Topics in particle systems and singular SDEs

Dissertation

zur Erlangung des Grades eines

Doktors der Naturwissenschaften

(Dr. rer. nat.)

am Fachbereich Mathematik und Informatik

der Freien Universität Berlin

vorgelegt von

Helena Katharina Kremp

Berlin, 2022

Erstgutachter:	Prof. Dr. Nicolas Perkowski
Zweitgutachter:	Prof. Dr. Xue-Mei Li

Tag der Disputation: 14. April 2023

Abstract

This thesis studies irregular stochastic (partial) differential equations arising in fluctuating hydrodynamics or regularization by noise, and homogenization limits thereof.

In the first part, we consider a model for particles on a biological membrane. The membrane is given by an ultra-violet cutoff of the quasi-planar Helfrich surface, that is subject to space-time fluctuations. We study the homogenization limits of the Itô and Stratonovich rough paths lifts of the diffusion in different scaling regimes. As an outlook on the construction of the diffusion on the Helfrich membrane without cutoff, we prove convergence of the rescaled surface measures.

Moreover, we study nonlinear approximations of the Dean-Kawasaki SPDE, a model for the dynamics of the empirical density of independent Brownian particles. We approximate this highly irregular SPDE such that the physical constraints of the particle system are preserved and derive weak error estimates. We prove well-posedness and a comparison principle for a class of nonlinear regularized Dean-Kawasaki equations.

The second part of this thesis deals with the weak well-posedness of multidimensional singular SDEs with Besov drift in the rough regularity regime and additive stable jump noise. We first solve the associated fractional parabolic Kolmogorov equation. To that end, we employ the paracontrolled ansatz and furthermore generalize to irregular terminal conditions, that are itself paracontrolled.

We then prove existence and uniqueness of a solution to the martingale problem. Motivated by the equivalence between probabilistic weak solutions of SDEs with bounded, measurable drift and solutions of the martingale problem, we define a rough-path-type weak solution concept for singular Lévy diffusions, proving moreover equivalence to the martingale solution in the Young and rough regime. To this end, we construct a rough stochastic sewing integral. In particular, we show that canonical weak solutions are in general non-unique in the rough case. We apply our theory to construct the Brox diffusion with Lévy noise.

Finally, we combine the theory of periodic homogenization with the solution theory for singular SDEs with stable noise. For the martingale solution projected onto the torus, we prove existence of a unique invariant probability measure. We solve the singular Fokker-Planck equation and prove a strict maximum principle. Furthermore, we solve the singular resolvent and Poisson equation. Using Kipnis-Varadhan methods, we prove a central limit theorem and obtain a Brownian motion with constant diffusion matrix. In the pure stable noise case, we rescale differently and encounter no diffusivity enhancement. We conclude on the periodic homogenization result for the singular parabolic PDE via Feynman-Kac formula.

Zusammenfassung

Diese Arbeit untersucht irreguläre stochastische (partielle) Differentialgleichungen, die in der fluktuierenden Hydrodynamik oder in der Regularisierung durch Rauschen auftauchen, sowie Homogenisierungsgrenzwerte diesbezüglich.

Im ersten Teil betrachten wir ein Modell eines Partikels auf einer biologischen Membran. Die Membran ist gegeben durch den ultra-violetten Cutoff einer quasi-planaren Helfrich-Fläche, die Raum-Zeit-Fluktuationen ausgesetzt ist. Wir untersuchen den Homogenisierungsgrenzwert des Itô- und Stratonovich-Lifts der Diffusion in verschiedenen Skalierungsregimen. Als Ausblick auf die Konstruktion einer Diffusion auf der Helfrich-Fläche ohne Cutoff, zeigen wir die Konvergenz der reskalierten Oberflächenmaße.

Darüber hinaus betrachten wir nichtlineare Approximationen der Dean-Kawasaki SPDE, ein Modell für die Dynamik des empirischen Maßes von unabhängigen Brown'schen Partikeln. Wir approximieren die irreguläre SPDE so, dass die physikalischen Eigenschaften des Partikelsystems erhalten bleiben und zeigen die schwachen Konvergenzabschätzungen. Wir beweisen Wohlgestelltheit und ein Maximumsprinzip für eine Klasse von nichtlinearen regularisierten Dean-Kawasaki-Gleichungen.

Der zweite Teil der Arbeit befasst sich mit der schwachen Wohlgestelltheit für multidimensionale singuläre SDEs mit Besov-Drift im irregulären Regularitätsregime und additivem stabilen Rauschen. Wir lösen zunächst die assoziierte fraktionale parabolische Kolmogorov-Gleichung. Dazu nutzen wir den parakontrollierten Ansatz und verallgemeinern außerdem zu irregulären, selbst parakontrollierten Endbedingungen.

Wir beweisen die Existenz und Eindeutigkeit von Lösungen zum Martingalproblem. Die Äquivalenz von stochastisch schwachen Lösungen von SDEs mit beschränkten, messbaren Drifts und Lösungen des Martingaleproblems motiviert uns, ein schwaches Lösungskonzept für singulare SDEs zu entwickeln und die Äquivalenz zur Martingallösung im Young-Fall sowie im irregulären Regularitätsregime zu zeigen. Dafür konstruieren wir ein irreguläres stochastisches Integral. Insbesondere zeigen wir, dass kanonische schwache Lösungen im irregulären Regularitätsregime im Allgemeinen nicht eindeutig sind. Als Anwendung präsentieren wir die Brox-Diffusion mit Lévy-Rauschen.

Schlussendlich verbinden wir die Theorie der periodischen Homogenisierung mit der Lösungstheorie für singuläre SDEs. Für die Martingallösung projiziert auf den Torus können wir die Existenz eines eindeutigen invarianten Maßes beweisen. Wir lösen die singuläre Fokker-Planck-Gleichung und beweisen ein starkes Maximumsprinzip. Weiterhin lösen wir die singuläre Resolventengleichung und die Poisson-Gleichung. Mithilfe der Kipnis-Varadhan-Methode beweisen wir einen zentralen Grenzwertsatz und erhalten eine Brown'sche Bewegung mit konstanter Kovarianzmatrix. Im rein stabilen Fall reskalieren wir anders und bemerken, dass es zu keiner Diffusionserweiterung kommt. Wir schließen auf das periodische Homogenisierungsresultat für die singuläre parabolische PDE mithilfe der Feynman-Kac-Formel.

"If you always do what you've always done,
you'll always get what you've always got."
– Henry Ford

Acknowledgements

First and foremost, I would like to express my deep gratitude to Nicolas Perkowski for his excellent supervison of this PhD project. He was always pleased to discuss matters of our research and brought forward major parts of this thesis with fresh ideas. During the PhD project he opened up a lot of opportunities for me to enhance my competences as a researcher and had various valuable advices. The casual atmosphere he creates in the research group is motivating great research and stimulating the joy in mathematics. I am also grateful, that he gave me the chance to literally attend as many conferences as I wish to.

Many thanks go to Xue-Mei Li for agreeing to co-referee this thesis.

Through the PhD project I got to know Ana. During the past years she became my mentor, but more importantly a true friend. I would like to thank Ana for her ongoing support in research and in life.

I owe deep gratitude to my parents and Peter for supporting me in every phase of my studies. About eight years ago, my father encouraged me to study mathematics and motivated me to work hard – in his own amusing way.

Let me thank all friends and colleges at Freie Universität who accompanied my research.

Last, but not least, I gratefully acknowledge funding by the DFG (Deutsche Forschungsgemeinschaft) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689). I would also like to acknowledge travel funding from the Berlin Mathematical School (BMS) and the Berlin-Oxford IRTG 2544 "Stochastic Analysis in Interaction".

To Peter
and
Jutta Cäcilia and Stefan Kremp.

Contents

Introduction

Fluctuating hydrodynamics deals with the macroscopic fluctuations of particle systems in equilibrium (or local equilibrium) thermodynamic models, cf. e.g. [LL59, Spo91, DZS06, DFVE14, BDSG+15]). The origins of the theory date back to Landau, Lifshitz [LL59]. At the micro scale one observes a "chaotic" random *system of interacting particles* (e.g. in a gas or fluid). Zooming out in the right scaling, structure becomes apparent in the form of a so-called hydrodynamic limit (the analogue of the law of large numbers), that is here given by a deterministic PDE. The fluctuations, observed on a finer scale, around the hydrodynamic limit of the particle system are described by means of a suitable stochastic partial differential equation (SPDE) with Gaussian noise (in analogy to the central limit theorem). In the physics literature on fluctuating hydrodynamics, the Gaussian noise correlations are formally determined by the so-called fluctuation-dissipation relation (cf. e.g. [DZS06, OSL09]). We refer to the book [Spo91] for an in-depth analysis of interacting particle systems from the microscopic to the macroscopic point of view and physical background. The book [KL98] coveres scaling limits for interacting particle systems in a broader context.
Fluctuating hydrodynamics can moreover be seen as a source for *singular SPDEs*, that are mathematically interesing from the well-posedness and numerical point of view. For numerical schemes for fluctuation hydrodynamic equations we refer to [BUBDB+12] and the references therein. An example is the Dean-Kawasaki equation ([Dea96, Kaw94]) for the empirical density of particles (i.e. at the microscopic scale), which is a singular SPDE whose well-posedness is mathematically challenging. Different solution concepts were developed in recent years in [KLvR19, KLvR20, FG21]. The equation is useful, as it correctly predicts the law of large numbers, central limit theorem and large deviations of the particle system, cf. [FG22].
To facilitate a numerical analysis of the equation, we consider a nonlinear approximation of the Dean-Kawasaki equation and prove weak error bounds between our approximation and the martingale solution from [KLvR19] utilizing duality arguments à la [Myt96]. Furthermore, we prove well-posedness and a comparison principle for a class of regularized Dean-Kawasaki equations.
Similar mathematical problems arise in the study of *particle-membrane models*. These model interacting particle systems posed on a hypersurface. The motivation comes from cell biology. The interplay of proteins and curvature of a biological membrane are well-known to regulate cell functions (cf. e.g. [MG05]), for example in neurobiology in the context of signal transmisson with neurotransmitters at a postsynapic membrane. The optimal mathematical model would consider membrane-mediated interactions between the particles (e.g. proteins) and interactions of the particles with the membrane. We refer to the article [KGSK20], that studies the modelling perspective. Physical

investigations (cf. e.g. [NB07]) suggest the biological membrane to behave like a typical sample of a Gaussian field with a covariance operator given by the linearized Canham-Helfrich energy (cf. [Can70, Hel73]). The sample paths of such a Gaussian field are typically non-differentiable, which imposes difficulties for the construction of a diffusion on that Gaussian hypersurface. Most articles therefore consider the ultra-violet cutoff of the Helfrich surface (cf. e.g. [NB07, Dun13, DEPS15]). Furthermore, we assume the membrane to be subject to space-time fluctuations that occur for thermal reasons. Here, homogenization theory comes into play (cf. the classical monographs [BLP78, PS08]). The question is whether the effects of the oscillations of the membrane for the diffusion are averaged out, respectively homogenized, in the limit. [Dun13, DEPS15] already studied the homogenization limit for the diffusion on the ultra-violet cutoff of the fluctuating Helfrich membrane. We generalize their results to convergence of the rough paths lifts of the diffusion. We refer to the book [FH20] for an introduction to rough paths theory and the articles [KM17, CFK$^+$19] and refereces therein for the connection to homogenization. Considering different space-time scaling regimes, we prove that an area correction term in the limit of the Itô and Statonovich lifts appears. Furthermore, we study the limit of the rescaled surface measure when the ultra-violet cutoff is taken to infinity, which gives an indication on the limit behaviour for the diffusion on the rough Helfrich surface.

Regularization by noise is a generic concept to restore well-posedness of certain ill-posed ODEs or PDEs by adding a random pertubation. The resulting SDEs or SPDEs give rise to pathwise and probabilistic solution theories.

The ideas underlying regularization by noise for SDEs were first introduced by Zvonkin [Zvo74] (in one dimension) and Veretennikov [Ver81], who in particular prove strong well-posedness for multidimensional SDEs with bounded, measurable drift and additive Brownian noise. In the case of bounded, measurable drift a Girsanov transformation (cf. e.g. [RY99]) yields probabilistic weak existence of solutions. Pathwise uniqueness can be established with the so-called Zvonkin transformation, that removes the irregular drift. Via Yamada-Watanabe arguments (cf. [YW71]) strong existence and uniqueness follows. [KR05] generalized the strong well-posedness to drifts in $L^q([0,T], L^p(\mathbb{R}^d))$ for $\frac{d}{p}+\frac{2}{q}<1$, $p,q\geqslant 2$. We refer to the monograph [Fla11] and the summary paper [Ges19] for a detailed overview on the topic of regularization by noise, also in the context of SPDEs originating in fluid dynamics. Regularization by noise for SPDEs is a recent and promising development, but so far mostly limited to linear SPDEs and a few other equations, see e.g. [GP93, Fla11, DPFRV16, GG19, ABLM22, Lan22]. We focus on SDEs with distributional drift and additive noise. Morally, the rougher the noise, the more irregular coefficients can be handled, cf. [CG16, HP21].

The noise we consider is an α-stable jump process, which exhibits weaker regularization compared to the Brownian noise. We assume $\alpha>1$ and cover simultaneously the Brownian noise case, which corresponds to $\alpha=2$. Our methods employ a probabilistic approach towards regularization by noise (cf. the original articles [Fig08, BL08]). Pathwise approaches à la Catellier and Gubinelli were investigated e.g. in [CG16, HP21, GG21, Gal22]. However, combining pathwise techniques with probabilistic methods turned out to be highly beneficial, in particular to tackle critical equations,

see e.g. [Ger22, GG22]. The tool that made this possible is the stochastic sewing lemma from [Lê20]. In our setting, we use a combination of rough paths techniques and probabilistic properties to construct a dynamical weak solution concept for SDEs with rough drifts. The probabilistic methods are based on the correspondence of a Markov diffusion with its singular generator via the martingale problem. We refer to the book [EK86] for an in-depth analysis of the martingale problem.
To define a sufficiently large subset of the domain of the generator, which makes the martingale problem well-posed, we solve the backward Kolmogorov equation. We consider singular drifts beyond the so-called Young regime where products with the distributional drift are well-defined in the classical sense. Therefor we employ the paracontrolled ansatz of [GIP15]. We furthermore generalize to singular paracontrolled terminal conditions for the Kolmogorov equation. This is motivated by the observation that Schauder and commutator estimates for the semigroup of the diffusion process can be inferred from regularity properties of the solution of the Kolmogorov equation with singular terminal conditions. This will be relevant for example to “sew” (in the sense of [Lê20]) additive functionals of the diffusion process with functionals that are irregular or (paracontrolled by) the drift itelf.
Then, we prove existence and uniqueness of solutions of the singular martingale problem associated to the SDE with rough drift, extending the articles [DD16, CC18] to the stable noise case. In addition, we introduce a dynamical weak solution concept (called “rough weak solutions”) and prove equivalence to the concept of solutions of the martingale problem. In the Young regularity regime, the problem of well-posed weak solutions was already tackled in [ABM20, IR22]. In the rough regime, a well-posed dynamical weak solution concept has so far been an open problem. The canonical weak solution concept of [ABM20] turns out to allow – even in the one-dimensional Brownian noise setting – for non-unique solutions in the rough case. For rough weak solutions we prove a generalized Itô formula.
Finally, we extend the techniques for central limit theorems of additive functionals of Markov processes from [KV86, KLO12] to singular diffusions in the setting of periodic coefficients. This yields a periodic homogenization result for the singular fractional Kolmogorov PDE with oscillating and unbounded drift. Extensions to the setting of diffusions in random singular environments are conceivable, but left for future research. Applications to the construction and periodic homogenization for the Brox diffusion with Lévy noise are outlined.
In the following, we give more details on each chapter.

Part I. Stochastic analysis of particle and particle-membrane models

Chapter 1: Rough homogenization for diffusions on fluctuating membranes

Sections 1.1 to 1.4 are based on the submitted manuscript [DKP22], that is a joint work together with Ana Djurdjevac and Nicolas Perkowski. Section 1.5 is based on an unpublished joint work with Nicolas Perkowski.

The lateral diffusion of particles is crucial for cellular processes, including signal transmission, cellular organization and the transport of matter (cf. [MP11, MHS14, LP13, AV95]). Motivated by these applications of diffusion on cell membranes, we consider the diffusion on a curved domain - a hypersurface $\mathscr{S}$, cf. [Sei97]. The Brownian motion on the surface, whose generator in local coordinates is the Laplace-Beltrami operator, provides a simple example of a diffusing particle on a biological surface. In physics it is known as the overdamped Langevin dynamics on a Helfrich membrane (cf. [NB07]).
We will restrict our considerations to the classical situation of the so-called "essentially flat surfaces" $\mathscr{S}$. A standard way of representing the essentially flat surface is the Monge-gauge parametrization, where we specify the height H of the hypersurface as a function of the coordinates from the flat base, namely over $[0, L]^2$.
Moreover, these membranes are fluctuating, both in time and space, due to the spatial microstructure and thermal fluctuations of active proteins. The analysis of the macroscopic behavior of a laterally diffusive process on surfaces possessing microscopic space and time scales was derived in [Dun13, DEPS15]. Based on classical methods from homogenization theory, the authors prove that under the assumption of scale separation between the characteristic length and time scales of the membrane fluctuations and the characteristic scale of the diffusing particle, the lateral diffusion process can be well approximated by a Brownian motion on the plane with constant diffusion tensor D. In particular, they show that D depends in a highly nonlinear way on the detailed properties of the surface.
In this chapter, we prove a rough homogenization result for the diffusion on the ultraviolet cutoff of the Helfrich membrane. Specifically, we extend the results from [Dun13, DEPS15] by proving the convergence towards a particular lift of the homogenization limit in rough paths topology for different time-space fluctuation scaling regimes (α, β). Interestingly, in some regimes, the rough paths lift of (X^ε) converges to a non-trivial lift of the limiting Brownian motion X, in the sense that, additionally, an area correction to the iterated integrals appears. In different settings, such a phenomenon was also observed in [LL05, FGL15]. [FGL15, KM16, KM17, CFK$^+$19, GL20, DOP21] already fruitfully combined rough paths with homogenization techniques.
The underlying model for the surface is based on the Helfrich elasticity membrane model. We consider random hypersurfaces whose typical paths can be represented as a graph of a sufficiently smooth field $H : [0, L]^d \times [0, \infty) \to \mathbb{R}$, the so-called Monge gauge parametrization. The classical description of fluid membranes $\mathscr{S}$ in equilibrium state, here in the quasi-planer case modelled by H, is based on the linearized Canham-

Helfrich free energy (cf. [Can70, Hel73]) given by

$$\mathscr{E}[H] = \frac{1}{2} \int_{[0,L]^2} \kappa(\Delta H(x))^2 + \sigma |\nabla H(x)|^2 dx$$

where the constant κ denotes the (bare) bending modulus and σ denotes the surface tension. For more details about the description of fluid lipid membranes, we refer to [Des15]. The dynamics, that correspond to the formal invariant measure $\exp(-\mathscr{E}[H])dH$, are described by the stochastic partial differential equation (SPDE)

$$\partial_t H = -RAH(t) + \xi(t) \tag{0.1}$$

where $AH := -\kappa \Delta^2 H + \sigma \Delta H$ is the restoring force for the free energy associated to $\mathscr{E}[\mathscr{S}]$. Moreover, the operator R characterizes the effect of nonlocal interactions of the membrane through the medium. Above, ξ is a Gaussian field, that is white in time and whose spatial fluctuations have mean zero and covariance operator $2(k_B T)R$ with Boltzmann constant k_B.
In this chapter, we consider the ultra-violet cutoff of H given by

$$H(x,t) = h(x, \eta_t) = \sum_{0<|k|\leqslant \tilde{K}} \eta_t^k e_k(x), \quad (x,t) \in \mathbb{T}^2 \times \mathbb{R}^+, \tag{0.2}$$

for the Fourier basis $(e_k)_{k\in\mathbb{Z}^2}$ on the torus $\mathbb{T}^2$, independent Ornstein-Uhlenbeck processes $\eta = (\eta^k)$, whose joint dynamics we specify below and a fixed cutoff $\tilde{K} \in \mathbb{N}$. Using the expansion (0.2), we define the Brownian motion X on H, as the diffusion X, for which (X, η) solves the following system of SDEs

$$\begin{aligned} dX_t &= F(X_t, \eta_t)dt + \sqrt{2\Sigma(X_t, \eta_t)}dB_t \\ d\eta_t &= -\Gamma \eta_t dt + \sqrt{2\Gamma\Pi}dW_t \end{aligned}$$

with independent standard Brownian motions B, W, explicit smooth coefficients Σ, F, that are periodic in x and grow at most linearly in η and symmetric, positive definite diagonal matrices Π, Γ. The matrices Γ, Π depend on the covariance structure of H and Σ is the inverse metric tensor. F depends on the curvature of H. The generator of X is the so-called Laplace-Beltrami operator.
We consider fluctuations of the surface H in space and time. That is, we replace H by the surface $H^\varepsilon(x,t) := \varepsilon^\alpha H(\varepsilon^{-\alpha}x, \varepsilon^{-\beta}t)$. The corresponding diffusion X^ε on H^ε is given by

$$\begin{aligned} dX_t^\varepsilon &= \frac{1}{\varepsilon^\alpha} F\left(\frac{X_t^\varepsilon}{\varepsilon^\alpha}, \eta_t^\varepsilon\right) dt + \sqrt{2\Sigma\left(\frac{X_t^\varepsilon}{\varepsilon^\alpha}, \eta_t^\varepsilon\right)} dB_t \\ d\eta_t^\varepsilon &= -\frac{1}{\varepsilon^\beta}\Gamma \eta_t^\varepsilon dt + \sqrt{\frac{2\Gamma\Pi}{\varepsilon^\beta}} dW_t. \end{aligned} \tag{0.3}$$

For a detailed derivation of the system see [Dun13, section 2.3.3].

Considering space-time scaling regimes (α, β) with only temporal fluctuations, comparable spatial and temporal fluctuations and temporal fluctuations twice as fast as spatial fluctuations, we prove the convergence of the Itô and Stratonovich rough paths lift of X^ε and identify the limit.
As in [Dun13, DEPS15, DOP21], we utilize martingale methods for additive functionals of Markov processes (cf. [KV86, KLO12]). That is, we identify a stationary, ergodic Markov process and exploit the solution of the associated Poisson equation for the generator of that Markov process to rewrite the (in general unbounded) drift term of X^ε. As already observed in [DOP21], for the Stratonovich lift we expect an area correction to appear if and only if the underlying Markov process is non-reversible.
Indeed, in the regime of comparable spatial and temporal fluctuations ($(\alpha, \beta) = (1, 1)$) the underlying Markov process $Y^\varepsilon := \varepsilon^{-1} X^\varepsilon$ is reversible (for each fixed, stationary realization of η) and the Stratonovich limit is the usual Stratonovich lift of the Brownian motion X.
In contrast, in the regime of doubly as fast temporal fluctuations ($(\alpha, \beta) = (1, 2)$) an area correction for the Stratonovich lift appears.
In the regime of purely temporal fluctuations ($(\alpha, \beta) = (0, 1)$) the limit is obtained by averaging over the invariant measure of the Ornstein Uhlenbeck process η. The rough limit is given by the canonical lift of the Brownian motion since the uniform controlled variation (UCV) condition is satisfied for (X^ε).
Other space-time scaling regimes (α, β) are less interesting, because it turns out that the regimes $(\alpha, \beta) \in \{(0, 1), (1, 2), (1, 1)\}$ yield, together with the quenched regime $(\alpha, \beta) = (1, -\infty)$, the four different limit behaviors, that can occur (cf. [Dun13, Theorem 6.0.1]). It is however worth mentioning, that in the regime $\alpha = 1$ and $\beta \in (2, 3]$, the limit for the process (and thus also for the rough paths lift) is open, as certain Poisson equations might not have solutions (cf. [Dun13, section 6]).
In the quenched regime with deterministic η_0, one consideres a non-random periodic surface H. Thus X is a diffusion with periodic coefficients. Due to the functional central limit theorem for Itô and Stratonovich rough paths lifts of diffusions with periodic coefficients proven in [DOP21, section 4.3], the rough paths limit of (X^ε) is known and given by a nontrivial lift of the limiting Brownian motion X.
The chapter is structured as follows. Section 1.1 formally defines the model and the surface H. We also recall the definition of the space of α-Hölder rough path and the uniform controlled variation condition. Sections 1.2 to 1.4 treat the scaling regimes $(\alpha, \beta) \in \{(0, 1), (1, 2), (1, 1)\}$. In each case, we prove tightness in γ-Hölder rough path topology for $\gamma \in (\frac{1}{3}, \frac{1}{2})$ and derive the rough homogenization limit.
Section 1.5 gives an outlook on the construction of the diffusion on the Helfrich membrane without ultra-violet cutoff in dimension $d = 2$. In that section, we assume the surface to be time-independent and we prove convergence of the rescaled surface measure to the Lebesgue measure utilizing chaos expansions in Gaussian Hilbert spaces. This suggests that the appropriately damped diffusion converges, in the limit $\tilde{K} \to \infty$ and in local coordinates, to a standard Brownian motion. A proof for the convergence of the processes is an open problem, that we leave for future research.

Chapter 2: Weak error bounds for a nonlinear approximation of the Dean-Kawasaki equation

This chapter is based on an unpublished paper, that is a joint work together with Ana Djurdjevac and Nicolas Perkowski.

Initially studied in [Dea96] and [Kaw94], the Dean-Kawasaki (DK) equation is a coarse-grained SPDE model for the empirical measure of N particles following the Langevin dynamics with pairwise interaction. There has been a huge interest in Dean-Kawasaki-type models in both physics (e.g. [MT00, VCCK08, GNS+12, DOL+16, DCG+18]) and mathematics (e.g. [DFVE14, CSZ19, FG19, FG21, CF21, FG22]). The equation arises as a model of diffusions in liquids with correlations among the diffusing particles. For the physical background we refer to [DFVE14]. A mathematical well-posedness theory is a challenging problem, cf. [DFVE14, page 6], with partial answers that were developed in the recent years, see e.g. [KLvR19, KLvR20, FG21]. The equation also has applications in agend-based modelling, cf. [DCKD22].
The DK equation with pairwise interaction potential W reads

$$\partial_t u = \frac{1}{2}\Delta u + \frac{1}{\alpha}\nabla \cdot (u(\nabla W * u)) + \frac{1}{\sqrt{\alpha}}\nabla \cdot (\sqrt{u}\xi), \tag{0.4}$$

where ξ denotes space-time white noise and $\alpha > 0$.
Let $(X^i)_{i=1}^N$ solve the Langevin dynamics with interaction potential W, that is

$$dX_t^i = -\frac{1}{N^2}\sum_{j=1}^N \nabla W(X_t^i - X_t^j)dt + dB_t^i$$

for N independent Brownian motions $(B^i)_{i=1}^N$. The recent papers [KLvR19, KLvR20] show that equation (0.4) posseses a unique solution to the measure-valued martingale problem if and only if the parameter $\alpha = N$ for a natural number N and the initial condition is atomic, that is $u_0 = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$ for $x^i \in \mathbb{R}$. In this case, the empirical measure $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is the unique solution to (0.4).
For regularized Dean-Kawasaki equations, which means imposing a cutoff on the noise, well-posedness was investigated in [CSZ19, CSZ20] which establish existence of mild solutions with high probability. The regularized model is derived from the mollified empirical densities.
In the pathwise sense, well-posedness for the DK equation (0.4) is an open problem even in the case without interaction potential, $W = 0$. The difficulty arises from the divergence operator at the noise term and due to irregularity of the space-time white noise ξ, as well as from the product with the square-root of the solution. Even in dimension $d = 1$, the equation is supercritical in the language of regularity structures ([Hai14]) and paracontrolled distributions ([GIP15]), which are techniques to tackle subcritical (and some critical) singular SPDEs. Supercriticality can be infered from a scaling argument. Heuristically, it means that the regularization induced by the

Laplacian does not dominate the irregular product with ξ.
Entropy solutions of porous media equations (replacing Δu by $\Delta(|u|^{m-1}u)$ with exponent $m \in (1,\infty)$ and a noise $\sum_k \sigma^k(x,u)dB_t^k$), which does not include the Dean-Kawasaki case $m=1$, were studied in [DGG19] and in [DG20] (with nonlinear gradient noise). Using a pathwise approach via rough paths techniques and considering Stratonovich noise, [FG19] established well-posedness for porus media equations with exponent $m \in (0,\infty)$. However, these techniques require highly regular diffusion coefficients σ in the case of $m=1$. The problem of square-root diffusion coefficient was recently solved in [FG21], where the authors establish existence and uniqueness of stochastic kinetic solutions to generalized Dean-Kawasaki equations with Stratonovich noise. The Stratonovich noise enables to obtain a-priori entropy-type estimates on the kinetic solutions (cf. [FG21, Section 5.1]), which yield the compactness. We refer to [FG21] for more details on the well-posedness of related equations and the references therein. Furthermore, the Dean-Kawasaki equation is related to scaling limit for interacting particle systems, see e.g. [GLP98]. [FG22] show that the Stratonovich DK equation correctly predicts the large deviation rate function for the non-equilibrium fluctuations of the zero range process.
From a numerical perspective, the results of [KLvR19, KLvR20] are not stable with respect to changes in the parameter N. That is, slight changes in N yield an ill-posed problem and possible large numerical errors. However, simulating the particle system is computationally expensive for increasing number of particles N, see e.g. [HCD$^+$21, section 4.4.1]. For large N, it is thus more efficient to discretize the SPDE (0.4), than to simulate the particle system. Numerical schemes for a class of stochastic porous media equations where investigated e.g. in [BGV20] and for Dean-Kawasaki-type equations in [CS22, CF21]. [CS22] introduce a discontinuous Galerkin scheme for the regularized DK equation that was considered in [CSZ19, CSZ20]. [CF21] consider finite element and finite difference approximations for (0.4) without interaction ($W=0$) and prove weak error estimates. The weak distance is parametrized by the Sobolev regularity of the test functions and the rate measured in their distance can be arbitrarily high, only limited by the numerical error and the error coming from the negative part of the approximation. However, the authors do not prove positivity for the approximations (hence the consideration of the negative part). Additionally they impose a strong assumption ([CF21, Assumption FD4]) on the existence of a lower bound for the solution of the discrete heat equation.
In this chapter, we consider an approximation of the DK equation (0.4) in the case of independent particles, that satisfies the physical contraints of the particle system, i.e. mass preservation and non-negativity. Then, we derive weak error estimates that relate our approximation with the martingale solution from [KLvR19]. In the classical theory, one would consider as an approximation the Gaussian fluctuations around the hydrodynamic limit of the particle system (i.e. the solution of the heat equation), which are described by a *linear* SPDE. That leads to a Gaussian approximation of the empirical measure, which is neither positive nor a probability measure, but achieves a weak convergence rate of $N^{-3/2}$. We consider an approximation which respects the constraints of the particle system. This approximation will be described by a *nonlinear*

SPDE, for which well-posedness is also a nontrivial problem. We replace the square root by a Lipschitz function f_{δ_N} with diverging Lipschitz constant δ_N^{-1} when $\delta_N \to 0$ and consider a noise cutoff $M_N \to \infty$. Explicitly, our approximation is given by

$$du_t^N = \frac{1}{2}\Delta u_t^N dt + \frac{1}{\sqrt{N}} \sum_{|k| \leqslant M_N} \nabla \cdot (f_{\delta_N}(u_t^N)\phi_k) dB_t^k \tag{0.5}$$

for independent Brownian motions $(B^k)_{k\in\mathbb{Z}^d}$ and the Fourier basis $(\phi_k)_{k\in\mathbb{Z}^d}$ on the torus $\mathbb{T}^d$. For regularized DK equations of the form (0.5), the variational theory (cf. [LR15]) does not apply, as the local monotonicity is violated due to the gradient noise. Because we consider Itô noise and analytically weak solutions, to later on compare with the martingale solution of the DK equation from [KLvR19], the theory from [CSZ19, CSZ20, DG20, FG21] does also not apply for well-posedness of (0.5). We prove well-posedness for (0.5) through a suitable transformation of the equation and a combination with a priori energy bounds. Furthermore, we prove a comparison principle that yields non-negativity of the solution for non-negative initial data and the conservation of the L^1-norm. For (0.5) with optimally tuned parameters δ_N, M_N, we can prove a weak error rate of $N^{-1-c_d}\log(N)^{1/2}$ with $c_d = 1/(d+2)$ decreasing in the dimension d. To derive the estimates, we apply duality arguments (cf. [Myt96]) using the solution of the Hamilton-Jacobi-Bellman equation. We believe that this approach is quite powerful and can possibly be generalized to the interaction case.

The chapter is structured as follows. Section 2.1 introduces the nonlinear Dean-Kawasaki approximation. In Sections 2.2 and 2.3, we prove the well-posedness and a comparison principle for regularized Dean-Kawasaki-type equations, that in particular applies for our approximation (0.5). Section 2.4 provides the weak error estimates.

Part II. Regularization by noise for singular Lévy SDEs

Chapter 3: Kolmogorov equations with singular paracontrolled terminal conditions

The results of this chapter are generalizations of the corresponding results in [KP22, Sections 2 and 3]. The chapter is based on an unpublished paper with Nicolas Perkowski.

Kolmogorov equations are second order parabolic differential equations. Their connection with SDEs was already investigated by Kolmogorov in the seminal work [Kol31]. There exist analytic and probabilistic methods to study Kolmogorov equations. We refer to the books [KZR99, DP04, Kry08, BKRS15] for an overview on Kolmogorov equations in both finite and infinite dimensional spaces. In the finite dimensional setting, Kolmogorov equations with bounded and measurable coefficients and uniformly elliptic diffusion coefficients can be treated as a special case of the infinite dimensional Dirichlet form methods of [MR95], see also [KZR99, Section 2.4.1] and the connection to the martingale problem in [KZR99, Section 6.1.2]. We remain in the

finite-dimensional setting, but consider distributional drifts in Besov spaces. Besov spaces play well with the paracontrolled calculus that defines products of distributions, cf. the Littlewood-Paley theory in [BCD11]. Previous articles that consider distributional drifts are [FRW03, FIR17], as well as [CC18] in the setting of rougher distributional drifts. Heat kernel estimates for the solution to the Kolmogorov equation were established in [ZZ17, PvZ22].
In the article [KP22], that underlies this chapter, the Laplace operator is replaced by a generalized fractional Laplacian. We extend our previous results from [KP22] to allow for irregular terminal conditions. That is, we consider the fractional parabolic Kolmogorov backward equation

$$\big(\partial_t - \mathscr{L}_\nu^\alpha + V \cdot \nabla\big)u = f, \quad u(T,\cdot) = u^T,$$

on $[0,T] \times \mathbb{R}^d$, where $\mathscr{L}_\nu^\alpha$ generalizes the fractional Laplace operator $(-\Delta)^{\alpha/2}$ for $\alpha \in (1,2]$ and V is a vector-valued Besov drift with negative regularity $\beta \in (\frac{2-2\alpha}{3}, 0)$, i.e. $V \in C([0,T], (B^\beta_{\infty,\infty})^d) =: C_T\mathscr{C}^\beta_{\mathbb{R}^d}$ for short. Since V is a distribution, we need to be careful with well-definedness of the product $V \cdot \nabla u$. The regularity obtained from $-(-\Delta)^{\alpha/2}$ suggests that $u(t,\cdot) \in \mathscr{C}^{\alpha+\beta}$ if right-hand side f and terminal condition u^T are regular enough. Therefore we have $\nabla u(t,\cdot) \in \mathscr{C}^{\alpha+\beta-1}$. Since the product $V(t,\cdot) \cdot \nabla u(t,\cdot)$ is well-defined if and only if the sum of the regularities of the factors is strictly positive, we obtain the condition $\alpha + 2\beta - 1 > 0$, equivalently $\beta > (1-\alpha)/2$. We call this the *Young regime*, in analogy to the regularity requirements that are needed for the construction of the Young integral. However, we go beyond the Young regime, considering also the so-called *rough regime* $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$. In the rough case, we employ paracontrolled distributions (cf. [GIP15]) to solve the equation. The idea is to gain some regularity by treating u as a perturbation of the solution of the linearized equation with additive noise, $\partial_t w = \mathscr{L}_\nu^\alpha w - V$. The techniques work as long as the nonlinearity $V \cdot \nabla u$ is of lower order than the linear operator $\mathscr{L}_\nu^\alpha$, i.e. for $\alpha > 1$ or equivalently $(2-2\alpha)/3 < (1-\alpha)/2$. The price one has to pay to go beyond the Young regime is a stronger assumption on V. That is, we assume that certain resonant products involving V are a priori given. Those play the role of the iterated integrals in rough paths theory (cf. [FH20]). We then enhance V by that resonant product component and call the enhancement $\mathscr{V}$.
In [CC18, KP22], only regular terminal conditions were considered, i.e. $u^T \in \mathscr{C}^{\alpha+\beta}$ in the Young regime and $u^T \in \mathscr{C}^{2(\alpha+\beta)-1}$ in the rough regime. The right-hand side f can either be an element of $C_T L^\infty$ or $f = V^i$ for $i = 1, \ldots, d$. There are techniques available to treat less regular terminal conditions, cf. [GP17, Section 6]). With the help of those techniques, one can allow for terminal conditions $u^T \in \mathscr{C}^{(1-\gamma)\alpha+\beta}$ in the Young case and $u^T \in \mathscr{C}^{(2-\gamma)\alpha+2\beta-1}$ in the rough case for $\gamma \in [0,1)$, obtaining a solution $u_t \in \mathscr{C}^{\alpha+\beta}$ for $t < T$ and blow-up γ for $t \to T$. In this chapter, consider moreover *singular paracontrolled* right-hand sides f as well as *singular paracontrolled terminal condition* u^T, which includes all cases mentioned above. Moreover, we can consider f and u^T more generally as elements of Besov spaces $\mathscr{C}^\theta_p = B^\theta_{p,\infty}$ with integrability parameter $p \in [1,\infty]$. Examples for terminal conditions that we cover in the rough

case include the Dirac measure, that is $u^T = \delta_0 \in \mathscr{C}_1^0$, and $u_T = V(T, \cdot)$. To be more precise, in the rough regime, we assume paracontrolled right-hand sides and terminal conditions,

$$f = f^\sharp + f' \otimes V, \quad u^T = u^{T,\sharp} + u^{T,\prime} \otimes V_T,$$

with $u^{T,\prime}, f_t' \in \mathscr{C}_p^{\alpha+\beta-1}$ and remainders $f_t^\sharp \in \mathscr{C}_p^{\alpha+2\beta-1}$, $u^{T,\sharp} \in \mathscr{C}_p^{(2-\gamma)\alpha+2\beta-1}$ for $\gamma \in [0,1)$. For f_t' and $f_t^\sharp$ we also allow a blow-up γ for $t \to T$. We prove existence and uniqueness of mild solutions of the Kolmogorov backward equation for singular paracontrolled data (f, u^T). The paracontrolled solution is an element of the solution space with blow-up γ at terminal time T. As a byproduct, we prove a new commutator estimate for the $(-\mathscr{L}_\nu^\alpha)$-semigroup, cf. Lemma 3.14, that allows to gain not only space regularity, but also time regularity. Thanks to Lemma 3.14 there is no need for the so-called "modified paraproduct" from [GP17, Section 6.1]. Moreover, we prove continuity of the Kolmogorov solution map and a uniform bound for the solutions considered on subintervals of $[0, T]$ for bounded sets of terminal conditions and right-hand sides. The techniques we develop in this chapter are not limited to that particular equation and can possibly be used to treat other singular PDEs, that can be tackled with the paracontrolled ansatz.
The chapter is structured as follows. In Section 3.1 we introduce the generalized fractional Laplacian $\mathscr{L}_\nu^\alpha$ and its semigroup. We prove semigroup and commutator estimates and relate $-\mathscr{L}_\nu^\alpha$ with the generator of an α-stable Lévy process, a connection that will become relevant in Chapter 4. In Section 3.2 we introduce the solution spaces and prove generalized Schauder and commutator estimates thereon. Finally, we solve the Kolmogorov equation with singular paracontrolled data (f, u^T) in Section 3.3 and prove continuity of the solution map, as well as a uniform bound for the solutions on subintervals.

Chapter 4: Weak solution concepts for singular Lévy SDEs

Chapter 4 is based on joint work with Nicolas Perkowski. Section 4.1 relies on [KP22, Section 4]. Sections 4.2 to 4.5 are based on an unpublished paper together with Nicolas Perkowski. Section 4.6 is based on [KP22, Section 5].

We solve multidimensional SDEs with distributional drift driven by symmetric α-stable Lévy processes for $\alpha \in (1, 2]$ via the associated (singular) martingale problem using the solution theory for Kolmogorov backward equations from Chapter 3. We allow for drifts of regularity $\beta > (2 - 2\alpha)/3$ and in particular we go beyond the – by now well understood – *Young regime*, where the drift must have higher regularity than $(1 - \alpha)/2$. This generalizes the existing results [DD16] and [CC18] from the Brownian case to the case of α-stable noise. As an application of our results we construct a Brox diffusion with Lévy noise. Furthermore, we define a non-canonical weak solution concept for singular Lévy diffusions and prove its equivalence to martingale solutions in both the

Young and the *rough regime*. This turns out to be highly non-trivial in the rough case and requires to make sense of certain rough stochastic sewing integrals involved. In particular, we show that the canonical weak solution concept already introduced in [ABM20], which is well-posed in the Young case, yields non-uniqueness of solutions in the rough case.
More precisely, Chapter 4 studies the weak well-posedness of SDEs

$$dX_t = V(t, X_t)dt + dL_t, \qquad X_0 = x \in \mathbb{R}^d, \tag{0.6}$$

driven by non-degenerate symmetric α-stable Lévy noise L for $\alpha \in (1, 2]$ and with drift $V(t, \cdot)$ that is a Besov distribution in the space variable.
The special case where L is a Brownian motion has received lots of attention in recent years, since such *singular diffusions* arise as models for stochastic processes in random media. Examples are random directed polymers [AKQ14, DD16, CSZ17], self-attracting Brownian motion in a random medium [CC18], or a continuum analogue of Sinai's random walk in random environment (Brox diffusion, [Bro86]). Singular diffusions also arise as "stochastic characteristics" of singular SPDEs, for example the KPZ equation, cf. [GP17], or the parabolic Anderson model, cf. [CC18].
SDEs with distributional drifts were first considered in [BC01, FRW03] in the one-dimensional time-homogeneous setting. Of course, for distributional V the point evaluation $V(t, X_t)$ is not meaningful, so a priori it is not clear how to make sense of (0.6). The appropriate perspective is not to consider $V(t, X_t)$ at fixed time t, but rather to work with the integral $\int_0^t V(s, X_s)ds$. The intuition is that, because of small scale oscillations of X induced by the oscillations of L, we only "see an averaged version" of V and this gives rise to some regularization, at least for a Brownian motion or a sufficiently "wild" Lévy jump process. On the other hand we would not expect any regularization from a Poisson process. In the Brownian case, this intuition can be made rigorous in different ways. For example via a Zvonkin transform which removes the drift, cf. [Zvo74, Ver81, BC01, KR05, FGP10, FIR17], by considering the associated martingale problem and by constructing a domain for the singular generator, cf. [FRW03, DD16, CC18], or by Dirichlet forms as in [Mat94]. In the one-dimensional case it is also possible to apply an Itô-McKean construction based on space and time transformations, cf. [Bro86].
Here we follow the martingale problem approach in the spirit of [DD16, CC18] who considered the Brownian case. Formally, X solves (0.6) if and only if it solves the martingale problem for the generator $\mathscr{G}^V = \partial_t - \mathscr{L}_\nu^\alpha + V \cdot \nabla$, where the fractional Laplacian $(-\mathscr{L}_\nu^\alpha)$ is the generator of L. That is, for all functions u in the domain of $\mathscr{G}^V$, the process $u(t, X_t) - u(0, x) - \int_0^t \mathscr{G}^V u(s, X_s)ds$, $t \geqslant 0$, is a martingale. One difficulty is that the domain of $\mathscr{G}^V$ necessarily has a trivial intersection with the set of smooth functions: If u is smooth, then $(\partial_t - \mathscr{L}_\nu^\alpha)u$ is smooth as well, while for non-constant u the product $V \cdot \nabla u$ is only a distribution, but not a continuous function. If we want $\mathscr{G}^V u$ to be a continuous function, then u has to be non-smooth, so that $(\partial_t - \mathscr{L}_\nu^\alpha)u$ is also a distribution which has appropriate cancellations with $V \cdot \nabla u$.
We can find such u by solving the Kolmogorov backward equation for continuous

right-hand sides and regular terminal conditions, such that $\mathscr{G}^{\mathscr{V}} u = f$ by construction, as carried out in Chapter 3.

There have been several results on singular Lévy SDEs in the Young regime in recent years. [ABM20] consider the time-homogeneous one-dimensional case and construct weak solutions via a Zvonkin transform. They additionally establish strong uniqueness and existence by a Yamada-Watanabe type argument, which in particular is restricted to $d = 1$. Related to that, but for Hölder continuous drift of regularity at least $1 - \alpha/2$, [Pri12] proves pathwise uniqueness in the multidimensional, time-homogeneous case. Two nearly simultaneous articles, [LZ19] and [dRM19], consider the multidimensional case (time-homogeneous, respectively time-inhomogeneous) and prove existence and uniqueness for the martingale problem. They consider $V \in C([0,T], B^{\beta}_{p,q})$ for general p, q (subject to suitable conditions), where for $p = q = \infty$ they require the Young regime, $\beta > (1-\alpha)/2$. Let us also mention [HL20], who prove pathwise regularization by noise results for SDEs driven by (very irregular) fractional Lévy noise, based on the methods of [CG16, HP21].

We treat the multidimensional time-inhomogeneous case with drifts in the rough regime. However, we concentrate on $B_{\infty,\infty}$ Besov spaces and do not consider $B_{p,q}$ for general p, q. Reaching the rough regularity regime is important for our main application, the construction of a "Brox jump diffusion" with α-stable Lévy noise. Here, $d = 1$ and V is a typical path of a (periodic) space white noise. So in particular we can only take $\beta = -1/2 - \varepsilon$ for $\varepsilon > 0$, which is never in the Young regime, not even in the Brownian case $\alpha = 2$. We moreover show that the periodic white noise can be enhanced (in the sense of Chapter 3) and that the mollified resonant products converge almost surely without renormalization in the sense of subtracting divering constants. We also indicate, how to adapt our constructions in order to treat non-periodic white noise, or the gradient of the Brownian sheet in higher dimensions.

The second main contribution of this chapter is the derivation of a well-posed rough weak solution concept. In the case of bounded and measurable coefficients the equivalence of probabilitic weak solutions and solutions of the martingale problem is by now classical, cf. [SV06] (in the Brownian noise case) and [KC11] (in the Lévy noise case). It has so far been an open problem (in both the Brownian and Lévy noise case), whether these results can be generalized to distributional drifts in the rough regime. In the Young regime with time-independent drift [ABM20] introduces a canonical weak solution concept, replacing the singular drift by the limit of smooth drift terms. The same concept can be considered in multiple dimensional, with time-depending drift. A canonical weak solution is a tuple of stochastic processes (X, L) on some probability base space, such that L is a symmetric α-stable Lévy process and X is given by

$$X = x + Z + L, \tag{0.7}$$

for a continuous drift process Z that is given as a limiting object (in probability)

$$Z = \lim_{n\to\infty} \int_0^{\cdot} V^n(r, X_r) dr =: \lim_{n\to\infty} Z^n$$

for smooth $V^n \to V$. Furthermore, Z satisfies the following Hölder-type bound: there exists $C > 0$, such that for all $0 \leqslant s < t \leqslant T$,

$$\mathbb{E}[|Z_t - Z_s|^2] \leqslant C|t-s|^{2(\alpha+\beta)/\alpha}. \tag{0.8}$$

The particular Hölder-regularity in (0.8) originates from the time regularity of the solution of the Kolmogorov backward equation. This implies that the solution X is a Dirichlet process and Itô formulas are available.
The recent article [IR22] consideres the multidimensional Brownian noise case and proves equivalence of martingale solutions and so-called B-solutions from [IR22, Definition 6.2] with $B = C_T\mathscr{C}^{\beta}_{\mathbb{R}^d}$ for $\beta \in (-1/2, 0)$, that additionally satisfy the "reinforced local time property" (cf. [IR22, Definition 6.7]). The reinforced local time property boils down to stability of the integral

$$\int_0^t \nabla u(s, X_s) dZ_s, \tag{0.9}$$

when approximating the drift term by (Z^n) and u, the solution of the Kolmogorov equation, by $(u^n) \subset C^{1,2}([0,T] \times \mathbb{R}^d)$.
In the Young case we show that the regularity of the PDE solution, stability of the PDE solution map and the bound (0.8) together yield stability for the stochastic integral (0.9) using the stochastic sewing lemma by [Lê20]. However, this fails in the rough case as the PDE solution is too irregular. In the setting of general dimension, time-depending drift and α-stable noise for $\alpha \in (1, 2]$, we prove that canonical weak solutions are equivalent to martingale solutions in the Young regime. In the rough case, we prove that canonical weak solutions are in general non-unique in law. Heuristically, this comes from the fact that the canonical weak solution does not uniquely determine the enhancement $\mathscr{V}$ of V. In the one-dimensional Brownian case the situation becomes particularily interesting. While for any smooth approximating sequence (V^n) of the singular drift V, the strong solutions X^n of

$$dX^n_t = V^n(t, X^n_t)dt + dB_t$$

converge in distribution to the same limit, given by the solution of the $\mathscr{G}^{\mathscr{V}}$- martingale problem, it is however not the case that there exists a unique canonical weak solution in the above sense.
Our approach to obtain a well-posed weak solution concept in the rough regime is to impose further assumptions that ensure the uniqueness of the extension of the integral (0.9). To that aim, we use ideas from rough paths theory, specifically the construction of rough stochastic integrals from [FHL21]. Lifting Z to a rough *stochastic* integrator process $(Z, \mathbb{Z}^V)$ enables to extend the integral from smooth integrands ∇u^n in a stable manner to (para-)controlled integrands ∇u. The lift is chosen in such a way as to correct for the most irregular terms in the paracontrolled decomposition of ∇u. We define a rough weak solutions accordingly. That is, we require a weak solution X to satisfy the conditions of a canonical weak solution and to be furthermore such that

the iterated integrals $\mathbb{Z}^V$ are well-defined and satisfy suitable Hölder-moment bounds. These are not the iterated integrals of Z itself, but $\mathbb{Z}^V$ formally corresponds to the resonant product component in the enhancement $\mathscr{V}$. Those Hölder bounds are such that $(Z, \mathbb{Z}^V)$ is a so-called *rough stochastic integrator*, while the process $(\nabla u(t, X_t))$ with the solution u of the Kolmogorov equation for regular right-hand side, is *stochastically controlled*.
The terms "stochastically controlled" and "rough (stochastic) integrator" are motivated by [FHL21], but our definitions and the integral we construct differ. The difficulty here does not arise due to a low regularity integrator, but due to a integrand of low regularity, that is however controlled. Further difficulties arise due to the integrability issue in the pure stable noise case, which means that we can construct the integral in $L^2(\mathbb{P})$, but possibly not in $L^p(\mathbb{P})$ for $p > 2$. That means, for a rough weak solution X we obtain a stable rough stochastic integral

$$\int_0^t \nabla u(s, X_s) d(Z, \mathbb{Z}^V)_s \tag{0.10}$$

in $L^2(\mathbb{P})$. For regular integrands the rough stochastic integral (0.10) coincides with the stochastic integral against Z (0.9). More generally, we construct the rough stochastic integral $\int_0^\cdot f_t d(Z, \mathbb{Z}^A)_t$, for rough stochastic integrators $(Z, \mathbb{Z}^A)$ for a given stochastic process A (formally $\mathbb{Z}^A_{st} = \int_s^t A_{s,r} dZ_r$) and an integrand f, that is stochastically controlled by A.
The stability of the rough stochastic integral (0.10) then enables to prove that a rough weak solution is indeed a solution of the $\mathscr{G}^{\mathscr{V}}$-martingale problem, in particular unique. To prove that a martingale solution is a rough weak solution we need to prove the existence of the iterated integrals $\mathbb{Z}^V$ satisfying suitable bounds. We show that the bounds on $\mathbb{Z}^V$ are implied by regularity properties and fortunate cancellations between the solutions of Kolmogorov backward equations for the *singular* terminal conditions given by V_r and $\mathscr{V}_r^2$, for $r \leqslant T$ and the enhancement $\mathscr{V} = (V, \mathscr{V}^2)$, whose existence follows from Chapter 3.
Chapter 4 is structured as follows. Our main Theorem 4.2 concerning existence and uniqueness of a solution of the martingale problem is proven in Section 4.1. Section 4.2 introduces our rough weak solution concept, while in Section 4.3 we construct the general rough stochastic integral. In Section 4.4 we prove in Theorems 4.32 and 4.35 equivalence of the weak solution concepts. Section 4.5 investigates the canonical weak solution concept. We prove well-posedness in the Young case and the ill-posedness in the rough case. In Section 4.6 we construct the Brox diffusion with Lévy noise.

Chapter 5: Periodic homogenization for singular SDEs

This chapter is based on an unpublished joint paper with Nicolas Perkowski.

Periodic homogenization decribes the limit procedure from microscopic boundary-value problems posed on periodic structures to a macroscopic equation. Such periodic media

are for example composite materials or polymer structures. The theory originated from engineering purposes in material sciences in the 1970s, cf. [BLP78] and the references therein. Mathematically, this leads to the study of the limit of periodic operators with rapidly oscillating coefficients. There exist analytic and probabilistic methods to determine the limit equation. We refer to the classical works [BLP78, PS08] for the background on homogenization theory. We employ a probabilistic method using the Feynman-Kac formula (cf. [Øks03]). Via the Feynman-Kac formula, the periodic homogenization result for the Kolmogorov PDE with fluctuating and unbounded drift corresponds to a central limit theorem for the diffusion process.
In this chapter, we generalize the theory of periodic homogenization for SDEs with additive Brownian noise, respectively stable Lévy noise, from [BLP78], respectively [Fra07], from the setting of regular coefficients to singular Besov drifts $F \in (\mathscr{C}^\beta(\mathbb{T}^d))^d$ for $\beta \in ((2-2\alpha)/3, 0)$ ($\mathbb{T}^d$ denotes the d-dimensional torus).
In [BLP78, Section 3.4.2], the periodic drift coefficient is assumed to be C^1 with Hölder-continuous derivative and the periodic diffusion coefficient is assumed to be symmetric and uniformly elliptic, as well as C^2 with Hölder-continuous first derivative and bounded second derivative. The assumption of uniform ellipticity can be relaxed to allow for some degeneracy, which was investigated in [HP08] using Malliavin calculus techniques.
In [Fra07] the multiplicative symmetric α-stable noise case for $\alpha \in (1, 2)$ is studied and the coefficients are assumed to be even more regular, namely C^3. The regularity assumptions were relaxed in [HDS18, HDS22], where the authors more generally consider the periodic homogenization for the generator of an α-stable-like Feller process. In [HDS22], using a Zvonkin transformation to remove the drift (cf. [Zvo74]), the authors can consider drifts that are bounded and β-Hölder continuous for $\beta \in (1 - \alpha/2, 1)$. They also consider a non-linear intensity function σ and therefore a multiplicative noise term of the form $\sigma(X_t, dL^\alpha_t)$, see [HDS22, Equation (2.1)] with an isotropic α-stable process L^α, whereas in [Fra07] the intensity function $\sigma(x, y)$ is linear in y.
In the recent article [CCKW21] the authors further generalize the assumption on the drift coefficient to bounded, measurable drifts and consider the solution of the martingale problem associated to the SDE. The operator they consider is a Lévy-type operator that in particular includes all stable Lévy noise generators, symmetric and non-symmetric. They prove the homogenization result with the corrector method, an analytical method in homogenization theory, and show that different limit phenomena occur in the cases $\alpha \in (0, 1)$, $\alpha = 1$, $\alpha \in (1, 2)$, $\alpha = 2$ and $\alpha \in (2, \infty)$.
With analytical methods, the papers [KPZ19, Ari10, Sch10] deal with Lévy-type operators with oscillating coefficients for $\alpha \in (0, 2)$, but without drift part.
In the mixed jump-diffusion case, [San16] investigates the periodic homogenization for zero-drift diffusions with small jumps. The homogenized process in this case is also a Brownian motion.
We focus on the addive α-stable symmetric noise case, where different limit behaviours occure for $\alpha = 2$ (the Brownian noise case) and $\alpha \in (1, 2)$. Our contribution is the generalization to distributional drifts, not only in the Young, but also in the rough regime.

For the homogenization result, we rely on Kipnis-Varadhan martingale methods (cf. [KV86] and [KLO12]). Those methods require to solve the Poisson equation for the generator of the diffusion (or more generally the resolvent equations and imposing additional assumption) and to rewrite the additive functional in terms of that solution and Dynkin's martingale. Poisson equations for generators of diffusions with regular coefficients were studied in the classical article [Par98].
Following [KLO12], e generalize those techniques to much less regular drift coefficient. In particular this includes bounded measurable drifts or distributional drifts in the Young regime, where classical PDE techniques apply. More interestingly, our theory applies in the setting of singular drifts such as a typical realization of the periodic spatial white noise, cf. Remark 5.30. In order to apply the SDE solution theory from Chapter 4, we restrict to additive noise.
To be more precise, we study the functional central limit theorem for the solution X of the martingale problem associated to the SDE

$$dX_t = F(X_t)dt + dL_t$$

with $F \in (\mathscr{C}^\beta(\mathbb{T}^d))^d$ and a symmetric α-stable process L for $\alpha \in (1,2]$. The singular generator $\mathfrak{L}$ of X is given by

$$\mathfrak{L} = -\mathscr{L}_\nu^\alpha + F \cdot \nabla.$$

The first step is to prove existence and uniqueness of an invariant probability measure π on $\mathbb{T}^d$ for $\mathfrak{L}$ with strictly positive Lebesgue density. We achieve this by solving the singular Fokker-Planck equation with singular initial condition $\mu \in \mathscr{C}_1^0$,

$$(\partial_t - \mathfrak{L}^*)\rho_t = 0, \quad \rho_0 = \mu,$$

with formal Lebesgue adjoint $\mathfrak{L}^*$ of $\mathfrak{L}$ and proving a strict maximum principle on compacts. Furthermore, we prove spectral gap estimates on the semigroup of the diffusion projected onto the torus and solve the singular resolvent equation for $\mathfrak{L}$. This enables, through a limiting argument in a Sobolev-type space $\mathscr{H}^1(\pi)$ with respect to π, to solve the Poisson equation (0.11) with singular right-hand side $F - \langle F\rangle_\pi$. Here, we define $\langle F\rangle_\pi = \int F d\pi$ in a stable manner.
For the homogenization, we distinguish between the cases $\alpha = 2$ (Brownian noise case) and $\alpha \in (1,2)$, as the scaling and the limit behaviour differs. In the standard Brownian noise case, we prove weak convergence

$$\left(\frac{1}{\sqrt{n}}(X_{nt} - nt\langle F\rangle_\pi)\right)_{t\in[0,T]} \Rightarrow (\sqrt{D}B_t)_{t\in[0,T]},$$

where B is a standard Brownian motion and D is the constant diffusion matrix with entries

$$D(i,j) := \int_{\mathbb{T}^d} (e_i + \nabla\chi^i(x))(e_j + \nabla\chi^j(x))^T \pi(dx),$$

for $i, j = 1, \dots, d$ and e_i denoting the i-th euclidean unit vector. The limit is motivated by the result from [BLP78, Section 3.4.2]. Furthermore, $\chi \in (L^2(\pi))^d$ solves the Poisson equation with singular right-hand side $F - \langle F \rangle_\pi$:

$$(-\mathfrak{L})\chi^i = F^i - \langle F^i \rangle_\pi, \tag{0.11}$$

for $i = 1, \dots, d$. In the pure Lévy noise case $\alpha \in (1,2)$ we rescale in the α-stable scaling $n^{-1/\alpha}$ instead of $n^{-1/2}$. In this scaling we show, that the Dynkin martingale vanishes and thus we obtain weak convergence towards the stable process itself,

$$\left(\frac{1}{n^{1/\alpha}} (X_{nt} - nt\langle F \rangle_\pi) \right)_{t \in [0,T]} \Rightarrow (L_t)_{t \in [0,T]}.$$

In particular, compared to the Brownian noise case, there is no diffusivity enhancement in the limit (analogously to the regular coefficient case, cf. [Fra07]).
The chapter is structured as follows. Preliminaries and the strategy to prove the central limit theorem are outlined in Section 5.1. In Section 5.2 we solve the singular Fokker-Planck equation with the paracontrolled approach. The singular resolvent equation for $\mathfrak{L}$ is solved in Section 5.3. We show in Section 5.4 existence and uniqueness of the invariant measure π. Section 5.4 furthermore yields a characterization of the domain of the generator $\mathfrak{L}$ in $L^2(\pi)$, cf. Theorem 5.17. In Section 5.5, we solve the Poisson equation with singular right-hand side $F - \langle F \rangle_\pi$. Finally, we prove the CLT in Section 5.6 and relate to the periodic homogenization result for the parabolic PDE with oscillating operator $\mathfrak{L}^\varepsilon = -\mathscr{L}^\alpha_\nu + \varepsilon^{1-\alpha} F(\varepsilon^{-1}\cdot) \cdot \nabla$, cf. Corollary 5.27.

Part I.

Stochastic analysis of particle and particle-membrane systems

1. Rough homogenization for diffusions on fluctuating membranes

In this chapter, we prove a rough homogenization result for a Brownian particle on a fluctuating Gaussian hypersurface with covariance given by (the ultra-violet cutoff of) the Helfrich energy, that will be introduced in detail in Section 1.1 below. Specifically, we extend the results from [Dun13, DEPS15] by proving the convergence to a particular lift of the homogenization limit in rough path topology. Considering the time-space scaling regimes $(\alpha, \beta) \in \{(0,1),(1,2),(1,1)\}$, we prove in Sections 1.2 to 1.4 the convergence of the Itô and Stranonovich rough path lift of the diffusion and identify the limit. Interestingly, in some regimes, the rough path lift converges to a non-trivial lift of the limit, in the sense that, additionally, an area correction to the iterated integrals appears. Sections 1.1 to 1.4 are based on [DKP22].
Section 1.5 yields a first result in the direction of constructing the diffusion on the Helfrich membrane (not depending on time) when the cutoff diverges to infinity in dimension $d = 2$.

1.1. Langevin dynamics on a fluctuating Helfrich membrane

In order to give a better understanding of the form of the considered system of SDEs, we introduce the Helfrich elasticity memebrane model in the following. The describtion of the model is based on [Dun13, Section 2.2], see also [DEPS15, DE88]. Afterwards, we rigorously define in Definition 1.1 the diffusion X on the membrane H given by the ultra-violet cutoff of the Helfrich membrane and introduce the oscillating model (1.15). We also define the Laplace-Beltrami operator, which is the generator of X and gather properties of the operator and the coefficients.

We consider a random hypersurface that can be represented by a graph of a sufficiently smooth field $H : [0, L]^d \times [0, \infty) \to \mathbb{R}$, the so-called Monge gauge parametrization. More precisely, we assume that for each $t > 0$ and every fixed realization of the field, $x \mapsto H(x, t)$ is smooth and periodic with period L_H. Without loss of generality, $L_H = 1$. Furthermore, we assume the existence of a characteristic timescale $T_H = T$, which describes the observation time of the system. The hypersurface $\mathscr{S}(t)$ is given by the graph $J : [0,1]^d \times [0,\infty) \to \mathbb{R}^{d+1}$ with

$$J(x,t) = (x, H(x,t)). \tag{1.1}$$

The metric tensor of $\mathscr{S}(t)$ in local coordinates $x \in \mathbb{R}^d$ is given by

$$G(x,t) = I + \nabla H(x,t) \otimes \nabla H(x,t).$$

We define

$$|G|(x,t) := \det G(x,t) = 1 + |\nabla H(x,t)|^2.$$

For the physical application we consider dimension $d = 2$. Nontheless, our results apply for general dimension $d \in \mathbb{N}$. The dimension becomes relevant in Section 1.5 if the cutoff goes to infinity, as the surface becomes rougher with increasing dimension.
The classical description of fluid membranes $\mathscr{S}$ in equilibrium state, here in the quasi-planer case modelled by H, is based on the Canham [Can70] - Helfrich [Hel73] free energy

$$\mathscr{E}[H] = \frac{1}{2} \int_{[0,L]^2} (\kappa K^2(x) + \sigma) \sqrt{|G|(x)} dx,$$

with the mean curvature K, the (bare) bending modulus constant κ and the surface tension σ. We omit the term with Gaussian curvature, since we consider fluctuations of the membrane which do not change its topology. For small deformations of H, that is $|\nabla H(x)| \ll 1$, one can interpret $K(x)$ and $\sqrt{|G|(x)}$ as an approximation of $\Delta H(x)$, respectively $1 + |\nabla H(x)|^2$, so that $\mathscr{E}[H]$ can be approximated by

$$\mathscr{E}[H] = \frac{1}{2} \int_{[0,1]^2} \kappa(\Delta H(x))^2 + \sigma |\nabla H(x)|^2 dx + \frac{\sigma L^2}{2}.$$

By possibly changing the constant κ, we absorbed the term $\kappa(\Delta H(x))^2 |\nabla H(x)|^2$ for small $|\nabla H(x)|^2$ into the term $\kappa(\Delta H(x))^2$. The constant term can also be omitted. For more details about description of fluid lipid membranes we refer to [Des15].

The dynamics that correspond to the formal invariant measure $\exp(-\mathscr{E}[H])dH$ are described by the stochastic partial differential equation (SPDE)

$$\partial_t H = -RAH + \xi, \tag{1.2}$$

where $AH := -\kappa \Delta^2 H + \sigma \Delta H$ is the restoring force for the free energy associated to $\mathscr{E}[H]$. Moreover, R is the operator that characterizes the effect of nonlocal interactions of the membrane through the medium. For more details, see [DEPS15, section 4] or [Dun13, section 2.2], where R is defined as $Rf := \Lambda * f$ for $\Lambda(x) := (8\pi\lambda|x|)^{-1}$, $f \in L^2_{\text{per}}([0,L]^2)$ and λ is the viscosity of the surrounding medium. The last term ξ is a Gaussian field, that is white in time and whose spatial fluctuations have mean zero and covariance operator $2(k_B T)R$ with Boltzmann constant k_B.

Consider the Galerkin-projection of H given by

$$H^{\tilde{K}}(x,t) = h(x,\eta_t) = \sum_{0<|k|\leqslant \tilde{K}} \eta_t^k e_k(x), \quad (x,t) \in \mathbb{T}^2 \times \mathbb{R}^+, \tag{1.3}$$

for the Fourier basis $(e_k)_{k\in\mathbb{Z}^2}$ on the torus $\mathbb{T}^2 \simeq [0,1]^2$, i.e. $e_k(x) = \exp(2\pi i k \cdot x) \in C^\infty(\mathbb{T}^2)$, and cutoff $\tilde{K} \in \mathbb{N}$. Substituting (1.3) into (1.2), we see that the SPDE diagonalizes and that the coefficients $\eta = (\eta^k)_{|k|\leqslant \tilde{K}}$ with $K := \#\{k \in \mathbb{Z}^2 \mid 0 < |k| \leqslant \tilde{K}\}$ are independent Ornstein-Uhlenbeck processes with joint dynamics

$$d\eta_t = -\Gamma \eta_t dt + \sqrt{2\Gamma\Pi} dW_t,$$

where W is a K-dimensional complex-valued standard Brownian motion with constraint $\overline{W^k} = W^{-k}$, i.e. $W = \frac{1}{\sqrt{2}}(\mathrm{Re}(W) + i\,\mathrm{Im}(W))$ with independent $\mathbb{R}^K$-valued Brownian motions $\mathrm{Re}(W), \mathrm{Im}(W)$, and where Γ, Π are real diagonal matrices, such that $\overline{\eta^k} = \eta^{-k}$, which implies that H is real-valued, as $\overline{H} = H$.

We consider the membrane $H = H^{\tilde{K}}$ given by the ultra-violet cutoff of the Helfrich membrane in (1.3) and the cutoff $\tilde{K}$ will be fixed throughout the chapter (except in the outlook Section 1.5). [Stu10, Lemma 6.25] yields that, without the ultra-violet cutoff $\tilde{K}$, the surface H is almost surely Hölder-continuous with exponent $\alpha < 1$ (in $d = 2$), but not for $\alpha = 1$. Due to this irregularity, we can not define a diffusion on H using classical methods (cf. [Hsu02]). This is the reason why most papers that deal with the diffusion on H assume a fixed ultra-violet cutoff $\tilde{K}$.

Since we work in the real-valued setting, we identify η with $(\mathrm{Re}(\eta), \mathrm{Im}(\eta))$ that is a $2K$-dimensional real-valued Ornstein-Uhlenbeck process with the above dynamics for a $2K$-dimensional real-valued standard Brownian motion W and with the property that $\mathrm{Re}(\eta^k) = \mathrm{Re}(\eta^{-k})$ and $\mathrm{Im}(\eta^k) = -\mathrm{Im}(\eta^{-k})$. The matrices Γ, Π are symmetric and positive definite and defined by $\Gamma := \mathrm{diag}(\Gamma_k)$, $\Pi = \mathrm{diag}(\Pi_k)$ with

$$\Gamma_k = \frac{\kappa^*|2\pi k|^4 + \sigma^*|2\pi k|^2}{|2\pi k|}, \quad \Pi_k = \frac{1}{\kappa^*|2\pi k|^4 + \sigma^*|2\pi k|^2} \tag{1.4}$$

where $\kappa^* = \kappa/(2k_B T), \sigma^* = \sigma/(2k_B T)$.

Since the matrices Γ, Π commute, the normal distribution

$$N(0,\Pi) =: \rho_\eta \tag{1.5}$$

is the invariant measure for the Ornstein Uhlenbeck process η. Then we have

$$\rho_\eta(d\eta) = \rho_\eta d\eta = \frac{1}{\sqrt{(2\pi)^2|\Pi|}} \exp\Big(-\frac{1}{2}\eta \cdot \Pi^{-1} \cdot \eta\Big) d\eta, \tag{1.6}$$

with $|\Pi| := \det(\Pi)$. We use the same notation ρ_η for the measure and its density. The generator $\mathscr{L}_\eta$ of η is given by

$$\mathscr{L}_\eta = -\Gamma\eta \cdot \nabla_\eta + \Pi\Gamma : \nabla_\eta \nabla_\eta. \tag{1.7}$$

Here and later on, we will use the notation

$$A : \nabla_x \nabla_x f(x) := \sum_{i,j=1}^{n} A_{i,j} \partial_{x_i} \partial_{x_j} f(x) \quad \text{for } A \in \mathbb{R}^{n\times n}, f \in C^2(\mathbb{R}^n, \mathbb{R}), n \in \mathbb{N}.$$

The generator $\mathscr{L}_\eta$ is a closed, unbounded operator on $L^2(\rho_\eta)$ with domain $dom(\mathscr{L}_\eta) = \{f \in L^2(\rho_\eta) \mid \mathscr{L}_\eta f \in L^2(\rho_\eta)\}$. Observe that, since $\mathscr{L}_\eta^* \rho_\eta = 0$ for the Lebesgue adjoint $\mathscr{L}_\eta^*$, invariance of ρ_η, i.e.

$$\langle \mathscr{L}_\eta f \rangle_{\rho_\eta} := \int \mathscr{L}_\eta f(x) \rho_\eta(x) dx = 0, \quad \text{for all } f \in dom(\mathscr{L}_\eta),$$

can be checked easily. Consider the Sobolev-type space

$$H^1(\rho_\eta) := \{f \in dom(\mathscr{L}_\eta) \mid \|f\|^2_{H^1(\rho_\eta)} := \langle (-\mathscr{L}_\eta) f, f \rangle_{\rho_\eta} < \infty\} \tag{1.8}$$

for the scalar-product $\langle \cdot, \cdot \rangle_{\rho_\eta}$ in $L^2(\rho_\eta)$. Furthermore, notice that for the Ornstein-Uhlenbeck generator $\mathscr{L}_\eta$, spectral gap estimates hold true. That is, there exists a constant $C > 0$, such that

$$\|f\|^2_{H^1(\rho_\eta)} = \langle (-\mathscr{L}_\eta) f, f \rangle_{\rho_\eta} \geqslant C \|f - \langle f \rangle_{\rho_\eta}\|^2_{\rho_\eta} \tag{1.9}$$

for any $f \in H^1(\rho_\eta)$ with $\langle f \rangle_{\rho_\eta} = \int f(\eta) d\rho_\eta$. Indeed, from a simple calculation using invariance for f^2, it follows that $\langle (-\mathscr{L}_\eta) f, f \rangle_{\rho_\eta} = \langle \Pi\Gamma \nabla_\eta f, \nabla_\eta f \rangle_{\rho_\eta}$. Then (1.9) follows from $\min_{\mathbb{T}^2} \rho_\eta > 0$ and the Poincaré-inequality for the Laplacian on the Sobolev space $H^1(\mathbb{T}^2)$.

In particular, ρ_η is an ergodic measure for the Ornstein-Uhlenbeck process η. Furthermore, if f is centered under ρ_η, then $P_t^\eta f$ is centered by invariance and thus (1.9) applied to $P_t^\eta f$ together with $\partial_t \langle P_t^\eta f, P_t^\eta f \rangle_{\rho_\eta} = 2 \langle \mathscr{L}_\eta P_t^\eta f, P_t^\eta f \rangle_{\rho_\eta}$ yields the spectral gap estimates for the semigroup $(P_t^\eta)_{t\geqslant 0}$ of the Ornstein-Uhlenbeck process, that is,

$$\|P_t^\eta f - \langle f \rangle_{\rho_\eta}\|_{L^2(\rho_\eta)} \leqslant e^{-Ct} \|f\|_{L^2(\rho_\eta)}$$

for all $t \geqslant 0$ and $f \in L^2(\rho_\eta)$ (sometimes also called exponential ergodicity).

We will to consider a Brownian motion X on the Helfrich membrane H given in (1.3). This diffusion will be driven by an independent Brownian motion B. For each fixed realization of the membrane, X is the Markov process that has in local coordinates the Laplace-Beltrami operator $\mathscr{L}^H = \mathscr{L}$ as a generator. As we assume the expansion (1.3) with coefficients η, we obtain a system of SDEs for (X, η) describing the dynamics of the diffusion X on the membrane H. Following [DEPS15], we define X as follows.

Definition 1.1. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space and* B *a two-dimensional standard Brownian motion independent of a* $2K$*-dimensional standard Brownian motion* W*. Let* x_0 *be a random variable with values in* $\mathbb{T}^2$ *independent of* B *and* W*. Let* (X, η) *be the solution of the following system of SDEs*

$$\begin{aligned} dX_t &= F(X_t, \eta_t)dt + \sqrt{2\Sigma(X_t, \eta_t)}dB_t, \quad X_0 = x_0 \\ d\eta_t &= -\Gamma\eta_t dt + \sqrt{2\Gamma\Pi}dW_t, \quad \eta_0 \sim \rho_\eta \end{aligned} \tag{1.10}$$

with $\Sigma : \mathbb{T}^2 \times \mathbb{R}^{2K} \to \mathbb{R}^{2\times 2}_{sym}$*, where* $\Sigma(x, \eta) = g^{-1}(x, \eta)$ *is the inverse of the metric tensor matrix* $g(x, \eta) \in \mathbb{R}^{2\times 2}$ *defined by*

$$g(x, \eta) := \mathrm{Id} + \nabla_x h(x, \eta) \otimes \nabla_x h(x, \eta) \tag{1.11}$$

and $F : \mathbb{T}^2 \times \mathbb{R}^{2K} \to \mathbb{T}^2$ *with*

$$F(x, \eta) := \frac{1}{\sqrt{|g|(x, \eta)}} \nabla_x \cdot (\sqrt{|g|} g^{-1}(x, \eta)). \tag{1.12}$$

Then we call X *a Brownian motion on the Helfrich membrane* H *given by* (1.3) *started in* x_0*.*

As shown in [Dun13, Proposition 2.3.1], the solution (X, η) exists and is a Markov process with generator on smooth, compactly supported test functions $f : \mathbb{T}^2 \times \mathbb{R}^{2K} \to \mathbb{R}$ given by

$$\begin{aligned} (\mathscr{L} + \mathscr{L}_\eta) f(x, \eta) &= \frac{1}{\sqrt{|g|(x, \eta)}} \nabla_x \cdot (\sqrt{|g|(x, \eta)} \Sigma(x, \eta) \nabla_x f(x, \eta)) + \mathscr{L}_\eta f(x, \eta) \\ &= F(x, \eta) \nabla_x f(x, \eta) + \Sigma(x, \eta) : \nabla_x \nabla_x f(x, \eta) + \mathscr{L}_\eta f(x, \eta), \end{aligned}$$

where $\mathscr{L}_\eta$ is the generator of the Ornstein Uhlenbeck process η given with (1.7) and $\mathscr{L}$ is the Laplace-Beltrami operator. Moreover, the proof of [Dun13, Proposition 2.3.1] provides the following uniform bounds: there exists a constant $C_1 > 0$ such that

$$|\Sigma(x, \eta)|_F \leqslant C_1, \qquad \forall (x, \eta) \in \mathbb{T}^2 \times \mathbb{R}^{2K}, \tag{1.13}$$

where $|\cdot|_F$ denotes the Frobenius-norm (or any other equivalent matrix norm) and there exists a constant $C_2 > 0$ such that

$$|F(x, \eta)| \leqslant C_2(1 + |\eta|), \qquad \forall (x, \eta) \in \mathbb{T}^2 \times \mathbb{R}^{2K}, \tag{1.14}$$

where $|\cdot|$ denotes the usual euclidean norm.

We are interested in considering fluctuations of the membrane in time (ε^β, $\beta \geqslant 0$ or $\beta = -\infty$) and space (ε^α, $\alpha \geqslant 0$) with different speeds α, β. More precisely, we consider instead of $H(x, t)$ the fluctuating surface $\varepsilon^\alpha H(\frac{x}{\varepsilon^\alpha}, \frac{t}{\varepsilon^\beta}) = \varepsilon^\alpha h(\varepsilon^{-\alpha} x, \eta_{\varepsilon^{-\beta} t})$, which transforms the system of equations (1.10) into

$$dX_t^\varepsilon = \frac{1}{\varepsilon^\alpha} F\left(\frac{X_t^\varepsilon}{\varepsilon^\alpha}, \eta_t^\varepsilon\right) dt + \sqrt{2\Sigma\left(\frac{X_t^\varepsilon}{\varepsilon^\alpha}, \eta_t^\varepsilon\right)} dB_t,$$
$$d\eta_t^\varepsilon = -\frac{1}{\varepsilon^\beta}\Gamma\eta_t^\varepsilon dt + \sqrt{\frac{2\Gamma\Pi}{\varepsilon^\beta}} d\tilde{W}_t, \quad \eta_0^\varepsilon \sim \rho_\eta. \tag{1.15}$$

Here we define the Brownian motion $\tilde{W}_t := \varepsilon^{\beta/2} W_{\varepsilon^{-\beta}t}$ and thus $(\eta_{\varepsilon^{-\beta}t})_{t\geqslant 0} = (\eta_t^\varepsilon)_{t\geqslant 0}$. Since we are only interested in convergence in distribution, we may replace $\tilde{W}$ by W. We refer to [Dun13, section 2.3.3] for the derivation of the system and the physical background. Furthermore, note that stationarity and the Gaussian distribution imply boundedness of all moments of η_t^ε in ε, t.
In the following, we study the convergence of the Itô and Stratonovich rough path lift of (X^ε) for different speeds α, β as $\varepsilon \to 0$ in γ-Hölder rough path topology. Here, we briefly recall the definition of a γ-Hölder rough path from [FH20, Definition 2.1], and for more details we refer the reader to [FH20]. We write $X_{s,t} := X_t - X_s$ and we define the triangle

$$\Delta_T := \{(s,t) \in [0,T]^2 \mid s \leqslant t\}.$$

Definition 1.2. *For $\gamma \in (1/3, 1/2]$ we call $(X, \mathbb{X}) \in C([0,T], \mathbb{R}^d) \times C(\Delta_T, \mathbb{R}^{d\times d})$ a γ-Hölder rough path if:*

i) Chen's relation holds, that is

$$\mathbb{X}_{r,t} - \mathbb{X}_{r,s} - \mathbb{X}_{s,t} = X_{s,u} \otimes X_{u,t}$$

for all $0 \leqslant r \leqslant s \leqslant t \leqslant T$ with $\mathbb{X}_{t,t} = 0$,

ii) the (inhomogeneous) γ-Hölder norms are finite, that is

$$\|(X,\mathbb{X})\|_\gamma := \|X\|_{\gamma,T} + \|\mathbb{X}\|_{2\gamma,T} := \sup_{0\leqslant s<t\leqslant T} \frac{|X_{s,t}|}{|t-s|^\gamma} + \sup_{0\leqslant s<t\leqslant T} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\gamma}} < \infty.$$

We denote the nonlinear space of all such γ-Hölder rough paths by $C_{\gamma,T}$ equipped with distance

$$\|(X^1,\mathbb{X}^1);(X^2,\mathbb{X}^2)\|_\gamma := \|X^1 - X^2\|_{\gamma,T} + \|\mathbb{X}^1 - \mathbb{X}^2\|_{2\gamma,T}.$$

Remark 1.3. *For a two-dimensional Brownian motion B, the Itô lift $(B, \mathbb{B}_{\text{Ito}})$, where $\mathbb{B}_{\text{Ito}}(s,t) := \int_s^t B_{s,r} \otimes dB_r$ are Itô-integrals, as well as the Stratonovich lift $(B, \mathbb{B}_{\text{Strato}})$ for $\mathbb{B}_{\text{Strato}}(s,t) := \int_s^t B_{s,r} \otimes \circ dB_r$ being Stratonovich integrals, are almost surely γ-Hölder rough path for $\gamma = 1/2 - \varepsilon$ for any $\varepsilon > 0$, cf. [FH20, Chapter 3]. But also $(B, (s,t) \mapsto \mathbb{B}_{s,t} + A(t-s))$ is a γ-rough path for a matrix $A \in \mathbb{R}^{2\times 2}$ and $\mathbb{B} = \mathbb{B}_{\text{Ito}}$ or $\mathbb{B} = \mathbb{B}_{\text{Strato}}$. The latter will be the lift of the Brownian motion that we encounter below.*

We finalize this section by recalling the concept of uniform controlled variations by Kurtz and Protter ([KP96, Definition 7.5]; here for continuous semimartingales without the need for stopping times).

Definition 1.4. *A sequence $(X^\varepsilon)_\varepsilon$ of $\mathbb{R}^d$-valued continuous semi-martingales on $[0,T]$ with $X^\varepsilon = M^\varepsilon + A^\varepsilon$, where M^ε is a local martingale and A^ε is of finite variation, satisfies the UCV (Uniformly Controlled Variations) condition if and only if*

$$(\langle M^{\varepsilon,i}\rangle_T)_\varepsilon \quad \text{and} \quad (\mathrm{Var}_{1,[0,T]}(A^\varepsilon))_\varepsilon \quad \text{are tight in } \mathbb{R}, \tag{1.16}$$

for $i = 1, ..., d$ and $\mathrm{Var}_{1,[0,T]}(f) := \lim_{|\pi|\to 0} \sum_{s,t\in\pi} |f_t - f_s|$ denotes the one-variation of a function $f : [0,T] \to \mathbb{R}^d$ and the limit is taken over all finite partitions π of $[0,T]$ with mesh size $|\pi| = \max_{s,t\in\pi}|t-s| \to 0$.

Remark 1.5. *The tightness* (1.16) *follows from*

$$\max_{i=1,\dots,d} \sup_\varepsilon \big(\mathbb{E}[\langle M^{\varepsilon,i}\rangle_T] + \mathbb{E}[\mathrm{Var}_{1,[0,T]}(A^\varepsilon)]\big) < \infty. \tag{1.17}$$

Furthermore, we are in the situation in which the (X^ε) are defined on the same probability space.

We state a version of [KP96, Theorem 7.7. and 7.10], that will be repeatedly exploited in order to prove the distributional convergence of certain Itô integrals.

Proposition 1.6. *A sequence $(X^\varepsilon)_\varepsilon$ of $\mathbb{R}^d$-valued continuous semi-martingales on $[0,T]$ satisfies the UCV condition if and only if for all sequences $(Y^\varepsilon)_\varepsilon$ with $(Y^\varepsilon, X^\varepsilon) \Rightarrow (Y, X)$ jointly in distribution in $C([0,T], \mathbb{R}^{2d})$ and with Y^ε integrable against X^ε and Y against X in the Itô sense, it follows that $(Y^\varepsilon, X^\varepsilon, \int_0^\cdot Y_s^\varepsilon \otimes dX_s^\varepsilon) \Rightarrow (Y, X, \int_0^\cdot Y_s \otimes dX_s)$ in distribution in $C([0,T], \mathbb{R}^{2d+d\times d})$.*

Let us furthermore state a version of the Kolmogorov criterion for rough paths, [FH20, Theorem 3.1].

Lemma 1.7. *Let $(X^\varepsilon, \mathbb{X}^\varepsilon)_\varepsilon$ be a family of γ-Hölder rough path, $\gamma < 1/2$. Assume that for any $p > 2$, there exist constants $C_1, C_2 > 0$ such that,*

$$\sup_\varepsilon \mathbb{E}[|X_t^{\varepsilon,i} - X_s^{\varepsilon,i}|^p] \leqslant C_1|t-s|^{p/2}, \quad \forall s,t \in [0,T] \tag{1.18}$$

and

$$\sup_\varepsilon \mathbb{E}\left[\left|\int_s^t X_{s,r}^{\varepsilon,i} dX_r^{\varepsilon,j}\right|^{p/2}\right] \leqslant C_2|t-s|^{p/2}, \quad \forall s,t \in \Delta_T \tag{1.19}$$

for $i, j \in \{1, ..., d\}$. Then for any $\gamma' < 1/2$,

$$\sup_\varepsilon \mathbb{E}[\|(X^\varepsilon, \mathbb{X}^\varepsilon)\|_{\gamma'}^p] < \infty.$$

If furthermore $\sup_\varepsilon \mathbb{E}[|X_0^\varepsilon|] < \infty$ *holds true, then it follows that* $(X^\varepsilon, \mathbb{X}^\varepsilon)_\varepsilon$ *is tight in* $C_{\gamma,T}$.

Proof. The proof follows from the proof of [FH20, Theorem 3.1] applied to $\beta = 1/2$ and $\gamma' = \beta - 1/q - \varepsilon$, $\varepsilon > 0$, using the fact that, according to assumption (1.18), (1.19) holds for any $p > 2$. Tightness in $C_{\gamma,T}$ then follows utilizing the compact embedding $C_{\gamma',T} \hookrightarrow C_{\gamma,T}$ for $\gamma < \gamma'$. □

Below, we use the following convention. The notation $a \lesssim b$ shall indicate, that there exists a constant $C > 0$, such that $a \leqslant Cb$. This constant does not depend on the relevant parameters at hand, unless we indicate the dependence by $\lesssim_k$ if the bound $C = C(k)$ depends on a parameter k.

1.2. Membrane with purely temporal fluctuations

In this section, we consider the scaling regime $\alpha = 0$, $\beta = 1$ in (1.15) and thus obtain the slow-fast system,

$$\begin{aligned} dX_t^\varepsilon &= F(X_t^\varepsilon, \eta_t^\varepsilon)dt + \sqrt{2\Sigma(X_t^\varepsilon, \eta_t^\varepsilon)}dB(t), \\ d\eta_t^\varepsilon &= -\frac{1}{\varepsilon}\Gamma\eta_t^\varepsilon dt + \sqrt{\frac{2}{\varepsilon}\Gamma\Pi}dW_t, \quad \eta_0^\varepsilon \sim \rho_\eta, \end{aligned} \tag{1.20}$$

where B and W are independent Brownian motions.
From classical stochastic averaging, see also [DEPS15, Theorem 4], we know that $X^\varepsilon \Rightarrow X$ in distribution in $C([0,T], \mathbb{R}^2)$ as $\varepsilon \to 0$. The limit X is the solution of the averaged system

$$dX_t = \overline{F}(X_t)dt + \sqrt{2\overline{\Sigma}(X_t)}dB_t,$$

with

$$\overline{F}(x) := \int_{\mathbb{R}^{2K}} F(x, \eta)\rho_\eta(d\eta), \tag{1.21}$$

$$\overline{\Sigma}(x) := \int_{\mathbb{R}^{2K}} \Sigma(x, \eta)\rho_\eta(d\eta), \tag{1.22}$$

where the invariant measure $\rho_\eta(d\eta)$ of η is given by (1.6). Utilizing the linear growth of F in η uniformly in x, cf. (1.14), boundedness of all moments of η_t^ε in ε, t and boundedness of Σ, cf. (1.13), we can conclude that $(X^\varepsilon)_\varepsilon$ satisfies the UCV condition (1.17); see below for the detailed proof. Thus, according to Proposition 1.6, the iterated Itô integrals of X^ε will converge to the iterated Itô integrals of the limit X. More precisely, the following theorem holds:

Theorem 1.8. *Let* $\gamma < 1/2$ *and* X^ε *and* X *be as above. Let* $\mathbb{X}_{s,t}^\varepsilon := \int_s^t (X_r^\varepsilon - X_s^\varepsilon) \otimes dX_r^\varepsilon$ *and* $\mathbb{X}_{s,t} := \int_s^t (X_r - X_s) \otimes dX_r$, *where the stochastic integrals are understood in the Itô* *sense. Then it follows that*

$$(X^\varepsilon, \mathbb{X}^\varepsilon) \Rightarrow (X, \mathbb{X}) \tag{1.23}$$

in distribution in γ-Hölder rough path topology.

Proof. We first prove the weak convergence of the iterated integrals $(\mathbb{X}^\varepsilon_{0,t})$ in $\mathbb{R}^{2\times 2}$ for any $t \geqslant 0$ and then show tightness of $(X^\varepsilon, \mathbb{X}^\varepsilon)$ in γ-Hölder rough path topology.
The first aim is to apply Proposition 1.6 and show that the sequence $(X^\varepsilon)_\varepsilon$ satisfies the UCV condition. We have that $X^\varepsilon = A^\varepsilon + M^\varepsilon$ for

$$\begin{aligned} A^\varepsilon_t &:= \int_0^t F(X^\varepsilon_s, \eta^\varepsilon_s) ds, \\ M^\varepsilon_t &:= \int_0^t \sqrt{2\Sigma(X^\varepsilon_s, \eta^\varepsilon_s)} dB_s, \end{aligned}$$

where A^ε is of finite variation and M^ε is a martingale. For $(X^\varepsilon)_\varepsilon$ to satisfy the UCV condition, we thus have to show the bound (1.17). By boundedness of Σ it is immediate that the expected quadratic variation of M^ε is also uniformly bounded in ε. For the bound on the total variation of A^ε we use that (1.14) holds uniformly in $x \in \mathbb{T}^2$, such that:

$$\begin{aligned} \sup_\varepsilon \mathbb{E}\big[\mathrm{Var}_{1,[0,T]}(A^\varepsilon)\big] &\le \sup_\varepsilon \mathbb{E}\bigg[\int_0^T |F(X^\varepsilon_s, \eta^\varepsilon_s)| ds\bigg] \\ &\le C \sup_\varepsilon \mathbb{E}\bigg[\int_0^T (1 + |\eta^\varepsilon_s|) ds\bigg] \\ &= C(1 + \mathbb{E}[|\eta_0|])T, \end{aligned}$$

where in the last equality, we used the stationarity of η^ε. As we have weak convergence of $(X^\varepsilon)_\varepsilon$ to X in $C([0,T], \mathbb{R}^2)$, this implies, according to Proposition 1.6, that we also have weak convergence of the Itô integrals $\mathbb{X}^\varepsilon := \int X^\varepsilon \otimes dX^\varepsilon$ to $\mathbb{X} := \int X \otimes dX$ in $C(\Delta_T, \mathbb{R}^{2\times 2})$.
To prove tightness in γ-Hölder rough path topology for $\gamma < 1/2$, we utilize Lemma 1.7. Here (1.18) follows immediately from the linear growth of F in η and bounded moments of η^ε_t in ε, t by stationarity, as well as Burkholder-Davis-Gundy inequality for the martingale part and boundedness of Σ. Moreover, (1.19) follows from (1.18) and the estimate

$$\begin{aligned} &\mathbb{E}\bigg[\bigg|\int_s^t X^{\varepsilon,i}_{s,r} dX^{\varepsilon,j}_r\bigg|^{p/2}\bigg] \\ &\quad\lesssim \mathbb{E}\bigg[\bigg|\int_s^t X^{\varepsilon,i}_{s,r} dM^{\varepsilon,j}_r\bigg|^{p/2}\bigg] + \mathbb{E}\bigg[\bigg|\int_s^t X^{\varepsilon,i}_{s,r} dA^{\varepsilon,j}_r\bigg|^{p/2}\bigg] \\ &\quad\lesssim \mathbb{E}\bigg[\bigg(\int_s^t |X^{\varepsilon,i}_{s,r}|^2 dr\bigg)^{p/4}\bigg] + \mathbb{E}\bigg[\bigg(\int_s^t |X^{\varepsilon,i}_{s,r} F^j(X^\varepsilon_r, \eta^\varepsilon_r)| dr\bigg)^{p/2}\bigg] \end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\int_s^t \mathbb{E}[|X_{s,r}^{\varepsilon,i}|^{p/2}]^{4/p}dr\right)^{p/4} + \left(\int_s^t \mathbb{E}[|X_{s,r}^{\varepsilon,i}F^j(X_r^\varepsilon,\eta_r^\varepsilon)|^{p/2}]^{2/p}dr\right)^{p/2}\\
&\lesssim |t-s|^{(2\times\frac{1}{2}+1)\times\frac{p}{4}} + \left(\int_s^t \mathbb{E}[|X_{s,r}^{\varepsilon,i}|^p]^{1/p}\mathbb{E}[|F^j(X_r^\varepsilon,\eta_r^\varepsilon)|^p]^{1/p}dr\right)^{p/2}\\
&\lesssim |t-s|^{p/2} + \left(\int_s^t \mathbb{E}[|X_{s,r}^{\varepsilon,i}|^p]^{1/p}\mathbb{E}[(1+|\eta_r^\varepsilon|)^p]^{1/p}dr\right)^{p/2}\\
&\lesssim |t-s|^{p/2} + \left(\int_s^t \mathbb{E}[|X_{s,r}^{\varepsilon,i}|^p]^{1/p}dr\right)^{p/2}\\
&\lesssim |t-s|^{p/2} + |t-s|^{p/4+p/2} \lesssim_T |t-s|^{p/2}
\end{aligned}
$$

using the Burkholder-Davis-Gundy inequality for the martingale part, boundedness of Σ in the second line and the generalized Minkowski's inequality for integrals for both summands in the third line (and the linear growth of F and stationarity of η).
Combining distributional convergence of $(X^\varepsilon, \mathbb{X}^\varepsilon)_\varepsilon$ to $(X, \mathbb{X})$ in $C(\Delta_T, \mathbb{R}^{d+d\times d})$ and tightness in $C_{\gamma,T}$, we conclude on distributional convergence in γ-Hölder rough path topology for $\gamma < 1/2$. □

1.3. Membrane with temporal fluctuations twice as fast as spatial fluctuations

In this section, we consider the scaling regime $\alpha = 1$, $\beta = 2$ in (1.15), that is, temporal fluctuations occur twice as fast as spatial ones. We introduce the fast process

$$
Y_t^\varepsilon := \frac{X_t^\varepsilon}{\varepsilon} \mod \mathbb{Z}^2.
$$

The operation $\mod \mathbb{Z}^2$ projects onto the torus $\mathbb{T}^2$, such that Y^ε is a Markov process with compact state space $\mathbb{T}^2$. Then the general SDE system can be written as

$$
\begin{cases}
dX_t^\varepsilon = \dfrac{1}{\varepsilon}F(Y_t^\varepsilon,\eta_t^\varepsilon)dt + \sqrt{2\Sigma(Y_t^\varepsilon,\eta_t^\varepsilon)}dB_t,\\
dY_t^\varepsilon = \dfrac{1}{\varepsilon^2}F(Y_t^\varepsilon,\eta_t^\varepsilon)dt + \sqrt{\dfrac{2}{\varepsilon^2}\Sigma(Y_t^\varepsilon,\eta_t^\varepsilon)}dB_t,\\
d\eta_t^\varepsilon = -\dfrac{1}{\varepsilon^2}\Gamma\eta_t^\varepsilon dt + \sqrt{\dfrac{2}{\varepsilon^2}\Gamma\Pi}dW_t,
\end{cases}
\tag{1.24}
$$

for independent Brownian motions B and W, where B is a two-dimensional and W is a $2K$-dimensional standard Brownian motion.
Utilizing Itô's formula, one can easily check that on smooth, compactly supported functions $f \in C_c^\infty(\mathbb{T}^2 \times \mathbb{R}^{2K}, \mathbb{R})$, the infinitesimal generator of the fast process $(Y^\varepsilon, \eta^\varepsilon)$

is $\varepsilon^{-2}\mathscr{G}$, where

$$\mathscr{G} = \mathscr{L}_0 + \mathscr{L}_\eta \tag{1.25}$$

for

$$\mathscr{L}_0 f(y,\eta) = F(y,\eta) \cdot \nabla_y f(y,\eta) + \Sigma(y,\eta) : \nabla_y \nabla_y f(y,\eta), \tag{1.26}$$

which is the generator of Y (for fixed η) and $\mathscr{L}_\eta$ is the generator of the Ornstein-Uhlenbeck process η, given by (1.7), cf. also [Dun13, section 5.3]. We also write $\mathscr{L}_0(\eta)$ to denote the operator $\mathscr{L}_0$ acting on functions $f : \mathbb{T}^2 \to \mathbb{R}$, stressing the dependence on fixed $\eta \in \mathbb{R}^{2K}$. The reason for introducing the fast process Y^ε is that the drift term of X^ε is given as an (unbounded) additive functional of the Markov process $(Y^\varepsilon, \eta^\varepsilon)$. Moreover, we have the equality in law, $(Y_t, \eta_t)_{t\geqslant 0} \stackrel{d}{=} (Y^\varepsilon_{\varepsilon^2 t}, \eta^\varepsilon_{\varepsilon^2 t})_{t\geqslant 0}$, where (Y, η) is the Markov process with generator $\mathscr{G}$.

As shown in [Dun13, Prop. 5.3.1], there exists a unique invariant measure ρ for the Markov process (Y, η), whose density is the unique, normalized solution of

$$\mathscr{G}^* \rho = 0, \tag{1.27}$$

$\mathscr{G}^*$ being the adjoint operator of $\mathscr{G}$ with respect to $L^2(dyd\eta)$. As ρ is the unique invariant measure, it is in particular ergodic for (Y, η). Furthermore, we can extend the semigroup $(P_t^{(Y,\eta)})_{t\geqslant 0}$ of the Markov process (Y, η), with $P_t^{(Y,\eta)} f(y,\eta) = \mathbb{E}[f(Y_t, \eta_t) \mid (Y_0, \eta_0) = (y, \eta)]$ for $f \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^{2K})$, uniquely to a strongly continuous contraction semigroup on $L^2(\rho)$ (that is possible by invariance of ρ, cf. [Yos95, Theorem 1, p. 381]) and define the generator $\mathscr{G} : dom(\mathscr{G}) \subset L^2(\rho) \to L^2(\rho)$ with $dom(\mathscr{G}) = \{u \in L^2(\rho) \mid \mathscr{G}u \in L^2(\rho)\}$ and $\mathscr{G}u := \lim_{t\to 0} t^{-1}(P_t^{(Y,\eta)} u - u)$ with limit in $L^2(\rho)$.
Let us define

$$V(\eta) := 1 + \frac{1}{2}|\eta|^2.$$

Then, according to the proof of [Dun13, Prop. 5.3.1], V is a Lyapunov function for the fast process $(Y^\varepsilon, \eta^\varepsilon)$ and we have the pointwise spectral-gap-type estimates of the form

$$|P_t^{(Y,\eta)} f(y,\eta) - \int f d\rho|^2 \leqslant K e^{-ct} |V(\eta)|^2 \quad \text{for all } t \geqslant 0, \tag{1.28}$$

for constants $K, c > 0$ (not depending on f) and for all $f : \mathbb{T}^2 \times \mathbb{R}^{2K} \to \mathbb{R}$ such that $|f(y,\eta)| \leqslant V(\eta)$, $(y,\eta) \in \mathbb{T}^2 \times \mathbb{R}^{2K}$. If we integrate the pointwise inequality (1.28) over (y, η) with respect to ρ, we obtain the $L^2(\rho)$-spectral-gap-type estimates for all such f, assuming $V \in L^2(\rho)$,

$$\|P_t^{(Y,\eta)} f - \int f d\rho\|^2_{L^2(\rho)} \leqslant K e^{-ct} \|V\|^2_{L^2(\rho)} \quad \text{for all } t \geqslant 0. \tag{1.29}$$

We will in particular apply the spectral gap estimates to $f = F$, which satisfies (1.14). Let us, similarly as in the previous section, define the H^1 space with respect to the generator $\mathscr{G}$ using the symmetric part $\mathscr{G}^S := \frac{1}{2}(\mathscr{G} + \mathscr{G}^\star)$, where $\mathscr{G}^\star$ is the $L^2(\rho)$-adjoint of $\mathscr{G}$,

$$H^1(\rho) := \{u \in dom(\mathscr{G}) \mid \langle(-\mathscr{G})u, u\rangle_\rho = \langle(-\mathscr{G}^S)u, u\rangle_\rho < \infty\}.$$

The scalar product in $H^1(\rho)$ is given by $\langle f, g\rangle_{H^1(\rho)} = \langle(-\mathscr{G}^S)f, g\rangle_{H^1(\rho)}$.
Then, as a consequence of the spectral gap estimates, we can solve the Poisson equation

$$(-\mathscr{G})u = g \tag{1.30}$$

explicitly with right-hand side g that has mean zero under ρ, $\langle g\rangle_\rho = 0$, and satisfies $|g| \leqslant V$ with $V \in L^2(\rho)$. The unique solution $u \in H^1(\rho)$ is given by $u = \int_0^\infty P_t^{(Y,\eta)} g dt \in L^2(\rho)$. In fact, for our tightness arguments, we will need a stronger integrability condition on the solution u and $\nabla_\eta u, \nabla_y u$, that is given by the following proposition.

Proposition 1.9. *Let $p \geqslant 2$ and let $g \in C^\infty(\mathbb{T}^2 \times \mathbb{R}^{2K}, \mathbb{R}^2)$ with*

$$|g(y, \eta)| \leqslant V(\eta) \tag{1.31}$$

(i.p. $g \in L^p(\rho)$) and with $\langle g\rangle_\rho = \int_{\mathbb{T}^2\times\mathbb{R}^{2K}} g(y, \eta)\rho(d(y, \eta)) = 0$. Then the Poisson equation

$$(-\mathscr{G})u = g$$

has a unique strong solution $u \in C^\infty(\mathbb{T}^2 \times \mathbb{R}^{2K}, \mathbb{R}^2)$ with the property that $\langle u\rangle_\rho = 0$. Moreover, there exists a constant $C > 0$, such that the solution satisfies $|u(y, \eta)| \leqslant CV(\eta)$ and

$$|\nabla_{(y,\eta)}u(y, \eta)| \leqslant |\nabla_y u(y, \eta)| + |\nabla_\eta u(y, \eta)| \leqslant 2CV(\eta). \tag{1.32}$$

In particular, it follows that $u \in W^{1,p}(\rho) = \{u \in L^p(\rho) \mid \nabla_{(y,\eta)}u \in L^p(\rho)\}$.

Proof. The solution $u = \int_0^\infty P_t^{(Y,\eta)} g dt$ is smooth, as g is assumed to be smooth, cf. also [Dun13, Proposition A.3.1]. It satisfies an analogue growth bound as g by the pointwise spectral gap estimates (1.28) with constant $C = K\int_0^\infty e^{-ct}dt \in (0, \infty)$, in particular $u \in L^p(\rho)$ for any $p \geqslant 1$. For the bound on the derivative, we proceed as in part (e) of the proof of [PV01, Theorem 1]. That is, we apply Sobolev embedding, the estimate (9.40) from [GT01] and the bound on g and u, such that for $p > d + 2K$ (notation: $B_{x,R} = \{z \in \mathbb{R}^d \times \mathbb{R}^{2K} \mid |z - x| \leqslant R\}$)

$$|\nabla_{(y,\eta)}u(y, \eta)| \leqslant C(\|u\|_{L^p(B_{(y,\eta),2})} + \|\mathscr{G}u\|_{L^p(B_{(y,\eta),2})}) \leqslant 2CV(\eta).$$

Notice also that, compared to [PV01], in our situation we have compactness in the y variable and the bound on g, u and ∇u is uniform in $y \in \mathbb{T}^d$. $\square$

In what follows, we will always assume the system (1.24) starts in stationarity, i.e. $(Y_0^\varepsilon, \eta_0^\varepsilon) \sim \rho$. In [DEPS15, Theorem 7] the authors prove the homogenization result for the process X^ε (see also [Dun13, chapter 5]), namely

$$X^\varepsilon \Rightarrow \sqrt{2D} Z,$$

where the convergence is in distribution in $C([0,T], \mathbb{R}^2)$ for $\varepsilon \to 0$, with a standard two-dimensional Brownian motion Z and

$$\begin{aligned} D = & \int (\mathrm{Id} + \nabla_y \chi(y,\eta))^T \Sigma(y,\eta)(\mathrm{Id} + \nabla_y \chi(y,\eta))\rho(dy, d\eta) \\ & + \int \nabla_\eta \chi(y,\eta)^T \Gamma \Pi \nabla_\eta \chi(y,\eta) \rho(dy, d\eta). \end{aligned}$$

Here $\mathrm{Id} \in \mathbb{R}^{2\times 2}$ is the identity matrix and χ is the solution of the Poisson equation $(-\mathscr{G})\chi = F$.
In order to obtain the homogenization result for the rough path lift of the process X^ε, we will use martingale methods (cf. [KLO12, Ch. 2]) applied to the stationary, ergodic Markov process $(Y^\varepsilon, \eta^\varepsilon)$ started in ρ. In addition we will exploit the decomposition of the additive functional in terms of Dynkin's martingale and the boundary term involving the solution of the Poisson equation (1.30).
To solve the Poisson equation with right-hand side F, we furthermore need that F is centered with respect to ρ. This was proven in [Dun13, Proposition 5.3.4]. We state that result in the following lemma.

Lemma 1.10. *For F from* (1.12) *and the invarinat probability measure ρ for $\mathscr{G}$, the following centering condition holds true*

$$\int_{\mathbb{T}^2 \times \mathbb{R}^{2K}} F(y,\eta)\rho(y,\eta) d(y,\eta) = 0. \tag{1.33}$$

We prove in the following lemma, that the density ρ that solves (1.27) is given by $\rho(y,\eta) = g_\eta(y) f(\eta)$, where f is the density of the normal distribution invariant for η and g solves the equation (1.34) below.

Lemma 1.11. *Let ρ be the probability measure with the density denoted also by ρ, such that it solves $\mathscr{G}^*\rho = 0$. Moreover, let $g_\eta(y)$ be the unique solution, satisfying $\int_{\mathbb{T}^2} g_\eta(y)dy = 1$ and $g_\eta(y) \geqslant 0$, to the equation*

$$(\mathscr{L}_0^* + \mathscr{L}_\eta) g_\eta(y) = 0 \tag{1.34}$$

for the adjoint operator $\mathscr{L}_0^* = \mathscr{L}_0^*(\eta)$ of $\mathscr{L}_0 = \mathscr{L}_0(\eta)$ with respect to $L^2(dy)$. Then the density ρ fulfills the disintegration formula

$$\rho(y,\eta) = g_\eta(y) f(\eta),$$

where

$$f(\eta) = \frac{1}{2\pi\sqrt{|\Pi|}} \exp\left(-\frac{1}{2}\eta^T \Pi^{-1}\eta\right).$$

In particular, the marginal distribution of ρ in the η-variable is the normal distribution $N(0, \Pi)$.

Proof. First we show that the form of the density ρ follows from the disintegration theorem from measure theory (see for example [DM78, chapter 3, 70 and 71]) and the invariance of ρ. Let $\pi : \mathbb{R}^{2K} \times \mathbb{T}^2 \to \mathbb{R}^{2K}, (\eta, y) \mapsto \eta$ be the projection and $\nu := \rho \circ \pi^{-1}$ the push-forward under ρ. Then the disintegration theorem implies that there exists a family of measures $(\mu_\eta)_{\eta \in \mathbb{R}^{2K}}$ on $\mathbb{T}^2$, such that:

- $\eta \mapsto \mu_\eta(A)$ is Borel measurable for each Borel measurable set $A \in \mathscr{B}(\mathbb{T}^2)$
- for every Borel measurable function $h : \mathbb{T}^2 \times \mathbb{R}^{2K} \to \mathbb{R}$,

$$\int_{\mathbb{T}^2 \times \mathbb{R}^{2K}} h(y, \eta)\rho(d(y, \eta)) = \int_{\mathbb{R}^{2K}} \int_{\mathbb{T}^2} h(y, \eta)\mu_\eta(dy)\nu(d\eta). \tag{1.35}$$

Since by assumption ρ has a density, which we also denote by ρ, it follows that ν has a density given by

$$\eta \mapsto \int \rho(y, \eta)dy =: f(\eta).$$

Consequently, also μ_η has a density, namely the conditional density

$$y \mapsto \mathbf{1}_{\{f>0\}}\rho(y, \eta)/f(\eta) =: g_\eta(y).$$

In order to prove that the marginal distribution ν under ρ is the normal distribution $N(0, \Pi)$, consider $h \in C_b(\mathbb{T}^2 \times \mathbb{R}^K, \mathbb{R})$ with $h(y, \eta) = h(\eta)$ not depending on y. Then, for $(Y_0, \eta_0) \sim \rho$ we have for any $t \geqslant 0$:

$$\mathbb{E}[h(\eta_t)] = \mathbb{E}[h(Y_t, \eta_t)] = \int h(y, \eta)d\rho = \int h(\eta)f(\eta)d\eta.$$

Hence f is given by the density of the $N(0, \Pi)$ distribution, as this is the unique invariant distribution for $(\eta_t)_t$.

It is left to derive the equation (1.34) for the density $g_\eta(y)$. For that we use the invariance of ρ and write

$$\begin{aligned} 0 = \mathscr{G}^*\rho &= (\mathscr{L}_0^* + \mathscr{L}_\eta^*)(g_\eta(y)f(\eta)) \\ &= f(\eta)\mathscr{L}_0^* g_\eta(y) + \mathscr{L}_\eta^*(g_\eta(y)f(\eta)) \\ &= f(\eta)\big(\mathscr{L}_0^* g_\eta(y) + \mathscr{L}_\eta g_\eta(y)\big), \end{aligned}$$

where we used that for any $h \in C^2(\mathbb{R}^{2K}, \mathbb{R})$,

$$\begin{aligned}
\mathscr{L}_\eta^*(h(\eta)f(\eta)) &= \nabla_\eta \cdot (h(\eta)\Gamma\eta f(\eta)) + \Gamma\Pi : \nabla_\eta\nabla_\eta(h(\eta)f(\eta)) \\
&= h(\eta)[\nabla_\eta \cdot (\Gamma\eta f(\eta)) + \Gamma\Pi : \nabla_\eta\nabla_\eta f(\eta)] \\
&\qquad + f(\eta)[\Gamma\eta \cdot \nabla_\eta h(\eta) + \Gamma\Pi : \nabla_\eta\nabla_\eta h(\eta)] \\
&\qquad + 2\nabla_\eta h(\eta) \cdot \Gamma\Pi\nabla_\eta f(\eta) \\
&= h(\eta)\mathscr{L}_\eta^* f(\eta) + f(\eta)\mathscr{L}_\eta h(\eta) \\
&\qquad + 2(\nabla_\eta h(\eta) \cdot \Gamma\Pi\nabla_\eta f(\eta) + f(\eta)\Gamma\eta \cdot \nabla_\eta h(\eta)) \\
&= h(\eta)\mathscr{L}_\eta^* f(\eta) + f(\eta)\mathscr{L}_\eta h(\eta) \\
&= f(\eta)\mathscr{L}_\eta h(\eta),
\end{aligned}$$

where we added and subtracted the term $f(\eta)\Gamma\eta \cdot \nabla_\eta h(\eta)$ and then used that $\nabla_\eta f(\eta) = -f(\eta)\Pi^{-1}\eta$ and $\mathscr{L}_\eta^* f = 0$. As $f > 0$, the above implies the equation (1.34) for $g_\eta(y)$. The uniqueness of the solution $g_\eta(y)$ in the class of probability densities in the y-variable follows from the uniqueness of the density ρ solving $\mathscr{G}^*\rho = 0$. Indeed, let $g^1, g^2 \in C^2(\mathbb{R}^{2K} \times \mathbb{T}^2, \mathbb{R})$ be positive such that $\int_{\mathbb{T}^2} g^i(\eta, y)dy = 1$ and they solve

$$(\mathscr{L}_0^* + \mathscr{L}_\eta)g^i(\eta, y) = 0 \quad \text{for } i = 1, 2.$$

Then, setting $\rho^i(\eta, y) := g^i(\eta, y)f(\eta)$ for $i = 1, 2$, we obtain probability densities of a probability measure on $\mathbb{R}^{2K} \times \mathbb{T}^2$ solving $\mathscr{G}^*\rho^i = 0$ for $i = 1, 2$. As a consequence, $\rho^1 = \rho^2 = \rho$, which implies $g^1 = g^2$. □

1.3.1. Determining the limit rough path

In this subsection, we prove convergence of the Itô integrals

$$\int_s^t (X_r^\varepsilon - X_s^\varepsilon) \otimes dX_r^\varepsilon = \left(\int_s^t (X_r^{\varepsilon,i} - X_s^{\varepsilon,i}) dX_r^{\varepsilon,j} \right)_{i,j=1,2}$$

and determine the limit. In order to obtain the limit, we will use a decomposition of $X^\varepsilon(t)$ via the solution χ of the Poisson equation $\mathscr{G}\chi = -F$, which exists by Proposition 1.9. Rewriting the drift term using Itô-formula for $\chi(Y_t^\varepsilon, \eta_t^\varepsilon)$, we obtain,

$$\frac{1}{\varepsilon} \int_0^t F(Y_s^\varepsilon, \eta_s^\varepsilon) ds = -\Big(\varepsilon\big(\chi(Y_t^\varepsilon, \eta_t^\varepsilon) - \chi(Y_0^\varepsilon, \eta_0^\varepsilon)\big) - \tilde{M}_t^\varepsilon \Big),$$

where

$$\begin{aligned}
\tilde{M}_t^{\varepsilon,i} &:= \tilde{M}_1^{\varepsilon,i}(t) + \tilde{M}_2^{\varepsilon,i}(t) \\
&:= \int_0^t \nabla_y \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon) \cdot \sqrt{2\Sigma}(Y_s^\varepsilon, \eta_s^\varepsilon) dB_s + \int_0^t \nabla_\eta \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon) \cdot \sqrt{2\Gamma\Pi} dW_s
\end{aligned}$$

for $i = 1, 2$. As a consequence, we have

$$X_t^\varepsilon = X_0^\varepsilon + \varepsilon \left(\chi\left(Y_0^\varepsilon, \eta_0^\varepsilon\right) - \chi(Y_t^\varepsilon, \eta_t^\varepsilon)\right) + M_1^\varepsilon(t) + M_2^\varepsilon(t), \tag{1.36}$$

where martingale terms $M^\varepsilon := M_1^\varepsilon + M_2^\varepsilon$ are given by

$$M_1^{\varepsilon,i}(t) := \int_0^t (\nabla_y \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon) + e_i) \cdot \sqrt{2\Sigma}(Y_s^\varepsilon, \eta_s^\varepsilon) dB_s, \tag{1.37}$$

$$M_2^{\varepsilon,i}(t) := \int_0^t \nabla_\eta \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon) \cdot \sqrt{2\Gamma\Pi} dW_s, \tag{1.38}$$

for $i = 1, 2$. Using the dynamics of X^ε, we decompose the iterated integrals for $i, j \in \{1, 2\}$,

$$\int_0^t (X_s^{\varepsilon,i} - X_0^{\varepsilon,i}) dX_s^{\varepsilon,j} = \int_0^t (X_s^{\varepsilon,i} - X_0^{\varepsilon,i}) \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_s^\varepsilon, \eta_s^\varepsilon) dB_s^l \tag{1.39}$$

$$+ \int_0^t (X_s^{\varepsilon,i} - X_0^{\varepsilon,i}) \frac{1}{\varepsilon} F^j(Y_s^\varepsilon, \eta_s^\varepsilon) ds. \tag{1.40}$$

The next step is to rewrite the terms (1.39) and (1.40) collecting the vanishing and non-vanishing terms.

First we consider the term (1.39) and plug in the decomposition (1.36) of X^ε. We obtain

$$\begin{aligned}
&\int_0^t (X_s^{\varepsilon,i} - X_0^{\varepsilon,i}) \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_s^\varepsilon, \eta_s^\varepsilon) dB_s^l \\
&= \varepsilon \int_0^t (\chi^i(Y_0^\varepsilon, \eta_0^\varepsilon) - \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_s^\varepsilon, \eta_s^\varepsilon) dB_s^l \\
&\quad + \int_0^t M_s^{\varepsilon,i} \sum_{l=1}^2 \sqrt{2\Sigma_{j,l}(Y_s^\varepsilon, \eta_s^\varepsilon)} dB_s^l \\
&= \varepsilon \int_0^t (\chi^i(Y_0^\varepsilon, \eta_0^\varepsilon) - \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_s^\varepsilon, \eta_s^\varepsilon) dB_s^l \\
&\quad + M_t^{\varepsilon,i} \left(\int_0^t \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_r^\varepsilon, \eta_r^\varepsilon) dB_r^l \right) \\
&\quad - \int_0^t \left(\int_0^s \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_r^\varepsilon, \eta_r^\varepsilon) dB_r^l \right) dM_s^{\varepsilon,i} \\
&\quad - \left\langle \int_0^\cdot \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_r^\varepsilon, \eta_r^\varepsilon) dB_r^l, M^{\varepsilon,i} \right\rangle_t .
\end{aligned} \tag{1.41}$$

By stationarity of $(Y^\varepsilon,\eta^\varepsilon)$ and the boundedness (1.13) of Σ, we notice that the first summand in the decomposition (1.41) will converge in $L^2(\mathbb{P})$ to zero. Moreover, for the quadratic variation term, we can argue with the ergodic theorem for (Y,η), [DPZ96, Theorem 3.3.1], obtaining the convergence in probability

$$\begin{aligned}
&\mathbb{P}\left(\left|\left\langle \int_0^\cdot \sum_{l=1}^2 \sqrt{2\Sigma}(j,l)(Y_r^\varepsilon,\eta_r^\varepsilon)dB_r^l, M^{\varepsilon,i}\right\rangle_t - t\int e_j\cdot 2\Sigma(e_i+\nabla_y\chi^i)d\rho\right|>\delta\right)\\
&\quad=\mathbb{P}\left(\left|\int_0^t e_j\cdot 2\Sigma(e_i+\nabla_y\chi^i)(Y_s^\varepsilon,\eta_s^\varepsilon)ds - t\int e_j\cdot 2\Sigma(e_i+\nabla_y\chi^i)d\rho\right|>\delta\right)\\
&\quad=\tilde{\mathbb{P}}\left(\left|\varepsilon^2\int_0^{\varepsilon^{-2}t} e_j\cdot 2\Sigma(e_i+\nabla_y\chi^i)(Y_s,\eta_s)ds - t\int e_j\cdot 2\Sigma(e_i+\nabla_y\chi^i)d\rho\right|>\delta\right)\\
&\quad\to 0, \qquad (1.42)
\end{aligned}$$

as $\varepsilon\to 0$, for any $\delta>0$, using that $(Y_t^\varepsilon,\eta_t^\varepsilon)_{t\geqslant 0}\overset{d}{=}(Y_{\varepsilon^{-2}t},\eta_{\varepsilon^{-2}t})_{t\geqslant 0}$, where $(Y_t,\eta_t)_{t\geqslant 0}$ is the Markov process with generator $\mathscr{G}$ with respect to some base probability measure $\tilde{\mathbb{P}}$. To deduce the convergence of the remaining two martingale terms in (1.41), we will add them up with the decomposition of the term (1.40) below.

We decompose the term (1.40) in the following way

$$\begin{aligned}
\int_0^t (X_s^{\varepsilon,i}-X_0^{\varepsilon,i})\frac{1}{\varepsilon}F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds &= \int_0^t(\chi^i(Y_0^\varepsilon,\eta_0^\varepsilon)-\chi^i(Y_s^\varepsilon,\eta_s^\varepsilon))F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds \qquad (1.43)\\
&\quad+\int_0^t M_s^{\varepsilon,i}\frac{1}{\varepsilon}F^j(Y_s^\varepsilon,\eta_s^\varepsilon)dt. \qquad (1.44)
\end{aligned}$$

For the first term in (1.43) we again apply the ergodic theorem for (Y,η), yielding the convergence in probability, analogously as above,

$$\begin{aligned}
&\int_0^t(\chi^i(Y_0^\varepsilon,\eta_0^\varepsilon)-\chi^i(Y_s^\varepsilon,\eta_s^\varepsilon))F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds\\
&\quad\overset{d}{=}\chi^i(Y_0^\varepsilon,\eta_0^\varepsilon)\varepsilon^2\int_0^{\varepsilon^{-2}t}F^j(Y_r,\eta_r)dr-\varepsilon^2\int_0^{\varepsilon^{-2}t}\chi^iF^j(Y_r,\eta_r)dr\\
&\quad\to t(\chi^i(Y_0^\varepsilon,\eta_0^\varepsilon)E_\rho[F^j]-E_\rho[\chi^iF^j])\\
&\quad= tE_\rho[\chi^i(-F)^j]=:ta_F(i,j),
\end{aligned}$$

where we used that F has mean zero under ρ by Lemma 1.10 and we introduced the notation $E_\rho[f]:=\int f(y,\eta)\rho(y,\eta)d(y,\eta)$.

For the second term (1.44), we apply the integration by parts formula to further rewrite

$$\int_0^t M_s^{\varepsilon,i}\varepsilon^{-1}F^j\left(Y_s^\varepsilon,\eta_s^\varepsilon\right)ds = M_t^{\varepsilon,i}\int_0^t\varepsilon^{-1}F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds-\int_0^t\left(\int_0^s\varepsilon^{-1}F^j(Y_r^\varepsilon,\eta_r^\varepsilon)dr\right)dM_s^{\varepsilon,i}. \qquad (1.45)$$

Let a_F^ε be defined as

$$a_F^\varepsilon(i,j) := \varepsilon \int_0^t (\chi^i(Y_0^\varepsilon, \eta_0^\varepsilon) - \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) d\Big(\int_0^\cdot \sqrt{2\Sigma}(Y^\varepsilon, \eta^\varepsilon) dB\Big)_s^j + \int_0^t (\chi^i(Y_0^\varepsilon, \eta_0^\varepsilon) - \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) F^j(Y_s^\varepsilon, \eta_s^\varepsilon) ds.$$

Then, using the definition of a_F^ε and summing up the two remaining terms in (1.41) and the terms in (1.45), we get

$$\int_0^t (X_s^{\varepsilon,i} - X_0^{\varepsilon,i}) dX_s^{\varepsilon,j} = a_F^\varepsilon(i,j) + M_t^{\varepsilon,i}(X_t^{\varepsilon,j} - X_0^{\varepsilon,j}) - \int_0^t (X_s^{\varepsilon,j} - X_0^{\varepsilon,j}) dM_s^{\varepsilon,i} - \Big\langle M^{\varepsilon,i}, \Big(\int_0^\cdot \sqrt{2\Sigma}(Y_r^\varepsilon, \eta_r^\varepsilon) dB_r\Big)^j \Big\rangle_t. \tag{1.46}$$

To obtain the limit in distribution, we utilize the convergence of a_F^ε in probability proven above (with same limit as for the term in (1.43)), the convergence of the quadratic variation term in (1.42) and Proposition 1.6 for the remaining terms. Then Slutzky's lemma ensures that the sum of a random variable converging in distribution and a random variable converging in probability, converges in distribution to the sum of the limits. To apply Proposition 1.6, we check that the UCV condition is satisfied for $(M^{\varepsilon,i})$ and that $(M^{\varepsilon,i}, X^{\varepsilon,j}) \Rightarrow (X^i - X_0^i, X^j)$ jointly in distribution. Here, the joint convergence is due the decomposition (1.36) and the convergence for the process by [DEPS15, Theorem 7]. To show the UCV condition, we utilize the stationarity of (Y, η) and $(Y_t^\varepsilon, \eta_t^\varepsilon)_t \overset{d}{=} (Y_{\varepsilon^{-2}t}, \eta_{\varepsilon^{-2}t})_t$, such that

$$\mathbb{E}[\langle \tilde{M}^{\varepsilon,i} \rangle_t] = t \int [(\nabla_y \chi^i)^T 2\Sigma \nabla_y \chi^i + (\nabla_\eta \chi^i)^T 2\Gamma\Pi \nabla_\eta \chi^i] d\rho < \infty.$$

Here, the right-hand side is finite due to Proposition 1.9 and boundedness of Σ from (1.13) and the bound does not depend on ε, such that the UCV condition for $(M^{\varepsilon,i})_\varepsilon$ is satisfied.

Altogether, we thus obtain the distributional convergence,

$$\begin{aligned}\int_0^t (X_s^{\varepsilon,i} - X_0^{\varepsilon,i})dX_s^{\varepsilon,j} &= a_F^\varepsilon(i,j) + M_t^{\varepsilon,i}(X_t^{\varepsilon,j} - X_0^{\varepsilon,j}) - \int_0^t (X_s^{\varepsilon,j} - X_0^{\varepsilon,j})dM_s^{\varepsilon,i} \\ &\quad - \Big\langle M^{\varepsilon,i}, \Big(\int_0^\cdot \sqrt{2\Sigma}(Y_r^\varepsilon, \eta_r^\varepsilon)dB_r\Big)^j\Big\rangle_t \\ &\Rightarrow ta_F(i,j) + (X_t^i - X_0^i)(X_t^j - X_0^j) - \int_0^t (X_s^j - X_0^j)dX_s^i \\ &\quad - t\int e_j \cdot 2\Sigma(e_i + \nabla_y\chi^i)d\rho \\ &= ta_F(i,j) + \langle X^i, X^j\rangle_t - t\int (e_i + \nabla_y \chi^i)\cdot 2\Sigma e_j d\rho \\ &\quad + \int_0^t (X_s^i - X_0^i)dX_s^j.\end{aligned}$$

The arguments can also be generalized to a different base-point $s > 0$ in the same manner as for $s = 0$ above, such that we obtain the weak limit of the iterated integrals $\mathbb{X}_{s,t}^\varepsilon(i,j)$, which decomposes in the iterated integrals $\mathbb{X}_{s,t}(i,j)$ of the Brownian motion $X = \sqrt{2D}Z$ plus an area correction term. Furthermore, the joint distributional convergence of $((\mathbb{X}_{s,t}^\varepsilon(i,j))_{i,j=1,2})_\varepsilon$ follows from the decomposition (1.46) and joint distributional convergence of $((M^{\varepsilon,i}, X^{\varepsilon,j})_{i,j=1,2})_\varepsilon$, which relies on joint convergence of $((X^{\varepsilon,i})_{i=1,2})_\varepsilon$ by [DEPS15, Theorem 7].
The following proposition summarizes our findings.

Proposition 1.12. *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$ solve the system (1.24) for $(Y_0^\varepsilon, \eta_0^\varepsilon) \sim \rho$. Then for all $s, t \in \Delta_T$, the iterated Itô-integrals $(\mathbb{X}_{s,t}^\varepsilon)$ convergence weakly in $\mathbb{R}^{2\times 2}$ as $\varepsilon \to 0$, i.e. for $i, j \in \{1, 2\}$,*

$$\begin{aligned}\mathbb{X}_{s,t}^\varepsilon(i,j) := \int_s^t X_{s,r}^{\varepsilon,i}dX_r^{\varepsilon,j} \Rightarrow \mathbb{X}_{s,t}(i,j) + (t-s)\Big(&\langle X^i, X^j\rangle_1 + \langle \chi^i, (\mathscr{G}\chi)^j\rangle_\rho \\ &- \int (e_i + \nabla_y\chi^i)\cdot 2\Sigma e_j d\rho\Big),\end{aligned} \tag{1.47}$$

where

$$X = \sqrt{2D}Z, \quad \mathscr{G}\chi = -F, \quad e_1 = (1,0), \quad e_2 = (0,1),$$

for a standard two-dimensional Brownian motion Z, $\mathbb{X}_{s,t}(i,j) := \int_s^t X_{s,r}^i dX_r^j$ and

$$D = \int (\mathrm{Id} + \nabla_y\chi)^T \Sigma(\mathrm{Id} + \nabla_y \chi)\rho(dy, d\eta) + \int (\nabla_\eta \chi)^T \Gamma\Pi\nabla_\eta\chi\rho(dy, d\eta). \tag{1.48}$$

Corollary 1.13. *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$ be as in Proposition 1.12. Then for all $s, t \in \Delta_T$ also the iterated Stratonovich integrals $(\tilde{\mathbb{X}}_{s,t}^\varepsilon)$ converge weakly in $\mathbb{R}^{2\times 2}$ as $\varepsilon \to 0$, i.e.*

for $i,j\in\{1,2\}$,

$$\tilde{\mathbb{X}}^{\varepsilon}_{s,t}(i,j):=\int_s^t X^{\varepsilon,i}_{s,r}\circ dX^{\varepsilon,j}_r \Rightarrow \tilde{\mathbb{X}}_{s,t}(i,j)+(t-s)\tilde{A}(i,j), \tag{1.49}$$

where

$$X=\sqrt{2D}Z,\quad \tilde{\mathbb{X}}_{s,t}(i,j):=\int_s^t X^i_{s,r}\circ dX^j_r$$

for a standard two-dimensional Brownian motion Z *and* D *is given by* (1.48). *Furthermore, the area correction is given by*

$$\tilde{A}(i,j)=\langle\chi^i,\mathscr{G}^A\chi^j\rangle_\rho+\int e_i\cdot\Sigma\nabla_y\chi^j d\rho-\int\nabla_y\chi^i\cdot\Sigma e_j d\rho.$$

for $\mathscr{G}^A:=\frac{1}{2}(\mathscr{G}-\mathscr{G}^\star)$ *with* $L^2(\rho)$*-adjoint* $\mathscr{G}^\star$ *of* $\mathscr{G}$.

Proof. Recall the relation between the Itô and Stratonovich integral

$$\tilde{\mathbb{X}}^{\varepsilon}_{0,t}(i,j)=\mathbb{X}^{\varepsilon}_{0,t}(i,j)+\frac{1}{2}\langle X^{\varepsilon,i},X^{\varepsilon,j}\rangle_t.$$

The ergodic theorem for (Y,η), [DPZ96, Theorem 3.3.1], together with $(Y^\varepsilon_t,\eta^\varepsilon_t)_{t\geqslant 0}\overset{d}{=}(Y_{\varepsilon^{-2}t},\eta_{\varepsilon^{-2}t})_{t\geqslant 0}$, implies convergence in probability of the quadratic variation:

$$\frac{1}{2}\langle X^{\varepsilon,i},X^{\varepsilon,j}\rangle_t=\int_0^t e_i\cdot\Sigma e_j(Y^\varepsilon_s,\eta^\varepsilon_s)ds\to t\int e_i\cdot\Sigma e_j d\rho.$$

Thus we obtain, from Proposition 1.12 and Lemma 1.11, the following convergence in distribution:

$$\begin{aligned}\tilde{\mathbb{X}}^{\varepsilon}_{0,t}(i,j)&=\mathbb{X}^{\varepsilon}_{0,t}(i,j)+\frac{1}{2}\langle X^{\varepsilon,i},X^{\varepsilon,j}\rangle_t\\&\Rightarrow\mathbb{X}_{0,t}(i,j)+t\langle\chi^i,(\mathscr{G}\chi)^j\rangle_\rho+\langle X^i,X^j\rangle_t\\&\qquad-t\int(\nabla_y\chi^i+e_i)\cdot 2\Sigma e_j d\rho+t\int e_i\cdot\Sigma e_j d\rho\\&=\mathbb{X}_{0,t}(i,j)+\frac{1}{2}\langle X^i,X^j\rangle_t\\&\qquad+t\Big(\langle\chi^i,(\mathscr{G}\chi)^j\rangle_\rho+\frac{1}{2}\langle X^i,X^j\rangle_1\\&\qquad\qquad-\int\nabla_y\chi^i\cdot 2\Sigma e_j d\rho-\int e_i\cdot\Sigma e_j d\rho\Big)\\&=\tilde{\mathbb{X}}_{0,t}(i,j)+t\tilde{A}(i,j).\end{aligned}$$

The area correction can furthermore be written as

$$\begin{aligned}\tilde{A}(i,j) &= \langle \chi^i, \mathscr{G}\chi^j\rangle_\rho + D(i,j) - \int \nabla_y \chi^i \cdot 2\Sigma e_j d\rho - \int e_i \cdot \Sigma e_j d\rho \\ &= \langle \chi^i, \mathscr{G}^A \chi^j\rangle_\rho + \int e_i \cdot \Sigma \nabla_y \chi^j d\rho - \int \nabla_y \chi^i \cdot \Sigma e_j d\rho,\end{aligned}$$

using that $\mathscr{G} = \mathscr{G}^A + \mathscr{G}^S$ for $\mathscr{G}^S := \frac{1}{2}(\mathscr{G} + \mathscr{G}^\star)$ and $\mathscr{G}^A := \frac{1}{2}(\mathscr{G} - \mathscr{G}^\star)$ and that by [KLO12, section 2.4], we have a correspondence between the quadratic variation of Dynkin's martingale $\tilde{M}^\varepsilon$ and the $\mathscr{H}^1(\rho)$-norm of χ (utilizing again stationarity of (Y,η)), such that

$$\begin{aligned}\langle \chi^i, \mathscr{G}^S \chi^j\rangle_\rho = -\langle \chi^i, \chi^j\rangle_{H^1(\rho)} &= -\frac{1}{2}\mathbb{E}[\langle \tilde{M}^{\varepsilon,i}, \tilde{M}^{\varepsilon,j}\rangle_1] \\ &= -\int [\nabla_y \chi^i \cdot \Sigma \nabla_y \chi^j + \nabla_\eta \chi^i \cdot \Gamma\Pi \nabla_\eta \chi^j] d\rho.\end{aligned}$$

For a base-point $s > 0$ we can pursue with an analogue argument. □

Remark 1.14. *We expect (without proof) that in fact $\tilde{A}(i,j) = \langle \chi^i, \mathscr{G}^A \chi^j\rangle_\rho$ holds true and that $\mathscr{G}^A$ is a nontrivial operator, such that the Stratonovich area correction is truely non-vanishing. The difficulty in verifying this is that the density $g_\eta(y)$ remains non-explicit. Typically (cf. [DOP21]), the area correction in the Stratonovich case can be expressed in terms of the asymmetric part of the generator of the underlying Markov process, which is a non-trivial operator for a non-reversible Markov process.*

1.3.2. Tightness

For convergence in distribution of the lift $(X^\varepsilon, \mathbb{X}^\varepsilon)$ to the respective lift of X, it remains to prove tightness in the rough path space utilizing Lemma 1.7. We verify the moment bounds in the next proposition in order to apply the Kolmogorov criterion, Lemma 1.7.

Proposition 1.15. *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$ be as in Proposition 1.12. Then the following moment bounds hold true for any $p \geqslant 2$, $i,j \in \{1,2\}$,*

$$\sup_\varepsilon \mathbb{E}[|X_t^{\varepsilon,i} - X_s^{\varepsilon,i}|^p] \lesssim |t-s|^{p/2}, \quad \textit{for } s,t \in \Delta_T \tag{1.50}$$

and

$$\sup_\varepsilon \mathbb{E}\left[\left|\int_s^t X_{s,r}^{\varepsilon,i} dX_r^{\varepsilon,j}\right|^{p/2}\right] \lesssim |t-s|^{p/2}, \quad \textit{for } s,t \in \Delta_T. \tag{1.51}$$

In particular, tightness of $(X^\varepsilon, \mathbb{X}^\varepsilon)$ in $C_{\gamma,T}$ for $\gamma < 1/2$ follows.

Proof. Let $p \geqslant 2$. First, utilizing the growth condition (1.14) on F, finiteness of moments of η, Burkholder-Davis-Gundy inequality and boundedness of Σ in (1.13), we obtain

$$\begin{aligned}\mathbb{E}[|X_t^\varepsilon - X_s^\varepsilon|^p] &\leqslant \mathbb{E}\left[\left|\varepsilon^{-1}\int_s^t F(Y_s^\varepsilon,\eta_s^\varepsilon)ds\right|^p\right] + \mathbb{E}\left[\left|\int_s^t \sqrt{2\Sigma}(Y_s^\varepsilon,\eta_s^\varepsilon)dB_s\right|^p\right]\\ &\lesssim \varepsilon^{-p}|t-s|^p + |t-s|^{p/2}. \end{aligned} \tag{1.52}$$

Secondly, using the representation (1.46), we conclude

$$\mathbb{E}[|X_t^{\varepsilon,i} - X_s^{\varepsilon,i}|^p] \leqslant \varepsilon^p \mathbb{E}[|\chi^i(Y^\varepsilon(t),\eta^\varepsilon(t)) - \chi^i(Y^\varepsilon(s),\eta^\varepsilon(s))|^p] + \mathbb{E}[|M^{\varepsilon,i}(t) - M^{\varepsilon,i}(s)|^p],$$

where the martingale is given by $M^\varepsilon = M_1^\varepsilon + M_2^\varepsilon$ with

$$\begin{aligned} M_1^{\varepsilon,i}(t) &:= \int_0^t (\nabla_y \chi^i + e_i)\cdot\sqrt{2\Sigma}(Y^\varepsilon(s),\eta^\varepsilon(s))dB_s\\ M_2^{\varepsilon,i}(t) &:= \int_0^t \nabla_\eta\chi^i(Y^\varepsilon(s),\eta^\varepsilon(s))\cdot\sqrt{2\Gamma\Pi}dW_s. \end{aligned}$$

Burkholder-Davis-Gundy and the Minkowski inequality, together with stationarity of $(Y^\varepsilon,\eta^\varepsilon)$ yield

$$\mathbb{E}[|M^\varepsilon(t) - M^\varepsilon(s)|^p] \lesssim |t-s|^{p/2}\int |(\nabla_y\chi + \mathrm{Id})^T\Sigma(\nabla_y\chi+\mathrm{Id}) + (\nabla_\eta\chi)^T\Gamma\Pi\nabla_\eta\chi|^{p/2}d\rho.$$

Boundedness of Σ, stated in (1.13), and Proposition 1.9, i.e. $\nabla_y\chi + \nabla_\eta\chi \in L^p(\rho)$ for any $p \geqslant 2$, imply finiteness of the expectation on the right-hand side.
Moreover, according to Proposition 1.9, χ satisfies a growth condition in η, which we use to estimate the boundary term (as well as stationarity of η^ε and that η_0 has all moments under ρ, as it is the normal distribution $N(0,\Pi)$), obtaining

$$\mathbb{E}[|X_t^{\varepsilon,i} - X_s^{\varepsilon,i}|^p] \lesssim \varepsilon^p + |t-s|^{p/2}. \tag{1.53}$$

By combining the estimates (1.52) and (1.53), using the first one for $\varepsilon > |t-s|^{1/2}$ and the latter for $\varepsilon < |t-s|^{1/2}$, we conclude

$$\sup_{\varepsilon\in[0,1]} \mathbb{E}[|X_t^{\varepsilon,i} - X_s^{\varepsilon,i}|^p] \lesssim |t-s|^{p/2}.$$

The estimate for the iterated integrals is then immediate by the estimate on the moments of $X^{\varepsilon,j}$ and the decomposition of the iterated integral in (1.46), as well as the boundedness of the quadratic variation of the martingale $M^{\varepsilon,i}$ in $L^{p/2}(\rho)$ for any $p \geqslant 2$.
The conclusion on tightness in $C_{\gamma,T}$ for $\gamma < 1/2$ follows from Lemma 1.7. □

1.3.3. Rough homogenization limit

In this subsection, we prove our first main theorem of this chapter.

Theorem 1.16 (Itô lift). *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$, X and $\mathbb{X}^\varepsilon, \mathbb{X}$ be as in Proposition 1.12. Then for any $\gamma < 1/2$, $(X^\varepsilon, \mathbb{X}^\varepsilon)$ weakly converges in the γ-Hölder rough paths space as $\varepsilon \to 0$,*

$$(X^\varepsilon, \mathbb{X}^\varepsilon) \Rightarrow (X, (s,t) \mapsto \mathbb{X}_{s,t} + A(t-s)), \tag{1.54}$$

with matrix $A = (A(i,j))_{i,j\in\{1,2\}}$ where

$$A(i,j) = \langle \chi^i, (\mathscr{G}\chi)^j \rangle_\rho + \langle X^i, X^j \rangle_1 - \int (e_i + \nabla_y \chi^i) \cdot 2\Sigma e_j d\rho \tag{1.55}$$

and the solution χ of $\mathscr{G}\chi = -F$.

Proof. From Proposition 1.12 the convergence of the one dimensional distributions of $(X^\varepsilon, \mathbb{X}^\varepsilon)_\varepsilon$, that is of (X^ε_t) and $(\mathbb{X}^\varepsilon_{s,t})_\varepsilon$ for any $0 \leqslant s < t \leqslant T$, follows. By weak convergence of (X^ε), it follows in particular convergence of the finite dimensional distributions. For the finite dimensional distributions of $\mathbb{X}^\varepsilon$, we use the same argument as for the one dimensional distributions. Indeed, for (1.46) above, the convergence of the part for which we applied the UCV condition is also true in the sense of weak convergence of processes in $C(\Delta_T, \mathbb{R}^{2\times 2})$ by Proposition 1.6 and the remaining terms converge in probability. Furthermore, Proposition 1.15 yields tightness in the rough path space $C_{\gamma,T}$ for $\gamma < 1/2$. Together, we obtain the weak convergence in $C_{\gamma,T}$ for $\gamma < 1/2$:

$$(X^\varepsilon, \mathbb{X}^\varepsilon) \Rightarrow (X, (s,t) \mapsto \mathbb{X}_{s,t} + A(t-s)),$$

as claimed. □

Corollary 1.17 (Stratonovich lift). *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$, X and $\tilde{\mathbb{X}}^\varepsilon, \tilde{\mathbb{X}}$ be as in Corollary 1.13. Then for any $\gamma < 1/2$, we have the weak convergence in the rough path space $C_{\gamma,T}$,*

$$(X^\varepsilon, \tilde{\mathbb{X}}^\varepsilon) \Rightarrow (X, (s,t) \mapsto \tilde{\mathbb{X}}_{s,t} + \tilde{A}(t-s)), \tag{1.56}$$

as $\varepsilon \to 0$, where

$$\tilde{A}(i,j) = \langle \chi^i, \mathscr{G}^A \chi^j \rangle_\rho + \int e_i \cdot \Sigma \nabla_y \chi^j d\rho - \int \nabla_y \chi^i \cdot \Sigma e_j d\rho, \quad \mathscr{G}\chi = -F.$$

Proof. The proof follows immediately from Corollary 1.13 and Theorem 1.16. □

1.4. Membrane with comparable spatial and temporal fluctuations

In this section we consider the space-time scaling regime for $\alpha = \beta = 1$, which means that we observe spatial and temporal fluctuations of the membrane with comparable size. In this case, the general system (1.15) becomes

$$\begin{cases} dX_t^\varepsilon = \dfrac{1}{\varepsilon}F(Y_t^\varepsilon,\eta_t^\varepsilon)dt + \sqrt{2\Sigma(Y_t^\varepsilon,\eta_t^\varepsilon)}dB_t, \\ dY_t^\varepsilon = \dfrac{1}{\varepsilon^2}F(Y_t^\varepsilon,\eta_t^\varepsilon)dt + \sqrt{\dfrac{2}{\varepsilon^2}\Sigma(Y_t^\varepsilon,\eta_t^\varepsilon)}dB_t, \\ d\eta_t^\varepsilon = -\dfrac{1}{\varepsilon}\Gamma\eta_t^\varepsilon dt + \sqrt{\dfrac{2}{\varepsilon}\Gamma\Pi}dW_t, \end{cases} \tag{1.57}$$

where B and W are independent Brownian motions and we define again $Y^\varepsilon := \varepsilon^{-1}X^\varepsilon \bmod \mathbb{Z}^2$.
To determine the limit in this regime, the difficulty is that Y and η now fluctuate at different scales, which means that, compared to the previous section, we no longer obtain a generator with fluctuations of the same order for the joint Markov process $(Y^\varepsilon,\eta^\varepsilon)$, but the generator $\varepsilon^{-2}\mathscr{L}_0+\varepsilon^{-1}\mathscr{L}_\eta$ for $\mathscr{L}_0=\mathscr{L}_0(\eta)$ from (1.26) and $\mathscr{L}_\eta$ from (1.7). The idea is to first deduce a quenched result for each fixed environment η and afterwards average over the invariant measure ρ_η of η.
Let, for $\eta\in\mathbb{R}^{2K}$,

$$\rho_Y(dy,\eta) := c^{-1}\sqrt{|\Sigma^{-1}|(y,\eta)}dy$$

be a probability measure on $\mathbb{T}^2$ with normalizing constant $c:=\int_{\mathbb{T}^2}\sqrt{|\Sigma^{-1}|(y,\eta)}dy>0$. It is straightforward to verify that $\rho_Y(\cdot,\eta)$ is invariant for $\mathscr{L}_0(\eta)$, i.e. $\langle\mathscr{L}_0(\eta)f\rangle_{\rho_Y(\cdot,\eta)}=0$ for all $f\in dom(\mathscr{L}_0)\subset L^2(\rho_Y(\cdot,\eta))$, and that $\int F(y,\eta)\rho_Y(dy,\eta)=0$ by the definition (1.12) of F.
Let now for any $\eta\in\mathbb{R}^K$, $\chi(\cdot,\eta)\in C^2(\mathbb{T}^2,\mathbb{R}^2)$ be the unique solution of

$$\mathscr{L}_0(\eta)\chi(\cdot,\eta) = -F(\cdot,\eta), \tag{1.58}$$

with $\int\chi(y,\eta)\rho_Y(dy,\eta)=0$. Existence of χ follows from the $L^2(\rho_Y(\cdot,\eta))$-spectral gap estimates for $\mathscr{L}_0(\eta)$ from [Dun13, Lemma A.2.1, Proposition A.2.2]. That is, there exists a constant $\lambda(\eta)>0$ such that for all $f\in L^2(\rho_Y(\cdot,\eta))$,

$$\|P_t^0 f-\langle f\rangle_{\rho_Y(\cdot,\eta)}\|_{L^2(\rho_Y(\cdot,\eta))} \leqslant e^{-\lambda(\eta)t}\|f\|_{L^2(\rho_Y(\cdot,\eta))},$$

where $(P_t^0)_{t\geqslant 0}=(P_t^0(\eta))_{t\geqslant 0}$ denotes the semigroup on $L^2(\rho_Y(\cdot,\eta))$ associated to $\mathscr{L}_0(\eta)$. Furthermore, according to [Dun13, Proposition A.2.2], the solution χ will be smooth in the η-variable (as F is smooth) and satisfies $|\nabla_\eta^k\chi(y,\eta)|\leqslant C_k(1+|\eta|^{l_k})$ for $k=0,1,2$ with $l_k\geqslant 1$ and constant $C_k>0$ (since F satisfies such a growth condition with a possibly different constant C_k).

Then, we can decompose the drift part of X^ε with the help of χ, yielding

$$\begin{aligned}\varepsilon^{-1}\int_0^t F^i(Y_s^\varepsilon,\eta_s^\varepsilon)ds &= \varepsilon(\chi^i(Y_0^\varepsilon,\eta_0^\varepsilon)-\chi^i(Y_t^\varepsilon,\eta_t^\varepsilon))+\sqrt{\varepsilon}\int_0^t \nabla_\eta\chi^i(Y_r^\varepsilon,\eta_r^\varepsilon)\cdot\sqrt{2\Gamma\Pi}dW_r \\ &+\int_0^t \nabla_y\chi^i(Y_r^\varepsilon,\eta_r^\varepsilon)\cdot\sqrt{2\Sigma(Y_r^\varepsilon,\eta_r^\varepsilon)}dB_r \\ &+\int_0^t (\mathscr{L}_\eta\chi)^i(Y_s^\varepsilon,\eta_s^\varepsilon)ds \end{aligned}\tag{1.59}$$

for $i=1,2$. Plugging (1.59) into the dynamics for X^ε, one can deduce that (X^ε) converges in distribution to a Brownian motion with variance $2Dt$ for the matrix D given below, plus a constant drift C. This was proven in [Dun13, Theorem 5.2.2], namely

$$(X_t^\varepsilon)_{t\geqslant 0}\Rightarrow(tC+\sqrt{2D}Z_t)_{t\geqslant 0}$$

for a standard Brownian motion Z and where

$$D=\int_{\mathbb{R}^{2K}}\int_{\mathbb{T}^2}(\mathrm{Id}+\nabla_y\chi)^T\Sigma(\mathrm{Id}+\nabla_y\chi)\rho_Y(dy,\eta)\rho_\eta(d\eta)$$

and

$$C=\int_{\mathbb{R}^{2K}}\int_{\mathbb{T}^2}\mathscr{L}_\eta\chi(y,\eta)\rho_Y(dy,\eta)\rho_\eta(d\eta).$$

Indeed, the drift term C arises from the convergence of the last term in (1.59), that is

$$\int_0^t\mathscr{L}_\eta\chi(Y_s^\varepsilon,\eta_s^\varepsilon)ds\to t\iint\mathscr{L}_\eta\chi(y,\eta)\rho_Y(dy,\eta)\rho_\eta(d\eta)=:tC \tag{1.60}$$

in probability, which motivates the ergodic theorem for $(Y^\varepsilon,\eta^\varepsilon)$ in Proposition 1.18 below. In the following, we denote by $E_\rho[\cdot]$ integration with respect to a probability measure ρ on $\mathbb{T}^2$ or on $\mathbb{R}^{2K}$.
For completeness, we state the ergodic theorem here, its proof follows from [Dun13, Lemma A.2.3]. One can also deduce the claim from decomposing the additive functional in terms of the solution $G(\cdot,\eta)$ of the Poisson equation (analogously as in (1.59))

$$\mathscr{L}_0G(\cdot,\eta)=b(\cdot,\eta)-E_{\rho_Y(\cdot,\eta)}[b(\cdot,\eta)]$$

for fixed $\eta\in\mathbb{R}^{2K}$ (existence follows by the $L^2(\rho_Y(\cdot,\eta))$-spectral gap estimates on $\mathscr{L}_0(\eta)$, [Dun13, Lemma A.2.1]) and utilizing the ergodic theorem for the Ornstein-Uhlenbeck process $(\eta_t)_{t\geqslant 0}$ started in ρ_η. The growth assumption on b in the following proposition is needed to obtain an analogue growth condition on G, such that the martingale term and the drift part involving $\mathscr{L}_\eta G$ in the decomposition vanish in $L^2(\mathbb{P})$.

Proposition 1.18. *Let $b:\mathbb{T}^2\times\mathbb{R}^{2K}\to\mathbb{R}$ be such that $b(y,\cdot)\in C^2(\mathbb{R}^{2K})$ satisfies the* *growth assumption*

$$|\nabla_\eta^k b(y,\eta)| \lesssim_k 1 + |\eta|^{l_k} \tag{1.61}$$

for suitable $l_k \geqslant 1$ and all $k = 0, 1, 2$.
Then the following convergence in probability holds true for $\varepsilon \to 0$:

$$\int_0^t b(Y_s^\varepsilon, \eta_s^\varepsilon) ds \to t \int E_{\rho_Y(\cdot,\eta)}[b(\cdot,\eta)]_{|\eta=\tilde{\eta}}\, \rho_\eta(d\tilde{\eta}). \tag{1.62}$$

Remark 1.19. *In particular Proposition 1.18 applies for $b(\cdot,\tilde{\eta}) := \mathscr{L}_\eta \chi(\cdot,\tilde{\eta})$, where $\mathscr{L}_\eta$ and χ are defined by* (1.7) *and* (1.58)*. This is due to the fact that the derivatives in η of the solution χ also satisfy a growth condition like* (1.61)*. This was proven in [Dun13, Proposition A.2.2] using $|(\nabla_\eta)^m F(y,\eta)| \lesssim 1 + |\eta|^{q_m}$ for some $q_m \geqslant 0$.*

1.4.1. Determining the limit rough path

Similarly to Section 1.3.1, we will represent X_t^ε via the solution of the Poisson equation (1.58). More precisely, from (1.59) we obtain

$$\begin{aligned} X_t^{\varepsilon,i} - X_0^i = {} & \varepsilon(\chi^i(Y_0^\varepsilon, \eta_0^\varepsilon) - \chi^i(Y_t^\varepsilon, \eta_t^\varepsilon)) + \sqrt{\varepsilon} \int_0^t (\nabla_\eta \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon))^T \sqrt{2\Gamma\Pi} dW_r \\ & + \int_0^t (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon))^T \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r \\ & + \int_0^t (\mathscr{L}_\eta \chi)^i (Y_s^\varepsilon, \eta_s^\varepsilon) ds \end{aligned} \tag{1.63}$$

for $i = 1, 2$. We utilize the decomposition (1.63) to represent the iterated Itô-integrals as follows for $i, j \in \{1, 2\}$,

$$\mathbb{X}_{0,t}^\varepsilon(i,j) = \int_0^t (X_s^{\varepsilon,i} - X_0^i) dX_s^{\varepsilon,j}$$

$$= \int_0^t (\chi^i(Y_0^\varepsilon, \eta_0^\varepsilon) - \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) F^j(Y_s^\varepsilon, \eta_s^\varepsilon) ds \tag{1.64}$$

$$+ \sum_{l=1,2} \int_0^t \varepsilon(\chi^i(Y_0^\varepsilon, \eta_0^\varepsilon) - \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) \sqrt{2\Sigma(Y_s^\varepsilon, \eta_s^\varepsilon)}(j,l) dB_s^l \tag{1.65}$$

$$+ \sum_{l=1,2} \int_0^t \sqrt{\varepsilon} \left(\int_0^s \nabla_\eta \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon) \cdot \sqrt{2\Gamma\Pi} dW_r \right) \sqrt{2\Sigma(Y_s^\varepsilon, \eta_s^\varepsilon)}(j,l) dB_s^l \tag{1.66}$$

$$+ \int_0^t \varepsilon^{-1/2} \left(\int_0^s \nabla_\eta \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon) \cdot \sqrt{2\Gamma\Pi} dW_r \right) F^j(Y_s^\varepsilon, \eta_s^\varepsilon) ds \tag{1.67}$$

$$+\sum_{l=1,2}\int_0^t\Big(\int_0^s (e_i+\nabla_y\chi^i(Y_r^\varepsilon,\eta_r^\varepsilon))\cdot\sqrt{2\Sigma(Y_r^\varepsilon,\eta_r^\varepsilon)}dB_r$$

$$+\int_0^s(\mathscr{L}_\eta\chi)^i(Y_r^\varepsilon,\eta_r^\varepsilon)dr\Big)\sqrt{2\Sigma(Y_s^\varepsilon,\eta_s^\varepsilon)}(j,l)dB_s^l \tag{1.68}$$

$$+\int_0^t\varepsilon^{-1}\Big(\int_0^s (e_i+\nabla_y\chi^i(Y_r^\varepsilon,\eta_r^\varepsilon))\cdot\sqrt{2\Sigma(Y_r^\varepsilon,\eta_r^\varepsilon)}dB_r$$

$$+\int_0^s(\mathscr{L}_\eta\chi)^i(Y_r^\varepsilon,\eta_r^\varepsilon)dr\Big)F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds. \tag{1.69}$$

We immediately see that terms (1.65) and (1.66) converge in $L^2(\mathbb{P})$ to zero by Burkholder-Davis-Gundy inequality and the growth conditions on $\nabla_\eta^k\chi$ for $k=0,1$ from [Dun13, Proposition A.2.2]. Furthermore, the fourth term (1.67) can be written using integration by parts as

$$\int_0^t\varepsilon^{-1/2}\Big(\int_0^s\nabla_\eta\chi^i(Y_r^\varepsilon,\eta_r^\varepsilon)\cdot\sqrt{2\Gamma\Pi}dW_r\Big)F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds$$

$$=\int_0^t\Big(\int_0^s\varepsilon^{1/2}\nabla_\eta\chi^i(Y_r^\varepsilon,\eta_r^\varepsilon)\cdot\sqrt{2\Gamma\Pi}dW_r\Big)\varepsilon^{-1}F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds$$

$$=\int_0^t\Big(\int_0^s\varepsilon^{-1}F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds\Big)d\Big(\int_0^\cdot\varepsilon^{1/2}\nabla_\eta\chi^i(Y_r^\varepsilon,\eta_r^\varepsilon)\cdot\sqrt{2\Gamma\Pi}dW_r\Big)_s$$

$$+\Big(\int_0^t\varepsilon^{-1}F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds\Big)\Big(\int_0^t\varepsilon^{1/2}\nabla_\eta\chi^i(Y_s^\varepsilon,\eta_s^\varepsilon)\cdot\sqrt{2\Gamma\Pi}dW_s\Big).$$

Utilizing Proposition 1.6, we deduce that the term (1.67) converges to zero in probability. Indeed, we have the convergence

$$\Big(\varepsilon^{-1}\int_0^\cdot F^j(Y_s^\varepsilon,\eta_s^\varepsilon)ds,\int_0^\cdot\varepsilon^{1/2}\nabla_\eta\chi^i\sqrt{2\Gamma\Pi}(Y_s^\varepsilon,\eta_s^\varepsilon)dW_s\Big)\Rightarrow\big((\tilde Z_t^j+tC^j)_t,0\big),$$

jointly in distribution by the decomposition (1.59) and the arguments from [Dun13, Theorem 5.2.2]. In the limiting process, $\tilde Z$ denotes the Brownian motion with variance $t\iint(\nabla_y\chi)^T2\Sigma\nabla_y\chi d\rho_Y d\rho_\eta$. Furthermore, the UCV condition holds for the martingale, as $\nabla_\eta\chi\in L^2(\rho_Y(\eta)\rho_\eta)$ by the growth condition proven in [Dun13, Proposition A.2.2], with which the expected quadratic variation can be bounded. Together, this then yields that (1.67) converges to zero in probability.

Applying Proposition 1.18 for $b=\chi^iF^j$ and using that $\int F(y,\eta)d\rho_Y(dy,\eta)=0$ by the definition of F and ρ_Y, we obtain that the first term (1.64) converges in probability to

$$t\iint\chi^iF^j\rho_Y(dy,\eta)\rho_\eta(d\eta)=:ta_F(i,j).$$

In order to deal with the remaining terms (1.68) and (1.69), we rewrite them using integration by parts or Itô's formula. For (1.69) we obtain

$$\begin{aligned}
&\int_0^t \varepsilon^{-1}\Big(\int_0^s (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r \\
&\qquad\qquad + \int_0^s (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big) F^j(Y_s^\varepsilon, \eta_s^\varepsilon) ds \\
&\quad = \Big(\int_0^t \varepsilon^{-1} F^j(Y_s^\varepsilon, \eta_s^\varepsilon)) ds\Big)\Big(\int_0^t (e_i + \nabla_y \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) \cdot \sqrt{2\Sigma(Y_s^\varepsilon, \eta_s^\varepsilon)} dB_s \\
&\qquad\qquad\qquad\qquad + \int_0^t (\mathscr{L}_\eta \chi)^i (Y_s^\varepsilon, \eta_s^\varepsilon) ds\Big) \\
&\qquad - \int_0^t \Big(\int_0^s \varepsilon^{-1} F^j(Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big) d\Big(\int_0^\cdot (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r) \\
&\qquad\qquad\qquad\qquad + \int_0^\cdot (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big)_s.
\end{aligned}$$

According to Itô's formula for (1.68) we have

$$\begin{aligned}
&\int_0^t \Big(\int_0^s (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r \\
&\qquad\qquad + \int_0^s (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big) d\left(\int_0^\cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r\right)_s^j \\
&\quad = \Big(\int_0^t (e_i + \nabla_y \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) \cdot \sqrt{2\Sigma(Y_s^\varepsilon, \eta_s^\varepsilon)} dB_s \\
&\qquad\qquad + \int_0^t (\mathscr{L}_\eta \chi)^i (Y_s^\varepsilon, \eta_s^\varepsilon) ds\Big) \left(\int_0^t \sqrt{2\Sigma(Y_s^\varepsilon, \eta_s^\varepsilon)} dB_s\right)^j \\
&\qquad - \int_0^t \left(\int_0^s \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r\right)^j d\Big(\int_0^\cdot (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r \\
&\qquad\qquad\qquad\qquad + \int_0^\cdot (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big)_s \\
&\qquad - \int_0^t (e_i + \nabla_y \chi^i(Y_s^\varepsilon, \eta_s^\varepsilon)) \cdot 2\Sigma(Y_s^\varepsilon, \eta_s^\varepsilon) e_j ds.
\end{aligned}$$

By combining the previous two terms, we obtain that overall

$$\begin{aligned}
&\int_0^t X_s^{\varepsilon,i} dX_s^{\varepsilon,j} \\
&= a_F^\varepsilon(i,j) + X_t^{\varepsilon,j}\Big(\int_0^t (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r + \int_0^t (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big) \\
&\quad - \int_0^t X_s^{\varepsilon,j} d\Big(\int_0^\cdot (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r + \int_0^\cdot (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big)_s \\
&\quad - \int_0^t (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot 2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon) e_j dr, \qquad (1.70)
\end{aligned}$$

where $a_F^\varepsilon(i,j)$ denotes the sum of the terms (1.64), (1.65), (1.66) and (1.67), that will converge in probability to

$$t a_F(i,j) = t \iint \chi^i F^j \rho_Y(dy, \eta) \rho_\eta(d\eta).$$

For the stochastic integral and the product term in (1.70), we again apply Proposition 1.6. Now, we consider the joint distributional convergence of

$$\Big(X^{\varepsilon,j}, \int_0^\cdot (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r + \int_0^\cdot (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big) \Rightarrow (X^j, X^i),$$

which holds true by [Dun13, Theorem 5.2.2] and using that the UCV condition for the semi-martingales

$$\Big(\int_0^\cdot (e_i + \nabla_y \chi^i(Y_r^\varepsilon, \eta_r^\varepsilon)) \cdot \sqrt{2\Sigma(Y_r^\varepsilon, \eta_r^\varepsilon)} dB_r + \int_0^\cdot (\mathscr{L}_\eta \chi)^i (Y_r^\varepsilon, \eta_r^\varepsilon) dr\Big)_\varepsilon$$

holds by boundedness of Σ and the bounds (A.7) and (A.8) of [Dun13, Proposition A.2.2]. Let us define

$$c^{i,j} := \iint (e_i + \nabla_y \chi^i) \cdot 2\Sigma e_j d\rho_Y(\cdot, \eta) d\rho_\eta.$$

Then according to Proposition 1.18, the last term in (1.70) (the quadratic variation term) converges in probability to $tc^{i,j}$. Hence, utilizing Slutzky's Lemma, we obtain the distributional convergence

$$\begin{aligned}\int_0^t & X_s^{\varepsilon,i} dX_s^{\varepsilon,j} \\ &\Rightarrow t a_F(i,j) + X_t^j X_t^i - \int_0^t X_s^j dX_s^i - \frac{t}{2} c^{i,j} \\ &= t(a_F(i,j) - c^{i,j} + \langle X^i, X^j \rangle_1) + \int_0^t X_s^i dX_s^j.\end{aligned}$$

We summarize our findings about the convergence of the iterated Itô integrals, as well as the iterated Stratonovich integrals, in the next proposition and corollary.

Proposition 1.20. *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$ solve the system (1.57) started in $(Y_0^\varepsilon, \eta_0^\varepsilon) \sim \rho_Y(dy,\eta)\rho_\eta(d\eta)$, $X_0^\varepsilon = \varepsilon Y_0^\varepsilon$. Moreover, let*

$$X_t = \sqrt{2D} Z_t + tC$$

for a standard two-dimensional Brownian motion Z and

$$D = \iint (\mathrm{Id} + \nabla_y \chi(y,\eta))^T \Sigma(y,\eta)(\mathrm{Id} + \nabla_y \chi(y,\eta))\rho_Y(dy,\eta)\rho_\eta(d\eta)$$

and

$$C = \iint \mathscr{L}_\eta \chi(y,\eta)\rho_Y(dy,\eta)\rho_\eta(d\eta),$$

where for each $\eta \in \mathbb{R}^K$, $\chi(\cdot,\eta)$ is the solution of $\mathscr{L}_0\chi(\cdot,\eta) = -F(\cdot,\eta)$.
Then for all $s,t \in \Delta_T$, weak convergence of the iterated Itô-integrals $(\mathbb{X}_{s,t}^\varepsilon)$ in $\mathbb{R}^{2\times 2}$ holds true, where for $i,j \in \{1,2\}$,

$$\begin{aligned}\mathbb{X}_{s,t}^\varepsilon(i,j) &:= \int_s^t X_{s,r}^{\varepsilon,i} dX_r^{\varepsilon,j} \\ &\Rightarrow \mathbb{X}_{s,t}(i,j) + (t-s)\Big(\langle X^i, X^j\rangle_1 + \langle \chi^i, \mathscr{L}_0 \chi^j \rangle_{\rho_Y(\cdot,\eta)\rho_\eta} \\ &\quad - \iint (e_i + \nabla\chi^i(y,\eta) \cdot 2\Sigma(y,\eta) e_j \rho_Y(dy,\eta)\rho_\eta(d\eta)\Big)\end{aligned} \tag{1.71}$$

as $\varepsilon \to 0$ with $\mathbb{X}_{s,t}(i,j) := \int_s^t X_{s,r}^i dX_r^j$.

Corollary 1.21. *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$ be as in Proposition 1.20. Then for all $s,t \in \Delta_T$, weak convergence in $\mathbb{R}^{2\times 2}$ of the iterated Stratonovich-integrals $(\tilde{\mathbb{X}}_{s,t}^\varepsilon)$ holds true, where for $i,j \in \{1,2\}$,*

$$\tilde{\mathbb{X}}_{s,t}^\varepsilon(i,j) := \int_s^t X_{s,r}^{\varepsilon,i} \circ dX_r^{\varepsilon,j} \Rightarrow \tilde{\mathbb{X}}_{s,t}(i,j) + (t-s)\tilde{A}(i,j) \tag{1.72}$$

as $\varepsilon \to 0$ *with*

$$X_t = \sqrt{2D} Z_t + tC$$

for a standard two-dimensional Brownian motion Z *and* $\tilde{\mathbb{X}}_{s,t}(i,j) := \int_s^t X^i_{s,r} \circ dX^j_r$. D, L *are defined as in Proposition 1.20. The area correction is given by*

$$\begin{aligned}
\tilde{A}(i,j) &= \iint e_i \cdot \Sigma \nabla_y \chi^j(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta) - \iint \nabla_y \chi^i(y,\eta) \cdot \Sigma e_j \rho_y(dy,\eta)\rho_\eta(d\eta) \\
&= \iint \chi^i(y,\eta) F^j(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta) - \iint F^i(y,\eta)\chi^j(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta)
\end{aligned}$$

Proof. The corollary follows from Proposition 1.20 with analogue arguments as in Corollary 1.13. We have that

$$\begin{aligned}
\frac{1}{2}\langle X^{\varepsilon,i}, X^{\varepsilon,j}\rangle_t &= \int_0^t e_i \cdot \Sigma(Y^\varepsilon_s, \eta^\varepsilon_s) e_j ds \\
&\to t \int e_i \cdot \Sigma(y,\eta) e_j \rho_Y(dy,\eta)\rho_\eta(d\eta)
\end{aligned}$$

by Proposition 1.18. Furthermore, we have that

$$\begin{aligned}
\tilde{\mathbb{X}}^\varepsilon_{0,t}(i,j) &= \mathbb{X}^\varepsilon_{0,t}(i,j) + \frac{1}{2}\langle X^{\varepsilon,i}, X^{\varepsilon,j}\rangle_t \\
&\Rightarrow \mathbb{X}_{0,t}(i,j) + t\langle \chi^i, (\mathscr{L}_0\chi)^j\rangle_{\rho_y(\cdot,\eta)\rho_\eta} + \langle X^i, X^j\rangle_t \\
&\quad - t\int (\nabla_y \chi^i + e_i) \cdot 2\Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta) + t\int e_i \cdot \Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta) \\
&= \mathbb{X}_{0,t}(i,j) + \frac{1}{2}\langle X^i, X^j\rangle_t \\
&\quad + t\Bigg(\langle \chi^i, (\mathscr{L}_0\chi)^j\rangle_{\rho_y(\cdot,\eta)\rho_\eta} + \frac{1}{2}\langle X^i, X^j\rangle_t \\
&\qquad - \int \nabla_y \chi^i \cdot 2\Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta) - \int e_i \cdot \Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta)\Bigg) \\
&= \tilde{\mathbb{X}}_{0,t}(i,j) + t\tilde{A}(i,j).
\end{aligned}$$

The area correction can be written as

$$\begin{aligned}
\tilde{A}(i,j) &= \langle \chi^i, \mathscr{L}_0\chi^j\rangle_{\rho_y(\cdot,\eta)\rho_\eta} + D(i,j) - \int \nabla_y \chi^i \cdot 2\Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta) \\
&\quad - \int e_i \cdot \Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta) \\
&= \int e_i \cdot \Sigma \nabla_y \chi^j d(\rho_y(\cdot,\eta)\rho_\eta) - \int \nabla_y \chi^i \cdot \Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta),
\end{aligned}$$

using furthermore that

$$\langle \chi^i, \mathscr{L}_0 \chi^j \rangle_{\rho_y(\cdot,\eta)\rho_\eta} = -\int \nabla_y \chi^i \cdot \Sigma \nabla_y \chi^j d(\rho_y(\cdot,\eta)\rho_\eta)$$

by the definition of F and the invariant measure $\rho_y(dy,\eta) = c^{-1}\sqrt{|\Sigma(y,\eta)|}dy$. By integrating ∇_y by parts and using once more the definition of F and the invariant measure, we obtain

$$\int \nabla_y \chi^i \cdot \Sigma e_j d(\rho_y(\cdot,\eta)\rho_\eta) = -\int_{\mathbb{R}^{2K}} \int_{\mathbb{T}^2} \chi^i(y,\eta) F^j(y,\eta) \rho_y(dy,\eta)\rho_\eta(d\eta).$$

Hence, the claim for $\tilde{A}$ follows. □

Corollary 1.22. *For $\tilde{A}$ from Corollary 1.21, it follows that $\tilde{A}(i,j) = 0$ for all $i,j = 1,2$.*

Proof. Using $(-\mathscr{L}_0)\chi = F$, we obtain

$$\begin{aligned}
\tilde{A}(i,j) &= \iint \chi^i(y,\eta) F^j(y,\eta) \rho_y(dy,\eta)\rho_\eta(d\eta) - \iint F^i(y,\eta) \chi^j(y,\eta) \rho_y(dy,\eta)\rho_\eta(d\eta) \\
&= \iint \chi^i(y,\eta)(-\mathscr{L}_0)\chi^j(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta) \\
&\quad - \iint (-\mathscr{L}_0)\chi^i(y,\eta)\chi^j(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta) \\
&= \iint (-\mathscr{L}_0^\star)\chi^i(y,\eta)\chi^j(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta) \\
&\quad - \iint (-\mathscr{L}_0)\chi^i(y,\eta)\chi^j(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta) \\
&= 0
\end{aligned}$$

and the claim follows as $\mathscr{L}_0$ is symmetric with respect to the measure $\rho_y(dy,\eta)\rho_\eta(d\eta)$, that is $\mathscr{L}_0^\star = \mathscr{L}_0$, where $\mathscr{L}_0^\star$ denotes the adjoint with respect to $L^2(\rho_y(dy,\eta)\rho_\eta(d\eta))$. □

Remark 1.23. *Corollary 1.22 means that for the limit of the Stratonovich integrals there appears no area correction. This is due to the fact that the underlying Markov process $(Y_t)_{t\geqslant 0}$, for fixed $\eta \in \mathbb{R}^{2K}$, is reversible when started in $\rho_Y(\cdot,\eta)$. Equivalently, the generator $\mathscr{L}_0(\eta) = \mathscr{L}_0(\eta)^\star$ is symmetric with respect to $L^2(\rho_Y(\cdot,\eta))$. This is a phenomenon that was already observed in previous articles, see also the discussion in the introduction of [DOP21] about the relation of a vanishing Stratonovich area correction and an underlying reversible Markov process. [DOP21] conjecture that the Stratonovich area correction vanishes if and only if the underlying Markov process is reversible.*

Remark 1.24. *Via a symmetry argument utilizing the Fourier expansion of the Helfrich surface H, one can show that the constant limit drift C actually vanishes, $C = 0$. For a proof see [Dun13, Proposition 5.2.4].*

1.4.2. Tightness

The tightness is again a consequence of Lemma 1.7, once we have verified the moment bounds.

Proposition 1.25. *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$ be as in Proposition 1.20. Then the following moment bounds hold true for any $p \geqslant 2$, $i, j \in \{1, 2\}$,*

$$\sup_\varepsilon \mathbb{E}[|X^{\varepsilon,i}(t) - X^{\varepsilon,i}(s)|^p] \lesssim |t-s|^{p/2}, \quad s,t \in \Delta_T \tag{1.73}$$

and

$$\sup_\varepsilon \mathbb{E}\left[\left|\int_s^t X^{\varepsilon,i}_{s,r} dX^{\varepsilon,j}_r\right|^{p/2}\right] \lesssim |t-s|^{p/2}, \quad s,t \in \Delta_T. \tag{1.74}$$

In particular, tightness of $(X^\varepsilon, \mathbb{X}^\varepsilon)$ in $C_{\gamma,T}$ for $\gamma < 1/2$ follows.

Proof. The arguments are analogous to the proof of Proposition 1.15. For the estimate on X^ε we use, similarly as in Proposition 1.15, a trade-off argument for the drift term using the decomposition (1.63). For the iterated integrals, we use the bound (1.73) on X^ε and the decomposition (1.70). □

1.4.3. Rough homogenization limit

In this subsection, we state our second main theorem, which follows directly from Proposition 1.20 and Proposition 1.25. The corollary on the Statonovich lift then follows from the result for the Itô lift (Theorem 1.26), Corollary 1.21 and Corollary 1.22.

Theorem 1.26 (Itô lift)**.** *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$, X and $\mathbb{X}^\varepsilon, \mathbb{X}$ be as in Proposition 1.20. Then for all $\gamma < 1/2$, weak convergence in the rough path space $C_{\gamma,T}$ of*

$$(X^\varepsilon, \mathbb{X}^\varepsilon) \Rightarrow (X, (s,t) \mapsto \mathbb{X}_{s,t} + A(t-s)) \tag{1.75}$$

as $\varepsilon \to 0$ follows, where $A = (A(i,j))_{i,j\in\{1,2\}}$ denotes the matrix with

$$\begin{aligned} A(i,j) = \langle \chi^i, \mathscr{L}_0 \chi^j \rangle_{\rho_Y(\cdot,\eta)\rho_\eta} + \langle X^i, X^j \rangle_1 \\ - \iint (e_i + \nabla_y \chi^i(y,\eta)) \cdot 2\Sigma(y,\eta) e_j \rho_Y(dy,\eta)\rho_\eta(d\eta). \end{aligned} \tag{1.76}$$

Corollary 1.27 (Stratonovich lift)**.** *Let $(X^\varepsilon, Y^\varepsilon, \eta^\varepsilon)$, X and $\tilde{\mathbb{X}}^\varepsilon, \tilde{\mathbb{X}}$ be as in Corollary 1.21. Then for all $\gamma < 1/2$, weak convergence in the rough paths space $C_{\gamma,T}$ of*

$$(X^\varepsilon, \tilde{\mathbb{X}}^\varepsilon) \Rightarrow (X, \tilde{\mathbb{X}}) \tag{1.77}$$

as $\varepsilon \to 0$ follows.

1.5. Outlook – Construction of a diffusion on a rough Gaussian membrane

This section gives an outlook on work in progress and discusses arising difficulties. The overall goal is to construct the Brownian motion from the previous section on the Helfrich membrane (1.3) without ultra-violet cutoff, formally $K = \infty$. Contrary to the previous sections, there is no averaging or homogenization procedure in this section, although it would certainly be interesting to study homogenization scaling regimes and simultaneously considering $K \to \infty$.

Below, ε denotes the vanishing inverse cutoff. We moreover assume the surface $\mathbb{T}^2 \ni x \mapsto H(x)$ to be time-independent in this section. Classical methods (defining the associated Laplace-Beltrami operator, cf. [Hsu02]) do not apply due to the roughness of the Gaussian surface H, that is almost surely γ-Hölder continuous for $\gamma < 1$ in dimension 2 (as was proven in [Stu10, Lemma 6.25]).

After mollifying the surface, that is, considering H^ε with cutoff $K = \varepsilon^{-1}$, [Hsu02, Proposition 3.2.1] yields existence of the diffusion on H^ε, whose generator in local coordinates is given by the Laplace-Beltrami operator

$$L^\varepsilon = \frac{1}{\sqrt{|G^\varepsilon|}} \nabla \cdot (\sqrt{|G^\varepsilon|} (G^\varepsilon)^{-1} \nabla), \tag{1.78}$$

with metric tensor matrix

$$G^\varepsilon(x) = \begin{pmatrix} 1 + |\partial_1 H^\varepsilon(x)|^2 & \partial_1 H^\varepsilon(x) \partial_2 H^\varepsilon(x) \\ \partial_1 H^\varepsilon(x) \partial_2 H^\varepsilon(x) & 1 + |\partial_2 H^\varepsilon(x)|^2 \end{pmatrix}. \tag{1.79}$$

Let $|G^\varepsilon(x)| := 1 + |\nabla H^\varepsilon(x)|^2$. The operator L^ε is defined for every fixed realization of the surface H. Here, H^ε is a complex-valued Gaussian field with paths in $C^\infty(\mathbb{T}^2)$ given by

$$H^\varepsilon(x) = \sum_{0 < |k| \leqslant \varepsilon^{-1}} e_k(x) \hat{H}(k), \quad x \in \mathbb{T}^2$$

for $e_k(x) := e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^2$, where $(\hat{H}(k))_{k \in \mathbb{Z}^2}$ is a family of centered Gaussians with $\mathbb{E}[\hat{H}(k)\hat{H}(l)] = \delta_{k,-l} |2\pi k|^{-4}$. The covariance structure, we consider here is a simplification of the covariance of the Helfrich membrane given by (1.4). Due to

$$\mathbb{E}[\|H\|^2_{W^{2,\alpha}}] = \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^\alpha \mathbb{E}[|\hat{H}(k)|^2] \lesssim \sum_{k \in \mathbb{Z}^2} |k|^{2\alpha - 4} < \infty$$

for $\alpha < 1$, one sees that H is almost surely in the fractional Sobolev space $W^{2,\alpha}$ of regularity $\alpha < 1$. Note also that H^ε is a stationary Gaussian field, as the covariance $\mathbb{E}[H^\varepsilon(x) H^\varepsilon(y)] = \sum_{|k| \leqslant \varepsilon^{-1}} e_k(x-y) |2\pi k|^{-4}$ only depends on $x - y$.

Take the surface measure μ^ε on the torus defined by

$$\mu^\varepsilon(dx) := \sqrt{|G^\varepsilon(x)|}dx. \tag{1.80}$$

One easily sees that the measure μ^ε is invariant for the generator L^ε and that the Dirichlet form of the Markov process X^ε with generator L^ε is given by

$$\mathscr{E}^\varepsilon(f,g) = \langle (G^\varepsilon)^{-1}\nabla f, \nabla g\rangle_{\mu^\varepsilon} = \int_{\mathbb{T}^2} (G^\varepsilon(x))^{-1}\nabla f(x)\cdot\nabla g(x)\mu^\varepsilon(dx)$$

for smooth test functions f, g.
We aim to make sense out of the limit of the operators L^ε, respectively $\mathscr{E}^\varepsilon$, for $\varepsilon \to 0$. Below, we justify that it makes sense to consider instead the rescaled Dirichlet form $\sigma_\varepsilon^{-1}\mathscr{E}^\varepsilon$ for $\sigma_\varepsilon \to \infty$ as $\varepsilon \to 0$, which corresponds to the slowed-down process.
A tool called Mosco convergence for Dirichlet forms (for definition, see [Kol06, Definition 2.5]) yields convergence of the Markov semigroup, cf. [Kol06, Theorem 2.6] and [KS03, section 5]. However proving that the (rescaled) Dirichlet forms above converge in the Mosco sense using [Kol06, Theorem 3.6], turns out to be nontrivial due to the absence of uniform lower bounds for the Dirichlet forms. Thus, following this technique, a convergence result for the processes is not yet in reach.
Another approach for the construction of the diffusion on H might be to use additive functionals analoguous to the construction of the Liouville Brownian motion from [Ber15]. However this method does not directly apply to our situation due to the the arguments in [Ber15] being in the one-dimensional setting.
Nonethless, studying convergence of the surface measures μ^ε for H^ε can hint on a limiting behaviour for the process. Indeed, in this section, we prove that the rescaled surface measures converge weakly to the Lebesgue measure. This then indicates that the appropriately rescaled/slowed-down diffusion converges in local coordinates to the standard Brownian motion. Hence, the slowed-down process is not influenced by the roughness of surface, while heuristically the non-slowed-down process moves "infinitely fast". Our arguments for the convergence of the surface measures utilize the Gaussian distribution of the surface and Wiener chaos decompositions.
We start by defining the rescaled surface measures ν^ε by

$$\nu^\varepsilon(dx) := \sigma_\varepsilon^{-1}\mu^\varepsilon(dx)$$

with rescaling constant $\sigma_\varepsilon^2 := \sum_{|k|\leqslant\varepsilon^{-1}} |2\pi k|^{-2} = \mathbb{E}[|\nabla H^\varepsilon(x)|^2]$. The idea is similar to renormalization by σ_ε^2. This means, while $(|\nabla H^\varepsilon|^2)_\varepsilon$ diverges, subtracting the diverging constant σ_ε^2 and exploiting randomness might lead to a almost sure finite limit for

$$(|\nabla H^\varepsilon|^2 - \sigma_\varepsilon^2)_\varepsilon.$$

Notice that, due to stationarity of H^ε, for every $x \in \mathbb{T}^2$,

$$|G^\varepsilon(x)| = |\nabla H^\varepsilon(x)|^2 + 1 \overset{d}{=} \sigma_\varepsilon^2|Z_1|^2 + \sigma_\varepsilon^2|Z_2|^2 + 1 \tag{1.81}$$

for a standard two-dimensional normal random variable $Z = (Z_1, Z_2) \sim N(0, \mathrm{Id}_{2\times 2})$. Here, independence of Z_1 and Z_2 follows from

$$\mathbb{E}[\partial_1 H^\varepsilon(x)\partial_2 H^\varepsilon(x)] = \sum_{|k|\leqslant \varepsilon^{-1}} \frac{(2\pi i k_1)(-2\pi i k_2)}{|2\pi k|^4} = 0,$$

which vanishes by antisymmetry of the summands under the substitution $k_1 \rightsquigarrow -k_1$. Equipped with these properties, we can derive the following lemma.

Lemma 1.28. *The family of finite, positive random measures $(\nu^\varepsilon)_\varepsilon$ is tight.*

Proof. Consider the compact set K_M in the space of finite, positive measures with the topology of weak convergence of measures,

$$K_M := \{\mu \mid \mu(\mathbb{T}^2) \leqslant M\}.$$

K_M is compact, since any sequence of measures in K_M is tight due to compactness of $\mathbb{T}^2$, which implies the weak convergence along a subsequence by Prohorov theorem. To obtain tightness of (ν^ε), it is thus enough, due Chebyshev's inequality, to prove the moment bound $\sup_\varepsilon \mathbb{E}[|\nu^\varepsilon(\mathbb{T}^2)|] < \infty$. The bound follows from

$$\mathbb{E}[|\nu^\varepsilon(\mathbb{T}^2)|] \leqslant \sigma_\varepsilon^{-1}\mathbb{E}[(\sigma_\varepsilon^2|Z_1|^2 + \sigma_\varepsilon^2|Z_2|^2 + 1)^{1/2}]$$

and

$$\mathbb{E}[(\sigma_\varepsilon^2|Z_1|^2 + \sigma_\varepsilon^2|Z_2|^2 + 1)^{1/2}] \lesssim \mathbb{E}[|Z|]\sigma_\varepsilon + 1. \qquad \square$$

Notice that (1.81) implies

$$\mathbb{E}[\nu^\varepsilon(\varphi)] = \mathbb{E}[\sigma_\varepsilon^{-1}\sqrt{1 + |\sigma_\varepsilon Z|^2}] \int_{\mathbb{T}^2} \varphi(x)dx \to \mathbb{E}[|Z|] \int_{\mathbb{T}^2} \varphi(x)dx.$$

This suggests the following convergence of the measures.

Lemma 1.29. *For $\varphi \in C(\mathbb{T}^2)$, the following convergence holds true as $\varepsilon \to 0$,*

$$\mathbb{E}\left[\left|\nu^\varepsilon(\varphi) - c\int_{\mathbb{T}^2} \varphi(x)dx\right|^2\right] \to 0.$$

Herein, $c = \mathbb{E}[\sqrt{|Z|^2 + 1}]$ for a standard normal random variable $Z = (Z_1, Z_2) \sim N(0, \mathrm{Id}_{2\times 2})$.

Proof. The proof uses ideas from [GP16, section 3.1]. We write

$$\nu^\varepsilon(\varphi) - c\int_{\mathbb{T}^2} \varphi(x)dx = \sigma_\varepsilon^{-1}\int_{\mathbb{T}^2} \varphi(x)(\sqrt{|G^\varepsilon(x)|} - \sigma_\varepsilon c)dx$$

with $|G^\varepsilon(x)| = |\nabla H^\varepsilon(x)|^2 + 1$, such that

$$\mathbb{E}\left[\left|\nu^\varepsilon(\varphi) - c\int_{\mathbb{T}^2}\varphi(x)dx\right|^2\right] = \mathbb{E}\left[\left|\int_{\mathbb{T}^2}\varphi(x)(\sqrt{|\sigma_\varepsilon^{-1}\partial_1 H^\varepsilon(x)|^2 + |\sigma_\varepsilon^{-1}\partial_2 H^\varepsilon(x)|^2 + 1} - c)dx\right|^2\right].$$

To prove the claim, we use the chaos expansion on the space $L^2(\rho)$ with respect to $\rho := \mathrm{Law}(N(0, \mathrm{Id}_{2\times 2}))$. We have that $N(0,\mathrm{Id})\sim Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon) \overset{d}{=} (\sigma_\varepsilon^{-1}\partial_1 H^\varepsilon(x), \sigma_\varepsilon^{-1}\partial_2 H^\varepsilon(x))$. We define

$$\phi_\varepsilon(Z_1^\varepsilon(x), Z_2^\varepsilon(x)) := \sqrt{|Z_1^\varepsilon(x)|^2 + |Z_2^\varepsilon(x)|^2 + 1} - c$$

and expand with respect to the Hermite basis $(h_j)_{j\in\mathbb{N}^2}$ given by $h_j(y) = h_{j_1}(y_1)h_{j_2}(y_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ for the Hermite polynomials $h_n(y) := (-1)^n e^{y^2/2}\partial_y^n e^{-y^2/2}$, $n \in \mathbb{N}$, $h_0(y) := 1$, $y \in \mathbb{R}$. Then by [Nua06, Theorem 1.1.1 and Example 1.1.1], we obtain

$$\phi_\varepsilon(Z_1^\varepsilon(x), Z_2^\varepsilon(x)) = \sum_{j\in\mathbb{N}^2} c_j(\phi_\varepsilon)h_j(Z_1^\varepsilon(x), Z_2^\varepsilon(x)) = \sum_{j\in\mathbb{N}^2\setminus\{(0,0)\}} c_j(\phi_\varepsilon)h_j(Z_1^\varepsilon(x), Z_2^\varepsilon(x))$$

for the coefficients $c_j(\phi_\varepsilon) = \frac{1}{j!}\mathbb{E}[\phi_\varepsilon(Z_1^\varepsilon(x), Z_2^\varepsilon(x))h_j(Z_1^\varepsilon(x), Z_2^\varepsilon(x))]$ with $j! := j_1!j_2!$ (the expectation does not depend on x due to stationarity). Note that our definition of the Hermite polynomial differs by a factor of $1/n!$ compared to [Nua06]. We have that $c_j(\phi_\varepsilon) = 0$ for $j = (0,0)$, as $c = \mathbb{E}[\phi_\varepsilon(Z_1^\varepsilon(x), Z_2^\varepsilon(x))]$. With the expansion of H^ε, we can write for $m = 1, 2$, $k = (k^1, k^2)$

$$\sigma_\varepsilon^{-1}\partial_m H^\varepsilon(x) = \sigma_\varepsilon^{-1}\sum_{0<|k|\leqslant\varepsilon^{-1}} e_k(x)\frac{(2\pi i k^m)}{|2\pi k|^2}\hat{\xi}(k),$$

for a white noise ξ (that is, ξ being the centered Gaussian process with $\mathbb{E}[\hat{\xi}(k)\hat{\xi}(l)] = \delta_{k,-l}$). Thus $h_j(Z_1^\varepsilon(x), Z_2^\varepsilon(x))$ can be understood as a Hermite polynomial of the variables $(\langle\xi, e_{-k}\rangle)_{|k|\leqslant\varepsilon^{-1}}$ and thus for the projection Π_n onto the n-th homogeneous Wiener chaos generated by ξ (cf. [Nua06, section 1.1.2]), we obtain

$$\begin{aligned} h_j(Z_1^\varepsilon(x), Z_2^\varepsilon(x)) &= \Pi_{j_1}\big((Z_1^\varepsilon(x))^{j_1}\big)\Pi_{j_2}\big((Z_2^\varepsilon(x))^{j_2}\big) \\ &= \sigma_\varepsilon^{-(j_1+j_2)}\sum_{|k_1|,|l_1|,\ldots,|k_{j_1}|,|l_{j_2}|\leqslant\varepsilon^{-1}} e_{k_1+\cdots+k_{j_1}+l_1+\cdots+l_{j_2}}(x)\frac{(2\pi i k_1^1)\cdots(2\pi i k_{j_1}^1)(2\pi i l_1^2)\cdots(2\pi i l_{j_2}^2)}{|2\pi k_1|^2\cdots|2\pi k_{j_1}|^2|2\pi l_1|^2\cdots|2\pi l_{j_2}|^2}\times \\ &\qquad \Pi_{j_1}(\hat{\xi}(k_1)\cdots\hat{\xi}(k_{j_1}))\Pi_{j_2}(\hat{\xi}(l_1)\cdots\hat{\xi}(l_{j_2})). \end{aligned}$$

Hence, with the notation $[k_1 : k_j] := k_1 + \cdots + k_j$ we obtain

$$\begin{aligned}
&\langle \phi_\varepsilon(Z_1^\varepsilon, Z_2^\varepsilon), \varphi\rangle \\
&\quad = \sum_{(j_1,j_2)\in\mathbb{N}^2\setminus\{(0,0)\}} c_j(\phi_\varepsilon)\sigma_\varepsilon^{-(j_1+j_2)} \times \\
&\qquad \sum_{|k_1|,|l_1|,\ldots,|k_{j_1}|,|l_{j_2}|\leqslant\varepsilon^{-1}} \hat{\varphi}\big(-([k_1 : k_{j_1}] + [l_1 : l_{j_2}])\big) \frac{(2\pi i k_1^1)\cdots(2\pi i k_{j_1}^1)(2\pi i l_1^2)\cdots(2\pi i l_{j_2}^2)}{|2\pi k_1|^2\cdots|2\pi k_{j_1}|^2|2\pi l_1|^2\cdots|2\pi l_{j_2}|^2} \times \\
&\qquad\qquad \Pi_{j_1}(\hat{\xi}(k_1)\cdots\hat{\xi}(k_{j_1}))\Pi_{j_2}(\hat{\xi}(l_1)\cdots\hat{\xi}(l_{j_2})).
\end{aligned}$$

Moreover for the projections, we have with $j = (j_1, j_2), \tilde{j} = (\tilde{j}_1, \tilde{j}_2)$ due to the isometry of the n-th chaos and $L_s^2((\mathbb{T}^d)^n)$ (symmetric L^2-functions), that

$$\begin{aligned}
&\mathbb{E}[\Pi_{j_1}(\hat{\xi}(k_1)\cdots\hat{\xi}(k_{j_1}))\Pi_{j_2}(\hat{\xi}(l_1)\cdots\hat{\xi}(l_{j_2}))\Pi_{\tilde{j}_1}(\hat{\xi}(\tilde{k}_1)\cdots\hat{\xi}(\tilde{k}_{j_1}))\Pi_{\tilde{j}_2}(\hat{\xi}(\tilde{l}_1)\cdots\hat{\xi}(\tilde{l}_{j_2}))] \\
&\ = \mathbb{E}[\Pi_{j_1}(\hat{\xi}(k_1)\cdots\hat{\xi}(k_{j_1}))\Pi_{\tilde{j}_1}(\hat{\xi}(\tilde{k}_1)\cdots\hat{\xi}(\tilde{k}_{j_1}))]\ \mathbb{E}[\Pi_{j_2}(\hat{\xi}(l_1)\cdots\hat{\xi}(l_{j_2}))\Pi_{\tilde{j}_2}(\hat{\xi}(\tilde{l}_1)\cdots\hat{\xi}(\tilde{l}_{j_2}))] \\
&\ = \delta_{j,\tilde{j}}\, j_1!\, \langle e_{-k_1}\otimes\cdots\otimes e_{-k_{j_1}}, e_{-\tilde{k}_1}\otimes\cdots\otimes e_{-\tilde{k}_{j_1}}\rangle\ j_2!\langle e_{-l_1}\otimes\cdots\otimes e_{-l_{j_2}}, e_{-\tilde{l}_1}\otimes\cdots\otimes e_{-\tilde{l}_{j_2}}\rangle \\
&\ = \delta_{j,\tilde{j}}\, j_1!\, j_2!\, \delta_{k_1,-\tilde{k}_1}\cdots\delta_{k_{j_1},-\tilde{k}_{j_1}}\delta_{l_1,-\tilde{l}_1}\cdots\delta_{l_{j_2},-\tilde{l}_{j_2}},
\end{aligned}$$

where in the second line, we used independence of $\partial_1 H^\varepsilon(x)$ and $\partial_2 H^\varepsilon(x)$. This yields

$$\begin{aligned}
&\mathbb{E}[|\langle \phi_\varepsilon(Z_1^\varepsilon, Z_2^\varepsilon), \varphi\rangle|^2] \\
&\quad = \sum_{(j_1,j_2)\in\mathbb{N}^2\setminus\{(0,0)\}} |c_j(\Phi_\varepsilon)|^2\, j_1! j_2!\, \sigma_\varepsilon^{-2(j_1+j_2)} \times \\
&\qquad \sum_{|k_1|,|l_1|,\ldots,|k_{j_1}|,|l_{j_2}|\leqslant\varepsilon^{-1}} |\hat{\varphi}(-([k_1 : k_{j_1}] + [l_1 : l_{j_2}]))|^2 \frac{|2\pi k_1^1|^2\cdots|2\pi k_{j_1}^1|^2|2\pi l_1^2|^2\cdots|2\pi l_{j_2}^2|^2}{|2\pi k_1|^4\cdots|2\pi k_{j_1}|^4|2\pi l_1|^4\cdots|2\pi l_{j_2}|^4}.
\end{aligned}$$

The inner sum we estimate as follows, renaming $m_1 := k_1 + \cdots + k_{j_1} + l_1 + \cdots + l_{j_2}, m_i := k_i, i = 2, \ldots, j_1$,

$$\begin{aligned}
&\sum_{|k_1|,|l_1|,\ldots,|k_{j_1}|,|l_{j_2}|\leqslant\varepsilon^{-1}} |\hat{\varphi}(-([k_1 : k_{j_1}] + [l_1 : l_{j_2}]))|^2 \frac{|2\pi k_1^1|^2\cdots|2\pi k_{j_1}^1|^2|2\pi l_1^2|^2\cdots|2\pi l_{j_2}^2|^2}{|2\pi k_1|^4\cdots|2\pi k_{j_1}|^4|2\pi l_1|^4\cdots|2\pi l_{j_2}|^4} \\
&\quad = \sum_{\substack{m_1\in\mathbb{Z}^2, |m_2|,\ldots,|m_{j_1}|\leqslant\varepsilon^{-1} \\ |l_1|,\ldots,|l_{j_2}|\leqslant\varepsilon^{-1}}} |\hat{\varphi}(-m_1)|^2 \frac{|2\pi m_2^1|^2\cdots|2\pi m_{l_1}^1|^2|2\pi l_1^2|^2\cdots|2\pi l_{j_2}^2|^2}{|2\pi m_2|^4\cdots|2\pi m_{j_1}|^4|2\pi l_1|^4\cdots|2\pi l_{j_2}|^4} \times \\
&\qquad\qquad \frac{|2\pi(m_1^1 - m_2^1 - \cdots - m_{l_1}^1)|^2 \mathbf{1}_{|m_1 - m_2 - \cdots - m_{l_1}|\leqslant\varepsilon^{-1}}}{|2\pi(m_1 - m_2 - \cdots - m_{l_1})|^4}
\end{aligned}$$

$$\leqslant \sum_{\substack{m_1\in\mathbb{Z}^2,|m_2|,\ldots,|m_{j_1}|\leqslant\varepsilon^{-1}\\ |l_1|,\ldots,|l_{j_2}|\leqslant\varepsilon^{-1}}} |\hat{\varphi}(-m_1)|^2 \frac{|2\pi m_2^1|^2\cdots|2\pi m_{l_1}^1|^2|2\pi l_1^2|^2\cdots|2\pi l_{j_2}^2|^2}{|2\pi m_2|^4\cdots|2\pi m_{j_1}|^4|2\pi l_1|^4\cdots|2\pi l_{j_2}|^4}$$
$$= \|\varphi\|_{L^2}^2\Big(\sum_{|k|\leqslant\varepsilon^{-1}}\frac{|2\pi k^1|^2}{|2\pi k|^4}\Big)^{j_1+j_2-1} = \|\varphi\|_{L^2}^2\Big(\frac{\sigma_\varepsilon^2}{2}\Big)^{j_1+j_2-1}.$$

Furthermore, as $(h_j(Z)/\sqrt{j!})$ is an orthonormal basis of $L^2(\mathrm{Law}(Z))$, and using again stationarity, we obtain

$$\sum_{j=(j_1,j_2)\in\mathbb{N}^2\setminus\{(0,0)\}} |c_j(\Phi_\varepsilon)|^2\, j! = \mathbb{E}[|\phi_\varepsilon(Z_1^\varepsilon(x),Z_2^\varepsilon(x))|^2] = \mathbb{E}[|\sqrt{|Z|^2+1}-c|^2] < \infty.$$

Together, this yields

$$\mathbb{E}[|\langle\phi_\varepsilon(Z_1^\varepsilon,Z_2^\varepsilon),\varphi\rangle|^2] \lesssim \|\varphi\|_{L^2}^2\sigma_\varepsilon^{-2}\to 0. \qquad \square$$

Proposition 1.30. *For the rescaled surface measures* (ν^ε)*, the convergence in distribution to the Lebesgue measure on the torus in the space of finite positive measures follows:*

$$\nu^\varepsilon \Rightarrow c\lambda_{\mathbb{T}^2}. \tag{1.82}$$

Proof. The proof follows from an application of [Wal86, Theorem 6.15] together with the tightness from Lemma 1.28 and the distributional convergence (even in probability) of the finite dimensional distributions $(\nu^\varepsilon(\varphi_1),\ldots,\nu^\varepsilon(\varphi_n))$ to $(c\lambda_{\mathbb{T}^2}(\varphi_1),\ldots,c\lambda_{\mathbb{T}^2}(\varphi_n))$ for $\varphi_1,\ldots,\varphi_n\in C(\mathbb{T}^2)$, $n\in\mathbb{N}$, by Lemma 1.29. $\square$

2. Weak error bounds for a nonlinear approximation of the Dean-Kawasaki equation

In this chapter, we consider suitable approximations of the Dean-Kawasaki (DK) equation ([Dea96, Kaw94]). The DK equation, without interaction, reads

$$\partial_t u = \frac{1}{2}\Delta u + \frac{1}{\sqrt{N}}\nabla \cdot (\sqrt{u}\xi), \tag{2.1}$$

where ξ denotes space-time white noise. The empirical density of N independent Brownian particles is a solution to the DK equation (in the sense we specify below), if $u_0 = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ for the Dirac measures δ_{x_i} in x_i. From an SPDE perspective, equation (2.1) is highly irregular and ill-behaved. In applications, typically $N \gg 1$ is large and simulating directly the particle system is computationally expensive, cf. [HCD$^+$21, Section 4.4.1]. We thus seek to approximate (2.1) in a way that respects the physical constraints of the particle system and prove weak error estimates of a desirable rate. We introduce our nonlinear approximation in Section 2.1 and prove the error estimates in Section 2.4. In Sections 2.2 and 2.3, we prove well-posedness and a comparison principle for a class of regularized Dean-Kawasaki SPDEs, that include our approximation. In particular, non-negativity and mass preservation of the approximation follow.

2.1. Preliminaries and approximations

Consider the particle system of $N \in \mathbb{N}$ independent standard Brownian motions $(X^i)_{i=1}^N$ projected onto the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (with generator being the Laplacian with periodic boundary conditions) started at $X_0^i = x^i \in \mathbb{T}^d$ and its empirical measure

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}. \tag{2.2}$$

Utilizing Itô's formula for $\langle \mu_t^N, \varphi\rangle = \frac{1}{N}\sum_{i=1}^N \varphi(X_t^i)$, $\varphi \in C^\infty(\mathbb{T}^d)$, one sees that μ^N formally solves the Dean-Kawasaki equation without with atomic initial condition,

$$\begin{aligned} \partial_t \mu^N &= \frac{1}{2}\Delta\mu^N + \frac{1}{\sqrt{N}}\nabla\cdot(\sqrt{\mu^N}\xi) \\ \mu_0^N &= \frac{1}{N}\sum_{i=1}^N \delta_{x^i}, \end{aligned} \tag{2.3}$$

where $\xi := (\xi^j)_{j=1,\dots,d}$ with independent space-time white noise processes ξ^j. A formal derivation can be found in the original work [Dea96]. As in [KLvR19, Definition 2.1], we will consider the solution to the associated martingale problem. The state space of the solution process is the space $\mathscr{M}$ of probability measures on $\mathbb{T}^d$. We equip $\mathscr{M}$ with the topology of weak convergence of probability measures, metrized by $d(\pi,\nu) = \sum_k 2^{-k}(|\pi(f_k) - \nu(f_k)| \wedge 1)$ for a dense sequence $(f_k)_{k\in\mathbb{N}} \subset C(\mathbb{T}^d)$. Here, for $\mu \in \mathscr{M}$, $\varphi \in C(\mathbb{T}^d)$, we write $\mu(\varphi) = \langle \mu, \varphi\rangle = \int_{\mathbb{T}^d} \varphi d\mu$.

Definition 2.1. *We call a stochastic process $(\mu_t^N)_{t\geqslant 0}$ on a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ with values in $\mathscr{M}$ a solution to* (2.3) *if for every $t \in [0,T]$ and for all test functions $\varphi \in C^\infty(\mathbb{T}^d)$, the process*

$$t \mapsto \langle \mu_t^N, \varphi\rangle - \langle \mu_0^N, \varphi\rangle - \int_0^t \langle \mu_s^N, \frac{1}{2}\Delta\varphi\rangle ds$$

is an $(\mathscr{F}_t)$–adapted martingale with quadratic variation

$$\frac{1}{N}\int_0^\cdot \langle \mu_s^N, |\nabla\varphi|^2\rangle ds. \tag{2.4}$$

From [KLvR19, Theorem 2.2], it follows in particular that the stated martingale problem has a unique (in law) solution given by (2.2). Replacing in (2.3), μ_0^N by a general initial condition $\mu_0 \in \mathscr{M}$ and N by a constant $\alpha \in \mathbb{R}^{>0} \setminus \mathbb{N}$, the authors moreover prove, that there exists no solution to the equation in the sense of martingale solutions. In this sense, the result is not very robust with respect to changes of the parameter N and the equation is not suitable for a stable numerical approximation. Our goal is to approximate (2.3) in a controlled manner with an equation that has stability properties and that preserves the physical contraints (positivity and unit mass).

Our approach is to replace the non-Lipschitz square root function with a Lipschitz approximation that will depend on a parameter δ and replace the noise by its ultra-violet cutoff. The parameter δ will influence the order of the approximation and will be chosen subsequently in the error estimate depending on N. Let for now $\delta \equiv \delta_N > 0$.

We define the Lipschitz function $f \equiv f^\delta$ as follows:

$$f(x) = \begin{cases} \frac{1}{\sqrt{\delta}}x & |x| \leqslant \delta/2, \\ \text{smooth} & \delta/2 \leqslant |x| \leqslant \delta, \\ \text{sign}(x)\sqrt{|x|} & |x| \geqslant \delta. \end{cases} \tag{2.5}$$

The smooth interpolation should be such that $f \in C^1(\mathbb{R})$ satisfies

$$\|f'\|_{L^\infty} \lesssim \frac{1}{\sqrt{\delta}}, \quad |f'(x)| \lesssim \frac{1}{\sqrt{x}}, \text{ for all } x > 0 \tag{2.6}$$

and

$$|f(x)| \lesssim \sqrt{|x|}, \quad |f(x)^2 - x| \lesssim \delta, \text{ for all } x \geqslant 0. \tag{2.7}$$

The short-hand notation $a \lesssim b$ means that there exists a constant $C > 0$ (not depending on the relevant parameters; above the parameters are δ, x), such that the bound $a \leqslant Cb$ holds. In the case that we want to indicate the dependence of the constant $C(d)$ on a parameter d, we write $a \lesssim_d b$.
Any C^1 approximation of the square root satisfying those bounds works for our analysis and a particular example of such a function is given in the following remark.

Remark 2.2 (Example)**.** *Consider, for example,*

$$f(x) = \begin{cases} \frac{1}{\sqrt{\delta}}x & |x| \leqslant \delta/2, \\ -\frac{2\sqrt{\delta}}{\delta^3}x^3 + \text{sign}(x)\frac{4}{\delta\sqrt{\delta}}x^2 - \frac{3}{2\sqrt{\delta}}x + \text{sign}(x)\frac{\sqrt{\delta}}{2} & \delta/2 \leqslant |x| \leqslant \delta, \\ \text{sign}(x)\sqrt{|x|} & |x| \geqslant \delta. \end{cases} \tag{2.8}$$

One easily varifies that $f \in C^1$ and that f satisfies the bounds (2.6) *and* (2.7)*.*

We denote by $\tilde{\mu}^N$ the solution of the approximated equation

$$\begin{aligned} d\tilde{\mu}_t^N &= \frac{1}{2}\Delta\tilde{\mu}_t^N dt + \frac{1}{\sqrt{N}}\nabla \cdot (f(\tilde{\mu}_t^N)dW_t^N), \\ \tilde{\mu}_0^N &= \rho^N * \mu_0^N \in L^2(\mathbb{T}^d), \end{aligned} \tag{2.9}$$

where $(\rho^N)_N$ is an appropriate approximation of the identity, that we will choose later. Moreover, the truncated noise $(W_t^N)_t$ is given by

$$W_t^N(x) := \sum_{0 \leqslant |k| \leqslant M_N} e_k(x)B_t^k := \sum_{0 \leqslant |k| \leqslant M_N} \exp(2\pi i k \cdot x)B_t^k \tag{2.10}$$

for $x \in \mathbb{T}^d$ and independent d-dimensional complex-valued Brownian motions $(B^k)_{k\in\mathbb{Z}^d}$ (that is, $B^k = B^{k,1} + iB^{k,2}$ for independent $\mathbb{R}^d$-valued Brownian motions $B^{k,1}, B^{k,2}$) with constraint $\overline{B^k} = B^{-k}$, and a truncation parameter $M_N \in \mathbb{N}$, that will be chosen

optimally depending on N for the error estimate.
Well-posedness of equation (2.9), as well as non-negativity of the strong solution and the preservation of mass property follow from the results of the following two sections.

2.2. Well-posedness for regularized DK-type SPDEs

In this section, we prove well-posedness for the class of SPDEs of the type

$$\begin{aligned} \partial_t u_t &= \frac{1}{2}\Delta u_t + \nabla \cdot (b(u_t)W_t^\phi) \\ u_0 &\in L^2(\mathbb{T}^d) \end{aligned} \tag{2.11}$$

under the following assumptions.

Assumption 2.3 (Assumption on the noise)**.** *The noise $(W_t^\phi)_t$ can be written as*

$$W_t^\phi(x) := \sum_{k\in\mathbb{Z}^d} \phi_k(x) B_t^k$$

for $(\phi_k)_k \subset C^1(\mathbb{T}^d, \mathbb{C})$ with $\overline{\phi_k} = \phi_{-k}$ and independent d-dimensional complex-valued Brownian motions $(B^k)_{k\in\mathbb{Z}^d}$ with $\overline{B^k} = B^{-k}$. Moreover, the expansion is such that

$$C_1 := \sum_{k\in\mathbb{Z}^d} \|\phi_k\|_\infty^2 < \infty \quad \text{and} \quad C_2 := \sum_{k\in\mathbb{Z}^d} \|\nabla\phi_k\|_\infty^2 < \infty, \tag{2.12}$$

with supremum norms $\|\phi_k\|_\infty^2 := \sup_{x\in\mathbb{T}^d}|\phi_k(x)|^2 = \sup_{x\in\mathbb{T}^d} \phi_k(x)\phi_{-k}(x)$ and $\|\nabla\phi_k\|_\infty^2 := \sup_{x\in\mathbb{T}^d}\sum_{i=1}^d |\partial_i\phi_k(x)|^2$.

Remark 2.4. *An example for a noise expansion satisfying Assumption 2.3 is the Fourier expansion with cut-off $M_N \in \mathbb{N}$ from* (2.10)*. The summability assumptions, $C_1 + C_2 < \infty$, are trivially satisfied due to the finite cut-off. Similar to [FG21, Remark 2.3], we may as well consider a noise*

$$\xi^a(x) = \sum_{k\in\mathbb{Z}^d} a_k \exp(2\pi i k \cdot x) B^k,$$

for a real sequence $(a_k) \subset \mathbb{R}$ with $\sum_{k\in\mathbb{Z}^d}|k|^2 a_k^2 < \infty$, which also satisfies Assumption 2.3.

Assumption 2.5 (Assumption on the diffusion)**.** *The diffusion coefficient $b \in C^1(\mathbb{R}, \mathbb{R})$ has a bounded derivative with bound $L > 0$,*

$$\|b'\|_\infty \leqslant L, \tag{2.13}$$

and is of linear growth with constant $C > 0$,

$$|b(x)|^2 \leqslant C^2(1 + |x|^2), \qquad x \in \mathbb{R}. \tag{2.14}$$

Assumption 2.6 (Assumption on the constants)**.** *The parameters from Assumptions 2.3 and 2.5 satisfy* $C_1 \max(L, C)^2 < 1/2$.

From Assumption 2.5, it follows that b is Lipschitz with bound L, that is,

$$|b(x) - b(y)| \leq L|x - y|, \quad \text{for all } x, y \in \mathbb{R}. \tag{2.15}$$

In particular, the well-posedness theory of this section applies to the approximated Dean-Kawasaki equation (2.9). In that case, $b = \frac{1}{\sqrt{N}} f$ for the regularized square root function f from (2.5) with $L = C = 1/\sqrt{N\delta}$ and the noise expansion is given with respect to the Fourier basis $(e_k)_k$ with cut-off $M_N \in \mathbb{N}$, so that we have $C_1 \leqslant (2M_N)^d$ and $C_2 \leqslant (2M_N)^d (2\pi M_N)^2$.
Note that due to the gradient noise term, the local monotonicity condition is violated for SPDEs of the form (2.11) and standard variational theory cannot be applied directly. Instead, we transform equation (2.11) (multiplying by $(\Delta - 1)^{1/2}$) into an equation for which the variational theory can be applied. We then conclude on well-posedness for the original equation using a priori energy bounds.

Remark 2.7. *The form of equation* (2.11) *is similar to the SPDEs studied in [Bec21]. Specifically to the case of "critical unboundedness" from [Bec21, Section 4]. However, a direct application of their methods to our setting would only yield a probabilistically weak solution with paths in* $C([0, T], H^{-\varepsilon}) \cap L^2([0, T], H^1)$ *for any* $\varepsilon > 0$. *Still, by a pathwise uniqueness argument using a priori energy bound, one could in addition show strong existence and uniqueness of a solution with paths in* $C([0, T], H^{-\varepsilon}) \cap L^2([0, T], H^1)$ *for any* $\varepsilon > 0$. *Instead, in what follows, we directly show the stronger statement about strong existence and uniqueness of a solution with path in* $C([0, T], L^2) \cap L^2([0, T], H^1)$.

The setting for the variational theory is defined as follows. For notation and concepts cf. [LR15]. Consider the Sobolev space $V = H^1(\mathbb{T}^d)$ with $V^* = H^{-1}(\mathbb{T}^d)$ and $H = L^2(\mathbb{T}^d)$, such that we have the Gelfand triple $V \subset H \subset V^*$. We consider the Laplacian with periodic boundary conditions, that is, $\Delta : V = H^1(\mathbb{T}^d) \to (H^1(\mathbb{T}^d))^* = V^*$ with $\Delta u(v) := \langle \Delta u, v \rangle_{V^*,V}$, $u \in V^*, v \in V$. For the duality pairing, we have $\langle \Delta u, v \rangle_{V^*,V} = -\langle \nabla u, \nabla v \rangle_H = -\int \nabla u \cdot \nabla v$ for $u, v \in V$. Equipped with these prerequisites, we define a solution to the equation (2.11) as follows.

Definition 2.8. *Let* $(B^k)_k$ *and* $(\phi_k)_k$ *satisfying Assumptions 2.3 and 2.5. We call a stochastic process* $(u_t)_{t \geqslant 0}$ *with paths in* $L^2([0, T], H^1(\mathbb{T}^d)) \cap C([0, T], L^2(\mathbb{T}^d))$ *a (probabilistically strong and analytically weak) solution to equation* (2.11) *for initial condition* $u_0 \in L^2(\mathbb{T}^d)$, *if for all* $\varphi \in H^1(\mathbb{T}^d)$,

$$\begin{aligned} \langle u_t, \varphi \rangle &= \langle u_0, \varphi \rangle - \int_0^t \frac{1}{2} \langle \nabla u_s, \nabla \varphi \rangle ds + \sum_{k \in \mathbb{Z}^d} \int_0^t \langle \nabla (b(u_t)\phi_k), \varphi \rangle \cdot dB_s^k \\ &= \langle u_0, \varphi \rangle - \sum_{i=1}^d \int_0^t \frac{1}{2} \langle \partial_i u_s, \partial_i \varphi \rangle ds + \sum_{i=1}^d \sum_{k \in \mathbb{Z}^d} \int_0^t \langle \partial_i (b(u_t)\phi_k), \varphi \rangle dB_s^{k,i}. \end{aligned} \tag{2.16}$$

First we will show an auxiliary result for the transformed equation that is based on the variational approach.

Lemma 2.9. *Let Assumptions 2.3, 2.5 and 2.6 hold and let* $v_0 \in L^2(\mathbb{T}^d)$. *Then there exists a unique probabilistically strong solution* $v \in L^2([0,T], H^1(\mathbb{T}^d)) \cap C([0,T], L^2(\mathbb{T}^d))$ *of*

$$d\langle v_t, \varphi\rangle = -\frac{1}{2}\langle \nabla v_t, \nabla\varphi\rangle dt + \sum_{k\in\mathbb{Z}^d} \langle G_k(v_t), \varphi\rangle \cdot dB_t^k, \quad \text{for all } \varphi \in H^1, \tag{2.17}$$

where $G_k(v) := (1-\Delta)^{-1/2}\nabla(b((1-\Delta)^{1/2}v)\phi_k)$.

Proof. We will check the conditions of [LR15, Section 4] to apply the variational theory. Existence and uniqueness of the solution then follows form [LR15, Theorem 4.2.4]. Let $G(v)w := \sum_{k\in\mathbb{Z}^d}(1-\Delta)^{-1/2}\nabla(b((1-\Delta)^{1/2}v)\phi_k)w_k$ for $w \in l^2(\mathbb{Z}^d)$ and $v \in H^1$. Using (2.14), we obtain the following coercivity bound [LR15, Section 4, Assumption (H3)]:

$$\begin{aligned}
&\langle \Delta v, v\rangle_{V^*,V} + \|G(v)\|^2_{L_2(l^2(\mathbb{Z}^d),H)} \\
&\quad = -\|\nabla v\|^2_{L^2} + \sum_k \|(1-\Delta)^{-1/2}\nabla(b((1-\Delta)^{1/2}v)\phi_k)\|^2_{L^2} \\
&\quad \leqslant -\|\nabla\cdot v\|^2_{L^2} + \sum_k \|(b((1-\Delta)^{1/2}v)\phi_k)\|^2_{L^2} \\
&\quad \leqslant -\|\nabla v\|^2_{L^2} + \sum_k \|\phi_k\|^2_\infty \|b((1-\Delta)^{1/2}v)\|^2_{L^2} \\
&\quad \leqslant -\|\nabla v\|^2_{L^2} + C_1C^2(1 + \|(1-\Delta)^{1/2}v\|^2_{L^2}) \\
&\quad = (C_1C^2 - 1)\|\nabla v\|^2_{L^2} + C_1C^2,
\end{aligned}$$

by Assumption 2.12. Here, the first inequality follows from the isometry of $L^2(\mathbb{T}^d)$ and $l^2(\mathbb{Z}^d)$ by the Fourier transform $\mathscr{F}_{\mathbb{T}^d}$ given by $\mathscr{F}_{\mathbb{T}^d}f(k) := \int_{\mathbb{T}^d} e^{-2\pi ik\cdot x}f(x)dx$, such that for $f \in L^2(\mathbb{T}^d)$,

$$\begin{aligned}
\|(1-\Delta)^{-1/2}\nabla f\|^2_{L^2(\mathbb{T}^d)} &= \sum_{k\in\mathbb{Z}^d}(1+|2\pi k|^2)^{-1}|2\pi k|^2|\mathscr{F}_{\mathbb{T}^d}(f)(k)|^2 \\
&\leqslant \sum_{k\in\mathbb{Z}^d}|\mathscr{F}_{\mathbb{T}^d}(f)(k)|^2 \\
&= \|f\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

Coercivity then follows by Assumption 2.6 on the parameters. The weak monotonicity condition [LR15, Section 4, Assumption (H2)] follows from an analogue estimate as the one above, using the global Lipschitz bound L from (2.13):

$$\begin{aligned}
&\big\langle \Delta(v^1-v^2), v^1-v^2\big\rangle_{V^*,V} + \|G(v^1)-G(v^2)\|^2_{L^2(l^2(\mathbb{Z}^d),H)} \\
&\quad = -\|\nabla(v^1-v^2)\|^2_{L^2} + \sum_k \Big\|(1-\Delta)^{-1/2}\nabla\Big(\big(b\big((1-\Delta)^{1/2}v^1_t\big)-b\big((1-\Delta)^{1/2}v^2_t\big)\big)\phi_k\Big)\Big\|^2_{L^2} \\
&\quad \leqslant -\|\nabla(v^1-v^2)\|^2_{L^2} + \sum_k \big\|\big(b((1-\Delta)^{1/2}v^1_t)-b((1-\Delta)^{1/2}v^2_t)\big)\phi_k\big\|^2_{L^2} \\
&\quad \leqslant -\|\nabla(v^1-v^2)\|^2_{L^2} + \sum_k \|\phi_k\|^2_\infty L^2\|(1-\Delta)^{1/2}(v^1_t-v^2_t)\|^2_{L^2} \\
&\quad = -\|\nabla(v^1-v^2)\|^2_{L^2} + C_1L^2(\|v^1_t-v^2_t\|^2_{L^2} + \|\nabla(v^1_t-v^2_t)\|^2_{L^2}) \\
&\quad \leqslant C_1L^2\|v^1_t-v^2_t\|^2_{L^2}.
\end{aligned}$$

Due to our assumption on the parameters, which implies $C_1L^2 < 1/2$, we thus obtain the weak monotonicity. The hemicontinuity from [LR15, Section 4, Assumption (H1)] follows by linearity of the Laplacian. The boundedness [LR15, Section 4, Assumption (H4)] is trivially satisfied due to continuity, that is, $\|\Delta u\|_{V^*} \leqslant \|u\|_V$. □

Remark 2.10. *Notice that, if u solves* (2.16)*, then $v := (1-\Delta)^{-1/2}u$ solves* (2.17)*. As* (2.17) *has a unique strong solution by [LR15, Theorem 4.2.4], it follows that the solution to* (2.16) *is unique.*

To prove existence of a solution for (2.16) according to Definition 2.8, we utilize a priori energy estimates, that is Lemma 2.11 below.
Given the solution v of (2.17), we can define $u_t := (1-\Delta)^{1/2}v_t$. Then it follows that $u \in C([0,T], H^{-1}(\mathbb{T}^d)) \cap L^2([0,T], L^2(\mathbb{T}^d))$ almost surely and that u is a solution to the (very weak) equation

$$d\langle u_t,\varphi\rangle = \frac{1}{2}\langle u_t,\Delta\varphi\rangle dt - \sum_k \langle b(u_t)\phi_k,\nabla\varphi\rangle \cdot dB^k_t, \qquad \text{for all } \varphi \in C^\infty(\mathbb{T}^d). \tag{2.18}$$

Define the orthogonal projection $\Pi_R : H^1 \to V_R$ with $V_R := \operatorname{span}\{e_r \mid r \in \mathbb{Z}^d, |r| \leqslant R\}$ for an orthonormal basis $(e_r)_{r\in\mathbb{Z}^d}$ of $H^1(\mathbb{T}^d)$ with $e_r \in C^\infty(\mathbb{T}^d)$. Let v^R solve (2.17) with $v^R_0 = \Pi_R v_0$ and G_k replaced by G^R_k with

$$G^R_k(v) := \Pi_R(1-\Delta)^{-1/2}\nabla\big(b(\Pi_R(1-\Delta)^{1/2}v)\phi_k\big), \quad v \in H^1. \tag{2.19}$$

That is, v^R solves

$$\begin{aligned}
d\langle v^R_t,\varphi\rangle &= -\frac{1}{2}\langle\nabla v_t,\nabla\varphi\rangle dt + \sum_{k\in\mathbb{Z}^d}\langle G^R_k(v_t),\varphi\rangle \cdot dB^k_t, \qquad \text{for all } \varphi \in H^1, \\
v^R_0 &= \Pi_R v_0.
\end{aligned} \tag{2.20}$$

Then, as the coefficients for the equation for v^R also satisfy the assumptions of [LR15, Section 4], v^R is a strong solution of (2.20). Furthermore, if Assumption 2.6 is satisfied, using [LR15, Theorem 4.2.5] applied for $\|v_T^R - v_T\|_{L^2}^2$, one can prove the following estimate for a constant $\tilde{\lambda} > 0$,

$$\begin{aligned}\mathbb{E}[\|v_T^R - v_T\|_{L^2}^2] + \lambda \int_0^T \mathbb{E}[\|v_s^R - v_s\|_{H^1}^2] ds \\ \leqslant 2C_2CT\|v_0^R - v_0\|_{L^2}^2 \exp(T\tilde{\lambda}) + C_1C^2\|\mathrm{Id} - \Pi_R\|_{L(L^2)}^2(1 + \|v\|_{L^2(\Omega\times[0,T],H^1)}^2).\end{aligned}$$

The proof of the above estimate is analogous to the proof of Lemma 2.11 below. In particular, we obtain that $v^R \to v$ in $L^2(\Omega \times [0,T], H^1)$ for $R \to \infty$. Let now

$$u^R := (1-\Delta)^{1/2}\Pi_R v^R = \Pi_R(1-\Delta)^{1/2}v^R. \tag{2.21}$$

The following lemma proves an energy estimate, that yields tightness of the sequence of Galerkin-type projected solutions $(u^R)_R$.

Lemma 2.11. *Let Assumptions 2.3, 2.5 and 2.6 hold. Let $u_0 \in L^2(\mathbb{T}^d)$ and $v_0 := (1-\Delta)^{-1/2}u_0$. Let v^R solve (2.20) with initial condition $v_0^R = \Pi_R v_0$. Let u^R be defined as in (2.21). Then, the following energy bound holds true:*

$$\mathbb{E}[\|u_t^R\|_{L^2}^2] + \lambda \int_0^t \mathbb{E}[\|u_s^R\|_{H^1}^2] ds \leqslant 2C_2C^2t\|u_0\|_{L^2}^2 \exp(t\tilde{\lambda}), \tag{2.22}$$

with $\lambda := 1 - C_1L^2 > 0$ and $\tilde{\lambda} := \lambda + C_2C^2$.

Proof. As $e_r \in C^\infty(\mathbb{T}^d)$, it follows that $\Pi_R v^R \in C([0,T], C^\infty)$ almost surely and thus in particular $u^R \in C([0,T], C^\infty)$ almost surely. Furthermore, by the equation (2.20) for v^R, u^R solves

$$\langle u_t^R, \varphi\rangle = \langle u_0^R, \varphi\rangle + \int_0^t \frac{1}{2}\langle \Delta u_s^R, \varphi\rangle ds + \sum_k \int_0^t \langle \Pi_R \nabla(b(u_t^R)\phi_k), \varphi\rangle \cdot dB_s^k,\ \forall \varphi \in C^\infty(\mathbb{T}^d). \tag{2.23}$$

Due to $u^R \in C([0,T], C^\infty)$, this implies that for $x \in \mathbb{T}^d$,

$$u_t^R(x) = u_0^R(x) + \int_0^t \frac{1}{2}\Delta u_s^R(x) ds + \sum_k \int_0^t \Pi_R \nabla(b(u_s^R)\phi_k)(x) \cdot dB_s^k.$$

By applying Itô's formula to $(u_t^R(x))^2$, we obtain

$$\begin{aligned} d\|u_t^R\|_{L^2}^2 &= \int d(u_t^R(x))^2 dx \\ &= 2\int u_t^R(x)\Big(\frac{1}{2}\Delta u_t^R(x) dt + \sum_k \Pi_R \nabla\big(b(u_t^R(x))\phi_m(x)\big) \cdot dB_t^m\Big) dx \\ &\quad + \sum_k \int \big|\Pi_R \nabla\big(b(u_t^R(x))\phi_m(x)\big)\big|^2 dx dt. \end{aligned}$$

Taking the expectation, the martingale vanishes. Using that $\|\Pi_R v\|_{L^2} \leqslant \|v\|_{L^2}$ and (2.13), we thus obtain

$$\begin{aligned}\mathbb{E}[\|u_t^R\|_{L^2}^2] \leqslant \|u_0^R\|_{L^2}^2 - &\int_0^t \mathbb{E}\Big[\int_{\mathbb{T}^d} \nabla u_s^R(x)\cdot\nabla u_s^R(x)dx\Big]ds \\ &+ \int_0^t \sum_k \mathbb{E}[\|b'(u_s^R)\phi_k \nabla u_s^R\|_{L^2}^2]ds + \int_0^t \sum_k \mathbb{E}[\|b(u_s^R)\nabla\phi_k\|_{L^2}^2]ds \\ &\leqslant \|u_0^R\|_{L^2}^2 - \int_0^t \mathbb{E}[\|\nabla u_s^R\|_{L^2}^2]ds + C_1 L^2 \int_0^t \mathbb{E}[\|\nabla u_s^R\|_{L^2}^2]ds \\ &+ C_2 \int_0^t \mathbb{E}[\|b(u_s^R)\|_{L^2}^2]ds. \end{aligned} \tag{2.24}$$

With $\lambda = 1 - C_1 L^2$, we add and subtract $\lambda \int_0^t \mathbb{E}[\|\mu_s^R\|_{L^2}^2]ds$ to (2.24). Furthermore, utilizing the linear growth assumption on b given by (2.14), we obtain

$$\mathbb{E}[\|u_t^R\|_{L^2}^2] \leqslant \|u_0^R\|_{L^2}^2 - \lambda \int_0^t \mathbb{E}[\|u_s^R\|_{H^1}^2]ds + (C_2C^2 + \lambda)\int_0^t \mathbb{E}[\|u_s^R\|_{L^2}^2]ds + C_2C^2 t. \tag{2.25}$$

Using $\lambda > 0$, we thus obtain by Gronwall's inequality

$$\mathbb{E}[\|u_t^R\|_{L^2}^2] \leqslant C_2C^2 t \|u_0^R\|_{L^2}^2 \exp(t\tilde{\lambda}) \tag{2.26}$$

for $\tilde{\lambda} = C_2C^2 + \lambda$. Hence, plugging (2.26) into (2.25), yields

$$\mathbb{E}[\|u_t^R\|_{L^2}^2] + \lambda \int_0^t \mathbb{E}[\|u_s^R\|_{H^1}^2]ds \leqslant C_2C^2 t \|u_0^R\|_{L^2}^2 \exp(t\tilde{\lambda}),$$

which implies (2.22), as $\|u_0^R\|_{L^2}^2 = \|\Pi_R u_0\|_{L^2}^2 \leqslant \|u_0\|_{L^2}^2$. □

Remark 2.12. *Once we know that the sequence of mollifications $(u^R)_R$ converges in $L^2([0,T], H^1)$ almost surely, we can apply Fatou's lemma and obtain the energy bound (2.22) for the limit u.*

Remark 2.13 (Energy estimate)**.** *If we take $b = \frac{1}{\sqrt{N}} f$ for the regularized square root function f given by (2.5), we can improve the energy estimate using that by (2.7), $|f(x)| \lesssim \sqrt{|x|}$, in order to estimate the L^2-norm of $b(u_s^R)$ in (2.24). Utilizing also the mass conservation property of the solution u, that we later prove in Proposition 2.19, we then obtain the following a priori energy bound*

$$\begin{aligned}\mathbb{E}[\|u_t\|_{L^2}^2] + \lambda \int_0^t \mathbb{E}[\|\nabla u_s\|_{L^2}^2]ds &\lesssim \|u_0\|_{L^2}^2 + \frac{C_2}{N}\int_0^t \mathbb{E}[\|u_s\|_{L^1}]ds \\ &\lesssim \|u_0\|_{L^2}^2 + \frac{C_2}{N} t \|u_0\|_{L^1}. \end{aligned} \tag{2.27}$$

Theorem 2.14. *Let Assumptions 2.3, 2.5 and 2.6 hold and let $u_0 \in L^2(\mathbb{T}^d)$. Then there exists a unique solution u with paths in $L^2([0,T], H^1(\mathbb{T}^d)) \cap C([0,T], L^2(\mathbb{T}^d))$ of the equation* (2.11) *in the sense of Definition 2.8.*

Proof. From Lemma 2.9 it follows that for $v_0 := (1-\Delta)^{-1/2} u_0 \in H^1 \subset L^2$, there exists a unique strong solution $v \in L^2([0,T], H^1) \cap C([0,T], L^2)$ of (2.17). Let $u_t := (1-\Delta)^{1/2} v_t$. By the regularity of v, we obtain that almost surely

$$u \in L^2([0,T], L^2) \cap C([0,T], H^{-1}).$$

Furthermore, from the equation of v, testing against $\varphi \in C^\infty(\mathbb{T}^d)$, we obtain that u solves (2.18). By Lemma 2.11, the Galerkin projected solutions $(u^R)_R$ satisfy the energy bound (2.22).
Since $L^2(\Omega \times [0,T], H^1))$ is reflexive, we thus obtain that $(u^R)_R$ converges weakly in $L^2(\Omega \times [0,T], H^1))$ along a subsequence. As $v^R \to v$ in $L^2(\Omega \times [0,T], H^1)$ (see above), it follows that for $R \to \infty$,

$$u^R = \Pi_R (1-\Delta)^{1/2} v^R \to (1-\Delta)^{1/2} v = u$$

in $L^2(\Omega \times [0,T], L^2)$. Thus, the limit of each such subsequence is given by u and we can conclude that the whole sequence $(u^R)_R$ converges to u, weakly in $L^2(\Omega \times [0,T], H^1)$. In particular, the limit u satisfies

$$u \in L^2([0,T], H^1)$$

almost surely. Due to $u \in L^2([0,T], H^1) \cap C([0,T], H^{-1})$ a.s., the mapping $t \mapsto u_t \in L^2$ is almost surely weakly continuous. Since $u \in L^2([0,T], H^1)$ a.s. and $u_0 \in L^2$, and because (2.18) is equivalent to u solving

$$du_t = -\frac{1}{2}\Delta u_t dt + \sum_k \nabla[b(u_t)\phi_k] \cdot dB_t^k \in (H^1)^* = H^{-1},$$

we can apply [LR15, Theorem 4.2.5] to conclude that an Itô formula for $d\|u_t\|_{L^2}^2$ follows. Almost sure continuity of the integrals in time then implies almost sure continuity of the mapping

$$t \mapsto \|u_t\|_{L^2}^2. \tag{2.28}$$

From continuity of (2.28) and continuity of $t \mapsto u_t \in L^2$ in the weak topology follows that $u \in C([0,T], L^2)$ almost surely. Hence, together, we indeed have that $u \in L^2([0,T], H^1) \cap C([0,T], L^2)$ almost surely. By the regularity of u and as u solves (2.18), it follows that u solves (2.16) (for all $\varphi \in C^\infty(\mathbb{T}^d)$ and thus, by density for all $\varphi \in H^1(\mathbb{T}^d)$). Uniqueness of the solution follows from Remark 2.10. □

Remark 2.15. *Using the Itô formula for $\|u_t^R - u_t\|_{L^2}^2$ (cf. [LR15, Theorem 4.2.5]) and $u_0^R = \Pi_R u_0 \to u_0$ in L^2, we see that $u^R \to u$ strongly in $L^2(\Omega \times [0,T], H^1)$ and that the energy estimate holds true for the limit u.*

Remark 2.16. *The regularity, $u \in C([0,T],L^2)$ almost surely, will be used to prove the comparison principle, Theorem 2.17, below.*

2.3. A comparison principle for regularized DK-type SPDEs

In this section we prove a comparison principle for the class of SPDEs (2.11), that will in particular imply positivity and mass preservation of the solution.

Theorem 2.17. *Let Assumptions 2.3 and 2.5 hold. Let the parameters from those assumptions moreover satisfy $8C_1L^2 < 1/2$. Furthermore, let u^+ and u^- be two solutions of (2.16) with initial conditions $u_0^+, u_0^- \in L^2$, respectively, such that $u_0^+(x) \geqslant u_0^-(x)$ for λ-almost all $x \in \mathbb{T}^d$. Then it follows that*

$$(\mathbb{P} \otimes \lambda)\big(u_t^+ \geqslant u_t^- \ \forall t \in [0,T]\big) = 1.$$

Proof. The proof is similar to the proof presented in [DMP93, Theorem 2.1]. The main idea is an application of Itô's formula to a suitable C^2 approximation of the map $x \mapsto \max(x,0)^2$, applied to the difference of the solutions. More precisely, let for $p > 0$, $\varphi_p \in C^2(\mathbb{R},\mathbb{R})$ be defined by

$$\varphi_p(x) := \mathbf{1}_{[0,\infty)}(x) \int_0^x \int_0^y [2pz\mathbf{1}_{[0,\frac{1}{p}]}(z) + 2\,\mathbf{1}_{(\frac{1}{p},\infty)}(z)]dzdy.$$

Note that φ_p satisfies

$$0 \leqslant \varphi_p'(x) \leqslant 2\max(x,0) \quad \text{and} \quad 0 \leqslant \varphi_p''(x) \leqslant 2\,\mathbf{1}_{x\geqslant 0}.$$

Next, we define

$$\Phi_p(h) := \int_{\mathbb{T}^d} \varphi_p(h(x))dx.$$

Let $t > 0$ and $w_t := u_t^- - u_t^+$. Since $\varphi_p(x) \uparrow \max(x,0)^2$ for $p \to \infty$, by monotone convergence we conclude that $\Phi_p(w_t) \uparrow \|\max(w_t,0)\|_{L^2}^2$ for $p \to \infty$. Moreover, Φ_p is twice Fréchet differentiable and we obtain by the Itô formula from [Par80, Theorem 1.2] that

$$d\Phi_p(w_t) = -\frac{1}{2}\langle\varphi_p''(w_t), |\nabla w_t|^2\rangle dt + \sum_k \big\langle \varphi_p'(w_t), \nabla\big((b(u_t^-) - b(u_t^+))\phi_k\big)\big\rangle \cdot dB_t^k \tag{2.29}$$

$$+ \frac{1}{2}\sum_k \int \big|\nabla\big((b(u_t^-) - b(u_t^+))\phi_k\big)\big|^2 \varphi_p''(w_t)dxdt. \tag{2.30}$$

Notice that $\Phi_p(w_0) = 0$, since by assumption $w_0 \leq 0$ a.e., and that the martingale term in (2.29) is a indeed a martingale. We can estimate the quadratic variation term

(2.30) using Assumption 2.5 on b and the bound on φ_p'':

$$\begin{aligned}&\left(\nabla\left((b(u_t^-)-b(u_t^+))\phi_k\right)\right)^2\varphi_p''(w_t)\\&\leqslant 2\left[\left((b(u_t^-)-b(u_t^+))\nabla\phi_k\varphi_p''(w_t)\right)^2+\left(\nabla(b(u_t^-)-b(u_t^+))\phi_k\varphi_p''(w_t)\right)^2\right]\\&\leqslant 2\left[L^2|w_t|^2|k|^2 4\,\mathbf{1}_{w_t\geqslant 0}+4\|b'\|_\infty^2|\nabla w_t|^2\right].\end{aligned}$$

Therefore, altogether we obtain

$$\begin{aligned}\mathbb{E}[\phi_p(w_t)]&\leqslant\left(8L^2C_1-\frac{1}{2}\right)\int_0^t\mathbb{E}\left[\varphi_p''(w_s),|\nabla w_s|^2\right]ds+8L^2C_2\int_0^t\mathbb{E}\left[\|\max(w_s,0)\|_{L^2}^2\right]ds\\&\leqslant 8\|b'\|_\infty^2C_2\int_0^t\mathbb{E}\left[\|\max(w_s,0)\|_{L^2}^2\right]ds\end{aligned}$$

due to the assumption $8L^2C_1<\frac{1}{2}$. Taking $p\to\infty$ and applying Gronwall's inequality yields

$$\mathbb{E}\left[\|\max(w_s,0)\|_{L^2}^2\right]=0.$$

Hence, $(\mathbb{P}\otimes\lambda)(w_t\leqslant 0)=1$ for all $t\geqslant 0$. By continuity, $w\in C([0,T],L^2)$, the claim follows. □

In what follows, we assume on b that $b\in C^1(\mathbb{R})$ with

$$b(0)=0,\quad \|b'\|_\infty\leqslant L, \tag{2.31}$$

which implies Assumption 2.5 for $C=L$. Note that Equation (2.31) is satisfied for the regularized square root f given by (2.5).

Corollary 2.18 (Non-negativity of the solution). *Let $8L^2C_1<\frac{1}{2}$ and Assumptions 2.3 and* (2.31) *be satisfied. Let u be a solution of* (2.11) *with initial condition $u_0\geqslant 0$ almost everywhere. Then it follows that* $(\mathbb{P}\otimes\lambda)(u_t\geqslant 0\ \forall t\in[0,T])=1$.

Proof. Since $b(0)=0$, it follows that the zero function is a solution of (2.11). Then the claim directly follows from Theorem 2.17. □

Proposition 2.19 (Conservation of mass). *Let $8L^2C_1<\frac{1}{2}$ and Assumptions 2.3 and* (2.31) *be satisfied. Let u solve* (2.11) *with non-negative initial condition u_0. Then it follows that almost surely, $\int|u_t|(x)dx=\int|u_0|(x)dx$.*

Proof. Using non-negativity of the solution u obtained by Corollary 2.18 and testing the equation against $\varphi=1\in C^\infty(\mathbb{T}^d)$, we have that, for almost all $\omega\in\Omega$,

$$\int|u_t|(x)dx=\int u_t(x)dx=\langle u_t,1\rangle=\langle u_0,1\rangle=\int|u_0|(x)dx.$$

□

2.4. Weak error estimate

In this section we estimate the weak error between the martingale solution μ^N of the Dean-Kawasaki equation and the strong solution $\tilde{\mu}^N$ of the approximate Dean-Kawasaki equation (2.9). More precisely, we aim for a bound of the form

$$|\tilde{\mathbb{E}}[F(\tilde{\mu}_t^N)] - \mathbb{E}[F(\mu_t^N)]| \lesssim_{F,t} N^{-\alpha_d},$$

for $t > 0$ and $F(\mu) := \exp(\langle \mu, \varphi \rangle)$, $\varphi \in C^\infty(\mathbb{T}^d)$, $\mu \in \mathscr{M}$. Here, $\mathbb{E}[\cdot]$ and $\tilde{\mathbb{E}}[\cdot]$ denote the expectations with respect to $\mathrm{Law}(\mu^N) = \mathbb{P}$ and $\mathrm{Law}(\tilde{\mu}^N) = \tilde{\mathbb{P}}$, respectively. In the following, we ease notation and simply write $\mathbb{E}$ instead of $\tilde{\mathbb{E}}$, but we keep in mind, that the error compares $\mathrm{Law}(\mu^N)$ and $\mathrm{Law}(\tilde{\mu}^N)$. Due to the factor $1/N$ in front on the quadratic variation, a direct estimate shows that the hydrodynamic limit, that is, the solution ρ of the heat equation

$$\partial_t \rho = \frac{1}{2}\Delta \rho$$

achieves a weak rate of $\alpha_d = 1$. Furthermore, for the Gaussian approximation $\hat{\rho}$ of the Dean-Kawasaki equation given by the fluctuations of the particle system around the hydrodynamic limit, that is,

$$\partial_t \hat{\rho} = \frac{1}{2}\Delta \hat{\rho} + \frac{1}{\sqrt{N}} \nabla \cdot (\sqrt{\rho} \xi^\phi)$$

one can prove a weak rate of $\alpha_d = 3/2$. As opposed to the approximation $\tilde{\mu}^N$, the Gaussian approximation $\hat{\rho}$ does not satisfy non-negativity. We prove in Theorem 2.21 that the approximation $\tilde{\mu}^N$ achieves a weak rate $\alpha_d \in (1, 3/2)$.

Below, $B^s_{p,q}$ denotes the periodic Besov space of regularity $s \in \mathbb{R}$, integrability $p \in [1, \infty]$ and summability $q \in [1, \infty)$, as in [ST87, Section 3.5.1]:

$$B^s_{p,q}(\mathbb{T}^d) := \{u \in \mathscr{S}'(\mathbb{T}^d) \mid \|(2^{js}\|\Delta_j u\|_{L^p(\mathbb{T}^d)})_{j \geqslant -1}\|_{l^q(\mathbb{Z}^d)} < \infty\} \tag{2.32}$$

for the space of tempered distributions $\mathscr{S}'(\mathbb{T}^d)$ and Littlewood-Paley blocks $\Delta_j u = \mathscr{F}^{-1}_{\mathbb{T}^d}(\rho_j \mathscr{F}_{\mathbb{T}^d} u)$ with Fourier transform $\mathscr{F}_{\mathbb{T}^d} f(k) = \hat{f}(k) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i k \cdot x} dx$, $k \in \mathbb{Z}^d$, and a dyadic partition of unity $(\rho_j)_{j \geqslant -1}$ in the sense of [BCD11, Section 2.2] (cf. also [ST87, Section 3.4.4]). In the case $q = \infty$, we rather work with the separable Besov space, and thus define

$$B^s_{p,\infty} = B^s_{p,\infty}(\mathbb{T}^d) := \{u \in \mathscr{S}'(\mathbb{T}^d) \mid \lim_{j \to \infty} 2^{js}\|\Delta_j u\|_{L^p} = 0\}. \tag{2.33}$$

We write $B^s_{p,q} = B^s_{p,q}(\mathbb{T}^d)$ for short, as we only work on the torus in this section. Since we represent the torus as $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, we can associate to any distribution $u \in \mathscr{S}'(\mathbb{T}^d)$ a periodic distribution on $\mathbb{R}^d$ given by $u^{\mathbb{R}^d}(\phi) = u(\sum_{k \in \mathbb{Z}^d} \phi(\cdot + k))$. Furthermore, we denote by $(P_t)_{t\geqslant 0}$ the heat-semigroup on the torus,

$$P_t\varphi := \mathscr{F}_{\mathbb{T}^d}^{-1}(e^{-t|2\pi\cdot|^2}\mathscr{F}_{\mathbb{T}^d}\varphi) \tag{2.34}$$

for $\varphi \in C^\infty(\mathbb{T}^d)$, which may be extended to Besov distributions via $P_t u(\varphi) := u(P_t\varphi)$, $\varphi \in C^\infty(\mathbb{T}^d)$, $u \in B^\theta_{p,q}$. We have that $P_t u = p(t,\cdot) * u = \int_{\mathbb{T}^d} p(t, x-y)u(y)dy$ for the heat kernel

$$p(t,x) = \sum_{k\in\mathbb{Z}^d} e^{2\pi i k\cdot x}e^{-|2\pi k|^2 t} = \sum_{k\in\mathbb{Z}^d} (4\pi t)^{-d/2}\exp\Big(-\frac{|x+k|^2}{4t}\Big), \quad x\in\mathbb{T}^d,$$

where the second equality follows from the Poisson summation formula.
In order to obtain a bound for the approximation, that does not depend on the $L^2(\mathbb{T}^d)$ norm of the initial condition $\tilde{\mu}_0^N = \rho^N * \mu_0^N$ (as it is the case for the energy estimate from Remark 2.13), which explodes for $N\to\infty$, we prove the entropy estimate (2.35). The main advantage is that instead of the L^2 norm of $\tilde{\mu}_0^N$, we obtain an estimate by $\int \tilde{\mu}_0^N \log(\tilde{\mu}_0^N)$, which improves the error estimates. To be precise, we have that $\int \tilde{\mu}_0^N \log(\tilde{\mu}_0^N) \lesssim \log(N)$, while $\|\tilde{\mu}_0^N\|_{L^2}^2 \lesssim N^d$ explodes fast with increasing dimension d, cf. Remark 2.23 below.
Here and below, we use the short hand notation for space integrals, $\int \mu := \int_{\mathbb{T}^d}\mu(x)dx$.

Proposition 2.20 (Entropy estimate)**.** *Let $\tilde{\mu}^N$ be solve the approximate Dean-Kawasaki equation (2.16) with the initial condition $\tilde{\mu}_0^N := \rho^N * \mu_0^N$ for $\mu_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$ and let $(\rho^N)_{N\geqslant 1}$ be a mollifying sequence such that $\tilde{\mu}_0^N \geqslant 0$ and $\|\rho^N * \mu_0^N\|_{L^1} = 1$. Furthermore, assume the coercivity condition $\frac{(2M_N)^d}{N\delta} \leqslant \frac{1}{2}$.*
Then the following entropy estimate holds

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\Big[\int \tilde{\mu}_t^N \log(\tilde{\mu}_t^N)\Big] + \lambda\int_0^T \mathbb{E}\Big[\int \frac{|\nabla\tilde{\mu}_t^N|^2}{\tilde{\mu}_t^N}\mathbf{1}_{\tilde{\mu}_t^N>0}\Big]dt \\ \lesssim \int \tilde{\mu}_0^N\log(\tilde{\mu}_0^N) + \frac{T\pi^2(2M_N)^{d+2}}{N},\end{aligned} \tag{2.35}$$

for $\lambda := 1 - \frac{(2M_N)^d}{N\delta}$.

Proof. We will first prove the bound for the projected solutions (u^R) from (2.21) with $u^R \in C([0,T], C^\infty)$. Utilizing Fatou's lemma and $u^R \to \tilde{\mu}^N$ in $L^2([0,T],H^1)$ almost surely by Remark 2.15, we can then conclude on the entropy bound for $\tilde{\mu}^N$. Here, $u^R(x)$ solves, for $x\in\mathbb{T}^d$,

$$u_t^R(x) = u_0^R(x) + \int_0^t \frac{1}{2}\Delta u_s^R(x)ds + \sum_k \int_0^t \Pi_R\nabla\big(f(u_s^R)e_k\big)(x)\cdot dB_s^k$$

with $u_0^R := \Pi_R\tilde{\mu}_0^N$. Then, it follows that $u^R \geqslant 0$ almost surely, as the comparison principle from the last section holds true for the equation for u^R and $\|u_t^R\|_{L^1} = \|u_0^R\|_{L^1}$ almost surely. Let $\gamma\in(0,1)$ and $g^\gamma(y) := (\gamma+y)\log(\gamma+y), y\in[0,\infty)$. Then $g \in$

$C^2([0,\infty),\mathbb{R})$ and we can apply Itô's formula to $(\gamma+u_t^R(x))\log(\gamma+u_t^R(x)) = g^\gamma(u_t^R(x))$, for $x\in\mathbb{T}^d$, with $(g^\gamma)'(y) = \log(\gamma+y)+1$ and $(g^\gamma)''(y) = \frac{1}{\gamma+y}$. After integration over x, we obtain

$$
\begin{aligned}
d&\left(\int (\gamma+u_t^R)\log(\gamma+u_t^R)\right)_t \\
&= \int (\log(\gamma+u_t^R)+1)\Delta(\gamma+u_t^R)dt \\
&\quad + \frac{1}{\sqrt{N}}\sum_{|k|\leqslant M_N}\int (\log(\gamma+u_t^R)+1)\Pi_R\nabla(f(u_t^R)e_k)dB_t^k \\
&\quad + \frac{1}{N}\sum_{|k|\leqslant M_N}\int \frac{|\Pi_R\nabla(f(u_t^R)e_k)|^2}{(\gamma+u_t^R)}dt \\
&= \int \log(\gamma+u_t^R)\Delta u_t^R dt + dM_t + \frac{1}{N}\sum_{|k|\leqslant M_N}\int \frac{|\Pi_R\nabla(f(u_t^R)e_k)|^2}{\gamma+u_t^R}dt \\
&= -\int \frac{|\nabla u_t^R|^2}{\gamma+u_t^R}dt + dM_t + \frac{1}{N}\sum_{|k|\leqslant M_N}\int \frac{|\Pi_R\nabla(f(u_t^R)e_k)|^2}{\gamma+u_t^R}dt,
\end{aligned}
$$

where M denotes the local martingale term. Next, we justify that M is a true martingale and that we can estimate the quadratic variation term by

$$
\begin{aligned}
&\frac{1}{N}\sum_{|k|\leqslant M_N}\int_0^t\int \frac{|\Pi_R\nabla(f(u_s^R)e_k)|^2}{\gamma+u_s^R}ds \\
&\quad\leqslant \frac{1}{N}\sum_{|k|\leqslant M_N}\int_0^t\int \frac{|\nabla(f(u_s^R)e_k)|^2}{\gamma+u_s^R}ds + \frac{1}{N}\sum_{|k|\leqslant M_N}\int_0^t\int \frac{|(\Pi_R-\mathrm{Id})\nabla(f(u_s^R)e_k)|^2}{\gamma+u_s^R}ds \\
&\quad\lesssim \frac{t\pi^2(2M_N)^{d+2}}{N} + \frac{(2M_N)^d}{\delta N}\int_0^t\int \frac{|\nabla u_s^R|^2}{\gamma+u_s^R}ds \\
&\qquad + \frac{1}{N\gamma}\sum_{|k|\leqslant M_N}\int_0^t \|(\Pi_R-\mathrm{Id})\nabla(f(u_s^R)e_k)\|_{L^2}^2 ds.
\end{aligned}
$$

The above estimate follows from

$$
|\nabla(f(u_s^R)e_k)|^2 \leqslant 2[|f(u_s^R)|^2\|\nabla e_k\|_\infty^2 + |f'(u_s^R)\nabla u_s^R|^2\|e_k\|_\infty^2]
$$

and using the properties (2.6) and (2.7) of f. Namely, that $|f(u)|^2\lesssim u$ for $u\geqslant 0$, $|\nabla e_k|^2\leqslant t|2\pi k|^2$ and $\|f'\|_\infty^2\lesssim\frac{1}{\delta}$. The martingale property follows from $|f(u)|^2\lesssim|u|$, since this implies

$$\begin{aligned}
&\sum_{|k|\leqslant M_N}\int_0^t \mathbb{E}\big[\big|\big\langle \log(\gamma+u_s^R)+1, \Pi_R\nabla(f(\tilde{\mu}_s^N)e_k)\big\rangle\big|^2\big]ds \\
&\quad\leqslant \sum_{|k|\leqslant M_N}\int_0^t \mathbb{E}\bigg[\bigg|\int \frac{\Pi_R(f(u_s^R)e_k)\nabla u_s^R}{\gamma+u_s^R}\bigg|^2\bigg]ds \\
&\quad\leqslant \sum_{|k|\leqslant M_N}\frac{1}{\gamma}\int_0^t \mathbb{E}[\|\Pi_R(f(u_s^R)e_k)\|_{L^2}^2\|\nabla u_s^R\|_{L^2}^2]ds \\
&\quad\leqslant \sum_{|k|\leqslant M_N}\frac{1}{\gamma}\int_0^t \mathbb{E}[\|f(u_s^R)e_k\|_{L^2}^2\|\nabla u_s^R\|_{L^2}^2]ds \\
&\quad\lesssim \frac{(2M_N)^d}{\gamma}\|u_0^R\|_{L^1}\int_0^t \mathbb{E}[\|\nabla u_s^R\|_{L^2}^2]ds<\infty.
\end{aligned}$$

Together, the above yields the bound

$$\begin{aligned}
&\mathbb{E}\bigg[\int(\gamma+u_t^R)\log(\gamma+u_t^R)\bigg]+\lambda\int_0^t\mathbb{E}\bigg[\int\frac{|\nabla u_s^R|^2}{\gamma+u_s^R}\bigg]ds \\
&\quad\lesssim \int(\gamma+u_0^R)\log(\gamma+u_0^R)+\frac{t\pi^2(2M_N)^{d+2}}{N} \\
&\qquad+\frac{1}{N\gamma}\sum_{|k|\leqslant M_N}\int_0^T\mathbb{E}[\|(\Pi_R-\mathrm{Id})\nabla(f(u_s^R)e_k)\|_{L^2}^2]ds.
\end{aligned}$$

Letting $R\to\infty$ and using Fatou's Lemma and $u_0^R\to\tilde{\mu}_0^N$ in L^2, $\Pi_R\varphi\to\varphi$ for $\varphi\in L^2$ and $\sup_R\mathbb{E}[\|\nabla u^R\|_{L^2}^2]<\infty$, we thus obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\int(\gamma+\tilde{\mu}_t^N)\log(\gamma+\tilde{\mu}_t^N)\bigg]+\lambda\int_0^t\mathbb{E}\bigg[\int\frac{|\nabla\tilde{\mu}_s^N|^2}{\gamma+\tilde{\mu}_s^N}\bigg]ds \\
&\quad\lesssim\int(\gamma+\tilde{\mu}_0^N)\log(\gamma+\tilde{\mu}_0^N)+\frac{t\pi^2(2M_N)^{d+2}}{N}. \qquad (2.36)
\end{aligned}$$

As $[0,\infty)\ni x\to x\log(x)$ is bounded from below, we can apply Fatou's lemma to the left-hand side of (2.36), and since $|x\log(x)|\leqslant x^2+1$ for $x\geqslant 0$ and $\|\tilde{\mu}_0^N\|_{L^2}^2<\infty$, we can apply the dominated convergence theorem to the right-hand side, such that we obtain for $\gamma\to 0$,

$$\mathbb{E}\bigg[\int\tilde{\mu}_t^N\log(\tilde{\mu}_t^N)\bigg]+\lambda\int_0^t\mathbb{E}\bigg[\int\frac{|\nabla\tilde{\mu}_s^N|^2}{\tilde{\mu}_s^N}\mathbf{1}_{\tilde{\mu}_s^N>0}\bigg]ds\lesssim\int\tilde{\mu}_0^N\log(\tilde{\mu}_0^N)+\frac{t(2\pi)^2M_N^{d+2}}{N}.$$

□

Theorem 2.21. *Let μ^N be the martingale solution of the Dean-Kawasaki equation in* *the sense of Definition 2.1 with initial condition* $\mu_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$*. Let* $(\rho^N)_{N\geqslant 1}$ *be a mollifying sequence, such that* $\tilde{\mu}_0^N = \rho^N * \mu_0^N \geqslant 0$*,*

$$\|\rho^N * \mu_0^N - \mu_0^N\|_{B^{-\kappa}_{1,\infty}} \lesssim N^{-\kappa} \tag{2.37}$$

for any $\kappa \in (1,2]$*,*

$$\int (\rho^N * \mu_0^N)\log(\rho^N * \mu_0^N) \lesssim_d \log(N) \tag{2.38}$$

and $\|\rho^N * \mu_0^N\|_{L^1} = 1$*. Let furthermore* $\frac{M_N^d}{N\delta} \leqslant \frac{1}{2}$ *(coercivity assumption) and let* $\tilde{\mu}^N$ *be the solution of the approximate Dean-Kawasaki equation* (2.8) *with* $f \in C^1$ *as in* (2.5) *satifying* (2.6)–(2.7) *and with initial condition* $\tilde{\mu}_0^N = \rho^N * \mu_0^N$*.*
Then for any $t > 0$*,* $\varphi \in C^\infty(\mathbb{T}^d)$ *and* $F(\mu) := \exp(\langle \mu, \varphi\rangle)$ *for* $\mu \in \mathscr{M}$*, the following weak error bound holds:*

$$|\mathbb{E}[F(\tilde{\mu}_t^N)] - \mathbb{E}[F(\mu_t^N)]| \lesssim_{\varphi,t,d} N^{-\kappa} + N^{-1-\frac{1}{d/2+1}} + N^{-1-\frac{1}{d+2}}(1+\log(N)^{1/2}). \tag{2.39}$$

Remark 2.22. *Considering the functions* $F(\mu) = \exp(-\langle \mu, \varphi\rangle)$ *for* $\varphi \in C^\infty$*, one could build an appropriate metric for the topology of weak convergence of probability measures on* $\mathscr{M}$ *using [EK86, Theorem 3.4.5] and replace the left-hand side of* (2.39) *by the distance of* $\tilde{\mu}_t^N$ *and* μ_t^N *in this weak metric.*

Remark 2.23. *Consider* $\rho \in C^\infty(\mathbb{T}^d)$ *with* $\int_{\mathbb{T}^d}\rho(y)dy = 1$*, such that* $\rho \geqslant 0$ *and* ρ *is symmetric in the sense that* $\rho(y) = \rho(-y)$ *for* $y \in \mathbb{T}^d$*. Then any mollification* $(\rho^N)_N$ *with* $\rho^N(x) = \rho(Nx)N^d$ *satisfies the assumptions of the above theorem.*
Indeed, by non-negativity of ρ *and* $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$*, we obtain* $\rho * \mu_0^N \geqslant 0$*. As* $\int_{\mathbb{T}^d}\rho(y)dy = 1$*, we see that with Fubini,*

$$\|\rho^N * \mu_0^N\|_{L^1} = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\rho(N(x-y))N^d\mu_0^N(dy)dx = 1.$$

Furthermore, we can trivially bound

$$\|\rho^N * \mu_0^N\|_{L^\infty} \leqslant N^d\|\rho\|_{L^\infty} \lesssim N^d.$$

Hence, we obtain the following bound

$$\begin{aligned}\int (\rho^N * \mu_0^N)\log(\rho^N * \mu_0^N) &\leqslant \int (\rho^N * \mu_0^N)\log(\rho^N * \mu_0^N)\mathbf{1}_{\rho^N * \mu_0^N \geqslant 1} \\ &\leqslant \|\rho^N * \mu_0^N\|_{L^1}\|\log(\rho^N * \mu_0^N)\mathbf{1}_{\rho^N * \mu_0^N \geqslant 1}\|_{L^\infty} \\ &= \|\log(\rho^N * \mu_0^N)\mathbf{1}_{\rho^N * \mu_0^N \geqslant 1}\|_{L^\infty} \\ &\leqslant \log(\|\rho^N * \mu_0^N\|_{L^\infty}) \\ &\lesssim \log(N^d) = d\log(N).\end{aligned}$$

This is a much better estimate than for the L^2 norm of $\rho^N * \mu_0^N$, which we can only bound by N^d.
Furthermore, for $h \in B^{\kappa}_{\infty,1}(\mathbb{T}^d) \subset B^{\kappa}_{\infty,\infty}(\mathbb{T}^d)$, with $\kappa \in (1,2)$ we have that $h \in C^{\kappa}(\mathbb{T}^d)$, that is h being continuously differentiable with $(\kappa-1)$-Hölder continuous derivative (cf. [ST87, Theorem in Section 3.5.4, part (ii)]) we can bound, using that the symmetry of ρ so that $\int_{\mathbb{T}^d} \rho(y) y dy = 0$,

$$\begin{aligned}
\|\rho^N * h - h\|_{L^\infty} &= \left\| \int_{\mathbb{T}^d} \rho(Ny) N^d (h(x-y) - h(x)) dy \right\|_{L^\infty} \\
&= \left\| \int_{\mathbb{T}^d} \rho(Ny) N^d (h(x-y) - h(x) - h'(x)y) dy \right\|_{L^\infty} \\
&= \left\| \int_0^1 \int_{\mathbb{T}^d} \rho(Ny) N^d y (h'(x-\lambda y) - h'(x)) dy d\lambda \right\|_{L^\infty} \\
&\leqslant \int_{\mathbb{T}^d} \rho(Ny) N^d |y|^{1+(\kappa-1)} dy \\
&= N^{-\kappa} \int_{\mathbb{T}^d} \rho(y) |y|^{\kappa} dy \lesssim N^{-\kappa}
\end{aligned}$$

and analogously for $h \in C^2$, we obtain $\|\rho^N * h - h\|_{L^\infty} \lesssim N^{-2}$. Then, we obtain that, as ρ is symmetric and by the above estimate (the first inequality follows from duality, cf. [ST87, Theorem in Section 3.5.6] and [BCD11, Proposition 2.76])

$$\|\rho^N * \mu_0^N - \mu_0^N\|_{B^{-\kappa}_{1,\infty}} \lesssim \sup_{\|h\|_{B^{\kappa}_{\infty,1}}=1} |\langle \rho^N * \mu_0^N - \mu_0^N, h\rangle| = \sup_{\|h\|_{B^{\kappa}_{\infty,1}}=1} |\langle \rho^N * h - h, \mu_0^N\rangle| \lesssim N^{-\kappa}.$$

Proof of Theorem 2.21. To prove the weak error bound (2.39), we apply a duality argument as is typically used to prove weak uniqueness, cf. [Myt98]. For that purpose, let v solve the Hamilton-Jacobi backward equation with initial condition $\varphi \in C^\infty(\mathbb{T}^d)$,

$$\partial_t v_t = \frac{1}{2}\Delta v_t + \frac{1}{2N}|\nabla v_t|^2, \quad v_0 = \varphi. \tag{2.40}$$

Due to Cole-Hopf transformation, the solution is explicitly given by

$$v_t = -\log(P_t e^{-\varphi}) =: V_t \varphi,$$

where $(P_t)_{t\geqslant 0}$ denotes the heat-semigroup, see (2.34). Note, that $P_t e^{-\varphi} > 0$ by to the strong maximum principle for the heat equation. In particular, $v \in C([0,T], C^\infty(\mathbb{T}^d))$ as $\varphi \in C^\infty(\mathbb{T}^d)$. Trivially, we have $F(\mu_t^N) = \exp(\langle \mu_t^N, V_{t-t}\varphi\rangle)$. By [KLvR19, Theorem 2.2], μ^N satisfies, for $\phi \in C^\infty$,

$$\langle \mu_t^N, \phi\rangle = \langle \mu_0, \phi\rangle + \int_0^t \langle \mu_s^N, \frac{1}{2}\Delta\phi\rangle ds + M_t^\phi, \tag{2.41}$$

where M^ϕ is a martingale with quadratic variation $\frac{1}{N}\int_0^\cdot \langle \mu_s^N, |\nabla\phi|^2\rangle ds$. Using (2.40) and (2.41), we obtain

$$
\begin{aligned}
&d\big(\exp\big(\langle \mu_s^N, V_{t-s}\varphi\rangle\big)\big)_s \\
&\quad= \exp(\langle \mu_s^N, v_{t-s}\rangle)\big(\langle \mu_s^N, -\partial_s v_{t-s}\rangle ds + \Big\langle \mu_s^N, \frac{1}{2}\Delta v_{t-s} + \frac{1}{2N}|\nabla v_{t-s}|^2\Big\rangle ds + dM_s^{v_{t-\cdot}}\big) \\
&\quad= \exp(\langle \mu_s^N, v_{t-s}\rangle) dM_s^{v_{t-\cdot}}.
\end{aligned}
$$

This yields

$$
F(\mu_t^N) = \exp(\langle \mu_0^N, V_t\varphi\rangle) + \int_0^t \exp(\langle \mu_s^N, v_{t-s}\rangle) dM_s^{v_{t-\cdot}}. \tag{2.42}
$$

Proceeding analogously for $\tilde{\mu}^N$ for which it holds

$$
\langle \tilde{\mu}_t^N, \phi\rangle = \langle \tilde{\mu}_0^N, \phi\rangle + \frac{1}{2}\int_0^t \langle \tilde{\mu}_s^N, \Delta\phi\rangle ds - \frac{1}{\sqrt{N}} \sum_{|m|\leqslant M_N} \int_0^t \langle f(\tilde{\mu}_t^N) e_m, \nabla\phi\rangle dB_t^k \tag{2.43}
$$

and using equation (2.40) for v, we arrive at

$$
\begin{aligned}
&d\big(\exp(\langle \tilde{\mu}_s^N, V_{t-s}\varphi\rangle)\big)_s \\
&\quad= -\frac{1}{\sqrt{N}} \sum_{|m|\leqslant M_N} \exp(\langle \tilde{\mu}_s^N, V_{t-s}\varphi\rangle)\langle f(\tilde{\mu}_s^N) e_m, \nabla V_{t-s}\varphi\rangle dB_s^m \qquad (2.44)\\
&\qquad+ \frac{1}{2N}\exp(\langle \tilde{\mu}_s^N, V_{t-s}\varphi\rangle)\Bigg(\sum_{1\leqslant |m|\leqslant M_N} |\langle f(\tilde{\mu}_s^N) e_m, \nabla V_{t-s}\varphi\rangle|^2 - \langle \tilde{\mu}_s^N, |\nabla V_{t-s}\varphi|^2\rangle\Bigg) ds.
\end{aligned}
$$

If we denote by $\tilde{M}^{v_{t-\cdot}}$ the martingale term in (2.44), we get

$$
\begin{aligned}
F(\tilde{\mu}_t^N) &= \exp(\langle \tilde{\mu}_0^N, V_t\phi\rangle) + \tilde{M}_s^{v_{t-\cdot}} \\
&\quad+ \int_0^t \frac{1}{2N}\exp(\langle \tilde{\mu}_s^N, V_{t-s}\varphi\rangle)\times \\
&\qquad\Bigg(\sum_{1\leqslant |m|\leqslant M_N} |\langle f(\tilde{\mu}_s^N) e_m, \nabla V_{t-s}\varphi\rangle|^2 - \langle \tilde{\mu}_s^N, |\nabla V_{t-s}\varphi|^2\rangle\Bigg) ds. \qquad (2.45)
\end{aligned}
$$

Note that $\tilde{M}^{v_{t-\cdot}}$ is indeed a martingale (not only a local martingale). This can be concluded from the bound $|f(u)|^2 \leqslant \|f'\|_\infty^2 |u|^2$ that implies

$$
\sum_{|m|\leqslant M_N} \int_0^t \mathbb{E}[|\langle f(\tilde{\mu}_s^N) e_m, \nabla v_{t-s}\rangle|^2] ds \leqslant C_1 \sup_{s\in[0,t]} \|\nabla v_{t-s}\|_{L^\infty}^2 \|f'\|_\infty^2 \int_0^t \mathbb{E}[\|\tilde{\mu}_s^N\|_{L^2}^2] ds,
$$

where the right-hand side is finite.
Combining (2.42) and (2.45) and taking the expectation, we obtain for the weak error

the bound

$$\begin{aligned}&|\mathbb{E}[F(\tilde{\mu}_t^N)] - \mathbb{E}[F(\mu_t^N)]|\\&\leqslant |\exp(\langle \mu_0^N, V_t\varphi\rangle) - \exp(\langle \mu_0^N, V_t\varphi\rangle)|\\&+ \frac{1}{2N}\mathbb{E}\left[\int_0^t \exp(\langle \tilde{\mu}_s^N, V_{t-s}\varphi\rangle)\left|\sum_{1\leqslant|m|\leqslant M_N} |\langle f(\tilde{\mu}_s^N)e_m, \nabla V_{t-s}\varphi\rangle|^2 - \langle \tilde{\mu}_s^N, |\nabla V_{t-s}\varphi|^2\rangle\right| ds\right].\end{aligned} \tag{2.46}$$

In what follows, we abbreviate $V_{t-s}\varphi = \psi_s$. Due to the conservation of mass stated in Proposition 2.19 and since $\|\tilde{\mu}_0^N\|_{L^1} = 1$, we obtain, almost surely,

$$\exp(\langle \tilde{\mu}_s^N, \psi_s\rangle) \leqslant \exp(\|\tilde{\mu}_s^N\|_{L^1}\|\psi_s\|_{L^\infty}) = \exp(\|\tilde{\mu}_0^N\|_{L^1}\|\psi_s\|_{L^\infty}) = \exp(\|\psi_s\|_{L^\infty}) \lesssim_\varphi 1.$$

Hence, it is left to estimate the term in the brackets in (2.46). To that aim, we define

$$I_s := \frac{1}{2N}\sum_{1\leqslant|m|\leqslant M_N}\iint f(\tilde{\mu}_s^N(x))f(\tilde{\mu}_s^N(y))\nabla\psi_s(x)\nabla\psi_s(y)e_m(x)e_{-m}(y)dxdy - \langle \tilde{\mu}_s^N, |\nabla\psi_s|^2\rangle.$$

and let $K_{M_N}(x-y) := \sum_{|m|\leqslant M_N} e_m(x)e_{-m}(y)$. Next we consider the decomposition

$$I_s = \frac{A_s + C_s}{2N},$$

where

$$A_s = \langle f(\tilde{\mu}_s^N)^2, |\nabla\psi_s|^2\rangle - \langle \tilde{\mu}_s^N, |\nabla\psi_s|^2\rangle \tag{2.47}$$

and

$$C_s = \iint f(\tilde{\mu}_s^N(x))f(\tilde{\mu}_s^N(y))\nabla\psi_s(x)\nabla\psi_s(y)\left(K_{M_N}(x-y) - \delta(x-y)\right)dxdy. \tag{2.48}$$

To estimate A, we use positivity of $\tilde{\mu}^N$ from Corollary 2.18, as well as (2.7), such that $|x - f(x)^2| \lesssim \delta$ for $x \geqslant 0$. This implies the following estimate on A:

$$|A_s| = |\langle \tilde{\mu}_s^N, |\nabla\psi_s|^2\rangle - \langle f(\tilde{\mu}_s^N)^2, |\nabla\psi_s|^2\rangle| \lesssim \delta\int |\nabla\psi_s|^2(x)dx, \tag{2.49}$$

and thus

$$\int_0^t |A_s|ds \lesssim \delta\|\psi\|^2_{L^2([0,T],\dot{B}^1_{2,2})}.$$

In order to estimate the term C_s, let $g(x) := f(\tilde{\mu}_s^N(x))\nabla\psi_s(x)$ and write:

$$\begin{aligned}\mathbb{E}[|C_s|] &:= \mathbb{E}\Big[\Big|\int\int g(x)(g(x) - K_{M_N}(x-y)(g(y))dxdy\Big|\Big] \\ &= \mathbb{E}\Big[\Big|\int g(x)((K_{M_N} * g)(x) - g(x))dx\Big|\Big] \\ &= \mathbb{E}\Big[\Big|\int g(x)(K_{>M_N} * g)(x)dx\Big|\Big],\end{aligned}$$

where $K_{>M_N} * g(x) := \sum_{|m|>M_N} e_m(x)\langle e_{-m}, g\rangle$. Using $f(x)^2 \lesssim |x|$ by (2.7) and $\|\tilde{\mu}_s^N\|_{L^1} = \|\tilde{\mu}_0^N\|_{L^1} = 1$ by conservation of mass, we get

$$\begin{aligned}\mathbb{E}[|C_s|] &\leqslant \|\nabla\psi\|_{L^\infty}\mathbb{E}[\|f(\tilde{\mu}_s^N)\|_{L^2}\|K_{>M_N} * g\|_{L^2}] \\ &\leqslant \|\tilde{\mu}_0^N\|_{L^1}^{1/2}\mathbb{E}[\|K_{>M_N} * g\|_{L^2}] \\ &= \mathbb{E}[\|K_{>M_N} * g\|_{L^2}] \\ &= \mathbb{E}\Big[\Big(\sum_{|m|>M_N} |\langle e_m, g\rangle|^2\Big)^{1/2}\Big].\end{aligned}$$

We estimate the series of Fourier coefficients as follows

$$\begin{aligned}\sum_{|m|>M_N} |\langle e_m, g\rangle|^2 &\leqslant M_N^{-2} \sum_{|m|>M_N} |m|^2|\hat{g}(m)|^2 \\ &\leqslant M_N^{-2}\|g\|_{H^1}^2 \\ &\lesssim \|\psi_s\|_{B^2_{\infty,\infty}} M_N^{-2}(\|f(\tilde{\mu}_s^N)\|_{L^2}^2 + \|f'(\tilde{\mu}_s^N)\nabla\tilde{\mu}_s^N \mathbf{1}_{\tilde{\mu}_s^N>0}\|_{L^2}^2) \\ &\lesssim \|\psi_s\|_{B^2_{\infty,\infty}} M_N^{-2}\Big(\|\tilde{\mu}_s^N\|_{L^1}^2 + \int \frac{|\nabla\tilde{\mu}_s^N|^2}{\tilde{\mu}_s^N}\mathbf{1}_{\tilde{\mu}_s^N>0}\Big),\end{aligned}$$

where we used that $f(x)^2 \lesssim |x|$ and $f'(x)^2 \lesssim \frac{1}{x}$ for $x > 0$ by (2.6). Integrating over time, using the entropy estimates from Proposition 2.20 to bound the second term and conservation of mass, $\|\tilde{\mu}_s^N\|_{L^1} = 1$, combined with Jensen's inequality implies

$$\int_0^t \mathbb{E}[|C_s|]ds \lesssim_\psi \frac{1}{M_N}\Big(1 + \lambda^{-1}\frac{tM_N^{d+2}}{N} + \lambda^{-1}\int \tilde{\mu}_0^N \log(\tilde{\mu}_0^N)\Big)^{1/2}.$$

The error from the initial condition we estimate with the mean value theorem as follows:

$$
\begin{aligned}
&|\exp(\langle\tilde{\mu}_0^N, V_t\varphi\rangle) - \exp(\langle\mu_0^N, V_t\varphi\rangle)| \\
&\quad \leqslant |\langle\tilde{\mu}_0^N - \mu_0^N, V_t\varphi\rangle| \exp(\|V_t\varphi\|_{L^\infty}\|\tilde{\mu}_0^N\|_{L^1} + \|V_t\varphi\|_{L^\infty}) \\
&\quad = |\langle\tilde{\mu}_0^N - \mu_0^N, V_t\varphi\rangle| \exp(2\|V_t\varphi\|_{L^\infty}) \\
&\quad \lesssim_\varphi \|\tilde{\mu}_0^N - \mu_0^N\|_{B^{-\kappa}_{1,\infty}} \|V_t\varphi\|_{B^{\kappa}_{\infty,1}} \\
&\quad \lesssim_\varphi \|\tilde{\mu}_0^N - \mu_0^N\|_{B^{-\kappa}_{1,\infty}} \\
&\quad \lesssim_\varphi N^{-\kappa},
\end{aligned}
$$

using also the duality estimate for Besov spaces from [BCD11, Proposition 2.76] (that also holds true for Besov spaces on the torus). Furthermore, we used that $\varphi \in C^\infty(\mathbb{T}^d)$ and the choice of the mollification (ρ^N) for the last bound.
Together, we obtain for the weak error:

$$
\begin{aligned}
&|\mathbb{E}[F(\tilde{\mu}_t^N)] - \mathbb{E}[F(\mu_t^N)]| \\
&\lesssim_{t,\varphi} |\exp(\langle\tilde{\mu}_0^N, V_t\varphi\rangle) - \exp(\langle\mu_0^N, V_t\varphi\rangle)| \\
&\quad + \frac{\delta}{N} + \frac{1}{M_N N} + N^{-1}\left(\lambda^{-1}\frac{tM_N^d}{N} + \lambda^{-1}M_N^{-2}\int \tilde{\mu}_0^N \log(\tilde{\mu}_0^N)\right)^{1/2} \\
&\lesssim_{t,\varphi} N^{-\kappa} + \frac{\delta}{N} + \frac{1}{M_N N} + \frac{M_N^{d/2}}{N^{3/2}} + \frac{\left(\int \tilde{\mu}_0^N \log(\tilde{\mu}_0^N)\right)^{1/2}}{M_N N}
\end{aligned}
$$

where we used that the coercivity assumption implies $\lambda \geqslant 1/2$. By assumption on $(\rho^N)_{N\geq 1}$, we have $\|\tilde{\mu}_0^N - \mu_0^N\|_{B^{-\kappa}_{1,\infty}} \lesssim N^{-\kappa}$ and $\int \tilde{\mu}_0^N \log(\tilde{\mu}_0^N) \lesssim_d \log(N)$. Overall, we obtain

$$
|\mathbb{E}[F(\tilde{\mu}_t^N)] - \mathbb{E}[F(\mu_t^N)]| \lesssim_{t,\varphi,d} N^{-\kappa} + \frac{\delta}{N} + \frac{1}{M_N N} + \frac{M_N^{d/2}}{N^{3/2}} + \frac{\log(N)^{1/2}}{M_N N}. \tag{2.50}
$$

The coercivity assumption dictates $M_N^d \lesssim \delta N$. Hence, if we substitute $M_N^{d/2}$ by $(\delta N)^{1/2}$ in the forth term of (2.50), we get

$$
|\mathbb{E}[F(\tilde{\mu}_t^N)] - \mathbb{E}[F(\mu_t^N)]| \lesssim_{t,\varphi,d} N^{-\kappa} + \frac{\delta}{N} + \frac{1}{M_N N} + \frac{\delta^{1/2}}{N} + \frac{\log(N)^{1/2}}{M_N N}. \tag{2.51}
$$

Choosing $M_N = \delta^{-1/2}$, together with the coercivity assumption yields $\delta = N^{-\frac{1}{d/2+1}}$. Altogether, we obtain the estimate

$$
|\mathbb{E}[F(\tilde{\mu}_t^N)] - \mathbb{E}[F(\mu_t^N)]| \lesssim_{t,\varphi,d} N^{-\kappa} + N^{-1-\frac{1}{d/2+1}} + N^{-1-\frac{1}{d+2}}(1 + \log(N)^{1/2}).
$$

□

Part II.

Regularization by noise for singular Lévy SDEs

3. Kolmogorov equations with singular paracontrolled terminal conditions

The purpose of this chapter is to solve the fractional Kolmogorov backward equation

$$(\partial_t - \mathscr{L}_\nu^\alpha)u = -V \cdot \nabla u + f, \quad u(T, \cdot) = u_T$$

for $V \in C_T\mathscr{C}^\beta_{\mathbb{R}^d} := C([0,T], (\mathscr{C}^\beta)^d)$, $\beta \in ((2-2\alpha)/3, 0)$. Here, $\mathscr{L}_\nu^\alpha$ denotes a generalization of the fractional Laplacian $(-\Delta)^{\alpha/2}$, that we define in Section 3.1 below. The regime $\beta \in ((1-\alpha)/2, 0)$ corresponds to the so-called Young regime, where the product $V \cdot \nabla u$ is well-defined in the classical sense, while the rough regime, $\beta \in ((2-2\alpha)/3, (1-\alpha)/2]$, requires tools from paracontrolled analysis (cf. [GIP15]) and the assumption of an enhanced drift $\mathscr{V}$ (cf. Definition 3.20). This chapter generalizes the PDE solution theory from [KP22, Section 3] to singular terminal conditions u^T, that are paracontrolled by V_T, as well as right-hand sides f, that are paracontrolled by V (cf. Section 3.3).
Section 3.1 is concerned with preliminaries, including the definition of $\mathscr{L}_\nu^\alpha$ and its semigroup (P_t). In Section 3.2 we define the solution space and prove Schauder and commutator estimates. Our commutator estimate from Lemma 3.14 allows to gain not only space regularity, but also time regularity, compared to both summands. Theorem 3.19 and Theorem 3.25 in Section 3.3 prove the existence and uniqueness of mild solutions in the Young, respectively rough case. Theorem 3.30 shows continuity of the solution map. Furthermore, Corollary 3.32 proves a uniform bound on solutions considered on subintervals of $[0,T]$, that will be employed in Chapter 4.

3.1. Paracontrolled analysis for the generalized fractional Laplacian

In this section, we introduce some technical ingredients about Besov spaces and paraproducts, that we will need in the sequel. Moreover, we collect properties of the α-stable Lévy process and relate the process with its generator $(-\mathscr{L}_\nu^\alpha)$, that will also be relevant for Chapter 4. We study estimates for the generalized fractional Laplacian and its semigroup, as well as, commutator estimates involving the paraproducts and the fractional semigroup.
The results (and notation) of this section will be used in all of the following chapters.

Let $(p_j)_{j\geqslant -1}$ be a smooth dyadic partition of unity, i.e. a family of functions $p_j \in C_c^\infty(\mathbb{R}^d)$ for $j \geqslant -1$, such that

1.) p_{-1} and p_0 are non-negative radial functions (they just depend on the absolute value of $x \in \mathbb{R}^d$), such that the support of p_{-1} is contained in a ball and the support of p_0 is contained in an annulus;

2.) $p_j(x) := p_0(2^{-j}x)$, $x \in \mathbb{R}^d$, $j \geqslant 0$;

3.) $\sum_{j=-1}^\infty p_j(x) = 1$ for every $x \in \mathbb{R}^d$; and

4.) $\operatorname{supp}(p_i) \cap \operatorname{supp}(p_j) = \emptyset$ for all $|i-j| > 1$.

We then define the Besov spaces for $p, q \in [1,\infty]$,

$$B^\theta_{p,q} := \{u \in \mathscr{S}' : \|u\|_{B^\theta_{p,q}} = \big\|(2^{j\theta}\|\Delta_j u\|_{L^p})_{j\geqslant -1}\big\|_{\ell^q} < \infty\}, \tag{3.1}$$

where $\Delta_j u = \mathscr{F}^{-1}(p_j \mathscr{F} u)$ are the Littlewood-Paley blocks, and the Fourier transform is defined with the normalization $\hat{\varphi}(y) := \mathscr{F}\varphi(y) := \int_{\mathbb{R}^d} \varphi(x) e^{-2\pi i\langle x,y\rangle} dx$ (and $\mathscr{F}^{-1}\varphi(x) = \hat{\varphi}(-x)$); moreover, $\mathscr{S}$ are the Schwartz functions and $\mathscr{S}'$ are the Schwartz distributions. Let $C_b^\infty = C_b^\infty(\mathbb{R}^d, \mathbb{R})$ denote the space of bounded and smooth functions with bounded partial derivatives. For $q = \infty$, the space $B^\theta_{p,\infty}$ has the unpleasant property that $C_b^\infty \subset B^\theta_{p,\infty}$ is not dense. Therefore, we rather work with the following space:

$$B^\theta_{p,\infty} := \{u \in \mathscr{S}' \mid \lim_{j\to\infty} 2^{j\theta}\|\Delta_j u\|_{L^p} = 0\}, \tag{3.2}$$

for which C_b^∞ is a dense subset (cf. [BCD11, Remark 2.75]). We also use the notation $\mathscr{C}^\theta_{\mathbb{R}^d} := (\mathscr{C}^\theta)^d = \mathscr{C}^\theta(\mathbb{R}^d, \mathbb{R}^d)$, $\mathscr{C}^{\theta-} := \bigcap_{\gamma<\theta} \mathscr{C}^\gamma$ and $\mathscr{C}^{\theta+} = \bigcup_{\gamma>\theta}\mathscr{C}^\gamma$. Furthermore, we introduce the notation $\mathscr{C}^\theta_p := B^\theta_{p,\infty}$ for $\theta \in \mathbb{R}$ and $p \in [1,\infty]$, where $\mathscr{C}^\theta := \mathscr{C}^\theta_\infty$ with norm denoted by $\|\cdot\|_\theta := \|\cdot\|_{\mathscr{C}^\theta}$.
For $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ and $s \in \mathbb{R}$, the Besov space $B^s_{p_1,q_1}$ is continuously embedded in $B^{s-d(1/p_1-1/p_2)}_{p_2,q_2}$ (cf. [BCD11, Proposition 2.71]). Furthermore, we will use that for $u \in B^s_{p,q}$ and a multi-index $n \in \mathbb{N}^d$, $\|\partial^n u\|_{B^{s-|n|}_{p,q}} \lesssim \|u\|_{B^s_{p,q}}$, which follows from the more general multiplier result from [BCD11, Proposition 2.78].
We recall from Bony's paraproduct theory (cf. [BCD11, Section 2]) that in general for $u \in \mathscr{C}^\theta$ and $v \in \mathscr{C}^\beta$ with $\theta, \beta \in \mathbb{R}$, the product $uv := u \varolessthan v + u \varogreaterthan v + u \odot v$, is well defined in $\mathscr{C}^{\min(\theta,\beta,\theta+\beta)}$ if and only if $\theta + \beta > 0$. Denoting $S_i u = \sum_{j=-1}^{i-1} \Delta_j u$, the paraproducts are defined as follows

$$u \varolessthan v := \sum_{i\geqslant -1} S_{i-1}u\Delta_i v, \quad u \varogreaterthan v := v \varolessthan u, \quad u \odot v := \sum_{|i-j|\leqslant 1} \Delta_i u \Delta_j v.$$

Here, we use the notation of [MP19, MW17] for the para- and resonant products $\varolessthan, \varogreaterthan$ and $\odot$.
In estimates we often use the notation $a \lesssim b$, which means, that there exists a constant

$C > 0$, such that $a \leqslant Cb$. In the case that we want to stress the dependence of the constant $C(d)$ in the estimate on a parameter d, we write $a \lesssim_d b$.
The paraproducts satisfy the following estimates for $p, p_1, p_2 \in [1, \infty]$ with $\frac{1}{p} = \min(1, \frac{1}{p_1} + \frac{1}{p_2})$ and $\theta, \beta \in \mathbb{R}$ (cf. [PvZ22, Theorem A.1] and [BCD11, Theorem 2.82, Theorem 2.85])

$$\begin{aligned} \|u \odot v\|_{\mathscr{C}_p^{\theta+\beta}} &\lesssim \|u\|_{\mathscr{C}_{p_1}^{\theta}} \|v\|_{\mathscr{C}_{p_2}^{\beta}}, && \text{if } \theta + \beta > 0, \\ \|u \otimes v\|_{\mathscr{C}_p^{\beta}} \lesssim \|u\|_{L^{p_1}} \|v\|_{\mathscr{C}_{p_2}^{\beta}} &\lesssim \|u\|_{\mathscr{C}_{p_1}^{\theta}} \|v\|_{\mathscr{C}_{p_2}^{\beta}}, && \text{if } \theta > 0, \\ \|u \otimes v\|_{\mathscr{C}_p^{\beta+\theta}} &\lesssim \|u\|_{\mathscr{C}_{p_1}^{\theta}} \|v\|_{\mathscr{C}_{p_2}^{\beta}}, && \text{if } \theta < 0. \end{aligned} \tag{3.3}$$

So if $\theta + \beta > 0$, we have $\|uv\|_{\mathscr{C}_p^{\gamma}} \lesssim \|u\|_{\mathscr{C}_{p_1}^{\theta}} \|v\|_{\mathscr{C}_{p_2}^{\beta}}$ for $\gamma := \min(\theta, \beta, \theta + \beta)$.
Next, we collect some facts about α-stable Lévy processes and their generators and semigroups. For $\alpha \in (0, 2]$, a symmetric α-stable Lévy process L is a Lévy process, that moreover satisfies the scaling property $(L_{kt})_{t \geqslant 0} \overset{d}{=} k^{1/\alpha}(L_t)_{t \geqslant 0}$ for any $k > 0$ and $L \overset{d}{=} -L$, where $\overset{d}{=}$ denotes equality in law. These properties determine the jump measure μ of L, see [Sat99, Theorem 14.3]. That is, if $\alpha \in (0, 2)$, the Lévy jump measure μ of L is given by

$$\mu(A) := \mathbb{E}\Big[\sum_{0 \leqslant t \leqslant 1} \mathbf{1}_A(\Delta L_t)\Big] = \int_S \int_{\mathbb{R}^+} \mathbf{1}_A(k\xi) \frac{1}{k^{1+\alpha}} dk \tilde{\nu}(d\xi), \quad A \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}), \tag{3.4}$$

where $\tilde{\nu}$ is a finite, symmetric, non-zero measure on the unit sphere $S \subset \mathbb{R}^d$. Furthermore, we also define for $A \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$ and $t \geqslant 0$ the Poisson random measure

$$\pi(A \times [0, t]) = \sum_{0 \leqslant s \leqslant t} \mathbf{1}_A(\Delta L_s),$$

with intensity measure $dt\mu(dy)$. Denote the compensated Poisson random measure of L by $\hat{\pi}(dr, dy) := \pi(dr, dy) - dr\mu(dy)$. We refer to the book by Peszat and Zabczyk [PZ07] for the integration theory against Poisson random measures and for the Burkholder-Davis-Gundy inequality [PZ07, Lemma 8.21 and 8.22], which we will both use in the sequel. The generator A of L satisfies $C_b^\infty(\mathbb{R}^d) \subset \mathrm{dom}(A)$ and is given by

$$A\varphi(x) = \int_{\mathbb{R}^d} \big(\varphi(x + y) - \varphi(x) - \mathbf{1}_{\{|y| \leqslant 1\}}(y) \nabla\varphi(x) \cdot y\big)\mu(dy) \qquad \text{for } \varphi \in C_b^\infty(\mathbb{R}^d). \tag{3.5}$$

If $(P_t)_{t \geqslant 0}$ denotes the semigroup of L, the convergence $t^{-1}(P_t f(x) - f(x)) \to Af(x)$ is uniform in $x \in \mathbb{R}^d$ (see [PZ07, Theorem 5.4]).
To derive Schauder estimates for (P_t) it will be easier to work with another representation of the generator A. For that purpose we first introduce an operator $\mathscr{L}_\nu^\alpha$ via Fourier analysis, and then we show that it agrees with A.

Definition 3.1. *Let $\alpha \in (0, 2)$ and let ν be a symmetric (i.e. $\nu(A) = \nu(-A)$), finite* *and non-zero measure on the unit sphere* $S \subset \mathbb{R}^d$*. We define the operator* $\mathscr{L}_\nu^\alpha$ *as*

$$\mathscr{L}_\nu^\alpha \mathscr{F}^{-1}\varphi = \mathscr{F}^{-1}(\psi_\nu^\alpha \varphi) \qquad \text{for } \varphi \in C_b^\infty, \tag{3.6}$$

where $\psi_\nu^\alpha(z) := \int_S |\langle z, \xi\rangle|^\alpha \nu(d\xi)$*. For* $\alpha = 2$*, we set* $\mathscr{L}_\nu^\alpha := -\frac{1}{2}\Delta$*.*

Remark 3.2. *If we take* ν *as a suitable multiple of the Lebesgue measure on the sphere, then* $\psi_\nu^\alpha(z) = |2\pi z|^\alpha$ *and thus* $\mathscr{L}_\nu^\alpha$ *is the fractional Laplace operator* $(-\Delta)^{\alpha/2}$*.*

Lemma 3.3. *Let* $\alpha \in (0,2)$ *and let again* ν *be a symmetric, finite and non-zero measure on the unit sphere* $S \subset \mathbb{R}^d$*. Then for* $\varphi \in C_b^\infty$ *we have* $-\mathscr{L}_\nu^\alpha \varphi = A\varphi$*, where* A *is the generator of the symmetric,* α*-stable Lévy process* L *with characteristic exponent* $\mathbb{E}[\exp(2\pi i \langle z, L_t\rangle)] = \exp(-t\psi_\nu^\alpha(z))$*. The process* L *has the jump measure* μ *as defined in Equation* (3.4)*, with* $\tilde{\nu} = C\nu$ *for some constant* $C > 0$*.*

Proof. By Fourier inversion, L_t has the density $\rho_t = \mathscr{F}^{-1}(\exp(-t\psi_\nu^\alpha))$ with respect to the Lebesgue measure (note that $\psi_\nu^\alpha(z) = \psi_\nu^\alpha(-z)$). So for the semigroup (P_t) of L we have $P_t\varphi(x) = \int \rho_t(y)\varphi(x+y)dy$ with $\partial_t P_t \varphi_{|t=0} = \mathscr{F}^{-1}(-\psi_\nu^\alpha \hat{\varphi}) = -\mathscr{L}_\nu^\alpha \varphi$ for any $\varphi \in C_b^\infty$. The identity $\tilde{\nu} = C\nu$ is shown in the proof of [Sat99, Theorem 14.10]. □

If $\alpha = 2$, then the generator of the symmetric, α-stable process coincides with $\sum_{i,j} C(i,j)\partial_{x_i}\partial_{x_j}$ for an explicit covariance matrix C (cf. [Sat99, Theorem 14.2]), that is, the generator of $\sqrt{2C}B$ for a standard Brownian motion B. To ease notation, we consider here $C = \frac{1}{2}\operatorname{Id}_{d\times d}$ and whenever we refer to the case $\alpha = 2$, we mean the standard Brownian motion noise case and $\mathscr{L}_\nu^\alpha = -\frac{1}{2}\Delta$.

Assumption 3.4. *Throughout the work, we assume that the measure* ν *from Definition 3.1 has d-dimensional support, in the sense that the linear span of its support is* $\mathbb{R}^d$*. This means that the process* L *can reach every open set in* $\mathbb{R}^d$ *with positive probability.*

An α-stable, symmetric Lévy process, that satisfies Assumption 3.4, we also call non-degenerate.

So far we defined $\mathscr{L}_\nu^\alpha$ on C_b^∞, so in particular on Schwartz functions. But the definition of $\mathscr{L}_\nu^\alpha$ on Schwartz distributions by duality is problematic, because for $\alpha \in (0,2)$ the function ψ_ν^α has a singularity in 0. This motivates the next proposition.

Proposition 3.5 (Continuity of the operator $\mathscr{L}_\nu^\alpha$)**.** *Let* $\alpha \in (0,2]$*. Then for* $\beta \in \mathbb{R}$ *and* $u \in C_b^\infty$*,* $p \in [1,\infty]$*, we have*

$$\|\mathscr{L}_\nu^\alpha u\|_{\mathscr{C}_p^{\beta-\alpha}} \lesssim \|u\|_{\mathscr{C}_p^\beta}.$$

In particular, $\mathscr{L}_\nu^\alpha$ *can be uniquely extended to a continuous operator from* $\mathscr{C}_p^\beta$ *to* $\mathscr{C}_p^{\beta-\alpha}$*.*

Proof. For $j \geqslant 0$ it follows from [BCD11, Lemma 2.2] the estimate $\|\mathscr{L}_\nu^\alpha \Delta_j u\|_{L^p} \lesssim 2^{-j(\beta-\alpha)}\|u\|_{\mathscr{C}_p^\beta}$. This uses that ψ_ν^α is infinitely continuously differentiable in $\mathbb{R}^d \setminus \{0\}$ with $|\partial^\mu \psi_\nu^\alpha(z)| \lesssim |z|^{\alpha-|\mu|}$ for a multi-index $\mu \in \mathbb{N}_0^d$ with $|\mu| := \mu_1 + \cdots + \mu_d \leqslant \alpha$ and that $\Delta_j u$ has a Fourier transform, which is supported in $2^j\mathscr{A}$, where $\mathscr{A}$ is the annulus, where p_0 is supported. For $j = -1$ and $p = \infty$, we use that $-\mathscr{L}^\alpha_\nu = A$ for A as in Equation (3.5), and therefore

$$\begin{aligned}
-\mathscr{L}^\alpha_\nu \Delta_{-1} u(x) &= \int_{\mathbb{R}^d} \left(\Delta_{-1}u(x+y) - \Delta_{-1}u(x) - \nabla\Delta_{-1}u(x)\cdot y\mathbb{1}_{\{|y|\leqslant 1\}}\right)\mu(dy) \\
&\lesssim \int_{B(0,1)} \|D^2\Delta_{-1}u\|_{L^\infty}|y|^2\mu(dy) + \|\Delta_{-1}u\|_{L^\infty}\mu(B(0,1)^c) \\
&\lesssim \|u\|_\beta,
\end{aligned} \tag{3.7}$$

where $B(0,1) = \{|y| \leqslant 1\}$ and the last step follows from the Bernstein inequality in [BCD11, Lemma 2.1].
For $p \in [1,\infty)$, we use that $-\mathscr{L}^\alpha_\nu \Delta_{-1}u = -\mathscr{L}^\alpha_\nu \mathscr{F}^{-1}p_{-1} * \Delta_{-1}u$, where p_{-1} is compactly supported in a ball, and Young's inequality, such that

$$\|-\mathscr{L}^\alpha_\nu \Delta_{-1}u\|_{L^p} \leqslant \|-\mathscr{L}^\alpha_\nu \mathscr{F}^{-1}p_{-1}\|_{L^1}\|\Delta_{-1}u\|_{L^p}.$$

Then, we estimate $-\mathscr{L}^\alpha_\nu \mathscr{F}^{-1}p_{-1}$ as in (3.7):

$$\begin{aligned}
\|-\mathscr{L}^\alpha_\nu \mathscr{F}^{-1}p_{-1}\|_{L^1} &\leqslant \|-\mathscr{L}^\alpha_\nu \mathscr{F}^{-1}p_{-1}\|_{L^\infty} \\
&\lesssim \|D^2\mathscr{F}^{-1}p_{-1}\|_{L^\infty}\int_{B(0,1)}|y|^2\mu(dy) + \|\mathscr{F}^{-1}p_{-1}\|_{L^\infty}\mu(B(0,1)^c) \\
&\lesssim 1.
\end{aligned}$$

□

Remark 3.6. *One can show that the operators A and $-\mathscr{L}^\alpha_\nu$ even agree on $\bigcup_{\varepsilon>0}\mathscr{C}^{2+\varepsilon}$. Indeed, for $\varphi \in \bigcup_{\varepsilon>0}\mathscr{C}^{2+\varepsilon}$ we have that φ and its partial derivatives up to order 2 are uniformly continuous, and thus it follows from [PZ07, Theorem 5.4] that $A\varphi$ has the same expression as in* (3.5)*. Then we can use that C^∞_b is dense in $\mathscr{C}^{2+\varepsilon}$ for all $\varepsilon > 0$ and apply a continuity argument to deduce that $A\varphi = -\mathscr{L}^\alpha_\nu\varphi$ for $\varphi \in \bigcup_{\varepsilon>0}\mathscr{C}^{2+\varepsilon}$.*

For $z \in \mathbb{R}^d \setminus \{0\}$, we also have

$$\psi^\alpha_\nu(z) = |z|^\alpha \int_S \left|\left\langle \frac{z}{|z|}, \xi\right\rangle\right|^\alpha \nu(d\xi) \geqslant |z|^\alpha \min_{|y|=1}\int_S |\langle y,\xi\rangle|^\alpha\nu(d\xi),$$

and by Assumption 3.4 the minimum on the right hand side is strictly positive. Otherwise, there exists some $y_0 \neq 0$ with $\int_S |\langle y_0,\xi\rangle|^\alpha\nu(d\xi) = 0$ and this would mean that the support of ν (and thus also its span) is contained in the orthogonal complement of span(y_0). Therefore, $e^{-\psi^\alpha_\nu}$ decays faster than any polynomial at infinity and outside of 0 it even behaves like a Schwartz function.

Lemma 3.7 (Semigroup estimates)**.** *Let ν be a finite, symmetric measure on the sphere $S \subset \mathbb{R}^d$ satisfying Assumption 3.4. Let $P_t\varphi := \mathscr{F}^{-1}(e^{-t\psi^\alpha_\nu}\hat\varphi) = \rho_t * \varphi$, where $t > 0$, $\rho_t = \mathscr{F}^{-1}e^{-t\psi^\alpha_\nu} \in L^1$, and $\varphi \in C^\infty_b$. Then we have for $\vartheta \geqslant 0$, $\beta \in \mathbb{R}$, $p \in [1,\infty]$*

$$\|P_t\varphi\|_{\mathscr{C}^{\beta+\vartheta}_p} \lesssim (t^{-\vartheta/\alpha} \vee 1)\|\varphi\|_{\mathscr{C}^\beta_p}, \tag{3.8}$$

and for $\vartheta \in [0, \alpha]$

$$\|(P_t - \mathrm{Id})\varphi\|_{\mathscr{C}_p^{\beta-\vartheta}} \lesssim t^{\vartheta/\alpha} \|\varphi\|_{\mathscr{C}_p^{\beta}}. \tag{3.9}$$

Furthermore, for $\beta \in (0, 1)$, $p = \infty$,

$$\|(P_t - \mathrm{Id})\varphi\|_{L^\infty} \lesssim t^{\beta/\alpha} \|\varphi\|_{\mathscr{C}^{\beta}}. \tag{3.10}$$

Therefore, if $\vartheta \geqslant 0$, *then* P_t *has a unique extension to a bounded linear operator in* $L(\mathscr{C}^{\beta}, \mathscr{C}^{\beta+\vartheta})$ *and this extension satisfies the same bounds.*

Proof. In the case $\theta \in [0, \alpha)$, this follows from [GIP15, Lemma A.5], see also [GIP15, Lemma A.7], whose generalization to integrability $p \in [1, \infty]$ is immediate. For the case $\vartheta = \alpha$ in (3.9), we estimate

$$\begin{aligned} \|(P_t - \mathrm{Id})\varphi\|_{\mathscr{C}_p^{\beta-\alpha}} &= \left\| \int_0^t (-\mathscr{L}_\nu^\alpha) P_r \varphi dr \right\|_{\mathscr{C}_p^{\beta-\alpha}} \\ &\leqslant \int_0^t \|(-\mathscr{L}_\nu^\alpha) P_r \varphi\|_{\mathscr{C}_p^{\beta-\alpha}} dr \\ &\lesssim \int_0^t \|P_r \varphi\|_{\mathscr{C}_p^{\beta}} dr \lesssim t \|\varphi\|_{\mathscr{C}_p^{\beta}} \end{aligned}$$

using Proposition 3.5 and (3.8) for $\vartheta = 0$. (3.10) follows from [GIP15, Lemma A.8]. □

The next three lemmas deal with commutators between the $(-\mathscr{L}_\nu^\alpha)$ operator, its semigroup and the paraproduct. The proofs can be found in Appendix A.

Lemma 3.8. *Let* $\alpha \in (1, 2]$, $f \in \mathscr{C}_p^\sigma$ *and* $g \in \mathscr{C}^\varsigma$ *with* $\sigma \in (0, 1)$ *and* $\varsigma \in \mathbb{R}$, $p \in [1, \infty]$. *Then the commutator for* $(-\mathscr{L}_\nu^\alpha)$ *follows:*

$$\|(-\mathscr{L}_\nu^\alpha)(f \varolessthan g) - f \varolessthan (-\mathscr{L}_\nu^\alpha) g\|_{\mathscr{C}_p^{\sigma+\varsigma-\alpha}} \lesssim \|f\|_{\mathscr{C}_p^\sigma} \|g\|_{\mathscr{C}^\varsigma}.$$

Lemma 3.9. *Let* (P_t) *be as in Lemma 3.7. Then, for* $\gamma < 1$, $\beta \in \mathbb{R}$, $p \in [1, \infty]$ *and* $\vartheta \geqslant -\alpha$ *the following commutator estimate holds true:*

$$\|P_t(u \varolessthan v) - u \varolessthan P_t v\|_{\mathscr{C}_p^{\gamma+\beta+\vartheta}} \lesssim (t^{-\vartheta/\alpha} \vee 1) \|u\|_{\mathscr{C}_p^\gamma} \|v\|_{\mathscr{C}^\beta}. \tag{3.11}$$

Lemma 3.10. *Let* $\mathscr{L}_\nu^\alpha$ *and* $(P_t)_{t \geqslant 0}$ *be defined as in Definition 3.1 and Lemma 3.7 and let* $\alpha \in (1, 2]$. *Let* $T > 0$, $\sigma \in (0, 1)$, $\varsigma \in \mathbb{R}$, $p \in [1, \infty]$ *and* $\theta \geqslant 0$. *Then the commutator on the operator* $(-\mathscr{L}_\nu^\alpha) P_t$ *follows:*

$$\|(-\mathscr{L}_\nu^\alpha) P_t(u \varolessthan v) - u \varolessthan (-\mathscr{L}_\nu^\alpha) P_t v\|_{\mathscr{C}_p^{\sigma+\varsigma-\alpha+\theta}} \lesssim (t^{-\theta/\alpha} \vee 1) \|u\|_{\mathscr{C}_p^\sigma} \|v\|_{\mathscr{C}^\varsigma}.$$

The mild formulation of the Kolmogorov equation is given by

$$u_t = P_{T-t}u_T + \int_t^T P_{r-t}(V_r \cdot \nabla u_r - f_r)dr =: P_{T-t}u_T + J^T(V \cdot \nabla u - f)(t). \quad (3.12)$$

Due to the Schauder estimates, considering a singular terminal condition with $u_T \in \mathscr{C}_p^{\beta+}$, we obtain that $\|P_{T-t}u_T\|_{\mathscr{C}_p^{\alpha+\beta}}$ blows up for $t \to T$ and the blow-up is of order $\gamma \in (0,1)$. This motivates the definition of blow-up spaces below, from which we can build the solution space in the next section.
For $\gamma \in (0,1)$, $T > 0$ and $\overline{T} \in (0,T]$, and a Banach space X, let us define the blow-up space

$$\mathscr{M}^{\gamma}_{\overline{T},T}X := \{u : [T-\overline{T},T) \to X \mid t \mapsto (T-t)^{\gamma}u_t \in C([T-\overline{T},T),X)\},$$

with $\|u\|_{\mathscr{M}^{\gamma}_{\overline{T},T}X} := \sup_{t\in[T-\overline{T},T)}(T-t)^{\gamma}\|u_t\|_X$ and $\mathscr{M}^{0}_{\overline{T},T}X := C([T-\overline{T},T),X)$. For $\overline{T} = T$, we use the notation $\mathscr{M}^{\gamma}_{T}X := \mathscr{M}^{\gamma}_{T,T}X$. For $\vartheta \in (0,1]$, $\gamma \in (0,1)$, we furthermore define

$$C^{\gamma,\vartheta}_{\overline{T},T}X := \left\{ u : [T-\overline{T},T) \to X \;\middle|\; \|f\|_{C^{\gamma,\vartheta}_T X} := \sup_{0\leqslant s<t<T} \frac{(T-t)^{\gamma}\|f_t - f_s\|_X}{|t-s|^{\vartheta}} < \infty \right\}$$

and $C^{\gamma,\vartheta}_{T}X := C^{\gamma,\vartheta}_{T,T}X$. Let us also define for $\vartheta \in (0,1]$, $\overline{T} \in (0,T]$, the space of ϑ-Hölder continuous functions on $[T-\overline{T},T]$ with values in X,

$$C^{\vartheta}_{\overline{T},T}X := \left\{ u : [T-\overline{T},T] \to X \;\middle|\; \|u\|_{C^{\vartheta}_T X} := \sup_{T-\overline{T}\leqslant s<t\leqslant T} \frac{\|u_t - u_s\|_X}{|t-s|^{\vartheta}} < \infty \right\}$$

and $C^{\vartheta}_{T}X := C^{\vartheta}_{T,T}X$. We set $C^{0,\vartheta}_{\overline{T},T}X := C^{\vartheta}([T-\overline{T},T),X)$.
We have the trivial estimates

$$\|u\|_{\mathscr{M}^{\gamma_1}_{\overline{T},T}X} \leqslant \overline{T}^{\gamma_1-\gamma_2}\|u\|_{\mathscr{M}^{\gamma_2}_{\overline{T},T}X}, \quad \|u\|_{C^{\gamma_1,\vartheta_1}_{\overline{T},T}X} \leqslant \overline{T}^{(\gamma_1-\gamma_2)+(\vartheta_2-\vartheta_1)}\|u\|_{C^{\gamma_2,\vartheta_2}_{\overline{T},T}X} \quad (3.13)$$

for $0 \leqslant \gamma_2 \leqslant \gamma_1 < 1$ and $0 < \vartheta_1 \leqslant \vartheta_2 \leqslant 1$. Moreover, we have that for a subinterval $[T-2\overline{T}, T-\overline{T}] \subset [0,T]$ with $0 < \overline{T} \leqslant \frac{T}{2}$,

$$\|u\|_{\mathscr{M}^{0}_{\overline{T},T-\overline{T}}X} \leqslant \overline{T}^{-\gamma}\|u\|_{\mathscr{M}^{\gamma}_{T}X}. \quad (3.14)$$

3.2. Schauder theory and commutator estimates for blow-up spaces

In this section, we define the solution space $\mathscr{L}^{\gamma,\alpha+\beta}_{T}$ and prove Schauder and commutator estimates. We conclude the section with interpolation estimates for the solution spaces. Heuristically, the solution space shall combine maximal space regularity (i.e. $\alpha+\beta$) in

a time-blow-up space with maximal time regularity (i.e. Lipschitz) in a space of low space regularity. By interpolation, the solution will then also admit all time and space regularities "in between".
Let us thus define for $\gamma \in (0,1)$ and $\theta \in \mathbb{R}$, $p \in [1,\infty]$, the space

$$\mathscr{L}_T^{\gamma,\theta} := \mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta} \cap C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha} \cap C_T^{\gamma,1}\mathscr{C}_p^{\theta-\alpha}. \tag{3.15}$$

We moreover define for $\gamma = 0$,

$$\mathscr{L}_T^{0,\theta} := C_T^1\mathscr{C}_p^{\theta-\alpha} \cap C_T\mathscr{C}_p^{\theta}, \tag{3.16}$$

where $C_T^1 X$ denotes the space of 1-Hölder or Lipschitz functions with values in X.
For $\overline{T} \in (0,T)$, we define $\mathscr{L}_{\overline{T},T}^{\gamma,\theta} := \mathscr{M}_{\overline{T},T}^{\gamma}\mathscr{C}_p^{\theta} \cap C_{\overline{T},T}^{1-\gamma}\mathscr{C}_p^{\theta-\alpha} \cap C_{\overline{T},T}^{\gamma,1}\mathscr{C}_p^{\theta-\alpha}$.
The spaces $\mathscr{L}_T^{\gamma,\theta}$ are Banach spaces equipped with the norm

$$\begin{aligned}\|u\|_{\mathscr{L}_T^{\gamma,\theta}} &:= \|u\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}} + \|u\|_{C_T^{\gamma,1}\mathscr{C}_p^{\theta-\alpha}} + \|u\|_{C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha}} \\ &= \sup_{t\in[0,T)} (T-t)^{\gamma}\|u_t\|_{\mathscr{C}_p^{\theta}} + \sup_{0\leqslant s<t<T} \frac{(T-t)^{\gamma}\|u_t-u_s\|_{\mathscr{C}_p^{\theta-\alpha}}}{|t-s|} + \sup_{0\leqslant s<t\leqslant T} \frac{\|u_t-u_s\|_{\mathscr{C}_p^{\theta-\alpha}}}{|t-s|^{1-\gamma}}.\end{aligned}$$

Notice, that $u \in \mathscr{L}_T^{\gamma,\theta}$ in particular implies that $t \mapsto \|u_t\|_{\mathscr{C}^{\theta-\alpha}}$ is $(1-\gamma)$-Hölder continuous at $t = T$.
The next corollary proves estimates for the semigroup (P_t) of $(-\mathscr{L}_\nu^{\alpha})$ acting on the spaces $\mathscr{L}_T^{\gamma,\theta}$. We will need the following auxillary lemma. In particular, the lemma can be applied, to show that the inverse fractional Laplacian improves space regularity by α (and not only by $\theta < \alpha$). It is a slight generalization of [GIP15, Lemma A.9, (A.1)]. Its proof can be found in Appendix A.

Lemma 3.11. *Let $\sigma \in \mathbb{R}$, $p \in [1,\infty]$, $\gamma \in [0,1)$, $\varepsilon \in (0,1)$ and $\varsigma \geqslant 0$. Let moreover $f : \mathring{\Delta}_T \to \mathscr{S}'$, $\mathring{\Delta}_T := \{(t,r) \in [0,T]^2 \mid t < r\}$, be such that there exists $C > 0$ such that for all $j \geqslant -1$ and $0 \leqslant t < r \leqslant T$, for the Littlewood-Paley blocks holds*

$$\|\Delta_j f_{t,r}\|_{L^p} \leqslant C(T-r)^{-\gamma}\min(2^{-j\sigma}, 2^{-j(\sigma+\varsigma+\varepsilon\varsigma)}(r-t)^{-(1+\varepsilon)}).$$

Then it follows that for all $t \in [0,T]$

$$\left\|\int_t^T f_{t,r}dr\right\|_{\mathscr{C}_p^{\sigma+\varsigma}} \leqslant [2C\max(\varepsilon^{-1},(1-\gamma)^{-1})](T-t)^{-\gamma}. \tag{3.17}$$

Corollary 3.12 (Schauder estimates)**.** *Let (P_t) and ν be as in Lemma 3.7. Let $T > 0$, $\overline{T} \in (0,T]$. For $t \in [T-\overline{T},T]$ we define $J^T v(t) = J^T(v)(t) := \int_t^T P_{r-t}v(r)dr$. Then we have for $\beta \in \mathbb{R}$, $\vartheta \in [0,\alpha]$, $\gamma \in [\vartheta/\alpha, 1]$,*

$$\|P_{T-\cdot}w\|_{\mathscr{L}_{\overline{T},T}^{\gamma,\beta+\vartheta}} \lesssim \overline{T}^{(\gamma\alpha-\vartheta)/\alpha}\|w\|_{\mathscr{C}_p^{\beta}} \tag{3.18}$$

and for $0 \leqslant \gamma' \leqslant \gamma < 1$,

$$\|J^T v\|_{\mathscr{L}^{\gamma,\beta+\alpha}_{\overline{T},T}} \lesssim \overline{T}^{\gamma-\gamma'} \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{\beta}_p}. \tag{3.19}$$

Remark 3.13. *Recall the definition of $\mathscr{L}^{0,\theta}_T$ and that $v \in \mathscr{L}^{0,\theta}_T$ does not imply continuity of $t \mapsto \|v_t\|_{\mathscr{C}^\theta_p}$ at $t = T$. However, continuity of $t \mapsto \|P_{T-t}w\|_{\mathscr{L}^{0,\beta}_{\overline{T},T}}$ and $t \mapsto \|J^T v\|_{\mathscr{L}^{0,\beta+\alpha}_{\overline{T},T}}$ at $t = T$ can be inferred from the Schauder estimates and separability of $\mathscr{C}^\theta_p$ (cf. the definition in (3.2)). Indeed, approximating v in $C_T\mathscr{C}^\beta_p$ by $(v^n) \subset C_T C^\infty_b$ and using (3.19) yields that $\|J^T(v) - J^T(v^n)\|_{\mathscr{M}^0_T\mathscr{C}^{\beta+\alpha}} \lesssim \|v - v^n\|_{C_T\mathscr{C}^\beta_p} \to 0$ and $J^T(v^n) \in C_T C^\infty_b$ for all n. As the uniform limit of continuous functions is a continuous function, we obtain that indeed $J^T(v) \in C_T\mathscr{C}^{\beta+\alpha}_p$, analogously $P_{T-\cdot}w \in C_T\mathscr{C}^\beta_p$. In particular, continuity at $t = T$ follows.*
Moreover, for $w \in \mathscr{C}^\beta_p$ follows $\frac{d}{dt}P_t w = (-\mathscr{L}^\alpha_\nu)P_t w$ in the distributional sense and we have that $(-\mathscr{L}^\alpha_\nu)P_t w \in C_T\mathscr{C}^{\beta-\alpha}_p$. Hence, $P_t w$ and $J^T(v)$ are not only Lipschitz continuous in time with values in $\mathscr{C}^{\beta-\alpha}_p$, respectively $\mathscr{C}^\beta_p$, but also continuously differentiable in time.
Together, this shows that we can replace $\mathscr{L}^{0,\theta}_{\overline{T},T}$ in the Schauder estimates, and the following estimates involving the semigroup, by $C([T-\overline{T},T],\mathscr{C}^\theta_p) \cap C^1([T-\overline{T},T],\mathscr{C}^{\theta-\alpha}_p)$.

Proof. For (3.18) we only prove the estimate in $C^{1-\gamma}_{\overline{T},T}\mathscr{C}^{\beta+\vartheta-\alpha}_p$ and in $C^{\gamma,1}_{\overline{T},T}\mathscr{C}^{\beta+\vartheta-\alpha}_p$, the estimate in $\mathscr{M}^\gamma_{\overline{T},T}\mathscr{C}^{\beta+\vartheta}_p$ follows from a direct application of Lemma 3.7.
Therefore we write $P_{T-t}w - P_{T-s}w = P_{T-t}(\mathrm{Id} - P_{t-s})w$ for $T - \overline{T} \leqslant s < t \leqslant T$ and use Lemma 3.7 to conclude

$$\begin{aligned}\|P_{T-t}w - P_{T-s}w\|_{\mathscr{C}^{\beta+\vartheta-\alpha}_p} \lesssim \|(\mathrm{Id} - P_{t-s})w\|_{\mathscr{C}^{\beta+\vartheta-\alpha}_p} &\lesssim (t-s)^{1-\vartheta/\alpha}\|w\|_{\mathscr{C}^\beta_p} \\ &\lesssim \overline{T}^{(\gamma\alpha-\vartheta)/\alpha}(t-s)^{1-\gamma}\|w\|_{\mathscr{C}^\beta_p}\end{aligned}$$

using $0 \leqslant \vartheta \leqslant \alpha$ and $\gamma \geqslant \vartheta/\alpha$. This controls $\|P_{T-\cdot}w\|_{C^{1-\gamma}_{\overline{T},T}\mathscr{C}^{\beta+\gamma-\alpha}_p}$. To bound the norm $\|P_{T-\cdot}w\|_{C^{\gamma,1}_{\overline{T},T}\mathscr{C}^{\beta+\gamma-\alpha}_p}$, we note that

$$\begin{aligned}\|P_{T-t}w - P_{T-s}w\|_{\mathscr{C}^{\beta+\vartheta-\alpha}_p} &\lesssim (T-t)^{-\vartheta/\alpha}\|(\mathrm{Id} - P_{t-s})w\|_{\mathscr{C}^{\beta-\alpha}_p} \\ &\lesssim (T-t)^{-\vartheta/\alpha}(t-s)\|w\|_{\mathscr{C}^\beta_p} \\ &\lesssim \overline{T}^{(\gamma\alpha-\vartheta)/\alpha}(T-t)^{-\gamma}(t-s)\|w\|_{\mathscr{C}^\beta_p}.\end{aligned}$$

To estimate the $\mathscr{M}^\gamma_{\overline{T},T}\mathscr{C}^{\beta+\alpha}_p$-norm in (3.19), we use Lemma 3.11 with $f_{t,r} = P_{r-t}v_r$ and $\sigma = \beta$, $\varsigma = \alpha$, to obtain for $t \in [T - \overline{T}, T]$

$$(T-t)^\gamma \|J^T v(t)\|_{\mathscr{C}^{\beta+\alpha}_p} = (T-t)^{\gamma-\gamma'}(T-t)^{\gamma'}\left\|\int_t^T P_{r-t}v_r dr\right\|_{\mathscr{C}^{\beta+\alpha}_p} \lesssim \overline{T}^{\gamma-\gamma'}\|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^\beta_p}.$$

To prove the bounds on the time regularity in (3.19) we write

$$J^T(v)_t - J^T(v)_s = \int_s^t P_{r-s} v_r dr - (P_{t-s} - \mathrm{Id})\bigg(\int_t^T P_{r-t} v_r dr\bigg),$$

for $T - \overline{T} \leqslant s < t \leqslant T$. We can estimate by Lemma 3.7

$$\begin{aligned}\bigg\|\int_s^t P_{r-s} v_r dr\bigg\|_{\mathscr{C}_p^\beta} &\leqslant \int_s^t \|P_{r-s} v_r\|_{\mathscr{C}_p^\beta} dr \\ &\lesssim \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta} \int_s^t |T-r|^{-\gamma'} dr \\ &\lesssim \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta} (|T-s|^{1-\gamma'} - |T-t|^{1-\gamma'}) \\ &\leqslant \overline{T}^{\gamma-\gamma'} |t-s|^{1-\gamma} \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta},\end{aligned}$$

using that $0 \leqslant \gamma' \leqslant \gamma < 1$ and the estimate

$$|T-t|^{1-\gamma'} - |T-s|^{1-\gamma'} \leqslant |t-s|^{1-\gamma'} \leqslant \overline{T}^{\gamma-\gamma'} |t-s|^{1-\gamma}.$$

On the other hand, we can also estimate that term by

$$\begin{aligned}\bigg\|\int_s^t P_{r-s} v_r dr\bigg\|_{\mathscr{C}_p^\beta} &\lesssim \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta} \int_s^t |T-r|^{-\gamma'} dr \\ &\leqslant \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta} |T-t|^{-\gamma'} \int_s^t dr \\ &\leqslant \overline{T}^{\gamma-\gamma'} \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta} |T-t|^{-\gamma} |t-s|.\end{aligned}$$

Moreover, by Lemma 3.7 for $\vartheta = \alpha$ and Lemma 3.11, we obtain that

$$\begin{aligned}\bigg\|(P_{t-s} - \mathrm{Id})\bigg(\int_t^T P_{r-t} v_r dr\bigg)\bigg\|_{\mathscr{C}_p^\beta} &\lesssim |t-s| \bigg\|\int_t^T P_{r-t} v_r dr\bigg\|_{\mathscr{C}_p^{\beta+\alpha}} \\ &\lesssim |t-s| \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta} (T-t)^{-\gamma'} \\ &\lesssim |t-s| \overline{T}^{\gamma-\gamma'} \|v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\beta} (T-t)^{-\gamma},\end{aligned}$$

and on the other hand we can estimate by Lemma 3.7 for $\vartheta = (1-\gamma)\alpha$,

$$\begin{aligned}
&\left\|(P_{t-s}-\mathrm{Id})\left(\int_t^T P_{r-t}v_r dr\right)\right\|_{\mathscr{C}_p^\beta} \\
&\lesssim |t-s|^{(\alpha-\gamma\alpha)/\alpha}\left\|\int_t^T P_{r-t}v_r dr\right\|_{\mathscr{C}_p^{\beta+\alpha-\gamma\alpha}} \\
&\lesssim |t-s|^{(\alpha-\gamma\alpha)/\alpha}\|v\|_{\mathscr{M}_{\overline{T},T}^{\gamma'}\mathscr{C}_p^\beta}\int_t^T (T-r)^{-\gamma'}(t-r)^{(\gamma\alpha-\alpha)/\alpha}dr \\
&\lesssim |t-s|^{1-\gamma}\|v\|_{\mathscr{M}_{\overline{T},T}^{\gamma'}\mathscr{C}_p^\beta}(T-t)^{\gamma-\gamma'} \\
&\lesssim |t-s|^{1-\gamma}\|v\|_{\mathscr{M}_{\overline{T},T}^{\gamma'}\mathscr{C}_p^\beta}\overline{T}^{\gamma-\gamma'},
\end{aligned}$$

where we used that $\gamma > 0$ and that $\gamma' \leqslant \gamma < 1$ (if $\gamma = 0$, we can use the previous estimate instead). □

Next, we prove a commutator estimate for the J^T-operator and the paraproduct.

Lemma 3.14 (Commutator estimates)**.** *Let $T > 0$ and $\overline{T} \in (0,T]$ and let $\varsigma \in \mathbb{R}$, $\sigma \in (0,1)$ and $p \in [1,\infty]$. Let $\alpha \in (1,2]$ and $\gamma \in [0,1)$. Then for $u \in \mathscr{C}_p^\sigma$, $v \in \mathscr{C}^\varsigma$ the following semigroup commutator estimate holds*

$$\|t \mapsto P_{T-t}(u \otimes v) - u \otimes P_{T-t}(v)\|_{\mathscr{L}_T^{\gamma,\sigma+\varsigma+\gamma\alpha}} \lesssim \|u\|_{\mathscr{C}_p^\sigma}\|v\|_{\mathscr{C}^\varsigma}. \tag{3.20}$$

Furthermore, for $g \in \mathscr{L}_{\overline{T},T}^{\gamma',\sigma}$ with $0 \leqslant \gamma' \leqslant \gamma < 1$ and $h \in C_T\mathscr{C}^\varsigma$, we have

$$\|J^T(g \otimes h) - g \otimes J^T(h)\|_{\mathscr{L}_{\overline{T},T}^{\gamma,\sigma+\varsigma+\alpha}} \lesssim \overline{T}^{\gamma-\gamma'}\|g\|_{\mathscr{L}_{\overline{T},T}^{\gamma',\sigma}}\|h\|_{C_T\mathscr{C}^\varsigma}. \tag{3.21}$$

Remark 3.15. *It is known that the commutator for the J^T-operator from the lemma allows for more space regularity than both of its summands. The above commutator estimate moreover yields a gain in time regularity, i.e. $J^T(g \otimes h) - g \otimes J^T(h) \in C_T^{1-\gamma}\mathscr{C}_p^{\sigma+\varsigma} \cap C_T^{\gamma,1}\mathscr{C}_p^{\sigma+\varsigma}$, provided that $g \in \mathscr{L}_T^{\gamma,\sigma}$.*

Proof. Recall that $\mathscr{L}_T^{\gamma,\sigma+\varsigma+\gamma\alpha}$ is equipped with the sum of the norms in

$$\mathscr{M}_T^\gamma\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma}, \quad C_T^{1,\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha} \quad \text{and} \quad C_T^{1-\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha},$$

that we need to estimate below.
For (3.20), the estimate in $\mathscr{M}_T^\gamma\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma}$ follows directly by the semigroup commutator Lemma 3.9 applied to $\vartheta = \gamma\alpha$. For the estimate in $C_T^{1,\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha} \cap C_T^{1-\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha}$

we write for $0 \leqslant s \leqslant t \leqslant T$,

$$\begin{aligned}P_{T-t}(u \otimes v) - u \otimes P_{T-t}(v) - (P_{T-s}(u \otimes v) - u \otimes P_{T-s}(v)) \\ = (\mathrm{Id} - P_{t-s})[P_{T-t}(u \otimes v) - u \otimes P_{T-t}v] \\ + [u \otimes P_{t-s}P_{T-t}v - P_{t-s}(u \otimes P_{T-t}v)].\end{aligned}$$

The first summand we can estimate by the semigroup estimates (Lemma 3.7) for $\mathrm{Id} - P_{t-s}$ and the commutator estimate in $\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma}$, obtaining

$$\begin{aligned}&\|(\mathrm{Id} - P_{t-s})[P_{T-t}(u \otimes v) - u \otimes P_{T-t}v]\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha}} \\ &\lesssim |t-s| \|[P_{T-t}(u \otimes v) - u \otimes P_{T-t}v]\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma}} \\ &\lesssim (T-t)^{-\gamma}|t-s| \|u\|_{\mathscr{C}_p^{\sigma}} \|v\|_{\mathscr{C}^{\varsigma}}.\end{aligned}$$

This gives the estimate in $C_T^{1,\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha(\gamma-1)}$. Analogously we can estimate the norm in $C_T^{1-\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha(\gamma-1)}$ using the Schauder estimates for $\mathrm{Id} - P_{t-s}$ (obtaining a factor of $|t-s|^{1-\gamma}$) and the commutator in $C_{\overline{T},T}\mathscr{C}_p^{\sigma+\varsigma}$, i.e.

$$\begin{aligned}&\|(\mathrm{Id} - P_{t-s})[P_{T-t}(u \otimes v) - u \otimes P_{T-t}v]\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha}} \\ &\lesssim |t-s|^{1-\gamma} \|[P_{T-t}(u \otimes v) - u \otimes P_{T-t}v]\|_{\mathscr{C}_p^{\sigma+\varsigma}} \\ &\lesssim |t-s|^{1-\gamma} \|u\|_{\mathscr{C}_p^{\sigma}} \|v\|_{\mathscr{C}^{\varsigma}}.\end{aligned}$$

The second summand can be estimated using the semigroup commutator (Lemma 3.9) for $\vartheta = (\gamma-1)\alpha \geqslant -\alpha$ and the semigroup estimate (3.9), such that

$$\begin{aligned}&\left\|P_{t-s}(u \otimes P_{T-t}v) - u \otimes P_{t-s}P_{T-t}v\right\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha}} \\ &\lesssim |t-s|^{1-\gamma} \|u\|_{\mathscr{C}_p^{\sigma}} \|P_{T-t}v\|_{\mathscr{C}^{\varsigma}} \lesssim |t-s|^{1-\gamma} \|u\|_{\mathscr{C}_p^{\sigma}} \|v\|_{\mathscr{C}^{\varsigma}}.\end{aligned}$$

Using instead the semigroup commutator for $\vartheta = -\alpha \geqslant -\alpha$ and again the semigroup estimate (3.9) yields

$$\begin{aligned}&\left\|P_{t-s}(u \otimes P_{T-t}v) - u \otimes P_{t-s}P_{T-t}v\right\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha\gamma-\alpha}} \\ &\lesssim |t-s| \|u\|_{\mathscr{C}_p^{\sigma}} \|P_{T-t}v\|_{\mathscr{C}^{\varsigma+\alpha\gamma}} \\ &\lesssim |t-s|(T-t)^{-\gamma} \|u\|_{\mathscr{C}_p^{\sigma}} \|v\|_{\mathscr{C}^{\varsigma}}.\end{aligned} \tag{3.22}$$

Together, we obtain (3.20). For (3.21), we first prove that $C(g,h) := J^T(g \otimes h) - g \otimes J^T(h) \in \mathscr{M}_{\overline{T},T}^{\gamma}\mathscr{C}_p^{\sigma+\varsigma+\alpha}$. To that end, we write

$$C(g,h)_t = \int_t^T \big(P_{r-t}(g_r \otimes h_r) - g_r \otimes P_{r-t}h_r\big)dr + \int_t^T (g_r - g_t) \otimes P_{r-t}h_r dr =: I_1(t) + I_2(t).$$

To estimate I_1, we utilize Lemma 3.11 for $f_{t,r} = P_{r-t}(g_r \otimes h_r) - g_r \otimes P_{r-t}h_r$, where the assumptions of the lemma are satisfied by the semigroup commutator estimate (Lemma 3.9). Then, we obtain

$$\|I_1(t)\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha}} \lesssim \|g\|_{\mathscr{M}_{\overline{T},T}^{\gamma'}\mathscr{C}_p^{\sigma}} \|h\|_{C_T\mathscr{C}^{\varsigma}}(T-t)^{-\gamma'} \lesssim \overline{T}^{\gamma-\gamma'}\|g\|_{\mathscr{M}_{\overline{T},T}^{\gamma'}\mathscr{C}_p^{\sigma}} \|h\|_{C_T\mathscr{C}^{\varsigma}}(T-t)^{-\gamma}.$$

For I_2, we apply Lemma 3.11 for $f_{t,r} := (g_r - g_t) \otimes P_{r-t}h_r$. We check the assumptions on $f_{t,r}$ of that lemma, using the time regularity of g, as well as the paraproduct estimate (using $\sigma - \alpha < 0$) and the semigroup estimates. Then, choosing $\theta = 0$ or $\theta = (1+\varepsilon)\alpha$ for $\varepsilon \in [0,1]$, we estimate (the estimate is in fact valid for all $\theta \geqslant -\alpha$)

$$\begin{aligned}\|(g_r - g_t) \otimes P_{r-t}h_r\|_{\mathscr{C}_p^{\sigma+\varsigma+\theta}} &= \|(g_r - g_t) \otimes P_{r-t}h_r\|_{\mathscr{C}_p^{(\sigma-\alpha)+(\varsigma+\theta+\alpha)}} \\ &\lesssim (T-r)^{-\gamma'}(r-t)^{-\theta/\alpha}\|h\|_{C_T\mathscr{C}^{\varsigma}}\|g\|_{C_{\overline{T},T}^{\gamma',1}\mathscr{C}_p^{\sigma-\alpha}}\end{aligned}$$

Applying Lemma 3.11 yields then the estimate for I_2:

$$\|I_2(t)\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha}} \lesssim \overline{T}^{\gamma-\gamma'}\|g\|_{C_{\overline{T},T}^{\gamma',1}\mathscr{C}_p^{\sigma-\alpha}}\|h\|_{C_T\mathscr{C}^{\varsigma}_{\mathbb{R}^d}}(T-t)^{-\gamma}.$$

Next, we prove the time regularity estimates on the commutator $C(g,h)$. For that, we write for $T-\overline{T} \leqslant s \leqslant l \leqslant T$,

$$\begin{aligned}&J^T(g \otimes h)_l - g_l \otimes J^T(h)_l - (J^T(g\otimes h)_s - g_s \otimes J^T(h)_s) \\ &\quad = -\int_s^l P_{r-s}(g_r \otimes h_r)dr - (P_{l-s} - \mathrm{Id})\Big(\int_l^T P_{r-l}(g_r \otimes h_r)dr\Big) \\ &\qquad + g_s \otimes \int_s^l P_{r-s}h_r dr - g_{s,l} \otimes \int_l^T P_{r-l}h_r dr + g_s \otimes (P_{l-s} - \mathrm{Id})\Big(\int_l^T P_{r-l}h_r dr\Big) \\ &\quad = A_{sl} + B_{sl} + C_{sl},\end{aligned}$$

where we define

$$A_{sl} := g_s \otimes \int_s^l P_{r-s}h_r dr - \int_s^l P_{r-s}(g_r \otimes h_r)dr$$

and

$$B_{sl} := -g_{s,l} \otimes \int_l^T P_{r-l}h_r dr,$$

where $g_{s,l} := g_l - g_s$ and

$$C_{sl} := g_s \otimes P_{l-s}\Big(\int_l^T P_{r-l}h_r dr\Big) - P_{l-s}\Big(\int_l^T P_{r-l}(g_r \otimes h_r)dr\Big).$$

We will consider the terms A_{sl}, B_{sl} and C_{sl} separately and estimate each term in the

$C^{1-\gamma}_{\overline{T},T}\mathscr{C}^{\sigma+\varsigma}$-norm and in the $C^{\gamma,1}_{\overline{T},T}\mathscr{C}^{\sigma+\varsigma}$-norm.
We start with B_{sl}, using the time regularity of g, obtaining on the one hand

$$\begin{aligned}\|B_{sl}\|_{\mathscr{C}_p^{\sigma+\varsigma}} &= \left\| g_{s,l} \otimes \int_l^T P_{r-l}h_r dr \right\|_{\mathscr{C}_p^{(\sigma-\alpha)+(\alpha+\varsigma)}} \\ &\lesssim \|g\|_{C^{1-\gamma'}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}} |l-s|^{1-\gamma'} \left\| \int_l^T P_{r-l}h_r dr \right\|_{\mathscr{C}^{\alpha+\varsigma}} \\ &\lesssim |l-s|^{1-\gamma'} \|g\|_{C^{1-\gamma'}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}} \|h\|_{C_T\mathscr{C}^{\varsigma}} \\ &\lesssim \overline{T}^{\gamma-\gamma'} |l-s|^{1-\gamma} \|g\|_{C^{1-\gamma'}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}} \|h\|_{C_T\mathscr{C}^{\varsigma}},\end{aligned}$$

using $\sigma-\alpha<0$ and Lemma 3.11 for $f_{l,r} = P_{r-l}h_r$ to bound the time integral. On the other hand, along the same lines, using instead $g \in C^{\gamma',1}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}$, we can estimate B_{sl} by

$$\|B_{sl}\|_{\mathscr{C}_p^{\sigma+\varsigma}} \lesssim \overline{T}^{\gamma-\gamma'} \|h\|_{C_T\mathscr{C}^{\varsigma}} \|g\|_{C^{\gamma',1}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}} |l-s|(T-l)^{-\gamma}.$$

For A_{sl}, we use the semigroup commutator (Lemma 3.9) for $\vartheta=0$, as well as the time regularity of g, which yields

$$\begin{aligned}&\|A_{sl}\|_{\mathscr{C}_p^{\sigma+\varsigma}} \\ &\quad= \left\| \int_s^l P_{r-s}(g_r \otimes h_r)dr - g_s \otimes \int_s^l P_{r-s}h_r dr \right\|_{\mathscr{C}_p^{\sigma+\varsigma}} \\ &\quad\leqslant \left\| \int_s^l (P_{r-s}(g_r \otimes h_r) - g_r \otimes P_{r-s}h_r)dr \right\|_{\mathscr{C}_p^{\sigma+\varsigma}} + \left\| \int_s^l (g_r - g_s) \otimes P_{r-s}h_r dr \right\|_{\mathscr{C}_p^{(\sigma-\alpha)+(\varsigma+\alpha)}} \\ &\quad\leqslant \|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{\sigma}} \|h\|_{C_T\mathscr{C}_p^{\varsigma}} \int_s^l (T-r)^{-\gamma'}dr + \|h\|_{C_T\mathscr{C}^{\varsigma}} \|g\|_{C^{1-\gamma'}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}} \int_s^l |r-s|^{-\gamma'}dr \\ &\quad\lesssim \|g\|_{\mathscr{L}^{\gamma',\sigma}_{\overline{T},T}} \|h\|_{C_T\mathscr{C}^{\varsigma}} \big((T-s)^{1-\gamma'} - (T-l)^{1-\gamma'} + |l-s|^{1-\gamma'} \big) \\ &\quad\lesssim \|g\|_{\mathscr{L}^{\gamma',\sigma}_{\overline{T},T}} \|h\|_{C_T\mathscr{C}^{\varsigma}} \overline{T}^{\gamma-\gamma'} |l-s|^{1-\gamma}.\end{aligned}$$

We can also estimate the term A_{sl} by

$$\begin{aligned}\|A_{sl}\|_{\mathscr{C}_p^{\sigma+\varsigma}} &\leqslant \|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^{\sigma}} \|h\|_{C_T\mathscr{C}^{\varsigma}} \int_s^l (T-r)^{-\gamma'}dr + \|h\|_{C_T\mathscr{C}^{\varsigma}} \|g\|_{C^{\gamma',1}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}} \int_s^l (T-r)^{-\gamma'}dr \\ &\lesssim (T-l)^{-\gamma'} |l-s| \|h\|_{C_T\mathscr{C}^{\varsigma}} \left(\|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^{\sigma}} + \|g\|_{C^{\gamma',1}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}} \right) \\ &\lesssim \overline{T}^{\gamma-\gamma'} (T-l)^{-\gamma} |l-s| \|h\|_{C_T\mathscr{C}^{\varsigma}} \|g\|_{\mathscr{L}^{\gamma',\sigma}_{\overline{T},T}},\end{aligned}$$

using $(T-r)^{-\gamma} \leqslant (T-l)^{-\gamma}$ for $r \in [s,l]$. It is left to estimate the term C_{sl}, that we first rewrite:

$$\begin{aligned}C_{sl} &= P_{l-s}\Big(\int_l^T P_{r-l}(g_r \otimes h_r)dr\Big) - g_s \otimes P_{l-s}\Big(\int_l^T P_{r-l}h_r dr\Big) \\ &= (P_{l-s} - \mathrm{Id})\Big(\int_l^T P_{r-l}(g_r \otimes h_r)dr - g_s \otimes \int_l^T P_{r-l}h_r dr\Big) && (3.23) \\ &\quad + P_{l-s}\Big(g_s \otimes \int_l^T P_{r-l}h_r dr\Big) - g_s \otimes P_{l-s}\Big(\int_l^T P_{r-l}h_r dr\Big). && (3.24)\end{aligned}$$

To estimate the term in line (3.23), we use Lemma 3.7 and the estimate for $I_1(l)+I_2(l)$ from above to obtain

$$\begin{aligned}&\Big\|(P_{l-s} - \mathrm{Id})\Big(\int_l^T (P_{r-l}(g_r \otimes h_r) - g_s \otimes P_{r-l}h_r)dr\Big)\Big\|_{\mathscr{C}_p^{\varsigma+\sigma+\alpha-\alpha}} \\ &\quad\lesssim |l-s|\Big\|\int_l^T (P_{r-l}(g_r \otimes h_r) - g_s \otimes P_{r-l}h_r)dr\Big\|_{\mathscr{C}_p^{\varsigma+\sigma+\alpha}} \\ &\quad\lesssim |l-s|(T-l)^{-\gamma}\overline{T}^{\gamma-\gamma'}\|g\|_{\mathscr{L}^{\gamma',\sigma}_{\overline{T},T}}\|h\|_{C_T\mathscr{C}^\varsigma}.\end{aligned}$$

The term in line (3.23), we can also estimate differently using Lemma 3.7 and an easier estimate for $I_1(l), I_2(l)$ using the semigroup estimates and $\alpha(1-\gamma') < \alpha$ to obtain

$$\begin{aligned}&\Big\|(P_{l-s} - \mathrm{Id})\Big(\int_l^T (P_{r-l}(g_r \otimes h_r) - g_s \otimes P_{r-l}h_r)dr\Big)\Big\|_{\mathscr{C}_p^{\varsigma+\sigma}} \\ &\quad\lesssim |l-s|^{1-\gamma'}\Big\|\int_l^T (P_{r-l}(g_r \otimes h_r) - g_s \otimes P_{r-l}h_r)dr\Big\|_{\mathscr{C}_p^{\varsigma+\sigma+\alpha(1-\gamma')}} \\ &\quad\lesssim |l-s|^{1-\gamma'}(\|I_1(l)\|_{\mathscr{C}_p^{\varsigma+\sigma+\alpha(1-\gamma')}} + \|I_2(l)\|_{\mathscr{C}_p^{\varsigma+\sigma+\alpha(1-\gamma')}}) \\ &\quad\lesssim |l-s|^{1-\gamma'}\|h\|_{C_T\mathscr{C}^\varsigma}\Big([\|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\sigma} + \|g\|_{C^{\gamma',1}_{\overline{T},T}\mathscr{C}_p^{\sigma-\alpha}}]\int_l^T (T-r)^{-\gamma'}(r-l)^{-1+\gamma'}dr\Big) \\ &\quad\lesssim \overline{T}^{\gamma-\gamma'}|l-s|^{1-\gamma}\|h\|_{C_T\mathscr{C}^\varsigma}\|g\|_{\mathscr{L}^{\gamma',\sigma}_{\overline{T},T}}\int_0^1 (1-r)^{-\gamma'}r^{-1+\gamma'}dr \\ &\quad\lesssim \overline{T}^{\gamma-\gamma'}|l-s|^{1-\gamma}\|h\|_{C_T\mathscr{C}^\varsigma}\|g\|_{\mathscr{L}^{\gamma',\sigma}_{\overline{T},T}}.\end{aligned}$$

To estimate the term in line (3.24), we use the commutator for P_{l-s} for $\vartheta = -\alpha$ and again Lemma 3.11 for $f_{l,r} = P_{r-l}h_r$, yielding

$$\begin{aligned}
&\Big\| P_{l-s}\Big(g_s \otimes \int_l^T P_{r-l}h_r dr\Big) - g_s \otimes P_{l-s}\Big(\int_l^T P_{r-l}h_r dr)\Big)\Big\|_{\mathscr{C}_p^{\sigma+\varsigma+\alpha-\alpha}} \\
&\quad\lesssim |l-s| \|h\|_{C_T\mathscr{C}^\varsigma} \|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\sigma} (T-s)^{-\gamma'} \Big\| \int_l^T P_{r-l}h_r dr \Big\|_{\mathscr{C}^{\varsigma+\alpha}} \\
&\quad\lesssim \overline{T}^{\gamma-\gamma'} |l-s| \|h\|_{C_T\mathscr{C}^\varsigma} \|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\sigma} (T-s)^{-\gamma}.
\end{aligned}$$

Applying instead the semigroup commutator for $\vartheta = -(1-\gamma')\alpha$ yields

$$\begin{aligned}
&\Big\| P_{l-s}\Big(g_s \otimes \int_l^T P_{r-l}h_r dr\Big) - g_s \otimes P_{l-s}\Big(\int_l^T P_{r-l}h_r dr)\Big)\Big\|_{\mathscr{C}_p^{\sigma+\varsigma}} \\
&\quad\lesssim |l-s|^{1-\gamma'} \|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\sigma} (T-s)^{-\gamma'} \Big\| \int_l^T P_{r-l}h_r dr \Big\|_{\mathscr{C}^{\varsigma+\alpha(1-\gamma')}} \\
&\quad\lesssim |l-s|^{1-\gamma'} \|h\|_{C_T\mathscr{C}^\varsigma} \|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\sigma} (T-s)^{-\gamma'} (T-l)^{\gamma'} \\
&\quad\lesssim \overline{T}^{\gamma-\gamma'} |l-s|^{1-\gamma} \|h\|_{C_T\mathscr{C}^\varsigma} \|g\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}_p^\sigma},
\end{aligned}$$

where to bound the time integral, we used that $\alpha(1-\gamma') < \alpha$ and $s \leqslant l$. Together we obtain the desired estimates for C_{sl}, which yield together with the estimates for A_{sl} and B_{sl} the claim. □

Remark 3.16. *The proof of the commutator estimate does not apply if we consider instead of $g \in \mathscr{L}_T^{\gamma,\sigma}$, a function $g \in \mathscr{M}_T^\gamma\mathscr{C}^\sigma \cap C_T^{1-\gamma}\mathscr{C}^{\sigma-\alpha}$. The reason is the estimate for the term I_2 in the above proof, for which we need to employ that $g \in C_T^{\gamma,1}\mathscr{C}^{\sigma-\alpha}$.*

We conclude this section with interpolation estimates for the spaces $\mathscr{L}_T^{\gamma,\theta}$.

Lemma 3.17 (Interpolation estimates)**.** *Let $\gamma \in [0,1)$, $\theta \in [0,\alpha]$, $p \in [1,\infty]$. Let moreover $v \in \mathscr{L}_T^{\gamma,\theta}$. Then the following estimates hold true:*
It follows that for $\theta \in [\alpha\gamma, \alpha)$,

$$\|v\|_{C_T^{\gamma(1-\gamma),\theta/\alpha-\gamma}L^p} \lesssim (T^{\gamma(1-\gamma)} \vee 1)\|v\|_{\mathscr{L}_T^{\gamma,\theta}}. \tag{3.25}$$

Furthermore, for $\tilde\theta \in [0,\alpha]$, it holds that

$$\|v\|_{C_T^{\gamma,\tilde\theta/\alpha}\mathscr{C}_p^{\theta-\tilde\theta}} \lesssim \|v\|_{\mathscr{L}_T^{\gamma,\theta}} \tag{3.26}$$

and

$$\|v\|_{\mathscr{M}_T^{\gamma(1-\tilde\theta/\alpha)}\mathscr{C}_p^{\theta-\tilde\theta}} \lesssim \|v\|_{\mathscr{L}_T^{\gamma,\theta}}. \tag{3.27}$$

If $v_T \in \mathscr{C}_p^\theta$ and $\tilde\theta \in [\alpha\gamma, \alpha]$, then the following estimate holds true

$$\|v_t\|_{\mathscr{C}_p^{\theta-\tilde\theta}} \lesssim (T-t)^{\tilde\theta/\alpha-\gamma}[\|v\|_{\mathscr{L}_T^{\gamma,\theta}} + \|v_T\|_{\mathscr{C}_p^\theta}^{1-\tilde\theta/\alpha}] + \|v_T\|_{\mathscr{C}_p^{\theta-\tilde\theta}}. \tag{3.28}$$

Remark 3.18. *For $\gamma = 0$, $\theta \in (0,1]$ and a Banach space X, we recall that $C_T^{0,\theta}X = C_T^\theta X$.*

Proof. To prove (3.25) we let $0 \leqslant s \leqslant t \leqslant T$ and we first assume that $(T-t) \leqslant |t-s|$. In that case, we utilize the estimate (3.27) for $\tilde\theta = \alpha\gamma$, such that $v \in \mathscr{L}_T^{\gamma,\theta}$ implies $v \in \mathscr{M}_T^{\gamma(1-\gamma)}\mathscr{C}_p^{\theta-\alpha\gamma}$ and furthermore estimate

$$\begin{aligned}\|v_t - v_s\|_{L^p} \leqslant \sum_j \|\Delta_j(v_t - v_s)\|_{L^p} &\lesssim \sum_{j:\,2^{-j}\leqslant|t-s|^{1/\alpha}} 2^{-j(\theta-\alpha\gamma)}(T-t)^{-\gamma(1-\gamma)}\|v\|_{\mathscr{M}_T^{\gamma(1-\gamma)}\mathscr{C}_p^{\theta-\alpha\gamma}} \\ &\quad + \sum_{j:\,2^{-j}>|t-s|^{1/\alpha}} 2^{-j(\theta-\alpha)}|t-s|^{1-\gamma}\|v\|_{C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha}} \\ &\lesssim (T-t)^{-\gamma(1-\gamma)}|t-s|^{\theta/\alpha-\gamma}\|v\|_{\mathscr{M}_T^{\gamma(1-\gamma)}\mathscr{C}_p^{\theta-\alpha\gamma}} \\ &\quad + |t-s|^{\theta/\alpha-\gamma}\|v\|_{C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha}},\end{aligned}$$

using that $\theta > \alpha\gamma$ for the convergence of the geometric sum and that $\theta < \alpha$. If $|t-s| \leqslant (T-t)$, we estimate

$$\begin{aligned}\|v_t - v_s\|_{L^p} \leqslant \sum_j \|\Delta_j(v_t - v_s)\|_{L^p} &\lesssim \sum_{j:\,2^{-j}\leqslant|t-s|^{1/\alpha}} 2^{-j\theta}(T-t)^{-\gamma}\|v\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}} \\ &\quad + \sum_{j:\,2^{-j}>|t-s|^{1/\alpha}} 2^{-j(\theta-\alpha)}|t-s|^{1-\gamma}\|v\|_{C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha}} \\ &\lesssim \sum_{j:\,2^{-j}\leqslant|t-s|^{1/\alpha}} 2^{-j\theta}|t-s|^{-\gamma}\|v\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}} \\ &\quad + \sum_{j:\,2^{-j}>|t-s|^{1/\alpha}} 2^{-j(\theta-\alpha)}|t-s|^{1-\gamma}\|v\|_{C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha}} \\ &\lesssim |t-s|^{\theta/\alpha-\gamma}\|v\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}} + |t-s|^{\theta/\alpha-\gamma}\|v\|_{C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha}} \\ &\lesssim |t-s|^{\theta/\alpha-\gamma}\|v\|_{\mathscr{L}_T^{\gamma,\theta}}.\end{aligned}$$

Together, we obtain (3.25).
To prove (3.26) and (3.27), we let $\tilde\theta \in [0,\alpha]$. Then we estimate for $s < t$,

$$\|\Delta_j(v_t - v_s)\|_{L^p} \lesssim (T-t)^{-\gamma}\min\left(2^{-j\theta}\|v\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}}, 2^{-j(\theta-\alpha)}|t-s|\|v\|_{C_T^{\gamma,1}\mathscr{C}_p^{\theta-\alpha}}\right)$$

and for $t \in [0,T)$,

$$\|\Delta_j v_t\|_{L^p} \lesssim \min(2^{-j\theta}(T-t)^{-\gamma}\|v\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}}, 2^{-j(\theta-\alpha)}\|v\|_{C_T\mathscr{C}_p^{\theta-\alpha}}).$$

Thus by interpolation (that is, $\min(a,b) \leqslant a^{\varepsilon} b^{1-\varepsilon}$ for $a, b \geqslant 0$, $\varepsilon \in [0,1]$) and using that $\|v\|_{C_T \mathscr{C}_p^{\theta-\alpha}} \lesssim \|v\|_{\mathscr{L}_T^{\gamma,\theta}}$, we obtain

$$\begin{aligned}\|\Delta_j(v_t - v_s)\|_{L^p} &\lesssim (T-t)^{-\gamma} 2^{-j\theta(1-\tilde{\theta}/\alpha)} 2^{-j(\theta-\alpha)\tilde{\theta}/\alpha} |t-s|^{\tilde{\theta}/\alpha} \|v\|_{\mathscr{L}_T^{\gamma,\theta}} \\ &= (T-t)^{-\gamma} 2^{-j(\theta-\tilde{\theta})} |t-s|^{\tilde{\theta}/\alpha} \|v\|_{\mathscr{L}_T^{\gamma,\theta}},\end{aligned}$$

from which (3.26) follows, and

$$\begin{aligned}\|\Delta_j v_t\|_{L^p} &\lesssim 2^{-j\theta(1-\tilde{\theta}/\alpha)} (T-t)^{-\gamma(1-\tilde{\theta}/\alpha)} \|v\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}}^{1-\tilde{\theta}/\alpha} \, 2^{-j(\theta-\alpha)\tilde{\theta}/\alpha} \|v\|_{C_T\mathscr{C}_p^{\theta-\alpha}}^{\tilde{\theta}/\alpha} \\ &\leqslant 2^{-j(\theta-\tilde{\theta})} \|v\|_{\mathscr{L}_T^{\gamma,\theta}} (T-t)^{-\gamma(1-\tilde{\theta}/\alpha)},\end{aligned}$$

which yields (3.27). Finally, if $(T-t) \geqslant 1$, then (3.28) follows from (3.27) as

$$\begin{aligned}\|v_t\|_{\mathscr{C}_p^{\theta-\tilde{\theta}}} \lesssim \|v\|_{\mathscr{L}_T^{\gamma,\theta}} (T-t)^{-\gamma(1-\tilde{\theta}/\alpha)} &\leqslant \|v\|_{\mathscr{L}_T^{\gamma,\theta}} \\ &\leqslant (T-t)^{\tilde{\theta}/\alpha-\gamma} [\|v\|_{\mathscr{L}_T^{\gamma,\theta}} + \|v_T\|_{\mathscr{C}_p^{\theta-\tilde{\theta}}}^{1-\tilde{\theta}/\alpha}] + \|v_T\|_{\mathscr{C}_p^{\theta-\tilde{\theta}}}\end{aligned}$$

using that $\tilde{\theta}/\alpha \geqslant \gamma$. If $(T-t) \leqslant 1$, then (3.28) follows from

$$\|v_t\|_{\mathscr{C}_p^{\theta-\tilde{\theta}}} \leqslant \|v_t - v_T\|_{\mathscr{C}_p^{\theta-\tilde{\theta}}} + \|v_T\|_{\mathscr{C}_p^{\theta-\tilde{\theta}}}$$

and using that $v_T \in \mathscr{C}_p^{\theta}$, we obtain

$$\begin{aligned}&\|\Delta_j(v_t - v_T)\|_{L^p} \\ &\quad \lesssim \min\Big(2^{-j\theta}(T-t)^{-\gamma}[\|v\|_{\mathscr{M}_T^{\gamma}\mathscr{C}_p^{\theta}} + \|v_T\|_{\mathscr{C}_p^{\theta}}], 2^{-j(\theta-\alpha)}(T-t)^{1-\gamma}\|v\|_{C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha}}\Big).\end{aligned}$$

By interpolation as above, we thus have

$$\|\Delta_j(v_t - v_T)\|_{L^p} \lesssim 2^{-j(\theta-\tilde{\theta})}(T-t)^{\tilde{\theta}/\alpha-\gamma}[\|v\|_{\mathscr{L}_T^{\gamma,\theta}} + \|v_T\|_{\mathscr{C}_p^{\theta}}^{1-\tilde{\theta}/\alpha}],$$

such that together (3.28) follows. □

3.3. Solving the Kolmogorov backward equation

In this section, we develop a concise solution theory that simultaneously treats singular and non-singular terminal condition for the Kolmogorov backward equation.
We start by solving the Kolmogorov equation in the Young regime, that is $\beta > (1-\alpha)/2$.

Theorem 3.19. [1] *Let* $\alpha \in (1,2]$, $\beta \in (\frac{1-\alpha}{2}, 0)$ *and* $p \in [1,\infty]$. *Let* $V \in C_T\mathscr{C}^\beta_{\mathbb{R}^d}$, $f \in C_T\mathscr{C}^\beta_p$ *and* $u^T \in \mathscr{C}^{\alpha+\beta}_p$. *Then the PDE*

$$\partial_t u = \mathscr{L}^\alpha_\nu u - V \cdot \nabla u + f, \quad u(T,\cdot) = u^T, \tag{3.29}$$

has a unique mild solution $u \in C_T\mathscr{C}^{\alpha+\beta} \cap C^1_T\mathscr{C}^\beta$ *(i.p. by* (3.25)*,* $u \in C_T^{(\alpha+\beta)/\alpha}L^p$*). Moreover, the solution map*

$$\mathscr{C}^{\alpha+\beta}_p \times C_T\mathscr{C}^\beta_p \times C_T\mathscr{C}^\beta_{\mathbb{R}^d} \ni (u^T, f, V) \mapsto u \in \mathscr{L}^{0,\alpha+\beta}_T$$

is continuous.
Furthermore, for a singular terminal conditions $u^T \in \mathscr{C}^{(1-\gamma)\alpha+\beta}_p$ *for* $\gamma \in [0,1)$*, the solution* u *is obtained in* $\mathscr{L}^{\gamma,\alpha+\beta}_T$.

Proof. Let $u^T \in \mathscr{C}^{\alpha+\beta}_p$. We first prove, that the solution exists in $\mathscr{L}^{\gamma,\alpha+\beta}_T$ for any $\gamma \in (0,1)$. Then we argue, that $u \in \mathscr{L}^{0,\alpha+\beta}_T$, and with Remark 3.13 the continuity at $t = T$ follows.
The proof follows from the Banach fixed point theorem applied to the map

$$\mathscr{L}^{\gamma,\alpha+\beta}_{\overline{T},T} \ni u \mapsto \Phi^{\overline{T},T}(u) \in \mathscr{L}^{\gamma,\alpha+\beta}_{\overline{T},T} \text{ with } \Phi^{\overline{T},T}u(t) = P_{T-t}u^T + J^T(\nabla u \cdot V - f)(t),$$

where $J^T(v)(t) = \int_t^T P_{r-t}v(r)dr$. We show below, that for $\overline{T} \in (0,T]$ small enough, the map is a contraction. By the Schauder estimates (Corollary 3.12), we obtain that $t \mapsto P_{T-t}u^T \in \mathscr{L}^{0,\alpha+\beta}_{\overline{T},T}$ and $J^T(f) \in \mathscr{L}^{0,\alpha+\beta}_{\overline{T},T}$. Furthermore, the Schauder estimates (Corollary 3.12) and the interpolation estimate (3.27) from Lemma 3.17 yield that for $\gamma' \in (0,\gamma)$ chosen, such that $\gamma = \gamma'(1-\theta/\alpha)$ for a $\theta \in (0, \alpha+2\beta-1)$,

$$\begin{aligned}
\|J^T(\nabla u \cdot V)\|_{\mathscr{L}^{\gamma,\alpha+\beta}_{\overline{T},T}} &\lesssim \overline{T}^{\gamma-\gamma'}\|\nabla u \cdot V\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^\beta_p} \\
&\lesssim \overline{T}^{\gamma-\gamma'}\|\nabla u\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}(\mathscr{C}^{\alpha+\beta-1-\theta}_p)^d}\|V\|_{C_T\mathscr{C}^\beta_{\mathbb{R}^d}} \\
&\lesssim \overline{T}^{\gamma-\gamma'}\|u\|_{\mathscr{L}^{\gamma,\alpha+\beta}_{\overline{T},T}}\|V\|_{C_T\mathscr{C}^\beta_{\mathbb{R}^d}}.
\end{aligned}$$

Notice that due to the choice of θ the regularity of the resonant product $\nabla u \odot V$ is strictly positive. Thus, for $\overline{T} \in (0,T]$ sufficiently small, $\Phi^{\overline{T},T}$ is a contraction on $\mathscr{L}^{\gamma,\alpha+\beta}_{\overline{T},T}$ and we obtain a solution $u \in \mathscr{L}^{\gamma,\alpha+\beta}_{\overline{T},T}$ (i.e. the fixed point of the map).
By plugging the solution back in the contraction map and using the interpolation

[1] The theorem is a generalization of [KP22, Theorem 3.1] to regularity $\theta = \alpha+\beta$ and integrability $p \in [1,\infty]$.

estimate (3.28) with $\gamma \in (0, (\alpha+2\beta-1)/\alpha)$, we then obtain

$$\begin{aligned}\|u\|_{\mathscr{L}^{0,\alpha+\beta}_{\overline{T},T}} &= \|\Phi^{\overline{T},T}(u)\|_{\mathscr{L}^{0,\alpha+\beta}_{\overline{T},T}} \\ &\lesssim \|P_{T-\cdot}u^T + J^T(f)\|_{\mathscr{L}^{0,\alpha+\beta}_{\overline{T},T}} + \|u\|_{C_{\overline{T},T}\mathscr{C}^{\alpha+\beta-\gamma\alpha}}\|V\|_{C_T\mathscr{C}^{\beta}_{\mathbb{R}^d}} \\ &\lesssim \|P_{T-\cdot}u^T + J^T(f)\|_{\mathscr{L}^{0,\alpha+\beta}_{T}} + [\|u\|_{\mathscr{L}^{\gamma,\alpha+\beta}_{\overline{T},T}} + \|u_T\|^{1-\gamma}_{\mathscr{C}^{\alpha+\beta}_p} + \|u_T\|_{\mathscr{C}^{\alpha+\beta}_p}]\|V\|_{C_T\mathscr{C}^{\beta}_{\mathbb{R}^d}}. \end{aligned} \tag{3.30}$$

This implies that indeed $u \in \mathscr{L}^{0,\alpha+\beta}_{\overline{T},T}$ and we constructed the solution on $[T-\overline{T}, T]$. Moreover, the choice of $\overline{T}$ does not depend on the terminal condition u^T and therefore we can iterate the construction of the solution on subintervals $[T-k\overline{T}, T-(k-1)\overline{T}]$ for $k \in 1, \dots, n$ and $n \in \mathbb{N}$ such that $T - n\overline{T} \leqslant 0$. Here, we choose the terminal condition of the solution on $[T-k\overline{T}, T-(k-1)\overline{T}]$ equal to the initial value of the solution constructed in the previous iteration step. We then obtain the solution $u \in \mathscr{L}^{0,\alpha+\beta}_{T}$ on $[0,T]$ by patching the solutions on the subintervals together. Indeed, u is the fixed point of $\Phi^{T,T}$, due to the semigroup property $P_tP_s = P_{t+s}$ for $t, s \geqslant 0$.
The continuity of the solution map follows from the linearity of the equation, (3.30) and from Gronwall's inequality for locally finite measures, cf. [EK86, Appendix, Theorem 5.1] applied to

$$\begin{aligned}\|u_t\|_{\mathscr{C}^{\alpha+\beta-\gamma\alpha}_p} &\lesssim \int_t^T \|P_{r-t}(\nabla u_r \cdot V_r)\|_{\mathscr{C}^{\alpha+\beta-\gamma\alpha}} dr \\ &\lesssim \|V\|_{C_T\mathscr{C}^{\beta}_{\mathbb{R}^d}} \int_t^T (r-t)^{\gamma-1}\|u_r\|_{\mathscr{C}^{\alpha+\beta-\gamma\alpha}_p} dr,\end{aligned}$$

for $\gamma \in (0, (\alpha+2\beta-1)/\alpha)$, if $u^T = 0$, $f = 0$.
For a terminal condition $u^T \in \mathscr{C}^{(1-\gamma)\alpha+\beta}_p$, the above arguments show that we obtain a solution in $\mathscr{L}^{\gamma,\alpha+\beta}_{T}$. Notice that the blow-up just occures for the solution on the last subinterval $[T-\overline{T}, T]$. That is, the solutions on $[T-k\overline{T}, T-(k-1)\overline{T}]$ for $k = 2, \dots, n$ have a regular terminal condition in $\mathscr{C}^{\alpha+\beta}_p$. □

Next, we define the space of enhanced distributions and afterwards the solution space for solving the generator equation with paracontrolled terminal condition and right hand side in the rough regime $\beta \leqslant \frac{1-\alpha}{2}$. For that, we define for a Banach space X, the blow up space

$$\mathscr{M}^{\gamma}_{\mathring{\Delta}_T} X = \{g : \mathring{\Delta}_T \to X \mid \sup_{0 \leqslant s < t \leqslant T} (t-s)^{\gamma}\|g(s,t)\|_X < \infty\}$$

for the triangle without diagonal $\mathring{\Delta}_T := \{(s,t) \in [0,T]^2 \mid s < t\}$. Below we take $g(s,t) = P_{t-s}(\partial_j \eta^i_t) \odot \eta^j_s$ for $\eta \in C_T C^{\infty}_b(\mathbb{R}^d, \mathbb{R}^d)$ and $i, j \in \{1, ..., d\}$.

Definition 3.20 (Enhanced drift). *Let $T > 0$. For $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$ and $\gamma \in [\frac{2\beta+2\alpha-1}{\alpha}, 1)$,* *we define the space of enhanced drifts* $\mathscr{X}^{\beta,\gamma}$ *as the closure of*

$$\{(\eta,\mathcal{K}(\eta)) := (\eta,(P_{\cdot}(\partial_j\eta^i)\odot\eta^j)_{i,j=1,\dots,d}) : \eta\in C_TC_b^\infty(\mathbb{R}^d,\mathbb{R}^d)\}$$

in $C_T\mathscr{C}_{\mathbb{R}^d}^{\beta+(1-\gamma)\alpha}\times\mathscr{M}_{\mathring{\Delta}_T}^{\gamma}\mathscr{C}_{\mathbb{R}^{d\times d}}^{2\beta+\alpha-1}$*. We say that* $\mathscr{V}$ *is a lift or an enhancement of* V *if* $\mathscr{V}_1=V$ *and we also write* $V\in\mathscr{X}^{\beta,\gamma}$ *identifying* V *with* $(\mathscr{V}_1,\mathscr{V}_2)$*.*
For $\beta\in(\frac{1-\alpha}{2},0)$ *and* $\gamma\in[\frac{\beta-1}{\alpha},1)$*, we set* $\mathscr{X}^{\beta,\gamma}=C_T\mathscr{C}^{\beta+(1-\gamma)\alpha}$*.*

Remark 3.21. *For* $\mathscr{V}\in\mathscr{X}^{\beta,\gamma}$*, we assume on the first component* $\mathscr{V}_1\in C_T\mathscr{C}^{\beta+\alpha(1-\gamma)}$*. We think of* $\gamma\sim 1$*, that is* $\gamma<1$*, but very close to* 1*. The assumptions on* $\mathscr{V}$ *in particular imply by the semigroup estimates, that* $t\mapsto P_{T-t}V_T^i\in\mathscr{M}_T^{\gamma}\mathscr{C}^{\alpha+\beta}$*. Furthermore, from* $P(\partial_iV^j)\odot V^i\in\mathscr{M}_{\mathring{\Delta}_T}^{\gamma}\mathscr{C}^{2\beta+\alpha-1}$ *follows that* $t\mapsto J^T(\partial_iV^j)_t\odot V_t^i=\int_t^T P_{r-t}(\partial_iV_r^j)\odot V_t^idr\in C_T\mathscr{C}^{2\beta+\alpha-1}$*. Indeed, as* $\gamma<1$*, we can estimate*

$$\begin{aligned}\sup_{t\in[0,T]}\|J^T(\partial_iV^j)_t\odot V_t^i\|_{\alpha+2\beta-1}&\leqslant\sup_{t\in[0,T]}\int_t^T\|P_{u-t}(\partial_iV_u^j)\odot V_t^i\|_{\alpha+2\beta-1}du\\&\leqslant\|P_{\cdot}(\partial_iV^j)\odot V^i\|_{\mathscr{M}_{\mathring{\Delta}_T}^{\gamma}\mathscr{C}_{\mathbb{R}^d}^{2\beta+\alpha-1}}\sup_{t\in[0,T]}\int_t^T(u-t)^{-\gamma}du\\&\lesssim\|P_{\cdot}(\partial_iV^j)\odot V^i\|_{\mathscr{M}_{\mathring{\Delta}_T}^{\gamma}\mathscr{C}_{\mathbb{R}^d}^{2\beta+\alpha-1}}\times T^{1-\gamma},\end{aligned}$$

using that $\gamma<1$*. Analogously we obtain that* $J^r(\partial_iV^j)\odot V^i\in C_{[0,r]}\mathscr{C}^{\alpha+2\beta-1}$ *with a uniform bound in* $r\in(0,T]$*. The assumptions on the enhancement will become handy, as soon as we consider paracontrolled solutions on subintervals of* $[0,T]$*.*

Remark 3.22. *We assume the lower bound on* γ *to ensure, that the regularity of* V*, respectively the regularity of the resonant products* $J^T(\partial_iV^j)_t\odot V_t^i$ *are negative. That is, for* $\gamma<(2\beta+2\alpha-1)/\alpha$*, we obtain that* $J^T(\partial_iV^j)_t\odot V_t^i\in C_T\mathscr{C}^{2\beta+(2-\gamma)\alpha-1}$ *due to* $V\in C_T\mathscr{C}^{\beta+(1-\gamma)\alpha}$ *with* $2\beta+(2-\gamma)\alpha-1\geqslant 0$*. In this case,* V *has enough regularity, so that the Kolmogorov PDE can be solved with the classical approach. We exclude this case here, as we explicitly treat the singular case.*

Definition 3.23. *Let* $\alpha\in(1,2]$ *and* $\beta\in(\frac{2-2\alpha}{3},\frac{1-\alpha}{2}]$*. Let* $T>0$ *and* $V\in\mathscr{X}^{\beta,\gamma'}$ *for* $\gamma'\in[\frac{2\beta+2\alpha-1}{\alpha},1)$ *and let* $u^{T,\prime}\in\mathscr{C}_p^{\alpha+\beta-1}$*. For* $\gamma\in(\gamma',\frac{\alpha}{2-\alpha-3\beta}\gamma')$ *and* $\overline{T}\in(0,T]$*, we define the space of paracontrolled distributions* $\mathscr{D}_{\overline{T},T}^{\gamma}=\mathscr{D}_{\overline{T},T}^{\gamma,\gamma'}(\mathscr{V},u^{T,\prime})$ *as the set of tuples* $(u,u')\in\mathscr{L}_{\overline{T},T}^{\gamma',\alpha+\beta}\times(\mathscr{L}_{\overline{T},T}^{\gamma,\alpha+\beta-1})^d$*, such that*

$$u^\sharp := u-u'\otimes J^T(V)-u^{T,\prime}\otimes P_{T-\cdot}V_T\in\mathscr{L}_{\overline{T},T}^{\gamma,2(\alpha+\beta)-1}.$$

We define a metric on $\mathscr{D}^{\gamma}_{\overline{T},T}$ *by*

$$d_{\mathscr{D}^{\gamma}_{\overline{T},T}}((u,u'),(v,v')) := \|u-v\|_{\mathscr{D}^{\gamma}_{\overline{T},T}} := \|u-v\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} + \|u'-v'\|_{(\mathscr{L}^{\gamma,\alpha+\beta-1}_{\overline{T},T})^d} + \|u^\sharp - v^\sharp\|_{\mathscr{L}^{\gamma,2(\alpha+\beta)-1}_{\overline{T},T}}.$$

Then, $(\mathscr{D}^{\gamma}_{\overline{T},T}, d_{\mathscr{D}^{\gamma}_{\overline{T},T}})$ *is a complete metric space. If moreover* $(v,v') \in \mathscr{D}^{\gamma,\gamma'}_{\overline{T},T}(\mathscr{W}, v^{T,\prime})$ *for different data* $(\mathscr{W}, v^{T,\prime}) \in \mathscr{X}^{\beta,\gamma'} \times \mathscr{C}^{\alpha+\beta-1}_p$*, then we use the same definition for* $\|u-v\|_{\mathscr{D}^{\gamma}_{\overline{T},T}}$*, despite the fact that* (u,u') *and* (v,v') *do not live in the same space.*

Remark 3.24. *The intuition behind the paracontrolled ansatz is as follows. Assume for simplicitiy regular data* $(u^T, f) \in \mathscr{C}^{2(\alpha+\beta)-1}_p \times \mathscr{L}^{0,\alpha+2\beta-1}_T$*. Assume also that we found a solution* $u \in \mathscr{L}^{0,\alpha+\beta}_T$ *and that we can make sense of the resonant product* $\nabla u \odot V$ *in such a way that it has its natural regularity* $C_T\mathscr{C}^{2\beta+\alpha-1}_p$*, despite the fact that* $2\beta+\alpha-1 \leqslant 0$*. Then we would get that*

$$\begin{aligned} u^\sharp :&= u - \nabla u \otimes J^T(V) \\ &= P_{T-\cdot}u^T - J^T(f) + J^T(\nabla u \otimes V) + J^T(\nabla u \odot V) + (J^T(\nabla u \otimes V) - \nabla u \otimes J^T(V)) \end{aligned}$$

is more regular than u*. Indeed, by the Schauder estimates for the first four terms and by the commutator estimate from Lemma 3.14, we obtain that* $u^\sharp \in \mathscr{L}^{0,2(\alpha+\beta)-1}_T$*. This explains why the paracontrolled ansatz might be justified. The reason why the ansatz is useful is that it isolates the singular part of* u *in a paraproduct, that we can handle by commutator estimates and the assumptions on* V*.*

Our main theorem of this section is the following. We give its proof after the corollary below.

Theorem 3.25. *Let* $T > 0$*,* $\alpha \in (1,2]$*,* $p \in [1,\infty]$ *and* $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$ *and* $\mathscr{V} \in \mathscr{X}^{\beta,\gamma'}$ *for* $\gamma' \in [\frac{2\beta+2\alpha-1}{\alpha}, 1)$*. Let*

$$f = f^\sharp + f' \otimes V$$

for $f^\sharp \in \mathscr{L}^{\gamma',\alpha+2\beta-1}_T$*,* $f' \in (\mathscr{L}^{\gamma',\alpha+\beta-1}_T)^d$ *and*

$$u^T = u^{T,\sharp} + u^{T,\prime} \otimes V_T$$

for $u^{T,\sharp} \in \mathscr{C}^{(2-\gamma')\alpha+2\beta-1}_p$*,* $u^{T,\prime} \in (\mathscr{C}^{\alpha+\beta-1}_p)^d$*.*
Then for $\gamma \in (\gamma', \frac{\alpha}{2-\alpha-3\beta}\gamma')$ *there exists a unique mild solution* $(u,u') \in \mathscr{D}^{\gamma}_T(\mathscr{V}, u^{T,\prime})$ *of the singular Kolmogorov backward PDE*

$$\mathscr{G}^{\mathscr{V}} u = f, \qquad u(T,\cdot) = u^T.$$

Remark 3.26. *As* $\mathscr{L}^{\tilde\gamma,\theta}_T \subset \mathscr{L}^{\gamma',\theta}_T$ *and* $\mathscr{C}^{(2-\tilde\gamma)\alpha+2\beta-1}_p \subset \mathscr{C}^{(2-\gamma')\alpha+2\beta-1}_p$ *for* $\tilde\gamma \in [0,\gamma']$*, we can in particular treat* $f^\sharp \in \mathscr{L}^{\tilde\gamma,\alpha+2\beta-1}_T$*,* $f' \in \mathscr{L}^{\tilde\gamma,\alpha+\beta-1}_T$ *and* $u^{T,\sharp} \in \mathscr{C}^{(2-\tilde\gamma)\alpha+2\beta-1}_p$*.*

Remark 3.27. *Examples for right-hand-sides and terminal conditions, which are paracontrolled by V, respectively V_T, are the following. Clearly we can take as a right-hand side $f = V^i$, i.e. $f' = e_i$ for the i-th unit vector e_i. Another example would be $f = J^T(\nabla V^i) \cdot V$ for $i \in \{1, \dots, d\}$, where $f^\sharp = J^T(\nabla V^i) \odot V + J^T(\nabla V^i) \varogreaterthan V$ and $f' = J^T(\nabla V^i)$. Furthermore, as a terminal condition, we can take $u^T = V_T^i$, i.e. $u^{T,\prime} = e_i$.*

In the case of $u^{T,\prime} = 0$, the terminal condition can still be irregular, but is such that $t \mapsto P_{T-t}u^T = P_{T-t}u^{T,\sharp} \in \mathscr{M}_T^{\gamma'}\mathscr{C}_p^{2(\alpha+\beta)-1}$. As $\frac{2\alpha+2\beta-1}{\alpha} \leqslant \gamma'$ and thus $(2-\gamma')\alpha + 2\beta - 1 \leqslant 0$, another example for a terminal condition, that can be treated with our approach would be a distribution $u^T = u^{T,\sharp} \in \mathscr{C}_p^0$. An example would be $u^T = \delta_0 \in \mathscr{C}_1^0$, where δ_0 denotes the Dirac measure at $x = 0$.
In the case of $u^{T,\prime} = 0$ and $u^{T,\sharp} \in \mathscr{C}^{2(\alpha+\beta)-1}$, the terminal condition is sufficiently regular, such that we can prove, that the solution of the equation is an element of the solution space without blow-up (provided, that f admits zero blow-up). We define, in the case of $u^{T,\prime} = 0$ and $u^{T,\sharp} \in \mathscr{C}^{2(\alpha+\beta)-1}$, the paracontrolled solution space as

$$D_T := \mathscr{D}_T^0 = \{(u, u') \in \mathscr{L}_T^{0,\alpha+\beta} \times (\mathscr{L}_T^{0,\alpha+\beta-1})^d \mid u^\sharp := u - u' \varolessthan J^T(V) \in \mathscr{L}_T^{0,2(\alpha+\beta)-1}\}.$$

Corollary 3.28 (Regular terminal condition)**.** *Let $T > 0$, $\alpha \in (1, 2]$, $p \in [1, \infty]$ and $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$ and $\mathscr{V} \in \mathscr{X}^{\beta,\gamma'}$ for $\gamma' \in [\frac{2\beta+2\alpha-1}{\alpha}, 1)$. Let $f = f^\sharp + f' \varolessthan V$ for $f^\sharp \in \mathscr{L}_T^{0,\alpha+2\beta-1}$ and $f' \in \mathscr{L}_T^{0,\alpha+\beta-1}$ and let $u^T = u^{T,\sharp} \in \mathscr{C}_p^{2\alpha+2\beta-1}$ be non-singular. Then, there exists a unique mild solution $u \in D_T$ of the generator equation*

$$\mathscr{G}^{\mathscr{V}} u = f, \qquad u(T, \cdot) = u^T.$$

The proof is deferred to page 112.

Remark 3.29. *The proof of the corollary only uses that $\mathscr{V} = (V, (J^T(\partial_i V^j) \odot V^i)_{i,j}) \in C_T\mathscr{C}_{\mathbb{R}^d}^\beta \times C_T\mathscr{C}_{\mathbb{R}^{d\times d}}^{2\beta+\alpha-1}$, which is implied by the stronger assumption $\mathscr{V} \in \mathscr{X}^{\beta,\gamma'}$ (cf. Remark 3.21).*

Proof of Theorem 3.25. Let $\mathscr{V} \in \mathscr{X}^{\beta,\gamma'}$ with $\mathscr{V}_1 = V$, $\mathscr{V}_2 = (P_\cdot(\partial_i V^j) \odot V^i)_{i,j}$ for $\gamma' \in [\frac{2\beta+2\alpha-1}{\alpha}, 1)$. Let $\overline{T} \in (0, T]$ to be chosen later and $\gamma \in (\gamma', \frac{\alpha}{2-\alpha-3\beta}\gamma')$. Then we define the contraction mapping as

$$\phi = \phi^{\overline{T},T} : \mathscr{D}_{\overline{T},T}^\gamma \to \mathscr{D}_{\overline{T},T}^\gamma, \quad (u, u') \mapsto (\psi(u), \nabla u - f') \tag{3.31}$$

for

$$\begin{aligned} \psi(u)(t) &= P_{T-t}u^T + J^T(-f)(t) + J^T(\nabla u \cdot \mathscr{V})(t), \quad t \in [T - \overline{T}, T] \\ &= P_{T-\cdot}u^{T,\sharp} + J^T(-f^\sharp) + J^T(\nabla u \odot \mathscr{V}) + J^T(V \varolessthan \nabla u) \\ &\quad + C_1(u^{T,\prime}, V_T) + C_2(-f', V) + C_2(\nabla u, V) \\ &\quad + (\nabla u - f') \varolessthan J^T(V) + u^{T,\prime} \varolessthan P_{T-\cdot}V_T, \end{aligned} \tag{3.32}$$

where we define

$$\begin{aligned}\nabla u \odot \mathscr{V} &= \sum_{i=1}^{d} \partial_i u \odot \mathscr{V}^i \\ &:= \sum_{i=1}^{d} [u' \cdot (J^T(\partial_i V) \odot V^i) + C_3(u', J^T(\partial_i V), V^i) + U^\sharp \odot V^i \qquad (3.33) \\ &\quad + u^{T,\prime} \otimes (P_{T-\cdot}\partial_i V_T \odot V^i) + C_3(u^{T,\prime}, P_{T-\cdot}\partial_i V_T, V^i)], \qquad (3.34)\end{aligned}$$

with $U^\sharp := \partial_i u^\sharp + \partial_i u' \otimes J^T(V) + \partial_i u^{T,\prime} \otimes P_{T-\cdot}V_T$. The commutators are defined as follows:

$$C_1(f,g) := P_{T-\cdot}(f \otimes g) - f \otimes P_{T-\cdot}g, \quad C_2(u,v) := J^T(u \otimes v) - u \otimes J^T(v),$$

where C_1 denotes the commutator on the semigroup $P_{T-\cdot}$ and C_2 is the commutator from Lemma 3.14. Furthermore, C_3 denotes the commuator from [GIP15, Lemma 2.4], that is

$$C_3(f,g,h) := (f \otimes g) \odot h - f(g \odot h).$$

For the terms in (3.33), we obtain with Remark 3.21, the paraproduct estimates and [GIP15, Lemma 2.4] using that $3\beta + 2\alpha - 2 > 0$ and $2\beta + \alpha - 1 \leqslant 0$,

$$\begin{aligned}&\|u' \cdot (J^T(\partial_i V) \odot V^i) + C_3(u', J^T(\partial_i V), V^i) + U^\sharp \odot V^i\|_{\mathscr{M}^{\gamma'}_{T,T}\mathscr{C}^{\alpha+2\beta-1}} \\ &\quad \lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma}}(1 + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma}})[\|u^\sharp\|_{\mathscr{M}^{\gamma'}_{T,T}\mathscr{C}_p^{2(\alpha+\beta)-1}} + \|u'\|_{\mathscr{M}^{\gamma'}_{T,T}(\mathscr{C}_p^{\alpha+\beta-1})^d}]\end{aligned}$$

For the terms in (3.34), we have by the estimate on the paraproduct and the definition of the enhanced distribution space $\mathscr{X}^{\beta,\gamma'}$

$$\begin{aligned}\|u^{T,\prime} \otimes (P_{T-\cdot}\partial_i V_T \odot V^i)\|_{\mathscr{M}^{\gamma'}_{T,T}\mathscr{C}_p^{2\beta+\alpha-1}} &\lesssim \|u^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|P_{T-\cdot}\partial_i V_T \odot V^i\|_{\mathscr{M}^{\gamma'}_{T,T}(\mathscr{C}^{2\beta+\alpha-1})^d} \\ &\lesssim \|u^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}},\end{aligned}$$

where we used that $\alpha + \beta - 1 > 0$. By the commutator estimate for C_3 from [GIP15, Lemma 2.4] and the estimates for the semigroup to control $P_{T-\cdot}\nabla V_T$, we obtain

$$\|C_3(u^{T,\prime}, P_{T-\cdot}\partial_i V_T, V^i)\|_{\mathscr{M}^{\gamma'}_{T,T}\mathscr{C}_p^{3\beta+2\alpha-2}} \lesssim \|u^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|\mathscr{V}\|^2_{\mathscr{X}^{\beta,\gamma'}},$$

using again $2\alpha + 3\beta - 2 > 0$ by the assumption on β.
Define $\varepsilon := \alpha - \alpha\frac{\gamma'}{\gamma}$. Then it follows that $\varepsilon \in (0, 3\beta + 2\alpha - 2)$ by the assumption on γ. Subtracting ε regularity for u' and $u^\sharp$, we can estimate the resonant product along the

same lines as above, due to $3\beta + 2\alpha - 2 - \varepsilon > 0$, obtaining

$$\begin{aligned}
&\|\nabla u \odot \mathscr{V}\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{2\beta+\alpha-1}_p} \\
&\quad\lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\big(\|u^\sharp\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{2(\alpha+\beta)-1-\varepsilon}_p} + \|u'\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}(\mathscr{C}^{\alpha+\beta-1-\varepsilon}_p)^d}\big) \\
&\qquad + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\|u^{T,\prime}\|_{(\mathscr{C}^{\alpha+\beta-1}_p)^d} \\
&\quad\lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})[\|(u,u')\|_{\mathscr{D}^{\gamma}_{\overline{T},T}} + \|u^{T,\prime}\|_{(\mathscr{C}^{\alpha+\beta-1}_p)^d}]. \qquad (3.35)
\end{aligned}$$

In (3.35), we moreover used the interpolation bound (3.27) for the norm of u', that is

$$\|u'\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{\alpha+\beta-1-\varepsilon}_p} = \|u'\|_{\mathscr{M}^{\gamma(1-\varepsilon/\alpha)}_{\overline{T},T}\mathscr{C}^{\alpha+\beta-1-\varepsilon}_p} \lesssim \|u'\|_{\mathscr{L}^{\gamma,\alpha+\beta-1}_{\overline{T},T}}$$

by the definition of ε, and analogously for $u^\sharp$. For $(u,u'),(v,v') \in \mathscr{D}^{\gamma}_{\overline{T},T}(\mathscr{V}, u^{T,\prime})$, this also implies the Lipschitz bound:

$$\begin{aligned}
&\|\nabla u \odot \mathscr{V} - \nabla v \odot \mathscr{V}\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{2\beta+\alpha-1}_p} \\
&\quad\lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\|(u,u')-(v,v')\|_{\mathscr{D}^{\gamma}_{\overline{T},T}}.
\end{aligned}$$

Next, we show that indeed $\phi(u,u') = (\psi(u), \nabla u - f') \in \mathscr{D}^{\gamma}_{\overline{T},T}$ and that ϕ is a contraction for small enough $\overline{T}$.
Towards the first aim, we note that by (3.32),

$$\begin{aligned}
\phi(u,u')^\sharp &= \psi(u) - (\nabla u - f') \otimes J^T(V) - u^{T,\prime} \otimes P_{T-\cdot}V_T \\
&= P_{T-\cdot}u^{T,\sharp} + J^T(-f^\sharp) + J^T(\nabla u \odot V) + J^T(V \otimes \nabla u) \\
&\quad + C_1(u^{T,\prime}, V_T) + C_2(-f', V) + C_2(\nabla u, V).
\end{aligned}$$

By the Schauder estimates, we obtain $P_{T-\cdot}u^{T,\sharp} + J^T(f^\sharp) \in \mathscr{L}^{\gamma',2\alpha+2\beta-1}_{\overline{T},T}$ and

$$\begin{aligned}
&\|J^T(\nabla u \odot \mathscr{V}) + J^T(V \otimes \nabla u)\|_{\mathscr{L}^{\gamma,2\alpha+2\beta-1}_{\overline{T},T}} \\
&\quad\lesssim \overline{T}^{\gamma-\gamma'}[\|\nabla u \odot \mathscr{V}\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{\alpha+2\beta-1}_p} + \|V \otimes \nabla u\|_{\mathscr{M}^{\gamma'}_{\overline{T},T}\mathscr{C}^{\alpha+2\beta-1}_p}] \\
&\quad\lesssim \overline{T}^{\gamma-\gamma'}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\|(u,u')\|_{\mathscr{D}^{\gamma}_{\overline{T},T}} \\
&\qquad + \overline{T}^{\gamma-\gamma'}\|u\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}}\|V\|_{C_T\mathscr{C}^{\beta}_{\mathbb{R}^d}}
\end{aligned}$$

using the estimate for the resonant product from above. Utilizing the commutator estimate (Lemma 3.14), we obtain

$$\|C_2(\nabla u, V)\|_{\mathscr{L}^{\gamma,2(\alpha+\beta)-1}_{\overline{T},T}} \lesssim \overline{T}^{\gamma-\gamma'}\|V\|_{C_T\mathscr{C}^{\beta}_{\mathbb{R}^d}}\|u\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} \lesssim \overline{T}^{\gamma-\gamma'}\|V\|_{C_T\mathscr{C}^{\beta}_{\mathbb{R}^d}}\|(u,u')\|_{\mathscr{D}^{\gamma}_{\overline{T},T}}.$$

By $V_T \in \mathscr{C}^{\beta+(1-\gamma')\alpha}_{\mathbb{R}^d}$ and $u^{T,\prime} \in (\mathscr{C}_p^{\alpha+\beta-1})^d$ and the commutator estimate (3.9) for C_1 for $\vartheta = \gamma'\alpha$ and $\alpha+\beta-1 \in (0,1)$ and again Lemma 3.14 for C_2, we have that

$$\begin{aligned}&\|C_1(u^{T,\prime}, V_T) + C_2(f', V)\|_{\mathscr{L}^{\gamma',2(\alpha+\beta)-1}_{\overline{T},T}} \\ &\quad\lesssim \|V\|_{C_T\mathscr{C}^{\beta+(1-\gamma')\alpha}_{\mathbb{R}^d}}(\|u^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d} + \|f'\|_{\mathscr{L}_T^{\gamma',\alpha+\beta-1}}).\end{aligned}$$

Hence, together we obtain $\phi(u,u')^\sharp \in \mathscr{L}^{\gamma,2\alpha+2\beta-1}_{\overline{T},T}$.
Next, we show that $\psi(u) \in \mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}$.
Define $\gamma'' := \gamma'(1-\varepsilon_1/\alpha)$ for a fixed $\varepsilon_1 \in (0, (\alpha+\beta-1) \wedge \frac{(1-\gamma')\alpha}{2-\alpha-3\beta}) = (0, \frac{(1-\gamma')\alpha}{2-\alpha-3\beta})$ and define $\varepsilon_2 := \alpha - \alpha\frac{\gamma''}{\gamma}$. Then it follows that $\varepsilon_2 \in (0, 3\beta+2\alpha-2+(1-\gamma')\alpha)$. Using that $V \in C_T\mathscr{C}^{\beta+(1-\gamma')\alpha}_{\mathbb{R}^d}$ and applying twice the interpolation bound (3.27) (once for u and once for $u^\sharp$ and u'), an analogue estimate as for the resonant product $\|J^T(\nabla u \odot \mathscr{V})\|_{\mathscr{L}_T^{\gamma,2\alpha+2\beta-1}}$ yields that

$$\begin{aligned}&\|J^T(\nabla u \cdot \mathscr{V})\|_{\mathscr{L}^{\gamma',\beta+\alpha}_{\overline{T},T}} \\ &\quad\lesssim \|J^T(\nabla u \olessthan V)\|_{\mathscr{L}^{\gamma',\beta+\alpha}_{\overline{T},T}} + \|J^T(\nabla u \ogreaterthan V + \nabla u \odot \mathscr{V})\|_{\mathscr{L}^{\gamma',2\alpha+\beta-1}_{\overline{T},T}} \\ &\quad\lesssim \overline{T}^{\gamma'-\gamma''}[\|\nabla u \olessthan V\|_{\mathscr{M}^{\gamma''}_{\overline{T},T}\mathscr{C}_p^\beta} + \|\nabla u \ogreaterthan V + \nabla u \odot \mathscr{V}\|_{\mathscr{M}^{\gamma''}_{\overline{T},T}\mathscr{C}_p^{\alpha+2\beta-1}}] \\ &\quad\lesssim \overline{T}^{\gamma'-\gamma''}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}) \\ &\qquad\times [\|u\|_{\mathscr{M}^{\gamma''}_{\overline{T},T}\mathscr{C}_p^{\alpha+\beta-\varepsilon_1}} + \|u^\sharp\|_{\mathscr{M}^{\gamma''}_{\overline{T},T}\mathscr{C}_p^{2(\alpha+\beta)-1-\varepsilon_2}} + \|u'\|_{\mathscr{M}^{\gamma''}_{\overline{T},T}(\mathscr{C}_p^{\alpha+\beta-1-\varepsilon_2})^d}] \\ &\quad\lesssim \overline{T}^{\gamma'-\gamma''}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})[\|u\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} + \|u^\sharp\|_{\mathscr{L}^{\gamma,2(\alpha+\beta)-1}_{\overline{T},T}} + \|u'\|_{(\mathscr{L}^{\gamma,\alpha+\beta-1}_{\overline{T},T})^d}] \\ &\quad= \overline{T}^{\gamma'-\gamma''}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\|u\|_{\mathscr{D}^{\gamma,\alpha+\beta}_{\overline{T},T}}.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}&\|\psi(u)\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} \\ &\quad= \|P_{T-\cdot}u^T + J^T(f) + J^T(\nabla u \cdot \mathscr{V})\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} \\ &\quad\leqslant \|P_{T-\cdot}u^T\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} + \|f\|_{\mathscr{L}_T^{\gamma',\beta}} + \|J^T(\nabla u \cdot \mathscr{V})\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} \\ &\quad\lesssim \|u^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}} + \|u^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}} + \|C_1(u^{T,\prime}, V_T)\|_{\mathscr{M}_T^{\gamma'}\mathscr{C}_p^{2\alpha+2\beta-1}} \\ &\qquad + \|f\|_{\mathscr{L}_T^{\gamma',\beta}} + \overline{T}^{\gamma'-\gamma''}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\|u\|_{\mathscr{D}^{\gamma,\alpha+\beta}_{\overline{T},T}},\end{aligned}$$

which yields in particular $\psi(u) \in \mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}$. The Gubinelli derivative $\phi(u,u')' = \nabla u - f'$, we estimate as follows

$$\begin{aligned}
&\|\nabla u - f'\|_{(\mathscr{L}^{\gamma,\alpha+\beta-1}_{\overline{T},T})^d} \\
&\quad\lesssim \|\nabla u\|_{\mathscr{M}^{\gamma}_{\overline{T},T}(\mathscr{C}^{\alpha+\beta-1}_p)^d} + \|\nabla u\|_{C^{1-\gamma}_{\overline{T},T}(\mathscr{C}^{\beta-1}_p)^d} + \|\nabla u\|_{C^{\gamma,1}_{\overline{T},T}(\mathscr{C}^{\beta-1}_p)^d} + \|f'\|_{(\mathscr{L}^{\gamma,\alpha+\beta-1}_{\overline{T},T})^d} \\
&\quad\lesssim \overline{T}^{\gamma-\gamma'}\big(\|u\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} + \|f'\|_{(\mathscr{L}^{\gamma',\alpha+\beta-1}_{\overline{T},T})^d}\big) \\
&\quad\lesssim \overline{T}^{\gamma-\gamma'}\big(\|u\|_{\mathscr{D}^{\gamma}_{\overline{T},T}} + \|f'\|_{(\mathscr{L}^{\gamma',\alpha+\beta-1}_{\overline{T},T})^d}\big),
\end{aligned}$$

where we exploit the fact that $\gamma-\gamma'>0$ to obtain a non-trivial factor depending on $\overline{T}$. Together with the estimate for $\psi(u)$ and $\phi(u,u')^\sharp$, this yields $\phi(u,u')=(\psi(u),\nabla u-f')\in\mathscr{D}^{\gamma}_{\overline{T},T}$.
The contraction property follows using the above estimates for $\psi(u)$, $\phi(u,u')^\sharp$ and $\phi(u,u')'$, utilizing linearity of ϕ and ψ (for $u^T=0, f=0$), such that

$$\begin{aligned}
&\|(\psi(u),\nabla u - f') - (\psi(v),\nabla v - f')\|_{\mathscr{D}^{\gamma}_{\overline{T},T}} \\
&\quad= \|\psi(u)-\psi(v)\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} + \|\nabla u-\nabla v\|_{(\mathscr{L}^{\gamma,\alpha+\beta-1}_{\overline{T},T})^d} + \|\phi(u,u')^\sharp-\phi(v,v')^\sharp\|_{\mathscr{L}^{\gamma,2\alpha+2\beta-1}_{\overline{T},T}} \\
&\quad\lesssim (\overline{T}^{\gamma-\gamma'}\vee\overline{T}^{\gamma'-\gamma''})\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\|(u,u')-(v,v')\|_{\mathscr{D}^{\gamma}_{\overline{T},T}}.
\end{aligned} \tag{3.36}$$

Now, we can choose $\overline{T}$ small enough, such that the implicit constant times the factor $(\overline{T}^{\gamma-\gamma'}\vee\overline{T}^{\gamma'-\gamma''})\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})$ is strictly less than 1, such that $\phi=\phi^{\overline{T},T}$ is a contraction on the corresponding space $\mathscr{D}^{\gamma}_{\overline{T},T}$. It is left to show, that we can obtain a paracontrolled solution in $\mathscr{D}^{\gamma}_{T}$ on the whole interval $[0,T]$. The solution on $[0,T]$ is obtained by patching the solutions on the subintervals of length $\overline{T}$ together. Indeed, let inductively $u^{[T-\overline{T},T]}$ be the solution on the subinterval $[T-\overline{T},T]$ with terminal condition u^T and $u^{[T-k\overline{T},T-(k-1)\overline{T}]}$ be the solution on $[T-k\overline{T},T-(k-1)\overline{T}]$ with terminal condition $u^{[T-k\overline{T},T-(k-1)\overline{T}]}_{T-(k-1)\overline{T}}=u^{[T-(k-1)\overline{T},T-(k-2)\overline{T}]}_{T-(k-1)\overline{T}}$ for $k=2,\dots,n$ and $n\in\mathbb{N}$, such that $T-n\overline{T}\leqslant 0$. There is a small subtlety, as we consider the solution on $[T-k\overline{T},T-(k-1)\overline{T}]$, that is paracontrolled by $J^T(V)$ (and not by $J^{T-(k-1)\overline{T}}(V)$). That is, for $k=2,\dots,n$, the solution has the paracontrolled structure,

$$\begin{aligned}
&u_t^{[T-k\overline{T},T-(k-1)\overline{T}],\sharp} \\
&\quad= u_t^{[T-k\overline{T},T-(k-1)\overline{T}]} - (\nabla u_t^{[T-k\overline{T},T-(k-1)\overline{T}]} - f'_t)\varolessthan J^T(V)_t - u^{T,\prime}\varolessthan P_{T-t}V_T \in \mathscr{L}^{\gamma,2(\alpha+\beta)-1}_{\overline{T},T-(k-1)\overline{T}}
\end{aligned}$$

Notice, that for $k\geqslant 2$, $u^{T,\prime}\varolessthan P_{T-t}V_T\in\mathscr{L}^{\gamma,2(\alpha+\beta)-1}_{\overline{T},T-(k-1)\overline{T}}$, so that term can also be seen as a part of the regular paracontrolled remainder.
By assumption we have that $f^\sharp\in\mathscr{L}^{\gamma',\alpha+2\beta-1}_{T}$ and $f'\in(\mathscr{L}^{\gamma',\alpha+\beta-1}_{T})^d$. This implies by

(3.14) that

$$f^\sharp \in \mathscr{M}^0_{\overline{T},T-(k-1)\overline{T}}\mathscr{C}_p^{\alpha+2\beta-1} \cap C^1_{\overline{T},T-(k-1)\overline{T}}\mathscr{C}_p^{2\beta-1},$$
$$f' \in \mathscr{M}^0_{\overline{T},T-(k-1)\overline{T}}(\mathscr{C}_p^{\alpha+\beta-1})^d \cap C^1_{\overline{T},T-(k-1)\overline{T}}(\mathscr{C}_p^{\beta-1})^d$$

for $k = 2, \dots, n$. If $u^{[T-\overline{T},T]}$ denotes the solution on $[T-\overline{T},T]$, then $u^{[T-\overline{T},T]}_{T-\overline{T}} \in \mathscr{C}_p^{\alpha+\beta}$ and $u^{[T-\overline{T},T],\sharp}_{T-\overline{T}} \in \mathscr{C}_p^{2\alpha+2\beta-1}$. Thus, for the solution on $[T-2\overline{T},T-\overline{T}]$ follows

$$\begin{aligned} u^{[T-2\overline{T},T-\overline{T}],\sharp}_{T-\overline{T}} &= u^{[T-2\overline{T},T-\overline{T}]}_{T-\overline{T}} - (\nabla u^{[T-2\overline{T},T-\overline{T}]}_{T-\overline{T}} - f'_{T-\overline{T}}) \otimes J^T(V)_{T-\overline{T}} - u^{T,\prime} \otimes P_{\overline{T}}V \\ &= u^{[T-\overline{T},T],\sharp}_{T-\overline{T}} \in \mathscr{C}_p^{2\alpha+2\beta-1}. \end{aligned}$$

Because we can trivially bound,

$$\sup_{t\in[T-2\overline{T},T-\overline{T}]} \|P_{T-\overline{T}-t}u^{[T-2\overline{T},T-\overline{T}],\sharp}_{T-\overline{T}}\|_{\mathscr{C}_p^{2(\alpha+\beta)-1}} \lesssim \|u^{[T-2\overline{T},T-\overline{T}],\sharp}_{T-\overline{T}}\|_{\mathscr{C}_p^{2(\alpha+\beta)-1}},$$

there is no blow-up for the solution on $[T-2\overline{T},T-\overline{T}]$ at time $t = T-\overline{T}$. Hence, the Banach fixed point argument for the map $\phi^{\overline{T},T-\overline{T}}$ yields a solution $u^{[T-2\overline{T},T-\overline{T}]} \in \mathscr{D}^{\hat{\gamma}}_{\overline{T},T-\overline{T}}$ for any small $\hat{\gamma} > 0$. By plugging the solution back in the fixed point map and using the interpolation estimates (cf. the arguments in the proof of Theorem 3.19 above and Corollary 3.28 below), we obtain that indeed $u^{[T-2\overline{T},T-\overline{T}]} \in \mathscr{D}^0_{\overline{T},T-\overline{T}}$. Proceeding iteratively, we thus obtain solutions

$$u^{[T-k\overline{T},T-(k-1)\overline{T}]} \in \mathscr{D}^0_{\overline{T},T-(k-1)\overline{T}} \quad \text{for} \quad k = 2, \dots, n$$

and $u^{[T-\overline{T},T]} \in \mathscr{D}^\gamma_{\overline{T},T-(k-1)\overline{T}}$. Then, the solution u, which is patched together on the subintervals ($u_t := u_t^{[T-k\overline{T},T-(k-1)\overline{T}]}$ for $t \in [T-k\overline{T}, T-(k-1)\overline{T}]$, $k = 1, \dots, n$), is indeed a fixed point of the map $\phi = \phi^{0,T}$ considered on $[0,T]$ and an element of $\mathscr{D}^\gamma_T$. □

Proof of Corollary 3.28. By assumption, we have that $u^{T,\prime} = 0$ and $u^{T,\sharp} = u^T \in \mathscr{C}_p^{2(\alpha+\beta)-1}$ and $f^\sharp, f'$ have no blow-up. By the assumption on $\mathscr{V}$, it follows that $J^T(\partial_i V^j) \odot V^i \in C_T\mathscr{C}^{\alpha+2\beta-1}$ due to $\gamma' \in (0,1)$. Furthermore due to $u^{T,\prime} = 0$ the paraproduct $u^{T,\prime} \otimes P_{T-\cdot}V_T$ in (3.32) vanishes, which previously was the term that introduced a blow-up of at least γ' for the solution. Thus, we have that $P_{T-\cdot}u^T \in C_T\mathscr{C}^{2(\alpha+\beta)-1}$. Hence, the arguments from Theorem 3.25 yield a paracontrolled solution $u \in \mathscr{D}^\gamma_T$ for any small $\gamma > 0$, i.p. $u \in \mathscr{L}_T^{\gamma,\alpha+\beta}$. It remains to justify that $u \in D_T$. By the regular terminal condition $u^T \in \mathscr{C}_p^{2(\alpha+\beta)-1} \subset \mathscr{C}_p^{\alpha+\beta}$ and the interpolation estimate (3.28), we obtain that

$$\sup_{t\in[0,T]} \|u_t\|_{\mathscr{C}_p^{\alpha+\beta-\alpha\gamma}} \lesssim \|u\|_{\mathscr{L}_T^{\gamma,\alpha+\beta}} + \|u_T\|^{1-\gamma}_{\mathscr{C}_p^{\alpha+\beta}} + \|u_T\|_{\mathscr{C}_p^{\alpha+\beta-\alpha\gamma}}$$

and since $u^\sharp_T = u_T \in \mathscr{C}_p^{2(\alpha+\beta)-1}$,

$$\sup_{t\in[0,T]} \|u^\sharp_t\|_{\mathscr{C}_p^{2(\alpha+\beta)-1-\alpha\gamma}} \lesssim \|u^\sharp\|_{\mathscr{L}_T^{\gamma,2(\alpha+\beta)-1}} + \|u_T\|^{1-\gamma}_{\mathscr{C}_p^{2(\alpha+\beta)-1}} + \|u_T\|_{\mathscr{C}_p^{2(\alpha+\beta)-1-\alpha\gamma}}$$

for any small $\gamma > 0$. If γ is small enough, that is $\gamma \in (0, (3\beta + 2\alpha - 2)/\alpha)$, we can estimate

$$\begin{aligned}\sup_{t\in[0,T]} &\|\nabla u \cdot \mathscr{V}(t)\|_\beta \\ &\lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1 + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\Big(\sup_{t\in[0,T]} \|u_t\|_{\mathscr{C}_p^{\alpha+\beta-\alpha\gamma}} + \sup_{t\in[0,T]} \|u^\sharp_t\|_{\mathscr{C}_p^{2(\alpha+\beta)-1-\alpha\gamma}}\Big). \quad (3.37)\end{aligned}$$

Plugging now the solution u back in the contraction map using the fixed point, i.e. $u = P_{T-\cdot}u^T + J^T(\nabla u\cdot \mathscr{V})$, and (3.37), we can use the Schauder estimates for $\gamma = \gamma' = 0$, such that we obtain that indeed $u \in \mathscr{L}_T^{0,\alpha+\beta}$. By the commutator estimate (3.21) for $\gamma = \gamma' = 0$ and $u \in \mathscr{L}_T^{0,\alpha+\beta}$, we then also obtain that $u^\sharp \in \mathscr{L}_T^{0,2(\alpha+\beta)-1}$. □

The next theorem proves the continuity of the solution map. The proof is similar to [KP22, Theorem 3.8], but adapted to the generalized setting for singular paracontrolled data. There are a few subtleties. First, the space $\mathscr{D}_T^\gamma(V, u^{T,\prime})$ depends on $V, u^{T,\prime}$. Furthermore due to the blow-up $\gamma > 0$, one cannot simply estimate the norm $\mathscr{M}_T^\gamma \mathscr{C}^\theta$ on the inteval $[0, T]$ by the sum of the respective blow-up norms on subintervals of $[0, T]$. In the case of regular terminal condition, that splitting issue does not occure, but we aim for continuity of the solution map in $\mathscr{L}_T^{0,\alpha+\beta}$. This we establish by first proving continuity of the map with values in $\mathscr{L}_T^{\gamma,\alpha+\beta}$ for any small $\gamma > 0$ and conclude from there together with the interpolation estimates.

Theorem 3.30. *In the setting of Theorem 3.25, the solution map*

$$(u^T = u^{T,\sharp} + u^{T,\prime} \otimes V_T,\ f = f^\sharp + f' \otimes V,\ \mathscr{V}) \mapsto (u, u^\sharp) \in \mathscr{L}_T^{\gamma',\alpha+\beta} \times \mathscr{L}_T^{\gamma,2(\alpha+\beta)-1},$$

is locally Lipschitz continuous, that is,

$$\begin{aligned}\|u - v\|_{\mathscr{L}_T^{\gamma',\alpha+\beta}} &+ \|u^\sharp - v^\sharp\|_{\mathscr{L}_T^{\gamma,2(\alpha+\beta)-1}} \\ &\leqslant C[\|u^{T,\sharp} - v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}} + \|u^{T,\prime} - v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d} \\ &\qquad + \|f^\sharp - g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}} + \|f' - g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d} + \|\mathscr{V} - \mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}] \quad (3.38)\end{aligned}$$

for a constant $C = C(T, \|\mathscr{V}\|, \|\mathscr{W}\|, \|u^T\|, \|v^T\|, \|f\|, \|g\|) > 0$.
Furthermore, in the setting of Corollary 3.28, the solution map

$$(u^T = u^{T,\sharp},\ f = f^\sharp + f' \otimes V,\ \mathscr{V}) \mapsto (u, u^\sharp) \in \mathscr{L}_T^{0,\alpha+\beta} \times \mathscr{L}_T^{0,2(\alpha+\beta)-1},$$

is locally Lipschitz continuous allowing for an analogue bound (3.38) with $\gamma' = 0$ for the norms of $u^{T,\sharp} - v^{T,\sharp}, f^\sharp - g^\sharp, f' - g'$ and $u^{T,\prime} = v^{T,\prime} = 0$.

Proof. We first prove the continuity in the case of singular paracontrolled data. Let u be the solution of the PDE for $\mathscr{V} \in \mathscr{X}^{\beta,\gamma'}$, $f = f^\sharp + f' \otimes V$ and $u^T = u^{T,\sharp} + u^{T,\prime} \otimes V_T$ and v the solution corresponding to the data $\mathscr{W}$, g and v^T. By the fixed point property we have $\phi(u,u') = (u,u')$ and $\phi(v,v') = (v,v')$ and thus $u' = \nabla u - f'$ and $v' = \nabla v - g'$. Hence, we can estimate

$$\|u' - v'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d} \lesssim \|u - v\|_{\mathscr{L}_T^{\gamma',\alpha+\beta}} + \|f' - g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}. \tag{3.39}$$

We estimate the terms in (3.38) by itself times a factor less than 1, plus a term depending on $\|f-g\|$, $\|\mathscr{V}-\mathscr{W}\|$ and $\|u^T - v^T\|$. Here we keep in mind that $u \in \mathscr{D}_T^\gamma(V, u^{T,\prime})$, whereas $v \in \mathscr{D}_T^\gamma(W, v^{T,\prime})$, but we explained the notation of $\|u-v\|_{\mathscr{D}_T^\gamma}$ in Definition 3.23. For that purpose, we estimate the product using re-bracketing like $ab - cd = a(b-d) + (a-c)d$ and the estimate (3.35) for the product, where $\gamma'' < \gamma'$,

$$\begin{aligned}
&\|\nabla u \cdot \mathscr{V} - \nabla v \cdot \mathscr{W}\|_{\mathscr{M}_T^{\gamma''}\mathscr{C}_p^\beta} \\
&\quad \lesssim (1 + \|\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}})\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\|u - v\|_{\mathscr{D}_T^\gamma} + (1 + \|\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}})\|\mathscr{V} - \mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}\|v\|_{\mathscr{D}_T^\gamma} \\
&\qquad + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\|u\|_{\mathscr{D}_T^\gamma}\|\mathscr{V} - \mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}} \\
&\qquad + \tilde{C}(\|\mathscr{V}\|, \|\mathscr{W}\|, \|u^{T,\prime}\|, \|v^{T,\prime}\|)\big(\|\mathscr{V} - \mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}} + \|u^{T,\prime} - v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\big).
\end{aligned} \tag{3.40}$$

Since the solution u can be bounded in terms of $u^T, f, \mathscr{V}$ by Gronwall's inequality for locally finite measures using that $\gamma, \gamma' \in (0,1)$ (cf. [EK86, Appendix, Theorem 5.1]), and similarly for v, we conclude that

$$\begin{aligned}
&\|\nabla u \cdot \mathscr{V} - \nabla v \cdot \mathscr{W}\|_{\mathscr{M}_T^{\gamma''}\mathscr{C}_p^\beta} \\
&\quad \lesssim \big((\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}} + \|v\|_{\mathscr{D}_T^\gamma})(1 + \|\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}) + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\|u\|_{\mathscr{D}_T^\gamma}\big) \times \\
&\qquad\qquad \Big(\|u - v\|_{\mathscr{D}_T^\gamma} + \|\mathscr{V} - \mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}\Big) \\
&\qquad + \tilde{C}\big(\|\mathscr{V} - \mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}} + \|u^{T,\prime} - v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\big) \\
&\quad \lesssim C\Big(\|u - v\|_{\mathscr{D}_T^\gamma} + \|\mathscr{V} - \mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}} + \|u^{T,\prime} - v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\Big),
\end{aligned}$$

where $C = C(\|\mathscr{V}\|, \|\mathscr{W}\|, \|u^T\|, \|v^T\|, \|f\|, \|g\|)$ is a constant, that depends on the norms of the input data on $[0,T]$. Therefore, we obtain by the fixed point and using the estimate for $\|\phi(u,u')\|_{\mathscr{L}_T^{\gamma',\alpha+\beta}}$ from the proof of Theorem 3.25 with $\gamma''<\gamma'$,

$$\begin{aligned}&\|u-v\|_{\mathscr{L}_T^{\gamma',\alpha+\beta}}\\&\quad\lesssim \|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|u^{T,\prime}-v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\\&\qquad+\|u^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}+\|f'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}\\&\qquad+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\\&\qquad+T^{\gamma'-\gamma''}\|\nabla u\cdot\mathscr{V}-\nabla v\cdot\mathscr{W}\|_{\mathscr{M}_T^{\gamma''}\mathscr{C}_p^\beta}.\end{aligned}$$

Moreover, using the fixed point and the estimate for $\|\phi(u,u')^\sharp\|_{\mathscr{L}_T^{\gamma,2(\alpha+\beta)-1}}$, we obtain

$$\begin{aligned}&\|u^\sharp-v^\sharp\|_{\mathscr{L}_T^{\gamma,2(\alpha+\beta)-1}}\\&\quad\lesssim \|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|u^{T,\prime}-v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\\&\qquad+\|u^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}+\|f'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}\\&\qquad+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\\&\qquad+T^{\gamma-\gamma'}\|\nabla u\cdot\mathscr{V}-\nabla v\cdot\mathscr{W}\|_{\mathscr{M}_T^{\gamma'}\mathscr{C}_p^\beta}\\&\qquad+T^{\gamma-\gamma'}\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}\|u\|_{\mathscr{D}_T^\gamma}+T^{\gamma'-\gamma}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\|u-v\|_{\mathscr{D}_T^\gamma}.\end{aligned}\tag{3.41}$$

To shorten notation, let us abbreviate the term in (3.38), that we aim to estimate, in the following by

$$\|u-v\|_{\gamma,\alpha+\beta}:=\|u-v\|_{\mathscr{L}_T^{\gamma',\alpha+\beta}}+\|u^\sharp-v^\sharp\|_{\mathscr{L}_T^{\gamma,2(\alpha+\beta)-1}}.$$

Then overall, using also (3.39), we obtain

$$\begin{aligned}\|u-v\|_{\gamma,\alpha+\beta}\leqslant C[&\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|u^{T,\prime}-v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\\&+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}+\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}]\\&+(T^{\gamma-\gamma'}\vee T^{\gamma'-\gamma''})C\|u-v\|_{\gamma,\alpha+\beta},\end{aligned}$$

where $C>0$ is again a (possibly different) constant depending on the norms of the input data. Assume for the moment that T is small enough so that $(T^{\gamma-\gamma'}\vee T^{\gamma'-\gamma''})C$ times the implicit constant on the right-hand side is <1. Then we can take the last term to the other side and divide by a positive factor, obtaining

$$\begin{aligned}\|u-v\|_{\gamma,\alpha+\beta}\leqslant C[&\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|u^{T,\prime}-v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\\&+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}+\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}],\end{aligned}\tag{3.42}$$

where $C = C(T, \|\mathscr{V}\|, \|\mathscr{W}\|, \|u^T\|, \|v^T\|, \|f\|, \|g\|) > 0$ is a constant that depends on the norms of the input data. Thus, the map $(u^T, f, \mathscr{V}) \mapsto (u, u^\sharp)$ is locally Lipschitz continuous, which implies the claim.
If T is such that $(T^{\gamma-\gamma'} \vee T^{\gamma'-\gamma''})C$ times the implicit constant is at least 1, then we want to apply the estimates above on the subintervals $[T - k\overline{T}, T - (k-1)\overline{T}]$ of length $\overline{T}$, where $\overline{T}$ is chosen, such that $(\overline{T}^{\gamma-\gamma'} \vee \overline{T}^{\gamma'-\gamma''})C$ times the implicit constant is strictly less than 1 and where $k = 1, \ldots, n$ for $n \in \mathbb{N}$ with $T - n\overline{T} \leqslant 0$. To obtain the continuity in $\mathscr{D}_T^\gamma$, we consider the solutions $u^{[T-k\overline{T}, T-(k-1)\overline{T}]}, v^{[T-k\overline{T}, T-(k-1)\overline{T}]}$ on the subintervals $[T - k\overline{T}, T - (k-1)\overline{T}]$ for $k = 1, \ldots n$, where the terminal condition of the solution $u^{[T-k\overline{T}, T-(k-1)\overline{T}]}$ is the initial value of the solution $u^{[T-(k-1)\overline{T}, T-(k-2)\overline{T}]}$ (analogously for v), such that, patched together, we obtain the solutions u, v on $[0, T]$.
Let $\varepsilon > 0$ to be chosen below.
For $k = 2, \ldots, n$, we have that $u^{[T-k\overline{T}, T-(k-1)\overline{T}]}, v^{[T-k\overline{T}, T-(k-1)\overline{T}]} \in \mathscr{D}^{0,\alpha+\beta}_{T-(k-1)\overline{T}}$ (see the argument in the proof of Theorem 3.25), such that we can estimate

$$
\begin{aligned}
&\|u - v\|_{\mathscr{M}_T^{\gamma'} \mathscr{C}_p^{\alpha+\beta-\varepsilon\alpha}} \\
&\quad \leqslant T^{\gamma'} \|u - v\|_{\mathscr{M}^0_{T-\overline{T}} \mathscr{C}_p^{\alpha+\beta-\varepsilon\alpha}} + \|u - v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T} \mathscr{C}_p^{\alpha+\beta}} \\
&\quad \leqslant T^{\gamma'} \sum_{k=2}^{n} \|u - v\|_{\mathscr{M}^0_{\overline{T}, T-(k-1)\overline{T}} \mathscr{C}_p^{\alpha+\beta-\varepsilon\alpha}} + \|u - v\|_{\mathscr{M}^{\gamma'}_{\overline{T},T} \mathscr{C}_p^{\alpha+\beta}}.
\end{aligned} \tag{3.43}
$$

Furthermore, we can estimate for $\varepsilon \in (0, \gamma']$,

$$
\begin{aligned}
\|u - v\|_{C_T^{1-\gamma'} \mathscr{C}_p^\beta} &\leqslant T^{\gamma'-\varepsilon} \|u - v\|_{C^{1-\varepsilon}_{T-\overline{T}} \mathscr{C}_p^\beta} + \|u - v\|_{C^{1-\gamma'}_{\overline{T},T} \mathscr{C}_p^\beta} \\
&\leqslant T^{\gamma'-\varepsilon} \sum_{k=2}^{n} \|u - v\|_{\mathscr{L}^{\varepsilon,\beta+\alpha}_{\overline{T}, T-(k-1)\overline{T}}} + \|u - v\|_{\mathscr{L}^{\gamma',\beta+\alpha}_{\overline{T},T}}.
\end{aligned} \tag{3.44}
$$

Subtracting the terminal condition for each of the terms with $k = 2, \ldots, n$ and applying the interpolation bound (3.28) for $\theta = \alpha + \beta$, $\tilde{\theta} = \varepsilon\alpha$ yields for $k = 2, \ldots, n$,

$$
\begin{aligned}
&\|(u - u_{T-(k-1)\overline{T}}) - (v - v_{T-(k-1)\overline{T}})\|_{\mathscr{M}^0_{\overline{T}, T-(k-1)\overline{T}} \mathscr{C}_p^{\alpha+\beta-\varepsilon\alpha}} \\
&\quad \leqslant \|(u - u_{T-(k-1)\overline{T}}) - (v - v_{T-(k-1)\overline{T}})\|_{\mathscr{M}^\varepsilon_{\overline{T}, T-(k-1)\overline{T}} \mathscr{C}_p^{\alpha+\beta}}.
\end{aligned} \tag{3.45}
$$

Together with (3.43), (3.44) and (3.45), this then yields

$$
\begin{aligned}
&\|u-v\|_{\mathscr{L}_T^{\gamma',\alpha+\beta-\varepsilon\alpha}} \\
&\lesssim T^{\gamma'}\sum_{k=2}^{n}\big(\|(u-u_{T-(k-1)\overline{T}})-(v-v_{T-(k-1)\overline{T}})\|_{\mathscr{L}^{0,\alpha+\beta-\varepsilon\alpha}_{\overline{T},T-(k-1)\overline{T}}}+\|u_{T-(k-1)\overline{T}}-v_{T-(k-1)\overline{T}}\|_{\mathscr{C}_p^{\alpha+\beta}}\big) \\
&\qquad +\|u-v\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} \\
&\lesssim T^{\gamma'}\sum_{k=2}^{n}\big(\|(u-u_{T-(k-1)\overline{T}})-(v-v_{T-(k-1)\overline{T}})\|_{\mathscr{L}^{\varepsilon,\alpha+\beta}_{\overline{T},T-(k-1)\overline{T}}}+\|u_{T-(k-1)\overline{T}}-v_{T-(k-1)\overline{T}}\|_{\mathscr{C}_p^{\alpha+\beta}}\big) \\
&\qquad +\|u-v\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}} \\
&\lesssim T^{\gamma'}\sum_{k=2}^{n}\|u-v\|_{\mathscr{L}^{\varepsilon,\alpha+\beta}_{\overline{T},T-(k-1)\overline{T}}}+\|u-v\|_{\mathscr{L}^{\gamma',\alpha+\beta}_{\overline{T},T}},
\end{aligned}
\tag{3.46}
$$

where in the last estimate, we estimated the norm of the terminal conditions by the norm of the solutions in the previous iteration step.
Analogously, we can argue for $u^\sharp - v^\sharp$, obtaining

$$
\begin{aligned}
&\|u^\sharp-v^\sharp\|_{\mathscr{L}_T^{\gamma,2(\alpha+\beta)-1-\varepsilon\alpha}} \\
&\qquad\lesssim T^{\gamma}\sum_{k=2}^{n}\|u^\sharp-v^\sharp\|_{\mathscr{L}^{\varepsilon,2(\alpha+\beta)-1}_{\overline{T},T-(k-1)\overline{T}}}+\|u^\sharp-v^\sharp\|_{\mathscr{L}^{\gamma,2(\alpha+\beta)-1}_{\overline{T},T}}.
\end{aligned}
\tag{3.47}
$$

Now, taking $\varepsilon := (\gamma-\gamma') \in (0,\gamma')$, we can apply the above estimate (3.42) for each of the terms on the right-hand side of the inequalities (3.46) and (3.47). That is, for each of the terms for $k = 2,\dots,n$, we obtain

$$
\begin{aligned}
&\|u-v\|_{\mathscr{L}^{\varepsilon,\alpha+\beta}_{\overline{T},T-(k-1)\overline{T}}}+\|u^\sharp-v^\sharp\|_{\mathscr{L}^{\varepsilon,2(\alpha+\beta)-1}_{\overline{T},T-(k-1)\overline{T}}} \\
&\quad\lesssim \frac{1}{1-\overline{T}^{\varepsilon}\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})}\Big[\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|u^{T,\prime}-v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d} \\
&\qquad +\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}+\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}\Big].
\end{aligned}
$$

This uses that by the choice of ε, $\overline{T}^{\varepsilon} = \overline{T}^{\gamma-\gamma'} \leqslant \overline{T}^{\gamma-\gamma'} \vee \overline{T}^{\gamma'-\gamma''}$ and that $u, v \in \mathscr{D}^{0,\alpha+\beta}_{T-(k-1)\overline{T}}$ for $k = 2,\dots,n$. For $k = 1$, we replace ε by γ, respectively γ' for $u^\sharp - v^\sharp$, and obtain the estimate (3.42) on the subinterval $[T-\overline{T}, T]$. Together, this then yields the following estimate on the whole interval $[0,T]$ (with a possibly different constant C):

$$
\begin{aligned}
\|u-v\|_{\gamma,\alpha+\beta-\varepsilon\alpha} \leqslant C[&\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|u^{T,\prime}-v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d} \\
&+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d}+\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}].
\end{aligned}
\tag{3.48}
$$

Plugging now $u-v$ back in the contraction map on $[0,T]$, we can remove the loss $\varepsilon\alpha$ in regularity. That is, we can estimate for ε small enough,

$$\begin{aligned}\|u-v\|_{\gamma,\alpha+\beta} \lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\|u-v\|_{\gamma,\alpha+\beta-\alpha\varepsilon}\\ + C[\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}} + \|u^{T,\prime}-v^{T,\prime}\|_{(\mathscr{C}_p^{\alpha+\beta-1})^d}\\ + \|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{\gamma',\alpha+2\beta-1}} + \|f'-g'\|_{(\mathscr{L}_T^{\gamma',\alpha+\beta-1})^d} + \|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}].\end{aligned}$$

Thus the local Lipschitz continuity (3.38) on $[0,T]$ follows.
In the setting of Corollary 3.28, we obtain from the above, that the Lipschitz estimate (3.38) holds true with $\gamma'=0$ for the norms of $u^{T,\sharp}-v^{T,\sharp}$, $f^\sharp-g^\sharp$, $f'-g'$ and $u^{T,\prime}=v^{T,\prime}=0$ on the right-hand side and any small $\gamma>0$ on the left-hand-side of the estimate. Similar as in the proof of Corollary 3.28, we can use the fixed point property and the estimate (3.37) for small enough $\gamma>0$, together with the Schauder estimates and the interpolation bound (3.28), to obtain that

$$\begin{aligned}&\|u-v\|_{\mathscr{L}_T^{0,\alpha+\beta}} + \|u^\sharp-v^\sharp\|_{\mathscr{L}_T^{0,2(\alpha+\beta)-1}}\\ &\lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\Big(\|u-v\|_{\mathscr{L}_T^{0,\alpha+\beta-\gamma\alpha}} + \|u^\sharp-v^\sharp\|_{\mathscr{L}_T^{0,2(\alpha+\beta)-1-\gamma\alpha}}\Big)\\ &\quad + C[\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{2\alpha+2\beta-1}}+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{0,\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{0,\alpha+\beta-1})^d}+\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}]\\ &\lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\times\\ &\qquad\Big(\|(u-v)-(u^{T,\sharp}-v^{T,\sharp})\|_{\mathscr{L}_T^{0,\alpha+\beta-\gamma\alpha}}+\|(u^\sharp-v^\sharp)-(u^{T,\sharp}-v^{T,\sharp})\|_{\mathscr{L}_T^{0,2(\alpha+\beta)-1-\gamma\alpha}}\Big)\\ &\quad + C[\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{2\alpha+2\beta-1}}+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{0,\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{0,\alpha+\beta-1})^d}+\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}]\\ &\lesssim \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})\Big(\|u-v\|_{\mathscr{L}_T^{\gamma,\alpha+\beta}} + \|u^\sharp-v^\sharp\|_{\mathscr{L}_T^{\gamma,2(\alpha+\beta)-1}}\Big)\\ &\quad + C[\|u^{T,\sharp}-v^{T,\sharp}\|_{\mathscr{C}_p^{2\alpha+2\beta-1}}+\|f^\sharp-g^\sharp\|_{\mathscr{L}_T^{0,\alpha+2\beta-1}}+\|f'-g'\|_{(\mathscr{L}_T^{0,\alpha+\beta-1})^d}+\|\mathscr{V}-\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}].\end{aligned}$$

Notice that, to apply the interpolation bound (3.28) in the last estimate above, we subtracted the terminal condition $u^T-v^T=u^{T,\sharp}-v^{T,\sharp}$, so that $(u-v)_T-(u^T-v^T)=0$. The constant C above changes in each line. Thus together the Lipschitz continuity of the solution map with values in $\mathscr{L}_T^{0,\alpha+\beta}\times\mathscr{L}_T^{0,2(\alpha+\beta)-1}$ follows. □

Remark 3.31 (Super-exponential dependency of the Lipschitz constant on $\mathscr{V},\mathscr{W}$)**.** *The Lipschitz constant of the solution map on $[0,T]$ depends super-exponentially on the norms $\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}, \|\mathscr{W}\|_{\mathscr{X}^{\beta,\gamma'}}$. Indeed, to obtain the Lipschitz estimate of the solution map on $[0,T]$, we have to apply the estimate in (3.41) on every subinterval $[T-(k+1)\overline{T}, T-k\overline{T}]$, where we have to choose $\overline{T}$ small enough so that $\overline{T}^\kappa < C^{-1}$ for $\kappa=\gamma-\gamma'\wedge\gamma'-\gamma''$ and for the constant $C=C(\|\mathscr{V}\|,\|\mathscr{W}\|,\|u^T\|,\|v^T\|,\|f\|,\|g\|)$. This means that in (3.42) we have to iterate the estimate at least $T/C^{-\kappa}=TC^\kappa$ times, and each time we multiply with the constant $\tilde{C}$ in (3.42), leading roughly speaking to a factor $\tilde{C}^{TC^\kappa}$. By doing the analysis more carefully we can show that there is (super-)exponential dependence only on $\|\mathscr{V}\|,\|\mathscr{W}\|$ and that the Lipschitz constant actually depends linearly on $\|u^T\|,\|v^T\|,\|f\|,\|g\|$. But the super-exponential dependence on $\|\mathscr{V}\|,\|\mathscr{W}\|$ is inherent* *to the problem and we expect that it cannot be significantly improved. By similar arguments, we also see that the norm of the solution u to the Kolmogorov backward equation in Theorem 3.25 depends super-exponentially on* $\|\mathscr{V}\|$.

This will be relevant when we take $\mathscr{V}$ *random, as in our application of the Brox diffusion (cf. Section 4.6). If we do not have super-exponential moments for* $\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}$, *then we do not know if u has finite moments. And if V is Gaussian, then the second component of the lift* $\mathscr{V}$ *is a second order polynomial of a Gaussian and therefore it does not have super-exponential moments.*

But note that this only concerns Hölder norms of the Kolmogorov backward equation. If we are only interested in the L^p *norm,* $p \in [1,\infty]$, *we can always use the trivial bound* $\|u\|_{L^p} \leq \|u^T\|_{L^p} + T\|f\|_{L^p}$, *provided that the right-hand side is finite, which holds for smooth* V, f, u^T *by the stochastic representation (Feynman-Kac) of the Kolmogorov backward equation, and which extends by approximation to the general setting.*

In the following chapters, we consider solutions of the Kolmorov PDE for $\mathscr{G}^{\mathscr{V}}$ (for fixed $\mathscr{V}$) on subintervals $[0,r]$ of $[0,T]$ for bounded sets of terminal conditions $(y^r)_{r\in[0,T]}$ and right-hand-sides $(f^r)_{r\in[0,T]}$. Examples that we encounter are $y^r \equiv 0$, $y^r = V_r$ or $y^r = J^T(\partial_i V^j)_r \cdot V_r^i$ for $i,j \in \{1,\dots,d\}$ and $f^r \equiv 0$ or $f^r = V^i_{|[0,r]}$.
The solution u^r on $[0,r]$ has the following paracontrolled structure

$$u^r = u^{r,\sharp} + (\nabla u^r - f^{r,\prime}) \varolessthan J^r(V) + y^{r,\prime} \varolessthan P_{r-\cdot}V_r \tag{3.49}$$

with

$$\begin{aligned} u^{r,\sharp} &= P_{r-\cdot}y^{r,\sharp} + J^r(-f^\sharp) + J^r(\nabla u^r \odot V) + J^r(V \varogreaterthan \nabla u^r) \\ &\quad + C_1(y^{r,\prime}, V_r) + C_2(-f', V) + C_2(\nabla u^r, V), \end{aligned}$$

for the commutators from the proof of Theorem 3.25.
We conclude this chapter by proving a uniform bound for the solutions (u^r).

Corollary 3.32. *Let* $T > 0$ *and* $\mathscr{V} \in \mathscr{X}^{\beta,\gamma'}$ *for* β, γ' *as in Theorem 3.25. Let* $\gamma \in (\gamma', 1)$ *and* $\gamma'' \in (0,\gamma')$ *be as in the proof of Theorem 3.25. Let* $(y^r = y^{r,\sharp} + y^{r,\prime} \varolessthan V_r)_{r\in[0,T]}$ *be a bounded sequence of singular paracontrolled terminal conditions, that is,*

$$C_y := \sup_{r\in[0,T]} [\|y^{r,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}} + \|y^{r,\prime}\|_{\mathscr{C}_p^{\alpha+\beta-1}}] < \infty.$$

Let $(f^r = f^{r,\sharp} + f^{r,\prime} \varolessthan V)_{r\in[0,T]}$ *be a sequence of right-hand-sides with*

$$C_f := \sup_{r\in[0,T]} [\|f^{r,\sharp}\|_{\mathscr{L}_r^{\gamma',\alpha+2\beta-1}} + \|f^{r,\prime}\|_{\mathscr{L}_r^{\gamma',\alpha+\beta-1}}] < \infty.$$

Let for $r \in [0,T]$, $(u_t^r)_{t\in[0,r]}$ *be the solution of the backward Kolmogorov PDE for* $\mathscr{G}^{\mathscr{V}}$ *with terminal condition* $u_r^r = y^r$ *and right-hand side* f^r.

Then, the following uniform bound for the solutions (u^r) *holds true*

$$\begin{aligned}\sup_{r\in[0,T]}[\|u^{r,\sharp}\|_{\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1}}+\|u^r\|_{\mathscr{L}_r^{\gamma',\alpha+\beta}}]\\ \lesssim_T \lambda_{\overline{T},\mathscr{V}}^{-1}\Bigg(\sup_{r\in[0,T]}[\|y^{r,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|f^{r,\sharp}\|_{\mathscr{L}_r^{\gamma',\alpha+2\beta-1}}]\\ +\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\sup_{r\in[0,T]}[\|y^{r,\prime}\|_{\mathscr{C}_p^{\alpha+\beta-1}}+\|f^{r,\prime}\|_{\mathscr{L}_r^{\gamma',\alpha+\beta-1}}]\Bigg),\end{aligned} \tag{3.50}$$

where $\lambda_{\overline{T},\mathscr{V}}:=1-(\overline{T}^{\gamma-\gamma'}\vee\overline{T}^{\gamma'-\gamma''})\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}})>0$.
In particular, replacing y^r *by* $y_1^r-y_2^r$ *and* f^r *by* $f_1^r-f_2^r$ *with analogue bounds, a uniform Lipschitz bound for the solutions* $u_1^r-u_2^r$ *follows.*
In the setting of Corollary 3.28, the bound (3.50) *holds true with* $\gamma=\gamma'=0$, *under the assumption, that* $C_f+C_y<\infty$ *for* $\gamma'=0$.

Remark 3.33. *In setting of Theorem 3.19 for* β *in the Young regime and considering bounded sets of terminal conditions* $(y^r)_r\subset\mathscr{C}_p^{(1-\gamma)\alpha+\beta}$ *and right-hand-sides* $f^r\subset\mathscr{L}_r^{\gamma,\beta}$ *for* $\gamma\in[0,1)$, *an analogue uniform Lipschitz bound for the solutions* (u^r) *on* $[0,r]$ *holds true. The proof is similar except much easier.*

Proof. The proof follows from Theorem 3.30 replacing T by r and considering paracontrolled solutions on $[0,r]$ in the sense of (3.49). Then, by (3.40) and (3.41) from the proof of Theorem 3.30 for $\mathscr{V}=\mathscr{W}$ and splitting the interval $[0,r]$ in subintervals of length $\overline{T}$, we obtain for every $r\leqslant T$,

$$\begin{aligned}\|u^{r,\sharp}\|_{\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1}}+\|u^r\|_{\mathscr{L}_r^{\gamma',\alpha+\beta}}\\ \lesssim_T C(r)\lambda_{\overline{T},\mathscr{V}}^{-1}\Bigg(\sup_{r\in[0,T]}[\|y^{r,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|f^{r,\sharp}\|_{\mathscr{L}_r^{\gamma',\alpha+2\beta-1}}]\\ +\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}\sup_{r\in[0,T]}[\|y^{r,\prime}\|_{\mathscr{C}_p^{\alpha+\beta-1}}+\|f^{r,\prime}\|_{\mathscr{L}_r^{\gamma',\alpha+\beta-1}}]\Bigg).\end{aligned}$$

The dependence of the constant $C(r)$ on $r\leqslant T$ is as follows: $C(r)\lesssim\frac{r}{\overline{T}}\leqslant\frac{T}{\overline{T}}$. Notice that the choice of $\overline{T}$ only depends on $\|\mathscr{V}\|$, which is fixed here. Thus we obtain (3.50). As the solution u^r depends linearly on the terminal condition y^r and the right-hand side f^r, the uniform Lipschitz bound follows. □

Remark 3.34. *Let* (V^m) *be such that* $\mathscr{V}^m:=(V^m,(P(\partial_iV^{m,j})\odot V^i)_{i,j})\overset{m\to\infty}{\to}\mathscr{V}$ *in* $\mathscr{X}^{\beta,\gamma'}$. *Let* (f^r), (y^r) *be as in the corollary. Moreover, let* $(y^{r,m})$ *with* $y^{r,m}=y^{r,\sharp,m}+y^{r,\prime}\otimes V_r^m$ *be such that* $\sup_{r\in[0,T]}\|y^{r,\sharp,m}-y^{r,\sharp}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}\to 0$ *for* $m\to\infty$. *Analogously, let* $(f^{r,m})$ *with* $f^{r,m}=f^{r,\sharp,m}+f^{r,\prime}\otimes V^m$ *and convergence of* $(f^{r,\sharp,m})_m$. *Let* u^r *and* $u^{r,m}$ *be the solutions for* $\mathscr{G}^{\mathscr{V}}$ *with right-hand side* f^r *and terminal conditions* y^r

and $y^{r,m}$*, respectively. Then the proof of the corollary furthermore shows that*

$$\begin{aligned}\sup_{r\in[0,T]}&[\|u^{r,\sharp}-u^{r,\sharp,m}\|_{\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1}}+\|u^r-u^{r,m}\|_{\mathscr{L}_r^{\gamma',\alpha+\beta}}]\\&\lesssim_T\lambda_{T,\mathscr{V}}^{-1}\Big(\sup_{r\in[0,T]}[\|y^{r,\sharp}-y^{r,\sharp,m}\|_{\mathscr{C}_p^{(2-\gamma')\alpha+2\beta-1}}+\|f^{r,\sharp}-f^{r,\sharp,m}\|_{\mathscr{L}_r^{\gamma',\alpha+2\beta-1}}]\\&\qquad+\|\mathscr{V}-\mathscr{V}^m\|_{\mathscr{X}^{\beta,\gamma'}}\sup_{r\in[0,T]}[\|y^{r,\prime}\|_{\mathscr{C}_p^{\alpha+\beta-1}}+\|f^{r,\prime}\|_{\mathscr{L}_r^{\gamma',\alpha+\beta-1}}]\Big)\\&\to0,\end{aligned}$$

for $m\to\infty$.

4. Weak solution concepts for singular Lévy SDEs

This chapter is devoted to prove existence and uniqueness of weak solutions to singular SDEs of the form

$$dX_t = V(t, X_t)dt + dL_t, \quad X_0 = x \in \mathbb{R}^d. \tag{4.1}$$

Herein, L is a symmetric α-stable Lévy process for $\alpha \in (1, 2]$, that satisfies the non-degeneracy Assumption 3.4, and $V \in \mathscr{X}^{\beta,\gamma}$ is an enhanced distribution in the sense of Definition 3.20 with regularity $\beta \in (\frac{2-2\alpha}{3}, 0)$. In the case of $\alpha = 2$, we consider additive standard Brownian noise $L = B$. The first notion of a weak solution, that we introduce in Section 4.1, is the concept of solutions of the martingale problem. Section 4.1 is based on the results from [KP22, Section 4] and we prove existence and uniqueness of martingale solutions in Theorem 4.2. In Section 4.2, we develop the concept of weak rough-path-type solutions (we call them below rough weak solutions), that are proven to be equivalent to martingale solutions in Theorem 4.18 in the Young ($\beta \in ((1-\alpha)/2, 0)$) and the rough regularity regime ($\beta \in ((2-2\alpha)/3, (1-\alpha)/2]$). Theorem 4.18 follows from Theorem 4.35 and Theorem 4.32 in Section 4.4. To show the equivalence of the solution concepts in the rough regime, we construct in Section 4.3 a rough stochastic sewing integral. In the Young regime, we introduce canonical weak solutions and prove equivalence to rough weak solutions in Section 4.5. Moreover, we prove in Section 4.5 ill-posedness of the canonical weak solution concept in the rough regime. Section 4.6 is based on the results from [KP22, Section 5] and we apply our theory to construct the solution of the so-called Brox diffusion with Lévy noise, where the drift is a typical realization of periodic white noise.

4.1. Solutions of the martingale problem

In this section, we prove in Theorem 4.2 existence and uniqueness of solutions of the martingale problem associated to the singular SDE (4.1).
Formally, the singular generator of X is given by

$$\mathscr{G}^V = \partial_t - \mathscr{L}_\nu^\alpha + V \cdot \nabla.$$

In the case of regular drift, this is rigorous in the following sense. If $V \in C_T C_b^\infty$, an application of Itô's formula to test functions $\varphi \in C_b^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ shows that the generator of the diffusion with drift V is given by $\mathscr{G}^V \varphi = \partial_t \varphi - \mathscr{L}_\nu^\alpha \varphi + V \cdot \nabla \varphi$ and

the set of those functions is a subset of the domain of the generator. For a definition and properties of infinitesimal generators of (Markov) semigroups, we refer to [EK86, Sections 1 and 4]. In the case of distributional drift V, it turns out, that the set of those test functions has trivial intersection with the domain of the singular generator (mapping into a space of bounded functions) and is thus not a proper function space to formulate the martingale problem. However, the set of solutions of the backward Kolmogorov equations $\mathscr{G}^V u = f$, $u(T,\cdot) = u^T$, for right-hand-sides $f \in C_T\mathscr{C}^\varepsilon$ and regular terminal conditions $u^T \in \mathscr{C}^{2(\alpha+\beta)-1}$, whose existence follows from Corollary 3.28, is a rich enough class of functions to formulte the martingale problem, so that the martingale problem uniquely determines the law of the solution process.
In the next definition, $(\Omega, \mathscr{F}) := (D([0,T], \mathbb{R}^d), \mathscr{B}(D([0,T], \mathbb{R}^d)))$ denotes the Skorokhod space with canonical filtration $(\mathscr{F}_t)_{t\geqslant 0}$, i.e. $\mathscr{F}_t = \sigma(X_s : s \leqslant t)$ where $(X_t)_{t\geqslant 0}$ is the canonical process with $X_t = \omega(t)$ for $\omega \in \Omega$.

Definition 4.1 (Martingale problem)**.** *Let $\alpha \in (1,2]$ and $\beta \in (\frac{2-2\alpha}{3}, 0)$, and let $T > 0$ and $V \in \mathscr{X}^{\beta,\gamma}$. Then, we call a probability measure $\mathbb{P}$ on the Skorokhod space $(\Omega, \mathscr{F})$ a solution of the martingale problem for $(\mathscr{G}^V, \delta_x)$, if*

1.) $\mathbb{P}(X_0 \equiv x) = 1$ *(i.e. $\mathbb{P}^{X_0} = \delta_x$), and*

2.) *for all $f \in C_T\mathscr{C}^\varepsilon$ with $\varepsilon > 2-\alpha$ and for all $u^T \in \mathscr{C}^3$, the process $M = (M_t)_{t\in[0,T]}$ is a martingale under $\mathbb{P}$ with respect to $(\mathscr{F}_t)$, where*

$$M_t = u(t, X_t) - u(0,x) - \int_0^t f(s, X_s)ds \tag{4.2}$$

and where u solves the Kolmogorov backward equation $\mathscr{G}^V u = f$ with terminal condition $u(T,\cdot) = u^T$.

This is a generalization of the classical notion of a weak solution for regular drifts, in the sense that if V^n is a bounded and measurable function, then $(X_t^n)_{t\in[0,T]}$ is a weak solution of

$$dX_t^n = V^n(t, X_t^n)dt + dL_t, \qquad X_0^n = x, \tag{4.3}$$

if and only if it solves the martingale problem of Definition 4.1.
The first main theorem of this chapter proves the existence and uniqueness of martingale solutions.

Theorem 4.2. *Let $\alpha \in (1,2]$ and L be a symmetric, α-stable Lévy process, such that the measure ν satisfies Assumption 3.4. Let $T > 0$ and $\beta \in ((2-2\alpha)/3, 0)$ and let $V \in \mathscr{X}^{\beta,\gamma}$ be as in Definition 3.20. Then for all $x \in \mathbb{R}^d$, there exists a unique solution $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ of the martingale problem for $(\mathscr{G}^V, \delta_x)$. Under $\mathbb{Q}$ the canonical process is a strong Markov process.*

To prove the theorem, we first establish an auxiliary lemma based on Cambell's formula, with which we can establish moment estimates on the jump martingale in Lemma 4.4 below. It's proof can be found in Appendix A.

Lemma 4.3. *Let $\alpha \in (1,2)$ and let π be the Poisson random measure of the α-stable Lévy process L. We define for a multi-index $\omega \in \mathbb{N}_0^n$ with $n \in \mathbb{N}$:*

$$|\omega| := \omega_1 + 2\omega_2 + \cdots + n\omega_n.$$

For $\lambda \in \mathbb{R}$ we furthermore define the following moment generating function:

$$\Phi(\lambda) := \mathbb{E}\left[\exp\left(\int_r^t \int_{|y|\leqslant C} \lambda |y|^2 \pi(ds, dy)\right)\right].$$

Then the derivatives of Φ satisfy

$$\Phi^{(n)}(\lambda) = \Phi(\lambda) \sum_{\omega \in \mathbb{N}_0^n : |\omega| = n} c(n, \omega) \prod_{i=1}^n \left((t-r)\int_{|y|\leqslant C} |y|^{2i} e^{\lambda |y|^2} \mu(dy)\right)^{\omega_i} \tag{4.4}$$

for suitable integers $c(n,\omega)$. In particular, we have for all $C > 0$ and $t > r$:

$$\mathbb{E}\left[\left(\int_r^t \int_{|y|\leqslant C} |y|^2 \pi(ds, dy)\right)^n\right] \lesssim \sum_{\omega \in \mathbb{N}_0^n : |\omega| = n} \prod_{i=1}^n \left((t-r)\int_{|y|\leqslant C} |y|^{2i} \mu(dy)\right)^{\omega_i}. \tag{4.5}$$

Lemma 4.4. *Let $\alpha \in (1,2)$, let $\theta \in (1,\alpha)$ and $u \in C_T\mathscr{C}^\theta \cap C_T^{\theta/\alpha} L^\infty$, and let $\rho \in 2\mathbb{N}$. Let moreover $\hat{\pi}$ be the compensated Poisson random measure of the α-stable Lévy process L. Then we have, uniformly in $0 \leqslant r \leqslant t \leqslant T$:*

$$\begin{aligned} &\mathbb{E}\left[\left|\int_r^t \int_{\mathbb{R}^d} \big((u(t, X_{s-}+y) - u(t, X_{s-})) - (u(s, X_{s-}+y) - u(s, X_{s-}))\big)\hat{\pi}(ds, dy)\right|^\rho\right] \\ &\quad \lesssim |t-r|^{\rho\theta/\alpha}. \end{aligned}$$

Proof. To abbreviate the notation we write $\Delta_y u(s,x) := u(s, x+y) - u(s,x)$. By the Burkholder-Davis-Gundy inequality together with [PZ07, Lemma 8.21] we get for any $\rho \geqslant 1$ and for $C > 0$ to be chosen later

$$\begin{aligned} &\mathbb{E}\left[\left|\int_r^t \int_{\mathbb{R}^d} (\Delta_y u(t, X_{s-}) - \Delta_y u(s, X_{s-}))\hat{\pi}(ds, dy)\right|^\rho\right] \\ &\quad \lesssim \mathbb{E}\left[\left|\int_r^t \int_{\mathbb{R}^d} (\Delta_y u(t, X_{s-}) - \Delta_y u(s, X_{s-}))^2 \pi(ds, dy)\right|^{\rho/2}\right] \\ &\quad \lesssim \mathbb{E}\left[\left|\int_r^t \int_{|y|\leqslant C} (\Delta_y u(t, X_{s-}) - \Delta_y u(s, X_{s-}))^2 \pi(ds, dy)\right|^{\rho/2}\right] \\ &\qquad + \mathbb{E}\left[\left|\int_r^t \int_{|y|> C} (\Delta_y u(t, X_{s-}) - \Delta_y u(s, X_{s-}))^2 \pi(ds, dy)\right|^{\rho/2}\right]. \end{aligned} \tag{4.6}$$

Since π is a positive measure, the second term on the right hand side is bounded by

$$\mathbb{E}\left[\left|\int_r^t\int_{|y|>C}(\Delta_y u(t,X_{s-})-\Delta_y u(s,X_{s-}))^2\pi(ds,dy)\right|^{\rho/2}\right]$$
$$\lesssim |t-r|^{\rho\theta/\alpha}\|u\|^{\rho}_{C_T^{\frac{\theta}{\alpha}}L^\infty}\mathbb{E}\left[\left|\int_r^t\int_{|y|>C}\pi(ds,dy)\right|^{\rho/2}\right].$$

The integral inside the expectation is a Poisson distributed random variable with the parameter $(t-r)\mu(\{y:|y|>C\})\simeq(t-r)C^{-\alpha}$. This motivates the choice $C=(t-r)^{1/\alpha}$, for which this term is of the claimed order. For the first term on the right hand side of (4.6), we estimate by the mean value theorem and using the time regularity of ∇u:

$$\mathbb{E}\left[\left|\int_r^t\int_{|y|\leqslant C}(\Delta_y u(t,X_{s-})-\Delta_y u(s,X_{s-}))^2\pi(ds,dy)\right|^{\rho/2}\right]$$
$$\lesssim |t-r|^{\rho(\theta-1)/\alpha}\|\nabla u\|^{\rho}_{C_T^{(\theta-1)/\alpha}L^\infty}\mathbb{E}\left[\left|\int_r^t\int_{|y|\leqslant C}|y|^2\pi(ds,dy)\right|^{\rho/2}\right].$$

Now by Lemma 4.3 and by the choice $C=(t-r)^{1/\alpha}$, we obtain

$$\mathbb{E}\left[\left(\int_r^t\int_{|y|\leqslant C}|y|^2\pi(ds,dy)\right)^{\rho/2}\right]\lesssim\sum_{\omega\in\mathbb{N}_0^n:|\omega|=\rho/2}\prod_{i=1}^{\rho/2}\left((t-r)\int_{|y|\leqslant C}|y|^{2i}\mu(dy)\right)^{\omega_i}$$
$$\lesssim|t-r|^{\rho/\alpha},$$

where we used that $\int_{|y|\leqslant C}|y|^k\mu(dy)\simeq C^{k-\alpha}$ for $k\geqslant 2$. Together this yields for any $\rho\in 2\mathbb{N}$,

$$\mathbb{E}\left[\left|\int_r^t\int_{\mathbb{R}^d}(\Delta_y u(t,X_{s-})-\Delta_y u(s,X_{s-}))\hat{\pi}(ds,dy)\right|^{\rho}\right]$$
$$\lesssim|t-r|^{\rho\theta/\alpha}+|t-r|^{\rho(\theta-1)/\alpha}|t-r|^{\rho/\alpha}$$
$$\simeq|t-r|^{\rho\theta/\alpha}.\qquad\square$$

Corollary 4.5. *In the setting of Theorem 4.2, let* $(V^n)_{n\in\mathbb{N}}\subset C_TC_b^\infty(\mathbb{R}^d,\mathbb{R}^d)$ *be a smooth approximation with* $(V^n,\mathcal{K}(V^n))\to\mathscr{V}$ *in* $\mathscr{X}^{\beta,\gamma}$. *Let* $(X_t^n)_{t\in[0,T]}$ *be the strong solution of the SDE*

$$dX_t^n=V^n(t,X_t^n)dt+dL_t,\qquad X_0=x\in\mathbb{R}^d.$$

Let[1] $\theta=\alpha+\beta$ *and* $\rho\in 2\mathbb{N}$.

[1] In [KP22], the result was proven for $\theta<\alpha+\beta$, with θ being the regularity of the PDE solution from [KP22, section 3]. Due to Corollary 3.28, we can indeed generalize to $\theta=\alpha+\beta$.

Then, there exists $N \in \mathbb{N}$, such that uniformly in $0 \leqslant r \leqslant t \leqslant T$:

$$\sup_{n \geqslant N} \mathbb{E}\left[\left|\int_r^t V^n(s, X_s^n) ds\right|^\rho\right] \lesssim |t-r|^{\theta\rho/\alpha}. \tag{4.7}$$

Proof. [2] To prove the claim, we apply Corollary 3.32 and use the time regularity θ/α with $\theta = \alpha + \beta$ of the solution of the Kolmogorov backward equation with right-hand side V.
Let $N \in \mathbb{N}$ be large enough, such that $\sup_{n \geqslant N} \|V^n\|_{\mathscr{X}^{\beta,\gamma'}} \leqslant 2\|V\|_{\mathscr{X}^{\beta,\gamma'}}$.
Let $t \in (0, T]$ and consider the solution $u^{n,t} \in C_t C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ of the system of equations

$$\mathscr{G}^{V^n} u^{n,t,i} = V^{n,i}, \qquad u^{n,t,i}(t, \cdot) = 0, \qquad \text{for } i = 1, ..., d,$$

whose existence follows from Corollary 3.28 and which converges by the continuity of the solution map (and the interpolation bound (3.25)) in $C_t \mathscr{C}^\theta \cap C_t^{\theta/\alpha} L^\infty$ to the solution $u^{t,i} \in D_t$ of

$$\mathscr{G}^V u^{t,i} = V^i, \qquad u^{t,i}(t, \cdot) = 0, \qquad \text{for } i = 1, ..., d.$$

By the uniform bound on the solutions $u^{n,t}$ from Corollary 3.32 in $t \in [0, T]$ and the choice of N follows that,

$$\sup_{n \geqslant N, t \in [0,T]} \|u^{n,t}\|_{C_t \mathscr{C}^\theta_{\mathbb{R}^d}} + \|u^{n,t}\|_{C_t^{\theta/\alpha} L^\infty_{\mathbb{R}^d}} < \infty. \tag{4.8}$$

Let now first $\alpha \in (1, 2)$. Then we apply Itô's formula to $u^{n,t}(t, X_t^n) - u^{n,t}(r, X_r^n)$ and we use that X^n solves the SDE with drift V^n and that $\mathscr{G}^{V^n} u^n = V^n$ to obtain

$$\begin{aligned}\int_r^t V^n(s, X_s^n) ds &= u^{n,t}(t, X_t^n) - u^{n,t}(r, X_r^n) \\ &\quad + \int_r^t \int_{\mathbb{R}^d} \big(u^{n,t}(s, X_{s-}^n + y) - u^{n,t}(s, X_{s-}^n)\big) \hat{\pi}(ds, dy).\end{aligned}$$

As $u^{n,t}(t) = 0$ and by (4.8) we obtain

$$|u^{n,t}(t, X_t^n) - u^{n,t}(r, X_r^n)| = |u^{n,t}(t, X_r^n) - u^{n,t}(r, X_r^n)| \leqslant |t-r|^{\theta/\alpha} \|u^t\|_{C_t^{\theta/\alpha} L^\infty}.$$

[2] We simplified the proof compared to [KP22] by an application of Corollary 3.32.

Using once more that $u^{n,t}(t)=0$, we obtain from Lemma 4.4:

$$\mathbb{E}\Big[\Big|\int_r^t\int_{\mathbb{R}^d\setminus\{0\}}\big(u^{n,t}(s,X^n_{s-}+y)-u^{n,t}(s,X^n_{s-})\big)\hat{\pi}(ds,dy)\Big|^\rho\Big]$$
$$=\mathbb{E}\Big[\Big|\int_r^t\int_{\mathbb{R}^d\setminus\{0\}}\big(u^{n,t}(s,X^n_{s-}+y)-u^{n,t}(s,X^n_{s-})-(u^{n,t}(t,X^n_{s-}+y)-u^{n,t}(t,X^n_{s-}))\big)\hat{\pi}(ds,dy)\Big|^\rho\Big]$$
$$\lesssim |t-r|^{\theta\rho/\alpha},$$

so (4.7) holds for $\alpha\in(1,2)$. For $\alpha=2$ the argument is essentially the same, except much easier: then we only have to replace the jump martingale

$$\int_r^t\int_{\mathbb{R}^d}\big(u^{n,t}(s,X^n_{s-}+y)-u^{n,t}(s,X^n_{s-})\big)\hat{\pi}(ds,dy)$$

by $\int_r^t \nabla u^{n,t}(s,X^n_s)dB_s$ and apply the Burkholder-Davis-Gundy inequality. □

Proof of Theorem 4.2. Let $(V^n)_{n\in\mathbb{N}}\subset C_TC_b^\infty(\mathbb{R}^d,\mathbb{R}^d)$ be such that $(V^n,\mathcal{K}(V^n))\to\mathscr{V}$ in $\mathscr{X}^{\beta,\gamma}$ and let X^n be the unique strong solution of the SDE

$$dX^n_t=V^n(t,X^n_t)dt+dL_t,\qquad X^n_0=x. \tag{4.9}$$

To prove the existence of a solution of the martingale problem for $(\mathscr{G}^V,\delta_x)$ we follow the usual strategy: we show tightness of $(X^n)_{n\in\mathbb{N}}$, and then we show that every limit point solves the martingale problem for $(\mathscr{G}^V,\delta_x)$. Moreover, we prove that the solution to that martingale problem is unique in law, and therefore (X^n) converges weakly. The limit equals the solution of the martingale problem.

Step 1: Tightness of $(\mathbb{P}^{X^n})$ *on* $D([0,T],\mathbb{R}^d)$.
We apply (4.7) from Corollary 4.5 for $\rho=2$, so that $2(\alpha+\beta)/\alpha>1$, which shows that the drift term $A^n:=\int_0^\cdot V^n(s,X^n_s)ds$ satisfies Kolmogorov's tightness criterion. Therefore, (A^n) is tight in $C([0,T],\mathbb{R}^d)$ and thus in particular C-tight in $D([0,T],\mathbb{R}^d)$ (meaning that every limit point is continuous). By [JS03, Corollary VI.3.33], we thus obtain the tightness of the tuple (A^n,L) and of $X^n=x+A^n+L$.

Step 2: Any weak limit solves the martingale problem for $(\mathscr{G}^V,\delta_x)$.
We consider a weakly convergent subsequence, also denoted by $(\mathbb{P}^{X^n})$, and we write $\mathbb{Q}$ for its limit. Let X be the canonical process on $D([0,T],\mathbb{R}^d)$ and let $\mathbb{E}_n[\cdot]$ (resp. $\mathbb{E}_\mathbb{Q}[\cdot]$) denote integration w.r.t. $\mathbb{P}^{X^n}$ (resp. $\mathbb{Q}$). Let $f\in C_T\mathscr{C}^\varepsilon$ and $u^T\in\mathscr{C}^3$, and let $(f^n)_{n\in\mathbb{N}}\subset C_TC_b^\infty$ be such that f^n converges to f in $C_T\mathscr{C}^\varepsilon$. Let u^n be the solution of $\mathscr{G}^{V^n}u^n=f^n$ with terminal condition $u^n(T,\cdot)=u^T$. Since f^n and V^n are smooth we have $u^n\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$ and u^n is a strong solution of the Kolmogorov backward equation. We can thus apply Ito's formula for càdlàg processes to $u^n(t,X_t)$ under the measure $\mathbb{P}^{X_n}$ and obtain as the operators $-\mathscr{L}^\alpha_\nu$ and A from (3.5) agree on C_b^∞ (and in

fact $u^n_t \in C^\infty_b$), that in the jump case $\alpha \in (1,2)$

$$M^n_t := u^n(t, X_t) - u^n(0, x) - \int_0^t f^n(s, X_s)ds \tag{4.10}$$

$$= u^n(t, X_t) - u^n(0, x) - \int_0^t \mathscr{G}^{V^n} u^n(s, X_s)ds \tag{4.11}$$

$$= \int_0^t \int_{\mathbb{R}^d} \big(u^n(r, X_{r-} + y) - u^n(r, X_{r-})\big)\hat{\pi}(dr, dy) \tag{4.12}$$

is a martingale in the canonical filtration. Indeed, M^n is a local martingale because it is a stochastic integral against a compensated Poisson random measure, and it is a true martingale because $u^n(s, X_{s-} + y) - u^n(s, X_{s-})$ is square-integrable w.r.t. $\mathbb{P} \otimes dr \otimes \mu$, where we use the boundedness of u^n for the big jump part and the boundedness of ∇u^n for the small jump part. In the Brownian case ($\alpha = 2$) we have $M^n = \int_0^\cdot \nabla u^n(s, X^n_s)dB_s$, which is a martingale because ∇u^n is bounded.

Let now u be the solution to $\mathscr{G}^V u = f$ with terminal condition $u(T) = u^T$. By the continuity of the solution map, (u^n) converges to u in the spaces $C_T\mathscr{C}^\theta$ and $C^{\theta/\alpha}_T L^\infty$ for $\theta = \alpha + \beta$. We show that $(M_t)_{t\in[0,T]}$ is a martingale under $\mathbb{Q}$, where

$$M_t = u(t, X_t) - u(0, x) - \int_0^t f(s, X_s)ds. \tag{4.13}$$

For that purpose let $0 \leqslant r \leqslant t \leqslant T$ and let $F : D([0, r], \mathbb{R}^d) \to \mathbb{R}$ be continuous and bounded. Since M^n is a martingale under $\mathbb{P}^{X^n}$, we have

$$\mathbb{E}_n[(M^n_t - M^n_r)F((X_u)_{u\leqslant r})] = 0. \tag{4.14}$$

We define for $x \in D := D([0, T], \mathbb{R}^d)$,

$$M^n_{r,t}(x) := \Big(u^n(t, x(t)) - u^n(r, x(r)) - \int_r^t f^n(u, x(u))du\Big),$$

and $M_{r,t}(x)$ analogously with u^n, f^n replaced by u, f. Let $M^n_t(x) := M^n_{0,t}(x)$ and $M_t(x) := M_{0,t}(x)$. We aim to send $n \to \infty$ in (4.14). Therefore, observe that $\sup_{x\in D}|M^n_t(x) - M_t(x)| \to 0$ for $n \to \infty$, because (u^n, f^n) converges to (u, f) in $C_TC_b \times C_TC_b \subset C_T\mathscr{C}^\theta \times C_T\mathscr{C}^\varepsilon$. Thus, we obtain, by boundedness of F, that

$$\lim_{n\to\infty} \mathbb{E}_n[M_{r,t}F((X_u)_{u\leqslant r})] = 0.$$

Now, by [JS03, Proposition VI.2.1], we know that the map $D \ni x \mapsto \int_0^t f(s, x(s))ds$ is continuous w.r.t. the J_1-topology and it is bounded by boundedness of f. Moreover, if we know that $\mathbb{Q}(\Delta X_t = \Delta X_r = 0) = 1$, then by [JS03, Proposition VI.3.14] and since $X^n \to X$ in distribution in D, we have that $X^n_t \to X_t$ and $X^n_r \to X_r$ in distribution.

Together, this gives (as $\mathbb{R} \ni y \mapsto u(t,y) - u(r,y)$ is continuous and bounded)

$$0 = \lim_{n\to\infty} \mathbb{E}_n[M_{t,r}F((X_u)_{u\in[0,T]})] = \mathbb{E}_{\mathbb{Q}}[M_{t,r}F((X_u)_{u\in[0,T]})],$$

and since $0 \leqslant r \leqslant t \leqslant T$ and F were arbitrary, we obtain that $\mathbb{Q}$ solves the martingale problem for $(\mathscr{G}^V, \delta_x)$. So it remains to show that indeed $\mathbb{Q}(\Delta X_t = \Delta X_r = 0) = 1$. Since the map $C([0,T],\mathbb{R}^d) \times D \ni (x,y) \mapsto x + y \in D$ is continuous by [JS03, Section VI.1b, Proposition VI.1.23] and since (A^n, L) is tight by Step 1, we obtain (possibly along a further subsequence)

$$X \leftarrow X^n = x + \int_0^\cdot V^n(s, X^n_s)ds + L \to x + A + L \quad \text{in distribution in } D,$$

where A denotes the continuous limit of the drift term. Hence we conclude

$$\mathbb{Q}(\Delta X_t = \Delta X_r = 0) = \mathbb{P}(\Delta L_t = \Delta L_r = 0) = 1.$$

Step 3: Uniqueness for the martingale problem and strong Markov property.
Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be two solutions of the martingale problem for $\mathscr{G}^V$ with the same initial distribution $\mu = \mathbb{Q}_1^{X_0} = \mathbb{Q}_2^{X_0}$. Let $f \in C_T\mathscr{C}^\varepsilon$ and let u be the solution of $\mathscr{G}^V u = f$, $u(T) = 0$. Then we obtain for $i = 1, 2$,

$$\int_{\mathbb{R}^d} u(0,x)\mu(dx) = \mathbb{E}_{\mathbb{Q}_i}\Big[u(T, X_T) - \int_0^T f(s, X_s)ds\Big] = -\mathbb{E}_{\mathbb{Q}_i}\Big[\int_0^T f(s, X_s)ds\Big].$$

Thus, we have for all $f \in C_T\mathscr{C}^\varepsilon$

$$\mathbb{E}_{\mathbb{Q}_1}\Big[\int_0^T f(s, X_s)ds\Big] = \mathbb{E}_{\mathbb{Q}_2}\Big[\int_0^T f(s, X_s)ds\Big].$$

Therefore, $\mathbb{Q}_1^{X_t} = \mathbb{Q}_2^{X_t}$ for all $t \in [0,T]$, that is, the one dimensional marginal distributions of $\mathbb{Q}_1$ and $\mathbb{Q}_2$ agree. Indeed, this follows by taking $f_\delta(s,x) = \delta^{-1}h_\delta(s)g(x)$ for $h_\delta \simeq \mathbf{1}_{[t,t+\delta]}$ and $g \in \mathscr{C}^\varepsilon$ and letting $\delta \to 0$. Now [EK86, Theorem 4.4.3] shows that $\mathbb{Q}_1 = \mathbb{Q}_2$ and that under the solution $\mathbb{Q}$ to the martingale problem for $(\mathscr{G}^V, \delta_x)$ the canonical process is a strong Markov process. □

Remark 4.6. *To solve the martingale problem, we only used that there exists a sequence* (V^n) *with* $V^n \in C_T(C_b^\infty)^d$ *and* $(V^n, (J^T(\partial_i V^{n,j}) \odot V^{n,i})_{i,j}) \to (V, (J^T(\partial_i V^j) \odot V^i)_{i,j}) \in C_T\mathscr{C}^\beta_{\mathbb{R}^d} \times C_T\mathscr{C}^{2\beta+\alpha-1}_{\mathbb{R}^{d\times d}}$*, which is implied by the stronger assumption* $V \in \mathscr{X}^{\beta,\gamma'}$*. This follows, because we only need to consider solutions of the Kolmogorov equation with regular terminal conditions, cf. Remark 3.29, to solve the martingale problem.*

4.2. Weak rough-path-type solutions

In this section, we introduce our rough weak solution concept in Definition 4.10 and state the second main Theorem 4.18, that proves equivalence of rough weak solutions and martingale solutions from the previous section. The proof of Theorem 4.18 follows from Theorem 4.32 and Theorem 4.35 in Section 4.4. We furthermore state the stochastic sewing lemma from [Lê20]. An application of the stochastic sewing lemma yields in Lemma 4.15 existence of the stochastic integral of $f(t, X_t)$, for regular enough functions f, against the drift Z of the diffusion X. Moreover, we define a class $\mathscr{K}^{\vartheta}$ of processes in Definition 4.16, which are processes with a certain regularity requirement, that will be used in Section 4.4.
Let us start by stating the stochastic sewing lemma from [Lê20, Theorem 2.1].

Lemma 4.7 (Stochastic sewing lemma)**.** *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete probability space. Let $(\Xi_{s,t})_{0\leqslant s\leqslant t\leqslant T}$ be a two-parameter stochastic process with values in $\mathbb{R}^d$, that is adapted ($\Xi_{s,t}$ being $\mathscr{F}_t$-measurable for $s \leqslant t$) and $L^2(\mathbb{P})$-integrable. Let $\delta\Xi_{s,u,t} := \Xi_{st} - \Xi_{su} - \Xi_{ut}$, $0 \leqslant s \leqslant u \leqslant t \leqslant T$. Suppose that there are constants $\Gamma_1, \Gamma_2, \varepsilon_1, \varepsilon_2 > 0$, such that for all $0 \leqslant s \leqslant u \leqslant t \leqslant T$,*

$$\|\mathbb{E}[\delta\Xi_{s,u,t} \mid \mathscr{F}_s]\|_{L^2(\mathbb{P})} \leqslant \Gamma_1|t-s|^{1+\varepsilon_1}, \quad \|\delta\Xi_{s,u,t}\|_{L^2(\mathbb{P})} \leqslant \Gamma_2|t-s|^{\frac{1}{2}+\varepsilon_2}. \tag{4.15}$$

Then, there exists a unique (up to modifications) stochastic process $(I_t)_{t\in[0,T]} := (I_t(\Xi))_{t\in[0,T]}$ with values in $\mathbb{R}^d$ satisfying the following properties

- *$I_0 = 0$, $(I_t)_{t\in[0,T]}$ is $(\mathscr{F}_t)$-adapted and $L^2(\mathbb{P})$-integrable and*
- *there exist constants $C_1 = C(\varepsilon_1), C_2 = C(\varepsilon_2) > 0$, such that for all $0 \leqslant s \leqslant t \leqslant T$,*

$$\begin{aligned}
&\|I_t - I_s - \Xi_{s,t}\|_{L^2(\mathbb{P})} \leqslant C_1\Gamma_1|t-s|^{1+\varepsilon_1} + C_2\Gamma_2|t-s|^{\frac{1}{2}+\varepsilon_2},\\
&\|\mathbb{E}[I_t - I_s - \Xi_{s,t} \mid \mathscr{F}_s]\|_{L^2(\mathbb{P})} \leqslant C_1\Gamma_1|t-s|^{1+\varepsilon_1}.
\end{aligned} \tag{4.16}$$

Furthermore, for every $t \in [0,T]$ and any partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$, the Riemann sums $I_t^{\Pi} = \sum_{i=0}^{N-1} \Xi_{t_i,t_{i+1}}$ converge to I_t in $L^2(\mathbb{P})$ for vanishing mesh size $|\Pi| := \max_i |t_{i+1} - t_i| \to 0$.

Remark 4.8. *We can apply the stochastic sewing lemma to a germ $\Xi_{st} := f_s Y_{st} = f_s(Y_t - Y_s)$ for stochastic processes $(f_t), (Y_t)$. Then, typically the constants Γ_1, Γ_2 depend linearly on the Hölder-type moment bounds of the stochastic processes f and Y. The bounds from Lemma 4.7 then imply that $\sup_{0\leqslant s<t\leqslant T}\|I_t - I_s - \Xi_{s,t}\|_{L^2(\mathbb{P})}|t-s|^{-(1/2+\varepsilon_2)} \lesssim T^{1/2+\varepsilon_1-\varepsilon_2}\Gamma_1 + \Gamma_2$ (assuming $\varepsilon_2 \in (0, 1/2)$), which yields the stability of the stochastic sewing integral.*

Next, we define a (rough) weak solution to the singular SDE

$$dX_t = V(t, X_t)dt + dL_t, \quad X_0 = x \in \mathbb{R}^d \tag{4.17}$$

for an enhanced Besov drift $V \in \mathscr{X}^{\beta,\gamma}$ with $\beta \in (\frac{2-2\alpha}{3}, 0)$ and $\gamma \in [\frac{2\beta+2\alpha-1}{\alpha}, 1)$ in the rough case, respectively $\gamma \in (\frac{1-\beta}{\alpha}, 1)$ in the Young case (cf. Definition 3.20).
In the case of (locally) bounded drifts V the equivalence of solutions to the martingale problem and weak solutions is known by [SV06, Theorem 8.1.1] in the Brownian noise case and by [KC11, Theorem 1.1] in the Lévy noise setting. The first attempt to define weak solutions in the singular drift case, is to replace the singular drift by the limit Z of the drift terms $\int_0^\cdot V^n(t, X_t)dt$ for a smooth sequence (V^n) approximating V. In this way, one obtains a *canonical* weak solution concept. In the Young case, this yields a well-posed solution concept (if moreover requiring regularity bounds on the drift Z). However, it turns out that canonical weak solutions are in general non-unique in the rough regime (cf. Section 4.5 below). In this section, we thus adapt the canonical weak solution concept, imposing additional assumptions to ensure well-posedness. The idea for a (rough) weak solution is to impose rough-paths-type assumptions on certain iterated integrals $\mathbb{Z}^V$, that formally correspont to the resonant component in the enhancement $\mathscr{V}$. This motivates Definition 4.10 below.
Let us introduce the notation

$$\mathscr{K}(\eta_1, \eta_2) := [\mathring{\Delta}_T \ni (s,t) \mapsto P_{t-s}\partial_j \eta_1^i(t) \odot \eta_2^j(s)],$$

for $\eta_1, \eta_2 \in C_T C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $\mathring{\Delta}_T := \{(s,t) \in [0,T]^2 \mid s < t\}$.
Let $\Delta_T := \{(s,t) \in [0,T]^2 \mid s \leqslant t\}$.
Moreover, for a sequence $(a_{m,n})_{m,n\in\mathbb{N}}$ in a Banach space X, for which the convergence

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = a = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}$$

holds, we use the short-hand notation $a^{m,n} \to a$ for $m, n \to \infty$ or $\lim_{m,n\to\infty} a^{m,n} = a$.

Assumption 4.9. *In the following, we will assume that for $\mathscr{V} = (V, \mathscr{V}_2) \in \mathscr{X}^{\beta,\gamma}$, there exists a sequence $(V^n) \subset C_T(C_b^\infty)^d$, such that*

$$(V^n, \mathscr{K}(V^n, V^m)) \to \mathscr{V} = (V, \mathscr{V}_2) \text{ in } \mathscr{X}^{\beta,\gamma} \tag{4.18}$$

for $n, m \to \infty$, i.e. that the mixed resonant products $(\mathscr{K}(V^n, V^m))$ converge to $\mathscr{V}_2$.

Whenever we write $V \in \mathscr{X}^{\beta,\gamma}$ in this section and Section 4.4, we mean that additionally Assumption 4.9 is satisfied. Assumption 4.9 implies in particular that $\mathscr{K}(V^n, V) \to \mathscr{V}_2$ for $n \to \infty$.

Furthermore, we call a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F})_{t\in[0,T]}, \mathbb{P})$ a stochastic basis, if $(\Omega, \mathscr{F}, (\mathscr{F})_{t\in[0,T]}, \mathbb{P})$ is complete and the filtration $(\mathscr{F}_t)$ is right-continuous. We call a process L a $(\mathscr{F}_t)$-Lévy process, if L is adapted to $(\mathscr{F}_t)$ with $L_0 = 0$, $L_t - L_s$ being independent of $\mathscr{F}_s$ and $L_t - L_s \overset{d}{=} L_{t-s}$ for all $0 \leqslant s < t \leqslant T$. As before, we call an α-stable Lévy process non-degenerate, if Assumption 3.4 is satisfied.
In the following, we use the notation $Z_{st} = Z_{s,t} = Z_t - Z_s$, $s \leqslant t$, for the increment

of a stochastic process (Z_t) and $\mathbb{E}_s[\cdot] := \mathbb{E}[\cdot \mid \mathscr{F}_s]$ for the conditional expectation. A stochastic process $(\Xi_{st})_{(s,t)\in\Delta_T}$ indexed by Δ_T we call adapted to $(\mathscr{F}_t)$ if Ξ_{st} is $\mathscr{F}_t$-measurable for all $(s,t) \in \Delta_T$. For $p \in [2,\infty]$, $m \in \mathbb{N}$ and an adapted $\mathbb{R}^m$-valued stochastic process $(\Xi_{st})_{(s,t)\in\Delta_T}$ indexed by Δ_T, we define for $\theta \in (0,1)$,

$$\|\Xi\|_{\theta,p} := \sup_{(s,t)\in\Delta_T} \frac{\|\Xi_{st}\|_{L^p(\mathbb{P},\mathbb{R}^m)}}{|t-s|^\theta}, \quad \|\Xi \mid \mathscr{F}\|_{\theta,p} := \sup_{(s,t)\in\Delta_T} \frac{\|\mathbb{E}_s[\Xi_{st}]\|_{L^p(\mathbb{P},\mathbb{R}^m)}}{|t-s|^\theta}. \tag{4.19}$$

If $(Z_t)_{t\in[0,T]}$ is an adapted process indexed by $[0,T]$, we use the same notation (4.19) for $\Xi_{st} = Z_{st} = Z_t - Z_s$.

Definition 4.10 (Rough weak solution)**.** *Let $V \in \mathscr{X}^{\beta,\gamma}$ for $\beta \in (\frac{2-2\alpha}{3}, 0)$ (with $\gamma \in [\frac{2\beta+2\alpha-1}{\alpha}, 1)$ in the rough case, respectively $\gamma \in [\frac{1-\beta}{\alpha}, 1)$ in the Young case). Let $x \in \mathbb{R}^d$. We call a triple $(X, L, \mathbb{Z}^V)$ a weak solution to the SDE (4.17) starting at $X_0 = x \in \mathbb{R}^d$, if there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$, such that L is an α-stable symmetric non-degenerate $(\mathscr{F}_t)$-Lévy process and almost surely*

$$X = x + Z + L,$$

where Z is a continuous and $(\mathscr{F}_t)$-adapted process with the property that

$$\|Z\|_{\frac{\alpha+\beta}{\alpha},2} + \|Z \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty} < \infty. \tag{4.20}$$

Moreover $(\mathbb{Z}^V_{st})_{(s,t)\in\Delta_T}$ is a continuous, $(\mathscr{F}_t)$-adapted, $\mathbb{R}^{d\times d}$-valued stochastic process indexed by Δ_T with

$$\|\mathbb{Z}^V\|_{\frac{\alpha+\beta}{\alpha},2} + \|\mathbb{Z}^V \mid \mathscr{F}\|_{\frac{2\alpha+2\beta-1}{\alpha},\infty} < \infty. \tag{4.21}$$

Furthermore, Z and $\mathbb{Z}^V$ are given as follows. There exists a sequence $(V^n) \subset C_T(C_b^\infty)^d$ with $(V^n, \mathscr{K}(V^m, V^n)) \to (V, \mathscr{V}_2)$ in $\mathscr{X}^{\beta,\gamma}$ for $m, n \to \infty$ with the following properties:

1.) $Z^n := \int_0^\cdot V^n(s, X_s)ds$ converges to Z in the sense that

$$\lim_{n\to\infty} [\|Z^n - Z\|_{\frac{\alpha+\beta}{\alpha},2} + \|Z^n - Z \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty}] = 0 \tag{4.22}$$

and

2.) $\mathbb{Z}^{m,n}_{st} = (\mathbb{Z}^{m,n}_{st}(i,j))_{i,j} = \left(\int_s^t [J^T(\partial_i V^{m,j})(r, X_r) - J^T(\partial_i V^{m,j})(s, X_s)]dZ^{n,i}_r\right)_{i,j}$ converges to $\mathbb{Z}^V$ in the sense that

$$\lim_{m,n\to\infty} [\|\mathbb{Z}^{m,n} - \mathbb{Z}^V\|_{\frac{\alpha+\beta}{\alpha},2} + \|\mathbb{Z}^{m,n} - \mathbb{Z}^V \mid \mathscr{F}\|_{\frac{2\alpha+2\beta-1}{\alpha},\infty}] = 0. \tag{4.23}$$

We also call X a (rough) weak solution, if there exists a stochastic basis, an α-stable symmetric non-degenerate Lévy process L and a stochastic process $\mathbb{Z}^V$, such that $(X, L, \mathbb{Z}^V)$ is a rough weak solution.

Remark 4.11 (Notation). *We also use the following abbreviations. Let*

$$\begin{aligned}\mathbb{Z}^{m,n}_{st}(i,j) &= \int_s^t J^T(\partial_i V^{m,j})(r,X_r)dZ^{n,i}_r - J^T(\partial_i V^{m,j})(s,X_s)Z^{n,i}_{st}\\ &=: \mathbb{A}^{m,n}_{st}(i,j) - J^T(\partial_i V^{m,j})(s,X_s)Z^{n,i}_{st}.\end{aligned}$$

Let furthermore $\mathbb{A}^i_{st} := \lim_{m,n\to\infty}(\mathbb{A}^{m,n}_{st}(i,j))_{j=1,\dots,d} =: \lim_{m,n\to\infty}\mathbb{A}^{m,n,i}_{st}$ *for* $i=1,\dots,d$. *Here, the convergence with respect to* $\|\cdot\|_{\frac{\alpha+\beta}{\alpha},2}$ *follows from the convergences* 1.) *and* 2.) *and* $J^T(\partial_i V^{m,j}) \to J^T(\partial_i V^j)$ *in* $C_T L^\infty$. *Moreover,* $(\mathbb{A}^{m,n})_{m,n}, \mathbb{A}$ *satisfy the same bounds as* Z, *that is* (4.20).
We write $\mathbb{Z}^{V,i}_{st} := \lim_{m,n\to\infty}(\mathbb{Z}^{m,n}_{st}(i,j))_{j=1,\dots,d} =: \lim_{m,n\to\infty}\mathbb{Z}^{m,n,i}_{st}$ *for* $i=1,\dots,d$. *And for fixed* $m\in\mathbb{N}$, *we define* $\mathbb{A}^{m,\infty,i}_{st} := \int_s^t J^T(\partial_i V^m)(r,X_r)dZ^i_r$, *and analogously we define* $\mathbb{Z}^{m,\infty,i}_{st} := \int_s^t [J^T(\partial_i V^m)(r,X_r) - J^T(\partial_i V^m)(s,X_s)]dZ^i_r$.

Remark 4.12 (X is a Dirichlet process). *By the* L^2*-moment bound on* Z *from* (4.20), Z *has zero quadratic variation as* $2(\alpha+\beta)/\alpha > 1$ *(but* Z *is not necessarily of finite one-variation). Thus,* $X = x+Z+L$ *is a Dirichlet process, i.e. the sum of a local martingale and a zero quadratic variation process, cf. [CJMS06, Definition 2.4]. In particular, for* $F \in C^{1,2}_b([0,T]\times\mathbb{R}^d,\mathbb{R})$, *the Itô-formula for* $F(t,X_t)$ *from [CJMS06, Theorem 3.1] (which can be extended to time depending* F, *that are* C^1 *in time) holds. Notice that in our case* X *is not only a weak Dirichlet process, but a Dirichlet process and* Z *is continuous. Thus in the Itô-formula the terms involving the quadratic variation and the pure jump part of* Z *vanish. In particular, the stochastic integral* $\int_0^\cdot \nabla F(s,X_s)\cdot dZ_s$ *is the limit, in probability, of the classical Riemann sums.*

Remark 4.13 (Stochastic integrals against Z and well-definedness of $\mathbb{A}^{m,n}, \mathbb{A}^{m,\infty}$). *For any* $f = \partial_i F$ *for* $F \in C^{1,2}_b([0,T]\times\mathbb{R}^d,\mathbb{R})$, *we have that the integral* $\int_0^t f(s,X_s)dZ^i_s$ *for* $i=1,\dots,d$, $t \leqslant T$, *is defined via Itô's formula as the limit of the classical Riemann sums*

$$\sum_{s,r\in\Pi} f(s,X_s)(Z^i_r - Z^i_s) \tag{4.24}$$

for partitions Π *of* $[0,t]$ *with mesh-size* $|\Pi| \to 0$, *cf. Remark 4.12. Thus, in particular, for fixed* $m \in \mathbb{N}$, $\mathbb{A}^{m,\infty}_{0,t}(i,j)$ *is defined as the limit of the Riemann sums* $\sum_{s,r\in\Pi} J^T(\partial_i V^{m,j})(s,X_s)Z^i_{s,r}$, *analogously for* $\mathbb{A}^{m,n}_{0,t}(i,j)$.
Thanks to Lemma 4.15 below and the moment bounds (4.20) *of* Z, *existence of the* $L^2(\mathbb{P})$*-limit of the sums* (4.24) *can even be shown for* $f \in C^{\theta/\alpha}_T L^\infty \cap C_T\mathscr{C}^\theta$ *for* $\theta\in(0,1)$ *with* $(\theta+\alpha+\beta)/\alpha > 1$.
In particular, the stability of the stochastic sewing integral yields that, for fixed $m\in\mathbb{N}$, $\lim_{n\to\infty}\mathbb{A}^{m,n}_{st} = \mathbb{A}^{m,\infty}_{st}$ *in* $L^2(\mathbb{P})$, *uniformly in* $(s,t)\in\Delta_T$.
In the Young case, we can take $f = J^T(\partial_i V^j) \in C^{(\alpha+\beta-1)/\alpha}_T L^\infty \cap C_T\mathscr{C}^{\alpha+\beta-1}$, *that is* $\theta = \alpha+\beta-1$, *which satisfies the assumptions of Lemma 4.15. Thus, existence of the integral* $\mathbb{A}_{st}(i,j)$ *follows. In the rough case, the regularity of* $f = J^T(\partial_i V^j)$ *does not*

suffice to take the limit in (4.24)*, because* $2(\alpha+\beta)-1 \leqslant \alpha$ *if* $\beta \leqslant (1-\alpha)/2$. *Thus, the bounds* (4.21) *on* $\mathbb{Z}^V$ *and the convergence in* 2.) *are non-trivial assumptions in the rough case.*

Remark 4.14 (One and all sequences (V^n))**.** *If* $(X, L, \mathbb{Z}^V)$ *is a weak solution and* $\mathscr{V} \in \mathscr{X}^{\beta,\gamma}$*, then the convergences in* 1.) *and* 2.) *are true for* <u>*all*</u> *sequences* (V^n) *with* $(V^n, \mathscr{K}(V^n, V^m)) \to \mathscr{V}$ *in* $\mathscr{X}^{\beta,\gamma}$. *This means, that* (Z^n), $(\mathbb{Z}^{m,n})$ *converge and the limit is the same for any such sequence* (V^n). *Indeed, this will follow from the equivalence proof of the solution concepts below (specifically Theorem 4.35).*

The following lemma is an application of Lemma 4.7, cf. the discussion in Remark 4.13. Lemma 4.15 will moreover be used to prove (together with Theorem 4.18 and Proposition 4.33 below), that in the Young case rough weak solutions are equivalent to canonical weak solutions, defined in Section 4.5.

Lemma 4.15. *Let* $(X, L, \mathbb{Z}^V)$ *be a weak solution and let* $f \in C_T^{\theta/\alpha} L^\infty \cap C_T \mathscr{C}^\theta$ *for* $\theta \in (0,1)$ *with* $(\theta+\alpha+\beta)/\alpha > 1$. *Let* $i \in \{1, \ldots, d\}$.
Then, for every $t \in [0,T]$*, the stochastic sewing integral*

$$I(f,Z)_t := \lim_{|\Pi| \to 0} \sum_{s,r \in \Pi} f(s, X_s)(Z_r^i - Z_s^i) \in L^2(\mathbb{P})$$

for finite partitions Π *of* $[0,t]$ *with* $|\Pi| = \max_{s,r \in \Pi} |r-s| \to 0$*, is well-defined and allows for the bound,*

$$\begin{aligned}
&\|I(f,Z)_t - I(f,Z)_s - f(s,X_s) Z_{s,t}^i\|_{L^2(\mathbb{P})} \\
&\quad \lesssim \|f\|_{C_T L^\infty} \|Z\|_{\frac{\alpha+\beta}{\alpha},2} |t-s|^{(\alpha+\beta)/\alpha} \\
&\qquad + \|f\|_{C_T^{\theta/\alpha} L^\infty \cap C_T \mathscr{C}^\theta} \|Z\|^\theta_{\frac{\alpha+\beta}{\alpha},2} \|Z \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty} |t-s|^{(\theta+\alpha+\beta)/\alpha}.
\end{aligned}$$

In particular, for $\beta \in (\frac{1-\alpha}{2}, 0)$ *(Young regime), the existence of the integral* $\mathbb{A}_{0,t} = \left(\int_0^t J^T(\partial_i V^j)(s,X_s) dZ_s^i\right)_{i,j} \in L^2(\mathbb{P}, \mathbb{R}^{d \times d})$, $t \in [0,T]$ *and the bound* $\|\mathbb{Z}^V\|_{(\alpha+\beta)/\alpha,2} < \infty$, *as well as the convergence* $\|\mathbb{Z}^{m,n} - \mathbb{Z}^V\|_{\frac{\alpha+\beta}{\alpha},2} \to 0$ *in* 2.) *follow. Furthermore, the convergence* $\|\mathbb{Z}^{m,n} - \mathbb{Z}^V \mid \mathscr{F}\|_{\frac{2\alpha+2\beta-1}{\alpha},2} \to 0$ *follows.*

Proof. We apply the stochastic sewing lemma, Lemma 4.7, to the germ

$$\Xi_{st} = f(s, X_s)(Z_t^i - Z_s^i).$$

We have that

$$\delta \Xi_{srt} = \Xi_{st} - \Xi_{sr} - \Xi_{rt} = [f(r, X_r) - f(s, X_s)] Z_{rt}^i.$$

To prove the bound on the expectation in the stochastic sewing lemma, we will use the trivial estimate $|f(r,X_r) - f(s,X_s)| \leqslant 2\|f\|_{C_T L^\infty} \lesssim 2\|f\|_{C_T \mathscr{C}^\theta}$ as $\theta > 0$ and the bound on Z from (4.20), such that

$$\mathbb{E}[|\delta\Xi_{srt}|^2] \leqslant \|f\|_{C_T L^\infty}^2 \mathbb{E}[|Z_{rt}^i|^2] \leqslant \|f\|_{C_T \mathscr{C}^\theta}^2 \|Z\|_{\frac{\alpha+\beta}{\alpha},2}^2 |t-s|^{2(\alpha+\beta)/\alpha}$$

with $(\alpha+\beta)/\alpha > 1/2$. For bound on the conditional expectation, we utilize the bounds (4.20) on Z and the time and space regularity of f, such that

$$\begin{aligned}|f(r,X_r)-f(s,X_s)| &\leqslant |f(r,X_r)-f(s,X_r)| + |f(s,X_s)-f(s,X_r)| \\ &\lesssim \|f\|_{C_T^{\theta/\alpha}L^\infty}|r-s|^{\theta/\alpha} + \|f\|_{C_T\mathscr{C}^\theta}|X_r-X_s|^\theta \\ &\lesssim \|f\|_{C_T^{\theta/\alpha}L^\infty\cap C_T\mathscr{C}^\theta}[|r-s|^{\theta/\alpha} + |X_r-X_s|^\theta],\end{aligned}$$

where we used that the norm in $\mathscr{C}^\theta$ is equivalent to the norm in the Hölder space C_b^θ of bounded, θ-Hölder continuous functions for $\theta \in (0,1)$ (cf. [BCD11, Section 2.7, Examples]). Hence, with $|X_r - X_s| \leqslant |Z_r - Z_s| + |L_r - L_s|$, we can estimate

$$\begin{aligned}&\mathbb{E}[|\mathbb{E}_s[\delta\Xi_{srt}]|^2] \\ &\quad= \mathbb{E}[|\mathbb{E}_s[f(r,X_r)-f(s,X_s))\mathbb{E}_r[Z_{rt}^i]]|^2] \\ &\quad\leqslant \mathbb{E}[\mathbb{E}_s[|f(r,X_r)-f(s,X_s)||\mathbb{E}_r[Z_{rt}^i]|]^2] \\ &\quad\leqslant \|\mathbb{E}_r[Z_{rt}^i]\|_{L^\infty(\mathbb{P})}^2 \mathbb{E}[\mathbb{E}_s[|f(r,X_r)-f(s,X_s)|]^2] \\ &\quad\lesssim \|Z \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty}^2 |t-r|^{2(\alpha+\beta)/\alpha} \|f\|_{C_T^{\theta/\alpha}L^\infty\cap C_T\mathscr{C}^\theta}^2 \times \\ &\qquad\qquad \big(|r-s|^{2\theta/\alpha} + \mathbb{E}[|Z_r-Z_s|^{2\theta}] + \mathbb{E}[\mathbb{E}_s[|L_r-L_s|^\theta]^2]\big) \\ &\quad\lesssim_T \|Z \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty}^2 \|Z\|_{\frac{\alpha+\beta}{\alpha},2}^{2\theta} \|f\|_{C_T^{\theta/\alpha}L^\infty\cap C_T\mathscr{C}^\theta}^2 |t-s|^{2(\theta+\alpha+\beta)/\alpha}\end{aligned}$$

In the last estimate above, we used stationarity, scaling and independence of the increment $L_r - L_s$ of $\mathscr{F}_s$ for $s \leqslant r$, such that almost surely

$$\mathbb{E}_s[|L_r-L_s|^\theta] = \mathbb{E}[|L_r-L_s|^\theta] = |r-s|^{\theta/\alpha}\mathbb{E}[|L_1|^\theta] \lesssim |r-s|^{\theta/\alpha}$$

and the expectation is finite due to $\theta < \alpha$. Furthermore, as $\theta \in (0,1)$, we used above Jensen's inequality and the following estimate on Z:

$$\mathbb{E}[|Z_r-Z_s|^{2\theta}] \leqslant \mathbb{E}[|Z_r-Z_s|^2]^\theta \leqslant \|Z\|_{\frac{\alpha+\beta}{\alpha},2}^{2\theta}|r-s|^{2\theta(\alpha+\beta)/\alpha} \lesssim_T \|Z\|_{\frac{\alpha+\beta}{\alpha},2}^{2\theta}|r-s|^{2\theta/\alpha}.$$

Due to $(\theta+\alpha+\beta)/\alpha > 1$, Lemma 4.7 applies and yields the bound for $I(f, Z)$.
In the Young case, $\beta > (1-\alpha)/2$, we can take $f = J^T(\partial_i V^j) \in C_T^{(\alpha+\beta-1)/\alpha}L^\infty \cap C_T\mathscr{C}^{\alpha+\beta-1}$ with $(2\alpha+2\beta-1)/\alpha > 1$, which yields existence of the integral $I(\Xi)_t = \mathbb{A}_{0,t}(i,j)$. Let $I(\Xi^{m,n})$ be the sewing integral with germ

$$\Xi_{sr}^{m,n} = J^T(\partial_i V^{m,j})(s,X_s)(Z_r^{n,i} - Z_s^{n,i})$$

and let $f^m := J^T(\partial_i V^{m,j})$. By the Schauder and interpolation estimates we furthermore

have that $f^m \to f$ in $\mathscr{L}_T^{0,\alpha+\beta-1}$ and in $C_T^{(\alpha+\beta-1)/\alpha} L^\infty \cap C_T \mathscr{C}^{\alpha+\beta-1}$.
Let $\mathbb{Z}^V_{st} = \mathbb{A}_{st} - \Xi_{st} = I(\Xi)_{st} - \Xi_{st}$. Then, Lemma 4.7 yields that $\|\mathbb{Z}^V\|_{\frac{\alpha+\beta}{\alpha},2} < \infty$ and $\|\mathbb{Z}^V \mid \mathscr{F}\|_{\frac{2\alpha+2\beta-1}{\alpha},2} < \infty$. The stability of the sewing integral implies the convergence $\|\mathbb{Z}^{m,n} - \mathbb{Z}^V\|_{\frac{\alpha+\beta}{\alpha},2} \to 0$, since by the estimates above we obtain for $(s,t) \in \Delta_T$ that

$$\begin{aligned}
&\|\mathbb{Z}^{m,n}_{st} - \mathbb{Z}^V_{st}\|_{L^2} \\
&\quad \leqslant \|I(\Xi^{m,n})_{st} - \Xi^{m,n}_{s,t} - (I(\Xi)_{st} - \Xi_{s,t})\|_{L^2} \\
&\quad \lesssim_T |t-s|^{(\alpha+\beta)/\alpha} \Big(\sup_n \|Z^n \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty} \|Z\|^\theta_{\frac{\alpha+\beta}{\alpha},2} \|f^m - f\|_{\mathscr{L}_T^{0,\alpha+\beta-1}} \\
&\qquad\qquad + \|Z^n - Z \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty} \|Z\|^\theta_{\frac{\alpha+\beta}{\alpha},2} \|f\|_{\mathscr{L}_T^{0,\alpha+\beta-1}} \\
&\qquad\qquad + \|f^m - f\|_{\mathscr{L}_T^{0,\alpha+\beta-1}} \sup_n \|Z^n\|_{\frac{\alpha+\beta}{\alpha},2} \\
&\qquad\qquad + \|f\|_{\mathscr{L}_T^{0,\alpha+\beta-1}} \|Z^n - Z\|_{\frac{\alpha+\beta}{\alpha},2} \Big) \\
&\quad \to 0
\end{aligned}$$

for $n, m \to \infty$. Analogously we can show that $\|\mathbb{Z}^{m,n} - \mathbb{Z}^V \mid \mathscr{F}\|_{\frac{2\alpha+2\beta-1}{\alpha},2} \to 0$. □

Let $\mathscr{V} = (V, \mathscr{V}_2) \in \mathscr{X}^{\beta,\gamma}$ and let $(X, L, \mathbb{Z}^V)$ be a weak solution. Then, the proof of Theorem 4.35 below shows the following representations of Z and $\mathbb{A}$ (and thus of $\mathbb{Z}^V$):

$$\mathbb{E}_s[Z^i_{s,t}] = \mathbb{E}_s[u^{t,i}(t,X_t) - u^{t,i}(s,X_s)] \text{ and } \mathbb{E}_s[\mathbb{A}_{s,t}(i,j)] = \mathbb{E}_s[v^{t,i,j}(t,X_t) - v^{t,i,j}(s,X_s)]$$

for the solutions $u^t = (u^{t,i})_{i=1,\dots,d}$, $v^t = (v^{t,i,j})_{i,j=1,\dots,d}$ of the backward PDEs

$$\mathscr{G}^{\mathscr{V}} u^{t,i} = V^i, \quad u^{t,i}(t,\cdot) = 0. \tag{4.25}$$

and

$$\begin{aligned}
\mathscr{G}^{\mathscr{V}} v^{t,i,j} &= J^T(\partial_i V^j) \cdot V^i \\
&= \mathscr{V}_2(i,j) + J^T(\partial_i V^j) \varolessthan V^i + J^T(\partial_i V^j) \circledcirc V^i, \quad v^{t,i,j}(t,\cdot) = 0.
\end{aligned} \tag{4.26}$$

Let for $i \in \{1,\dots,d\}$, $v^{t,i} := (v^{t,i,j})_{j=1,\dots,d}$.
Together with the bound $\|\mathbb{Z}^V \mid \mathscr{F}\|_{\frac{2\alpha+2\beta-1}{\alpha},\infty} < \infty$ from (4.21), this motivates the definition of the class of processes $\mathscr{K}^\vartheta = \mathscr{K}^\vartheta(\mathscr{V})$.

Definition 4.16 (Class $\mathscr{K}^\vartheta$)**.** *Let $\beta \in (\frac{2-2\alpha}{3}, 0)$, $\mathscr{V} \in \mathscr{X}^{\beta,\gamma}$ and $\vartheta \in ((\alpha+\beta)/\alpha, 1]$. Let for $t \in (0,T]$, v^t be the solution of the PDE* (4.26) *and u^t be the solution of* (4.25)*. An adapted càdlàg stochastic process $(X_t)_{t\in[0,T]}$ is said to be of class $\mathscr{K}^\vartheta(\mathscr{V})$ if for all $i = 1,\dots,d$:*

$$\begin{aligned}
&\|\mathbb{E}_s[(v^{t,i}(t,X_t) - v^{t,i}(s,X_s)) - J^T(\partial_i V)(s,X_s)(u^{t,i}(t,X_t) - u^{t,i}(s,X_s))]\|_{(L^\infty(\mathbb{P}))^d} \\
&\quad \lesssim |t-s|^\vartheta.
\end{aligned} \tag{4.27}$$

Remark 4.17. *Using* $v^{t,i,j}, u^{t,i} \in C_t^{(\alpha+\beta)/\alpha} L^\infty$ *with norm uniformly bounded for* $t \in [0,T]$ *by Corollary 3.32, a trivial estimate yields* (4.27) *for* $\vartheta = (\alpha+\beta)/\alpha$. *We are interested in the case* $\vartheta > (\alpha+\beta)/\alpha$.

As an intermediate step in the equivalence proof of the solution concepts we will prove in Proposition 4.33 that the martingale solution for $\mathscr{G}^{\mathscr{V}}$ is a process of class $\mathscr{K}^\vartheta(\mathscr{V})$ for $\vartheta = (2\alpha+2\beta-1)/\alpha$.
Let us now state our second main theorem of this chapter. The proof follows directly from Theorems 4.32 and 4.35 in Section 4.4.

Theorem 4.18. *Let* $\alpha \in (1,2]$, $\beta \in (\frac{2-2\alpha}{3}, 0)$ *and* $V \in \mathscr{X}^{\beta,\gamma}$. *Let* $x \in \mathbb{R}^d$. *Then,* X *is a (rough) weak solution in the sense of Definition 4.10 if and only if* X *solves the* $(\mathscr{G}^{\mathscr{V}}, \delta_x)$*-martingale problem. In particular, (rough) weak solutions are unique in law.*

To prove Theorem 4.18, it turns out that for a weak solution X we need to make sense out of the limit of the stochastic integrals

$$\int_0^t \nabla u^m(s, X_s) \cdot dZ_s = \sum_{i=1}^d \int_0^t \partial_i u^m(s, X_s) dZ_s^i,$$

as $m \to \infty$, where $u^m \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$ with $u^m \to u$ and u solving the Kolmogorov backward equation (for regular data f, u^T). In the rough regime, $\partial_i u$ won't have enough regularity, such that we can define the integral of $\partial_i u(s, X_s)$ against Z in a stable manner utilizing Lemma 4.15. Indeed, a regularity counting argument yields that the time regularity of $\partial_i u$, i.e. $\partial_i u \in C_T^{(\alpha+\beta-1)/\alpha} L^\infty$, together with the $(\alpha+\beta)/\alpha$-Hölder regularity of Z in $L^2(\mathbb{P})$ sum up to $(2\alpha+2\beta-1)/\alpha \leqslant 1$ for $\beta \leqslant (1-\alpha)/2$. The idea is thus to enhance Z by $\mathbb{Z}^V$ from Definition 4.10 in order to correct for the irregular terms in $\partial_i u$ and to define the stable rough stochastic integral

$$\int_0^t \nabla u(s, X_s) \cdot d(Z, \mathbb{Z}^V) = \sum_{i=1}^d \int_0^t \partial_i u(s, X_s) d(Z^i, \mathbb{Z}^{V,i})_s.$$

For regular integrands ∇u^m the stochastic integral against Z and the rough stochastic integral against $(Z, \mathbb{Z}^V)$ coincide. This motivates the general theory in the next section.

4.3. A rough stochastic sewing integral

In this section, we construct in Theorem 4.24 a rough stochastic integral using the stochastic sewing Lemma 4.7. The theory is inspired by [FHL21]. Nonetheless, the results from [FHL21] do not apply in our setting and our rough stochastic integral differs from the one constructed there. Instead of considering a γ-rough paths as an integrator, we consider a lifted stochastic process $(Z, \mathbb{Z}^A)$ with bounds with respect to the semi-norms $\|\cdot\|_{\theta,2}$ and $\|\cdot \mid \mathscr{F}\|_{\theta,\infty}$, cf. (4.19). The Hölder exponent σ of Z is assumed to satisfy $\sigma > 1/2$, since this is the case for the drift term Z in the section above. However, the integrand is rough, but stochastically controlled by a given stochastic process A. We construct the integral in $L^2(\mathbb{P})$, but replacing 2 by $p \geqslant 2$ in the bounds below, we could more generally construct the integral in $L^p(\mathbb{P})$. We will not bother doing so, as we aim for the situation where the integrand is given by a ϑ-Hölder function of the α-stable process for $\vartheta \in (0, \alpha/2)$, which lacks arbitrarily high moments, i.e. $\mathbb{E}[|L_1|^{2\vartheta}] < \infty$ as $2\vartheta < \alpha$, but we may not necessarily have $\mathbb{E}[|L_1|^{p\vartheta}] < \infty$ for $p > 2$. Let us start with the definition of stochastically controlled processes. Let here and below, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space.

Definition 4.19 (Stochastically controlled processes)**.** *Let $(A_t)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued adapted stochastic process with $A \in C_T L^\infty(\mathbb{P}, \mathbb{R}^d)$. We call an adapted stochastic process $(f_t, f'_t)_{t\in[0,T]}$ with $(f, f') \in C_T L^\infty(\mathbb{P}) \times C_T L^\infty(\mathbb{P}, \mathbb{R}^d)$ stochastically controlled by A, if for $\varsigma, \varsigma' \in (0, 1)$ the following bounds hold true*

$$\|f\|_{\varsigma,2} + \|f'\|_{\varsigma',2} < \infty$$

and the remainder $(R^f_{st})_{(s,t)\in\Delta_T}$ defined by

$$R^f_{st} := f_{st} - f'_s \cdot A_{st},$$

satisfies

$$\|R^f\|_{\varsigma+\varsigma',2} < \infty.$$

The remainder is then an adapted process indexed by Δ_T. We denote the space of all such (f, f'), that are stochastically controlled by A by $D^{\varsigma,\varsigma'}_T(A)$ and we define the complete norm

$$\|f - g\|_{D^{\varsigma,\varsigma'}_T(A)} := \sup_{t\in[0,T]} [\|(f - g)_t\|_{L^\infty(\mathbb{P})} + \|(f' - g')_t\|_{L^\infty(\mathbb{P},\mathbb{R}^d)}] + \|f - g\|_{\varsigma,2} + \|f' - g'\|_{\varsigma',2} + \|R^f - R^g\|_{\varsigma+\varsigma',2}.$$

Remark 4.20. *In particular, as $f' \in C_T L^\infty(\mathbb{P}, \mathbb{R}^d)$, the bounds on f and f' imply the following bound on A: $\|A\|_{\varsigma,2} < \infty$.*

Definition 4.21 (Rough stochastic integrator)**.** *Let $(A_t)_{t\in[0,T]}$ be an adapted $\mathbb{R}^d$-valued stochastic process, which satisfies $A \in C_T L^\infty(\mathbb{P}, \mathbb{R}^d)$ and $\|A\|_{\varsigma,2} < \infty$ for $\varsigma \in (0, 1)$. Let $\sigma \in (1/2, 1)$. Then we call $(Z, \mathbb{Z}^A)$ a rough integrator, if $(Z_t)_{t\in[0,T]}$, $(\mathbb{Z}^A_{st})_{(s,t)\in\Delta_T}$ are adapted stochastic processes with Z being $\mathbb{R}$-valued and $\mathbb{Z}^A$ being $\mathbb{R}^d$-valued, such that $Z_0 = 0$ and for all $0 \leqslant s \leqslant l \leqslant t \leqslant T$, the following algebraic relation holds*

$$\mathbb{Z}^A_{st} = \mathbb{Z}^A_{sl} + \mathbb{Z}^A_{lt} + A_{sl} Z_{lt}.$$

Furthermore the following Hölder-type moment bounds hold:

$$\|(Z,\mathbb{Z}^A)\|_{\mathscr{R}^\sigma_T(A)} := \|Z\|_{\sigma,2} + \|Z \mid \mathscr{F}\|_{\sigma,\infty} + \|\mathbb{Z}^A\|_{\sigma,2} + \|\mathbb{Z}^A \mid \mathscr{F}\|_{\sigma+\varsigma,\infty} < \infty.$$

We call the space of stochastic processes, that are rough integrators $\mathscr{R}^\sigma_T(A)$*. Equipped with the norm*

$$\begin{aligned}\|((Z,\mathbb{Z}^A)-(W,\mathbb{W}^A))\|_{\mathscr{R}^\sigma_T(A)} := {}& \|Z-W\|_{\sigma,2} + \|Z-W \mid \mathscr{F}\|_{\sigma,\infty} \\ &+ \|\mathbb{Z}^A - \mathbb{W}^A\|_{\sigma,2} + \|\mathbb{Z}^A - \mathbb{W}^A \mid \mathscr{F}\|_{\sigma+\varsigma,\infty}\end{aligned}$$

the space becomes a Banach space.

Remark 4.22. *We refer to [FH20] for the connection to rough paths. The space of rough stochastic integrators is a vector space, because the algebraic relation here is linear. Furthermore, we assume that* $Z_0 = 0$ *to obtain a norm.*

Remark 4.23. *We can relax the boundedness assumption on the process* A*. Instead we can assume that* A *is such that, there exists* $C > 0$*, such that for all* $0 \leqslant s \leqslant l \leqslant t \leqslant T$*,* $\|A_{sl}Z_{lt}\|_{L^2} \leqslant C|t-s|^\sigma$*. Then, in principle the choice* $A = Z$ *would be possible, because* $\|Z_{sl}Z_{lt}\|^2_{L^2} \leqslant \frac{1}{2}[\|Z_{sl}\|^2_{L^2} + \|Z_{lt}\|^2_{L^2}] \leqslant \|Z\|^2_{\sigma,2}|t-s|^{2\sigma}$*. But if* $A = Z$*, then* $\varsigma = \sigma$ *with* $2\sigma > 1$ *and the assumptions on* $\mathbb{Z}^A$ *are superfluous, as the iterated integrals of* Z *can be sewed due to* $\sigma > 1/2$*. To be precise, Lemma 4.7 would yield the bound* $\|\mathbb{Z}^A\|_{\sigma,2} + \|\mathbb{Z}^A \mid \mathscr{F}\|_{\sigma+\varsigma,2} < \infty$*, but not* $\|\mathbb{Z}^A \mid \mathscr{F}\|_{\sigma+\varsigma,\infty} < \infty$*.*

Theorem 4.24. *Let* $(A_t)_{t\in[0,T]}$ *be an adapted* $\mathbb{R}^d$*-valued stochastic process with* $A \in C_T L^\infty(\mathbb{P},\mathbb{R}^d)$ *and let* f *be stochastically controlled by* A*. Let* $(Z,\mathbb{Z}^A)$ *be a rough integrator. Let the parameters be such that*

$$\sigma + \varsigma' + \varsigma > 1.$$

Then for $t \in (0,T]$*, the rough stochatic sewing integral*

$$I(f,Z)_t := \int_0^t f_s d(Z,\mathbb{Z}^A)_s = \lim_{|\Pi|\to 0} \sum_{r,l\in\Pi} [f_r Z_{rl} + f'_r \cdot \mathbb{Z}^A_{rl}],$$

exists in $L^2(\mathbb{P})$*, where the limit ranges over partitions* Π *of* $[0,t]$ *with mesh size* $|\Pi| \to 0$*. Moreover, the following Lipschitz bound holds:*

$$\sup_{(s,t)\in\Delta_T} \frac{\|I(f,Z)_{s,t}\|_{L^2(\mathbb{P})}}{|t-s|^\sigma} \lesssim \|f\|_{D^{\varsigma,\varsigma'}_T(A)} \|(Z,\mathbb{Z}^A)\|_{\mathscr{R}^\sigma_T(A)}.$$

Furthermore, if $\sigma + \varsigma > 1$*, then the rough stochastic integral* $I(f,Z)_t$ *almost surely agrees with the integral* $\tilde{I}(f,Z)_t = \lim_{|\Pi|\to 0}\sum_{r,l\in\Pi} f_r Z_{rl} \in L^2(\mathbb{P})$*.*

Proof. Define for $r < s$, $\Xi_{rs} := f_r Z_{rs} + f'_r \cdot \mathbb{Z}^A_{rs}$. Then we have that for $r < l < s$, by the algebraic relation of $Z, \mathbb{Z}^A$ and the definition of the remainder $R = R^f$,

$$\delta\Xi_{r,l,s} = \Xi_{rs} - \Xi_{rl} - \Xi_{ls} = -R_{r,l}Z_{l,s} - f'_{r,l} \cdot \mathbb{Z}^A_{l,s}.$$

To apply the stochastic sewing Lemma 4.7, we need to show the bounds on the expectation and conditional expectation. We start with the bound on the expectation. As $\sigma > 1/2$ it suffices to trivially estimate the L^2-norm using

$$\sup_{r,l\in\Delta_T} \|R_{r,l}\|_{L^\infty} \leqslant 2 \sup_{t\in[0,T]} [\|f_t\|_{L^\infty(\mathbb{P})} + \|f'_t\|_{L^\infty(\mathbb{P},\mathbb{R}^d)}\|A_t\|_{L^\infty(\mathbb{P},\mathbb{R}^d)}] < \infty,$$

such that

$$\begin{aligned}\mathbb{E}[|\delta\Xi_{r,l,s}|^2]^{1/2} &\lesssim \sup_{r,l\in\Delta_T} \|R_{r,l}\|_{L^\infty}\|Z_{l,s}\|_{L^2} + \sup_{t\in[0,T]} \|f'_t\|_{L^\infty(\mathbb{P},\mathbb{R}^d)}\|\mathbb{Z}^A_{l,s}\|_{L^2} \\ &\lesssim \|f\|_{D^{\varsigma,\varsigma'}_T(A)}\|(Z,\mathbb{Z}^A)\|_{\mathscr{R}^\sigma_T(A)}|s-r|^\sigma.\end{aligned}$$

For the conditional expectation, we have

$$\begin{aligned}\mathbb{E}[|\mathbb{E}_r[\delta\Xi_{r,l,s}]|^2] &\leqslant \mathbb{E}\big[\mathbb{E}_r[|R_{r,l}\mathbb{E}_l[Z_{l,s}]|]^2\big] + \mathbb{E}\big[\mathbb{E}_r[|f'_{r,l} \cdot \mathbb{E}_l[\mathbb{Z}^A_{l,s}]|]^2\big] \\ &\leqslant \|R_{r,l}\|^2_{L^2(\mathbb{P})}\|\mathbb{E}_l[Z_{l,s}]\|^2_{L^\infty} + \|f'_{r,l}\|^2_{L^2(\mathbb{P},\mathbb{R}^d)}\|\mathbb{E}_l[\mathbb{Z}^A_{l,s}]\|^2_{L^\infty(\mathbb{P},\mathbb{R}^d)} \\ &\leqslant \|f\|_{D^{\varsigma,\varsigma'}_T(A)}\|(Z,\mathbb{Z}^A)\|_{\mathscr{R}^\sigma_T(A)}|s-r|^{2(\varsigma'+\sigma+\varsigma)},\end{aligned}$$

with $\varsigma'+\sigma+\varsigma > 1$ by assumption. Thus Lemma 4.7 applies for the existence of $I(f,Z)_t$, $t \in [0,T]$. The prescribed Lipschitz bound follows from the bound on $\|I(f,Z)_{st} - \Xi_{st}\|_{L^2}$ from Lemma 4.7 and $\|\Xi_{st}\|_{L^2} \leqslant \|f\|_{D^{\varsigma,\varsigma'}_T(A)}[\|Z\|_{\sigma,2} + \|\mathbb{Z}^A\|_{\sigma,2}]|t-s|^\sigma$.

If $\sigma + \varsigma > 1$, then by the uniqueness of the stochastic sewing integral $\tilde{I}(f,Z)_t = \lim_{|\Pi|\to 0}\sum_{r<l\in\Pi} f_r Z_{rl} \in L^2(\mathbb{P})$, we obtain that $\tilde{I}(f,Z)_t = I(f,Z)_t$ almost surely. Indeed, this uses that I satisfies the bounds for the germ $\tilde{\Xi}_{st} = f_s Z_{st}$ of $\tilde{I}$, that is

$$\|I(f,Z)_{st} - f_s Z_{st}\|_{L^2} \leqslant \|I(f,Z)_{st}\|_{L^2} + \|f_s Z_{st}\|_{L^2} \lesssim |t-s|^\sigma,$$

with $\sigma > 1/2$ and using the bound of $I(f,Z)$ from Lemma 4.7,

$$\begin{aligned}\|\mathbb{E}_s[I(f,Z)_{s,t} - f_s Z_{st}]\|_{L^2} &\leqslant \|\mathbb{E}_s[I(f,Z)_{s,t} - f_s Z_{st} - f'_s\mathbb{Z}^A_{st}]\|_{L^2} + \|\mathbb{E}_s[f'_s\mathbb{Z}^A_{st}]\|_{L^2} \\ &\leqslant \|\mathbb{E}_s[I(f,Z)_{s,t} - f_s Z_{st} - f'_s\mathbb{Z}^A_{st}]\|_{L^2} + \|\mathbb{E}_s[f'_s\mathbb{Z}^A_{st}]\|_{L^\infty} \\ &\lesssim |t-s|^{\sigma+\varsigma},\end{aligned}$$

where $\sigma + \varsigma > 1$ by assumption. □

4.4. Equivalence of weak solution concepts

In this section, we prove in Theorems 4.32 and 4.35 that the weak solution concept from Definition 4.10 yields an equivalent notion of solution to the martingale solutions. Furthermore, we show in Theorem 4.36 an extension of Itôs formula for rough weak solutions.
To prove that a weak solution solves the martingale problem we employ the stability of the rough stochastic integral, which extends the integral against Z for regular integrands $f(s, X_s)$, i.e. for $f \in C_b^{1,2}$, to paracontrolled integrands:

$$D_T(V) \times \mathscr{R}_T(V) \ni (u, (Z, \mathbb{Z}^V)) \mapsto \int_0^t \nabla u(s, X_s) \cdot d(Z, \mathbb{Z}^V)_s \in L^2(\mathbb{P}). \tag{4.28}$$

Above we define

$$\mathscr{R}_T(V) := \bigtimes_{i=1}^{d} \mathscr{R}_T^{(\alpha+\beta)/\alpha}(A^i),$$

for $A^i = (J^T(\partial_i V)(t, X_t))_{t\in[0,T]}$ and $\mathscr{R}_T^{\varsigma}(A)$ from Definition 4.21 in Section 4.3. $D_T(V)$ denotes the space of paracontrolled distributions from Corollary 3.28, we recall its definition below.
To obtain stability of the rough integral (4.28), we apply Theorem 4.24 to $f = (\partial_i u(t, X_t))_{t\in[0,T]}$ and the rough integrator $(Z^i, \mathbb{Z}^{V,i})$ that is given by the definition of a weak solution (Definition 4.10), for $i = 1, \dots, d$.
Lemma 4.29 below shows that, if $u \in D_T(V)$, then $(\partial_i u(t, X_t))$ is stochastically controlled by $(A_t^i) = (J^T(\partial_i V)(t, X_t))$ for $\varsigma = \varsigma' = (\alpha + \beta - 1)/\alpha$. The following lemma verifies that $(Z^i, \mathbb{Z}^{V,i})$ is a rough stochastic integrator in the sense of Definition 4.21.

Lemma 4.25. *Let $\beta \in (\frac{2-2\alpha}{3}, 0)$ and $V \in \mathscr{X}^{\beta,\gamma}$. Let $(X, L, \mathbb{Z}^V)$ be a weak solution. Then, $(Z, \mathbb{Z}^V) \in \mathscr{R}_T(V)$, i.e. $(Z^i, \mathbb{Z}^{V,i}) \in \mathscr{R}_T^{(\alpha+\beta)/\alpha}(A^i)$ for $A^i = (J^T(\partial_i V)(t, X_t))_{t\in[0,T]}$ and $i = 1, \dots, d$. Moreover, the following convergence holds for $n, m \to \infty$*

$$\|(Z^n, \mathbb{Z}^{m,n}) - (Z, \mathbb{Z}^V)\|_{\mathscr{R}_T(V)} \to 0.$$

Remark 4.26. *Notice that we use the notation $\|(Z^n, \mathbb{Z}^{m,n}) - (Z, \mathbb{Z}^V)\|_{\mathscr{R}_T(V)}$ despite the fact, that $(Z^n, \mathbb{Z}^{m,n})$ and $(Z, \mathbb{Z}^V)$ do not live in the same space. The notation means that the four semi-norms in Definition 4.21 converge.*

Proof. Let $i = 1, \dots, d$. To prove that $(Z^i, \mathbb{Z}^{V,i})$ is a rough integrator, notice that for $\mathbb{A}^i = \lim_{n,m\to\infty} \mathbb{A}^{m,n,i}$ the following additivity holds: $\mathbb{A}_{st}^i = \mathbb{A}_{sl}^i + \mathbb{A}_{lt}^i$ (using that additivity holds for $\mathbb{A}^{m,n,i}$). Thus for $\mathbb{Z}_{st}^{V,i} = \mathbb{A}_{st}^i - J^T(\partial_i V)(s, X_s) Z_{st}^i$, we obtain the algebraic relation

$$\mathbb{Z}_{st}^{V,i} = \mathbb{Z}_{sl}^{V,i} + \mathbb{Z}_{lt}^{V,i} + [J^T(\partial_i V)(l, X_l) - J^T(\partial_i V)(s, X_s)] Z_{lt}^i.$$

Furthermore, (4.20) and (4.21) yield the bounds on Z^i and $\mathbb{Z}^{V,i}$ and hence $(Z^i, \mathbb{Z}^{V,i}) \in$

$\mathscr{R}_T(V^i)$. We have that $\mathbb{Z}^{m,n,i}_{st} = \mathbb{A}^{m,n,i}_{st} - J^T(\partial_i V^m)(s, X_s)Z^{n,i}_{st}$. Due to bounds (4.20) and (4.21) and the convergence in 2.), it follows that $(Z^{n,i}, \mathbb{Z}^{m,n,i}) \in \mathscr{R}_T(V^{m,i}) := \mathscr{R}^{(\alpha+\beta)/\alpha}_T((J^T(\partial_i V^m)(t, X_t))_t)$. The convergence with respect to $\|\cdot\|_{\mathscr{R}_T(V)}$ follows directly from the convergence in 2.). □

Remark 4.27. *For a stochastic process Z satisfying the bounds* (4.20)*, there is in general no unique choice for $\mathbb{Z}^A$, such that $(Z, \mathbb{Z}^A)$ is a rough integrator. But if Z is given by a weak solution $(X, L, \mathbb{Z}^V)$, i.e. given by $Z = X - x - L$, and $A = (J^T(\partial_i V)(t, X_t))_{t\in[0,T]}$, then there is a unique choice for $\mathbb{Z}^V$ such that $(Z, \mathbb{Z}^V) \in \mathscr{R}_T(V)$. This follows from the fact, that for each fixed $n \in \mathbb{N}$ and $A^n = (J^T(\partial_i V^n)(t, X_t))_{t\in[0,T]}$, there is a unique choice for $\mathbb{Z}^{A^n}$ such that $(Z, \mathbb{Z}^{A^n}) \in \mathscr{R}_T(V^n)$ given by $\mathbb{Z}^{A^n}_{st} = \mathbb{Z}^{n,\infty}_{st} = \mathbb{A}^{n,\infty}_{st} - J^T(\partial_i V^n)(s, X_s)Z_{st}$, where (V^n) is the smooth sequence from Definition 4.10. By the convergence in 2.), it follows that $\mathbb{Z}^{n,\infty}_{st} \to \mathbb{Z}_{st}$ in $L^2(\mathbb{P})$, which yields that the unique choice $\mathbb{Z}^V$ is given by $\mathbb{Z}^V_{st} = \mathbb{A}_{st} - J^T(\partial_i V)(s, X_s)Z_{st}$. This is the same story as for geometric rough paths.*

To prove that $(\partial_i u(t, X_t))_t$ is stochastically controlled by $A = (J^T(\partial_i V)(t, X_t))_t$, we will need the following auxillary lemma, which is also of independent interest. Its proof only relies on regularity properties of the solution of the Kolmogorov backward equation. The lemma proves the bound (4.29) on the time-space differences of the paracontrolled remainder. The proof of the lemma can be found in Appendix A.

Lemma 4.28. *Let $\alpha \in (1, 2]$, $\beta \in (\frac{2-2\alpha}{3}, 0)$, $\mathscr{V} \in \mathscr{X}^{\beta,\gamma}$ and*

$$u \in D_T(V) = \{(u, u') \in \mathscr{L}^{0,\alpha+\beta}_T \times \mathscr{L}^{0,\alpha+\beta-1}_T \mid u^\sharp := u - u' \otimes J^T(V) \in \mathscr{L}^{0,2(\alpha+\beta)-1}_T\}.$$

Then, we have the following time-space Hölder bound:

$$\begin{aligned} &|\partial_i u(r, x) - \partial_i u(s, y) - \nabla u(s, y) \cdot (J^T(\partial_i V)(r, x) - J^T(\partial_i V)(s, y))| \\ &\quad \lesssim \|u\|_{D_T(V)}(1 + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma}})[|r - s|^{(2(\alpha+\beta)-2)/\alpha} + |x - y|^{2(\alpha+\beta)-2}], \end{aligned} \tag{4.29}$$

for $i = 1, \ldots, d$.

Lemma 4.29. *Let $\beta \in ((2 - 2\alpha)/3, (1 - \alpha)/2]$, $\alpha \in (1, 2]$. Let $u \in D_T(V)$ and let $(X, L, \mathbb{Z}^V)$ be a weak solution. Then, for every $i = 1, \ldots, d$, $(\partial_i u(t, X_t))_{t\in[0,T]}$ is stochastically controlled by $(J^T(\partial_i V)(t, X_t))_{t\in[0,T]}$ with $\varsigma = \varsigma' = (\alpha + \beta - 1)/\alpha$.*

Remark 4.30. *In the pure stable noise case, $\alpha \in (1, 2)$, the proof of Lemma 4.29 does not apply for $\beta \in ((1 - \alpha)/2, 0)$ (i.e. in the Young case), while for $\alpha = 2$, the statement of the lemma is also valid in the Young regime. The reason is the integrability issue for the α-stable process. Indeed, in the proof we need that $4(\alpha + \beta) - 4 < \alpha$, that is $\beta < (4 - 3\alpha)/4$. If $\alpha < 2$ and $\beta \leqslant (1 - \alpha)/2$, then in particular $\beta < (4 - 3\alpha)/4$. However, the latter doesn't need to be satisfied in the Young regime, unless $\alpha < 4/3$.*

Proof of Lemma 4.29. By Lemma 4.28, we obtain that

$$\begin{aligned}|R_{st}| &:= |\partial_i u(t,X_t) - \partial_i u(s,X_s) - \nabla u_s \cdot [J^T(\partial_i V)(t,X_t) - J^T(\partial_i V)(s,X_s)]| \\ &\lesssim \|u\|_{D_T}(1+\|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma}})[|t-s|^{(2(\alpha+\beta)-2)/\alpha} + |X_t - X_s|^{2(\alpha+\beta)-2}]. \end{aligned} \tag{4.30}$$

Using the triangle inequality and $u \in \mathscr{L}_T^{0,\alpha+\beta}$ as well as Lemma 3.17, we can bound

$$\begin{aligned}&\mathbb{E}[|\partial_i u(t,X_t) - \partial_i u(s,X_s)|^2] \\ &\quad\lesssim \|\partial_i u\|^2_{C_T^{(\alpha+\beta-1)/\alpha}L^\infty}|t-s|^{2(\alpha+\beta-1)/\alpha} + \|\partial_i u\|^2_{C_T\mathscr{C}^{\alpha+\beta-1}}\mathbb{E}[|X_t - X_s|^{2(\alpha+\beta-1)}] \\ &\quad\lesssim_T \|u\|^2_{D_T(V)}|t-s|^{2(\alpha+\beta-1)/\alpha},\end{aligned}$$

and the same bound is also valid for ∇u. In the last estimate above, we used that due to the bound (4.20) on Z in L^2,

$$\begin{aligned}&\mathbb{E}[|X_t - X_s|^{2(\alpha+\beta-1)}] \\ &\quad\leqslant \mathbb{E}[|Z_t - Z_s|^{2(\alpha+\beta-1)}] + \mathbb{E}[|L_t - L_s|^{2(\alpha+\beta-1)}] \\ &\quad\leqslant \mathbb{E}[|Z_t - Z_s|^2]^{(\alpha+\beta-1)} + \mathbb{E}[|L_t - L_s|^{2(\alpha+\beta-1)}] \\ &\quad\leqslant \|Z\|^{2(\alpha+\beta-1)}_{\frac{\alpha+\beta}{\alpha},2}|t-s|^{2(\alpha+\beta)(\alpha+\beta-1)/\alpha} + |t-s|^{2(\alpha+\beta-1)/\alpha}\mathbb{E}[|L_1|^{2(\alpha+\beta-1)}] \\ &\quad\lesssim_T |t-s|^{2(\alpha+\beta-1)/\alpha},\end{aligned}$$

due to Jensen's inequality with $\alpha+\beta-1 \in (0,1)$ and due to $(L_t - L_s) \overset{d}{=} (t-s)^{1/\alpha}L_1$ for $s \leqslant t$, and $2(\alpha+\beta-1) < \alpha$, such that $\mathbb{E}[|L_1|^{2(\alpha+\beta-1)}] < \infty$. For the remainder, we employ the bound (4.30) to obtain

$$\begin{aligned}\mathbb{E}[|R_{st}|^2] &\lesssim \|u\|^2_{D_T}\big(|t-s|^{(4(\alpha+\beta)-4)/\alpha} + \mathbb{E}[|X_t - X_s|^{4(\alpha+\beta)-4}]\big) \\ &\lesssim_T \|u\|^2_{D_T}|t-s|^{4\varsigma/\alpha},\end{aligned}$$

using an analogoue argument to estimate $\mathbb{E}[|X_t - X_s|^{4(\alpha+\beta)-4}]$ as above, where now $\mathbb{E}[|L_1|^{4(\alpha+\beta)-4}] < \infty$ since $4(\alpha+\beta)-4 < \alpha$ in the case of $\alpha < 2$, as $\beta \leqslant (1-\alpha)/2$. In the case of $\alpha = 2$, we have all moments on the Brownian motion $B_1 = L_1$. Thus, we obtain that $(\partial_i u(t,X_t))_t$ is stochastically controlled by $(J^T(\partial_i V)(t,X_t))_t$ with $\varsigma = \varsigma' = (\alpha+\beta-1)/\alpha$ and $(\partial_i u)' = \nabla u$. □

Proposition 4.31. *Let $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$ and $V \in \mathscr{X}^{\beta,\gamma}$. Let $(X, L, \mathbb{Z}^V)$ be a weak solution.*
Then for all $0 \leqslant t \leqslant T$ and for $(Z, \mathbb{Z}^V) \in \mathscr{R}_T(V)$ given by Definition 4.10 and for $u \in D_T(V)$, the rough stochastic integral

$$\int_0^t \nabla u(s,X_s)\cdot d(Z,\mathbb{Z}^V)_s := \sum_{i=1}^d \lim_{|\Pi|\to 0}\sum_{r,l\in\Pi}[\partial_i u(r,X_r)Z^i_{rl} + \nabla u(r,X_r)\cdot \mathbb{Z}^{V,i}_{rl}] \in L^2(\mathbb{P}),$$

where the limit ranges over all partitions Π *of* $[0,t] \subset [0,T]$ *with mesh-size* $|\Pi| := \max_{r,s\in\Pi}|r-s| \to 0$*, is well-defined and allows for the bound*

$$\left\| \int_s^t \nabla u(r, X_r) \cdot d(Z, \mathbb{Z}^V)_r \right\|_{L^2(\mathbb{P})} \lesssim_T |t-s|^{(\alpha+\beta)/\alpha} \|u\|_{D_T(V)} \|(Z, \mathbb{Z}^V)\|_{\mathscr{R}_T(V)}.$$

In particular, for the sequence (V^n) *from Definition 4.10 and* $(Z^n, \mathbb{Z}^{m,n}), (Z, \mathbb{Z}^{m,\infty}) \in \mathscr{R}_T^\alpha(V^m)$ *and for a sequence* $(u^m)_m \subset D_T(V^m)$ *with* $\|u^m - u\|_{D_T(V)} \to 0$*, it follows, that almost surely*

$$\int_0^t \nabla u^m(r, X_r) \cdot d(Z^n, \mathbb{Z}^{m,n})_r = \int_0^t \nabla u^m(r, X_r) \cdot dZ_r^n,$$
$$\int_0^t \nabla u^m(r, X_r) \cdot d(Z, \mathbb{Z}^{m,\infty})_r = \int_0^t \nabla u^m(r, X_r) \cdot dZ_r,$$

where the stochastic integrals on the right-hand side are defined as the $L^2(\mathbb{P})$*-limit of the classical Riemann sums. And the following convergence in* $L^2(\mathbb{P})$*, uniformly in* $t \in [0,T]$*, holds for* $n, m \to \infty$*, respectively* $m \to \infty$*,*

$$\int_0^t \nabla u^m(r, X_r) \cdot d(Z^n, \mathbb{Z}^{m,n})_r \to \int_0^t \nabla u(r, X_r) \cdot d(Z, \mathbb{Z}^V)_r,$$
$$\int_0^t \nabla u^m(r, X_r) \cdot d(Z, \mathbb{Z}^{m,\infty})_r \to \int_0^t \nabla u(r, X_r) \cdot d(Z, \mathbb{Z}^V)_r.$$

Proof. Recall that u^m is paracontrolled by $J^T(V^m)$ (i.e. $u^m \in D_T(V^m)$),

$$\mathbb{Z}_{st}^{m,n,i} = \int_s^t [J^T(\partial_i V^m)(r, X_r) - J^T(\partial_i V^m)(s, X_s)] dZ_r^{n,i} \quad \text{and}$$
$$\mathbb{Z}_{st}^{m,\infty,i} = \int_s^t [J^T(\partial_i V^m)(r, X_r) - J^T(\partial_i V^m)(s, X_s)] dZ_r^{i}.$$

Then the proof follows from Theorem 4.24, Lemma 4.29 and Lemma 4.25. □

Proposition 4.31 extendes the stochastic integral to the stable rough stochastic integral, which finally enables to prove the following theorem.

Theorem 4.32. *Let* $\mathscr{V} \in \mathscr{X}^{\beta,\gamma}$ *and* $\beta \in ((2-2\alpha)/3, 0)$*. Let* $(X, L, \mathbb{Z}^V)$ *be a weak solution, starting at* $x \in \mathbb{R}^d$*. Then* X *solves the martingale problem for the generator* $\mathscr{G}^{\mathscr{V}}$*, starting at* $x \in \mathbb{R}^d$*.*

Proof. Let $f \in C_T\mathscr{C}^\varepsilon$, $\varepsilon > 2 - \alpha$, $u^T \in \mathscr{C}^3$ and u be the solution of $\mathscr{G}^{\mathscr{V}} u = f$, $u(T, \cdot) = u^T$. Let $(X, L, \mathbb{Z}^V)$ be a weak solution starting at x on the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$. Then the goal is to show that the process $(M_t)_{t\in[0,T]}$ with

$$M_t = u(t, X_t) - u(0, x) - \int_0^t f(s, X_s) ds \tag{4.31}$$

is a martingale with respect to $(\mathscr{F}_t^X) \subset (\mathscr{F}_t)$ under $\mathbb{P}$, where $\mathscr{F}_t^X := \sigma(X_s \mid s \leqslant t)$ is the canonical filtration. Because X is a weak solution, there exists a sequence $(V^n) \subset C_T(C_b^\infty)^d$ satisfying $(V^n, \mathscr{K}(V^m, V^n)) \to \mathscr{V}$ in $\mathscr{X}^{\beta,\gamma}$ and such that the convergences from 1.), 2.) from Definition 4.10 hold. Consider the solution u^n of $\mathscr{G}^{V^n} u^n = f$, $u^n(T, \cdot) = u^T$, which converges to u in $D_T(V)$ by the continuity of the solution map from Theorem 3.30. Then as $u^n \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$ and X is a Dirichlet process, we can apply the Itô formula from [CJMS06, Theorem 3.1] to $u^n(t, X_t)$ (see also Remark 4.12), such that for $n \in \mathbb{N}$,

$$\begin{aligned} u^n(t, X_t) &- u^n(0, x) \\ &= \int_0^t (\partial_t - \mathscr{L}_\nu^\alpha) u^n(s, X_s) ds + \int_0^t \nabla u^n(s, X_s) \cdot dZ_s + M_t^n \\ &= \int_0^t f(s, X_s) ds + M_t^n \\ &\quad + \Big(\int_0^t \nabla u^n(s, X_s) \cdot dZ_s - \int_0^t \nabla u^n(s, X_s) \cdot V^n(s, X_s) ds \Big), \end{aligned} \tag{4.32}$$

where we furthermore used the equation for u^n and abbreviate the martingale, in the case $\alpha \in (1,2)$, by

$$M_t^n := \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (u^n(s, X_{s-} + y) - u^n(s, X_{s-})) \hat{\pi}(ds, dy). \tag{4.33}$$

In the Brownian noise case, $\alpha = 2$, we have that $M_t^n = \int_0^t \nabla u^n(s, X_s) \cdot dB_s$. It follows that M^n is a $(\mathscr{F}_t)$-martingale (cf. the argument in the proof of Theorem 4.2) and it is $(\mathscr{F}_t^X)$-adapted by (4.31). Thus M^n is a $(\mathscr{F}_t^X)$-martingale. We claim that $(M^n)_n$ converges in $L^2(\mathbb{P})$, uniformly in $t \in [0,T]$, to a martingale $\tilde{M}$ given by the expression (4.33) for u^n replaced by u. Indeed, this uses the Burkholder-Davis-Gundy-type inequality from [PZ07, Lemma 8.21] and an analogoue argument as in the proof of Lemma 4.4.

Since $u^n \to u$ in $C_T L^\infty$, we thus obtain that the process M in (4.31) is a martingale with $M = \tilde{M}$, provided that the remainder in (4.32) vanishes in $L^2(\mathbb{P})$. That is, for $Z^n = \int_0^\cdot V^n(s, X_s) ds$, we claim that when $n \to \infty$,

$$\sup_{t \in [0,T]} \Big\| \int_0^t \nabla u^n(s, X_{s-}) \cdot dZ_t^n - \int_0^t \nabla u^n(s, X_{s-}) \cdot dZ_s \Big\|_{L^2(\mathbb{P})} \to 0. \tag{4.34}$$

The stochastic integrals are given by the limit of the classical Riemann sums in $L^2(\mathbb{P})$. If $\beta > (1-\alpha)/2$ (i.e. in the Young case), the convergence follows from the stability of the stochastic sewing integral from Lemma 4.15 applied for $f = \partial_i u$ and using only that $\|Z^n - Z\|_{\frac{\alpha+\beta}{\alpha},2} \to 0$, $\|Z^n - Z \mid \mathscr{F}\|_{\frac{\alpha+\beta}{\alpha},\infty} \to 0$ and $u^n \to u$ in $D_T(V)$. Thus, in the Young case, the remainder vanishes.

If $\beta \in ((2-2\alpha)/3, (1-\alpha)/2]$, then an application of Proposition 4.31 yields (4.34). □

In what follows, we prove the reverse implication: a martingale solution is a (rough) weak solution in the sense of Definition 4.10. The first step is to prove that a martingale solution is of class $\mathscr{K}^{\vartheta}$ for $\vartheta = (2(\alpha+\beta)-1)/\alpha$, cf. Definition 4.16. With that, the bound on $\mathbb{Z}^V$ in (4.21) follows after identifying Z and $\mathbb{A}$ with the respective solutions u^t, v^t of the backward PDEs in (4.25), (4.26). The latter will be established in the following proposition.

Proposition 4.33. *Let $V \in \mathscr{X}^{\beta,\gamma}$ for $\beta \in ((2-2\alpha)/3, 0)$. Let X be the solution of the martingale problem for the generator $\mathscr{G}^{\mathscr{V}}$, starting at $x \in \mathbb{R}^d$. Then X is of class $\mathscr{K}^{\vartheta}(\mathscr{V})$ (cf. Definition 4.16) for $\vartheta = (2(\alpha+\beta)-1)/\alpha$.*

Remark 4.34. *For a martingale solution X, we prove below that*

$$\begin{aligned}\mathbb{Z}^{V,i}_{st} &:= \int_s^t [J^T(\partial_i V)(r, X_r) - J^T(\partial_i V)(s, X_s)] dZ^i_s \\ &= v^{t,i}(t, X_t) - v^{t,i}(s, X_s) - J^T(\partial_i V)(s, X_s)(u^{t,i}(t, X_t) - u^{t,i}(s, X_s)) + M^i_{st} \quad (4.35)\end{aligned}$$

for a martingale differences M^i_{st} and the solutions $v^{t,i}, u^{t,i}$ from (4.26), (4.25)*. Then, Proposition 4.33 shows $\|\mathbb{Z}^V \mid \mathscr{F}\|_{\vartheta,\infty} < \infty$ for $\vartheta = (2(\alpha+\beta)-1)/\alpha$. It remains open, if one can prove the bound $\|\mathbb{Z}^V\|_{\vartheta,2} < \infty$, which is a stronger bound than $\|\mathbb{Z}^V\|_{(\alpha+\beta)/\alpha,2} < \infty$ (the latter can be inferred from the time regularity of v^t, u^t).*
It is not straightforward to adjust the proof of Proposition 4.33 to show $\|\mathbb{Z}^V\|_{\vartheta,2} < \infty$. Indeed, the proof exploits the conditional expectation in two ways. On the one hand, it removes the martingale differences, for which the estimates would be more involved. On the other hand, the conditional expectation enables to use the Markov property of X and thus to transform the problem into a question on Schauder and commutator estimates for the semigroup $(T_{s,r})_{s\leqslant r}$ of X. In the proof below, we infer those estimates on $(T_{s,r})_{s\leqslant r}$ from regularity properties of the solutions of the generator PDE with singular terminal conditons from Chapter 3.

Proof of Proposition 4.33. Let $0 \leqslant s < t \leqslant T$. By Theorem 4.18 the martingale solution X for $\mathscr{G}^{\mathscr{V}}$ is a strong Markov process. Replacing $v^{t,i}$ by the solution $v^{n,i}$ of $\mathscr{G}^{(V,\mathscr{V}_2)} v^{n,i} = J^T(\partial_i V) \cdot V^{n,i}$, $v^{n,i}(t,\cdot) = 0$ and $u^{t,i}$ by $u^{n,i}$ with $\mathscr{G}^{(V,\mathscr{V}_2)} u^{n,i} = V^{n,i}$, $u^{n,i}(t,\cdot) = 0$ and rewriting the conditional expectation in (4.27) with the semigroup $(T_{r,l})_{0\leqslant r\leqslant l\leqslant T}$ of the (time-inhomogeneous) Markov process X, we obtain

$$\begin{aligned}&\mathbb{E}_s[v^{t,i}(t, X_t) - v^{t,i}(s, X_s) - J^T(\partial_i V)(s, X_s)(u^{t,i}(t, X_t) - u^{t,i}(s, X_s))] \\ &\quad = \lim_{n\to\infty} \mathbb{E}_s[v^{n,i}(t, X_t) - v^{n,i}(s, X_s) - J^T(\partial_i V)(s, X_s)(u^{n,i}(t, X_t) - u^{n,i}(s, X_s))] \quad (4.36) \\ &\quad = \lim_{n\to\infty} \mathbb{E}_s\left[\int_s^t (J^T(\partial_i V)(r, X_r) - J^T(\partial_i V)(s, X_s)) \cdot V^{n,i}(r, X_r) dr\right] \quad (4.37) \\ &\quad = \lim_{n\to\infty} \int_s^t T_{s,r}(J^T(\partial_i V) \cdot V^{n,i})_r(X_s) - J^T(\partial_i V)(s, X_s) T_{s,r}(V^{n,i}_r)(X_s) dr\end{aligned}$$

$$= \lim_{n\to\infty} \int_s^t T_{s,r}(J^T(\partial_i V)\cdot V^{n,i})_r(X_s) - J^T(\partial_i V)(r,X_s)T_{s,r}(V_r^{n,i})(X_s)dr$$
$$+ \lim_{n\to\infty} \int_s^t (J^T(\partial_i V)(s,X_s) - J^T(\partial_i V)(r,X_s))T_{s,r}V_r^{n,i}(X_s)dr$$
$$= \int_s^t w_s^{r,i}(X_s) - J^T(\partial_i V)(r,X_s)y_s^{r,i}(X_s)dr \tag{4.38}$$
$$+ \int_s^t (J^T(\partial_i V)(s,X_s) - J^T(\partial_i V)(r,X_s))y_s^{r,i}(X_s)dr. \tag{4.39}$$

The convergence above is in $L^\infty(\mathbb{P})$, uniformly in s,t. The convergence in (4.36) follows from $\sup_{t\in[0,T]}\|v^{n,i}-v^{t,i}\|_{C_tL^\infty_{\mathbb{R}^d}} \to 0$ and $\sup_{t\in[0,T]}\|u^{n,i}-u^{t,i}\|_{C_tL^\infty} \to 0$ by Corollary 3.32 (taking $f^{t,n} = V^{n,i}_{|[0,t]}$ and $y^{n,t} = u^{n,i}(t,\cdot) = 0$). For the equality (4.37), we used that X solves the martingale problem for $\mathscr{G}^{\mathscr{V}}$, such that

$$u^{n,i}(t,X_t) - u^{n,i}(s,X_s) = \int_s^t V^{n,i}(r,X_r)dr + M^{n,i}_{r,t},$$

for martingale differences $M^{n,i}_{r,t}$, that vanishes after taking the conditional expectation, analogously for $v^{n,i}$. Moreover, y^r, w^r are defined as follows. Let

$$T_{s,r}V_r^{n,i} =: y_s^{r,n,i}, \quad T_{s,r}(J^T(\partial_i V)\cdot V_r^{n,i}) =: w_s^{r,n,i}, \quad s\in[0,r]. \tag{4.40}$$

Then we have that $y^{r,n,i}$ solves $\mathscr{G}^{\mathscr{V}} y^{r,n,i} = 0$ with singular terminal condition $y_r^{n,r} = V_r^{n,i}$ at time r and $w^{r,n,i} = (w^{r,n,i,j})_{j=1,\dots,d}$ solves $\mathscr{G}^{\mathscr{V}} w^{n,r,i,j} = 0$ with $w_r^{r,n,i,j} = J^T(\partial_i V^j)_r\cdot V_r^{n,i}$. By the continuity of the PDE solution map (Theorem 3.30), we conclude that

$$(y^{r,n,i}, y^{r,n,i,\sharp}) \to (y^{r,i}, y^{r,i,\sharp}),$$
$$(w^{r,n,i,j}, w^{r,n,i,j,\sharp}) \to (w^{r,i,j}, w^{r,i,j,\sharp}) \in \mathscr{L}_r^{\gamma,\alpha+\beta}\times\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1},$$

for $n\to\infty$, where $y^{r,i}, w^{r,i}$ solve $\mathscr{G}^{\mathscr{V}} y^{r,i} = 0$, $\mathscr{G}^{\mathscr{V}} w^{r,i,j} = 0$ with terminal conditions $y_r^r = V_r^i$, respectively $w_r^{r,i,j} = J^T(\partial_i V^j)_r\cdot V_r^i$, and $\gamma\in(0,1)$. The convergence is uniform in $r\in[0,T]$ by Corollary 3.32. In particular, it follows that $y^{r,n,i}\to y^{r,i}$ and $w^{r,n,i,j}\to w^{r,i,j}$ in $\mathscr{M}_r^\gamma L^\infty$, uniformly in $r\in[0,T]$, which implies the convergence of the integrals (4.38) and (4.39) in $L^\infty(\mathbb{P})$, uniformly in $0\leqslant s<t\leqslant T$.
Our goal is now to estimate both integrals (4.38) and (4.39) in $L^\infty(\mathbb{P})$ by $|t-s|^\vartheta$.
To estimate the integral (4.38), we apply again Corollary 3.32 for the solutions $(y^{r,i})$, $(w^{r,i,j})$. The uniform bound from the corollary together with the interpolation bound (3.27) from Lemma 3.17 for $\tilde{\theta} = \alpha+\beta\gamma^{-1}$ (note $\beta(1-\gamma^{-1})>0$) , $\theta=\alpha+\beta$ and the embedding $\mathscr{C}^{(1-\gamma')\alpha}\hookrightarrow L^\infty$ yield that

$$\sup_{r\in[0,T]}\|y^{r,i}\|_{\mathscr{M}_r^{-\beta/\alpha}L^\infty} \lesssim \sup_{r\in[0,T]}\|y^{r,i}\|_{\mathscr{M}_r^{-\beta/\alpha}\mathscr{C}^{\beta(1-\gamma^{-1})}} \lesssim \sup_{r\in[0,T]}\|y^{r,i}\|_{\mathscr{L}_r^{\gamma,\alpha+\beta}} \lesssim_{T,\mathscr{V}} 1.$$

Thus, we obtain

$$\begin{aligned}
&\left\|\int_s^t (J^T(\partial_i V)_s - J^T(\partial_i V)_r) y_s^{r,i} dr\right\|_{L^\infty} \\
&\quad\lesssim \sup_{r\in[0,T]} \|y^{r,i}\|_{\mathscr{M}_r^{-\beta/\alpha}L^\infty} \|J^T(\partial_i V)\|_{C_T^{(\alpha+\beta-1)/\alpha}L^\infty_{\mathbb{R}^d}} \int_s^t |r-s|^{(\alpha+2\beta-1)/\alpha} dr \\
&\quad\lesssim_T \|V\|_{C_T\mathscr{C}^\beta_{\mathbb{R}^d}} |t-s|^{(2\alpha+2\beta-1)/\alpha} \\
&\quad= \|V\|_{C_T\mathscr{C}^\beta_{\mathbb{R}^d}} |t-s|^{\vartheta},
\end{aligned}$$

using that $2\alpha+2\beta-1>0$ and $J^T(\partial_i V)\in C_T^{(\alpha+\beta-1)/\alpha}L^\infty_{\mathbb{R}^d}$.
To estimate the term in (4.39), we use cancellations between the solution $w^{r,i}$ and $y^{r,i}$. For the argument, we need to distinguish between the cases $\beta\in((1-\alpha)/2,0)$ and $\beta\in((2-2\alpha)/3,(1-\alpha)/2]$.

First, we consider the Young case, $\beta\in((1-\alpha)/2,0)$. Then, we have that $\alpha+2\beta-1>0$. We can write the difference of the solutions as follows:

$$\begin{aligned}
&w_s^{r,i} - J^T(\partial_i V)_r y_s^{r,i} \\
&\quad= [P_{r-s}(J^T(\partial_i V)_r\cdot V_r^i) - J^T(\partial_i V)_r\cdot P_{r-s}V_r^i] \\
&\qquad + J_s^r(\nabla w^{r,i}\cdot V) - J_s^r(\nabla y^{r,i}\cdot V) \\
&\quad= (P_{r-s}-\mathrm{Id})(J^T(\partial_i V)_r \odot V_r^i + J^T(\partial_i V)_r \varogreaterthan V_r^i) && (4.41)\\
&\qquad - J^T(\partial_i V)_r \odot (P_{r-s}-\mathrm{Id})V_r^i + J^T(\partial_i V)_r \varogreaterthan (P_{r-s}-\mathrm{Id})V_r^i && (4.42)\\
&\qquad + [P_{r-s}(J^T(\partial_i V)_r \varolessthan V_r^i) - J^T(\partial_i V)_r \varolessthan P_{r-s}V_r^i] && (4.43)\\
&\qquad + J_s^r(\nabla w^{r,i}\cdot V) - J^T(\partial_i V)_r J_s^r(\nabla y^{r,i}\cdot V). && (4.44)
\end{aligned}$$

Above and below we use the notation $J_r^t(v) := J^t(v)_r = J^t(v)(r,\cdot)$ for $r\leqslant t$. The term in (4.41), we estimate with the semigroup estimate (3.10) from Lemma 3.7, using that $\alpha+2\beta-1\in(0,1)$,

$$\begin{aligned}
&\|(P_{r-s}-\mathrm{Id})(J^T(\partial_i V)_r\odot V_r^i + J^T(\partial_i V)_r \varogreaterthan V_r^i)\|_{L^\infty_{\mathbb{R}^d}} \\
&\quad\lesssim |r-s|^{(\alpha+2\beta-1)/\alpha} \|J^T(\partial_i V)_r\odot V_r^i + J^T(\partial_i V)_r\varogreaterthan V_r^i)\|_{\mathscr{C}^{\alpha+2\beta-1}_{\mathbb{R}^d}} \\
&\quad\lesssim |r-s|^{(\alpha+2\beta-1)/\alpha} \|V\|^2_{C_T\mathscr{C}^\beta_{\mathbb{R}^d}}.
\end{aligned}$$

The terms in (4.42), we also estimate with the semigroup and Schauder estimates using that $V\in C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d$ for $\gamma'\in[(1-\beta)/\alpha,1)$ and again that $\alpha+2\beta-1\in(0,1]$

and the embedding $\mathscr{C}^{(1-\gamma')\alpha} \hookrightarrow L^\infty$,

$$\begin{aligned}
&\|J^T(\partial_i V)_r \varogreaterthan (P_{r-s} - \mathrm{Id})V_r^i\|_{L^\infty} \\
&\quad\lesssim \|J^T(\partial_i V)_r \varogreaterthan (P_{r-s} - \mathrm{Id})V_r^i\|_{\mathscr{C}^{(1-\gamma')\alpha}_{\mathbb{R}^d}} \\
&\quad\lesssim \|J^T(\partial_i V)_r\|_{\mathscr{C}^{(2-\gamma')\alpha+\beta-1}_{\mathbb{R}^d}} \|(P_{r-s} - \mathrm{Id})V_r^i\|_{\mathscr{C}^{-(\alpha+\beta-1)}} \\
&\quad\lesssim \|V\|_{C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d} |r-s|^{(\alpha+2\beta-1)/\alpha} \|V\|_{C_T(\mathscr{C}^\beta)^d}.
\end{aligned}$$

The argument for the $\odot$-product is analogous. Moreover, the term in (4.43) equals the semigroup commutator from Lemma 3.9 and using that $\alpha + 2\beta - 1 < \alpha$, we obtain

$$\begin{aligned}
&\|[P_{r-s}(J^T(\partial_i V)_r \varolessthan V_r^i) - J^T(\partial_i V)_r \varolessthan P_{r-s}V_r^i]\|_{L^\infty} \\
&\quad\lesssim \|[P_{r-s}(J^T(\partial_i V)_r \varolessthan V_r^i) - J^T(\partial_i V)_r \varolessthan P_{r-s}V_r^i]\|_{\mathscr{C}^{(1-\gamma')\alpha}_{\mathbb{R}^d}} \\
&\quad\lesssim |r-s|^{(\alpha+2\beta-1)/\alpha} \|V\|_{C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d} \|V\|_{C_T(\mathscr{C}^\beta)^d}.
\end{aligned}$$

For the last term (4.44), we give the argument for the term involving $y^{r,i}$ and the argument for the term with $w^{r,i}$ is analogue. We decompose

$$\begin{aligned}
J_s^r(\nabla y^{r,i} \cdot V) &= J_s^r(\nabla y^{r,i} \odot V + \nabla y^{r,i} \varogreaterthan V) + \nabla y_s^{r,i} \varolessthan J_s^r(V) \\
&\quad + [J_s^r(\nabla y^{r,i} \varolessthan V) - \nabla y_s^{r,i} \varolessthan J_s^r(V)].
\end{aligned}$$

Due to the interpolation bound (3.27), we obtain that $\nabla y^{r,i} \in \mathscr{M}_r^{(1-\beta)/\alpha} \mathscr{C}^{(1-\beta)(\frac{\alpha}{\gamma'}-1)}_{\mathbb{R}^d} \hookrightarrow \mathscr{M}_r^{(1-\beta)/\alpha} L^\infty_{\mathbb{R}^d}$ as $\gamma' < 1 \leqslant \alpha$. This yields together with Corollary 3.12 that

$$\begin{aligned}
\|\nabla y_s^{r,i} \varolessthan J_s^r(V)\|_{L^\infty_{\mathbb{R}^d}} &\lesssim \|\nabla y^{r,i}\|_{\mathscr{M}_r^{(1-\beta)/\alpha} L^\infty_{\mathbb{R}^d}} |r-s|^{(\beta-1)/\alpha} \|J_s^r(V)\|_{L^\infty_{\mathbb{R}^d}} \\
&\lesssim \|\nabla y^{r,i}\|_{\mathscr{M}_r^{(1-\beta)/\alpha} L^\infty_{\mathbb{R}^d}} |r-s|^{(\beta-1)/\alpha} \|J_s^r(V)\|_{\mathscr{C}^{(1-\gamma')\alpha}_{\mathbb{R}^d}} \\
&\lesssim |r-s|^{(\alpha+2\beta-1)/\alpha} \|\nabla y^{r,i}\|_{\mathscr{M}_r^{(1-\beta)/\alpha} L^\infty_{\mathbb{R}^d}} \|V\|_{C_T \mathscr{C}^{\beta+(1-\gamma')\alpha}_{\mathbb{R}^d}}. \qquad (4.45)
\end{aligned}$$

Corollary 3.12 furthermore yields that $J^r(\nabla y^{r,i} \odot V) \in \mathscr{L}_r^{\gamma, 2\beta+\alpha-1+(2-\gamma')\alpha}$ using that $V \in C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d$, which implies by the interpolation bound (3.28) and $J_r^r(v) = 0$ that $J^r(\nabla y^{r,i} \odot V) \in C_r^{(2\beta+\alpha-1+(2-\gamma')\alpha-\gamma\alpha)/\alpha} L^\infty$ (and analogously for the $\varogreaterthan$-product). Thus, as $J_r^r(v) = 0$ and $2 - \gamma' - \gamma > 0$ as $\gamma \in (0,1)$, $\gamma' \in (0,1)$, we obtain that

$$\|J_s^r(\nabla y^{r,i} \odot V + \nabla y^{r,i} \varogreaterthan V)\|_{L^\infty} \lesssim |r-s|^{(\alpha+2\beta-1)/\alpha} \|y^{r,i}\|_{\mathscr{L}_r^{\gamma,\alpha+\beta}} \|V\|_{C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d}.$$

Due to the commutator Lemma 3.14, we obtain that

$$[J^r(\nabla y^{r,i} \varolessthan V) - \nabla y^{r,i} \varolessthan J^r(V)] \in \mathscr{L}_r^{\gamma, 2\beta+\alpha-1+(2-\gamma')\alpha}$$

and by an interpolation argument as above we thus find

$$\|J^r_s(\nabla y^{r,i} \varolessthan V) - \nabla y^{r,i}_s \varolessthan J^r_s(V)\|_{L^\infty} \lesssim |r-s|^{(\alpha+2\beta-1)/\alpha}\|y^{r,i}\|_{\mathscr{L}^{\gamma,\alpha+\beta}_r}\|V\|_{C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d}.$$

Together, we obtain for the term (4.44):

$$\|J^T(\partial_i V)_r J^r_s(\nabla y^{r,i}\cdot V)\|_{L^\infty} \lesssim |r-s|^{(\alpha+2\beta-1)/\alpha}\|y^r\|_{\mathscr{L}^{\gamma,\alpha+\beta}_r}\|V\|^2_{C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d}.$$

Together with the uniform bound of the norm of y^r, w^r from Corollary 3.32 the above estimates yield

$$\int_s^t \|w^{r,i}_s - J^T(\partial_i V)_r y^{r,i}_s\|_{L^\infty_{\mathbb{R}^d}} dr \lesssim |t-s|^{(2\alpha+2\beta-1)/\alpha},$$

which, together with the estimate for the integral (4.38) yields the claim in the Young case.

If $\beta \in ((2-2\alpha)/3, (1-\alpha)/2]$, we need to estimate the integral (4.39) differently. In order to rewrite the difference of the solutions $w^{r,i}_s - J^T(\partial_i V)y^{r,i}_s$ in a way so that we can prove the claimed bound, we first specify the paracontrolled structure of y^r, w^r. The terminal condition of $w^{r,i}$ has the paracontrolled structure

$$\begin{aligned} w^{r,i,j}_r = J^T(\partial_i V^j)_r \cdot V^i_r &= J^T(\partial_i V^j)_r \odot V^i_r + J^T(\partial_i V^j)_r \varogreaterthan V^i_r + J^T(\partial_i V^j)_r \varolessthan V^i_r \\ &=: w^{R,\sharp,j} + w^{R,\prime,j} \varolessthan V_r \end{aligned}$$

with $w^{R,\sharp,j} \in \mathscr{C}^{(2-\gamma')\alpha+2\beta-1}$ (due to $V \in C_T\mathscr{C}^{\beta+(1-\gamma')\alpha}_{\mathbb{R}^d}$) and $w^{R,\prime,j} = J^T(\partial_i V^j)_r e_i \in \mathscr{C}^{\alpha+\beta-1}_{\mathbb{R}^d}$ (e_i denoting the i-th unit vector). By Theorem 3.25, the solution has the following paracontrolled structure

$$w^{r,i,j}_s = w^{r,\sharp,i,j}_s + \nabla w^{r,i,j}_s \varolessthan J^r(V)_s + J^T(\partial_i V^j)_r e_i \varolessthan P_{r-s}V_r \tag{4.46}$$

with $w^{r,\sharp,i,j} \in \mathscr{L}^{\gamma,2(\alpha+\beta)-1}_r$. Then, again by the interpolation estimate (3.27) applied for $\theta = \tilde{\theta} = 2\alpha+2\beta-1 \in (0,\alpha)$, since $\beta \leqslant (1-\alpha)/2$, and for $\theta = \tilde{\theta} = \alpha+\beta-1 \in (0,\alpha)$ we obtain together with Corollary 3.32 the uniform bound

$$\begin{aligned} &\sup_{r\in[0,T]} [\|w^{r,\sharp,i}\|_{\mathscr{M}^{(1-\alpha-2\beta)/\alpha}_r L^\infty_{\mathbb{R}^d}} + \|\nabla w^{r,i}\|_{\mathscr{M}^{(1-\beta)/\alpha}_r L^\infty_{\mathbb{R}^{d\times d}}}] \\ &\quad\lesssim \sup_{r\in[0,T]} [\|w^{r,\sharp,i}\|_{(\mathscr{L}^{\gamma,2(\alpha+\beta)-1}_r)^d} + \|w^{r,i}\|_{(\mathscr{L}^{\gamma,\alpha+\beta}_r)^d}] \lesssim_{T,\mathscr{V}} 1. \end{aligned} \tag{4.47}$$

Thus, using an estimate as for (4.45), we can estimate

$$\|\nabla w^{r,i,j}_s \varolessthan J^r(V)_s\|_{L^\infty} \lesssim |r-s|^{(\alpha+2\beta-1)/\alpha}\|V\|_{C_T\mathscr{C}^{\beta+(1-\gamma')\alpha}_{\mathbb{R}^d}} \sup_{r\in[0,T]}\|\nabla w^{r,i}\|_{\mathscr{M}^{(1-\beta)/\alpha}_r L^\infty_{\mathbb{R}^{d\times d}}}. \tag{4.48}$$

Furthermore, (4.47) implies that

$$\int_s^t \|w_s^{r,\sharp,i,j}\|_{L^\infty} dr \lesssim \sup_{r\in[0,T]} \|w^{r,\sharp,i}\|_{\mathscr{M}_r^{(1-\alpha-2\beta)/\alpha} L^\infty_{\mathbb{R}^d}} \int_s^t |r-s|^{(2\beta+\alpha-1)/\alpha} dr$$
$$\lesssim |t-s|^{(2\alpha+2\beta-1)/\alpha} \sup_{r\in[0,T]} \|w^{r,\sharp,i}\|_{\mathscr{M}_r^{(1-\alpha-2\beta)/\alpha} L^\infty_{\mathbb{R}^d}}, \tag{4.49}$$

since $(2\alpha+2\beta-1)/\alpha > 0$.
Moreover, for the solution $y^{r,i}$ from above, we have the paracontrolled structure

$$y_s^{r,i} = y_s^{r,\sharp,i} + e_i \varolessthan P_{r-s} V_r + \nabla y_s^{r,i} \varolessthan J^r(V)_s.$$

The bounds (4.48), (4.49) hold analogously for $\nabla w^{i,j}$, $w^{r,\sharp,i,j}$ replaced by $\nabla y^{r,i}$, $y^{r,i,\sharp}$. Furthermore, the interpolation estimate (3.27) (again for $\theta = \alpha+\beta$, $\tilde{\theta} = 2\alpha+2\beta-1 \in (0,\alpha)$ as $\beta \leqslant (1-\alpha)/2$) yields that

$$\sup_{r\in[0,T]} [\|y^{r,i}\|_{\mathscr{M}_r^{(1-\alpha-2\beta)/\alpha}\mathscr{C}^{-(\alpha+\beta-1)}} + \|y^{r,\sharp,i}\|_{\mathscr{M}_r^{(1-\alpha-2\beta)/\alpha} L^\infty}] \lesssim_{T,\mathscr{V}} 1.$$

With the latter bound, we can estimate,

$$\begin{aligned}
&\|J^T(\partial_i V)_r \odot y_s^{r,i} + J^T(\partial_i V)_r \varogreaterthan y_s^{r,i}\|_{L^\infty_{\mathbb{R}^d}} \\
&\quad\lesssim \|J^T(\partial_i V)_r \odot y_s^{r,i} + J^T(\partial_i V)_r \varogreaterthan y_s^{r,i}\|_{\mathscr{C}^{(1-\gamma)\alpha}_{\mathbb{R}^d}} \\
&\quad\lesssim \|J^T(\partial_i V)_r\|_{(\mathscr{C}^{\alpha+\beta+(1-\gamma')\alpha-1})^d} \|y_s^{r,i}\|_{\mathscr{C}^{-(\alpha+\beta-1)}} \\
&\quad\lesssim |r-s|^{(2\beta+\alpha-1)/\alpha} \|y^{r,i}\|_{\mathscr{M}_r^{(1-\alpha-2\beta)/\alpha}\mathscr{C}^{-(\alpha+\beta-1)}} \|V\|_{C_T(\mathscr{C}^{\beta+(1-\gamma')\alpha})^d}.
\end{aligned}$$

Using the paracontrolled structure of the solutions $w^{r,i,j}, y^{r,i}$, we obtain for $j = 1, \ldots, d$,

$$\begin{aligned}
&w_s^{r,i,j} - J^T(\partial_i V^j)_r \varolessthan y_s^{r,i} \\
&\quad= w_s^{r,\sharp,i,j} + J^T(\partial_i V^j)_r \varolessthan y_s^{r,\sharp,i} + J^T(\partial_i V^j)_r \varolessthan (\nabla y^{r,i} \varolessthan J^r(V))_s \\
&\qquad + (\nabla w^{r,i,j} \varolessthan J^r(V))_s.
\end{aligned} \tag{4.50}$$

Finally, we can estimate the integral (4.39) using the bounds derived above and (4.50):

$$\begin{aligned}
&\Big\|\Big(\int_s^t (w_s^{r,i,j} - J^T(\partial_i V^j)_r \cdot y_s^{r,i}) dr\Big)_j\Big\|_{L^\infty_{\mathbb{R}^d}} \\
&\quad\leqslant \Big\|\Big(\int_s^t (w_s^{r,i,j} - J^T(\partial_i V^j)_r \varolessthan y_s^{r,i}) dr\Big)_j\Big\|_{L^\infty_{\mathbb{R}^d}} \\
&\qquad + \Big\|\Big(\int_s^t [J^T(\partial_i V^j)_r \odot y_s^{r,i} + J^T(\partial_i V^j)_r \varogreaterthan y_s^{r,i}] dr\Big)_j\Big\|_{L^\infty_{\mathbb{R}^d}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left\| \int_s^t w_s^{r,\sharp,i} dr \right\|_{L^\infty_{\mathbb{R}^d}} + \| J^T(\partial_i V)\|_{L^\infty_{\mathbb{R}^d}} \left\| \int_s^t y_s^{r,\sharp,i} dr \right\|_{L^\infty} \\
&\qquad + \| J^T(\partial_i V)\|_{L^\infty_{\mathbb{R}^d}} \left\| \int_s^t (\nabla y^{r,i} \circledcirc J^r(V))_s dr \right\|_{L^\infty} \\
&\qquad + \left\| \left(\int_s^t (\nabla w^{r,i,j} \circledcirc J^r(V))_s dr \right)_j \right\|_{L^\infty_{\mathbb{R}^d}} \\
&\qquad + \left\| \left(\int_s^t [J^T(\partial_i V^j)_r \odot y_s^{r,i} + J^T(\partial_i V^j)_r \circledcirc y_s^{r,i}] dr \right)_j \right\|_{L^\infty_{\mathbb{R}^d}} \\
&\lesssim |t-s|^\vartheta \sup_{r\in[0,T]} \Big[\|w^{r,i}\|_{(\mathscr{L}_r^{\gamma,\alpha+\beta})^d} + \|w^{r,i,\sharp}\|_{(\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1})^d} \\
&\qquad\qquad + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}} (\|y^{r,i}\|_{\mathscr{L}_r^{\gamma,\alpha+\beta}} + \|y^{r,i,\sharp}\|_{\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1}}) \Big]. \qquad (4.51)
\end{aligned}
$$

In the last estimate, we used also that $\vartheta = (2\alpha + 2\beta - 1)/\alpha > 0$. Together with the bound for (4.38), this yields the claim in the rough case. □

Theorem 4.35. *Let $\mathscr{V} \in \mathscr{X}^{\beta,\gamma}$ for $\beta \in ((2-2\alpha)/3, 0)$. Let X be the solution of the martingale problem for the generator $\mathscr{G}^{\mathscr{V}}$, starting at $x \in \mathbb{R}^d$.*
Then, there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ and an α-stable symmtric non-degenerate $(\mathscr{F}_t)$-Lévy process L, such that $(X, L, \mathbb{Z}^V)$ is a weak solution starting at $x \in \mathbb{R}^d$. Furthermore, the following representations for $Z = X - x - L$ and $\mathbb{Z}^V$ follow for $(s,t) \in \Delta_T$

$$
\begin{aligned}
Z_{st} &= \mathbb{E}_s[u^t(t, X_t) - u^t(s, X_s)], \\
\mathbb{Z}_{st}^{V,i} &= \mathbb{E}_s[v^{t,i}(t, X_t) - v^{t,i}(s, X_s) - J^T(\partial_i V)(s, X_s)(u^{t,i}(t, X_t) - u^{t,i}(s, X_s))]
\end{aligned}
$$

almost surely, for the solutions $v^{t,i}, u^{t,i}$ from (4.26), (4.25).

Proof. Let $(V^n) \subset C_T C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ with $(V^n, \mathscr{K}(V^n, V^m)) \to \mathscr{V}$ in $\mathscr{X}^{\beta,\gamma}$ for $n, m \to \infty$ (existence by Assumption 4.9). Let X be the solution of the $(\mathscr{G}^{\mathscr{V}}, \delta_x)$-martingale problem. Let, as in the proof of Theorem 4.2, X^n be the strong solution of

$$
X^n = x + \int_0^\cdot V^n(s, X_s^n) ds + L =: x + Z^{n,n} + L
$$

and let for $l, m, n \in \mathbb{N}$,

$$
\begin{aligned}
Z_t^{m,n} &:= \int_0^t V^m(s, X_s^n) ds, \quad t \in [0,T] \\
\mathbb{Z}_{st}^{l,m,n} &:= \left(\int_s^t [J^T(\partial_i V^{j,l})(r, X_r^n) - J^T(\partial_i V^{j,l})(s, X_s^n)] dZ_r^{i,m,n} \right)_{i,j}, \quad (s,t) \in \Delta_T.
\end{aligned}
$$

Theorem 4.2 proves the distributional convergence $(X^n, Z^{n,n}) \Rightarrow (X, Z)$, where Z is a continuous process. Let $u^{t,m,n} = (u^{t,m,n,i})_{i=1,\dots,d}$ and $v^{t,l,m,n} = (v^{t,l,m,n,i,j})_{i,j=1,\dots,d}$ solve

$$\mathscr{G}^{V^n} u^{t,m,n,i} = V^{m,i}, \quad \mathscr{G}^{V^n} v^{t,l,m,n,i,j} = J^T(\partial_i V^{l,j}) \cdot V^{m,i}$$

with zero terminal condition at time $t \in [0,T]$. By convergence of the mixed resonant products, i.e. $(V^n, \mathscr{K}(V^n, V^m)) \to \mathscr{V}$ in $\mathscr{X}^{\beta,\gamma}$, and by continuity of the PDE solution map (Theorem 3.30), we obtain that $u^{t,m,n,i} \to u^{t,i}$ and $v^{t,l,m,n,i,j} \to v^{t,i,j}$ in D_t, where u^t, v^t solve the PDEs (4.25), (4.26). An application of Itô's formula (cf. in the proof of Theorem 4.2) then yields the representations

$$Z^{m,n}_{st} = u^{t,m,n}(t, X^n_t) - u^{t,m,n}(s, X^n_s) + M^{u^{t,m,n}}_{st} \tag{4.52}$$

for the martingale differences $M^{u^{t,m,n}}_{st}$ defined as in (4.10) with u^n replaced by $u^{t,m,n}$ (respectively $M^{u^{t,m,n}}_{st} := \int_s^t \nabla u^{t,m,n}(r, X^n_r) dB_r$ in the case $\alpha = 2$) and

$$\begin{aligned}\mathbb{Z}^{l,m,n}_{st}(i,j) = {} & v^{t,l,m,n,i,j}(t, X^n_t) - v^{t,l,m,n,i,j}(s, X^n_s) \\ & - J^T(\partial_i V^{j,l})(s, X^n_s)\big(u^{t,m,n,i}(t, X^n_t) - u^{t,m,n,i}(s, X^n_s)\big) \\ & + M^{l,m,n}_{st}(i,j)\end{aligned} \tag{4.53}$$

for the martingale differences defined by

$$M^{l,m,n}_{st}(i,j) := M^{v^{t,l,m,n}}_{st} - J^T(\partial_i V^{j,l})(s, X^n_s) M^{u^{t,m,n}}_{st}. \tag{4.54}$$

Notice that $\mathbb{E}_s[M^{l,m,n}_{st}] = 0$. By convergence of the PDE solutions, we obtain that for all large enough l, m, n,

$$\sup_{t \in [0,T]} [\|u^{t,m,n}\|_{D^d_t} + \|v^{t,l,m,n}\|_{D^{d\times d}_t}] \leqslant 2 \sup_{t \in [0,T]} [\|u^t\|_{D^d_t} + \|v^t\|_{D^{d\times d}_t}] < \infty, \tag{4.55}$$

where the right-hand side is finite due to Corollary 3.32.
By (4.55), $J^T(\partial_i V^j) \in C_T L^\infty$ and Lemma 4.4 applied to both $M^{u^{t,m,n}}_{st}, M^{v^{t,l,m,n}}_{st}$ we can estimate $\|(s,t) \mapsto M^{l,m,n}_{st}\|_{(\alpha+\beta)/\alpha,p}$ for $p \in 2\mathbb{N}$. Together with Corollary 4.5 we thus obtain, for any $p \in 2\mathbb{N}$ the following uniform bound for large enough l, n, m:

$$\sup_{l,m,n} [\|Z^{m,n}\|_{(\alpha+\beta)/\alpha,p} + \|\mathbb{Z}^{l,m,n}\|_{(\alpha+\beta)/\alpha,p}] < \infty. \tag{4.56}$$

Kolmogorov's continuity criterion, (4.56) yields tightness of the laws of the processes $(Z^{m,n}, \mathbb{Z}^{l,m,n})_{l,m,n}$ on $C(\Delta_T, \mathbb{R}^{d+d\times d})$. Furthermore, the uniform bound

$$\sup_{m,n} \|Z^{m,n} \mid \mathscr{F}\|_{(\alpha+\beta)/\alpha,\infty} < \infty \tag{4.57}$$

follows from the representation (4.52) and $u^{t,m,n}(t, X^n_t) = 0 = u^{t,m,n}(t, X^n_s)$ and thus for m, n large enough,

$$\begin{aligned}\|\mathbb{E}_s[Z^{m,n}_{st}]\|_{L^\infty(\mathbb{P})} &\leqslant \sup_{t\in[0,T]} \|u^{t,m,n}\|_{C_t^{(\alpha+\beta)/\alpha} L^\infty_{\mathbb{R}^d}} |t-s|^{(\alpha+\beta)/\alpha} \\ &\leqslant 2 \sup_{t\in[0,T]} \|u^t\|_{(\mathscr{L}_t^{0,\alpha+\beta})^d} |t-s|^{(\alpha+\beta)/\alpha}\end{aligned}$$

and the $\mathscr{L}_t^{0,\alpha+\beta}$-norm of u^t is bounded in $t \in [0,T]$ due to Corollary 3.32. By Proposition 4.33, X is of class $\mathscr{K}^\vartheta$. The bound from that proposition applied to the solutions $u^{t,m,n}, v^{t,l,m,n}$ yields together with (4.55), that for large enough l, m, n

$$\sup_{l,m,n} \|\mathbb{Z}^{l,m,n} \mid \mathscr{F}\|_{(2(\alpha+\beta)-1)/\alpha,\infty} < \infty. \tag{4.58}$$

By tightness of $(Z^{m,n}, \mathbb{Z}^{l,m,n})_{l,m,n}$, uniqueness of the limit, since X is the unique solution of the martingale problem (cf. Theorem 4.2), and continuity of $(Z, \mathbb{Z}^V)$ we can thus deduce the distributional convergence

$$(Z^{m,n}, \mathbb{Z}^{l,m,n}, L) \overset{l,m,n\to\infty}{\Longrightarrow} (Z, \mathbb{Z}^V, L) \qquad \text{in} \quad C(\Delta_T, \mathbb{R}^{d+d\times d}) \times D([0,T], \mathbb{R}^d).$$

Hence, we obtain the distributional convergence

$$(X^n, Z^{m,n}, \mathbb{Z}^{l,m,n}) \overset{l,m,n\to\infty}{\Longrightarrow} (X, Z, \mathbb{Z}^V) \qquad \text{in} \quad D([0,T], \mathbb{R}^d) \times C(\Delta_T, \mathbb{R}^{d+d\times d})$$

as $X^n = x + Z^{n,n} + L$, i.e. X^n is given by a continuous map of $(Z^{n,n}, L)$ (using [JS03, Proposition VI.1.23]).

An application of Skorokhods representation theorem (cf. [Bil99, Theorem 6.7]) then yields that there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and random variables

$$(Y^n, W^{m,n}, \mathbb{W}^{l,m,n})_{l,m,n} \quad \text{and} \quad (Y, W, \mathbb{W})$$

on $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathrm{Law}((Y^n, W^{m,n}, \mathbb{W}^{l,m,n})) = \mathrm{Law}((X^n, Z^{m,n}, \mathbb{Z}^{l,m,n}))$ for $l, m, n \in \mathbb{N}$ and $\mathrm{Law}((Y, W, \mathbb{W})) = \mathrm{Law}((X, Z, \mathbb{Z}^V))$, such that the convergence

$$(Y^n, W^{m,n}, \mathbb{W}^{l,m,n}) \overset{l,m,n\to\infty}{\longrightarrow} (Y, W, \mathbb{W}) \tag{4.59}$$

holds almost surely with respect to the topology on $D([0,T], \mathbb{R}^d) \times C(\Delta_T, \mathbb{R}^{d+d\times d})$ (i.e. J_1-topology on the Skorokhod space and uniform topology on the space of continuous functions on Δ_T).

We define the filtration as the completion of the canonical filtration of Y, $(\mathscr{F}_t) := (\mathscr{F}_t^Y) \subset \mathscr{F}$. The filtration is right-continuous, since Y is càdlàg and by construction complete. It follows that $L := Y - x - W$ is an α-stable symmetric non-degenerate $(\mathscr{F}_t^{(Y,W)})$-Lévy process, because $\mathrm{Law}(X, Z) = \mathrm{Law}(Y, W)$. Below, we show that W is the almost sure limit of $(\mathscr{F}_t^Y)$-adapted processes $(W^m)_m$. This then implies, using also completeness of the filtration, that L is $(\mathscr{F}_t^Y)$-measurable and thus also a $(\mathscr{F}_t^Y)$-Lévy process.

Moreover, we have that $0 = Z^{m,n} - \int_0^{\cdot} V^m(r, X_r^n)dr \overset{d}{=} W^{m,n} - \int_0^{\cdot} V^m(r, Y_r^n)dr$. This implies that $W^{m,n} = \int_0^{\cdot} V^m(r, Y_r^n)dr$ almost surely. Analogously, we deduce the representation $\mathbb{W}_{st}^{l,m,n}(i,j) = \int_s^t [J^T(\partial_i V^{j,l})(r, Y_r^n) - J^T(\partial_i V^{j,l})(s, Y_s^n)]dW_r^{i,m,n}$. Let W^m, $\mathbb{W}^{l,m}$ be defined analogously with Y^n replaced by Y. This yields the represenations (4.52), (4.53) for $W^{m,n}, \mathbb{W}^{l,m,n}$. By almost sure convergence of (4.59), letting first $n \to \infty$, we obtain that for $(s,t) \in \Delta_T$,

$$W_{st} = u^t(t, Y_t) - u^t(s, Y_s) + M_{st}^{u^t},$$
$$W_{st}^m = u^{t,m}(t, Y_t) - u^{t,m}(s, Y_s) + M_{st}^{u^{t,m}},$$

and

$$\begin{aligned}\mathbb{W}_{st}(i,j) = v^{t,i,j}(t, Y_t) - v^{t,i,j}(s, Y_s) \\ - J^T(\partial_i V^{j,l})(s, Y_s)\big(u^{t,i}(t, Y_t) - u^{t,i}(s, Y_s)\big) \\ + M_{st}^{v^{t,i,j}} - J^T(\partial_i V^{j,l})(s, Y_s) M_{st}^{u^{t,i}},\end{aligned}$$

and

$$\begin{aligned}\mathbb{W}_{st}^{l,m}(i,j) = v^{t,l,m,i,j}(t, Y_t) - v^{t,l,m,i,j}(s, Y_s) \\ - J^T(\partial_i V^{j,l})(s, Y_s)\big(u^{t,m,i}(t, Y_t) - u^{t,m,i}(s, Y_s)\big) \\ + M_{st}^{v^{t,l,m,i,j}} - J^T(\partial_i V^{j,l})(s, Y_s) M_{st}^{u^{t,m,i}}.\end{aligned}$$

Herein, we used that convergence in the J_1-topology implies in particular convergence Lebesgue almost everywhere in $[0,T]$ and that Y almost surely does not jump at fixed times t (cf. in the proof of Theorem 4.2). The differences $M_{st}^{u^t}, M_{st}^{u^{t,m}}, M_{st}^{v^t}, M_{st}^{v^{t,l,m}}$ are defined analogously as above and $u^{t,m} = (u^{t,m,i})_{i=1,\dots,d}$ and $v^{t,l,m} = (v^{t,l,m,i,j})_{i,j=1,\dots,d}$ solve

$$\mathscr{G}^{\mathscr{V}} u^{t,m,i} = V^{m,i}, \quad \mathscr{G}^{\mathscr{V}} v^{t,l,m,i,j} = J^T(\partial_i V^{l,j}) \cdot V^{m,i}$$

with zero terminal condition at $t \in [0,T]$.

It remains to prove that for $l, m \to \infty$

$$\|W^m - W\|_{(\alpha+\beta)/\alpha,2} + \|W^m - W \mid \mathscr{F}\|_{(\alpha+\beta)/\alpha,\infty} \to 0 \tag{4.60}$$

and

$$\|\mathbb{W}^{l,m} - \mathbb{W}\|_{(\alpha+\beta)/\alpha,2} + \|\mathbb{W}^{l,m} - \mathbb{W} \mid \mathscr{F}\|_{(2(\alpha+\beta)-1)/\alpha,\infty} \to 0. \tag{4.61}$$

The bounds (4.20), (4.21) for the limits $W, \mathbb{W}$ then follow from the convergences (4.60) and (4.61) and the uniform bounds (4.56), (4.57) and (4.58) above.

The proof of Corollary 4.5 shows the bound

$$\|W^m - W\|_{(\alpha+\beta)/\alpha,2} + \|W^m - W \mid \mathscr{F}\|_{(\alpha+\beta)/\alpha,\infty} \lesssim \sup_{t\in[0,T]} \|u^{t,m} - u^t\|_{(\mathscr{L}_t^{0,\alpha+\beta})^d}.$$

The right-hand side vanishes if $m \to \infty$ by the uniform Lipschitz continuity from Corollary 3.32 (cf. also Remark 3.34). Together this yields (4.60). Analogously, we can argue for the convergence of $\|\mathbb{W}^{l,m} - \mathbb{W}\|_{(\alpha+\beta)/\alpha,2}$, using Corollary 3.32 and that

$$\|\mathbb{W}^{l,m} - \mathbb{W}\|_{(\alpha+\beta)/\alpha,2} \lesssim \sup_{t\in[0,T]} [\|v^{t,l,m} - v^t\|_{(\mathscr{L}_t^{0,\alpha+\beta})^{d\times d}} + \|V\|_{C_T\mathscr{C}^\beta_{\mathbb{R}^d}} \|u^{t,m} - u^t\|_{(\mathscr{L}_t^{0,\alpha+\beta})^d}].$$

Moreover, the estimate (4.51) from the proof of Proposition 4.33 yields the bound

$$\begin{aligned}
&\|\mathbb{W}^{l,m} - \mathbb{W} \mid \mathscr{F}\|_{(2(\alpha+\beta)-1)/\alpha,\infty} \\
&\quad\lesssim \sup_{r\in[0,T]} \Big[\|w^{r,l,m} - w^r\|_{(\mathscr{L}_r^{\gamma,\alpha+\beta})^{d\times d}} + \|w^{r,l,m,\sharp} - w^{r,\sharp}\|_{(\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1})^{d\times d}} \\
&\qquad\qquad + \|\mathscr{V}\|_{\mathscr{X}^{\beta,\gamma'}}(\|y^{r,m} - y^r\|_{(\mathscr{L}_r^{\gamma,\alpha+\beta})^d} + \|y^{r,m} - y^r\|_{(\mathscr{L}_r^{\gamma,2(\alpha+\beta)-1})^d})\Big],
\end{aligned}$$

where $w^r = (w^{r,i,j})_{i,j}$ denotes the solution of $\mathscr{G}^{\mathscr{V}} w^{r,i,j} = 0$ with terminal condition $w_r^{r,i,j} = J^T(\partial_i V^j)_r \cdot V_r^i$ at time r and $y^r = (w^{r,i})_i$ denotes the solution of $\mathscr{G}^{\mathscr{V}} y^{r,i} = 0$ with $y_r^{r,i} = V_r^i$. The right-hand side vanishes by the uniform Lipschitz bound from Corollary 3.32. Thus, together (4.61) follows. Finally, we can of course rename $(Y, W, \mathbb{W})$ as $(X, Z, \mathbb{Z}^V)$. $\square$

The following theorem generalizes Itô's formula for rough weak solutions X.

Theorem 4.36. *Let $\mathscr{V} \in \mathscr{X}^{\beta,\gamma}$ for $\beta \in (\frac{2-2\alpha}{3}, 0)$ and let $(X, L, \mathbb{Z}^V)$ be a weak solution started in $x \in \mathbb{R}^d$. Let $u \in D_T$ be such that $v := (\partial_t - \mathscr{L}_\nu^\alpha)u$ is paracontrolled by V in the sense that $v = v^\sharp + v' \otimes V$ with $v^\sharp \in C_T\mathscr{C}^{(2-\gamma)\alpha+2\beta-1}$ and $v' \in C_T\mathscr{C}^{\alpha+\beta-1}_{\mathbb{R}^d}$ and such that $u \in C^1([0,T], \mathscr{C}^\beta)$ (i.e. continuously differentiable in time with values in $\mathscr{C}^\beta$). Then, if $\alpha \in (1,2)$, the following Itô-formula holds true:*

$$\begin{aligned}
u(t, X_t) = u(0, x) &+ \int_0^t (\partial_s - \mathscr{L}_\nu^\alpha)u(s, X_s)ds + \int_0^t \nabla u(s, X_s) \cdot d(Z, \mathbb{Z}^V)_s \\
&+ \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [u(s, X_{s-} + y) - u(s, X_{s-})]\hat{\pi}(ds, dy),
\end{aligned}$$

where $Z = X - x - L$ and $\hat{\pi}$ denotes the compensated Poisson random measure of L. If $\alpha = 2$ and $L = B$ for a Brownian motion B, the martingale is replaced by $\int_0^t \nabla u(s, X_s) \cdot dB_s$. In the formula above, $\int_0^t (\partial_s - \mathscr{L}_\nu^\alpha)u(s, X_s)ds$ is defined as the limit of $\big(\int_0^t (\partial_s - \mathscr{L}_\nu^\alpha)u^n(s, X_s)ds\big)_n$ in $L^2(\mathbb{P})$ for the smooth mollifications $(u^n = P_{n^{-1}}u)_n$.

Remark 4.37. *The assumptions on u are satisfied for solutions u of Kolmogorov backward equations with regular terminal conditions $u^T \in \mathscr{C}^{2\alpha+2\beta-1}$ and right-hand sides $f \in C_T L^\infty$ (cf. also Remark 3.13).*

Proof of Theorem 4.36. We give the proof in the case $\alpha \in (1,2)$, $\alpha = 2$ is similar. Since $u \in C^1([0,T], \mathscr{C}^\beta)$ the mollification satisfies for $n \in \mathbb{N}$,

$$u_t^n = P_{n^{-1}} u_t \in C^1([0,T], C_b^\infty)$$

In particular, $u^n \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$ and applying [CJMS06, Theorem 3.1] to $u^n(t, X_t)$ yields

$$\begin{aligned} u^n(t,X_t) = u^n(0,x) &+ \int_0^t (\partial_s - \mathscr{L}_\nu^\alpha) u^n(s,X_s) ds + \int_0^t \nabla u^n(s,X_s) \cdot dZ_s \\ &+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u^n(s, X_{s-} + y) - u^n(s, X_{s-})] \hat{\pi}(ds, dy). \end{aligned}$$

Due to convergence of the mollifications $u^n \to u$ in $\mathscr{L}_T^{0,\alpha+\beta}$ and analogue arguments as for Lemma 4.4 (with Burkholder-Davis-Gundy inequality and separating in large and small jumps), we obtain that the $\hat{\pi}$-martingales converge in $L^2(\mathbb{P})$ to the one with u^n replaced by u.
By Theorem 4.24 it follows that almost surely,

$$\int_0^t \nabla u^n(s,X_s) \cdot dZ_s = \int_0^t \nabla u^n(s,X_s) \cdot d(Z, \mathbb{Z}^V)_s.$$

The stability of the rough stochastic integral from Proposition 4.31 for the rough stochastic integral yields convergence of the integrals if $(u^n, u') \to (u, u')$ in $D_T(V)$. The latter follows from the convergence of the mollification $u^n \to u$ in $\mathscr{L}_T^{0,\alpha+\beta}$ and

$$\begin{aligned} u^{n,\sharp} &= u^n - u' \otimes J^T(V) \\ &= P_{n^{-1}} u^\sharp + \big(P_{n^{-1}}[u' \otimes J^T(V)] - u' \otimes P_{n^{-1}} J^T(V)\big) \\ &\to u^\sharp \end{aligned}$$

in $\mathscr{L}_T^{0,2(\alpha+\beta)-1}$ due to Lemma 3.7, the semigroup commutator (Lemma 3.9) and Lemma 3.11. It remains to show that the additive functional

$$\int_0^t (\partial_s - \mathscr{L}_\nu^\alpha) u^n(s,X_s) ds$$

converges in $L^2(\mathbb{P})$. For $r \in [0,T]$ we have by assumption for $v_r := (\partial_r - \mathscr{L}_\nu^\alpha) u_r$, that

$$v_r = v_r^\sharp + v_r' \otimes V_r,$$

for $v_r^\sharp \in \mathscr{C}^{(2-\gamma)\alpha+2\beta-1}$ and $v_r' \in \mathscr{C}^{\alpha+\beta-1}$. Thus v_r is an admissible terminal condition (in the sense of Theorem 3.25) for solving the Kolmogorov equation on $[0,r]$ with paracontrolled terminal condition v_r.
By Theorem 4.32, X is a martingale solution and in particular a strong Markov process.

Denote again by $(T_{s,r})_{0\leqslant s\leqslant r\leqslant T}$ its semigroup. Utilizing the ideas of [Lê20, section 3], we define,

$$\Xi_{s,t} := \int_s^t T_{s,r}v_r(X_s)dr.$$

We let $y_s := T_{s,r}v_r$ for $s \in [0,r]$. Then y is a mild solution of the Kolmogorov equation with terminal condition v_r.
We apply the stochastic sewing Lemma 4.7 to Ξ. Therefore, we write for $s \leqslant u \leqslant t$,

$$\Xi_{s,u,t} = \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t} = \int_u^t [T_{s,r}v_r(X_s) - T_{u,r}v_r(X_u)]dr$$

Due to the Markov property of X, we see that $\mathbb{E}_s[\Xi_{s,u,t}] = 0$. It is left to estimate $\|\Xi_{s,u,t}\|_{L^2(\mathbb{P})}$. To this end, we use that $y = T_{\cdot,r}v_r \in \mathscr{M}_r^{-\beta/\alpha}L^\infty$ with a uniform bound in $r \in [0,T]$ due to Corollary 3.32 and the interpolation estimate (3.27) from Lemma 3.17, such that we can estimate,

$$\|y_u\|_{L^\infty} = \|T_{u,r}v_r\|_{L^\infty} \lesssim |r-u|^{\beta/\alpha} \sup_{r\in[0,T]} \|y\|_{\mathscr{M}_r^{-\beta/\alpha}L^\infty}$$

and thus it follows that

$$\|\Xi_{s,u,t}\|_{L^2(\mathbb{P})} \lesssim |t-u|^{1+\beta/\alpha} = |t-u|^{(\alpha+\beta)/\alpha},$$

with $(\alpha+\beta)/\alpha > 1/2$. Thus the stochastic sewing lemma applies and we obtain existence of the integral $I_t^X(v) := \lim_{|\Pi|\to 0}\sum_{s,t\in\Pi}\Xi_{s,t} \in L^2(\mathbb{P})$. Furthermore, if we consider $v^n := (\partial_t - \mathscr{L}_\nu^\alpha)u^n$, we obtain that $I_t^X(v^n) \to I_t^X(v)$ due to stability of the stochastic sewing integral and continuity of the Kolmogorov solution map. The last step is then to see that almost surely $I_t^X(v^n) = \int_0^t v^n(s,X_s)ds$. Indeed, this can be deduced from uniqueness of the stochastic sewing integral and $\mathbb{E}_s[v_r^n(X_r)] = T_{s,r}v_r^n(X_s)$ as $v_r^n \in C_b$ (cf. also [Lê20, Proposition 3.7]). □

4.5. Ill-posedness of the canonical weak solution concept in the rough regime

Below, we introduce so-called canonical weak solutions. We prove in Corollary 4.42, that in the Young case the canonical weak solution concept is well-posed (and equivalent to martingale solutions). We finalize by proving in Theorem 4.44 that canonical weak solutions are in general non-unique. We construct a counterexample, that justifies the latter.
We start by defining the concept of canonical weak solutions.

Definition 4.38 (Canonical weak solution). *Let $\alpha \in (1,2]$, $V \in C_T\mathscr{C}^\beta_{\mathbb{R}^d}$ for $\beta \in (\frac{2-2\alpha}{3}, 0)$. Let $x \in \mathbb{R}^d$. We call a tuple (X, L) a canonical weak solution starting at*

$X_0 = x \in \mathbb{R}^d$, if there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$, such that L is an α-stable symmetric non-degenerate $(\mathscr{F}_t)$-Lévy process and almost surely

$$X = x + Z + L,$$

where Z is an $(\mathscr{F}_t)$-adapted, continuous process with

$$\|Z\|_{(\alpha+\beta)/\alpha,2} + \|Z \mid \mathscr{F}\|_{(\alpha+\beta)/\alpha,\infty} < \infty.$$

Moreover there exists a sequence $(V^n) \subset C_T C_b^\infty$ with $V^n \to V$ in $C_T \mathscr{C}^\beta$, such that

$$\lim_{n\to\infty} \int_0^{\cdot} V^n(s, X_s) ds = Z, \tag{4.62}$$

with convergence in $L^2(\mathbb{P})$, uniformly in $[0,T]$.

Remark 4.39. *The definition of canonical weak solutions is similar to [ABM20, Definition 2.1]. The difference is the assumption that $\|Z \mid \mathscr{F}\|_{(\alpha+\beta)/\alpha,\infty} < \infty$. However, boths bounds on Z are natural to assume and motivated by Corollary 4.5.*

Remark 4.40 (One-dimensional, time-homogeneous case with $\alpha = 2$)**.** *In this case, for any approximation (V^n) of V, the weak limit of the strong solutions (X^n) with*

$$dX_t^n = V^n(X_t^n)dt + dB_t$$

is the same and given by the solution of the $\mathscr{G}^{\mathscr{V}}$-martingale problem. The one-dimensional case is special in this sense. The above is true, because in $d = 1$ the resonant products $(J^T(\partial_x V^n) \odot V^n)$ converge to the same limit for any approximation (V^n). This can be seen with Leibniz rule considering $((-\Delta)^{-1}(\partial_x V^n) \odot V^n)_n = (J^\infty(\partial_x V^n) \odot V^n)_n$. Indeed, we have with v^n, such that $V^n = \partial_x v^n$

$$\lim_{n\to\infty} (-\Delta)^{-1}(\partial_x V^n) \odot V^n = -\frac{1}{2} \lim_{n\to\infty} \partial_x (v^n \odot v^n) = -\frac{1}{2} \partial_x (v \odot v)$$

using that $(-\Delta_x)^{-1}(\partial_x V^n) = -v^n$ and $v^n \odot v^n \to v \odot v$ as products of functions. Notice moreover that this does not imply, that the limit of the mixed resonant products $((-\Delta)^{-1}(\partial_x V^n) \odot V)_n$ is uniquely determined. In fact in the rough case the latter limit is in general not unique (cf. Lemma 4.43 below). This fact will imply that, even in the one-dimensional case, a weak solution in the sense of Definition 4.38 is in general non-unique in law.

Remark 4.41 (One and all sequences (V^n))**.** *One may ask, if requiring the convergence (4.62) to hold for all (instead of one) approximating sequences (V^n) renders the solution concept well-posed. Though, if we require (4.62) to hold for all such sequences (V^n), then in the rough case the solution of the $\mathscr{G}^{\mathscr{V}}$-martingale problem won't be a solution. This follows also from the fact that the limit of the mixed resonant products is non-unique. Thus, we would expect non-existence of solutions in the rough case. Hence, requiring (4.62) to hold for all such sequences (V^n) makes the definition too restrictive.*

The results from the previous section imply the following corollary.

Corollary 4.42. *Let $\beta \in ((1-\alpha)/2, 0)$ (Young regime) and $V \in C_T\mathscr{C}^\beta$. Then, X is a solution of the $(\mathscr{G}^V, \delta_x)$-martingale problem if and only if X is a canonical weak solution starting at $x \in \mathbb{R}^d$. In particular, the canonical weak solution concept is well-posed in the Young regime.*

Proof. If X is a martingale solution, then X is a canonical weak solution by Theorem 4.35, since a weak solution in the sense of Definition 4.10 is in particular a canonical weak solution. For the reverse implication, notice that (4.62) and the assumption $Z \in C_T^{(\alpha+\beta)/\alpha} L^2(\mathbb{P})$ imply that $\|Z^n - Z\|_{\theta,2} \to 0$ for any $\theta < (\alpha+\beta)/\alpha$. Then, Lemma 4.15 and the arguments in the proof of Theorem 4.32 for β in the Young regime (that do not use the bounds (4.21) on $\mathbb{Z}^V$ and the convergence 2.)) imply that X is a solution of the $(\mathscr{G}^V, \delta_x)$-martingale problem. □

If $\beta \leqslant \frac{1-\alpha}{2}$, we show that the solution concept from Definition 4.38 is ill-posed. The idea is as follows. We construct two different lifts of V with two different solutions of the respective Kolmogorov backward equations, which yields two different weak solutions X in law. The next lemma proves the existence of such desired lifts. We give its proof after the proof of Theorem 4.44.

Lemma 4.43. *Let $d = 1$ and $\alpha = 2$. Let $(P_t)_{t\geqslant 0}$ be the heat semigroup, that is $P_t f = \mathscr{F}^{-1}(\exp(\frac{t}{2}|2\pi\cdot|^2)\hat{f})$. Then, there exists $V \in C_T\mathscr{C}^\beta$ and two sequences $(V^n) \subset C_T C_b^\infty$, $(W^n) \subset C_T C_b^\infty$ such that $V^n \to V \in C_T\mathscr{C}^\beta$ and $W^n \to V \in C_T\mathscr{C}^\beta$ and such that $J^T(\partial_x V^n) \odot V \to \mathscr{V}_2$ and $J^T(\partial_x W^n) \odot V \to \mathscr{W}_2$ in $C_T\mathscr{C}^{2\beta+1}$, where $\mathscr{V}_2 = \mathscr{W}_2 + C$ for a constant $C \neq 0$.*

Theorem 4.44. *Let $\beta \leqslant \frac{1-\alpha}{2}$ and $V \in C_T\mathscr{C}^\beta_{\mathbb{R}^d}$. Then, canonical weak solutions in the sense of Definition 4.38 are in general non-unique in law.*

Proof of Theorem 4.44. We construct two solutions that are not equal in law. To that end, we let $d = 1$, $\alpha = 2$. By Lemma 4.43, there exists $V \in C_T\mathscr{C}^\beta$ and two sequences $(V^{n,i})$ for $i = 1, 2$ with $(V^{n,1}, J^T(\partial_x V^{n,1}) \odot V) \to (V, \mathscr{V}_2)$ and $(V^{n,2}, J^T(\partial_x V^{n,2}) \odot V) \to (V, \mathscr{W}_2)$ in $C_T\mathscr{C}^\beta \times C_T\mathscr{C}^{2\beta+\alpha-1}$ and $\mathscr{V}_2 = \mathscr{W}_2 + C$ for $C \neq 0$.
Let u be the solution of $\mathscr{G}^{(V,\mathscr{V}_2)}u = V$ with $u(T,\cdot) = 0$ and $\tilde{u}$ be the solution of $\mathscr{G}^{(V,\mathscr{W}_2)}\tilde{u} = V$, $\tilde{u}(T,\cdot) = 0$. Then, it follows that

$$\tilde{u}(t,y) = u(t, y - (T-t)C), \quad (t,y) \in [0,T] \times \mathbb{R}.$$

In particular, there exists $(s,x) \in [0,T) \times \mathbb{R}$, such that $u(s,x) \neq \tilde{u}(s,x)$ (otherwise u is constant, which is a contradiction, as $V \neq 0$). Consider the shifted solutions $v(t,x) := u(t+s,x)$ and $\tilde{v}(t,x) := \tilde{u}(t+s,x)$, $(t,x) \in [0,T] \times \mathbb{R}$. Then we have that $v(0,x) \neq \tilde{v}(0,x)$.
Let X^1 be the solution of the $(\mathscr{G}^{(V,\mathscr{V}_2)}, \delta_x)$ martingale problem and X^2 be the solution of the $(\mathscr{G}^{(V,\mathscr{W}_2)}, \delta_x)$ martingale problem (cf. also Remark 4.6).

By Theorem 4.35, there exist stochastic basis $(\Omega_1, \mathscr{F}^1, (\mathscr{F}^1_t), \mathbb{P}_1)$ and $(\Omega_2, \mathscr{F}^2, (\mathscr{F}^2_t), \mathbb{P}_2)$, such that X^i for $i = 1, 2$ satisfy

$$X^i = x + Z^i + B^i, \text{ a.s., where } Z^i = \lim_{n\to\infty} \int_0^\cdot V^{n,i}(s, X^i_s)ds \in L^2(\mathbb{P}_i)$$

and such that $\|Z^i\|_{(\alpha+\beta)/\alpha, L^2(\mathbb{P}_i)} + \|Z^i \mid \mathscr{F}^i\|_{(\alpha+\beta)/\alpha, L^\infty(\mathbb{P}_i)} < \infty$. In particular, X^1 and X^2 are canonical weak solutions in the sense of Definition 4.38.
We prove that $\mathrm{Law}(X^1) \neq \mathrm{Law}(X^2)$. But this is clear, if we show that

$$\mathbb{E}_1[Z^1_{T-s}] \neq \mathbb{E}_2[Z^2_{T-s}].$$

Let $u_{n,1}$ be the solution of $\mathscr{G}^{(V,\mathscr{V}_2)} u_{n,1} = V^{n,1}$ with $u_{n,1}(T, \cdot) = 0$ and $u_{n,2}$ be the solution of $\mathscr{G}^{(V,\mathscr{W}_2)} u_{n,2} = V^{n,2}$ with $u_{n,2}(T, \cdot) = 0$. Let $v^{n,1}, v^{n,2}$ be the shifted solutions, that solve the same equations as $u^{n,1}, u^{n,2}$ with $v^{n,1}(T - s, \cdot) = v^{n,2}(T - s, \cdot) = 0$. Then $u_{n,1} \to u$ and $u_{n,2} \to \tilde{u}$ in $C_T L^\infty_{\mathbb{R}^d}$. As X^1 solves the $(\mathscr{G}^{(V,\mathscr{V}_2)}, \delta_x)$ martingale problem and X^2 solves the $(\mathscr{G}^{(V,\mathscr{W}_2)}, \delta_x)$ martingale problem, we have (abbreviating the martingale term by $M^{v_{n,1}}$)

$$\begin{aligned}
\mathbb{E}_1[Z^1_{T-s}] &= \lim_{n\to\infty} \mathbb{E}_1\Big[\int_0^{T-s} V^{n,1}(r, X^1_r)dr\Big] \\
&= \lim_{n\to\infty} \mathbb{E}_1[v_{n,1}(T - s, X^1_T) - v_{n,1}(0, x) - M^{v_{n,1}}_{0,T-s}] \\
&= \lim_{n\to\infty} u_{n,1}(s, x) \\
&= u(s, x) \neq \tilde{u}(s, x) = \mathbb{E}_2[Z^2_{T-s}],
\end{aligned}$$

such that the claim follows. □

Proof of Lemma 4.43. Recall that $d = 1$ and $L = B$ for a standard Brownian motion B. We construct a distribution V, that is time independent and can thus integrate out time in $J^T(\partial_x V)$, i.e. instead of

$$\begin{aligned}
J^T(\partial_x V)(r) &= \int_r^T \mathscr{F}^{-1}\big(\exp\big((s - r)\tfrac{1}{2}|2\pi\cdot|^2\big)\mathscr{F}(\partial_x V)\big)ds \\
&= \mathscr{F}^{-1}\Big((1 - \exp((r - T)\tfrac{1}{2}|2\pi\cdot|^2))\frac{1_{\cdot\neq 0}}{\frac{1}{2}|2\pi\cdot|^2}\mathscr{F}(\partial_x V)\Big) \\
&=: J^\infty(\partial_x V) - \varphi_{r,T} * J^\infty(\partial_x V),
\end{aligned}$$

we consider w.l.o.g. $J^\infty(\partial_x V) = \mathscr{F}^{-1}(\frac{1_{\cdot\neq 0}}{\frac{1}{2}|2\pi\cdot|^2}\mathscr{F}(\partial_x V)) = (-\frac{1}{2}\Delta)^{-1}(\mathrm{Id} - \Pi_0)\partial_x V$, where Π_0 is the projection onto the zero-order Fourier mode.
Inspired by [CF14], we set

$$f(x) = \sum_{k>0} a_k e^{2\pi i 2^k x}$$

for $(a_k) \subset \mathbb{C}$ to be determined. We define

$$F := J^\infty(\partial_x f) = \left(-\frac{1}{2}\Delta\right)^{-1} (\mathrm{id} - \Delta_0)\partial_x f = \sum_{k>0} a_k \frac{2\pi i 2^k}{\frac{1}{2}|2\pi 2^k|^2} e^{2\pi i 2^k x} = \sum_{k>0} \frac{i a_k}{\pi 2^k} e^{2\pi i 2^k x}.$$

Then, it follows that $\Delta_k f(x) = a_k e^{2\pi i 2^k x}$ and $\Delta_k F(x) = \frac{i a_k}{2\pi 2^k} e^{2\pi i 2^k x}$, and therefore

$$F \odot f = \sum_{k>0} \frac{i a_k^2}{\pi 2^k} e^{2\pi i 2^{k+1} x}, \qquad \overline{F} \odot \overline{f} = -\sum_{k>0} \frac{i \overline{a_k}^2}{\pi 2^k} e^{-2\pi i 2^{k+1} x},$$

$$F \odot \overline{f} = \sum_{k>0} \frac{i |a_k|^2}{\pi 2^k}, \qquad \overline{F} \odot f = -\sum_{k>0} \frac{i |a_k|^2}{\pi 2^k}.$$

Letting $a_k := 2^{-k/2}$, we obtain that $f \in \mathscr{C}^{-1/2}$. Moreover, taking $V := \mathrm{Re}(f) = \frac{1}{2}\left(f + \overline{f}\right)$ and $J^\infty(\partial_x V) = \mathrm{Re}(F) = \frac{1}{2}\left(F + \overline{F}\right)$, we obtain for the resonant product

$$\begin{aligned}
\mathscr{V}_2 := J^\infty(\partial_x V) \odot V &= \frac{1}{4}\left(F \odot f + \overline{F} \odot \overline{f} + F \odot \overline{f} + \overline{F} \odot f\right) \\
&= \frac{1}{4} \sum_{k>0} \frac{i a_k^2}{\pi 2^k} (e^{2\pi i 2^{k+1} x} - e^{-2\pi i 2^{k+1} x}) \\
&= \frac{1}{4} \sum_{k>0} \frac{i}{\pi} 2i \sin(2\pi 2^{k+1} x) = -\sum_{k>0} \frac{1}{2\pi} \sin(2\pi 2^{k+1} x).
\end{aligned}$$

The latter is a distribution in $\mathscr{C}^{0-}$. Clearly, we have that $V^n \to V$ in $\mathscr{C}^{-1/2}$ for $V^n(x) := \mathrm{Re}\left(\sum_{k=1}^n a_k e^{2\pi i 2^k x}\right)$. Then it follows

$$J^\infty(\partial_x V^n) \odot V \to J^\infty(\partial_x V) \odot V =: \mathscr{V}_2,$$

in $\mathscr{C}^{0-}$ for $n \to \infty$.
Let $(c_n) \subset \mathbb{C}$ to be determined. The other sequence (W^n), we then define as follows

$$W^n(x) := \mathrm{Re}\left(\sum_{k=1}^n a_k e^{2\pi i 2^k x}\right) + \mathrm{Re}(c_n e^{2\pi i 2^n x}) = V^n(x) + \mathrm{Re}(c_n e^{2\pi i 2^n x}),$$

$$\begin{aligned}
J^\infty(\partial_x W^n)(x) &= \mathrm{Re}\left(\sum_{k=1}^n \frac{i a_k}{\pi 2^k} e^{2\pi i 2^k x}\right) + \mathrm{Re}\left(\frac{i c_n}{\pi 2^n} e^{2\pi i 2^n x}\right) \\
&= J^\infty(\partial_x V^n)(x) + \mathrm{Re}\left(\frac{i c_n}{\pi 2^n} e^{2\pi i 2^n x}\right).
\end{aligned}$$

Writing $G_n = \frac{ic_n}{\pi 2^n} e^{2\pi i 2^n x}$, we have (recall that $a_n \in \mathbb{R}$)

$$G_n \odot f = \frac{ia_n c_n}{\pi 2^n} e^{2\pi i 2^{n+1} x}, \qquad \overline{G_n} \odot \overline{f} = -\frac{ia_n \overline{c_n}}{\pi 2^n} e^{-2\pi i 2^{n+1} x},$$
$$G_n \odot \overline{f} = \frac{ia_n c_n}{\pi 2^n}, \qquad \overline{G_n} \odot f = -\frac{ia_n \overline{c_n}}{\pi 2^n},$$

and thus for $c_n := 2^{n/2} d_n$ for $(d_n) \subset \mathbb{C}$ to be determined below,

$$\begin{aligned}
\operatorname{Re}(G_n) \odot V &= \frac{1}{4} \frac{ia_n}{\pi 2^n} \left(c_n e^{2\pi i 2^{n+1} x} - \overline{c_n} e^{-2\pi i 2^{n+1} x} \right) + \frac{1}{4} \frac{a_n}{\pi 2^n} \left(ic_n - i\overline{c_n} \right) \\
&= \frac{1}{4} \frac{i}{\pi} \left(d_n e^{2\pi i 2^{n+1} x} - \overline{d_n} e^{-2\pi i 2^{n+1} x} \right) - \frac{1}{2} \frac{\operatorname{Im} d_n}{\pi} \\
&= -\frac{1}{2\pi} [\operatorname{Im}(d_n e^{2\pi i 2^{n+1} x}) + \operatorname{Im}(d_n)].
\end{aligned}$$

We now set $d_n := -2\pi C i$ for a constant $C \in \mathbb{R} \setminus \{0\}$. Then we have $W^n \to V$ in $\mathscr{C}^{-1/2}$ and the following convergences in $\mathscr{C}^{0-}$:

$$\operatorname{Re}(G_n) \odot V \to C, \qquad J^\infty(\partial_x W^n) \odot V \to J^\infty(\partial_x V) \odot V + C =: \mathscr{W}_2,$$

for $n \to \infty$. □

4.6. Application – Brox diffusion with Lévy noise

SDEs with distributional drift have applications in various situations. This includes the study of stochastic processes in random media, such as the so-called Brox diffusion ([Bro86]), that we consider below, and the construction of random polymer measures (cf. [DD16, CC18]). Furthermore singular SDEs arise as stochastic characteristics of singular SPDEs such as KPZ and PAM, cf. [KPvZ21]. In what follows, we apply the theory of Section 4.1 to the construction of the solution to the Brox diffusion with stable noise. The section is based on [KP22, Section 5].
Let us start with the model, that Brox studied in [Bro86]. The Brox diffusion is the solution X of the following SDE

$$dX_t = \dot{W}(X_t) dt + dB_t, \qquad X_0 = x \in \mathbb{R}, \tag{4.63}$$

where B is a standard Brownian motion and $(W(x))_{x \in \mathbb{R}}$ is a two-sided standard Brownian motion that is independent of B. This model was introduced in [Bro86] as a continuous analogue of Sinai's random walk, with the motivation that when studying X we can exploit the scaling properties of W and B. Brox's construction is based on time and space transformations as in the Itô-McKean construction of diffusions. It is natural to replace W or B by α-stable Lévy processes, which also have nice scaling properties. The construction of the process with W replaced by a Lévy process is not much of a problem, as the Itô-McKean approach still works [Tan87, Car97, KTT17].

On the other hand, replacing B by an α-stable Lévy process is more delicate and it is not obvious if the Itô-McKean construction could work. But using our approach we can hope to solve the martingale problem for the SDE

$$dX_t = \dot{W}(X_t)dt + dL_t, \qquad X_0 = x \in \mathbb{R}. \tag{4.64}$$

To be precise, the white noise $\dot{W}$ is not actually an element of any Besov space, but only of weighted Besov spaces: With $\langle x\rangle = (1+|x|^2)^{1/2}$ we have $\langle\cdot\rangle^{-\kappa}\dot{W} \in \mathscr{C}^{-1/2-}$ for all $\kappa > 0$. It is possible to extend our analysis of the martingale problem to allow for a drift term in a suitable weighted Besov space, and at the end of this section we discuss how this could be done. But to simplify the presentation we consider a periodic white noise $\dot{W}$ instead, which is in the unweighted space $\mathscr{C}^{-1/2-}$. Note that this regularity is not in the Young regime, no matter which $\alpha \in (1,2]$ we choose, and therefore the methods of [ABM20, dRM19] do not apply and we are not aware of any other way of constructing X, apart from the approach we present here. Although we should point out that in $d = 1$ it would be possible to extend the methods of Delarue and Diel [DD16], who treat the Brownian case, to the Lévy setting, i.e. to replace our paracontrolled analysis by rough path analysis. As we discuss in Remark 4.49 below, the paracontrolled approach has the advantage that it allows us to construct a multidimensional variant of the Lévy Brox diffusion.

So let $\xi = \dot{W}$ be a 1-periodic white noise, that is, ξ is a centered Gaussian process with values in $\mathscr{S}'(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus and $\mathscr{S}'(\mathbb{T})$ is the space of Schwartz distributions on $\mathbb{T}$, i.e. the topological dual of $C^\infty(\mathbb{T})$. The covariance of ξ is $\mathbb{E}[\xi(\varphi)\xi(\psi)] = \langle\varphi,\psi\rangle_{L^2(\mathbb{T})}$ for $\varphi,\psi \in C^\infty(\mathbb{T})$. To any $u \in \mathscr{S}'(\mathbb{T})$ we associate a periodic distribution on the real line by setting $u^{\mathbb{R}}(\varphi) = u(\sum_{k\in\mathbb{Z}}\varphi(\cdot+k))$, $\varphi \in \mathscr{S}$. If $u \in \mathscr{C}^\beta(\mathbb{T})$, then $u^{\mathbb{R}} \in \mathscr{C}^\beta$. Here $\mathscr{C}^\beta(\mathbb{T})$ is a Besov space on the torus, which is defined in the same way as on the real line, except using the Fourier transform $\mathscr{F}_{\mathbb{T}}$ on $\mathbb{T}$ and inverse $\mathscr{F}_{\mathbb{T}}^{-1}$ Fourier transform on $\mathbb{Z}$.

We choose ξ independently of the Lévy process L, and we consider a fixed "typical" realization $\xi(\omega)$. To apply the theory that we developed in this paper, we need to construct a canonical enhancement of $\xi(\omega)^{\mathbb{R}}$ in such a way that we obtain an enhanced drift in the sense of Definition 3.20.

We first note that almost surely $\xi \in C_T\mathscr{C}^{-1/2-}(\mathbb{T})$ (so we let $\beta = -1/2-\varepsilon$ for some very small $\varepsilon > 0$), see e.g. [GP15, Exercise 11]. Therefore, $\xi(\omega)^{\mathbb{R}} \in C_T\mathscr{C}^{-1/2-}$ for almost all ω. In fact, we require $\xi \in \mathscr{C}^{\beta+(1-\gamma')\alpha}$. Hence, we let $\gamma' \in (0,1)$ be such that $\beta + (1-\gamma')\alpha < -1/2$. It remains to construct $t \mapsto (P_t(\nabla\xi)\odot\xi)(\omega) \in \mathscr{M}_T^\gamma\mathscr{C}^{(\alpha\gamma-2)-}(\mathbb{T})$ for $\gamma < 1$ (cf. Definition 3.20) for almost all ω, which we will do in the next lemma. In particular, by choosing γ larger if necessary, we can take $\gamma = \gamma'$ and have $\gamma\alpha - 2 = \gamma\alpha - 1 - 1/2 - 1/2 > \gamma\alpha - 1 + \beta + \beta + (1-\gamma)\alpha = 2\beta + \alpha - 1$, such that indeed $V = \xi(\omega) \in \mathscr{X}^{\beta,\gamma'}$.

Lemma 4.45. *Let $\alpha \in (3/2, 2]$, $\gamma \in (\frac{3}{2\alpha}, 1)$ and $\vartheta < \alpha\gamma - 2$. Let (P_t) be the semigroup generated by $(-\mathscr{L}_\nu^\alpha)$, $P_t\phi = \mathscr{F}_{\mathbb{T}}^{-1}(e^{-t\psi_\nu^\alpha}\mathscr{F}_{\mathbb{T}}\phi)$. Let $\xi^n = \sum_{|k|\leqslant n}\hat{\xi}(k)e_k$, where $(e_k)_{k\in\mathbb{Z}} = (e^{-2\pi i k\cdot})_{k\in\mathbb{Z}}$ is the Fourier basis of $L^2(\mathbb{T})$. Then $(t \mapsto P_t(\nabla\xi^n)\odot\xi^n)_n$ converges in* *probability in* $\mathscr{M}_T^\gamma \mathscr{C}^\vartheta(\mathbb{T})$ *to a limit denoted by* $(t \mapsto P_t(\nabla\xi) \odot \xi) \in \mathscr{M}_T^\gamma \mathscr{C}^\vartheta(\mathbb{T})$.

Proof. [3] We carry out the computations for $n = \infty$ and show that $P(\nabla\xi) \odot \xi \in \mathscr{M}_T^\gamma \mathscr{C}^\vartheta(\mathbb{T})$ can be constructed as a random variable in the second Wiener-Itô chaos generated by ξ. Since the kernel appearing in the definition of $P(\nabla\xi) \odot \xi$ provides a uniform bound for the kernels that appear in the chaos representation of $(P(\nabla\xi^n) \odot \xi^n)_n$, the claimed convergence then follows from the dominated convergence theorem.
To bound $P(\nabla\xi) \odot \xi$, note that $P(\nabla\xi)(t) = \varrho_t * \xi$, where $\varrho_t = \nabla\mathscr{F}_{\mathbb{T}}^{-1}(e^{-t\psi_\nu^\alpha})$. We first derive a bound on the expectation of the $B_{p,p}^\zeta$-norm (for ζ to be chosen afterwards) of the increment $(\varrho_t * \xi) \odot \xi - (\varrho_s * \xi) \odot \xi = ((\varrho_t - \varrho_s) * \xi) \odot \xi$. Using this bound, our claim will follow from the Besov embedding theorem together with Kolmogorov's continuity criterion. Let us abbreviate $\tilde{\varrho}_t := t^\gamma \varrho_t$. Then, we have

$$\begin{aligned}\mathbb{E}\Big[\|(\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi\|_{B_{p,p}^\zeta}^p\Big] &= \mathbb{E}\Big[\sum_j 2^{j\zeta p}\|\Delta_j((\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi)\|_{L^p}^p\Big] \\ &= \sum_j 2^{j\zeta p}\int_{\mathbb{T}} \mathbb{E}[|\Delta_j((\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi)(x)|^p]dx \\ &\lesssim \sum_j 2^{j\zeta p}\int_{\mathbb{T}} \mathbb{E}[|\Delta_j((\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi)(x)|^2]^{p/2}dx,\end{aligned}$$

where in the last step we used that the random variable $\Delta_j((\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi)(x)$ is in the second (inhomogeneous) Wiener-Itô chaos and therefore all its moments are comparable by Gaussian hypercontractivity [Jan97, Theorem 5.10].
It remains to estimate

$$\mathbb{E}[|\Delta_j((\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi)(x)|^2] = \mathbb{E}[|((\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi)(\kappa_j(x - \cdot))|^2], \tag{4.65}$$

where $\kappa_j = \mathscr{F}_{\mathbb{T}}^{-1} p_j = \sum_{k\in\mathbb{Z}} e^{2\pi i k \cdot} p_j(k)$. Let now $\psi_\odot(x, y) = \sum_{|l_1 - l_2| \leqslant 1} \kappa_{l_1}(x)\kappa_{l_2}(y)$. Then with formal notation we have

$$(\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi(x) = \iint \psi_\odot(x - y_1, x - y_2)((\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi)(y_1)\xi(y_2)dy_1dy_2,$$

and thus

$$\begin{aligned}&(\tilde{\varrho}_t - \tilde{\varrho}_s) * \xi \odot \xi(\kappa_j(x - \cdot)) \\ &\quad= \iiint \kappa_j(x - z)\psi_\odot(z - y_1, z - y_2)\xi((\tilde{\varrho}_t - \tilde{\varrho}_s)(y_1 - \cdot))\xi(\delta(y_2 - \cdot))dy_1dy_2dz.\end{aligned}$$

[3] We adjusted the proof to account for Definition 3.20, which is a stronger assumption on enhanced distributions compared to [KP22].

To derive the chaos decomposition of the right hand side, we introduce the kernel

$$A_j^{t,s}(x,r_1,r_2) = \iiint \kappa_j(x-z)\psi_\odot(z-y_1,z-y_2)(\tilde{\varrho}_t-\tilde{\varrho}_s)(y_1-r_1)\delta(y_2-r_2)dy_1dy_2dz,$$

with which

$$(\tilde{\varrho}_t-\tilde{\varrho}_s)*\xi\odot\xi(\kappa_j(x-\cdot)) = W_2(A_j^{t,s}(x,\cdot,\cdot)) + \mathbb{E}[(\tilde{\varrho}_t-\tilde{\varrho}_s)*\xi\odot\xi(\kappa_j(x-\cdot))], \quad (4.66)$$

where W_2 denotes a second order Wiener-Itô integral. We start by estimating the first term on the right hand side: Using the symmetrization $\tilde{A}_j^{t,s}(x,r_1,r_2) = \frac{1}{2}(A_j^{t,s}(x,r_1,r_2)+A_j^{t,s}(x,r_2,r_1))$, we have

$$\begin{aligned}\mathbb{E}[|W_2(A_j^{t,s}(x,\cdot,\cdot))|^2] &= 2\|\tilde{A}_j^{t,s}(x,\cdot,\cdot)\|_{L^2(\mathbb{T}^2)}^2 \leqslant 2\|A_j^{t,s}(x,\cdot,\cdot)\|_{L^2(\mathbb{T}^2)}^2 \\ &= \sum_{k_1,k_2\in\mathbb{Z}}\left|\iint A_j^{t,s}(x,r_1,r_2)e^{-2\pi i(k_1r_1+k_2r_2)}dr_1dr_2\right|^2, \end{aligned} \quad (4.67)$$

where the last equality is Parseval's identity. Now, we obtain by computing each integral iteratively

$$\begin{aligned}&\iint A_j(x,r_1,r_2)e^{-2\pi i(k_1r_1+k_2r_2)}dr_1dr_2 \\ &\quad = \hat{\kappa}_j(-(k_1+k_2))e^{-2\pi i(k_1+k_2)x}\hat{\psi_\odot}(-k_1,-k_2)(\widehat{\tilde{\varrho}_t-\tilde{\varrho}_s})(-k_1),\end{aligned}$$

where $\hat{f}(k) = \int_{\mathbb{T}} f(x)e^{-2\pi ikx}dx$ is the Fourier transform on the torus and

$$\hat{\psi_\odot}(k_1,k_2) := \iint \psi_\odot(y_1,y_2)e^{-2\pi i(k_1y_1+k_2y_2)}dy_1dy_2 = \sum_{|l_1-l_2|\leqslant 1} p_{l_1}(k_1)p_{l_2}(k_2).$$

As $|\psi_\nu^\alpha(k)| \geqslant |k|^\alpha$ and $1-e^{-x}\leqslant x^\varepsilon$ for $x\geqslant 0$ and $\varepsilon\in[0,1]$, we have that for $\varepsilon\in[0,\gamma]$, $s\leqslant t$ and $\gamma\in(0,1]$

$$\begin{aligned}|\widehat{\varrho_t-\varrho_s}| = |\mathscr{F}_{\mathbb{T}}(t^\gamma\varrho_t - s^\gamma\varrho_s)| &= |2\pi k||t^\gamma e^{-t\psi_\nu^\alpha(k)} - s^\gamma e^{-s\psi_\nu^\alpha(k)}| \\ &= |2\pi k||(t^\gamma-s^\gamma)e^{-t\psi_\nu^\alpha(k)} + s^\gamma e^{-t\psi_\nu^\alpha(k)}(1-e^{(t-s)\psi_\nu^\alpha(k)})| \\ &\lesssim (t^\gamma-s^\gamma)t^{-\gamma+\varepsilon}|k|^{1-\alpha(\gamma-\varepsilon)} + s^\gamma t^{-\gamma}|k|^{1-\alpha\gamma}|t-s|^\varepsilon|k|^{\alpha\varepsilon} \\ &\lesssim |t-s|^\varepsilon|k|^{1-\alpha(\gamma-\varepsilon)},\end{aligned}$$

using $t^\varepsilon - s^\varepsilon \leqslant |t-s|^\varepsilon$, $s^\gamma t^{-\gamma} \leqslant 1$ and $s^{\gamma-\varepsilon}t^{-\gamma+\varepsilon} \leqslant 1$. This leads to

$$\Big|\iint A_j^{t,s}(x,r_1,r_2)e^{-2\pi i(k_1r_1+k_2r_2)}dr_1dr_2\Big|^2 \\ \lesssim |t-s|^{2\varepsilon}|p_j(k_1+k_2)|^2\Big|\sum_{|l_1-l_2|\leqslant 1} p_{l_1}(k_1)p_{l_2}(k_2)\Big|^2|k_1|^{2-2\alpha(\gamma-\varepsilon)}.$$

Let now $\tilde{p}_{l_1} := \sum_{l:|l-l_1|\leqslant 1} p_l$. Since for fixed k_1 there are at most three l_1 with $p_{l_1}(k_1) \neq 0$, we can bound $\left|\sum_{l_1} p_{l_1}(k_1)\tilde{p}_{l_1}(k_2)\right|^2 \lesssim \sum_{l_1} p_{l_1}(k_1)^2\tilde{p}_{l_1}(k_2)^2$ and thus we obtain in (4.67)

$$\begin{aligned}\mathbb{E}\big[|W_2(A_j^{t,s}(x,\cdot,\cdot))|^2\big] &\lesssim |t-s|^{2\varepsilon}\sum_{k_1,k_2}\sum_{l_1} p_j(k_1+k_2)^2p_{l_1}(k_1)^2\tilde{p}_{l_1}(k_2)^2|k_1|^{2-2\alpha(\gamma-\varepsilon)} \\ &= |t-s|^{2\varepsilon}\sum_{l_1:2^j\lesssim 2^{l_1}}\sum_{k_1} 2^j p_{l_1}(k_1)^2|k_1|^{2-2\alpha(1-\varepsilon)} &(4.68) \\ &\lesssim |t-s|^{2\varepsilon}\sum_{l_1:2^j\lesssim 2^{l_1}} 2^j2^{l_1}2^{l_1(2-2\alpha(\gamma-\varepsilon))} &(4.69) \\ &\lesssim |t-s|^{2\varepsilon}2^{j(4-2\alpha(\gamma-\varepsilon))},\end{aligned}$$

where we used that $p_i(k) \neq 0$ for $O(2^i)$ values of k, with $i=j$ respectively $i=l_1$, and we choose $\varepsilon \in (0,1)$ so that $3-2\alpha(\gamma-\varepsilon) < 0$ to obtain the convergence of the series in the last estimate (recall that we assume $\gamma > 3/2\alpha$).
Moreover, the expectation on the right hand side of (4.66) vanishes. Indeed, using that for $e_k(x) = e^{2\pi ikx}$, we have $\int e_k(x)e_l(x)dx = \delta_{k,-l}$ (Kronecker delta), we obtain

$$\begin{aligned}&\mathbb{E}[(\tilde{\varrho}_t-\tilde{\varrho}_s)*\xi\odot\xi(\kappa_j(x-\cdot))] \\ &= \iiint \kappa_j(x-z)\psi_\odot(z-y_1,z-y_2)(\tilde{\varrho}_t-\tilde{\varrho}_s)(y_1-y_2)dy_1dy_2dz \\ &= \sum_{k,l,k',l'}\hat{\kappa}_j(k)\hat{\psi}_\odot(k',l')(\widehat{\tilde{\varrho}_t-\tilde{\varrho}_s})(l)\iiint e_k(x-z)e_{k'}(z-y_1)e_{l'}(z-y_2)e_l(y_1-y_2)dy_1dy_2dz \\ &= \sum_{k'}\hat{\kappa}_j(0)\hat{\psi}_\odot(k',-k')(\widehat{\tilde{\varrho}_t-\tilde{\varrho}_s})(k') = 0.\end{aligned}$$

Here, the last equality is due to $\widehat{\tilde{\varrho}_t-\tilde{\varrho}_s}$ being an odd function, i.e. $\widehat{\tilde{\varrho}_t-\tilde{\varrho}_s}(-k) = -(\widehat{\tilde{\varrho}_t-\tilde{\varrho}_s})(k)$ for $k \in \mathbb{Z}$, and $\hat{\psi}_\odot(k,-k) = \hat{\psi}_\odot(-k,k) = 1$. To make this formal argument rigorous we have to include the regularization, which only has the effect of restricting the sum to $|k'| \leqslant n$. Therefore, also the expectation of the regularized resonant product vanishes.

Combining this with (4.66) and (4.68), we get via the Besov embedding theorem that for all $\vartheta' < \alpha\gamma - 2$ there exists $\varepsilon > 0$ such that for all $p > 1$ (by taking $\zeta = \vartheta'$),

$$\mathbb{E}[\|(\tilde{\varrho}_t-\tilde{\varrho}_s)*\xi\odot\xi\|^p_{\mathscr{C}^{\vartheta'-1/p}}] \lesssim \mathbb{E}[\|(\tilde{\varrho}_t-\tilde{\varrho}_s)*\xi\odot\xi\|^p_{B^{\vartheta'}_{p,p}}] \lesssim |t-s|^{\varepsilon p}.$$

After choosing p large enough so that $\varepsilon p > 1$ we obtain from Kolmogorov's continuity criterion that $P(\nabla\xi) \odot \xi \in \mathscr{M}_T^\gamma \mathscr{C}^{\vartheta' - 1/p}$. Given $\vartheta < \alpha\gamma - 2$ as in the statement of the theorem, it now suffices to take $\vartheta' \in (\vartheta, \alpha\gamma - 2)$ and then p large enough so that $\vartheta' - 1/p \geqslant \vartheta$. □

By freezing a "typical" realization of $\xi(\omega)$, we obtain the following corollary of Lemma 4.45 and Theorem 4.2.

Theorem 4.46. *Let $\alpha \in (7/4, 2]$ and let ξ be a periodic white noise on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Then for almost all ω there exists a unique solution to the "quenched martingale problem" associated to the Brox diffusion with symmetric, α-stable Lévy process L,*

$$dX_t = \xi(\omega)(X_t)dt + dL_t, \qquad X_0 = x \in \mathbb{R}.$$

If we denote the distribution of X by P_ω, then the "annealed measure" $\int P_\omega(\cdot)\mathbb{P}(d\omega)$ is the distribution of a Brox diffusion in a white noise potential, driven by an independent symmetric α-stable Lévy process L.

Remark 4.47. *By analogy with rough path regularities, the constraint $\alpha > 7/4$ corresponds to an "$\alpha > 1/3$ condition" in rough paths, and we expect that it is possible to treat $\alpha \in (3/2, 7/4]$ by considering higher order expansions of the Kolmogorov backward equation. To carry out this analysis we would need to use regularity structures [Hai14] or the higher order paracontrolled calculus of [BB19]. The constraint $\alpha > 3/2$ appears in the construction of the resonant product $P(\nabla\xi) \odot \xi$, so it seems to be of a similar nature as the constraint $H > 1/4$ for the Hurst index of a fractional Brownian motion that is required to construct its iterated integrals (cf. [CQ02]). But in fact not only the probabilistic construction fails at $\alpha = 3/2$: at that value the equation is* critical *in the sense of [Hai14] and we cannot solve it with perturbative techniques such as paracontrolled distributions or regularity structures.*

Remark 4.48. *To avoid dealing with weighted function spaces, we restricted our attention to periodic ξ. But we expect that it is also possible to treat the white noise ξ on $\mathbb{R}$ with our approach, at the price of a slightly more involved analysis. In that case we have $\langle\cdot\rangle^{-\kappa}\xi \in \mathscr{C}^{-1/2-}$ and $\langle\cdot\rangle^{-\kappa}P(\nabla\xi) \odot \xi \in \mathscr{M}_T^\gamma \mathscr{C}^{(\alpha\gamma-2)-}$ for all $\kappa > 0$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$. With the techniques of [DD16, HL15, MP19] it is still possible to solve the Kolmogorov backward equation for such ξ, by working in weighted function spaces with a time-dependent weight. Roughly speaking, if the terminal condition u_T grows like $\exp(l|x|^\delta)$ as $x \to \infty$, where $\delta \in (0,1)$ and $l \in \mathbb{R}$, then $u(T-t)$ grows like $\exp((l+t)|x|^\delta)$. This might look dangerous because for $\alpha < 2$ our Lévy noise does not even have finite second moments, let alone finite (sub-)exponential moments. But we can take $l \in \mathbb{R}$ arbitrary, and in particular $l \leqslant -T$ is allowed and then $u(t)$ is bounded for all t. In that way it should be possible to extend our results to construct a Brox diffusion with Lévy noise in a non-periodic white noise potential.*

Remark 4.49. *With our approach we can more generally solve the multidimensional SDE*

$$dX_t = \nabla W(X_t)dt + dL_t, \qquad X_0 = x \in \mathbb{R}^d$$

where W is the Brownian sheet on $\mathbb{R}^d$. In the Brownian noise case $\alpha = 2$, the construction of this SDE, which coincides with the Brox diffusion in $d = 1$, can be done via Dirichlet form techniques and was already carried out in [Mat94]. But to the best of our knowledge in the Lévy noise case the Dirichlet form approach is not applicable. Let us argue in the periodic case why our theory applies. Since $W \in \mathscr{C}^{1/2-}(\mathbb{T}^d)$ we have $\nabla W \in \mathscr{C}^{-1/2-}(\mathbb{T}^d)$, and therefore we only have to lift ∇W to an element of $\mathscr{X}^{-1/2-,\gamma}$. This can be done using similar arguments as for Lemma 4.45. The periodic Brownian sheet is the centered Gaussian process on $\mathbb{T}^d$ with $\mathbb{E}[\hat{W}(k)\hat{W}(l)] = \delta_{k,-l}(2\pi)^{-2d}(k_1 \cdots k_d)^{-2}\mathbf{1}_{k\in\mathbb{Z}_0^d}$ for the Fourier transform $\hat{f} = \mathscr{F}_{\mathbb{T}^d} f$ on the torus and $\mathbb{Z}_0^d := (\mathbb{Z} \setminus \{0\})^d$. Under the assumption that $\alpha > 3/2$, $\gamma \in (\frac{3}{2\alpha}, 1)$, one can show that $P(\partial_j\partial_i W) \odot \partial_i W$ for $i, j = 1, \ldots, d$, exists as limit of the mollified resonant products $P(\partial_j\partial_i W^n) \odot \partial_i W^n$ in $\mathscr{M}_T^\gamma \mathscr{C}^\vartheta$ for $\vartheta < \alpha\gamma - 2$. In the case $j \neq i$, one can even show that the resonant product $P(\partial_j\partial_i W) \odot \partial_i W$ lies in $\mathscr{M}_T^\gamma \mathscr{C}^{\tilde{\vartheta}}$ with $0 < \tilde{\vartheta} < \alpha\gamma - 3/2$. Thus, our methods allow us to solve the multidimensional SDE with drift given by the gradient of the Brownian sheet on $\mathbb{R}^d$ and driven by α-stable Lévy noise for $\alpha > 7/4$.

5. Periodic homogenization for singular SDEs

We prove a functional central limit theorem for the solution to singular SDEs of the form

$$dX_t = F(X_t)dt + dL_t, \quad X_0 = x \tag{5.1}$$

where L is a d-dimensional α-stable symmetric and non-degenerate Lévy process for $\alpha \in (1,2]$ and periodic drift $F \in (\mathscr{C}^\beta)^d =: \mathscr{C}^\beta_{\mathbb{R}^d}$ for $\beta < 0$. The drift does not depend on a time variable in this chapter. Sections 5.1 to 5.5 prepare the theoretical foundation that leads to the main Theorem 5.26 in Section 5.6.

5.1. Preliminaries

This section gives an introduction to periodic Besov spaces and Schauder and exponential Schauder estimates on such. Furthermore, we introduce the projected solution $X^{\mathbb{T}^d}$ of X onto the torus and its generator $\mathfrak{L}$ and semigroup. We define the space of enhanced distributions $\mathscr{X}^{\beta,\gamma}_\infty$, which differs from the space considered in the previous chapters. This section finishes with a summary on our strategy in proving the convergence results (5.7) and (5.8).

A periodic (or 1-periodic) distribution u satisfies $u(\varphi(\cdot+1)) = u(\varphi)$ for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Let $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ denote the torus and define the Besov space on the torus as $B^\theta_{p,q}(\mathbb{T}^d)$ as in (2.32) and (2.33), i.e. by replacing the Fourier transform on $\mathbb{R}^d$ by the one on $\mathbb{T}^d$ in the definition (3.1) (cf. [ST87, Section 3.5]). The Fourier transform on the torus is defined by $\hat{f}(k) := \mathscr{F}_{\mathbb{T}^d} f(k) := \int_{\mathbb{T}^d} e^{-2\pi i k\cdot x} f(x)dx$, $k \in \mathbb{Z}^d$, for $f \in L^2(\mathbb{T}^d)$.
To a distribution $u \in \mathscr{S}'(\mathbb{T}^d)$, we can associate a periodic distribution on $\mathbb{R}^d$ via $u^{\mathbb{R}^d}(\phi) := u(\sum_{z\in\mathbb{Z}^d} \phi(\cdot + z))$, $\phi \in \mathscr{S}(\mathbb{R}^d)$ and vice versa, cf. [ST87, Section 3.2]. If $u \in \mathscr{C}^\theta(\mathbb{T}^d) = B^\theta_{\infty,\infty}(\mathbb{T}^d)$, then we have $u^{\mathbb{R}^d} \in \mathscr{C}^\theta$. Thus, $B^\theta_{\infty,\infty}(\mathbb{T}^d)$ is simply the space of periodic distributions on $\mathbb{R}^d$, that are in $B^\theta_{\infty,\infty}$. The periodic Besov space $B^\theta_{\infty,\infty}(\mathbb{T})$ we also introduced in Section 4.6.
In the following, we will not distinguish between $F \in (\mathscr{C}^\beta(\mathbb{T}^d))^d$ and the periodic version on $\mathbb{R}^d$, $F^{\mathbb{R}^d} \in (\mathscr{C}^\beta)^d$, whenever there is no danger of confusion. We understand (5.1) as a singular SDE with periodic coefficient $F^{\mathbb{R}^d}$ and in particular the existence results from Chapter 4 apply. In the following, we will also consider the projected process $(X^{\mathbb{T}^d}_t) = (\iota(X_t))$ for the canonical projection $\iota : \mathbb{R}^d \to \mathbb{T}^d$, $x \mapsto [x] = x \mod \mathbb{Z}^d$,

with generator $\mathfrak{L}$ defined below, acting on functions $f:\mathbb{T}^d\to\mathbb{R}$.
For the homogenization, we need to distinguish between the cases $\alpha\in(1,2)$ and $\alpha=2$, due to the different scaling and limit behaviour. For $\alpha\in(1,2)$, we consider a symmtric α-stable Lévy process satifying the non-degeneracy Assumption 3.4. If $\alpha=2$, the symmetric α-stable process L has the same law as the Brownian motion $(\sqrt{C}B_t)_{t\geqslant 0}$ for some covariance matrix $C\in\mathbb{R}^{d\times d}$ (cf. [Sat99, Theorem 14.2]). Thus in the case $\alpha=2$ we set, without loss of generality and to ease notation, $-\mathscr{L}_\nu^\alpha:=\frac{1}{2}\Delta$, which is the generator of the standard Brownian motion, whereas in the general case, one would consider the generator $\sum_{i,j=1,\dots,d}C(i,j)\partial_{x_i}\partial_{x_j}$.
This chapter moreover yields a characterization of the domain $\operatorname{dom}(\mathfrak{L})$ of the generator $\mathfrak{L}$, cf. Theorem 5.17, with

$$\mathfrak{L}f:=-\mathscr{L}_\nu^\alpha f+F\cdot\nabla f$$

acting on functions $f:\mathbb{T}^d\to\mathbb{R}$. That is, the generator of the Markov process $(X_t^{\mathbb{T}^d})_{t\geqslant 0}$ with compact state space $\mathbb{T}^d$. We denote its semigroup by $(T_t^{\mathbb{T}^d})_{t\geqslant 0}$ with $T_t^{\mathbb{T}^d}f:=T_tf^{\mathbb{R}^d}$, $f\in L^\infty(\mathbb{T}^d)$, with the semigroup $(T_t)_{t\geqslant 0}$ of the Markov process (X_t) on $\mathbb{R}^d$ with periodic drift $F^{\mathbb{R}^d}$.
The semigroup $(P_t^{\mathbb{T}^d})$ of the generalized fractional Laplacian $(-\mathscr{L}_\nu^\alpha)$ acting on functions on the torus, is analogously defined as $P_t^{\mathbb{T}^d}f:=P_tf^{\mathbb{R}^d}$ and the semigroup estimates for (P_t) imply the estimates for $(P_t^{\mathbb{T}^d})$ on the periodic Besov spaces $\mathscr{C}^\theta(\mathbb{T}^d)=\mathscr{C}^\theta_\infty(\mathbb{T}^d)$ (due to $u\in L^\infty(\mathbb{T}^d)$ implying $u^{\mathbb{R}^d}\in L^\infty(\mathbb{R}^d)$ and vice versa). The following lemma states the semigroup estimates for $(P_t^{\mathbb{T}^d})$ on $\mathscr{C}_2^\theta(\mathbb{T}^d)$, that will be employed in this chapter. The proof can be found in Appendix A. For $\alpha\in(1,2)$, the fractional Laplacian on smooth test functions $f\in C^\infty(\mathbb{T}^d)$ is defined via $\mathscr{L}_\nu^\alpha f=\mathscr{F}^{-1}_{\mathbb{T}^d}(\mathbb{Z}^d\ni k\mapsto\psi_\nu^\alpha(k)\hat{f}(k))$ and ψ_ν^α as in (3.1). Lemma 5.1 in particular proves the extension of $\mathscr{L}_\nu^\alpha$ to Besov spaces $\mathscr{C}_2^\beta(\mathbb{T}^d)$.

Lemma 5.1. *Let $u\in\mathscr{C}_2^\beta(\mathbb{T}^d)$ for $\beta\in\mathbb{R}$. Then the following estimates hold true*

$$\|\mathscr{L}_\nu^\alpha u\|_{\mathscr{C}_2^{\beta-\alpha}(\mathbb{T}^d)}\lesssim\|u\|_{\mathscr{C}_2^\beta(\mathbb{T}^d)}. \tag{5.2}$$

Moeover, for any $\theta\geqslant 0$ and $\vartheta\in[0,\alpha]$,

$$\|P_tu\|_{\mathscr{C}_2^{\beta+\theta}(\mathbb{T}^d)}\lesssim(t^{-\theta/\alpha}\vee 1)\|u\|_{\mathscr{C}_2^\beta(\mathbb{T}^d)},\quad\|(P_t-\mathrm{Id})u\|_{\mathscr{C}_2^{\beta-\vartheta}(\mathbb{T}^d)}\lesssim t^{\vartheta/\alpha}\|u\|_{\mathscr{C}_2^\beta(\mathbb{T}^d)}. \tag{5.3}$$

For functions with vanishing zero-order Fourier mode, we can improve the Schauder estimates for large $t>0$. This is established in the following lemma.

Lemma 5.2. *Let (P_t) be the $(-\mathscr{L}_\nu^\alpha)$-semigroup on the torus $\mathbb{T}^d$ as defined above. Then for $g\in\mathscr{C}_p^\beta$, $\beta\in\mathbb{R}$, $p\in[1,\infty]$, with $\hat{g}(0)=\mathscr{F}_{\mathbb{T}^d}(g)(0)=0$, exponential Schauder estimates hold true. That is, for any $\theta\geqslant 0$, there exists $c>0$, such that*

$$\|P_tg\|_{\mathscr{C}_p^{\beta+\theta}(\mathbb{T}^d)}\lesssim t^{-\theta/\alpha}e^{-ct}\|g\|_{\mathscr{C}_p^\beta(\mathbb{T}^d)}.$$

Proof. By the assumption of vanishing zero-order Fourier mode, we have

$$P_t g = \sum_{k\neq 0} \exp(-t|2\pi k|^\alpha)\hat{g}(k)e_k.$$

Thus, for any $\theta \geqslant 0$, we obtain an estimate for the Littlewood-Paley blocks:

$$\|\Delta_j(P_t g)\|_{L^p} \lesssim \|g\|_{\mathscr{C}_p^\beta(\mathbb{T}^d)} \min\big(2^{-j\beta}\exp(-t|2\pi|^\alpha),\, 2^{-j(\beta+\theta)}(t^{-\theta/\alpha}\vee 1)\big).$$

The claim thus follows by interpolation. □

In the sequel, we will employ the following duality result for Besov spaces on the torus. For Besov spaces on $\mathbb{R}^d$, the result is proven in [BCD11, Proposition 2.76]. The same proof applies for Besov spaces on the torus (cf. also [ST87, Theorem in Section 3.5.6]).

Lemma 5.3. *Let $\theta \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{T}^d)$. Then we have the duality estimate:*

$$|\langle f, g\rangle| \lesssim \|f\|_{B^\theta_{2,2}(\mathbb{T}^d)}\|g\|_{B^{-\theta}_{2,2}(\mathbb{T}^d)}. \tag{5.4}$$

In particular, the mapping $(f,g)\mapsto \langle f,g\rangle$ can be extended uniquely to $f \in B^\theta_{2,2}(\mathbb{T}^d)$, $g \in B^{-\theta}_{2,2}(\mathbb{T}^d)$.

Motivated by the corresponding characterization of periodic Besov spaces from [ST87, Section 3.5.4], we define the homogeneous Besov space on the torus for $\theta \in (0,1)$ with notation $\Delta_h u(x) := u(x+h) - u(x)$, $h, x \in \mathbb{T}^d$ as follows:

$$\dot{B}^\theta_{2,2}(\mathbb{T}^d) := \left\{ u \in L^2(\mathbb{T}^d) \;\middle|\; \|u\|^2_{\dot{B}^\theta_{2,2}(\mathbb{T}^d)} := \int_{\mathbb{T}^d} |h|^{-2\theta}\|\Delta_h u\|^2_{L^2(\mathbb{T}^d)}\frac{dh}{|h|^d} < \infty \right\}. \tag{5.5}$$

Using derivatives of u, one can define homogeneous periodic Besov spaces in that way also for $\theta > 1$ (cf. [ST87, Section 3.5.4]), but we will not need them below. Let us also define the periodic Bessel-potential space or fractional Sobolev space for $s \in \mathbb{R}$,

$$H^s(\mathbb{T}^d) = \left\{ u \in \mathscr{S}'(\mathbb{T}^d) \;\middle|\; \|u\|^2_{H^s(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^s|\hat{f}(k)|^2 < \infty \right\},$$

and the homogeneous periodic Bessel-potential space

$$\dot{H}^s(\mathbb{T}^d) = \left\{ u \in \mathscr{S}'(\mathbb{T}^d) \;\middle|\; \|u\|^2_{H^s(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d}|k|^{2s}|\hat{f}(k)|^2 < \infty \right\}.$$

We refer to [ST87, (iv) of Theorem, Section 3.5.4] for an equivalent characterization of spaces $B^\theta_{2,2}(\mathbb{T}^d)$ for $\theta \in (0,1]$ in terms of the differences $\Delta_h u$.
For time-independent drifts, the definition of enhanced distributions Definition 3.20 from Chapter 3 simplifies. That is, for $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$, we assume that $(F_1 = F, F_2) \in$

$\mathscr{X}_{\infty}^{\beta,\gamma}(\mathbb{T}^d)$, i.e. $(F^{\mathbb{R}^d}, F_2^{\mathbb{R}^d}) \in \mathscr{X}_{\infty}^{\beta,\gamma}$, where

$$\mathscr{X}_{\infty}^{\beta,\gamma} := cl(\{\big(\eta, (P_t(\partial_i\eta^j) \odot \eta^k)_{i,j,k\in\{1,\ldots,d\}}\big) \mid \eta \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)\}) \tag{5.6}$$

for the closure in $\mathscr{C}^{\beta+(1-\gamma)\alpha} \times \mathscr{M}_{\infty,0}^{\gamma}\mathscr{C}^{2\beta+\alpha-1}$ with $\gamma \in (0,1)$, where

$$\mathscr{M}_{\infty,0}^{\gamma}X = \{u : (0,\infty) \to X \mid \exists C > 0, \forall t > 0, \|u_t\|_X \leqslant C[t^{-\gamma} \vee 1]\}.$$

The notation $\mathscr{M}_{\infty,0}^{\gamma}X$ shall indicate that the blow-up occures at time $t = 0$. Furthermore, the assumption on the enhanced distribution in (5.6) is stronger compared to Chapter 3. First, we do not fix a $T > 0$, but instead require that F is an enhanced distribution for any $T > 0$. This assumption will be needed in (5.3) to solve the resolvent equation. Second, we allow for three different indices i, j, k in (5.6). This assumption is due to the fact that we also solve the adjoint equation, i.e. the Fokker-Planck equation. For the Fokker-Planck equation, we will encounter the products $P_t(\partial_i F^i) \odot F^j$ for $i, j = 1, \ldots, d$, whereas for the Kolmogorov equation, we saw $P_t(\partial_i F^j) \odot F^i$ for i, j. To cover both products, we assume in this chapter (5.6).

Strategy to prove the main result

To prove the CLT in Theorem 5.26, we distinguish between the cases $\alpha = 2$ and $\alpha \in (1,2)$. Let us first consider the case $\alpha = 2$. As mentioned above, $\alpha = 2$ shall refer to taking $L = B$ for a standard Brownian motion B. Then, motivated by results from periodic homogenization for SDEs with periodic and C_b^2 coeffients (cf. [BLP78, Chapter 3, Section 4.2]), we prove in Section 5.4 existence and uniqueness of an invariant, ergodic measure π for X and in Theorem 5.26 the following weak convergence

$$\left(\frac{1}{\sqrt{n}}(X_{nt} - nt\langle F\rangle_\pi)\right)_{t\in[0,T]} \xrightarrow{d} (\sqrt{D}W_t)_{t\in[0,T]}, \tag{5.7}$$

where W is a standard d-dimensional Brownian motion and a constant diffusion matrix D with entries

$$D(i,j) := \int_{\mathbb{T}^d} (e_i + \nabla\chi^i(x))(e_j + \nabla\chi^j(x))^T \pi(dx).$$

for $i, j = 1, \ldots, d$ and e_i denoting the i-th euclidean unit vector. Herein, $\chi \in (L^2(\pi))^d$ solves the singular Poisson equation

$$(-\mathfrak{L})\chi^i = F^i - \langle F^i\rangle_\pi$$

for $i = 1, \ldots, d$.

In the pure Lévy noise case $\alpha \in (1,2)$, to observe a non-trivial limit, we need to consider the scaling $n^{-1/\alpha}$ for the fluctuations around the mean $\langle F\rangle_\pi$. In analogy to [Fra07], we prove in Theorem 5.26 the following weak convergence

$$\left(\frac{1}{n^{1/\alpha}}(X_{nt} - nt\langle F\rangle_\pi)\right)_{t\in[0,T]} \xrightarrow{d} (L_t)_{t\in[0,T]}, \tag{5.8}$$

where L is a stable process with generator $(-\mathscr{L}_\nu^\alpha)$. Compared to the Brownian case, in the pure stable noise case, there is no diffusivity enhancement in the limit.
In the following, we briefly summarize our strategy to prove (5.7) and (5.8).
For existence of π, we solve in Section 5.2 the singular Fokker-Planck equation with the paracontrolled approach in $\mathscr{C}_1^{\alpha+\beta-1}$, yielding a continuous (as $\alpha+\beta-1>0$) Lebesgue-density. Furthermore, we prove a strict maximum principle for the Fokker-Planck equation. In Section 5.4 an application of Doeblin's theorem then yields existence and uniqueness of the invariant ergodic probability measure π for $\mathfrak{L}$ with a strictly positive Lebesgue density ρ_∞. Doeblin's theorem also gives L^∞-spectral gap estimates on the semigroup $(T_t^{\mathbb{T}^d})_{t\geqslant 0}$ associated to $\mathfrak{L}$, which means that the process $X^{\mathbb{T}^d}$ is exponentially ergodic.
We then extend those spectral gap estimates to $L^2(\pi)$-spectral gap estimates. This enables to solve the Poisson equation in Corollary 5.19 for right-hand sides that are in $L^2(\pi)$ and that have vanishing mean under π. In particular, we can solve the Poisson equation with right-hand side $F^m - \langle F^m\rangle_\pi$ for $F^m \in C^\infty(\mathbb{T}^d)$ with $F^m \to F$ in $\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$, denoting the solution by χ^m.
We then prove convergence of $(\chi^m)_m$ in $L^2(\pi)$ utilizing a Poincaré-type estimate for the opertor $\mathfrak{L}$ and combining with the theory from [KLO12]. Via solving the resolvent equation $(\lambda-\mathfrak{L})g = G$ in Section 5.3 with the paracontrolled approach for right-hand-sides in $G \in L^2(\pi)$ or $G = F^i$, $i = 1, ..., d$, we then obtain in Section 5.5 convergence of $(\chi^m)_m$ in $(\mathscr{C}_2^{\alpha+\beta}(\mathbb{T}^d))^d$ to a limit χ which solves the Poisson equation $(-\mathfrak{L})\chi = F - \langle F\rangle_\pi$ with singular right-hand side $F - \langle F\rangle_\pi$. Here, $\langle F\rangle_\pi$ can be defined in a stable manner using the regularity, respectively the paracontrolled structure, of the density ρ_∞, cf. Lemma 5.20.
Decomposing the drift in terms of the solution to the Poisson equation and Dynkin's martingale, we can finally prove the functional CLT in Section 5.6.
Via Feynman-Kac formula, the CLT yields the periodic homogenization result of Corollary 5.27 for the solution to the associated Cauchy problem with operator $\mathfrak{L}^\varepsilon$ as $\varepsilon\to 0$, where formally $\mathfrak{L}^\varepsilon f = -\mathscr{L}_\nu^\alpha f + \varepsilon^{1-\alpha}F(\varepsilon^{-1}\cdot)\cdot\nabla f$.

5.2. Singular Fokker-Planck equation and a strict maximum principle

This section features the results on the Fokker-Planck equation, Theorem 5.4 and Proposition 5.8, that will be of use in Section 5.4 below.

Let us define the blow-up spaces for $\gamma \in (0,1)$,

$$\mathscr{M}_{T,0}^{\gamma} X := \{ u : (0,T] \to X \mid \sup_{t\in[0,T]} t^{\gamma} \|u_t\|_X < \infty \}$$

and

$$C_{T,0}^{1,\gamma} X := \left\{ u : (0,T] \to X \;\middle|\; \sup_{0\leqslant s<t\leqslant T} \frac{s^{\gamma}\|u_t - u_s\|_X}{|t-s|} < \infty \right\}$$

with blow-up at $t = 0$.
The solution to the Fokker-Planck eqution with initial condition equal to a Dirac measure, will have a blow-up at time $t = 0$ due to the singularity of the initial condition. A direct computation shows that the Dirac measure in $x \in \mathbb{R}^d$ satisfies $\delta_x \in \mathscr{C}_p^{-d(1-\frac{1}{p})}$ for any $p \in [1,\infty]$, in particular $\delta_x \in \mathscr{C}_1^0$. Moreover, one can show that the map $x \mapsto \delta_x \in \mathscr{C}_1^{-\varepsilon}$ is continuous for any $\varepsilon > 0$. The next theorem proves existence of a mild solution to the Fokker-Planck equation

$$(\partial_t - \mathfrak{L}^*)\rho_t = 0, \quad \rho_0 = \mu,$$

with initial condition $\mu \in \mathscr{C}_1^{-\varepsilon}$ for small $\varepsilon > 0$. Here, $\mathfrak{L}^*$ denotes the formal Lebesgue-adjoint to $\mathfrak{L}$,

$$\mathfrak{L}^* f := -\mathscr{L}_{\nu}^{\alpha} f - \nabla \cdot (Ff) = -\mathscr{L}_{\nu}^{\alpha} f - div(Ff).$$

We refer to Chapter 3 for the paraproducts and the notation. The proof of Theorem 5.4 is similar to Theorem 3.25.

Theorem 5.4. *Let $T > 0$, $\alpha \in (1,2]$ and $p \in [1,\infty]$. Let either $\beta \in (\frac{1-\alpha}{2}, 0)$ and $F \in \mathscr{C}_{\mathbb{R}^d}^{\beta}$ or $F \in \mathscr{X}_{\infty}^{\beta,\gamma'}$ for $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$, $\gamma' \in (\frac{2\beta+2\alpha-1}{\alpha}, 1)$.*
Then, for any small enough $\varepsilon > 0$ and any initial condition $\mu \in \mathscr{C}_p^{-\varepsilon}$, there exists a unique mild solution ρ to the Fokker-Planck equation in $\mathscr{M}_{T,0}^{\gamma}\mathscr{C}_p^{\alpha+\beta-1} \cap C_T^{1-\gamma}\mathscr{C}_p^{\beta} \cap C_{T,0}^{1,\gamma}\mathscr{C}_p^{\beta}$ for $\gamma \in (C(\varepsilon), 1)$ (for some $C(\varepsilon) \in (0,1)$) in the Young regime and $\gamma \in (\gamma', \frac{\alpha\gamma'}{2-\alpha-3\beta})$ in the rough regime, i.e.

$$\rho_t = P_t\mu + \int_0^t P_{t-s}(-\nabla \cdot (F\rho_s))ds, \tag{5.9}$$

where $(P_t)_{t\geqslant 0}$ denotes the $(-\mathscr{L}_{\nu}^{\alpha})$-semigroup.
In the rough case, the solution satisfies

$$\rho_t = \rho_t^{\sharp} + \rho_t \varolessthan I_t(-\nabla \cdot F) \tag{5.10}$$

where $\rho_t^{\sharp} \in \mathscr{M}_{T,0}^{\gamma}\mathscr{C}_p^{2(\alpha+\beta)-2} \cap C_T^{1-\gamma}\mathscr{C}_p^{2\beta-2+\alpha} \cap C_{T,0}^{1,\gamma}\mathscr{C}_p^{2\beta-2+\alpha}$ and $I_t(v) := \int_0^t P_{t-s}v_s ds$. Moreover, the solution depends continuously on the data $(F,\mu) \in \mathscr{X}_{\infty}^{\beta,\gamma'} \times \mathscr{C}_p^{-\varepsilon}$. Furthermore, for any fixed $t > 0$, the solution satisfies $(\rho_t, \rho_t^{\sharp}) \in \mathscr{C}^{\alpha+\beta-1} \times \mathscr{C}^{2(\alpha+\beta)-2}$.

If (F,μ) are 1-periodic distributions, then the solution ρ_t is 1-periodic.

Proof. We will prove that we can solve the Fokker-Planck equation for initial conditions $\mu \in \mathscr{C}_p^{-\varepsilon}$ for $\varepsilon = -((1-\tilde{\gamma})\alpha + \beta)$ for $\tilde{\gamma} \in [\frac{\alpha+\beta}{\alpha}, 1)$ in the Young regime and for $\varepsilon = -((2-\tilde{\gamma})\alpha + 2\beta - 1)$ for $\tilde{\gamma} \in [\frac{2\beta+2\alpha-1}{\alpha} \vee 0, \gamma']$ in the rough regime. In the Young regime, we obtain a solution $\rho \in \mathscr{M}_{T,0}^{\gamma}\mathscr{C}_p^{\alpha+\beta-1} \cap C_T^{1-\gamma}\mathscr{C}_p^{\beta} \cap C_{T,0}^{\gamma,1}\mathscr{C}_p^{\beta}$ for $\gamma = \tilde{\gamma}$ and the proof is analogous to Theorem 3.19. We thus only give the proof in the rough regime. To that aim, let us define, analogously as in the proof of Theorem 3.25 for $\gamma \in (\gamma', 1)$ as there,

$$\mathscr{L}_{T,p}^{\gamma,\theta} := \mathscr{M}_{T,0}^{\gamma}\mathscr{C}_p^{\theta} \cap C_T^{1-\gamma}\mathscr{C}_p^{\theta-\alpha} \cap C_{T,0}^{1,\gamma}\mathscr{C}_p^{\theta-\alpha}$$

and the paracontrolled solution space

$$\mathscr{D}_{T,p}^{\gamma} := \{(u,u') \in \mathscr{L}_{T,p}^{\gamma',\alpha+\beta-1} \times (\mathscr{L}_{T,p}^{\gamma,\alpha+\beta-1})^d \mid u_t^{\sharp} = u_t - u_t' \otimes I_t(-\nabla\cdot F) \in \mathscr{L}_{T,p}^{\gamma,2(\alpha+\beta)-2}\}$$

for $p \in [1,\infty]$, equipped with the norm

$$\|u-w\|_{\mathscr{D}_{T,p}^{\gamma}} := \|u-w\|_{\mathscr{L}_{T,p}^{\gamma',\alpha+\beta-1}} + \|u'-w'\|_{(\mathscr{L}_{T,p}^{\gamma,\alpha+\beta-1})^d} + \|u^{\sharp}-w^{\sharp}\|_{\mathscr{L}_{T,p}^{\gamma,2(\alpha+\beta)-1}},$$

which makes the space a Banach space.
For $\mu \in \mathscr{C}_p^{-\varepsilon}$, $\varepsilon = -((2-\tilde{\gamma})\alpha + 2\beta - 2)$, we first prove that we obtain a paracontrolled solution $\rho \in \mathscr{D}_{T,p}^{\gamma}$. The proof is similar to Theorem 3.25 and we only give the essential arguments of the proof. Notice that compared to Theorem 3.25, here we consider the operator $\mathfrak{L}^*$ instead of $\mathfrak{L}$ and initial conditions in $\mathscr{C}_p^{-\varepsilon}$ for $\varepsilon = -((2-\tilde{\gamma})\alpha + 2\beta - 2)$, hence $\rho_0 = \rho_0^{\sharp}$.
For $\rho \in \mathscr{D}_{T,p}^{\gamma}$ the resonant product $F \odot \rho = (F^i \odot \rho)_{i=1,..,d}$ is well-defined and satisfies

$$F^i \odot \rho = F^i \odot \rho^{\sharp} + \rho' \cdot (F^i \odot I_t(\nabla\cdot F)) + C_1(\rho', I_t(\nabla\cdot F), F^i)$$

for the paraproduct commutator

$$C_1(f,g,h) := (f \otimes g) \odot h - f \cdot (g \odot h).$$

Using the paraproduct estimates from the preliminaries of Chapter 3, we obtain Lipschitz dependence of the product on $(F,\rho) \in \mathscr{X}_{\infty}^{\beta,\gamma'} \times \mathscr{D}_{T,p}^{\gamma}$, that is,

$$\begin{aligned}
&\|F \odot \rho\|_{\mathscr{M}_T^{\gamma'}\mathscr{C}_p^{\alpha+2\beta-1}} \\
&\quad\lesssim \|F\|_{\mathscr{X}_{\infty}^{\beta,\gamma'}}(1+\|F\|_{\mathscr{X}_{\infty}^{\beta,\gamma'}})\big(\|\rho\|_{\mathscr{M}_T^{\gamma'}\mathscr{C}_p^{\alpha+\beta-1}} + \|\rho'\|_{(\mathscr{M}_T^{\gamma'}\mathscr{C}_p^{\alpha+\beta-1-\delta})^d} + \|\rho^{\sharp}\|_{\mathscr{M}_T^{\gamma'}\mathscr{C}_p^{2(\alpha+\beta)-2-\delta}}\big) \\
&\quad\lesssim \|F\|_{\mathscr{X}_{\infty}^{\beta,\gamma'}}(1+\|F\|_{\mathscr{X}_{\infty}^{\beta,\gamma'}})\big(\|\rho\|_{\mathscr{M}_T^{\gamma'}\mathscr{C}_p^{\alpha+\beta-1}} + \|\rho'\|_{(\mathfrak{L}_{T,p}^{\gamma,\alpha+\beta-1})^d} + \|\rho^{\sharp}\|_{\mathfrak{L}_{T,p}^{\gamma,2(\alpha+\beta)-2}}\big) \\
&\quad\lesssim \|F\|_{\mathscr{X}_{\infty}^{\beta,\gamma'}}(1+\|F\|_{\mathscr{X}_{\infty}^{\beta,\gamma'}})\|\rho\|_{\mathscr{D}_{T,p}^{\gamma}}
\end{aligned}$$

for $\delta = \alpha - \alpha\frac{\gamma'}{\gamma}$, using moreover the interpolation estimates from Lemma 3.17.

The contraction map will be defined as

$$\mathscr{D}^{\gamma}_{\overline{T},p} \ni (\rho,\rho') \mapsto (\phi(\rho),\rho) \in \mathscr{D}^{\gamma}_{\overline{T},p}$$

with

$$\phi(\rho)_t := P_t\mu + I_t(-\nabla\cdot(F\rho)).$$

Here, $\overline{T}$ will be chosen small enough, such that the above map becomes a contraction. Afterwards the solutions on the subintervals of length $\overline{T}$ are patched together. Notice that the fixed point satisfies $\rho' = \rho$.
As $\varepsilon = -((2-\tilde{\gamma})\alpha + 2\beta - 2)$, we obtain by the semigroup estimates from Lemma 3.7, that

$$\|P_t\mu\|_{\mathscr{C}_p^{2(\alpha+\beta)-2}} \lesssim t^{-\tilde{\gamma}}\|\mu\|_{\mathscr{C}_p^{-\varepsilon}}. \tag{5.11}$$

Utilizing Corollary 3.12 (which applies by a time change also for blow-up-spaces with blow-up at $t=0$ instead of blow-ups at $t=T$) and the estimate for the resonant product yields

$$\begin{aligned}\|I(\nabla\cdot(F\rho))\|_{\mathscr{L}_{T,p}^{\gamma,\alpha+\beta-1}} &\lesssim T^{\gamma-\gamma'}\|\nabla\cdot(F\rho)\|_{\mathscr{M}_{T,0}^{\gamma'}\mathscr{C}_p^{\beta-1}}\\ &\lesssim T^{\gamma-\gamma'}\|F\|_{\mathscr{X}_\infty^{\beta,\gamma'}}(1+\|F\|_{\mathscr{X}_\infty^{\beta,\gamma'}})\|\rho\|_{\mathscr{D}_{T,p}^{\gamma}}.\end{aligned}$$

Moreover, we have that for a solution ρ,

$$\rho_t^\sharp = P_t\mu + C_2(\rho,\nabla\cdot F)_t + I_t(-\nabla\cdot(\rho\odot F)) + I_t(-\nabla\cdot(\rho\circledcirc F)) + I_t(-\nabla\rho\otimes F)$$

for the semigroup commutator

$$C_2(u,v) = I(u\otimes v) - u\otimes I(v).$$

Using (5.11) and Lemma 3.14, we obtain

$$\|\rho^\sharp\|_{\mathscr{L}_{T,p}^{\gamma,2(\alpha+\beta)-2}} \lesssim \|\mu\|_{\mathscr{C}_p^{-\varepsilon}} + T^{\gamma-\gamma'}\|F\|_{\mathscr{X}_\infty^{\beta,\gamma'}}\|\rho\|_{\mathfrak{L}_{T,p}^{\gamma',\alpha+\beta-1}}.$$

Hence, as $\gamma > \gamma'$, replacing T by $\overline{T} \leqslant T$ small enough, we obtain a paracontrolled solution in $\mathscr{D}^{\gamma}_{\overline{T},p}$. Then, we paste the solutions on the subintervals together to obtain a solution on $[0,T]$, cf. in the proof of Theorem 3.25.
It remains to justify that the solution at fixed times $t>0$ satisfies $(\rho_t,\rho_t^\sharp) \in \mathscr{C}^{\alpha+\beta-1} \times \mathscr{C}^{2(\alpha+\beta-1)}$, i.e. that we can increase the integrability from p to ∞. From the above, we obtain $(\rho,\rho^\sharp) \in C([t,T],\mathscr{C}_p^{\alpha+\beta-1}) \times C([t,T],\mathscr{C}_p^{2(\alpha+\beta-1)})$. Then, we can apply the argument to increase the integrability, that was carried out in the end of the proof of [PvZ22, Proposition 2.4], to obtain that indeed $(\rho,\rho^\sharp) \in C([t,T],\mathscr{C}^{\alpha+\beta-1}) \times C([t,T],\mathscr{C}^{2(\alpha+\beta-1)})$ for any $t \in (0,T)$.

The continuous dependence of the solution on the data (F, μ) follows analogously as in Theorem 3.30, with the above estimates and a Gronwall-type argument.
If (F, μ) are 1-periodic distributions, then $P_t\mu = p_t * \mu$ is 1-periodic, as the convolution with the fractional heat-kernel p_t with a periodic distribution yields a periodic function and the fixed point argument can be carried out in the periodic solution space $\mathscr{D}^{\gamma}_{T,p}(\mathbb{T}^d)$. □

Corollary 5.5. *Let X be the unique martingale solution of the singular periodic SDE* (5.1) *for $\mathfrak{L}$ (acting on functions $f : \mathbb{R}^d \to \mathbb{R}$), starting at $x \in \mathbb{R}^d$. Let $(t, y) \mapsto \rho_t(x, y)$ be the mild solution of the Fokker-Planck equation with $\rho_0 = \delta_x$ from Theorem 5.4. Then for any $t > 0$, the map $(x, y) \mapsto \rho_t(x, y)$ is continuous.*
Furthermore, for any $f \in L^\infty(\mathbb{R}^d)$,

$$\mathbb{E}_{X_0=x}[f(X_t)] = \int_{\mathbb{R}^d} f(y)\rho_t(x, y)dy, \tag{5.12}$$

that is, $\rho_t(x, \cdot)$ is the density of $Law(X_t)$, if $X_0 = x$, with respect to the Lebesgue measure. In particular, for the projected solution $X^{\mathbb{T}^d}$ with drift $F \in \mathscr{X}^{\beta,\gamma'}_{\infty}(\mathbb{T}^d)$ and $f \in L^\infty(\mathbb{T}^d)$ and $z \in \mathbb{T}^d$,

$$\mathbb{E}_{X_0^{\mathbb{T}^d}=z}[f(X_t^{\mathbb{T}^d})] = \int_{\mathbb{T}^d} f(w)\rho_t(z, w)dw, \tag{5.13}$$

where, by abusing notation to not introduce a new symbol for the density on the torus, $\rho_t(z, w) := \rho_t(x, y)$ for $(x, y) \in \mathbb{R}^d$ with $(\iota(x), \iota(y)) = (z, w)$, $\iota : \mathbb{R}^d \to \mathbb{T}^d$ denoting the canonical projection.

Remark 5.6. *Let $\rho(x, \cdot)$ be the solution of the Fokker-Planck equation started in δ_x from Theorem 5.4 and u^y solve the Kolmogorov backward equation with terminal condition $u_T = \delta_y$ from Theorem 3.25. Then due to* (5.12) *and the Feynman-Kac formula (approximating F and utilizing the continuity of the solutions maps) we see the equality $\rho_t(x, y) = u^y_{T-t}(x)$.*

Remark 5.7. *If $F \in \mathscr{X}^{\beta,\gamma'}_{\infty}(\mathbb{T}^d)$, then by definition of $(P_t^{\mathbb{T}^d})$, $\rho(z, \cdot)$ is the mild solution of the Fokker-Planck equation on the torus (that is, (P_t) replaced by $(P_t^{\mathbb{T}^d})$ in* (5.9)*) with $\rho_0(z, \cdot) = \delta_z$.*

Proof. Continuity in y follows from $\rho_t(x, \cdot) \in \mathscr{C}^{\alpha+\beta-1}$ and $\alpha + \beta - 1 > 0$. Continuity in x follows from the continuous dependence of the solution on the initial condition δ_x and continuity of the map $x \mapsto \delta_x \in \mathscr{C}_1^{-\varepsilon}$ for $\varepsilon > 0$.
That ρ_t is the density of $\mathrm{Law}(X_t)$ follows by approximation of F by $F^m \in C_b^\infty(\mathbb{R}^d)$ with $F^m \to F$ in $\mathscr{C}^{\beta}_{\mathbb{R}^d}$, respectively in $\mathscr{X}^{\beta,\gamma'}_{\infty}$, using that ρ depends continuously on the data (F, μ) and that $X^m \to X$ in distribution, where X^m is the strong solution to the SDE with drift term F^m (cf. the proof of Theorem 4.2) and the Feynman-Kac formula for classical SDEs. Indeed, for $m \in \mathbb{N}$, we have that for $f \in C_b^2$ (and thus for $f \in L^\infty$

by approximation),

$$u^m_{T-t}(x) = \mathbb{E}_{X^m_0=x}[f(X^m_t)] = \int f(y)\rho^m_t(x,y)dy$$

with $(\partial_t + \mathfrak{L}^m)u^m = \mathscr{G}^{F^m}u^m = 0$, $u^m_T = f$, and $(\partial_t - (\mathfrak{L}^m)^*)\rho = 0$, $\rho_0 = \delta_x$. Now, we let $m \to \infty$ to obtain (5.12). In particular, $\rho_t \geqslant 0$ and $\rho_t \in L^1(dx)$. That ρ_t is well-defined follows as ρ_t is periodic (due to the periodicity assumption on F). Equality (5.13) follows from (5.12) considering $f \circ \iota$ instead of f. □

Proposition 5.8. *Let $\mu \in \mathscr{C}^0_1$ be a positive, nontrivial ($\mu \neq 0$) measure. Let ρ be the mild solution of the Fokker-Planck equation $(\partial_t - \mathfrak{L}^*)\rho_t = 0$ with $\rho_0 = \mu$. Then for any compact $K \subset \mathbb{R}^d$ and any $t > 0$, there exists $c > 0$ such that*

$$\min_{x\in K} \rho_t(x) \geqslant c > 0.$$

Let ρ_t be as in Remark 5.7. Then, in particular, for any $z \in \mathbb{T}^d$, $t > 0$, there exists $c > 0$ such that

$$\min_{x\in \mathbb{T}^d} \rho_t(z,x) \geqslant c > 0.$$

Proof. In the Brownian case, $\alpha = 2$, this follows from the proof of [CFG17, Theorem 5.1]. We give the adjusted argument for $\alpha \in (1, 2]$.
Let p_t be the α-stable density of L_t. Without loss of generality, we assume $\mu = u \in C_b(\mathbb{R}^d)$ with $u \geqslant 0$ and with $u \geqslant 1$ on a ball $B(0,\kappa)$, $\kappa > 0$. Otherwise, we may consider ρ_s for $s > 0$ as an initial condition, for which we know that $\rho_s \in \mathscr{C}^{\alpha+\beta-1} \subset C_b(\mathbb{R}^d)$ and that $\rho_s \geqslant 0$ by Corollary 5.5. Then by continuity there exists a ball $B(x,\kappa)$ where $\rho_s > 0$. Dividing by the lower bound and shifting ρ_s, we can assume that $\rho_s > 1$ on $B(0,\kappa)$.
Let now $\kappa > 0$ and $u \in C_b(\mathbb{R}^d)$ with $u \geqslant 0$ and with $u \geqslant 1$ on the ball $B(0,\kappa)$. Then by the scaling property, we have that

$$p_t * u(y) \geqslant \mathbb{P}(|y + t^{1/\alpha}L_1| \leqslant \kappa) = \mathbb{P}(L_1 \in B(yt^{-1/\alpha}, \kappa t^{-1/\alpha}))$$

Let $y = (\kappa + t\rho)z$ for $z \in B(0,1)$, $\rho \geqslant 0$, so that $y \in B(0,\kappa + t\rho)$. Then we obtain

$$\begin{aligned}
&\mathbb{P}(L_1 \in B(yt^{-1/\alpha}, \kappa t^{-1/\alpha})) \\
&\quad = \mathbb{P}(L_1 \in B(z(\kappa t^{-1/\alpha} + \rho t^{1-1/\alpha}), \kappa t^{-1/\alpha})) \\
&\quad \geqslant \mathbb{P}(2z\cdot L_1 \geqslant |L_1|^2(\kappa t^{-1/\alpha} + \rho t^{1-1/\alpha})^{-1} + (|z|^2-1)[\kappa t^{-1/\alpha} + \rho t^{1-1/\alpha}]) \\
&\quad \geqslant \inf_{|z|\leqslant 1} \mathbb{P}(2z\cdot L_1 \geqslant |L_1|^2(\kappa t^{-1/\alpha} + \rho t^{1-1/\alpha})^{-1} + (|z|^2-1)[\kappa t^{-1/\alpha} + \rho t^{1-1/\alpha}])
\end{aligned}$$

$$= \inf_{|z|=1} \mathbb{P}(2z \cdot L_1 \geqslant |L_1|^2(\kappa t^{-1/\alpha} + \rho t^{1-1/\alpha})^{-1})$$
$$\to \inf_{|z|=1} \mathbb{P}(z \cdot L_1 \geqslant 0) = \frac{1}{2}$$

for $t \to 0$. Here we used that $\alpha > 1$ and that by symmetry of L, for any $z \in B(0,1)$ with $|z| = 1$, $\mathbb{P}(z \cdot L_1 \geqslant 0) = \mathbb{P}(z \cdot L_1 \leqslant 0) = 1 - \mathbb{P}(z \cdot L_1 \geqslant 0)$, because $\mathbb{P}(z \cdot L_1 = 0) = \mathbb{P}(L_1 = 0) = 0$.
Thus, we conclude, that there exists $t_\rho > 0$, such that for all $t \in [0, t_\rho]$ and all $y \in B(0, \kappa + t\rho)$, $p_t * u(y) \geqslant \frac{1}{4}$.
Moreover, we have

$$\rho_t = P_t u + \int_0^t P_{t-s}(-\nabla \cdot (F\rho_s))ds$$

with $P_t u = p_t * u$ and

$$\left\| \int_0^t P_{t-s}(-\nabla \cdot (F\rho_s))ds \right\|_{L^\infty} \leqslant Ct^{(\alpha+\beta-1-\varepsilon)/\alpha}$$

for $\varepsilon \in (0, \alpha + \beta - 1)$ by the semigroup estimates, Lemma 3.7, with $\alpha + \beta - 1 > 0$. Hence, for small enough t, we can achieve

$$\left\| \int_0^t P_{t-s}(-\nabla \cdot (F\rho_s))ds \right\|_{L^\infty} < \frac{1}{8}.$$

Together with the lower bound for $p_t * u$, we obtain that there exists $t_\rho > 0$, such that for all $t \in [0, t_\rho]$ and all $y \in B(0, \kappa + t\rho)$, it holds that

$$\rho_t(y) \geqslant \frac{1}{8}.$$

Using linearity of the equation, we can repeat that argument on $[t_\rho, 2t_\rho]$ etc. Because K is compact, finitely many steps suffice (for large enough t, the ball $B(0, \kappa + t\rho)$ will cover K) to conclude that for all $T > 0$ there exists $c > 0$ such that for all $y \in K$ and all $t \in [0, T]$,

$$\rho_t(y) \geqslant c > 0.$$

□

5.3. Singular resolvent equation

In this and all subsequent sections of this chapter, we write (P_t), respectively (T_t), for the semigroups acting on the periodic Besov spaces $\mathscr{C}_p^\theta(\mathbb{T}^d)$, $p = 2, \infty$, omitting the supercript $\mathbb{T}^d$ that we introduced earlier.
We solve the resolvent equation in Theorem 5.10 for the singular operator $\mathfrak{L}$ and for singular paracontrolled right-hand sides $G = G^\sharp + G' \otimes F$, $G^\sharp \in \mathscr{C}_2^0(\mathbb{T}^d)$, $G' \in$

$(\mathscr{C}_2^{\alpha+\beta-1}(\mathbb{T}^d))^d$, that is

$$(\lambda - \mathfrak{L})g = G,$$

obtaining a solution $g \in \mathscr{C}_2^{\alpha+\beta}(\mathbb{T}^d)$.
The next Lemma proves semigroup and commutator estimates for the I_λ-operator.

Lemma 5.9. *Let $\lambda \geqslant 1$, $\delta \in \mathbb{R}$ and $v \in \mathscr{C}_2^\delta$. Let again $I_\lambda(v) := \int_0^\infty e^{-\lambda t} P_t v dt$. Then, $I_\lambda(v)$ is well-defined in $\mathscr{C}_2^{\beta+\vartheta}(\mathbb{T}^d)$ for $\vartheta \in [0,\alpha]$ and the following estimate holds true*

$$\|I_\lambda(v)\|_{\mathscr{C}_2^{\delta+\vartheta}(\mathbb{T}^d)} \lesssim \lambda^{-(1-\vartheta/\alpha)} \|v\|_{\mathscr{C}_2^{\delta}(\mathbb{T}^d)}. \tag{5.14}$$

Furthermore, for $v \in \mathscr{C}_2^\sigma(\mathbb{T}^d)$, $\sigma < 1$, $u \in \mathscr{C}^\beta(\mathbb{T}^d)$, $\beta \in \mathbb{R}$, and $\vartheta \in [0,\alpha]$, the following commutator estimate holds true:

$$\begin{aligned} \|C_\lambda(v,u)\|_{\mathscr{C}_2^{\sigma+\beta+\vartheta}(\mathbb{T}^d)} &:= \|I_\lambda(v \varodot u) - v \varodot I_\lambda(u)\|_{\mathscr{C}_2^{\sigma+\beta+\vartheta}(\mathbb{T}^d)} \\ &\lesssim \lambda^{-(1-\vartheta/\alpha)} \|v\|_{\mathscr{C}_2^{\sigma}(\mathbb{T}^d)} \|u\|_{\mathscr{C}^{\beta}(\mathbb{T}^d)}. \end{aligned} \tag{5.15}$$

Proof. The proof of (5.14) follows from the semigroup estimates, Lemma 5.1. Indeed, we have

$$\begin{aligned} \|I_\lambda(v)\|_{\mathscr{C}_2^{\delta+\vartheta}(\mathbb{T}^d)} &\leqslant \int_0^\infty e^{-\lambda t} \|P_t v\|_{\mathscr{C}_2^{\delta+\vartheta}(\mathbb{T}^d)} dt \\ &\lesssim \|v\|_{\mathscr{C}_2^{\vartheta}(\mathbb{T}^d)} \int_0^\infty e^{-\lambda t} [t^{-\vartheta/\alpha} \vee 1] dt \\ &= \|v\|_{\mathscr{C}_2^{\vartheta}(\mathbb{T}^d)} \left(\lambda^{-(1-\vartheta/\alpha)} \int_0^1 e^{-t} t^{-\vartheta/\alpha} dt + \lambda^{-1} \int_1^\infty e^{-t} dt \right) \\ &\lesssim \lambda^{-(1-\vartheta/\alpha)} \|v\|_{\mathscr{C}_2^{\vartheta}(\mathbb{T}^d)}, \end{aligned}$$

since $\lambda \geqslant 1$ and where we use that $\int_0^1 e^{-t} t^{-\vartheta/\alpha} dt \leqslant \int_0^1 t^{-\vartheta/\alpha} dt < \infty$ if $\vartheta \in [0,\alpha)$ and $\int_1^\infty e^{-t} dt < \infty$. The bound in the case $\vartheta = \alpha$ follows with

$$\begin{aligned} \|I_\lambda(v)\|_{\mathscr{C}_p^{\delta+\alpha}(\mathbb{T}^d)} &\leqslant \left\| \int_0^1 e^{-\lambda t} P_t v dt \right\|_{\mathscr{C}_2^{\delta+\alpha}(\mathbb{T}^d)} + \int_1^\infty e^{-\lambda t} \|P_t v\|_{\mathscr{C}_p^{\delta+\alpha}(\mathbb{T}^d)} dt \\ &\lesssim \|v\|_{\mathscr{C}_p^{\delta}(\mathbb{T}^d)}, \end{aligned}$$

using Lemma 3.11 to estimate the integral over $[0,1]$ (with, in the notation of that lemma, $T = 1$, $\gamma = 0$, $\sigma = \delta$, $\varsigma = \alpha$, $f_{0,t} = e^{-\lambda t} P_t v$).
The commutator (5.15) is proven analogously using Lemma 3.9. □

Theorem 5.10. *Let $\alpha \in (1,2]$ and $F \in \mathscr{C}^\beta(\mathbb{T}^d)$ for $\beta \in (\frac{1-\alpha}{2}, 0)$ or $F \in \mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$ for $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$ and $\gamma \in (\frac{2\beta+2\alpha-1}{\alpha}, 1)$.*

Then, for $\lambda > 0$ large enough, the resolvent equation

$$R_\lambda g = (\lambda - \mathfrak{L})g = G \tag{5.16}$$

with right-hand side $G = G^\sharp + G' \varolessthan F$, $G^\sharp \in \mathscr{C}_2^0(\mathbb{T}^d)$, $G' \in (\mathscr{C}_2^{\alpha+\beta-1}(\mathbb{T}^d))^d$, possesses a unique solution $g \in \mathscr{C}_2^\theta(\mathbb{T}^d)$, $\theta \in ((2-\beta)/2, \beta+\alpha)$.
If $\beta \in (\frac{2-2\alpha}{3}, \frac{1-\alpha}{2}]$, the solution is paracontrolled, that is,

$$g = g^\sharp + (G' + \nabla g) \varolessthan I_\lambda(F), \quad g^\sharp \in \mathscr{C}_2^{2\theta-1}(\mathbb{T}^d). \tag{5.17}$$

Proof. Consider the paracontrolled solution space

$$\mathscr{D}_2^\theta := \{(g, g') \in \mathscr{C}_2^\theta(\mathbb{T}^d) \times (\mathscr{C}_2^{\theta-1}(\mathbb{T}^d))^d \mid g^\sharp := g - g' \varolessthan I_\lambda(F) \in \mathscr{C}_2^{2\theta-1}(\mathbb{T}^d)\} \tag{5.18}$$

with norm $\|g - h\|_{\mathscr{D}_2^\theta} := \|g - h\|_{\mathscr{C}_2^\theta(\mathbb{T}^d)} + \|g^\sharp - h^\sharp\|_{\mathscr{C}_2^{2\theta-1}(\mathbb{T}^d)} + \|g' - h'\|_{\mathscr{C}_2^{\theta-1}(\mathbb{T}^d)}$, which makes it a Banach space.
The solution g satisfies

$$g = \int_0^\infty e^{-\lambda t} P_t(G + F \cdot \nabla g) dt,$$

i.e. it is the fixed point of the map $\mathscr{C}_2^\theta(\mathbb{T}^d) \ni g \mapsto \phi_\lambda(g) := \int_0^\infty e^{-\lambda t} P_t(G + F \cdot \nabla g) dt \in \mathscr{C}_2^\theta(\mathbb{T}^d)$, respectively, in the rough case $\beta \in ((2-2\alpha)/3, (1-\alpha)/2]$, of the map

$$\mathscr{D}_2^\theta \ni (g, g') \mapsto (\phi_\lambda(g), G' + \nabla g) =: \Phi_\lambda(g, g') \in \mathscr{D}_2^\theta.$$

The product is defined as $F \cdot \nabla g := F \odot \nabla g + F \varolessthan \nabla g + F \varogreaterthan \nabla g$, where for $F \in \mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$ and $g \in \mathscr{D}_2^\theta$,

$$F \odot \nabla g = \sum_{i=1}^d F^i \odot \partial_i g := \sum_{i=1}^d \Big[F^i \odot [\partial_i g^\sharp + \partial_i g' \varolessthan I_\lambda(F)] + g(I_\lambda(\partial_i F) \odot F^i) + C_1(g, I_\lambda(\partial_i F), F^i) \Big],$$

with paraproduct commutator

$$C_1(g, f, h) := (g \varolessthan f) \odot h - g(f \odot h) \tag{5.19}$$

from [GIP15, Lemma 2.4]. Analogously as before, the product of $F \in \mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$ and $g \in \mathscr{D}_2^\theta$ with $\theta > (2-\beta)/2$ can thus be estimated by

$$\|F \cdot g\|_{\mathscr{C}_2^\beta(\mathbb{T}^d)} \lesssim \|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)} (1 + \|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)}) \|g\|_{\mathscr{D}_2^\theta}.$$

The unique fixed point is obtained by the Banach fixed point theorem, where, in the Young case the map ϕ, and in the rough case, $\Phi_\lambda^2 = \Phi_\lambda \circ \Phi_\lambda$ are contractions for large enough $\lambda > 0$. This can be seen by estimating

$$\begin{aligned}\|\phi_\lambda(g) - \phi_\lambda(h)\|_{\mathscr{C}_2^\theta(\mathbb{T}^d)} &\lesssim \lambda^{(\theta-\beta-\alpha)/\alpha}\|F \cdot \nabla(g-h)\|_{\mathscr{C}_2^\beta(\mathbb{T}^d)} \\ &\lesssim \lambda^{(\theta-\beta-\alpha)/\alpha}\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)}(1+\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)})\|g-h\|_{\mathscr{D}_2^\theta}\end{aligned}$$

using (5.14) and the estimate for the product. Thus a contraction is obtained by choosing λ large enough, such that $\lambda^{(\theta-\beta-\alpha)/\alpha}\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)}(1+\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)}) < 1$, using $\theta < \alpha + \beta$. To check that indeed $\Phi_\lambda(g, g') \in \mathscr{D}_2^\theta$, we note that

$$\begin{aligned}\Phi_\lambda(g,g')^\sharp &= \phi_\lambda(g) - [G'e_i + \nabla g] \otimes I_\lambda(F) \\ &= I_\lambda(G^\sharp + F \odot g + F \otimes \nabla g) + C_\lambda(G'e_i + \nabla g, F)\end{aligned}$$

for the commutator C_λ from (5.15). Notice that, if $\beta < (1-\alpha)/2$, for $G^\sharp \in \mathscr{C}_2^0(\mathbb{T}^d)$, $I_\lambda(G^\sharp) \in \mathscr{C}_2^\alpha(\mathbb{T}^d) \subset \mathscr{C}_2^{2\theta-1}(\mathbb{T}^d)$ as $\theta > (1+\alpha)/2$. Hence, together with Lemma 5.9, it follows that $\Phi_\lambda(g,g')^\sharp \in \mathscr{C}^{2\theta-1}(\mathbb{T}^d)$. Thereby we also get the small factor of $\lambda^{(\theta-\alpha-\beta)/\alpha}$ in the estimate. To see that $\Phi_\lambda^2 = \Phi_\lambda \circ \Phi_\lambda$ is a contraction, we furthermore check

$$\begin{aligned}&\|\Phi_\lambda(\Phi_\lambda(g,g'))' - \Phi_\lambda(\Phi_\lambda(h,h'))'\|_{\mathscr{C}_2^{\theta-1}(\mathbb{T}^d)} \\ &\quad = \|\nabla\phi_\lambda(g) - \nabla\phi_\lambda(h)\|_{\mathscr{C}_2^{\theta-1}(\mathbb{T}^d)} \\ &\quad \lesssim \|\phi_\lambda(g) - \phi_\lambda(h)\|_{\mathscr{C}_2^\theta(\mathbb{T}^d)} \\ &\quad \lesssim \lambda^{(\theta-\beta-\alpha)/\alpha}\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)}(1+\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)})\|g-h\|_{\mathscr{D}_2^\theta},\end{aligned}$$

by the above estimate. □

5.4. Existence of an invariant measure and spectral gap estimates

In this section, we prove with Theorem 5.12 existence and uniqueness of an invariant, ergodic probability measure for the process $X^{\mathbb{T}^d}$ with state space $\mathbb{T}^d$, in the following for short denoted by X. The theorem moreover shows that X is exponentially ergodic, in the sense that pointwise spectral gap estimates for its semigroup (T_t) hold. Furthermore, we characterize the domain of $\mathfrak{L}$ in $L^2(\pi)$ in Theorem 5.17 and define the mean of $F \in \mathscr{X}_\infty^{\beta,\gamma}$ with respect to the invariant measure π in Lemma 5.20.
Existence and uniqueness of the invariant measure together with the pointwise spectral gap estimates on the semigroup are obtained by an application of Doeblin's theorem (see e.g. [BLP78, Theorem 3.1, Chapter 3, Section 3, p. 365]), that we state here in the continuous time setting.

Lemma 5.11 (Doeblin's theorem). *Let $(X_t)_{t\geqslant 0}$ be a time-homogeneous Markov process with state space (S, Σ) for a compact metric space S and its Borel-sigma-field Σ. Let $(T_t)_{t\geqslant 0}$ be the associated semigroup, $T_t f(x) := \mathbb{E}[f(X_t) \mid X_0 = x]$ for $x \in S$ and $f : S \to \mathbb{R}$ bounded measurable. Assume further, that there exists a probability measure*

μ on (S, Σ) and, for any $t > 0$, a continuous function $\rho_t : S \times S \to \mathbb{R}^+$, such that $T_t \mathbf{1}_E(x) = \int_E \rho_t(x,y)\mu(dy)$, $E \in \Sigma$. Assume moreover that, for any $t > 0$, there exists an open ball U_0, such that $\mu(U_0) > 0$ and $\rho_t(x,y) > 0$ for all $x \in S$ and $y \in U_0$.
Then, there exists a unique invariant probability measure π (i.e. $\int_S T_t \mathbf{1}_E(x)\pi(dx) = \pi(E)$ for all $E \in \Sigma$ and all $t \geqslant 0$) on (S, Σ) with the property that there exist constants $K, \nu > 0$, such that for all $t \geqslant 0$, $x \in S$ and $\phi : S \to \mathbb{R}$ bounded measurable,

$$\left| T_t\phi(x) - \int_S \phi(y)\pi(dy) \right| \leqslant K|\phi|e^{-\nu t} \tag{5.20}$$

where $|\phi| := \sup_{x \in S}|\phi(x)|$.

Proof. For discrete time Markov chains, the result follows immediately from [BLP78, Theorem 3.1, p. 365]. For continuous time Markov processes, the proof is similar. Indeed, in the same manner one proves that if π is such that (5.20) holds, then π is unique and π is invariant for (T_t). Furthermore, using the assumptions on the density ρ and the same proof steps as in [BLP78, Theorem 3.1, p. 365], one obtains existence of an invariant measure π with $\pi(E)$ given as the limit of $(T_n \mathbf{1}_E(x))_n$ for any $x \in S$ and with (5.20) for t replaced by $n \in \mathbb{N}$. Then, using the semigroup property, we also obtain (5.20) for any $t \geqslant 0$, with a possibly different constant $K > 0$. Indeed, let $t > 0$ and $n = \lfloor t \rfloor$. Then for bounded measurable ϕ with $\int_S \phi d\pi = 0$, we obtain

$$|T_t\phi(x)| = |T_n T_{t-n}\phi(x)| \leqslant K|T_{t-n}\phi|e^{-\nu n} \leqslant K|\phi|e^{-\nu n} = Ke^{\nu(t-n)}|\phi|e^{-\nu t} \leqslant Ke^{\nu}|\phi|e^{-\nu t}.$$

Now, by changing the constant K, we obtain (5.20) for all $t \geqslant 0$. □

Theorem 5.12. *Let X be the martingale solution to the singular periodic SDE* (5.1) *projected onto $\mathbb{T}^d$ with contraction semigroup $(T_t)_{t \geqslant 0}$ on bounded measurable functions $f : \mathbb{T}^d \to \mathbb{R}$.*
Then there exists a unique invariant probability measure π for (T_t). In particular, π is ergodic for X. Furthermore there exist constants $K, \mu > 0$ such that for all $f \in L^\infty(\mathbb{T}^d)$,

$$\|T_t f - \langle f \rangle_\pi\|_{L^\infty} \leqslant K\|f\|_{L^\infty} e^{-\mu t}. \tag{5.21}$$

That is, L^∞-spectral gap estimates for the associated Markov semigroup (T_t) hold true. In particular, π is absolutely continuous with respect to the Lebesgue measure on the torus, with density denoted by ρ_∞.

Proof. The proof is an application of Doeblin's theorem. We check, that the assumptions of Lemma 5.11 are satisfied. To that aim, note that for the Fokker-Planck density $\rho_t(x, \cdot)$ with $\rho_0 = \delta_x$, the map $(x,y) \mapsto \rho_t(x,y)$ is continuous by Theorem 5.4. It remains to show that there exists an open ball U_0 and a constant $c > 0$, such that ρ_t is bounded from below by c on $\mathbb{T}^d \times U_0$. We choose $U_0 = \mathbb{T}^d$ and obtain

$$\min_{x \in \mathbb{T}^d, y \in U_0} \rho_t(x,y) = \rho_t(x^*, y^*) \geqslant c > 0.$$

Indeed, this follows from the strict maximum principle for $y \mapsto \rho_t(x^*, y)$ by Proposition 5.8 with $c = c(x^*) > 0$.
The spectral gap estimates also imply absolute continuity, as

$$\langle 1_A \rangle_\pi = \lim_{t\to\infty} \mathbb{E}_{X_0=x}[1_A(X_t)] = \lim_{t\to\infty} \int 1_A(y)\rho_t(x,y)dy$$

and thus any Lebesgue nullset A is also a π-nullset. The existence of the density thus follows by the Radon-Nikodym theorem. □

Corollary 5.13. *Let ρ_∞ be the Lebesgue density of the invariant measure π. Then $\rho_\infty \in \mathscr{C}^{\alpha+\beta-1}(\mathbb{T}^d)$ and it follows the paracontrolled structure*

$$\rho_\infty = \rho_\infty^\sharp + \rho_\infty \varolessthan I_\infty(\nabla \cdot F),$$

where $\rho_\infty^\sharp \in \mathscr{C}^{2(\alpha+\beta)-2}(\mathbb{T}^d)$ and $I_\infty(\nabla \cdot F) := \int_0^\infty P_s(\nabla \cdot F)ds$.
Furthermore, the density is strictly positive,

$$\min_{x\in\mathbb{T}^d} \rho_\infty(x) > 0.$$

In particular, π is equivalent to the Lebesgue measure.

Proof. Let $t > 0$. By invariance of π, i.e. $\langle T_t f\rangle_\pi = \langle f \rangle_\pi$ for all $f \in L^\infty(\mathbb{T}^d)$, and $d\pi = \rho_\infty dx$, we obtain that almost surely

$$\rho_\infty = T_t^* \rho_\infty,$$

where T_t^* denotes the adjoint of T_t with respect to $L^2(\lambda)$. Here λ denotes the Lebesgue measure and $\langle f\rangle_\pi := \int_{\mathbb{T}^d} f(x)\pi(dx)$.
Denote $y_t(x) := T_t^*\rho_\infty(x)$. Then we show that y is a mild solution of the Fokker-Planck equation started in ρ_∞, that is

$$(\partial_t - \mathfrak{L}^*)y = 0, \quad y_0 = \rho_\infty. \tag{5.22}$$

Here the density satisfies $\rho_\infty \in L^1(\lambda)$, i.p. $\rho_\infty \in \mathscr{C}_1^0(\mathbb{T}^d)$. Indeed, that $y_t = T_t^*\rho_\infty$ is a mild solution of the Fokker-Planck equation follows from approximation of F by $F^m \in C^\infty(\mathbb{T}^d)$ with $F^m \to F$ in $\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$ using that for $m \in \mathbb{N}$, $y^m = (T_t^m)^*\rho_\infty$ solves $(\partial_t - (\mathfrak{L}^m)^*)y^m = 0$, $y^m = \rho_\infty$ by the classical Fokker-Planck theory, where T^m denotes the semigroup for the strong solution of the SDE with drift F^m and generator $\mathfrak{L}^m := -\mathscr{L}_\nu^\alpha + F^m \cdot \nabla$. By continuity of the Fokker-Planck solution map from Theorem 5.4 for converging data $(F^m, \rho_\infty) \to (F, \rho_\infty)$ in $\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d) \times \mathscr{C}_1^0(\mathbb{T}^d)$, we deduce $y^m \to y$ in the paracontrolled solution space, where y is the mild solution of (5.22).
The lower bound away from zero then also follows from Theorem 5.4, as well as the

paracontrolled structure

$$\rho_\infty = \rho_\infty^\sharp + \rho_\infty \otimes I_t(\nabla \cdot F),$$

where $\rho_\infty^\sharp := y_t^\sharp \in \mathscr{C}^{2(\alpha+\beta)-2}(\mathbb{T}^d)$ and $I_t(\nabla \cdot F) := \int_0^t P_{t-s}(\nabla \cdot F)ds$.
Due to $\mathscr{F}_{\mathbb{T}^d}(\nabla \cdot F)(0) = 0$, we have that, for any $\theta \geqslant 0$, there exists $c > 0$, such that, uniformly in $s > 0$,

$$\|P_s(\nabla \cdot F)\|_{\mathscr{C}^{\beta-1+\theta}(\mathbb{T}^d)} \lesssim s^{-\theta/\alpha} e^{-cs} \|\nabla \cdot F\|_{\mathscr{C}^{\beta-1}(\mathbb{T}^d)}. \tag{5.23}$$

Indeed, this follows from Lemma 5.2. Thus we obtain, for $t > 0$ and any $\theta \geqslant 0$, that

$$I_t(\nabla \cdot F) - I_\infty(\nabla \cdot F) = \int_t^\infty P_s(\nabla \cdot F)ds \in \mathscr{C}^\theta(\mathbb{T}^d).$$

That is, the remainder is smooth and thus can be absorbed into $\rho_\infty^\sharp$. Notice, that $\int_0^t P_s(\nabla \cdot F)ds \in \mathscr{C}^{\alpha+\beta-1}(\mathbb{T}^d)$ by (5.23) and Lemma 3.14 and in particular that $I_\infty(\nabla \cdot F) \in \mathscr{C}^{\alpha+\beta-1}(\mathbb{T}^d)$ is well-defined. □

Corollary 5.14. *Let X and (T_t) be as before. Then, the semigroup $(T_t)_{t\geqslant 0}$ can be uniquely extended to a strongly continuous contraction semigroup on $L^2(\pi)$, i.e. $T_{t+s} = T_t T_s$, $T_t 1 = 1$, $T_t f \to f$ for $t \downarrow 0$ and $f \in L^2(\pi)$ and $\|T_t f\|_{L^2(\pi)} \leqslant \|f\|_{L^2(\pi)}$, such that and for (possibly different) constants $K, \mu > 0$, the $L^2(\pi)$-spectral gap estimates hold true:*

$$\|T_t f - \langle f\rangle_\pi\|_{L^2(\pi)} \leqslant K\|f\|_{L^2(\pi)} e^{-\mu t} \quad \textit{for all } f \in L^2(\pi).$$

Proof. That the semigroup $(T_t)_{t\geqslant 0}$ can be uniquely extended to a contraction semigroup on $L^2(\pi)$ follows from Jensen's inequality,

$$\|T_t f\|_{L^2(\pi)}^2 = \int |\mathbb{E}_{X_0=x}[f(X_t)]|^2 \pi(dx) \leqslant \int \mathbb{E}_{X_0=x}[|f(X_t)|^2]\pi(dx) = \|f\|_{L^2(\pi)}^2,$$

for $f \in L^\infty$, using the invariance of π (by Theorem 5.12). By approximation, we then also obtain for the extension, that $T_t f(x) = \mathbb{E}_{X_0=x}[f(X_t)]$ for $f \in L^2(\pi)$.
We check strong continuity of the semigroup on $L^2(\pi)$. Using the contraction property in $L^2(\pi)$, we obtain

$$\|T_t f - f\|_{L^2(\pi)}^2 = \|T_t f\|_{L^2(\pi)}^2 + \|f\|_{L^2(\pi)}^2 - 2\langle T_t f, f\rangle_\pi \leqslant 2\|f\|_{L^2(\pi)}^2 - 2\langle T_t f, f\rangle_\pi. \tag{5.24}$$

It is left to prove that the right-hand side vanishes as $t \downarrow 0$. By Fatou's lemma and using that X is almost surely càdlàg, we have that for $x \in \mathbb{T}^d$ and $f \in C(\mathbb{T}^d, \mathbb{R})$,

$$\lim_{t\downarrow 0} |T_t f(x) - f(x)| \leqslant \mathbb{E}_{X_0=x}[\lim_{t\downarrow 0} |f(X_t) - f(X_0)|] = 0. \tag{5.25}$$

Furthermore, we can bound uniformly in $x \in \mathbb{T}^d$ and $t > 0$,

$$|T_t f(x)| = \left| \int \rho_t(x,y) f(y) dy \right| \leqslant \frac{\sup_{t>0} \max_{x,y\in\mathbb{T}^d} \rho_t(x,y)}{\min_{y\in\mathbb{T}^d} \rho_\infty(y)} \|f\|_{L^1(\pi)} \leqslant C \|f\|_{L^2(\pi)} \quad (5.26)$$

where $C > 0$ is a constant (not depending on t, f) and $\rho_t(x,y)$ denotes the Fokker-Planck density with $\rho_0(x,y) = \delta_x$. Here, we have $\min_{y\in\mathbb{T}^d} \rho_\infty(y) > 0$ by Corollary 5.13. Furthermore we have

$$\sup_{t>0} \max_{x,y\in\mathbb{T}^d} \rho_t(x,y) < \infty. \quad (5.27)$$

Indeed, by the L^∞-spectral gap estimates, it follows that

$$\sup_{t\geqslant 0} \|\rho_t * f\|_{L^\infty} \leqslant K \|f\|_{L^\infty} + |\langle f\rangle_\pi|,$$

with convolution $(\rho_t * f)(x) := \int_{\mathbb{T}^d} \rho_t(x,y) f(y) dy$. We can apply this bound for $f^{\varepsilon,\tilde{y}}(y) := \mathbf{1}_{|y-\tilde{y}|<\varepsilon}$ for $\tilde{y} \in \mathbb{T}^d$ and $\varepsilon > 0$ and let $\varepsilon \downarrow 0$. By continuity of $y \mapsto \rho_t(x,y)$ and the dominated convergence theorem, $(\rho_t * f^{\varepsilon,\tilde{y}})(x) \to \rho_t(x,\tilde{y})\lambda(\mathbb{T}^d)$, which yields (5.27).
In particular, by (5.26), $\sup_{t>0} \|T_t f\|_{L^\infty} \lesssim \|f\|_{L^2(\pi)}$ and an application of the dominated convergence theorem using (5.25), yields that for $f \in C(\mathbb{T}^d, \mathbb{R})$,

$$\lim_{t\downarrow 0} \langle T_t f, f\rangle_\pi = \|f\|^2_{L^2(\pi)}.$$

We conclude with (5.24), that for all $f \in C(\mathbb{T}^d, \mathbb{R})$, $\|T_t f - f\|_{L^2(\pi)} \to 0$ as $t \downarrow 0$.
As (T_t) is a contraction semigroup on $L^2(\pi)$, the operator norm is trivially bounded, that is $\sup_{t\geqslant 0} \|T_t\|_{L(L^2(\pi))} \leqslant 1$. Above, we proved that (T_t) is strongly continuous on a dense subset of $L^2(\pi)$. Thus, together with boundedness of the operator norm, (T_t) is also strongly continuous on $L^2(\pi)$ as a consequence of the Banach-Steinhaus theorem. It remains to prove that the $L^2(\pi)$-spectral gap estimates follow from the L^∞-spectral gap estimates and the bound (5.26). Indeed, we obtain for $f \in L^2(\pi)$ with $\langle f\rangle_\pi = 0$ and all $t > 1$,

$$\|T_t f\|_{L^2(\pi)} = \|T_{t-1} T_1 f\|_{L^2(\pi)} \leqslant K e^{-\mu(t-1)} \|T_1 f\|_{L^\infty} \leqslant e^\mu C K e^{-\mu t} \|f\|_{L^2(\pi)}.$$

For $t \in [0,1]$, we trivially estimate, using the contraction property,

$$\|T_t f\|_{L^2(\pi)} \leqslant \|f\|_{L^2(\pi)} \leqslant e^\mu e^{-\mu t} \|f\|_{L^2(\pi)}.$$

□

Remark 5.15. *The argument in the above proof of Corollary 5.14 (using the bound* (5.27) *and* $\rho_\infty > 0$*) can be adapted to prove the stronger estimate (for constants*

$K, \mu > 0$)

$$\|T_t f - \langle f\rangle_\pi\|_{L^\infty} \leqslant K e^{-\mu t}\|f - \langle f\rangle_\pi\|_{L^1(\pi)},$$

which in particular implies the $L^2(\pi)$-$L^2(\pi)$-bound from the corollary.

Remark 5.16. *More generally, one can show the Feller property, that is (T_t) is strongly continuous on $C(\mathbb{T}^d)$. Using [RY99, Proposition III.2.4] and (5.25), it is left to show $T_t f \in C(\mathbb{T}^d)$ for $f \in C(\mathbb{T}^d) \subset \mathscr{C}^0(\mathbb{T}^d)$. But this follows from Theorem 3.25, since for $R > t$, $y_t = T_{R-t} f$ solves the backward Kolmogorov equation with periodic terminal condition $y_R = f \in \mathscr{C}^0$ and $y \in \mathscr{M}_R^\gamma \mathscr{C}^{\alpha+\beta}$, such that in particular $x \mapsto y_t(x)$ is continuous.*

The next theorem relates the semigroup $(T_t)_{t\geqslant 0}$ from above with the generator $\mathfrak{L}$ and gives an explicit representation of its domain in terms of paracontrolled solutions of singular resolvent equations.

Theorem 5.17. *Let (T_t) be the contraction semigroup on $L^2(\pi)$ from Corollary 5.14 and denote its generator by $(A, dom(A))$ with $A : dom(A) \subset L^2(\pi) \to L^2(\pi)$ and domain $dom(A) := \{f \in L^2(\pi) \mid \lim_{t\to 0}(T_t f - f)/t =: Af$ exists in $L^2(\pi)\}$. Let $\theta \in ((1+\alpha)/2, \alpha+\beta)$ and*

$$D := \{g \in \mathscr{D}_2^\theta \mid R_\lambda g = G \textit{ for some } G \in L^2(\pi) \textit{ and } \lambda > 0\},$$

where $R_\lambda := (\lambda - \mathfrak{L})$.
Then it follows $D = dom(A)$ and $(A, D) = (\mathfrak{L}, D)$. In particular, $(\mathfrak{L}, D)$ is the generator of the Markov process X with state space $\mathbb{T}^d$ and transition semigroup (T_t).

Remark 5.18. *Since the drift F does not depend on a time variable, one could reformulate the martingale problem for X in terms of the elliptic generator $\mathfrak{L}$ and the domain $D \subset L^2(\pi)$.*

Proof. We first show that $D \subset dom(A)$. To this aim, note that for $f \in D$, we obtain $R_\lambda f = G$ for $G \in L^2(\pi)$. For a mollification $(G^n) \subset C^\infty(\mathbb{T}^d)$ of G and $(f^n) \subset C^\infty(\mathbb{T}^d)$, such that $R_\lambda f^n = G^n$, we obtain that in particular f^n is a mild solution of the Kolmogorov backward equation on the torus for $\mathscr{G} = \partial_t + \mathfrak{L}$ with right-hand side $\lambda f^n - G^n \in L^\infty$ and terminal condition $f^n \in \mathscr{C}^3$. Equivalently, its periodic version is the periodic solution of the Kolmogorov backward equation on $\mathbb{R}^d$. As X equals the projected solution of the $(\mathscr{G}, x)$-martingale problem onto the torus, we have, for $n \in \mathbb{N}$ and $x \in \mathbb{T}^d$, that

$$\begin{aligned} T_t f^n(x) - f^n(x) &= \mathbb{E}_{X_0=x}[f^n(X_t) - f^n(X_0)] \\ &= \mathbb{E}_{X_0=x}\Big[\int_0^t (\lambda f^n - G^n)(X_s) ds\Big] = \int_0^t T_s(\lambda f^n - G^n)(x) ds. \end{aligned}$$

Using that $f^n \to f$ in $L^2(\pi)$ as $G^n \to G$ by continuity of the resolvent solution map, we obtain that for $f \in D$,

$$T_t f - f = \int_0^t T_s(\lambda f - G) ds.$$

By continuity of the map $s \mapsto T_s(\lambda f - G) \in L^2(\pi)$, since T is strongly continuous on $L^2(\pi)$, we obtain that for $f \in D$, $\lim_{t\to 0}(T_t f - f)/t$ exists in $L^2(\pi)$ and

$$Af = \lambda f - G = \lambda f - R_\lambda f = \mathfrak{L} f.$$

To prove that also $dom(A) \subset D$, we use that for $\chi \in dom(A)$, there trivially exists $f \in L^2(\pi)$ with $A\chi = f$. Notice that by Theorem 5.10, we can solve the resolvent equation for $\lambda > 0$ large enough,

$$R_\lambda \tilde{\chi} = \lambda \chi - f,$$

with right-hand side $\lambda\chi - f \in L^2(\pi) \subset \mathscr{C}_2^0$, obtaining a solution $\tilde{\chi} \in D$. By the above, we have that $A_{|D} = \mathfrak{L}_{|D}$, such that $\mathfrak{L}\tilde{\chi} = A\tilde{\chi}$. This yields by inserting in the equation for $\tilde{\chi}$ and since $f = A\chi$, that $A(\tilde{\chi} - \chi) = \lambda(\tilde{\chi} - \chi)$. As $\lambda > 0$, by uniqueness of the solution of the resolvent equation for the generator A, we obtain $\tilde{\chi} = \chi$. Thus with the equation for $\tilde{\chi}$ this yields $\chi \in D$ and $\mathfrak{L}\chi = f$. □

Corollary 5.19. *Let $f \in L^2(\pi)$ with $\langle f \rangle_\pi = 0$. Then there exists a unique solution $\chi \in D$ of the Poisson equation $\mathfrak{L}\chi = f$ such that $\langle \chi \rangle_\pi = 0$.*

Proof. This follows from the $L^2(\pi)$-spectral gap estimates. We can solve the Poisson equation in $L^2(\pi)$ for the given right-hand side $f \in L^2(\pi)$ with $\langle f \rangle_\pi = 0$. The solution is explicitly given by $\chi = \int_0^\infty T_t f dt \in L^2(\pi)$.
We check that χ is indeed a solution. By [EK86, Proposition 1.1.5 part a)], we have that for $f \in L^2(\pi)$, $\int_0^t T_s f ds \in dom(A)$ and

$$T_t f - f = A \int_0^t T_s f ds,$$

where $(A, dom(A))$ denotes again the generator of (T_t) on $L^2(\pi)$. By the L^2-spectral gap estimates and $\langle f \rangle_\pi = 0$, we obtain that $(\int_0^t T_s f ds)_t$ converges in $L^2(\pi)$ for $t \to \infty$ to a limit χ, and that $(T_t f)_t$ converges to zero in $L^2(\pi)$ for $t \to \infty$. Hence, since A is a closed operator (cf. [EK86, Corollary 1.1.6]), we obtain in the limit $t \to \infty$, that $f = A \int_0^\infty T_t f dt = A\chi$ and $\chi \in dom(A)$. Now, using $dom(A) = D$ and $(A, D) = (\mathfrak{L}, D)$ by Theorem 5.17, this yields $\chi \in D$ and $\mathfrak{L}\chi = f$. □

Thanks to the regularity of the density of the invariant measure π, we can finally define the mean of the singular drift F under π, $\langle F \rangle_\pi = \langle F, \rho_\infty \rangle_\lambda$, respectively the product $F \cdot \rho_\infty$.

Lemma 5.20. *Let ρ_∞ be the density of π. Let $\langle F\rangle_\pi = (\langle F^i\rangle_\pi)_{i=1,\dots,d}$ for*

$$\begin{aligned}\langle F^i\rangle_\pi &= (F^i\cdot\rho_\infty)(\mathbf{1})\\ &:= [(F^i\cdot\rho_\infty^\sharp) + (F^i\odot I_\infty(\nabla\cdot F))\cdot\rho_\infty + C_1(\rho_\infty, I_\infty(\nabla\cdot F), F^i)](\mathbf{1}),\end{aligned}$$

where $\mathbf{1}\in C^\infty(\mathbb{T}^d)$ is the constant test function and C_1 denotes the paraproduct commutator defined in (5.19).
Then, $\langle F^i\rangle_\pi$ is well-defined and continuous, that is, $\langle F^m\rangle_\pi\to\langle F\rangle_\pi$ for $F^m\to F$ in $\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$. Moreover, the following Lipschitz bound holds true

$$\|F\cdot\rho_\infty\|_{\mathscr{C}^\beta(\mathbb{T}^d)}\lesssim\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)}(1+\|F\|_{\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)})[\|\rho_\infty\|_{\mathscr{C}^{\alpha+\beta-1}}+\|\rho_\infty^\sharp\|_{\mathscr{C}^{2(\alpha+\beta-1)}}].$$

Proof. The proof follows directly from Theorem 5.4 and Corollary 5.13. □

5.5. Solving the Poisson equation with singular right-hand side

To prove the central limit theorem for the solution of the martingale problem X, we utilize the classical approach of decomposing the additive functional in terms of a martingale and a boundary term, using the solution of the Poisson equation for $\mathfrak{L}$ with singular right-hand side $F-\langle F\rangle_\pi$. For solving the Poisson equation in Theorem 5.24 below, Corollary 5.19 is not applicable, as F is a distribution and therefore not an element of $L^2(\pi)$. Consider an approximation $(F^m)\subset C^\infty(\mathbb{T}^d)$ with $F^m\to F$ in $\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$. Then, we can apply Corollary 5.19 for the right-hand sides $F^m-\langle F^m\rangle_\pi\in L^2(\pi)$, $m\in\mathbb{N}$. This way we obtain solutions $\chi^m=(\chi^{m,i})_{i=1,\dots,d}\in D^d\subset L^2(\pi)^d$ of the Poisson equations

$$(-\mathfrak{L})\chi^{m,i}=F^{m,i}-\langle F^{m,i}\rangle_\pi \tag{5.28}$$

for $m\in\mathbb{N}$.
In this section, we show that the sequence $(\chi^m)_m$ converges in a space of sufficient regularity to a the limit χ that indeed solves the Poisson equation

$$(-\mathfrak{L})\chi=F-\langle F\rangle_\pi. \tag{5.29}$$

Let us define the space $\mathscr{H}^1(\pi)$ as in [KLO12, Section 2.2],

$$\mathscr{H}^1(\pi):=\{f\in D\mid\|f\|^2_{\mathscr{H}^1(\pi)}:=\langle(-\mathfrak{L})f,f\rangle_\pi<\infty\}, \tag{5.30}$$

which is the Sobolev space for the operator $\mathfrak{L}$ with respect to $L^2(\pi)$. Its dual is defined by

$$\mathscr{H}^{-1}(\pi) := \{F : \mathscr{H}^1(\pi) \to \mathbb{R} \mid F \text{ linear with } \|F\|_{\mathscr{H}^{-1}(\pi)} := \sup_{\|f\|_{\mathscr{H}^1(\pi)}=1} |F(f)| < \infty\}. \tag{5.31}$$

The space $\mathscr{H}^1(\pi)$ is related to the quadratic variation of Dynkin's martingale, see [KLO12, Section 2.4], which motivates the definition.
To prove convergence of $(\chi^m)_m$ in $L^2(\pi)^d$, we first establish in Corollary 5.23 convergence of $(\chi^m)_m$ in the space $\mathscr{H}^1(\pi)^d$ and utilize a Poincaré-type bound on the operator $\mathfrak{L}$. A standard argument as in [GZ02, Property 2.4] shows that the $L^2(\pi)$-spectral gap estimates from Corollary 5.14 for the constant $K = 1$, imply the Poincaré estimate for the operator $\mathfrak{L}$:

$$\|f - \langle f\rangle_\pi\|^2_{L^2(\pi)} \leqslant \mu\langle(-\mathfrak{L})f, f\rangle_\pi = \mu\|f\|^2_{\mathscr{H}^1(\pi)}, \qquad \text{for all } f \in D.$$

In general, the constant $K > 0$ in the spectral gap estimates from Corollary 5.14 does not need to satisfy $K = 1$ and the above argument breaks down for $K \neq 1$. Hence, we show below in (5.34) that $\|f - \langle f\rangle_\pi\|^2_{L^2(\pi)} \leqslant C\|f\|^2_{\mathscr{H}^1(\pi)}$ holds true for some constant $C > 0$. That constant may differ from the constant μ and may not be optimal, but the bound suffices for our purpose of concluding on $L^2(\pi)^d$ convergence given $\mathscr{H}^1(\pi)^d$ convergence of $(\chi^m)_m$.
An optimal estimate, that however applies for a much more general situation of weak Poincaré inequalities and slower than exponential convergences, can be found in [RW01, Theorem 2.3].
The $\mathscr{H}^1(\pi)^d$ convergence of $(\chi^m)_m$ follows from $\mathscr{H}^{-1}(\pi)^d$-convergence of $(F^m)_m$ for the approximating sequence $F^m \to F$ in $\mathscr{X}^{\beta,\gamma}_\infty(\mathbb{T}^d)$. Convergence of $(F^m)_m$ in $\mathscr{H}^{-1}(\pi)^d$ is established in Theorem 5.22. The following lemma is an auxiliary result, which proves that the semi-norms in $\mathscr{H}^1(\pi)$ and the homogeneous Besov space $\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)$, cf. (5.5), are equivalent.

Lemma 5.21. *Let $\alpha \in (1, 2]$ and $\theta \in (1, \alpha)$. Define the carré-du-champ operator of the generalized fractional Laplacian as $\Gamma^\alpha_\nu(f) = \Gamma^\alpha_\nu(f, f) := \frac{1}{2}((-\mathscr{L}^\alpha_\nu)f^2 - 2f(-\mathscr{L}^\alpha_\nu)f)$. Then, there exist constants $c, C > 0$, such that for all $f \in \dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)$,*

$$c\|f\|^2_{\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)} \leqslant \langle\Gamma^\alpha_\nu(f)\rangle_\lambda \leqslant C\|f\|^2_{\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)}. \tag{5.32}$$

Proof. By [ST87, part (v) of Theorem, Section 3.5.4] we obtain that the periodic Lizorkin space $F^s_{2,2}(\mathbb{T}^d)$ coincides with the periodic Bessel-potential space $H^s(\mathbb{T}^d)$. Furthermore $F^s_{2,2}(\mathbb{T}^d)$ coincides with $B^s_{2,2}(\mathbb{T}^d)$ (cf. [ST87, Section 3.5.1, Remark 4]). Thus, we obtain that in particular

$$\dot{B}^s_{2,2}(\mathbb{T}^d) = \dot{H}^s(\mathbb{T}^d).$$

It remains to show (5.32) with $\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)$ replaced by $\dot{H}^s(\mathbb{T}^d)$. To that aim, we calculate, using the definition of $\mathscr{L}^\alpha_\nu$ for a Schwartz function $f \in \mathscr{S}(\mathbb{T}^d)$ and $\psi^\alpha_\nu(0) = 0$,

$$\begin{aligned}\langle \Gamma^\alpha_\nu(f)\rangle_\lambda = \int_{\mathbb{T}^d} \Gamma^\alpha_\nu(f)(x)dx &= \mathscr{F}_{\mathbb{T}^d}(\Gamma^\alpha_\nu(f))(0)\\ &= \frac{1}{2}\mathscr{F}_{\mathbb{T}^d}((-\mathscr{L}^\alpha_\nu)f^2)(0) - \mathscr{F}_{\mathbb{T}^d}(f(-\mathscr{L}^\alpha_\nu)f)(0)\\ &= -\frac{1}{2}\psi^\alpha_\nu(0)(\hat{f} * \hat{f})(0) + (\hat{f} * \psi^\alpha_\nu \hat{f})(0)\\ &= \sum_{k\in\mathbb{Z}^d} \hat{f}(-k)\hat{f}(k)\psi^\alpha_\nu(k)\\ &= \sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2\psi^\alpha_\nu(k).\end{aligned}$$

By Assumption 3.4 on the spherical component of the jump measure ν, we obtain, that there exist constants $c, C > 0$ with

$$c|k|^\alpha \leqslant \psi^\alpha_\nu(k) = \int_S |\langle k, \xi\rangle|^\alpha \nu(d\xi) \leqslant C|k|^\alpha.$$

Thus it follows that

$$c\|f\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} = c\sum_{k\in\mathbb{Z}^d} |k|^\alpha|\hat{f}(k)|^2 \leqslant \langle \Gamma^\alpha_\nu(f)\rangle_\lambda \leqslant C\sum_{k\in\mathbb{Z}^d} |k|^\alpha|\hat{f}(k)|^2 = C\|f\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}.$$

By a density argument, the claim follows for all $f \in \dot{H}^{\alpha/2}(\mathbb{T}^d) = \dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)$. □

Theorem 5.22. *Let $F \in \mathscr{X}^{\beta,\gamma}_\infty(\mathbb{T}^d)$ for $\beta \in (\frac{2-2\alpha}{3}, 0)$ and $\alpha \in (1, 2]$. Then, equivalence of the semi-norms $\|\cdot\|_{\mathscr{H}^1(\pi)} \simeq \|\cdot\|_{\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)}$ follows and $\overline{F} := F - \langle F\rangle_\pi \in \mathscr{H}^{-1}(\pi)^d$. In particular, $F^m \to F$ in $\mathscr{X}^{\beta,\gamma}_\infty(\mathbb{T}^d)$ implies $\overline{F}^m \to \overline{F}$ in $\mathscr{H}^{-1}(\pi)^d$.*

Proof. By invariance of π we obtain $\langle \mathfrak{L}g\rangle_\pi = 0$ for $g \in D$, because for $g \in D$, $(\frac{d}{dt}T_t)_{|t=0}f = \mathfrak{L}f \in L^2(\pi)$. We now apply this for $g = f^2$ for which we need to check that if $f \in D$, then $\mathfrak{L}f^2$ is well-defined and $\mathfrak{L}f^2 \in L^1(\pi)$. This follows by calculating

$$f^2 = (f^\sharp + \nabla f \varolessthan I_\lambda(F))^2 = g^\sharp + g' \varolessthan I_\lambda(F),$$

where

$$\begin{aligned}g^\sharp = (f^\sharp)^2 + 2f^\sharp \odot (\nabla f \varolessthan I_\lambda(F)) + 2f^\sharp \varogreaterthan (\nabla f \varolessthan I_\lambda(F))\\ + (\nabla f \varolessthan I_\lambda(F)) \odot (\nabla f \varolessthan I_\lambda(F)) \in \mathscr{C}^{2\theta-1}_1(\mathbb{T}^d)\end{aligned}$$

and

$$g' = 2f^\sharp \varolessthan \nabla f + \nabla f \varolessthan I_\lambda(F) \varolessthan \nabla f + I_\lambda(F) \varolessthan \nabla f \varolessthan I_\lambda(F) \in (\mathscr{C}^{\theta-1}_1(\mathbb{T}^d))^d.$$

Hence, we conclude that for $f \in D$, f^2 admits a paracontrolled structure with $g^\sharp \in \mathscr{C}_1^{2\theta-1}(\mathbb{T}^d)$ and $g' \in (\mathscr{C}_1^{\theta-1}(\mathbb{T}^d))^d$, such that $\mathfrak{L}f^2$ is well-defined and

$$\mathfrak{L}f^2 = 2f\mathfrak{L}f + 2\Gamma_\nu^\alpha(f) = 2\lambda f^2 - 2fR_\lambda f + 2\Gamma_\nu^\alpha(f) \in L^1(\pi).$$

Herein we used that $2\lambda f^2 - 2fR_\lambda f \in L^1(\pi)$ as $f \in D$ and $\Gamma_\nu^\alpha(f) = \Gamma_\nu^\alpha(f,f) = \frac{1}{2}(\mathscr{L}_\nu^\alpha f^2 - 2f\mathscr{L}_\nu^\alpha f) \in L^1(\pi)$ for $f \in \mathscr{C}_2^\theta(\mathbb{T}^d)$ by Lemma 5.21 as θ can be chosen close to $\alpha+\beta$, such that $\theta > \alpha/2$.
Analogously, if we denote the domain of $\mathfrak{L}$ with integrability p by D_p, then for $f,g \in D_2$, we concluded that $f \cdot g \in D_1$, which in particular implies that the carré-du-champ operator

$$\Gamma^{\mathfrak{L}}(f,g) = \frac{1}{2}(\mathfrak{L}(fg) - f\mathfrak{L}g - g\mathfrak{L}f) \in L^1(\pi)$$

for $f,g \in D$ is well-defined in $L^1(\pi)$.
Applying invariance of π for $g = f^2$, we can add $\frac{1}{2}\langle \mathfrak{L}f^2\rangle_\pi = 0$ yielding

$$\|f\|^2_{\mathscr{H}^1(\pi)} = \langle (-\mathfrak{L})f, f\rangle_\pi = \langle \Gamma^{\mathfrak{L}}(f)\rangle_\pi = \langle \Gamma_\nu^\alpha(f)\rangle_\pi.$$

where $\Gamma^{\mathfrak{L}}(f) = \frac{1}{2}\mathfrak{L}f^2 - f\mathfrak{L}f = \Gamma_\nu^\alpha(f)$. Thus, we obtain

$$\|f\|^2_{\mathscr{H}^1(\pi)} = \langle \Gamma_\nu^\alpha(f)\rangle_\pi \simeq \langle \Gamma_\nu^\alpha(f)\rangle_\lambda \simeq \|f\|^2_{\dot{B}^{\alpha/2}_{2,2}}, \tag{5.33}$$

where $\simeq$ denotes that the norms are equivalent.
Here, we used that absolute continuity of π with respect to the Lebesgue-measure, with density ρ_∞ that is uniformly bounded from above and from below, away from zero by Corollary 5.13. Moreover, note that the carré-du-champ is non-negative, $\Gamma_\nu^\alpha(f) \geqslant 0$. Furthermore we utilized (5.32) from Lemma 5.21.
Thus applying the duality estimate from Lemma 5.3 (for functions $f - \langle f\rangle_\lambda, g - \langle g\rangle_\lambda$ to obtain the result for the homogeneous Besov spaces), we get for $\overline{F} := F - \langle F\rangle_\pi$ with mean $\langle F\rangle_\pi$ from Lemma 5.20,

$$\begin{aligned}
|\langle \overline{F}^i, g\rangle_\pi| &= |\langle \overline{F}^i \rho_\infty, g\rangle| \\
&\lesssim \|\overline{F}^i\rho_\infty\|_{\dot{B}^{-\alpha/2}_{2,2}(\mathbb{T}^d)} \|g\|_{\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)} \\
&\lesssim \|\overline{F}^i\rho_\infty\|_{\dot{B}^{\beta}_{2,2}(\mathbb{T}^d)} \|g\|_{\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)} \\
&\lesssim \|\overline{F}^i\rho_\infty\|_{\dot{B}^{\beta}_{2,2}(\mathbb{T}^d)} \|g\|_{\mathscr{H}^1(\pi)},
\end{aligned}$$

for $i = 1, ..., d$, using $\beta > -\alpha/2$ and (5.33). Hence, we find

$$\begin{aligned}\|\overline{F}^i\|_{\mathscr{H}^{-1}(\pi)} &\lesssim \|\overline{F}^i\rho_\infty\|_{\dot{B}^{\beta}_{2,2}(\mathbb{T}^d)}\\ &\lesssim \|\overline{F}^i\rho_\infty\|_{B^{\beta}_{2,2}(\mathbb{T}^d)}\\ &\lesssim \|\overline{F}^i\rho_\infty\|_{\mathscr{C}^{\beta+(1-\gamma)\alpha}(\mathbb{T}^d)}\\ &\lesssim \|F\|_{\mathscr{X}^{\beta,\gamma}_{\infty}(\mathbb{T}^d)}(1+\|F\|_{\mathscr{X}^{\beta,\gamma}_{\infty}(\mathbb{T}^d)})[\|\rho_\infty\|_{\mathscr{C}^{\alpha+\beta-1}} + \|\rho^\sharp_\infty\|_{\mathscr{C}^{2(\alpha+2\beta-1)}}],\end{aligned}$$

where the estimate for the product of $\overline{F}^i$ and ρ_∞ follows from Lemma 5.20.
This proves that $\overline{F} \in \mathscr{H}^{-1}(\pi)^d$. Convergence follows by the same estimate. □

Corollary 5.23. *Let $F \in \mathscr{X}^{\beta,\gamma}_{\infty}$ and $F^m \in C^\infty(\mathbb{T}^d)$ with $F^m \to F$ in $\mathscr{X}^{\beta,\gamma}_{\infty}(\mathbb{T}^d)$. Let $\chi^m = (\chi^{m,i})_{i=1,...,d} \in L^2(\pi)^d$ denote the unique solution of*

$$(-\mathfrak{L})\chi^{m,i} = F^{m,i} - \langle F^{m,i}\rangle_\pi =: \overline{F}^{m,i}$$

with $\langle\chi^{m,i}\rangle_\pi = 0$. Then $(\chi^m)_m$ converges in $\mathscr{H}^1(\pi)^d \cap L^2(\pi)^d$ to a limit χ.

Proof. Convergence in $\mathscr{H}^1(\pi)$ follows from the estimate

$$\begin{aligned}\|\chi^{m,i} - \chi^{m',i}\|^2_{\mathscr{H}^1(\pi)} &= \langle(-\mathfrak{L})(\chi^{m,i} - \chi^{m',i}), \chi^{m,i} - \chi^{m',i}\rangle_\pi\\ &= \langle\overline{F}^{m,i} - \overline{F}^{m',i}, \chi^{m,i} - \chi^{m',i}\rangle_\pi\\ &\leqslant \|\overline{F}^{m,i} - \overline{F}^{m',i}\|_{\mathscr{H}^{-1}(\pi)}\|\chi^{m,i} - \chi^{m',i}\|_{\mathscr{H}^1(\pi)}.\end{aligned}$$

Thus we obtain

$$\|\chi^{m,i} - \chi^{m',i}\|_{\mathscr{H}^1(\pi)} \leqslant \|\overline{F}^{m,i} - \overline{F}^{m',i}\|_{\mathscr{H}^{-1}(\pi)}.$$

And indeed the $\mathscr{H}^{-1}(\pi)$-norm on the right-hand side is small, when m, m' are close, by Theorem 5.22.
It remains to conclude on $L^2(\pi)$ convergence. By Theorem 5.22, we also obtain the seminorm equivalences, $\|\cdot\|_{\mathscr{H}^1(\pi)} \simeq \|\cdot\|_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} \simeq \|\cdot\|_{\dot{B}^{\alpha/2}_{2,2}(\mathbb{T}^d)}$. Combining with the fractional Poincaré inequality on the torus,

$$\|u - \langle u\rangle_\lambda\|^2_{L^2} = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\hat{u}(k)|^2 \leqslant \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^\alpha|\hat{u}(k)|^2 = \|u\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)},$$

with Lebesgue measure λ on $\mathbb{T}^d$, we can thus estimate

$$\|\chi - \langle\chi\rangle_\lambda\|_{L^2(\pi)} \lesssim \|\chi - \langle\chi\rangle_\lambda\|_{L^2(\lambda)} \leqslant \|\chi\|_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} \lesssim \|\chi\|_{\mathscr{H}^1(\pi)}. \tag{5.34}$$

Furthermore, as $\langle\chi\rangle_\pi = 0$, we obtain $\|\chi - \langle\chi\rangle_\lambda\|^2_{L^2(\pi)} = \|\chi\|^2_{L^2(\pi)} + \langle\chi\rangle^2_\lambda$. Together, we thus find

$$\|\chi\|^2_{L^2(\pi)} \lesssim \|\chi\|^2_{L^2(\pi)} + \langle\chi\rangle^2_\lambda \lesssim \|\chi\|^2_{\mathscr{H}^1(\pi)}. \tag{5.35}$$

In particular, we conclude that $\mathscr{H}^1(\pi)$-convergence implies $L^2(\pi)$-convergence of the sequence (χ^m). □

Theorem 5.24. *Let $(F^m)_m$, $(\chi^m)_m$ and χ be as in Corollary 5.23.*
Then, $(\chi^m)_m$ converges to χ in $(\mathscr{C}_2^\theta(\mathbb{T}^d))^d$, $\theta \in ((1-\beta)/2, \alpha+\beta)$ and there exists $\lambda > 0$, such that

$$\chi = \chi^\sharp + \nabla\chi \varolessthan I_\lambda(\overline{F}) \tag{5.36}$$

for $\chi^\sharp \in (\mathscr{C}_2^{2\theta-1}(\mathbb{T}^d))^d$.
Furthermore, the limit χ solves the singular Poisson equation with singular right-hand side $\overline{F}$,

$$(-\mathfrak{L})\chi = \overline{F}. \tag{5.37}$$

Proof. Trivially, for $\lambda > 0$, χ^m solves the resolvent equation

$$R_\lambda\chi^m = (\lambda - \mathfrak{L})\chi^m = \lambda\chi^m + \overline{F}^m$$

with right-hand side $G^m := \lambda\chi^m + \overline{F}^m$. The right-hand sides (G^m) converge in $(\mathscr{C}_2^\beta(\mathbb{T}^d))^d$ to $G = \lambda\chi + \overline{F}$, because $\chi^m \to \chi$ in $L^2(\pi)^d$ by Corollary 5.23 and, thanks to the equivalence of π and the Lebesgue measure $\lambda_{\mathbb{T}^d}$, thus also in $L^2(\lambda_{\mathbb{T}^d})^d$. Choosing $\lambda > 1$ big enough, by Theorem 5.10, we can solve the resolvent equation

$$R_\lambda g^i = G^i = G^{\sharp,i} + G^{\prime,i} \varolessthan F, \tag{5.38}$$

with $G^{\sharp,i} := \lambda\chi^i \in L^2(\lambda) \subset \mathscr{C}_2^0(\mathbb{T}^d)$ and $G^{\prime,i} := (1 - \langle F^i\rangle_\pi)e_i \in \mathscr{C}^{\alpha+\beta-1}(\mathbb{T}^d)$. Thereby we obtain a paracontrolled solution $g^i \in \mathscr{D}_2^\theta$ for $\theta < \alpha+\beta$, with $g^i = g^{\sharp,i} + \nabla g^i \varolessthan I_\lambda(F)$, $g^{\sharp,i} \in \mathscr{C}_2^{2\theta-1}(\mathbb{T}^d)$ and $I_\lambda(F) := \int_0^\infty e^{-\lambda t}P_tF dt \in \mathscr{C}^{\alpha+\beta}(\mathbb{T}^d)$. By continuity of the solution map for the resolvent equation, we obtain convergence of $\chi^{m,i} \to g^i$ in $\mathscr{D}_2^\theta$ for $m \to \infty$. Convergence of (χ^m) to g in $(\mathscr{D}_2^\theta)^d$ in particular implies convergence in $L^2(\lambda_{\mathbb{T}^d})^d$ and thus in $L^2(\pi)^d$, which implies that almost surely $g = \chi$ and hence, by (5.38), that $\chi \in (\mathscr{D}_2^\theta)^d$ solves $(-\mathfrak{L})\chi = \overline{F}$. □

5.6. Fluctuations in the Brownian and pure Lévy noise case

In this section, we prove the central limit Theorem 5.26 for the diffusion X with periodic coefficients. In the following, we again explicitly distinguish between X and the projected process $X^{\mathbb{T}^d}$. Of course, the central limit theorem in particular implies

that for $t>0$, $\frac{1}{n}X_{nt}\to t\langle F\rangle_\pi$ with convergence in probability for $n\to\infty$, i.e. a weak law of large numbers. The central limit theorem then quantifies the fluctuations around the mean $t\langle F\rangle_\pi$.
Due to ergodicity of π, it follows by the von Neumann ergodic theorem that, if the projected process is started in $X_0^{\mathbb{T}^d}\sim\pi$, $\frac{1}{n}\int_0^{nt}b(X_s^{\mathbb{T}^d})ds\to t\langle b\rangle_\pi$ in $L^2(\mathbb{P}_\pi)$ as $n\to\infty$ for $b\in L^\infty(\mathbb{T}^d)$. As $\mathbb{P}_\pi=\int_{\mathbb{T}^d}\mathbb{P}_x\pi(dx)$, this implies in particular the convergence (along a subsequence) in $L^2(\mathbb{P}_x)$ for π-almost all x.
The L^∞-spectral gap estimates yield the following slightly stronger ergodic theorem for the process started in $X_0^{\mathbb{T}^d}=x$ for any $x\in\mathbb{T}^d$. In particular, in the periodic homogenization result for the PDE, Corollary 5.27 below, pointwise convergence (for every $x\in\mathbb{T}^d$) of the PDE solutions can be proven.

Lemma 5.25. *Let $b\in L^\infty(\mathbb{T}^d)$ and $x\in\mathbb{T}^d$. Let $X^{\mathbb{T}^d}$ be the projected solution of the $\mathscr{G}=\partial_t+\mathfrak{L}$- martingale problem on the torus $\mathbb{T}^d$ started in $X_0^{\mathbb{T}^d}=x\in\mathbb{T}^d$.*
Then the following convergence holds in $L^2(\mathbb{P})$:

$$\frac{1}{n}\int_0^{nt}b(X_s^{\mathbb{T}^d})ds\to t\langle b\rangle_\pi.$$

Proof. Without loss of generality, we assume that $\langle b\rangle_\pi=0$, otherwise we subtract the mean. With the Markov property we obtain

$$\begin{aligned}\left\|\frac{1}{n}\int_0^{nt}b(X_s^{\mathbb{T}^d})ds\right\|_{L^2(\mathbb{P})}^2&=\frac{1}{n^2}\int_0^{nt}\int_0^{nt}\mathbb{E}[b(X_s^{\mathbb{T}^d})b(X_r^{\mathbb{T}^d})]dsdr\\&=\frac{2}{n^2}\int_0^{nt}\int_0^{nt}\mathbf{1}_{s\leqslant r}\mathbb{E}\Big[b(X_s^{\mathbb{T}^d})\mathbb{E}_s[b(X_r^{\mathbb{T}^d})]\Big]dsdr\end{aligned}$$

Using the spectral gap estimate (5.21), we can estimate

$$\begin{aligned}\left|\frac{2}{n^2}\int_0^{nt}\int_0^{nt}\mathbf{1}_{s\leqslant r}T_s(bT_{r-s}b)(x)dsdr\right|&\leqslant\frac{2K^2\|b\|_{L^\infty}^2}{n^2}\int_0^{nt}\int_0^{nt}e^{-\mu s}e^{-\mu(r-s)}dsdr\\&=\frac{tK^2\|b\|_{L^\infty}^2}{n\mu}(1-e^{-\mu nt})\to 0,\end{aligned}$$

for $n\to\infty$. ☐

Theorem 5.26. *Let $\alpha\in(1,2]$ and $F\in\mathscr{C}^\beta(\mathbb{T}^d)$ for $\beta\in(\frac{1-\alpha}{2},0)$ or $F\in\mathscr{X}_\infty^{\beta,\gamma}(\mathbb{T}^d)$ for $\beta\in(\frac{2-2\alpha}{3},\frac{1-\alpha}{2}]$ and $\gamma\in(\frac{2\beta+2\alpha-1}{\alpha},1)$. Let X be the solution of the $\mathscr{G}=(\partial_t+\mathfrak{L})$-martingale problem started in $X_0=x\in\mathbb{R}^d$.*
In the case $\alpha=2$ and $L=B$ for a standard Brownian motion B, the following functional central limit theorem holds:

$$\left(\frac{1}{\sqrt{n}}(X_{nt}-nt\langle F\rangle_\pi)\right)_{t\in[0,T]}\Rightarrow\sqrt{D}(W_t)_{t\in[0,T]},$$

with convergence in distribution in $C([0,T],\mathbb{R}^d)$, a d-dimensional standard Brownian motion W and constant diffusion matrix D given by

$$D(i,j) := \int_{\mathbb{T}^d} (e_i + \nabla\chi^i(x))^T (e_j + \nabla\chi^j(x))\pi(dx)$$

for $i,j = 1,\dots,d$ and the i-th euclidean unit vector e_i. Here, χ solves the singular Poisson equation $(-\mathfrak{L})\chi^i = F^i - \langle F^i\rangle_\pi$, $i = 1,...,d$, according to Theorem 5.24.
In the case $\alpha \in (1,2)$, the following non-Gaussian central limit theorem holds:

$$\left(\frac{1}{n^{1/\alpha}}(X_{nt} - nt\langle F\rangle_\pi)\right)_{t\in[0,T]} \Rightarrow (\tilde{L}_t)_{t\in[0,T]},$$

with convergence in distribution in $D([0,T],\mathbb{R}^d)$, where $\tilde{L}$ is a d-dimensional symmetric α-stable nondegenerate Lévy process (with generator $-\mathscr{L}^\alpha_\nu$).

Proof. The martingale solution X of the singular SDE started in $X_0 = x$ is a weak solution by Theorem 4.35. In particular there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with an α-stable symmetric non-degenerate process L, such that $X = x + Z + L$, where Z is given by

$$Z_t = \lim_{m\to\infty} \int_0^t F^m(X_s)ds \tag{5.39}$$

for a sequence (F^m) of smooth functions F^m with $F^m \to F$ in $\mathscr{X}^{\beta,\gamma}_\infty(\mathbb{T}^d)$ and where the limit is taken in $L^2(\mathbb{P})$, uniformly in $t \in [0,T]$.
We write the additive functional $\int_0^\cdot (\overline{F^m})^{\mathbb{R}^d}(X_s)ds = \int_0^\cdot \overline{F^m}(X^{\mathbb{T}^d}_s)ds$ in terms of the periodic solution χ^m of the Poisson equation (5.28) with right hand side $F^m - \langle F^m\rangle_\pi =: \overline{F^m}$, such that

$$\begin{aligned}
X_t - t\langle F\rangle_\pi &= X_0 + (Z_t - t\langle F\rangle_\pi) + L_t && (5.40)\\
&= X_0 + \lim_{m\to\infty}\int_0^t \overline{F^m}(X^{\mathbb{T}^d}_s)ds + L_t && (5.41)\\
&= X_0 + \lim_{m\to\infty}\left([\chi^m(X^{\mathbb{T}^d}_0) - \chi^m(X^{\mathbb{T}^d}_t)] + M^m_t\right) + L_t && (5.42)\\
&= X_0 + [\chi(X^{\mathbb{T}^d}_0) - \chi(X^{\mathbb{T}^d}_t)] + M_t + L_t. && (5.43)
\end{aligned}$$

Here, the limit is again taken in $L^2(\mathbb{P})$ and χ is the solution of the Poisson equation (5.29) with right-hand side $\overline{F}$, which exists by Theorem 5.24.
To justify (5.41), we use the convergence from (5.39) and $\langle F\rangle_\pi = \lim_{m\to\infty}\langle F^m\rangle_\pi$ by Lemma 5.20. In (5.42), we applied Itô's formula to $(\chi^m)^{\mathbb{R}^d}(X_t)$ for $m \in \mathbb{N}$. For the equality (5.43), we utilized that $\chi^m \to \chi$ in $L^\infty(\mathbb{T}^d)$ by Theorem 5.24 and that the sequence of martingales (M^m) converges in $L^2(\mathbb{P})$ uniformly in time in $[0,T]$ to the martingale M. Here, for $\alpha \in (1,2)$, the martingales are given by (notation: $[y] := y$

mod $\mathbb{Z}^d = \iota(y)$)

$$M_t^m = \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [\chi^m(X_{s-}^{\mathbb{T}^d} + [y])) - \chi^m(X_{s-}^{\mathbb{T}^d})]\hat{\pi}(ds, dy),$$

where $\hat{\pi}(ds, dy) = \pi(ds, dy) - ds\mu(dy)$ is the compensated Poisson random measure associated to L. M is given by an analogue expression, where we replace χ^m by χ.
In the Brownian noise case, $\alpha = 2$, we have that $M_t^m = \int_0^t \nabla\chi^m(X_s^{\mathbb{T}^d}) \cdot dB_s$ and M_t is defined analogously with χ^m replaced by χ. Indeed, convergence of the martingales in $L^2(\mathbb{P})$ follows from the convergence of (χ^m) to χ in $\mathscr{C}_2^\theta(\mathbb{T}^d)$ with $\theta \in (1, \alpha+\beta)$ by Theorem 5.24, which in particular implies uniform convergence of (χ^m) and $(\nabla\chi^m)$ (cf. also the arguments for Lemma 4.4).

Let now first $\alpha = 2$ and $L = B$ for a standard Brownian motion B. Then we have by the above, almost surely,

$$\frac{1}{\sqrt{n}}(X_{nt} - nt\langle F\rangle_\pi) = \frac{1}{\sqrt{n}}X_0 + \frac{1}{\sqrt{n}}[\chi(X_0^{\mathbb{T}^d}) - \chi(X_{nt}^{\mathbb{T}^d})] + \frac{1}{\sqrt{n}}(M_{nt} + B_{nt})$$

with $M_t = \int_0^t \nabla\chi(X_s^{\mathbb{T}^d}) \cdot dB_s$.
To obtain the central limit theorem, we will apply the functional martingale central limit theorem, [EK86, Theorem 7.1.4], to

$$\left(\frac{1}{\sqrt{n}}(M_{nt} + B_{nt})\right)_{t\in[0,T]}.$$

To that aim, we check the convergence of the quadratic variation

$$\frac{1}{n}\langle M^i + B^i, M^j + B^j\rangle_{nt} = \frac{1}{n}\int_0^{nt} (\mathrm{Id} + \nabla\chi(X_s^{\mathbb{T}^d}))^T(\mathrm{Id} + \nabla\chi(X_s^{\mathbb{T}^d}))(i,j)ds$$

in probability to

$$t\int_{\mathbb{T}^d} (\mathrm{Id} + \nabla\chi(x))^T(\mathrm{Id} + \nabla\chi(x))(i,j)\pi(dx) = tD(i,j).$$

This is a consequence of Lemma 5.25.
The boundary term $\frac{1}{\sqrt{n}}[\chi(X_0^{\mathbb{T}^d}) - \chi(X_{nt}^{\mathbb{T}^d})]$ vanishes when $n \to \infty$ as $\chi \in L^\infty(\mathbb{T}^d)$. Furthermore, as a processes,

$$\left(\frac{1}{\sqrt{n}}[\chi(X_0^{\mathbb{T}^d}) - \chi(X_{nt}^{\mathbb{T}^d})]\right)_{t\in[0,T]}$$

converges to the constant zero process almost surely with respect to the uniform topology in $C([0,T],\mathbb{R}^d)$.
Using Slutsky's lemma and combining with the functional martingale central limit

theorem above, we obtain weak convergence of $(n^{-1/2}X_{nt})_{t\in[0,T]}$ to the Brownian motion $\sqrt{D}W$ with the constant diffusion matrix D stated in the theorem.

Let now $\alpha \in (1,2)$. We rescale by $n^{-1/\alpha}$ and claim that the martingale $n^{-1/\alpha}M_{nt}$ vanishes in $L^2(\mathbb{P})$ for $n \to \infty$. Indeed, in this case the martingale M is given by

$$M_t = \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [\chi(X_{s-}^{\mathbb{T}^d} + [y]) - \chi(X_{s-}^{\mathbb{T}^d})]\hat{\pi}(ds, dy).$$

Using the estimate from [PZ07, Lemma 8.22] and the mean-value theorem, we obtain

$$\begin{aligned}
\mathbb{E}[\sup_{t\in[0,T]} |M_{nt}|^2] &\lesssim \int_0^{nT} \int_{\mathbb{R}^d\setminus\{0\}} \mathbb{E}[|\chi(X_{s-}^{\mathbb{T}^d} + [y]) - \chi(X_{s-}^{\mathbb{T}^d})|^2]\mu(dy)ds \\
&= \int_0^{nT} \int_{\mathbb{R}^d\setminus\{0\}} \mathbb{E}[|\chi(X_{s}^{\mathbb{T}^d} + [y]) - \chi(X_{s}^{\mathbb{T}^d})|^2]\mu(dy)ds \\
&\leqslant \int_0^{nT} \int_{B(0,1)^c} \mathbb{E}[|\chi(X_{s}^{\mathbb{T}^d} + [y]) - \chi(X_{s}^{\mathbb{T}^d})|^2]\mu(dy)ds \\
&\quad + \int_0^{nT} \int_{B(0,1)\setminus\{0\}} \mathbb{E}[|\chi(X_{s}^{\mathbb{T}^d} + [y]) - \chi(X_{s}^{\mathbb{T}^d})|^2]\mu(dy)ds \\
&\leqslant 2nT\mu(B(0,1)^c)\|\chi\|^2_{L^\infty(\mathbb{T}^d)^d} + 2nT\|\nabla\chi\|^2_{L^\infty(\mathbb{T}^d)^{d\times d}} \int_{B(0,1)\setminus\{0\}} |y|^2\mu(dy) \\
&\lesssim nT.
\end{aligned}$$

Hence, we conclude

$$\mathbb{E}[\sup_{t\in[0,T]} |n^{-1/\alpha}M_{nt}|^2] \lesssim Tn^{1-2/\alpha} \tag{5.44}$$

and since $\alpha < 2$, we obtain the claimed convergence to zero.
As the J_1-metric (for definition, see [JS03, Chapter VI, Equation 1.26]) can be bounded by the uniform norm, (5.44) implies in particular, that the process $(n^{-1/\alpha}M_{nt})_{t\in[0,T]}$ converges to the constant zero process in probability with respect to the J_1-topology on the Skorokhod space $D([0,T],\mathbb{R}^d)$. Furthermore, $(n^{-1/\alpha}L_{nt})_{t\geqslant 0} \stackrel{d}{=} (L_t)_{t\geqslant 0}$. Using [JS03, Chapter VI, Proposition 3.17] and that the constant process is continuous, we thus obtain that $(n^{-1/\alpha}X_{nt})_{t\geqslant 0}$ convergences in distribution in $D([0,T],\mathbb{R}^d)$ to the α-stable process $(\tilde{L}_t)_{t\in[0,T]}$, that has the same law as $(L_t)_{t\in[0,T]}$. □

Utilizing the correspondence of the solution of the SDE (i.e. the solution of the martingale problem) to the parabolic generator PDE via Feynman-Kac, we can now show the corresponding periodic homogenization result for the PDE as a corollary.

Corollary 5.27. *Let F and $F^{\mathbb{R}^d}$ be as in Theorem 5.26. Assume moreover that $\langle F\rangle_\pi = 0$ and let $f \in C_b(\mathbb{R}^d)$. Let $T > 0$ and let $u \in D_T$ (D_T denotes the paracontrolled solution* *space from Corollary 3.28) be the mild solution of the singular parabolic PDE*

$$(\partial_t - \mathfrak{L})u = 0, \quad u_0 = f^\varepsilon,$$

where $f^\varepsilon(x) := f(\varepsilon x)$. *Let* $u^\varepsilon(t,x) := u(\varepsilon^{-\alpha}t, \varepsilon^{-1}x)$ *with* $u^\varepsilon(0,\cdot) = f$.
Let furthermore, for $\alpha = 2$ *and* $-\mathscr{L}_\nu^\alpha = \frac{1}{2}\Delta$, $\overline{u}$ *be the solution of*

$$(\partial_t - D : \nabla\nabla)\overline{u} = 0, \quad \overline{u}_0 = f,$$

with notation $D : \nabla\nabla := \sum_{i,j=1,\dots,d} D(i,j)\partial_{x_i}\partial_{x_j}$,
and for $\alpha \in (1,2)$, *let* $\overline{u}$ *be the solution of*

$$(\partial_t + \mathscr{L}_\nu^\alpha)\overline{u} = 0, \quad \overline{u}_0 = f.$$

Then, for any $t \in (0,T]$, $x \in \mathbb{R}^d$, *we have the convergence* $u_t^\varepsilon(x) \to \overline{u}_t(x)$ *for* $\varepsilon \to 0$.

Remark 5.28. *Note that* u^ε *solves* $(\partial_t - \mathfrak{L}^\varepsilon)u^\varepsilon = 0$, $u_0^\varepsilon = f$ *with operator* $\mathfrak{L}^\varepsilon g = -\mathscr{L}_\nu^\alpha g + \varepsilon^{1-\alpha}F(\varepsilon^{-1}\cdot)\nabla g$.

Remark 5.29. *If* $\alpha = 2$ *and* F *is of gradient-type, that is,* $F = \nabla f$ *for* $f \in \mathscr{C}^{1+\beta}$ *(*f *is a continuous function, as* $1+\beta > 0$*), the invariant measure is explicitly given by* $d\pi = c^{-1}e^{-f(x)}dx$ *with suitable normalizing constant* $c > 0$, *since the operator is of divergence form,* $\mathfrak{L} = e^f\nabla\cdot(e^{-f}\nabla\cdot)$. *Then it follows that* $\langle F\rangle_\pi = \int_{\mathbb{T}^d}\nabla e^{-f(x)}dx = 0$. *Thus,* F *satisfies the assumptions of Corollary 5.27.*

Proof of Corollary 5.27. Notice that $(\tilde{u}_s := u_{t-s})_{s\in[0,t]}$ solves the backward Kolmogorov equation $(\partial_s + \mathfrak{L})\tilde{u} = 0, \tilde{u}(t,\cdot) = f^\varepsilon$. Approximating f by $\mathscr{C}^3(\mathbb{R}^d)$ functions and using that X solves the $(\partial_t + \mathfrak{L}, x)$-martingale problem, we obtain

$$u^\varepsilon(t,x) = \mathbb{E}_{X_0=\varepsilon^{-1}x}[f(\varepsilon X_{\varepsilon^{-\alpha}t})].$$

The stated convergence then follows from Theorem 5.26. Indeed, if $X_0 = \varepsilon^{-1}x$, then $\varepsilon X_{\varepsilon^{-2}\cdot} \to W^x$ in distribution, where W^x is the Brownian motion started in x with covariance D, respectively $\varepsilon X_{\varepsilon^{-\alpha}\cdot} \to L^x$ if $\alpha \in (1,2)$ for the α-stable process L with generator $(-\mathscr{L}_\nu^\alpha)$ and $L_0 = x$. The Feynman-Kac formula for the limit process then gives that the limit of $(u^\varepsilon(t,x))$ equals $\overline{u}(t,x) = \mathbb{E}[f(W^x)]$ if $\alpha = 2$, respectively $\overline{u}(t,x) = \mathbb{E}[f(L^x)]$ if $\alpha \in (1,2)$. □

Remark 5.30 (Brox diffusion with Lévy noise)**.** *We can apply our theory to obtain the long-time behaviour of the periodic Brox diffusion with Lévy noise (see Section 4.6 for the construction). As* $\alpha \in (1,2]$, *Theorem 5.26 yields that* $|X_t| \sim t^{1/\alpha}$ *for* $t \to \infty$. *In the non-periodic situation, the long-time behaviour of the Brox diffusion with Brownian noise is however very different. Brox [Bro86] proved, that the diffusion gets trapped in local minima of the white noise environment and thus slowed down (that is, for almost all environments:* $|X_t| \sim \log(t)^2$ *for* $t \to \infty$, *cf. [Bro86, Theorem 1.4]). In the non-periodic pure stable noise case, the long-time behaviour of the Brox diffusion is an open problem, that we leave for future research.*

Remark 5.31 (Homogenization for diffusions in singular random environments)**.** *To prove the homogenization limit Theorem 5.26, we used the Kipnis-Varadhan approach and some of the techniques developed in [KLO12]. Those techniques can be applied to diffusions in random stationary environments (cf. [KLO12, Chapter 9]) of which the diffusion with periodic coefficients is a special case. Parts of the theory developed in this chapter can thus also be of use when generalizing to singular, stationary random fields F with path space $\mathscr{C}^{\beta}_{\mathbb{R}^d}$.*

A. Appendix

Appendix for Chapter 3

Proof of Lemma 3.8. The proof of the lemma uses ideas from the proof of [Per14, Lemma 5.3.20]. Let $\psi = p_0 \in C_c^\infty$ and let $j \geqslant 0$ (for $j = -1$, $\Delta_j(f \otimes g) = 0$, so there is nothing to estimate). Then we estimate (notation: $S_{j-1}u := \sum_{l \leqslant j-1} \Delta_l u$)

$$\begin{aligned}
&\|\Delta_j[(-\mathscr{L}_\nu^\alpha)(f \otimes g) - f \otimes (-\mathscr{L}_\nu^\alpha)g]\|_{L^p} \\
&\quad = \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathscr{F}^{-1}(-\psi_\nu^\alpha p_j)(x-y)(S_{j-1}f(y) - S_{j-1}f(x))\Delta_j g(y) dy \right|^p dx \right)^{1/p} \\
&\quad \lesssim \sum_{|\eta|=1} \|[z \mapsto z^\eta \mathscr{F}^{-1}(-\psi_\nu^\alpha p_j)(z)] * \partial^\eta(S_{j-1}f)\|_{L^p} \|\Delta_j g\|_{L^\infty} \\
&\quad \lesssim \sum_{|\eta|=1} \|z \mapsto z^\eta \mathscr{F}^{-1}(-\psi_\nu^\alpha p_j)(z)\|_{L^1} \|\partial^\eta S_{j-1} f\|_{L^p} \|\Delta_j g\|_{L^\infty}
\end{aligned}$$

for a multi-index η and using that $S_{j-1}f(x) - S_{j-1}f(y) = \int_0^1 DS_{j-1}f(\lambda x + (1-\lambda)y))(x-y)d\lambda$ with $\lambda x + (1-\lambda)y = (1+\lambda)x - \lambda y - (x-y)$ (and substituting $y \to x - y$, $x \to (1+\lambda)x - \lambda y$) and Young's inequality for the last estimate. We have that, as $\sigma < 1$,

$$\|\partial^\eta S_{j-1}f\|_{L^p} \|\Delta_j g\|_{L^\infty} \lesssim 2^{-j(\sigma-1+\varsigma)} \|\partial^\eta f\|_{\mathscr{C}_p^{\sigma-1}} \|g\|_{\mathscr{C}^\varsigma} \lesssim 2^{-j(\sigma-1+\varsigma)} \|f\|_{\mathscr{C}_p^\sigma} \|g\|_{\mathscr{C}^\varsigma}.$$

Moreover, we obtain

$$\begin{aligned}
\|z \mapsto z^\eta \mathscr{F}^{-1}(-\psi_\nu^\alpha p_j)(z)\|_{L^1} &= 2^{j\alpha} \|z \mapsto z^\eta \mathscr{F}^{-1}(\psi_\nu^\alpha(2^{-j}\cdot)p_0(2^{-j}\cdot))(z)\|_{L^1} \\
&= 2^{j\alpha} 2^{-j} \|\mathscr{F}^{-1}(\partial^\eta[\psi_\nu^\alpha p_0](2^{-j}\cdot))\|_{L^1} \\
&\lesssim 2^{j(\alpha-1)}
\end{aligned}$$

using that

$$\|\mathscr{F}^{-1}(\partial^\eta[\psi_\nu^\alpha p_0](2^{-j}\cdot))\|_{L^1} = \|2^{jd} \mathscr{F}^{-1}(\partial^\eta[\psi_\nu^\alpha p_0])(2^{-j}\cdot)\|_{L^1} = \|\mathscr{F}^{-1}(\partial^\eta[\psi_\nu^\alpha p_0])\|_{L^1} < \infty.$$

Together we have

$$\|\Delta_j[(-\mathscr{L}_\nu^\alpha)(f \otimes g) - f \otimes (-\mathscr{L}_\nu^\alpha)g]\|_{L^p} \lesssim 2^{-j(\sigma+\varsigma-\alpha)} \|f\|_{\mathscr{C}_p^\sigma} \|g\|_{\mathscr{C}^\varsigma},$$

which yields the claim. □

Proof of Lemma 3.9. For $\vartheta \in [-1, \infty)$, the claim follows from [Per14, Lemma 5.3.20 and Lemma 5.5.7], applied to $\varphi(z) = \exp(-\psi_\nu^\alpha(z))$. Here, [Per14, Lemma 5.3.20] can be generalized, with the notation from that lemma, to $u \in \mathscr{C}_p^\alpha$ for $p \in [1, \infty]$ arguing analoguously as in the proof of Lemma 3.8.
It remains to prove the commutator for $\vartheta \in [-\alpha, -1)$. For that we note that

$$\begin{aligned} P_t(u \varolessthan v) - u \varolessthan P_t(v) &= (P_t - \mathrm{Id})(u \varolessthan v) - u \varolessthan (P_t - \mathrm{Id})v \\ &= \int_0^t [(-\mathscr{L}_\nu^\alpha)P_r(u \varolessthan v) - u \varolessthan (-\mathscr{L}_\nu^\alpha)P_r v]dr. \end{aligned}$$

For the operator $(-\mathscr{L}_\nu^\alpha)P_r$ we have by Lemma 3.10 below (whose claim follows from (3.11) for $\vartheta \geqslant 0$ and Lemma 3.8), that for $\theta \geqslant 0$ (uniformly in $r \in [0, t]$)

$$\|(-\mathscr{L}_\nu^\alpha)P_r(u \varolessthan v) - u \varolessthan (-\mathscr{L}_\nu^\alpha)P_r v\|_{\mathscr{C}_p^{\sigma+\varsigma-\alpha+\theta}} \lesssim r^{-\theta/\alpha}\|u\|_{\mathscr{C}_p^\sigma}\|v\|_{\mathscr{C}^\varsigma}$$

holds true and thus we obtain (taking $\theta = \vartheta + \alpha \geqslant 0$)

$$\begin{aligned} &\|P_t(u \varolessthan v) - u \varolessthan P_t(v)\|_{\mathscr{C}_p^{\varsigma+\sigma+\vartheta}} \\ &\quad\leqslant \int_0^t \|(-\mathscr{L}_\nu^\alpha)P_r(u \varolessthan v) - u \varolessthan (-\mathscr{L}_\nu^\alpha)P_r v\|_{\mathscr{C}_p^{(\varsigma+\sigma-\alpha)+(\vartheta+\alpha)}}dr \\ &\quad\lesssim \|u\|_{\mathscr{C}_p^\sigma}\|v\|_{\mathscr{C}^\varsigma}\int_0^t r^{-(\vartheta+\alpha)/\alpha}dr \lesssim t^{-\vartheta/\alpha}\|u\|_{\mathscr{C}_p^\sigma}\|v\|_{\mathscr{C}^\varsigma}, \end{aligned}$$

where the last two estimates are valid for $\vartheta \in [-\alpha, 0)$. □

Proof of Lemma 3.10. We have that

$$\begin{aligned} (-\mathscr{L}_\nu^\alpha)P_t(u \varolessthan v) - u \varolessthan (-\mathscr{L}_\nu^\alpha)P_t v &= (-\mathscr{L}_\nu^\alpha)\big(P_t(u \varolessthan v) - u \varolessthan P_t v\big) \\ &\quad + (-\mathscr{L}_\nu^\alpha)(u \varolessthan P_t v) - u \varolessthan (-\mathscr{L}_\nu^\alpha)P_t v. \end{aligned}$$

The first summand, we estimate by the commutator for (P_t) from Lemma 3.9, and continuity of the operator $(-\mathscr{L}_\nu^\alpha)$ from Proposition 3.5, which gives

$$\begin{aligned} \|(-\mathscr{L}_\nu^\alpha)\big(P_t(u \varolessthan v) - u \varolessthan P_t v\big)\|_{\mathscr{C}_p^{\sigma+\varsigma+\theta-\alpha}} &\lesssim \|P_t(u \varolessthan v) - u \varolessthan P_t v\|_{\mathscr{C}_p^{\sigma+\varsigma+\theta}} \\ &\lesssim t^{-\vartheta/\alpha}\|u\|_{\mathscr{C}_p^\sigma}\|v\|_{\mathscr{C}^\varsigma}. \end{aligned}$$

The second summand follows from the commutator for $(-\mathscr{L}_\nu^\alpha)$. If $\alpha = 2$, then the estimate is immediate due to Leibnitz rule, $\sigma < 1$ and Schauder estimates for P_t as $\theta \geqslant 0$. If $\alpha \in (1, 2)$, then we apply Lemma 3.8 with $f = u$ and $g = P_t v$ and use the Schauder estimates with $\theta \geqslant 0$, Lemma 3.7, to obtain

$$\|(-\mathscr{L}_\nu^\alpha)(u \varolessthan P_t v) - u \varolessthan (-\mathscr{L}_\nu^\alpha)P_t v\|_{\mathscr{C}_p^{\sigma+\varsigma-\alpha+\theta}} \lesssim \|u\|_{\mathscr{C}_p^\sigma}\|P_t v\|_{\mathscr{C}^{\varsigma+\theta}} \lesssim t^{-\theta/\alpha}\|u\|_{\mathscr{C}_p^\sigma}\|v\|_{\mathscr{C}^\varsigma}.$$

Altogether, we obtain the desired bound. □

Proof of Lemma 3.11. The proof of the lemma uses the ideas from the proof of [GIP15, Lemma A.9]. Let $\delta \in (0, \frac{T-t}{2})$ to be chosen later. Then we have that for $j \geqslant -1$

$$\Delta_j \int_t^T f_{t,r} dr = \int_t^T \Delta_j f_{t,r} dr = \int_{t+\delta}^T \Delta_j f_{t,r} dr + \int_t^{t+\delta} \Delta_j f_{t,r} dr.$$

The first summand we estimate as follows, using Minkowski's inequality,

$$\begin{aligned}
\left\| \int_{t+\delta}^T \Delta_j f_{t,r} dr \right\|_{L^p} &\leqslant \int_{t+\delta}^T \|\Delta_j f_{t,r}\|_{L^p} dr \\
&\leqslant C 2^{-j(\sigma+\varsigma+\varepsilon\varsigma)} \int_{t+\delta}^T (T-r)^{-\gamma} (r-t)^{-(1+\varepsilon)} dr \\
&= C 2^{-j(\sigma+\varsigma+\varepsilon\varsigma)} (T-t)^{-\gamma-\varepsilon} \int_{\delta/(T-t)}^1 (1-r)^{-\gamma} r^{-(1+\varepsilon)} dr \\
&\leqslant [2\max(\varepsilon^{-1}, (1-\gamma)^{-1}) C] 2^{-j(\sigma+\varsigma+\varepsilon\varsigma)} (T-t)^{-\gamma-\varepsilon} (\delta/(T-t))^{-\varepsilon} \\
&= [2\max(\varepsilon^{-1}, (1-\gamma)^{-1}) C] 2^{-j(\sigma+\varsigma)} (T-t)^{-\gamma} (2^{j\varsigma}\delta)^{-\varepsilon},
\end{aligned}$$

where we used that for $\sigma \in (0, \frac{1}{2})$, as $\varepsilon > 0$ and $\gamma < 1$,

$$\begin{aligned}
\int_\sigma^1 (1-r)^{-\gamma} r^{-(1+\varepsilon)} dr &= \int_\sigma^{1/2} (1-r)^{-\gamma} r^{-(1+\varepsilon)} dr + \int_{1/2}^1 (1-r)^{-\gamma} r^{-(1+\varepsilon)} dr \\
&\leqslant [(\frac{1}{2})^{-\gamma} \varepsilon^{-1} + (\frac{1}{2})^{-\gamma} (1-\gamma)^{-1}] \sigma^{-\varepsilon} \leqslant 2\max(\varepsilon^{-1}, (1-\gamma)^{-1}) \sigma^{-\varepsilon}.
\end{aligned}$$

For the second summand, we have

$$\begin{aligned}
\left\| \int_t^{t+\delta} \Delta_j f_{t,r} dr \right\|_{L^p} &\leqslant C 2^{-j\sigma} \int_t^{t+\delta} (T-r)^{-\gamma} dr \\
&= \frac{C}{1-\gamma} 2^{-j\sigma} [(T-t)^{1-\gamma} - (T-t-\delta)^{1-\gamma}] \\
&= \frac{C}{1-\gamma} 2^{-j\sigma} (T-t)^{-\gamma} [(T-t) - (T-t-\delta) \big(\frac{T-t}{T-t-\delta} \big)^{\gamma}] \\
&\leqslant \frac{C}{1-\gamma} 2^{-j\sigma} (T-t)^{-\gamma} \delta.
\end{aligned}$$

The goal is to estimate $\sup_{j \geqslant -1} 2^{j(\sigma+\varsigma)} \big\| \Delta_j \int_t^T f_{t,r} dr \big\|_{L^p}$. For that purpose, we use for j such that $2^{-j\varsigma} \leqslant \frac{T-t}{2}$ the above estimates for $\delta = 2^{-j\varsigma}$. If j is such that $2^{-j\varsigma} > \frac{T-t}{2}$, then we trivially estimate

$$\left\| \Delta_j \int_t^T f_{t,r} dr \right\|_{L^p} \leqslant C 2^{-j\sigma} \int_t^T (T-r)^{-\gamma} dr = \frac{C}{1-\gamma} 2^{-j\sigma} (T-t)^{1-\gamma}$$
$$\leqslant \frac{C}{1-\gamma} 2^{-j(\sigma+\varsigma)} (T-t)^{-\gamma}$$

using $\gamma < 1$. Together we thus obtain uniformly in $t \in [0,T]$

$$\sup_{j \geqslant -1} 2^{j(\sigma+\varsigma)} \left\| \Delta_j \int_t^T f_{t,r} dr \right\|_{L^p} \leqslant [2 \max(\varepsilon^{-1}, (1-\gamma)^{-1}) C] (T-t)^{-\gamma},$$

which yields the claim. □

Appendix for Chapter 4

Proof of Lemma 4.3. We prove this by induction. For $n = 0$ the claim is true, so we assume that it holds for n and establish it also for $n+1$. We get with Campbell's formula (see [Kin93, Section 3.2]):

$$\Phi(\lambda) = \exp\left(\int_r^t \int_{|y|\leqslant C} (e^{\lambda|y|^2} - 1)\mu(dy)ds\right) = \exp\left((t-r)\int_{|y|\leqslant C} (e^{\lambda|y|^2} - 1)\mu(dy)\right),$$

and therefore

$$\begin{aligned}
&\Phi^{(n+1)}(\lambda)\\
&\quad = \partial_\lambda \Phi^{(n)}(\lambda)\\
&\quad = \partial_\lambda \left(\Phi(\lambda) \sum_{\omega\in\mathbb{N}_0^n:|\omega|=n} c(n,\omega) \prod_{i=1}^n \left((t-r)\int_{|y|\leqslant C} |y|^{2i} e^{\lambda|y|^2}\mu(dy)\right)^{\omega_i}\right)\\
&\quad = \Phi(\lambda)(t-r)\int_{|y|\leqslant C} |y|^2 e^{\lambda|y|^2}\mu(dy) \sum_{\omega\in\mathbb{N}_0^n:|\omega|=n} c(n,\omega) \prod_{i=1}^n \left((t-r)\int_{|y|\leqslant C} |y|^{2i} e^{\lambda|y|^2}\mu(dy)\right)^{\omega_i}\\
&\qquad + \Phi(\lambda) \sum_{\omega\in\mathbb{N}_0^n:|\omega|=n} c(n,\omega)\partial_\lambda \left(\prod_{i=1}^n \left((t-r)\int_{|y|\leqslant C} |y|^{2i} e^{\lambda|y|^2}\mu(dy)\right)^{\omega_i}\right).
\end{aligned}$$

The first term on the right-hand side is of the claimed form with the choice $\tilde{\omega} = (\omega_1 + 1, \omega_2, \ldots, \omega_n, 0) \in \mathbb{N}_0^{n+1}$ such that $|\tilde{\omega}| = n+1$. For the second term on the right-hand side we get by Leibniz's rule

$$\begin{aligned}
&\partial_\lambda \left(\prod_{i=1}^n \left((t-r)\int_{|y|\leqslant C} |y|^{2i} e^{\lambda|y|^2}\mu(dy)\right)^{\omega_i}\right)\\
&\quad = \sum_{j=1}^n \prod_{i\neq j}^n \left((t-r)\int_{|y|\leqslant C} |y|^{2i} e^{\lambda|y|^2}\mu(dy)\right)^{\omega_i} \times \omega_j \left((t-r)\int_{|y|\leqslant C} |y|^{2j} e^{\lambda|y|^2}\mu(dy)\right)^{\omega_j - 1}\\
&\qquad\qquad \times (t-r)\int_{|y|\leqslant C} |y|^{2(j+1)} e^{\lambda|y|^2}\mu(dy)\\
&\quad = \sum_{j=1}^{n+1} \omega_j \prod_{i=1}^{n+1} \left((t-r)\int_{|y|\leqslant C} |y|^{2i} e^{\lambda|y|^2}\mu(dy)\right)^{\tilde{\omega}_i^j},
\end{aligned}$$

with $\tilde{\omega}_i^j \in \mathbb{N}_0^{n+1}$ defined by

$$\tilde{\omega}_i^j = \begin{cases} \omega_i, & i \neq j, j+1,\\ \omega_j - 1, & i = j,\\ \omega_{j+1} + 1, & i = j+1. \end{cases}$$

As required we have $|\tilde{\omega}^j| = |\omega| - j + (j+1) = |\omega| + 1 = n+1$, and thus the proof is complete. By plugging $\lambda = 0$ into (4.4), we obtain (4.5). □

Proof of Lemma 4.28. We use the paracontrolled structure of u to prove the bound (4.29), but nonetheless the bound does not trivially follow from that. We abbreviate $\theta := \alpha + \beta$. Let us first make some observations. By the Schauder estimates, Corollary 3.12, we have that $J^T(\partial_i V) \in (\mathscr{L}_T^{0,\theta-1+(1-\gamma)\alpha})^d$ as $V \in C_T\mathscr{C}_{\mathbb{R}^d}^{\beta+(1-\gamma)\alpha}$, and thus by the interpolation estimates, specifically (3.26) from Lemma 3.17 (applied for $\tilde{\theta} = 2(\theta-1) \in (0,\alpha)$), we obtain that $J^T(\partial_i V) \in C_T^{2(\theta-1)/\alpha}\mathscr{C}_{\mathbb{R}^d}^{(1-\gamma)\alpha-(\theta-1)}$ and $\nabla u \in C_T^{2(\theta-1)/\alpha}\mathscr{C}_{\mathbb{R}^d}^{-(\theta-1)}$. With that, we can estimate the resonant and paraproduct product as $(1-\gamma)\alpha > 0$ and the following notation $\nabla u_{sr}(x) := \nabla u(r,x) - \nabla u(s,x)$ (and analogously for $J^T(\partial_i V)$)

$$\begin{aligned}
&\|\nabla u_{sr} \odot J^T(\partial_i V)_r\|_{L^\infty} + \|\nabla u_{sr} \circledcirc J^T(\partial_i V)_r\|_{L^\infty} \\
&\quad\lesssim \|\nabla u_{sr} \odot J^T(\partial_i V)_r\|_{\mathscr{C}^{(1-\gamma)\alpha}} + \|\nabla u_{sr} \circledcirc J^T(\partial_i V)_r\|_{\mathscr{C}^{(1-\gamma)\alpha}} \\
&\quad\lesssim \|J^T(\partial_i V)\|_{C_T\mathscr{C}^{\theta-1+(1-\gamma)\alpha}} \|\nabla u\|_{C_T^{(2\theta-2)/\alpha}\mathscr{C}^{-(\theta-1)}} |r-s|^{(2\theta-2)/\alpha}. \qquad \text{(A.1)}
\end{aligned}$$

Furthermore, using that by assumption $(1-\gamma)\alpha - (\theta-1) = -\gamma\alpha - \beta + 1 < 0$ since $\gamma \geqslant (2\beta + 2\alpha - 1)/\alpha > (1-\beta)/\alpha$ in the rough case, and $\gamma > (1-\beta)/\alpha$ in the Young case (cf. Definition 3.20), we obtain

$$\begin{aligned}
&\|J^T(\partial_i V)_{sr} \odot \nabla u_s\|_{L^\infty} + \|J^T(\partial_i V)_{sr} \circledcirc \nabla u_s\|_{L^\infty} \\
&\quad\lesssim \|J^T(\partial_i V)_{sr} \odot \nabla u_s\|_{\mathscr{C}^{(1-\gamma)\alpha}} + \|J^T(\partial_i V)_{sr} \circledcirc \nabla u_s\|_{\mathscr{C}^{(1-\gamma)\alpha}} \\
&\quad\lesssim \|J^T(\partial_i V)\|_{C_T^{2(\theta-1)/\alpha}\mathscr{C}^{(1-\gamma)\alpha-(\theta-1)}} \|\nabla u\|_{C_T\mathscr{C}^{\theta-1}} |r-s|^{(2\theta-2)/\alpha}. \qquad \text{(A.2)}
\end{aligned}$$

Notice that we do not have the symmetric estimate for the time difference in the upper part of the product.
Moreover, using the paracontrolled structure we have that

$$\begin{aligned}
&\partial_i u(r,x) - \partial_i u(s,y) \\
&\quad= (\nabla u \circledcirc J^T(\partial_i V))(r,x) - (\nabla u \circledcirc J^T(\partial_i V))(s,y) + \partial_i u^\sharp(r,x) - \partial_i u^\sharp(s,y) \\
&\qquad + (\partial_i \nabla u \circledcirc J^T(V))(r,x) - (\partial_i \nabla u \circledcirc J^T(V))(s,y) \\
&\quad= (\nabla u \circledcirc J^T(\partial_i V))(r,x) - (\nabla u \circledcirc J^T(\partial_i V))(s,y) + g(r,x) - g(s,y) \qquad \text{(A.3)}
\end{aligned}$$

for g defined as

$$g := \partial_i u^\sharp + \partial_i \nabla u \circledcirc J^T(V).$$

By $\partial_i \nabla u \circledcirc J^T(V) \in C_T^{(2\theta-2)/\alpha} L^\infty \cap C_T\mathscr{C}^{2\theta-2}$, $\partial_i u^\sharp \in C_T^{(2\theta-2)/\alpha} L^\infty \cap C_T\mathscr{C}^{2\theta-2}$ and the interpolation estimates, we obtain the desired estimate for g:

$$\begin{aligned}
|g(r,x) - g(s,y)| &\leqslant |g(r,y) - g(s,y)| + |g(r,x) - g(r,y)| \\
&\lesssim |r-s|^{(2\theta-2)/\alpha} + |x-y|^{2\theta-2}.
\end{aligned}$$

Thus it is left to show that

$$|(\nabla u \otimes J^T(\partial_i V))(r,x) - (\nabla u \otimes J^T(\partial_i V))(s,y) - \nabla u(s,y)\cdot(J^T(\partial_i V)(r,x) - J^T(\partial_i V)(s,y))| \lesssim |t-s|^{(2\theta-2)/\alpha} + |x-y|^{2\theta-2}.$$

To that aim, we replace r by s and subtract the remainder, such that we obtain

$$\begin{aligned} &\big|(\nabla u \otimes J^T(\partial_i V))(r,x) - (\nabla u \otimes J^T(\partial_i V))(s,y) \\ &\quad - \nabla u(s,y)\cdot(J^T(\partial_i V)(r,x) - J^T(\partial_i V)(s,y))\big| \\ &\leqslant \big|(\nabla u \otimes J^T(\partial_i V))(s,x) - (\nabla u \otimes J^T(\partial_i V))(s,y) \\ &\quad - \nabla u(s,y)\cdot(J^T(\partial_i V)(s,x) - J^T(\partial_i V)(s,y))\big| \\ &\quad + |(\nabla u \otimes J^T(\partial_i V))_{sr}(x) - \nabla u(s,y) J^T(\partial_i V)_{sr}(x)|. \end{aligned} \tag{A.4}$$

For the first term in (A.4), we abbreviate $v = J^T(\partial_i V)_s \in \mathscr{C}^{\theta-1}_{\mathbb{R}^d}$ and $u = \nabla u_s \in \mathscr{C}^{\theta-1}_{\mathbb{R}^d}$ and utilize the space regularities to prove the claim

$$|u \otimes v(x) - u \otimes v(y) - u(y)(v(x) - v(y))| \lesssim |x-y|^{2\theta-2}. \tag{A.5}$$

To prove (A.5), we use ideas from the proof of [GIP15, Lemma B.2]. We write

$$\begin{aligned} &u \otimes v(x) - u \otimes v(y) - u(y)(v(x) - v(y)) \\ &\quad = \sum_{j\geqslant -1} (S_{j-1}u(x) - u(y))(\Delta_j v(x) - \Delta_j v(y)) + \sum_j (S_{j-1}u(y) - S_{j-1}u(x))\Delta_j v(y) \end{aligned} \tag{A.6}$$

with the notation $S_{j-1}u := \sum_{-1\leqslant i\leqslant j-1} \Delta_i u$. The first summand, we estimate in two different ways, once using that $\mathscr{C}^{\theta-1} \subset C^{\theta-1}$, where $C^{\theta-1}$ is the Hölder space with $\theta - 1 \in (0,1)$ and on the other hand using the mean value theorem, such that

$$\begin{aligned} &|(S_{j-1}u(x) - u(y))(\Delta_j v(x) - \Delta_j v(y))| \\ &\quad \lesssim 2^{-j(\theta-1)}\|u\|_{\theta-1}\big(|x-y|^{\theta-1}\|v\|_{\theta-1} \wedge |x-y|\|D\Delta_j v\|_{L^\infty}\big) \\ &\quad \lesssim 2^{-j(\theta-1)}\|u\|_{\theta-1}\big(|x-y|^{\theta-1}\|v\|_{\theta-1} \wedge |x-y| 2^{-j(\theta-2)}\|v\|_{\theta-1}\big). \end{aligned}$$

The second summand, we estimate analogously using $\theta - 1 \in (0,1)$ (and thus $\theta - 2 < 0$)

$$\begin{aligned} &|(S_{j-1}u(y) - S_{j-1}u(x))\Delta_j v(y)| \\ &\quad \lesssim \big(|x-y|^{\theta-1}\|u\|_{\theta-1} \wedge |x-y|\|DS_{j-1}u\|_{L^\infty}\big)2^{-j(\theta-1)}\|v\|_{\theta-1} \\ &\quad \lesssim \big(|x-y|^{\theta-1}\|u\|_{\theta-1} \wedge |x-y|2^{-j(\theta-2)}\|u\|_{\theta-1}\big)2^{-j(\theta-1)}\|v\|_{\theta-1}. \end{aligned}$$

We can w.l.o.g. assume that $|x-y| \leqslant 1$. Otherwise the estimate (A.5) is trivial as $u, v \in L^\infty$. Then we let j_0 such that $2^{-j_0} \sim |x-y|$ and decompose both sums in (A.6) in the part with $j > j_0$, such that $2^{-j} < 2^{-j_0} \leqslant |x-y|$, and in the part with $j \leqslant j_0$

(i.p. a finite sum), such that $2^{-j} \geqslant |x-y|$, and perform an analogous estimate for the two. We write down the estimate for the first sum. That is, we have

$$\begin{aligned}
&\sum_{j\geqslant -1} |(S_{j-1}u(x) - u(y))(\Delta_j v(x) - \Delta_j v(y))| \\
&\quad\lesssim \|u\|_{\theta-1}\|v\|_{\theta-1}\Big(\sum_{j>j_0} |x-y|^{\theta-1}2^{-j(\theta-1)} + \sum_{j\leqslant j_0} |x-y| 2^{-j(\theta-2)}2^{-j(\theta-1)}\Big) \\
&\quad\lesssim \|u\|_{\theta-1}\|v\|_{\theta-1}\Big(|x-y|^{\theta-1}2^{-j_0(\theta-1)} + \sum_{j\leqslant j_0} |x-y||x-y|^{2\theta-3}\Big) \\
&\quad\lesssim \|u\|_{\theta-1}\|v\|_{\theta-1}\Big(|x-y|^{\theta-1}|x-y|^{(\theta-1)} + |x-y||x-y|^{2\theta-3}\Big) \\
&\quad= \|u\|_{\theta-1}\|v\|_{\theta-1}|x-y|^{2\theta-2}
\end{aligned}$$

using $\theta-1>0$, such that the first series converges, and $2\theta-3<0$ and that the second sum is a finite sum. Hence the claim (A.5) follows. Rewritten in the previous notation, we thus obtain a bound uniformly in s,x,y, that is,

$$\begin{aligned}
&\big|(\nabla u \otimes J^T(\partial_i V))(s,x) - (\nabla u \otimes J^T(\partial_i V))(s,y) \\
&\qquad - \nabla u(s,y)\cdot (J^T(\partial_i V)(s,x) - J^T(\partial_i V)(s,y))\big| \\
&\quad\lesssim |x-y|^{2\theta-2}.
\end{aligned}$$

We are left with the second term in (A.4), that we furthermore decompose as follows

$$\begin{aligned}
&|(\nabla u \otimes J^T(\partial_i V))_{sr}(x) - \nabla u(s,y)\cdot J^T(\partial_i V)_{sr}(x)| \\
&\quad\leqslant |\nabla u_s \otimes J^T(\partial_i V)_{sr}(x) - \nabla u(s,x)\cdot J^T(\partial_i V)_{sr}(x)| + |\nabla u_{sr} \otimes J^T(\partial_i V)_r(x)| \\
&\qquad + |(\nabla u(s,x) - \nabla u(s,y))\cdot J^T(\partial_i V)_{sr}(x)| \\
&\quad\leqslant |\nabla u_s \circledcirc J^T(\partial_i V)_{sr}(x)| + |\nabla u_s \odot J^T(\partial_i V)_{sr}(x)| + |\nabla u_{sr} \otimes J^T(\partial_i V)_r(x)| \\
&\qquad + |(\nabla u(s,x) - \nabla u(s,y))\cdot J^T(\partial_i V)_{sr}(x)| \\
&\quad\lesssim |r-s|^{(2\theta-2)/\alpha} + |(\nabla u(s,x) - \nabla u(s,y))\cdot J^T(\partial_i V)_{sr}(x)| \\
&\quad\lesssim |r-s|^{(2\theta-2)/\alpha} + |x-y|^{\theta-1}|r-s|^{(\theta-1)/\alpha},
\end{aligned}$$

where we used the estimates from the beginning, that is (A.1) and (A.2). Thus altogether we obtain the estimate for (A.4):

$$\begin{aligned}
&\big|(\nabla u \otimes J^T(\partial_i V))(r,x) - (\nabla u \otimes J^T(\partial_i V))(s,y) \\
&\qquad - \nabla u(s,y)\cdot (J^T(\partial_i V)(r,x) - J^T(\partial_i V)(s,y))\big| \\
&\quad\lesssim |r-s|^{(2\theta-2)/\alpha} + |x-y|^{2\theta-2} + |x-y|^{\theta-1}|r-s|^{(\theta-1)/\alpha},
\end{aligned}$$

where for $|x-y| < |r-s|^{1/\alpha}$ as well as for $|x-y| \geqslant |r-s|^{1/\alpha}$ (i.e. $|r-s| \leqslant |x-y|^\alpha$), we obtain the desired estimate (4.29). □

Appendix for Chapter 5

Proof of Lemma 5.1. To show (5.2), we notice that by the isometry of the spaces $L^2(\mathbb{T}^d)$, $l^2(\mathbb{Z}^d)$ by the Fourier transform,

$$\|\Delta_j \mathscr{L}_\nu^\alpha u\|_{L^2(\mathbb{T}^d)}^2 = \sum_{k\in\mathbb{Z}^d} |\rho_j(k)\psi_\nu^\alpha(k)\hat{u}(k)|^2.$$

Due to $\rho_j(k) \neq 0$ only if $|k| \sim 2^j$ and $|\psi_\nu^\alpha(k)| \lesssim |k|^\alpha$, we obtain that

$$\|\Delta_j \mathscr{L}_\nu^\alpha u\|_{L^2(\mathbb{T}^d)}^2 \lesssim 2^{2j\alpha} \sum_{k\in\mathbb{Z}^d} |\rho_j(k)\hat{u}(k)|^2 = 2^{2j\alpha}\|\Delta_j u\|_{L^2(\mathbb{T}^d)}^2$$

and thus

$$\|\mathscr{L}_\nu^\alpha u\|_{\mathscr{C}_2^{\beta-\alpha}(\mathbb{T}^d)} = \sup_j 2^{j(\beta-\alpha)}\|\Delta_j \mathscr{L}_\nu^\alpha u\|_{L^2(\mathbb{T}^d)} \lesssim \sup_j 2^{j\beta}\|\Delta_j u\|_{L^2(\mathbb{T}^d)} = \|u\|_{\mathscr{C}_2^{\beta}(\mathbb{T}^d)}.$$

To show (5.3), we again use the isometry, such that

$$\|\Delta_j P_t u\|_{L^2(\mathbb{T}^d)}^2 = \sum_{k\in\mathbb{Z}^d} |\rho_j(k)\exp(-t\psi_\nu^\alpha(k))\hat{u}(k)|^2.$$

For $j = -1$, ρ_j is supported in a ball around zero and as $|\exp(-t\psi_\nu^\alpha(k))| \leqslant 1$, the estimate $\|\Delta_j P_t u\|_{L^2(\mathbb{T}^d)}^2 \lesssim (t^{-\theta/\alpha} \vee 1)2^\theta \sum_{k\in\mathbb{Z}^d}|\rho_j(k)\hat{u}(k)|^2$ holds trivially for $\theta \geqslant 0$. For $j > -1$, p_j is supported away from zero and we can use that $\exp(-t\psi_\nu^\alpha(\cdot))$ is a Schwartz function away from 0 and thus, for $|k| > 0$, $|\exp(-t\psi_\nu^\alpha(k))| \lesssim (t\psi_\nu^\alpha(k) + 1)^{-\theta/\alpha} \lesssim t^{-\theta/\alpha}|k|^{-\theta}$, for any $\theta \geqslant 0$. Thus, for $j > -1$, we obtain

$$\|\Delta_j P_t u\|_{L^2(\mathbb{T}^d)}^2 \leqslant 2^{-2j\theta}t^{-\theta/\alpha} \sum_{k\in\mathbb{Z}^d} |\rho_j(k)\hat{u}(k)|^2 = 2^{-2j\theta}t^{-\theta/\alpha}\|\Delta_j u\|_{L^2(\mathbb{T}^d)}^2,$$

such that together (5.3) follows. To obtain the remaining estimate, we argue in a similar manner using that, due to Hölder-continuity of the exponential function, for $\theta/\alpha \in [0,1]$, $|\exp(-t\psi_\nu^\alpha(k)) - 1| \leqslant |t\psi_\nu^\alpha(k)|^{\theta/\alpha} \leqslant t^{\theta/\alpha}|k|^\theta$. □

Selbstständigkeitserklärung

Name: Kremp
Vorname: Helena Katharina

Ich erkläre gegenüber der Freien Universität Berlin, dass ich die vorliegende Dissertation selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe. Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Schriften entnommen sind, habe ich als solche kenntlich gemacht. Diese Dissertation wurde in gleicher oder ähnlicher Form noch in keinem früheren Promotionsverfahren eingereicht.
Mit einer Prüfung meiner Arbeit durch ein Plagiatsprüfungsprogramm erkläre ich mich einverstanden.

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