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Author(s):

F. Hahn, A. Dahlberg, J. Eisert, and A. Pappa

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Limitations of nearest-neighbour quantum networks

F. Hahn,^{1,2} A. Dahlberg,³ J. Eisert,^{1,4} and A. Pappa^{2,5}

¹*Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*

²*Electrical Engineering and Computer Science Department, Technische Universität Berlin, 10587 Berlin, Germany*

³*QuTech, TU Delft, 2628CJ Delft, The Netherlands*

⁴*Fraunhofer Heinrich Hertz Institute, 10587 Berlin, Germany*

⁵*Fraunhofer Institute for Open Communication Systems - FOKUS, 10589 Berlin, Germany*

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Quantum communication research has in recent years shifted to include multi-partite networks for which questions of quantum network routing naturally emerge. To understand the potential for multi-partite routing, we focus on the most promising architectures for future quantum networks – those connecting nodes close to each other. Nearest-neighbour networks such as rings, lines, and grids, have been studied under different communication scenarios to facilitate the sharing of quantum resources, especially in the presence of bottlenecks. We here analyze the potential of nearest-neighbour quantum networks and identify some serious limitations, by demonstrating that rings and lines cannot overcome common bottleneck communication problems.

INTRODUCTION

The idea of communication in multi-partite quantum networks beyond point-to-point quantum key distribution has gained substantial momentum in recent years. Such applications suggest that it should be possible to distribute quantum data stably over a wide area [1–6]. In this context, new and challenging questions of how to pursue routing and quantum network coding arise [7–9].

A key multi-partite feature is the promise to solve communication bottlenecks in quantum networks. The best known and practically motivated example is perhaps the butterfly network [10–12], where two pairs of nodes intend to send quantum messages between them, bypassing the existing bottleneck in the network. To understand the potential and limitations of quantum network coding, a sound level of abstraction to discuss such delicate challenges is to capture the quantum state as a *graph state* [6, 13–17].

But what is the underlying property that enables bypassing existing bottlenecks in the network? In the case of the butterfly cluster state, an instance of a graph state, X -measurements on the two ‘additional’ nodes lead to the creation of two crossing maximally-entangled pairs that enable further quantum communication via teleportation. This is equivalent to a more widely-applicable technique that uses graph transformations called *local complementations* (LCs). As shown in Ref. [6], LCs can reveal ‘hidden’ properties of the shared graph state and allow us to optimise the quantum resources available.

It is generally more common to have a bottleneck in sparse networks, where the connectivity between the different nodes is limited. A particular case of sparse networks that is of great importance for quantum communication is that of nearest-neighbour architectures. Such networks allow quantum information to travel only over short distances and therefore aim to minimise the noise and losses in the transmission. The butterfly network is one of the smallest instances of a grid network, while other common nearest-neighbour architectures are lines and rings. However, except for the case of the butterfly and

related examples [10, 18], not much is known to date about what is possible in this type of network architectures.

In this work, we examine in detail whether we can extend the prominent example of the butterfly network to other nearest-neighbour architectures. We specifically ask whether simultaneous communication of two pairs of nodes is possible in bottleneck scenarios when the underlying architecture is a ring or a line. We conclude that these nearest-neighbour networks are unsuitable for bypassing bottlenecks, and a long-distance communication link is required. Finally, our techniques can find application to more general network topologies and bottleneck settings.

METHODS

Throughout this work, we stay in the framework where quantum systems held by the respective parties of the network are qubits and are – on a level of abstraction – seen as being in pure quantum states. The connectivity pattern of the network is captured by a suitable graph, following the mindset of Refs. [6, 15–17]. A graph $G = (V, E)$ consists of a finite set of vertices $V \subseteq \mathbb{N}$ and of edges $E \subseteq V \times V$. We consider *simple* graphs that neither contain edges connecting a vertex to itself nor multiple edges between the same pair of vertices. The set of vertices sharing an edge with vertex v is called its *neighbourhood* and denoted as N_v . The graph’s adjacency matrix is

$$(\Gamma_G)_{i,j} := \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E. \end{cases}$$

A *graph state* vector [13] $|G\rangle$ is defined by $|V|$ qubits in $|+\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$ entangled via CZ gates for each edge,

$$|G\rangle := \prod_{(i,j) \in E} CZ_{i,j} |+\rangle^{\otimes |V|}.$$

Physically important are *local Clifford* operations, i.e. local unitaries from the single-qubit Clifford group. Such operations on a graph state are interestingly reflected by simple

transformations of the respective graph, namely local complementations [13, 19]. We define the following.

Definition 1 (Local complementation). *A graph $G = (V, E)$ and vertex $v \in V$ define a locally complemented graph $\tau_v(G)$ with adjacency matrix*

$$\Gamma_{\tau_v(G)} := \Gamma_G + \Theta_v \pmod{2},$$

where Θ_v is the complete graph of the neighbourhood N_v .

The graph state vector obtained from local complementation with respect to node v of graph G , is defined by $|\tau_v(G)\rangle := U_v^\tau |G\rangle$, where $U_v^\tau := (-iX_v)^{1/2}(iZ_{N_v})^{1/2}$. Deciding whether or not two graphs can be transformed into each other via sequential LCs is possible in polynomial time [20]. As we only consider local Clifford operations and Pauli measurements, the resulting states remain graph states and can be described in terms of the pre-measurement graph with LCs and vertex deletions [13, 14].

We hence define local Pauli measurements P_v with respect to this graph action, i.e., $P_v \in \{X_v, Y_v, Z_v\}$ maps a graph with n vertices to one with $n - 1$ by removing v . Note that local Pauli measurements on $|G\rangle$ result in a different graph state up to local unitary corrections (cf. Prop. 7 in Ref. [14]). We will omit these corrections for the sake of clarity.

Definition 2 (Pauli measurements). *The graph action of Z_v is $Z_v(G) := (\tilde{V}, E \cap \tilde{V} \times \tilde{V})$ with $\tilde{V} := V \setminus \{v\}$, that is, deleting the row and column of v from Γ_G gives $\Gamma_{Z_v(G)}$. With local complementations we further have $Y_v(G) := Z_v \circ \tau_v(G)$ and $X_v(G) := Z_v \circ \tau_w \circ \tau_v \circ \tau_w(G)$, where $w \in N_v$.*

A graph H that can be obtained from a graph G via a sequence of local complementations and vertex deletions is called a *vertex-minor* of G [21]. We will denote this as $H < G$ and call a graph v -minor of another graph, if v is the single vertex that has been deleted, e.g., $X_v(G), Y_v(G)$ and $Z_v(G)$ are v -minors of G . Deciding whether or not a graph H is a vertex-minor of G on some subset of vertices is *NP-complete* [16]. Likewise, extracting a set of Bell pairs on a fixed set of vertices from a general graph is NP-complete [15]. Exactly because solving general problems in quantum network routing is provably hard, we focus on impossibility results for widely used network architectures: rings and lines. Due to their symmetry, we consider the former first. Theorems 1 and 3 eliminate specific instances of – what we expect to be – commonly-encountered problems in future quantum networks.

RING GRAPHS

We first consider graph state vectors $|R_n\rangle$ corresponding to ring graphs, i.e. $R_n := (V_n, E_n)$ with $V_n := \{1, \dots, n\}$ and $E_n := \{(1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}$ (Fig. 1). Our goal is to obtain two maximally entangled pairs between qubits $\{a_1, a_2\}$ and $\{b_1, b_2\}$ via local Clifford operations and

Pauli measurements. That is, we want to determine if the most simple graph with two connected components

$$K_2 \cup K_2 := (\{a_1, a_2, b_1, b_2\}, \{(a_1, a_2), (b_1, b_2)\})$$

is a vertex-minor of R_n . Without loss of generality we can restrict to $a_1 < a_2, b_1 < b_2$ and set $a_1 = 1$. In order to show that it is not possible to achieve our goal if $a_1 < b_1 < a_2 < b_2$, we will make use of the following lemmas.

Lemma 1 (Theorem 3.1 [17]). *Let G and H be two graphs and (v_1, v_2, \dots, v_k) be an ordered tuple of vertices that contains each element of $V_G \setminus V_H$ exactly once. We define the corresponding set of possible Pauli operations as*

$$\mathcal{P}_{(v_1, v_2, \dots, v_k)} := \{P_{v_k} \circ P_{v_{k-1}} \circ \dots \circ P_{v_1} | P_v \in \{X_v, Y_v, Z_v\}\}.$$

Then H is a vertex-minor of G if and only if there exists an operation $P \in \mathcal{P}_{(v_1, v_2, \dots, v_k)}$ such that H can be obtained from $P(G)$ via a sequence of local complementations.

It will also be useful to single out two specific types of vertices, namely *leaves* and *axils*.

Definition 3 (Leaf and axil). *A leaf is a vertex with degree one. An axil is the unique neighbour of a leaf.*

For leaves and axils, we have the following lemma regarding the relevant vertex-minors.

Lemma 2 (Theorem 2.7 [22]). *Let G and H be graphs and v be a vertex in V_G but not in V_H . Then it holds that:*

- (a) *If v is a leaf: $H < G \Leftrightarrow H < G \setminus v$.*
- (b) *If v is an axil: $H < G \Leftrightarrow H < \tau_w \circ \tau_v(G) \setminus v$, where w is the leaf associated with v .*

Note that (b) follows from (a) as leaf and associated axil can be transformed into each other via local complementation. With Lemmas 1 and 2 we can prove our no-go results (Theorems 1 and 3). In combination with Theorem 2, we provide a tool that can find application to more general network architectures that are not limited to nearest-neighbour ones.

Theorem 1 (No crossing on a ring). *It is not possible to extract two maximally entangled pairs from $|R_n\rangle$ if $a_1 = 1 < b_1 < a_2 < b_2$ for any $n \in \mathbb{N}$ with local Clifford operations, local Pauli measurements and classical communication.*

Proof. The proof works by induction. The base case is trivial for $1 \leq n \leq 4$. For $n = 5$ and $n = 6$ it can be derived by Propositions 1 and 2 in Ref. [6], since ring graphs have a bottleneck with respect to communication requests of the type $a_1 < b_1 < a_2 < b_2$.

For the inductive step, we now assume that Theorem 1 holds up to a given n . We can then build an argument on three case distinctions to show the same follows for $n + 1$. In order to see this, note that $K_2 \cup K_2$ is a vertex-minor of R_{n+1} if and only if it is a vertex-minor of at least one of

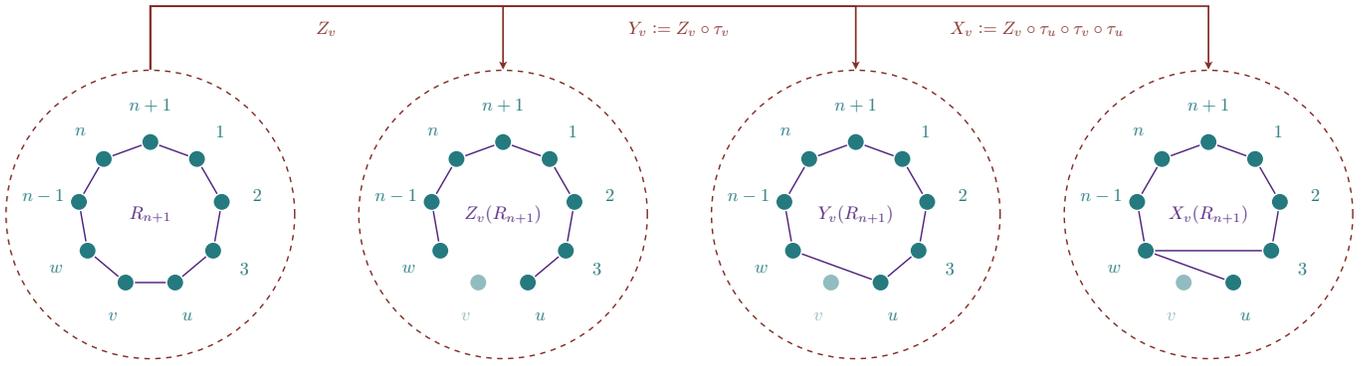


Figure 1: **Measurements on a ring.** From left to right: Ring R_{n+1} with $n + 1$ vertices, node v is measured in the Z -, Y - and X -basis.

$X(R_{n+1})$, $Y(R_{n+1})$ or $Z(R_{n+1})$ (Lemma 1). In the following we will show that it is not a vertex-minor of any of them. More specifically, any v -minor H of R_{n+1} is, according to Lemma 1, equivalent to either $X_v(R_{n+1})$, $Y_v(R_{n+1})$ or $Z_v(R_{n+1})$ via a sequence of local complementations. As the vertex-minor relationship is inherited via local complementations, $K_2 \cup K_2$ can only be a vertex-minor of H if it is a vertex-minor of at least one of $X(R_{n+1})$, $Y(R_{n+1})$ or $Z(R_{n+1})$. It is therefore enough to show that $K_2 \cup K_2 \not\prec P_v(R_{n+1})$ for all $P_v \in \{X_v, Y_v, Z_v\}$, where we have $v \notin \{a_1, a_2, b_1, b_2\}$. We consider the three cases separately.

Ring 1. $K_2 \cup K_2 \not\prec Z_v(R_{n+1})$.

$Z_v(R_{n+1})$ is the line graph L_n with n vertices as depicted in Fig. 1. With Theorem 3 we find that maximally entangled pairs (a_1, a_2) and (b_1, b_2) can not be extracted from the corresponding graph state vector $|L_n\rangle$, i.e., $K_2 \cup K_2 \not\prec Z_v(R_{n+1})$.

Ring 2. $K_2 \cup K_2 \not\prec Y_v(R_{n+1})$.

Since $Y_v(R_{n+1}) = R_n$ (see also Fig. 1), we can use our induction hypothesis to infer that $|Y_v(R_{n+1})\rangle$ does not allow for the extraction of maximally entangled pairs (a_1, a_2) and (b_1, b_2) , that is, we have $K_2 \cup K_2 \not\prec Y_v(R_{n+1})$.

Ring 3. $K_2 \cup K_2 \not\prec X_v(R_{n+1})$.

$X_v(R_{n+1})$ is the ring graph R_n with an additional leaf as depicted in Fig. 1. The former neighbours of v within the graph R_{n+1} constitute leaf u and axil w – note that the roles of u, w are reversed if the other vertex is chosen as a special neighbour in the sense of Def. 2.

If both leaf and axil are part of the target graph and constitute one of the target Bell pairs, i.e., $\{u, w\} \in \{\{a_1, a_2\}, \{b_1, b_2\}\}$, this contradicts the assumption $a_1 < b_1 < a_2 < b_2$, since there is no vertex between u and w . Both leaf and axil can also not be part of the target graph while being in different Bell pairs: Leaf-axil pairs either remain such pairs under local complementations or turn into twins (cf. Theorem 2, Fig. 2 and Defs. 4, 5). This is a contradiction to u and w being in different Bell pairs of the target graph, since measuring the neighbourhood of axils or twins

can never result in a graph with two connected components. If just the axil w is part of the target graph, Lemma 2 (a) with $v = u$ reduces the problem to R_n and we can use our induction hypothesis. If just the leaf u is part of the target graph, we can use Lemma 2 (b) with $v = w$. \square

Definition 4 (Twin). A twin is a vertex v that has the same neighbourhood as a second vertex $w \neq v$ in the sense that

$$N_v \setminus \{w\} = N_w \setminus \{v\}.$$

Definition 5 (Foliage). The set containing all the leaves, axils and twins of a graph is called the foliage of that graph.

Theorem 2 (Foliage is LC invariant). The foliage of a graph G is invariant under local complementation.

Proof. For $G = K_2 = \tau_v(K_2)$ the statement is trivial as both vertices are leaves, axils, and twins at the same time. For all other graphs note that a twin can be transformed into a leaf (or an axil) via local complementations: If twins v and w are neighbours, τ_w disconnects v from all its other neighbours, that is, in $\tau_w(G)$ the vertex v is a leaf and w its axil. If twins v and w are not neighbours, note that a twin pair always has a common neighbour u (unless $G = K_2$). In the graph $\tau_u(G)$, the two vertices are then neighbouring twins. With the above argument we know that in $\tau_w \circ \tau_u(G)$ vertex v is a leaf and w the corresponding axil. Conversely, given a pair of leaf v and axil w , the local complementation τ_w connects v to every vertex in N_w , i.e., a twin pair is created. Again, choosing a common neighbour u allows us to go to the graph $\tau_u \circ \tau_v(G)$ in which w is a leaf and v its axil. In fact, Fig. 2 shows that all LCs on u, v, w with $u \in N_v$ and/or $u \in N_w$ leave the foliage invariant. Since LCs can only transform the neighbourhood, we thus have shown that leaves, axils and twins can *only* be transformed into each other with local complementations.

As LCs are self-inverse this concludes the proof: Assume that a node in the foliage can be created via LCs out of a node that is not in the foliage. Then the reverse sequence of LCs would transform a node in the foliage to one that is not – contradicting the last sentence of the previous paragraph. \square

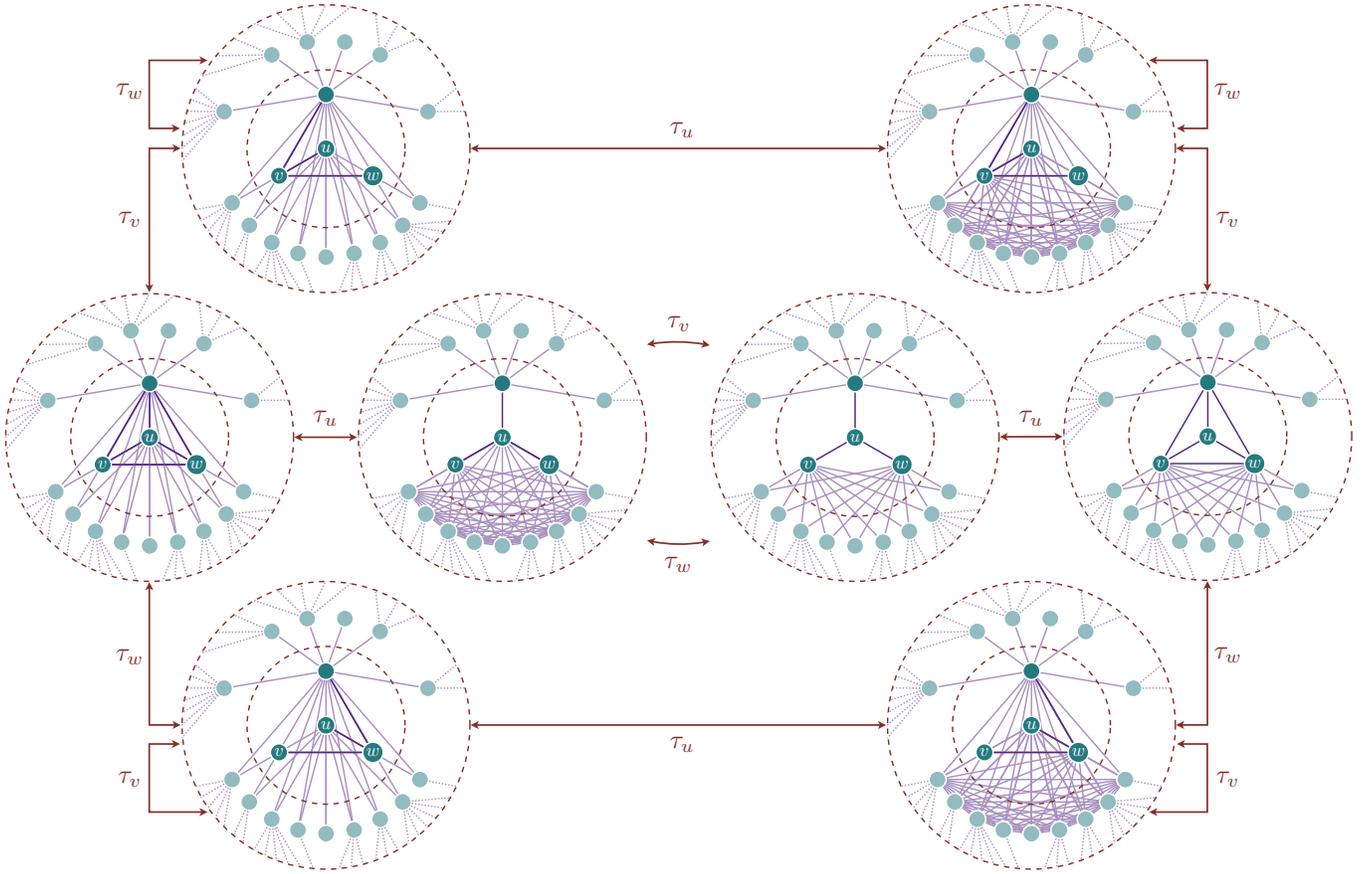


Figure 2: **The foliage is LC invariant.** The arrows indicate how foliage nodes u, w transform under LCs. The central four graphs show twins v, w that can be (dis-)connected via a LC with respect to one of their neighbours u . The LCs τ_v and τ_w transform the outer graphs with the connected twins into graphs where either v (upper row) or w (lower row) is an axil and their former twin partner a leaf.

LINE GRAPHS

In analogy to Theorem 1, we can prove a no-go theorem for line graphs, i.e., $L_n := (V_n, E_n)$ with vertices $V_n := \{1, \dots, n\}$ and edges $E_n := \{(1, 2), (2, 3), \dots, (n-1, n)\}$.

Theorem 3 (No crossing on a line). *It is not possible to extract two maximally entangled pairs from $|L_n\rangle$ if $a_1 < b_1 < a_2 < b_2$ for any $n \in \mathbb{N}$ with local Clifford operations, local Pauli measurements and classical communication.*

Proof. Again, the base case for $1 \leq n \leq 4$ is trivial and for $n = 5$ and $n = 6$ given by [6]. For the inductive step, we assume that Theorem 3 holds up to a given n . By the same argument as above, $K_2 \cup K_2$ is a vertex-minor of L_{n+1} if and only if it is a vertex-minor of at least one of $X(L_{n+1}), Y(L_{n+1})$ or $Z(L_{n+1})$. We will now show that $K_2 \cup K_2 \not\prec P_v(L_{n+1})$ for all $P_v \in \{X_v, Y_v, Z_v\}$, where we have $v \notin \{a_1, a_2, b_1, b_2\}$. Again, we consider three cases.

Line 1. $K_2 \cup K_2 \not\prec Z_v(L_{n+1})$.

If $v = 1$ or $v = n + 1$, we find $Z_v(L_{n+1}) = L_n$ and can use our induction hypothesis. Otherwise, the Z_v -measurement

splits the line into two line segments L_i and L_j with $i + j = n$. This implies $i, j \in \{1, 2, \dots, n-1\}$ and we can again use the induction hypothesis.

Line 2. $K_2 \cup K_2 \not\prec Y_v(L_{n+1})$.

We have $Y_v(L_{n+1}) = L_n$, a graph on the n vertices $\{1, 2, \dots, v-1, v+1, \dots, n+1\}$, since local complementation with respect to v connects $v-1$ to $v+1$ but leaves the remaining graph unchanged. Again, $K_2 \cup K_2$ cannot be a vertex-minor of $Y_v(L_{n+1})$ by our induction hypothesis.

Line 3. $K_2 \cup K_2 \not\prec X_v(L_{n+1})$.

If $v = 1$ or $v = n + 1$, we find $X_v(L_{n+1}) = L_{n-1}$ and can use our induction hypothesis. Similarly, in the cases $v = 2$ and $v = n$ we get $X_v(L_{n+1}) = L_n$. In all other cases, we have $X_v(L_{n+1})$ equal to L_{n-1} with an additional leaf, where the leaf-axil pair is made up by the set $\{v-1, v+1\}$. Using the same argument as in the proof of Ring 3 – involving Theorem 2 and Lemma 2 – we can conclude our proof. \square

OUTLOOK

In this work, we build upon quantum network routing research to examine whether commonly used nearest-neighbour architectures can aid with bypassing bottlenecks. We establish two no-go results for ring and line topologies. We show that, unlike the grid, whose smallest instance is the butterfly network, these two architectures are not suitable for bypassing bottlenecks without additional longer communication links. This would be possible if we did allow for longer distance 2-local operations; as shown in Fig. 3 we can indeed transform a ring graph state to a butterfly graph state and thereby enable the generation of crossing maximally entangled pairs.

Our investigation aims to increase our understanding of the potential and limitations of quantum network routing, in times when these settings are moving closer to experimental reality. The techniques that we have developed can potentially be used to eliminate more general quantum communication scenarios with existing bottlenecks. Relevant studies [23] show how to generate the underlying graph states by sharing maximally entangled pairs between the nodes, while a proof-of-concept implementation using the IBM Quantum Experience has also been demonstrated [24]. Finally, the long-distance communication links that are necessary in order to bypass the bottlenecks shown in this work, can in principle be built following the approach of Ref. [5].

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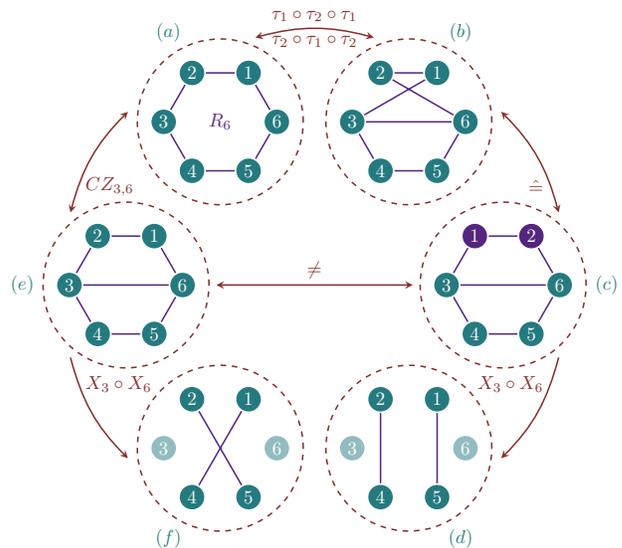


Figure 3: **Ring (a) and butterfly (e) network.** While ring and butterfly networks are not LC-equivalent, a ring graph (a) can be transformed via LCs into a butterfly-like graph (c), where two nodes, here 1 and 2, are swapped ($\hat{=}$). The ring R_6 is transformed via a local complementation sequence (a), (b) $\hat{=}$ (c). From the resulting graph (c), one can obtain maximally entangled pairs (d) between nodes (2, 4) and (1, 5) by measuring 3 and 6. When allowing for 2-local operations (specifically CZ -gates) between nodes 3 and 6 (e) one can obtain crossing Bell pairs (1, 4) and (2, 5) (f).

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