{aligned} \tag{5.3}$$

For $g(x) := \sqrt{\frac{\ln x}{\ln\ln x}}$ it is straight-forward to check that for $x \geqslant 28$ (and hence $\ln\ln x > 1$)

(i) $g'(x) = \left(1 - \frac{1}{\ln\ln x}\right) \cdot \frac{1}{2x\sqrt{\ln x \cdot \ln\ln x}} > 0$ and

(ii) $g''(x) = -\frac{2\ln x(\ln\ln x - 1)(\ln\ln x) + (\ln\ln x)^2 - 3}{4x^2(\ln x)^{3/2}(\ln\ln x)^{5/2}} < 0.$

Therefore, for $x \geqslant 28$, $g(x)$ is increasing and concave. It follows that $g(x+1) - g(x) \leqslant g'(x)$. Now for $\ln\ln x > 1$,

$$g'(x) = \left(1 - \frac{1}{\ln\ln x}\right) \cdot \frac{1}{2x\sqrt{\ln x \cdot \ln\ln x}} \leqslant \frac{1}{x+1}.$$

It follows that

$$\sqrt{\frac{\ln r}{\ln\ln r}} - \sqrt{\frac{\ln(r-1)}{\ln\ln(r-1)}} \leqslant \frac{1}{r}.$$

Thus, (5.3) implies that $s_r \geqslant c\,r^2 \sqrt{\frac{\ln r}{\ln\ln r}}$ if

$$\left(c_f r\sqrt{\frac{c}{2}} - 3cr\right) \geqslant 0,$$

which is true by condition (2) on the constant c. So, (5.1) follows for r. □

We want to remark at this point that a similar argument works for $k = 3$. However, using the bound $f_{2,3}(n) \geqslant (1/2 - o(1))\sqrt{n \ln n}$ mentioned above and similar calculations as in the previous proof would give a lower bound of $cr^2\sqrt{\ln r}$ on $s_r(K_3)$. To obtain the extra factor of $\sqrt{\ln r}$ desired for Theorem 1.2.4 we need to work more. Recall that for $k \geqslant 4$ we used lower bounds on the Erdős-Rogers function $f_{k-1,k}(n)$ which gives a universal lower bound on $\alpha_{k-1}(G_i)$ for *all* colour classes G_i in the neighbourhood of our critical vertex v in our r-minimal graph G. However, when we want to use Lemma 5.1.3, we require a good lower bound on $\alpha_{k-1}(G_i)$ only for *one* of the colour classes G_i in $N(v)$. When $k = 3$, we use the fact that one of the colour classes in $N(v)$ has particularly few edges, and hence small average degree. This then forces the independence number to be larger. The following well-known result about the independence number in triangle-free graphs is due to Shearer.

Theorem 5.1.6. *[51, Theorem 1] Let F be a triangle-free graph on n vertices with average degree d. Let $g(d) = (d\ln d - d + 1)/(d-1)^2$, $g(0) = 1$, $g(1) = 1/2$. Then $\alpha(F) \geqslant ng(d)$.*

Note first that g is continuous for $0 \leqslant d < \infty$, and that for $0 < d < \infty$, $0 < g(d) < 1$ and $g'(d) < 0$. In particular, $g(d)$ is decreasing on $(0,\infty)$. Furthermore, it is straightforward to check that $g(d) \geqslant \frac{\ln d}{2d}$ for all $d > 1$. The following recursion is a straight-forward application of the simple observation in Lemma 5.1.3 and Shearer's Inequality.

Corollary 5.1.7. *For $r \geqslant 3$, let G be an r-Ramsey graph for K_3, and let v be a vertex in G of degree $d_v = \deg(v)$ such that $G-v \nrightarrow (K_3)_r$. Then $d_v \geqslant s_{r-1}(K_3)+d_v\cdot g\left(\frac{d_v-1}{r}\right)$. In particular, if G has minimum degree $s_r := s_r(K_3)$ and $\deg(v) = s_r$, then $s_r \geqslant s_{r-1}(K_3) + s_r \cdot g\left(\frac{s_r-1}{r}\right)$.*

Proof. Let χ be a colouring of $G - v$ without a monochromatic copy of K_3, and let the G_i's be defined as above, i.e. G_i is the subgraph of colour i in $N(v)$. By the pigeonhole principle, there exists a colour, say colour i, of density at most $1/r$ in $N(v)$. That is, the subgraph G_i has at most $\binom{d_v}{2}/r$ edges in $N(v)$. Therefore, G_i has average degree $d \leqslant (d_v-1)/r$. Since G_i is triangle-free by assumption, we can apply Shearer's Inequality. Since g is decreasing, it therefore follows that $\alpha(G_i) \geqslant d_v \cdot g\left(\frac{d_v-1}{r}\right)$. The claim follows then with Lemma 5.1.3. □

Again, we want to use induction on r. In the proof of of the lower bound in Theorem 1.2.5 we used the simple lower bound $s_r(K_k) \geqslant r$ from Lemma 5.1.1 in the induction step. This linear lower bound will not be enough, so we first prove the following.

Lemma 5.1.8. *For all $r \geqslant 2$ we have that $s_r(K_3) > \frac{r^2}{16}$.*

Proof by induction on r. Note that for $2 \leqslant r \leqslant 16$, it holds that $r \geqslant \frac{r^2}{16}$, so the claim follows by Lemma 5.1.1. Let now $r > 16$ and assume that for $r-1$ it holds that $s_{r-1}(K_3) > \frac{(r-1)^2}{16}$. Again, we use that $s_r(K_3) \geqslant s_{r-1}(K_3)$. Recall that $f_{2,3}(n) \geqslant \lfloor\sqrt{n}\rfloor$. By Corollary 5.1.4 and since $r > 16$, we therefore get that

$$s_r(K_3) \geqslant s_{r-1}(K_3) + f_{2,3}(s_r(K_3)) > \frac{1}{16}(r-1)^2 + \left\lfloor\sqrt{\frac{(r-1)^2}{16}}\right\rfloor \geqslant \frac{r^2}{16}. \quad \square$$

We are ready to prove the lower bound in Theorem 1.2.4.

Proof of lower bound in Theorem 1.2.4. Induction on r. Set $s_r := s_r(K_3)$ for brevity and fix $c = 1/100$. First let $2 \leqslant r \leqslant e^6$. Then $s_r \geqslant \frac{r^2}{16} \geqslant cr^2 \ln r$ by Lemma 5.1.8, by choice of c and since $\ln r \leqslant 6$. Now assume that $r > e^6$, and assume that for $r-1$ it holds that $s_{r-1} \geqslant c(r-1)^2 \ln(r-1)$. By Corollary 5.1.7,

$$s_r \geqslant s_{r-1} + s_r \cdot g\left(\frac{s_r - 1}{r}\right).$$

Now, $(s_r - 1)/r \geqslant 1$ by the simple bound in Lemma 5.1.1. Therefore, we can use that $g(x) \geqslant \frac{\ln x}{2x}$ for $x \geqslant 1$. It follows that

$$\begin{aligned} s_r &\geqslant s_{r-1} + s_r \cdot \frac{r \ln\left(\frac{s_r-1}{r}\right)}{2(s_r-1)} \\ &\geqslant c(r-1)^2 \ln(r-1) + \frac{r}{2} \ln\left(\frac{r}{20}\right) \\ &= cr^2 \ln r - cr^2 \ln\left(\frac{r}{r-1}\right) - 2cr \ln(r-1) + c \ln(r-1) \\ &\quad + \frac{r}{2} \ln r - \frac{r}{2} \ln 20, \end{aligned}$$

where the second inequality follows from the induction hypothesis and since $\frac{s_r-1}{r} \geqslant \frac{r}{16} - \frac{1}{r} \geqslant \frac{r}{20}$ by Lemma 5.1.8 and since $r \geqslant 9$. Therefore, we get that $s_r \geqslant cr^2 \ln r$ if

$$\frac{r}{2} \ln r \geqslant cr^2 \ln\left(\frac{r}{r-1}\right) + 2cr \ln(r-1) + \frac{r}{2} \ln 20. \tag{5.4}$$

But

$$cr^2 \ln\left(\frac{r}{r-1}\right) \leqslant \frac{cr^2}{r-1} \leqslant 2cr \leqslant 2cr \ln r,$$

since $\ln(1+x) \leqslant x$ and $\ln r \geqslant 1$. Also, $\ln 20 \leqslant 3$, so

$$cr^2 \ln\left(\frac{r}{r-1}\right) + 2cr \ln(r-1) + \frac{r}{2} \ln 20 \leqslant 4cr \ln r + \frac{3}{2} r.$$

Hence, (5.4) follows if

$$4c < \frac{1}{4} \quad \text{and} \quad \frac{3}{2} \leqslant \frac{1}{4} \ln r,$$

which is true by choice of c and since $r \geqslant e^6$. □

5.2 Upper bounds on $s_r(K_k)$

Let us first motivate the idea to our proofs, and introduce necessary concepts. This will be common to all three upper bounds in Theorem 1.2.4, 1.2.5, and 1.2.6. Recall that by Observation 5.1.2, we need to construct a graph G with a vertex v of degree at most u such that $G \to (K_k)_r$, but $G - v \nrightarrow (K_k)_r$ to prove an upper bound of u on $s_r(K_k)$.

The graph $H := G[N(v)]$ needs to have special properties. Suppose we constructed a graph G such that $G - v \nrightarrow (K_k)_r$, and let an r-colouring of $G - v$ without a monochromatic copy of K_k be given. Then we want to ensure that no matter how we colour the edges incident to v, we find a monochromatic copy of K_k in some colour. By the pigeonhole principle, no matter how the edges incident to v are coloured, at least one colour, say red, appears at least n/r times, where $n = |N(v)|$. Let $S \subseteq N(v)$ be the subset of neighbours of v such that vs is red for all $s \in S$. Then we would like to deduce that the red subgraph G_{red} of G contains a copy of K_{k-1} in $G_{red}[S]$. Since we do not know which colour appears n/r times at vertex v, and on which edges, we therefore crave for a statement like the following. For a colouring χ of $G - v$ let $G_i(\chi)$ be the subgraph of $G[N(v)]$ in colour i. Then for every colouring χ of $G - v$ without a monochromatic copy of K_k, *every* $G_i(\chi)$ as defined contains a copy of K_{k-1} in *every* subset of size at least n/r.

But rather than defining the graphs $G_i(\chi)$ depending on the colouring, we fix graphs G_i on vertex set $[n]$ at the beginning, and equip them with certain *sender graphs* that "send" colour i to G_i. With this in mind, we use the following notation. We call a collection $G_1, \ldots, G_r$ of graphs a *colour pattern* if $V(G_1) = \ldots = V(G_r)$ and for all $1 \leqslant i < j \leqslant r$ we have that $E(G_i) \cap E(G_j) = \emptyset$. Each graph G_i is then called a *colour class*.

Before making the idea of "sending colours" precise, let us dwell on the structure of the graphs G_i. Certainly, when G_i is the witness graph of colour i in $N(v)$ (in our big graph G) and we want that $G - v \nrightarrow (K_k)_r$, then we better make sure that $K_k \nsubseteq G_i$. On the other hand, we want that in any subset S of the vertices of G_i of size at least n/r, we have that $K_{k-1} \subseteq G_i[S]$. Since we will refer to such graphs frequently, we call a graph F on n vertices (n, r, k)-*critical* if $K_k \nsubseteq F$ and $\alpha_{k-1}(F) < n/r$. Note that the definition is meaningful even for general real-valued r; however, for us, r denotes the number of colours, and therefore we always assume it is an integer.

Recall from the previous section that the Erdős-Rogers function was defined as $f_{k-1,k}(n) = \min\{\alpha_{k-1}(F)\}$, where the minimum is taken over all K_k-free graphs F on n vertices. By definition we have for all $u \in \mathbb{R}$ that

$$f_{k-1,k}(n) < u \quad \Longleftrightarrow \quad \text{there exists an } (n, n/u, k)\text{-critical graph } F. \tag{5.5}$$

So the question whether (n, r, k)-critical graphs exist is equivalent to the question whether $f_{k-1,k}(n) < n/r$. We want to construct a colour pattern $G_1, \ldots, G_r$ on vertex set $[n]$ such that each G_i is (n, r, k)-critical. Since $n = |N(v)| = \deg(v)$, for a good upper bound on $s_r(K_k)$, one would like to have (n, r, k)-critical graphs where $n = n(r, k)$ is as small as possible.

When $k = 3$, the problem of finding an $(n, r, 3)$-critical graph translates to finding a triangle-free graph with independence number less than n/r. This is related to the Ramsey number $R(3, k)$ in the following way. There exists an $(n, r, 3)$-critical graph G if and only if $n < R(3, n/r)$. It is known[1] that $R(3, k)$ is of the order $\Theta\left(\frac{k^2}{\log k}\right)$. Therefore, if G is an $(n, r, 3)$-critical graph then $n \geqslant c \cdot r^2 \log r$ for some constant $c > 0$; and $(n, r, 3)$-critical graphs do exist for $n = C \cdot r^2 \log r$ for some constant $C > 0$.

For our purpose, we need to pack r copies of $(n, r, 3)$-critical graphs edge-disjointly in K_n. The next Lemma states that we can do so on the expense of a factor of $\log r$.

Lemma 5.2.1. *Let r be an integer. Then there exists a colour pattern $G_1, \dots, G_r$ on vertex set $[n]$, where $n = O(r^2 \log^2 r)$, such that each G_i is $(n, r, 3)$-critical.*

We will prove this lemma in Section 5.3. For $k \geqslant 4$, Dudek, Retter and Rödl [25] recently showed that $f_{k-1,k}(n) < O\left((\log n)^{4(k-1)^2}\sqrt{n}\right)$. That is, they constructed a K_k-free graph F on n vertices (where n is large enough) such that every subset of $c(\log n)^{4(k-1)^2}\sqrt{n}$ vertices contains a copy of K_{k-1}, that is an (n, r, k)-critical graph F where $n = c^2\big(2 \log r(1 + o(1))\big)^{8(k-1)^2} r^2$. Again, we would like to pack r of those graphs into K_n. But rather than taking a fixed (n, r, k)-critical graph F and pack it into K_n, we construct r (edge-disjoint) (n, r, k)-critical graphs $G_1, \dots, G_r$ simultaneously as subgraphs of K_n. As it turns out, this simultaneous construction is only little harder than the construction itself in [25]. We prove the following in Section 5.3.

Lemma 5.2.2. *For all integers $k \geqslant 3$ there exist a constant $C = C(k) > 0$ and $r_0 \in \mathbb{N}$ such that for all $r \geqslant r_0$ the following holds. There exists a colour pattern $G_1, \dots, G_r$ on vertex set $[n]$, where $n \leqslant C\,(2 \ln r)^{8(k-1)^2} r^2$, such that each G_i is (n, r, k)-critical.*

This Lemma will be used to prove Theorem 1.2.5. For the upper bound in Theorem 1.2.6, we resort to graphs constructed by Dudek and Rödl in [26]. The graph F on n vertices constructed in [26] is (n, r, k)-critical with $n = O(k^6 r^3)$. Here, it is not so straight-forward anymore that we can do a "simultaneous" construction. So we will start the construction from scratch and provide all the details needed.

Lemma 5.2.3. *Let $k, r \geqslant 3$. Then there exists a colour pattern $G_1, \dots, G_r$ on vertex set $[n]$, where $n \leqslant 8(k-1)^6 r^3$, such that each G_i is (n, r, k)-critical.*

[1] An upper bound of order $\Theta\left(\frac{k^2}{\log k}\right)$ was first proven by Ajtai, Komlos and Szemerédi in [2]. The constant factor was later improved to $1 + o(1)$ by Shearer [51]. A lower bound of the same order of magnitude was first proven by Kim in [45]; and then improved by a constant factor by Fiz-Pontiveros, Griffiths and Morris [36] and independently by Bohman and Keevash [13]. These recent results differ by a factor of $4 + o(1)$ from Shearer's upper bound.

We now turn to the concept of "sending colour i to the graph G_i" for a given colour pattern $G_1, \dots, G_r$. The idea of certain sender graphs was first introduced by Burr, Erdős and Lovász [18].

Definition 5.2.4. *Let H be a graph. A* negative signal sender $S = S^-(r, H, e, f)$ *is a graph S with two distinct edges $e, f \in E(S)$ such that*

(*a*) *$S \nrightarrow (H)_r$, and*

(*b*) *in every r-colouring of $E(S)$ without a monochromatic copy of H, the edges e and f have different colours.*

Similarly, a positive signal sender $S = S^+(r, H, e, f)$ *is a graph S with two distinct edges $e, f \in E(S)$ such that*

(*a*) *$S \nrightarrow (H)_r$, and*

(*b*) *in every r-colouring of $E(S)$ without a monochromatic copy of H, the edges e and f have the same colour.*

The reason for the word "sender" is clear: In every critical colouring (without monochromatic H), the edge e sends either the same or a different colour to the edge f. We call e and f the *signal edges* of the sender graph.

Signal senders are useful to construct certain minimal r-Ramsey graphs and control the colour patterns in them. As mentioned, they were introduced in [18] where Burr, Erdős and Lovász showed that positive and negative signal senders $S^-(2, K_k, e, f)$ and $S^+(2, K_k, e, f)$ exist, i.e. in the special case when the number of colours is two, and the graph H is a clique on at least three vertices. Moreover, they proved the existence of such senders in which the two signal edges are either adjacent or arbitrarily far apart. These senders were crucial for proving the upper bound $s_2(K_k) \leqslant (k-1)^2$.

Later, Burr, Nešetřil and Rödl [17] extended these results and showed that positive and negative signal senders $S^-(2, H, e, f)$ and $S^+(2, H, e, f)$ exist whenever H is 3-connected. Again, it was shown that the signal edges may be either adjacent or arbitrarily far away.

Finally, in 2008, Rödl and Siggers [50] extended the study of signal senders to more than two colours. They showed that positive and negative signal senders $S^-(r, H, e, f)$ and $S^+(r, H, e, f)$ exist for any $r \geqslant 3$ as long as H is 3-connected. Their argument easily extends to H being the triangle, though they do not explicitely mention it. Since we need signal senders specifically for $H = K_3$ and $H = K_4$ for $r \geqslant 2$ colours, we include the proof of the existence of such graphs in Section 5.4. Moreover, since it is only little harder to prove for general 4-connected H, we do so. Our proof is based on the original proofs in [17] and [18], and is therefore also similar to the proof in [50].

Once we established the existence of signal senders in which the signal edges may be arbitrarily far apart, we can join them together and create almost any colour pattern which we desire. The following graphs play a key role in our proofs of upper bounds on $s_r(K_k)$. Note that the acronym BEL refers to Burr, Erdős, and Lovász.

Definition 5.2.5. *Let $r \geqslant 2$ and let H be a graph. Further, let $G_1, \dots, G_r$ be a given colour pattern, and let $F := \bigcup_{i=1}^{r} G_i$ denote the edge-union of the colour classes. Then a graph $B = B(r, H, G_1, \dots, G_r)$ is called an H-*BEL gadget *(or short* BEL gadget*) if*

- *(a) $B \nrightarrow (H)_r$, and*
- *(b) B contains F as an induced subgraph such that in every r-colouring of the edges of B without a monochromatic copy of H, the edges of each G_i, $1 \leqslant i \leqslant r$, are monochromatic, no two $G_i \neq G_j$ having the same colour.*

Note that for a BEL-gadget $B = B(r, H, G_1, \dots, G_r)$ to exist the G_i trivially need to be H-free. However, assuming that H is well-connected enough, this is the only requirement we need. Let Γ_4 denote the class of all 4-connected graphs together with K_3 and K_4.

Lemma 5.2.6. *Let $r \geqslant 2$ be an integer, let $H \in \Gamma_4$ be a graph, and let $G_1, \dots, G_r$ be a given colour pattern such that all colour classes satisfy $H \nsubseteq G_i$. Then there exists a BEL-gadget graph $B(r, H, G_1, \dots, G_r)$.*

In Section 5.4, we will prove the existence of negative and positive signal senders $S(r, H, e, f)$ in which the two edges are either adjacent or arbitrarily far apart. We will then use those "building bricks" to show Lemma 5.2.6.

Before we prove the Lemmas 5.2.1, 5.2.2 and 5.2.3, and the existence of BEL-gadgets in Lemma 5.2.6, we show precisely how they imply our upper bounds on $s_r(K_k)$.

Proof of upper bound in Theorem 1.2.5. Let $k \geqslant 3$ and let $r \geqslant 2$ be integers. To give an upper bound u on $s_r(K_k)$ we need to find a graph G with a vertex $v \in V(G)$ of degree $\deg(v) \leqslant u$ such that $G \to (K_k)_r$ and $G - v \nrightarrow (K_k)_r$.

Let $G_1, \dots, G_r$ be the colour pattern on vertex set $[n]$ given by Lemma 5.2.2. Then $n \leqslant C\,(2\ln r)^{8(k-1)^2}\, r^2$ and each colour class G_i is an (n, r, k)-critical graph. Again, let $F := \dot{\bigcup}_{i=1}^{r} G_i$ be the edge-disjoint union of these graphs, and let $B = B(r, K_k, G_1, \dots, G_r)$ be a K_k-BEL-gadget. Since the G_i are (n, r, k)-critical, they are K_k-free, and therefore, by Lemma 5.2.6, B exists. That is, B contains F as an induced subgraph, $B \nrightarrow (K_k)_r$, and in

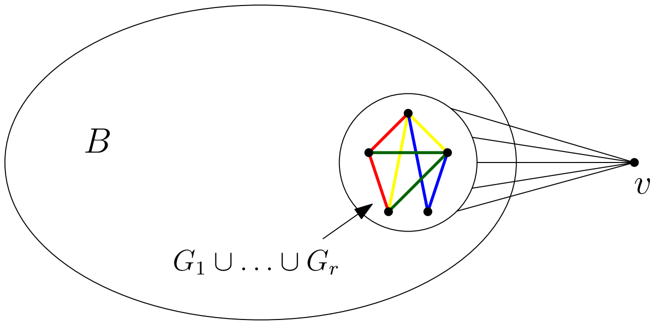


Figure 5.2: An illustration of the graph G.

any r-colouring of the edges of B without a monochromatic K_k all G_i are monochromatic, all G_i having distinct colours. Now, we add a vertex v to the graph B, and add all edges vi for $i \in [n]$. Call the resulting graph G. An illustration is shown in Figure 5.2. By construction and Lemma 5.2.2, $\deg_G(v) = n \leqslant C\,(2\ln r)^{8(k-1)^2}\, r^2$. By definition of a BEL-gadget $G - v \nrightarrow (K_k)_r$. We claim that $G \rightarrow (K_k)_r$. To that end, suppose χ was an r-colouring of $E(G)$ without a monochromatic copy of K_k. By the definition of a BEL-gadget, all G_i are monochromatic, no two having the same colour. Assume without loss of generality that G_i has colour i. By pigeonhole, there exists a colour i such that v is incident to at least n/r edges in colour i. Let $U \subseteq [n]$ be the set of vertices adjacent to v via edges in colour i. Then $|U| \geqslant n/r$, so U contains a copy of K_{k-1} in G_i since G_i is (n,r,k)-critical, and thus $\alpha_{k-1}(G_i) < n/r$. That is we find a copy of K_k in colour i. □

Proof of upper bound in Theorem 1.2.6. Use Lemma 5.2.3 instead of Lemma 5.2.2 in the above proof. □

Proof of upper bound in Theorem 1.2.4. Use Lemma 5.2.1 instead of Lemma 5.2.2 in the above proof. □

5.3 Packing (n, r, k)-critical graphs

In this section, we prove the lemmas 5.2.1, 5.2.2 and 5.2.3, each concerned with packing (edge-disjointly) r graphs $G_1, \ldots, G_r$ which are all (n, r, k)-critical. Recall that we said a graph F is (n, r, k)-critical if it is K_k-free and $\alpha_{k-1}(F) < n/r$, that is, every subset of size n/r contains a copy of K_{k-1}.

Packing many K_3-free graphs with small independence number

Here, we prove Lemma 5.2.1. To that end, we will show the existence of a graph F on

$n := Cr^2 \ln^2 r$ vertices, where $C = 1000$, which can be written as a union of edge-disjoint graphs $G_1, \ldots, G_r$, which are all K_3-free and without independent sets of size n/r. We will find the graphs G_i successively as subgraphs of K_n, using the Local Lemma (see [3, Lemma 5.1.1]). Given r, set $m := n/r = Cr \ln^2 r$ and $q := \binom{m}{2}/(2r)$. For a graph H on n vertices, we define $e_{\min}(m, H)$ ($e_{\max}(m, H)$) to be the smallest (largest) number of edges that appear in any subset $S \subseteq V(H)$ of size $|S| = m$. The following inductive lemma is the crucial step to find the graphs G_i.

Lemma 5.3.1. *Let $H = (V, E)$ be a graph on n vertices, where $n \geqslant n_0$ is large enough, and assume $e_{\min}(m, H) \geqslant \binom{m}{2}/2$. Then there is some subgraph $H' \subseteq H$ on the same vertex set such that $H' = (V, E')$ is triangle free, has no independent set on m vertices, and $e_{\max}(m, H') \leqslant q$.*

Proof. Let c_1, c_2 be constants such that

(*i*) $c_1^3 < c_2/e$ and

(*ii*) $c_1 > \frac{4}{\sqrt{C}} + 2c_2$.

These two conditions are fulfilled e.g. for $c_1 = 1/4$ and $c_2 = 1/20$. Now choose H' by taking each edge of H with probability $p := c_1 n^{-1/2}$, all choices being independent. For a subset $S \subseteq V$, let $e(S)$ and $e'(S)$ denote the number of edges in $H[S]$ and $H'[S]$, respectively. We want to show that H' is triangle-free, $e_{\min}(m, H') \geqslant 1$ and $e_{\max}(m, H') \leqslant q$ with positive probability. To that end, we want to apply the asymmetric version of the Lovász Local Lemma, and therefore, we define the set of bad events in the natural way. Namely, for every $S \in \binom{V}{3}$ that forms a triangle in H, we set T_S to be the event that $H'[S]$ is a triangle as well. Clearly, the probability of such an event is $p_T := p^3$. Further, for every $S \in \binom{V}{m}$, we set I_S to be the event that either S is an independent set in H' or satisfies $e'(S) > q$. That is, $I_S = \{e'(S) = 0 \text{ or } e'(S) > q\}$ and

$$\begin{aligned}
\mathbb{P}(I_S) &\leqslant \mathbb{P}(e'(S) = 0) + \mathbb{P}(e'(S) \geqslant q) \\
&\leqslant (1-p)^{e(S)} + \binom{e(S)}{q} p^q \\
&\leqslant (1-p)^{\binom{m}{2}/2} + \left(\frac{\binom{m}{2} ep}{q}\right)^q \\
&= (1-p)^{\binom{m}{2}/2} + (2epr)^q.
\end{aligned}$$

Note that $(1-p)^{\binom{m}{2}/2} = \exp\left[-p\binom{m}{2}/2\,(1 + o(1))\right] = e^{-pqr(1+o(1))}$ and $(2epr)^q = o(e^{-pqr(1+o(1))})$, since $pr \to 0$, so that for n large enough

$$\mathbb{P}(I_S) \leqslant 2(1-p)^{\binom{m}{2}/2} =: p_I.$$

Let $\mathcal{E}$ be the collection of bad events. That is, $\mathcal{E} = \{T_S : H[S] \cong K_3\} \cup \{I_S : S \in \binom{V}{m}\}$. In the auxiliary dependency graph D, we connect two of the events $A_S, A_{S'} \in \mathcal{E}$ if $|S \cap S'| \geqslant 2$. Then $A_S \in \mathcal{E}$ is mutually independent from the family of all $A_{S'}$ for which $A_S A_{S'}$ is not an edge in this dependency graph. To apply the Lovász Local Lemma, we now bound the degrees in D. If $|S| = 3$ we have

$$\left|d(T_S) \cap \{T_{S'} : \left|S'\right| = 3\}\right| \leqslant 3n \qquad \text{and}$$
$$\left|d(T_S) \cap \{I_{S'} : \left|S'\right| = m\}\right| \leqslant \binom{n}{m}.$$

If $|S| = m$ we have

$$\left|d(I_S) \cap \{T_{S'} : \left|S'\right| = 3\}\right| \leqslant \binom{m}{2}(n-2) < \binom{m}{2}n, \qquad \text{and}$$
$$\left|d(I_S) \cap \{I_{S'} : \left|S'\right| = m\}\right| \leqslant \binom{n}{m}.$$

Therefore, by the general Local Lemma, see Lemma 5.1.1 in [3], if there exist real numbers $x, y \in [0, 1)$ s.t.

$$p_T \leqslant x(1-x)^{3n}(1-y)^{\binom{n}{m}} \tag{5.6}$$
$$p_I \leqslant y(1-x)^{\binom{m}{2}n}(1-y)^{\binom{n}{m}}, \tag{5.7}$$

then there exists a graph H' such that none of the events in $\mathcal{E}$ occurs. We show that these two conditions are fulfilled for $x = c_2 n^{-3/2}$ and $y = \binom{n}{m}^{-1}$. First note that for n large enough

$$x(1-x)^{3n}(1-y)^{\binom{n}{m}} = c_2 n^{-3/2} e^{-1}(1+o(1)) > p^3,$$

since $c_1^3 < \frac{c_2}{e}$, so inequality (5.6) holds. Now (5.7) is equivalent to

$$2^{2/\binom{m}{2}}(1-p) \leqslant y^{2/\binom{m}{2}}(1-x)^{2n}(1-y)^{2\binom{n}{m}/\binom{m}{2}}.$$

We use $1 - p \leqslant e^{-p}$ and $1 - z \geqslant e^{-z-z^2}$ for $z \leqslant 0.6$ to claim (5.7) holds if

$$\exp\left[\frac{2\ln 2}{\binom{m}{2}} - p\right] \leqslant \exp\left[\frac{2\ln y}{\binom{m}{2}} - 2n(x+x^2) - \frac{2\binom{n}{m}}{\binom{m}{2}}(y+y^2)\right].$$

Now, $\frac{2\ln y}{\binom{m}{2}} \geqslant -\frac{4}{\sqrt{C}}n^{-1/2}(1+o(1))$ and $1/m^2 = o\left(n^{-1/2}\right)$. So (5.7) holds if

$$\exp\left[-c_1 n^{-1/2}(1+o(1))\right] \leqslant \exp\left[-(4/\sqrt{C} + 2c_2)n^{-1/2}(1+o(1))\right],$$

which is satisfied by condition (ii) on the constants. Applying the Local Lemma yields the existence of a subgraph H' such that none of the events in $\mathcal{E}$ hold, i.e. H' has the desired properties. □

Proof of Lemma 5.2.1. Let r large enough be given, and set $m := n/r = Cr\ln^2 r$ and $q := \binom{m}{2}/(2r)$ as before. Define $H_1 := K_n$. We choose our graphs inductively as subgraphs of H_1; given H_i for $i \leqslant r$ such that $e_{\min}(m, H_i) \geqslant \binom{m}{2} - (i-1)q$, we have since $i \leqslant r$ that

$$e_{\min}(m, H_i) > \binom{m}{2} - rq = \frac{1}{2}\binom{m}{2}.$$

So by Lemma 5.3.1, we may find G_i as a subgraph of H_i with $e_{\max}(m, G_i) \leqslant q$ such that G_i is triangle-free and has no independent set on n/r vertices. Then take $H_{i+1} = H_i - G_i$. The graph H_{i+1} will be edge-disjoint from G_i (and, inductively, from $G_1, \ldots, G_{i-1}$), and

$$e_{\min}(m, H_{i+1}) \geqslant e_{\min}(m, H_i) - e_{\max}(m, G_i) \geqslant \binom{m}{2} - (i-1)q - q = \binom{m}{2} - iq$$

as desired. □

An upper bound quadratic in r

Here, we prove Lemma 5.2.2. As mentioned in the Section 5.2, we will rely heavily on the graphs constructed in [25] and use a big part of their construction as a black box.

Proof (sketch) of Lemma 5.2.2. Fix $k \geqslant 3$ and let r be large enough. For simplicity in notation we switch to index $s = k - 1$. We need to construct r graphs on $n = O(r^2 \operatorname{polylog}(r))$ vertices that are K_{s+1}-free, but every subset of size n/r contains a K_s. Let q be the smallest prime power such that

$$q \geqslant 64s(\log q)^{4s^2} r.$$

By Bertrand's postulate, $q \leqslant 128s(\log q)^{4s^2} r$, and therefore, $q \leqslant 128s(2\log r)^{4s^2} r$ since r is large enough compared to s. Consider the affine plane of order q. It has $n := q^2$ points and $q^2 + q$ lines such that any two points lie on a unique line, every line contains q points, and every point lies on $q + 1$ lines. It is a well-known fact that affine planes exist whenever q is a prime power. We call two lines L and L' in the affine plane *parallel* if $L \cap L' = \emptyset$. In the affine plane of order q, there exist $q+1$ sets of q pairwise disjoint lines. Let $(V, \mathcal{L})$ be a hypergraph where the vertex set V is the point set of the affine plane of order q, and the hyperedges are lines of the affine plane, with one set of parallel lines removed. Then $(V, \mathcal{L})$ is a q-uniform hypergraph on q^2 vertices such that any two hyperedges meet in at most one vertex.

In [25], Dudek et al. consider a random subhypergraph $(V, \mathcal{L}')$ of $(V, \mathcal{L})$ and show that they can embed the required graph G "along the hyperedges" of $(V, \mathcal{L}')$. For our purposes, let us call a hypergraph $(V, \mathcal{H})$ *good* if there exists a graph G on vertex set V such that

- $K_{s+1} \not\subseteq G$,
- every subset of size $64s(\log q)^{4s^2} q$ of V contains a K_s in G, and

- any edge of G lies inside a hyperedge of $\mathcal{H}$, i.e. for every $e \in E(G)$ there is some $h \in \mathcal{H}$ such that $e \subseteq h$.

Clearly, by choice of q and n, any such graph G is $(n, r, s+1)$-critical. Though it is not explicitely stated as a lemma, the following is proven in Lemma 2.2 in [25].

Lemma 5.3.2 ([25] Lemma 2.2*). *Let $(V, \mathcal{L}')$ be the (random) hypergraph obtained by picking each hyperedge of $(V, \mathcal{L})$ with probability $\frac{\log^2 q}{q}$. Then $(V, \mathcal{L}')$ is* good *with probability at least $1/2 - o(1)$.*

For our purpose, it would be enough to find r hypergraphs $\mathcal{L}_1, \ldots, \mathcal{L}_r$ which are *good*, and such that the hyperedges of different hypergraphs intersect in at most one vertex. Let G_i be the graph associated with hypergraph $\mathcal{L}_i$. Then as mentioned above, all the graphs G_i are $(n, r, s+1)$-critical. Furthermore, they are edge-disjoint, since for $i \neq j$, the edges of G_i (G_j) lie inside hyperedges of $\mathcal{L}_i$ ($\mathcal{L}_j$), and hyperedges of $\mathcal{L}_i$ and $\mathcal{L}_j$ intersect in at most one vertex.

To find the r hypergraphs $\mathcal{L}_1, \ldots, \mathcal{L}_r$ which are *good*, choose a c-edge-colouring of $(V, \mathcal{L})$ at random, where $c := \frac{q}{\log^2 q}$. Note that since $s \geqslant 2$ and by choice of q, c satisfies $c > 4r$. Let $\mathcal{L}_i$ be the sub-hypergraph in colour i ($1 \leqslant i \leqslant c$). Clearly, no two hypergraphs $\mathcal{L}_i$ and $\mathcal{L}_j$ contain the same hyperedge. Moreover, since hyperedges are lines in the affine plane, no two hyperedges intersect in more than one vertex. The probability that a line $\ell \in \mathcal{L}$ is in $\mathcal{L}_i$ is $\frac{\log^2 q}{q}$. So $\mathcal{L}_i$ has the same distribution as the random hypergraph $(V, \mathcal{L}')$ in Lemma 5.3.2. Therefore, $\mathcal{L}_i$ is *good* with probability at least $1/4$, provided q is large enough. Hence, the expected number of *good* hypergraphs $\mathcal{L}_i$ is at least $c/4 > r$. So, there exists a c-colouring of $(V, \mathcal{L})$ such that at least r sub-hypergraphs are *good*. This finishes the proof (sketch) of Lemma 5.2.2. □

An upper bound polynomial in both k and r

Here, we prove Lemma 5.2.3. Let $r, k \geqslant 3$. For $n \leqslant 8(k-1)^6 r^3$ we need to construct r (n, r, k)-critical graphs G_i on n vertices which are edge-disjoint. We will define incidence structures $\mathcal{I}_i = (\mathcal{P}, \mathcal{L}_i)$ on the same set of points such that the families of lines $\mathcal{L}_i$ are disjoint for distinct i. Further, any three lines within one $\mathcal{L}_i$ do not form a triangle. We will then enrich the lines in $\mathcal{L}_i$ randomly as done by Dudek and Rödl in the proof of Theorem 1.1 in [26], and show that the resulting graphs have the desired property with (constant) non-zero probability. For simplicity in notation let us switch to $s = k - 1$ as in the previous proof. So, we are looking for K_{s+1}-free, edge-disjoint graphs such that any subset of the vertices of size n/r contains a K_s, i.e. $\alpha_s(G_i) < n/r$.

Proof of Lemma 5.2.3. First, let us define the incidence structures $\mathcal{I}$. Let q be the smallest prime power such that $s^2 r \leqslant q$, and let $\mathbb{F}_q$ be the finite field of order q. The common vertex set of our graphs is $V := \mathbb{F}_q^3$, i.e. $n = |V| \leqslant 8s^6r^3$. For every $\lambda \in \mathbb{F}_q \setminus \{0\}$, we will define an incidence structure $\mathcal{I}_\lambda = (V, \mathcal{L}_\lambda)$ where $\mathcal{L}_\lambda$ is a family of lines in $\mathbb{F}_q^3$. For $\lambda \in \mathbb{F}_q \setminus \{0\}$ set

$$M_\lambda := \Big\{(1, \lambda\alpha, \lambda\alpha^2) : \alpha \in \mathbb{F}_q \setminus \{0\}\Big\}.$$

We call M_λ the λ-moment curve. In [57], Wenger used the usual moment curve M_1 to construct dense C_6-free graphs. Note that for non-zero $\lambda_1 \neq \lambda_2$ the two curves M_{λ_1} and M_{λ_2} do not intersect. An important and crucial property is that for any $\lambda \neq 0$ any three vectors from M_λ are linearly independent, that is for distinct $\alpha_1, \alpha_2, \alpha_3$,

$$\det \begin{pmatrix} 1 & \lambda\alpha_1 & \lambda\alpha_1^2 \\ 1 & \lambda\alpha_2 & \lambda\alpha_2^2 \\ 1 & \lambda\alpha_3 & \lambda\alpha_3^2 \end{pmatrix} = \lambda^2(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1) \neq 0.$$

In general, a line in $\mathbb{F}_q^3$ is a set of the form $\ell_{\mathbf{s},\mathbf{v}} = \{\beta\mathbf{s} + \mathbf{v} : \beta \in \mathbb{F}_q\}$, where $\mathbf{s} \in \mathbb{F}_q^3 \setminus \{0\}$ is called the *slope*. We define

$$\mathcal{L}_\lambda := \{\ell_{\mathbf{s},\mathbf{v}} : \mathbf{s} \in M_\lambda, \mathbf{v} \in \mathbb{F}_q^3\},$$

that is in the incidence structure $\mathcal{I}_\lambda = (\mathbb{F}_q^3, \mathcal{L}_\lambda)$ we only allow lines with slope vectors from the λ-moment curve. Clearly, $|\mathcal{L}_\lambda| = |M_\lambda| \frac{q^3}{q} = q^2(q-1)$ since each line contains q points. We establish the following properties about each structure $\mathcal{I}_\lambda$, $\lambda \neq 0$.

(1) Every point $v \in V$ is contained in $q-1$ lines from $\mathcal{L}_\lambda$, every line $\ell \in \mathcal{L}_\lambda$ contains q points.

(2) Any two points lie in at most one line.

(3) No three lines in $\mathcal{L}_\lambda$ intersect pairwise in three distinct points (i.e. form a triangle).

Further, we have for $\lambda_1 \neq \lambda_2$,

(4) $\mathcal{L}_{\lambda_1} \cap \mathcal{L}_{\lambda_2} = \emptyset$.

For (1), note that every slope vector in M_λ gives rise to exactly one line through a given point $v \in V$. The second part of (1) follows from the definition of a line. Property (2) holds because lines are affine subspaces of dimension 1 in the vector space $\mathbb{F}_q^3$. For (3), suppose three lines in $\mathcal{L}_\lambda$ intersect pairwise in three distinct points. Then their three slope vectors would be linearly dependent, a contradiction to the linear independence of any three vectors in L_λ we established above. Property (4) simply follows from $M_{\lambda_1} \cap M_{\lambda_2} = \emptyset$ for $\lambda_1 \neq \lambda_2$.

Now, we are ready to define our graphs $G_1, \ldots, G_{q-1}$. Let $\lambda \in \mathbb{F}_q \setminus \{0\}$. We partition every line $\ell \in \mathcal{L}_\lambda$ randomly into s sets $L_1^{(\ell)}, \ldots, L_s^{(\ell)}$ each of cardinality $l_1 := \lfloor \frac{q}{s} \rfloor$ or $l_2 := \lfloor \frac{q}{s} \rfloor + 1$. Note that $l_1, l_2 \geqslant rs$, since rs is an integer. To be precise, between all partitions of a line

$\ell = \dot{\bigcup}_{j=1}^{s} L_j^{(\ell)}$ where $\left|L_1^{(\ell)}\right| = \cdots = \left|L_{s'}^{(\ell)}\right| = l_1$ and $\left|L_{s'}^{(\ell)}\right| = \cdots = \left|L_s^{(\ell)}\right| = l_2$ we choose one uniformly at random, choices for distinct lines in $\mathcal{L}_\lambda$ being independent. The graph G_λ on the vertex set $V = \mathbb{F}_q^3$ is defined as follows. For every $\ell \in \mathcal{L}_\lambda$ and any $i \neq j$, we introduce a complete bipartite graph between the vertex sets $L_i^{(\ell)}$ and $L_j^{(\ell)}$ on ℓ. That is, the graph G_λ consists of a collection of Turán graphs on q vertices with s parts. Each Turán part "lives" along one of the lines $\ell \in \mathcal{L}_\lambda$. By property (2), these parts are edge-disjoint. Further, by property (3), G_λ is K_{s+1}-free. Also, for distinct $\lambda \in \mathbb{F}_q^3$, by property (4), the graphs G_λ are edge disjoint. To finish the proof, we show that $\alpha_s(G_\lambda) < n/r$ with probability at least $1/2$ (and in fact with probability tending to one as q tends to infinity).

These calculations are almost identical to those in [26], so we just briefly sketch them. Let $U \subset V(G)$ of size $|U| = \left\lfloor \frac{n}{r} \right\rfloor$, and let $\mathcal{A}(U)$ denote the event that $G_\lambda[U]$ contains no copy of K_s. Then, since by property (3) any K_s can only appear within a line $\ell \in \mathcal{L}_\lambda$,

$$\mathcal{A}(U) \subseteq \bigcap_{\ell \in \mathcal{L}_\lambda} \mathcal{A}(U \cap \ell),$$

and therefore, since all the events $\mathcal{A}(U \cap \ell)$ are independent,

$$\mathbb{P}(\mathcal{A}(U)) \leqslant \prod_{\ell \in \mathcal{L}_\lambda} \mathbb{P}(\mathcal{A}(U \cap \ell)).$$

Now, for an individual line $\ell \in \mathcal{L}_\lambda$, set $u_\ell := |U \cap \ell|$, and let $\ell = \bigcup_{j=1}^{s} L_j^{(\ell)}$ be the partition we chose at random. Then the event $\mathcal{A}(U \cap \ell)$ is equivalent to the existence of a $j \in [s]$ such that $U \cap L_j^{(\ell)} = \emptyset$. But for fixed $j \in [s]$,

$$\mathbb{P}\left(U \cap L_j^{(\ell)} = \emptyset\right) = \frac{\binom{q-u_\ell}{\left|L_j^{(\ell)}\right|}}{\binom{q}{\left|L_j^{(\ell)}\right|}} \leqslant \left(1 - \frac{u_\ell}{q}\right)^{\left|L_j^{(\ell)}\right|} \leqslant \exp\left(-\frac{l_1 u_\ell}{q}\right).$$

Therefore,

$$\begin{aligned}
\mathbb{P}(\mathcal{A}(U)) &\leqslant \prod_{\ell \in \mathcal{L}_\lambda} \mathbb{P}\left(\exists j \in [s] : U \cap L_j^{(\ell)} = \emptyset\right) \\
&\leqslant s^{|\mathcal{L}_\lambda|} \exp\left(-\sum_{\ell \in \mathcal{L}_\lambda} \frac{l_1 u_\ell}{q}\right) \\
&= s^{|\mathcal{L}_\lambda|} \exp\left(-\frac{q-1}{q} l_1 |U|\right),
\end{aligned}$$

since every point in U belongs to exactly $q-1$ lines (property (1)), and therefore $\sum_{\ell \in \mathcal{L}_\lambda} u_\ell =$

$\sum_{\ell \in \mathcal{L}_\lambda} |U \cap \ell| = (q-1)|U|$. We obtain,

$$\begin{aligned}
\mathbb{P}\left(\exists U \in \binom{V}{\lfloor \frac{n}{r} \rfloor} : \mathcal{A}(U)\right) &\leqslant \binom{n}{\lfloor \frac{n}{r} \rfloor} s^{|\mathcal{L}_\lambda|} \exp\left(-\frac{q-1}{q} l_1 \left\lfloor \frac{n}{r} \right\rfloor\right) \\
&\leqslant (re)^{n/r} s^{q^2(q-1)} \exp\left(-\frac{q-1}{q}(rs) \left\lfloor \frac{n}{r} \right\rfloor\right) \\
&\leqslant \exp\left[q^3\left(\frac{\ln r}{r} + \frac{1}{r} + \ln s - \frac{3}{4}s\right)\right] \\
&< \frac{1}{2}
\end{aligned}$$

for $s \geqslant 2$ and $r \geqslant 3$. Therefore, there exists an instance of G_λ such that every subset U of size at least $\lfloor \frac{n}{r} \rfloor$ contains a copy of K_s in G_λ. □

5.4 Signal senders and BEL-gadgets

Throughout this section, let H be a fixed graph. All colourings are r-colourings, for $r \geqslant 2$. We prove the existence of negative and positive signal senders in a series of lemmas where the first ones are the building blocks like basic lego bricks, followed by lemmas which say that we can combine the basic lego bricks to build more elaborate structures, culminating in the last lemma which says, that we can combine all those elaborate structures to a lego castle (the BEL gadget). Though we can show existence of such castle only when $H \in \Gamma_4$, we state in each lemma explicitely what is needed from H.

The following definitions will prove to be useful in constructing negative and positive signal senders. Let $G_{\max} = G_{\max}(r, H)$ be an edge-maximal graph on $r_r(H)$ vertices subject to the following constraints: $G_{\max}$ is a clique on $r_r(H) - 1$ vertices along with some other vertex v and $G_{\max} \nrightarrow (H)_r$.

For a graph G, we call a colouring of $E(G)$ *critical* if it does not contain a monochromatic copy of H. By definition of $G_{\max}$, such a critical colouring of $E(G_{\max})$ exists. We want to study the possible colour patterns that occur on edges incident to v in a critical colouring in $G_{\max}$. Given the graph $G_{\max}$ and the vertex v, for a given colouring $\chi : E(G_{\max}) \to [r]$, we say that it has *colour pattern* $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$ if v is incident to a_i edges of colour i. We say a colour pattern $\mathbf{a} \in \mathbb{N}^r$ is *admissible* if there exists a critical colouring $\chi : E(G_{\max}) \to [r]$ with colour pattern $\mathbf{a}$. Throughout this section, let $\mathcal{A}(H) = \{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$ be the collection of all admissible colour patterns .

The first simple lemma puts a restriction on the form of an admissible colour pattern when $\delta(H) \geqslant 2$. It is a version of which appeared in [17].

Lemma 5.4.1. *If $\delta(H) \geqslant 2$, then in any colouring of $G_{\max}$ without a monochromatic copy of H, v is incident to edges of colour C, for* every *$C \in [r]$.*

Proof. First note that by definition of $r_r(H)$ and since $G_{\max} \nrightarrow (H)_r$, $G_{\max}$ cannot be complete. Pick a vertex w that is not adjacent to v and let χ be an r-colouring of the edge set of $G_{\max}$ without a monochromatic copy of H. Suppose for a contradiction that v is not incident to any edge in colour C, for some $C \in [r]$. Now add the edge vw to $G_{\max}$ and extend χ by colouring this edge with colour C. Since $G_{\max}$ was edge-maximal, there must be a monochromatic copy H_1 of H in $G_{\max} + vw$. Since G did not contain a monochromatic copy of H, H_1 must use the edge vw and be of colour C. However, v is incident to only one edge of colour C, namely vw, contradicting that $\delta(H) \geqslant 2$. □

Given an edge-colouring of $G_{\max}$, it induces a vertex-colouring of the neighbours of v by colouring a neighbour w with the colour of the edge vw. With this in mind, the following definition and lemma will prove useful in constructing negative and positive signal senders.

Definition 5.4.2. *For a hypergraph $\mathcal{H}$, an r-colouring of $V(\mathcal{H})$, and an edge $h \in \mathcal{H}$, we say h has* colour pattern $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$ *if it has a_i vertices of colour i.*

Recall, that by $\mathbf{a}_1, \ldots, \mathbf{a}_q$ we denoted the collection of admissible colour patterns, that is colour patterns that appear when $G_{\max}$ is coloured without a monochromatic copy of H. For a hypergraph $\mathcal{H}$, we say an r-colouring of the vertices $V(\mathcal{H})$ is *H-critical* if each edge $h \in \mathcal{H}$ has one of the colour patterns $\mathbf{a}_k$, $1 \leqslant k \leqslant q$. We call two vertices x and y in a hypergraph $\mathcal{H}$ adjacent if there exists a hyperedge $h \in \mathcal{H}$ such that $x, y \in h$. Otherwise, we say x and y are non-adjacent.

Lemma 5.4.3. *Let t be the degree of v in $G_{\max}$. If $\delta(H) \geqslant 2$, then there exists a t-uniform hypergraph $\mathcal{H}$ and two non-adjacent vertices x, y in $V(\mathcal{H})$ such that*

(i) there exists an H-critical r-colouring of the vertices of $\mathcal{H}$;

(ii) for every H-critical r-colouring of $V(\mathcal{H})$, x and y have different colours;

(iii) $\mathcal{H}$ contains no circuits of length $|V(H)|$ or less.

Proof. For a t-uniform hypergraph $\widetilde{\mathcal{H}}$ define *Property* $(*)$*:* For every r-colouring of $V(\widetilde{\mathcal{H}})$ there exists a hyperedge in $\widetilde{\mathcal{H}}$ with colour pattern different from all of $\mathbf{a}_1, \ldots, \mathbf{a}_q$.
Let $\mathcal{H}'$ be a t-uniform hypergraph which contains no circuits of $|V(H)|$ or shorter, and which is $(r+1)$-chromatic. Such a hypergraph exists as was shown in [33] by a now standard application of the probabilistic method [3].

The hypergraph $\mathcal{H}'$ has property $(*)$; given an r-colouring of the vertices of $\mathcal{H}$, since $\mathcal{H}'$ has chromatic number at least $r+1$, there must exist an edge $h \in \mathcal{H}'$ so that all of its vertices have colour i for some i. But such an edge cannot have any of the patterns $\mathbf{a}_1, \ldots, \mathbf{a}_q$ by Lemma 5.4.1.

Now, let $\mathcal{H}'' \subseteq \mathcal{H}'$ be edge-minimal with respect to property $(*)$, and let $h = \{x_1, \ldots, x_t\}$ be an arbitrary edge in $\mathcal{H}''$ (since there is at least one colour pattern $\mathbf{a}_1$, $\mathcal{H}''$ must contain an edge). Add t new vertices $y_1, \ldots, y_t$ and call this new hypergraph $\mathcal{H}_0$. Set $\mathcal{H}_i := \mathcal{H}_0 - \{h\} + \{h_i\}$ where $h_i = \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_t\}$ for $1 \leqslant i \leqslant t$. Clearly, $\mathcal{H}_0$ has property $(*)$, while $\mathcal{H}_t$ has not, since $\mathcal{H}''$ was edge-minimal with respect to $(*)$ and h_t is independent of all the edges of $\mathcal{H}''$. Therefore, there must exist an index $1 \leqslant i \leqslant t$ such that $\mathcal{H}_{i-1}$ has $(*)$, whereas $\mathcal{H}_i$ has not. Then taking $\mathcal{H} := \mathcal{H}_i$, $x := x_i$ and $y := y_i$ fulfills the conditions of the lemma. First, note that by construction, x and y are non-adjacent (the only edge containing y is h_i, and $x \notin h_i$). Now, (i) simply follows because $\mathcal{H}$ does not have property $(*)$.

For (ii), suppose there was an r-colouring χ of $V(\mathcal{H})$ in which every edge has one of the colour patterns $\mathbf{a}_1, \ldots, \mathbf{a}_q$ and x_i and y_i have the same colour. Then one could take the same colouring for $\mathcal{H}_{i-1}$ to see that $\mathcal{H}_{i-1}$ does not have property $(*)$, a contradiction.

To see that (iii) holds, note that any circuit of $\mathcal{H}$ corresponds to a circuit of $\mathcal{H}''$ by replacing any use of the hyperedge h_i by h_0. Then, since $\mathcal{H}''$ is a subgraph of $\mathcal{H}'$ and $\mathcal{H}'$ has no circuits of length $|V(H)|$ or less, we get that $\mathcal{H}$ has has no circuits of length $|V(H)|$ or less. □

We now prove the existence of a negative (r, H, e, f)-signal sender when H is either a triangle or 3-connected, in which the two signal edges are adjacent.

Lemma 5.4.4. *If $H \in \Gamma_3$ then there exists a negative (r, H, e, f)-signal sender G_{ad} in which the two signal edges e and f form an induced path of length 2. That is, e and f are adjacent, though they do not form a K_3.*

Proof. Take $G_{\max}$ and v as defined at the beginning of this section. Recall that $G_{\max} - v$ is a clique, that $G_{\max} \nrightarrow (H)_r$, and that any r-colouring of $G_{\max}$ without a monochromatic copy of H has v incident to edges of at least two different colours.

As before, let $\mathbf{a}_1, \ldots, \mathbf{a}_q$ be the admissible colour patterns, i. e. the possible colour patterns that appear in an r-colouring of $G_{\max}$ without a monochromatic H and take t to be the degree of v in G.

Take $\mathcal{H}$ to be the t-uniform hypergraph with distinguished (non-adjacent) vertices x, y as in the previous lemma.

G_{ad} is constructed as follows. It has vertex set equal to the vertex set of $\mathcal{H}$ along with $r_r(H) - t - 1$ new vertices for each hyperedge of $\mathcal{H}$ and a new distinguished vertex w. Formally, we may take its vertex set to be $V(\mathcal{H}) \cup (E(\mathcal{H}) \times [r(H)_r - t - 1]) \cup \{w\}$. Connect w to all

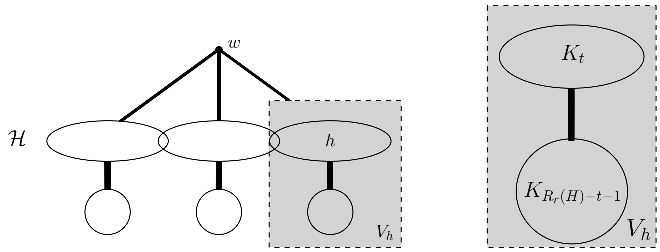


Figure 5.3: An illustration of the graph G_{ad} in Lemma 5.4.4. A thick line indicates that the vertices of the corresponding sets are pairwise connected.

vertices of $V(\mathcal{H})$. For every edge h of $\mathcal{H}$, we put in G_{ad} a clique on vertex set $V_h := h \cup (\{h\} \times [r_r(H) - t - 1])$. That is, we put a clique on the $r_r(H) - 1$ vertices that correspond to h. The construction of G_{ad} is illustrated in Figure 5.3. Set e and f to be $e := \{wx\}$ and $f := \{wy\}$. Note that since x and y are non-adjacent in $\mathcal{H}$, e and f form an induced path of length two in $G_{\max}$. We say an edge $g \in G_{\text{ad}}$ corresponds to a hyperedge h if $g \subseteq V_h$. Note that since the shortest circuit in $\mathcal{H}$ has length greater than $|V(H)| > 2$, any edge in $G_{\text{ad}} - w$ corresponds to exactly one edge in $\mathcal{H}$. Furthermore, w along with V_h for any hyperedge h induces a copy of $G_{\max}$.

We claim that any copy of H has to be contained in $V_h \cup \{w\}$ for some hyperedge $h \in \mathcal{H}$. To see this, assume first that $H \neq K_3$. Assume for a contradiction that there is a copy of some H' contained in $V(G_{\text{ad}}) - w$ but not contained in V_h for any h, where H' is obtained from H by removing a vertex. Note that since H is 3-connected, H' must be 2-connected and use at most $|V(H)|$ vertices in $G_{\text{ad}} - w$. Therefore, there must be some pair of edges $f_1 = \{v_1, v_2\}, f_2 = \{v_2, v_3\}$ of G_{ad} contained in that copy of H' so that $v_1 \in V_h$, $v_2 \in V_h \cap V_{h'}$, $v_3 \in V_{h'}$ with $h \neq h'$. Then there must be a path in H' between v_1 and v_3, say with edges $e_1, \ldots, e_j$, that does not use v_2. Any such path in H' contains at most $|V(H')| - 2 \leqslant |V(H)| - 2$ edges; thus, considering the sequence of hyperedges containing the respective e_i along with f_1, f_2, this induces a circuit in $\mathcal{H}$ of length at most $|V(H)|$, contradicting choice of $\mathcal{H}$.
Assume now that $H = K_3$. Then either a copy of H uses the vertex w. But deleting a vertex in H leaves an edge, which must belong to a single hyperedge h as said above. Hence, this copy of H lies in $V_\cup \{w\}$. Or a copy of H does not use w. But then the claim follows as before since $H' = H = K_3$ is 2-connected on at most $|V(H)|$ vertices.

For property (a) of negative signal senders, we need to show that $G_{\text{ad}} \nrightarrow (H)_r$. To that end, colour the vertices of $\mathcal{H}$ with r colours such that in every hyperedge $h \in \mathcal{H}$ one of the colour patterns $\mathbf{a}_1, \ldots, \mathbf{a}_q$ appears. We define an r-colouring on $E(G_{\text{ad}})$ as follows. Colour each edge $\{w, z\}$ with the colour of z assigned by the hypergraph colouring. For every hyperedge $h \in \mathcal{H}$, extend this colouring to an r-colouring of $G_{\text{ad}}[\{w\} \cup V_h] \cong G$ without creating a monochromatic H. This is possible because h has one of the colour patterns $\mathbf{a}_1, \ldots, \mathbf{a}_q$. Further, there will not be any conflicts, since any two of the V_h intersect in at most one vertex (and so don't share edges). This defines a colouring of $E(G_{\text{ad}})$ without a monochromatic H, as any copy of H must be contained in $G_{\text{ad}}[\{w\} \cup V_h]$ for some h. In a similar fashion, any r-colouring of $E(G)$ without any monochromatic H induces a vertex colouring of $V(\mathcal{H})$ in which every edge $h \in \mathcal{H}$ has one of the colour patterns $\mathbf{a}_1, \ldots, \mathbf{a}_q$. So, x and y have different colours by (ii) of the previous lemma, and thus $\{w, x\}$ and $\{w, y\}$ have different colours in the edge-colouring. □

We now know of the existence of negative signal senders, with the limitation that the two signal edges need to be adjacent. However, now that we have it, proving the existence of the various other gadgets is straightforward. We first create a version which is easier to apply; it says we can combine the signal senders from Lemma 5.4.4 with little restrictions and without concerns about creating monochromatic copies of H. Recall that we call a colouring of any given graph G critical if it does not contain a monochromatic copy of H.

Lemma 5.4.5. *Let G_0 be a graph with a collection $\mathcal{C}$ of pairs of edges, $\mathcal{C} = \{(e_1, f_1), \ldots, (e_k, f_k)\}$ where each pair (e_i, f_i) is an induced path of length two in G_0. If $H \in \Gamma_4$, then there is a graph G with an induced copy of G_0 so that:*

1. *Any critical colouring of G_0 which colours every e_i with a different colour than f_i extends to a critical colouring of G.*
2. *Any critical colouring of G must satisfy that for every pair of edges (e_i, f_i), e_i and f_i have different colours.*

Proof. Consider the graph obtained by taking G_0 and, for each pair of edges $(e_i, f_i) \in \mathcal{C}$, insert a copy G_i of the negative (r, H, e, f)-signal sender G_{ad} from Lemma 5.4.4 and identify the two signal edges e and f of G_{ad} with e_i and f_i. This is the graph G. Note that the copy of G_0 in G is an induced one since e and f form an induced path of length two in G_{ad}. By construction of G_{ad}, property (2) must hold; any colouring of G without a monochromatic copy of H must satisfy that every pair of edges (e_i, f_i) have different colours.

To show property (1), let any colouring of G_0 without a monochromatic copy of H satisfying that every pair of edges (e_i, f_i) have different colours be given. By property (b) of negative signalers, there is a critical colouring of G_{ad} so that the two signal edges e and$f$ have different colours. By symmetry of colours, given any pair of different colours for the signal edges of G_{ad}, it can be extended to a colouring of G_{ad} without a monochromatic copy of H. Therefore, for any $G_i \cong G_{\text{ad}}$ in G, there is some colouring of it without a monochromatic copy of H that extends the given colouring of e_i and f_i. Taking for each G_i a colouring without a monochromatic copy of H gives a colouring of G; we claim this colouring has no monochromatic copy of H. To see this, assume the opposite and let H' be a monochromatic copy of H in G. Then H' must use some edge of $G_i \setminus G_j$ and some edge of $G_j \setminus G_i$, for some $i \neq j$. Since for $1 \leqslant i < j \leqslant k$, G_i and G_j intersect in at most one edge, and since the pairs (e_i, f_i) and (e_j, f_j) form induced P_2 in G_0 and in G_i (G_j) (and hence in G), G_i and G_j intersect in at most 2 vertices. Since $H = K_3$ or H is 3-connected, we may therefore assume that $G_j = G_0$. Again, since (e_i, f_i) forms an induced P_2 in G, there exists $w_0 \in V(G_0) \setminus V(G_i)$ and $w_i \in V(G_i) \setminus V(G_0)$ such that both vertices are contained in H', the copy of H in G. Therefore, $w_0 w_i \notin E(G)$, which is a contradiction when $H = K_3$ or $H = K_4$. Otherwise, since $H \in \Gamma_4$, H is 4-connected. A contradiction again, since removing the three vertices in $V(G_0) \cap V(G_i) = e_i \cup f_i$ disconnects w_0 from w_i. So G contains no monochromatic copy of H, as desired. □

We are now ready to prove the existence of positive signal senders.

Lemma 5.4.6. *If $H \in \Gamma_4$, then positive (r, H, e, f)-signal senders exist, where e and f are independent.*

Proof. We will construct a K_3-free graph G_0 with two distinguished independent edges e and f. We then choose a collection $\mathcal{C}$ of adjacent edges of G_0 so that there is a critical colouring of G_0 (i.e. without a monochromatic copy of H) in which every pair of edges in $\mathcal{C}$ has different colours, and in any r-colouring of G_0 in which each pair of edges in $\mathcal{C}$ has different colours, e and f must have the same colour. Applying Lemma 5.4.5 to G_0 will complete the proof.

To construct G_0, take two stars $K_{1,r}$. Label the centers by x and y, and the leaves by $x_1, \ldots, x_r$ and by $y_1, \ldots, y_r$ respectively. Add the edge xy to G_0. Set $e := xx_1$ and $f := yy_1$. Now add the following pairs of adjacent edges of G_0 to $\mathcal{C}$: For all $1 \leqslant i < j \leqslant r$, add the pairs (xx_i, xx_j) and (yy_i, yy_j). For all $2 \leqslant i \leqslant r$ add the pairs (xy, xx_i) and (xy, yy_i). If we are given a colouring of G_0 so that every pair of edges in $\mathcal{C}$ has different colours, then we claim e and f must have the same colour. Note that all of the edges of the first star have different colours and use all r colours. Therefore, since xy has a colour different from all of the edges of the first star except e, we must have that xy and e have the same colour. By symmetry, xy and f have the same colour. Therefore, e and f have the same colour, as desired. Note further that there is a colouring in which every pair of edges in $\mathcal{C}$ has different colours, namely the colouring that assigns to e, f, and xy colour 1 and assigns to each edge xx_i (yy_i) the colour i. Note that G_0 is a tree and contains no copy of H (and therefore, no colouring contains a monochromatic copy of H). Furthermore, for the same reason, each pair (e_i, f_i) forms an induced path of length two, and we can apply Lemma 5.4.5. □

Remark 5.4.7. *Note that the proof also gives easily the existence of positive signal senders* *in which the two signal edges are adjacent. However, this is not crucial for the remaining lemmas.*

Signal senders are not by themselves particularly useful due to concerns about creating monochromatic copies of H. By stringing together positive signal senders, however, we can alleviate these concerns by creating an (r, H, e, f)-signal sender in which e and f are far apart. Let us say that two edges e and f have distance at least k in G, denoted by $\mathrm{dist}(e, f) \geqslant k$, if any path P in G containing both e and f has length at least k.

Lemma 5.4.8. *If $H \in \Gamma_3$ and there exists a positive (r, H, e, f)-signal sender with independent edges e, f, then for any $k \geqslant 1$ there is a positive (r, H, e, f)-signal sender G such that dist$(e, f) \geqslant k$.*

Proof. Induction on k. For $k = 3$, take G_3 to be a positive (r, H, e, f)-signal sender with independent signal edges e and f, which exists by assumption. Then any path containing both e and f has length at least 3. For $k \geqslant 3$, let G_k be a positive (r, H, e_k, f_k)-signal sender such that $\mathrm{dist}(e_k, f_k) \geqslant k$ in G_k, and let G_3 be as before. We construct G_{k+1} by taking the disjoint union of G_k and G, and identify e with f_k. Set now $e_{k+1} := e_k$ and $f_{k+1} := f$. Clearly, any colouring of G_{k+1} without a monochromatic copy of H must have that e_{k+1} and f_{k+1} have the same colour. Also, any path containing both e_{k+1} and f_{k+1} must have length at least $k + 1$. Note that removing the two vertices of e disconnects G_k from G_3, so any copy of H in G_{k+1} is either in G_k or in G_3. To see that $G_{k+1} \nrightarrow H$, take any colouring of G_k without a monochromatic copy of H. This only colours one of the edges in the copy of G_3 (namely e), so by symmetry of the colours and the assumption that $G_3 \nrightarrow H$, this may be extended to a colouring of G_{k+1} without a monochromatic copy of H. □

Combining the previous two lemmas gives the existence of a useful version of positive signal senders.

Corollary 5.4.9. *If $H \in \Gamma_4$, then there is a positive (r, H, e, f)-signal sender G so that dist$(e, f) \geqslant |V(H)| + 3$.*

This Corollary also easily gives the existence of a useful version of negative signal senders.

Corollary 5.4.10. *If $H \in \Gamma_4$, then there is a negative (r, H, e, f)-signal sender G so that dist$(e, f) \geqslant |V(H)| + 3$.*

Proof. Let G_0 be a positive (r, H, e_0, f_0)-signal sender given by Corollary 5.4.9. Let G_1 be a negative (r, H, e_1, f_1)-signal sender given by Lemma 5.4.4. Let G be the union of G_0 and G_1 and identify e_1 with f_0. Set $e := e_0, f := f_1$. As in the previous lemma, since H is 3-connected or $H = K_3$, $G \nrightarrow (H)_r$. Furthermore, since in G_0 any path containing e_0 and f_0 had length at least $|V(H)| + 3$, in G any path containing $e = e_0$ and $f = f_1$ must have length at least $|V(H)|+3$. Finally, in any colouring of G without a monochromatic copy of H, we must have that e_0 and f_0 have the same colour, and e_1 and f_1 have different colours. Since f_0 and e_1 are the same edge, e_0 and e_1 have the same colour and so $e = e_0$ and $f = f_1$ have different colours, as desired. □

These two corollaries allow us to prove a version of Lemma 5.4.5 where the edges on which we impose restrictions may be independent.

Lemma 5.4.11. *Let G_0 be a graph with collections $\mathcal{C}, \mathcal{C}'$ of pairs of independent edges of G_0, $\mathcal{C} = \{(e_1, f_1), \ldots, (e_k, f_k)\}$ and $\mathcal{C}' = \{(e'_1, f'_1), \ldots, (e'_{k'}, f'_{k'})\}$. If $H \in \Gamma_4$, then there is a graph G with an induced copy of G_0 so that:*

1. *Any critical colouring of G_0 that colours each pair of edges e_i, f_i with the same colour and every pair of edges e'_i, f'_i with different colours extends to a critical colouring of G.*
2. *Any critical colouring of G must satisfy that for every pair of edges (e_i, f_i), e_i and f_i have the same colour, and every pair of edges (e'_i, f'_i), e'_i and f'_i have different colours.*

Proof. Let G' be a positive (r, H, e, f)-signal sender given by Corollary 5.4.9, and let G'' be a negative (r, H, e', f')-signal sender given by Corollary 5.4.10. Construct the graph G as follows. Take G_0 and, for each pair of edges $(e_i, f_i) \in \mathcal{C}$, insert a copy $G_i \cong G'$ and identify the two signal edges e and f with e_i and f_i; and for each pair of edges $(e'_i, f'_i) \in \mathcal{C}'$, insert a copy $G'_i \cong G''$ and identify the two signal edges e' and f' with e'_i and f'_i. Note that no G_i (G'_i) adds edges between e_i and f_i (e'_i and f'_i) since $\mathrm{dist}_{G_i}(e_i, f_i) > 3$ ($\mathrm{dist}_{G'_i}(e'_i, f'_i) > 3$), so G_0 is an unduced subgraph of G. By constructions of G', G'', property (2) must hold.

To show property (1), let any colouring of G_0 without a monochromatic copy of H be given such that for every pair of edges (e_i, f_i), e_i and f_i have the same colour, and such that for every pair of edges (e'_i, f'_i), e'_i and f'_i have different colours. Note that since G' (G'') is a positive (negative) signal sender, there is some colouring of G' (G'') without a monochromatic copy of H so that the signal edges have the same (a different) colour. By symmetry of colours, given any pair of same (different) colours for the signal edges, it can be extended to a colouring of G' (G'') without a monochromatic copy of H. Therefore, for any copy G_i of G' (G'_i of G'') there is some colouring of it without a monochromatic copy of H that extends the given colouring of e_i and f_i (e'_i and f'_i). Taking for each G_i (G'_i) such a colouring gives a colouring of G. We claim this has no monochromatic copy of H. In fact, any copy H' of H in G is either a subgraph of G_i for $i \geqslant 0$ or of G'_i for $i \geqslant 1$. To see this, assume some copy H' of H is given so that it is not a subgraph of G_0 and not a subgraph of any of the G_i or G'_i. Then, since H is connected, H' must use some pair of adjacent edges g, g' so that g is contained in some G_i or G'_i but not in G_0, and g' is not contained in that G_i (or G'_i). Assume without loss of generality that g is contained in G_i. By construction, e_i and f_i are the signal edges of G_i, and since g is adjacent to the edge $g' \notin G_i$, we may assume without loss of generality that g and e_i are adjacent (though not equal since $e_i \in G_0$ and we assumed g is not). Let v be the vertex of g not contained in e_i, and let w be the vertex of g' not contained in e_i. Assume first that $H \neq K_3$. We claim that removing the vertices of e_i disconnects this copy of H, which contradicts that H is 3-connected. If removing these vertices does not disconnect H', then there must be some path from v to w in H'. Thus, this path has length at most $|V(H)| - 1$. Furthermore, since $v \in V(G_i) \setminus V(G_0)$, it must use a vertex of f_i. Taking the path from v to the first time it intersects f_i and then adding e_i, g, and f_i to it forms a path of length at most $|V(H)| + 2$ inside G_i, contradicting that $\text{dist}_{G_i}(e_i, f_i) \geqslant |V(H)| + 3$. When $H = K_3$, the claim follows similarly. Indeed, assume the edge vw was present in G, then since v is in $V(G_i)$ solely, $vw \in G_i$. Moreover, since $g' \notin G_i$, we have that $w \in V(G_i) \cap V(G_0)$, that is, w is one of the vertices of f_i. Then the path f_i, vw, g, e_i is again a path in G_i of length less than $|V(H)| + 3$, a contradiction. □

The above lemma will easily give the existence of BEL-gadgets in the case that H is 4-connected or $H = K_3$ or $H = K_4$; and the G_i are H-free.

Proof of Lemma 5.2.6. Let $G_1, \ldots, G_r$ be a given colour pattern (that is they are pairwise edge-disjoint and $V(G_1) = \ldots = V(G_r)$) so that none of $G_1, \ldots, G_r$ contains a copy of H. We construct G_0 (to apply Lemma 5.4.11) in the following way. Take the edge-union of $G_1, \ldots, G_r$ and add to this $2r$ new vertices. On the $2r$ vertices, add r independent edges $e_1, \ldots, e_r$. This will be our graph G_0. Add every pair (e_i, e_j) to a collection $\mathcal{C}'$ (the collection of *negative* pairs). For every $1 \leqslant i \leqslant r$, and every $f \in G_i$ add the pair (e_i, f) to a collection $\mathcal{C}$ (the collection of *positive* pairs). Applying Lemma 5.4.11 to G_0 with collections $\mathcal{C}$ and $\mathcal{C}'$ gives us some graph G. We claim that $G \nrightarrow (H)_r$. To show this, it is sufficient to give a colouring of G_0 that satisfies all the conditions imposed by $\mathcal{C}$ and $\mathcal{C}'$ and contains no monochromatic copy of H. We achieve this by colouring edge e_i and every edge of G_i with colour i. Any two edges e_i, e_j have different colours, so the conditions imposed by $\mathcal{C}'$ are satisfied, and any edge f of G_i has the same colour as e_i, namely i, so the conditions imposed by $\mathcal{C}$ are satisfied. Note that G_0 has no monochromatic copy of H in this colouring, as such a copy cannot use any of the isolated edges e_i and therefore must be contained in some G_i. But $H \nsubseteq G_i$ for all i, so $G \nrightarrow (H)_r$. To complete the proof, it is sufficient to show that given any colouring satisfying the conditions imposed by $\mathcal{C}$ and $\mathcal{C}'$, it satisfies that each G_i is monochromatic and no two G_i, G_j share a colour. Note that any edge of G_i must have the same colour as e_i by the conditions of $\mathcal{C}$, so indeed each G_i is monochromatic. Furthermore, no two e_i, e_j may have the same colour by the conditions of $\mathcal{C}'$, so no two G_i, G_j have the same colour, as desired. □

5.5 Concluding remarks

We have seen in Lemma 5.1.3, that it is fairly simple to show that $s_r(K_k) > s_{r-1}(K_k)$. However, as mentioned in the introduction, it is not that clear that $s_r(K_k)$ is also increasing in k. It would be surprising though if that was not the case.

Question 5.5.1. *Is it true that for all $r \geqslant 2$, $k \geqslant 3$ we have that $s_r(K_k) \geqslant s_r(K_{k-1})$?*

We also saw that the Erdős-Rogers function defined as $f_{s,k}(n) = \min\{\alpha_s(F) : |V(F)| = n \text{ and } K_k \nsubseteq F\}$ is tightly connected to the study of $s_r(K_k)$. For our lower bounds in Section 5.1, we heavily used the recursion $s_r \geqslant s_{r-1} + \max_i \alpha_{k-1}(G_i)$, and the fact that the G_i are K_k-free and thus *all* have $(k-1)$-independence number at least $f_{k-1,k}(n)$. On the other hand, we saw in Section 5.3 that the known constructions for K_k-free graphs with small independence number can be modified to constructions of r edge-disjoint such graphs on the same vertex set. Therefore, we believe that tightening the known bounds on $f_{k-1,k}(n)$ will directly contribute to tightening the bounds on $s_r(K_k)$. The currently best known bounds on the Erdős-Rogers function are

$$\Omega\left(\sqrt{\frac{n\log n}{\log\log n}}\right) = f_{k-1,k}(n) = O\left((\log n)^{4(k-1)^2}\sqrt{n}\right),$$

see Theorem 5.1.5 and [25]. That is $f_{k-1,k}(n)$ is of the order of $n^{1/2+o(1)}$. We wonder whether the upper bound can be strengthened in the following way.

Question 5.5.2. *Does there exist a universal constant C (independent of k) such that $f_{k-1,k}(n) = O\left((\log n)^C\sqrt{n}\right)$? And does the construction of such a K_k-free graph on n vertices with $(k-1)$-independence number less than $O\left((\log n)^C\sqrt{n}\right)$ generalize to a packing of such graphs?*

The affirmative of both questions would then imply that there is a universal constant $C > 0$ such that $s_r(K_k) = O\left(r^2(\log r)^C\right)$.

In the special case when $k = 3$, recall that we used the Lovász Local Lemma iteratively to successively find (edge-disjoint) triangle-free subgraphs $G_i \subseteq K_n$ with independence number less than $\Theta(\sqrt{n}(\log n)^2)$. This way (and using the power of BEL-gadgets) we proved the upper bound of order $\Theta\left(r^2(\log r)^2\right)$ in Theorem 1.2.4. Indeed, this well-known application of the Local Lemma which first appeared in [53] by Spencer simplified an earlier proof of Erdős of the lower bound $R(3,k) \geqslant c\,(k/\log k)^2$ on the off-diagonal Ramsey numbers (applying the Local Lemma only once gives a triangle-free graph on n vertices with independence number less than $\Theta(\sqrt{n}\log n)$, but we needed the extra log-factor to show that we can actually pack these graphs we find with the Local Lemma). In 1990, Erdős and Bollobás suggested the triangle-free process as a strategy to find better lower bounds on $R(3,k)$, or reversly, triangle-free graphs on n vertices with an even smaller independence number. In [45], Kim proved the existence of a triangle-free graph G on n vertices with independence number at most $O\left(\sqrt{n\log n}\right)$, showing that $R(3,k) \geqslant \Omega\left(k^2/\log k\right)$. Later Bohman [12], reproved this result, using the triangle-free process. Very recently, Fiz Pontiveros, Griffiths and Morris [36], and independently Bohman and Keevash [13], improved the constant factor and showed, using the triangle-free process again, that $R(3,k) \geqslant (1/4 - o(1))k^2/\ln k$. We are optimistic that one can apply the triangle-free process iteratively, with some modifications, as we did with the Local Lemma, and thus find a packing of graphs $G_1,\dots,G_r$ on n vertices, all being triangle-free and having smaller independence number than $\Theta(\sqrt{n}(\log n)^2)$. However, we are not sure which power in the exponent of the log-factor is needed, since some freedom in packing the graphs G_i seems to be necessary. We therefore pose the following problem.

Problem 5.5.3. *Determine the constant c such that $s_r(K_3) = \Theta\left(r^2(\log r)^c\right)$.*

Our bounds on $s_r(K_3)$ imply that $1 \leqslant c \leqslant 2$, and we strongly believe that the latter should be a strict inequality for the aforementioned reasons.

Zusammenfassung

Die Dissertation besteht im Wesentlichen aus zwei Teilen, die unabhängig voneinander sind.

Im ersten Teil befassen wir uns mit Orientierungsspielen, die unter anderem bereits von Aigner, Alon, Beck, Ben-Shimon, Bollobás, Krivelevich, Sudakov, Szabó und Tuza studiert wurden. Zwei Spieler, genannt OMaker und OBreaker, richten abwechselnd bisher ungerichtete Kanten des K_n, dem vollständigen Graphen auf n Knoten. OMaker gewinnt, wenn der resultierende Digraph (ein Turnier) eine gewisse, vorher bestimmte, Eigenschaft $\mathcal{P}$ besitzt. Andernfalls gewinnt OBreaker. Für ein gegebenes Turnier T_k auf k Knoten betrachten wir das Orientierungsturnierspiel $Or(T_k)$, bei dem OMaker gewinnt, wenn das finale Turnier eine Kopie von T_k enthällt. Wir zeigen, dass OMaker dieses Spiel gewinnen kann, solange $k \leqslant (2-o(1)) \log_2 n$, während OBreaker eine Gewinnstrategie hat, sobald k ungefähr die Größenordnung $4 \log_2 n$ besitzt. Für die untere Schranke betrachten wir die Spielvariante, in der OMaker gewinnt, wenn der Digraph, der *nur* aus ihren gerichteten Kanten besteht, eine Kopie von T_k enthällt. Dieses *Turnierspiel* wurde bereits von Beck und Gebauer studiert, und unsere untere Schranke verbessert bisherige Ergebnisse um einen konstanten Faktor. Darüberhinaus ist sie für das Turnierspiel scharf, wie das Kriterium von Erdős und Selfridge impliziert. Das zweite Orientierungsspiel, das wir betrachten ist das "Oriented-cycle game", in dem OMaker gewinnt, falls das finale Turnier einen gerichteten Kreis enthällt. Kürzlich zeigten Ben-Shimon, Krivelevich und Sudakov, dass OMaker gewinnt, selbst wenn OBreaker bis zu $n/2$ Kanten in jeder Runde richten darf. Sei b die Anzahl der Kanten, die OBreaker in einer Runde richten darf. Wie schon Bollobás und Szabó beobachteten, gewinnt OBreaker sobald $b \geqslant n-2$. Wir verbessern die triviale obere Schranke und zeigen, dass OBreaker eine Gewinnstrategie hat, wenn $b \geqslant 5n/6+2$. Weiterhin passen wir die Strategie an für den Fall, dass OBreaker *genau* b Kanten in jeder Runde richten muss und widerlegen somit eine Vermutung von Bollobás und Szabó.

Im zweiten Teil studieren wir minimale Ramseygraphen. Dabei ist ein Graph G Ramsey für einen Graphen H, falls jede Zweifärbung der Kanten von G eine einfarbige Kopie von H enthällt. Der Graph G wird dann Ramsey-minimal genannt, falls er Ramsey für H ist, aber kein echter Untergraph von G diese Eigenschaft besitzt. Sei $s(H)$ der kleinste Minimalgrad, den ein Graph G haben kann, der Ramsey-minimal für H ist. Dieser Parameter wurde erstmals von Burr, Erdős, und Lovász studiert, die zeigten, dass $s(K_k) = (k-1)^2$. In dieser Arbeit beantworten wir eine Frage von Szabó, Zumstein, und Zürcher und zeigen, dass $s(K_k \cdot K_2) = k - 1$, wobei $K_k \cdot K_2$ der Graph auf $k + 1$ Knoten ist, bestehend aus einem K_k und einer angehängten Kante. Dieses Resultat impliziert interessanterweise, im Zusammenspiel mit einem bekannten Resultat von Nešetřil und Rödl, dass jeder Graph, der Ramsey-äquivalent zu K_k ist, die disjunkte Vereinigung von K_k und einem Graph ohne K_k sein muss. Wir studieren die maximale Anzahl an Cliquen K_t, die zu K_k hinzugefügt werden können, sodass der resultierende Graph Ramsey-äquivalent zu K_k ist. Eine obere Schranke, die wir erhalten, ist ungefähr um einen Faktor zwei größer als eine untere Schranke von Szabó et al. Weiterhin verallgemeinern wir die Konzepte für r Farben und betrachten das Verhalten des entsprechenden Paramenters $s_r(K_k)$ in Abhängigkeit von r. Unsere Schranken sind scharf bis auf einen Faktor, der polylogarithmisch in r ist.

Eidesstattliche Erklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den

Anita Liebenau

References

[1] H. Abbott and D. Hanson, *A Problem of Schur and its Generalizations*, Acta Arithmetica **20** (1972), no. 2, 175–187.

[2] M. Ajtai, J. Komlós, and E. Szemerédi, *A note on Ramsey numbers*, Journal of Combinatorial Theory, Series A **29** (1980), no. 3, 354–360.

[3] N. Alon and J.H. Spencer, *The Probabilistic Method, 3^{rd} ed.*, vol. 73, Wiley-Interscience, 2008.

[4] J. Beck, *Random graphs and positional games on the complete graph*, Annals of Discrete Math **118** (1985), 7–13.

[5] ———, *Achievement games and the probabilistic method*, Combinatorics, Paul Erdős is Eighty **1** (1993), 51–78.

[6] ———, *Deterministic graphs games and a probabilistic intuition*, Combinatorics, Probability & Computing **3** (1994), 13–26.

[7] ———, *Foundations of positional games*, Random Structures & Algorithms **9** (1996), no. 1-2, 15–47.

[8] ———, *Combinatorial games: Tic-Tac-Toe theory*, vol. 114, Cambridge University Press, 2008.

[9] M. Bednarska and T. Łuczak, *Biased positional games for which random strategies are nearly optimal*, Combinatorica **20** (2000), no. 4, 477–488.

[10] ———, *Biased positional games and the phase transition*, Random Structures and Algorithms **18** (2001), no. 2, 141–152.

[11] I. Ben-Eliezer, M. Krivelevich, and B. Sudakov, *Biased orientation games*, Discrete Mathematics **312** (2012), no. 10, 1732–1742.

[12] T. Bohman, *The triangle-free process*, Advances in Mathematics **221** (2009), no. 5, 1653–1677.

[13] T. Bohman and P. Keevash, *Dynamic concentration of the triangle-free process*, arXiv:1302.5963 (2013).

[14] B. Bollobás, *Random graphs*, vol. 73, Cambridge University Press, 2001.

[15] B. Bollobás and H.R. Hind, *Graphs without large triangle-free subgraphs*, Discrete Mathematics **87** (1991), no. 2, 119–131.

[16] B. Bollobás and T. Szabó, *The oriented cycle game*, Discrete mathematics **186** (1998), no. 1, 55–67.

[17] S. A. Burr, J. Nešetřil, and V. Rödl, *On the use of senders in generalized Ramsey theory for graphs*, Discrete mathematics **54** (1985), no. 1, 1–13.

[18] S.A. Burr, P. Erdős, and L. Lovász, *On graphs of Ramsey type*, Ars Combinatoria **1** (1976), 167–190.

[19] V. Chvátal and P. Erdős, *Biased positional games*, Annals of Discrete Math **2** (1978), 221–228.

[20] V. Chvátal and F. Harary, *Generalized Ramsey theory for graphs, I. Diagonal numbers*, Periodica Mathematica Hungarica **3** (1973), no. 1, 115–124.

[21] D. Clemens, H. Gebauer, and A. Liebenau, *The random graph intuition for the tournament game*, arXiv preprint arXiv:1307.4229 (2013).

[22] D. Clemens and A. Liebenau, *A non-trivial upper bound on the threshold bias of the oriented-cycle game*, arXiv preprint arXiv:1404.4529 (2014).

[23] D. Conlon, *A new upper bound for diagonal Ramsey numbers*, Annals of Mathematics **170** (2009), 941–960.

[24] A. Dudek and D. Mubayi, *On Generalized Ramsey Numbers for 3-uniform Hypergraphs*, Journal of Graph Theory **76** (2014), no. 3, 217–223.

[25] A. Dudek, T. Retter, and V. Rödl, *On generalized Ramsey numbers of Erdős and Rogers*, arXiv preprint arXiv:1309.4521 (2013).

[26] A. Dudek and V. Rödl, *On K_s-free subgraphs in K_{s+k}-free graphs and vertex Folkman numbers*, Combinatorica **31** (2011), no. 1, 39–53.

[27] ———, *On the function of Erdős and Rogers*, Ramsey Theory, Springer, 2011, pp. 63–76.

[28] P. Erdős and T. Gallai, *On the minimal number of vertices representing the edges of a graph*, Magyar Tud. Akad. Mat. Kutató Int. Közl **6** (1961), 181–203.

[29] P. Erdős and C.A. Rogers, *The construction of certain graphs*, Can. J. Math **14** (1962), 702–707.

[30] P. Erdős and J.L. Selfridge, *On a combinatorial game*, Journal of Combinatorial Theory, Series A **14** (1973), no. 3, 298–301.

[31] P. Erdős, *Some remarks on the theory of graphs*, Bulletin of the American Mathematical Society **53** (1947), 292–294.

[32] P. Erdős, R.J. Faudree, C.C. Rousseau, and R.H. Schelp, *The size Ramsey number*, Periodica Mathematica Hungarica **9** (1978), no. 1-2, 145–161.

[33] P. Erdős and A. Hajnal, *On chromatic number of graphs and set-systems*, Acta Mathematica Hungarica **17** (1966), 61–99.

[34] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compositio Mathematica **2** (1935), 463–470.

[35] R.J. Faudree and R.H. Schelp, *A survey of results on the size Ramsey number*, Paul Erdős and his mathematics, II (Budapest, 1999) **11** (2002), 291–309.

[36] G. Fiz Pontiveros, S. Griffiths, and R. Morris, *The triangle-free process and R(3,k)*, arXiv:1302.6279 (2013).

[37] J. Fox, A. Grinshpun, A. Liebenau, Y. Person, and T. Szabó, *Minimum degrees of minimal Ramsey graphs for bipartite graphs*, in preparation.

[38] ———, *On the minimimum degree of minimal Ramsey graphs for multiple colours*, in preparation.

[39] ———, *What is Ramsey-equivalent to the clique?*, 2014.

[40] J. Fox and K. Lin, *The minimum degree of Ramsey-minimal graphs*, Journal of Graph Theory **54** (2006), 167–177.

[41] H. Gebauer, *On the clique-game*, European Journal of Combinatorics **33** (2012), no. 1, 8–19.

[42] H. Gebauer and T. Szabó, *Asymptotic random graph intuition for the biased connectivity game*, Random Structures & Algorithms **35** (2009), no. 4, 431–443.

[43] R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math **7** (1955), no. 1, 7.

[44] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó, *Planarity, colorability, and minor games*, SIAM Journal on Discrete Mathematics **22** (2008), no. 1, 194–212.

[45] J. H. Kim, *The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$*, Random Structures & Algorithms **7** (1995), no. 3, 173–207.

[46] M. Krivelevich, *K^s-Free graphs without large K^r-free subgraphs*, Combinatorics, Probability and Computing **3** (1994), 349–354.

[47] J. Nešetřil and V. Rödl, *The Ramsey property for graphs with forbidden complete subgraphs*, Journal of Combinatorial Theory, Series B **20** (1976), 243–249.

[48] S.P. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin **1** (1994), no. 7, revised version #13 (2011), available at `http://www.cs.rit.edu/ spr/ElJC/ejcram13.pdf`.

[49] F. P. Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society **30** (1930), 264–286.

[50] V. Rödl and M. Siggers, *On Ramsey minimal graphs*, SIAM Journal on Discrete Mathematics **22** (2008), no. 2, 467–488.

[51] J. B. Shearer, *A note on the independence number of triangle-free graphs*, Discrete Mathematics **46** (1983), no. 1, 83–87.

[52] ———, *On the independence number of sparse graphs*, Random Structures & Algorithms **7** (1995), no. 3, 269–271.

[53] J.H. Spencer, *Asymptotic lower bounds for Ramsey functions*, Discrete Mathematics **20** (1977), 69–76.

[54] M. Stojaković and T. Szabó, *Positional games on random graphs*, Random Structures & Algorithms **26** (2005), no. 1-2, 204–223.

[55] T. Szabó, P. Zumstein, and S. Zürcher, *On the minimum degree of minimal Ramsey graphs*, Journal of Graph Theory **64** (2010), no. 2, 150–164.

[56] H. Wan, *Upper bounds for Ramsey numbers $R(3,3,\ldots,3)$ and Schur numbers*, Journal of Graph Theory **26** (1997), no. 3, 119–122.

[57] R. Wenger, *Extremal graphs with no C_4's, C_6's, or C_{10}'s*, Journal of Combinatorial Theory, Series B **52** (1991), no. 1, 113–116.

[58] X. Xiaodong, X. Zheng, G. Exoo, and S.P. Radziszowski, *Constructive lower bounds on classical multicolor Ramsey numbers*, Journal of Combinatorics **11** (2004), no. 2, R35.

Curriculum Vitae

Anita Liebenau

1998-2002	Carl-Friedrich-Gauß-Gymnasium, Schwedt/Oder
2002-2005	Hermann-Hesse-Schule, Berlin Friedrichshain-Kreuzberg
2006-2010	Technische Universität Darmstadt (2008-2010 on leave) Mathematics with Computer Science
2008-2010	University of Cambridge Part II & Part III of the Mathematical Tripos Certificate of Advanced Studies in Mathematics
2010-3013	PhD student at Freie Universität Berlin