

RESEARCH ARTICLE

Polyhedra, lattice structures, and extensions of semigroups

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Abstract

For an arbitrary rational polyhedron, we consider its decompositions into Minkowski summands and, dual to this, the so-called free extensions of the associated pair of semigroups. Being free for a pair of semigroups is equivalent to flatness for the corresponding algebras. The main result is phrased in this dual setup: the category of free extensions always contains an initial object, which we describe explicitly. This provides a canonical free extension of the original pair of semigroups provided by the given polyhedron. Our motivation comes from the deformation theory of the associated toric singularity.

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1 | INTRODUCTION

The present paper deals entirely with objects from discrete mathematics, like semigroups, convex polyhedra and polyhedral cones, and their relations to lattice points. Nevertheless, most of the motivation comes from algebraic deformation theory of affine toric varieties.

1.1 | The main result and a first example

Our main result deals with rational polyhedra P and their associated finitely generated semigroups $\text{cone}_{\mathbb{Z}}(P)^{\vee}$ consisting of all integral linear forms which are non-negative on P . They contain a distinguished sub-semigroup $\mathbb{N} \subseteq \text{cone}_{\mathbb{Z}}(P)^{\vee}$ collecting the affine linear forms having constant (integral) value on P . Our main result, Theorem 9.2, states that these pairs of semigroups admit universal, hence canonical, extensions which can be described in terms of the polyhedra; they are, in particular, related to the set of Minkowski decompositions of the given P . Actually, we insist in so-called free extensions; this notion will be explained in Definition 3.7.

Let us start with a 1-dimensional example, namely, with the line segment $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$, a detailed treatment of which can be found at the end of the paper in Example 11.1. The associated $\text{cone}_{\mathbb{Z}}(P)^{\vee} = \{[a, b] \in \mathbb{Z}^2 : aP + b \geq 0\}$ is, as a semigroup, generated by

$$\{-2, 1\}, \{-1, 1\}, [0, 1], [1, 1], [2, 1]\},$$

and the sub-semigroup \mathbb{N} materializes in the central element $R = [0, 1]$. Both parts are reflected in the gray gadgets of Figure 1. This example shows that the theory of (free) extensions is richer than that of Minkowski summands. For a line segment the latter is quite boring, while for the former we obtain two different free extensions: Both result from doubling the central point, that is, by replacing R by some R' and R'' . However, both ways differ in the behavior of the remaining elements of the semigroup. This is depicted in Figure 1 via using the blue and green color, respectively. Finally, both ‘1-dimensional’ extensions fit into a common ‘2-dimensional’ one (the dimension counts refer to the figures in the left gray box of Figure 1), depicted in red. Note that, in the picture, only the bullets matter; the lines in between are included for a better visualization.

Our theorem is a full generalization of [3, 4], in the sense that no restrictions are imposed on the polyhedron. We tried to keep our notation and terminology as close as possible to those two papers. One important exception is that we have decided to use here the terminology which seems to be standard in discrete geometry papers, that is, we use the word *recession cone* for what was called *tail cone* in [3, 4].

Immediate applications of this paper to deformations of toric singularities are gathered in [1, 9]. There, the construction of a graded component of the versal deformation relies entirely on the universal object we introduce here.

1.2 | Structure of the paper

Before proceeding with the introduction in Section 2, let us summarize briefly the structure of the paper. In Section 3, we introduce free pairs of semigroups, and present their basic properties. Section 4 deals with extensions of semigroups, and introduces the category for which we will construct an initial object. We then proceed by analyzing two special classes of semigroups, which

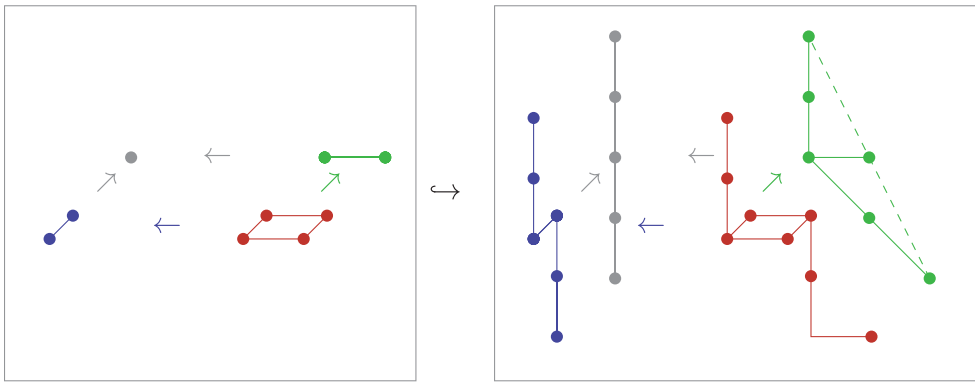


FIGURE 1 The generators of the semigroups, and the way they fit together

are most relevant for toric geometry: *The cone setup* (Section 5) looks at semigroups which are polyhedral cones in some finite-dimensional real vector space; *The discrete setup* (Section 6) deals with finitely generated semigroups. We give an overview of the relation between the two setups below, in Subsection 2.3. Section 7 is the key technical part behind both the construction of the initial object, and for the future applications related to obstruction maps and versality. In Section 8, we define the initial object, which will be the pair of semigroups denoted by (\tilde{T}, \tilde{S}) , and prove its most basic properties: it is an extension and the semigroups are finitely generated. We state and prove our main result in Section 9. The proof of Theorem 9.2 is by far the longest part of the paper. Finally, in Section 10, we connect our initial object to Minkowski decompositions and with the Kodaira–Spencer map.

2 | PRELIMINARIES

2.1 | Minkowski sums of polyhedra

A central notion of the present paper is the Minkowski sum of two convex polyhedra A, B in some real vector space $N_{\mathbb{R}} \cong \mathbb{R}^d$. It is a very classical notion, cf. [17, pp. 28, 198]. It is simply defined as

$$A + B := \{a + b : a \in A, b \in B\}.$$

It is easy to see that the result is again a convex polyhedron. In analogy to this, the ambient vector space is defined as $A - A := \{a - a' : a, a' \in A\}$. Recall that Minkowski decomposition can be used to write every convex polyhedron P as a Minkowski sum of a polytope, that is, a bounded polyhedron, and a polyhedral cone, namely, its recession cone

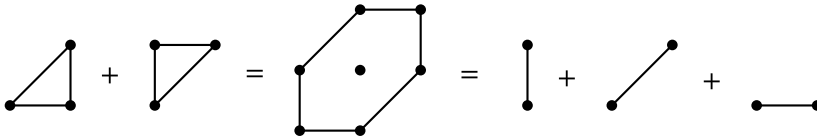
$$\text{recc}(P) := \{a \in P - P : a + P \subseteq P\}.$$

However, this is exactly the type of situation we will *not* consider. Instead, for all Minkowski sums and decompositions in this paper we will assume that all participating polyhedra share the same recession cone. For example, if this recession cone is 0, then we speak about polytopes. An advantage of this general assumption is that the Minkowski addition allows cancellation, that is, $A + B = A' + B$ implies $A = A'$.

Starting with a polyhedron P , one might look at all possibilities of splitting P into a Minkowski sum

$$P = P_0 + \dots + P_k.$$

Even if one looks only at the most elementary or extreme decompositions, they are far from being unique. They do rather behave like a non-unique prime factorization. Arguably the most convincing example is the following.



It is well-known that the set of Minkowski summands of scalar multiples of P (see Definition 5.6) carries the structure of a convex, polyhedral cone $C(P)$, that is, each $\xi \in C(P)$ represents a Minkowski summand P_ξ [4]. For the previous hexagon example, it is the 4-dimensional cone over a double tetrahedron. Its vertices, that is, the fundamental rays of the cone, correspond to the five summands displayed in the figure above.

2.2 | Considering families

The concept of studying Minkowski summands of scalar multiples of P can be reformulated into a relative setting. We may look at homomorphisms $p_+ : \tilde{C} \rightarrow C$ of polyhedral cones such that $p_+^{-1}(\xi + \xi') = p_+^{-1}(\xi) + p_+^{-1}(\xi')$ for all $\xi, \xi' \in C$, where the common recession cone of all the fibers $p_+^{-1}(\xi)$ is $p_+^{-1}(0)$. A trivial example of this can be obtained by taking the affine cone over P in $N_{\mathbb{R}} \oplus \mathbb{R}$ (with P embedded in height 1) and considering its natural height function $\text{cone}(P) \rightarrow \mathbb{R}_{\geq 0}$. Another example is the projection $\tilde{C}(P) \rightarrow C(P)$ with

$$\tilde{C}(P) := \{(\xi, v) : \xi \in C(P), v \in P_\xi\}.$$

The latter is even universal, namely, it is the terminal object in the category of all those families around $\text{cone}(P) \rightarrow \mathbb{R}_{\geq 0}$, cf. Proposition 5.8. However, while this might just look like an arming of language, the striking point consists of the combination of the following two observations.

- (i) One may dualize these notions, looking at injections $p_+^\vee : C^\vee \hookrightarrow \tilde{C}^\vee$. Then, the property of Minkowski linearity translates into an interesting property we call freeness, cf. Proposition 5.4. This property addresses the splitting of \tilde{C}^\vee into a product of C^\vee and a boundary part.
- (ii) The advantage of (i) is that it allows porting the whole setup into the category of finitely generated semigroups. Doing so, one can again ask for universal (so, after dualizing, initial) objects of the appropriate categories.

2.3 | Extensions of semigroups

We take the observations of Subsection 2.2 as our starting point of the whole paper. We will begin in Sections 3 and 4 from scratch with developing the appropriate notions in the category

of semigroups. Then, insisting on finite generation, the general approach naturally splits into two different setups. We have called them the *cone* and the *discrete* setup, and we will focus on them in Sections 5 and 6, respectively. While the cone setup will recover the (duals of the) cones $C(P)$ and $\tilde{C}(P)$, the comparison of both setups will lead to a new vector space $\mathcal{T}(P)$ together with a lattice $\mathcal{T}_{\mathbb{Z}}(P) \subset \mathcal{T}(P)$, and a rational, polyhedral cone $\mathcal{T}_+(P) \subset \mathcal{T}(P)$ generalizing $C(P)$. Studying their dual level, we obtain a finer structure, that is, there is a (unique) finitely generated sub-semigroup \tilde{T} of the dual Abelian group $\mathcal{T}_{\mathbb{Z}}^*(P)$ fulfilling the universal property (ii) above, that is, it is the base for a universal free extension.

The existence of a universal object in the discrete setup is our main result. It is formulated in Theorem 9.2, the proof of which occupies the whole of Section 9. It seems to be an interesting question if the existence and structure of initial extensions is linked to results like [11, Proposition 3.38] addressing unique liftings in log geometry; see the remark after Proposition 4.3. Note that unique liftings in log geometry were important for producing smoothings in [8, 14], see also [7, 12, 15].

2.4 | Involving a lattice structure

Let us return to Subsection 2.1 and let us assume that we have fixed a lattice structure in our ambient \mathbb{R} -vector space. For instance, let us start with a free Abelian group N of rank d , that is, $N \cong \mathbb{Z}^d$, and take $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ as our ambient vector space. If P is a lattice polyhedron, that is, if all vertices belong to N , then it is a natural question to look for all lattice decompositions, that is, for those Minkowski decompositions such that the summands P_{ν} are lattice polytopes, too. One might expect that those P_{ν} correspond to special points inside the parameterizing cone $C(P)$.

However, in the present paper, we go far beyond lattice polyhedra. Instead, we will deal with arbitrary rational polyhedra, but we study their interaction with the lattice. In particular, lattice decompositions do no longer make sense. Instead, in Subsection 10.3, we introduce the weaker notion of lattice-friendly decompositions, cf. Definition 10.6. Then, it is our second main result of this paper that the parameters $\xi \in \mathcal{T}_+(P) \cap \mathcal{T}_{\mathbb{Z}}(P)$ introduced in Subsection 2.3 correspond to exactly those Minkowski summands P_{ξ} occurring in lattice-friendly decompositions, cf. Theorem 10.12. Moreover, similarly to the definition of $\tilde{C}(P)$ in Subsection 2.2, we have combined in Theorem 10.5 all Minkowski summands P_{ξ} , lattice friendly or not, into a common polyhedral, so-called tautological cone $\tilde{\mathcal{T}}_+(P)$ fibered over $\mathcal{T}_+(P)$. That is, the lattice $\mathcal{T}_{\mathbb{Z}}^*(P)$ occurs twice in this paper — as the ambient space of some universal object \tilde{T} , but also as the right tool to check Minkowski decompositions for the lattice friendly property.

3 | FREE PAIRS OF SEMIGROUPS

3.1 | Relative boundaries of semigroups within two different setups

Let $T \subseteq S$ be two commutative and cancellative ($a + c = b + c \Rightarrow a = b$) semigroups with identity ($0 + a = a + 0 = a$), satisfying $S \cap (-S) = \{0\}$, that is, S (and hence T) is pointed. This situation gives rise to the following notion of a relative boundary.

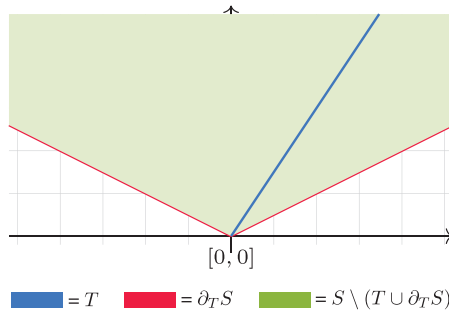


FIGURE 2 $T = \mathbb{R}_{\geq 0} \cdot [2, 3] \subseteq S_{\mathbb{R}}$

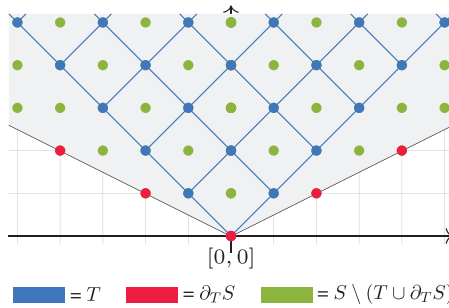


FIGURE 3 $T = \text{span}_{\mathbb{N}}\{-1, 1\}, [1, 1] \subseteq S$

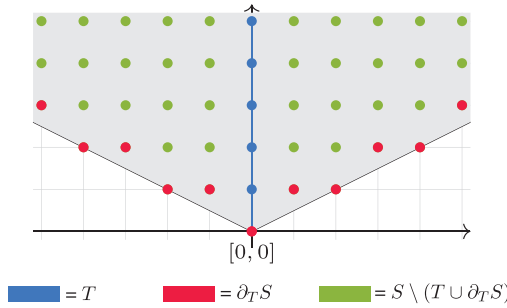


FIGURE 4 $T_0 = \text{span}_{\mathbb{N}}\{[0, 1]\} \subseteq S$

Definition 3.1. The boundary of S relative to T is defined as

$$\partial_T S = \{s \in S : (s - T) \cap S = \{s\}\}.$$

This setting comes with a natural addition map $a : \partial_T S \times T \rightarrow S$.

Example 3.2. In the context of numerical semigroups, that is sub-semigroups of \mathbb{N} , the so-called Apéry sets are relative boundaries with respect to the subgroup generated by the smallest element.

The following examples illustrate that the relative boundary is almost never a semigroup itself.

Example 3.3. Consider the real cone $S_{\mathbb{R}} := \mathbb{R}_{\geq 0} \cdot [-2, 1] + \mathbb{R}_{\geq 0} \cdot [2, 1] \subset \mathbb{R}^2$, and the finitely generated semigroup $S = S_{\mathbb{R}} \cap \mathbb{Z}^2$. In the following, we consider the boundary of $S_{\mathbb{R}}$ with respect to

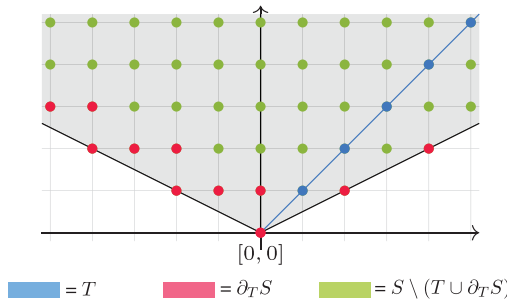


FIGURE 5 $T_1 = \text{span}_{\mathbb{N}}\{[1, 1]\} \subseteq S$

an internal ray, and boundaries of S relative to different sub-semigroups; Figures 4 and 5 show how different embeddings of \mathbb{N} in S give rise to different boundaries (see also Example 3.5 for more details).

There are two quite different classes of semigroups we have in mind. Both are, in their own way, finitely generated.

3.1.1 | The cone setup

Here we take for $T \subseteq S$ polyhedral cones in some finitely dimensional real vector space. These gadgets are finitely generated by their fundamental rays as ‘ $\mathbb{R}_{\geq 0}$ -modules’, but not at all finitely generated as semigroups.

Example 3.4. Assume that $T \subseteq S$ is a ray, that is, $T = \mathbb{R}_{\geq 0} \cdot R$ for some $R \in S \setminus \{0\}$. Then, there is unique face $F = F(T) \leq S$ such that R and hence $T \setminus \{0\}$ is contained in the relative interior $\text{int}(F)$. Then, $S \setminus \partial_T S = \text{star}(F) := \bigcup_{F \leq G \leq S} \text{int}(G)$. Note that both $\text{int}(S)$ and $\text{int}(F)$ are parts of this set, that is, $\partial_T S \subseteq \partial S \setminus \text{int}(F)$ with $\partial S := S \setminus \text{int}(S)$ denoting the classical topological boundary.

For the special case $R \in \text{int}(S)$, as in Figure 2, we even have that $\partial_T S = \partial S$. In particular, in this situation the relative boundary does not depend on a further specification of R .

3.1.2 | The discrete setup

Here we suppose that both T and S are finitely generated as semigroups, that is, as ‘ \mathbb{N} -modules’. In this case, the so-called Hilbert basis, consisting of all irreducible elements, provides even a minimal, hence canonical, finite generating system. A typical example of this situation is the intersection of a cone setup with an underlying lattice.

Example 3.5. Let $S := \text{span}_{\mathbb{R}_{\geq 0}}\{[-2, 1], [2, 1]\} \cap \mathbb{Z}^2$ as in the three discrete figures of Example 3.3, and in Figure 10, the Hilbert basis of this semigroup is

$$H = \{[-2, 1], [-1, 1], [0, 1], [1, 1], [2, 1]\}.$$

The semigroup S contains the inner ‘discrete rays’ $T_0 = \mathbb{N} \cdot [0, 1]$ and $T_1 = \mathbb{N} \cdot [1, 1]$, and their respective relative boundaries are

$$\partial_{T_0} S = \{[\pm 2b, b] : b \in \mathbb{N}\} \cup \{[\pm(2b - 1), b] : b \in \mathbb{N}_{\geq 1}\}$$

and

$$\partial_{T_1} S = \{[\pm 2b, b] : b \in \mathbb{N}\} \cup \{[-2b + 1, b], [-2b + 2, b] : b \in \mathbb{N}_{\geq 1}\}.$$

That is, while both T_0 and T_1 come from the ‘interior’ of S , they lead to different relative, discrete boundaries; see Figures 4 and 5, respectively.

3.2 | Freeness

We will use the addition map $a : \partial_T S \times T \rightarrow S$ for decomposing elements of the semigroup S . In general, that is, if we are in the cone setup (3.1.1) or the discrete setup (3.1.2), the existence of those decompositions is not a problem. This is established by the following lemma.

Lemma 3.6. *Assume that we are either in the cone or the discrete setup. Then, the canonical addition map $a : \partial_T S \times T \rightarrow S$ is automatically surjective.*

Proof. Let $T = \text{span}\{t_1, \dots, t_k\}$ and $S = \text{span}\{s_1, \dots, s_r\}$, which we consider either as \mathbb{N} -modules, or as $\mathbb{R}_{\geq 0}$ -modules. For each $s \in S$ write $s = a_1 s_1 + \dots + a_r s_r$, and for each $i = 1, \dots, k$ write $t_i = b_{i1} s_1 + \dots + b_{ir} s_r$. By the pointedness assumption, we have that for every $n \in \mathbb{N}$ with $a_i < n b_i$, for all i , we get $s - n t \notin S$. So in both setups, there exists a maximal $n^* \in \mathbb{R}_{\geq 0}$, respectively, $\in \mathbb{N}$, with $s - n^* t \in S$. Continuing this process with all generators of T eventually leads to an element $s^* \in S$ which cannot be decreased via T . □

The injectivity of a is less common, but, as we will see, very powerful. Therefore, we introduce the following key terminology.

Definition 3.7. The semigroups $T \subseteq S$ form a *free pair* (T, S) (or $\iota : T \hookrightarrow S$ is called a free embedding) if the addition map $a : \partial_T S \times T \rightarrow S$ is bijective.

Example 3.8. Let $S := \text{span}_{\mathbb{R}_{\geq 0}}\{-2, 1\}, [2, 1]\}$ and $T := \text{span}_{\mathbb{R}_{\geq 0}}\{-1, 1\}, [1, 1]\}$. We are thus in the cone setup, and $\partial_T S = \partial S$ as in the situation at the end of Example 3.4. However, the surjective map $a : \partial_T S \times T \rightarrow S$ is not injective. For instance, $[0, 0] + [2, 2] = [2, 1] + [0, 1]$ displays two different decompositions of $[2, 2] \in S$. Applying Proposition 3.11 will make this even more obvious: We obtain $M = \mathbb{R}^2 / \mathbb{R}^2 = 0$, hence $q : \partial_T S \rightarrow M$ has no chance to become injective.

Note that literally the same remains true if we intersect everything with the lattice \mathbb{Z}^2 . This yields a non-free example in the discrete setup, too. Alternatively, in Figure 3, where we have that $T := \text{span}_{\mathbb{N}}\{-1, 1\}, [1, 1]\}$ and $S := \text{span}_{\mathbb{N}}\{-2, 1\}, [-1, 1], [0, 1], [1, 1], [2, 1]\}$ we can take for a non-unique decomposition $[0, 0] + [4, 4] = [4, 2] + [0, 2]$.

In log-geometry, there is the notion of an integral homomorphism of semigroups. In fact, our concept of freeness means exactly this. See Subsection 4.3 for a discussion of this relation.

3.3 | The decomposition operators

By definition, free pairs (T, S) allow a unique decomposition of every element $s \in S$ into a sum

$$s = \partial(s) + \lambda(s) \quad \text{with} \quad \partial(s) \in \partial_T S \quad \text{and} \quad \lambda(s) \in T.$$

In other words, there are retraction maps $\partial : S \rightarrow \partial_T S$ and $\lambda : S \rightarrow T$ with $\partial + \lambda = \text{id}$ satisfying

$$\partial|_{\partial_T S} = \text{id}, \quad \partial|_T = 0 \quad \text{and} \quad \lambda|_{\partial_T S} = 0, \quad \lambda|_T = \text{id}.$$

Note that λ is in general not linear, that is, not a semigroup homomorphism. Moreover, for ∂ , linearity does not even make sense, since the target $\partial_T S$ is not a semigroup. Finally, in the discrete setup, the Hilbert basis H of S hosting a free pair (T, S) splits into two parts, namely,

$$H = (H \cap \partial_T S) \sqcup (H \cap T).$$

3.4 | Rays yield free pairs

While Example 3.8 has shown that freeness is not always satisfied, there is, nevertheless, a standard situation where this property is guaranteed.

Definition 3.9. In both setups, we call T a ray if it is saturated in the ambient Abelian group $S - S$ and if its canonical poset structure $(t \leq t' : \iff t' - t \in T)$ is a total order.

In the cone setup (3.1.1), this means $T \cong \mathbb{R}_{\geq 0}$; in the discrete setup (3.1.2), the ray property implies that $T \cong \mathbb{N}$. In both situations, there exists an $R \in S$ such that $T \subseteq S$ consists of all ‘allowed’ multiples of R , that is, using $\mathbb{R}_{\geq 0}$ or \mathbb{N} as coefficients, respectively.

Proposition 3.10. *If T is a ray, then a is injective, that is, (T, S) is a free pair.*

Proof. Let $b, b' \in \partial_T S$ and $t, t' \in T$ with $b + t = b' + t'$. We may, without loss of generality, assume that $t \geq t'$. Then, the cancellation property implies that $b + (t - t') = b' \in \partial_T S$ with $t - t' \in T$. By definition of the relative boundary, this means that $t - t' = 0$, that is, $t = t'$ and hence $b = b'$. \square

3.5 | Involving the ambient Abelian groups

Since $T \subseteq S$ are both cancellative, we may embed them into their respective linear hulls

$$W := T - T \subseteq S - S =: V.$$

These ambient objects W, V are torsion-free Abelian groups. In the cone or in the discrete setup, they are finitely generated \mathbb{R} -, respectively, \mathbb{Z} -modules. That is, W and V are finitely dimensional vector spaces or free Abelian groups of finite rank. We denote by $M := V/W$ the quotient (which

might have torsion in the discrete setup). This leads to the quotient map

$$q : S \twoheadrightarrow \bar{S} \subseteq M \quad \text{with} \quad \bar{S} := S/T := (S - T)/(T - T)$$

denoting its image. The quotient S/T had been built via the equivalence relation saying that $s \sim s'$ if and only if there are $t, t' \in T$ such that $s + t = s' + t'$ in S . Very often, namely, if T contains ‘interior’ points of S , this semigroup is already a group, that is, it equals M . The usage of the ambient groups and their quotient M yields the following criterion of freeness in terms of the injectivity of $q|_{\partial S}$.

Lemma 3.11. *Let (T, S) be a pair of semigroups such that $a : \partial_T S \times T \rightarrow S$ is surjective. We have the following.*

- (i) *The restriction $q|_{\partial S} : \partial_T S \rightarrow \bar{S}$ is surjective.*
- (ii) *The map $q|_{\partial S} : \partial_T S \rightarrow M$ is injective if and only if (T, S) is free.*

Proof.

- (i) The surjectivity of $q|_{\partial S}$ is a direct consequence from the surjectivity of the addition map a .
- (ii) The direct implication is obvious. For the converse, assume that (T, S) is free and that $q(b) = q(b')$ for some $b, b' \in \partial_T S$. This implies $b - b' \in T - T$, so there are $t, t' \in T$ with $b + t = b' + t'$. The latter displays two decompositions of the same element into summands from $\partial_T S$ and T . Hence, freeness implies $b = b'$. □

4 | EXTENDING FREE PAIRS

4.1 | Extending semigroups

Starting with a free pair $T \hookrightarrow S$ we are going to consider all possibilities to put this in relation with other free pairs $\tilde{T} \hookrightarrow \tilde{S}$ having isomorphic boundaries.

Definition 4.1. We call a semigroup homomorphism $\pi : \tilde{S} \rightarrow S$ an *extension* if it has trivial kernel, that is if $\ker \pi = \{\tilde{s} \in \tilde{S} : \pi(\tilde{s}) = 0\} = 0$, which is equivalent[†] to

$$\pi(\tilde{S} \setminus \{0\}) \subseteq S \setminus \{0\}.$$

Let $T \hookrightarrow S$ be a pair of semigroups (not necessary free). A commutative diagram of semigroup maps

$$\begin{array}{ccc} \tilde{T} & \hookrightarrow & \tilde{S} \\ \pi_T \downarrow & & \downarrow \pi_S \\ T & \hookrightarrow & S \end{array}$$

[†]Note that for semigroup homomorphisms, having a trivial kernel does not imply injectivity.

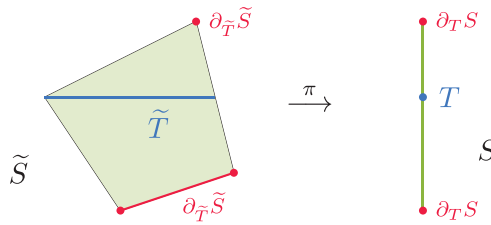


FIGURE 6 π does not always map boundary to boundary

is an *extension of the pair* (T, S) if π_S (and thus also π_T) is an extension. An extension is called *iso-bounded* if the following two conditions are satisfied:

- (i) the addition maps a and \tilde{a} are surjective, and
- (ii) π induces a bijection on the boundaries: $\partial_{\tilde{T}}\tilde{S} \xrightarrow{\sim} \partial_T(S)$.

We will frequently denote both vertical maps simply by π . Note that the above diagram alone immediately implies that $\pi(\tilde{S} \setminus \partial_{\tilde{T}}\tilde{S}) \subseteq S \setminus \partial_T S$. On the other hand, π generally fails to map $\partial_{\tilde{T}}\tilde{S}$ into $\partial_T S$, cf. Example 4.2.2.

Example 4.2.

- (1) A trivial possibility for extending pairs is to first define $\tilde{S} := S \times F$ with F any semigroup of the scenario in question. However, the plain projection $\text{pr}_S : S \times F \rightarrow S$ does not meet our requirements, because its kernel equals F . This can be corrected by choosing any semigroup map $\ell : F \rightarrow S$ with trivial kernel and defining $\pi_\ell := \text{pr}_S + \ell$, that is, $\pi_\ell(s, f) := s + \ell(f)$. Using this notation, the forbidden plain projection corresponds to the forbidden $\ell = 0$. To obtain an extension of the pair, take $\ell : F \rightarrow T \subseteq S$ with $\ker \ell = 0$, and define $\tilde{T} := T \times F$. Note that $\partial_{T \times F}(S \times F) = \partial_T(S) \times \{0\}$. Hence, the freeness property of (T, S) is equivalent to the similar one for (\tilde{T}, \tilde{S}) .
- (2) We consider an example in the cone setup (3.1.1). To be able to draw what is going on, we intersect both cones \tilde{S} and S with affine hyperplanes — displaying convex polytopes (the origin of the cones being behind the screen):

Extending the formula of Example 3.4, we obtain that, in the cone setup,

$$S \setminus \partial_T S = \bigcup_{\substack{R \in T \\ R \neq 0}} (S \setminus \partial_R S) = \bigcup_{\substack{R \in T \\ R \neq 0}} \left(\bigcup_{\substack{R \in G \\ G \leq S}} \text{int}(G) \right) = \bigcup_{\substack{G \leq S \\ G \cap T \neq 0}} \text{int}(G).$$

In particular, in Figure 6, we have that $\pi^{-1}(T) = \tilde{T}$, but $\partial_{\tilde{T}}\tilde{S}$ does not map to $\partial_T S$. Note that the pair (T, S) is free, but (\tilde{T}, \tilde{S}) is not.

4.2 | Keeping it free

The main point of the present subsection is to keep track of the freeness property along extensions of pairs. The next result shows two important consequences of a diagram being iso-bounded, and

that if the vertical maps are surjective, each of these consequences are also sufficient. Let the following diagram define an extension of the free pair (T, S)

$$\begin{array}{ccccc}
 \tilde{T} & \hookrightarrow & \partial_{\tilde{T}}(\tilde{S}) \times \tilde{T} & \xrightarrow{\tilde{a}} & \tilde{S} \\
 \pi_T \downarrow & & & & \downarrow \pi_S \\
 T & \hookrightarrow & \partial_T(S) \times T & \xrightarrow{a} & S
 \end{array}$$

with the addition map \tilde{a} surjective. Denote by $\tilde{M} := (\tilde{S} - \tilde{S})/(\tilde{T} - \tilde{T})$ and by $M := (S - S)/(T - T)$. Consider the following three conditions.

- (C1) The extension is iso-bounded.
- (C2) The pair (\tilde{T}, \tilde{S}) is free and $\bar{\pi} : \tilde{M} \rightarrow M$ is an isomorphism.
- (C3) For all $\tilde{s}_1, \tilde{s}_2 \in \tilde{S}$ with $\pi(\tilde{s}_1) = \pi(\tilde{s}_2)$, there exist $\tilde{t}_1, \tilde{t}_2 \in \tilde{T}$ such that $\tilde{s}_1 - \tilde{t}_1 = \tilde{s}_2 - \tilde{t}_2 \in \tilde{S}$.

Proposition 4.3.

(i) *In the above situation we have the following logical relations:*

$$(C1) \xLeftrightarrow{\neq} (C2) \xLeftrightarrow{\neq} (C3)$$

(ii) *If the maps π_S and π_T are surjective, then*

$$(C3) \implies (C1),$$

thus the three conditions are equivalent in this case.

Proof. (C1) \implies (C2) *The decomposability of (\tilde{T}, \tilde{S}) . Assume that $\tilde{b}_1 + \tilde{t}_1 = \tilde{b}_2 + \tilde{t}_2$, with $\tilde{b}_i \in \partial_{\tilde{T}}\tilde{S}$ and $\tilde{t}_i \in \tilde{T}$. Applying π , we obtain*

$$\pi(\tilde{b}_1) + \pi(\tilde{t}_1) = \pi(\tilde{b}_2) + \pi(\tilde{t}_2).$$

By (C1), we have $\pi(\tilde{b}_1), \pi(\tilde{b}_2) \in \partial_T S$, and the diagram condition implies $\pi(\tilde{t}_1), \pi(\tilde{t}_2) \in T$. So, by the decomposability of (T, S) , that $\pi(\tilde{b}_1) = \pi(\tilde{b}_2)$. Again by (C1), we obtain $\tilde{b}_1 = \tilde{b}_2$, and thus $\tilde{t}_1 = \tilde{t}_2$, so the decomposition is unique.

The group isomorphism. Since the addition maps are surjective, every element of M and of \tilde{M} can be represented by a corresponding boundary element. So, the surjectivity of the restriction to the boundary implies the surjectivity of the map $\bar{\pi} : \tilde{M} \rightarrow M$. By Lemma 3.11, we have $q|_{\partial_S} : \partial_T S \xrightarrow{\sim} \bar{S} \subseteq M$ on both levels, \tilde{M} and M . Hence, $\bar{\pi} : \tilde{M} \rightarrow M$ is an isomorphism on the images of \bar{S} and S in \tilde{M} and M , respectively. Since these images generate the two groups, we are done.

(C2) \implies (C3) Let $\tilde{s}_1, \tilde{s}_2 \in \tilde{S}$ with $\pi(\tilde{s}_1) = \pi(\tilde{s}_2)$ in S , hence in M . Then, the second part of (C2), that is, the fact that $\bar{\pi}$ is an isomorphism, implies that \tilde{s}_1 and \tilde{s}_2 become equal in \tilde{M} , that is, $\tilde{q}(\tilde{s}_1) = \tilde{q}(\tilde{s}_2)$. Now, we consider the unique decompositions

$$\tilde{s}_1 = \tilde{s}'_1 + \tilde{t}_1 \quad \text{and} \quad \tilde{s}_2 = \tilde{s}'_2 + \tilde{t}_2 \quad \text{within} \quad \partial_{\tilde{T}}(\tilde{S}) \times \tilde{T}.$$

We still have $\tilde{q}(\tilde{s}'_1) = \tilde{q}(\tilde{s}'_2)$, but now we can use the decomposability of (\tilde{T}, \tilde{S}) in the way provided by Lemma 3.11, namely, as the injectivity of $\tilde{q} : \partial_{\tilde{T}}\tilde{S} \rightarrow \tilde{M}$. This implies $\tilde{s}'_1 = \tilde{s}'_2 =: \tilde{s}'$, hence $\tilde{s}_1 - \tilde{t}_1 = \tilde{s}_2 - \tilde{t}_2 = \tilde{s}' \in \tilde{S}$.

(C2) \Rightarrow (C1) Take the following extension with surjective addition maps:

$$\begin{array}{ccc} 0 & \hookrightarrow & \langle 2, 3 \rangle \subseteq \mathbb{N} \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & \langle 1 \rangle = \mathbb{N} \end{array}$$

with the second vertical map being the canonical inclusion. The two pairs on the rows are free, and even the first projection is surjective. Also, the groups \tilde{M} and M are both isomorphic to \mathbb{Z} , and π_S induces the identity as isomorphism. However, the restriction to the boundary is a strict inclusion.

(C3) \Rightarrow (C2) Take the following extension with surjective addition maps:

$$\begin{array}{ccc} 0 & \hookrightarrow & \mathbb{R}_{\geq 0} \\ \downarrow & & \downarrow \\ \mathbb{R}_{\geq 0} & \hookrightarrow & \mathbb{R}_{\geq 0}^2 \end{array}$$

with both maps $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}^2$ given by $t \mapsto (t, t)$. Even if the two pairs are free and the groups M and \tilde{M} are isomorphic to \mathbb{R} , the map induced by the vertical one is the zero map, so not an isomorphism.

(C3) \Rightarrow (C1) if π_S is surjective. *The restriction $\pi_{\partial(\tilde{S})}$ is well-defined.* Let $\tilde{b} \in \partial_{\tilde{T}}(\tilde{S})$ and $b = \pi(\tilde{b})$. Then, b admits a unique decomposition into $s + t \in \partial_T(S) \times T$ and, by surjectivity, we may lift both summands to $\tilde{s} \in \tilde{S}$ and $\tilde{t} \in \tilde{T}$, respectively. Thus, \tilde{b} and $\tilde{s} + \tilde{t}$ have the same image under π , and (C3) implies the existence of $\tilde{t}_1, \tilde{t}_2 \in \tilde{T}$ with

$$\tilde{b} - \tilde{t}_1 = \tilde{s} + \tilde{t} - \tilde{t}_2 \in \tilde{S}.$$

The hypothesis $\tilde{b} \in \partial_{\tilde{T}}(\tilde{S})$ enforces $\tilde{t}_1 = 0$. Hence, $\tilde{b} = \tilde{s} + \tilde{t} - \tilde{t}_2$. After applying π , this means

$$b = \pi(\tilde{b}) = \pi(\tilde{s}) + \pi(\tilde{t}) - \pi(\tilde{t}_2) = s + t - \pi(\tilde{t}_2).$$

Comparing with our original equation $b = s + t$, this implies $\pi(\tilde{t}_2) = 0$, that is, $\tilde{t}_2 \in \ker \pi_T = \{0\}$. Hence, $\tilde{s} + \tilde{t} = \tilde{b} \in \partial_{\tilde{T}}(\tilde{S})$ which again enforces $\tilde{t} = 0$. Finally, we apply π to the equation $\tilde{b} = \tilde{s}$, leading to $b = s \in \partial_T(S)$.

Injectivity. Let $\tilde{b}_1, \tilde{b}_2 \in \partial_{\tilde{T}}(\tilde{S})$ with $\pi(\tilde{b}_1) = \pi(\tilde{b}_2)$. By (C3), we obtain elements $\tilde{t}_1, \tilde{t}_2 \in \tilde{T}$ with

$$\tilde{b}_1 - \tilde{t}_1 = \tilde{b}_2 - \tilde{t}_2 \in \tilde{S}.$$

Again, the defining property of $\partial_{\tilde{T}}(\tilde{S})$ implies $\tilde{t}_1 = \tilde{t}_2 = 0$.

Surjectivity. Let $b \in \partial_T(S)$. By the surjectivity of π_S , this may be lifted to an element $\tilde{b} = \tilde{s} + \tilde{t} \in \partial_{\tilde{T}}(\tilde{S}) + \tilde{T}$. Applying π yields $b = s + t \in \partial_T(S) + T$, thus $\pi(\tilde{t}) = t = 0$. Again we conclude that $\tilde{t} \in \ker \pi_T = \{0\}$. □

4.3 | The relation among the notions of being free, integral, and iso-bounded

In this subsection, we are going to explain relations between the notions defined so far and also to the classical notion of integrality from log-geometry.

4.3.1 | Integrality

There are several notions of integrality among monoids, that is, commutative semigroups with neutral element, cf. [13], [11, chapter 3]. First of all, such a monoid is called integral if it is cancellative. In a second step, a homomorphism $\iota : T \rightarrow S$ of cancellative semigroups is called integral if for each $T \rightarrow T'$ the pushout $S \oplus_T T'$ is cancellative, too.

In [13, Proposition 4.1], it was shown that integrality of $\iota : T \rightarrow S$ is equivalent to the following property: For all $s_1, s_2 \in S$ and $t_1, t_2 \in T$ with $s_1 + \iota(t_1) = s_2 + \iota(t_2)$ there exist $t'_1, t'_2 \in T$ and $s \in S$ such that

$$s_i = \iota(t'_i) + s \quad (i = 1, 2) \quad \text{and} \quad t_1 + t'_1 = t_2 + t'_2.$$

Moreover, Kato has shown that, for injective ι (what we always assume), this is also equivalent to the flatness of $\mathbb{Z}[T] \rightarrow \mathbb{Z}[S]$. See [10, section 5] for a comprehensive theory of modules over monoids dealing with related notions.

4.3.2 | Freeness

The characterization of integrality of homomorphisms just mentioned gives rise to the following equivalent description.

Lemma 4.4. *If $\iota : T \hookrightarrow S$ is an embedding of cancellative semigroups such that $a : \partial_T S \times T \rightarrow S$ is surjective, then this embedding is integral if and only if ι is free.*

Proof. Since ι is injective, we will omit this map in the notation.

(\Rightarrow) Assume that $\bar{s}_1 + t_1 = \bar{s}_2 + t_2$ with $\bar{s}_i \in \partial_T S \subseteq S$ and $t_i \in T$ for $i = 1, 2$. By the condition mentioned in (4.3.1), there exist $t'_1, t'_2 \in T$ and $s \in S$ such that

$$\bar{s}_i = \iota(t'_i) + s \quad (i = 1, 2) \quad \text{and} \quad t_1 + t'_1 = t_2 + t'_2.$$

However, since $\bar{s}_i \in \partial_T S \subseteq S$, this implies $t'_i = 0$, hence $\bar{s}_i = s$, that is, $\bar{s}_1 = \bar{s}_2$.

(\Leftarrow) Let $s_1, s_2 \in S$ and $t_1, t_2 \in T$ with $s_1 + t_1 = s_2 + t_2$ as in (4.3.1). We decompose $s_i = \bar{s}_i + t'_i$ with $\bar{s}_i \in \partial_T S$ and $t'_i \in T$ for $i = 1, 2$. Hence,

$$\bar{s}_1 + t'_1 + t_1 = \bar{s}_2 + t'_2 + t_2,$$

and freeness implies $\bar{s}_1 = \bar{s}_2$. □

4.3.3 | The notion ‘iso-bounded’ is a relative version of freeness

There is a striking similarity between Condition (C3) of Subsection 4.2 and the definition of integrality in (4.3.1). This observation can be made precise when considering the following commutative diagram

$$\begin{array}{ccc} T & \hookrightarrow & S \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & S/T \end{array}$$

where the construction of S/T was discussed in Subsection 3.5. Then, Condition (C3) for this particular diagram translates as the definition of integrality from (4.3.1). Thus, while this (relative) notion of integrality for injective homomorphisms $T \rightarrow S$ of monoids corresponds to the absolute notion of freeness of pairs (T, S) , the notion ‘iso-bounded’ becomes the relative version of the latter via extensions of pairs.

4.3.4 | Despite freeness

It could have been that both (T, S) and (\tilde{T}, \tilde{S}) are free, but the diagram is not iso-bounded. For example, take $\tilde{T} = T = S = \mathbb{R}_{\geq 0}$ and $\tilde{S} = \mathbb{R}_{\geq 0}^2$ containing \tilde{T} as the ray $\mathbb{R}_{\geq 0} \cdot (1, 1)$ with $\pi = \frac{1}{2}(1, 1)$. By Proposition 3.10, we know that (T, S) and (\tilde{T}, \tilde{S}) are free but $\bar{\pi} : \tilde{M} \rightarrow M$ is not an isomorphism, that is, even (C2) fails.

4.3.5 | Cartesian diagrams

Any iso-bounded diagram is automatically Cartesian, that is, it follows that $\tilde{T} = \pi_S^{-1}(T) \subseteq \tilde{S}$. However, as it can be seen in Figure 6 of Example 4.2(2), this condition does not suffice. In view of Lemma 4.6, it might be interesting to know if this, however, does suffice whenever we start with a surjective extension consisting of free pairs.

4.3.6 | Co-Cartesian diagrams

Similarly, we can compare the iso-bounded property with being co-Cartesian, that is, with the property $S = \tilde{S} \oplus_{\tilde{T}} T$.

Example 4.5. Consider the diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{(1,0)^T} & \mathbb{N}^2 \\ (1,0)^T \downarrow & & \downarrow \text{id} \\ \mathbb{N}^2 & \xrightarrow{\tilde{\tau}} & \mathbb{N}^2 \end{array}$$

with $\tilde{\tau} = \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. It is iso-bounded, but not co-Cartesian.

However, we have the following positive result.

Lemma 4.6. *Suppose the following diagram is an extension among two free pairs.*

$$\begin{array}{ccc}
 \tilde{T} & \hookrightarrow & \tilde{S} \\
 \pi_T \downarrow & & \downarrow \pi_S \\
 T & \hookrightarrow & S
 \end{array}$$

Then, it is iso-bounded if and only if it is co-Cartesian, that is, if $S = \tilde{S} \oplus_{\tilde{T}} T$.

Proof. (\Leftarrow) We assume that $S = \tilde{S} \oplus_{\tilde{T}} T$ and will first show that $\partial_T S = (\partial_{\tilde{T}} \tilde{S}, 0)$. While the inclusion \subseteq is clear, we start with an element $(\tilde{s}, 0)$ from the right-hand side, that is, with $\tilde{s} \in \partial_{\tilde{T}} \tilde{S}$. If there was a pair (\tilde{s}', t) equal to $(\tilde{s}, 0)$ in the pushout with $\tilde{s}' \in \tilde{S}$ and $t \in T \setminus \{0\}$, then this would mean that there are $\tilde{t}_1, \tilde{t}_2 \in \tilde{T}$ with

$$\tilde{s} + \tilde{t}_1 = \tilde{s}' + \tilde{t}_2 \quad \text{and} \quad 0 + \pi_T(\tilde{t}_2) = t + \pi_T(\tilde{t}_1).$$

Hence, since $\tilde{s} \in \partial_{\tilde{T}} \tilde{S}$ and (\tilde{T}, \tilde{S}) is free, there must be a $\tilde{t}_3 \in \tilde{T}$ such that $\tilde{s}' = \tilde{s} + \tilde{t}_3$ and $\tilde{t}_1 = \tilde{t}_2 + \tilde{t}_3$. This implies $\pi_T(\tilde{t}_2) = t + \pi_T(\tilde{t}_2) + \pi_T(\tilde{t}_3)$, that is, $t + \pi_T(\tilde{t}_3) = 0$. Using $T \cap (-T) = 0$ and $\ker \pi_T = 0$, we obtain $t = 0$ in T and $\tilde{t}_3 = 0$ in \tilde{T} . While we just need the former, the latter translates into $\tilde{s}' = \tilde{s}$.

It remains to check that the map $\pi_S : \partial_{\tilde{T}} \tilde{S} \rightarrow \partial_T S$ is injective. For this, we assume that $\tilde{s}, \tilde{s}' \in \partial_{\tilde{T}} \tilde{S}$ give rise to $(\tilde{s}, 0) = (\tilde{s}', 0)$ in $S = \tilde{S} \oplus_{\tilde{T}} T$. But this leads to the same arguments as before and ends with $\tilde{s}' = \tilde{s}$.

(\Rightarrow) This is a special case of Proposition 4.9. □

4.4 | Boundary independence

The goal of Section 8 is to construct a universal iso-bounded extension for any free pair. To this aim, we have to identify the essential structures and concepts that a iso-bounded extension has to preserve. The first is the concept of independence. This is defined for tuples of elements in the boundary. The second is a family of special elements in the smaller semigroup $(T, \text{respectively, } \tilde{T})$ which can be defined in terms of the boundary, and has to be compatible with the bijection on the boundary.

Definition 4.7. Let (T, S) be a free pair of semigroups. A collection of r (not necessarily distinct) boundary elements $b_1, \dots, b_r \in \partial_T S$ is called *boundary independent* if their sum is still in the boundary, that is, if

$$b_1 + \dots + b_r = \partial(b_1 + \dots + b_r).$$

In contrast, such a collection is called *boundary dependent*, if it is not boundary independent, and *minimally dependent* if it is dependent and every proper subset is independent.

Let (T, S) be a free pair, and (\tilde{T}, \tilde{S}) be a iso-bounded extension of it. We thus have an induced bijection $\pi_{\partial}^{-1} : \partial_T(S) \xrightarrow{\sim} \partial_{\tilde{T}}(\tilde{S})$, and for every $b \in \partial_T S$ we simply denote

$$\tilde{b} := \pi_{\partial}^{-1}(b).$$

Let us denote the retractions upstairs by $\tilde{\partial} : \tilde{S} \rightarrow \partial_{\tilde{T}}\tilde{S}$ and $\tilde{\lambda} : \tilde{S} \rightarrow \tilde{T}$, respectively.

Proposition 4.8. *For any iso-bounded extension $\pi : (\tilde{T}, \tilde{S}) \rightarrow (T, S)$ and for any (not necessarily distinct) elements $b_1, \dots, b_r \in \partial_T S$ we have*

$$\begin{aligned} \tilde{\partial}(\tilde{b}_1 + \dots + \tilde{b}_r) &= \pi_{\partial}^{-1}(\partial(b_1 + \dots + b_r)) \\ \pi_T(\tilde{\lambda}(\tilde{b}_1 + \dots + \tilde{b}_r)) &= \lambda(b_1 + \dots + b_r) \\ \tilde{b}_1, \dots, \tilde{b}_r \text{ are boundary independent} &\iff b_1, \dots, b_r \text{ are boundary independent.} \end{aligned}$$

In particular,

$$\pi_{\partial}^{-1}(b_1) + \dots + \pi_{\partial}^{-1}(b_r) - \pi_{\partial}^{-1}(\partial(b_1 + \dots + b_r)) \in \tilde{T}.$$

Furthermore, if we choose $\tilde{s}_1, \dots, \tilde{s}_r \in \tilde{S}$ and denote by $s_i := \pi(\tilde{s}_i)$, then the first two relations above still hold.

Proof. From $\pi(\tilde{b}_1 + \dots + \tilde{b}_r) = b_1 + \dots + b_r$, using the unique boundary decompositions we get

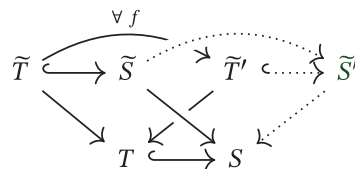
$$\pi_{\partial}(\tilde{\partial}(\tilde{b}_1 + \dots + \tilde{b}_r)) + \pi_T(\tilde{\lambda}(\tilde{b}_1 + \dots + \tilde{b}_r)) = \partial(b_1 + \dots + b_r) + \lambda(b_1 + \dots + b_r).$$

By the uniqueness of the decomposition, by the bijectivity of π_{∂} and by $\ker \pi_T = 0$, we conclude. □

4.5 | The category of free extensions of a pair

Assume that (T, S) is a free pair. Then, the iso-bounded extensions $\pi : (\tilde{T}, \tilde{S}) \rightarrow (T, S)$ form a category $\mathcal{E}_{(T,S)}$ where the morphisms are defined in the obvious way. Moreover, we have the following construction imitating base change from algebraic geometry and equipping $\mathcal{E}_{(T,S)}$ with the structure of being fibered in groupoids.

Proposition 4.9. *Assume that $(\tilde{T}, \tilde{S}) \in \mathcal{E}_{(T,S)}$ and that $\pi'_T : \tilde{T}' \rightarrow T$ is another extension of T . Then, for any semigroup homomorphism $f : \tilde{T} \rightarrow \tilde{T}'$ over T , there is a unique extension $\pi'_S : \tilde{S}' \rightarrow S$ such that $\pi' = (\pi'_T, \pi'_S) : (\tilde{T}', \tilde{S}') \rightarrow (T, S)$ belongs to $\mathcal{E}_{(T,S)}$ and that f extends to a morphism $(\tilde{T}, \tilde{S}) \rightarrow (\tilde{T}', \tilde{S}')$ in this category.*



Proof. It suffices to prove that the canonical map $\partial_{\tilde{T}}\tilde{S} \times \tilde{T}' \rightarrow \tilde{S} \oplus_{\tilde{T}} \tilde{T}'$ is a bijection where the latter denotes the pushout $\tilde{S} \oplus_{\tilde{T}} \tilde{T}' := (\tilde{S} \times \tilde{T}') / \sim$ defined by modding out the equivalence relation

$$(\tilde{s}_1, \tilde{t}'_1) \sim (\tilde{s}_2, \tilde{t}'_2) : \iff \exists \tilde{t}_1, \tilde{t}_2 \in \tilde{T} : \tilde{s}_1 + \tilde{t}_2 = \tilde{s}_2 + \tilde{t}_1 \text{ and } \tilde{t}'_1 + f(\tilde{t}_1) = \tilde{t}'_2 + f(\tilde{t}_2).$$

However, it is straightforward to check that the assignment $(\tilde{s}, \tilde{t}') \mapsto (\partial\tilde{s}, f(\lambda\tilde{s}) + \tilde{t}')$ yields a correctly defined inverse map $\tilde{S} \oplus_{\tilde{T}} \tilde{T}' \rightarrow \partial_{\tilde{T}}\tilde{S} \times \tilde{T}'$. □

4.6 | Initial objects in $\mathcal{E}_{(T,S)}$

The main result of this paper is the following.

Theorem 4.10. *Let us assume that we are in the discrete setup, cf. Subsection 3.1.2. The category of iso-bounded extensions of (T, S) contains an initial object.*

We will provide a very explicit construction of this universal object in the discrete setup. We start analyzing first the cone setup in Section 5, where we get a terminal object, cf. Proposition 5.8. Just to get an impression of what this initial object may look like we provide the following example.

Example 4.11. Let us return to Example 3.5 and Figure 4, that is, $S = \langle [-2, 1], [-1, 1], [0, 1], [1, 1], [2, 1] \rangle$ with $T = \mathbb{N} \cdot R$ and $R = [0, 1]$. In Example 6.4 this semigroup will be understood starting from the 1-dimensional polytope $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$; the link between these two approaches is that the polyhedral cone σ over $P \times \{1\} \subset \mathbb{R}^2$ is dual to $S_{\mathbb{R}} = \sigma^\vee$ from Example 3.3 which contains S as the set of lattice points. Anyway, in algebraic geometry, this setup gives rise to the toric singularity $X = \mathbb{TV}(\sigma) \subseteq \mathbb{A}_k^5$ which can, alternatively, be understood as the vanishing set of the six minors encoded by the condition

$$\text{rank} \begin{pmatrix} z_{-2} & z_{-1} & z_0 & z_1 \\ z_{-1} & z_0 & z_1 & z_2 \end{pmatrix} \leq 1.$$

The elements $[k, 1] \in S$ can be recovered as the multidegrees of the variables z_i . In [1], we discuss the deformation theory of those toric singularities. In this context, the present example became famous in the last century, because Pinkham detected that the deformation space of X admits two different components. In [1], we recall that this corresponds to two different lattice-friendly decompositions of P as we will meet them here in Section 10, cf. Example 10.10. Finally, it comes full circle by the fact that these two decompositions correspond to the following two iso-bounded extensions of (T, S) , which we represent in Figures 7 and 8 only through the generators. The blue points correspond to T and \tilde{T} , respectively. The semigroups are recovered from the pictures by taking the cone over the convex hull. Figure 10 in Example 6.4 depicts this explicitly for S .

Now, by Theorem 4.10, we know that both extensions can be merged to a common one. This leads to a 4-dimensional semigroup, that is, to a semigroup filling a 4-dimensional polyhedral cone where its 3-dimensional crosscut is depicted in Figure 9. In the introduction, as a preview, we had presented Figure 1 as an alternative representation of this situation. See Example 7.12 for the detailed calculations. Note that this establishes a remarkable difference to the algebro-geometric

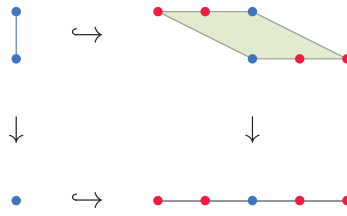


FIGURE 7 The Artin component

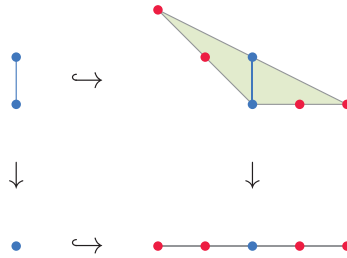


FIGURE 8 The qG component

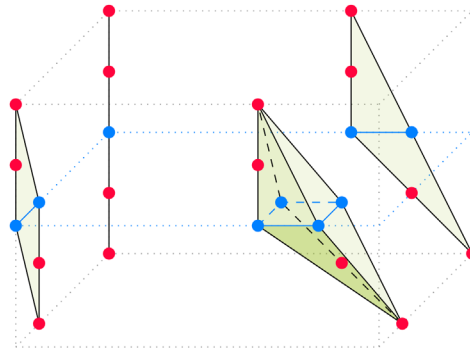


FIGURE 9 The full picture: Artin- and qG-components as projections of the initial object

setup: There, the two deformation components cannot be dominated by a higher dimensional joint deformation of the same kind, that is, with an irreducible base space. See also Subsection 9.1.

5 | THE CONE SETUP

5.1 | Dualizing the cone setup

In the present section, we deal exclusively with the situation introduced in (3.1.1). The main result is the construction of a terminal object in a certain category (Proposition 5.8). This proposition is a much easier to prove analog of Theorem 4.10. One of the striking features of the cone setup is that it allows dualization of both the cones T and S and their ambient vector spaces $V := S - S = \text{span}_{\mathbb{R}}\{S\}$ and $W := T - T = \text{span}_{\mathbb{R}}\{T\}$, respectively. In particular, considering $\iota : W \hookrightarrow V$ with

$\iota(T) \subseteq S$, there is a dual linear map

$$p : V^* \rightarrow W^* \quad \text{with} \quad p(S^\vee) \subseteq T^\vee.$$

While p is always surjective on the level of vector spaces, the surjectivity of its restriction $p_+ : S^\vee \rightarrow T^\vee$ to the level of semigroups is equivalent to the property $S \cap W = T$. This equivalence follows since the mutually dual relations

$$T \subseteq S \cap W \quad \iff \quad T^\vee \supseteq S^\vee + T^\perp$$

become equalities simultaneously. This is, however, automatically fulfilled for free pairs, cf. Definition 3.7.

Lemma 5.1. *If (T, S) is free, then $S \cap \text{span}_{\mathbb{R}}\{T\} = T$.*

Proof. If $t_1 - t_2 \in S$ (with $t_i \in T$), then this element can be decomposed into $t_1 - t_2 = b + t$ with $b \in \partial_T(S)$ and $t \in T$. Hence, $0 + t_1 = b + (t + t_2)$, but this displays two decompositions within $S = \partial_T(S) + T$. Thus, $0 = b$, and this means $t_1 - t_2 = 0 + t \in T$. □

5.2 | A dual characterization of freeness

In the cone setup, freeness can be characterized by the following enhancement of the surjectivity of $p_+ : S^\vee \rightarrow T^\vee$.

Proposition 5.2. *The pair (T, S) is free if and only if $p_+ : S^\vee \rightarrow T^\vee$ is surjective and maps faces onto faces.*

Proof. Step 1. For each $s \in S$ we define $T_s := (s + \text{span}_{\mathbb{R}}\{T\}) \cap S$, that is, it is a polyhedron in (a translate of) $\text{span}_{\mathbb{R}}\{T\}$. Note that, for every $s' \in T_s$, one has $T_s = T_{s'}$. By Lemma 5.1, we may and will assume that $p_+ : S^\vee \rightarrow T^\vee$ is surjective. Hence, every $\xi \in T^\vee$ is bounded from below on T_s . This implies that the normal fan $\mathcal{N}(T_s)$ has exactly T^\vee as its support. Dually, this means that all recession cones equal

$$\text{recc } T_s = T,$$

which is also the content of [6, Lemma 2.3].

Step 2. (T, S) is free $\iff T_s$ is a cone for all $s \in S$:

(\Rightarrow) If T_s is not a cone, then there exist at least two distinct vertices $s, s' \in T_s$. In particular, these two elements belong to $\partial_T S$. On the other hand, since T is a full-dimensional cone within $W = \text{span}_{\mathbb{R}}\{T\}$, we have $(s + T) \cap (s' + T) \neq \emptyset$. Choosing an element from this intersection leads to elements $t, t' \in T$ with

$$s + t = s' + t',$$

that is, we have got a contradiction.

(\Leftrightarrow) Any equation $s + t = s' + t'$ with $s, s' \in \partial_T S$ and $t, t' \in T$ leads to $s \equiv s' \pmod{W}$, hence $s, s' \in T_s$. However, if $T_s = \bar{s} + T$, then, unless $s = s' = \bar{s}$, the elements $s, s' \in \bar{s} + T$ cannot belong to the T -boundary of S .

Step 3. T_s is a cone $\iff \mathcal{N}(s', T_s) \leq T^\vee$ is a face for all $s' \in T_s$, where $\mathcal{N}(s', T_s)$ denotes the normal cone of T_s in s' or, equivalently, in the smallest face $f \leq T_s$ containing s' as an interior point.

(\Rightarrow) If T_s is a cone, then its normal fan equals $\mathcal{N}(T_s) = T^\vee$. In particular, its elements are faces of T^\vee .

(\Leftarrow) If T_s is not a cone, then it contains a compact edge $f := \overline{ss''}$. However, the normal cone $\mathcal{N}(f, T_s)$ is not a face of T^\vee . And, for any $s' \in \text{int } f$, we have $\mathcal{N}(s', T_s) = \mathcal{N}(f, T_s)$.

Step 4. The definition of the normal cones of $s' \in \text{int } f$ with $f \leq T_s$ is

$$\mathcal{N}(s', T_s) = \mathcal{N}(f, T_s) = \{\xi \in T^\vee : \langle s', \xi \rangle = \min \langle T_s, \xi \rangle\}$$

where, strictly speaking, the previous description requires the usage of some lift $\tilde{\xi} \in V^*$ of $\xi \in W^*$. Similarly, if $F \leq S$ is a face, then

$$\mathcal{N}(F, S) = \{\tilde{\xi} \in S^\vee : \langle F, \tilde{\xi} \rangle = 0 = \min \langle S, \tilde{\xi} \rangle\} = S^\vee \cap F^\perp.$$

That is, since the ambient polyhedron S is a cone (in contrast to the polyhedra T_s), the normal cones are true faces of S^\vee . The surjection $p_+ : S^\vee \rightarrow T^\vee$ preserves the normal cones, that is, for each $s \in S$ with $f_s := F \cap T_s \neq \emptyset$ we have

$$p_+(\mathcal{N}(F, S)) \subseteq \mathcal{N}(f_s, T_s).$$

It follows from [6, Lemma 2.5] that for $\text{int}(F) \cap T_s \neq \emptyset$ these inclusions are actually equalities. Here, $\text{int}(\cdot)$ denotes the relative interior, and the previous condition does also imply that this non-empty intersection equals $\text{int}(f_s)$.

Step 5. Now, we can conclude the proof as follows: If (T, S) is free, then by Steps 2 and 3, all normal cones $\mathcal{N}(s', T_s) \leq T^\vee$ are faces of T^\vee . On the other hand, every face of S^\vee is of the form $S^\vee \cap F^\perp = \mathcal{N}(F, S)$ for some face $F \leq S$. Hence, choosing some $s \in \text{int } F \subseteq S$, we obtain that the image of this face equals $p_+(\mathcal{N}(F, S)) = \mathcal{N}(f_s, T_s) \leq T^\vee$ by Step 4.

For the reverse implication, let $s \in S$ and $s' \in T_s$. Then, there is a unique $F \leq S$ with $s' \in \text{int } F$. In particular, $s' \in \text{int } f_s$, and we obtain

$$\mathcal{N}(s', T_s) = \mathcal{N}(f_s, T_s) = p_+(\mathcal{N}(F, S)) = p_+(S^\vee \cap F^\perp)$$

by Step 4. Now, if p_+ sends faces to faces, then this means that all normal cones of T_s are faces of T^\vee . Hence, the claim follows by Steps 2 and 3 again. \square

5.3 | Freeness and Minkowski linearity

We have already seen that it was important to distinguish between the map $p : V^* \rightarrow W^*$ on the level of vector spaces and its restriction $p_+ : S^\vee \rightarrow T^\vee$. This is particularly relevant when we deal

with fibers. For an element $\xi \in T^\vee$, we will call

$$p_+^{-1}(\xi) := p^{-1}(\xi) \cap S^\vee$$

the positive fiber of ξ (under p).

Definition 5.3. We call the pair (T, S) *Minkowski linear* if $p_+ : S^\vee \rightarrow T^\vee$ is surjective, and if for each $\xi, \xi' \in T^\vee$ we have $p_+^{-1}(\xi) + p_+^{-1}(\xi') = p_+^{-1}(\xi + \xi')$.

Note that the inclusion ‘ \subseteq ’ as well as equality on the level of vector spaces, that is, replacing p_+ by p , is always satisfied. However, the linearity among the positive fibers becomes equivalent to freeness.

Proposition 5.4. *The pair (T, S) is free if and only if it is Minkowski linear.*

Proof. Let us visualize the present situation. Denoting $\bar{S} := q(S) \subseteq V/W$, we obtain the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & W & \longrightarrow & V & \xrightarrow{q} & V/W & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & T_s & \longrightarrow & S \supseteq \partial_T S & \twoheadrightarrow & \bar{S} \ni q(s) & & \end{array}$$

where, for an element $s \in S$, the positive fiber $q_+^{-1}(q(s)) := q^{-1}(q(s)) \cap S$ is just another way of writing T_s from the proof of Proposition 5.2. In particular, for a fixed s , the linearity of the function

$$\Phi_s : \varphi \in V^* \mapsto \min \langle q_+^{-1}(q(s)), \varphi \rangle$$

is equivalent to the fact that T_s is a shifted copy of T . Hence, the linearity of Φ_s for all $s \in S$ is equivalent to the freeness of the pair (T, S) . On the other hand, the dual picture is

$$\begin{array}{ccccccc} 0 & \longleftarrow & W^* & \xleftarrow{p} & V^* & \longleftarrow & W^\perp & \longleftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \xi \in T^\vee & \xleftarrow{p_+} & S^\vee & \longleftarrow & p_+^{-1}(\xi) & & \end{array}$$

Writing $\xi = p(\varphi)$ and $\xi' = p(\varphi')$, the linearity of

$$\Psi_s : \varphi \mapsto \min \langle s, p_+^{-1}(p(\varphi)) \rangle$$

means that the polyhedra $p_+^{-1}(\xi + \xi')$ and $p_+^{-1}(\xi) + p_+^{-1}(\xi')$ provide the same values after applying $\min \langle s, \cdot \rangle$ for all $s \in S$, that is, that both polyhedra coincide. That is, the linearity of Ψ_s for all $s \in S$ is equivalent to the Minkowski linearity of the pair (T, S) .

Finally, starting with two elements $s \in S$ and $\varphi \in S^\vee$, we obtain from the proof of [2, Proposition 8.5] the equality

$$\min \langle q_+^{-1}(q(s)), p_+^{-1}(p(\varphi)) \rangle = 0,$$

or the equivalent version

$$\Phi_s(\varphi) + \Psi_s(\varphi) = \min \langle q_+^{-1}(q(s)), \varphi \rangle + \min \langle s, p_+^{-1}(p(\varphi)) \rangle = \langle s, \varphi \rangle.$$

It follows that the linearity of Φ_s is equivalent to the linearity of Ψ_s . □

5.4 | The cone of Minkowski summands

Let us assume that (T, S) is free, meaning that the map $p_+ : S^\vee \rightarrow T^\vee$ is Minkowski linear and, in particular, surjective. We fix a ray in the interior of the cone $T^\vee \subseteq W^*$, that is we fix a linear map $e : \mathbb{R}_{\geq 0} \hookrightarrow T^\vee$ such that $e(1) \in \text{int } T^\vee$. The preimage $P := p_+^{-1}(e(1)) \subseteq S^\vee$ can be understood, well-defined up to some shift, as a polyhedron in $W^\perp =: N_{\mathbb{R}}$. Consequently, the preimage of the whole ray $e^*(S^\vee) := p_+^{-1}(e(\mathbb{R}_{\geq 0}))$ equals $\sigma := \text{cone}(P)$.

Remark 5.5. In Sections 3 and 4, we had originally denoted $M = (S - S)/(T - T)$. This fits perfectly well for the discrete setup: M is then a finitely generated Abelian group, and in Subsection 4.6 we had denoted by $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ the associated vector space. However, in the cone setup, $(S - S)/(T - T) = V/W$ is already an \mathbb{R} -vector space, and it seems appropriate to denote it by $M_{\mathbb{R}}$ instead of M . The same applies for the dual gadgets N and $N_{\mathbb{R}}$, that is, in particular W^\perp becomes $N_{\mathbb{R}}$.

The recession cone $\text{recc}(P) = \{a \in N_{\mathbb{R}} : a + P \subseteq P\}$ equals $\ker(p_+) = S^\vee \cap N_{\mathbb{R}} = \overline{S}^\vee$, and, more general, this is the common recession cone of all other positive fibers $p_+^{-1}(\xi)$. In particular, \overline{S} is the common support of their normal fans $\mathcal{N}(p_+^{-1}(\xi))$.

Definition 5.6. A (convex) polyhedron Q with $\text{recc}(Q) = \text{recc}(P)$ is called a *Minkowski summand* of $\mathbb{R}_{\geq 0} \cdot P$ if there is another polyhedron Q' and a scalar $\lambda \in \mathbb{R}_{\geq 0}$ such that $Q + Q' = \lambda \cdot P$.

It is well-known fact that Q is a Minkowski summand of $\mathbb{R}_{\geq 0} \cdot P$ if and only if the normal fan $\mathcal{N}(P)$ is a refinement, that is, a subdivision of $\mathcal{N}(Q)$. Exactly this property applies to all positive fibers $Q = p_+^{-1}(\xi)$. For interior points $\xi \in \text{int } T^\vee$, we even have $\mathcal{N}(p_+^{-1}(\xi)) = \mathcal{N}(P)$. In [4], we have constructed a linear surjective map of polyhedral cones

$$p_C : \tilde{C}(P) \twoheadrightarrow C(P),$$

where the elements $\xi \in C(P)$ of the target parameterize the set of (translation classes of) Minkowski summands P_ξ of P . The summands P_ξ are encoded via the associated dilation factors $t_{ij}(\xi) \in \mathbb{R}_{\geq 0}$ of the bounded edges $d_{ij} = v^j - v^i$ connecting the vertices $v^i, v^j \in P$. For instance, $t_{ij}(\xi) = 1$ for all i, j leads to $\underline{1} \in C(P)$ with $P_{\underline{1}} = P$. In general, the parameters $t_{ij}(\xi)$ are supposed to meet the closing conditions $\sum_i t_{i,i+1}(\xi) \cdot d_{i,i+1} = 0$ along the oriented boundaries of all 2-dimensional, compact faces of P . The source cone of p_C is the ‘universal Minkowski summand’; it is defined as

$$\tilde{C}(P) := \{(\xi, w) : \xi \in C(P), w \in P_\xi\}, \quad \text{that is, } p_C^{-1}(\xi) = \{\xi\} \times P_\xi.$$

While the Minkowski summands are only well-defined up to translation, one can make the previous definition precise by fixing a vertex $v_* \in P$ and placing all associated vertices $(v_*)_\xi \in P_\xi$ in

the origin. In general, every vertex $v \in P$ provides a linear section $v : C(P) \hookrightarrow \tilde{C}(P)$ of p_C sending $\xi \mapsto v_\xi$. The formula $v_\xi^j - v_\xi^i = t_{ij}(\xi) \cdot (v^j - v^i)$ summarizes the situation.

Remark 5.7.

- (i) The special element $\underline{1} \in C(P)$, representing P itself, provides a linear embedding $e_C : \mathbb{R}_{\geq 0} \hookrightarrow C(P)$. As in the beginning of this subsection, the fiber product, that is, the preimage of this ray $p_C^{-1}(e_C(\mathbb{R}_{\geq 0}))$ equals $\sigma = \text{cone}(P)$.

$$\begin{array}{ccc}
 \text{cone}(P) & \longrightarrow & \tilde{C}(P) \\
 \downarrow & & \downarrow p_C \\
 \mathbb{R}_{\geq 0} & \xrightarrow{e_C} & C(P)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{cone}(P_0 * \dots * P_k) & \longrightarrow & \tilde{C}(P) \\
 \downarrow & & \downarrow p_C \\
 \mathbb{R}_{\geq 0}^{k+1} & \longrightarrow & C(P)
 \end{array}$$

- (ii) If $P = P_0 + \dots + P_k$ is a Minkowski decomposition of P , then the summands induce elements $[P_0], \dots, [P_k] \in C(P)$, and thus a linear map $\mathbb{R}_{\geq 0}^{m+1} \rightarrow C(P)$ sending the i th unit vector $e^i \mapsto [P_i]$. The fiber product becomes equal to the cone over the so-called *Cayley product* $P_0 * \dots * P_k$.

5.5 | A terminal object in the cone setup

This subsection is the dual of the cone setup variant of Subsection 4.6. Let P be a rational, convex polyhedron in some \mathbb{R} -vector space $N_{\mathbb{R}}$; for instance, it could arise from the situation in Subsection 5.4. Taking the height induces a natural map $R : \text{cone}(P) \rightarrow \mathbb{R}_{\geq 0}$. Then, the pairs (p_+, e) consisting of a surjective homomorphism of polyhedral cones $p_+ : \tilde{C} \twoheadrightarrow C$ and an embedding $e : \mathbb{R}_{\geq 0} \hookrightarrow C$ form a *category* \mathcal{F}_P . The embedding e gives thus an element $e(1) \in C$, such that

- (i) for $\xi, \xi' \in C$ one has $p_+^{-1}(\xi) + p_+^{-1}(\xi') = p_+^{-1}(\xi + \xi')$ and
- (ii) $R : \text{cone}(P) \rightarrow \mathbb{R}_{\geq 0}$ is obtained from $p_+ : \tilde{C} \twoheadrightarrow C$ via base change $e : \mathbb{R}_{\geq 0} \hookrightarrow C$, that is, $\text{cone}(P) = \tilde{C} \times_C \mathbb{R}_{\geq 0}$.

The two examples $[R : \text{cone}(P) \rightarrow \mathbb{R}_{\geq 0} \ni 1]$ and $[p_C : \tilde{C}(P) \twoheadrightarrow C(P) \ni \underline{1}]$ yield two objects in \mathcal{F}_P . Moreover, if P arises from Subsection 5.4, then also $[p_+ : S^\vee \rightarrow T^\vee \ni e(1)]$ becomes an object in \mathcal{F}_P .

By Proposition 5.4, \mathcal{F}_P equals the opposite category $\mathcal{E}_P^{\text{opp}}$ of the cone setup variant of $\mathcal{E}_P = \mathcal{E}_{(T,S)}$ from Subsection 4.6. Amplifying this comparison, the (dual) analog to Proposition 4.9 is just base change. Moreover, it is clear that $[R, 1] \in \mathcal{F}_P$ is an initial object. However, the true analog to Theorem 4.10 (restricted to this setting) is the existence of a terminal object in \mathcal{F}_P .

Proposition 5.8. *The pair $[p_C, \underline{1}]$ is a terminal object in \mathcal{F}_P . That is, for any $[p_+ : \tilde{C} \twoheadrightarrow C \ni e(1)]$ in the category \mathcal{F}_P , there is a unique linear $e' : C \rightarrow C(P)$ such that p_+ is induced from p_C via e'*

$$\begin{array}{ccccc}
 & & \text{cone}(P) & \longrightarrow & \tilde{C} & \longrightarrow & \tilde{C}(P) \\
 & & \downarrow R & & \downarrow p_+ & & \downarrow p_C \\
 \mathbb{R}_{\geq 0} & \xrightarrow{e} & C & \xrightarrow{e'} & C(P) & & \\
 & & \downarrow e_C & & & & \\
 & & \mathbb{R}_{\geq 0} & & & &
 \end{array}$$

and $e_C(1) = \underline{1} = (e' \circ e)(1)$. Moreover, the map $\tilde{C} \rightarrow \tilde{C}(P)$ is supposed to induce the identity map $\text{id}_P : p_+^{-1}(e(1)) \rightarrow P_{\underline{1}}$ on the two distinguished fibers.

Proof. Let $\xi \in C$. Then, since we have that $e(1)$ is an interior point of C , there is an $n \in \mathbb{N}$ such that $\xi' := e(n) - \xi \in C$. That is, by Minkowski linearity, the decomposition $e(n) = \xi + \xi'$ within C provides a Minkowski decomposition $n \cdot P = p_+^{-1}(\xi) + p_+^{-1}(\xi')$, that is, $p_+^{-1}(\xi)$ is a Minkowski summand of a scalar multiple of P . Now, since the points of $C(P)$ are in a one-to-one correspondence to the Minkowski summands of scalar multiples of P , the polyhedron $p_+^{-1}(\xi)$ corresponds to a unique $e'(\xi) \in C(P)$. This establishes the map e' . It is clearly additive, and one easily checks the remaining properties. □

6 | THE DISCRETE SETUP

6.1 | The pair of semigroups associated to a polyhedron

Let N be a lattice of rank d , that is a finitely generated free Abelian group $N \simeq \mathbb{Z}^d$, and let $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. We denote the ambient real vector spaces by $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^d$, respectively, by $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $P \subset N_{\mathbb{R}}$ be a rational polyhedron, which means that P is the intersection of finitely many halfspaces defined by linear inequalities with rational coefficients. Recall from Subsection 2.1 that polyhedra are not necessarily bounded, but we will assume that they have at least one vertex. Then, they split into a Minkowski sum

$$P = P^c + \text{recc}(P),$$

where P^c is the (bounded) convex hull of the vertices of P , and $\text{recc}(P)$ is a pointed polyhedral cone. We associate a cone to the polyhedron P by embedding it in the hyperplane of height one of $N_{\mathbb{R}} \oplus \mathbb{R}$, and taking the (closure of) the cone over it:

$$\sigma := \text{cone}(P) \subseteq N_{\mathbb{R}} \oplus \mathbb{R}.$$

The dual cone will be denoted by $\sigma^\vee = \text{cone}(P)^\vee \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$. We call R the canonical projection $N_{\mathbb{R}} \oplus \mathbb{R} \rightarrow \mathbb{R}$. Alternatively, we can understand this map as the element $R = [0, 1] \in M \oplus \mathbb{Z}$, which defines a ray $R : \mathbb{R}_{\geq 0} \hookrightarrow \sigma^\vee$. Note that the intersections $\sigma \cap R^\perp := \sigma \cap [R = 0]$ and $\sigma \cap [R = 1]$ recover $\text{recc}(P)$ and P , respectively. The pair of discrete semigroups associated to P is given by

$$T := \mathbb{N}, \quad S := \text{cone}_{\mathbb{Z}}(P)^\vee := \text{cone}(P)^\vee \cap (M \oplus \mathbb{Z}), \quad \text{and} \quad R : T \hookrightarrow S. \tag{6.1}$$

By Proposition 3.10, this forms a free pair. One goal of this work is to construct explicitly the universal iso-bounded extension of $T \hookrightarrow S$ (cf. Theorem 9.2). The inspiration for the constructions to come was the knowledge of the space of infinitesimal deformations T^1 of the toric singularity $\text{TV}(\sigma) = \text{Spec } \mathbb{C}[S]$ in the multidegree $-R \in M$.

6.2 | The structure of σ^\vee

We want to understand the relative boundaries of both σ^\vee and $\sigma^\vee \cap (M \oplus \mathbb{Z})$, with respect to the rays given by R . To this aim, we introduce the following.

Definition 6.1. For every linear form $c \in \text{recc}(P)^\vee \subseteq M_\mathbb{R}$ define

$$\begin{aligned} \eta(c) &:= -\min_{v \in P} \langle c, v \rangle = -\min \langle c, P \rangle \in \mathbb{R}, \\ \eta_{\mathbb{Z}}(c) &:= \lceil \eta(c) \rceil \in \mathbb{Z}, \end{aligned}$$

where $\lceil \eta \rceil$ denotes the ceiling, that is, the least integer not smaller than the real number η . It is easy to see that the set $f_c := \{v \in P : -\langle c, v \rangle = \eta(c)\}$ is a face and contains at least one vertex. We choose and fix one such vertex and denote it by $v(c)$. So, we have

$$\eta(c) = -\langle c, v(c) \rangle \leq \eta_{\mathbb{Z}}(c).$$

The reason for not taking any $c \in M_\mathbb{R}$ is that we want c to be bounded below on P . Obviously, when P is compact, $\text{recc}(P)^\vee = M_\mathbb{R}$. The function η is piecewise linear and usually called the *support function* of P . It is the Legendre transform of the 0 function on P .

Notation 6.2. We denote the set of all vertices of P by $\text{Vert}(P)$. We will need to distinguish between lattice and non-lattice vertices, and for this use the notation

$$\text{Vert}_{\in \mathbb{Z}}(P) := \text{Vert}(P) \cap N \quad \text{Vert}_{\notin \mathbb{Z}}(P) := \text{Vert}(P) \setminus \text{Vert}_{\in \mathbb{Z}}(P).$$

Moreover, for real numbers $z \in \mathbb{R}$ we will use the following notation:

$$\{z\} := \lceil z \rceil - z. \tag{6.2}$$

For $c \in \text{recc}(P)^\vee \cap M$ and $v(c) \in \text{Vert}_{\in \mathbb{Z}}(P)$ we have $\eta(c) = \eta_{\mathbb{Z}}(c)$. If $\eta(c) \notin \mathbb{Z}$, then the integer $\eta_{\mathbb{Z}}(c)$ equals the value of $-\langle c, \cdot \rangle$ at some point sitting on a moving affine c -hyperplane *before* it reaches our polyhedron P .

Remark 6.3.

- (i) If $0 \in P$, then $\eta(c) \geq 0$.
- (ii) The elements $[c, \eta(c)]$ form the relative boundary $\partial_{(\mathbb{R}_{\geq 0}R)}\sigma^\vee \subseteq \partial\sigma^\vee$ (the inclusion is strict if the recession cone is not trivial). This implies

$$\sigma^\vee = \{[c, \eta(c)] : c \in \text{recc}(P)^\vee\} + \mathbb{R}_{\geq 0} \cdot [0, 1].$$

- (iii) The semigroup $S := \sigma^\vee \cap (M \oplus \mathbb{Z})$ is generated by the Hilbert basis of σ^\vee , which has the form

$$\{[c_1, \eta_{\mathbb{Z}}(c_1)], \dots, [c_k, \eta_{\mathbb{Z}}(c_k)], [0, 1]\},$$

with uniquely determined elements $c_i \in \text{recc}(P)^\vee \cap M$. We will use the font ‘ c ’ only for the Hilbert basis elements.

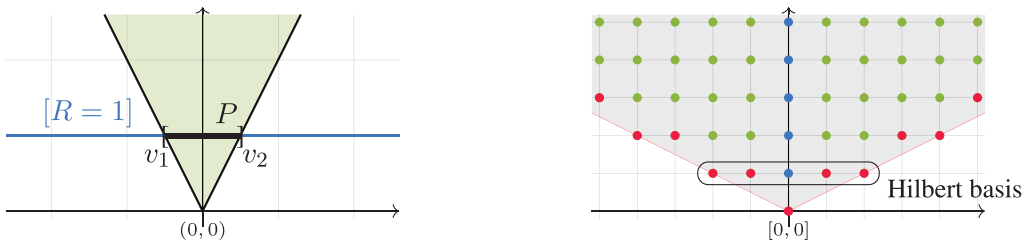


FIGURE 10 The cone and the semigroups for the 1-dimensional $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$

(iv) We defined $T := \mathbb{N}$ and embedded it in S by $1 \mapsto [0, 1]$. For $c \in \text{recc}(P)^\vee \cap M$, the elements $[c, \eta_{\mathbb{Z}}(c)]$ are always in $\partial_T S$, but not in $\partial_{(\mathbb{R}_{\geq 0} R)} \sigma^\vee$ whenever $\eta(c) \notin \mathbb{Z}$. Note that the latter implies that $v(c)$ does not belong to the lattice, that is, that $v(c) \in \text{Vert}_{\neq \mathbb{Z}}(P)$.

Example 6.4. Let $P = \text{conv}(-\frac{1}{2}, \frac{1}{2}) \subseteq \mathbb{R}$. Then, $\sigma \subseteq \mathbb{R}^2$ is spanned by the rays $\mathbb{R}_{\geq 0} \cdot (-1, 2)$ and $\mathbb{R}_{\geq 0} \cdot (1, 2)$. The dual σ^\vee is spanned by $\mathbb{R}_{\geq 0} \cdot [-2, 1]$ and $\mathbb{R}_{\geq 0} \cdot [2, 1]$, see Figure 10 continuing the story of Figure 4 and of Example 4.11. We obtain:

c	...	-2	-1	0	1	2	...
$v(c)$...	v_2	v_2	v_2 or v_1	v_1	v_1	...
$\eta(c)$...	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	...
$\eta_{\mathbb{Z}}(c)$...	1	1	0	1	1	...

6.3 | The ambient space of the universal extension

In this section, we will define $\mathcal{T}(P)$. The two subgroups which will later give the initial object live in the dual spaces of $\mathcal{T}(P)$ and $N_{\mathbb{R}} \times \mathcal{T}(P)$, respectively.

Notation 6.5. The set of compact edges of P is $\text{Edge}^c(P) = \{d_1, \dots, d_r\}$. We write $d = [v^i, v^j]$ for the edge connecting v^i and v^j and we write $[v^i, v^j]$ for the half-open edge. We will abuse notation and denote oriented edges also by $d := v^j - v^i \in N_{\mathbb{R}}$; in the few ambiguous situations we will use words to specify which of the two d refers to. We also use the convention that round brackets denote vectors: $(1, 2, 3) \in N_{\mathbb{R}}$, square brackets denote linear forms $[4, 5, 6] \in M_{\mathbb{R}}$, and pointed brackets denote the standard perfect pairing: $\langle [4, 5, 6], (1, 2, 3) \rangle = 32 \in \mathbb{R}$. We will also fix

$$r = |\text{Edge}^c(P)| \quad m = |\text{Vert}(P)|.$$

The \mathbb{R} -vector space $\mathcal{T}(P)$ will be a subspace of $\mathbb{R}^r \oplus \mathbb{R}^m$. To describe its equations we need to introduce the following notions.

Definition 6.6. For every compact 2-dimensional face F of P let $\varepsilon_F : \text{Edge}^c(P) \rightarrow \{-1, 0, 1\}$ be one of the two functions satisfying

$$\begin{aligned} \varepsilon_F(d) \in \{-1, 1\} &\iff d \subseteq F \\ \{\varepsilon_F(d) \cdot d : d \subseteq F\} &\text{ forms an oriented cycle along } \partial F. \end{aligned}$$

The last property implies (but is not equivalent to)

$$\sum_{d \in \text{Edge}^c(P)} \varepsilon_F(d) \cdot d = 0.$$

Definition 6.7.

(i) To each half open edge $d = [v, w)$ of P we associate the positive integer

$$g = g(d) := \min\{g \in \mathbb{Z}_{\geq 1} : \text{the affine line through } gv \text{ and } gw \text{ contains lattice points}\}.$$

(ii) We call $d = [v, w)$ a *short half open edge* if

$$|\{g \cdot [v, w) \cap N\}| \leq g - 1.$$

If this is the case, then it follows that $v \notin N$. Finally, we call d a *short edge* if both $[v, w)$ and $(w, v]$ are short half open edges.

Remark 6.8. If at least one of the half open edges $[v, w)$ or $(w, v]$ is short, then $\ell(w - v) < 1$, where ℓ denotes the lattice length[†]. While the property $\ell(w - v) < 1$ is responsible for the name ‘short’, $\ell < 1$ alone does not suffice for shortness. For example $d = [-\frac{1}{2}, \frac{1}{3}] \subset \mathbb{R}$ has lattice length $\ell = \frac{5}{6} < 1$, but still, neither of the two half open edges is short.

Definition 6.9. For each compact edge $d_i = [v^j, v^k]$ of P we introduce a parameter which we denote by t_{d_i} , t_i , or t_{jk} .[‡] We then collect all of these in a vector $\mathbf{t} \in \mathbb{R}^r$ and define the linear subspace

$$C^{\text{lin}}(P) := \left\{ \mathbf{t} \in \mathbb{R}^r : \sum_{d \subset F} \varepsilon_F(d) t_d \cdot d = 0 \text{ for all compact 2-faces } F \text{ of } P \right\}.$$

The intersection $C^{\text{lin}}(P) \cap R_{\geq 0}^r$ parameterizes the Minkowski summands of positive multiples of P , cf. [4, (2.2)].

Definition 6.10. For each vertex $v = v^i \in \text{Vert}(P)$, we introduce a parameter which we denote by s_v or s_i . We then define

$$\mathcal{T}(P) := \left\{ (\mathbf{t}, \mathbf{s}) \in C^{\text{lin}}(P) \oplus \mathbb{R}^m : \begin{aligned} s_i &= 0 && \text{if } v^i \in N, \\ s_i &= s_j && \text{if } [v^i, v^j] \in \text{Edge}^c(P) \text{ with } [v^i, v^j] \cap N = \emptyset, \text{ and} \\ s_i &= t_{ij} && \text{if } [v^i, v^j] \text{ is a half open short edge} \end{aligned} \right\}.$$

Note that the vector space $\mathcal{T}(P)$ contains a distinguished element $(\underline{1}; \underline{1}, \underline{0}) = [P]$ which is defined by $s_i := 0$ for $v^i \in N$ and $s_j := 1$ and $t_{ij} := 1$ for all remaining coordinates, cf. Remark 10.11. In

[†] Defined as the homogeneous function on $N_{\mathbb{R}}$ such that any primitive element of N has lattice length one.

[‡] Always choosing the most convenient one in the given context.

the upcoming sections, we will often deal with the dual vector space $\mathcal{T}^*(P)$. Then, its elements $s_i, t_{ij} \in \mathcal{T}^*(P)$ form a generating set of this space. We could easily omit the elements $s_i = 0$ for $v^i \in N$. However, while they are just zero, their existence will simplify some formulae.

The role of the parameters s_i remains mysterious. Their algebraic or geometric meaning is not clear for us — neither was it, for example, in the deformation theory of cyclic quotient singularities. From the combinatorial point of view, however, they serve as an instrument forcing some of the dilation parameters t_{ij} to remain unchanged when crossing non-lattice vertices.

6.4 | Relation to algebraic geometry, Part I

In Example 4.11, we have already mentioned that our theory of extensions of semigroups has strong links to deformation theory in algebraic geometry, cf. [1] for addressing this in detail. Nevertheless, we would like to mention here that for a singularity X there is the vector space T_X^1 of so-called infinitesimal deformations. In case of a toric singularity, it is M -graded, and we denote by $T_X^1(-R)$ the contribution in multidegree $-R$.

Proposition 6.11. *For $X = \mathbb{T}\mathbb{V}(\text{cone}(P))$ we obtain that $T_X^1(-R) = (\mathcal{T}(P) \otimes_{\mathbb{R}} \mathbb{C}) / \mathbb{C} \cdot (\underline{1}; \underline{1}, \underline{0})$.*

Proof. Essentially, this corresponds to [5, Theorem 2.5]. One has just to check that the equations called G_{jk} in [5, (2.6)] coincide with those in the definition of the \mathbb{R} -vector space $\mathcal{T}(P)$. □

Example 6.12.

- (1) The mother of all examples is $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$ from Example 4.11. Here we have only one edge $d = P$ with $g_P = 1$. The interval has length one, and it contains exactly one lattice point, that is, $|\{P \cap N\}| = 1$. In particular, it gives rise to two non-short half open edges.
- (2) The interval $P = [-\frac{1}{2}, \frac{1}{3}] \subset \mathbb{R}$ has lattice length $\ell = \frac{5}{6} < 1$, but still, neither of the two half open edges is short.
- (3) Since the interval $P = [\frac{1}{2}, \frac{3}{4}]$ does not contain any lattice points, but the affine line through P does, we obtain that $g = 1$, and both half open edges are short.
- (4) Take $P := \text{conv}\{(-\frac{1}{6}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2})\} \subset \mathbb{R}^2$. Here we need to multiply with $g = 2$ to produce lattice points on the affine line. The resulting interval $gP = [-\frac{1}{3}, \frac{4}{3}]$ has length $\frac{5}{3} < 2$ and $|\{gP \cap N\}| = 2$. That is, neither of the half open edges are short.
- (5) At last, we consider $P := \text{conv}\{(-\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{2})\} \subset \mathbb{R}^2$. We still have $g = 2$ and this leads to the interval $gP = [-1, \frac{2}{3}]$. In particular, one of the half open edges is short, the opposite one is not.

See Subsection 9.1 for a sequel of this discussion.

6.5 | Understanding $\mathcal{T}_+^{\vee}(P)$

By definition, we have $\mathcal{T}(P) \subseteq \mathbb{R}^r \oplus \mathbb{R}^m$, which allows us to define

$$\mathcal{T}_+(P) := \mathcal{T}(P) \cap (\mathbb{R}_{\geq 0}^r \oplus \mathbb{R}_{\geq 0}^m).$$

For the construction of the universal object from Section 8 positivity does not play an important role. However, positivity will be crucial for the correspondence with lattice-friendly Minkowski decompositions in Section 10. While positivity of the t -coordinates has a clear meaning, the necessity of positive s -coordinates becomes apparent in Example 10.14.

We will denote the dual of \mathbb{R}^{r+m} by the same symbol[†]: \mathbb{R}^{r+m} . Since $\mathcal{T}(P) \subseteq \mathbb{R}^{r+m}$, we have a canonical projection $\mathbb{R}^{r+m} \rightarrow \mathcal{T}^*(P)$ yielding elements $t_{ij}, s_v \in \mathcal{T}^*(P)$ and the equality

$$\mathcal{T}^*(P) = \mathbb{R}^{r+m} / \mathcal{T}(P)^\perp. \tag{6.3}$$

According to Definition 6.10, the subspace $\mathcal{T}(P)^\perp \subseteq \mathbb{R}^{r+m}$ is generated by the following four types of elements:

$$\chi(F) := \sum_{d \subset F} \varepsilon_F(d) t_d \cdot d \text{ for all } \ddagger \text{ compact 2-faces } F \text{ of } P, \tag{6.4}$$

$$s_i \text{ for all } v^i \in N. \tag{6.5}$$

$$s_i - s_j \text{ for all } [v^i, v^j] \in \text{Edge}^c(P) \text{ with } [v^i, v^j] \cap N = \emptyset, \text{ and} \tag{6.6}$$

$$t_{ij} - s_i \text{ for all short edges } [v^i, v^j]. \tag{6.7}$$

The relations $\chi(F)$ enable us to encode Minkowski summands via edge dilation, cf. Subsection 5.4 and Definition 6.9. Their importance for the extensions of semigroups, however, becomes apparent in the proof of Proposition 7.15. From Equation (6.3), we get that the dual cone of $\mathcal{T}_+(P)$ is

$$\mathcal{T}_+^\vee(P) = \text{Image}(\mathbb{R}_{\geq 0}^{r+m} \rightarrow \mathcal{T}^*(P)) = \frac{\mathbb{R}_{\geq 0}^{r+m} + \mathcal{T}(P)^\perp}{\mathcal{T}(P)^\perp}.$$

6.6 | The lattice structure in $\mathcal{T}(P)$

We start by defining a subgroup of $\mathcal{T}_\mathbb{Z}(P) \subset \mathcal{T}(P)$, and then prove that this is a lattice. In the case in which P is a lattice polytope with primitive edges, this lattice is simply $\mathbb{Z}^r \cap C^{\text{lin}}(P)$, cf. Example 6.17.

Definition 6.13. Define the subgroup $\mathcal{T}_\mathbb{Z}(P) \subset \mathcal{T}(P)$ by

$$(\mathbf{t}, \mathbf{s}) \in \mathcal{T}_\mathbb{Z}(P) \quad : \Leftrightarrow \quad \begin{cases} s_i \in \mathbb{Z}, & \forall v^i \in \text{Vert}(P), \text{ and} \\ (t_{ij} - s_i)v^i - (t_{ij} - s_j)v^j \in N, & \forall [v^i, v^j] \in \text{Edge}^c(P). \end{cases}$$

Clearly, $\mathcal{T}_\mathbb{Z}(P)$ is a subgroup of $\mathcal{T}(P)$, thus it is Abelian and torsion-free.

[†] That is, we will not add a star here.

[‡] Actually, these elements are in $\mathcal{T}(P)^\perp \otimes N_\mathbb{R}$ and need to be evaluated by some $c \in M_\mathbb{R}$.

Lemma 6.14. *The subgroup $\mathcal{T}_{\mathbb{Z}}(P)$ is a free Abelian group satisfying*

$$\mathcal{T}_{\mathbb{Z}}(P) \otimes_{\mathbb{Z}} \mathbb{R} = \mathcal{T}(P).$$

Proof. We first show that $\mathcal{T}_{\mathbb{Z}}(P)$ is a discrete subgroup, that is, that $0 \in \mathcal{T}_{\mathbb{Z}}(P)$ is an isolated point. If $(\mathbf{t}, \mathbf{s}) \in \mathcal{T}_{\mathbb{Z}}(P)$ has sufficiently small coordinates, then the integrality of s_i implies that $s_i = 0$. For the resulting $(\mathbf{t}, 0)$, we thus get

$$t_{ij}(v^i - v^j) \in N.$$

As we have finitely many compact edges, and as N is a lattice, thus not divisible, it follows that sufficiently small t_{ij} are forced to be zero as well.

Every rational element of $\mathcal{T}(P)$ admits an integral multiple contained in $\mathcal{T}_{\mathbb{Z}}(P)$. Together with the discrete property, this implies that $\mathcal{T}_{\mathbb{Z}}(P)$ is an Abelian group satisfying

$$\mathcal{T}_{\mathbb{Z}}(P) \otimes_{\mathbb{Z}} \mathbb{R} = \mathcal{T}(P). \quad \square$$

The dual lattice is by definition

$$\mathcal{T}_{\mathbb{Z}}^*(P) = \{f \in \mathcal{T}^*(P) : f(\mathcal{T}_{\mathbb{Z}}(P)) \subseteq \mathbb{Z}\}.$$

Thus, the dual lattice $\mathcal{T}_{\mathbb{Z}}^*(P)$ is generated by

$$\{s_v : v \in \text{Vert}(P)\} \cup \{L_{ij}(c) : c \in \text{recc}(P)^\vee \cap M\}$$

with $L_{ij}(c)$ being the evaluations of the L_{ij} explained below; see Notation 6.16.

We will regard all the t_{ij}, s_i, s_j as coordinate functions, that is as elements of $\mathcal{T}^*(P)$. The two conditions of Definition 6.13 can be thus rephrased as

$$\begin{aligned} s_i &\in \mathcal{T}_{\mathbb{Z}}^*(P), \\ L_{ij} &:= (t_{ij} - s_i) \otimes v^i - (t_{ij} - s_j) \otimes v^j \in \mathcal{T}_{\mathbb{Z}}^*(P) \otimes_{\mathbb{Z}} N. \end{aligned}$$

We will often group the summands as: $L_{ij} = t_{ij} \otimes (v^i - v^j) + s_j \otimes v^j - s_i \otimes v^i$.

Remark 6.15. The elements L_{ij} , together with $s_i \otimes N$, generate $\mathcal{T}_{\mathbb{Z}}(P)^* \otimes_{\mathbb{Z}} N$.

Note that $t_{ji} = t_{ij}$, but $L_{ji} = -L_{ij}$. For any oriented compact edge $d = [v^i, v^j]$ we will write L_d or $L_{v^i v^j}$ instead of L_{ij} when it is more convenient to do so. Finally, let us point out that the distinguished element $(\underline{1}; \underline{1}, \underline{0}) =: [P]$ belongs to the lattice.

Notation 6.16. As we are dealing with the tensor product of two linear forms, one on $\mathcal{T}(P)$ and one on M , it makes sense to apply L_{ij} to both types of elements separately. We will denote as consistently as possible the elements of $\mathcal{T}(P)$ by ξ and those of M by c . Therefore, we will use the same notation when we apply L_{ij} to either of them:

$$\begin{aligned} L_{ij}(\xi) &:= t_{ij}(\xi) \cdot (v^i - v^j) + s_j(\xi) \cdot v^j - s_i(\xi) \cdot v^i \in N_{\mathbb{R}}, \text{ for } \xi \in \mathcal{T}(P), \\ L_{ij}(c) &:= \langle c, v^i - v^j \rangle \cdot t_{ij} + \langle c, v^j \rangle \cdot s_j - \langle c, v^i \rangle \cdot s_i \in \mathcal{T}^*(P), \text{ for } c \in M. \end{aligned}$$

Example 6.17.

- (1) If $P = [v, w] \subset \mathbb{R}$ is a rational line segment, then we denote by $d(P) := w - v$ the length of P and by $\{v\} := \lceil v \rceil - v$ and $\{w\} := \lceil w \rceil - w$ the discrepancies for P to have integral limits. Then, besides $s_v, s_w \in \mathcal{T}_{\mathbb{Z}}^*(P)$, this lattice is characterized by the incidence

$$d(P) \cdot t - \{v\} \cdot s_v + \{w\} \cdot s_w \in \mathcal{T}_{\mathbb{Z}}^*(P)$$

which has a straightforward geometric interpretation. Indeed, this follows from

$$\begin{aligned} L_{d(P)} &= -d(P) \cdot t + w \cdot s_w - v \cdot s_v \\ &= -d(P) \cdot t - \{w\} \cdot s_w + \{v\} \cdot s_v + (w + \{w\}) \cdot s_w - (v + \{v\}) \cdot s_v, \end{aligned}$$

because the last two coefficients, $w + \{w\}$ and $-v - \{v\}$, are integers.

- (2) Whenever P is a lattice polyhedron with primitive compact edges, then $\mathcal{T}(P) = C^{\text{lin}}(P)$, and $\mathcal{T}_{\mathbb{Z}}(P) = C_{\mathbb{Z}}^{\text{lin}}(P)$ is determined by the integrality of all t_{ij} .

In Section 7, we will lift $\eta(c)$ and $\eta_{\mathbb{Z}}(c)$ along the map $\pi : \mathcal{T}_{\mathbb{Z}}^*(P) \rightarrow \mathbb{Z}$ (or its rational version), where $t_{ij} \mapsto 1$ for all edges and $s_v \mapsto 1$ if $v \notin N$. This gives the vertical maps in the following diagram.

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{Z}}^*(P) \otimes_{\mathbb{Z}} N & \xrightarrow{\langle c, \cdot \rangle} & \mathcal{T}_{\mathbb{Z}}^*(P) \\ \pi := \pi \otimes \text{id}_N \downarrow & & \downarrow \pi \\ N & \xrightarrow{\langle c, \cdot \rangle} & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} L_{ij} & \xrightarrow{\quad} & L_{ij}(c) \\ \downarrow & & \downarrow \\ (\delta^{\mathbb{Z}} v^i - \delta^{\mathbb{Z}} v^j) & \xrightarrow{\quad} & \langle c, \delta^{\mathbb{Z}} v^i - \delta^{\mathbb{Z}} v^j \rangle \end{array}$$

In the previous diagram, we have used the symbol

$$\delta^{\mathbb{Z}} v := \begin{cases} v & \text{if } v \in N \\ 0 & \text{if } v \in N_{\mathbb{R}} \setminus N. \end{cases}$$

In particular, $L_{ij} \in \ker(\pi)$ if $v^i, v^j \in \text{Vert}_{\notin \mathbb{Z}}(P)$.

7 | LIFTING THE η TO \mathcal{T}^*

In this section, we define liftings $[c, \eta_{\mathbb{Z}}(c)] \mapsto [c, \tilde{\eta}_{\mathbb{Z}}(c)]$ in the finite dimensional \mathbb{R} -vector space $\mathcal{T}^*(P)$ from Subsection 6.3. The main idea behind constructing the universal extension is to use these liftings of the relative boundary and define \tilde{T} using the relations among them: $\tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_{\ell})$ (cf. Definition 7.10). Then, \tilde{S} will be the sum of the lifted boundary with \tilde{T} .

7.1 | Lifting the boundary

We start by fixing a *reference vertex* $v_* \in \text{Vert}(P)$.

Convention 7.1. Whenever $v_* \in P$ belongs to the lattice, we will set $v_* = 0$.

As the c in $[c, \eta(c)]$ will be unchanged when lifting, we will only focus on the η part. Let us state Definition 6.1 of $\eta(c)$ from a different point of view. For $c \in \text{recc}(P)^\vee$ we choose a path $v_* = v^0, v^1, \dots, v^k = v(c)$ along the compact edges of P . Then

$$\begin{aligned} -\eta(c) &= \langle c, v(c) \rangle \\ &= \langle c, v_* \rangle + \langle c, v(c) - v_* \rangle \\ &= \langle c, v_* \rangle + \sum_{j=1}^k \langle c, v^j - v^{j-1} \rangle. \end{aligned}$$

In complete analogy to this, we define now the lifting $\tilde{\eta}(c)$ of $\eta(c)$.

Definition 7.2. For every $c \in \text{recc}(P)^\vee$, we define $\tilde{\eta}(c) \in \mathcal{T}^*(P)$ as the piecewise linear

$$\tilde{\eta}(c) := -\langle c, v_* \rangle \cdot s_{v_*} - \sum_{j=1}^k \langle c, v^j - v^{j-1} \rangle \cdot t_{j-1, j}.$$

Lemma 7.3. The definition of $\tilde{\eta}(c) \in \mathcal{T}^*(P)$ depends neither on the choice of the vertex $v(c)$, nor on the choice of the path connecting v_* and $v(c)$.

Proof. The independence on the choice of the path follows by the usual argument, namely, by the presence of the closing relations, that is, the elements $\chi(F)$ providing $\langle c, \chi(F) \rangle \in \mathcal{T}(P)^\perp$ mentioned in Subsection 6.5.

For the independence on $v(c)$, let us choose another candidate $v'(c)$. Then, $\langle c, v'(c) \rangle = \langle c, v(c) \rangle$, and we may connect both vertices by a path within the level face $f_c = \{v \in P : \langle c, v \rangle = \langle c, v(c) \rangle\}$, that is, via edges $d \in c^\perp$. Thus, the two paths connecting v_* with $v(c)$ or $v'(c)$, respectively, can be chosen to differ only by those edges. In particular, they produce the same result after being paired with c . □

Note that $\tilde{\eta}(c)$ is always a lifting of $\eta(c)$ via the map π due to our Convention 7.1, that is, it holds that

$$\pi(\tilde{\eta}(c)) = \eta(c).$$

Remark 7.4.

- (i) Sending a certain point of P to 0 means to shift the polyhedron by some vector w . This implies $\tilde{\eta}(c)_{P-w} = \tilde{\eta}(c)_P + \langle c, w \rangle \cdot s_{v_*}$. In particular, if $v_* \in N$, then shifting by $-v_*$ to meet our convention does not change $\tilde{\eta}(c)$. Actually, when shifting P , it is the original $\eta(c)$ which is altered in a linear way to meet the property $\pi(\tilde{\eta}(c)) = \eta(c)$.
- (ii) Besides shifting P , the functions η , $\eta_{\mathbb{Z}}$, $\tilde{\eta}$, and $\tilde{\eta}_{\mathbb{Z}}$ defined below in Definition 7.7 depend all on the choice of the reference vertex $v_* \in \text{Vert}(P)$. However, this dependence is an easy one, provided by adding a \mathbb{Z} -linear map $M \rightarrow \mathcal{T}_{\mathbb{Z}}^*(P)$ for both $\tilde{\eta}$ and $\tilde{\eta}_{\mathbb{Z}}$.

For integrality issues, it is important to express $\tilde{\eta}(c)$ in terms of the integral $L_{ij} \in \mathcal{T}_{\mathbb{Z}}(P)^* \otimes_{\mathbb{Z}} N$.

Lemma 7.5. *For every $c \in \text{recc}(P)^\vee \cap M$ and any path $v_* = v^0, v^1, \dots, v^k = v(c)$ along the compact edges of P we have*

$$\tilde{\eta}(c) = \eta(c) \cdot s_{v(c)} + \sum_{j=1}^k L_{j-1,j}(c).$$

Proof. We use the chosen path $v_* = v^0, v^1, \dots, v^k = v(c)$ to obtain

$$\begin{aligned} -\tilde{\eta}(c) &= \langle c, v^0 \rangle \cdot s_0 + \sum_{j=1}^k \langle c, v^j - v^{j-1} \rangle \cdot t_{j-1,j} \\ &= \sum_{j=1}^k (\langle c, v^j - v^{j-1} \rangle \cdot t_{j-1,j} - \langle c, v^j \rangle \cdot s_j + \langle c, v^{j-1} \rangle \cdot s_{j-1}) + \langle c, v^k \rangle \cdot s_k \\ &= \sum_{j=1}^k L_{j,j-1}(c) + \langle c, v(c) \rangle \cdot s_{v(c)}. \end{aligned}$$

□

Corollary 7.6. *For every $c \in \text{recc}(P)^\vee \cap M$, we have $\tilde{\eta}(c) \in \mathcal{T}_{\mathbb{Z}}^*(P)$ if and only if $\eta(c) \in \mathbb{Z}$.*

Proof. Since $\pi : \mathcal{T}_{\mathbb{Z}}^*(P) \rightarrow \mathbb{Z}$ maps $\tilde{\eta}(c)$ to $\eta(c)$, we obtain the first implication. The converse is a direct consequence of Lemma 7.5 and the integrality of L_{ij} . □

For $c \in \text{recc}(P)^\vee \cap M$, we recall $\eta_{\mathbb{Z}}(c) = \lceil \eta(c) \rceil \in \mathbb{Z}$ from Definition 6.1 and, for $z \in \mathbb{R}$, $\{z\} = \lceil z \rceil - z$ from Notation 6.2. Thus, Lemma 7.5 suggests the following possibility to lift this definition via π .

Definition 7.7. For every $c \in \text{recc}(P)^\vee \cap M$ we define $\tilde{\eta}_{\mathbb{Z}}(c) \in \mathcal{T}_{\mathbb{Z}}^*(P)$ as

$$\begin{aligned} \tilde{\eta}_{\mathbb{Z}}(c) &:= \eta_{\mathbb{Z}}(c) \cdot s_{v(c)} + \sum_{j=1}^k L_{j-1,j}(c) \\ &= \tilde{\eta}(c) + (\eta_{\mathbb{Z}}(c) - \eta(c)) \cdot s_{v(c)} \\ &= \tilde{\eta}(c) + \{ \eta(c) \} \cdot s_{v(c)}. \end{aligned}$$

Remark 7.8.

(i) By Convention 7.1, we have $\pi(\tilde{\eta}(c)) = \eta(c)$. We also have

$$\pi(\tilde{\eta}_{\mathbb{Z}}(c)) = \eta_{\mathbb{Z}}(c).$$

In particular, in contrast to η and $\tilde{\eta}$, the functions $\eta_{\mathbb{Z}}$ and $\tilde{\eta}_{\mathbb{Z}}$ are no longer piecewise linear.

(ii) The pair $[c, \tilde{\eta}_{\mathbb{Z}}(c)]$ is a quite natural lifting of $[c, \eta_{\mathbb{Z}}(c)]$ from $M \times \mathbb{N}$ to $M \times \mathcal{T}^*(P)$. However, even when asking for some positivity, it might be not the only lifting — see [3, 3.7] for an example.

Lemma 7.9. *The definition of $\tilde{\eta}_{\mathbb{Z}}(c)$ does not depend on the choice of the vertex $v(c)$.*

Proof. As we did in the proof of Lemma 7.3, we may connect $v(c)$ and $v'(c)$ by edges $d = [v^i, v^j]$ contained in the face $f_c := \{p \in P : \langle c, p \rangle = \min \langle c, P \rangle\}$. If the shared coefficient of our two heroes $s_{v(c)}$ and $s_{v'(c)}$ matters at all, that is, if $\eta_{\mathbb{Z}}(c) - \eta(c) \neq 0$, then $\eta(c) \notin \mathbb{Z}$, that is, the face f_c contains no lattice points. That is, any edge $d = [v^i, v^j]$ on the path between $v(c)$ and $v'(c)$ satisfies the property $[v^i, v^j] \cap N = \emptyset$. This property occurs in Definition 6.10, and implies $s_i = s_j$ as elements of $\mathcal{T}_{\mathbb{Z}}^*(P) \subseteq \mathcal{T}^*(P)$. Altogether, it means that $s_{v(c)} = s_{v'(c)}$. \square

7.2 | Relations

Having defined $\eta(c)$ as a minimum and $\eta_{\mathbb{Z}}(c) = \lceil \eta(c) \rceil$ (Definition 6.1), we get $\lceil \eta(c_1) \rceil + \dots + \lceil \eta(c_\ell) \rceil \geq \lceil \eta(c_1) + \dots + \eta(c_\ell) \rceil \geq \lceil \eta(c_1 + \dots + c_\ell) \rceil$ for any sequence c_1, \dots, c_ℓ of not necessarily distinct elements of $\text{recc}(P)^\vee \cap M$. This implies:

$$\eta_{\mathbb{Z}}(c_1) + \dots + \eta_{\mathbb{Z}}(c_\ell) \geq \eta_{\mathbb{Z}}(c_1 + \dots + c_\ell). \tag{7.1}$$

Definition 7.10. Let $\ell \geq 2$. For each sequence $c_1, \dots, c_\ell \in \text{recc}(P)^\vee \cap M$ of not necessarily distinct elements, and for each of the symbols $\eta, \eta_{\mathbb{Z}}, \tilde{\eta}$, or $\tilde{\eta}_{\mathbb{Z}}$, which we represent bellow by a \diamond , we define

$$\diamond(c_1, \dots, c_\ell) := \sum_{i=1}^{\ell} \diamond(c_i) - \diamond\left(\sum_{i=1}^{\ell} c_i\right).$$

A sequence c_1, \dots, c_ℓ is called \diamond -independent if $\diamond(c_1, \dots, c_\ell) = 0$. We use the convention that every sequence of length one is independent as well[†].

This definition does not depend on the order of the c_i , just on the multiset.

Remark 7.11.

(i) Convention 7.1 and Remark 7.8 extend to:

$$\begin{aligned} \pi(\tilde{\eta}(c_1, \dots, c_\ell)) &= \eta(c_1, \dots, c_\ell) \\ \pi(\tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_\ell)) &= \eta_{\mathbb{Z}}(c_1, \dots, c_\ell). \end{aligned}$$

(ii) The fact that $\eta(c_1, \dots, c_\ell) \geq 0$ is a trivial consequence of $\eta(c)$ being defined as some minimum. However, for the integral variant $\eta_{\mathbb{Z}}(c_1, \dots, c_\ell)$ one should keep in mind that this does *not* need to be the roundup of $\eta(c_1, \dots, c_\ell)$; even the inequality $\eta_{\mathbb{Z}}(c_1, c_2) \geq \eta(c_1, c_2)$ might fail. Nevertheless, the non-negativity of $\eta_{\mathbb{Z}}(c_1, \dots, c_\ell)$ is given by the Inequality (7.1), so $\eta_{\mathbb{Z}}(c_1, \dots, c_\ell) \in \mathbb{N}$.

(iii) For every $\diamond \in \{\eta, \eta_{\mathbb{Z}}, \tilde{\eta}, \tilde{\eta}_{\mathbb{Z}}\}$, for every $c_1, \dots, c_\ell \in \text{recc}(P)^\vee \cap M$ with $\ell \geq 2$, and for every $i = 2, \dots, \ell - 1$ we have

$$\begin{aligned} \diamond(c_1, \dots, c_\ell) &= \diamond(c_1) + \dots + \diamond(c_i) - \diamond(c_1 + \dots + c_i) \\ &\quad + \diamond(c_1 + \dots + c_i) + \diamond(c_{i+1}) + \dots + \diamond(c_\ell) - \diamond(c_1 + \dots + c_\ell) \\ &= \diamond(c_1, \dots, c_i) + \diamond(c_1 + \dots + c_i, c_{i+1}, \dots, c_\ell). \end{aligned}$$

[†] This make sense, as the definition of $\diamond(c_1, \dots, c_\ell)$ would give zero for $\ell = 1$. However, due to the overlap in notation for $\ell = 1$, we define $\diamond(c_1, \dots, c_\ell)$ only for $\ell \geq 2$.

(iv) In particular, the above recursive formula gives us the semigroup equality

$$\text{span}_{\mathbb{N}} \{ \tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_\ell) : \forall \ell \geq 2 \} = \text{span}_{\mathbb{N}} \{ \tilde{\eta}_{\mathbb{Z}}(c_1, c_2) \},$$

where the c_i run through all possible elements of $\text{recc}(P)^\vee \cap M$.

Example 7.12. Let us continue Example 6.4. Denoting the variables associated to the two non-lattice vertices $-\frac{1}{2}$ and $\frac{1}{2}$ by s_1 and s_2 , respectively, and denoting by t the variable referring to the one and only edge $d = P$, we obtain the following values:

c	$\eta(c)$	$\eta_{\mathbb{Z}}(c)$	$\tilde{\eta}(c)$	$\tilde{\eta}_{\mathbb{Z}}(c)$
-2	1	1	$-s_1 + 2t$	$-s_1 + 2t$
-1	$\frac{1}{2}$	1	$-\frac{1}{2}s_1 + t$	$\frac{1}{2}(s_2 - s_1) + t$
0	0	0	0	0
1	$\frac{1}{2}$	1	$\frac{1}{2}s_1$	s_1
2	1	1	s_1	s_1

Turning to the values for $\tilde{\eta}_{\mathbb{Z}}(c_1, c_2)$, this leads to

$$\begin{aligned} \tilde{\eta}_{\mathbb{Z}}(-1, 1) &= s_1 & \tilde{\eta}_{\mathbb{Z}}(-1, -1) &= s_2 \\ \tilde{\eta}_{\mathbb{Z}}(-1, 1) &= t + \frac{1}{2}(s_1 + s_2) & \tilde{\eta}_{\mathbb{Z}}(-2, 2) &= 2t \\ \tilde{\eta}_{\mathbb{Z}}(-1, 2) &= \frac{1}{2}(s_2 - s_1) + t & \tilde{\eta}_{\mathbb{Z}}(-2, 1) &= \frac{1}{2}(s_1 - s_2) + t. \end{aligned}$$

Our main goal in this section is to prove that the notions of $\eta_{\mathbb{Z}}$ -independence and $\tilde{\eta}_{\mathbb{Z}}$ -independence from Definition 7.10 are equivalent (Proposition 7.15).

Lemma 7.13. *The property of $\eta_{\mathbb{Z}}$ -independence is bequeathed to subsequences and to partitioning. Moreover, the latter is also true for the property of being ‘minimally $\eta_{\mathbb{Z}}$ -dependent’.*

Proof. The first part follows from the Equation (7.1) and Remark 7.11(iii). If $\ell \geq 3$ and c_1, \dots, c_ℓ is minimally dependent, then $\eta_{\mathbb{Z}}(c_1, \dots, c_\ell) > 0$ and $\eta_{\mathbb{Z}}(c_1, c_2) = 0$, so $(c_1 + c_2), c_3, \dots, c_\ell$ is dependent too. The minimality of this property is clear. □

The next lemma will not be directly applied later on. Its proof, however, can be seen as a warm up in which some notation is fixed for the proof the main result in this section.

Lemma 7.14. *Let $c_1, c_2 \in \text{recc}(P)^\vee \cap M$. If $\eta(c_1, c_2) = 0$, then $\tilde{\eta}(c_1, c_2) = 0$.*

Proof. If d_1^1, \dots, d_k^1 and d_1^2, \dots, d_l^2 are the oriented edges forming paths leading from $v(c_1 + c_2)$ to $v(c_1)$ and $v(c_2)$ with $\langle c_1, d_i^1 \rangle, \langle c_2, d_j^2 \rangle \leq 0$, respectively, we obtain

$$\tilde{\eta}(c_1, c_2) = - \sum_{i=1}^k \langle c_1, d_i^1 \rangle \cdot t_{d_i^1} - \sum_{j=1}^l \langle c_2, d_j^2 \rangle \cdot t_{d_j^2} \tag{7.2}$$

showing non-negative coefficients. This implies that the edges d_i^1 and d_j^2 have to be contained in c_1^\perp and c_2^\perp , respectively, because

$$\pi(\tilde{\eta}(c_1, c_2)) = - \sum_{i=1}^k \langle c_1, d_i^1 \rangle - \sum_{j=1}^l \langle c_2, d_j^2 \rangle = \eta(c_1, c_2) = 0.$$

Thus, $\langle c_1, v(c_1) \rangle = \langle c_1, v(c_1 + c_2) \rangle$ and $\langle c_2, v(c_2) \rangle = \langle c_2, v(c_1 + c_2) \rangle$. This means that we could have chosen, that is, that we can assume now, that $v(c_1) = v(c_1 + c_2) = v(c_2)$, which implies $\tilde{\eta}(c_1, c_2) = 0$. □

The next proposition is a key point for many arguments occurring in the rest of the paper. In its proof, the generators of $\mathcal{T}(P)^\perp$ described in (6.4)–(6.7) play a crucial role.

Proposition 7.15. *Being $\eta_{\mathbb{Z}}$ -independent is equivalent to being $\tilde{\eta}_{\mathbb{Z}}$ -independent.*

Proof. By Remark 7.11 $\tilde{\eta}_{\mathbb{Z}}$ -independence implies $\eta_{\mathbb{Z}}$ -independence. For the other direction we use induction on the length ℓ of the sequence c_1, \dots, c_ℓ . The case $\ell = 1$ is trivial. The essential step is $\ell = 2$.

So, let $c_1, c_2 \in \text{recc}(P)^\vee \cap M$ with $\eta_{\mathbb{Z}}(c_1, c_2) = 0$. Note first that we have

$$\{\eta(c)\} = \lceil \eta(c) \rceil - \eta(c) = \lceil -\langle c, v(c) \rangle \rceil - (-\langle c, v(c) \rangle) = \langle c, v(c) \rangle - \lfloor \langle c, v(c) \rangle \rfloor.$$

Combining the above relation with the definition of $\tilde{\eta}$ and with the formula (7.2) for $\tilde{\eta}(c_1, c_2)$ we obtain

$$\begin{aligned} \tilde{\eta}_{\mathbb{Z}}(c_1, c_2) &= \tilde{\eta}(c_1, c_2) + \{\eta(c_1)\} \cdot s_{v(c_1)} + \{\eta(c_2)\} \cdot s_{v(c_2)} - \{\eta(c_1 + c_2)\} \cdot s_{v(c_1 + c_2)} \\ &= - \sum_{i=1}^k \langle c_1, d_i^1 \rangle \cdot t_{d_i^1} + (\langle c_1, v(c_1) \rangle - \lfloor \langle c_1, v(c_1) \rangle \rfloor) \cdot s_{v(c_1)} \end{aligned} \tag{7.3}$$

$$\begin{aligned} &- \sum_{j=1}^l \langle c_2, d_j^2 \rangle \cdot t_{d_j^2} + (\langle c_2, v(c_2) \rangle - \lfloor \langle c_2, v(c_2) \rangle \rfloor) \cdot s_{v(c_2)} \\ &- (\langle c_1 + c_2, v(c_1 + c_2) \rangle - \lfloor \langle c_1 + c_2, v(c_1 + c_2) \rangle \rfloor) \cdot s_{v(c_1 + c_2)}, \end{aligned} \tag{7.4}$$

where the d_i^1 and d_j^2 are just as in the proof of Lemma 7.14. Our goal is to show that, assuming $\eta_{\mathbb{Z}}$ -independence, all the edges above are short. The proof is analogous for both paths, so we focus only on d_1^1, \dots, d_k^1 , and label the vertices with $v(c_1 + c_2) = v^0, \dots, v^k = v(c_1)$. From $\langle c_1, d_i^1 \rangle = \langle c_1, v^i - v^{i-1} \rangle \leq 0$ we get that $\langle c_1, v^i \rangle \leq \langle c_1, v^{i-1} \rangle$. Via a suitable choice of $v(c_1)$, we can even insist on strict inequalities:

$$\langle c_1, v^i \rangle < \langle c_1, v^{i-1} \rangle.$$

So, the vanishing of $\eta_{\mathbb{Z}}(c_1, c_2)$, which is obtained from $\tilde{\eta}_{\mathbb{Z}}(c_1, c_2)$ by sending $t, \dots, s \mapsto 1$, means that all the non-negative coefficients in rows (7.3) and (7.4) added up cancel with the single negative coefficient: that of $s_{v(c_1 + c_2)}$, which is contained in the half-open real interval $[0, 1)$. In particular,

the sum of all coefficients of $t_{\cdot, \cdot}$ and s_{\cdot} below is positive and strictly less than 1:

$$S_1 := - \sum_{i=1}^k \langle c_1, v^i - v^{i-1} \rangle \cdot t_{i-1, i} + (\langle c_1, v(c_1) \rangle - \lfloor \langle c_1, v(c_1) \rangle \rfloor) \cdot s_{v(c_1)}.$$

We would like to express the coefficient of $s_{v(c_1)}$ in a similar way as the coefficients of $t_{i-1, i}$. For this, we choose a c_1 -integral point[†] $v^{k+1} \in N_{\mathbb{R}}$ such that

$$\langle c_1, v^{k+1} \rangle = \lfloor \langle c_1, v(c_1) \rangle \rfloor \in \mathbb{Z}.$$

Note that integral points $v \in N$ are always c_1 -integral, but the opposite is far from being true. Moreover, note that, unless $\eta(c_1) \in \mathbb{Z}$, the new point v^{k+1} cannot be contained in the polyhedron P . On the other hand, if $\eta(c_1) \in \mathbb{Z}$, then we may and will choose $v^{k+1} := v^k$. Anyway, the $s_{v(c_1)}$ -coefficient of S_1 becomes $\langle c_1, v^k - v^{k+1} \rangle$, and we obtain

$$\pi(S_1) = - \sum_{i=1}^k \langle c_1, v^i - v^{i-1} \rangle + \langle c_1, v^k - v^{k+1} \rangle = \langle c_1, v^0 \rangle - \langle c_1, v^{k+1} \rangle < 1.$$

We want to deduce that

$$s_{v(c_1+c_2)} = t_{0,1} = s_{v^1} = t_{1,2} = \dots = s_{v^{k-1}} = t_{k-1,k} = s_{v(c_1)} \quad \text{in } \mathcal{T}^*(P), \tag{7.5}$$

where the last $s_{v(c_1)} = s_{v^k}$ has to be omitted if $\eta(c_1) \in \mathbb{Z}$. These equalities (together with those for the analogous c_2 -summand) obviously imply that

$$\tilde{\eta}_{\mathbb{Z}}(c_1, c_2) = \eta_{\mathbb{Z}}(c_1, c_2) \cdot s_{v(c_1+c_2)} = 0.$$

To obtain (7.5), we show that for $i = 1, \dots, k$ the edges $[v^{i-1}, v^i]$ are short edges, cf. Definition 6.7. Actually, if $\eta(c_1) \in \mathbb{Z}$, then only the one half open $[v^{k-1}, v^k]$ is needed to be short for the last segment. Assume that this fails for one of them. Then, we have

$$|[gv^{i-1}, gv^i] \cap N| \geq g \quad \text{or} \quad |(gv^{i-1}, gv^i] \cap N| \geq g,$$

where $g \in \mathbb{N}_{\geq 1}$ denotes the smallest number such that the line connecting gv^{i-1} and gv^i contains lattice points. In the first case, this implies that there are at least $(g + 1)$ c_1 -integral points along our path from gv^{i-1} to gv^{k+1} , hence, more than ever, from gv^0 to gv^{k+1} . In the second case, we obtain the same, unless $i = k$ and $v^k = v^{k+1}$ — however, this exactly means that we speak about the half open interval $(v^{k-1}, v^k]$ in the situation where $\eta(c_1) \in \mathbb{Z}$, which was excluded before.

Anyway, we do always get

$$|[gv^0, gv^{k+1}] \cap N| \geq g + 1,$$

hence, we obtain

$$g \cdot \pi(S_1) = \langle c_1, gv^0 \rangle - \langle c_1, gv^{k+1} \rangle \geq g,$$

a contradiction. This concludes the proof for $\ell = 2$. The inductive step follows from Lemma 7.13 and Remark 7.11(iii). □

[†] Meaning that $\langle c_1, v^{k+1} \rangle \in \mathbb{Z}$.

7.3 | Liftings and relations of the Hilbert basis

In the last part of this section, we will prepare the proof of the finite generation of the semigroup of all relations $\tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_\ell)$ given in Proposition 8.7. To this aim, let $\{[c_1, \eta_{\mathbb{Z}}(c_1)], \dots, [c_k, \eta_{\mathbb{Z}}(c_k)], [0, 1]\}$ be the Hilbert basis of $\sigma^\vee \cap (M \oplus \mathbb{Z})$ (cf. subsection 6.2, Remark 6.3). Multisets supported on $\{c_1, \dots, c_k\}$ correspond to elements of \mathbb{N}^k via the multiplicities of occurrence of each element: $m_1, \dots, m_k \in \mathbb{N}$. We denote them by $\{c_1^{m_1}, \dots, c_k^{m_k}\}$. So, we may speak of $\eta_{\mathbb{Z}}$ -dependent elements $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{N}^k$ via this correspondence, and write

$$\eta_{\mathbb{Z}}(\mathbf{m}) := \eta_{\mathbb{Z}}(c_1^{m_1}, \dots, c_k^{m_k}), \quad \forall \mathbf{m} \in \mathbb{N}.$$

Lemma 7.16. *The number of minimally $\eta_{\mathbb{Z}}$ -dependent elements of \mathbb{N}^k is finite.*

Proof. By Lemma 7.13, the set of dependent sequences supported on c_1, \dots, c_k is in order-preserving correspondence with a subset of \mathbb{N}^k representing a monomial ideal $\text{Dep}_{\eta_{\mathbb{Z}}} \subseteq \mathbb{Z}[t_1, \dots, t_k]$. So, the above statement follows from Dickson’s Lemma or simply by the fact that (for example, monomial) ideals are finitely generated. \square

8 | THE UNIVERSAL ISO-BOUNDED EXTENSION

The inspiration for the following definition comes from Proposition 4.3, which defines iso-bounded extensions by isomorphic relative boundaries, and Remark 6.3 which describes the relative boundary of our given object $(\mathbb{N}, \text{cone}_{\mathbb{Z}}(P)^\vee)$, where $\text{cone}_{\mathbb{Z}}(P)^\vee := \text{cone}(P)^\vee \cap (M \oplus \mathbb{Z})$. Denote by

$$\tilde{\mathcal{T}}_{\mathbb{Z}}^*(P) := M \oplus \mathcal{T}_{\mathbb{Z}}^*(P).$$

Definition 8.1. For any rational polyhedron $P \subseteq N_{\mathbb{R}}$, define the semigroups $\tilde{T}, \tilde{S} \subset \tilde{\mathcal{T}}_{\mathbb{Z}}^*(P)$ as

$$\begin{aligned} \tilde{T} &= \text{span}_{\mathbb{N}} \{ [0, \tilde{\eta}_{\mathbb{Z}}(c_1, c_2)] : c_1, c_2 \in \text{recc}(P)^\vee \cap M \}, \\ \tilde{S} &= \tilde{T} + \text{span}_{\mathbb{N}} \{ [c, \tilde{\eta}_{\mathbb{Z}}(c)] : c \in \text{recc}(P)^\vee \cap M \}. \end{aligned}$$

By Remark 7.11(iv), we have

$$\tilde{T} = \text{span}_{\mathbb{N}} \{ [0, \tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_\ell)] : \forall \ell \geq 2 \text{ and } \forall c_1, \dots, c_\ell \in \text{recc}(P)^\vee \cap M \}.$$

Further on, in Proposition 8.7, we will see that we could have chosen, in the latter version of defining \tilde{T} , only those c which appear in a Hilbert Basis of $\text{cone}_{\mathbb{Z}}(P)^\vee$.

8.1 | Belonging to the category

In this section, we will check that the semigroups $\tilde{T} \subset \tilde{S}$ form an iso-bounded extension of $(\mathbb{N}, \text{cone}_{\mathbb{Z}}(P)^\vee)$.

Proposition 8.2. *It holds that*

$$\partial_{\tilde{T}}\tilde{S} = \{[c, \tilde{\eta}_{\mathbb{Z}}(c)] \mid c \in \text{recc}(P)^\vee \cap M\}.$$

Proof. The main consequence of Proposition 7.15 is that $\ker \pi_T = 0$, from which it follows that $[c, \tilde{\eta}_{\mathbb{Z}}(c)] \in \partial_{\tilde{T}}\tilde{S}$. The other inclusion is obvious. \square

Remark 8.3. Since $[c, \tilde{\eta}_{\mathbb{Z}}(c)]$ are natural (but not the only) liftings of $[c, \eta_{\mathbb{Z}}(c)] \in \partial_{\mathbb{N}}\text{cone}_{\mathbb{Z}}(P)^\vee$, it is quite natural to put these elements in \tilde{S} . Independently of the shape of \tilde{T} , the required triviality of the kernel of $\pi|_{\tilde{T}} : \tilde{T} \rightarrow \mathbb{N}$ (cf. Definition 4.1) implies that $[c, \tilde{\eta}_{\mathbb{Z}}(c)] \in \partial_{\tilde{T}}\tilde{S}$. By the defining property of the relative boundary, it follows that

$$[0, \tilde{\eta}_{\mathbb{Z}}(c_1, c_2)] = [c_1, \tilde{\eta}_{\mathbb{Z}}(c_1)] + [c_2, \tilde{\eta}_{\mathbb{Z}}(c_2)] - [c_1 + c_2, \tilde{\eta}_{\mathbb{Z}}(c_1 + c_2)]$$

has to be contained in \tilde{T} . Thus, the Definition 8.1 was quite inevitable. At least, it was the minimal choice.

Proposition 7.15 is also crucial to prove the following.

Proposition 8.4. *For every rational polyhedron P , the diagram*

$$\begin{array}{ccc} \tilde{T} & \hookrightarrow & \tilde{S} \\ \pi_T \downarrow & & \downarrow \pi_S \\ \mathbb{N} & \hookrightarrow & \text{cone}_{\mathbb{Z}}(P)^\vee \end{array}$$

with vertical maps induced by $t_{\cdot, \cdot}, s_v \mapsto 1$ for $v \notin N$ and $s_v \mapsto 0$ for $v \in N$, is a iso-bounded extension. This means that the addition maps are surjective, π_S induces a bijection on the boundaries, and $\ker \pi_T = \ker \pi_S = 0$.

Proof. The addition map downstairs is surjective because the pair is free. The addition map upstairs is by Proposition 8.2 surjective. The restriction of π_S to the boundary maps $[c, \tilde{\eta}_{\mathbb{Z}}(c)] \mapsto [c, \eta_{\mathbb{Z}}(c)]$, which is obviously bijective.

We have that $\pi_T([0, \tilde{\eta}_{\mathbb{Z}}(c_1, c_2)]) = 0 \iff \eta_{\mathbb{Z}}(c_1, c_2) = 0$, which by Proposition 7.15 is equivalent to $\tilde{\eta}_{\mathbb{Z}}(c_1, c_2) = 0$. Since every element $\tilde{s} \in \tilde{S}$ can be written as $\tilde{s} = [c, \tilde{\eta}_{\mathbb{Z}}(c)] + \tilde{t}$ for some elements $c \in \text{recc}(P)^\vee \cap M$ and $\tilde{t} \in \tilde{T}$, we have that $\pi(\tilde{s}) = 0$ implies $c = 0$ and $\pi(\tilde{t}) = 0$, which implies $\tilde{s} = 0$ because $\ker \pi_T = 0$. \square

8.2 | The s and multiples of t are in \tilde{T}

The next result shows that the special elements s_i and some multiples of the t_{ij} are always in \tilde{T} .

Proposition 8.5. *For every $v^i \in \text{Vert}_{\neq \mathbb{Z}}(P)$ there exist some $c_1^i, c_2^i \in \text{cone}(P)^\vee \cap M$ such that $\tilde{\eta}_{\mathbb{Z}}(c_1^i, c_2^i) = s_i$, where $s_i = s_{v^i}$ is the corresponding coordinate. Furthermore, we can also find for each t_{ij} a positive integer $a_{ij} \in \mathbb{N}$ such that $a_{ij}t_{ij} \in \tilde{T}$.*

Proof. Let $v^i \in \text{Vert}_{\notin \mathbb{Z}}(P)$. Clearly, there exists a c such that $v(c) = v^i$ and $\eta(c) \notin \mathbb{Z}$. We may assume that

$$\eta(c) = z + q \text{ with } z \in \mathbb{Z}, q \in \mathbb{Q} \text{ and } 0 < q \leq \frac{1}{2} :$$

otherwise we replace c by kc with k being a positive integer such that $\eta(kc) + 1 - \lceil \eta(kc) \rceil \leq \frac{1}{2}$. This brings us to

$$\begin{aligned} \tilde{\eta}_{\mathbb{Z}}(c, c) &= \tilde{\eta}_{\mathbb{Z}}(c)s_i + \tilde{\eta}_{\mathbb{Z}}(c)s_i - \tilde{\eta}_{\mathbb{Z}}(2c)s_i \\ &= (z + 1)s_i + (z + 1)s_i - \lceil 2z + 2q \rceil s_i \\ &= s_i. \end{aligned}$$

For the second part, we look at one edge $[v^i, v^j]$. We can choose c_1, c_2 such that $v(c_1) = v^i, v(c_2) = v^j$ and furthermore such that $\langle c_2, v^j \rangle < \langle c_2, v^i \rangle$ and that $v(c_1 + c_2) = v(c_1)$. Finally, we can assume that all the brackets are integers. By Definition 7.2, we then have

$$\tilde{\eta}_{\mathbb{Z}}(c_1, c_2) = (\langle c_2, v^i \rangle - \langle c_2, v^j \rangle)t_{ij}$$

and by our assumptions the coefficient of t_{ij} is a positive integer. □

8.3 | Finite generation

A consequence of Proposition 7.15 is that lifting the Hilbert basis elements

$$[c_1, \eta_{\mathbb{Z}}(c_1)], \dots, [c_k, \eta_{\mathbb{Z}}(c_k)],$$

we obtain generators of \tilde{S} as a ‘ \tilde{T} -module’:

Corollary 8.6. *The following equality holds: $\tilde{S} = \tilde{T} + \text{span}_{\mathbb{N}}\{[c_1, \tilde{\eta}_{\mathbb{Z}}(c_1)], \dots, [c_k, \tilde{\eta}_{\mathbb{Z}}(c_k)]\}$.*

Proof. Let $c \in \text{recc}(P)^\vee \cap M$. Our goal is to prove that $[c, \tilde{\eta}_{\mathbb{Z}}(c)] \in \text{span}_{\mathbb{N}}\{[c_i, \tilde{\eta}_{\mathbb{Z}}(c_i)] : i = 1 \dots k\}$. By Remark 6.3 $[c, \eta_{\mathbb{Z}}(c)] \in \partial_{\mathbb{N}} \text{cone}_{\mathbb{Z}}(P)^\vee$, that is $[c, \eta_{\mathbb{Z}}(c)] \in \text{span}_{\mathbb{N}}\{[c_i, \eta_{\mathbb{Z}}(c_i)] : i = 1 \dots k\}$. So, there exists an $\eta_{\mathbb{Z}}$ -independent sequence consisting of c_i s which adds up to c , and we conclude by Proposition 7.15. □

Proposition 8.7. *The semigroup \tilde{T} is finitely generated. A finite set of generators is given by the minimally dependent sequences supported on c_1, \dots, c_k , yielding $\tilde{\eta}_{\mathbb{Z}}(\mathbf{m})$ for certain $\mathbf{m} \in \mathbb{N}^k$.*

Proof. We start by claiming that for any sequence c_1, \dots, c_ℓ of elements from $\text{recc}(P)^\vee \cap M$, there exists an $\mathbf{m} \in \mathbb{N}^k$ such that, with the notation introduced in 7.3, we have

$$\tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_\ell) = \tilde{\eta}_{\mathbb{Z}}(\mathbf{m}). \tag{8.1}$$

Indeed, for every c_i we have

$$[c_i, \eta_{\mathbb{Z}}(c_i)] = \sum_{j=1}^k m_{ij} [c_j, \eta_{\mathbb{Z}}(c_j)] = \left[\sum_{j=1}^k m_{ij} c_j, \sum_{j=1}^k m_{ij} \eta_{\mathbb{Z}}(c_j) \right],$$

so we can choose $\mathbf{m} = \mathbf{m}_1 + \dots + \mathbf{m}_\ell \in \mathbb{N}^k$:

$$\begin{aligned} \tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_\ell) &= \tilde{\eta}_{\mathbb{Z}}(c_1) + \dots + \tilde{\eta}_{\mathbb{Z}}(c_\ell) - \tilde{\eta}_{\mathbb{Z}}(c_1 + \dots + c_\ell) \\ &= \tilde{\eta}_{\mathbb{Z}}\left(\sum_{j=1}^k m_{1j}c_j\right) + \dots + \tilde{\eta}_{\mathbb{Z}}\left(\sum_{j=1}^k m_{\ell,j}c_j\right) - \tilde{\eta}_{\mathbb{Z}}\left(\sum_{i=1}^{\ell} \sum_{j=1}^k m_{ij}c_j\right) \\ &= \sum_{j=1}^k m_{1j}\tilde{\eta}_{\mathbb{Z}}(c_j) + \dots + \sum_{j=1}^k m_{\ell,j}\tilde{\eta}_{\mathbb{Z}}(c_j) - \tilde{\eta}_{\mathbb{Z}}\left(\sum_{i=1}^{\ell} \sum_{j=1}^k m_{ij}c_j\right) \\ &= \tilde{\eta}_{\mathbb{Z}}(\mathbf{m}), \end{aligned}$$

so (8.1) holds. Furthermore, all the \mathbf{m}_i are independent, but their sum \mathbf{m} is independent if and only if c_1, \dots, c_ℓ are independent.

Next we claim that for every sequence $c_1, \dots, c_\ell \in \text{recc}(P)^\vee \cap M$ we can even express $\tilde{\eta}_{\mathbb{Z}}(c_1, \dots, c_\ell)$ using a combination of $\tilde{\eta}_{\mathbb{Z}}(\cdot)$ with minimally dependent arguments from \mathbb{N}^k . From this second claim, we can immediately conclude. To prove this claim we use double induction: first with respect to $\eta_{\mathbb{Z}}(c_1, \dots, c_\ell) \in \mathbb{N}$, and, inside each induction step we use induction on $\text{deg}(\mathbf{m}) = \sum m_{ij}$. The key of the proof is Remark 7.8(iii) adapted to the language involving \mathbb{N}^k : if $\mathbf{m}' \leq \mathbf{m}$ component-wise, then

$$\tilde{\eta}_{\mathbb{Z}}(\mathbf{m}) = \tilde{\eta}_{\mathbb{Z}}(\mathbf{m}') + \tilde{\eta}_{\mathbb{Z}}\left(\sum_{i=1}^k m'_i c_i, c_1^{m_1 - m'_1}, \dots, c_k^{m_k - m'_k}\right). \tag{8.2}$$

The case $\eta_{\mathbb{Z}}(c_1, \dots, c_\ell) = 1$. If $\text{deg}(\mathbf{m}) = 2$ we are trivially done. Otherwise, assume \mathbf{m} is not minimally dependent and choose $\mathbf{m}' < \mathbf{m}$ which is also dependent. By (8.2), we have $\eta_{\mathbb{Z}}(\mathbf{m}) = \eta_{\mathbb{Z}}(\mathbf{m}') = 1$ so $\eta_{\mathbb{Z}}(\sum_{i=1}^k m'_i c_i, c_1^{m_1 - m'_1}, \dots, c_k^{m_k - m'_k}) = 0$. By Proposition 7.15 and (8.2), it follows that $\tilde{\eta}_{\mathbb{Z}}(\mathbf{m}) = \tilde{\eta}_{\mathbb{Z}}(\mathbf{m}')$, with $\text{deg}(\mathbf{m}') < \text{deg}(\mathbf{m})$, so we conclude by induction on $\text{deg}(\mathbf{m})$.

The inductive step follows very similarly from (8.2). □

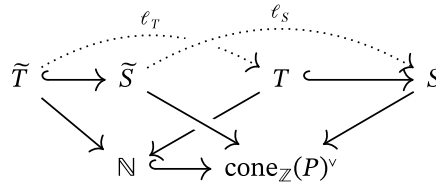
Question 8.8. Now that we know that \tilde{T} is a finitely generated semigroup, it might be interesting to ask for the polyhedral cone generated by \tilde{T} . What are its fundamental rays, and how do its facets look like? This will be answered partially in [1]. In that paper, we will use different techniques which will provide a new description of the generators of \tilde{T} as well as another proof of Proposition 8.7.

9 | THE INITIAL OBJECT PROPERTY

In Proposition 8.4, we showed that (\tilde{T}, \tilde{S}) belongs to the category of iso-bounded extensions of the pair $\mathbb{N} \hookrightarrow \text{cone}_{\mathbb{Z}}(P)^\vee$. For this, we have utilized the fact that (\tilde{T}, \tilde{S}) is ‘small enough’, that is, that the elements of the space $\mathcal{T}_{\mathbb{Z}}^*(P)$ satisfy sufficiently many relations, for example, the short edge relation, which lead us to Proposition 7.15. This was then crucial to Propositions 8.4 and 8.7. We are now going to show that (\tilde{T}, \tilde{S}) is an initial object in this category, so, in a sense, we care about the opposite: We have to show that all these relations within (\tilde{T}, \tilde{S}) (or $\mathcal{T}_{\mathbb{Z}}^*(P)$) are not arbitrarily but implicitly part of the structure of every other iso-bounded extension. This will allow us to construct a unique map from (\tilde{T}, \tilde{S}) to any other iso-bounded extension.

Notation 9.1. In contrast to Section 6, we no longer use (T, S) to denote the starting pair $(\mathbb{N}, \text{cone}_{\mathbb{Z}}(P)^{\vee})$. Instead, we assume that (T, S) is an arbitrary iso-bounded extension of $(\mathbb{N}, \text{cone}_{\mathbb{Z}}(P)^{\vee})$.

Our goal in this section is to define compatible maps $\ell_T : \tilde{T} \rightarrow T$ and $\ell_S : \tilde{S} \rightarrow S$ and prove the following theorem.



Theorem 9.2. *The pair (\tilde{T}, \tilde{S}) is an initial object in the category of iso-bounded extensions of the pair $(\mathbb{N}, \text{cone}_{\mathbb{Z}}(P)^{\vee})$. Down to earth, this means that, for any given (T, S) inducing a diagram as above, there exists a unique pair (ℓ_T, ℓ_S) of compatible semigroup homomorphisms $\ell_T : \tilde{T} \rightarrow T$ and $\ell_S : \tilde{S} \rightarrow S$.*

The proof of this theorem will start in Subsection 9.2, filling the rest of Section 9.

9.1 | Relation to algebraic geometry, Part II

We continue with our comments concerning the relations to algebraic geometry from Subsection 6.4. There, we had started with a polyhedron P and gave a description of the infinitesimal deformation of $X = \text{TV}(\sigma)$ in degree $-R$ with $\sigma := \text{cone}(P) \subset N_{\mathbb{R}} \oplus \mathbb{R}$ and $R = [0, 1]$. On the other hand, we have just looked at the semigroups $\mathbb{N} \cdot R$ and $\sigma^{\vee} \cap M$ and have studied their iso-bounded extensions. Theorem 9.2 provides a very special one — it is the pair (\tilde{T}, \tilde{S}) which we have constructed in the sections before.

Now it is tempting to expect the associated $\tilde{f} := \text{Spec } \mathbb{C}[\tilde{S}] \rightarrow \text{Spec } \mathbb{C}[\tilde{T}]$ to be a deformation of X , but it is not. While it is flat, the problem is that X does not occur as the special fiber, but equals $\tilde{f}^{-1}(\text{Spec } \mathbb{C}[\mathbb{N} \cdot R])$. Since \tilde{f} is a flat extension of the (also flat) map

$$f : X = \text{Spec } \mathbb{C}[\text{cone}_{\mathbb{Z}}(P)^{\vee}] \rightarrow \text{Spec } \mathbb{C}[\mathbb{N} \cdot R] = \mathbb{C}^1,$$

the map \tilde{f} is a deformation of $Z := f^{-1}(0)$ instead. In other words, f is a one-parameter deformation of Z , and \tilde{f} extends this family in a universal way. However, this does not yield the versal deformation of Z at all. For instance, in the situation of Example 10.2, the special fiber Z equals the zero set of the ideal

$$(z_{-1}^2, z_0, z_1, z_2) \cap (z_{-2}, z_{-1}, z_0, z_1^2),$$

that is, it is the transversal union of two double lines. In particular, $T_{\mathbb{Z}}^1$ is infinite-dimensional.

On the other hand, to produce a valid deformation of X out of the map $\text{Spec } \mathbb{C}[\tilde{S}] \rightarrow \text{Spec } \mathbb{C}[\tilde{T}]$ or, more general, out of $\text{Spec } \mathbb{C}[S] \rightarrow \text{Spec } \mathbb{C}[T]$ for any iso-bounded extension (T, S) , one has to find a flat map from $\text{Spec } \mathbb{C}[T]$ to some pointed space such that $\text{Spec } \mathbb{C}[\mathbb{N} \cdot R]$ becomes the special fiber. This, however, is not always possible. For instance, looking at Example 4.11, it is possible for

both components separately — but it fails for the initial extension (\tilde{T}, \tilde{S}) . This is what was meant with the claim that both deformation components cannot be dominated by a joint one.

9.2 | Uniqueness

The fact that both $(\pi_{\tilde{T}}, \pi_{\tilde{S}})$ and (π_T, π_S) are iso-bounded extensions implies that we have vertical isomorphisms $\pi_{\tilde{\delta}} := \pi_{\tilde{S}}|_{\partial_{\tilde{T}}(\tilde{S})}$ and $\pi_{\delta} := \pi_S|_{\partial_T(S)}$.

$$\begin{array}{ccc}
 \partial_{\tilde{T}}(\tilde{S}) & \xrightarrow[\sim]{\ell_{\delta}} & \partial_T(S) \\
 \searrow \pi_{\tilde{\delta}} & & \swarrow \pi_{\delta} \\
 & \partial_{\mathbb{N}}(\text{cone}_{\mathbb{Z}}(P)^{\vee}) &
 \end{array}$$

In particular, we are allowed and forced to set

$$\ell_{\delta} = \pi_{\delta}^{-1} \circ \pi_{\tilde{\delta}}$$

which will become the unique restriction to $\partial_{\tilde{T}}(\tilde{S})$ of any possible $\ell_S : \tilde{S} \rightarrow S$. Moreover, it follows that, like $\pi_{\tilde{\delta}}$ and π_{δ} , the map ℓ_{δ} is bijective too.

Notation 9.3. In this section, c, c_i will always denote elements from the semigroup $\text{recc}(P)^{\vee} \cap M$. For every c , we write

$$\begin{aligned}
 \ell_{\delta}(c) &:= \ell_{\delta}([c, \tilde{\eta}_{\mathbb{Z}}(c)]) = \pi_{\delta}^{-1}([c, \eta_{\mathbb{Z}}(c)]), \\
 \ell(c_1, c_2) &:= \ell_{\delta}(c_1) + \ell_{\delta}(c_2) - \ell_{\delta}(c_1 + c_2) \in T - T.
 \end{aligned}$$

So, we can also regard ℓ_{δ} as a map $\ell_{\delta} : \text{recc}(P)^{\vee} \cap M \rightarrow \partial_T(S)$.

Recall that from Remark 6.3 and as a consequence of Proposition 8.4, we have

$$\begin{aligned}
 \partial_{\mathbb{N}}(\text{cone}_{\mathbb{Z}}(P)^{\vee}) &= \{[c, \eta_{\mathbb{Z}}(c)] : c \in \text{recc}(P)^{\vee} \cap M\} \quad \text{and} \\
 \partial_{\tilde{T}}\tilde{S} &= \{[c, \tilde{\eta}_{\mathbb{Z}}(c)] : c \in \text{recc}(P)^{\vee} \cap M\},
 \end{aligned}$$

with $\pi_{\tilde{\delta}} : [c, \tilde{\eta}_{\mathbb{Z}}(c)] \mapsto [c, \eta_{\mathbb{Z}}(c)]$. So, \tilde{T} is generated by a combination of elements from the boundary:

$$[0, \tilde{\eta}_{\mathbb{Z}}(c_1, c_2)] = [c_1, \tilde{\eta}_{\mathbb{Z}}(c_1)] + [c_2, \tilde{\eta}_{\mathbb{Z}}(c_2)] - [c_1 + c_2, \tilde{\eta}_{\mathbb{Z}}(c_1 + c_2)].$$

Thus, since ℓ_S is supposed to equal the unique ℓ_{δ} on the summands on the right-hand side, it is uniquely determined too.

9.3 | Defining the maps

There is thus not much choice in defining the maps ℓ : on the boundary it has to be $\ell_{\delta} = \pi_{\delta}^{-1} \circ \pi_{\tilde{\delta}}$ and $\ell_{\tilde{T}}$ it has to satisfy

$$\begin{aligned} \ell_{\tilde{T}}([0, \tilde{\eta}_{\mathbb{Z}}(c_1, c_2)]) &= \ell_{\partial}([c_1, \tilde{\eta}_{\mathbb{Z}}(c_1)]) + \ell_{\partial}([c_2, \tilde{\eta}_{\mathbb{Z}}(c_2)]) - \ell_{\partial}([c_1 + c_2, \tilde{\eta}_{\mathbb{Z}}(c_1 + c_2)]) \\ &= \ell_{\partial}(c_1) + \ell_{\partial}(c_2) - \ell_{\partial}(c_1 + c_2) = \ell(c_1, c_2), \end{aligned}$$

where we have used Notation 9.3. We can then define

$$\ell_{\tilde{S}}(\tilde{S}) := \ell_{\partial}(\tilde{\partial}(\tilde{S})) + \ell_{\tilde{T}}(\tilde{\lambda}(\tilde{S})).$$

We first have to check that the maps land where they are supposed to. For ℓ_{∂} this holds by definition. For $\ell_{\tilde{T}}$ this follows directly from Proposition 4.8. Furthermore, by the same proposition, the map ℓ_{∂} is as linear as it may be:

Proposition 9.4. *For all c_1, c_2 we have $\ell_{\tilde{T}}([0, \tilde{\eta}_{\mathbb{Z}}(c_1, c_2)]) = \ell(c_1, c_2) \in T$. Moreover, c_1, \dots, c_r are $\eta_{\mathbb{Z}}$ -independent if and only if $\ell_{\partial}(c_1), \dots, \ell_{\partial}(c_r)$ are boundary independent.*

Remark 9.5. If $\ell_{\tilde{T}}$ is a well-defined semigroup homomorphism, then so is $\ell_{\tilde{S}}$. Being well-defined follows from the uniqueness of the decomposition $\tilde{s} = \tilde{\partial}(\tilde{S}) + \tilde{\lambda}(\tilde{S})$. The fact that $\ell_{\tilde{S}}(\tilde{S}_1 + \tilde{S}_2) = \ell_{\tilde{S}}(\tilde{S}_1) + \ell_{\tilde{S}}(\tilde{S}_2)$ is a consequence of Proposition 4.8 combined with the easy remark that for any free pair (\tilde{T}, \tilde{S}) and any $s_1, s_2 \in \tilde{S}$ we have

$$\begin{aligned} \partial(s_1 + s_2) &= \partial(\partial(s_1) + \partial(s_2)), \quad \text{and} \\ \lambda(s_1 + s_2) &= \lambda(s_1) + \lambda(s_2) + \lambda(\partial(s_1) + \partial(s_2)). \end{aligned}$$

The hard part is to show that $\ell_{\tilde{T}}$ is well-defined, that is, that it depends only on the element $\tilde{\eta}_{\mathbb{Z}}(c_1, c_2)$ but not on the individual c_1, c_2 . This will be a consequence of Lemma 9.12. So, for most of the remainder of this section we will work toward this goal. We will use the s and t coordinates introduced in Subsection 6.3 and prove that there are corresponding elements in T as well, and then show that these corresponding elements satisfy the relations from Definitions 6.9 and 6.10. The idea is to recover ℓ_T from a linear map $\mathcal{T}^*(P) \rightarrow (T - T) \otimes_{\mathbb{Z}} \mathbb{R}$. So, the elements s_v and t_{ij} are important because they generate $\mathcal{T}^*(P)$, and because the relations (such as those arising from the short edges) are formulated in terms of the elements s_v and t_{ij} . This is the rough idea of the next sections.

9.4 | Recovering the s -parameters

Recall from Definition 7.2 that the elements $\tilde{\eta}(c)$ depend linearly on c whenever the vertex $v(c)$ is not changing. That means that, fixing some vertex v of P , the map

$$\tilde{\eta}(\cdot) : \mathcal{N}(v, P) \subseteq \text{recc}(P)^\vee \rightarrow \mathcal{T}^*(P)$$

is linear on the normal cone $\mathcal{N}(v, P) \subseteq \text{recc}(P)^\vee \subseteq M_{\mathbb{R}}$; it defines some element $\tilde{\eta}_v \in N_{\mathbb{R}} \otimes \mathcal{T}^*(P)$. Let us simplify further the notation introduced in (6.2) by setting

$$\{c\} := \{\eta(c)\} = \eta_{\mathbb{Z}}(c) - \eta(c) \in [0, 1) \subset \mathbb{R}.$$

Then, Definition 7.2 turns into $\tilde{\eta}_{\mathbb{Z}}(c) = \tilde{\eta}(c) + \{c\} \cdot s_{v(c)} \in \mathcal{T}_{\mathbb{Z}}^*(P)$, and, via π , this element maps to $\eta_{\mathbb{Z}}(c) = \eta(c) + \{c\} \in \mathbb{Z}$. In Proposition 8.5, we have used elements c with $\{c\} \in [\frac{1}{2}, 1)$ to represent

$s_{v(c)} = \tilde{\eta}_{\mathbb{Z}}(c, c)$. This generalizes to the fact that

$$\tilde{\eta}_{\mathbb{Z}}(c_1, c_2) = \begin{cases} s_v & \text{if } \{c_1\} + \{c_2\} \geq 1 \\ 0 & \text{if } \{c_1\} + \{c_2\} < 1, \end{cases}$$

whenever $c_i \in \mathcal{N}(v, P) \cap M$, that is, whenever v can be chosen as $v(c_i)$ ($i = 1, 2$). Now, the first step into the direction of establishing the map ℓ is that this independence on the special choice of elements $c_i \in \mathcal{N}(v, P)$ remains true in S .

Proposition 9.6.

(i) Assume that $v \in P$ is a non-lattice vertex. Then, there is an element $\ell_s(v) \in S$ such that for all $c_1, c_2 \in \mathcal{N}(v, P) \cap M$ we have

$$\ell(c_1, c_2) = \begin{cases} \ell_s(v) & \text{if } \{c_1\} + \{c_2\} \geq 1 \\ 0 & \text{if } \{c_1\} + \{c_2\} < 1. \end{cases}$$

(ii) If $c \in \mathcal{N}(v, P) \cap M$ with $n \in \mathbb{N}$ being the smallest positive integer such that $n \cdot \{c\} \geq 1$, for example, if $\{c\} = 1/n$, then we obtain $\ell_s(v) = n \cdot \ell_\delta(c) - \ell_\delta(nc)$.

Proof. (i) *Step 1.* We check first that $\ell(c_1, c_2) = 0$ whenever $\{c_1\} + \{c_2\} < 1$. This inequality is equivalent to the equality

$$\{c_1\} + \{c_2\} = \{c_1 + c_2\},$$

that is, it yields

$$\eta_{\mathbb{Z}}(c_1) + \eta_{\mathbb{Z}}(c_2) = \eta_{\mathbb{Z}}(c_1 + c_2).$$

Hence, $\ell(c_1, c_2) = 0$ follows from Proposition 9.4. Note that the assumption of the just proven claim is trivially fulfilled if, $\{c_1\} = 0$, that is, if $\eta(c_1)$ is an integer. We will use this in the next step.

Step 2. Assume that $c, c' \in \mathcal{N}(v, P) \cap M$ such that $\{c\} = 1/n$ and that $\{c'\} = (n - k)/n$ for some not necessarily coprime natural numbers $n \in \mathbb{N}$ and $k \in \{1, \dots, n - 1\}$. Then

$$\ell(c', k \cdot c) = n \cdot \ell_\delta(c) - \ell_\delta(nc) = \ell(a \cdot c, b \cdot c) \text{ for all } a, b \in \mathbb{Z}_{\geq 1} \text{ with } a + b = n.$$

Step 1 immediately implies the second equality. To check the first one, we have to show that

$$\ell_\delta(c') + \ell_\delta(kc) - \ell_\delta(c' + kc) = n \cdot \ell_\delta(c) - \ell_\delta(nc).$$

Since Step 1 yields $\ell_\delta(kc) = k \cdot \ell_\delta(c)$, this reduces to the claim

$$\ell_\delta(c') + \ell_\delta(nc) = (n - k) \cdot \ell_\delta(c) + \ell_\delta(c' + kc).$$

However, since $\{nc\} = \{c' + kc\} = 0$, the expression ℓ_δ behaves linearly on both sides, that is, both sides are equal to $\ell_\delta(c' + nc)$.

Step 3. Assume that $\{c_1\} + \{c_2\} \geq 1$; in particular, that both summands are positive. For the present Step 3, we suppose that we have found an element $c \in \mathcal{N}(v, P) \cap M$ such that it satisfies the assumption made in Step 2 with respect to both $c' := c_1, c_2$. That is, $\{c\} = 1/n$ and

$\{c_i\} = (n - k_i)/n$ with $k_i \in \{1, \dots, n - 1\}$ for $i = 1, 2$. This leads to the equalities

$$\ell_\partial(c_i) + k_i \cdot \ell_\partial(c) - \ell_\partial(c_i + k_i \cdot c) = \ell(c_1, k_i \cdot c) = n \cdot \ell_\partial(c) - \ell_\partial(nc),$$

hence

$$\ell_\partial(c_i) = \ell_\partial(c_i + k_i \cdot c) + (n - k_i) \cdot \ell_\partial(c) - \ell_\partial(nc).$$

Alternatively, we could also take $c' := c_1 + c_2$ instead of the single c_i . Since $k_1 + k_2 < n$, we have to replace the coefficients k_i by $(k_1 + k_2)$. This leads to

$$\ell_\partial(c_1 + c_2) = \ell_\partial(c_1 + c_2 + (k_1 + k_2) \cdot c) + (n - k_1 - k_2) \cdot \ell_\partial(c) - \ell_\partial(nc).$$

Using these equations, we obtain

$$\begin{aligned} \ell(c_1, c_2) &= \ell_\partial(c_1) + \ell_\partial(c_2) - \ell_\partial(c_1 + c_2) \\ &= \ell_\partial(c_1 + k_1 \cdot c) + \ell_\partial(c_2 + k_2 \cdot c) - \ell_\partial(c_1 + c_2 + (k_1 + k_2) \cdot c) + n \cdot \ell_\partial(c) - \ell_\partial(nc). \end{aligned}$$

The arguments of the first two summands have the property that $\{\cdot\} = 0$, that is, $\eta_{\mathbb{Z}}(\cdot) = \eta(\cdot)$. In particular, since ℓ_∂ is linear in this case, their sum cancels with the third summand. Altogether this yields

$$\ell(c_1, c_2) = n \cdot \ell_\partial(c) - \ell_\partial(nc).$$

Step 4. Since v is a rational vertex of P , we know that the denominators of all $\eta(c)$ and hence that of all fractional parts $\{c\}$ with $c \in \mathcal{N}(v, P) \cap M$ are bounded. If n is the maximal denominator among them, then we can find a special $c \in \mathcal{N}(v, P) \cap M$ with $\{c\} = 1/n$. We will fix this element and set

$$\ell_s(v) := n \cdot \ell_\partial(c) - \ell_\partial(nc).$$

And now we can apply Step 3 for any given $c_1, c_2 \in \mathcal{N}(v, P) \cap M$ and our fixed c .

(ii) This is a direct consequence of the first part of the proposition and of Proposition 9.4. \square

Remark 9.7. The meaning of the elements $\ell_s(v)$ is that ℓ_T will map s_v onto $\ell_s(v)$. Hence, the existence of $\ell_s(v)$ is, on the one hand, a necessary condition for the existence of ℓ_T , but, on the other, it will also help to establish ℓ_T at all.

9.5 | Recovering the s -equations (6.6) for lattice-disjoint edges

In Definition 6.10, we had imposed the equations $s_i = s_j$ on the vector space $\mathcal{T}(P)$ for compact edges $d = [v^i, v^j]$ with $[v^i, v^j] \cap N = \emptyset$. These impose the equality $s_i = s_j \in \mathcal{T}^*(P)$. Hence, for the well-definition of the map $\ell_T : \tilde{T} \rightarrow T$, we have to check that this leads to the equality $\ell_s(v^i) = \ell_s(v^j)$ inside T too.

Recall from Definition 6.7 that $g_d \in \mathbb{Z}_{\geq 1}$ is minimal such that the affine line $\overline{g_d \cdot d}$ spanned by $g_d \cdot d$ contains lattice points. If $g_d = 1$, then we may choose some $w \in \overline{d} \cap N$, and for any integral

$$c \in \mathcal{N}(d, P) = \mathcal{N}(v^i, P) \cap \mathcal{N}(v^j, P),$$

we obtain that

$$\eta(c) = -\langle v^i, c \rangle = -\langle v^j, c \rangle = -\langle w, c \rangle \in \mathbb{Z},$$

that is, that $\{c\} = 0$. That means that those c do not qualify to determine neither $\ell_s(v^i)$, nor $\ell_s(v^j)$ via Proposition 9.6(ii). While this is bad news, the point is that the reverse implication works as well: Assume that $g_d \geq 2$. Considering the projection

$$N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}/\mathbb{R}(v^j - v^i) =: \overline{N_{\mathbb{R}}}$$

the polyhedron P maps to a polyhedron \overline{P} , and the edge d becomes a vertex \overline{d} of \overline{P} . Within the dual setup, the injection $\overline{M_{\mathbb{R}}} \hookrightarrow M_{\mathbb{R}}$ sends $\mathcal{N}(\overline{d}, \overline{P})$ isomorphically to $\mathcal{N}(d, P)$. The assumption $g_d \geq 2$ means that \overline{d} is not a lattice point in $\overline{N_{\mathbb{R}}}$. In particular, there are integral $c \in \mathcal{N}(\overline{d}, \overline{P}) \xrightarrow{\sim} \mathcal{N}(d, P)$ such that $\langle \overline{d}, c \rangle \notin \mathbb{Z}$. However, this number equals $\langle c, v^i \rangle = \langle c, v^j \rangle = -\eta(c)$. As a direct consequence, we obtain the following.

Proposition 9.8. *If $d = [v^i, v^j]$ is an edge with $g_d \geq 2$, then $\ell_s(v^i) = \ell_s(v^j)$ inside T .*

Proof. Using the element $c \in \mathcal{N}(d, P) \cap M$ with $\{c\} \neq 0$ constructed right before the proposition, we denote by $n \geq 2$ the smallest positive integer such that $n \cdot \{c\} \geq 1$. Then, Proposition 9.6(ii) implies that $\ell_s(v^i) = n \cdot \ell_{\delta}(c) - \ell_{\delta}(nc) = \ell_s(v^j)$. □

The task mentioned at the beginning of the present subsection is not fulfilled yet — it remains to show that $\ell_s(v^i) = \ell_s(v^j)$ for the lattice-disjoint edges $d = [v^i, v^j]$ with $g_d = 1$. While we have already indicated that the method of the proof of Proposition 9.8 does not work here, we are saved by the fact that, supposed that $g_d = 1$ and $v^i, v^j \notin N$, the property $d \cap N = \emptyset$ is equivalent to d being a short edge (see Definition 6.7). Thus, we can and will postpone this case until we have studied the elements $\ell(t_{ij})$ where t_{ij} is the dilation parameter.

9.6 | Recovering the t -parameters

In Subsection 9.4, we have utilized the fact that $\tilde{\eta}(\cdot)$ is linear on the normal cones $\mathcal{N}(v, P)$ for vertices $v \in P$. In the present subsection, however, we start with an edge $d = [v^1, v^2]$ leading to the normal cones

$$\mathcal{N}(d, P) = \mathcal{N}(v^1, P) \cap \mathcal{N}(v^2, P).$$

Here, we have to pay attention that the function $\tilde{\eta}(\cdot)$ is linear on each individual $\mathcal{N}(v^i, P)$, but not on their union. In particular, the function $c \mapsto \{c\}$ ceases to be linear (even mod \mathbb{Z}) when crossing the boundaries of normal cones.

Definition 9.9. We call an element $c \in \text{recc}(P)^\vee$ super integral if it belongs to M and has integral values on all (rational, but not necessarily integral) vertices of P . In particular, super integral elements c satisfy $\eta(c) \in \mathbb{Z}$, hence $\{c\} = 0$. This notion is additive, that is, the set of super integral elements form a sublattice $M_{\mathbb{Z}\mathbb{Z}} \subseteq M$.

Now, assume that $c_i \in \mathcal{N}(v^i, P)$ ($i = 1, 2$) are super integral such that $c_1 + c_2 \in \mathcal{N}(v^1, P) \cup \mathcal{N}(v^2, P)$. Note that the latter condition is automatic if $\mathcal{N}(v^1, P) \cup \mathcal{N}(v^2, P)$ is convex. Denoting

$d := v^2 - v^1$, this implies that $\langle c_1, d \rangle, \langle c_2, -d \rangle \geq 0$. In Proposition 8.5, we have related the element $\tilde{\eta}_{\mathbb{Z}}(c_1, c_2)$ to the edge parameter $t = t_{12}$. The exact statement mentioned in the proof generalizes to the fact that

$$\tilde{\eta}_{\mathbb{Z}}(c_1, c_2) = \min\{\langle c_1, d \rangle, \langle c_2, -d \rangle\} \cdot t.$$

Now, the next step toward establishing the map ℓ is that this special dependence on the choice of elements $c_i \in \mathcal{N}(v^i, P)$ remains true in T .

Proposition 9.10. *There is an element $\ell_t(d) \in \mathbb{Q}_{>0} \cdot T$ such that for all super integral $c_i \in \mathcal{N}(v^i, P)$ ($i = 1, 2$) with $c_1 + c_2 \in \mathcal{N}(v^1, P) \cup \mathcal{N}(v^2, P)$ we have $\ell(c_1, c_2) = \min\{\langle c_1, d \rangle, \langle c_2, -d \rangle\} \cdot \ell_t(d)$.*

Proof. We may assume that $\langle c_1, d \rangle \geq \langle c_2, -d \rangle (\geq 0)$. In this case, the claim turns into the equation $\ell(c_1, c_2) = \langle c_2, -d \rangle \cdot \ell_t(d)$.

Step 1: Show that $\ell(c_1, c_2)$ does indeed not depend on c_1 , provided that it does not leave the range $\mathcal{N}(v^1, P) \cap [\langle \cdot, d \rangle \geq \langle c_2, -d \rangle]$. If c'_1 is another candidate, then we obtain

$$\ell(c_1, c_2) - \ell(c'_1, c_2) = \ell_{\delta}(c_1) - \ell_{\delta}(c_1 + c_2) - \ell_{\delta}(c'_1) + \ell_{\delta}(c'_1 + c_2).$$

Hence, as our goal is $\ell(c_1, c_2) = \ell(c'_1, c_2)$, we have to show that

$$\ell_{\delta}(c_1) + \ell_{\delta}(c'_1 + c_2) = \ell_{\delta}(c'_1) + \ell_{\delta}(c_1 + c_2).$$

The inequalities $\langle c_1, d \rangle, \langle c'_1, d \rangle \geq \langle c_2, -d \rangle$ imply that both $c_1 + c_2$ and $c'_1 + c_2$ belong to $\mathcal{N}(v^1, P)$, that is, to the same normal cone which already contains c_1 and c'_1 . In particular, since c_1, c'_1 are super integral, Proposition 9.4 shows that the map ℓ_{δ} behaves linearly on both sums, that is, adding up to $\ell_{\delta}(c_1 + c'_1 + c_2)$ in both cases.

Step 2: Fix a super integral $c_1 \in \mathcal{N}(v^1, P)$ and show that $\ell(c_1, \cdot)$ is an additive function on

$$B(c_1) := \{c_2 \in \mathcal{N}(v^2, P) : c_1 + c_2 \in \mathcal{N}(v^1, P) \text{ and } \langle c_2, v^j \rangle \in \mathbb{Z} \text{ for } j = 1, 2\}.$$

Note that $B(c_1)$ is not a cone. However, if $c_2, c'_2 \in B(c_1)$ with $c_2 + c'_2 \in B(c_1)$, then we obtain

$$\begin{aligned} \ell(c_1, c_2 + c'_2) - \ell(c_1, c_2) - \ell(c_1, c'_2) &= \ell_{\delta}(c_1) + \ell_{\delta}(c_2 + c'_2) - \ell_{\delta}(c_1 + c_2 + c'_2) - \ell_{\delta}(c_1) \\ &\quad - \ell_{\delta}(c_2) + \ell_{\delta}(c_1 + c_2) - \ell_{\delta}(c_1) - \ell_{\delta}(c'_2) + \ell_{\delta}(c_1 + c'_2) \\ &= \ell_{\delta}(c_1 + c_2) + \ell_{\delta}(c_1 + c'_2) + \ell_{\delta}(c_2 + c'_2) \\ &\quad - \ell_{\delta}(c_1) - \ell_{\delta}(c_2) - \ell_{\delta}(c'_2) - \ell_{\delta}(c_1 + c_2 + c'_2). \end{aligned}$$

Since $c_2, c'_2 \in \mathcal{N}(v^2, P)$ are super integral, we know that $\ell_{\delta}(c_2) + \ell_{\delta}(c'_2) = \ell_{\delta}(c_2 + c'_2)$. This transforms the previous expression into

$$\ell(c_1, c_2 + c'_2) - \ell(c_1, c_2) - \ell(c_1, c'_2) = \ell_{\delta}(c_1 + c_2) + \ell_{\delta}(c_1 + c'_2) - \ell_{\delta}(c_1) - \ell_{\delta}(c_1 + c_2 + c'_2),$$

and the additivity claim that we want to prove for $\ell(c_1, \cdot)$ is equivalent to the equality

$$\ell_{\delta}(c_1 + c_2) + \ell_{\delta}(c_1 + c'_2) = \ell_{\delta}(c_1) + \ell_{\delta}(c_1 + c_2 + c'_2).$$

The integrality of $\langle c_1 + c_2, v^1 \rangle$ (and similarly for c'_2) implies that the left-hand side equals the compact expression $\ell_\partial(2c_1 + c_2 + c'_2)$. The same argument applies for the right-hand side, yielding the same value.

Step 3: Define the map $\psi : \{c_2 \in \mathcal{N}(v^2, P) \cap M : \langle c_2, v^i \rangle \in \mathbb{Z}, i = 1, 2\} \longrightarrow T$ as

$$\psi(c_2) := \ell(c_1, c_2)$$

under use of any $c_1 \in \mathcal{N}(v^1, P) \cap M$ with $\langle c_1, v^i \rangle \in \mathbb{Z}$ ($i = 1, 2$) and $c_1 + c_2 \in \mathcal{N}(v^1, P)$, that is, such that $c_2 \in B(c_1)$ from Step 2.

While it is obvious that those elements c_1 exist, it is a consequence of Step 1 that the definition of $\psi(c_2)$ does not depend on their choice. Moreover, for any $c_2, c'_2 \in \mathcal{N}(v^2, P) \cap M$ with $\langle c_2, v^i \rangle, \langle c'_2, v^i \rangle \in \mathbb{Z}$ ($i = 1, 2$), we can find an element c_1 such that $c_2, c'_2, c_2 + c'_2 \in B(c_1)$. Hence, it follows from Step 2, that ψ is an additive function.

By definition, it is clear that $d \leq 0$ on $\mathcal{N}(v^2, P)$, and we may restrict ψ to the d -face

$$\mathcal{N}(v^2, P) \cap (d)^\perp = \mathcal{N}(d, P) \subseteq \mathcal{N}(v^1, P).$$

That means that both arguments from $\psi(c_2) = \ell(c_1, c_2)$ become super integral elements of $\mathcal{N}(v^1, P)$, that is, they satisfy the linearity relation $\eta_{\mathbb{Z}}(c_1) + \eta_{\mathbb{Z}}(c_2) = \eta_{\mathbb{Z}}(c_1 + c_2)$. Thus, Proposition 9.4 implies that $\ell(c_1, c_2) = 0$ on $(d)^\perp$. It follows that ψ extends to a linear map

$$\mathcal{N}(v^2, P)/(d)^\perp \rightarrow \mathbb{Q}_{\geq 0} \cdot T,$$

that is, it is of the form $\psi(c_2) = \langle c_2, -d \rangle \cdot \ell_t(d)$ for some element $\ell_t(d) \in \mathbb{Q}_{\geq 0} \cdot T$. □

9.7 | Recovering the s/t -equations (6.7) for short edges

Note that the vector $d = v^2 - v^1$ spans the 1-dimensional \mathbb{Q} -vector space associated to the edge $d = [v^1, v^2]$. While the affine line spanned by d might lack lattice points (that is, $g_d \geq 2$ from Definition 6.7), the intersection $(\mathbb{Q} \cdot d) \cap N$ can be identified with \mathbb{Z} . It is dual to $M/(d)^\perp = \mathbb{Z}$. Choose a representative $c_+ \in M$ lifting 1. Note that, in the case of $g_d \geq 2$, the choice might indeed matter, cf. Subsection 9.7.3.

We fix an element $w \in \text{int } \mathcal{N}(d, P)$, meaning that $\langle w, v^1 \rangle = \langle w, v^2 \rangle$ is strictly less than the value of w on all other vertices of P . We will, additionally, assume that it is super integral $w \in M_{\mathbb{Z}\mathbb{Z}}$, meaning that it has integral values on all, even on the non-integral, vertices of P . This allows us to choose and fix an $A \gg 0$ leading to elements

$$c_1 := c_+ + A \cdot w \in \mathcal{N}(v^1, P) \quad \text{and} \quad c_2 := -c_+ + A \cdot w \in \mathcal{N}(v^2, P).$$

In particular,

$$c_0 := c_1 + c_2 = 2A \cdot w \in \mathcal{N}(d, P).$$

In contrast to w , the values of c_+ on v^1, v^2 might be non-integral. Let $n \in g_d \cdot \mathbb{Z}_{\geq 1}$ such that

$$n \cdot \langle M, v^i \rangle \in \mathbb{Z} \quad (i = 1, 2).$$

The idea is now to frequently make use of Proposition 9.4 stating that any linearity from $\eta_{\mathbb{Z}}$ transfers directly to ℓ_∂ . On the other hand, when looking for linearity instances of $\eta_{\mathbb{Z}}$, having w (or an

integral multiple) as one argument does always help: First, since w is contained in $\mathcal{N}(d, P)$, the function η acts linear into both regions $\mathcal{N}(v^1, P)$ and $\mathcal{N}(v^2, P)$. Second, the integrality assumption for w implies $\eta_{\mathbb{Z}}(w) = \eta(w)$.

9.7.1 | The first recursion formula

For any $h \in \mathbb{Z}_{\geq 0}$ we consider the differences

$$\eta_{\mathbb{Z}}(hc_1, c_1) = \eta_{\mathbb{Z}}(hc_1) + \eta_{\mathbb{Z}}(c_1) - \eta_{\mathbb{Z}}((h+1) \cdot c_1) \in \mathbb{N}.$$

Since both arguments sit in the same normal cone, that is, η behaves linear, we know $\eta_{\mathbb{Z}}(hc_1, c_1) \in \{0, 1\}$.

Case 1: $\eta_{\mathbb{Z}}(hc_1, c_1) = 0$. Then, Proposition 9.4 implies $\ell_{\delta}(hc_1) + \ell_{\delta}(c_1) = \ell_{\delta}((h+1) \cdot c_1)$.

Case 2: $\eta_{\mathbb{Z}}(hc_1, c_1) = 1$. Now, Proposition 9.6 says that

$$\ell(hc_1, c_1) = \ell_{\delta}(hc_1) + \ell_{\delta}(c_1) - \ell_{\delta}((h+1) \cdot c_1) = \ell_s(v^1).$$

Hence, we can express

$$\ell_{\delta}((h+1) \cdot c_1) = \ell_{\delta}(hc_1) + \ell_{\delta}(c_1) - \begin{cases} 0 & \text{in Case 1} \\ \ell_s(v^1) & \text{in Case 2.} \end{cases}$$

Assume that, for $h = 0, \dots, n-1$, the Cases 1 and 2 occur $(n-k)$ and k times, respectively. Then, since $\ell_{\delta}(0 \cdot c_1) = 0$, these recursion formulae add up to $\ell_{\delta}(nc_1) = n \cdot \ell_{\delta}(c_1) - k \cdot \ell_s(v^1)$. Analogously, utilizing the corresponding k_2 replacing $k_1 := k$, we obtain the same formula for $\ell_s(v^2)$. Hence,

$$n \cdot \ell_{\delta}(c_i) - \ell_{\delta}(nc_i) = k_i \cdot \ell_s(v^i) \quad (i = 1, 2).$$

Finally, we use Proposition 9.10. Note that c_1, c_2 do not meet the assumptions, but nc_1, nc_2 do. Hence,

$$\ell(nc_1, nc_2) = \ell_{\delta}(nc_1) + \ell_{\delta}(nc_2) - \ell_{\delta}(nc_0) = \langle nc_1, d \rangle \cdot \ell_t(d).$$

9.7.2 | The relation between ℓ_{δ} - and $\eta_{\mathbb{Z}}$ -equations

Recall from Subsection 9.2 that we have an additive map π_{δ} sending $\ell_{\delta}(c) \mapsto [c, \eta_{\mathbb{Z}}(c)]$. Followed by the projection to \mathbb{Z} , this becomes $\ell_{\delta}(c) \mapsto \eta_{\mathbb{Z}}(c)$. That means that all equations among the $\ell_{\delta}(c) \in S$ we obtained so far (or in the upcoming text) induce the same equations among the integers $\eta_{\mathbb{Z}}(c)$. Actually, it is the point of claims as treated in Subsection 9.7.1 to deal with the reverse direction, that is, lifting certain $\eta_{\mathbb{Z}}$ -relations to ℓ_{δ} -relations.

Nevertheless, when transferring relations, via $\text{pr}_{\mathbb{Z}} \circ \pi_{\delta}$, from the elements $\ell_{\delta}(c)$ to the integers $\eta_{\mathbb{Z}}(c)$, then by Proposition 9.6 and Proposition 9.10 we obtain that $\ell_s(v), \ell_t(d) \mapsto 1$, provided that $v \notin N$. This fits well with the facts $s_v, t_d \mapsto 1$ under π (cf. section 6.3). In particular, the equations of Subsection 9.7.1 imply that

$$k_i = n \cdot \eta_{\mathbb{Z}}(c_i) - \eta_{\mathbb{Z}}(nc_i) = n \cdot (\eta_{\mathbb{Z}}(c_i) - \eta(c_i))$$

and

$$\eta_{\mathbb{Z}}(nc_1) + \eta_{\mathbb{Z}}(nc_2) - \eta_{\mathbb{Z}}(nc_0) = \langle nc_1, d \rangle = \langle nc_2, -d \rangle.$$

Note that $k_i \geq 1$ if and only if $\eta(c_i) = -\langle c_i, v^i \rangle \notin \mathbb{Z}$, that is, exactly when $v^i \notin N$. In the case of $v^i \in N$, that is, if the parameter s_i is set to 0, then we proceed with $\ell_s(v^i)$ in the very same way.

9.7.3 | A property of short edges

Here we will show that, whenever d is a short edge, then there is a choice of c_+ (lifting $1 \in M/e^1$) such that the associated special elements c_i lead to $\eta_{\mathbb{Z}}(c_1, c_2) = 1$, that is, we obtain $\eta_{\mathbb{Z}}(c_1) + \eta_{\mathbb{Z}}(c_2) = \eta_{\mathbb{Z}}(c_0) + 1$. The special choice of c_+ does only matter for $g = g_d \geq 2$.

Fix an element $p \in 1/g \cdot N$ of the affine line \bar{d} containing the edge d . That means that $\bar{d} - p \subseteq \mathbb{Q} \cdot d$, inducing a lattice structure on \bar{d} with p becoming the origin. Now, the striking point is that we may and will choose p such that $\langle c_+, p \rangle \in \mathbb{Z}$. Note that a different choice of the lifting $c_+ \in M$ of $1 \in M/(d)^\perp$ at the beginning of the present Subsection 9.7 leads to a different p .

Now we can write $v^i = p + v_0^i$ with $v_0^i \in \mathbb{Q} \cdot d$ for $i = 1, 2$. Then, we obtain $\eta(c_i) = -\langle c_i, p + v_0^i \rangle$ and $\eta(c_1 + c_2) = -\langle c_1 + c_2, p + v_0^{1 \wedge 2} \rangle$, where $v_0^{1 \wedge 2}$ stands to $c_1 + c_2$ as v_0^i stands to c_i ; that is, its shift by p produces the minimum value for $-\langle c_1 + c_2, \cdot \rangle$. Since $\langle c_+, p \rangle \in \mathbb{Z}$, we obtain that $\langle c_i, p \rangle \in \mathbb{Z}$ as well, hence

$$\begin{aligned} \eta_{\mathbb{Z}}(c_1, c_2) &= [-\langle c_1, p + v_0^1 \rangle] + [-\langle c_2, p + v_0^2 \rangle] - [-\langle c_1 + c_2, p + v_0^{1 \wedge 2} \rangle] \\ &= [-\langle c_1, v_0^1 \rangle] + [-\langle c_2, v_0^2 \rangle] - [-\langle c_1 + c_2, v_0^{1 \wedge 2} \rangle] \\ &= [-\langle c_+, v_0^1 \rangle] + [-\langle -c_+, v_0^2 \rangle] \\ &= [\langle c_+, v_0^2 \rangle] - [\langle c_+, v_0^1 \rangle]. \end{aligned}$$

If we identify $(\mathbb{Q} \cdot d) \cap N$ with \mathbb{Z} , thus also $\mathbb{Q} \cdot d$ with \mathbb{Q} , then c_+ becomes 1 again, so

$$\eta_{\mathbb{Z}}(c_1, c_2) = \lceil v_0^2 \rceil - \lfloor v_0^1 \rfloor.$$

Hence, our claim $\eta_{\mathbb{Z}}(c_1, c_2) = 1$ is equivalent to the lack of interior lattice points in d (which we identified with $d - p$). This is clearly satisfied for short edges with $g_d = 1$, but we have to take a closer look at the case of $g_d \geq 2$.

Assume that $g_d \geq 2$. Then, the shortness still implies the lack of interior lattice points on $d - p$, provided that $p \in (1/g \cdot N) \cap \bar{d}$ is chosen as close as possible to d . Thus, it remains to check that any desirable choice of p can be realized by a suitable choice of c_+ . For this, we start by choosing coordinates $N \xrightarrow{\sim} \mathbb{Z}^d$ such that $d \cdot \mathbb{Q}$ becomes the first coordinate axis $\mathbb{Q} \times \mathbb{0}^{d-1} = \{(\bullet, \mathbf{0})\}$. The dual picture is $\mathbb{Z}^d \xrightarrow{\sim} M$ with $(d)^\perp = 0 \times \mathbb{Q}^{d-1} = \{(0, \bullet)\}$. So, the affine line \bar{d} equals $\mathbb{Q} \times \{(\frac{k_2}{g}, \dots, \frac{k_d}{g})\} = \{(\bullet, \frac{k_2}{g}, \dots, \frac{k_d}{g})\}$ with $k_2, \dots, k_d \in \mathbb{Z}$ and $\gcd(k_2, \dots, k_d, g) = 1$. Thus, if $p = (\frac{p_1}{g}, \frac{k_2}{g}, \dots, \frac{k_d}{g})$, there are coefficients $\lambda_i \in \mathbb{Z}$ such that

$$\frac{p_1}{g} \equiv \sum_{i=2}^d \lambda_i \cdot \frac{k_i}{g} \pmod{\mathbb{Z}}.$$

Then, $c_+ := (1, -\lambda_2, \dots, -\lambda_d) \in \mathbb{Z}^d = M$ is a suitable initial choice allowing to take this special point p as an origin afterward.

9.7.4 | The second recursion formula

Here we assume that d is a short edge and use Subsection 9.7.3. We will first show that

$$\eta_{\mathbb{Z}}(hc_1, c_1) = 1 - \eta_{\mathbb{Z}}((h+1)c_1, c_2)$$

for all $h \in \mathbb{Z}$ (both positive and negative). This can be seen as follows:

$$\begin{aligned} \eta_{\mathbb{Z}}(hc_1) + \eta_{\mathbb{Z}}(c_1) - \eta_{\mathbb{Z}}((h+1) \cdot c_1) &= \eta_{\mathbb{Z}}(hc_1) + (\eta_{\mathbb{Z}}(c_0) + 1 - \eta_{\mathbb{Z}}(c_2)) - \eta_{\mathbb{Z}}((h+1) \cdot c_1) \\ &= 1 + (\eta_{\mathbb{Z}}(hc_1) + \eta_{\mathbb{Z}}(c_0)) - \eta_{\mathbb{Z}}(c_2) - \eta_{\mathbb{Z}}((h+1) \cdot c_1) \\ &= 1 + \eta_{\mathbb{Z}}(hc_1 + c_0) - \eta_{\mathbb{Z}}(c_2) - \eta_{\mathbb{Z}}((h+1) \cdot c_1). \end{aligned}$$

Let us recall Case 2 from Subsection 9.7.1. We had assumed $\eta_{\mathbb{Z}}(hc_1, c_1) = 1$, occurs for $(k_1 = k)$ values of $h \in \{0, \dots, n-1\}$. Using our new relation, this implies $\eta_{\mathbb{Z}}((h+1)c_1, c_2) = 0$. Hence, Proposition 9.4 implies that $\ell_{\partial}((h+1)c_1) + \ell_{\partial}(c_2) = \ell_{\partial}(hc_1 + c_0)$. So, if $k_1 \geq 1$, that is, if Case 2 occurs, then we can express

$$\begin{aligned} \ell_s(v^1) &= \ell_{\partial}(hc_1) + \ell_{\partial}(c_1) - \ell_{\partial}((h+1) \cdot c_1) \\ &= \ell_{\partial}(hc_1) + \ell_{\partial}(c_1) + \ell_{\partial}(c_2) - \ell_{\partial}(hc_1 + c_0) \\ &= \ell_{\partial}(hc_1) + \ell_{\partial}(c_1) + \ell_{\partial}(c_2) - \ell_{\partial}(hc_1) - \ell_{\partial}(c_0) \\ &= \ell_{\partial}(c_1) + \ell_{\partial}(c_2) - \ell_{\partial}(c_0). \end{aligned}$$

In Subsection 9.7.2, we have seen that $k_i \geq 1$ if and only if $v^i \notin N$. In particular, we obtain for these cases

$$\ell_s := \ell_s(v^i) = \ell_{\partial}(c_1) + \ell_{\partial}(c_2) - \ell_{\partial}(c_0).$$

If both $v^1, v^2 \notin N$, then this already shows that $\ell_s(v^1) = \ell_s(v^2)$. Anyways, it remains to compare ℓ_s with $\ell_t(d)$. At the end of Subsection 9.7.1, we already got

$$\langle nc_1, d \rangle \cdot \ell_t(d) = \ell_{\partial}(nc_1) + \ell_{\partial}(nc_2) - \ell_{\partial}(nc_0),$$

which is in the same spirit as the formula before. Applying π as explained in Subsection 9.7.2 and $\eta_{\mathbb{Z}}(c_1, c_2) = 1$ from Subsection 9.7.3, this yields

$$\begin{aligned} \langle nc_1, e \rangle &= \eta_{\mathbb{Z}}(nc_1) + \eta_{\mathbb{Z}}(nc_2) - \eta_{\mathbb{Z}}(nc_0) \\ &= n \cdot \eta_{\mathbb{Z}}(c_1) - k_1 + n \cdot \eta_{\mathbb{Z}}(c_2) - k_2 - n \cdot \eta_{\mathbb{Z}}(c_0) \\ &= n - (k_1 + k_2). \end{aligned}$$

Adding up the two equations from Subsection 9.7.1: $n \cdot \ell_{\partial}(c_i) - \ell_{\partial}(nc_i) = k_i \cdot \ell_s(v^i)$, for $i = 1, 2$, we obtain

$$n \cdot (\ell_{\partial}(c_1) + \ell_{\partial}(c_2)) - (\ell_{\partial}(nc_1) + \ell_{\partial}(nc_2)) = (k_1 + k_2) \cdot \ell_s.$$

Note that this is even correct if one of the vertices v^i belongs to N , that is, if $k_i = 0$. Now, we replace $\ell_{\partial}(c_1) + \ell_{\partial}(c_2)$ by $\ell_{\partial}(c_0) + \ell_s$ and $\ell_{\partial}(nc_1) + \ell_{\partial}(nc_2)$ by $\ell_{\partial}(nc_0) + (n - k_1 - k_2) \cdot \ell_t(d)$. We obtain

$$n \cdot (\ell_{\partial}(c_0) + \ell_s) - (\ell_{\partial}(nc_0) + (n - k_1 - k_2) \cdot \ell_t(d)) = (k_1 + k_2) \cdot \ell_s.$$

Reordering, this yields

$$(n - k_1 - k_2) \cdot \ell_t(d) = (n - k_1 - k_2) \cdot \ell_s,$$

and it remains to check that $n - (k_1 + k_2) \neq 0$. However, since we have seen before that

$$n - (k_1 + k_2) = \langle nc_1, e \rangle = \langle nc_2, -e \rangle,$$

the vanishing of $n - (k_1 + k_2)$ would imply $v^2 - v^1 = \langle 1, e \rangle = \langle c_+, e \rangle = \langle c_1, e \rangle = 0$, which leads to a contradiction.

9.8 | Recovering the closing conditions along 2-faces

In Subsection 9.6, we have looked at adjacent vertices $v, v' \in P$. If their oriented connecting edge is $d = v' - v$, then we may choose sufficiently integral $c \in \mathcal{N}(v, P)$, $c' \in \mathcal{N}(v', P)$ in M with $\langle c + c', d \rangle = 0$, that is, with $\langle c, d \rangle = -\langle c', d \rangle > 0$, and $c + c' \in \mathcal{N}(d, P) = \mathcal{N}(v, P) \cap \mathcal{N}(v', P)$ leading to $\ell(c, c') = \langle c, d \rangle \cdot \ell_t(d)$ by Proposition 9.10. For the whole subsection we could keep the assumption of being ‘sufficiently integral’ for all relevant elements from M — namely, we could entirely work within the super integral sublattice $M_{\mathbb{Z}\mathbb{Z}} \subseteq M$. Instead, we replace $\ell_\delta(c)$ by the following stabilized version:

$$\ell^{\text{st}}(c) := \frac{1}{A} \cdot \ell_\delta(A \cdot c) \quad \text{for } A \in \mathbb{N} \text{ with } A \gg 0.$$

In accordance to this, we replace $\ell(c, c') = \ell_\delta(c) + \ell_\delta(c') - \ell_\delta(c + c')$ by the stabilized version too: $\ell^{\text{st}}(c, c') := \ell^{\text{st}}(c) + \ell^{\text{st}}(c') - \ell^{\text{st}}(c + c')$. It extends the validity of the above formula for $\ell_t(d)$ to non-integral arguments.

Now we consider a compact 2-dimensional face $F \leq P$. Assume that its vertices and oriented edges are $v^i \in N_{\mathbb{R}}$ and $d_i = v^{i+1} - v^i$ ($i \in \mathbb{Z}/n\mathbb{Z}$), respectively. Then, the cones $\mathcal{N}(F, P) \subseteq \mathcal{N}(d_i, P) \subseteq \mathcal{N}(v^i, P)$ are part of the inner normal fan $\mathcal{N}(P)$. Projecting them down to the 2-dimensional vector space $M_{\mathbb{R}}/F^\perp =: F^*$ (dual to the vector space $F - F$ accompanying the affine space spanned by F) yields a 2-dimensional complete fan $\overline{\mathcal{N}}_F$ within F^* . We denote the image cones by

$$0 = \overline{\mathcal{N}}(F, P) \subseteq \overline{\mathcal{N}}(d_i, P) \subseteq \overline{\mathcal{N}}(v^i, P) \subseteq F^*.$$

The cones $\overline{\mathcal{N}}(d_i, P)$ form the rays, and their linear hull is $(d_i)^\perp/F^\perp$. The 2-dimensional cones $\overline{\mathcal{N}}(v^i, P)$ are spanned by the rays $\overline{\mathcal{N}}(d^{i-1}, P)$ and $\overline{\mathcal{N}}(d_i, P)$ (Figure 11).

Proposition 9.11. *Corresponding to (6.4), in $T \otimes_{\mathbb{Z}} N_{\mathbb{R}}$ we have the equation $\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \ell_t(d_i) \otimes d_i = 0$.*

Proof. We choose elements $a_i \in \text{int } \mathcal{N}(d_i, P)$ mapping to points \bar{a}_i on the rays $\overline{\mathcal{N}}(d_i, P)$. Since $d_i \subseteq F - F$, we know that $\langle F^\perp, d_i \rangle = 0$, that is, we may write $\langle a_j, d_i \rangle = \langle \bar{a}_j, d_i \rangle$. This yields

$$\langle \bar{a}_{i-1}, d_i \rangle > 0, \quad \langle \bar{a}_i, d_i \rangle = 0, \quad \langle \bar{a}_{i+1}, d_i \rangle < 0.$$

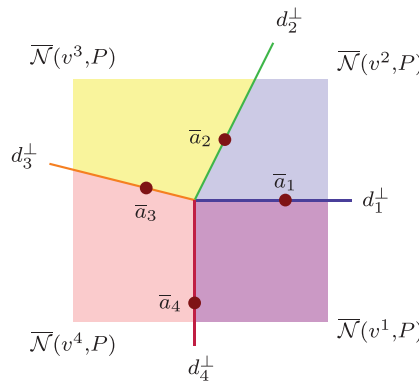


FIGURE 11 The normal fan of a 2-face with four edges

For a fixed $i \in \mathbb{Z}/n\mathbb{Z}$ and some $A \gg 0$, we will use

$$c := \frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \cdot a_{i-1} + A \cdot a_i \quad \text{and} \quad c' := \frac{-1}{\langle \bar{a}_{i+1}, d_i \rangle} \cdot a_{i+1} + A \cdot a_i$$

for which we have $\langle c', d_i \rangle = -\langle c, d_i \rangle$. Hence, we obtain that $c + c' \in d_i^\perp$ and, if A is large enough, even $c + c' \in \mathcal{N}(d_i, P)$. Thus, with $v := v^i, v' := v^{i+1}$, and $d = d_i$ we are exactly in the situation of the begin of this subsection. That is,

$$\ell_t(d_i) = \ell^{\text{st}}(c) + \ell^{\text{st}}(c') - \ell^{\text{st}}(c + c').$$

This equation remains valid if we alter $\ell^{\text{st}}(c)$ by a function that is linear in c . Thus, we may and will assume that ℓ^{st} vanishes on $\mathcal{N}(F, P)$. This implies that ℓ^{st} descends to a well-defined function $\ell^{\text{st}} : F^* \rightarrow (S - S) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is linear on the cones of $\overline{\mathcal{N}}_F$, and we still have

$$\begin{aligned} \ell_t(d_i) &= \ell^{\text{st}}(\bar{c}) + \ell^{\text{st}}(\bar{c}') - \ell^{\text{st}}(\bar{c} + \bar{c}') \\ &= \ell^{\text{st}}\left(\frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \cdot \bar{a}_{i-1} + A \cdot \bar{a}_i\right) + \ell^{\text{st}}\left(\frac{-1}{\langle \bar{a}_{i+1}, d_i \rangle} \cdot \bar{a}_{i+1} + A \cdot \bar{a}_i\right) \\ &\quad - \ell^{\text{st}}\left(\frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \cdot \bar{a}_{i-1} - \frac{1}{\langle \bar{a}_{i+1}, d_i \rangle} \cdot \bar{a}_{i+1} + 2A \cdot \bar{a}_i\right) \\ &= \frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \ell^{\text{st}}(\bar{a}_{i-1}) + A \ell^{\text{st}}(\bar{a}_i) - \frac{1}{\langle \bar{a}_{i+1}, d_i \rangle} \ell^{\text{st}}(\bar{a}_{i+1}) + A \ell^{\text{st}}(\bar{a}_i) - (\beta_i + 2A) \ell^{\text{st}}(\bar{a}_i) \\ &= \frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \ell^{\text{st}}(\bar{a}_{i-1}) - \frac{1}{\langle \bar{a}_{i+1}, d_i \rangle} \ell^{\text{st}}(\bar{a}_{i+1}) - \beta_i \ell^{\text{st}}(\bar{a}_i), \end{aligned}$$

where $\beta_i \in \mathbb{R}$ is defined by the equality

$$\frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \cdot \bar{a}_{i-1} - \frac{1}{\langle \bar{a}_{i+1}, d_i \rangle} \cdot \bar{a}_{i+1} - \beta_i \cdot \bar{a}_i = 0.$$

Now, we consider

$$\begin{aligned} \sum_i \ell_i(d_i) \otimes d_i &= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \left(\frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \ell^{\text{st}}(\bar{a}_{i-1}) - \frac{1}{\langle \bar{a}_{i+1}, d_i \rangle} \ell^{\text{st}}(\bar{a}_{i+1}) - \beta_i \ell^{\text{st}}(\bar{a}_i) \right) \otimes d_i \\ &= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \ell^{\text{st}}(\bar{a}_i) \otimes \left(\frac{1}{\langle \bar{a}_i, d_{i+1} \rangle} d_{i+1} - \frac{1}{\langle \bar{a}_i, d_{i-1} \rangle} d_{i-1} - \beta_i d_i \right) \end{aligned}$$

and check that all of the second factors vanish. This will be done in the following quick and dirty way via choosing coordinates, that is, fixing some isomorphism $(F - F) \xrightarrow{\sim} \mathbb{R}^2$ which determines a dual isomorphism $\mathbb{R}^2 \xrightarrow{\sim} F^*$ too. While the vectors $d_i \in (F - F) = \mathbb{R}^2$ are given by the choice of $F \leq P$, we have some freedom in choosing the $\bar{a}_i \in F^* = \mathbb{R}^2$ — one has just to ensure that $\bar{a}_i \perp d_i$ and that they have the right orientation. This can be obtained by

$$\bar{a}_i := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot d_i.$$

Doing so, the \bar{a}_i satisfy the same linear relations as the d_i do, that is, we obtain

$$\frac{1}{\langle \bar{a}_{i-1}, d_i \rangle} \cdot d_{i-1} - \frac{1}{\langle \bar{a}_{i+1}, d_i \rangle} \cdot d_{i+1} - \beta_i \cdot d_i = 0.$$

Thus, the claim follows from the equalities $\langle \bar{a}_i, d_{i-1} \rangle = -\langle \bar{a}_{i-1}, d_i \rangle$ for all indices $i \in \mathbb{Z}/n\mathbb{Z}$ and our special choices of $\bar{a}_i \in \mathbb{R}^2$. □

9.9 | Concluding the proof of the existence of ℓ

One might think that we are already done with the construction of the map ℓ — but it requires the following, seemingly paranoid conclusion of the proof. What do we have so far? First, we have well-defined elements $\ell_\beta(c) \in \partial_T(S)$ which have to become the images of $[c, \tilde{\eta}_{\mathbb{Z}}(c)] \in \partial_{\tilde{T}}(\tilde{S})$. Second, we have constructed the following elements.

- (i) If $v \in P$ is a vertex, then there is a well-defined $\ell_s(v) \in T$ planned to become the image $\ell(s_v)$, cf. Proposition 9.6 in Subsection 9.4.
- (ii) For each compact edge $d \leq P$ there is a well-defined $\ell_t(d) \in \mathbb{Q}_{>0} \cdot T$ planned to become the image $\ell(t_d)$, cf. Proposition 9.10 in Subsection 9.6.

In the Subsections 9.5, 9.7, and 9.8, we have shown that the new elements $\ell_s(v)$ and $\ell_t(d)$ satisfy the same linear relations as the original elements s_v and t_d . This gives rise to a well-defined linear map

$$\varphi : \mathcal{T}^*(P) = (\tilde{T} - \tilde{T}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow (T - T) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with $\varphi(s_v) = \ell_s(v)$ and $\varphi(t_d) = \ell_t(d)$.

Lemma 9.12. *For $c_1, c_2 \in \text{recc}(P)^\vee \cap M$, we have $\varphi(\tilde{\eta}_{\mathbb{Z}}(c_1, c_2)) = \ell(c_1, c_2)$.*

Proof. Step 1. First, by Proposition 9.6 and the preceding remarks in Subsection 9.4, the claim of the lemma follows for those pairs (c_1, c_2) where c_1, c_2 are contained in a common normal cone $\mathcal{N}(v, P)$ for some vertex $v \in P$. Second, by Proposition 9.10 and the preceding remarks in Subsection 9.6 the claim of the lemma does also follow for super integral $c_1, c_2 \in M_{\mathbb{Z}\mathbb{Z}}$ being contained in two adjacent normal cones $\mathcal{N}(v^1, P)$ and $\mathcal{N}(v^2, P)$, respectively. (That is, v^1 and v^2 have to be connected by an edge, and one has to suppose that $c_1 + c_2$ belongs to the union of these normal cones).

Step 2. If $c \in M$ and $n \in \mathbb{N}$, then we know from Subsection 9.4 that $n \cdot \tilde{\eta}_{\mathbb{Z}}(c) - \tilde{\eta}_{\mathbb{Z}}(nc) = (n\eta_{\mathbb{Z}}(c) - \eta_{\mathbb{Z}}(nc)) \cdot s_{v(c)}$ and $n \cdot \ell_{\partial}(c) - \ell_{\partial}(nc) = (n\eta_{\mathbb{Z}}(c) - \eta_{\mathbb{Z}}(nc)) \cdot \ell_s(v(c))$. Hence, for $c_1, c_2 \in M$ we obtain that $n \cdot \tilde{\eta}_{\mathbb{Z}}(c_1, c_2) - \tilde{\eta}_{\mathbb{Z}}(nc_1, nc_2)$ maps, via φ , to $n \cdot \ell(c_1, c_2) - \ell(nc_1, nc_2)$. Consequently, the fact that $\varphi(\tilde{\eta}_{\mathbb{Z}}(c_1, c_2)) = \ell(c_1, c_2)$ is equivalent to $\varphi(\tilde{\eta}_{\mathbb{Z}}(nc_1, nc_2)) = \ell(nc_1, nc_2)$. This means that it remains to show the claim for super integral (but not necessarily from adjacent cones) $c_1, c_2 \in M_{\mathbb{Z}\mathbb{Z}}$.

Step 3. We are going to use the inhomogeneous description of group cohomology, cf. [16, VII.3]. Both $\tilde{\eta}_{\mathbb{Z}}(\cdot, \cdot)$ and $\ell(\cdot, \cdot)$ are 2-coboundaries. Hence, the map $b : M_{\mathbb{Z}\mathbb{Z}} \times M_{\mathbb{Z}\mathbb{Z}} \rightarrow (T - T) \cdot \mathbb{Q}$ defined as

$$b(\cdot, \cdot) := \varphi(\tilde{\eta}_{\mathbb{Z}}(\cdot, \cdot)) - \ell(\cdot, \cdot)$$

is still a 2-cocycle for $H^2(M_{\mathbb{Z}\mathbb{Z}}, (T - T) \cdot \mathbb{Q})$. Since $(T - T) \cdot \mathbb{Q}$ is a divisible group, hence an injective \mathbb{Z} -module, we know that $H^2(M_{\mathbb{Z}\mathbb{Z}}, (T - T) \cdot \mathbb{Q}) = 0$. Thus, b is a 2-coboundary, that is, there is a map

$$b_{\partial} : M_{\mathbb{Z}\mathbb{Z}} \rightarrow (T - T) \cdot \mathbb{Q} \quad \text{with} \quad b(c_1, c_2) = b_{\partial}(c_1) + b_{\partial}(c_2) - b_{\partial}(c_1 + c_2).$$

From Step 1 we know that b_{∂} is linear on the full-dimensional normal cones $\mathcal{N}(v, P)$ or even on the union of adjacent ones $\mathcal{N}(v^1, P)$ and $\mathcal{N}(v^2, P)$ – provided that $c_1 + c_2$ belongs to this union. But this means that b_{∂} is globally linear, that is, $b = 0$. □

So, $\ell_T : \tilde{T} \rightarrow T$ is a well-defined linear map, and we conclude the existence part of Theorem 9.2 by Remark 9.5.

10 | MINKOWSKI DECOMPOSITIONS REVISITED

10.1 | Review of the case of lattice polytopes with primitive edges

In [4], we had treated a special case of the scenario described in Subsection 6.1. There it was assumed that P is a lattice polytope with primitive edges, that is, the edges did not contain any lattice points other than the vertices. In algebro-geometric terms, this means that $X = \mathbb{T}\mathbb{V}(\text{cone}(P)) = \text{Spec } \mathbb{C}[S]$ is Gorenstein and it is smooth in codimension two. Let us summarize the main results from [4] for this special case.

- (i) If $P = P_0 + \dots + P_k$ is Minkowski decomposition, then this corresponds to a decomposition $\underline{1} = \xi_0 + \dots + \xi_k$ within the cone $C(P)$. The summands P_ν are lattice polytopes if and only if the corresponding ξ_ν belong to the lattice $C_{\mathbb{Z}}^{\text{lin}}(P)$ within the vector space $C^{\text{lin}}(P) := C(P) - C(P)$. This lattice is defined by the integrality of all coordinates $t_{ij}(\xi)$.

- (ii) Since the coordinates t_{ij} are supposed to be non-negative on $C(P)$, they become elements of the dual cone $t_{ij} \in C(P)^\vee$. The sub-semigroup generated by these elements provides the base of the initial object from Theorem 9.2. That is, the present special case of a discrete setup shows tight parallels to the cone setup displayed in Proposition 5.8.
- (iii) Translated to the framework of algebraic geometry, lattice decompositions of P as in (i) correspond to components of the versal deformation of $X = \mathbb{T}\mathbb{V}(\text{cone}(P))$ in degree $-R$. The complexification of the vector space $C^{\text{lin}}(P)/\mathbb{1} \cdot \mathbb{R}$ equals the space of infinitesimal deformations $T_X^1(-R)$ of X in degree $-R$. Finally, the sub-semigroup $\text{span}_{\mathbb{N}}\{t_{ij}\} \subseteq C(P)^\vee$ of (ii) encodes the versal deformation itself, which is a much finer information than just its linear ambient space $T_X^1(-R)$.

Remark 10.1. In the Gorenstein case, that is, when P was a lattice polytope, then the smoothness of X in codimension two could be easily expressed by the primitivity of its lattice edges. In [3], we already got rid of the Gorenstein assumption, but we heavily depended on the assumption of smoothness in codimension two. In the non-Gorenstein case, this condition can still be expressed in the combinatorial language. It says that, for each bounded edge $[v^i, v^j]$ of $P \subseteq N_{\mathbb{R}}$, the polyhedral cone generated from $(v^i, 1), (v^j, 1) \in N_{\mathbb{R}} \oplus \mathbb{R}_{\geq 0}$ is \mathbb{Z} -linearly (!) isomorphic to the ordinary upper orthant $\mathbb{R}_{\geq 0}^2$.

Now, returning to the general discrete setup established in Subsection (6.1) and taking the points (i)–(iii) above as a guideline, we no longer assume that P is bounded, that is, P may have a non-trivial recession cone. More important, however, is that we do not require P to be a lattice polyhedron anymore, nor do we ask for any further restrictions (on the edges or anything else).

Example 10.2. Let us return to Example 4.11. In this case, we have $C^{\text{lin}}(P) = \mathbb{R}$, which shows that (10.1) (iii) is no longer valid in this case. Moreover, neither the lattice $C_{\mathbb{Z}}^{\text{lin}}(P)$, nor lattice decompositions $P = P_0 + \dots + P_k$ from Subsection 10.1 (i) make any sense here. So, these notions need to be replaced: $C_{\mathbb{Z}}^{\text{lin}}(P)$ by $\mathcal{T}_{+\mathbb{Z}}$ and lattice decompositions by lattice friendly decompositions (cf. Subsection 10.3).

10.2 | The universal Minkowski summand

We start by fixing a *reference vertex* $v_* \in \text{Vert}(P)$, and recall from Subsection 6.3 that

$$\mathcal{T}_+(P) := \mathcal{T}(P) \cap (\mathbb{R}_{\geq 0}^r \oplus \mathbb{R}_{\geq 0}^m).$$

For $\xi = (t, s) \in \mathcal{T}_+(P)$, we construct P_ξ by defining a map $\psi_{v_*}(\xi) : \text{Vert}(P) \rightarrow N_{\mathbb{R}}$ as. For the reference vertex, we set

$$\psi_{v_*}(\xi, v_*) := s_{v_*}(\xi) \cdot v_*.$$

Note that this definition implies that $\psi_{v_*}(\xi, v_*) = 0$ if $v_* \in N$. For every other $v \in \text{Vert}(P)$, choose a path $v_* = v^0, v^1, \dots, v^k = v$ along the compact edges of P . Denoting $d_i := v^i - v^{i-1}$, we define

$$\psi_{v_*}(\xi, v) = \psi_{v_*}(\xi, v_*) + \sum_{i=1}^k t_i(\xi) \cdot d_i.$$

It is a direct consequence of the closing conditions in the definition of $C^{\text{lin}}(P)$ and hence of $\mathcal{T}(P)$ in Subsection 6.3 that $\psi_{v_*}(\xi, v)$ does not depend on the special choice of the path, cf. (10.2.1). Now, we obtain the Minkowski summand associated to $\xi \in \mathcal{T}_+(P)$ as

$$P_\xi := \psi_{v_*}(\xi, P) := \text{conv}\{\psi_{v_*}(\xi, v) : v \in \text{Vert}(P)\} + \text{recc}(P).$$

Note that one can avoid the usage of the s -variables when there exists a lattice vertex in P to be chosen as v_* . However, even then the s -coordinates will play an important role in (10.2.2). Similarly to Subsection 5.4, we proceed with the following.

Definition 10.3. We define the *universal Minkowski summand* or the *tautological cone* as

$$\tilde{\mathcal{T}}_+^{v_*}(P) := \{(\xi, w) : \xi \in \mathcal{T}_+(P), w \in \psi_{v_*}(\xi, P)\} \subseteq \mathcal{T}_+(P) \times N_{\mathbb{R}}.$$

It comes with the natural projection $p_+ : \tilde{\mathcal{T}}_+^{v_*}(P) \rightarrow \mathcal{T}_+(P)$ onto the first factor.

Now we check that, up to consistent lattice translations in N , the previous definitions are independent of all choices, that is, that we may indeed call our Minkowski summands P_ξ and denote the tautological cone by $\tilde{\mathcal{T}}_+(P)$. In more detail, if $\xi \in \mathcal{T}_+(P) \cap \mathcal{T}_{\mathbb{Z}}(P)$, then the following (10.2.1) and (10.2.2) will imply that, for all v_*, v'_* , the polyhedron $\psi_{v_*}(\xi, P)$ is obtained from $\psi_{v'_*}(\xi, P)$ via a lattice isomorphism linearly depending on ξ .

10.2.1 | Independence on the path along the edges

This is a direct consequence of the closure conditions along the compact 2-faces $F \leq P$ which define $C^{\text{lin}}(P)$ or $\mathcal{T}(P)$, cf. Definition 6.10. Here, we are literally in the same situation as in [4].

10.2.2 | Independence on the reference vertex

Assume that v_* and v'_* are two different vertices of P which are connected by an (oriented) edge $d = v'_* - v_*$. Recall from Definition 6.13 that this situation gives rise to an element $L_d \in \mathcal{T}_{\mathbb{Z}}(P)^* \otimes_{\mathbb{Z}} N$. Again, we have to compare the Minkowski summands with respect to the same vertex, say v_* :

$$\psi_{v'_*}(\xi, v_*) - \psi_{v_*}(\xi, v_*) = s_{v'_*}(\xi) \cdot v'_* - t_d(\xi) \cdot (v'_* - v_*) - s_{v_*}(\xi) \cdot v_*$$

hence,

$$\begin{aligned} \psi_{v'_*}(v_*) - \psi_{v_*}(v_*) &= s_{v'_*} \otimes v'_* - t_d \otimes (v'_* - v_*) - s_{v_*} \otimes v_* \\ &= (t_d - s_{v_*}) \otimes v_* - (t_d - s_{v'_*}) \otimes v'_* = L_d \in \mathcal{T}_{\mathbb{Z}}^*(P) \otimes_{\mathbb{Z}} N. \end{aligned}$$

That is the two tautological cones differ via translation by an integral, linear section of p .

Convention 10.4. Unless P has at least one lattice vertex, we cannot assume that the reference vertex v_* is 0. Nevertheless, we will write $\tilde{\mathcal{T}}_+(P)$ for $\tilde{\mathcal{T}}_+^{v_*}(P)$ and keep in mind the dependence on v_* via[†] the shift by the p -section $\psi_{v_*'} - \psi_{v_*} = L_{v_* v_*'}$ and via $\tilde{\mathcal{T}}_+^{v_*'0}(P) = \tilde{\mathcal{T}}_+^{v_*0}(P) + L_{v_* v_*'}$.

Theorem 10.5. *The universal Minkowski summand $\tilde{\mathcal{T}}_+(P)$ is a convex, polyhedral cone.*

Proof. This follows because $\xi(v_*) := \psi(\xi, v_*)$ and hence $v_\xi = \xi(v) := \psi(\xi, v)$ depend, for every vertex $v \in P$, linearly on ξ . □

10.3 | Lattice-friendly Minkowski decompositions

In the situation of Subsection 10.1(i), each decomposition of a lattice polytope P into a sum of lattice polytopes $P = P_0 + \dots + P_k$ was encoding a component of the versal deformation. Independently on this interpretation, the lattice condition for the summands P_i was a discrete requirement reducing the number of admissible decompositions drastically; in particular, it becomes finite. In the general setup, however, that is, when P is no longer a lattice polytope, then lattice decompositions cannot exist at all. Inspired by [5, (3.2)], we nevertheless save this concept by defining the following weaker version.

Definition 10.6. A Minkowski decomposition $P = P_0 + \dots + P_k$ is called *lattice friendly* if all summands share the same recession cone, and if, for every $c \in \text{recc}(P)^\vee \cap M$, there is an index $\mu = \mu(c)$ such that all $\text{face}(P_i, c) \leq P_i$ with $i \in \{0, \dots, k\} \setminus \{\mu\}$ contain lattice points.

Recall that $\text{face}(P_i, c) := \{a \in P_i : \langle c, a \rangle = \min \langle c, P_i \rangle\}$ is the face of P_i where c attains its minimum. It suffices to check the condition of the previous definition for generic $c \in \text{recc}(P)^\vee \cap M$, that is, for those where $\text{face}(P, c)$ is a vertex of P . Since

$$\text{face}(P, c) = \text{face}(P_0, c) + \dots + \text{face}(P_k, c),$$

this implies that $\text{face}(P_i, c) \leq P_i$ are vertices, too. Hence, in this generic case, the above definition asks for

$$\text{face}(P_0, c), \dots, \text{face}(P_k, c) \in N$$

to be lattice vertices — with at most one exception, namely, for $\text{face}(P_\mu, c)$. This means that

- (i) any failure $\text{face}(P, c) \notin N$ stems from one single summand $\text{face}(P_\mu, c) \notin N$ where μ depends on the choice of the generic c , that is, on the choice of the vertex $\text{face}(P, c)$, and
- (ii) if $\text{face}(P, c) \in N$, then all summands $\text{face}(P_i, c)$ are lattice vertices, without any exception.

In particular, if P were a lattice polyhedron as in Subsection 10.1(i), then being lattice friendly just means being a lattice decomposition, that is, all summands P_i must be lattice polyhedra with $\text{recc}(P_i) = \text{recc}(P)$.

[†] This is not true literally. It maps $(\xi, w) \mapsto (\xi, w + L_{v_* v_*'})$.

10.4 | The Kodaira–Spencer map

Assume that $P = P_0 + \dots + P_k$ is any Minkowski decomposition with $\text{recc}(P_i) = \text{recc}(P)$. For each vertex $w = \text{face}(P, c)$ of P we will denote the corresponding vertex $\text{face}(P_i, c)$ by $w(P_i) = w_i \in P_i$. Note that it depends on w alone, that is, not on the special choice of $c \in \text{recc}(P)^\vee$. Actually, the associated normal cones, that is, the regions of those c providing the desired vertex, satisfy $\mathcal{N}(w, P) \subseteq \mathcal{N}(w_i, P_i)$ and

$$\mathcal{N}(w, P) = \mathcal{N}(w_0, P_0) \cap \dots \cap \mathcal{N}(w_k, P_k).$$

In accordance with Notation 6.5, we write $\text{Vert}(P) = \{v^1, \dots, v^m\}$ and $\text{Edge}^c(P) := \{d_1, \dots, d_r\}$, which gives rise to the \mathbb{R} -vector space $\mathbb{R}^r \oplus \mathbb{R}^m$ with coordinates (\mathbf{t}, \mathbf{s}) . We will define an evaluation $\rho : \{0, \dots, k\} \rightarrow \mathbb{R}_{\geq 0}^r \oplus \mathbb{R}^m$ of the Minkowski summands.

Definition 10.7. Let Q with $\text{recc}(Q) = \text{recc}(P)$ be a Minkowski summand of P . The *Kodaira–Spencer* evaluation $\rho(Q) = (t(Q), s(Q)) \in \mathbb{R}_{\geq 0}^r \oplus \mathbb{R}^m$ is defined by

$$t_d(Q) := (\text{the dilation factor of the edge } d \text{ inside } Q) \in [0, 1] \subset \mathbb{R}$$

$$s_v(Q) := \begin{cases} 0 & \text{if } v(Q) \in N \\ 1 & \text{if } v(Q) \notin N \end{cases} \text{ for any vertex } v \in P.$$

Recall from Subsection 5.4 that the dilation factor means the non-negative scalar transforming an edge of P into the associated edge of Q , that is, satisfying $v^j(Q) - v^i(Q) = t_{ij}(Q) \cdot (v^j - v^i)$ for vertices $v^i, v^j \in P$. Note that the values collected in $s(Q) \in \mathbb{R}^m$ do heavily depend on the position of Q , that is, in general, they do change after shifting Q along a vector from $N_{\mathbb{R}} \setminus N$. In particular, the Kodaira–Spencer map ρ is, in general, neither Minkowski-additive, nor is its image contained in the subspace $\mathcal{T}(P) \subseteq \mathbb{R}^r \oplus \mathbb{R}^m$ from Definition 6.10. Nevertheless, we have $\rho(0) = \underline{0}$ and $\rho(P) = (\underline{1}; \underline{1}, \underline{0}) = [P] \in \mathcal{T}(P)$ where the latter still denotes the distinguished element defined by $s_i := 0$ for $v^i \in N$ and $s_j := 1$ and $t_{ij} := 1$ for all remaining coordinates.

Example 10.8. Take $P = [\frac{1}{2}, \frac{3}{4}]$ from Example 6.12.3. and decompose it as

$$P_0 + P_1 = \left[0, \frac{1}{4}\right] + \left[\frac{1}{2}, \frac{1}{2}\right].$$

Using the coordinates $(t; s_1, s_2)$ of \mathbb{R}^3 , the Kodaira–Spencer map yields

$$\rho(P) = (1; 1, 1), \quad \rho(P_0) = (1; 0, 1), \quad \text{and} \quad \rho(P_1) = (0; 1, 1).$$

While this is clearly not additive, both summands $\rho(P_i)$ do also miss $\mathcal{T}(P)$: Since both half open edges induced from P are short, the equations for $\rho(P_i)$ involve $s_1 = t = s_2$, which is not satisfied.

10.5 | The Kodaira–Spencer map for lattice-friendly decompositions

While the Kodaira–Spencer map $\rho = (\mathbf{t}, \mathbf{s})$ behaves rather wildly for general Minkowski decompositions, it turns out to be the right tool to reflect lattice-friendly decompositions.

Theorem 10.9. *Let $P = P_0 + \dots + P_k$ be a Minkowski decomposition with $\text{recc}(P_i) = \text{recc}(P)$. Then, this decomposition is lattice friendly if and only if*

$$\rho(P) = (\underline{1}; \underline{1}, \underline{0}) = \rho(P_0) + \dots + \rho(P_k),$$

and this is a decomposition inside $\mathcal{T}_{\mathbb{Z}}(P)$, that is, for all summands we have $\rho(P_i) \in \mathcal{T}_{\mathbb{Z}}(P) \subset \mathbb{R}^r \oplus \mathbb{R}^m$.

Proof. (\Leftarrow) For each vertex $w \in P$, we obtain a decomposition $s_w(P) = s_w(P_0) + \dots + s_w(P_k)$ inside \mathbb{N} . Since $s_w(P) \in \{0, 1\}$, this means that there is at most one index $\mu = \mu(w)$ such that $s_w(P_\mu) = 1$. All remaining summands vanish, and this translates directly into the condition of Definition 10.6.

(\Rightarrow) Assume that the decomposition $P = P_0 + \dots + P_k$ is lattice friendly.

Step 1. Since there is never a problem with the dilation factors, let us focus on the s -parameters. If $w \in P$ is a vertex, then there is at most one index $\mu = \mu(w)$ such that $s_w(P_\mu) \neq 0$. Moreover, we know that

$$s_w(P_\mu) = 1 \iff s_w(P_\mu) \neq 0 \implies w \notin N \iff s_w(P) \neq 0 \iff s_w(P) = 1.$$

This shows the formula $\rho(P) = \sum_{i=0}^k \rho(P_i)$. Moreover, the integrality of the $s_w(P_i)$ is clear, too.

Step 2. Assume that $[v, w] \leq P$ is a compact edge with $[v, w] \cap N = \emptyset$. We may choose an element $c \in \text{recc}(P)^\vee \cap M$ such that $[v, w] = \text{face}(P, c)$. Since

$$\text{face}(P, c) = \text{face}(P_0, c) + \dots + \text{face}(P_k, c),$$

there is at least one summand $\text{face}(P_\mu, c)$ lacking lattice points, too. In particular, $v(P_\mu), w(P_\mu) \notin N$, and since the decomposition of P is lattice friendly, P_μ is the only summand with $v(P_\mu) \notin N$ or $w(P_\mu) \notin N$. Hence,

$$s_v(P_\mu) = 1 = s_w(P_\mu) \quad \text{and} \quad s_v(P_i) = 0 = s_w(P_i) \quad \text{for } i \neq \mu.$$

That is, the equation $s_v = s_w$ from the definition of $\mathcal{T}(P)$ is satisfied for all Minkowski summands.

Step 3. Assume that $[v, w]$ is a short half open edge of P . We are supposed to check the equality $s_v = t : = t_{vw}$ for all Minkowski summands. Since $v \notin N$, we know that $s_v(P) = 1$, that is, there is exactly one index $\mu = \mu(v)$ such that $s_v(P_\mu) = 1$. Since this means $s_v(P_i) = 0$ for $i \neq \mu$, it remains to show that $t(P_i) = 0$ for these indices; the equality $t(P_\mu) = 1$ follows then automatically. If we had $t(P_i) > 0$, then the equality

$$w(P_i) - v(P_i) = t(P_i) \cdot (w - v) \neq 0$$

would imply that $w(P_i) \neq v(P_i)$. On the other hand, both vertices $w(P_i)$ and $v(P_i)$ are lattice points. While this is clear for $v(P_i)$, we have to provide an extra argument for $w(P_i)$: If $w \in N$, then it is clear; if $w \notin N$, then it follows from the shortness of $[v, w]$ that $[v, w] \cap N = \emptyset$. Hence, we can use the equation $s_v = s_w$ obtained in Step 2. Now, since we know that $w(P_i), v(P_i) \in N$ do not coincide, we get a lower bound for the lattice lengths

$$\ell(w - v) \geq \ell(w(P_i) - v(P_i)) \geq 1,$$

which, by Remark 6.8, is not possible for short half open edges.

Step 4. So far, we have seen that $\rho(P_i) \in \mathcal{T}(P)$. To show that $\rho(P_i)$ is integral, that is, that $\rho(P_i) \in \mathcal{T}_{\mathbb{Z}}(P)$, we are supposed to check, for all $i = 0, \dots, k$, that

$$L_{vw}(P_i) = t(P_i) \cdot (v - w) - s_v(P_i) \cdot v + s_w(P_i) \cdot w \in N.$$

For this, we rewrite

$$L_{vw}(P_i) = (v(P_i) - s_v(P_i) \cdot v) - (w(P_i) - s_w(P_i) \cdot w)$$

and analyze the membership of N for both summands separately. If $v \in N$, then $v(P_i) \in N$ and $s_v(P_i) = 0$, hence,

$$v(P_i) - s_v(P_i) \cdot v \in N.$$

If $v \notin N$, then we denote by $\mu = \mu(v)$ the unique index with $v(P_\mu) \notin N$, and the previous argument survives for $i \neq \mu$. On the other hand, since $s_v(P_\mu) = 1$,

$$v(P_\mu) - s_v(P_\mu) \cdot v = v(P_\mu) - v = - \sum_{i \neq \mu} v(P_i) \in N.$$

The proof for the w -summand is the same, with a possibly different index $\mu = \mu(w)$. \square

Example 10.10. Let us continue our main example. The interval $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$ allows two non-trivial, lattice-friendly decompositions, namely,

$$\left[-\frac{1}{2}, \frac{1}{2}\right] = \left[-\frac{1}{2}, 0\right] + \left[0, \frac{1}{2}\right] = \left[-\frac{1}{2}, -\frac{1}{2}\right] + [0, 1].$$

Applying the Kodaira–Spencer map ρ , this decomposition looks like

$$(1, 1, 1) = \left(\frac{1}{2}, 1, 0\right) + \left(\frac{1}{2}, 0, 1\right) = (0, 1, 1) + (1, 0, 0) \quad \text{inside } \mathcal{T}_{\mathbb{Z}}(P).$$

According to Subsection 10.7, we can understand these two decompositions, for $i = 1, 2$, as two linear maps $\rho_i : \mathbb{Z}^2 \rightarrow \mathcal{T}_{\mathbb{Z}}(P)$; the dual maps $\rho_i^* : \mathcal{T}_{\mathbb{Z}}^*(P) \rightarrow \mathbb{Z}^2$ are given by the matrices

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The integrality of the target can be checked when ρ_i^* are applied to the generators of \tilde{T} :

$$s_1, \quad s_2, \quad A = t + \frac{1}{2}s_1 - \frac{1}{2}s_2, \quad B = t + \frac{1}{2}s_2 - \frac{1}{2}s_1.$$

Then, ρ_i^* yield two integral (2×4) -matrices mapping from $\mathbb{Z}s_1 \oplus \mathbb{Z}s_2 \oplus \mathbb{Z}A \oplus \mathbb{Z}B$ to \mathbb{Z}^2 :

$$\rho_1^* = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \rho_2^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

10.6 | lattice-friendly decompositions and the map ψ

In Subsection 10.2, we have defined for every $\xi \in \mathcal{T}_+(P)$ an associated Minkowski summand $P_\xi = \psi(\xi, P)$. While this construction depends on the choice of a reference vertex v_* , we have seen in (10.2.2) that, for integral $\xi \in \mathcal{T}_\mathbb{Z}(P)$, this dependence involves only lattice translations. Furthermore, ψ is linear in ξ , that is, for $\xi, \xi' \in \mathcal{T}_+(P)$ we have

$$P_\xi + P_{\xi'} = P_{\xi + \xi'}.$$

Remark 10.11. We do not always have the equality $\psi_{v_*}(\underline{(1; 1, 0)}, P) = P$, but we can be very precise about this:

$$P_{(\underline{1; 1, 0})} \neq P \iff v_* \in N \setminus \{0\}.$$

This issue can again be solved by a lattice translation of P . Hence, in accordance with Convention 10.4, we can and will assume that v_* is chosen such that $P_{(\underline{1; 1, 0})} = P$.

In Theorem 10.9, we have seen how the Kodaira–Spencer map ρ can detect whether a Minkowski decomposition is lattice friendly or not. The next result shows how the map $\psi(\cdot, P) = P$, fits into this relation.

Theorem 10.12. *Let $\xi_0, \dots, \xi_k \in \mathcal{T}_+(P)$ with $\xi_0 + \dots + \xi_k = (\underline{1; 1, 0})$. Then, the following three conditions are equivalent:*

- (i) $\xi_0, \dots, \xi_k \in \mathcal{T}_\mathbb{Z}(P)$,
- (ii) for each vertex $w \in P$ and index $i \in \{0, \dots, k\}$, we have

$$w(P_i) \notin N \iff w \notin N \text{ and } s_w(P_{\xi_i}) = 1, \text{ and}$$

- (iii) the decomposition $P_{\xi_0} + \dots + P_{\xi_k} = P$ is lattice friendly with $\rho(P_{\xi_i}) = \xi_i$ for $i = 0, \dots, k$.

Proof. (i) \Rightarrow (ii): Let $v \in P$ be a vertex and choose a path $v_* = v^0, v^1, \dots, v^k = v$ along the compact edges of P . Denoting $d_i := v^i - v^{i-1}$, we know from Subsection 10.2 that for each $\xi \in \mathcal{T}_+(P)$

$$\psi_{v_*}(\xi, v) = s_{v_*}(\xi) \cdot v_* + \sum_{i=1}^k t_i(\xi) \cdot d_i = \sum_{i=1}^k L_{i, i-1}(\xi) + s_v(\xi) \cdot v.$$

As $\xi \in \mathcal{T}_\mathbb{Z}(P)$ implies $L_{i, i-1}(\xi) \in N$ for every i , the equivalence in (ii) becomes evident.

(ii) \Rightarrow (iii): Since $P_{(\underline{1; 1, 0})} = P$, we obtain a Minkowski decomposition $P_{\xi_0} + \dots + P_{\xi_k} = P$. To check that it is lattice friendly, it suffices to check that for every vertex $w \in P$ we have at most one index $\mu \in \{0, \dots, k\}$ with $w(\xi_\mu) \notin N$. However, this follows directly from $s_w(\xi \in \mathcal{T}_\mathbb{Z}) \in \mathbb{N}$, from $\xi_0 + \dots + \xi_k = (\underline{1; 1, 0})$, hence from $s_w(\xi_0) + \dots + s_w(\xi_k) = 1$ and (ii).

Finally, the t -coordinates of $\rho(P_{\xi_i})$ and ξ_i are equal by definition. For the equality of the s -coordinates we use again $\xi_0 + \dots + \xi_k = (\underline{1; 1, 0})$ and (ii) in a straightforward manner.

- (iii) \Rightarrow (i): This follows from the direction (\Rightarrow) in Theorem 10.9. □

Remark 10.13.

- (i) On the one hand, since there are many non-lattice choices for ξ which produce the same polyhedron P_ξ , we cannot expect that ξ can be recovered from P_ξ . In particular, the equality

$\rho(P_\xi) = \xi$ cannot be true in general. Hence, it is not possible to erase the phrase ‘with $\rho(P_\xi) = \xi$ ’ from (iii) of the previous theorem.

- (ii) On the other hand, every lattice shift of a Minkowski summand produces the same value of ρ . So, we cannot expect to obtain $Q = P_{\rho(Q)}$ in general, either.
- (iii) Despite the negative claims above, we can consider the following two sets:

$$A := \{\text{polyhedra } Q \subseteq N_{\mathbb{R}} \text{ with } \text{recc}(Q) = \text{recc}(P) \text{ such that there is a polyhedron } Q' \text{ providing a lattice-friendly decomposition } P = Q + Q'\} \quad \text{and}$$

$$B := \{\xi \in \mathcal{T}_+(P) \cap \mathcal{T}_{\mathbb{Z}}(P) : (\underline{1}; \underline{1}, \underline{0}) - \xi \in \mathcal{T}_+(P)\}.$$

Then, dividing out integral translations, it follows from the theorems 10.9 and 10.12 that the two maps $\rho : A/N \rightarrow B$ and $\psi : B \rightarrow A/N$ are mutually inverse.

Example 10.14. While the construction $\xi \mapsto P_\xi$ from Subsection 10.2 does not make use of the non-negativity of s in $\xi = (t, s)$, the assumption $\xi \in \mathcal{T}_+(P)$ becomes really important for Theorem 10.12. To illustrate this, take $P = [-\frac{1}{3}, \frac{1}{4}]$ with $v_1 = -\frac{1}{3}$ and $v_2 = \frac{1}{4}$. So, this is not a short edge (none of the two half-open edges is), and the lattice conditions for $\xi = (t, s_1, s_2)$ are

$$s_1, s_2 \in \mathbb{Z} \quad \text{and} \quad \frac{7}{12}t - \frac{1}{3}s_1 - \frac{1}{4}s_2 \in \mathbb{Z}.$$

Choose $v_* = v_1$ and consider

$$\xi = \left(\frac{1}{7}, 1, -1\right), \quad \xi' = \left(\frac{6}{7}, 0, 2\right).$$

We see that $\xi, \xi' \in \mathcal{T}_{\mathbb{Z}}(P)$ with $\xi + \xi' = (1, 1, 1)$, but

$$P_\xi = \left[-\frac{1}{3}, -\frac{1}{4}\right] \quad \text{and} \quad P_{\xi'} = \left[0, \frac{1}{2}\right],$$

provides a non-lattice-friendly Minkowski decomposition of P .

10.7 | The Kodaira–Spencer map revisited

In Definition 10.6, we have introduced the notion of lattice-friendly decomposition $P = P_0 + \dots + P_k$. In [5, (3.2)], this notion was used to construct a k -parameter family, that is, a deformation $\tilde{X} \rightarrow \mathbb{A}_k^k$ of the associated toric singularity X . Its total space was built from the Cayley product mentioned in Remark 5.7(ii). Actually, similarly to $X = \mathbb{T}\mathbb{V}(\sigma)$, one defines it as $\tilde{X} = \mathbb{T}\mathbb{V}(\tilde{\sigma})$ with $\tilde{\sigma} := \text{cone}(P_0 * \dots * P_k)$.

In [5, (3.3) and (3.4)], it was shown that the Kodaira–Spencer map of this construction is exactly the map ρ we have defined in Subsection 10.4 — this is why we have called it like this even in the purely discrete, that is, non-algebraic setup. In [1, Proposition 4.3], we will connect the notion of a free pair to flatness, from which it follows that the corresponding inclusion $\mathbb{N}^{k+1} \hookrightarrow \tilde{\sigma}^\vee \cap (M \oplus \mathbb{N}^{k+1})$ is a iso-bounded extension of $\mathbb{N} \hookrightarrow \sigma^\vee \cap (M \oplus \mathbb{N})$. In particular, denoting by (\tilde{T}, \tilde{S}) the initial extension from Theorem 9.2, then this is induced from a semigroup homomorphism $\mathcal{T}_{\mathbb{Z}}^*(P) \supseteq \tilde{T} \rightarrow \mathbb{N}^{k+1}$. The dual map $\mathbb{Z}^{k+1} \rightarrow \mathcal{T}_{\mathbb{Z}}(P)$ equals ρ , and the fact that its target is $\mathcal{T}_{\mathbb{Z}}(P) \subset \mathcal{T}(P)$ illustrates Theorem 10.9.

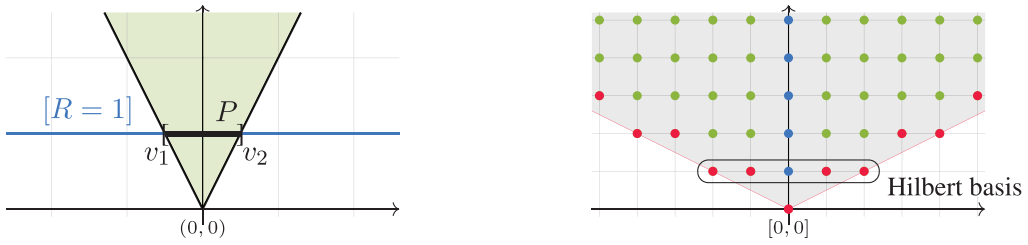


FIGURE 12 The cone and the semigroups for the 1-dimensional $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$

11 | EXAMPLES

For the convenience of the reader, we repeat the example of the cone over the -4 -curve, which is scattered through the previous sections, and add a new example of a lattice polytope with non-primitive edges. We also repeat Figure 9, with a few details added, and Figure 10.

Example 11.1. Consider the semigroup $S = \text{span}_{\mathbb{N}}\{-2, 1\}, [-1, 1], [0, 1], [1, 1], [2, 1]\}$ with the sub-semigroup $T = \mathbb{N} \cdot [0, 1]$. Let us first note that the relative boundary is

$$\partial_T S = \{[\pm 2b, b] : b \in \mathbb{N}\} \cup \{[\pm(2b - 1), b] : b \in \mathbb{N}_{\geq 1}\}.$$

The semigroup S is obtained from the 1-dimensional polytope $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$ by taking the the polyhedral cone σ over $P \times \{1\} \subset \mathbb{R}^2$, dualizing it and taking the set of lattice points (Figure 12). In algebraic geometry, this setup gives rise to the toric singularity $X = \mathbb{T}\mathbb{V}(\sigma) \subseteq \mathbb{A}_k^5$ which can, alternatively, be understood as the vanishing set of the six minors encoded by the condition

$$\text{rank} \begin{pmatrix} z_{-2} & z_{-1} & z_0 & z_1 \\ z_{-1} & z_0 & z_1 & z_2 \end{pmatrix} \leq 1.$$

The elements $[k, 1] \in S$ can be recovered as the multidegrees of the variables z_i . In [1] we discuss the deformation theory of those toric singularities.

We will focus now on the universal extension of this semigroup pair. As we shall see, this leads to a 4-dimensional semigroup, that is, to a semigroup filling a 4-dimensional polyhedral cone where its 3-dimensional crosscut is depicted in Figure 13. In the introduction, as a preview, we had presented Figure 1 as an alternative representation of this situation. Note that this establishes a remarkable difference to the algebro-geometric setup: There, the two deformation components cannot be dominated by a higher dimensional joint deformation of the same kind, that is, with an irreducible base space. See also Subsection 9.1.

We label the vertices of P as $v_1 = -\frac{1}{2}$ and $v_2 = \frac{1}{2}$, and associate the variables s_1 and s_2 to them. We denote by t the variable referring to the only edge $d = P$, with $g_p = 1$. The interval has length one, and it contains exactly one lattice point, that is, $|\{P \cap \mathbb{N}\}| = 1$. In particular, it gives rise to two non-short half open edges (cf. Definition 6.7) so there are no relations among s_1, s_2 , and t .

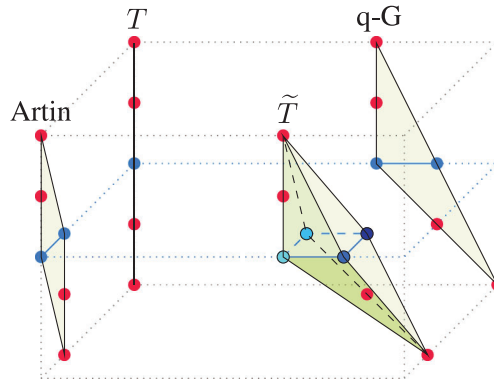


FIGURE 13 The full picture: Artin- and qG-components as projections of the initial object. In the 3-dimensional body, we have: $\bullet = [0, s_1]$, $\bullet = [0, s_2]$, $\bullet = [0, A]$, $\bullet = [0, B]$

For the computation of $\tilde{\eta}$ we choose $v_* = v_1$. As the first entries of the elements of the Hilbert Basis range from -2 to 2 , it is enough to compute the η s and $\tilde{\eta}$ s for them. We obtain:

	c	$v(c)$	$\eta(c)$	$\eta_{\mathbb{Z}}(c)$	$\tilde{\eta}(c)$	$\tilde{\eta}_{\mathbb{Z}}(c)$
c_5	2	v_1	1	1	s_1	s_1
c_4	1	v_1	$\frac{1}{2}$	1	$\frac{1}{2}s_1$	s_1
c_3	0	v_1 or v_2	0	0	0	0
c_2	-1	v_2	$\frac{1}{2}$	1	$-\frac{1}{2}s_1 + t$	$\frac{1}{2}(s_2 - s_1) + t$
c_1	-2	v_2	1	1	$-s_1 + 2t$	$-s_1 + 2t$

The elements $[c, \eta_{\mathbb{Z}}(c)]$ and $[c, \tilde{\eta}_{\mathbb{Z}}(c)]$ will be in the boundaries $\partial_T S$ and $\partial_{\tilde{T}} \tilde{S}$, respectively. They correspond to the red dots (\bullet) in Figure 13. The key step is to determine the generators of \tilde{T} , that is the dots ($\bullet \bullet \bullet$) in the 3-dimensional object of Figure 13. To this aim, there are two strategies.

- (i) Compute $\tilde{\eta}_{\mathbb{Z}}(c_1, c_2)$ for various c_1, c_2 until a pattern emerges.
- (ii) Compute $\{(m_1, \dots, m_5) \in \mathbb{N}^5 : \eta_{\mathbb{Z}}(m_1 c_1, \dots, m_5 c_5) \neq 0\}$, where $(c_i, \eta_{\mathbb{Z}}(c_i))$ are the Hilbert basis elements.

The latter is more systematic, as it boils down to determining monomial ideal by identifying the elements of \mathbb{N}^5 with exponents. This ideal has a finite minimal set of generators, and the corresponding $\tilde{\eta}_{\mathbb{Z}}(m_1 c_1, \dots, m_5 c_5)$ will then generate \tilde{T} . Describing this ideal and its relation with \tilde{T} is also an interesting problem, but we do not study the general case here. For this specific example, the six minimal generators of the monomial ideal from Lemma 7.16 correspond to the exponent vectors:

$$\begin{aligned}
 &(0, 0, 0, 2, 0) && (0, 2, 0, 0, 0) \\
 &(0, 1, 0, 1, 0) && (1, 0, 0, 0, 1) \\
 &(0, 1, 0, 0, 1) && (1, 0, 0, 1, 0),
 \end{aligned}$$

where the first entry is the number of copies of $c_1 = -2$, and so on. For instance, $(0,0,1,2,3)$ corresponds to $\tilde{\eta}_{\mathbb{Z}}(0, 1, 1, 2, 2)$. So, the above exponents give us the finite generating set of \tilde{T} :

$$\begin{aligned} \tilde{\eta}_{\mathbb{Z}}(1, 1) &= s_1 & \tilde{\eta}_{\mathbb{Z}}(-1, -1) &= s_2 \\ \tilde{\eta}_{\mathbb{Z}}(-1, 1) &= t + \frac{1}{2}(s_1 + s_2) & \tilde{\eta}_{\mathbb{Z}}(-2, 2) &= 2t \\ \tilde{\eta}_{\mathbb{Z}}(-1, 2) &= \frac{1}{2}(s_2 - s_1) + t & \tilde{\eta}_{\mathbb{Z}}(-2, 1) &= \frac{1}{2}(s_1 - s_2) + t. \end{aligned}$$

However, this is not a minimal generating set of \tilde{T} , as only four of them are needed:

$$s_1, s_2, A = t + \frac{1}{2}s_1 - \frac{1}{2}s_2, B = t + \frac{1}{2}s_2 - \frac{1}{2}s_1.$$

The two projections from the 3-dimensional object in Figure 13 $\rho_i^* : \mathcal{T}_{\mathbb{Z}}^*(P) \rightarrow \mathbb{Z}^2$ are given by the matrices

$$\rho_1^* = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}, \quad \rho_2^* = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The first one, ρ_1^* projects to the Artin component, and ρ_2^* projects to the q-G component. They correspond to the two non-trivial lattice-friendly decompositions of the interval $P = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$, namely,

$$\left[-\frac{1}{2}, \frac{1}{2}\right] = \left[-\frac{1}{2}, 0\right] + \left[0, \frac{1}{2}\right] = \left[-\frac{1}{2}, -\frac{1}{2}\right] + [0, 1].$$

Applying the Kodaira–Spencer map ρ , this decomposition looks like

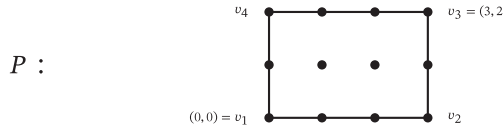
$$(1, 1, 1) = \left(\frac{1}{2}, 1, 0\right) + \left(\frac{1}{2}, 0, 1\right) = (0, 1, 1) + (1, 0, 0) \text{ inside } \mathcal{T}_{\mathbb{Z}}(P).$$

According to Subsection 10.7, we can understand these two decompositions, for $i = 1, 2$, as two linear maps $\rho_i : \mathbb{Z}^2 \rightarrow \mathcal{T}_{\mathbb{Z}}(P)$; the dual maps $\rho_i^* : \mathcal{T}_{\mathbb{Z}}^*(P) \rightarrow \mathbb{Z}^2$ being given above. The integrality of the target can be checked when ρ_i^* are applied to the generators of \tilde{T} : s_1, s_2, A , and B . This yields two integral (2×4) -matrices mapping from $\mathbb{Z}s_1 \oplus \mathbb{Z}s_2 \oplus \mathbb{Z}A \oplus \mathbb{Z}B$ to \mathbb{Z}^2 :

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Example 11.2. This second example is motivated by Gorenstein toric singularities which are not smooth in codimension two. These correspond to lattice polytopes whose edges are not necessarily primitive. This more general situation is not significantly different from the one with primitive edges which was treated in [4]: there are only t -parameters which correspond to edges and these encode Minkowski summands as dilation factors. The lattice structure changes, as rational dilation factors may still produce lattice polytopes. The most significant difference is that the map $\tilde{T} \rightarrow T$, which is obtained by mapping the t_i to 1, may fail to be surjective. We illustrate all these claims here with one simple example.

We consider the lattice rectangle:



The dual cone $\sigma^\vee = (\text{cone}(P))^\vee$ has ray generators: $[1, 0, 0], [0, 1, 0], [-1, 0, 3], [0, -1, 2]$. So, the Hilbert basis, that is, the generating set of the semigroup $\sigma^\vee \cap \mathbb{Z}^3$, is

$$[1, 0, 0], [0, 1, 0], [0, 0, 1], [-1, 0, 3], [0, -1, 2].$$

The sub-semigroup is thus $T = \text{span}_{\mathbb{N}}\{[0, 0, 1]\}$. To compute the relative boundary $\partial_T S$ we have to compute $\eta_{\mathbb{Z}}(c)$ for all $c \in \mathbb{Z}^2$. As P is a lattice polytope, $\eta_{\mathbb{Z}}(c) = \eta(c)$ for all $c \in \mathbb{Z}^2$. The normal fan of P has four cones: the four quadrants in \mathbb{R}^2 . Starting with the positive quadrant and going counter clockwise, the corresponding $v(c)$ are v_1, v_2, v_3, v_4 . In other words, we have for $c = (a, b) \in \mathbb{Z}^2$:

$$\eta((a, b)) = \begin{cases} 0, & \text{if } a, b \geq 0, \\ 3a, & \text{if } a \leq 0, b \geq 0, \\ 3a + 2b, & \text{if } a, b \leq 0, \\ 2b, & \text{if } a \geq 0, b \leq 0. \end{cases}$$

The ambient space of the cone of Minkowski summands and the lattice inside it are

$$\begin{aligned} \mathcal{T}(P) &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 \cdot (3, 0) + x_2 \cdot (0, 2) + x_3 \cdot (-3, 0) + x_4 \cdot (0, -2) = (0, 0)\} \\ &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3, x_2 = x_4\}. \end{aligned}$$

$$\mathcal{T}_{\mathbb{Z}}(P) = \left\{ \left(\frac{z_1}{3}, \frac{z_2}{2}, \frac{z_1}{3}, \frac{z_2}{2} \right) \in \mathbb{R}^4 : z_1, z_2 \in \mathbb{Z} \right\}.$$

Its dual space and lattice are

$$\begin{aligned} \mathcal{T}^*(P) &= \text{span}_{\mathbb{R}}\{t_1, t_2, t_3, t_4\} / \text{span}_{\mathbb{R}}\{t_1 - t_3, t_2 - t_4\}. \\ \mathcal{T}_{\mathbb{Z}}^*(P) &= \text{span}_{\mathbb{Z}}\{3t_1, 2t_2\}. \end{aligned}$$

To lift the boundary $\partial_T S$ to the boundary of $\partial_{\tilde{T}} \tilde{S}$, we compute the $\tilde{\eta}_{\mathbb{Z}}$, which are again equal to the $\tilde{\eta}$, because P is a lattice polytope. We have for $c = (a, b) \in \mathbb{Z}^2$:

$$\tilde{\eta}((a, b)) = \begin{cases} 0, & \text{if } a, b \geq 0, \\ 3a \cdot t_1, & \text{if } a \leq 0, b \geq 0, \\ 3a \cdot t_1 + 2b \cdot t_2, & \text{if } a, b \leq 0, \\ 2b \cdot t_2, & \text{if } a \geq 0, b \leq 0. \end{cases}$$

A quick computation shows that \tilde{T} is generated by $[0, 0, 3t_1]$ and $[0, 0, 2t_2]$. This brings us to the main difference from the Gorenstein-smooth-in-codimension-two case: the map $\tilde{T} \rightarrow T$, which is obtained by mapping the t_i to 1, is not always surjective. To summarize, the generators of the

semigroups in the universal extension of $T \subseteq S$ are

$$\begin{array}{ccc}
 \begin{matrix} [0, 0, 3t_1] \\ [0, 0, 2t_2] \end{matrix} & \hookrightarrow & [1, 0, 0] [0, 1, 0] \begin{matrix} [0, 0, 3t_1] \\ [0, 0, 2t_1] \end{matrix} \begin{matrix} [-1, 0, 3t_1] \\ [0, -1, 2t_2] \end{matrix} \\
 \downarrow & & \downarrow \\
 [0, 0, 1] & \hookrightarrow & [1, 0, 0] [0, 1, 0] [0, 0, 1] [-1, 0, 3] [0, -1, 2].
 \end{array}$$

The non-trivial lattice-friendly Minkowski decomposition of P is in this case

$$\begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} = 2 \times \begin{array}{|c|} \hline \bullet \\ \hline \bullet \\ \hline \end{array} + 3 \times \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array}$$

which corresponds to the lattice decomposition in $\mathcal{T}_{\mathbb{Z}}(P)$:

$$(1, 1, 1, 1) = 2 \cdot \left(0, \frac{1}{2}, 0, \frac{1}{2}\right) + 3 \cdot \left(\frac{1}{3}, 0, \frac{1}{3}, 0\right).$$

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