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Dissertation

"The future of some Bianchi A spacetimes with an ensemble of free falling particles"

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Dedicated to my aunts and uncles

Caco
Mageni
Manolo
Santi
Tote
Tuto
Uge

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1 Introduction

Cosmology describes the dynamics of the Universe as a whole. These dynamics can only be understood in terms of the theory of general relativity which was established by Albert Einstein in 1915 [19]-[20]. This theory is mathematically complicated enough to have given rise to a research field on its own, the area of Mathematical General Relativity.

One of the difficulties is the occurrence of singularities. Since the singularity theorems of Penrose and Hawking [37],[40],[69] we know that singularities can form under very general circumstances. To prove global existence is thus a fundamental question. It has been shown that the Einstein equations can be seen as an initial value problem [14], [27]. However the fundamental question of predictability is still open and requires a deep mathematical understanding of the problem.

Nevertheless one can look at simplified models and hope to learn something to be able in the future to understand more realistic ones. In some cases it is also physically reasonable to assume symmetries due to the physical situation one is modelling. One of the applications of the theory of general relativity is cosmology. With the observations and their interpretation Edwin Hubble [51] realized that there were strong arguments that the Universe is not static and now it is part of the standard model of cosmology that we live in a homogeneous and isotropic universe with a cosmological constant which was introduced by Einstein [21] but with a different purpose. There are strong indications that the cosmological constant exists. However there are serious problems for the physical interpretation of this entity which has led scientists to modify the standard model of cosmology. There are also suggestions that a cosmological constant is not needed if one would understand the dynamics of inhomogeneous models. The nature of the cosmological constant or whether this term is really constant is still very controversial. For instance Calogero has recently proposed that this term comes from velocity diffusion [10].

A starting point to understand general models are the homogeneous models. There are a lot of results concerning this subject and in particular two books [91], [102] which are an excellent introduction and a great summary of many of the results obtained. See also [43] for a critical discussion of the results obtained until now.

In general the focus has been on the fluid model since it appears (theoretically) relatively natural when dealing with isotropic universes and from observations we also know that the Universe is almost isotropic. However to have a deeper understanding of the dynamics one should go beyond the study of isotropic universes. General statements may vary then depending on the choice of the matter model. It is also important to note as is pointed out in [25] that a quasi-isotropic epoch is compatible with all Bianchi models and thus it is interesting to study the dynamics of all the different types.

We will deal with the *future asymptotics* of some homogeneous cosmological models within the so called Bianchi class A and the matter is described via an ensemble of free falling particles also called collisionless matter. For all the models treated here the fundamental questions are on a firm ground, i.e. future geodesic completeness has been shown for these models [71], [72].

Concerning the late time behaviour of the Universe one believes in the cosmic no hair conjecture [34], [39]. This conjecture states roughly speaking that all expanding cosmological models with a positive cosmological constant approach asymptotically the de Sitter solution (empty and flat universe with a positive cosmological constant).

Let us review some results which go in this direction. In [104] it was shown that all non-type IX Bianchi cosmologies which are initially expanding will eventually be locally indistinguishable from the de Sitter solution. It is remarkable that in this paper only the strong and the dominant energy condition are assumed for the matter model. However it has turned out that to obtain a more detailed analysis in general one has to deal with a concrete matter model. For the case with a positive cosmological constant global future stability in the class of solutions without symmetry could be shown for the vacuum [30], for Einstein-Maxwell, Einstein-Yang-Mills [31] and for the FLRW case with a perfect fluid if the fluid obeys a linear equation of state $P = (\gamma - 1)\rho$ with $1 < \gamma < \frac{4}{3}$ [89],[94] where P and ρ are the pressure and energy density of the fluid respectively. This last result could be extended recently to the case $\gamma = \frac{4}{3}$ [62]. For the Einstein-Vlasov system isotropization could be shown for all non-type IX Bianchi cosmologies [56]. For recent results on the quantum version of the cosmic no hair conjecture we refer to [49] and references therein.

In absence of a cosmological constant there are also different results concerning the future. For the Einstein-non-linear scalar field system future non-linear stability has been shown for a variety of scalar fields. For recent work on this subject we refer to [83], [84], [2] and [17]. The late-time behaviour of Bianchi spacetimes with a non-tilted fluid is well understood [101], [48]. In particular all non-tilted perfect fluid orthogonal Bianchi models except IX with a linear equation of state where $0 < \gamma < \frac{2}{3}$ are future asymptotic to the flat FL model [47]. Note the restriction on γ here. One cannot expect isotropization for most of the Bianchi models. However there are two important characteristics of the future asymptotics of some of the Bianchi models which have been the 'conjectures' of the present work:

1. The spacetimes considered tend to special (self-similar) solutions
2. For expanding models the dispersion of the velocities of the particles decays

This second 'conjecture' means that asymptotically there is a dust-like behaviour for collisionless matter which is the matter model we will use. This has been achieved already for locally rotationally symmetric (LRS) models in the cases of Bianchi I, II, III and IX [78],[79],[74],[11] and for reflection symmetric models in the case of Bianchi I [73]. These results have been obtained using dynamical systems theory.

The Einstein-Vlasov system remains a system of partial differential equations (PDE's) even if one assumes spatial homogeneity. The reason is that although the distribution function written in a suitable frame will not depend on the spatial point, the dependence with respect to the momenta remains (since the Vlasov equation is defined on the mass shell). However in the results mentioned a reduction to a system of ordinary differential equations was possible due to the additional symmetry assumptions. This is no longer possible if one drops some of these additional symmetries (see [63]-[64] for the reasons). Thus if one wants to generalize these results the theory of finite dimensional dynamical systems is not enough.

Most of the results obtained until now rely on the theory of dynamical systems. Thus one might be tempted to use techniques coming from the theory of infinite-dimensional dynamical systems. The first important difficulty would be to choose the suitable (weighted) norm. Another one is that important theorems which have been used for the finite-dimensional case cannot be used here. All this may work, but this is not the approach taken here.

Here the main tool used is a bootstrap argument which is often used in non-linear PDE's. We will present results concerning the late-time behaviour of some expanding Bianchi A spacetimes with collisionless matter where we have assumed small data. This assumption will be specified later, but roughly it means that the universe is close to the special self-similar solution mentioned earlier and that the velocity dispersion of the particles is small.

The results obtained are as follows. In the case of Bianchi I we could generalize to the non-diagonal case and thus generalize the results of [73] concerning the expanding direction. We have already published this result before ([67]; note that F and H are defined differently there). Here we present a simpler proof due to a compactness argument. For reflection symmetric Bianchi II and reflection symmetric Bianchi VI_0 we have been able to show that their late-time behaviour remains the same if the LRS condition is dropped. We will show that these spacetimes, reflection symmetric Bianchi II and reflection symmetric Bianchi VI_0, will tend to solutions which are even more symmetric. In the case of Bianchi II we will show that it will become LRS, a Bianchi model whose isometry group of the spatial metric is four-dimensional. In this case there exists a one-dimensional isotropy group and one can show that a spacetime of Bianchi class A admits a four-dimensional isometry group, if and only if two structure constants are equal and if the corresponding metric components are equal as well. Bianchi VI_0 cannot be LRS, however it is compatible with an additional discrete symmetry (Appendix B.1 of [12]). The analysis of the asymptotics shows that the Bianchi VI_0 spacetimes tend to this special class. Note that for VI_0 there is no corresponding LRS/previous result. Finally we were able to drop the reflection symmetry for Bianchi II.

All the results show that the dust model usually assumed in observational cosmology in the 'matter-dominated' Era is robust. Another way of saying the same is that asymptotically collisionless matter is well approximated by the dust system.

The thesis is organized as follows. In the following chapter we will present the general basic equations, namely the Einstein-Vlasov system. In chapters three and four we explain the symmetry assumptions and deduce the corresponding equations of the Einstein-Vlasov system which are then summarized in chapter five. Afterwards we present some special solutions and their linear stability as a pre-stage of our main argument in chapter eight: the bootstrap argument. This argument is refined in chapter nine and leads to our main results. In chapter ten we treat the non-diagonal case for Bianchi II. In our last chapter we have a discussion about our results and present some possible future directions.

2 Relativistic kinetic theory

We will consider as matter model collisionless matter. Before introducing it more formally we want to motivate first this restriction to that kind of matter instead of considering the full Boltzmann equation.

First of all it is of course an enormous simplifying assumption. In particular it is difficult to prove global existence and uniqueness for spatially homogeneous solutions even to the classical Boltzmann equation (see comments below proposition 4.3 of [72]). Nevertheless there are already some results concerning these questions [65] and concerning the future asymptotics. A dust-like behaviour is shown for Einstein-Boltzmann in the case of FLRW with a cosmological constant (theorem 3.2 of [98]). One assumes by modelling a galaxy as a particle that in a cosmological context the internal structure of the galaxy is irrelevant.

An important physical argument in favour of considering the collisionless model is that collisions between galaxies are not common and even if galaxies fly through each other not so many collisions between stars happen as one might expect. Also in stellar dynamics collisionless matter is often used since collisions between stars are very unlikely. Actually this led Eddington to state:

> "The apparent analogy with the kinetic theory of gases is rejected altogether, and it is taken as a fundamental principle that *the stars describe paths under the general attraction of the stellar system without interfering with one another*"(p. 254 of [18]; italics from Eddington)

Two pages later he continues:

> "A regular progression may be traced through rigid dynamics, hydrodynamics, gas-dynamics to stellar dynamics. In the first all the particles move in a connected manner; in the second there is continuity between the motions of contiguous particles ; in the third the adjacent particles act on one another by collision, so that, although there is no mathematical continuity, a kind of physical continuity remains; in the last the adjacent particles are entirely independent."(p. 256 of [18])

Later Jeans in the study of stellar dynamics referring to the collisionless Boltzmann equation writes:

> "This is the differential equation which must be satisfied by the distribution function f in every problem of stellar dynamics."(p. 230 of [52])

and on the same page as a footnote:

> "The student of the Kinetic Theory will recognise that it is simply Boltzmann's well-known equation with the collisions left out."

However this equation is usually named after Vlasov [100] in particular in the context of mathematical cosmology. Vlasov discovered in the context of plasma physics that pair collision terms do not describe correctly the plasma dynamics and also that these terms are not formally applicable since kinetic terms diverge. His point of view was that only the collective behaviour, i.e. the electromagnetic field created by the charged particles explains the dynamics of the individual particles.

Maybe the emphasis on the difference of taking one or the other point of view has been the reason that the Vlasov equation is named after him not only in the context of plasma physics (see [44] for a different point of view). Another maybe that although Boltzmann himself assumed that the particles interact only through:

- very long range forces which can be approximated by mean fields
- or very short range forces such as hard core interactions whose effect can be approximated by instantaneous collisions;

in practice the long range forces were often neglected since the gravitational force for instance is very weak.

Another fact which has attracted attention to the collisionless matter case is the discovery that analyzing the initial singularity of the Einstein-dust equations there arose singularities which are unphysical and not related to gravity but to the chosen matter model. One has seen that these problems do not occur when using collisionless matter as the matter model. For recent progress in this direction see [80]. Finally the Vlasov equation is used in astrophysics also to model dark matter where the particles are now elementary particles (see for instance [1] where these particles are conjectured to be 'sterile' neutrinos).

2.1 The Einstein-Vlasov system

A cosmological model represents a universe at a certain averaging scale. It is described via a Lorentzian metric $g_{\alpha\beta}$ (we will use signature $-+++$) on a manifold M and a family of fundamental observers. The metric is assumed to be time-orientable, which means that at each point of M the two halves of the light cone can be labelled past and future in a way which varies continuously from point to point. This enables to distinguish between future-pointing and past-pointing timelike vectors. This is a physically reasonable assumption from both a macroscopic point of view e.g. the increase of entropy and also from a microscopic point of view e.g. the kaon decay. As we already mentioned in the introduction one has also to specify the matter model and this we will do in the next section. The interaction between the geometry and the matter is described by the Einstein field equations (we use geometrized units, i.e. the gravitational constant G and the speed of light in vacuum c are set equal to one):

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ is the energy-momentum tensor. The Einstein tensor satisfies:

$$\nabla^{\alpha} G_{\alpha\beta} = 0.$$

Thus the energy-momentum tensor has to satisfy the same equation, which expresses the conservation of energy. For the matter model we will take the point of view of kinetic theory [95]. The sign conventions of [76] are used. Also the Einstein summation convention that repeated indices are to be summed over is used. Latin indices run from one to three and Greek ones from zero to three.

Consider a particle with non-zero rest mass which moves under the influence of the gravitational field. The mean field we mentioned in the introduction will be described now by the metric and the components of the metric connection. The wordline x^{α} of a particle is a timelike curve in spacetime. The unit future-pointing tangent vector to this curve is the 4-velocity v^{α} and $p^{\alpha} = mv^{\alpha}$ is the

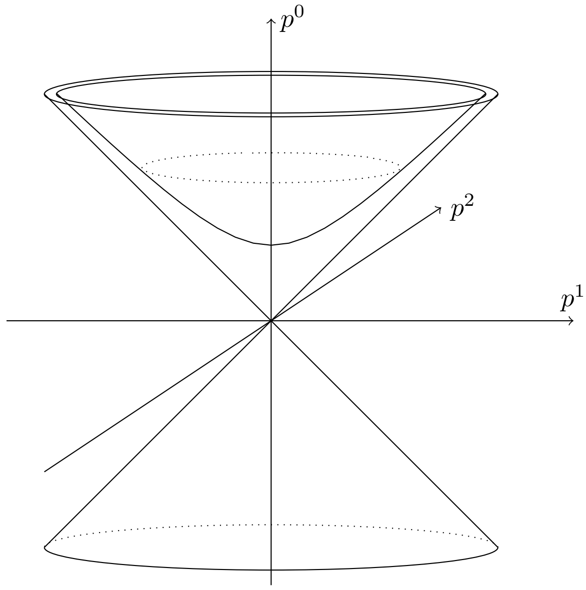

Figure 1. Sketch of the mass shell (hyperboloid $p^0 = \sqrt{(p^1)^2 + (p^2)^2}$) inside the forward light cone

4-momentum of the particle. Let T_x be the tangent space at a point x^α in the spacetime M, then we define the *phase space for particles of arbitrary rest masses* P to be the following set:

$$P = \{(x^\alpha, p^\alpha) : \; x^\alpha \in M, \; p^\alpha \in T_x, \; p_\alpha p^\alpha \leq 0, \; p^0 > 0\}$$

which is a subset of the tangent bundle $TM = \{(x^\alpha, p^\alpha) : \; x^\alpha \in M, \; p^\alpha \in T_x\}$. For particles of the same type and with the same rest mass m which is given by the mass shell relation:

$$p_\alpha p^\alpha = -m^2$$

we have the *phase space* P_m *for particles of mass* m:

$$P_m = \{(x^\alpha, p^\alpha) : \; x^\alpha \in M, \; p^\alpha \in T_x, \; p_\alpha p^\alpha = -m^2, \; p^0 > 0\}$$

We will consider from now on that all the particles have *equal* mass m. For how this relates to the general case of different masses see [11]. We will choose units such that $m = 1$ which means that a distinction between velocities and momenta is not necessary. We have then that the possible values for the 4-momenta are all future pointing unit timelike vectors. These values form the hypersurface:

$$P_1 = \{(x^\alpha, p^\alpha) : \; x^\alpha \in M, \; p^\alpha \in T_x, \; p_\alpha p^\alpha = -1, \; p^0 > 0\}$$

which we will call the mass shell. The collection of particles (galaxies or clusters of galaxies) will be described (statistically) by a non-negative real valued distribution function $f(x^\alpha, p^\alpha)$ on P_1. This function represents the density of particles at a given spacetime point with given four-momentum. A free particle travels along a geodesic. Consider now a future-directed timelike geodesic parametrized by proper time s. The tangent vector is then at any time future-pointing unit timelike. Thus the geodesic has a natural lift to a curve on P_1 by taking its position and tangent vector. The equations of motion thus define a flow on P_1 which is generated by a vector field L which is called geodesic spray or Liouville operator. The geodesic equations are:

$$\frac{dx^\alpha}{ds} = p^\alpha; \quad \frac{dp^\alpha}{ds} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma$$

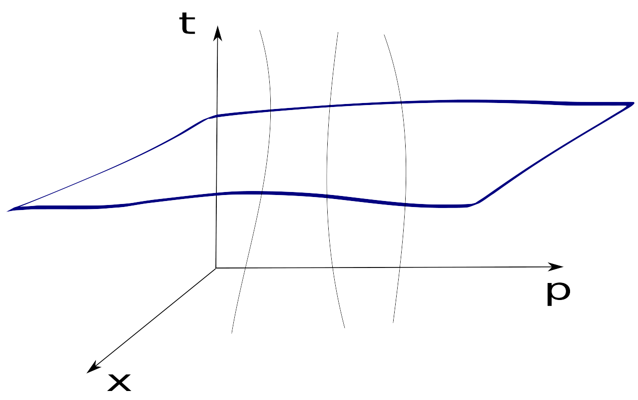

Figure 2. A 3-dimensional representation of the 7-dimensional phase space. A slice with $t = C$ which is cut by the worldlines corresponds to the classical 6-dim phase space

where the components of the metric connection, i.e. $\Gamma_{\alpha\beta\gamma} = g(e_\alpha, \nabla_\gamma e_\beta) = g_{\alpha\delta}\Gamma^\delta_{\beta\gamma}$ can be expressed in the vector basis e_α as [(1.10.3) of [96]]:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(e_\beta(g_{\alpha\gamma}) + e_\gamma(g_{\beta\alpha}) + e_\alpha(g_{\gamma\beta}) + \eta^\delta_{\gamma\beta}g_{\alpha\delta} + \eta^\delta_{\alpha\gamma}g_{\beta\delta} - \eta^\delta_{\beta\alpha}g_{\gamma\delta}) \tag{1}$$

The commutator of the vectors e_α can be expressed with the following formula:

$$[e_\alpha, e_\beta] = \eta^\gamma_{\alpha\beta}e_\gamma$$

where $\eta^\gamma_{\alpha\beta}$ are called commutation functions. The restriction of the Liouville operator to the mass shell is defined as:

$$L = \frac{dx^\alpha}{ds}\frac{\partial}{\partial x^\alpha} + \frac{dp^a}{ds}\frac{\partial}{\partial p^a}.$$

Using the geodesic equations it has the following form

$$L = p^\alpha\frac{\partial}{\partial x^\alpha} - \Gamma^a_{\beta\gamma}p^\beta p^\gamma\frac{\partial}{\partial p^a}.$$

This operator is sometimes also called geodesic spray. If we denote now the phase space density of collisions by $C(f)$, then the Boltzmann equation [9] in curved spacetime in our notation looks as follows:

$$L(f) = C(f)$$

describes the evolution of the distribution function. Between collisions the particles follow geodesics. We will consider the collisionless case which is described via the Vlasov equation:

$$L(f) = 0$$

2.2 Energy momentum tensor and characteristics

The unknowns of our system are a 4-manifold M, a Lorentz metric $g_{\alpha\beta}$ on this manifold and the distribution function f on the mass shell P_1 defined by the metric. We have the Vlasov equation defined by the metric for the distribution function and the Einstein equations. It remains to define the energy-momentum tensor $T_{\alpha\beta}$ in terms of the distribution and the metric. Before that we need a Lorentz invariant volume element on the mass shell. A point of a the tangent space has the

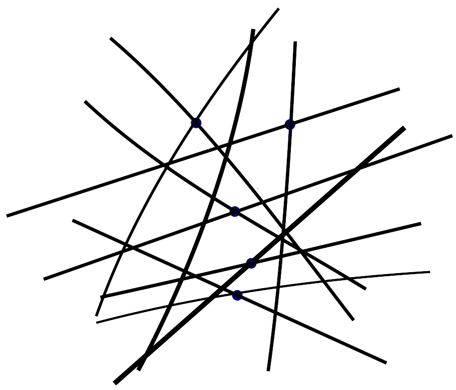

Figure 3. Schematic Space-time picture of a relativistic gas ('broken discrete complex' [97]). Between collisions the particles follow geodesics

volume element $|g^{(4)}|^{\frac{1}{2}} dp^0 dp^1 dp^2 dp^3$ ($g^{(4)}$ is the determinant of the spacetime metric) which is Lorentz invariant. Now considering p^0 as a dependent variable the induced (Riemannian) volume of the mass shell considered as a hypersurface in the tangent space at that point is

$$\varpi = 2H(p^\alpha)\delta(p_\alpha p^\alpha + m^2)|g^{(4)}|^{\frac{1}{2}} dp^0 dp^1 dp^2 dp^3$$

where δ is the Dirac distribution function and $H(p^\alpha)$ is defined to be one if p^α is future directed and zero otherwise. We can write this explicitly as:

$$\varpi = |p_0|^{-1}|g^{(4)}|^{\frac{1}{2}} dp^1 dp^2 dp^3$$

Now we define the energy momentum tensor as follows:

$$T_{\alpha\beta} = \int f(x^\alpha, p^a) p_\alpha p_\beta \varpi$$

One can show that $T_{\alpha\beta}$ is divergence-free and thus it is compatible with the Einstein equations. For collisionless matter all the energy conditions hold. In particular the dominant energy condition is equivalent to the statement that in any orthonormal basis the energy density dominates the other components of $T_{\alpha\beta}$, i.e. $T_{\alpha\beta} \leq T_{00}$ for each α, β (P. 91 of [38]). Using the mass shell relation one can see that this holds for collisionless matter. The non-negative sum pressures condition is in our case equivalent to $g_{ab}T^{ab} \geq 0$.

The Vlasov equation in a fixed spacetime can be solved by the method of characteristics (see chapter 3.2 of [26]):

$$\frac{dX^a}{ds} = P^a; \quad \frac{dP^a}{ds} = -\Gamma^a_{\beta\gamma} P^\beta P^\gamma$$

Let $X^a(s, x^\alpha, p^a)$, $P^a(s, x^\alpha, p^a)$ be the unique solution of that equation with initial conditions $X^a(t, x^\alpha, p^a) = x^a$ and $P^a(t, x^\alpha, p^a) = p^a$. Then the solution of the Vlasov equation can be written as:

$$f(x^\alpha, p^a) = f_0(X^a(0, x^\alpha, p^a), P^a(0, x^\alpha, p^a))$$

where f_0 is the restriction of f to the hypersurface $t = 0$. It follows that if f_0 is bounded the same is true for f. We will assume that f has compact support in momentum space for each fixed t (note that it is not possible in the Boltzmann case). This property holds if the initial datum f_0 has compact support and if each hypersurface $t = t_0$ is a Cauchy hypersurface [75].

2.3 The initial value problem

Before coming to our symmetry assumption we want to briefly introduce the initial value problem for the Einstein-Vlasov system. For a general introduction to the initial value problem in general relativity we refer to [85] and for the Einstein-Vlasov system in particular we refer to [75]. In general the initial data for the Einstein-matter equations consist of a metric g_{ab} on the initial hypersurface, the second fundamental form k_{ab} on that hypersurface and some matter data. Thus we have a Riemannian metric g_{ab}, a symmetric tensor k_{ab} and some matter fields defined on an abstract 3-dimensional manifold S.

Solving the initial value problem means embedding S into a 4-dimensional M on which are defined a Lorentzian metric $g_{\alpha\beta}$ and matter fields such that g_{ab} and k_{ab} are the pullbacks to S of the induced metric and second fundamental form of the image of the embedding of S while f is the pullback of the matter fields. Finally $g^{\alpha\beta}$ and f have to satisfy the Einstein-matter equations.

For the Einstein-Vlasov system it has been shown [13] that given an initial data set there exists a corresponding solution of the Einstein-Vlasov system and that this solution is locally unique up to diffeomorphism (see also theorem 1.1 of [75]). The extension to a global theorem has not been achieved yet. However if one assumes that the initial data have certain symmetry, this symmetry is inherited by the corresponding solutions (see 5.6 of [32] for a discussion). In particular for the case we will deal with, i.e. expanding Bianchi models (except type IX) coupled to dust or to collisionless matter the spacetime is future complete (theorem 2.1 of [72]).

3 Bianchi spacetimes

We start with a citation of the first article of Einstein about cosmology [21]:

> “Der metrische Charakter (Krümmung) des vierdimensionalen raumzeitlichen Kontinuums wird nach der allgemeinen Relativitätstheorie in jedem Punkte durch die daselbst befindliche Materie und deren Zustand bestimmt. Die metrische Struktur dieses Kontinuums muss daher wegen der Ungleichmässigkeit der Verteilung der Materie notwendig eine äusserst verwickelte sein. Wenn es uns aber nur auf die Struktur im grossen ankommt, dürfen wir uns die Materie als über ungeheure Räume gleichmässig ausgebreitet vorstellen, so dass deren Verteilungsdichte eine ungeheuer langsam veränderliche Funktion wird. Wir gehen damit ähnlich vor wie etwa die Geodäten, welche die im äusserst kompliziert gestaltete Erdoberfläche durch ein Ellipsoid approximieren (...) Der Skalar ρ der (mittleren) Verteilungsdichte kann a priori eine Funktion der räumlichen Koordinaten sein. Wenn wir aber die Welt als räumlich in sich geschlossen annehmen, so liegt die Hypothese nahe, dass ρ unabhängig vom Orte sei (...) Aus unserer Annahme über die Gleichmässigkeit der Verteilung der das Feld erzeugende Massen folgt, dass auch die Krümmung des gesuchten Messraumes eine konstante sein muss.”

We see here the assumption of homogeneity (and isotropy) as a simplifying assumption. We also see that Einstein has assumed a closed universe. He does this because he encounters serious difficulties to impose boundary conditions in an analogous way as one does with the Poisson equation. Today in mathematical cosmology it is standard to assume spatial compactness and thus one considers spacetimes possessing a compact Cauchy hypersurface which is a simplifying assumption.

3.1 Homogeneous spacetimes, isotropic spacetimes

We start with the definition of homogeneity and isotropy of spacetimes taken from chapter 5.1 of [105]. In that chapter it is also shown that isotropy implies homogeneity.

Definition 1 A spacetime $(M, g_{\alpha\beta})$ is said to be (spatially) *homogeneous* if there exists a one-parameter family of spacelike hypersurfaces S_t foliating the spacetime such that for each t

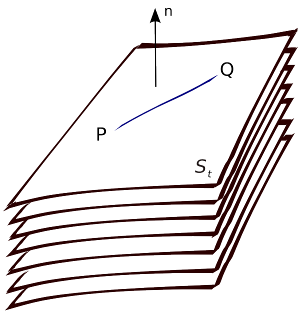

Figure 4. A spacetime is spatially homogeneous if it admits an isometry group whose orbits are spacelike hypersurfaces that foliate M

and for any points $P, Q \in S_t$ there exists an isometry of the spacetime metric, $g_{\alpha\beta}$, which takes P into Q.

Definition 2 A spacetime is said to be (spatially) *isotropic* at each point if there exists a congruence of timelike curves, with tangents denoted u^α filling the spacetime and satisfying the following property: Given any point P and any two unit "spatial" tangent vectors $s_1^\alpha, s_2^\alpha \in V_P$ (i.e., vectors at P orthogonal to u^α), there exists an isometry of $g_{\alpha\beta}$ which leaves P and u^α at P fixed but rotates s_1^α into s_2^α.

3.2 Left and right translations

It turns out that the group of isometries always yields a Lie group G of dimension m (see chapter 7.2 of [105] for details), i.e. a group which is also an m-dimensional manifold such that the inverse map $i(g) = g^{-1}$ and the multiplication map $f(g_1, g_2) = g_1 g_2$ are smooth. It follows that for each $h \in G$ the map

$$\psi_h(g) = hg$$

called *left translation* by h is a diffeomorphism. This map ψ_h induces another map ψ_h^* on tensors. If a vector field v^α satisfies

$$\psi_h^* v^\alpha = v^\alpha$$

for all $h \in G$ it is called *left invariant.* The left invariant vector fields on G form an m-dimensional vector space. If v^α and w^α are left invariant vector fields on a Lie group, the commutator is

$$[v, w]^\alpha = C^\alpha_{\beta\gamma} v^\beta w^\gamma$$

where $C^\alpha_{\beta\gamma}$ is called the structure constant tensor of the Lie group. From the definition it follows that $C^\alpha_{\beta\gamma} = -C^\alpha_{\gamma\beta}$. The Jacobi identity for commutators gives rise to another relation. A finite dimensional vector space with a structure tensor satisfying these two relations is called a *Lie algebra.* Analogously the map

$$\Xi_h(g) = gh$$

is called a *right translation* by h and is also a diffeomorphism. Right invariant vector fields are infinitesimal generators of left translations. The commutator of right invariant vector fields satisfies

$$[x, y]^\alpha = -C^\alpha_{\beta\gamma} x^\beta y^\gamma$$

i.e. the same relation as in the left invariant case, but with a minus sign.

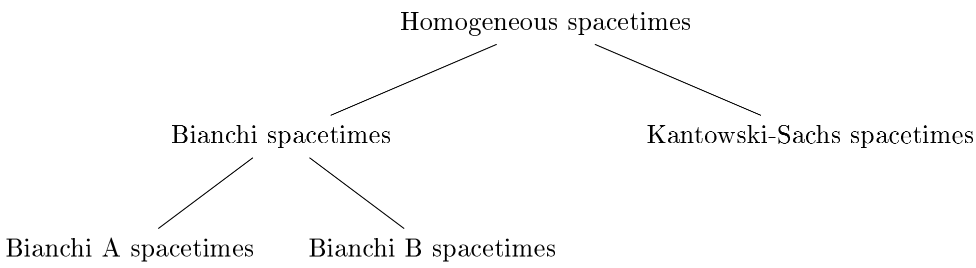

Figure 5. Subclasses of homogeneous spacetimes

3.3 Definition of Bianchi spacetimes

The basis for the classification of homogeneous spacetimes is the work of Bianchi [8] which was introduced to cosmology by Taub [99]. Here we will use the modern terminology and we define Bianchi spacetimes as follows:

Definition 3 A *Bianchi spacetime* is defined to be a spatially homogeneous spacetime whose isometry group possesses a three-dimensional subgroup G that acts simply transitively on the spacelike orbits.

Not all homogeneous spacetimes are Bianchi spacetimes. But the only case where G does not act simply transitively or does not possess a subgroup with simply transitive action are the so called Kantowski-Sachs models. The Bianchi models can be subclassified into two classes [23]: class A and B. Later we will only deal with Bianchi class A, however all the equations in this chapter are valid for Bianchi spacetimes in general.

3.4 Locally homogeneous spacetimes

The only Bianchi spacetimes which admit a compact Cauchy hypersurface are Bianchi I and IX. In order to be not that restrictive we will consider locally spatially homogeneous spacetimes. They are defined as follows. Consider an initial data set on a three-dimensional manifold S. Then this initial data set is called locally spatially homogeneous if the naturally associated data set on the universal covering $\tilde{S}$ is homogeneous. For Bianchi models the universal covering space $\tilde{S}$ can be identified with its Lie group G (see [71], [72] for details). As we said in the beginning of this chapter in mathematical cosmology it is standard to assume spatial compactness. It turns out that this assumption is problematic within the Bianchi class B. See [4],[33],[54] for details about these questions and relations to the Thurston classification and the Hamiltonian formulation.

3.5 Description of Bianchi spacetimes via the metric approach

A Bianchi spacetime admits a Lie algebra of Killing vector fields with basis $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and structure constants C^c_{ab}, such that:

$$[\mathbf{k}_a, \mathbf{k}_b] = -C^c_{ab}\mathbf{k}_c.$$

The Killing vector fields $\mathbf{k}_a$ are tangent to the group orbits which are called surfaces of homogeneity. If one chooses a unit vector field n normal to the group orbits we have a natural choice for the time coordinate t such that the group orbits are given by a constant t. This unit normal is invariant under the group, i.e:

$$[\mathbf{n}, \mathbf{k}_a] = 0$$

One can now choose a triad of spacelike vectors e_a that are tangent to the group orbits:

$$g(\mathbf{n}, \mathbf{e}_a) = 0$$

and that commute with the Killing vector fields:

$$[\mathbf{e}_a, \mathbf{k}_b] = 0$$

A frame $\{\mathbf{n}, \mathbf{e}_a\}$ chosen in this way is called a left invariant frame and it is generated by the right invariant Killing vector fields. Since $\mathbf{n}$ is hypersurface orthogonal the vector fields $\mathbf{e}_a$ generate a Lie algebra with structure constants η^c_{ab}. It can be shown that this Lie algebra is in fact equivalent to the Lie algebra of the Killing vector fields. Thus one can classify the Bianchi spacetimes using either the structure constants or the spatial commutation functions of the basis vectors. The remaining freedom in the choice of the frame is a time-dependent linear transformation, which can be used to introduce a set of time-independent spatial vectors $\mathbf{E}_a$:

$$[\mathbf{E}_a, \mathbf{n}] = 0.$$

The corresponding commutation functions are then constant in time and one can make them equal to the structure constants:

$$[\mathbf{E}_a, \mathbf{E}_b] = C^c_{ab}\mathbf{E}_c$$

This is our choice which is sometimes called the metric approach [24]. Often the orthonormal frame approach is used (see e.g. [102] and [85]). If $\mathbf{W}^a$ denote the 1-forms dual to the frame vectors $\mathbf{E}_a$ the metric of a Bianchi spacetime takes the form:

$$^4g = -dt^2 + g_{ab}(t)\mathbf{W}^a\mathbf{W}^b \tag{2}$$

where g_{ab} (and all other tensors) on G will be described in terms of the frame components of the left invariant frame which has been introduced. A dot above a letter will denote a derivative with respect to the cosmological time t.

3.6 3+1 Decomposition of the Einstein equations

We will use the 3+1 decomposition of the Einstein equations as made in [76]. Comparing our metric (2) with (2.28) of [76] we have that $\alpha = 1$ and $\beta^a = 0$ which means that the lapse function is the identity and the shift vector vanishes. There the abstract index notation is used. We can interpret the quantities as being frame components. For details we refer to chapter 2.3 of [76]. There are different projections of the energy momentum tensor which are important

$$\begin{aligned} \rho &= T^{00} \\ j_a &= T^0_a \\ S_{ab} &= T_{ab} \end{aligned}$$

where ρ is the energy density and j_a is the matter current.

The second fundamental form k_{ab} can be expressed as:

$$\dot{g}_{ab} = -2k_{ab}. \tag{3}$$

The Einstein equations:

$$\dot{k}_{ab} = R_{ab} + k\ k_{ab} - 2k_{ac}k^c_b - 8\pi(S_{ab} - \frac{1}{2}g_{ab}S) - 4\pi\rho g_{ab} \tag{4}$$

where we have used the notations $S = g^{ab}S_{ab}$, $k = g^{ab}k_{ab}$, and R_{ab} is the Ricci tensor of the three-dimensional metric. The evolution equation for the mixed version of the second fundamental form is (2.35) of [76]:

$$\dot{k}^a_b = R^a_b + k\ k^a_b - 8\pi S^a_b + 4\pi\delta^a_b(S - \rho) \tag{5}$$

From the constraint equations since k only depends on the time variable we have that:

$$R - k_{ab}k^{ab} + k^2 = 16\pi\rho \qquad (6)$$
$$\nabla^a k_{ab} = 8\pi j_b \qquad (7)$$

where R is the Ricci scalar curvature.

Another useful relation concerns the determinant g of the induced metric ((2.30) of [76]):

$$\frac{d}{dt}(\log g) = -2k \qquad (8)$$

Taking the trace of (5):

$$\dot{k} = R + k^2 + 4\pi S - 12\pi\rho \qquad (9)$$

With (6) one can eliminate the energy density and (9) reads:

$$\dot{k} = \frac{1}{4}(k^2 + R + 3k_{ab}k^{ab}) + 4\pi S \qquad (10)$$

Finally if one substitutes for the Ricci scalar with (6):

$$\dot{k} = k_{ab}k^{ab} + 4\pi(S + \rho) \qquad (11)$$

3.7 Time origin choice and new variables

Now with the 3+1 formulation our initial data are $(g_{ij}(t_0), k_{ij}(t_0), f(t_0))$, i.e. a Riemannian metric, a second fundamental form and the distribution function of the Vlasov equation, respectively, on a three-dimensional manifold $S(t_0)$. This is the initial data set at $t = t_0$ for the Einstein-Vlasov system.

We assume that $k < 0$ for all time following [71] (see comments below lemma 2.2 of [71]). This enables us to set without loss of generality $t_0 = -2/k(t_0)$. The reason for this choice will become clear later and is of technical nature.

We will now introduce several new variables in order to use the ones which are common in Bianchi cosmologies (e.g.[101]) and to be able to compare results. We can decompose the second fundamental form introducing σ_{ab} as the trace-free part:

$$k_{ab} = \sigma_{ab} - Hg_{ab} \qquad (12)$$

$$k_{ab}k^{ab} = \sigma_{ab}\sigma^{ab} + 3H^2 \qquad (13)$$

Using the Hubble parameter:

$$H = -\frac{1}{3}k$$

we define:

$$\Sigma_a^b = \frac{\sigma_a^b}{H} \qquad (14)$$

and

$$\Sigma_+ = -\frac{1}{2}(\Sigma_2^2 + \Sigma_3^3) \qquad (15)$$
$$\Sigma_- = -\frac{1}{2\sqrt{3}}(\Sigma_2^2 - \Sigma_3^3) \qquad (16)$$

Thus

$$\Sigma_a^b = \begin{pmatrix} 2\Sigma_+ & \Sigma_2^1 & \Sigma_3^1 \\ \Sigma_1^2 & -\Sigma_+ - \sqrt{3}\Sigma_- & \Sigma_3^2 \\ \Sigma_1^3 & \Sigma_2^3 & -\Sigma_+ + \sqrt{3}\Sigma_- \end{pmatrix}$$

The reason for using the variables Σ_+ and Σ_- is that the diagonal case has been very important to understand the non-diagonal case. Define also:

$$\Omega = 8\pi\rho/3H^2 \tag{17}$$

$$q = -1 - \frac{\dot{H}}{H^2} \tag{18}$$

$$\frac{d\tau}{dt} = H \tag{19}$$

The time variable τ is dimensionless and sometimes very useful. From (6) we obtain the constraint equation:

$$\frac{1}{6H^2}(R - \sigma_{ab}\sigma^{ab}) = \Omega - 1$$

and from (10) the evolution equation for the Hubble variable:

$$\partial_t(H^{-1}) = \frac{3}{2} + \frac{1}{12}(\frac{R}{H^2} + \frac{3}{H^2}\sigma_{ab}\sigma^{ab}) + \frac{4\pi S}{3H^2} \tag{20}$$

Combining the last two equations with (5) we obtain the evolution equations for Σ_- and Σ_+:

$$\dot{\Sigma}_+ = H[\frac{2R - 3(R_2^2 + R_3^3)}{6H^2} - \Sigma_+(3 + \frac{\dot{H}}{H^2}) + \frac{4\pi}{3H^2}(3S_2^2 + 3S_3^3 - 2S)] \tag{21}$$

$$\dot{\Sigma}_- = H[\frac{R_3^3 - R_2^2}{2\sqrt{3}H^2} - (3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi(S_2^2 - S_3^3)}{\sqrt{3}H^2}] \tag{22}$$

3.8 Vlasov equation with Bianchi symmetry

Since we use a left-invariant frame f will not depend on x^a and the Vlasov equation takes the form:

$$p^0\frac{\partial f}{\partial t} - \Gamma^a_{\beta\gamma}p^\beta p^\gamma \frac{\partial f}{\partial p^a} = 0$$

It turns out that the equation simplifies if we express f in terms of p_i instead of p^i what we can do due to the mass shell relation:

$$p^0\frac{\partial f}{\partial t} - \Gamma_{a\beta\gamma}p^\beta p^\gamma \frac{\partial f}{\partial p_a} = 0$$

Because of our special choice of frame the metric has the simple form (2). This has the consequence that only the spatial components of the metric connection remain and that the first three terms of (1) vanish. Due to the fact that we are contracting and the antisymmetry of the structure constant we finally arrive at:

$$\frac{\partial f}{\partial t} + (p^0)^{-1}C^d_{ba}p^b p_d \frac{\partial f}{\partial p_a} = 0 \tag{23}$$

From (23) it is also possible to define the characteristic curve V_a:

$$\frac{dV_a}{dt} = (V^0)^{-1}C^d_{ba}V^b V_d \tag{24}$$

Type	ν_1	ν_2	ν_3
I	0	0	0
II	1	0	0
VI_0	0	1	-1
VII_0	0	1	1
VIII	-1	1	1
IX	1	1	1

Table 1. Classification of Bianchi types class A

for each $V_i(\bar{t}) = \bar{v}_i$ given $\bar{t}$. Note that if we define:

$$V = g^{ij} V_i V_j \tag{25}$$

due to the antisymmetry of the structure constants we have with (24):

$$\frac{dV}{dt} = \frac{d}{dt}(g^{ij}) V_i V_j \tag{26}$$

Let us also write down the components of the energy momentum tensor in our frame:

$$T_{00} = \int f(t, p^a) p^0 \sqrt{g} dp^1 dp^2 dp^3 \tag{27}$$

$$T_{0j} = -\int f(t, p^a) p_j \sqrt{g} dp^1 dp^2 dp^3 \tag{28}$$

$$T_{ij} = \int f(t, p^a) p_i p_j (p^0)^{-1} \sqrt{g} dp^1 dp^2 dp^3 \tag{29}$$

4 Bianchi A spacetimes

4.1 Definition and classification of Bianchi A spacetimes

In the last chapter we have presented the Einstein-Vlasov system with Bianchi symmetry. However our results concern only a special class of the Bianchi spacetimes, namely that of class A.

Definition 4 A *Bianchi A spacetime* is a Bianchi spacetime whose three-dimensional Lie algebra has traceless structure constants, i.e. $C^a_{ba} = 0$.

In that case there is a unique symmetric matrix, called commutator matrix with components ν^{ij} such that the structure constants can be written as follows (lemma 19.3 of [85]):

$$C^a_{bc} = \varepsilon_{bcd} \nu^{da} \tag{30}$$

The transformation rule of the commutator matrix under a change of basis of the Lie algebra can be used to classify the Bianchi class A Lie algebras. It is possible to diagonalize ν and the diagonal elements ν_1, ν_2 and ν_3 can be used to classify the different Bianchi types of class A (for details see chapter 19.1 of [85]).

We will study Bianchi I, II and VI_0. The importance of these Bianchi types is that they play a fundamental role for higher Bianchi types. See [81] for an example concerning the direction of the singularity. In the following figure the relations between the different types are shown schematically.

4.2 The sign of the Ricci scalar and some consequences

Before coming to particular Bianchi types we want to obtain some interesting inequalities which are valid for all Bianchi A spacetimes except Bianchi type IX. The reason is that for those spacetimes

$$\begin{tikzcd}
B(\mathrm{VIII}) \arrow[d] \arrow[drr] & & B(\mathrm{IX}) \arrow[d] \\
B(\mathrm{VI}_0) \arrow[dr] & & B(\mathrm{VII}_0) \arrow[dl] \\
& B(\mathrm{II}) \arrow[d] & \\
& B(\mathrm{I}) &
\end{tikzcd}$$

Figure 6. Specialization diagram for the Bianchi invariant sets [101]

the Ricci scalar is non-positive (see corollary 19.12 of [85] for a proof). From this fact we can obtain an estimate for H. Since $\rho \geq S$ due to the energy condition we can conclude from (9) that

$$3H^2 \geq -\dot{H} \tag{31}$$

For all Bianchi spacetimes we can conclude from (11) and (13) that

$$-\dot{H} \geq H^2 \tag{32}$$

Integrating (31) and (32) we arrive at the estimate for H:

$$(3(t-t_0) + H^{-1}(t_0))^{-1} \leq H \leq (t - t_0 + H^{-1}(t_0))^{-1}$$

Another useful inequality which follows from the fact that the Ricci scalar is non-positive is an upper bound of the energy density in terms of the Hubble variable. If follows from (6) using the trace-free part of the second fundamental form that:

$$16\pi\rho = 6H^2 + R - \sigma_{ab}\sigma^{ab}$$

Thus we obtain:

$$16\pi\rho \leq 6H^2 \tag{33}$$

4.3 Lie groups and structure constants for Bianchi I, II and VI_0

Bianchi I has as the Lie group $\mathbb{R}^3$. Bianchi II has as Lie Group the Heisenberg group (chapter 23.4 of [85]): the subgroup of $GL(3, \mathbb{R})$ given by the matrices of the form:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

Finally Bianchi VI_0 has as the Lie group Sol (chapter 23.3 of [85]) where the underlying manifold is $\mathbb{R}^3$, but the group structure is given by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + e^{x_1} y_2 \\ z_1 + e^{-x_1} z_2 \end{pmatrix}$$

From (30) we see that the structure constants vanish for Bianchi I. For Bianchi II the only non-vanishing structure constants are:

$$C^1_{23} = 1 = -C^1_{32} \tag{34}$$

and in the case of Bianchi VI_0 these are:

$$C^2_{31} = 1 = -C^2_{13}, \quad C^3_{21} = 1 = -C^3_{12} \tag{35}$$

4.4 Reflection symmetry and vanishing tilt

In the case of Bianchi I we will treat the non-diagonal case and for this case the momentum constraint is automatically satisfied if the matter current vanishes. However for Bianchi II and VI_0 we will assume at some point an additional symmetry namely the reflection symmetry to ensure this. This will be a restriction to the diagonal case. The reflection symmetry has been defined in (2.10) of [73] for the case of Bianchi I, but one can define this for other Bianchi types as well with the difference that the distribution function now will depend in general on the time variable:

$$f(t,p_1,p_2,p_3) = f(t,-p_1,-p_2,p_3) = f(t,p_1,-p_2,-p_3)$$

One can easily see from (28)-(29) that the energy-momentum tensor is then diagonal. Thus from (3) and (4) we can see that if the metric and the second fundamental form are diagonal initially they will remain diagonal in the reflection symmetric case. This symmetry implies in particular that there is no matter current, which means that there is no 'tilt'. For Bianchi A in general the constraint equation (7) simplifies to:

$$8\pi T_{0j} = C^l_{ji} k^i_l$$

For Bianchi II we obtain thus

$$\begin{aligned} 8\pi T_{02} &= k^3_1 \\ 8\pi T_{03} &= -k^2_1 \end{aligned}$$

and for Bianchi VI_0:

$$\begin{aligned} 8\pi T_{01} &= -k^2_3 - k^3_2 \\ 8\pi T_{02} &= k^1_3 \\ 8\pi T_{03} &= k^1_2 \end{aligned}$$

4.5 Some formulas for the diagonal case

Now we will introduce formulas which are valid in the diagonal case. There (ijk) denotes a cyclic permutation of (123) and the Einstein summation convention is suspended for the first two formulas. Let us define:

$$n_i = \nu_i \sqrt{\frac{g_{ii}}{g_{jj}g_{kk}}}$$

The Ricci tensor is given by (11a) of [12]:

$$R^i_i = \frac{1}{2}[n_i^2 - (n_j - n_k)^2]$$

We define:

$$N_i = \frac{n_i}{H}$$

In the diagonal case we have:

$$\Sigma^2_+ + \Sigma^2_- = \frac{1}{6}\frac{\sigma_{ab}\sigma^{ab}}{H^2}$$

from which follows that the constraint equation can be written in the following way:

$$\Sigma^2_+ + \Sigma^2_- = \Omega - 1 - \frac{1}{6H^2}R \tag{36}$$

Now we proceed to use them for the cases of Bianchi II and VI_0.

4.5.1 Expressions for diagonal Bianchi II

For Bianchi II we have then:

$$R^1_1 = -R^2_2 = -R^3_3 = -R = \frac{1}{2}n_1^2$$

From the constraint equation (36) we obtain:

$$\Sigma_+^2 + \Sigma_-^2 = 1 - \Omega - \frac{1}{12}N_1^2$$

and from (20) we obtain the equation for the evolution of H:

$$\partial_t(H^{-1}) = \frac{3}{2} - \frac{N_1^2}{24} + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2}$$

From the definition (3) for the second fundamental form the evolution equation for n_1^2 follows:

$$\frac{d}{dt}(n_1^2) = 2(-4\sigma_+ - H)n_1^2$$

In terms of N_1^2:

$$\frac{d}{dt}(N_1^2) = -2N_1^2 H(4\Sigma_+ + 1 + \frac{\dot{H}}{H^2})$$

or

$$\dot{N}_1 \quad = \quad -N_1 H(4\Sigma_+ + 1 + \frac{\dot{H}}{H^2})$$

4.5.2 Expressions for diagonal Bianchi VI$_0$

For Bianchi VI$_0$ we have then:

$$R^1_1 = R = -\frac{1}{2}(n_2 - n_3)^2$$
$$R^2_2 = -R^3_3 = \frac{1}{2}(n_2^2 - n_3^2)$$

The constraint equation (36) is:

$$\Sigma_+^2 + \Sigma_-^2 = 1 - \Omega - \frac{1}{12}(N_2 - N_3)^2$$

$$\partial_t(H^{-1}) = \frac{3}{2} - \frac{1}{24}(N_2 - N_3)^2 + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2}$$

As in the Bianchi II case, using the equation (3) we arrive at:

$$\dot{N}_2 = -N_2 H(-2\Sigma_+ - 2\sqrt{3}\Sigma_- + 1 + \frac{\dot{H}}{H^2})$$
$$\dot{N}_3 = -N_3 H(-2\Sigma_+ + 2\sqrt{3}\Sigma_- + 1 + \frac{\dot{H}}{H^2})$$

5 Central equations

In this section we present or collect the main equations just to have them together.

5.1 Bianchi I

From (20) since $R = 0$ we have:

$$\partial_t(H^{-1}) = \frac{3}{2} + \frac{1}{4H^2}\sigma_{ab}\sigma^{ab} + \frac{4\pi S}{3H^2}$$

Defining $F = \frac{1}{4H^2}\sigma_{ab}\sigma^{ab}$, the evolution equation is

$$\dot{F} = -3H[F(1 - \frac{2}{3}F - \frac{8\pi S}{9H^2}) - \frac{4\pi}{3H^3}S_{ab}\sigma^{ab}] \tag{37}$$

and the constraint equation:

$$F = \frac{3}{2}(1 - \Omega)$$

The Vlasov equation in the chosen frame is trivial since the structure constants vanish

$$\frac{\partial f}{\partial t} = 0$$

5.2 Reflection symmetric Bianchi II

The evolution equations are:

$$\partial_t(H^{-1}) = \frac{3}{2} - \frac{N_1^2}{24} + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2} \tag{38}$$

$$\dot{\Sigma}_+ = H[\frac{1}{3}N_1^2 - (3 + \frac{\dot{H}}{H^2})\Sigma_+ + \frac{4\pi}{3H^2}(S_2^2 + S_3^3 - 2S_1^1)] \tag{39}$$

$$\dot{\Sigma}_- = H[-(3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi}{\sqrt{3}H^2}(S_2^2 - S_3^3)] \tag{40}$$

$$\dot{N}_1 = -N_1 H(4\Sigma_+ + 1 + \frac{\dot{H}}{H^2}) \tag{41}$$

and the constraint equation:

$$\Sigma_+^2 + \Sigma_-^2 = 1 - \Omega - \frac{1}{12}N_1^2$$

The Vlasov equation in this case using (34) is:

$$\frac{\partial f}{\partial t} + (p^0)^{-1}p_1(p^2\frac{\partial f}{\partial p_3} - p^3\frac{\partial f}{\partial p_2}) = 0$$

Let us write the equations (39)-(41) with τ and q:

$$\Sigma_+' = \frac{1}{3}N_1^2 - (2 - q)\Sigma_+ + \frac{4\pi}{3H^2}(S_2^2 + S_3^3 - 2S_1^1) \tag{42}$$

$$\Sigma_-' = -(2 - q)\Sigma_- + \frac{4\pi}{\sqrt{3}H^2}(S_2^2 - S_3^3) \tag{43}$$

$$N_1' = N_1(q - 4\Sigma_+) \tag{44}$$

Note that these equations are the same as (6.21) of [101] with $\gamma = 1$ if one sets $S = 0$ in (42)-(44).

5.3 Reflection symmetric Bianchi VI_0

For Bianchi VI_0 the evolution equations are:

$$\partial_t(H^{-1}) = \frac{3}{2} - \frac{1}{24}(N_2 - N_3)^2 + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2}$$
$$\dot{\Sigma}_+ = H[-\frac{1}{6}(N_2 - N_3)^2 - \Sigma_+(3 + \frac{\dot{H}}{H^2}) + \frac{4\pi}{3H^2}(S_2^2 + S_3^3 - 2S_1^1)]$$
$$\dot{\Sigma}_- = H[\frac{N_3^2 - N_2^2}{2\sqrt{3}} - (3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi(S_2^2 - S_3^3)}{\sqrt{3}H^2}]$$
$$\dot{N}_2 = -N_2 H(-2\Sigma_+ - 2\sqrt{3}\Sigma_- + 1 + \frac{\dot{H}}{H^2})$$
$$\dot{N}_3 = -N_3 H(-2\Sigma_+ + 2\sqrt{3}\Sigma_- + 1 + \frac{\dot{H}}{H^2})$$

and the constraint equation:

$$\Sigma_+^2 + \Sigma_-^2 = 1 - \Omega - \frac{1}{12}(N_2 - N_3)^2$$

For the Vlasov equation we obtain with (35)

$$\frac{\partial f}{\partial t} + (p^0)^{-1}[p_2(p^3\frac{\partial f}{\partial p_1} - p^1\frac{\partial f}{\partial p_3}) + p_3(p^2\frac{\partial f}{\partial p_1} - p^1\frac{\partial f}{\partial p_2})] = 0$$

In analogy to Bianchi II these equations can be compared to (6.9)-(6.10) of [101] setting N_1 to zero and using the definition of q (18).

6 Special solutions

In this section we present some special solutions which will be important, since we will show that the late time asymptotics of the Bianchi types considered behave like them in a sense which will be specified later. We start with the Kasner solution which is the general Bianchi I vacuum solution to motivate also the concept of generalized Kasner exponents.

6.1 The Kasner solution and generalized Kasner exponents

The Kasner solution [53] is the general Bianchi I vacuum solution, thus all components of the energy-momentum tensor and the scalar curvature R vanish. From the constraint equation one obtains:

$$\Sigma_+^2 + \Sigma_-^2 = 1$$

which is known as the Kasner circle. The metric components are:

$$g_{ij} = \mathrm{diag}(t^{2p_1}, t^{2p_2}, t^{2p_3})$$

where p_1, p_2 and p_3 satisfy:

$$p_1 + p_2 + p_3 = 1$$
$$p_1^2 + p_2^2 + p_3^2 = 1$$

One can easily compute that the Hubble variable is $H = \frac{1}{3}t^{-1}$.

For more general spacetimes let λ_i be the eigenvalues of k_{ij} with respect to g_{ij}, i.e., the solutions of:

$$\det(k^i_j - \lambda\delta^i_j) = 0 \qquad (45)$$

We define

$$p_i = \frac{\lambda_i}{k}$$

as the *generalized Kasner exponents*. They satisfy the first but in general not the second Kasner relation.

6.2 Friedman-Lemaître-Robertson-Walker Universes

The possible homogeneous and isotropic universes are called Friedman-Lemaître-Robertson-Walker (FLRW) universes ([28]- [29], [57]-[58], [86]-[88], [106]). Depending on the curvature the metric takes the form:

$$g = a^2(t) \begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \\ dx^2 + dy^2 + dz^2 \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \end{cases}$$

which can be written as:

$$g \quad = \quad a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

where a is positive (it cannot change the sign, if it does we have a singularity) and

$$\begin{cases} k = 1, \ r = \sin\psi \\ k = 0, \ r = \psi \\ k = -1, \ r = \sinh\psi \end{cases}$$

for the 3-sphere (closed, special case of Bianchi IX), flat space (special case of Bianchi I and Bianchi VII_0) and for the hyperboloid (open, special case of Bianchi V and VII_h with $h \neq 0$) respectively. For the Friedman solution with a flat geometry it is easy to calculate that $\Sigma_+ = \Sigma_- = 0$ which tells us that the solution is isotropic. For $a = t^{\frac{2}{3}}$ we have the *Einstein-de Sitter* solution [22] which corresponds to the dust case. The energy density in this case is $\rho_{ES} = \frac{1}{6\pi}t^{-2}$.

6.3 The Collins-Stewart solution

Another special solution which will play an important role is the Collins-Stewart solution ([15], p. 430) with dust ($\gamma = 1$) which has Bianchi II symmetry:

$$g_{CS} = \mathrm{diag}(2t, (2t)^{3/2}, (2t)^{3/2})$$

The Hubble parameter is $H = \frac{2}{3}t^{-1}$ and the energy density $8\pi\rho_{CS} = \frac{5}{4}t^{-2}$. The values of the variables which have been introduced previously are:

$$\Sigma_+ = \frac{1}{8}; \ \Sigma_- = 0; \ \Omega = \frac{15}{16}; \ N_1 = \frac{3}{4}$$

6.4 The Ellis-MacCallum solution

In the case of Bianchi VI_0 there is a dust solution with diagonal metric discovered by Ellis and MacCallum ([23], pp. 124-125):

$$g_{EM} = \mathrm{diag}(t^2, t^1, t^1)$$

The Hubble parameter is $H = \frac{2}{3}t^{-1}$ as in the Collins-Stewart solution, but the energy density $8\pi\rho_{EM} = t^{-2}$ is different. Here the values of the introduced variables are:

$$\Sigma_+ = -\frac{1}{4}; \ \Sigma_- = 0; \ N_1 = 0; \ N_2 = -N_3 = \frac{3}{4}; \ \Omega = \frac{3}{4}$$

The generalization of this solution to different values of γ is called Collins solution.

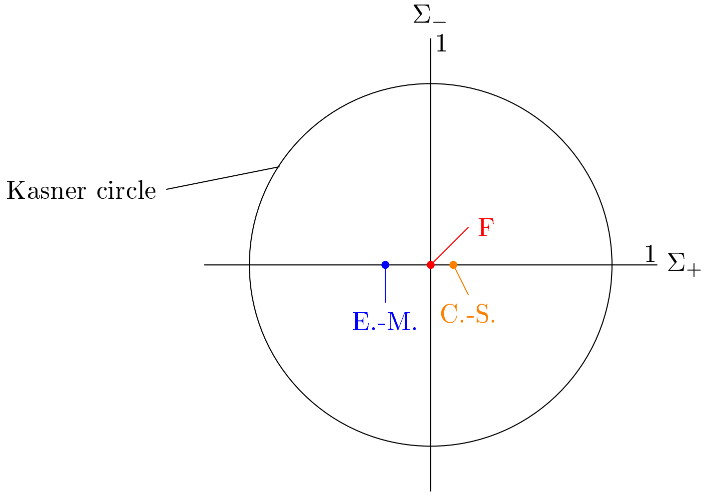

Figure 7. The different solutions projected to the $\Sigma_+\Sigma_-$-plane

7 Einstein-dust with small data

7.1 Einstein-dust system

Let us present briefly the Einstein-Euler system. We will consider the isentropic case where the matter fields are the energy density ρ and the four-velocity u^α which is a unit timelike vector. The equation of state $P = f(\rho)$ relates the pressure P with the energy density. The energy-momentum tensor is:

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + P g_{\alpha\beta}$$

and the equations of motion are equivalent to the condition that the energy-momentum tensor is divergence-free. The Einstein-dust system is then obtained via the condition $P = 0$. It can be seen as a very singular solution of the Einstein-Vlasov system. Formally the system can be obtained from the Einstein-Vlasov system choosing f to be of the form:

$$f(t, x^a, p^a) = |u_0||g^{(4)}|^{-\frac{1}{2}}\rho(t, x^a)\delta(p^a - u^a)$$

where u_0 is obtained via the mass shell relation. The relation of the Einstein-dust system to the Einstein-Vlasov system is in general subtle and we refer to [70] for more information on that. Here we will look at the special solutions of the corresponding Einstein-dust systems in order to obtain some intuition about the Einstein-Vlasov system. It is easy to see that the special solutions are equilibrium points. The stability of these equilibrium points has already been studied (see for instance [101]). For the case of Bianchi I there even exist a general expression [(11-1.12) of [41]], where one can see that isotropization occurs. Actually for all the Bianchi cases we study here, Liapunov functions have been found, such that besides the stability also the global behaviour is known.

7.2 Bianchi I with small data

In this section we will start dealing with estimates. C will denote an arbitrary constant and ϵ a small and strictly positive constant. They both may appear several times in different equations or inequalities without being the same constant.

Setting in (37) S_{ab} to zero we arrive at:

$$\dot{F} = -3H[F(1 - \frac{2}{3}F)] \tag{46}$$

Assuming now that $F \leq \epsilon$, it follows from the fact that $0 \leq F \leq \frac{3}{2}$ and (46) that

$$\partial_t(H^{-1}) \leq \frac{3}{2} + \epsilon$$

Integration (with $t_0 = \frac{2}{3}H^{-1}(t_0)$) leads to:

$$H \geq (\frac{2}{3} - \epsilon)t^{-1} \tag{47}$$

Using this inequality in (46) we obtain:

$$F = O(t^{-2+\epsilon}) \tag{48}$$

Integrating (20)

$$H(t) = \frac{1}{\frac{3}{2}t + I} = \frac{2}{3}t^{-1}\frac{1}{1 + \frac{2}{3}It^{-1}}$$

with

$$I = \int_{t_0}^{t} F(s)ds$$

Using (47) and (48) we obtain

$$H = \frac{2}{3}t^{-1}(1 + O(t^{-1}))$$

which used again in (46) leads to

$$F = O(t^{-2})$$

We expect that in the Vlasov case F will have a similar decay which is an indication for isotropization. However the estimate of F will turn out to be not sufficient. In addition to assume that we are close to the isotropic case, we will have to assume that we are close to the dust case. This is done via a momentum bound. We have a number (different from zero) of particles at possibly different momenta and we define P as the supremum of the absolute value of these momenta at a given time t:

$$P(t) = \sup\{|p| = (g^{ab}p_a p_b)^{\frac{1}{2}} | f(t,p) \neq 0\}$$

A bound on that quantity can be used for estimates on S/H^2 as we show now. Consider an orthonormal frame and denote the components of the spatial part of the energy-momentum tensor in this frame by $\widehat{S}_{ab}$. The components can be bounded by

$$\widehat{S}_{ab} \leq P^2(t)\rho$$

so we have that

$$\frac{\widehat{S}}{\rho} \leq 3P^2$$

from which follows with (33) that:

$$\frac{4\pi S}{3H^2} \leq \frac{3}{2}P^2.$$

Now if our idea of the asymptotics is correct we approach in a sense to be discussed the Einstein-de Sitter solution. Since in the Bianchi I case the structure constants vanish, the V_a are constant (see (24)). Putting these facts together, the expected decay of V is $O(t^{-\frac{4}{3}})$ and for P:

$$P = O(t^{-\frac{2}{3}})$$

7.3 Linearization of Einstein-dust around Collins-Stewart

Let us look at the stability of the Collins-Stewart solution with dust. For the Collins-Stewart solution we have ($\Sigma_+ = \frac{1}{8}, \Sigma_- = 0, N_1 = \frac{3}{4}$) which is an equilibrium point of the system (42)-(44) with $S = 0$. Let us translate the equilibrium point to the origin by introducing the variables:

$$\begin{aligned}\tilde{\Sigma}_+ &= \Sigma_+ - \frac{1}{8} \\ \tilde{\Sigma}_- &= \Sigma_- \\ \tilde{N}_1 &= N_1 - \frac{3}{4}\end{aligned}$$

The linearization is:

$$\begin{pmatrix}\tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{N}_1\end{pmatrix}' = \begin{pmatrix} -\frac{93}{64} & 0 & \frac{63}{128} \\ 0 & -\frac{3}{2} & 0 \\ -\frac{87}{32} & 0 & -\frac{3}{64}\end{pmatrix}\begin{pmatrix}\tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{N}_1\end{pmatrix}$$

The variable Σ_- decouples and we obtain:

$$\Sigma_- = \Sigma_-(\tau_0)e^{-\frac{3}{2}(\tau-\tau_0)}$$

The rest of the system:

$$\begin{pmatrix}\tilde{\Sigma}_+ \\ \tilde{N}_1\end{pmatrix}' = \begin{pmatrix} -93/64 & 63/128 \\ -87/32 & -3/64\end{pmatrix}\begin{pmatrix}\tilde{\Sigma}_+ \\ \tilde{N}_1\end{pmatrix}$$

has eigenvalues

$$\lambda_{1/2} = -\frac{3}{4}(1 \mp i\sqrt{\frac{3}{2}})$$

The matrix of eigenvectors and its inverse:

$$\mathbf{M_{II}} = \begin{pmatrix} 0 & 1 \\ \frac{32}{21}\sqrt{\frac{3}{2}} & \frac{10}{7}\end{pmatrix}; \mathbf{M_{II}^{-1}} = \begin{pmatrix} -\frac{5}{16}\sqrt{6} & \frac{21}{32}\sqrt{\frac{2}{3}} \\ 1 & 0\end{pmatrix}$$

which leads to:

$$\begin{pmatrix}\tilde{\Sigma}_+ \\ \tilde{N}_1\end{pmatrix} = e^{-\frac{3}{4}\tau}\begin{pmatrix} -\frac{5}{16}\sqrt{6}\sin\omega\tau + \cos\omega\tau & \frac{21}{32}\sqrt{\frac{2}{3}}\sin\omega\tau \\ -\frac{29}{24}\sqrt{6}\sin\omega\tau & \cos\omega\tau + \frac{15}{16}\sqrt{\frac{2}{3}}\sin\omega\tau\end{pmatrix}\begin{pmatrix}\tilde{\Sigma}_+(0) \\ \tilde{N}_1(0)\end{pmatrix}$$

with $\omega = \frac{3}{4}\sqrt{\frac{3}{2}}$.

Translated to our time variable means that the expected estimates for the Vlasov case are:

$$\begin{aligned}\Sigma_+ - \frac{1}{8} &= O(t^{-\frac{1}{2}}) \\ N_1 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\ \Sigma_- &= O(t^{-1})\end{aligned}$$

Whether this is true we do not know at this point. We will start proving estimates in the next chapter. Here we have obtained these estimates just in order to get a hint about the non-linear behaviour. For instance it could be the case that the variable Σ_- has the same decay as the other variables. For V due to (25) and introducing the Collins-Stewart solution we have:

$$\dot{V} = -t^{-1}g^{11}V_1^2 - \frac{3}{2}t^{-1}(g^{22}V_2^2 + g^{33}V_3^2)$$

We see that $\dot{V} \leq -t^{-1}V$ which implies that the following holds:

$$P = O(t^{-\frac{1}{2}})$$

7.4 Linearization of Einstein-dust around Ellis-MacCallum

For $S = 0$ we have:

$$\Sigma'_+ = -\frac{1}{6}(N_2 - N_3)^2 - \Sigma_+(2-q) \tag{49}$$

$$\Sigma'_- = \frac{N_3^2 - N_2^2}{2\sqrt{3}} - (2-q)\Sigma_- \tag{50}$$

$$N'_2 = N_2(2\Sigma_+ + 2\sqrt{3}\Sigma_- + q) \tag{51}$$

$$N'_3 = N_3(2\Sigma_+ - 2\sqrt{3}\Sigma_- + q) \tag{52}$$

with $q = \frac{1}{2} - \frac{1}{24}(N_2 - N_3)^2 + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2)$.

For the Ellis-MacCallum solution we have $(\Sigma_+ = -\frac{1}{4}, \Sigma_- = 0, N_2 = -N_3 = \frac{3}{4})$ which is an equilibrium point of the system (49)-(52). Introducing

$$\begin{aligned}
\tilde{\Sigma}_+ &= \Sigma_+ + \frac{1}{4} \\
\tilde{\Sigma}_- &= \Sigma_- \\
\tilde{N}_2 &= N_2 - \frac{3}{4} \\
\tilde{N}_3 &= N_3 + \frac{3}{4}
\end{aligned}$$

The linearization is:

$$\begin{pmatrix} \tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{N}_2 \\ \tilde{N}_3 \end{pmatrix}' = \begin{pmatrix} -\frac{21}{16} & 0 & -\frac{15}{32} & \frac{15}{32} \\ 0 & -\frac{3}{2} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} \\ \frac{15}{16} & \frac{3}{2}\sqrt{3} & -\frac{3}{32} & \frac{3}{32} \\ -\frac{15}{16} & \frac{3}{2}\sqrt{3} & \frac{3}{32} & -\frac{3}{32} \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{N}_2 \\ \tilde{N}_3 \end{pmatrix}$$

The eigenvalues are:

$$\begin{aligned}
\lambda_{1/2} &= -\frac{3}{4}(1 \pm i\sqrt{3}) \\
\lambda_{3/4} &= -\frac{3}{4}(1 \pm i)
\end{aligned}$$

and

$$\mathbf{M_{VI_0}} = \begin{pmatrix} 0 & 0 & \frac{3}{5} & -\frac{4}{5} \\ -\frac{1}{6}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M_{VI_0}^{-1}} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & \frac{1}{6}\sqrt{3} & \frac{1}{6}\sqrt{3} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{5}{4} & 0 & -\frac{3}{8} & \frac{3}{8} \end{pmatrix}$$

are the matrix of the eigenvectors and its inverse. Translated to our time variable the expected estimates for the Vlasov case are:

$$\begin{aligned}
\Sigma_+ + \frac{1}{4} &= O(t^{-\frac{1}{2}}) \\
\Sigma_- &= O(t^{-\frac{1}{2}}) \\
N_2 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\
N_3 + \frac{3}{4} &= O(t^{-\frac{1}{2}})
\end{aligned}$$

With the same procedure as in the Bianchi II case, using now the Ellis-MacCallum solution we arrive at:

$$P = O(t^{-\frac{1}{2}})$$

8 The bootstrap argument

The argument which will lead us to our main conclusions is a bootstrap argument, a kind of continuous induction argument. The argument will work as follows (see 10.3 of [76] for a detailed discussion). One has a solution of the evolution equations and assumes that the norm of that function depends continuously on the time variable. Assuming that one has small data initially at t_0, i.e. the norm of our function is small, one has to improve the decay rate of the norm such that the assumption that $[t_0, T)$ is the maximal interval with $T < \infty$ would lead to a contradiction. This is a way to obtain global existence for small data. In our case global existence is already clear but if the argument works we also obtain information about how the solution behaves asymptotically which is our goal. The interval we look at is $[t_0, t_1)$ and we will present the estimates assumed in the following for the different cases. All prefactors on the right hand side are positive and as small as we want.

8.1 Bootstrap assumptions

A first task is to find the suitable bootstrap assumptions. We choose a slightly slower decay for the anisotropy and the curvature variables than in the linearized cases with the hope that using the central equations, we are able to obtain the same decay as in the linearized case. For the estimate of P we start with a slower decay than the ones obtained in section 7 as well. The assumption of small data here is in the sense that our solutions are not “far away” from our special solutions. In general to improve an estimate the corresponding evolution equation will be integrated. The assumptions made for the different Bianchi cases exclude the vacuum case, since the values of Ω due to the constraint equation are near the corresponding values of Ω of the special solutions, thus far from being zero.

8.1.1 Bootstrap assumptions for Bianchi I

$$\begin{aligned} F &\leq A_I(1+t)^{-\frac{3}{2}} \\ P &\leq A_m(1+t)^{-\frac{7}{12}} \end{aligned}$$

8.1.2 Bootstrap assumptions for Bianchi II

$$\begin{aligned} |\Sigma_+ - \frac{1}{8}| &\leq A_+(1+t)^{-\frac{3}{8}} \\ |\Sigma_-| &\leq A_-(1+t)^{-\frac{3}{4}} \\ |N_1 - \frac{3}{4}| &\leq A_c(1+t)^{-\frac{3}{8}} \\ P &\leq A_m(1+t)^{-\frac{1}{3}} \end{aligned}$$

8.1.3 Bootstrap assumptions for Bianchi VI_0

$$
\begin{aligned}
|\Sigma_+ + \frac{1}{4}| &\leq A_+(1+t)^{-\frac{3}{8}} \\
|\Sigma_-| &\leq A_-(1+t)^{-\frac{3}{8}} \\
|N_2 - \frac{3}{4}| &\leq A_{c1}(1+t)^{-\frac{3}{8}} \\
|N_3 + \frac{3}{4}| &\leq A_{c2}(1+t)^{-\frac{3}{8}} \\
P &\leq A_m(1+t)^{-\frac{1}{3}}
\end{aligned}
$$

8.2 Estimate of the mean curvature

The first variable we estimate is the trace of the second fundamental form or equivalently the Hubble variable. Let us rewrite (20):

$$
\partial_t(H^{-1}) = \frac{3}{2} + D \tag{53}
$$

with

$$
D = \frac{1}{12}(\frac{R}{H^2} + \frac{3}{H^2}\sigma_{ab}\sigma^{ab}) + \frac{4\pi S}{3H^2}
$$

Integrating (20) and since $t_0 = \frac{2}{3}H^{-1}(t_0)$ (this choice was made in section (3.7)):

$$
H(t) = \frac{1}{\frac{3}{2}t + I} = \frac{2}{3}t^{-1}\frac{1}{1+\frac{2}{3}It^{-1}}
$$

with

$$
I = \int_{t_0}^{t} D(s)ds
$$

Now for Bianchi I we have:

$$
D_I = F + \frac{4\pi S}{3H^2}
$$

for Bianchi II:

$$
D_{II} = \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2} - \frac{N_1^2}{24}
$$

and for Bianchi VI_0:

$$
D_{VI_0} = \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2} - \frac{1}{24}(N_2 - N_3)^2
$$

It turns out that in all cases D is small, in particular from the different bootstrap assumptions we obtain for Bianchi I:

$$
D_I \leq \epsilon_1(1+t)^{-\frac{7}{6}}
$$

where $\epsilon_1 = C(A_I + A_m^2)$ and for Bianchi II and VI_0 respectively:

$$|D| \leq \epsilon_{2/3}(1+t)^{-\frac{3}{8}} \tag{54}$$

with

$$\epsilon_2 = C(A_+ + A_-^2 + A_c + A_m^2)$$
$$\epsilon_3 = C(A_+ + A_-^2 + A_{c1} + A_{c2} + A_m^2)$$

For Bianchi I we use the fact that I is bounded by ϵ_1 and for the other two cases we arrive at:

$$\frac{2}{3}t^{-1}I = O(\epsilon_{2/3}t^{-\frac{3}{8}})$$

The results for the Hubble variable in the case of Bianchi I:

$$\boxed{H = \frac{2}{3}t^{-1}(1 + O(\epsilon_1 t^{-1}))} \tag{55}$$

and in the other cases:

$$\boxed{H = \frac{2}{3}t^{-1}(1 + O(\epsilon_{2/3}t^{-\frac{3}{8}}))} \tag{56}$$

For all cases we also obtain an estimate for the determinant using the estimate of H and integrating (8) in both directions.

$$C(t_0)t^{4-\epsilon} \leq g(t) \leq C(t_0)t^{4+\epsilon} \tag{57}$$

8.3 Estimate of the metric

8.3.1 Matrix norms and related inequalities

For a matrix A its norm can be defined as:

$$\|A\| = \sup\{|Ax|/|x| : x \neq 0\}$$

Let B and C be $n \times n$ symmetric matrices with C positive definite. It is possible to define a *relative norm* by:

$$\|B\|_C = \sup\{|Bx|/|Cx| : x \neq 0\}$$

Clearly:

$$\|B\| \leq \|B\|_C\|C\|$$

It also true that:

$$\|B\|_C \leq \sqrt{\mathrm{tr}(C^{-1}BC^{-1}B)} \tag{58}$$

This can be shown as follows. Consider the common eigenbasis b_i of B and C. Then there exist α_i such that $Bb_i = \alpha_i Cb_i$ for each i. Then (58) is equivalent to the statement that the maximum modulus of any α_i is smaller than $\Sigma_i\alpha_i^2$. Using (58) we obtain in the sense of *quadratic forms*:

$$\sigma^{ab} \leq (\sigma_{cd}\sigma^{cd})^{\frac{1}{2}}g^{ab} \tag{59}$$

8.3.2 Estimate of the metric in the sense of quadratic forms for Bianchi I

Define:

$$\bar{g}^{ab} = t^{\frac{p}{q}} g^{ab}$$

Then

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) = t^{-\gamma-1}\bar{g}^{ab}(-\gamma + \frac{p}{q}) + 2t^{-\gamma+\frac{p}{q}}(\sigma^{ab} - Hg^{ab})$$

where we have introduced for technical reasons a small positive parameter γ. Using now the inequality (59)

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) \leq t^{-\gamma-1}\bar{g}^{ab}[-\gamma + \frac{p}{q} + 2tH((H^{-2}\sigma_{cd}\sigma^{cd})^{\frac{1}{2}} - 1)] \tag{60}$$

Introducing the estimate of H obtained for Bianchi I:

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) \leq t^{-\gamma-1}\bar{g}^{ab}[-\gamma + \frac{p}{q} + \frac{4}{3}(1 + O(\epsilon t^{-1}))((H^{-2}\sigma_{cd}\sigma^{cd})^{\frac{1}{2}} - 1)]$$

Choosing the constants ϵ smaller than γ and $\frac{p}{q} = \frac{4}{3}$ we will have that

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) \leq -\eta t^{-\gamma-1}\bar{g}^{ab}$$

with $\eta > 0$. From this it follows:

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) \leq 0$$

8.3.3 Estimate of the metric in the sense of components for Bianchi II and VI_0

For these cases consider the following equation in the sense of *components*:

$$\bar{g}^{ab} = t^{\frac{p}{q}} g^{ab}$$

In particular we will consider the components g^{22} and g^{33}, which means that for Bianchi II $\frac{p}{q} = \frac{3}{2}$ and for Bianchi VI_0 $\frac{p}{q} = 1$. We will show that:

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) = t^{-\gamma-1}\bar{g}^{ab}(-\gamma + \frac{p}{q} + \frac{\dot{g}^{ab}}{g^{ab}}t) \leq -\eta t^{-\gamma-1}\bar{g}^{ab}$$

with η positive with the help of the bootstrap assumptions and choosing γ in a suitable way. This means then that:

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) \leq 0$$

which implies what we wanted to show:

$$g^{ab}(t) \leq t_0^{-\gamma+\frac{p}{q}} g^{ab}(t_0) t^{-\frac{p}{q}+\gamma}. \tag{61}$$

For the covariant components one can do the same by defining $\bar{g}_{ab} = t^{-\frac{p}{q}} g_{ab}$. One obtains:

$$\frac{d}{dt}(t^\gamma \bar{g}_{ab}) = t^{\gamma-1}\bar{g}_{ab}(\gamma - \frac{p}{q} + \frac{\dot{g}_{ab}}{g_{ab}}t) \geq \eta t^{\gamma-1}\bar{g}^{ab}$$

For the last step one can actually use the same γ as for the contravariant components since $\dot{g}_{ab}g^{ab} = -g_{ab}\dot{g}^{ab}$. In other words once (61) is shown, we also have:

$$g_{ab}(t) \leq t_0^{\gamma-\frac{p}{q}} g_{ab}(t_0) t^{\frac{p}{q}-\gamma}$$

From the definitions made one can obtain:

$$\dot{g}^{11} = 2g^{11}H(-1+2\Sigma_+) \tag{62}$$
$$\dot{g}^{22} = 2g^{22}H(-1-\Sigma_+ - \sqrt{3}\Sigma_-) \tag{63}$$
$$\dot{g}^{33} = 2g^{33}H(-1-\Sigma_+ + \sqrt{3}\Sigma_-) \tag{64}$$

Then we have with (56) for the components g^{22} and g^{33}:

$$\begin{aligned} \eta &= \gamma + 2Ht(1+\Sigma_+ \pm \sqrt{3}\Sigma_-) - \frac{p}{q} \\ &= \gamma + \frac{4}{3}(1+O(\epsilon_{2/3}t^{-\frac{3}{8}}))(1+\Sigma_+ \pm \sqrt{3}\Sigma_-) - \frac{p}{q} \end{aligned}$$

In both Bianchi II and VI_0:

$$\frac{4}{3}(1+\Sigma_+) - \frac{p}{q} = O(A_+(1+t)^{-\frac{3}{8}})$$

which enables us to choose γ in such a way that η is positive. Different values of Σ_+ correspond to different exponents in the components of the metric. Using the estimates of g^{22} and g^{33} we obtain then the estimate for the other component of the metric g_{11} via the estimate of the determinant. We could also proceed directly from (62).

Summarizing this means that asymptotically up to a positive constant which depends only on t_0 the components (and their inverses) of the metrics g_{II} for Bianchi II and g_{VI_0} for Bianchi VI_0 have the same decay up to an ϵ as the corresponding components of the Collins-Stewart and Ellis-MacCallum solution respectively:

$$C(t_0)t^{-\epsilon} \leq \frac{g_{II}}{g_{CS}} \leq C(t_0)t^{+\epsilon}$$
$$C(t_0)t^{-\epsilon} \leq \frac{g_{VI_0}}{g_{EM}} \leq C(t_0)t^{+\epsilon}$$

8.4 Estimate of P

8.4.1 Bianchi I

For Bianchi I the characteristics are trivial, p_a is constant along the geodesics and we can conclude:

$$P(t) \leq t_0{}^{-\frac{\epsilon}{2}+\frac{2}{3}} P(t_0) t^{\frac{\epsilon}{2}-\frac{2}{3}}$$

Now since we can choose $P(t_0)$ and A_m independently as small as we want, let us choose $P(t_0)$ such that $t_0{}^{-\frac{\epsilon}{2}+\frac{2}{3}} P(t_0) \leq A_m$. Then

$$P(t) \leq A_m t^{\frac{\epsilon}{2}-\frac{2}{3}}$$

In order to improve the bootstrap assumption ϵ has to be smaller then $\frac{1}{6}$. Using the notation $\zeta = \frac{\epsilon}{2}$ the last inequality can be expressed as:

$$P(t) \leq A_m t^{-\frac{2}{3}+\zeta}$$

where $\zeta < \frac{1}{12}$.

8.4.2 Bianchi II and VI$_0$

We can express the derivative of the metric as follows:

$$\dot{g}^{bf} = 2H(\Sigma_a^b - \delta_a^b)g^{af}$$

It follows from (24) and using (26):

$$\dot{V} = \dot{g}^{bf}V_bV_f = 2H(\Sigma_a^b - \delta_a^b)g^{af}V_bV_f = 2H(\Sigma_1^1 g^{11}V_1^2 + \Sigma_2^2 g^{22}V_2^2 + \Sigma_3^3 g^{33}V_3^2) - 2HV$$

The maximum of Σ_1^1, Σ_2^2 and Σ_3^3 is for Bianchi II and VI$_0$ equal to $\frac{1}{4} + O(t^{-\frac{3}{8}})$. Thus:

$$\dot{V} \leq 2HV(-\frac{3}{4} + \epsilon t^{-\frac{3}{8}})$$

Using now the estimate of H and integrating :

$$V \leq V(t_0)(t/t_0)^{-1+\epsilon}$$

from which follows:

$$P \leq P(t_0)(t/t_0)^{-\frac{1}{2}+\epsilon}$$

Choosing $P(t_0) \leq A_m t_0^{\frac{1}{2}-\epsilon}$ we arrive at:

$$P \leq A_m t^{-\frac{1}{2}+\epsilon}$$

which is an improvement of the bootstrap assumption and which has the consequence that:

$$\boxed{\frac{S}{H^2} \leq Ct^{-1+\epsilon}} \tag{65}$$

8.5 Closing Bianchi I

Until now we have estimates for H and for P in the interval $[t_0, t_1)$. We need to improve the other variables. Although Bianchi II and Bianchi VI$_0$ are more complicated, the main argument will be the same as in Bianchi I. It is a kind of contradiction argument where one assumes that up to an ϵ the desired estimate is wrong and then one comes to a contradiction. For Bianchi I we have to improve the estimate for F coming from the bootstrap assumption. The desired estimate is $F(t_1) \leq A_I(1+t_1)^{-2+\epsilon}$. If this is the case (case I) the bootstrap argument will work and there is nothing more to do. Let us suppose now the opposite, that $F(t_1) > A_I(1+t_1)^{-2+\epsilon}$. Then define t_2 as the smallest number not smaller than t_0 with the property that $F(t_1) \geq A_I(1+t_1)^{-2+\epsilon}$. In this case, we have to distinguish between the case that $t_2 = t_0$ (Case IIa) and $t_2 > t_0$ (Case IIb). Let us look now at the evolution equation of F (37), in particular at the terms in square brackets. By using the bootstrap assumption for F:

$$F \leq CA_I(1+t)^{-\frac{3}{2}} \leq CA_I(1+t_0)^{-\frac{3}{2}} \leq \frac{\delta}{2}$$

where δ is a positive small constant. Using the constraint equation and the bootstrap assumption of P:

$$\frac{\widehat{S}}{H^2} \leq CP^2 \leq CA_m^2(1+t)^{-\frac{7}{6}} \leq CA_m^2(1+t_0)^{-\frac{7}{6}} \leq \frac{\delta}{4}$$

With the Cauchy-Schwarz inequality, the constraint equation again and supposing that $F > A_I(1+t)^{-2+\epsilon}$ in the interval $[t_2, t_1]$:

$$|C\frac{\sigma_{ab}\widehat{S}^{ab}}{H^3}| \leq F^{\frac{1}{2}}\frac{(\widehat{S}^{ab}\widehat{S}_{ab})^{\frac{1}{2}}}{\rho} \leq CFF^{-\frac{1}{2}}P^2 \leq CA_I^{-\frac{1}{2}}A_m^2(1+t)^{-\frac{1}{6}-\frac{\epsilon}{2}}F \leq \frac{\delta}{4}F$$

Note that although $A_I^{-\frac{1}{2}}$ may be a big quantity, since A_I and A_m are independent we can make A_m smaller to "correct" this. Using the estimate of H and the last three inequalities in (37) leads to:

$$\dot{F} \leq (-2+\varepsilon)(1-\delta)Ft^{-1}$$

where $\varepsilon = C(A_I + A_m^2)t_0^{-1}$. Now setting $\xi = \varepsilon + 2\delta - \varepsilon\delta$ we end up with:

$$\dot{F} \leq (-2+\xi)Ft^{-1}$$

which means that:

$$F(t_1) \leq F(t_2){t_2}^{2-\xi}{t_1}^{-2+\xi} \tag{66}$$

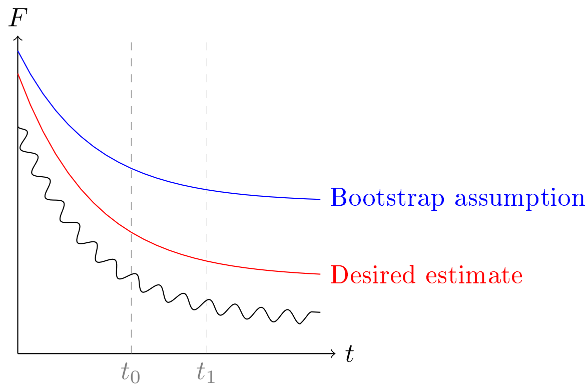

Figure 8. Schematic depiction of case I

8.5.1 Case IIa

In this case $t_2 = t_0$, so (66) means:

$$F(t_1) \leq F(t_0){t_0}^{2-\xi}{t_1}^{-2+\xi} \leq A_I{t_1}^{-2+\xi}$$

since we can choose $F(t_0)$ as small as we want. So it follows that in this case:

$$A_I{t_1}^{-2+\epsilon} \leq F(t_1) \leq A_I{t_1}^{-2+\xi}.$$

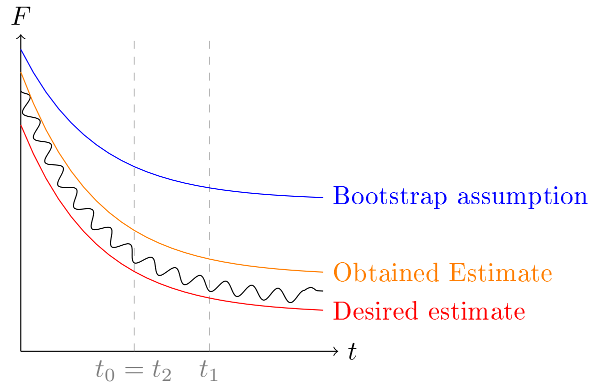

Figure 9. Schematic depiction of case IIa

8.5.2 Case IIb

In this case we can use the fact that by continuity $F(t_2) \leq A_I(1+t_2)^{-2+\epsilon}$ holds and then:

$$F(t_1) \leq A_I(1+t_2)^{-2+\epsilon}{t_2}^{2-\xi}{t_1}^{-2+\xi} \leq A_I(1+t_2)^{\epsilon-\xi}{t_1}^{-2+\xi}$$

The ϵ here is also a quantity which we can choose as small as we want and then it follows that in this case:

$$F(t_1) \leq A_I(1+t_0)^{\epsilon-\xi}{t_1}^{-2+\xi}$$

We can choose ϵ to be smaller than ξ and we obtain:

$$F(t_1) \leq A_I{t_1}^{-2+\xi}$$

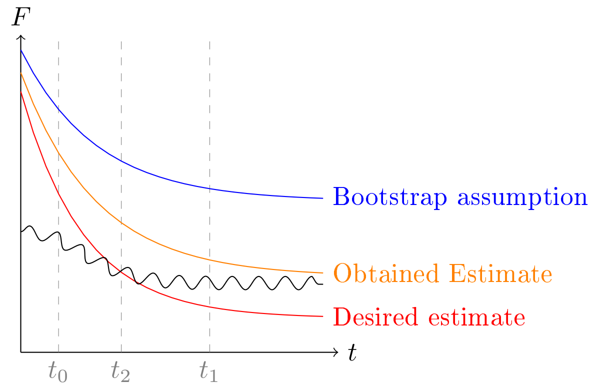

Figure 10. Schematic depiction of case IIb

8.6 Closing Bianchi II

8.6.1 Estimate for Σ_-

We will use the same argument as in the previous section for Bianchi I. If $|\Sigma_-| \leq A_-(1+t)^{-1+\epsilon}$ holds there nothing more to do, since this is a better estimate then the one assumed. Assume now $\Sigma_- > A_-(1+t)^{-1+\epsilon}$. Define again t_2 as the smallest number not smaller than t_0 with the property $\Sigma_- \geq A_-(1+t)^{-1+\epsilon}$. Since we are assuming that $\Sigma_- > 0$ we can divide (40) by Σ_-:

$$\frac{\dot{\Sigma}_-}{\Sigma_-} = H[-(3+\tfrac{\dot{H}}{H^2}) + \Sigma_-^{-1}\tfrac{8\pi}{2\sqrt{3}H^2}(S_2^2 - S_3^3)]$$

With (54),(56), the fact that $S_2^2 - S_3^3 \leq S$, (65) and our assumption:

$$\frac{\dot{\Sigma}_-}{\Sigma_-} \leq -t^{-1}(1-\xi) \tag{67}$$

where ξ is as small as we want. This variable ξ contains a term of type $A_-^{-1}A_m^2$, but A_- and A_m can be chosen independently as small as needed. Also the ϵ coming from S has to be chosen bigger than the ϵ coming from Σ_-. Note that it becomes clear here why we had to improve P to arrive at (65). Integrating (67) between t_1 and t_2:

$$\Sigma_-(t_1) \leq \Sigma_-(t_2)t_2^{1-\xi}t_1^{-1+\xi}$$

Assume $t_2 = t_0$ then:

$$\Sigma_-(t_1) \leq \Sigma_-(t_0)t_0^{1-\xi}t_1^{-1+\xi} \leq A_-t_1^{-1+\xi}$$

since $\Sigma_-(t_0)$ can be chosen in such a way that the last inequality holds. If $t_2 > t_0$ then by continuity $\Sigma_-(t_2) \leq A_-(1+t)^{-1+\epsilon}$ which means that:

$$\Sigma_-(t_1) \leq A_-(1+t)^{\epsilon-\xi}t_1^{-1+\xi} \leq A_-t_1^{-1+\xi}$$

if ϵ is chosen to be smaller than ξ. The argument for the case that Σ_- is negative is the same, just define $\bar{\Sigma}_- = -\Sigma_-$ and use $S_3^3 - S_2^2 \leq S$. This means that we could improve our bootstrap assumption to:

$$|\Sigma_-(t_1)| \leq A_-t_1^{-1+\xi}$$

8.6.2 Bootstrap assumptions for the other time variable

We have found that for the estimates in the following section it was useful, although not essential, to use the other time variable τ. Using the estimate of the Hubble variable (56) in the definition of τ (19) we have:

$$\tau - \tau_0 = \int_{t_0}^{t} \frac{2}{3}t^{-1}(1 + O(\epsilon t^{-\frac{3}{8}}))dt$$

After integrating and observing that τ_0 and t_0 are constants we arrive at:

$$t^{-\frac{2}{3}} = t_0^{-\frac{2}{3}}e^{-\tau+\tau_0+\xi}$$

where ξ is small: $\xi = O(\epsilon(t^{-\frac{3}{8}} + t_0^{-\frac{3}{8}}))$. So the bootstrap assumptions can be translated to the time variable τ. We obtain:

$$\begin{aligned}
|\tilde{\Sigma}_+| &< CA_+e^{-\frac{9}{16}\tau} \\
|\tilde{\Sigma}_-| &< CA_-e^{-\frac{9}{8}\tau} \\
|\tilde{N}_1| &< CA_ce^{-\frac{9}{16}\tau} \\
P &< CA_me^{-\frac{1}{2}\tau}
\end{aligned}$$

Since we have an estimate of H in both directions one can go also back from an estimate in terms of τ to an estimate of t just by a multiplication by a constant which will not be relevant.

8.6.3 Estimate for Σ_+ and N_1

Define

$$\begin{pmatrix} \hat{\Sigma}_+ \\ \hat{N}_1 \end{pmatrix} = \mathbf{M_{II}^{-1}} \begin{pmatrix} \tilde{\Sigma}_+ \\ \tilde{N}_1 \end{pmatrix}$$

Then we have:

$$\begin{pmatrix} \hat{\Sigma}_+ \\ \hat{N}_1 \end{pmatrix}' = -\frac{3}{4} \begin{pmatrix} 1 & \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} & 1 \end{pmatrix} \begin{pmatrix} \hat{\Sigma}_+ \\ \hat{N}_1 \end{pmatrix} + O(A_m^2 e^{-\tau}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

since $O(\tilde{\Sigma}_+^2 + \tilde{N}_1^2 + \tilde{\Sigma}_-^2 + P^2) = O(A_m^2 e^{-\tau})$. Multiplying the first equation by $\hat{\Sigma}_+$ and the second by $\hat{N}_1$ and adding both we obtain:

$$\frac{d}{dt}(\hat{\Sigma}_+^2 + \hat{N}_1^2) = -\frac{3}{2}(\hat{\Sigma}_+^2 + \hat{N}_1^2) + (\tilde{\Sigma}_+ + \tilde{N}_1)O(A_m^2 e^{-\tau})$$

$$\frac{d}{dt}[\log(\hat{\Sigma}_+^2 + \hat{N}_1^2)] = -\frac{3}{2} + (\tilde{\Sigma}_+ + \tilde{N}_1)(\hat{\Sigma}_+^2 + \hat{N}_1^2)^{-1} O(A_m^2 e^{-\tau})$$

Let us assume now that:

$$\tilde{\Sigma}_+^2 + \tilde{N}_1^2 > (A_+^2 + A_c^2) e^{(-\frac{3}{2}+\xi)\tau}$$

This implies:

$$\hat{\Sigma}_+^2 + \hat{N}_1^2 > C(A_+^2 + A_c^2) e^{(-\frac{3}{2}+\xi)\tau}$$

$$\frac{d}{d\tau}[\log(\hat{\Sigma}_+^2 + \hat{N}_1^2)] \leq -\frac{3}{2} + \epsilon e^{(-\frac{1}{16}-\xi)\tau}$$

From which follows that:

$$\begin{aligned} \hat{\Sigma}_+^2 + \hat{N}_1^2 &\leq (\hat{\Sigma}_+^2(\tau_0) + \hat{N}_1^2(\tau_0)) e^{(-\frac{3}{2}+\epsilon)(\tau-\tau_0)} \\ &\leq C(\hat{\Sigma}_+^2(t_0) + \hat{N}_1^2(t_0))(\frac{t}{t_0})^{-1+\epsilon} \end{aligned}$$

or:

$$\tilde{\Sigma}_+^2 + \tilde{N}_1^2 \leq C(\tilde{\Sigma}_+^2(t_0) + \tilde{N}_1^2(t_0))(\frac{t}{t_0})^{-1+\epsilon}$$

Making now the same argument as in the end of the estimate of Σ_- we arrive at improved estimates for $\tilde{\Sigma}_+$ and $\tilde{N}_1$:

$$|\tilde{\Sigma}_+| \leq \tilde{\Sigma}_+(t_0) t^{-\frac{1}{2}+\epsilon}$$
$$|\tilde{N}_1| \leq \tilde{N}_1(t_0) t^{-\frac{1}{2}+\epsilon}$$

i.e. for Σ_+ and N_1:

$$|\Sigma_+ - \frac{1}{8}| \leq A_+(1+t)^{-\frac{1}{2}+\epsilon}$$
$$|N_1 - \frac{3}{4}| \leq A_c(1+t)^{-\frac{1}{2}+\epsilon}$$

We have closed now the bootstrap argument. Note that for this last improvement of the estimates Σ_+ and N_1 we did not use the improved estimates for Σ_- and P.

8.7 Closing Bianchi VI_0

This case is analogous to Bianchi II. The bootstrap assumptions with the variable τ read:

$$\begin{aligned} |\tilde{\Sigma}_+| &< CA_+ e^{-\frac{9}{16}\tau} \\ |\tilde{\Sigma}_-| &< CA_- e^{-\frac{9}{16}\tau} \\ |\tilde{N}_2| &< CA_{c1} e^{-\frac{9}{16}\tau} \\ |\tilde{N}_3| &< CA_{c2} e^{-\frac{9}{16}\tau} \\ P &< CA_m e^{-\frac{1}{2}\tau} \end{aligned}$$

In this case what remains are the estimates for Σ_+, Σ_-, N_2 and N_3. In terms of the transformed linearization

$$\begin{pmatrix} \hat{\Sigma}_+ \\ \hat{\Sigma}_- \\ \hat{N}_2 \\ \hat{N}_3 \end{pmatrix} = \mathbf{M}_{\mathbf{VI0}}^{-1} \begin{pmatrix} \tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{N}_2 \\ \tilde{N}_3 \end{pmatrix}$$

we have:

$$\begin{pmatrix} \hat{\Sigma}_+ \\ \hat{\Sigma}_- \\ \hat{N}_2 \\ \hat{N}_3 \end{pmatrix}' = -\frac{3}{4} \begin{pmatrix} 1 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\Sigma}_+ \\ \hat{\Sigma}_- \\ \hat{N}_2 \\ \hat{N}_3 \end{pmatrix} + O(A_m^2 e^{-\tau}) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

since $O(\tilde{\Sigma}_+^2 + \tilde{N}_2^2 + \tilde{N}_3^2 + \tilde{\Sigma}_-^2 + P^2) = O(A_m^2 e^{-\tau})$. As in the Bianchi II case, we arrive with the same procedure at

$$\frac{d}{dt}[\log(\hat{\Sigma}_+^2 + \hat{\Sigma}_-^2)] = -\frac{3}{2} + (\tilde{\Sigma}_+ + \tilde{\Sigma}_-)(\hat{\Sigma}_+^2 + \hat{\Sigma}_-^2)^{-1} O(A_m^2 e^{-\tau})$$

$$\frac{d}{dt}[\log(\hat{N}_2^2 + \hat{N}_3^2)] = -\frac{3}{2} + (\tilde{N}_2 + \tilde{N}_3)(\hat{N}_2^2 + \hat{N}_3^2)^{-1} O(A_m^2 e^{-\tau})$$

and this means that:

$$\frac{d}{d\tau}[\log(\hat{\Sigma}_+^2 + \hat{\Sigma}_-^2)] \leq -\frac{3}{2} + \epsilon e^{(-\frac{1}{16}-\xi)\tau}$$

and a similar expression for N_2 and N_3 such that in the end we arrive at the estimates we wanted to obtain. In this case as well it was not necessary to use the improved estimate of P.

8.8 Results of the bootstrap argument

Let us summarize the results obtained in this chapter in the following three propositions:

Proposition 1. *Consider any C^∞ solution of the Einstein-Vlasov system with Bianchi I-symmetry and with C^∞ initial data. Assume that $F(t_0)$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$\begin{aligned} H(t) &= \frac{2}{3} t^{-1}(1 + O(t^{-1})) \\ F(t) &= O(t^{-2+\epsilon}) \\ P(t) &= O(t^{-\frac{2}{3}+\epsilon}) \end{aligned}$$

Proposition 2. *Consider any C^∞ solution of the Einstein-Vlasov system with reflection and Bianchi II symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0)-\frac{1}{8}|$, $|\Sigma_-(t_0)|$, $|N_1(t_0)-\frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$
\begin{aligned}
H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-\frac{1}{2}+\epsilon})) \\
\Sigma_+ - \frac{1}{8} &= O(t^{-\frac{1}{2}+\epsilon}) \\
\Sigma_- &= O(t^{-1+\epsilon}) \\
N_1 - \frac{3}{4} &= O(t^{-\frac{1}{2}+\epsilon}) \\
P(t) &= O(t^{-\frac{1}{2}+\epsilon})
\end{aligned}
$$

Proposition 3. *Consider any C^∞ solution of the Einstein-Vlasov system with reflection Bianchi VI_0 symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0)+\frac{1}{4}|$, $|\Sigma_-(t_0)|$, $|N_2(t_0)-\frac{3}{4}|$, $|N_3(t_0)+\frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$
\begin{aligned}
H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-\frac{1}{2}+\epsilon})) \\
\Sigma_+ + \frac{1}{4} &= O(t^{-\frac{1}{2}+\epsilon}) \\
\Sigma_- &= O(t^{-\frac{1}{2}+\epsilon}) \\
N_2 - \frac{3}{4} &= O(t^{-\frac{1}{2}+\epsilon}) \\
N_3 + \frac{3}{4} &= O(t^{-\frac{1}{2}+\epsilon}) \\
P(t) &= O(t^{-\frac{1}{2}+\epsilon})
\end{aligned}
$$

Proof: Consider the Bianchi I case. Let $[t_0, T^*)$, where T^* may be infinite, be the maximal interval on which a solution corresponding to the prescribed initial data exists and satisfies $F(t) \leq A_I(1+t)^{-\frac{3}{2}}$ and $P(t) \leq A_m(1+t)^{-\frac{7}{12}}$. If the data are small initially, then $F(t)$ and $P(t)$ are small and by continuity T^* is well-defined. Suppose that $T^* < \infty$. Since $F(t)$ and $P(t)$ are bounded the solution can be extended to a longer time interval $[t_0, T^*+\epsilon)$ for some $\epsilon > 0$. Since the data are sufficiently small we have shown in this chapter that in fact $F(t) \leq A_I(1+t)^{-2+\epsilon} < A_I(1+t)^{-\frac{3}{2}}$ and $P(t) \leq A_m(1+t)^{-\frac{2}{3}+\epsilon} < A_m(1+t)^{-\frac{7}{12}}$ on $[t_0, T*)$. Now by continuity $F(t)$ and $P(t)$ remain less than $A_I(1+t)^{-\frac{3}{2}}$ and $A_m(1+t)^{-\frac{7}{12}}$ respectively for a short time after T^*, but this is a contradiction. Thus $T^* = \infty$. The estimates obtained, i.e. $F \leq A_I(1+t)^{-2+\epsilon}$ and $P(t) \leq A_m(1+t)^{-\frac{2}{3}+\epsilon}$ thus hold globally. The estimate of H follows introducing the other estimates in (53). The same argument can be done analogously for Bianchi II and VI_0. □

In the next chapter we will improve the estimates such that we can get rid of the ϵ. However the results stated here in this section represent in fact the core of our results.

9 Main results

Until now we have obtained estimates which show that the decay rates of the different variables are up to an ϵ the decay rates one obtains from the linearization. The treatment of Bianchi I and the other two cases will be different since for the latter we already know that the components of the metric and the second fundamental form are bounded. Let us start with Bianchi I.

9.1 Bianchi I

We want to improve the estimate of F. For this reason we need an inequality in the other direction. From (37) we have:

$$\dot{F} \geq -3H(F + \sqrt{3}F^{\frac{1}{2}}P^2)$$

Implementing now the estimates of F and P coming from the results of the bootstrap argument:

$$\dot{F} \geq -3HF - 3HC(t_0)A_m^2 t^{-\frac{7}{3}+\epsilon}$$

Using now the estimate of H:

$$\dot{F} \geq -2t^{-1}F - C(t_0)A_m^2 t^{-\frac{10}{3}+\epsilon}$$

from which we obtain:

$$t^2F(t) \geq t_0^2 F(t_0) - \int_{t_0}^{t} C(t_0)A_m^2 s^{-\frac{4}{3}+\epsilon} ds$$

and

$$F(t) \geq t^{-2}(F(t_0)t_0^2 - A_m^2 C(t_0)t_0{}^{-\frac{1}{3}+\epsilon} + C(t_0)A_m^2 t^{-\frac{1}{3}+\epsilon}) \geq t^{-2}(F(t_0)t_0^2 - A_m^2 C(t_0)t_0{}^{-\frac{1}{3}+\epsilon})$$

Choosing now A_m small enough we have the following estimate:

$$F(t) \geq C(t_0)t^{-2} \tag{68}$$

Putting the last term of (37) with the help of (68) in the following manner:

$$|\frac{\sigma_{ab}\widehat{S}^{ab}}{H^3}| \leq CF(F^{-\frac{1}{2}}P^2) = FO(t^{-\frac{1}{3}+\gamma})$$

we can improve the estimate on F implementing the estimates in (37) with the result:

$$F = O(t^{-2})$$

For Bianchi I we want to prove that $|g_{ab}t^{-\frac{4}{3}}| \leq C$. Recall that:

$$\begin{aligned} \bar{g}_{ab} &= t^{-\frac{4}{3}} g_{ab} \\ \bar{g}^{ab} &= t^{+\frac{4}{3}} g^{ab}. \end{aligned}$$

We have also that:

$$\dot{\bar{g}}_{ab} = -\frac{2}{3}(2t^{-1} + k)\bar{g}_{ab} - 2t^{-\frac{4}{3}}\sigma_{ab}$$

Doing similar computations as in [56] we arrive at:

$$\|\bar{g}_{ab}(t)\| \leq \|\bar{g}_{ab}(t_0)\| + \int_{t_0}^{t} [\frac{2}{3}|2s^{-1} + k(s)| + 2(\sigma_{ab}\sigma^{ab}(s))^{\frac{1}{2}}]\|\bar{g}_{ab}(s)\| ds$$

and with Gronwall's inequality we obtain:

$$\|\bar{g}_{ab}(t)\| \leq \|\bar{g}_{ab}(t_0)\| \exp\{\int_{t_0}^{t} [\frac{2}{3}|2s^{-1} + k(s)| + 2(\sigma_{ab}\sigma^{ab}(s))^{\frac{1}{2}}] ds\} \leq C$$

Therefore $\bar{g}_{ab}$ is bounded for all $t \geq t_0$. The same holds for $\bar{g}^{ab}$ by similar computations. Thus:

$$|t^{-\frac{4}{3}}g_{ab}| \leq C$$
$$|t^{+\frac{4}{3}}g^{ab}| \leq C$$

From this we can conclude that:

$$\|\sigma_{ab}\| \leq Ct^{-\frac{2}{3}}$$

or

$$\sigma_{ab} = O(t^{-\frac{2}{3}})$$

Looking again at the derivative of $\bar{g}_{ab}$ and putting the facts which have been obtained together, we see that:

$$\dot{\bar{g}}_{ab} = O(t^{-2})$$

This is enough to conclude that:

Theorem 1. *Consider any C^∞ solution of the Einstein-Vlasov system with Bianchi I-symmetry and with C^∞ initial data. Assume that $F(t_0)$ and $P(t_0)$ are sufficiently small. Then:*

$$g_{ab} = t^{+\frac{4}{3}}[\mathcal{G}_{ab} + O(t^{-2})]$$
$$g^{ab} = t^{-\frac{4}{3}}[\mathcal{G}^{ab} + O(t^{-2})]$$

where $\mathcal{G}_{ab}$ and $\mathcal{G}^{ab}$ are independent of t.

Now putting together the estimates obtained:

Theorem 2. *Consider the same assumptions as in the previous theorem. Then at late times the following estimates hold:*

$$\begin{aligned} H(t) &= \frac{2}{3}t^{-1}(1 + O(t^{-1})) \\ F(t) &= O(t^{-2}) \\ P(t) &= O(t^{-\frac{2}{3}}) \end{aligned}$$

9.2 Arzela Ascoli

In the case of Bianchi I we are almost finished. Nevertheless we will include this case in the following for completeness and because it shows another way of obtaing the optimal estimate for F . We want to use the Arzela-Ascoli theorem. We will show the boundedness of the relevant variables and their derivatives. The variables F, Σ_-, Σ_+, N_1, N_2 and N_3 corresponding to the different Bianchi cases are bounded uniformly due to the constraint equation. In particular:

$$\begin{aligned} 0 \leq F &\leq \frac{3}{2} \\ \Sigma_+^2 + \Sigma_-^2 &\leq 1 \\ N_1^2 &\leq 12 \\ (N_2 - N_3)^2 &\leq 12 \end{aligned}$$

Note that N_3 is negative. The Hubble variable H is bounded for all Bianchi A types except IX and its derivative as well, since $3H^2 \geq -\dot{H} \geq H^2$ as we saw in section (4.2). We have also obtained with the bootstrap argument that P, which is non-negative, decays which means that S/H^2 is bounded. From the estimates obtained it is clear that g^{ab} and its derivative are bounded. Now having a look at the central equations we see that the derivatives of F, Σ_-, Σ_+, N_1, N_2 and N_3 are also bounded uniformly. If we can bound the derivative of S also the second derivatives of F, Σ_-, Σ_+, N_1, N_2, N_3 and H are bounded. For this purpose it is convenient to express the components of the energy momentum tensor in terms of integrals of the covariant momenta:

$$\begin{aligned} S_{ab}g^{ab} &= \int f(t,p)p_a p_b g^{ab}(1+g^{cd}p_c p_d)^{-\frac{1}{2}} g^{-\frac{1}{2}} dp_1 dp_2 dp_3 \\ &= \int f(t,p)V(1+V)^{-\frac{1}{2}} g^{-\frac{1}{2}} dp_1 dp_2 dp_3 \end{aligned}$$

The only term of the time derivative of S which could cause problems is the time derivative of the distribution function, since $\dot{V}$ and $\dot{g}^{ab}$ can be bounded by V and g^{ab} respectively and we know that S itself is bounded since S/H^2 is. The term with the time derivative of the distribution function can be handled with the Vlasov equation:

$$\begin{aligned} &\int \dot{f}(t,p)V(1+V)^{-\frac{1}{2}} g^{-\frac{1}{2}} dp_1 dp_2 dp_3 \\ = -&\int (p^0)^{-1} C^d_{ba} p^b p_d \frac{\partial f}{\partial p_a} V(1+V)^{-\frac{1}{2}} g^{-\frac{1}{2}} dp_1 dp_2 dp_3 \end{aligned}$$

Integrating by parts we obtain a term which can be bounded by S. Note that the momenta grow in the worst case with t^γ and that p_0 is also bounded from below since the particles are assumed to have mass. Now all the relevant quantities are bounded. Let $\{t_n\}$ be a sequence tending to infinity and let $F_n(t) = F(t+t_n)$, $(\Sigma_-)_n(t) = \Sigma_-(t+t_n)$, $(\Sigma_+)_n(t) = \Sigma_+(t+t_n)$, $(N_1)_n(t) = N_1(t+t_n)$, $(N_2)_n(t) = N_2(t+t_n)$, $(N_3)_n(t) = N_3(t+t_n)$, $H_n(t) = H(t+t_n)$ and $S_n(t) = S(t+t_n)$. Using the bounds already listed, the Arzela-Ascoli theorem [90] can be applied. This implies that, after passing to a subsequence, F_n, $(\Sigma_-)_n$, $(\Sigma_+)_n$ $(N_1)_n$, $(N_2)_n$, $(N_3)_n$, H_n and S_n converge uniformly on compact sets to a limit F_∞, $(\Sigma_-)_\infty$, $(\Sigma_+)_\infty$ $(N_1)_\infty$, $(N_2)_\infty$, $(N_3)_\infty$, H_∞ and S_∞ respectively. The first derivative of these variables converges to the corresponding derivative of the limits since we have been able to bound the derivative of S in the last section. Going to this limit it is easy to see that the variable D_∞ of section (4.2) is zero and consequently:

$$H_\infty = \frac{2}{3}t^{-1}$$

for all Bianchi types we are considering. From (62)-(64) we see that for Bianchi II and VI_0 we obtain the optimal decay rates for the metric and for its derivative. For Bianchi I we have the optimal decay of the metric as well. This implies that we obtain the optimal decay rates for P. Since S/H^2 is zero asymptotically we obtain the same estimates for F, Σ_-, Σ_+, N_1, N_2 and N_3 as in the Einstein-dust case. Introducing this estimates in (53), we also obtain the optimal estimate for H. Let us summarize the estimates.

9.3 Optimal estimates

Theorem 3. *Consider any C^∞ solution of the Einstein-Vlasov system with reflection and Bianchi II symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0) - \frac{1}{8}|$, $|\Sigma_-(t_0)|$, $|N_1(t_0) - \frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$\begin{aligned} H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-\frac{1}{2}})) \\ \Sigma_+ - \frac{1}{8} &= O(t^{-\frac{1}{2}}) \\ \Sigma_- &= O(t^{-1}) \\ N_1 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\ P(t) &= O(t^{-\frac{1}{2}}) \end{aligned}$$

Theorem 4. *Consider any C^∞ solution of the Einstein-Vlasov system with reflection Bianchi VI_0 symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0)+\frac{1}{4}|$, $|\Sigma_-(t_0)|$, $|N_2(t_0)-\frac{3}{4}|$, $|N_3(t_0)+\frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$\begin{aligned} H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-\frac{1}{2}})) \\ \Sigma_+ + \frac{1}{4} &= O(t^{-\frac{1}{2}}) \\ \Sigma_- &= O(t^{-\frac{1}{2}}) \\ N_2 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\ N_3 + \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\ P(t) &= O(t^{-\frac{1}{2}}) \end{aligned}$$

For the cases Bianchi II and VI_0 we are also able to obtain the optimal estimate for the metrics:

Corollary 1. *Consider the same assumptions as in the previous theorem concerning Bianchi II and VI_0 respectively. Then*

$$\begin{aligned} &g_{II} = t\,\mathrm{diag}(K_1, t^{1/2}K_2, t^{1/2}K_3) \\ &g_{VI_0} = t\,\mathrm{diag}(tK_4, K_5, K_6) \end{aligned}$$

with $K_n = C_n + O(t^{-\frac{1}{2}})$ and where C_1-C_6 are independent of time. The corresponding result for the inverse metric also holds.

We see that the error in the metrics comes from the error in Σ_+.

9.4 Kasner exponents

From (12) we see that the eigenvalues (45) of the second fundamental form with respect to the induced metric are also the solutions of:

$$\det(\sigma^i_j - [\lambda - \frac{1}{3}k]\delta^i_j) = 0$$

Let us define the eigenvalues of σ_{ij} with respect to g_{ij} by $\widehat{\lambda}_i$, we have that:

$$\widehat{\lambda}_i = \lambda_i - \frac{1}{3}k$$

Note that $\Sigma_i(\widehat{\lambda}_i)^2 = \sigma_{ab}\sigma^{ab}$. For Bianchi I we know that $\sigma_{ab}\sigma^{ab} = O(t^{-4})$, thus we see that the spacetime isotropizes at late times, in the sense that:

$$p_i = \frac{1}{3} + O(t^{-1})$$

where p_i are the generalized Kasner exponents. In the cases Bianchi II and VI_0 since everything is diagonal the Kasner exponents are easy to calculate. Using the optimal estimates for Σ_+, Σ_- and H and the fact that the sum of the generalized Kasner exponents is equal to one, we finally arrive at the generalized Kasner exponents for Bianchi II which are $(\frac{1}{4}, \frac{3}{8}, \frac{3}{8})$ and for Bianchi VI_0 $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ in both cases up to an error of order $O(t^{-\frac{1}{2}})$. Let us summarize these results:

Corollary 2. *Consider the same assumptions as in the previous theorem concerning Bianchi I. Then*

$$p_i = \frac{1}{3} + O(t^{-1})$$

Corollary 3. *Consider the same assumptions as in the previous theorem concerning Bianchi II and VI_0 respectively. Then:*

$$p_{II} = p_{CS} + O(t^{-\frac{1}{2}})$$
$$p_{VI_0} = p_{EM} + O(t^{-\frac{1}{2}})$$

We see that the error in the Kasner exponents comes from the error in H.

9.5 Estimates of the energy momentum tensor

Before coming to the estimates of the energy momentum tensor we show that for Bianchi II V_2 and V_3 become constants, something similar can be done for Bianchi VI$_0$. Define $E = g^{22}V_2^2 + g^{33}V_3^2$, then:

$$\dot{E} = \dot{g}^{22}V_2^2 + \dot{g}^{33}V_3^2 \leq 2H(-1 - \Sigma_+ + \sqrt{3}|\Sigma_-|)E$$

Integrating

$$\log[E/E(t_0)] = -\frac{3}{2}\log t/t_0 + O(\epsilon(t^{-\frac{1}{2}} + t_0^{-\frac{1}{2}}))$$

We have the following inequality for E:

$$E \leq Ct^{-\frac{3}{2}}$$

Since the components of the metric g^{22} and g^{33} tend to the corresponding components of the Collins-Stewart solution we see that V_2 and V_3 become constant asymptotically. The same is true in the case of the Ellis-MacCallum solution. Now since $f(t_0, p)$ has compact support on p, we obtain that there exists a constant C such that:

$$f(t, p) = 0 \quad |p_i| \geq C$$

Let us denote by $\hat{p}$ the momenta in an orthonormal frame. Since $f(t, \hat{p})$ is constant along the characteristics we have:

$$|f(t, \hat{p})| \leq \|f_0\| = \sup\{|f(t_0, \hat{p})|\}$$

Putting these facts together we arrive at the estimates which we summarize in the following:

Corollary 4. *Consider the same assumptions as in the previous theorem concerning Bianchi I. Then*

$$\rho = \rho_{ES}(1 + O(t^{-1}))$$
$$S_{ij} \leq C|f_0|t^{-\frac{10}{3}}$$

Corollary 5. *Consider the same assumptions as in the previous theorem concerning Bianchi II. Then*

$$\rho = \rho_{CS}(1 + O(t^{-\frac{1}{2}}))$$
$$S_{ij} \leq C|f_0|t^{-3}$$

Corollary 6. *Consider the same assumptions as in the previous theorem concerning Bianchi VI_0 respectively. Then*

$$\rho = \rho_{EM}(1 + O(t^{-\frac{1}{2}}))$$
$$S_{ij} \leq C|f_0|t^{-3}$$

The error in the energy density comes from the error in H.

Remark From the corollaries one can estimate the quotient S_{ij}/ρ which is $O(t^{-\frac{4}{3}})$ for Bianchi I and $O(t^{-1})$ for the other cases. That this quotient vanishes asymptotically means that the matter behaves as dust asymptotically as expected.

10 Non-diagonal Bianchi II

Before coming to the non-diagonal case we have a look at the tilted fluid models, since they are non-diagonal as well and they may help us to understand the non-diagonal case with collisionless matter. For the tilted Bianchi II we use the corresponding equations of [46] with $\gamma = 1$. We will not go into the details of the interpretation of the new variables which appear here, for this we refer to the mentioned work.

10.1 Tilted Bianchi II

The system in the tilted Bianchi II case is the following:

$$\begin{aligned}
\Sigma'_+ &= -(2-q)\Sigma_+ - 3\Sigma_3^2 + \frac{1}{3}N_1^2 + \frac{1}{2\sqrt{3}}\Sigma_3 N_1 v_3 \\
\Sigma'_- &= -(2-q)\Sigma_- + 2\sqrt{3}\Sigma_1^2 - \sqrt{3}\Sigma_3^2 - \frac{1}{2}\Sigma_3 N_1 v_3 \\
\Sigma'_1 &= -(2-q+2\sqrt{3}\Sigma_-)\Sigma_1 \\
\Sigma'_3 &= -(2-q-3\Sigma_+ - \sqrt{3}\Sigma_-)\Sigma_3 \\
N'_1 &= (q-4\Sigma_+)N_1 \\
v'_3 &= v_3(1-v_3^2)(-1-\Sigma_+ + \sqrt{3}\Sigma_-)
\end{aligned}$$

with

$$q = 2(1 - \frac{1}{12}N_1^2) - \frac{1}{2}(1 - \Sigma^2 - \frac{1}{12}N_1^2)(3 - v_3^2)$$

Let us have a look at the linearization around the Collins-Stewart solution which is an equilibrium point of the tilted system as well and which corresponds to $\Sigma_+ = \frac{1}{8}$, $N_1 = \frac{3}{4}$ and $\Sigma_- = \Sigma_1 = \Sigma_3 = v_3 = 0$:

$$\begin{pmatrix} \tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{\Sigma}_1 \\ \tilde{\Sigma}_3 \\ \tilde{N}_1 \\ \tilde{v}_3 \end{pmatrix}' = \begin{pmatrix} -\frac{93}{64} & 0 & 0 & 0 & \frac{63}{128} & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{9}{8} & 0 & 0 \\ -\frac{87}{32} & 0 & 0 & 0 & -\frac{3}{64} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{9}{8} \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{\Sigma}_1 \\ \tilde{\Sigma}_3 \\ \tilde{N}_1 \\ \tilde{v}_3 \end{pmatrix}$$

We see that the variables which did not appear in the diagonal case have decay rates which are between the ones considered previously. This is a good sign.

10.2 Equations of the non-diagonal case

Using (14) we arrive with (5) for $a \neq b$ to:

$$\dot{\Sigma}_a^b = H[\frac{R_a^b}{H^2} - \Sigma_a^b(3 + \frac{\dot{H}}{H^2}) - \frac{8\pi S_a^b}{H^2}]; \quad a \neq b$$

which together with (21)-(22), i.e.

$$\begin{aligned}
\dot{\Sigma}_+ &= H[\frac{2R - 3(R_2^2 + R_3^3)}{6H^2} - \Sigma_+(3 + \frac{\dot{H}}{H^2}) + \frac{4\pi}{3H^2}(3S_2^2 + 3S_3^3 - 2S)] \\
\dot{\Sigma}_- &= H[\frac{R_3^3 - R_2^2}{2\sqrt{3}H^2} - (3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi(S_2^2 - S_3^3)}{\sqrt{3}H^2}]
\end{aligned}$$

describe the evolution of Σ_b^a. With the formula (2c) of [12] for the Ricci tensor:

$$R_{ij} = -\frac{1}{2}C_{ki}^l(C_{lj}^k + g_{lm}g^{kn}C_{nj}^m) - \frac{1}{4}C_{nk}^m C_{ql}^p g_{jm} g_{ip} g^{kq} g^{ln} \qquad (69)$$

and

$$R_i^j = R_{ib}g^{bj} = -\frac{1}{2}C^l_{ki}g^{bj}(C^k_{lb} + g_{lm}g^{kn}C^m_{nb}) - \frac{1}{4}C^j_{nk}C^p_{ql}g_{ip}g^{kq}g^{ln} \tag{70}$$

We will now derive some expression concerning the derivative of (69):

$$\begin{aligned}\dot{R}_{ij} &= C^l_{ki}C^m_{nj}(k_{lm}g^{kn} - g_{lm}k^{kn}) + \\ &\quad \frac{1}{2}C^m_{nk}C^p_{ql}(k_{jm}g_{ip}g^{kq}g^{ln} + g_{jm}k_{ip}g^{kq}g^{ln} - g_{jm}g_{ip}k^{kq}g^{ln} - g_{jm}g_{ip}g^{kq}k^{ln})\end{aligned}$$

Thus:

$$\begin{aligned}g^{jr}\dot{R}_{ij} &= g^{jr}C^l_{ki}C^m_{nj}(k_{lm}g^{kn} - g_{lm}k^{kn}) + \\ &\quad \frac{1}{2}C^p_{ql}[C^m_{nk}k^r_m g_{ip}g^{kq}g^{ln} + C^r_{nk}(k_{ip}g^{kq}g^{ln} - g_{ip}(k^{kq}g^{ln} + g^{kq}k^{ln}))]\end{aligned}$$

For $r = i$ and relabelling the m with i for the terms with the prefactor $\frac{1}{2}$:

$$g^{ji}\dot{R}_{ij} = g^{ji}C^l_{ki}C^m_{nj}(k_{lm}g^{kn} - g_{lm}k^{kn}) + \frac{1}{2}C^p_{ql}C^i_{nk}[2k_{ip}g^{kq}g^{ln} - g_{ip}(k^{kq}g^{ln} + g^{kq}k^{ln}))]$$

Rearranging terms:

$$g^{ji}\dot{R}_{ij} = C^l_{ki}C^m_{nj}(k_{lm}g^{kn}g^{ji} - g_{lm}k^{kn}g^{ji}) + C^p_{ql}C^i_{nk}[k_{ip}g^{kq}g^{ln} - g_{ip}k^{kq}g^{ln}].$$

We see that the first with the third and the second with the fourth term cancel each other, hence:

$$g^{ji}\dot{R}_{ij} = 0 \tag{71}$$

The evolution equation for the Ricci scalar due to (71) is:

$$\dot{R} = 2R^i_jk^j_i = 2H(-R + R^i_j\Sigma^j_i)$$

Define

$$N_i^j = \frac{R_i^j}{H^2}$$

The derivative of this expression is:

$$\dot{N}_i^j = \frac{g^{pj}\dot{R}_{pi}}{H^2} + 2H(N_i^p\Sigma_p^j - (1 + \frac{\dot{H}}{H^2})N_i^j)$$

Consider the quantity $N = R/H^2$. Its evolution equation is:

$$\dot{N} = 2H[qN + N^i_j\Sigma^j_i]$$

10.3 Curvature expressions for Bianchi II

For bookkeeping reasons we define the following quantity

$$A = g^{22}g^{33} - (g^{23})^2 = \frac{g_{11}}{g}$$

Using (70) for Bianchi II:

$$R_i^j = \frac{1}{2}g_{11}[C^1_{2i}(g^{23}g^{2j} - g^{22}g^{3j}) + C^1_{i3}(g^{23}g^{3j} - g^{33}g^{2j})] + \frac{1}{2}g_{i1}C^j_{23}A$$

We obtain:

$$R = -\frac{1}{2}g_{11}A = -\frac{1}{2}\frac{(g_{11})^2}{g}$$

and as in the diagonal case:

$$\begin{aligned} R_1^1 &= -R = -R_2^2 = -R_3^3 \\ R_1^2 &= R_1^3 = R_2^3 = R_3^2 = 0 \end{aligned}$$

However in the non-diagonal case we have:

$$\begin{aligned} R_2^1 &= -2\frac{g_{12}}{g_{11}}R \\ R_3^1 &= -2\frac{g_{13}}{g_{11}}R \end{aligned}$$

Thus

$$\dot{N} = -2H[(1 + \frac{\dot{H}}{H^2} + 4\Sigma_+)N - W_{II}]$$

where $W_{II} = N_2^1\Sigma_1^2 + N_3^1\Sigma_1^3$. In order to calculate the derivative of N_2^1 we need the following expression:

$$R\frac{d}{dt}(-2\frac{g_{12}}{g_{11}}) = 2H[2\Sigma_2^1 R + (3\Sigma_+ + \sqrt{3}\Sigma_-)R_2^1 - \frac{1}{2R}((R_2^1)^2\Sigma_1^2 + R_3^1 R_2^1 \Sigma_1^3)]$$

Hence:

$$\begin{aligned} \dot{N}_2^1 &= H[4N\Sigma_2^1 - 2(\Sigma_+ + 1 - \sqrt{3}\Sigma_- + \frac{\dot{H}}{H^2})N_2^1 + W_2^1] \\ \dot{N}_3^1 &= H[4N\Sigma_3^1 - 2(\Sigma_+ + 1 + \sqrt{3}\Sigma_- + \frac{\dot{H}}{H^2})N_3^1 + W_3^1] \end{aligned}$$

where

$$\begin{aligned} W_2^1 &= -2\Sigma_2^3 N_3^1 + N_2^1 N^{-1}(\Sigma_1^2 N_2^1 + \Sigma_1^3 N_3^1) \\ W_3^1 &= -2\Sigma_3^2 N_2^1 + N_3^1 N^{-1}(\Sigma_1^2 N_2^1 + \Sigma_1^3 N_3^1) \end{aligned}$$

10.4 The non-diagonal asymptotics of Bianchi II

We will now discuss the asymptotics of the non-diagonal case for Bianchi II. The structure of the analysis is very similar to the diagonal case. We start with a bootstrap argument and end with applying Arzela-Ascoli.

10.4.1 Bootstrap assumptions

Next we will collect the bootstrap assumptions. The prefactors denoted by A and some index are small constants.

$$\begin{aligned}
|\Sigma_+ - \frac{1}{8}| &\leq A_+(1+t)^{-\frac{3}{8}} \\
|N + \frac{9}{32}| &\leq A_c(1+t)^{-\frac{3}{8}} \\
|\Sigma_2^1| &\leq A_{12} \\
|\Sigma_3^1| &\leq A_{13} \\
|N_2^1| &\leq A_{c12} \\
|N_3^1| &\leq A_{c13} \\
P &\leq A_m(1+t)^{-\frac{1}{3}} \\
|\Sigma_-| &\leq A_-(1+t)^{-\frac{3}{4}} \\
|\Sigma_3^2| &\leq A_{23}(1+t)^{-\frac{3}{4}} \\
|\Sigma_2^3| &\leq A_{32}(1+t)^{-\frac{3}{4}} \\
|\Sigma_1^2| &\leq A_{21}(1+t)^{-\frac{3}{4}} \\
|\Sigma_1^3| &\leq A_{31}(1+t)^{-\frac{3}{4}}
\end{aligned}$$

10.4.2 Mean curvature

Concerning the estimate of H there is no difference with respect to the diagonal case. The reason is that the estimate of D

$$D = \frac{1}{12}(N + \frac{3}{H^2}\sigma_{ab}\sigma^{ab}) + \frac{4\pi S}{3H^2}$$

is the same. Thus as in the diagonal case it follows from (20) that

$$\partial_t(H^{-1}) = \frac{3}{2} + O(\epsilon t^{-\frac{3}{8}})$$

and following the steps made for the diagonal case we arrive at:

$$\boxed{H = \frac{2}{3}t^{-1}(1 + O(\epsilon t^{-\frac{3}{8}}))}$$

will hold.

10.4.3 Estimate of the metric and P

Using the equation (60) and the estimate of H

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) \leq t^{-\gamma-1}\bar{g}^{ab}[-\gamma + \frac{p}{q} + \frac{4}{3}(1 + O(\epsilon t^{-\frac{3}{8}}))((H^{-2}\sigma_{cd}\sigma^{cd})^{\frac{1}{2}} - 1)]$$

We obtain decay for the metric (in the sense of quadratic forms) provided that $(H^{-2}\sigma_{cd}\sigma^{cd})^{\frac{1}{2}} \leq 1$. This holds for Bianchi II with for instance $\frac{p}{q} = 0.4$. Thus we have

$$g^{ab} \leq t^{-\frac{p}{q}} t_0^{\frac{p}{q}} g^{ab}(t_0)$$

This implies that the components of the metric are also bounded by some constant $C(t_0)$ which depends on the terms of $g^{ab}(t_0)$. Consider now

$$\dot{g}^{bf} = 2H(\Sigma_a^b - \delta_a^b)g^{af}$$

Since the metric components are bounded the non-diagonal terms will contribute only with an ϵ. Thus we have for every component g^{ij} (no summation over the indices in the following equation):

$$\dot{g}^{ij} = 2H(\Sigma_i^i - 1 + \epsilon)g^{ij} \leq 2H(\max(\Sigma_i^i) - 1 + \epsilon)g^{ij} = 2H(-\frac{3}{4} + \epsilon)g^{ij}$$

Using now the estimate of H

$$\dot{g}^{ij} \leq t^{-1}(-1 + \epsilon)g^{ij} \tag{72}$$

One can conclude that

$$\|g^{-1}\| \leq O(t^{-1+\epsilon})$$

From (72)

$$\dot{V} = \dot{g}^{bf} V_b V_f \leq t^{-1}(-1 + \epsilon)V$$

which means that

$$V = O(t^{-1+\epsilon})$$

which gives us the same decay for P as in the diagonal case:

$$P = O(t^{-\frac{1}{2}+\epsilon})$$

10.4.4 Closing the bootstrap argument for Bianchi II

It follows immediately by the same arguments as in the diagonal case:

$$\begin{aligned}
\Sigma_- &= O(t^{-1+\epsilon}) \\
\Sigma_1^2 &= O(t^{-1+\epsilon}) \\
\Sigma_1^3 &= O(t^{-1+\epsilon}) \\
\Sigma_2^3 &= O(t^{-1+\epsilon}) \\
\Sigma_3^2 &= O(t^{-1+\epsilon})
\end{aligned}$$

Defining $(N_1)^2 = -2N$ we arrive at:

$$\begin{aligned}
\dot{\Sigma}_+ &= H[\frac{(N_1)^2}{3} - \Sigma_+(3 + \frac{\dot{H}}{H^2}) + \frac{4\pi}{3H^2}(3S_2^2 + 3S_3^3 - 2S)] \\
\dot{\Sigma}_- &= H[-(3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi(S_2^2 - S_3^3)}{\sqrt{3}H^2}] \\
\dot{N}_1 &= H[(1 + \frac{\dot{H}}{H^2} + 4\Sigma_+)N_1 + 2\frac{W_{II}}{N_1}]
\end{aligned}$$

Since $2\frac{W_{II}}{N_1}$ decays like $t^{-1+\epsilon}$ we see that we can apply the same arguments as in the diagonal case to obtain an improvement of the bootstrap assumptions:

$$\begin{aligned}
\Sigma_+ - \frac{1}{8} &= O(t^{-\frac{1}{2}+\epsilon}) \\
\Sigma_- &= O(t^{-\frac{1}{2}+\epsilon}) \\
N_1 - \frac{3}{4} &= O(t^{-\frac{1}{2}+\epsilon})
\end{aligned}$$

The system which remains using the time variable τ is the following:

$$
\begin{aligned}
(\Sigma_2^1)' &= \Sigma_2^1(q-2) + N_2^1 - \frac{8\pi S_2^1}{H^2} \\
(\Sigma_3^1)' &= \Sigma_3^1(q-2) + N_3^1 - \frac{8\pi S_3^1}{H^2} \\
(N_2^1)' &= -2(N_1)^2\Sigma_2^1 - 2(\Sigma_+ - q - \sqrt{3}\Sigma_-)N_2^1 + W_2^1 \\
(N_3^1)' &= -2(N_1)^2\Sigma_3^1 - 2(\Sigma_+ - q + \sqrt{3}\Sigma_-)N_3^1 + W_3^1
\end{aligned}
$$

Let us focus on the $\Sigma_2^1 - N_2^1$-system. Using the estimates obtained we arrive at:

$$
\begin{pmatrix} \Sigma_2^1 \\ N_2^1 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{9}{8} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \Sigma_2^1 \\ N_2^1 \end{pmatrix} + O(\epsilon e^{(-\frac{3}{4}+\epsilon)\tau}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

Let us go to the basis of eigenvectors of the linear system via the linear transformation

$$
\begin{pmatrix} \check{\Sigma}_2^1 \\ \check{N}_2^1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{3}{2} & 2 \end{pmatrix} \begin{pmatrix} \Sigma_2^1 \\ N_2^1 \end{pmatrix}
$$

Thus we arrive at

$$
\begin{pmatrix} \check{\Sigma}_2^1 \\ \check{N}_2^1 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{4} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \check{\Sigma}_2^1 \\ \check{N}_2^1 \end{pmatrix} + O(\epsilon e^{(-\frac{3}{4}+\epsilon)\tau}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

Integrating and using the usual contradiction argument we obtain

$$
\begin{aligned}
|\check{\Sigma}_2^1| &= \check{\Sigma}_2^1(\tau_0) e^{(-\frac{3}{4}+\epsilon)\tau} \\
|\check{N}_2^1| &= \check{N}_2^1(\tau_0) + O(\epsilon e^{(-\frac{3}{4}+\epsilon)\tau})
\end{aligned}
$$

Going back to the variables Σ_2^1 and N_2^1 via

$$
\begin{pmatrix} \Sigma_2^1 \\ N_2^1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} \check{\Sigma}_2^1 \\ \check{N}_2^1 \end{pmatrix}
$$

$$
\begin{aligned}
\Sigma_2^1(\tau) &= [2\Sigma_2^1(\tau_0) - \frac{4}{3}N_2^1(\tau_0)]e^{(-\frac{3}{4}+\epsilon)\tau} + \frac{4}{3}N_2^1(\tau_0) - \Sigma_2^1(\tau_0) + O(\epsilon e^{(-\frac{3}{4}+\epsilon)\tau}) \\
N_2^1(\tau) &= [\frac{3}{2}\Sigma_2^1(\tau_0) - N_2^1(\tau_0)]e^{(-\frac{3}{4}+\epsilon)\tau} + 2N_2^1(\tau_0) - \frac{3}{2}\Sigma_2^1(\tau_0) + O(\epsilon e^{(-\frac{3}{4}+\epsilon)\tau})
\end{aligned}
$$

Changing back to the time variable t:

$$
\begin{aligned}
\Sigma_2^1(t) &= C(t_0)[2\Sigma_2^1(t_0) - \frac{4}{3}N_2^1(t_0)]t^{-\frac{1}{2}+\epsilon} + \frac{4}{3}N_2^1(t_0) - \Sigma_2^1(t_0) + O(\epsilon) \\
N_2^1(t) &= C(t_0)[\frac{3}{2}\Sigma_2^1(t_0) - N_2^1(t_0)]t^{-\frac{1}{2}+\epsilon} + 2N_2^1(t_0) - \frac{3}{2}\Sigma_2^1(t_0) + O(\epsilon)
\end{aligned}
$$

where C is a constant, in particular $C(t_0) = t_0^{\frac{1}{2}} e^{-\frac{3}{4}\tau_0}$. The only term which could prevent us from improving the estimates is the ϵ coming from the bootstrap assumptions of Σ_2^1, but note that it comes in combination with Σ_1^2 as a product of both, thus the last term $O(\epsilon)$ on the right hand side of the last two equations does not prevent us from improving our estimates. Thus if we wait long time enough and choose $N_2^1(t_0)$ and $\Sigma_2^1(t_0)$ small enough we will have an improvement for N_2^1 and Σ_2^1. There is no difference in the procedure for N_3^1 and Σ_3^1.

10.4.5 Arzela-Ascoli for Bianchi II

Since all estimates have been improved we can apply Arzela-Ascoli and we arrive for Σ_2^1 and N_2^1 to:

$$\begin{aligned}
\Sigma_2^1(t=\infty) &= \frac{4}{3}N_2^1(t_0) - \Sigma_2^1(t_0) \\
N_2^1(t=\infty) &= 2N_2^1(t_0) - \frac{3}{2}\Sigma_2^1(t_0)
\end{aligned}$$

Consider now the following transformation of the basis vector

$$\begin{aligned}
&\tilde{e}_1 = e_1 \\
&\tilde{e}_2 = e_2 + ae_1 \\
&\tilde{e}_3 = e_3 + be_1
\end{aligned}$$

It preserves the Lie-algebra, i.e. the Bianchi type. The following relation holds between the variables Σ_2^1 and Σ_3^1 in the different basis:

$$\begin{aligned}
\begin{pmatrix} \tilde{\Sigma}_1^1 & \tilde{\Sigma}_1^2 & \tilde{\Sigma}_1^3 \\ \tilde{\Sigma}_2^1 & \tilde{\Sigma}_2^2 & \tilde{\Sigma}_2^3 \\ \tilde{\Sigma}_3^1 & \tilde{\Sigma}_3^2 & \tilde{\Sigma}_3^3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_1^1 & 0 & 0 \\ \Sigma_2^1 & \Sigma_2^2 & 0 \\ \Sigma_3^1 & 0 & \Sigma_3^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Sigma_1^1 & 0 & 0 \\ \Sigma_2^1 + a(\Sigma_1^1 - \Sigma_2^2) & \Sigma_2^2 & 0 \\ \Sigma_3^1 + b(\Sigma_1^1 - \Sigma_3^3) & 0 & \Sigma_3^3 \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_1^1 & 0 & 0 \\ \Sigma_2^1 + a(3\Sigma_+ + \sqrt{3}\Sigma_-) & \Sigma_2^2 & 0 \\ \Sigma_3^1 + b(3\Sigma_+ - \sqrt{3}\Sigma_-) & 0 & \Sigma_3^3 \end{pmatrix}
\end{aligned}$$

We see that choosing $a = -\frac{8}{3}\Sigma_2^1(\infty)$ and $b = -\frac{8}{3}\Sigma_3^1(\infty)$ the transformed variables $\tilde{\Sigma}_2^1$, $\tilde{\Sigma}_3^1$ are zero asymptotically. By direct calculation one can see that the same is true for the transformed variables $\tilde{N}_2^1$ and $\tilde{N}_3^1$. Thus we obtain the same asymptotics as in the diagonal case and we can conclude:

Theorem 5. *Consider any C^∞ solution of the Einstein-Vlasov system with Bianchi II symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0) - \frac{1}{8}|$, $|\Sigma_-(t_0)|$, $|\Sigma_2^1(t_0)|$, $|\Sigma_3^1(t_0)|$, $|\Sigma_3^2(t_0)|$, $|\Sigma_2^3(t_0)|$, $|\Sigma_1^2(t_0)|$, $|\Sigma_1^3(t_0)|$, $|N_1(t_0) - \frac{3}{4}|$, $|N_2^1(t_0)|$, $|N_3^1(t_0)|$ and $P(t_0)$ are sufficiently small. Then at late times, after possibly a basis change, the following estimates hold:*

$$\begin{aligned}
H(t) &= \frac{2}{3}t^{-1}(1 + O(t^{-\frac{1}{2}})) \\
\Sigma_+ - \frac{1}{8} &= O(t^{-\frac{1}{2}}) \\
\Sigma_- &= O(t^{-1}) \\
\Sigma_2^1 &= O(t^{-\frac{1}{2}}) \\
\Sigma_3^1 &= O(t^{-\frac{1}{2}}) \\
\Sigma_3^2 &= O(t^{-1}) \\
\Sigma_2^3 &= O(t^{-1}) \\
\Sigma_1^2 &= O(t^{-1}) \\
\Sigma_1^3 &= O(t^{-1}) \\
N_1 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\
N_2^1 &= O(t^{-\frac{1}{2}}) \\
N_3^1 &= O(t^{-\frac{1}{2}}) \\
P(t) &= O(t^{-\frac{1}{2}})
\end{aligned}$$

11 Conclusions and Outlook

Our theorem concerning Bianchi I can be seen as a generalization of theorem 5.4 of [73] since we obtain the same the result, but a) we also obtain how fast the expressions converge b) we obtain an asymptotic expression for the spatial metric c) we do not assume any of the additional symmetries mentioned. However we used a different kind of restriction namely the small data assumptions.

The situation of several (tilted) fluids leads naturally to the non-diagonal case. The Bianchi I-symmetry implies the absence of a matter current such that a single *tilted* fluid is not compatible with this assumption. However there can be several tilted fluids such that the total current vanishes. In [93] and [92] this has been considered in the case of two fluids. What was found is that isotropization occurs if at least one of the two fluids has a speed of sound which is less or equal $\frac{1}{3}$ the speed of light. It is shown in particular for the case of two pressure free fluids in [92], which can be seen as a singular solution of the Einstein-Vlasov system.

The results concerning Bianchi II generalize the results obtained in [79]. For Bianchi VI_0 even for the reflection symmetric case there is no analogous previous result. The reason is that it is not compatible with the LRS-symmetry. Thus our result concerning Bianchi VI_0 shows clearly that the methods developed are powerful in the sense that one can obtain results which where out of reach with the techniques developed until now. Actually in our case one can see that there is no big difference between Bianchi II and VI_0. Maybe it is related to the fact, that the solutions which play the role of the attractors are self-similar. It would be interesting to investigate whether the work on homogeneous Ricci solitons [36] can help to understand the similarities between Bianchi II and VI_0 (in Thurstons classification Nil and Sol).

Another question is whether it is possible to remove the small data assumptions. Maybe it is possible to use the Liapunov functions discovered for the fluid model in a clever way. In the case of Bianchi II the future asymptotics are known globally even in the tilted case [45].

Of course there are many other ways of generalizing results which have been obtained for (single) perfect fluids. See for instance [55] and references therein for the inclusion of a Maxwell field in the Bianchi I case. In presence of a cosmological constant the results of [56] have been generalized even to the Einstein-Vlasov-Maxwell case [66].

In our argument we have used the Arzela-Ascoli theorem, but only at the end. Thus there exist a lot of estimates where one has control over the constants involved. Maybe this could help for a numerical analysis of the Einstein-Vlasov equation which is quite difficult.

We have discussed the future asymptotics of some Bianchi models, what about the higher types? First of all we believe that the non-diagonal case of Bianchi VI_0 can be done in a similar way to Bianchi II. We encountered some problems which we believe are of only technical nature. However the case of Bianchi VII_0 will probably be quite different. For instance in [103] it was discovered that the Bianchi VII_0 spacetimes with a non-tilted fluid are not asymptotically self-similar in the future and that some oscillations take place. It is shown that dynamics are dominated by the Weyl curvature. However for dust a bifurcation of the Weyl curvature takes place (theorem 2.4 of [103] and comments below). For this reason it is likely to expect difficulties when applying our techniques to this case. Something similar, but even more complicated happens in the case of Bianchi VIII spacetimes with a non-tilted fluid [50].

What about inhomogeneous models? Some direction to generalize our results could be to analyze the Gowdy model which is the simplest inhomogeneous case. In [77] different links between Bianchi and (twisted) Gowdy spacetimes are considered, in particular for Bianchi I, II, VI_0 and VII_0. For vacuum geodesic completeness was already shown [82]. The analysis of perturbations is another interesting approach towards the understanding of inhomogeneous models. See [3] for recent developments.

Another path of generalizing our results could be the extension to higher dimensions. For the vacuum case geodesic completeness was shown for some homogeneous models in higher dimensions [35]. The work on homogeneous spacetimes in higher dimensions may also shed some light on the inhomogeneous case in four spacetime dimensions.

We have been dealing with the expanding direction. To analyze the initial singularity is much more difficult. The singularity theorems were a very important breakthrough, however they do not tell us what happens dynamically at the singularity. The work of BKL [7]-[6] is an important step in this direction. See [5] and [59] for recent progress and [60],[61] for some surprising discoveries. As a first step one could analyze the Bianchi I case. In [42] the possible dynamical behavior towards the past has been determined assuming only the reflection symmetry and already there surprising new features like the existence of heteroclinic networks arose. Thus one can expect that the analysis towards the direction of the singularity will be a challenge. An interesting approach to understand the initial singularity is also to compare "geometric potential walls" with those which come from certain matter model (see for instance [68]).

Finally at some point one would like to compare with observations and for this purpose it will be necessary to combine different matter models, to use a multi-fluid etc. See [16] for recent developments.

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Wir haben gezeigt unter der Annahme, dass die Raumzeit in der Nähe von speziellen Lösungen ist, die die Rolle eines ω-limes gespielt haben und unter der Annahme, dass die maximale Geschwindigkeit der Teilchen klein ist: für Bianchi II und reflektionssymmetrische Bianchi VI_0 Raumzeiten mit stossfreier Materie kann das Langzeitverhalten durch Staub approximiert werden.

Hiermit erkläre ich, dass ich die vorgelegte Dissertation eigenständig verfasst und keine anderen als die im Literaturverzeichnis angegebenen Quellen benutzt habe.