Die Dissertation mit dem Titel

MONOTONICITY FOR SOME GEOMETRIC FLOWS

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vorgelegt.

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Potsdam, den 3. September 2014 Ahmad Afuni

Abstract

The aim of this thesis is to establish local monotonicity formulæ for solutions to Dirichlet-type flows, such as the harmonic map and Yang-Mills heat flows, and the mean curvature flow. In particular, for the former, we allow as domain an evolving Riemannian manifold and for the latter, we allow as target an evolving Riemannian manifold. The approach taken consists in first deriving divergence identities involving an appropriate evolving quantity, then integrating over superlevel sets (heat balls) of suitable kernels. A theory of heat balls analogous to that of Ecker, Knopf, Ni and Topping is developed in order to accomplish this. The main result is then that, provided certain integrals are finite, local monotonicity formulæ hold in this general setting, thus generalizing results for the mean curvature and harmonic map heat flows and establishing a new local monotonicity formula for solutions to the Yang-Mills flow.

Zusammenfassung

Das Ziel dieser Dissertation ist das Beweisen lokaler Monotonieformeln für Lösungen Dirichletartiger Flüsse, wie des harmonischen Abbildungs- und Yang-Mills-Flusses, und des mittleren Krümmungsflusses. Für die Ersteren darf die Metrik des Definitionsbereiches und für den Letzteren die der Zielmannigfaltigkeit eine Evolutionsgleichung lösen. Die gewählte Methode besteht darin, daß einige eine geeignete entwickelnde Größe umfassende Divergenzidentitäten erst hergeleitet werden, und daß diese dann über Superniveaumengen zulässiger Kerne integriert werden, zu welchem Zwecke eine zu der von Ecker, Knopf, Ni und Topping analoge Theorie der Wärmekugeln entwickelt wird. Das Hauptergebnis ist dann, daß lokale Monotonieformeln auch in diesem verallgemeinerten Rahmen gelten, solange gewisse Integrale endlich sind. Dieses Resultat verallgemeinert deshalb vorherige Ergebnisse für den mittleren Krümmungs- und harmonischen Abbildungsfluß, und führt eine neue lokale Monotonieformel für Lösungen des Yang-Mills-Flusses ein.

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Introduction

Monotonicity as a paradigm. Monotonicity is a property shared by many partial differential equations (PDE) arising in geometry in physics, its statement being that a one-parameter family of quantities, usually integral, depending on a solution to a certain PDE is monotone in that parameter. In this thesis, we are principally interested in *local* monotonicity formulæ. Roughly speaking, a local monotonicity formula is an inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^m}\int_{\Omega_r} f(u)\right) \geqslant 0,$$

on an interval of the form $]0, r_0[$, $r_0 > 0$, where $f(u)$ is a quantity depending on a solution u to a PDE, $m \in \mathbb{N}$ and Ω_r is an increasing one-parameter family of precompact subsets of the domain of u which tend in some sense to a point in the domain of u as $r \searrow 0$. Essentially, such a formula would permit us to deduce qualitative information about the local behaviour of solutions to PDE.

For instance, the most prominent such formula is the mean value property for harmonic functions due to Gauß (cf. [43, p. 223]): Let $\Omega \subset \mathbb{R}^n$ be open and fix $x_0 \in \Omega$. If $u : \Omega \to \mathbb{R}$ satisfies $-\Delta u = 0$ (Laplace's equation), then

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^n}\int_{B_r(x_0)} u\right) = 0 \tag{$*$}$$

on $]0, r_0[$ for any $r_0 > 0$ with $\overline{B}_{r_0}(x_0) \subset \Omega$. Since the differentiated quantity, the *average* of u in a neighbourhood of x_0, tends, as $r \searrow 0$, to $u(x_0)$, $(*)$ leads to many strong statements about the behaviour of solutions to Laplace's equation, such as the strong maximum principle.

More recently, an analogous formula has been derived by Watson [73] for solutions to the heat equation where the underlying domain of integration is more elaborate (see also [24]): Let $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}$ be open and fix $(x_0, t_0) \in \mathcal{D}$. Let $\Phi : \mathbb{R}^n \times]-\infty, t_0[\to \mathbb{R}^+$ denote the *backward heat kernel* centred at (x_0, t_0), i.e.

$$\Phi(x,t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right)$$

and define the *heat ball of radius r*, $r > 0$, by

$$E_r(x_0, t_0) := \left\{(x,t) \in \mathbb{R}^n \times]-\infty, t_0[\,:\, \Phi(x,t) > \frac{1}{r^n}\right\}.$$

If $u : \mathcal{D} \to \mathbb{R}$ satisfies $\partial_t u - \Delta u = 0$ (heat equation), then

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^n}\iint\limits_{E_r(x_0,t_0)} u(x,t)\frac{|x - x_0|^2}{4(t - t_0)^2}\mathrm{d}x\mathrm{d}t\right) = 0$$

on $]0, r_0[$ for any $r_0 > 0$ with $\overline{E}_{r_0}(x_0, t_0) \subset \mathcal{D}$. In this case, the monotone quantity, a *weighted* average of u in a neighbourhood of (x_0, t_0), tends, as $r \searrow 0$, to $u(x_0, t_0)$, leading to important consequences just as in the case of Laplace's equation.

In the above two examples, the quantities considered are more than just monotone – they are *conserved quantities.* For solutions to nonlinear PDE, this is too much to hope for, though similarly powerful monotonicity formulæ may still be obtained: if (M^n, g), $(N^m, \widetilde{g})$ are compact Riemannian manifolds ($n > 2$) and $u : M \to N$ is a *harmonic map*, i.e. if it is smooth and solves the system

$$\Delta' u := \Delta_M u^\alpha(x) + \sum_{i,j=1}^{n}\sum_{\beta=1}^{m} g^{ij}(x)\Gamma^\alpha_{\beta\gamma}(u(x))\partial_i u^\beta(x)\partial_j u^\gamma(x) = 0 \tag{$\#$}$$

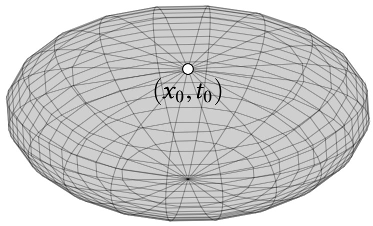

Figure: A heat ball in $\mathbb{R}^2 \times]-\infty, t_0[$.

for every $x \in M$, where $\Gamma_\alpha^{\beta\gamma}$ represent the Christoffel symbols obtained from the Levi-Civita connection on $(N, \tilde{g})$, then for fixed $x_0 \in M$, it satisfies the monotonicity property

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{\Lambda r}}{r^{n-2}}\int_{B_r(x_0)}\frac{1}{2}|\mathrm{d}u|^2\mathrm{dvol}_g\right) \geqslant 0$$

on $]0, \frac{1}{2}i_0[$, where for each $x \in M$, $(\mathrm{d}u)(x) \in T_{u(x)}N \otimes T_x^*M$ is the differential of u, i_0 is the injectivity radius of M at x_0 and $\Lambda \geqslant 0$ depends on the geometry of M in $B_{i_0}(x_0)$ and, if $M = \mathbb{R}^n$, $\Lambda = 0$ and this quantity is conserved iff u is *scale-invariant about* x_0, i.e. $u(x_0 + rx) = u(x_0 + x)$ for all $r > 0$ whenever both sides are defined. This was first established (in a slightly different form) by Schoen and Uhlenbeck [63] in a more general context for maps minimizing the Dirichlet energy

$$\int_M \frac{1}{2}|\mathrm{d}u|^2\mathrm{dvol}_g$$

in the appropriate Sobolev space and later established by Price [61] for maps which are critical (in the appropriate sense) for the Dirichlet energy which includes smooth maps satisfying (#). The monotonicity principle consequently states a law governing the behaviour of the local $(n-2)$-dimensional average energy of u which is a crucial ingredient in the regularity theory of harmonic maps due to Schoen and Uhlenbeck. Moreover, it has been applied by Schoen [64] to the study of compactness in the space of smooth solutions of uniformly bounded energy in dimensions greater than two.

Harmonic maps are a natural generalization of solutions to Laplace's equation. Similarly, a natural generalization of solutions to the heat equation exists— the harmonic map heat flow: a smooth map $u : M \times]0, T[\to N$ is said to evolve by the *harmonic map heat flow* if

$$\partial_t u - \Delta' u = 0.$$

Nonlocal monotonicity formulæ for such maps have been established by Struwe [68] for $M = \mathbb{R}^n$ and by Chen and Struwe [12] and Hamilton [33] in the case where M is a static compact manifold. A local counterpart of Struwe's formula has more recently been obtained by Ecker [20]: let $\gamma \in [0, n]$ and define the *weighted heat ball* of radius $r > 0$ centred at $(x_0, t_0) \in \mathbb{R}^n \times]0, T]$ by

$$E_r^\gamma(x_0, t_0) := \left\{(x, t) \in \mathbb{R}^n \times]-\infty, t_0[\,:\, (t_0 - t)^{\gamma/2}\Phi(x, t) > \frac{1}{r^{n-\gamma}}\right\}.$$

If $n > 2$ and u evolves by the harmonic map heat flow, then

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-2}}\iint_{E_r^2(x_0,t_0)}\frac{n-2}{2(t_0-t)}|\mathrm{d}u|^2 - \left\langle\sum_{i=1}^n\frac{(x-x_0)^i}{2(t-t_0)}\partial_i u, \partial_t u + \sum_{i=1}^n\frac{(x-x_0)^i}{2(t-t_0)}\partial_i u\right\rangle \mathrm{d}x\mathrm{d}t\right)$$
$$= \frac{n-2}{r^{n-1}}\iint_{E_r^2(x_0,t_0)}\left|\partial_t u + \sum_{i=1}^n\frac{(x-x_0)^i}{2(t-t_0)}\partial_i u\right|^2\mathrm{d}x\mathrm{d}t \geqslant 0$$

whenever these integrals make sense, i.e. whenever $E_r^2(x_0, t_0)$ is contained in the domain of u and the integrals are finite. Moreover, this quantity is conserved iff u is *parabolically scale-invariant about* (x_0, t_0), i.e. $u(x_0 + rx, t_0 + r^2 t) = u(x_0 + x, t_0 + t)$ for every $r > 0$ whenever both sides are defined. More recently, related formulæ have been established for other flows in settings involving evolving manifolds [22, 56].

Summary of results. In this thesis, an analogous formula to Ecker's local monotonicity formula for the harmonic map heat flow is proved for the Yang-Mills flow on (possibly evolving) Riemannian manifolds. In particular, a more general identity for k-forms with values in vector bundles which also implies a local monotonicity formula for the harmonic map heat flow between Riemannian manifolds is established, thus leading to a generalization of Ecker's result to curved ambient spaces. Moreover, a local monotonicity formula for the mean curvature flow with (possibly evolving) Riemannian target is established, generalizing another one of Ecker's results [18]. A brief sketch of these results follows.

Suppose that M is an oriented Riemannian manifold equipped with a family of metrics $\{g_t\}_{t\in]t_0-\delta_0,t_0[}$ ($t_0 \in \mathbb{R}$, $\delta_0 > 0$) with $\partial_t g = h$ and consider functions $\Phi : \mathcal{D} \subset M \times \mathbb{R} \to \mathbb{R}^+$ which are, in some sense, heat kernel-like (see Chapter 5). For such functions, an analogous notion of "heat ball" may be formulated; we set $E_r^\gamma := \{(x,t) \in \mathcal{D} : (t_0 - t)^{\gamma/2} \cdot \Phi(x,t) > \frac{1}{r^{n-\gamma}}\}$ and $\phi := \log \Phi$. Now, if P is a principal G-bundle (G as in §1.3) and the one-parameter family of connections $\{\omega_t = \omega_0 + a(t)\}_{t\in]t_0-\delta_0,t_0[}$ on P evolves by the Yang-Mills flow[1]

$$\partial_t a = \delta^\nabla \underline{\Omega}^\omega$$

with curvature two-form $\underline{\Omega}^\omega$ and codifferential δ^∇ induced by ω (see §1.5, §1.6 and §1.11 for details), then for sufficiently small $0 < r_1 < r_2$,

$$\left[\frac{1}{r^{n-4}}\iint_{E_r^4} \frac{1}{2}|\underline{\Omega}^\omega|^2\left(\partial_t\phi + |\nabla\phi|^2 - \frac{4}{2(t_0-t)}\right) - \left\langle \iota_{\nabla\phi}\underline{\Omega}^\omega, \iota_{\nabla\phi}\underline{\Omega}^\omega - \delta^\nabla\underline{\Omega}^\omega\right\rangle \mathrm{dvol}_g \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$= \int_{r_1}^{r_2}\left(\frac{n-4}{r^{n-3}}\iint_{E_r^4} -\frac{1}{2}|\underline{\Omega}^\omega|^2\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h\right)\right.$$
$$+ |\iota_{\nabla\phi}\underline{\Omega}^\omega - \delta^\nabla\underline{\Omega}^\omega|^2$$
$$\left. + \left\langle \nabla^2\phi + \frac{1}{2}h + \frac{1}{2(t_0-t)}g, \sum_{i,j}\left\langle \iota_{\partial_i}\underline{\Omega}^\omega, \iota_{\partial_j}\underline{\Omega}^\omega\right\rangle \mathrm{d}x^i \otimes \mathrm{d}x^j\right\rangle \mathrm{dvol}_g \mathrm{d}t\right)\mathrm{d}r$$

whenever $\frac{|\underline{\Omega}^\omega|^2}{t_0-t}$ is *summable*[2] over $E_{r_2}^4$ and these integrals make sense, where ι denotes the interior product of a vector field with a (vector bundle-valued) differential form (cf. §1.2) and $[f(r)]_{r=r_1}^{r=r_2} := f(r_2) - f(r_1)$ (Theorem 6.2.1). From the above formula, it may be read off that the first term on the right-hand side vanishes iff Φ solves the backward heat equation

$$\partial_t\Phi + \Delta\Phi + \frac{1}{2}\mathrm{tr}_g h \cdot \Phi = 0,$$

the second iff ω is self-similar in the sense that

$$\partial_t a = -\iota_{\nabla\phi}\underline{\Omega}^\omega$$

(see Chapter 2 for the case $M = \mathbb{R}^n$), and the final term, in the case where $h = -2\mathrm{Ric}$, i.e. when g evolves by the Ricci flow, iff

[1] Note that t is the same parameter on which g depends.

[2] Throughout this thesis, we say a real-valued function f (or n-form $f\mathrm{dvol}_g$) on an oriented Riemannian manifold (M^n, g) with volume form dvol_g is *summable* if $|f|$ is measurable and $\int_M |f|\mathrm{dvol}_g < \infty$.

$$\mathrm{Ric} = \nabla^2\phi + \frac{1}{2(t_0 - t)}g,$$

i.e. iff g is a *gradient shrinking soliton* (see [45]) This sort of structure is analogous to that exhibited by the monotonicity formula due to Magni, Mantegazza and Tsatis [53]. By applying the above identity to appropriate kernels, monotonicity formulæ may be obtained for solutions to the Yang-Mills heat flow in the case where (M,g) is compact and static (Theorem 6.3.6) or evolving and of locally bounded geometry about some $(x_0,t_0) \in M \times \mathbb{R}$ (Theorem 6.3.2), thus providing local counterparts of the nonlocal monotonicity formulæ due to Chen and Shen [11] and Hamilton [33].

Similarly, with $(M,(g_t)_{t\in]t_0-\delta_0,t_0[})$ as before, it is also shown that if $\{u(\cdot,t) : (M,g_t) \to N\}_{t\in]t_0-\delta,t_0[}$ evolves by the harmonic map heat flow, then for sufficiently small $r_2 > r_1 > 0$,

$$\left[\frac{1}{r^{n-2}}\iint\limits_{E_r^2}\frac{1}{2}|\mathrm{d}u|^2\left(\partial_t\phi + |\nabla\phi|^2 - \frac{2}{2(t_0-t)}\right) - \left\langle\partial_{\nabla\phi}u, \partial_{\nabla\phi}u + \partial_t u\right\rangle \mathrm{dvol}_g \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$

$$\begin{aligned}= \int_{r_1}^{r_2}\Bigg(\frac{n-2}{r^{n-1}}\iint\limits_{E_r^2} & -\frac{1}{2}|\mathrm{d}u|^2\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h\right) \\ & + |\partial_{\nabla\phi}u + \partial_t u|^2 \\ & + \left\langle\nabla^2\phi + \frac{1}{2}h - \frac{1}{2(t_0-t)}g, \sum_{i,j}\left\langle\partial_i u, \partial_j u\right\rangle \mathrm{d}x^i \otimes \mathrm{d}x^j\right\rangle \mathrm{dvol}_g \mathrm{d}t\Bigg)\mathrm{d}r\end{aligned}$$

whenever $\frac{|\mathrm{d}u|^2}{t_0-t}$ is summable over $E_{r_2}^2$ and these integrals make sense (Theorem 6.2.1). As may be seen, this formula (Theorem 6.2.1) coïncides with Ecker's in the case where $M = \mathbb{R}^n$ and Φ is replaced with $(t_0 - t)$ times the Euclidean backward heat kernel at (x_0,t_0) for any $x_0 \in \mathbb{R}^n$. Otherwise, it exhibits similar behaviour to that for the Yang-Mills flow and may similarly be used to establish local counterparts (Theorems 6.3.6 and 6.3.2) of the nonlocal monotonicity formulæ due to Chen and Struwe [12] and Hamilton [33].

Finally, it is established in Theorem 7.2.1 that if $\{F(\cdot,t) : N^m \to (M,g_t)\}_{t\in]t_0-\delta_0,t_0[}$ is a smooth one-parameter family of embeddings evolving by *mean curvature flow* in an evolving background manifold, i.e. $\partial_t F = H$ where H is the mean curvature vector of F and $\partial_t g = h$, *and* the map[3] $(F,\mathrm{pr}_2) : N \times]t_0-\delta_0,t_0[\to M \times]t_0-\delta_0,t_0[$ is *proper*[4], then for appropriate heat kernel-like $\underline{\Phi} := \Phi \circ (F,\mathrm{pr}_2) : (F,\mathrm{pr}_2)^{-1}(\mathcal{D}) \to \mathbb{R}^+$, $\underline{E}_r := \{(x,t) \in (F,\mathrm{pr}_2^{-1})(\mathcal{D}) : (t_0-t)^{\frac{n-m}{2}}\underline{\Phi}(x,t) > \frac{1}{r^m}\}$ is a "heat ball" for small enough r and for small enough $r_2 > r_1 > 0$,

$$\left[\frac{1}{r^m}\iint\limits_{E_r^m(\underline{\Phi})} u\left[|\nabla\underline{\phi}|^2 + \left(|H|^2 - \underline{\frac{1}{2}\mathrm{tr}_g^T h}\right)\left(\underline{\phi_r^m} + \frac{n-m}{2}\log(t_0-t)\right)\right]\mathrm{dvol}_{\mathfrak{I}}\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$

$$\begin{aligned}= \int_{r_1}^{r_2}\Bigg(\frac{m}{r^{m+1}}\iint\limits_{E_r^m(\underline{\Phi})} & -u\cdot\underline{\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h\right)} \\ & - \left(\underline{\phi_r^m} + \frac{n-m}{2}\log(t_0-t)\right)\cdot H_{\mathfrak{I}}u + u\left|H - \underline{\nabla^\perp\phi}\right|^2 \\ & + u\cdot\mathrm{tr}_g^\perp\underline{\left(\nabla^2\phi + \frac{1}{2}h - \frac{1}{2(t_0-t)}g\right)}\mathrm{dvol}_{\mathfrak{I}}\mathrm{d}t\Bigg)\mathrm{d}r\end{aligned}$$

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[3] Given two sets X, Y, $\mathrm{pr}_1 : X \times Y \to X$ is the projection onto the first component and $\mathrm{pr}_2 : X \times Y \to Y$ projection onto the second.

[4] By this it is meant that inverse images of compact subsets of the codomain of (F,pr_2) are compact.

whenever $\frac{u}{t_0-t}$ is summable over $\underline{E}_{r_2}$ and these integrals make sense, where $\phi_r^m = \phi + m \log r$ and $u : N \times]t_0 - \delta_0, t_0[$ is a smooth function, $H_{\mathfrak{Z}} u = \partial_t u - \Delta_{\mathfrak{Z}} u$ and $\underline{\cdot}$ denotes composition with (F, pr_2) (see §1.12 for details and notation). As may be seen here, the terms on the right-hand side vanish similarly to those in the monotonicity identity for the Yang-Mills heat flow above, most notably also when $H = \nabla^{\perp}\phi$, which represents a homothetically shrinking solution in $\mathbb{R}^n$ (cf. [19]). This is exactly the behaviour exhibited by the formula in [53]. This local monotonicity formula is a natural generalization (in the class of maps considered) of that due to Ecker [18] which is in turn a local counterpart of the (nonlocal) monotonicity formula due to Huisken [40]. On the other hand, the formula above yields local counterparts of nonlocal monotonicity formulæ established by Hamilton [33] for static compact M (Theorem 7.3.4) and Magni, Mantegazza and Tsatis [53] for general evolving M (Theorem 7.3.2).

Structure of the thesis. In Chapter 1, the stage is set by way of introducing the notation, geometric setup and kernels underlying the investigations to be carried out in the sequel, as well as a brief introduction to the PDE we shall be most interested in. Here it is also noted that both the harmonic map and Yang-Mills equations and their respective flows may be considered as special cases of a more general nonhomogeneous Laplace or heat-type equations, henceforth to be referred to as equations or flows of *Dirichlet type*.

In Chapter 2, the scaling behaviour of solutions to the Yang-Mills flow over $\mathbb{R}^n$ is investigated and then used to establish Price's monotonicity formula ([61]) for static solutions and a local monotonicity formula for the flow, thus providing— for the Yang-Mills flow over Euclidean space at least— alternative proofs of the more general theorems to be established later on in this thesis.

In Chapter 3, the metric structure of Dirichlet-type energies is expounded, culminating in a rigorous derivation of the so-called *energy-momentum tensor* and some useful identities, which are subsequently applied to establish monotonicity formulæ for solutions to equations of Dirichlet-type. Since it offers no additional difficulties, we consider the so-called p-Dirichlet-type energies in these two chapters, thus re-proving the monotonicity formula for p-harmonic maps (cf. [34, Lemma 4.1]) and establishing a monotonicity formula for p-Yang-Mills fields.

In Chapter 4, the identities derived from the energy-momentum tensor and the estimates in §1.8 are used to establish *nonlocal* monotonicity formulæ for Dirichlet-type flows. Such formulæ are natural analogues of the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u(x,\cdot)\Phi(x,\cdot)\mathrm{d}x = 0$$

for solutions u of the heat equation on $\mathbb{R}^n$ (and Φ as before) which behave appropriately at ∞ and are known to hold on static compact manifolds with both formal and canonical heat kernels. It is these formulæ that have heretofore been applied to questions regarding the structure of singularities occurring in flows [40, 68, 39]. The novelty of this chapter lies in the fact that these formulæ are established in the case of an evolving ambient space, much in the spirit of [53] and are subsequently used to establish estimates ensuring the finiteness of certain singular integrals, such as those occurring in the monotonicity formulæ for the Yang-Mills heat flow and Harmonic map heat flow stated above.

In Chapter 5, *heat balls* associated to kernels that satisfy certain properties are defined, of which a few examples are given, and integration formulæ analogous to those in [18] and [20] are established which are then applied in Chapter 6 to establish local monotonicity formulæ for Dirichlet-type flows. In Chapter 7, these formulæ are similarly applied to establish local monotonicity formulæ for mean curvature flow with Riemannian target. Finally, this dissertation is concluded by two appendices— one containing auxiliary geometric and another analytical lemmata used throughout the text.

For the convenience of the reader, an index containing terms, mathematical symbols and universal constants used throughout the text has been included, together with a reference to the first occurrence of each term or symbol.

— 1 —

Preliminaries

In this chapter the background underlying the following chapters is presented. After a summary of notation, some algebraic preliminaries underlying the calculations to be carried out in subsequent chapters are introduced. A summary of the results required from Lie theory for subsequent considerations is then presented, followed by a brief treatment of the theory of G-bundles and connections on them.[1] With these topics out of the way, the discussion moves on to a brief description of the evolving objects to be considered in the sequel, first starting with the notion of an evolving manifold, then proceeding onto heat kernel-type objects, then *Dirichlet-type problems* and finally the mean curvature flow.

1.1. Notation. Throughout this thesis, (M,g) shall denote a *smooth* oriented Riemannian manifold, TM its tangent bundle and T^*M its cotangent bundle. Moreover, $(\mathbb{R}^n,\delta)$ shall denote Euclidean space with the *flat* metric. If $f : M \to N$ is a once continuously differentiable map and $x \in M$, we write $\mathrm{d}_x f : T_x M \to T_{f(x)}M$ for its differential at x. Moreover, whenever $f : M \to V$ is a smooth map, where V is a finite-dimensional K-vector space[2], and $X \in TM$, we write $\partial_X f \in V$ for the directional derivative of f in direction X.

If N and P are smooth manifolds, we say that a map $f : M \times N \to P$ is $C^{k,l}$ (and write $f \in C^{k,l}(M \times N, P)$) if f is k-times continuously differentiable in M and l-times continuously differentiable in N. If $k = l$, we simply write C^k and we write C for C^0.

Given a collection of smooth manifolds $\{M_j\}_{j \in \{1,\ldots,n\}}$, the map $\mathrm{pr}_i : M_1 \times \cdots \times M_n \to M_i$ shall denote the *ith coördinate projection*, i.e. for $m_j \in M_j$, $\mathrm{pr}_i(m_1,\ldots,m_n) = m_i$.

If V is a K-vector space, we write $GL(V)$ for the set of all *invertible* K-linear maps $V \to V$ and $gl(V)$ for the set of *all* K-linear maps $V \to V$.

All integrals occurring in this thesis are *Lebesgue integrals*.

1.2. Algebraic preliminaries. We first begin with some algebraic preliminaries. Let V and $\{V_i\}_{i=1}^N$ ($N \in \mathbb{N}$) be finite dimensional $\mathbb{R}$-vector spaces equipped with positive-definite inner products $\langle\cdot,\cdot\rangle$ and $\{\langle\cdot,\cdot\rangle_i\}$ respectively.

Let $a_1,\ldots,a_k,b_1,\ldots,b_k \in V$ and $v_i, w_i \in V_i$ ($i \in \{1,\ldots,N\}$). The following table summarizes the vector spaces formed from these which we shall require in the sequel, together with the inner products induced on them:

Symbol	Designation	Inner Product
$V_1 \otimes \cdots \otimes V_k$	Tensor product space	$\langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k\rangle := \langle v_1,w_1\rangle \cdot \cdots \cdot \langle v_k,w_k\rangle$.
$\Lambda^k V$	kth exterior product space	$\langle a_1 \wedge \cdots \wedge a_k, b_1 \wedge \cdots \wedge b_k\rangle := \det\left(\left\langle a_i,b_j\right\rangle\right)_{i,j=1}^{k}$.
ΛV	Exterior algebra $(= \bigoplus_{k=1}^{\dim V} \Lambda^k V)$	Induced by inner products on $\left\{\Lambda^k V\right\}$ such that for $k \neq l$, $\Lambda^k V \perp \Lambda^l V$.

In particular, we make the identifications $\Lambda^0 V \cong V$, $\Lambda^0 V \cong \mathbb{R}$ and $\mathbb{R} \otimes V \cong V \otimes \mathbb{R} \cong V$. That these inner products are well-defined follows immediately from the characterizations of these spaces by means of universal mapping properties; we refer the reader to [52, §IX.8, §XVI.6] for definitions and proofs of these statements. Furthermore note that, if $\{\varepsilon_i^j\}_i$ form orthonormal bases for the $\{V_j\}_j$, then $\{\varepsilon_{i_1}^1 \otimes \cdots \otimes \varepsilon_{i_k^k}\}$ form an orthonormal basis for $V_1 \otimes \cdots \otimes V_k$ with respect to the induced inner product, where we vary over all $i_j \in \{1,\ldots,\dim V_j\}$. Moreover, if $\{\varepsilon_i\}_i$ form an orthonormal basis for V, then $\{\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}\}$ form an orthonormal basis for $\Lambda^k V$, where we vary over all increasing sequences $0 < i_1 < \cdots < i_k \leqslant \dim V$ of integers; we shall abbreviate such sequences using multi-index notation and write $I = (i_1,\ldots,i_k)$ (or I^k to emphasize that the multi-index is a k-multi-index).

[1]For a more detailed treatment of these topics, the reader is referred to [72, Chapter 2], [60] and [7].
[2]Throughout this thesis, $K = \mathbb{R}$ or $\mathbb{C}$.

Let V^* denote the dual space of V. Since the inner product on V is non-degenerate, the $\mathbb{R}$-linear map

$$\begin{aligned} \flat : V &\to V^* \\ v &\mapsto v^\flat := \langle v, \cdot \rangle \end{aligned}$$

defines an isomorphism of $\mathbb{R}$-vector spaces. We call it, together with its inverse $\sharp : V^* \to V$ the *musical isomorphisms* (cf. [60, §3.8]). If $\{\varepsilon_i\}$ is a basis for V and $\{\omega^i\}$ the dual basis[3] for V^*, then if we write $g_{ij} := \left\langle \varepsilon_i, \varepsilon_j \right\rangle$ and g^{ij} for the matrix inverse of $(g_{ij})_{i,j}$, these isomorphisms may be given explicitly by

$$\left(\sum_i v^i \varepsilon_i \right)^\flat = \sum_i \left(\sum_j g_{ij} v^j \right) \omega^i$$

and

$$\left(\sum_i v_i \omega^i \right)^\sharp = \sum_i \left(\sum_j g^{ij} v_j \right) \varepsilon_i.$$

On the one hand, $\flat$ induces an inner product on V^* given by $(v, w) \mapsto \left\langle v^\flat, w^\flat \right\rangle$. On the other hand, if we also write $\flat_i$ for the musical isomorphism defined on $(V_i, \langle \cdot, \cdot \rangle_i)$, we obtain isomorphisms

$$\begin{gathered} \flat_1 \otimes \cdots \otimes \flat_k : V_1 \otimes \cdots \otimes V_k \xrightarrow{\cong} V_1^* \otimes \cdots \otimes V_k^* \text{ and} \\ \underbrace{\flat \wedge \cdots \wedge \flat}_{k \text{ times}} : \Lambda^k V \xrightarrow{\cong} \Lambda^k V^*. \end{gathered}$$

Moreover, we note that the canonical non-singular bilinear pairing $V^* \times V \to \mathbb{R}$ (i.e. $(\nu, v) \mapsto \nu(v)$) naturally induces canonical non-singular *bilinear pairings* between tensor product spaces and the products of their duals, as well as exterior product spaces and exterior products of their duals. Let $v, a_1, \ldots, a_k \in V$, $\nu, \alpha_1, \ldots, \alpha_k \in V^*$ $v_i \in V_i$ and $\nu_i \in V_i^*$. We summarize these in the following table:

Pairing	**Characterization**
$\left(V \otimes V_1^* \otimes \cdots \otimes V_k^*\right) \times (V_1 \otimes \cdots \otimes V_k) \to V$	$(v \otimes \nu_1 \otimes \cdots \otimes \nu_k, v_1 \otimes \cdots \otimes v_k)$ $= \nu_1(v_1) \cdot \cdots \cdot \nu_k(v_k) v$
$\left(V_1 \otimes \Lambda^k V^*\right) \times \left(\Lambda^k V\right) \to V$	$(v_1 \otimes \alpha_1 \wedge \cdots \wedge \alpha_k, a_1 \wedge \ldots a_k)$ $= \det\left(\alpha_i(a_j)\right)_{i,j=1}^k v_1$
$(V_1 \otimes \Lambda V^*) \times (\Lambda V)$	Induced by the above pairing and extended such that the restriction to $(V_1 \otimes \Lambda^k V^*) \times (\Lambda^l V)$ is $0 \in V_1$ for $k \neq l$.

As before, well-definedness follows from the respective universal mapping properties (cf. [72, §2.8, §2.9]).

Finally, for each $v \in V$, we introduce two maps, both referred to as *interior products*: the first is the map[4] $\iota_v : V_1 \otimes \Lambda V^* \to V_1 \otimes \Lambda V^*$ dual to $v \wedge \cdot : \Lambda V \to \Lambda V$ with respect to the corresponding pairing introduced above, i.e. such that for every $\alpha \in \Lambda V^*$ and $a \in \Lambda V$,

$$(\iota_v \alpha, a) = (\alpha, v \wedge a).$$

[3]By this we mean $\omega^i(\varepsilon_j) = \delta^i_j$ for every i, j.

[4]This map is often denoted by the symbol $\neg$ or an appropriate reflection thereof depending on the choice of V, cf. e.g. [25].

ι_v is an $\mathbb{R}$-linear map and, moreover, an anti-derivation with respect to left wedge multiplication,[5] i.e. whenever $\alpha \in \Lambda^k V^*$ and $\beta \in V_1 \otimes \Lambda V^*$,

$$\iota_v (\alpha \wedge \beta) = \iota_v \alpha \wedge \beta + (-1)^k \alpha \wedge \iota_v \beta.$$

Note in particular that

$$\iota_v \left(V_1 \otimes \Lambda^k V^* \right) \subset V_1 \otimes \Lambda^{k-1} V^*$$

and, keeping in mind the identification $V_1 \cong V_1 \otimes \Lambda^0 V^*$,

$$\iota_v (V_1) = \{0\}.$$

The second interior product is the map $j_v : V_1 \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} \to V_1 \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k-1 \text{ times}}$ dual to $v \otimes \cdot : \underbrace{V \otimes \cdots \otimes V}_{k-1 \text{ times}} \to \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$ with respect to the corresponding pairing introduced above, i.e. such that for every $T \in V_1 \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}}$ and $S \in \underbrace{V \otimes \cdots \otimes V}_{k-1 \text{ times}}$,

$$(j_v T, S) = (T, v \otimes S).$$

j_v is $\mathbb{R}$-linear and, whenever $v_1 \in V_1$ and $\omega_i \in V^*$ ($i \in \{1, \ldots, k\}$), $j_v(v_1 \otimes \omega_1 \otimes \cdots \otimes \omega_k) = \omega_1(v) \cdot v_1 \otimes \omega_2 \otimes \cdots \otimes \omega_k$.

In the sequel, we shall write ι_v for *both* interior products, where the operand shall dictate which is to be used.

1.3. Some Lie theory. We now review the Lie theory necessary to discuss bundles with structure group and refer the reader to [72] and [36] for details. Recall that a *Lie group* is a group which is a differentiable manifold such that both its product map $\cdot : G \times G \to G$ and inverse map $\cdot^{-1} : G \to G$ are smooth. We denote the identity element of G by e and set $\mathfrak{g} = T_e G$. For fixed $g \in G$ and $X \in \mathfrak{g}$, we introduce the following notation.

Symbol	Signification
$\rho_g : G \to G$	Right multiplication by g
$\lambda_g : G \to G$	Left multiplication by g
$C_g : G \to G$	Conjugation by g ($:= \lambda_g \circ \rho_{g^{-1}}$)
$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$	Adjoint action of G on $\mathfrak{g}$ ($:= \mathrm{d}_e C_g$)
$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$	Adjoint action of $\mathfrak{g}$ on itself ($:= (Y \mapsto \partial_X(\mathrm{Ad}_\cdot Y))$)
$\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ or $\mathbb{C}$	The Killing form on $\mathfrak{g}$ ($:= ((X,Y) \mapsto \mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y))$)

The action Ad_g gives rise to the *adjoint representation* $\mathrm{Ad} : G \to GL(\mathfrak{g})$ of G on $\mathfrak{g}$, C_g the *conjugate action* $C : G \to C^\infty(G, G)$ of G on itself and ad_X the *Lie bracket* $(X, Y) \mapsto [X, Y] := \mathrm{ad}_X(Y)$ on $\mathfrak{g}$, making $\mathfrak{g}$ a Lie algebra in the algebraic sense which we refer to as *the* Lie algebra of G.

Now, the Killing form defines a symmetric bilinear form on $\mathfrak{g}$ invariant under the adjoint action of G on $\mathfrak{g}$, i.e. for every $g \in G$, $X, Y \in \mathfrak{g}$,

$$\left\langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \right\rangle = \langle X, Y \rangle.$$

Differentiating the map $g \mapsto \left\langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \right\rangle$ at e then implies the identity

$$\langle \mathrm{ad}_Z X, Y \rangle + \langle X, \mathrm{ad}_Z Y \rangle = 0 \Leftrightarrow \langle [X, Z], Y \rangle = \langle X, [Z, Y] \rangle$$

[5]Here, we may multiply elements of $V_1 \otimes \Lambda V^*$ from the left by elements of ΛV^* by wedge multiplication with the second entry of the tensor product. The universal mapping property implies that this is indeed well-defined.

by the skew-symmetry of the Lie bracket. It is negative-definite if topological and algebraic conditions are satisfied, viz. if G is connected, compact and semisimple. We shall *always* assume that these conditions hold.

Two such Lie groups with which we shall be principally concerned are

$$SO(N,\mathbb{R}) = \{A \in GL(N,\mathbb{R}) : A^T A = I \text{ and } \det A = 1\} \text{ and}$$
$$SU(N,\mathbb{C}) = \{A \in GL(N,\mathbb{C}) : \overline{A}^T A = I \text{ and } \det A = 1\},$$

where $GL(N,K)$ denotes the (real/complex) Lie group of invertible $N \times N$ matrices with entries in $K = \mathbb{R}$ or $\mathbb{C}$. The corresponding Lie algebras are, considered as $N \times N$ matrices,

$$\mathfrak{so}(N,\mathbb{R}) = \{X \in \mathfrak{gl}(N,\mathbb{R}) : X^T + X = 0\} \text{ and}$$
$$\mathfrak{su}(N,\mathbb{C}) = \{X \in \mathfrak{gl}(N,\mathbb{C}) : \overline{X}^T + X = 0\},$$

where $\mathfrak{gl}(N,K)$ is the K-vector space of $N \times N$ matrices with entries in $K = \mathbb{R}$ or $\mathbb{C}$. The Killing forms of these Lie algebras may both be written as

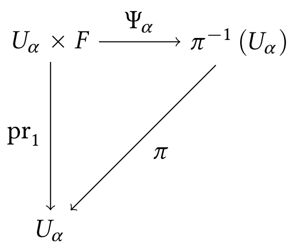
$$\langle X, Y\rangle = 2n\mathrm{tr}(XY) = -2n\mathrm{tr}(X\overline{Y}^T).$$

It can thus be seen that this form coïncides with the Hilbert-Schmidt inner product on these spaces of matrices up to a sign (and factor), which is known to be positive-definite.

1.4. G-bundles. We now proceed to use Lie groups to “glue” vector spaces (or Lie groups) associated to each point of M in a smooth manner. Let G be a Lie group. We begin with a definition.

Definition 1.4.1. [66, §2.3] [38, §3] A *G-bundle* is a quadruple $G \circlearrowright F \to E \xrightarrow{\pi} M$ of manifolds together with

- A *projection*, viz. a smooth surjection $\pi : E \to M$,
- a transformation group G acting effectively on F, viz. a smooth left G-action $G \times F \to F$ such that $g \cdot f = f$ for all $f \in F$ and some $g \in G$ implies $g = e$,
- an open cover $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ of M, and
- a *bundle atlas*, i.e. smooth functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}_{\alpha,\beta \in \mathrm{A}}$ as well as diffeomorphisms $\{\Psi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)\}_{\alpha \in \mathrm{A}}$ such that the diagram



commutes for every $\alpha \in \mathrm{A}$ and, for every $\alpha, \beta \in \mathrm{A}$, $x \in U_\alpha \cap U_\beta$ and $f \in F$, $\left(\Psi_\alpha^{-1} \circ \Psi_\beta\right)(x, f) = \left(x, g_{\alpha\beta}(x) \cdot f\right)$.

E is called the *total space of the bundle*, F the *standard fibre*, G the *structure group*, M the *base manifold* and for each $x \in M$, $E_x := \pi^{-1}(\{x\})$ is the *fibre over* x.

The bundle $G \circlearrowright F \to E \xrightarrow{\pi}\to M$ will henceforth simply be denoted E when context dictates the nature of the bundle.

Remark 1.4.2. We define a relation on the set of G-bundle atlases by

$$\begin{aligned}(\{U_\alpha\},\{\Psi_\alpha\},\{g_{\alpha\beta}\}) &\sim (\{V_\gamma\},\{\widetilde{\Psi}_\gamma\},\{\widetilde{g}_{\alpha\beta}\})\\ \Leftrightarrow \exists\{\tau_{\alpha\gamma} : U_\alpha \cap V_\gamma \to G \text{ smooth}\}_{\alpha\in \mathrm{A},\gamma\in\Gamma}&\\ \text{such that } \tilde{g}_{\gamma\delta} = \tau_{\alpha\gamma}^{-1} g_{\alpha\beta}\tau_{\beta\delta} \text{ on } U_\alpha \cap V_\gamma \ \forall \alpha,\beta \in \mathrm{A},\ \gamma,\delta\in\Gamma.&\end{aligned}$$

It is easy to check that this is an equivalence relation on the set of bundle atlases. We shall henceforth identify bundle atlases which are equivalent according to this relation.

For the sake of completeness, we also define what it means for G-bundles to be "isomorphic", the point being here that we may work with equivalent realizations of the same underlying bundle.

Definition 1.4.3. Let $G_1 \circlearrowright F_1 \to E_1 \xrightarrow{\pi_1} M$ and $G_2 \circlearrowright F_2 \to E_2 \xrightarrow{\pi_2} M$ be a G_1 and G_2 bundle respectively over M. A pair (ν,Φ) of maps $\nu : F_1 \to F_2$ and $\Phi : E_1 \to E_2$ is called a **strong bundle morphism** if whenever $\Psi,\widetilde{\Psi}$ are bundle charts over $U \subset M$ for E_1 and E_2 respectively, there is a smooth map $\tau : U \to G_2$ such that $\Phi\left(\Psi(x,f)\right) = \widetilde{\Psi}\left(x,\tau(x)\cdot\nu(f)\right)$ for every $x \in U$ and $f \in F$. If ν and Φ are diffeomorphisms, we say that E_1 and E_2 are *isomorphic* and write $E_1 \cong E_2$.

We are particularly interested in two sorts of G-bundles– vector bundles and principal bundles.

Example 1.4.4 (Vector Bundles). If F is a vector space and $\cdot$ is a group representation, then the fibres of E may be equipped with a vector space structure compatible with the differential structure of E according to the rules $\Psi_\alpha(x,v_1)+\Psi_\alpha(x,v_2) := \Psi_\alpha(x,v_1+v_2)$ and $c\cdot\Psi_\alpha(x,v) := \Psi_\alpha(x,cv)$ [66, §6.6] [38, §3.5]. Such a G-bundle E is called a *vector bundle*. By defining $\tilde{g}_{\alpha\beta}(x) := g_{\alpha\beta}(x)\cdot : F \to F$, it is clear that E is isomorphic to a $GL(F)$-bundle ($\nu = \tau = \mathrm{id}_F$, $\Phi = \mathrm{id}_E$). As we shall soon see, E may be seen to be isomorphic to an $O(F)$-bundle with respect to some inner product on F, and is said to be *orientable* if it may be made to be isomorphic to an $SO(F)$-bundle. If $G_0 \circlearrowright F_0 \to E_0 \xrightarrow{\pi_0} M$ is another vector bundle, a *vector bundle morphism* is a smooth map $\Phi : E_0 \to E$ such that $\Phi|_{(E_0)_x} : (E_0)_x \to E_x$ is linear. If $\Phi|_{(E_0)_x}$ is injective, E_0 is said to be a *vector subbundle* of E, written $E_0 < E$. In particular, given a vector bundle morphism $\Phi : E_0 \to E$ such that $\Phi|_{(E_0)_x}$ is of constant rank for each $x \in M$, it may easily be shown that $\ker\Phi < E_0$ and $\operatorname{im}\Phi < E$ (cf. [75, Prop. 2.10]).

Example 1.4.5 (Principal Bundles). If $F = G$ and $\cdot$ is group multiplication from the left, E is called a *principal G-bundle*. Such a bundle admits a smooth *global* right action $P \times G \ni (p,g) \mapsto pg =: R_g(p) =: L_p(g)$ defined such that

$$\Psi_\alpha(x,g)h := \Psi_\alpha(x,gh).$$

It is clear that this right action preserves fibres and, moreover, acts freely and transitively on them, i.e. $p\cdot g = p \Rightarrow g = e$ for any $p \in P$ and $E_x = pG$ for any $p \in E_x$.

The definition above implies that, given a G-bundle E, the functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathrm{A}}$ satisfy the *cocycle condition* $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ for every $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $\alpha,\beta,\gamma \in \mathrm{A}$, and we shall subsequently refer to such functions as *cocycles* relative to $\{U_\alpha\}$. It turns out that, in fact, given a transformation group G acting smoothly on F and a cover of M together with $\{g_{\alpha\beta}\}$, one can reconstruct the bundle E (up to isomorphism), as is evident from the

Theorem 1.4.6. [38, §3.2a] *If G is a transformation group acting smoothly on a manifold F, $\{U_\alpha\}_{\alpha\in\mathrm{A}}$ is an open cover of a manifold M and $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathrm{A}}$ are cocycles relative to this cover, then there exist a topology and differential structure making the set*

$$E_0 := \left(\bigcup_{\alpha\in A}\{\alpha\}\times U_\alpha \times F\right)\Big/ \sim,$$

where $(\alpha,x,f) \sim (\beta,\widetilde{x},\widetilde{f})$ iff $\widetilde{x} = x$ and $\widetilde{f} = g_{\beta\alpha}(x)\cdot f$, a differentiable manifold which defines a G-bundle $G \circlearrowright F \to E_0 \xrightarrow{\mathrm{pr}_2} M$.

In short, the cocycles and transformation group G contain all the information required to completely determine a G-bundle so that, for instance, given a G-bundle, one could isolate its cocycles and make use of a different group action of G to construct another G-bundle using the above theorem. Such a bundle is called an *associated bundle*. In particular, if P is a principal G-bundle and $\rho : G \to V$ is a group representation, we call the bundle obtained from the cocycles of P and ρ the *vector bundle associated to P and ρ* and denote it by $P \times_\rho V$. Likewise, given a vector bundle E, one may use its cocycles together with the left action of the structure group on itself to obtain the *principal bundle associated to E*.

Let V be an $\mathbb{R}$-vector space, $\{(V_\xi, x_\xi)\}_{\xi\in\Xi}$ an atlas for M, $f : N \to M$ a smooth map, where N is a smooth manifold, and $F \to E \to M$ and $F_0 \to E_0 \to M$ vector bundles with cocycles $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(F)\}_{\alpha,\beta\in\mathrm{A}}$ and $\{g^0_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(F_0)\}$ respectively, where in both cases we consider the vector bundles as $GL(\cdot)$ bundles with the standard representation of $GL(\cdot)$. The following table summarizes the $GL(\cdot)$-bundle constructions we shall require, where we make use of the canonical representation of $GL(\cdot)$:

Bundle	**Symbol**	**Cocycles**
Trivial bundle	V	$M \ni x \mapsto \mathrm{id}_V \in GL(V)$
Tangent bundle	TM	$h_{\xi\eta} := \left(D\left(x_\xi \circ x_\eta^{-1}\right)\right) \circ x_\eta : V_\xi \cap V_\eta \to GL(\mathbb{R}^n)$
Cotangent bundle	T^*M	$h^*_{\xi\eta} := {}^t\left(D\left(x_\eta \circ x_\xi^{-1}\right)\right) \circ x_\xi : V_\xi \cap V_\eta \to GL\left((\mathbb{R}^n)^*\right)$
Pullback bundle (over N)	$f^{-1}E$	$(f^*g)_{\alpha\beta} := g_{\alpha\beta} \circ f : f^{-1}\left(U_\alpha \cap U_\beta\right) \to GL(F)$
Dual bundle	E^*	$g^*_{\alpha\beta} := {}^t g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(F^*)$
Tensor product bundle	$E \otimes E_0$	$g^{\otimes}_{\alpha\beta} := g_{\alpha\beta} \otimes g^0_{\alpha\beta} : U_\alpha \cap U_\beta \to GL\left(F \otimes F_0\right)$
kth exterior product bundle	$\Lambda^k E$	$g^k_{\alpha\beta} := \underbrace{g_{\alpha\beta} \wedge \ldots \wedge g_{\alpha\beta}}_{k \text{ times}} : U_\alpha \cap U_\beta \to GL\left(\Lambda^k F\right)$
Exterior algebra bundle	ΛE	$g^{\cdot}_{\alpha\beta} := \oplus_k g^k_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\Lambda F)$

For convenience, whenever we make use of $V \otimes E$ with V trivial and E a vector bundle as above, we shall always assume that the representative point set of $V \otimes E$ is

$$V \otimes \bigcup_{\alpha\in\mathrm{A}} \{\alpha\} \times U_\alpha \times F$$

so that, for instance, $(V \otimes E)_x = V \otimes E_x$ for each $x \in M$. Moreover, we assume for convenience that the representative point set of $f^{-1}E$ is

$$\bigcup_{y\in N} \{y\} \times E_{f(y)}$$

so that $(f^{-1}E)_y = \{y\} \times E_{f(y)}$. We shall write T_xM, T^*_xM and $\Lambda^{\cdot} T^*_x M$ for the fibres of the respective bundles ($x \in M$). Finally, note that, by construction, all of the bundles above inherit canonical pairings from their linear algebraic counterparts and we shall use the same notation in all cases. For instance, the canonical dual pairing $F^* \times F \to \mathbb{R}$ defines the pairing

$$\begin{gathered} E^* \times E \to \mathbb{R} \\ ([\alpha, x, \varepsilon], [\alpha, x, e]) \mapsto (\varepsilon, e), \end{gathered}$$

which is well-defined, since

$$\left([\beta, x, {}^t g_{\alpha\beta}(x)\varepsilon], [\beta, x, g_{\beta\alpha}(x)e]\right) = ({}^t g_{\alpha\beta}(x)\varepsilon, g_{\beta\alpha}(x)e) = (\varepsilon, g_{\alpha\beta}(x)g_{\beta\alpha}(x)e) = (\varepsilon, e).$$

We now turn our attention to the maps from the base manifold to the total space. We shall only be concerned with sections in this regard.

Definition 1.4.7. Let $U \subset M$ be open and E a G-bundle. A (smooth) map $\sigma : U \to E$ is said to be a (smooth) *local section* over U if $\pi \circ \sigma = \mathrm{id}_U$ and the set of all smooth local sections of E over U is denoted by $\Gamma(E \to U)$, or simply $\Gamma(E)$ if $U = M$. If $E = TM$, we refer to σ as a *vector field*, and if $E = \Lambda T^*M$ we call σ a *differential form*. Finally, if $\sigma : U \times I \to E$ is a smooth function with $I \subset \mathbb{R}$ open such that $\sigma(\cdot,t)$ is a local section over U, we say that σ is a *time-dependent section* over $U \times I$.

Local sections exist in abundance; e.g. the map $U_\alpha \ni x \mapsto \Psi_\alpha(x,f)$ is a local section over U_α for fixed $f \in F$. The existence of a global section is not necessarily guaranteed, however. On the one hand, if E is a vector bundle, then the map $x \mapsto 0_x \in E_x$ can be seen to be a smooth section (0_x is the zero element of the vector space E_x). On the other hand,

Proposition 1.4.8. *Let P be a principal G-bundle. Then sections over $U \subset M$ open are in one-to-one correspondence with bundle charts over U.*

Proof. If $\Psi : U \times G \to \pi^{-1}(U)$ is a bundle chart, then, as above, $x \mapsto \Psi(x,e)$ is a local section over U, where e is the identity of G. On the other hand, if $\sigma : U \to P$ is a local section, then the map $U \times G \ni (x,g) \mapsto \sigma(x)g \in \pi^{-1}(U)$ is smooth and may be seen to be a diffeomorphism. □

The analogous statement for vector bundles is

Proposition 1.4.9. *Let E be a vector bundle. Then **frames** over $U \subset M$ open, i.e. a collection of $\dim F$ smooth sections $\{\varepsilon_i : U \to E\}$ with $E_x = \mathrm{span}\{\varepsilon_i(x)\}_{i=1}^{\dim F}$ for each $x \in U$, are in one-to-one correspondence with ($GL(F)$) bundle charts over U.*

Proof. If $\Psi : U \times F \to \pi^{-1}(U)$ is a $GL(F)$ bundle chart over U, then, by the above, $\Psi(x,\cdot) : F \to E_x$ is a vector space isomorphism for each $x \in U$, whence $\{x \mapsto \Psi(x,e_i)\}_{i=1}^{\dim F}$ yields a frame over U, where e_i is some basis for F. On the other hand, if $\{\varepsilon_i\}$ is a local frame over U, then $U \times F \ni \left(x, \sum_{i=1}^{\dim F} v^i e_i\right) \mapsto \sum_{i=1}^{\dim F} v^i \varepsilon_i(x)$ defines a bundle chart. □

To conclude this section, we recall a familiar example reformulated in the language of bundle theory.

Proposition 1.4.10. [72, §1.22] *Let M,N be smooth manifolds with charts $\{(U_\alpha,x_\alpha)\}$ and $\{(V_\xi,y_\xi)\}$ respectively, and suppose $f : M \to N$ is a smooth map. The map*

$$T_xM \ni [\alpha,x,v] \mapsto [\xi,f(x),D(y_\xi \circ f \circ x_\alpha^{-1})(x_\alpha(x))v] \in T_{f(x)}N$$

*defines a vector space homomorphism $\mathrm{d}_x f : T_xM \to T_{f(x)}N$, the **differential** of f, which may, alternatively, be viewed as the vector bundle morphism*

$$\begin{aligned} &\mathrm{d}f : TM \to f^{-1}TN \\ &[\alpha,x,v] \mapsto [\xi,x,D(y_\xi \circ f \circ x_\alpha^{-1})(x_\alpha(x))v] \end{aligned}$$

or the smooth section $\mathrm{d}f \in \Gamma\left(f^{-1}TN \otimes T^*M\right)$ defined by

$$x \mapsto [\xi\alpha, x, \sum_{i=1}^{n} D(y_\xi \circ f \circ x_\alpha^{-1})\,(x_\alpha(x))\, e_i \otimes \omega^i].$$

The differential of f induces the map

$$\delta_x f : T^*_{f(x)}N \to T^*_x M$$

*the **codifferential of** f, which in turn induces algebra homomorphisms $\Lambda T^*_{f(x)}N \to \Lambda T^*_x M$ and $\bigotimes T^*_{f(x)}N \to \bigotimes T^*_x M$, both denoted by $\delta_x f$. Given a section $\rho \in \Gamma(V \otimes \Lambda T^*N)$ ($\rho \in \Gamma(V \otimes \bigotimes T^*N)$), where V is a trivial vector bundle, the map*

$$M \ni x \mapsto (\mathrm{id}_V \otimes \delta_x f)\,(\rho_{f(x)}) \in V \otimes \Lambda T^*_x M \text{ (resp. } x \mapsto (\mathrm{id}_V \otimes \delta_x f)(\rho_{f(x)}) \in V \otimes \textstyle\bigotimes T^*_x M)$$

*defines a smooth section $f^*\rho \in \Gamma\left(V \otimes \Lambda T^*M\right)$ (resp. $f^*\rho \in \Gamma(V \otimes \bigotimes T^*M)$), the **pullback of** ρ **by** f.*

1.5. Connections and covariant derivatives. We now introduce the notion of a connection on a principal bundle $G \to P \xrightarrow{\pi} M$. To this end, let $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}_{\alpha,\beta \in \mathrm{A}}$ be cocycles for P. We shall adopt the following definition of a connection:[6]

Definition 1.5.1. [49, §4] A *connection* on P is a collection $\omega := \{\omega_\alpha \in \Gamma(\mathfrak{g} \otimes T^*U_\alpha)\}$ such that for any $\alpha, \beta \in \mathrm{A}$, $x \in U_\alpha \cap U_\beta$ and $v \in T_xM$,

$$(\omega_\beta, v) = \mathrm{Ad}_{g_{\beta\alpha}(x)}(\omega_\alpha, v) + \left(\mathrm{d}_{g_{\alpha\beta}(x)}\lambda_{g_{\beta\alpha}(x)} \circ \mathrm{d}_x g_{\alpha\beta}\right)(v).$$

Note that if it weren't for the latter term, we could appeal to Theorem 1.4.6 to conclude that a connection defines a section of $(P \times_{\mathrm{Ad}} \mathfrak{g}) \otimes T^*M$. However, if $\{\widetilde{\omega}_\alpha\}_\alpha$ is any other connection, then we have that

$$(\widetilde{\omega}_\beta - \omega_\beta, v) = \mathrm{Ad}_{g_{\beta\alpha}(x)}(\widetilde{\omega}_\alpha - \omega_\alpha, v)$$

so that the "difference" of two connections now defines a section of $(P \times_{\mathrm{Ad}} \mathfrak{g}) \otimes T^*M$. Therefore, the space of connections may be viewed as being parametrized by elements of $\Gamma\left(P \times_{\mathrm{Ad}} \mathfrak{g} \otimes T^*M\right)$ and, when we write $\widetilde{\omega} = \omega + a$ for $a \in \Gamma(P \times_{\mathrm{Ad}} \mathfrak{g} \otimes T^*M)$, we shall mean that a corresponds to the section given by $\{\widetilde{\omega}_\alpha - \omega_\alpha\}_{\alpha \in \mathrm{A}}$. This should be compared to the behaviour of Christoffel symbols in Riemannian geometry.

Given a connection and a section $g = [\alpha, \cdot, g_\alpha]$ of the associated bundle $G \to P \times_C G \to M$, one may form a new connection $g \cdot \omega = \{\omega_\alpha\}$ from ω by defining

$$(\widetilde{\omega}_\alpha, v) := \mathrm{Ad}_{g_\alpha(x)}(\omega_\alpha, v) + \left(\mathrm{d}_{g_\alpha^{-1}(x)}\lambda_{g_\alpha(x)} \circ \mathrm{d}_x g_\alpha^{-1}\right)(v)$$

for $\alpha \in A$, $x \in U_\alpha$ and $v \in T_xM$ [4, §2]. We refer to elements of $\Gamma(P \times_C G)$, where C is the conjugate action of G on itself (cf. §1.3), as *gauge transformations* [3, §2] and subsequently identify gauge-transformed connections. Such transformations also act on sections of $\Gamma(P \times_{\mathrm{Ad}} \mathfrak{g} \otimes \Lambda T^*M)$ by means of Ad, i.e. for $[\alpha, \cdot, f_\alpha] \in \Gamma(P \times_{\mathrm{Ad}} \mathfrak{g} \otimes \Lambda T^*M)$,

$$g \cdot [\alpha, \cdot, f_\alpha] := [\alpha, \cdot, \mathrm{Ad}_{g_\alpha} f_\alpha].$$

Connections induce covariant derivatives on associated bundles. We first recall the notion of a covariant derivative on a vector bundle.[7]

Definition 1.5.2. [60, §2.51] A *covariant derivative* on a vector bundle $E \to M$ is an $\mathbb{R}$-linear map $\Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*M)$ such that

$$\nabla(fs) = f\nabla s + s \otimes \mathrm{d}f \tag{1.1}$$

for every $f \in C^\infty M$ and $s \in \Gamma(E)$. We write $\nabla_X s$ for $(\nabla s, X)$ whenever $X \in \Gamma(TM)$.

Example 1.5.3. If E and F are vector bundles equipped with covariant derivatives, then $E \otimes F$ may be equipped with a covariant derivative such that, whenever $s_E \in \Gamma(E)$ and $s_F \in \Gamma(F)$,

$$\nabla_X(s_E \otimes s_F) := \nabla_X s_E \otimes s_F + s_E \otimes \nabla_X s_F.$$

for every $X \in \Gamma(TM)$ [60, §2.62]. That this operator is well defined has to be checked locally. Similarly, if $s_E^* \in \Gamma(E^*)$, then a covariant derivative may be defined on E^* [60, §2.61], just as is usually done for T^*M in Riemannian geometry, such that

[6] For a more geometric presentation of the theory of connections on principal bundles and a proof of the equivalence of the various definitions, the reader is referred to [60, Chapter 9] and [50, §2•2].

[7] Since there is a one-to-one correspondence between connections on principal $GL(N)$-bundles and covariant derivatives on vector bundles associated to them by the standard matrix representation $GL(N) \times \mathbb{R}^N \to \mathbb{R}^N$ [60, Theorem 9.3], covariant derivatives are usually also, by an abuse of terminology, referred to as connections.

$$\partial_X \left(s_E^*(s_E)\right) = (\nabla_X s_E^*)(s_E) + s_E^*(\nabla_X s_E).$$

Moreover, a covariant derivative on $\Gamma(\Lambda E)$ may be defined such that

$$\nabla_X(s_1 \wedge \cdots \wedge s_k) = \nabla_X s_1 \wedge s_2 \wedge \cdots \wedge s_k + s_1 \wedge \nabla_X s_2 \wedge s_3 \wedge \cdots \wedge s_k \\ + \cdots + s_1 \wedge \cdots \wedge s_{k-1} \wedge \nabla_X s_k.$$

whenever $s_1, \ldots, s_k \in \Gamma(E)$ [60, §2.62]. Finally, suppose $f : N \to M$ is smooth. A covariant derivative may be induced on $f^{-1}E$ as follows: if $U \subset M$ is open and $\{e_\alpha : U \to E\}$ form a local frame for E, then ∇ may be described locally by a collection of of local sections $\{\Gamma_\alpha^\beta\}$ of T^*M such that

$$\nabla e_\alpha = \sum_\beta e_\beta \otimes \Gamma_\alpha^\beta \tag{1.2}$$

on U [75, §III.1]. Since $\{e_\alpha^f := (f^{-1}(U) \ni y \mapsto (y, e_\alpha(y)))\}$ form a local frame for E, any section $s \in \Gamma(E)$ may be written in the form

$$s = \sum_\alpha s^\alpha e_\alpha^f.$$

We thus define a covariant derivative on $f^{-1}E$ by

$$\nabla s = \sum_\alpha \left(e_\alpha^f \otimes \mathrm{d}s^\alpha + s^\alpha \sum_\beta f^*\Gamma_\alpha^\beta \otimes e_\beta^f \right). \tag{1.3}$$

This expression may be shown to be independent of the choice of local frame.

Remark 1.5.4. For later purposes, we shall need to know how a Riemannian connection on $E \to M$ and the induced connection $f^{-1}E \to N$ ($f \in C^\infty(M, N)$) compare. To this end, let Y be a local section of $f^{-1}E$ and $\overline{Y}$ a local section of E such that

$$Y(x) = \left(x, \overline{Y}(f(x))\right) \tag{1.4}$$

for $x \in f^{-1}(U)$, where $U \subset M$ is open. Expanding in terms of a local frame as in Example 1.5.3, we write

$$\overline{Y} = \sum_\alpha \overline{Y}^\alpha e_\alpha$$

and

$$Y = \sum_\alpha Y^\alpha e_\alpha^f,$$

noting that the relation (1.4) implies that $\overline{Y}^\alpha \circ f = Y^\alpha$ on $f^{-1}(U_\alpha)$. Now, let $X \in T_{x_0}N$ for $x_0 \in f^{-1}U$. (1.3) implies that

$$\nabla_X Y = (x_0, \sum_\alpha \left((\partial_X Y^\alpha)\, e_\alpha(f(x_0)) + Y^\alpha(x_0) \sum_\beta (f^*\Gamma_\alpha^\beta)(X) e_\beta(f(x_0)) \right)).$$

Using $Y^\alpha = \overline{Y}^\alpha \circ f$ and a smooth curve $c :]-\varepsilon, \varepsilon[\to N$ such that $c(0) = x_0$ and $c'(0) = X$, we see that

$$\partial_X Y^\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (Y^\alpha \circ c) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\overline{Y}^\alpha \circ (f \circ c)\right) = \partial_{\mathrm{d}f(X)} \overline{Y}^\alpha.$$

We also note that $Y^\alpha(x_0) = \overline{Y}^\alpha(f(x_0))$

$$(f^*\Gamma_\alpha^\beta)(X) = \Gamma_\alpha^\beta(\mathrm{d}f(X))$$

so that, using (1.2) and the Leibniz rule (1.1), we have that

$$\nabla_X Y = (x_0, \nabla_{\mathrm{d}f(X)} \overline{Y}).$$

Example 1.5.5. [70, §2] Let $E = P \times_\rho V$ be a vector bundle associated to P and a representation ρ. The map

$$\Gamma(P \times_\rho V) \ni [\alpha, \cdot, s] \mapsto \sum_{i=1}^n [\alpha, \cdot, \partial_i s + \mathrm{d}_e \rho\left((\omega_\alpha, \partial_i)\right) s] \otimes \mathrm{d}x^i \in \Gamma(P \times_\rho V)$$

is well-defined. This is the canonical *induced covariant derivative* on the associated bundle E.

For later purposes, we shall need a differential operator on sections of $E \otimes \Lambda T^*M$ analogous to the exterior derivative. Recall that, whenever ∇ is a torsion-free covariant derivative on TM, the latter may be defined by

$$\mathrm{d}\omega = \sum_{i=1}^n \theta^i \wedge \nabla_{t_i} \omega$$

for $\omega \in \Gamma(\Lambda T^*M)$ and any local frame $\{t_i\}$ for TM with dual coframe $\{\theta^i\}$ for T^*M [60, §4.1]. In our case, such a differential operator may be similarly defined.

Definition 1.5.6. The *exterior covariant derivative* associated to a connection ∇ on E is defined by

$$\Gamma(E \otimes \Lambda T^*M) \to \Gamma(E \otimes \Lambda T^*M)$$
$$s \mapsto \mathrm{d}^\nabla s := \sum_{i=1}^n \mathrm{d}x^i \wedge \nabla_{\partial_i} s,$$

where ∇ is induced by the connections on E and TM and the connection on TM is torsion-free.

We now introduce a notion of curvature.

Proposition 1.5.7. *The collection $\Omega^\omega = \{\Omega_\alpha^\omega \in \Gamma(\mathfrak{g} \otimes \Lambda^2 T^* U_\alpha)\}$ defined by*[8]

$$\Omega_\alpha^\omega := \mathrm{d}\omega_\alpha + \frac{1}{2}[\omega_\alpha, \omega_\alpha]$$

*defines a global section $\underline{\Omega}^\omega \in \Gamma\left((P \times_{Ad} \mathfrak{g}) \otimes \Lambda^2 T^*M\right)$, referred to as the **curvature form** of ω. It satisfies the **Bianchi identity***

$$\mathrm{d}^\nabla \underline{\Omega}^\omega = 0,$$

where ∇ is the covariant derivative induced by ω on the associated bundle $P \times_{Ad} G$.

[8] Here, $[\cdot,\cdot]$ on $\mathfrak{g}$ is combined with wedge multiplication on T^*M to yield $[\cdot,\cdot]$ on $\mathfrak{g} \otimes \Lambda T^*M$.

Proof. By [7, Theorem 2.2.14], we have

$$\Omega^\omega_\beta = \mathrm{Ad}_{g_{\beta\alpha}} \Omega^\omega_\alpha$$

on $U_\alpha \cap U_\beta$. By Theorem 1.4.6, the map

$$x \mapsto \sum_{i<j} [x, \alpha, (\Omega^\omega_\alpha, \partial_i \wedge \partial_j)] \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j$$

is well-defined, thus establishing the first claim. The latter claim is a rephrasing of [7, Theorem 2.2.15]. □

1.6. Riemannian vector bundles. Let $E \to M$ be a vector bundle with projection $\pi : E \to M$. We denote the space of smooth sections of E over M, i.e. smooth maps $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$, by $\Gamma(E)$. If $s : M \times]a, b[\to E$ is a smooth map such that $s(\cdot, t) \in \Gamma(E)$ for each $t \in]a, b[$, we shall refer to s as a *time-dependent section* over $M \times]a, b[$.

Definition 1.6.1. [60, §3.1] If the fibres $E_p := \pi^{-1}(\{p\})$ ($p \in M$) of E are equipped with positive-definite inner products $\langle \cdot, \cdot \rangle_p$ such that, for any local frame $\{e_\alpha\}$ the map

$$p \mapsto \left\langle e_\alpha(p), e_\beta(p) \right\rangle_p$$

is smooth, we say that E is *Riemannian.*

Example 1.6.2. A manifold is Riemannian iff TM is a Riemannian vector bundle. □

Example 1.6.3. Let $\langle \cdot, \cdot \rangle : V \times V \to K = \mathbb{R}$ or $\mathbb{C}$ be a positive-definite inner product on a K-vector space V, P a principal G-bundle and $\rho : G \to GL(V)$ a group representation. If $\langle \cdot, \cdot \rangle$ is ρ-invariant, i.e.

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$$

for every $g \in G$ and $v, w \in V$, then it defines a Riemannian structure on the associated vector bundle $P \times_\rho V$ such that for each $x \in M$ and $v, w \in V$,

$$\langle [\alpha, x, v], [\alpha, x, w] \rangle_x := \langle v, w \rangle .$$

Thus, a ρ-invariant inner product on a vector bundle associated to P and ρ. □

Example 1.6.4. Given Riemannian structures on vector bundles $E, E_i \to M$, $i \in \{1, \ldots, k\}$, we may define Riemannian structures on ΛE, E^* and $E_1 \otimes \cdots \otimes E_k$ in a canonical manner just as was done when inducing inner products on the vector space counterparts of the above: define everything pointwise. Similarly, if $f : N \to M$ is smooth, then a Riemannian structure may be defined on $f^{-1}(E)$ such that for $(y, v_{f(y)}), (y, w_{f(y)}) \in (f^{-1}E)_y$,

$$\left\langle (y, v_{f(y)}), (y, w_{f(y)}) \right\rangle_y := \left\langle v_{f(y)}, w_{f(y)} \right\rangle_{f(y)} .$$

It follows from the definitions of these bundles that the resulting inner products are smooth [60, §3.6–§3.8]. □

A vector bundle may *always* be made to be Riemannian [60, Proposition 3.3] and a Riemannian vector bundle, viewed as a $GL(F)$-bundle, admits a canonical $O(F)$ (with respect to some fixed inner product on F) bundle atlas by taking the cocycles to be $\widetilde{g}_{\alpha\beta}(x) = \tau_\alpha(x)^{-1} g_{\alpha\beta}(x) \tau_\beta(x)$, where $\tau_\alpha(x) \in GL(F)$ is defined such that $\{[\alpha, x, \tau_\alpha(x)e_i]\}$ ($\{e_i\}$ orthonormal in F) form an orthonormal basis for E_x and $x \mapsto [\alpha, x, \tau_\alpha(x)e_i]$ is a smooth local section of E (e.g. by using the Gram-Schmidt algorithm). On the other hand, given an $O(F)$ bundle atlas (relative to an inner product $\langle \cdot, \cdot \rangle$ on F) for E, a Riemannian structure may be defined by the formula $(g_x, [\alpha, x, v] \otimes [\alpha, x, w]) := \langle v, w \rangle$, which defines a global object by virtue of the universal mapping property and the given bundle atlas.

Given a Riemannian structure, we may define the L^2 inner product

$$(s_1, s_2) := \int_M \langle s_1, s_2 \rangle \, \mathrm{dvol}_g \tag{1.5}$$

for sections[9] $s_1, s_2 \in \Gamma(E)$ such that (s_1, s_1) and (s_2, s_2) are finite and write $s \in L^2(E \to M)$ (or simply $s \in L^2(M)$ when E is understood) whenever $s \in \Gamma(E)$, $\langle s, s \rangle$ is measurable (with respect to the Borel measure induced by dvol_g) and $(s, s) < \infty$. We may similarly use the norm induced by the inner product $\langle \cdot, \cdot \rangle$ to define L^p norms $||\cdot||_p$ of sections *and* time-dependent sections, where the domain of integration is to be understood as the domain of definition of the section in question. For instance, if $(M, \{g_t\}_{t \in I})$ is an evolving Riemannian manifold and $\mathcal{D} \subset M \times I$ is open and $s : \mathcal{D} \to E$ is a time-dependent local section of E, i.e. $s(\cdot, t)$ is a local section of E for all $t \in \mathrm{pr}_2(\mathcal{D})$, then

$$||s||_p := \left(\iint_{\mathcal{D}} \langle s, s \rangle^{p/2} \, \mathrm{dvol}_{g_t} \mathrm{d}t \right)^{1/p}$$

and we write $s \in L^p(\mathcal{D})$ whenever $\langle s, s \rangle$ is measurable (with respect to the Borel measure on $M \times I$ induced by $\mathrm{dvol}_{g_t} \wedge \mathrm{d}t$) and $||s||_p < \infty$. Note that all of these considerations reduce to classical statements about L^p theory when $E = \mathbb{R} \to M$ is the trivial line bundle over M, in which case (time-dependent) sections are simply described by (time-dependent) functions on M.

We would like to consider differential operators on $\Gamma(E)$ in the sequel which, in a sense, are compatible with the Riemannian structure on E.

Definition 1.6.5 ([60, §3.19]). A *Riemannian connection* ∇ on E is a covariant derivative on E such that

$$\partial_X \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle$$

holds for every $X \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$.

Example 1.6.6. If $E \to M$ and $F \to M$ are Riemannian vector bundles equipped with Riemannian connections and $f : N \to M$ is smooth, then the induced covariant derivatives of Example 1.5.3 are all Riemannian connections [60, Proposition 3.23].

Example 1.6.7. Let $E = P \times_\rho V$ be a vector bundle associated to a principal bundle P and group representation $G \to GL(V)$ and suppose it is equipped with a Riemannian structure induced by some ρ-invariant inner product on V. Now, for fixed $Z \in \mathfrak{g}$ and $v, w \in V$,

$$0 = \partial_Z \langle \rho_\cdot(v), \rho_\cdot(w) \rangle = \langle \mathrm{d}_e \rho(Z) v, w \rangle + \langle v, \mathrm{d}_e \rho(Z) w \rangle .$$

Hence, if P is equipped with a connection ω, $s_1 = [\alpha, \cdot, \sigma_1], s_2 = [\alpha, \cdot, \sigma_2] \in \Gamma(E)$ and $X \in \Gamma(TM)$, we have

$$\begin{aligned}
\partial_X \langle s_1, s_2 \rangle &= \partial_x \langle \sigma_1, \sigma_2 \rangle \\
&= \langle \partial_X \sigma_1, \sigma_2 \rangle + \langle \sigma_1, \partial_X \sigma_2 \rangle \\
&= \langle \partial_X \sigma_1, \sigma_2 \rangle + \langle \mathrm{d}_e \rho((\omega_\alpha, X)) \sigma_1, \sigma_2 \rangle + \langle \sigma_1, \mathrm{d}_e((\omega_\alpha, X)) \sigma_2 \rangle + \langle \sigma_1, \partial_X \sigma_2 \rangle \\
&= \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle ,
\end{aligned}$$

where ∇ is the covariant derivative induced on E by ω, whence we see that ∇ is Riemannian. □

We may, as in §1.5, build an exterior covariant differential d^∇ from ∇. Using this and the L^2 inner product (1.5), we may form a "divergence-type" operator δ^∇ as in the following lemma.

[9] Here the sections in question needn't be smooth nor continuous.

Lemma 1.6.8. *The following hold:*

(i) $\mathrm{d}^\nabla : \Gamma(E \otimes \Lambda T^*M) \to \Gamma(E \otimes \Lambda T^*M)$ *is an* $\mathbb{R}$*-linear map such that* $\mathrm{d}^\nabla(fs) = df \wedge s + f\mathrm{d}^\nabla s$ *whenever* $f \in C^\infty(M)$ *and* $s \in \Gamma(E \otimes \Lambda T^*M)$.

(ii) Define the ***codifferential*** $\delta^\nabla : \Gamma(E \otimes \Lambda T^*M) \to \Gamma(E \otimes \Lambda T^*M)$ *as the* $\mathbb{R}$*-linear map such that*

$$\delta^\nabla s := -\sum_{i=1}^n \iota_{\varepsilon_i} \nabla_{\varepsilon_i} s =: -\mathrm{div}\, s$$

for every $s \in \Gamma(E \otimes \Lambda^k T^*M)$ *and* $k > 0$, *and* $\delta^\nabla\big|_{\Gamma(E)} \equiv 0$. *This operator satisfies the identity*

$$\left\langle s_1, \mathrm{d}^\nabla s_2 \right\rangle - \left\langle \delta^\nabla s_1, s_2 \right\rangle = \mathrm{div}\left(\sum_{i=1}^n \langle \iota_{\varepsilon_i} s_1, s_2 \rangle\, \omega^i \right)$$

for every $s_1 \in \Gamma(E \otimes \Lambda^{k+1} T^*M)$ *and* $s_2 \in \Gamma(E \otimes \Lambda^k T^*M)$. *In particular,* δ^∇ *is the formal adjoint of* d^∇ *restricted to compactly supported sections of* $E \otimes \Lambda^k T^*M$ *with respect to the* L^2*-inner product, i.e.*

$$\int_M \left\langle s_1, \mathrm{d}^\nabla s_2 \right\rangle \mathrm{dvol}_g = \int_M \left\langle \delta^\nabla s_1, s_2 \right\rangle$$

whenever $\mathrm{supp}\ s_i = \overline{\{x \in M : s_i(x) \neq 0 \in E_x \otimes \Lambda^{k+2-i} T_x^* M\}} \Subset M$.

Proof. (cf. [60, §2.76, §4.7])

1. The former claim is evident from the definition. As for the latter,

$$\begin{aligned} \mathrm{d}^\nabla(fs) &= \sum_{i=1}^n \omega^i \wedge \nabla_{\varepsilon_i}(fs) \\ &= \sum_{i=1}^n \partial_{\varepsilon_i} f \omega^i \wedge s + f\omega^i \wedge \nabla_{\varepsilon_i} s \\ &= \mathrm{d}f \wedge s + f\mathrm{d}^\nabla s. \end{aligned}$$

2. We compute in a frame adapted at $p \in M$:

$$\begin{aligned} &\left\langle s_1, \mathrm{d}^\nabla s_2 \right\rangle \\ &= \left\langle s_1, \sum_{i=1}^n \omega^i \wedge \nabla_{\varepsilon_i} s_2 \right\rangle \\ &= \sum_{i=1}^n \langle \iota_{\varepsilon_i} s_1, \nabla_{\varepsilon_i} s_2 \rangle \\ &= \sum_{i=1}^n \partial_{\varepsilon_i} \langle \iota_{\varepsilon_i} s_1, s_2 \rangle - \langle \nabla_{\varepsilon_i} (\iota_{\varepsilon_i} s_1), s_2 \rangle \\ &= \mathrm{div}\left(\sum_{i=1}^n \langle \iota_{\varepsilon_i} s_1, s_2 \rangle \right) + \left\langle \delta^\nabla s_1, s_2 \right\rangle, \end{aligned}$$

since $\nabla_{\varepsilon_i} \varepsilon_j = 0$ at p. The latter claim follows from Gauß' theorem.

□

Motivated by considerations in Hodge theory (cf. [72, Chapter 6]), we define the *Hodge Laplacian* Δ^∇ on $\Gamma(E \otimes \Lambda T^*M)$ associated to ∇ by

$$\Delta^\nabla := d^\nabla \delta^\nabla + \delta^\nabla d^\nabla.$$

It is noted that for compact M, the spectrum of this operator is *nonnegative* since, by Lemma 1.6.8 (ii),

$$\int_M \left\langle s, \Delta^\nabla s \right\rangle \mathrm{dvol}_g = \int_M |d^\nabla s|^2 + |\delta^\nabla s|^2 \mathrm{dvol}_g$$

for every $s \in \Gamma(E \otimes \Lambda^k T^*M)$. Thus, if $M = \mathbb{R}^n$, $E = \mathbb{R}^n \times \mathbb{R}$ is the trivial line bundle and $\nabla_X (x \mapsto (x, f(x))) = (p, \partial_X f)$ for $X \in T_p\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$, it follows (cf. [72, §6.1]) that

$$\left(\Delta^\nabla(\cdot, f)\right)(x) = (x, -\sum_{i=1}^n \partial_i^2 f(x)).$$

Similarly, the associated *heat operator*, acting on time-dependent sections of $E \otimes \Lambda T^*M$, is defined as the operator

$$\partial_t + \Delta^\nabla,$$

viz. if $M = \mathbb{R}^n$, and E is the trivial line bundle with ∇ as before, we have

$$\left(\partial_t + \Delta^\nabla\right)(\cdot, f)(x) = (x, (\partial_t f - \Delta f)(x))$$

which coïncides with the usual heat operator. We note that the above sign conventions for the Laplacian and heat operator shall only be retained for sections of bundles of the form $E \otimes \Lambda T^*M$ and that, for real-valued functions (cf. §1.7), different conventions shall be taken.

1.7. Geometric setup. The notion of an evolving Riemannian manifold is now introduced. This shall, for the most part, be the setting in which we work in this thesis.

Definition 1.7.1. A manifold M is said to be equipped with an *evolving Riemannian metric* $g = \{g_t\}_{t\in I}$ if $\{g_t \in \Gamma(T^*M \otimes T^*M)\}_{t\in I}$ is a smooth one-parameter family of Riemannian metrics indexed by an open interval I. Such a Riemannian manifold M is said to be *evolving*.

We shall *always* write

$$\partial_t g(x) = h(x),$$

where $\{h_t \in \Gamma(T^*M \otimes T^*M)\}_{t\in I}$ is a smooth one-parameter family of sections of $T^*M \otimes T^*M$ and agree to refer to the parameter t as *time*.

Example 1.7.2. If $I = \mathbb{R}$ and $h \equiv 0$, we say that g (or M) is *static*. □

Example 1.7.3. If $h = -2\mathrm{Ric}$, where Ric is the Ricci curvature of g, (M, g) is said to evolve by *Ricci flow*. □

The usual notions and quantities of Riemannian geometry translate to this setting by fixing time and considering the usual quantities. Noteworthy, however, is that the resulting quantities are *smooth* in time. The relevant notation is summarized in the following table:

Symbol	Signification
tr_{g_t}	Trace of a (time-dependent) section of $E \otimes T^*M \otimes T^*M$ (E a vector bundle) with respect to g_t

dvol_{g_t}	Volume form of g_t
Vol_{g_t}	Volume measure induced by dvol_{g_t}
$\exp_p^t$	Exponential map at p of (M, g_t)
ϑ_p^t	Exponential coördinates at p with respect to g_t as in Appendix B
inj_p^t	Injectivity radius at p of (M, g_t)
d^t	Geodesic distance on (M, g_t)
$B_r^t(p)$	Geodesic ball of radius r at p in (M, g_t)
∇_g	Gradient associated to g (time understood)
div_g	Divergence induced by the Levi-Civita connection ∇ associated to g
$\sec_M$	Sectional curvature of (M, g)
Ric	Ricci curvature. of (M, g)
∇_g^2	Hessian $:= \nabla \circ d$
Δ_g	Laplace-Beltrami operator ($:= \mathrm{div} \circ \nabla = \mathrm{tr}_g \circ \nabla^2$)
$\{\varepsilon_i(p,t)\}_{i=1}^n$	Local g_t-orthonormal frame for TM defined for (p,t) in some open subset of $M \times I$
$\{\omega^i(p,t)\}_{i=1}^n$	Local orthonormal (co)frame for T^*M dual to $\{\varepsilon_i\}_{i=1}^n$
ε_I	$\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}$ where $I^k = (i_1, \ldots, i_k)$ is an increasing k-multi-index
ω^I	$\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$ where $I^k = (i_1, \ldots, i_k)$ is an increasing k-multi-index

We adopt the sign conventions of [57] and refer to that book for definitions of the above objects, noting in particular that the sign convention of the Laplacian here is *opposite* to that of the Hodge-type Laplacian Δ^∇ introduced earlier. When the time parameter t is understood, we omit it from the above symbols and, when the metric is understood, we simply write div for div_g, ∇ for ∇_g and Δ for Δ_g. Fix $p \in M$ and set $\mathfrak{r}(x,t) = d^t(x,p)$. It is known that $\mathfrak{r}^2$ is smooth in some neighbourhood of (p,s) for any $p \in M$, $s \in I$ [14].Finally, we say a g_{t_0}-orthonormal frame $\{\varepsilon_i\}$ ($t_0 \in I$ fixed) is *adapted at $p \in M$* if $\nabla_{\varepsilon_i}\varepsilon_j = 0$ at p.

With the above notation in mind, we introduce for $r_x, r_t > 0$, $p \in M$ and $s \in I$ the **spacetime cylinder**

$$\mathcal{D}_{r_x,r_t}(p,s) = \bigcup_{t \in I \cap]s-r_t,s[} B_{r_x}^t(p) \times \{t\}.$$

In this thesis we shall be chiefly concerned with geometries that are suitably controlled locally, as in the following definition.

Definition 1.7.4. Let $(M, \{g_t\}_{t\in I})$ be a Riemannian manifold with evolving metric, $x_0 \in M$ and $t_0 \in \bar{I} \setminus \{\inf I\}$. M is said to be of **locally bounded geometry about** (x_0, t_0) if there exist $\kappa_{-\infty}, \kappa_\infty, \lambda_{-\infty}, \lambda_\infty \in \mathbb{R}$ and $\delta > 0$ such that $\mathrm{inj}_{x_0}^t \geqslant \frac{\mathrm{inj}_{x_0}^{t_0}}{2} =: \frac{i_0}{2}$ for every $t \in]t_0 - \delta, t_0[$ and the bounds

$$\kappa_{-\infty} \leqslant \sec \leqslant \kappa_\infty$$
$$\lambda_{-\infty} g \leqslant h \leqslant \lambda_\infty g$$

hold in a neighbourhood of $\mathcal{D}_{i_0/2,\delta}(x_0, t_0)$.[10]

Remark 1.7.5. If $t_0 \in I$, then M is always of locally bounded geometry about (x_0, t_0) as may be seen by writing the geodesic equations down and using the smooth dependence of the system of ODE on the parameter t (cf. e.g. [59, Theorem 4.17]) to conclude that for t sufficiently close to t_0, unit speed geodesics radiating from x_0 exist on the interval $[0, \frac{i_0}{2}]$. We shall, however, *require* that M be of locally bounded geometry about (x_0, t_0) and usually suppose I is an interval of the form $]t_0 - \delta_0, t_0[$.

[10]Note that if δ fulfils these conditions, then so does $\delta' < \delta$.

On the one hand, this then implies that $\left|\mathrm{tr}_g h\right| \leqslant n \cdot \max\{|\lambda_{-\infty}|, |\lambda_\infty|\} =: n\mu$ and $|\partial_t \mathfrak{r}| \leqslant \frac{\mu}{2} r$ on $\mathcal{D}_{i_0/2,\delta}(x_0, t_0) \backslash \{\mathfrak{r} = 0\}$ [14, §18.1.1].

Since we shall make use of comparison geometry results (see Appendix B), the spacetime neighbourhood of (x_0, t_0) we shall be working with will have to be small enough to accommodate for the geometries of the comparison spaces. To this end, we set[11]

$$j_0 = \min\left\{\frac{i_0}{2}, \frac{\pi}{2\sqrt{(\kappa_\infty)_+}}, \frac{\pi}{2\sqrt{(\kappa_{-\infty})_+}}\right\}. \tag{1.6}$$

We shall make use of this notation throughout this thesis and also assume that the flows to be considered are defined on $]t_0 - \delta, t_0[$.

For later purposes, we shall need to know how the family of metrics induced on T^*M, to be written (g_t^*), and dvol_g evolve. Write $\sharp^t$ and $\flat^t$ for the musical isomorphisms induced by g_t.

Proposition 1.7.6. $\partial_t g^* = -h^{\sharp^t}$.

Proof. We work in coördinates: write $g_{ij} = (g, \partial_i \otimes \partial_j)$ and note that, since $g^{ij} = (g^*, \mathrm{d}x^i \otimes \mathrm{d}x^j)$ is simply given by the components of the matrix inverse of (g_{ij}), $\sum_k g_{ik} g^{kj} = \delta_i^j$. Differentiating with respect to t and rearranging, we immediately see that $(\partial_t g^*, \mathrm{d}x^i \otimes \mathrm{d}x^j) = \partial_t g^{ij} = -\sum_{k,l} g^{ik} g^{lj} h_{kl} = -(h^{\sharp^t}, \mathrm{d}x^i \otimes \mathrm{d}x^j)$, whence the result follows. □

Proposition 1.7.7. $\partial_t \mathrm{dvol}_g = \frac{1}{2} tr_g h \mathrm{dvol}_g$

Proof. Retaining the notation of the preceding proposition, we note that in any coördinate neighbourhood dvol_g is given by $\mathrm{dvol}_g = \sqrt{\det(g_{ij})}\mathrm{d}x$. Since $(\partial_{ij} \det)(g_{ij}) = g^{ji} \det(g_{ij})$, we see that $\partial_t \mathrm{dvol}_g = \frac{1}{2\sqrt{\det(g_{ij})}} \sum_{i,j} g^{ji} \det(g_{ij}) h_{ij} \mathrm{d}x = \frac{1}{2}\sqrt{\det(g_{ij})} \sum_{i,j} g^{ij} h_{ij} \mathrm{d}x = \frac{1}{2} \mathrm{tr}_g h \mathrm{dvol}_g$. □

We shall also require information about how induced inner products on $E \otimes \Lambda^k T^*M$, where E is a Riemannian vector bundle, evolve.

Proposition 1.7.8. *Suppose $\psi_1, \psi_2 \in \Gamma(E \otimes T^*M)$. Then*

$$\partial_t \langle \psi_1, \psi_2 \rangle = -\left(h_t^{\sharp^t}, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i} \psi_1, \iota_{\varepsilon_j} \psi_2 \right\rangle \omega^i \otimes \omega^j\right) = -\left\langle h_t, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i} \psi, \iota_{\varepsilon_j} \psi \right\rangle \omega^i \otimes \omega^j \right\rangle$$

where $\{\varepsilon_i\} \leftrightarrow \{\omega^i\}$ are any local frame on M.

Proof. We fix a g_{t_0}-orthonormal basis $\{\varepsilon_i\}$ on TM for t_0 fixed and note that

$$\langle \psi_1, \psi_t \rangle_t = \sum_{I^k, J^k} \langle (\psi_1, \varepsilon_I), (\psi_2, \varepsilon_J) \rangle \cdot \det\left(g_t^*(\omega^{i_r}, \omega^{j_s})\right)_{r,s=1}^k$$

and

$$\begin{aligned} &\partial_t|_{t_0} \left(\det(g_\cdot^*(\omega^{i_r}, \omega^{j_s}))\right) \\ &= -\sum_{r,s=1}^k h^\sharp(\omega^{i_r}, \omega^{j_s}) \cdot (-1)^{r+s} \left\langle \omega_{i_1} \wedge \cdots \wedge \widehat{\omega^{i_r}} \wedge \cdots \wedge \omega^{i_k}, \omega^{j_1} \wedge \cdots \wedge \widehat{\omega^{j_s}} \wedge \cdots \wedge \omega^{j_k} \right\rangle, \end{aligned}$$

whence, noting that $\omega^{i_1} \wedge \cdots \wedge \widehat{\omega^{i_r}} \wedge \cdots \wedge \omega^{i_k} = (-1)^{r+1} \iota_{\varepsilon_{i_r}} \omega^I$,

$$\partial_t \langle \psi_1, \psi_2 \rangle$$

[11] Here we adopt the convention that $\frac{1}{0} = \infty$ and $\min\{a, \infty\} = a$ if $a \in \mathbb{R}$.

$$= -\sum_{I_k,J_k}\sum_{r,s=1}^{k} \langle(\psi_1,\varepsilon_I),(\psi_2,\varepsilon_J)\rangle \cdot (h^{\sharp t_0}_{t_0},\omega^{i_r}\otimes\omega^{j_s})\cdot \left\langle \iota_{\varepsilon_{i_r}}\omega^I, \iota_{\varepsilon_{j_s}}\omega^J\right\rangle_{t_0}.$$

The inner sum is invariant under permutations of I and J, i.e. under $I \to \sigma(I)$, $J \to \tau(J)$ for any $\sigma,\tau : \{1,\dots,k\} \to \{1,\dots,k\}$ bijective. We proceed with this in mind:

$$= -\frac{1}{(k!)^2}\sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}}\sum_{r,s=1}^{k} \langle(\psi_1,\varepsilon_I),(\psi_2,\varepsilon_J)\rangle \cdot (h^{\sharp t_0}_{t_0},\omega^{i_r}\otimes\omega^{j_s})\cdot \left\langle \iota_{\varepsilon_{i_r}}\omega^I, \iota_{\varepsilon_{j_s}}\omega^J\right\rangle_{t_0}. \tag{1.7}$$

Now, interchanging sums and fixing r,s, we note that the inner summand may be written as, writing $\sigma(I) = (i_r, i_1, \dots, \widehat{i_r}, \dots, i_k)$ and $\tau(J) = (j_s, j_1, \dots, \widehat{j_s}, \dots, j_k)$,

$$\begin{aligned}&\sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}} (-1)^{r+s}\left\langle(\psi_1,\varepsilon_{\sigma(I)}),(\psi_2,\varepsilon_{\tau(J)})\right\rangle\cdot(h^{\sharp t_0}_{t_0},\omega^{(\sigma(I))_1}\otimes\omega^{\tau(J)_1})\cdot(-1)^{r+s}\left\langle\iota_{\varepsilon_{\sigma(I)_1}}\omega^{\sigma(I)},\iota_{\varepsilon_{\tau(J)_1}}\omega^{\tau(J)}\right\rangle\\ &= \sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}}\left\langle(\iota_{\varepsilon_{i_1}}\psi_1,\varepsilon_{i_2}\wedge\cdots\wedge\varepsilon_{i_k}),(\iota_{\varepsilon_{j_1}}\psi_2,\varepsilon_{i_2}\wedge\cdots\wedge\varepsilon_{i_k})\right\rangle\cdot(h^{\sharp t_0}_{t_0},\omega^{i_1}\otimes\omega^{j_1})\cdot\left\langle\iota_{\varepsilon_{i_1}}\omega^I,\iota_{\varepsilon_{j_1}}\omega^J\right\rangle,\end{aligned}$$

where we have now made a change of variables. Noting now that this expression does not depend on r and s so that, summing over r and s we obtain k^2 of these sums and treating i_1 and j_1 as separate variables from the other $i_\cdot, j_\cdot$, we proceed from (1.7), rewriting the outer sum in terms of increasing multi-indices:

$$\begin{aligned}&= -\sum_{P^{k-1},Q^{k-1}}\sum_{i_1,j_1=1}^{n}\left\langle(\iota_{\varepsilon_{i_1}}\psi_1,\varepsilon_P),(\iota_{\varepsilon_{j_1}}\psi_2,\varepsilon_Q)\right\rangle(h^{\sharp t_0}_{t_0},\omega^{i_1}\otimes\omega^{j_1})\underbrace{\left\langle\omega^P,\omega^Q\right\rangle}_{\delta^{PQ}}\\ &= -\sum_{i,j=1}^{n}\left\langle\iota_{\varepsilon_i}\psi_1,\iota_{\varepsilon_j}\psi_2\right\rangle\left(h^{\sharp t_0}_{t_0},\omega^i\otimes\omega^j\right)\\ &= -\left(h^{\sharp t_0}_{t_0},\sum_{i,j=1}^{n}\left\langle\iota_{\varepsilon_i}\psi_1,\iota_{\varepsilon_j}\psi_2\right\rangle\omega^i\otimes\omega^j\right)_{t_0},\end{aligned}$$

which is independent of the choice of frame. □

1.8. Backward Heat Kernels. The kernels which play an important role in the monotonicity formulæ to follow are now introduced– the formal backward heat kernel and the canonical backward heat kernel.

Let $(M^n, (g_t)_{t\in I})$ be an evolving Riemannian manifold with $I \subset \mathbb{R}$ an open interval, fix $(x_0,t_0) \in M \times \bar{I}\backslash\{\inf I\}$ and assume M is of locally bounded geometry about (x_0,t_0) with bounds as in Definition 1.7.4 and j_0 as in (1.6) of §1.7.

Definition 1.8.1. [45, §33] The *canonical backward heat kernel* concentrated at (x_0,t_0) is the minimal function satisfying

$$\left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h\right)\mathrm{P}_{(x_0,t_0)} = 0 \text{ on } M\times I$$
$$\lim_{t\nearrow t_0}\mathrm{P}_{(x_0,t_0)}(\cdot,t) = \delta_{x_0}$$

where δ_{x_0} is the delta distribution at x_0 and this limit is to be considered in the (tempered) distributional sense.

Remark 1.8.2. The motivation for the introduction of these kernels is that, if $u,v \in C^\infty(M\times I)$, the following equality holds whenever Gauß' theorem may be applied [22]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M uv\mathrm{dvol}_g = \int_M [(\partial_t - \Delta)u]\, v + u\left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h\right) v\mathrm{dvol}_g. \tag{1.8}$$

Thus, if u solves the heat equation, viz. $(\partial_t - \Delta)\, u = 0$, and $v = \mathrm{P}_{(x_0,t_0)}$, we have that

$$\left(\int_M u\mathrm{P}_{(x_0,t_0)}\mathrm{dvol}_g\right)(t) = \lim_{\tau\nearrow t_0}\left(\int_M u\mathrm{P}_{(x_0,t_0)}\mathrm{dvol}_g\right)(\tau) = u(t_0)$$

for every $t \in I$. Therefore, $\mathrm{P}_{(x_0,t_0)}$ is the natural analogue of the usual backward heat kernel

$$(x,t) \mapsto \frac{1}{(4\pi(t_0-t))^{n/2}}\exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) \tag{1.9}$$

on Euclidean space.

Whereas there is an explicit formula for the backward heat kernel on $\mathbb{R}^n$, such a formula is not given on manifolds. For the purpose of comparison, we also introduce the kernel obtained by adapting the definition (1.9) of the Euclidean backward heat kernel to the manifold setting.

Definition 1.8.3. The *formal backward heat kernel concentrated at* (x_0,t_0) is the function

$$\begin{aligned}\Phi_{\mathrm{fml}} &: M \times I \to \mathbb{R}^+\\ (x,t) &\mapsto \frac{1}{4\pi(t_0-t)^{n/2}}\exp\left(\frac{d^t(x,x_0)^2}{4(t-t_0)}\right).\end{aligned}$$

Remark 1.8.4. Φ_{fml} is not everywhere smooth. However, we shall restrict our attention to the study of it in a neighbourhood of (x_0,t_0) of bounded geometry. On this set, Φ_{fml} is smooth.

It is well known that this kernel may, in the compact case, be used to construct the canonical backward heat kernel and furthermore draw conclusions about solutions to the (scalar) heat equation on M [6]. More generally, if M is complete, it may be shown that these kernels do not differ much from one another around (x_0,t_0) in the C^0 norm, as the following theorem states.

Theorem 1.8.5. *Suppose M is complete. For all $\varepsilon > 0$ there exist a relatively compact neighbourhood Ω of x_0, $\tau_0 \in]-\infty,t_0[$ and $\xi \in C^\infty(\Omega\times[\tau_0,t_0],\mathbb{R}^+)$ with $\xi(x_0,t_0) = 1$ such that on $\Omega\times[\tau_0,t_0[$,*

$$\left|\mathrm{P}_{(x_0,t_0)} - \xi\cdot\Phi_{\mathrm{fml}}\right| \leqslant \varepsilon.$$

Proof. See [22, Lemma 21] or [77, Proposition 5.1] in the case where sec, h and ∇h are bounded. □

If (M,g) is compact and static, many estimates on the spatial and temporal derivatives of $\mathrm{P}_{(x_0,t_0)}$ are known to hold. To this end, we introduce the *matrix Harnack expression* [32] associated to a $C^{2,1}$ function $f : \mathcal{D} \subset M\times I \to \mathbb{R}$ and $s \in \mathbb{R}$ by

$$\mathcal{H}_s(f) := \nabla^2 f + \frac{1}{2}\partial_t g + \frac{1}{2(s-t)}g$$

The following estimates are known to hold, where the matrix Harnack estimate is to be interpreted in the sense of bilinear forms, i.e. $(\mathcal{H}_{t_0}f)(x,t) \geqslant \lambda g_t(x)$ implies that $(\mathcal{H}_{t_0}f, v\otimes v) = \left\langle \mathcal{H}_{t_0}f, v^\flat\otimes v^\flat\right\rangle \geqslant \lambda|v|^2$ for every $v \in T_xM$.

Theorem 1.8.6. [48, 32] *Set $\rho_{(x_0,t_0)} = \log \mathrm{P}_{(x_0,t_0)}$. If M is closed, $h \equiv 0$ and $Ric \geqslant -Kg$, then there exist $B,C,F \in \mathbb{R}^+$ depending on the geometry of M such that the following hold on $M\times[t_0-1,t_0[$:*

$$(t_0-t)\left|\nabla\rho_{(x_0,t_0)}\right|^2 \leqslant C\log\left(\frac{B}{(4\pi(t_0-t))^{n/2}\mathrm{P}_{(x_0,t_0)}}\right) \tag{Gradient Estimate}$$

$$\partial_t \rho_{(x_0,t_0)} + e^{-2K(t_0-t)} \left|\nabla \rho_{(x_0,t_0)}\right|^2 - e^{2K(t_0-t)} \frac{n}{2(t_0-t)} \leqslant 0 \qquad \text{(Li-Yau Estimate)}$$

$$(t_0-t)\partial_t \rho_{(x_0,t_0)} \geqslant -F\left(1+\log\left(\frac{B}{(4\pi(t_0-t))^{n/2}\mathrm{P}_{(x_0,t_0)}}\right)\right) \qquad \text{(Lower Time Derivative Bound)}$$

$$\mathcal{H}_{t_0}\rho_{(x_0,t_0)} \geqslant -F\left(1+\log\left(\frac{B}{(4\pi(t_0-t))^{n/2}\mathrm{P}_{(x_0,t_0)}}\right)\right) g. \qquad \text{(Matrix Harnack Estimate)}$$

If $\sec \geqslant 0$ *and* $\mathrm{dRic} \equiv 0$*, then* $\mathcal{H}_{t_0}\rho_{(x_0,t_0)} \geqslant 0$.

As will be evident later on, it is the matrix Harnack estimate that is most crucial in establishing monotonicity formulæ, be they local or global. However, such estimates are *not* in abundance in more general settings, making it more difficult to deduce such formulæ in these cases. With enough work, coarser estimates on the gradient and time derivative of $\rho_{(x_0,t_0)}$ may help in establishing local monotonicity formulæ, but given Theorem 1.8.5 and Propositions 1.8.7 and 1.8.8, we shall be content with the formal backward heat kernel in these settings.

For later purposes, we compute the effect of applying the adjoint heat operator to Φ_{fml} on a neighbourhood of (x_0, t_0) of bounded geometry. We set $\mathfrak{r}(x,t) = d^t(x,x_0)$ as in §1.7.

Proposition 1.8.7. *The inequality*

$$-\left(\frac{n\mu}{2} + \frac{C_3\mathfrak{r}^2}{t_0-t}\right)\Phi_{\mathrm{fml}} \leqslant \partial_t\Phi_{\mathrm{fml}} + \Delta\Phi_{\mathrm{fml}} + \frac{1}{2}tr_g h\cdot\Phi_{\mathrm{fml}} \leqslant \left(\frac{n\mu}{2} + \frac{C_4\mathfrak{r}^2}{t_0-t}\right)\Phi_{\mathrm{fml}}$$

holds on $\mathcal{D}_{\frac{3j_0}{4},\delta}(x_0,t_0)$*, where* $C_3 = C_3(\kappa_{-\infty},\mu,j_0)$ *and* $C_4 = C_4(\kappa_{\infty},\mu,j_0)$ *are positive.*

Proof. We compute:

$$\begin{aligned}
\partial_t\Phi_{\mathrm{fml}} &= -\frac{n}{2(4\pi(t_0-t))^{(n+2)/2}}\cdot(-4\pi)\cdot\exp\left(\frac{\mathfrak{r}^2}{4(t-t_0)}\right) + \left(\frac{\partial_t(\mathfrak{r}^2)}{4(t-t_0)} - \frac{\mathfrak{r}^2}{4(t_0-t)^2}\right)\cdot\Phi_{\mathrm{fml}}\\
&= -\frac{n}{2\cdot(4\pi(t_0-t))}\cdot(-4\pi)\cdot\Phi_{\mathrm{fml}} + \left(\frac{\partial_t(\mathfrak{r}^2)}{4(t-t_0)} - \frac{\mathfrak{r}^2}{4(t_0-t)^2}\right)\cdot\Phi_{\mathrm{fml}}\\
&= \left(\frac{n}{2(t_0-t)} - \frac{\mathfrak{r}^2}{4(t_0-t)^2} + \frac{\partial_t(\mathfrak{r}^2)}{4(t-t_0)}\right)\Phi_{\mathrm{fml}}.
\end{aligned}$$

On the other hand, $\nabla\Phi_{\mathrm{fml}} = \Phi_{\mathrm{fml}}\cdot\frac{\mathfrak{r}}{2(t-t_0)}\nabla\mathfrak{r}$ so that

$$\begin{aligned}
\Delta\Phi_{\mathrm{fml}} &= \left\langle\nabla\Phi_{\mathrm{fml}}, \frac{\mathfrak{r}}{2(t-t_0)}\nabla\mathfrak{r}\right\rangle + \Phi_{\mathrm{fml}}\cdot\left(\frac{1}{2(t-t_0)} + \frac{\mathfrak{r}}{2(t-t_0)}\Delta\mathfrak{r}\right)\\
&= \left(\frac{\mathfrak{r}^2}{4(t-t_0)^2} + \frac{1}{2(t-t_0)} + \frac{\mathfrak{r}}{2(t-t_0)}\Delta\mathfrak{r}\right)\Phi_{\mathrm{fml}},
\end{aligned}$$

whence, writing H^* for $\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h$,

$$H^*\Phi_{\mathrm{fml}} = \Phi_{\mathrm{fml}}\cdot\left(\frac{n-1}{2(t_0-t)} + \frac{\mathfrak{r}}{2(t-t_0)}\Delta\mathfrak{r} + \frac{1}{4(t-t_0)}\partial_t\mathfrak{r}^2 + \frac{1}{2}\mathrm{tr}_g h\right).$$

Now, applying the Hessian comparison theorem (Theorem B.1) and taking the trace of the Hessian, we see that

$$(n-1)f_{\kappa_\infty}\circ\mathfrak{r} \leqslant \Delta\mathfrak{r} \leqslant (n-1)f_{\kappa_{-\infty}}\circ\mathfrak{r}$$

from which it is evident that

$$\left[\left(\frac{n-1}{2(t_0-t)}\cdot\left(1-\mathfrak{r}\cdot(f_{\kappa_{-\infty}}\circ\mathfrak{r})\right) + \frac{\partial_t(\mathfrak{r}^2)}{4(t-t_0)}\right) + \frac{1}{2}\mathrm{tr}_g h\right]\Phi_{\mathrm{fml}}$$

$$\leqslant H^*\Phi_{\mathrm{fml}}$$
$$\leqslant \left[\left(\frac{n-1}{2(t_0-t)}\cdot(1-\mathfrak{r}\cdot(f_{\kappa_\infty}\circ\mathfrak{r}))+\frac{\partial_t(\mathfrak{r}^2)}{4(t-t_0)}\right)+\frac{1}{2}\mathrm{tr}_g h\right]\Phi_{\mathrm{fml}}.$$

By Proposition B.3, $1-\mathfrak{r}\cdot(f_{\kappa_{-\infty}}\circ\mathfrak{r})\geqslant -C\mathfrak{r}^2$ with $C=C(j_0,\kappa_{-\infty})\geqslant 0$ and $1-\mathfrak{r}\cdot(f_{\kappa_\infty}\circ\mathfrak{r})\leqslant \tilde{C}\mathfrak{r}^2$ with $\tilde{C}=\tilde{C}(j_0,\kappa_\infty)\geqslant 0$. Using these bounds and the inequalities $|\mathrm{tr}_g h|\leqslant n\mu$ and $|\partial_t\mathfrak{r}|\leqslant\frac{\mu}{2}\mathfrak{r}$ then yields the result. □

Finally, we note that Φ_{fml} satisfies differential inequalities similar to those of Theorem 1.8.6 which shall suffice in applications in the sequel.

Proposition 1.8.8. *Let $\phi_{\mathrm{fml}}=\log\Phi_{\mathrm{fml}}$. The inequalities*

1. $|\nabla\phi_{\mathrm{fml}}|=\dfrac{\mathfrak{r}}{2(t_0-t)}$,
2. $|\partial_t\phi_{\mathrm{fml}}|\leqslant\dfrac{n}{2(t_0-t)}+\dfrac{\mathfrak{r}^2}{4(t_0-t)^2}+\dfrac{\mu\mathfrak{r}^2}{4(t_0-t)}$, *and*
3. $-\dfrac{C\mathfrak{r}^2}{2(t_0-t)}g_{\mathfrak{r}}+\dfrac{\lambda_{-\infty}}{2}g\leqslant\mathcal{H}_{t_0}\phi_{\mathrm{fml}}\leqslant\dfrac{\tilde{C}\mathfrak{r}^2}{2(t_0-t)}g_{\mathfrak{r}}+\dfrac{\lambda_\infty}{2}g$

hold on $\mathcal{D}_{\frac{3j_0}{4},\delta}(x_0,t_0)$, where $C=C(j_0,\kappa_{-\infty})$ and $\tilde{C}=\tilde{C}(j_0,\kappa_\infty)$ are as in the preceding proof.

Remark 1.8.9. If $(M,g)=(\mathbb{R}^n,\delta)$, then $\Phi_{\mathrm{fml}}=\mathrm{P}_{(x_0,t_0)}$ and these inequalities simplify to the equalities

1. $|\nabla\phi_{\mathrm{fml}}|(x,t)=\dfrac{|x-x_0|}{2(t_0-t)}$,
2. $\partial_t\phi_{\mathrm{fml}}(x,t)=\frac{n}{2(t_0-t)}-\frac{|x-x_0|^2}{4(t_0-t)^2}$, and
3. $\mathcal{H}_{t_0}\phi_{\mathrm{fml}}\equiv 0$.

Proof of Proposition 1.8.8. The first two assertions follow from the computations in the preceding proof, together with the bound on the time derivative of the distance function. For the third assertion, it is evident that

$$\nabla^2\phi_{\mathrm{fml}}=\nabla\left(\frac{\mathfrak{r}}{2(t-t_0)}\mathrm{d}\mathfrak{r}\right)$$
$$=\frac{\mathrm{d}\mathfrak{r}\otimes\mathrm{d}\mathfrak{r}}{2(t-t_0)}-\frac{\mathfrak{r}}{2(t_0-t)}\nabla^2\mathfrak{r},$$

and that

$$-\frac{\mathrm{d}\mathfrak{r}\otimes\mathrm{d}\mathfrak{r}}{2(t_0-t)}-\frac{\mathfrak{r}}{2(t_0-t)}(f_\kappa\circ\mathfrak{r})g_{\mathfrak{r}}+\frac{g}{2(t_0-t)}+\frac{1}{2}h=\frac{1-\mathfrak{r}\cdot(f_\kappa\circ\mathfrak{r})}{2(t_0-t)}g_{\mathfrak{r}}+\frac{1}{2}h$$

for every $\kappa\in\mathbb{R}$, whence an application of theorem B.1 yields

$$\frac{1-\mathfrak{r}\cdot(f_{\kappa_{-\infty}}\circ\mathfrak{r})}{2(t_0-t)}g_{\mathfrak{r}}+\frac{1}{2}h\leqslant\mathcal{H}_{t_0}(\phi_{\mathrm{fml}})\leqslant\frac{1-\mathfrak{r}\cdot(f_{\kappa_\infty}\circ\mathfrak{r})}{2(t_0-t)}g_{\mathfrak{r}}+\frac{1}{2}h.$$

The result then follows from Proposition B.3 (cf. proof of Proposition 1.8.7) and the bound on h. □

1.9. Problems of Dirichlet type. In this thesis, we are chiefly interested in *problems of Dirichlet type*. These are characterized as those problems giving rise to Riemannian vector bundle-valued k-forms, i.e. sections of $E\otimes\Lambda^kT^*M$ for some Riemannian vector bundle E, or time-dependent bundle-valued k-forms, which are either harmonic or satisfy the heat equation.

More precisely, if M is a static Riemannian manifold, $E \to M$ is a Riemannian vector bundle equipped with a Riemannian connection and a system may be written in the form

$$\mathrm{d}^\nabla \psi = 0 \text{ and } \delta^\nabla \psi = 0$$

for some section $\psi \in \Gamma(E \otimes \Lambda^k T^* M)$, we say that ψ solves a *static problem of Dirichlet type.* On the other hand, if $(M, \{g_t\}_{t \in I})$ is an evolving Riemannian manifolds, $E \to M$ is a Riemannian vector bundle equipped with a Riemannian connection and a system may be written in the form

$$(\partial_t + \Delta^\nabla)\psi = 0$$

for a time-dependent such section $\psi \in \Gamma(E \otimes \Lambda^k T^* M)$, we say that ψ evolves by a *flow of Dirichlet type.*

Associated to both problems is usually a Dirichlet-type energy of the form

$$\frac{1}{2}\int_M |\psi|^2 \mathrm{dvol}_g.$$

In the sequel, since it shall not offer us any additional difficulties, we shall consider *static problems of p-Dirichlet type* which are those systems that may be written in the form (cf. [41, §3])

$$\mathrm{d}^\nabla \psi = 0 \text{ and } \delta^\nabla \left(|\psi|^{p-2}\psi\right) = 0$$

for some section $\psi \in \Gamma(E \otimes \Lambda^k T^* M)$. Similarly associated to such a problem is a p-Dirichlet-type energy of the form

$$\frac{1}{p}\int_M |\psi|^p \mathrm{dvol}_g.$$

It is the structure of this energy which we shall investigate and subsequently make use of in later chapters for $p = 2$. In that case, we note that flows of Dirichlet type on compact manifolds enjoy the following energy decay property:

Proposition 1.9.1. *If $(M, \{g_t\}_{t \in I})$ is a complete evolving Riemannian manifold with $\partial_t g = h$ such that $|\mathrm{tr}_g h| \leqslant \mu$ and $(\psi_t)_{t \in I}$ is a one-parameter family of sections evolving by a flow of Dirichlet type such that the integral*

$$\int_M \left(|\psi|^2 + |\mathrm{d}^\nabla \psi|^2 + |\delta^\nabla \psi|^2 + |\Delta^\nabla \psi|^2 \mathrm{dvol}_g\right)(\cdot, t) \tag{1.10}$$

is finite for each $t \in I$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{\mu}{2}t}\int_M \frac{1}{2}|\psi|^2 \mathrm{dvol}_g\right) \leqslant -e^{-\frac{\mu}{2}t}\int_M |\delta^\nabla \psi|^2 + |\mathrm{d}^\nabla \psi|^2 \mathrm{dvol}_g$$

on I.

Proof. It is clear from Proposition 1.7.7 that

$$\begin{aligned}
\partial_t\left(\frac{1}{2}|\psi|^2 \mathrm{dvol}_g\right) &= \left(\langle \partial_t \psi, \psi\rangle + \frac{1}{2}|\psi|^2 \cdot \frac{1}{2}\mathrm{tr}_g h\right)\mathrm{dvol}_g \\
&= \left(-\left\langle \Delta^\nabla \psi, \psi\right\rangle + \frac{1}{2}|\psi|^2 \cdot \frac{1}{2}\mathrm{tr}_g h\right)\mathrm{dvol}_g \\
&= -\left(|\mathrm{d}^\nabla \psi|^2 + |\delta^\nabla \psi|^2 - \mathrm{div}\left(\sum_i \left(\left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla \psi, \psi\right\rangle - \left\langle \iota_{\varepsilon_i}\psi, \delta^\nabla \psi\right\rangle\right)\varepsilon_i\right)\right)\mathrm{dvol}_g
\end{aligned}$$

$$+\left(\frac{1}{2}\mathrm{tr}_g h\right)\frac{1}{2}|\psi|^2\mathrm{dvol}_g \tag{1.11}$$

where the last inequality is a consequence of Lemma 1.6.8 (ii). Now, by the triangle inequality, Cauchy-Schwarz inequality and Young's inequality,

$$\begin{aligned}
&\left|\sum_i \left(\left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi,\psi\right\rangle - \left\langle \iota_{\varepsilon_i}\psi,\delta^\nabla\psi\right\rangle\right)\varepsilon_i\right| \\
&\leqslant \sqrt{\sum_i \left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi,\psi\right\rangle^2} + \sqrt{\sum_i \langle \iota_{\varepsilon_i}\psi,\delta^\nabla\psi\rangle^2} \\
&\leqslant \frac{k+1}{2}\left(|\psi|^2 + |\mathrm{d}^\nabla\psi|^2\right) + \frac{1}{2}|\delta^\nabla\psi|^2.
\end{aligned}$$

Likewise, an application of the triangle inequality and Young's inequality implies that

$$\begin{aligned}
&\left|\mathrm{div}\left(\sum_i \left(\left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi,\psi\right\rangle - \left\langle \iota_{\varepsilon_i}\psi,\delta^\nabla\psi\right\rangle\right)\varepsilon_i\right)\right| \\
&\leqslant \left|\left\langle \Delta^\nabla\psi,\psi\right\rangle\right| + |\mathrm{d}^\nabla\psi|^2 + |\delta^\nabla\psi|^2 \\
&\leqslant \frac{1}{2}|\psi|^2 + \frac{1}{2}|\Delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2 + |\delta^\nabla\psi|^2
\end{aligned}$$

so that, by the bound $|\mathrm{tr}_g h| \leqslant \mu$ and the finiteness of the integral (1.10), all of the quantities under consideration are summable on M for fixed t. Thus, integrating both sides of (1.11), applying Gauß' theorem for complete manifolds [28] and standard integration theorems to interchange the integral and derivative, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{2}|\psi|^2\mathrm{dvol}_g &= -\int_M |\mathrm{d}^\nabla\psi|^2 + |\delta^\nabla\psi|^2\mathrm{dvol}_g + \int_M \left(\frac{1}{2}\mathrm{tr}_g h\right)\frac{1}{2}|\psi|^2\mathrm{dvol}_g \\
&\leqslant -\int_M |\mathrm{d}^\nabla\psi|^2 + |\delta^\nabla\psi|^2\mathrm{dvol}_g + \frac{\mu}{2}\int_M \frac{1}{2}|\psi|^2\mathrm{dvol}_g.
\end{aligned}$$

The result then follows from multiplying both sides by the integrating factor $e^{-\frac{\mu}{2}t}$. □

We now introduce the two problems of this type in which we shall be interested. In what follows, we assume that $(M,(g_t)_{t\in I})$, I an open interval, is an evolving Riemannian manifold, and t shall be understood as the same parameter appearing in the evolution equations to be discussed.

1.10. The theory of harmonic maps. Let $(N^m,\widetilde{g})$ be a smooth Riemannian manifold. If $u \in C^\infty(M,N)$, we write $\mathrm{d}u \in \Gamma\left(u^{-1}TN \otimes T^*M\right)$ and equip $u^{-1}TN\otimes T^*M$ with the metric induced by those on TM and TN.

We introduce the *p-Dirichlet energy* of a map $u : M \to N$ by

$$E_p(u) := \frac{1}{p}\int_M |\mathrm{d}u|^p\,\mathrm{dvol}_g.$$

The consideration of critical points of this energy over the class of smooth maps $u : M \to N$ for which $E_p(u)$ is finite is the starting point of the theory of p-harmonic maps.

Definition 1.10.1. A smooth map $u : M \to N$ is said to be *p-harmonic* if it satisfies

$$\delta^\nabla\left(|\mathrm{d}u|^{p-2}\,\mathrm{d}u\right) = 0,$$

where ∇ is the covariant derivative induced on $u^{-1}TN \otimes T^*M$ by the Levi-Civita connections on TM and TN.

Such maps have been widely studied [23, 63, 34, 26]. In particular, 2-harmonic maps, which we shall simply refer to as *harmonic maps*, arise as σ-models in physics [30] and as a natural generalization of harmonic functions in geometry [23].

A method for establishing the existence of harmonic maps initiated by Eells and Sampson [23], which has turned out to be highly successful for harmonic maps when $\sec_{\widetilde{g}} \leqslant 0$, is to instead study the corresponding heat equation.

Definition 1.10.2. A map $u : M \times I \to N$ is said to evolve by the *harmonic map heat flow* if

$$\partial_t u = -\delta^{\nabla} \mathrm{d}u \tag{HMHF}$$

on $M \times I$.

It was shown by Eells and Sampson [23] that, if (M, g) is compact and static and $(N, \widetilde{g})$ is complete, of nonpositive sectional curvature and admits an isometric embedding into Euclidean space satisfying appropriate growth conditions which are always satisfied by compact N, given smooth initial data $u(\cdot, 0) : M \to N$, a smooth solution to (HMHF) exists with $I = \left]0, \infty\right[$ and a subsequence of $\{u(\cdot, t)\}_{t \in]0,\infty[}$ uniformly converges to a harmonic map as $t \to \infty$; it was later shown by Hartman [35] that if N is compact, $u(\cdot, t)$ uniformly converges to a harmonic map as $t \to \infty$. More generally, however, even if smooth initial data is prescribed, in which case a solution exists for at least a short time T, u is expected to develop *singularities* in finite time. It was shown by Struwe [67] that if M and N are compact and (M, g) is a static Riemannian 2-manifold, then a solution u to (HMHF) on $M \times \left]0, \infty\right[$ is smooth away from finitely many points in space-time. Moreover, it was shown by Chen and Struwe [12] that if (M^n, g) is higher-dimensional, the set $\mathcal{S}$ of singularities of u, i.e. the points at which u is not smooth, is a closed subset of $M \times \mathbb{R}$ of locally finite n-dimensional Hausdorff measure with respect to a suitable parabolic metric; in fact, it was shown by Cheng [13] that for $t_0 > 0$ $\mathrm{pr}_1 (\mathcal{S} \cap (M \times \{t_0\})) \subset M$ has finite $n-2$-dimensional Hausdorff measure. Explicit examples of solutions to (HMHF) in various dimensions which develop singularities in finite time have been given e.g. by Coron and Ghidaglia [15] and Chang, Ding and Ye [10].

In the sequel, we shall make use of an alternative form of (HMHF) owing to the fact that $(\mathrm{d}u)(\cdot, t) \in \Gamma\left(u(\cdot, t)^{-1} TN \otimes T^*M\right)$, i.e. $\mathrm{d}u(\cdot, t)$ lives in a different bundle for each t, which makes $\mathrm{d}u$ a bit awkward to deal with geometrically. To overcome this difficulty,[12] we make use of an isometric embedding[13] $\jmath : N \hookrightarrow \mathbb{R}^K$ as follows: by writing $\widetilde{u} = \jmath \circ u : M \to \mathbb{R}^K$, we obtain a map into $\jmath(N)$, i.e. N considered as a submanifold of $\mathbb{R}^K$, for which $\mathrm{d}\widetilde{u} \in \Gamma\left(\mathbb{R}^K \otimes T^*M\right)$. By isometry, $\langle \mathrm{d}u, v\rangle = \langle \mathrm{d}\widetilde{u}, \mathrm{d}\jmath(v)\rangle$, so that the energies of both maps coïncide. (HMHF) may thus be shown to be equivalent to the equation [23, Lemma 7B]

$$\partial_t \widetilde{u} - \Delta_g \widetilde{u} \perp \jmath(TN),$$

where Δ_g acts on $\widetilde{u}$ componentwise. This may be written in the equivalent form

$$\partial_t \widetilde{u} + \delta^{\widetilde{\nabla}} \mathrm{d}\widetilde{u} \perp \jmath(TN)$$

where $\widetilde{\nabla}$ is the connection on $\mathbb{R}^K \otimes T^*M$ induced by the Levi-Civita connection on TM and the flat connection on $\mathbb{R}^K$. To see this, we note that

$$\mathrm{d}\widetilde{u} = \sum_{i=1}^n \partial_{\varepsilon_i} \widetilde{u} \otimes \omega^i = \sum_{\alpha=1}^K \sum_{i=1}^n \partial_{\varepsilon_i} \widetilde{u}^\alpha \cdot e_\alpha \otimes \omega^i,$$

where $\{e_\alpha\}_{\alpha=1}^K$ is a basis for $\mathbb{R}^K$ so that, since $\nabla e_\alpha \equiv 0$,

$$\delta^{\nabla} \mathrm{d}\widetilde{u} = -\sum_{i=1}^n \iota_{\varepsilon_i} \nabla_{\varepsilon_i} \left(\sum_{j=1}^n \sum_{\alpha=1}^K \partial_{\varepsilon_j} \widetilde{u}^\alpha e_\alpha \otimes \omega^j \right)$$

[12] An alternative to this option would be to instead view u as a map $M \times I \to N$ between manifolds, but in this case space and time would be treated on the same footing, which would make a few other things awkward.

[13] The existence of such an embedding is guaranteed by the Nash embedding theorem [55].

$$= -\sum_{\alpha=1}^{K}\left(\sum_{i=1}^{n}\iota_{\varepsilon_i}\nabla_{\varepsilon_i}\left(\sum_{j=1}^{n}\partial_{\varepsilon_j}\widetilde{u}^{\alpha}\omega^j\right)\right)e_\alpha$$
$$= -\sum_{\alpha=1}^{K}\left(\Delta_g u^{\alpha}\right)e_\alpha.$$

In this setting, $\mathrm{d}\widetilde{u}$ only satisfies an inhomogeneous heat-type equation. However, for our purposes, this shall be sufficient.

The following lemma summarizes the information we shall need about $\mathrm{d}u$:

Lemma 1.10.3. *Let $u : M \times I \to N$ be a smooth map with $\widetilde{u}$ as above. Then*

(i) $\mathrm{d}^{\widetilde{\nabla}}\mathrm{d}\widetilde{u} = 0$ *and*

(ii) If u solves (HMHF), then

$$\left\langle\left(\partial_t + \Delta^{\widetilde{\nabla}}\right)\mathrm{d}\widetilde{u}, \mathrm{d}\widetilde{u}\right\rangle = \left|\iota_X \mathrm{d}\widetilde{u} - \delta^{\widetilde{\nabla}}\mathrm{d}\widetilde{u}\right|^2 - \left|\partial_t\widetilde{u} + \iota_X\mathrm{d}\widetilde{u}\right|^2 \qquad \text{(Pythagoras-type identity)}$$

for any $X \in TM$.

(iii) If $(M, \{g_t\}_{t\in I})$ is a complete evolving Riemannian manifold with $\partial_t g = h$ such that $|tr_g h| \leqslant \mu$ and u solves (HMHF) such that the integral

$$\int_M \left(|\mathrm{d}u|^2 + |\partial_t u|^2 + |\delta^{\nabla}\mathrm{d}u|^2 + |\Delta^{\nabla}\mathrm{d}u|^2 \mathrm{dvol}_g\right)(\cdot, t)$$

is finite for each $t \in I$, then the energy decay estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{\mu}{2}t}\cdot\int_M \frac{1}{2}|\mathrm{d}u|^2\,\mathrm{dvol}_g\right) \leqslant -e^{-\frac{\mu}{2}t}\int_M |\partial_t u|^2 \tag{1.12}$$

holds on I.

Proof. To simplify notation, we write u for $\widetilde{u}$ and ∇ for $\widetilde{\nabla}$.

(i) This property is independent of the fact that u solves (HMHF). Indeed, if $v : M \to N \subset \mathbb{R}^K$, then

$$\mathrm{d}^{\nabla}\mathrm{d}u = \mathrm{d}^{\nabla}\left(\sum_{i=1}^{n}\partial_i u \otimes \mathrm{d}x^i\right)$$
$$= \sum_{i,j=1}^{n}\partial_j\partial_i u \otimes \mathrm{d}x^j \wedge \mathrm{d}x^i + \sum_{i=1}^{n}\partial_i u \otimes \underbrace{\mathrm{d}^2x^i}_{=0}.$$

Since mixed partial derivatives of u coïncide, separating the former sum out into sums over $i < j$, $i = j$ and $i > j$ immediately shows that the former sum also vanishes.

(ii) We thus compute, expanding the squares and noting that $\iota_X\mathrm{d}u = \partial_X u$:

$$\left|\iota_X\mathrm{d}u - \delta^{\nabla}\mathrm{d}u\right|^2 - |\partial_t u + \iota_X\mathrm{d}u|^2$$
$$= \left(|\iota_X\mathrm{d}u|^2 - 2\left\langle\iota_X\mathrm{d}u, \delta^{\nabla}\mathrm{d}u\right\rangle + |\delta^{\nabla}\mathrm{d}u|^2\right) - \left(|\partial_t u|^2 + 2\langle\partial_t u, \iota_X\mathrm{d}u\rangle + |\iota_X\mathrm{d}u|^2\right)$$
$$= -2\left\langle\partial_X u, \partial_t u + \delta^{\nabla}\mathrm{d}u\right\rangle + |\delta^{\nabla}\mathrm{d}u|^2 - |\partial_t u|^2.$$

Now, $-2\left\langle\partial_X u, \partial_t u + \delta^{\nabla}\mathrm{d}u\right\rangle = 0$, since $\partial_X u \in \iota(TN)$. On the other hand, $\partial_t u \in \iota(TN)$ so that $|\partial_t u|^2 = \langle\partial_t u, \partial_t u\rangle = -\left\langle\delta^{\nabla}\mathrm{d}u, \partial_t u\right\rangle$, whence we are left with

$$\left|\iota_X \mathrm{d}u - \delta^\nabla \mathrm{d}u\right|^2 - |\partial_t u + \iota_X \mathrm{d}u|^2 = \left\langle \delta^\nabla \mathrm{d}u, \partial_t u + \delta^\nabla \mathrm{d}u \right\rangle.$$

By Lemma 1.6.8 (ii),

$$\left\langle \delta^\nabla \mathrm{d}u, \partial_t u + \delta^\nabla \mathrm{d}u \right\rangle = \left\langle \mathrm{d}u, \mathrm{d}^\nabla \partial_t u + \mathrm{d}^\nabla \delta^\nabla \mathrm{d}u \right\rangle - \operatorname{div}\left(\sum_i \left\langle \iota_{\varepsilon_i} \mathrm{d}u, \partial_t u + \delta^\nabla \mathrm{d}u \right\rangle \varepsilon_i \right).$$

On the one hand, the latter term vanishes since $\iota_{\varepsilon_i} \mathrm{d}u = \partial_{\varepsilon_i} \mathrm{d}u \in \iota(TN)$. On the other, $\mathrm{d}^\nabla \mathrm{d}u = 0$ and $\mathrm{d}^\nabla \partial_t u = \partial_t \mathrm{d}u$, whence the result follows.

(iii) (cf. Proposition 1.9.1) We compute, using Proposition 1.7.7 and (ii) with $X = 0$:

$$\begin{aligned}
\partial_t \left(\frac{1}{2} |\mathrm{d}u|^2 \mathrm{dvol}_g \right) &= \left(\langle \partial_t \mathrm{d}u, \mathrm{d}u \rangle + \frac{1}{2} |\mathrm{d}u|^2 \cdot \frac{1}{2} \mathrm{tr}_g h \right) \mathrm{dvol}_g \\
&= \left(|\delta^\nabla \mathrm{d}u|^2 - |\partial_t u|^2 - \left\langle \Delta^\nabla \mathrm{d}u, \mathrm{d}u \right\rangle \right) \mathrm{dvol}_g + \frac{1}{2} |\mathrm{d}u|^2 \cdot \frac{1}{2} \mathrm{tr}_g h \mathrm{dvol}_g \\
&= -\left(|\partial_t u|^2 + \operatorname{div}\left(\sum_i \left\langle \iota_{\varepsilon_i} \mathrm{d}u, \delta^\nabla \mathrm{d}u \right\rangle \varepsilon_i \right) \right) \mathrm{dvol}_g \\
&\quad + \left(\frac{1}{2} \mathrm{tr}_g h \right) \cdot \frac{1}{2} |\mathrm{d}u|^2 \mathrm{dvol}_g. \qquad (1.13)
\end{aligned}$$

We now verify that we may integrate both sides of this expression and apply Gauß' theorem for complete manifolds [28]. Note that by the triangle inequality, the Cauchy-Schwarz inequality and Young's inequality,

$$\left(\left| \sum_i \left\langle \iota_{\varepsilon_i} \mathrm{d}u, \delta^\nabla \mathrm{d}u \right\rangle \varepsilon_i \right| \right)(\cdot, t) \leqslant \left(\frac{1}{2} \left| \delta^\nabla \mathrm{d}u \right|^2 + \frac{1}{2} \left| \mathrm{d}u \right|^2 \right)(\cdot, t) \in L^1(M)$$

for each $t \in I$ by the finiteness of 1.12. Similarly,

$$\left| \operatorname{div}\left(\sum_i \left\langle \iota_{\varepsilon_i} \mathrm{d}u, \delta^\nabla \mathrm{d}u \right\rangle \varepsilon_i \right) \right| \leqslant |\delta^\nabla \mathrm{d}u|^2 + \frac{1}{2} |\Delta^\nabla \mathrm{d}u|^2 + \frac{1}{2} |\mathrm{d}u|^2$$

which, for fixed t, is also in $L^1(M)$. Thus, in light of the bound $|\mathrm{tr}_g h| \leqslant \mu$, we may integrate both sides of (1.13) and interchange derivative and integral by standard integration theorems to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_M \frac{1}{2} |\mathrm{d}u|^2 \mathrm{dvol}_g &= -\int_M |\partial_t u|^2 \mathrm{dvol}_g + \int_M \left(\frac{1}{2} \mathrm{tr}_g h \right) \cdot \frac{1}{2} |\mathrm{d}u|^2 \mathrm{dvol}_g \\
&\leqslant -\int_M |\partial_t u|^2 \mathrm{dvol}_g + \frac{\mu}{2} \int_M \frac{1}{2} |\mathrm{d}u|^2 \mathrm{dvol}_g.
\end{aligned}$$

The result now follows from using the integrating factor $e^{-\frac{\mu}{2} t}$.

□

We shall henceforth *always* write u for $\widetilde{u}$ and ∇ for $\widetilde{\nabla}$ when it is made clear that N is isometrically embedded in $\mathbb{R}^K$.

1.11. Yang-Mills theory. Let $G \to P \to M$ be a principal G-bundle, where G is a real (complex) connected compact semisimple Lie group. We retain the notation of §1.3 and equip $P \times_{\mathrm{Ad}} \mathfrak{g}$ with the Riemannian structure induced by *minus* (the real part of) the Killing form.

We introduce the *p-Yang-Mills energy* of a connection ω on P by

$$YM_p(\omega) := \frac{1}{p}\int_M |\underline{\Omega}^\omega|^p \,\mathrm{dvol}_g. \tag{1.14}$$

The consideration of critical points of this energy over the class of smooth connections ω on P for which $YM_p(\omega)$ is finite is the starting point of the theory of p-Yang-Mills connections.

Definition 1.11.1. A connection ω on P is said to be a *p-Yang-Mills connection* if it satisfies

$$\delta^\nabla\left(|\underline{\Omega}^\omega|^{p-2}\,\underline{\Omega}^\omega\right) = 0,$$

where ∇ is the covariant derivative induced by ∇ on $P \times_{\mathrm{Ad}} G$ by ω.

Yang-Mills connections, i.e. 2-Yang-Mills connections, arise as models for certain elementary particles in mathematical physics [76] and have been used successfully in the study of the topology of 4-manifolds [16], whereas p-Yang-Mills connections have only more recently been studied [44, 58].

As is the case with harmonic maps, one way of establishing the existence of Yang-Mills connections, which has been successful in the case $n \leqslant 3$, is to consider the associated heat flow.

Definition 1.11.2. A smooth one-parameter family of connections $\{\omega_t = \widetilde{\omega} + a(t)\}_{t\in I}$, where $\widetilde{\omega}$ is some fixed connection on P, is said to evolve by the *Yang-Mills flow* if

$$\partial_t a = -\delta^\nabla \underline{\Omega}^\omega \tag{YMHF}$$

on $M \times I$.

Remark 1.11.3. The p-Yang-Mills equation and Yang-Mills flow system are invariant under gauge transformations, since, if $\widetilde{\omega} = g\cdot\omega$ with g a gauge transformation and $\widetilde{\nabla}$ is the corresponding covariant derivative, $\underline{\Omega}^{\widetilde{\omega}} = g\cdot\underline{\Omega}^\omega$ and $\delta^{\widetilde{\nabla}}(g\cdot\underline{\Omega}^\omega) = g\cdot\delta^\nabla\underline{\Omega}^\omega$.

This flow was first suggested by Atiyah and Bott [3]. It was subsequently shown by Råde [62] that if M is a static compact 2 or 3-manifold, given a smooth initial connection ω_0 on P, it may be made to smoothly evolve by (YMHF) on $]0,\infty[$ and, moreover, ω_t tends to a Yang-Mills connection in an appropriate Sobolev space. In higher dimensions, long-time existence is not guaranteed and the structure of the set of singularities that may develop is not as well-understood as in the case of (HMHF), though there is a theory of weak solutions on static compact 4-manifolds due to Struwe [69]. Examples of Yang-Mills connections developing singularities in higher dimensions have been constructed by Naito [54] and Grotowski [31].

We shall need only one property of the curvature two-form for our purposes.

Lemma 1.11.4. [62, §4] *Let ω be a connection on P and ∇ the induced covariant derivative on E. If ω evolves by the Yang-Mills flow, then $\partial_t\underline{\Omega}^\omega + \Delta^\nabla\underline{\Omega}^\omega = 0$.*

This lemma implies that $\underline{\Omega}^\omega$ evolves by a flow of Dirichlet type and, consequently, has the energy decay property (Proposition 1.9.1).

1.12. Mean curvature flow. We conclude this chapter with a discussion of the mean curvature flow. Much of the material is standard and may be found in [19], though here we allow the ambient space to be an *evolving Riemannian manifold*.

As before, we suppose $(M,(g_t)_{t\in I})$ with $I =]t_0-\delta_0,t_0[$ $(\delta > 0)$ and $\partial_t g = h$ is an evolving manifold of locally bounded geometry about (x_0,t_0) and assume the notation in and following Definition 1.7.4. Let N^m be a smooth oriented manifold and let $F : N\times I \to M$ be a smooth map[14] such that $\{F_t := F(\cdot,t) : N \to (M,g_t)\}_{t\in I}$ is a one-parameter family of embeddings and the map $(F,\mathrm{pr}_2) : N\times I \to M\times I$ is proper, i.e. $(F,\mathrm{pr}_2)^{-1}(K)$ is compact whenever $K \subset M\times I$ is.[15]

[14] Note that in contrast to the harmonic map heat flow, M is now the *target* manifold.

[15] This last assumption is made for technical reasons which shall only become apparent in Chapter 5, particularly in Examples 5.3.4 and 5.3.5. It is satisfied for instance if N is compact.

As in §1.7, all of the usual notions quantities of extrinsic differential geometry carry over to this setting by fixing time and considering the usual quantities. Let $U \subset M$ open, $f \in C^1(U \times I, \mathbb{R})$, $(X(\cdot,t) = X_t : U \to TM)_{t\in I}$ a one-parameter family of continuous (local) sections of TM, $(Z(\cdot,t) : U \to F_t^{-1}TM)_{t\in I}$ a one-parameter family of continuous (local) sections of $F_t^{-1}TM$ and $(Q(\cdot,t) = Q_t : U \to T^*M \otimes T^*M)_{t\in I}$ a one parameter family of continuous (local) sections of $T^*M \otimes T^*M$. For convenience, $F_t^{-1}N$ shall be realized as the point set

$$\bigcup_{x\in N} \{(x,t)\} \times T_{F_t(x)}M \subset F^{-1}TM.$$

The relevant notation is summarized in the following table:

Symbol	**Signification**
$\mathfrak{I}_t = \mathfrak{I}(\cdot,t)$	First fundamental form of F_t $(:= F_t^* g_t)$
II_t	Second fundamental form of F_t as a section of $F_t^{-1}TM \otimes T^*N \otimes T^*N$ (see discussion below)
$\underset{\frown}{H}_t = \underset{\frown}{H}(\cdot,t)$	Mean curvature of F_t $(:= \mathrm{tr}_{\mathfrak{I}_t} \mathrm{II}_t \in \Gamma(F_t^{-1}TM))$
$\underset{\frown}{X}(\cdot,t)$	$X(\cdot,t)$ as a section of $F_t^{-1}TM$ (i.e. such that $\underset{\frown}{X}(x,t) = ((x,t), X(F_t(x),t)))$
$(X_t(F_t(x)))^T$	Tangent part of $X_t(F_t(x))$ (Orthogonally projected onto im $\mathrm{d}_x F_t$ wrt. g_t)
$(X_t(F_t(x)))^\perp$	Normal part of $X_t(F_t(x))$ $(= X_t(F_t(x)) - (X_t(F_t(x)))^T)$
$\nabla^\perp f(\cdot,t)$	$\big(U \ni x \mapsto ((x,t),(\nabla f(F_t(x),t))^\perp) \in F_t^{-1}N\big)$ for differentiable f
$\underline{X}_t = (X(\cdot,t))$	The unique continuous local section $\underline{X}_t : F_t^{-1}(U) \to TN$ of TN such that $\mathrm{d}_x F_t(\underline{X}_t(x)) = (X_t(F_t(x)))^T$
$\underline{f}$	Pullback of f $(:=((F,\mathrm{pr}_2)^{-1}(U \times I) \ni (x,t) \mapsto f(F_t(x),t) \in \mathbb{R}))$
$\mathrm{tr}_g^\perp Q$	Normal trace $:= \big(N \times I \ni (x,t) \mapsto \big(\mathrm{tr}_{g_t} Q_t\big)(F_t(x)) - (\mathrm{tr}_{\mathfrak{I}_t} F_t^* Q_t)(x) \in \mathbb{R}\big)$
$\underline{B}_r^t(x_0)$	Pulled back geodesic ball $:= F_t^{-1}(B_r^t(x_0))$ $(x_0 \in M)$

According to the above setup, $\underset{\frown}{H}$, $\underset{\frown}{X}$ and $\nabla^\perp f$ also define (local) sections of $F^{-1}TM$ and we shall view them as such whenever t is not explicitly mentioned. Moreover, $F^{-1}TM$ shall be considered a Riemannian vector bundle with inner product defined such that for $((x,t),v),((x,t),w)$ with $v,w \in T_{F_t(x)}M$,

$$\langle ((x,t),v),((x,t),w) \rangle = (g_t(F_t(x)), v \otimes w)\,.$$

For fixed $t \in I$, this coïncides with the inner product induced on $F_t^{-1}TM$ by g_t. Note also that $\underset{\frown}{X}$ (resp. $\underline{f}$) is as regular as X (resp. f) is.

As a rule, we follow the sign conventions of [37] and refer there and to [47] for the notions of submanifold geometry used here. In particular, if $(X_t)_{t\in I}$ and $(Y_t)_{t\in I}$ are smooth one-parameter families of sections of TM such that $X_t(F_t(x)), Y_t(F_t(x)) \in \mathrm{im}\, \mathrm{d}_x F_t$ for each $(x,t) \in N \times I$, then

$$(\mathrm{II}_t(x), \underline{X}_t(x) \otimes \underline{Y}_t(x)) = \Big((x,t), \big(\nabla_{X_t(F_t(x))} Y_t\big)^\perp\Big) = \Big((x,t), \big(\nabla_{Y_t(F_t(x))} X_t\big)^\perp\Big)$$

for each $(x,t) \in N \times I$ and, if $(Z_t)_{t\in I}$ is a smooth one-parameter family of sections of TM such that $Z_t(F_t(x)) \perp \mathrm{im}\, \mathrm{d}_x F_t$ for each $(x,t) \in N \times I$, then

$$\Big\langle \big(\mathrm{II}_t(x), \underline{X}_t(x) \otimes \underline{Y}_t(x)\big), \underset{\frown}{Z}_t(x) \Big\rangle = -\Big\langle \nabla_{X_t(F_t(x))} Z_t, Y_t(F_t(x)) \Big\rangle$$

for each $(x,t) \in N \times I$. Moreover, writing $\nabla^{\mathfrak{I}}$ for the Levi-Civita connection of $\mathfrak{I}$ (t understood), we have the relation

$$(\nabla_{X_t(F_t(x))} Y_t)^T = \nabla^{\mathfrak{I}}_{\underline{X}_t(x)} \underline{Y}_t,$$

where the covariant derivative on the left-hand side is the Levi-Civita connection of g.

We write $H(x,t)$ for the mean curvature as an element of $T_{F_t(x)}M$, i.e. for the unique vector $H(x,t)$ such that for each $(x,t) \in N \times I$,

$$\underset{\smile}{H}\,(x,t) = (x, H(x,t))$$

and note in particular that both $\mathrm{II}_t(x)$ and $H(x,t)$ are normal to $\operatorname{im} \mathrm{d}_x F_t$ for each $(x,t) \in N \times I$.

Now, N may be made to be an evolving Riemannian manifold with family of metrics $\{\mathfrak{I}_t\}_{t \in I}$. We now restrict our attention to mean curvature flow.

Definition 1.12.1. The manifold N is said to evolve by *mean curvature flow* if

$$\partial_t F = H$$

in the sense that

$$\partial_t F(x,t) = H(x,t) \in T_{F_t(x)}M$$

for each $(x,t) \in N \times I$.

This flow was first introduced by Brakke [8] in the context of varifolds and subsequently studied in the smooth setting by Huisken [40], Ecker [19] and various others, especially in the case where $M = \mathbb{R}^{m+1}$.

For our purposes we shall need but a few basic properties of this flow. We first more explicitly describe how $\mathfrak{I}$ evolves.

Proposition 1.12.2. [53], [19, (B.2)] *The evolution equation*

$$(\partial_t \mathfrak{I}, v \otimes w) = (F_t^* h_t, v \otimes w) - 2\langle (\mathrm{II}_t, v \otimes w), \underset{\smile}{H} \rangle$$

holds for $v, w \in T_x N$. In particular,

$$tr_{\mathfrak{I}} \partial_t \mathfrak{I} = tr^T_{\mathfrak{I}} F_t^* h - 2 |\, \underset{\smile}{H}_t \,|^2.$$

Proof. Fix $t_0 \in I$ and let $\{\partial_i\}_{i=1}^m$ be a local coördinate frame for TN over $U \subset N$ open. An unwinding of definitions and an application of the product rule yields

$$\begin{aligned}
&\left(\partial_t|_{t_0} \underline{g}_t(x_0), \partial_i|_{x_0} \otimes \partial_j|_{x_0} \right) \\
&= \partial_t|_{t_0} \left(\underline{g}_t(x_0), \partial_i|_{x_0} \otimes \partial_j|_{x_0} \right) \\
&= \partial_t|_{t_0} \left(g_t(F_t(x_0)), \partial_i F(x_0,t) \otimes \partial_j F(x_0,t) \right) \\
&= \underbrace{\left(h_{t_0}(F_{t_0}(x_0)), \partial_i F(x_0,t_0) \otimes \partial_j F(x_0,t_0) \right)}_{=\left(F_{t_0}^* h_{t_0}(x_0), \partial_i|_{x_0} \otimes \partial_j|_{x_0}\right)} + \partial_t|_{t_0} \left(g_{t_0}(F_t(x_0)), \partial_i F(x_0,t) \otimes \partial_j F(x_0,t) \right) \qquad (1.15)
\end{aligned}$$

for each $x_0 \in U$. Now, we handle the latter term as follows: note first that for each $i \in \{1, \ldots, n\}$, $\partial_i F$ may be considered a local section $\underset{\smile}{\partial_i F}$ of $F^{-1}TM \to N \times I$ such that

$$\underset{\smile}{\partial_i F}\,(x,t) = (x, \partial_i F_t(x)).$$

Now, by Examples 1.6.4 and 1.5.3, g_{t_0} and its Levi-Civita connection induce a Riemannian structure and a Riemannian connection on the pullback bundle $F^{-1}TM$, which we denote by $\langle\cdot,\cdot\rangle_0$ and ∇^0 respectively. Using these, we write

$$(g_{t_0} \circ F_t, \partial_i F_t \otimes \partial_j F_t) = \left\langle \underset{\smile}{\partial_i F}, \underset{\smile}{\partial_j F} \right\rangle_0 (\cdot, t), \tag{1.16}$$

whence, using the fact that ∇^0 is compatible with $\langle\cdot,\cdot\rangle_0$,

$$\begin{aligned} &\partial_t|_{(x_0,t_0)} \langle \underset{\smile}{\partial_i F}, \underset{\smile}{\partial_j F} \rangle_0 \\ &= \langle \nabla^0_{\partial_t|_{(x_0,t_0)}} \underset{\smile}{\partial_i F}, \underset{\smile}{\partial_j F}\,(x_0,t_0) \rangle_0 + \langle \underset{\smile}{\partial_i F}\,(x_0,t_0), \nabla^0_{\partial_t|_{(x_0,t_0)}} \underset{\smile}{\partial_j F} \rangle_0, \end{aligned} \tag{1.17}$$

where here $\partial_t|_{(x_0,t_0)} = 0_{x_0} \oplus \partial_t|_{t_0} \in T_{(x_0,t_0)}(N \times I) \cong T_{x_0}N \bigoplus T_{t_0}I$. We proceed to use the local description of ∇^0 given in Example 1.5.3. Let $\{e_\alpha = \widetilde{\partial}_\alpha\}_{\alpha=1}^n$ be a local coördinate frame for TM in a neighbourhood $V \ni F_{t_0}(x_0)$ and write $e^F_\alpha(x,t) = ((x,t), e_\alpha(F_t(x)))$ as in Example 1.5.3. With this in mind, we note that

$$\underset{\smile}{\partial_i F}\,(x,t) = \sum_{\alpha=1}^n \partial_i F^\alpha_t(x) e^F_\alpha(x,t)$$

and note that, writing $\underset{\smile}{H}$ as

$$\underset{\smile}{H}\,(x,t) = \sum_{\alpha=1}^n H^\alpha(x,t) e^F_\alpha(x,t)$$

in this frame, the mean curvature flow equation reads

$$\partial_t F^\alpha(x,t) = H^\alpha(x,t)$$

for each $\alpha \in \{1,\ldots,n\}$. With all of this in mind, the local description (1.3) of ∇^0 in Example 1.5.3 implies that

$$\nabla^0_{\partial_t|_{(x_0,t_0)}} \underset{\smile}{\partial_i F} = \sum_{\alpha=1}^n \left(\partial_t \partial_i F^\alpha(x_0,t_0) e^F_\alpha(x_0,t_0) + \partial_i F^\alpha(x_0,t_0) \sum_{\beta=1}^n \Gamma^\beta_\alpha\,(\partial_t F(x_0,t_0))\, e^F_\beta(x_0,t_0) \right). \tag{1.18}$$

On the one hand,

$$\partial_t \partial_i F^\alpha = \partial_i \partial_t F^\alpha = \partial_i H^\alpha.$$

On the other,

$$\begin{aligned} &\Gamma^\beta_\alpha\,(\partial_t F(x_0,t_0)) \\ &= \sum_{\gamma=1}^n H^\alpha(x_0,t_0) \Gamma^\beta_\alpha(\partial_\gamma|_{F_{t_0}(x_0)}) \end{aligned}$$

and, since

$$\sum_{\beta=1}^n \Gamma^\beta_\alpha(\partial_\gamma|_{F_{t_0}(x_0)}\, e^F_\beta(x_0,t_0)$$

$$= \left((x_0,t_0), \sum_{\beta=1}^{n} \Gamma_\alpha^\beta (\partial_\gamma|_{F_{t_0}(x_0)} \, e_\beta(F_{t_0}(x_0))\right)$$
$$= ((x_0,t_0), \nabla_{\partial_\alpha|_{F_{t_0}(x_0)}} \partial_\gamma) = ((x_0,t_0), \nabla_{\partial_\gamma|_{F_{t_0}(x_0)}} \partial_\alpha),$$

where the torsion-free property of ∇ was used in the last line, we may write (1.18) as

$$\nabla^0_{\partial_t|_{(x_0,t_0)}} \underset{\frown}{\partial_i F} = \sum_{\alpha=1}^{n} \left(\partial_i H^\alpha(x_0,t_0) e_\alpha^F(x_0,t_0) + H^\alpha(x_0,t_0) \sum_{\beta=1}^{n} \Gamma_\alpha^\beta \left(\partial_i F(x_0,t_0)\right) e_\beta^F(x_0,t_0) \right)$$
$$= ((x_0,t_0), \nabla_{\partial_i F(x_0,t_0)} \overline{H})$$

where $\overline{H} \in \Gamma(TM)$ is a vector field locally agreeing with H in the sense that $\overline{H} \circ F = H$. Thus, (1.17) now reads, using the fact that $\langle \cdot, \cdot \rangle_0$ was induced by g_{t_0}, whose corresponding inner product we shall write simply as $\langle \cdot, \cdot \rangle$,

$$\partial_t|_{(x_0,t_0)} \langle \underset{\frown}{\partial_i F}, \underset{\frown}{\partial_j F} \rangle_0 = \left\langle \nabla_{\partial_i F(x_0,t_0)} \overline{H}, \partial_j F(x_0,t_0) \right\rangle + \left\langle \partial_i F(x_0,t_0), \nabla_{\partial_j F(x_0,t_0)} \overline{H} \right\rangle$$
$$= -2 \left\langle \left(\mathrm{II}_t(x_0), \partial_i|_{x_0} \otimes \partial_j|_{x_0} \right), H(x_0,t_0) \right\rangle.$$

The result now follows from (1.16) and (1.15). □

One reason for considering mean curvature flow is its tendency to decrease the volume of compact (N,I) when $h \equiv 0$, which is evident from the following proposition.

Proposition 1.12.3 (Area-Minimizing Property). [19, Corollary 4.3] *If N is compact and M is static, i.e. $h \equiv 0$, then*

$$\int_N \mathrm{dvol}_{\mathfrak{I}_{t_2}} = \int_N \mathrm{dvol}_{\mathfrak{I}_{t_1}} - \int_{t_1}^{t_2} \int_N |\, \underset{\frown}{H}\, |^2 \mathrm{dvol}_{\mathfrak{I}_t} \mathrm{d}t$$

where $t_1, t_2 \in I$ are such that $t_1 < t_2$.

Proof. In view of the compactness condition, the volume of N is finite and since $(x,t) \mapsto \mathrm{dvol}_{\mathfrak{I}_t}(x)$ is smooth, its $t$derivative is bounded and thus summable. Hence, by Proposition 1.12.2,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_N \mathrm{dvol}_{\mathfrak{I}} = \int_N \partial_t \mathrm{dvol}_{\mathfrak{I}} = - \int_N |\, \underset{\frown}{H}\, |^2 \mathrm{dvol}_{\mathfrak{I}_t}.$$ □

We are more interested in a local variant of this property which is applicable more generally. Before presenting that, we recall how the divergence of a vector field defined on M may be related to that of the induced vector field on N.

Proposition 1.12.4. *Let $X \in C^1(U \times I, TM)$ be a time-dependent local section of TM. Then*

$$\mathrm{div}_{\mathfrak{I}} \underline{X} = \underline{\mathrm{div}_g X} - tr_g^\perp \nabla X + \left\langle \underset{\frown}{X}, \underset{\frown}{H} \right\rangle.$$

In particular, if $f \in C^1(\mathcal{D})$ with $\mathcal{D} \subset M \times I$ open, then $\nabla^{\mathfrak{I}} \underline{f} = \underline{\nabla^g f}$ on $(F, \mathrm{pr}_2)^{-1}(\mathcal{D})$ and, taking $X = \nabla_g f$,

$$\Delta_{\mathfrak{I}} f = \underline{\Delta_g f} - tr_g^\perp \nabla_g^2 f + \left\langle \underset{\frown}{\nabla_g f}, \underset{\frown}{H} \right\rangle.$$

Proof. Let $\{\varepsilon_i\}_{i=1}^m$ be a local orthonormal frame for TN and set $\overline{\varepsilon}_i(F_t(x)) = \mathrm{d}F_t(\varepsilon_i(x)) \in T_{F_t(x)}M$ for each i ($t \in I$ fixed) and use the Gram-Schmidt algorithm to obtain an orthonormal basis $\{\overline{\varepsilon}_i(F_t(x))\}_{i=1}^m \cup \{\nu_i(F_t(x))\}_{i=m+1}^n$ for $T_{F_t(x)}M$. Now, by definition,

$$\left(\operatorname{div}_{g_t} X\right)(F_t(x)) = \sum_{i=1}^{m} \left\langle \nabla_{\overline{\varepsilon}_i} X, \overline{\varepsilon}_i \right\rangle + \underbrace{\sum_{i=m+1}^{n} \left\langle \nabla_{\nu_i} X, \nu_i \right\rangle}_{=\operatorname{tr}_g^{\perp} \nabla X}.$$

We deal with the first term as follows: extend $\{\overline{\varepsilon}_i\}$ and $\{\nu_i\}$ to a local frame on an open subset of M and write X as

$$X = \underbrace{\sum_{i=1}^{m} X^i \overline{\varepsilon}_i}_{=:X_1} + \underbrace{\sum_{i=m+1}^{n} X^i \nu_i}_{=:X_2}$$

on this set. Thus,

$$\begin{aligned}
\sum_{i=1}^{m} \left\langle \nabla_{\overline{\varepsilon}_i(F_t(x))} X, \overline{\varepsilon}_i(F_t(x)) \right\rangle &= \sum_{i=1}^{m} \left\langle \nabla_{\overline{\varepsilon}_i(F_t(x))} X_1, \overline{\varepsilon}_i(F_t(x)) \right\rangle + \sum_{i=1}^{m} \left\langle \nabla_{\overline{\varepsilon}_i(F_t(x))} X_2, \overline{\varepsilon}_i(F_t(x)) \right\rangle \\
&= \sum_{i=1}^{m} \left\langle \left(\nabla_{\overline{\varepsilon}_i(F_t(x))} X_1\right)^T, \overline{\varepsilon}_i(F_t(x)) \right\rangle + \sum_{i=1}^{m} \left\langle \left(\nabla_{\overline{\varepsilon}_i(F_t(x))} X_2\right)^T, \overline{\varepsilon}_i(F_t(x)) \right\rangle,
\end{aligned}$$

where we have used the fact that $\overline{\varepsilon}_i(F_t(x))$ is tangent to $F(M)$. We note that

$$\left(\nabla_{\overline{\varepsilon}_i(F_t(x))} X_1\right)^T = \mathrm{d}_x F_t \left(\nabla_{\varepsilon_i(x)} \underline{X}\right),$$

where the covariant derivative on the right-hand side is the Levi-Civita connection of $\mathfrak{I}$, since X_1 is an extension of the vector field $\underline{X}$, and by definition of $\overline{\varepsilon}_i$ and $\mathfrak{I}$, it is clear that

$$\sum_{i=1}^{m} \left\langle \left(\nabla_{\overline{\varepsilon}_i(F_t(x))} X_1\right)^T, \overline{\varepsilon}_i(F_t(x)) \right\rangle = \sum_{i=1}^{m} \left\langle \nabla_{\varepsilon_i(x)} \underline{X}, \varepsilon_i(x) \right\rangle = \operatorname{div}_{\mathfrak{I}} \underline{X}.$$

Similarly, since X_2 is an extension of the normal part of $X \circ F_t$,

$$\sum_{i=1}^{m} \left\langle \left(\nabla_{\overline{\varepsilon}_i(F_t(x))} X_2\right)^T, \overline{\varepsilon}_i(F_t(x)) \right\rangle = \sum_{i=1}^{m} \left\langle \utilde{X}_2\,(x,t), (\mathrm{II}_t(x), \varepsilon_i(x) \otimes \varepsilon_i(x)) \right\rangle = \left\langle \utilde{X}\,(x,t), \utilde{H}\,(x,t) \right\rangle.$$

□

Thus, assuming M is of locally bounded geometry about (x_0, t_0) and adopting the notation in and following Definition 1.7.4 and taking $f(x,t) = \mathfrak{r}(x,t)^2$, where $\mathfrak{r}(x,t) = d^t(x_0, x)$ and $\mathcal{D} = \mathcal{D}_{j_0,\delta}(x_0, t_0)$, we may compute

$$\begin{aligned}
(\partial_t - \Delta_{\mathfrak{I}_t})\underline{\mathfrak{r}}^2 &= \underline{\partial_t \mathfrak{r}^2} + \left\langle \utilde{\nabla_{g_t} \mathfrak{r}^2}, \utilde{H} \right\rangle - \underline{\Delta_{g_t} \mathfrak{r}^2} + \operatorname{tr}_{g_t}^{\perp} \nabla_{g_t}^2 \mathfrak{r}^2 - \left\langle \utilde{\nabla \mathfrak{r}^2}, \utilde{H} \right\rangle \\
&= \underline{\left(\partial_t - \Delta_{g_t}\right) \mathfrak{r}^2} + \operatorname{tr}_{g_t}^{\perp} \nabla_{g_t}^2 \mathfrak{r}^2. \qquad (1.19)
\end{aligned}$$

Now, using the sectional curvature bounds, theorem B.1, we see that

$$g - (1 - \mathfrak{r} \cdot (f_{\kappa_\infty} \circ \mathfrak{r}))\, g_{\mathfrak{r}} \leqslant \frac{1}{2} \nabla^2 \mathfrak{r}^2 \leqslant g - \left(1 - \mathfrak{r} \cdot (f_{\kappa_{-\infty}} \circ \mathfrak{r})\right) g_{\mathfrak{r}}$$

so that, on the one hand, by Proposition B.3,

$$\Delta_g \mathfrak{r}^2 \leqslant 2n - 2(n-1)\left(1 - \mathfrak{r} \cdot (f_{\kappa_{-\infty}} \circ \mathfrak{r})\right) \leqslant 2n + 2(n-1) C_{\kappa_{-\infty}} \mathfrak{r}^2.$$

On the other hand, $\mathrm{tr}_g^{\perp} g = n - m$ and $0 \leqslant g_{\mathrm{r}} \leqslant g$ on $(F, \mathrm{pr}_2)(N \times]t_0 - \delta, t_0[) \cap \mathcal{D}$, so that

$$\mathrm{tr}_g^{\perp} \nabla^2 \mathrm{r}^2 \geqslant 2(n-m) - 2C_{\kappa_\infty} \mathrm{r}^2 \mathrm{tr}_g^{\perp} g_{\mathrm{r}} \geqslant 2(n-m) - 2(n-m) C_{\kappa_{-\infty}} \mathrm{r}^2$$

on $(F, \mathrm{pr}_2)(N \times]t_0 - \delta, t_0[) \cap \mathcal{D}$. Hence, equation (1.19) leads to

$$(\partial_t - \Delta_{\mathfrak{I}_t}) \underline{\mathrm{r}}^2 \geqslant -2m - \big(\mu + 2C_{\kappa_{-\infty}}(n-1) + 2C_{\kappa_\infty}(n-m)\big) \underline{\mathrm{r}}^2 \geqslant -\gamma \tag{1.20}$$

on $(F, \mathrm{pr}_2)^{-1}\mathcal{D}$.

We are now in a position to state a localized form of Proposition 1.12.3, which is the analogue of the localized area identity of [8, §3.6] or [17, §1].

Theorem 1.12.5. *If $R \in]0, \min\{j_0, \sqrt{4\gamma\delta}\}[$ and $t \in [t_0 - \frac{R^2}{4\gamma}, t_0[$, then*

$$\int_{\underline{B}^t_{R/2}(x_0)} \mathrm{dvol}_{\mathfrak{I}_t} + \int_{t_0 - \frac{R^2}{4\gamma}}^{t} \int_{\underline{B}^s_{R/2}(x_0)} |\underset{\smile}{H}|^2 \mathrm{dvol}_{\mathfrak{I}_s} \mathrm{d}s \leqslant 16 \exp\left(\frac{m\lambda_\infty R^2}{8\gamma}\right) \left(\int_{\underline{B}^{\cdot}_R(x_0)} \mathrm{dvol}_{\mathfrak{I}_\cdot}\right) (t_0 - \frac{R^2}{4\gamma}).$$

Proof. We define (cf. Ex. A.2 for the definition of η)

$$\begin{aligned}\psi_R(x,t) &:= \eta\left(\frac{\underline{\mathrm{r}}(x,t)^2 + \gamma\left(t - \left(t_0 - \frac{R^2}{4\gamma}\right)\right)}{R^2}\right) \\ &= \left(1 - \frac{\underline{\mathrm{r}}(x,t)^2 + \gamma\left(t - \left(t_0 - \frac{R^2}{4\gamma}\right)\right)}{R^2}\right)_+^4.\end{aligned}$$

for $x \in N$, $t \in [t_0 - \frac{R^2}{4\gamma}, t_0[$. It is clear that ψ_R is twice differentiable, and $\mathrm{supp}\, \psi_R(\cdot, t) \subset \underline{B}^t_R(x_0) \subset \underline{B}^t_{j_0}(x_0)$, since

$$1 - \frac{\underline{\mathrm{r}}(x,t)^2 + \gamma\left(t - \left(t_0 - \frac{R^2}{4\gamma}\right)\right)}{R^2} \geqslant 0 \Leftrightarrow R^2 \geqslant \underline{\mathrm{r}}^2 + \gamma\left(t - \left(t_0 - \frac{R^2}{4\gamma}\right)\right) \geqslant \underline{\mathrm{r}}^2.$$

Also, $0 \leqslant \psi_R \leqslant 1$. Now,

$$(\partial_t - \Delta_{\mathfrak{I}_t})(\eta \circ f) = -4(\partial_t - \Delta_{\mathfrak{I}_t}) f \cdot (1-s)_+^3 - 12|\nabla f|^2 (1-f)^2$$

for $f : (F, \mathrm{pr}_2)^{-1}\mathcal{D} \to \mathbb{R}$, which together with (1.20) implies that $(\partial_t - \Delta_{\mathfrak{I}})\psi_R \leqslant 0$. Now, since F_t is a proper embedding for each t and both $\psi_R(\cdot, t)$ and $\partial_t \psi_R(\cdot, t)$ are compactly supported in N, we may compute using Proposition 1.12.2:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_N \psi_R \mathrm{dvol}_I &= \int_N \left(\partial_t \psi_R - |\underset{\smile}{H}|^2 \psi_R + \frac{1}{2} \mathrm{tr}_{\mathfrak{I}} F_t^* h\, \psi_R\right) \mathrm{dvol}_{\mathfrak{I}_t} \\ &\leqslant \int_N \Delta \psi_R \mathrm{dvol}_{\mathfrak{I}_t} - \int_N |\underset{\smile}{H}|^2 \psi_R \mathrm{dvol}_{\mathfrak{I}_t} + \frac{m}{2} \lambda_\infty \int_N \psi_R \mathrm{dvol}_{\mathfrak{I}_t} \\ &= -\int_N |\underset{\smile}{H}|^2 \psi_R \mathrm{dvol}_{\mathfrak{I}_t} + \frac{m\lambda_\infty}{2} \int_N \psi_R \mathrm{dvol}_{\mathfrak{I}_t},\end{aligned}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\frac{m\lambda_\infty}{2}\left(t - \left(t_0 - \frac{R^2}{4\gamma}\right)\right)\right) \int_N \psi_R \mathrm{dvol}_{\mathfrak{I}_t}\right)$$

$$\leqslant -\exp\left(-\frac{m\lambda_\infty}{2}\left(t-\left(t_0-\frac{R^2}{4\gamma}\right)\right)\right)\int_N |\underset{\sim}{H}|^2\psi_R \mathrm{dvol}_{\mathfrak{I}_t}$$
$$\leqslant -\exp\left(-\frac{m\lambda_\infty R^2}{8\gamma}\right)\int_N |\underset{\sim}{H}|^2\psi_R \mathrm{dvol}_{\mathfrak{I}_t}$$

so that, integrating from $t_0 - \frac{R^2}{4\gamma}$ to t and estimating $\exp\left(-\frac{m\lambda_\infty}{2}\left(t-\left(t_0-\frac{R^2}{4\gamma}\right)\right)\right)$ from below,

$$\exp\left(-\frac{m\lambda_\infty R^2}{8\gamma}\right)\cdot\left(\int_N \psi_R(\cdot,t)\mathrm{dvol}_{\mathfrak{I}_t} + \int_{t_0-\frac{R^2}{4\gamma}}^t\int_N |\underset{\sim}{H}|^2\psi_R(\cdot,s)\mathrm{dvol}_{\mathfrak{I}_s}\mathrm{d}s\right)$$
$$\leqslant \int_N \psi_R(\cdot,t_0-\frac{R^2}{4\gamma})\mathrm{dvol}_{\mathfrak{I}_{t_0-\frac{R^2}{4\gamma}}}. \tag{1.21}$$

Now, since $\psi_R(\cdot,t)\leqslant \chi_{\underline{B}^t_R(x_0)}$, the right-hand side of (1.21) may be bounded from above by

$$\left(\int_{\underline{B}^{\cdot}_R(x_0)}\mathrm{dvol}_{\mathfrak{I}_\cdot}\right)(t_0-\frac{R^2}{4\gamma}).$$

On the other hand, since

$$\underline{\mathrm{r}}<\frac{R}{2}\Rightarrow 1-\frac{\underline{\mathrm{r}}^2+\gamma\left(t-\left(t_0-\frac{R^2}{4\gamma}\right)\right)}{R^2}\geqslant 1-\frac{\frac{R^2}{4}+\gamma\cdot\frac{R^2}{4\gamma}}{R^2}=\frac{1}{2}$$

so that

$$\psi_R(\cdot,t)|_{\underline{B}^t_{R/2}(x_0)}\geqslant\left(\frac{1}{2}\right)^4=\frac{1}{16}. \tag{1.22}$$

Since the left-hand integrands of (1.21) are nonnegative, we may estimate their (spatial) integrals from below by the respective integrals on $\underline{B}^t_{R/2}(x_0)$, whence the result follows from (1.22). □

Finally, we shall need another variant of this result which bounds a pulled-back ball of (prescribed) variable radius by this radius to the appropriate power. In order to establish this, we shall need a monotonicity formula. The following formula was first established by Huisken [40] in the case where $u = 1$, $(M,g) = (\mathbb{R}^{m+1},\delta)$ and N is compact, then subsequently adapted to include u by Ecker and Huisken [21] and finally adapted to the curved setting by Hamilton [33] in the case where M and N are compact and M is static, by Lott [51] in the case where N is a compact hypersurface and M is a gradient steady Ricci soliton (cf. [45]) and finally by Magni, Mantegazza and Tsatis [53] in the case of general evolving manifolds.

Theorem 1.12.6 (Monotonicity Formula). *Let $(M,\{g_t\})_{t\in I}$ be an evolving Riemannian manifold with $\partial_t g = h$. If $u\in C^2(N\times I,\mathbb{R})$ is such that* $\operatorname{supp} u(\cdot,t)\Subset N$ *for each $t\in I$ and $f\in C^2(\mathcal{D},\mathbb{R}^+)$ with $\mathcal{D}\subset M\times I$ open with* $\operatorname{supp} u(\cdot,t)\subset(F,\mathrm{pr}_2)^{-1}\left(\mathcal{D}\cap\mathrm{pr}_2^{-1}(\{t\})\right)$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_N u\cdot\underline{f}\mathrm{dvol}_{\mathfrak{I}}\right)$$
$$=\int_N \underline{f}(\partial_t-\Delta_{\mathfrak{I}})u+u\cdot\underline{\left(\partial_t f+\Delta_g f+\frac{1}{2}tr_g h+\frac{n-m}{2(s-t)}f\right)}-u\underline{f}tr_g^{\perp}\mathcal{H}_s(\log f)$$
$$-u\underline{f}\left|\underset{\sim}{H}-\nabla^{\perp}\log f\right|^2\mathrm{dvol}_{\mathfrak{I}}$$

on I for every $s\geqslant t_0$.

Proof. (cf. [21] and [53]) We verify that the identity

$$\partial_t\left(u\cdot f\mathrm{dvol}_{\mathfrak{I}}\right)=\Big[\mathrm{div}_{\mathfrak{I}_t}\left(\underline{f}\nabla u-u\nabla\underline{f}\right)+\underline{f}\cdot\left(\partial_t-\Delta_{\mathfrak{I}_t}\right)u$$
$$+u\cdot\underline{\left(\partial_t f+\Delta_g f+\frac{1}{2}\mathrm{tr}_g h\cdot f+\frac{n-m}{2(s-t)}f\right)}$$
$$-u\underline{f\mathrm{tr}_g^\perp\mathcal{H}_s\left(\log f\right)}-u\underline{f}\left|H-\nabla^\perp\log f\circ F\right|^2\Big]\mathrm{dvol}_{\mathfrak{I}_t}\quad(1.23)$$

holds; an integration and an application of Gauß' theorem and standard integration theorems to justify interchanging the derivative and integral then imply the result.

Now, by Proposition 1.12.4, it is clear that

$$\begin{aligned}\mathrm{div}_{\mathfrak{I}}\left(\underline{f}\nabla_{\mathfrak{I}}u-u\nabla_{\mathfrak{I}}\underline{f}\right)&=\underline{f}\Delta_{\mathfrak{I}}u-u\Delta_{\mathfrak{I}}\underline{f}\\&=\underline{f}\Delta_{\mathfrak{I}}u-u[\underline{\Delta_g f}-\mathrm{tr}_g^\perp\nabla_g^2 f+\underbrace{\left\langle\nabla_g f,\underset{\smile}{H}\right\rangle}_{=\left\langle\nabla^\perp f,\underset{\smile}{H}\right\rangle}].\end{aligned}\quad(1.24)$$

On the other hand, using the fact that $\partial_t\underline{f}=\underline{\partial_t f}+\underbrace{\left\langle\nabla f,H\right\rangle}_{=\left\langle\nabla^\perp f,\underset{\smile}{H}\right\rangle}$ as well as Propositions 1.7.7 and 1.12.2, it is evident that

$$\begin{aligned}&\partial_t\left(u\cdot\underline{f}\mathrm{dvol}_{\mathfrak{I}}\right)\\&=\left(\partial_t u\cdot\underline{f}+u\cdot\underline{\partial_t f}+u\left\langle\nabla^\perp f,\underset{\smile}{H}\right\rangle+\frac{1}{2}\mathrm{tr}_{\mathfrak{I}_t}F_t^*h\cdot u\cdot\underline{f}-|\underset{\smile}{H}|^2\cdot u\cdot\underline{f}\right)\mathrm{dvol}_{\mathfrak{I}_t}.\end{aligned}\quad(1.25)$$

Using the fact that $\mathrm{tr}_{\mathfrak{I}_t}F_t^*h_t=\underline{\mathrm{tr}_{g_t}h_t}-\mathrm{tr}_{g_t}^\perp h_t$, (1.24) and (1.25) imply that

$$\begin{aligned}&\partial_t\left(u\cdot\underline{f}\mathrm{dvol}_{\mathfrak{I}}\right)-\mathrm{div}_{\mathfrak{I}_t}\left(\underline{f}\nabla_{\mathfrak{I}_t}u-u\nabla_{\mathfrak{I}_t}\underline{f}\right)\mathrm{dvol}_{\mathfrak{I}_t}\\&=\Bigg(\underline{f}\left(\partial_t u-\Delta_{\mathfrak{I}_t}u\right)+u\underline{\left(\partial_t f+\Delta_{g_t}f+\frac{1}{2}\mathrm{tr}_{g_t}h_t\cdot f\right)}\\&-u\underline{f}\mathrm{tr}_{g_t}^\perp\left(\frac{\nabla_{g_t}^2 f}{f}+\frac{1}{2}h_t\right)-|\underset{\smile}{H}|^2\cdot u\cdot\underline{f}+2u\cdot\underline{f}\left\langle\nabla^\perp\log f,\underset{\smile}{H}\right\rangle\Bigg)\mathrm{dvol}_{\mathfrak{I}_t}.\end{aligned}\quad(1.26)$$

Now, note that

$$\nabla_g^2\log f=\frac{\nabla_g^2 f}{f}-\frac{\nabla f\otimes\nabla f}{f^2}$$

and $\mathrm{tr}_g^\perp\frac{\nabla f\otimes\nabla f}{f^2}=\frac{|\nabla^\perp f|^2}{\underline{f}^2}=|\nabla^\perp\log f|^2$. On the other hand,

$$\mathrm{tr}_g^\perp\frac{g}{2(s-t)}=\frac{n-m}{2(s-t)}.$$

Incorporating these into (1.26) immediately implies (1.23). □

Theorem 1.12.7. *Let $\alpha=\sqrt{2\gamma/\pi}$, suppose $R<\min\{\frac{1}{2\alpha}R_0,\sqrt{4\pi\delta}\}$ for $R_0<\min\{j_0,\sqrt{4\gamma\delta}\}$ and set*

$$R_R^m(t)=\sqrt{2m(t-t_0)\log\left(\frac{4\pi(t_0-t)}{R^2}\right)}.$$

Then

$$\int_{\underline{B}^t_{R^m_R(t)}(x_0)} \mathrm{dvol}_{\mathfrak{J}_t} \leqslant R^m_R(t)^m \cdot \left(\rho_0 + \frac{\rho_1}{R^m}\right)\left(\int_{\underline{B}^{\cdot}_{R_0}(x_0)} \mathrm{dvol}_{\mathfrak{J}_\cdot}\right)(t_0 - \frac{R_0^2}{4\gamma}),$$

for $t \in]t_0 - \frac{\exp\left(-\frac{1}{2m}\right)}{4\pi}R^2, t_0[$, *where* $\rho_0, \rho_1 > 0$ *depend on* R_0, δ *and the geometry of* M *in* $\mathcal{D}_{j_0,\delta}(x_0,t_0)$.

Proof. Consider the *retarded*[16] weighted backward heat kernel

$$\begin{aligned} &{}^q_s\Phi : \mathcal{D}_{j_0,\delta}(x_0,t_0) \to \mathbb{R}^+ \\ &(x,t) \mapsto \frac{1}{(4\pi(s-t))^{q/2}} \exp\left(\frac{d^t(x,x_0)^2}{4(t-s)}\right), \end{aligned}$$

where $q \in \mathbb{N}$ and $s \in [t_0, s_0]$ with s and s_0 to be chosen later. The computation of Proposition 1.8.7 implies that

$$\begin{aligned} &\left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h + \frac{n-m}{2(s-t)}\right){}^m_s\Phi \\ &\leqslant \left(\frac{n\mu}{2} + C_4\frac{d^t(x,x_0)^2}{s-t}\right)\cdot {}^m_s\Phi \\ &= \left(\frac{n\mu}{2} + 4C_4 4\log\left(\frac{1}{(4\pi(s-t))^{\frac{m}{2}}\cdot {}^m_s\Phi}\right)\right)\cdot {}^m_s\Phi \\ &\leqslant \max\{\frac{n\mu}{2}, 4C_4\}\cdot\left(1+\log\left(\frac{1}{(4\pi(s-t))^{\frac{m}{2}}\cdot {}^m_s\Phi}\right)\right)\cdot {}^m_s\Phi \\ &\leqslant \max\{\frac{n\mu}{2}, 4C_4\}\left(1 + {}^m_s\Phi\cdot\log\left(\frac{1}{(4\pi(s-t))^{m/2}}\right)\right), \end{aligned} \tag{1.27}$$

where the definition of ${}^m_s\Phi$ was used in the third line and Lemma A.3 in the last line. Similarly, the computation of Proposition 1.8.8 together with Lemma A.3 implies that

$${}^m_s\Phi\cdot\mathcal{H}_s(\log{}^m_s\Phi) \geqslant -\max\{2C, -\lambda_{-\infty}\}\cdot\left(1 + {}^m_s\Phi\log\left(\frac{1}{(4\pi(s-t))^{m/2}}\right)\right)g, \tag{1.28}$$

where C depends on j_0 and $\kappa_{-\infty}$ and C_4 on j_0, κ_∞ and μ. We now apply Theorem 1.12.6 with $f = {}^m_s\Phi$ and $u = \psi_R$ where

$$\psi_R(x,t) := \eta\left(\frac{\underline{\mathfrak{r}}(x,t)^2 + \gamma\left(t - \left(t_0 - \frac{R^2}{4\pi}\right)\right)}{\alpha^2 R^2}\right),$$

$\alpha = \sqrt{2\frac{\gamma}{\pi}} > 2\frac{R^m_R(t)}{R}$, and $\mathfrak{r}(x,t) = d^t(F_t(x), x_0)$. As in the proof of Theorem 1.12.5, it is clear that $\mathrm{supp}\,\psi_R(\cdot,t) \subset \underline{B}^t_{\alpha R}(x_0) \Subset \underline{B}^t_{j_0}(x_0)$ for each $t \in [t_0 - \frac{R^2}{4\pi}, t_0[$ and $(\partial_t - \Delta_{\mathfrak{J}})\psi_R \leqslant 0$, whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_N \psi_R\cdot{}^m_s\Phi\mathrm{dvol}_{\mathfrak{J}}\right) \leqslant \varsigma_0\int_N \psi_R\mathrm{dvol}_{\mathfrak{J}_t} + \varsigma_1(s,t)\int_N \psi_R\cdot{}^m_s\Phi\mathrm{dvol}_{\mathfrak{J}_t}$$

where

$$\varsigma_0 = (n-m)\max\{2C, -\lambda_{-\infty}\} + \max\{\frac{n\mu}{2}, 4C_4\} \text{ and}$$

[16] The choice of words is motivated by considerations from physics.

$$\varsigma_1(s,t) = \Big((n-m)\max\{2C, -\lambda_{-\infty}\} + \max\{\frac{n\mu}{2}, 4C_4\}\Big)\log\left(\frac{1}{(4\pi(s-t))^{m/2}}\right).$$

Note that, since $s - t \in [t_0 - t, s_0 - t_0 + \frac{R^2}{4\pi}]$, the relation $\varsigma_1(s,t) \leqslant \varpi_1(t)$ holds with

$$\varpi_1(t) = \Big((n-m)\max\{2C, -\lambda_{-\infty}\} + \max\{\frac{n\mu}{2}, 4C_4\}\Big)\log\left(\frac{1}{(4\pi(t_0-t))^{m/2}}\right).$$

Moreover, $\varpi_1 \in L^1(]t_0 - \frac{R^2}{4\pi}, t_0[)$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(l(t)\right)\int_N \psi_R \cdot {}^m_s\Phi \mathrm{dvol}_{\mathfrak{I}}\right) \leqslant \varsigma_0 \exp(l(t))\int_N \psi_R \mathrm{dvol}_{\mathfrak{I}} \leqslant \varsigma_0 \exp\left(l(t_0 - \delta)\right)\int_N \psi_R \mathrm{dvol}_{\mathfrak{I}} \tag{1.29}$$

where $l(t) = \int_t^{t_0} \varpi_1(z)\mathrm{d}z$. Since $\psi_R(\cdot,t) \leqslant \chi_{\underline{B}^t_{\alpha R}(x_0)}$, an application of Theorem 1.12.5 yields

$$\left(\int_N \psi_R \mathrm{dvol}_{\mathfrak{I}}\right)(t) \leqslant \int_{\underline{B}^t_{\alpha R}(x_0)} \mathrm{dvol}_{\mathfrak{I}_t} \leqslant 16\exp\left(\frac{m\lambda_\infty R_0^2}{8\gamma}\right)\left(\int_{\underline{B}^{\cdot}_{R_0}} \mathrm{dvol}_{\mathfrak{I}}\right)(t_0 - \frac{R_0^2}{4\gamma}).$$

Substituting this into (1.29) and integrating from $t_0 - \frac{R^2}{4\pi}$ to $\bar{t} \in]t_0 - \frac{R^2}{4\pi}, t_0[$,

$$\begin{aligned}
&\exp(l(\bar{t}))\int_N \psi_R(\cdot,\bar{t}) \cdot {}^m_s\Phi(\cdot,\bar{t})\mathrm{dvol}_{\mathfrak{I}_{\bar{t}}} \\
&\leqslant \left(\exp(l(\cdot))\int_N \psi_R \cdot {}^m_s\Phi \mathrm{dvol}_{\mathfrak{I}_\cdot}\right)(t_0 - \frac{R^2}{4\pi}) \\
&\quad + 16a_0\exp\left(l(t_0-\delta) + \frac{m\lambda_\infty R_0^2}{8\gamma}\right)\left(\int_{\underline{B}^{\cdot}_{R_0}(x_0)} \mathrm{dvol}_{\mathfrak{I}}\right)(t_0 - \frac{R_0^2}{4\gamma}).
\end{aligned} \tag{1.30}$$

In light of the inequality

$${}^s_m\Phi(\cdot, t_0 - \frac{R^2}{4\pi}) \leqslant R^{-m},$$

we have that

$$\begin{aligned}
\left(\int_N \psi_R \cdot {}^m_s\Phi \mathrm{dvol}_{\mathfrak{I}_\cdot}\right)(t_0 - \frac{R^2}{4\pi}) &\leqslant R^{-m}\left(\int_{\underline{B}^{\cdot}_{\alpha R}(x_0)} \mathrm{dvol}_{\mathfrak{I}_\cdot}\right)(t_0 - \frac{R^2}{4\pi}) \\
&\leqslant 16\exp\left(\frac{m\lambda_\infty R_0^2}{8\gamma}\right)R^{-m}\left(\int_{\underline{B}^{\cdot}_{R_0}(x_0)} \mathrm{dvol}_{\mathfrak{I}_\cdot}\right)(t_0 - \frac{R_0^2}{4\gamma}).
\end{aligned}$$

On the other hand,

$$\int_N \psi_R(\cdot,\bar{t}) \cdot {}^m_s\Phi(\cdot,\bar{t})\mathrm{dvol}_{\mathfrak{I}_{\bar{t}}} \geqslant \frac{1}{(4\pi(s-\bar{t}))^{m/2}}\int_{\underline{B}^{\bar{t}}_{R^m_R(\bar{t})}(x_0)} \psi_R(\cdot,\bar{t})\exp\left(\frac{R^m_R(\bar{t})^2}{4(\bar{t}-s)}\right)\mathrm{dvol}_{\mathfrak{I}_{\bar{t}}}. \tag{1.31}$$

Note that for $\bar{t} \in [t_0 - R^2 \cdot \exp(-1/2m)/4\pi, t_0[$, we have $R^m_R(\bar{t})^2 + \bar{t} \geqslant t_0$, which is clearly bounded from above by some s_0 on this interval. Thus, we suppose $\bar{t}$ lies in this interval and set $s = R^m_R(\bar{t})^2 + \bar{t}$. Substituting this into (1.31), and noting that, in view of the definition of α,

$$\frac{\underline{\mathrm{r}}(\cdot,\bar{t})^2 + \gamma(\bar{t} - (t_0 - \frac{R^2}{4\pi}))}{\alpha^2 R^2} \leqslant \frac{R^m_R(\bar{t})^2 + \gamma\frac{R^2}{4\pi}}{\alpha^2 R^2} \leqslant \frac{1}{2}$$

$$\Rightarrow \psi_R(\cdot,\bar{t}) \geqslant \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

on $B^{\bar{t}}_{R_R^m(\bar{t})}(x_0)$, we see that the left hand side of (1.31) is greater than or equal to

$$\frac{\exp(1/4)}{16 \cdot (4\pi)^{m/2} \cdot R_R^m(\bar{t})^m} \int_{B^{\bar{t}}_{R_R^m(\bar{t})}(x_0)} \mathrm{dvol}_{\mathfrak{I}_{\bar{t}}}.$$

Substituting these inequalities into (1.30) then implies the result. □

— 2 —
Monotonicity for Yang-Mills over $\mathbb{R}^n$

In this chapter, we restrict our attention to Yang-Mills theory over $\mathbb{R}^n$ and provide alternative proofs of some monotonicity formulæ to be established more generally later on in this thesis. The existing local monotonicity formula for solutions to the Yang-Mills equation due to Price [61] is first proved before we establish the local monotonicity formula for the flow using methods along the lines of those available for solutions of the harmonic map heat equation and reaction-diffusion equations as developed by Ecker in [20]. Price's formula is local in nature, whereas the existing formula for the Yang-Mills flow due to Chen and Shen [11], itself an analogue of a monotonicity formula for solutions to the harmonic map heat equation due to Chen and Struwe [12], is not. However, it involves weighting the square norm of the curvature of the connection against an appropriate Gaußian. In contrast to this, the formula we prove is local in nature, where the domain of integration is a superlevel set of an appropriate Gaußian in space-time, a so-called "heat ball".

2.1. Simplifications. In what follows, we work over a *trivial* $SO(N)$-bundle over $\mathbb{R}^n$, whose Lie algebra $\mathfrak{g} = \mathfrak{so}(N)$ is equipped with the positive-definite rescaled *Killing inner product* (cf. §1.3)

$$(A,B) \mapsto \langle A,B\rangle := -\mathrm{tr}(AB).$$

We shall write $|A| = \sqrt{\langle A,A\rangle}$ for the induced norm. In this case, *Ad-invariance* takes the form[1]

$$\left\langle gAg^{-1}, gBg^{-1}\right\rangle = \langle A,B\rangle\,, \tag{2.1}$$

where $A,B \in \mathfrak{so}(N)$ and $g \in SO(N)$, which is easily verified. In particular, if $g :]-\varepsilon,\varepsilon[\to SO(N)$ is a curve with $g(0) = I$ and $\dot g(0) = X \in \mathfrak{so}(N)$, differentiating both sides of (2.1) at 0 yields

$$\begin{aligned}\langle XA,B\rangle - \langle AX,B\rangle + \langle A,XB\rangle - \langle A,BX\rangle &= 0\\ \Rightarrow \langle [A,X],B\rangle &= \langle A,[X,B]\rangle\end{aligned} \tag{2.2}$$

We note that a connection on the trivial bundle $\mathbb{R}^n \times SO(N)$ may be given by a single matrix-valued one-form $\omega : \mathbb{R}^n \to \mathfrak{so}(N) \otimes T^*\mathbb{R}^n$. Here we shall write $A_i = (\omega, \partial_i)$ and consider a connection to be a collection of matrix-valued functions

$$\{A_i : \mathbb{R}^n \to \mathfrak{so}(N)\}_{i=1}^n.$$

Similarly, we write $F_{ij} = (\underline{\Omega}^\omega, \partial_i \wedge \partial_j)$ so that the curvature two-form is represented by the matrix-valued functions

$$\{F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] : \mathbb{R}^n \to \mathfrak{so}(N)\}_{i,j=1}^n$$

so that the Yang-Mills energy density takes the form

$$e(A) = \frac{1}{2}\sum_{i<j}|F_{ij}|^2 = \frac{1}{4}\sum_{i,j}|F_{ij}|^2.$$

In this setting, the Bianchi identity (cf. Proposition 1.5.7) takes the form

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0,\;\; i,j,k \in \{1,\dots,n\}$$

where $\nabla_i F_{jk} = \partial_i F_{jk} + [A_i, F_{jk}]$, and the Yang-Mills flow system the form

[1] Here we treat elements of $\mathfrak{so}(N)$ and $SO(N)$ as matrices.

$$\partial_t A_j = \sum_{i=1}^{n} \nabla_i F_{ij} \quad j \in \{1, \ldots, n\}$$

for a *family* of connections $\{A_i(\cdot,t)\}_{t\in I}$ with $I \subset \mathbb{R}$ an open interval, Yang-Mills connections being static solutions ($\partial_t A_j \equiv 0$).

In the remainder of this chapter we shall assume that $I =]-\infty, 0[$ for the sake of simplicity. Furthermore, we shall be concerned with properties of A relative to $(0,0)$. By using the translation invariance of the Yang-Mills flow system, all of the formulæ may be stated relative to some $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ provided $\{A(\cdot,t)\}$ evolves by the Yang-Mills flow for $t < t_0$.

2.2. Scaling behaviour. We set $(A_r)_i(x,t) := r \cdot A_i(rx, r^2 t)$. If we consider $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ as being a function $(F_A)_{ij}$ of A, then

$$\begin{aligned}(F_{A_r})_{ij}(x,t) &= r^2 \cdot \left(\partial_i A_j(rx,r^2t) - \partial_j A_i(rx,r^2t) + [A_i(rx,r^2t), A_j(rx,r^2t)]\right)\\ &= r^2 F_{ij}(rx,r^2t).\end{aligned}$$

In particular, if A solves the Yang-Mills heat equation, then so does A_r, for

$$\begin{aligned}\partial_t (A_r)_j(x,t) = r^3 \partial_t A_j(rx,r^2t) &= r^3 \sum_{i=1}^{n} \partial_i F_{ij}(rx,r^2t) + [A_i(rx,r^2t), F_{ij}(rx,r^2t)]\\ &= \sum_{i=1}^{n} \partial_i (F_{A_r})_{ij}(x,t) + [(A_r)_i(x,t), (F_{A_r})_{ij}(x,t)].\end{aligned}$$

On the other hand,

$$e(A_r)(x,t) = \frac{1}{4}\sum_{i,j} \Big\langle (F_{A_r})_{ij}(x,t), (F_{A_r})_{ij}(x,t) \Big\rangle = r^4 \cdot e(A)(rx,r^2t), \tag{2.3}$$

which serves as motivation for how we shall weight the localized energy integrals to be introduced in the coming sections.

We abbreviate $(F_{A_r})_{ij}$ by $(F_r)_{ij}$. Now, note that

$$\frac{\mathrm{d}}{\mathrm{d}r}(F_r)_{ij}(x,t) = 2rF_{ij}(rx,r^2t) + r^2 \sum_{k=1}^{n} x^k \partial_k F_{ij}(rx,r^2t) + 2r^3 t \partial_t F_{ij}(rx,r^2t)$$

and that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}(F_r)_{ij}\left(\frac{x}{r}, \frac{t}{r^2}\right) &= r \cdot \left(2F_{ij}(x,t) + \sum_{k=1}^{n} x^k \partial_k F_{ij}(x,t) + 2t\partial_t F_{ij}(x,t)\right)\\ &= r \cdot \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=1} (F_r)_{ij}(x,t).\end{aligned}$$

2.3. Scale-invariant solutions. We call a family of connections $\{A(\cdot,t)\}_{t\in]-\infty,0[}$ *scale invariant* about $(0,0)$ if $(A_r)_i(x,t) = A_i(x,t)$ for all $r > 0$, $(x,t) \in \mathbb{R}^n \times]-\infty, 0[$ and $i \in \{1, \ldots, n\}$. Differentiating this identity with respect to r and evaluating at $r = 1$ (cf. [20, §1]), it is equivalent to the condition that

$$\begin{aligned}A_i(x,t) + \sum_j x^j \partial_j A_i(x,t) + 2t\partial_t A_i &= 0 \text{ or}\\ \partial_t A_i + \sum_j \frac{x^j}{2t} \partial_j A_i(x,t) + \frac{1}{2t} A_i(x,t) &= 0.\end{aligned} \tag{2.4}$$

On the other hand, we note that

$$\begin{aligned}\partial_t A_i + \sum_j \frac{x^j}{2t} F_{ji} &= \partial_t A_i + \sum_j \frac{x^j}{2t}\left(\partial_j A_i - \partial_i A_j + [A_i, A_j]\right)\\ &= \partial_t A_i + \sum_j \frac{x^j}{2t}\partial_j A_i + \frac{1}{2t}A_i - \partial_i\left(\sum_j \frac{x^j}{2t}A_j\right) + [A_i, \sum_j \frac{x^j}{2t}A_j]. \end{aligned} \tag{2.5}$$

For general families of connections, this expression may only be made to coïncide with the left hand side of (2.4) at some fixed time t_0 for which a radial gauge is chosen, i.e. in a gauge such that $\sum_j x^j A_j(x, t_0) = 0$ for all $x \in \mathbb{R}^n$. If, however, A is either scale-invariant or the expression (2.5) vanishes, then this gauge is preserved, wherefore it follows that both conditions are equivalent. For details of this argument, see e.g. [74].

We may thus characterize scale invariance by the condition that

$$\partial_t A_i + \sum_j \frac{x^j}{2t} F_{ji} = 0.$$

Such solutions are known to exist; see e.g. [31] and [29].

2.4. The static case. We now prove Price's monotonicity formula by scaling. This should be compared with the approach usually taken to prove the monotonicity formula for harmonic maps (see e.g. [64]) and contrasted with the approach taken in Chapter 3.

Theorem 2.4.1. *If A is a Yang-Mills connection with*[2] $A_i \in C^2(B_R; \mathfrak{so}(N))$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-4}}\int_{B_r} e(A)\right) = \frac{1}{r^{n-4}}\int_{\partial B_r}\sum_{j=1}^n \Big|\sum_{i=1}^n F_{ij}(x)\frac{x^i}{r}\Big|^2 \mathrm{d}S_x \geqslant 0$$

holds on $]0, R[$.

Proof. We first simplify the expression that is to be differentiated:

$$\begin{aligned}\frac{1}{r^{n-4}}\int_{B_r} e(A)(x)\mathrm{d}x &\overset{x=ry}{=} \frac{1}{r^{n-4}}\int_{B_1} e(A)(ry)\cdot r^n \mathrm{d}y\\ &\overset{2.3}{=} \int_{B_1} e(A_r)(y)\mathrm{d}y. \end{aligned} \tag{2.6}$$

Differentiating (2.6) with respect to r and making use of the scaling identities derived in the preceding section, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-4}}\int_{B_r} e(A)\right) &= \int_{B_1}\frac{1}{2}\sum_{i,j}\left\langle \frac{\mathrm{d}}{\mathrm{d}r}(F_r)_{ij}(y), (F_r)_{ij}(y)\right\rangle \mathrm{d}y\\ &\overset{y=\frac{x}{r}}{=} \frac{1}{2r^n}\int_{B_r}\sum_{i,j}\left\langle \frac{\mathrm{d}}{\mathrm{d}r}(F_r)_{ij}(\frac{x}{r}), (F_r)_{ij}(\frac{x}{r})\right\rangle \mathrm{d}x\\ &= \frac{1}{2r^{n-3}}\int_{B_r}\sum_{i,j}\left\langle \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=1}(F_r)_{ij}(x), F_{ij}(x)\right\rangle \mathrm{d}x\\ &= \frac{1}{r^{n-3}}\int_{B_r}\sum_{i,j}\left\langle F_{ij}(x) + \frac{1}{2}\sum_k x^k\partial_k F_{ij}(x), F_{ij}(x)\right\rangle \mathrm{d}x. \end{aligned} \tag{2.7}$$

Using (2.2), we may rewrite the integrand of (2.7) as

[2] We write B_r for $B_r(0)$.

$$\sum_{i,j}\left\langle F_{ij}(x)+\frac{1}{2}\sum_k x^k\left(\partial_k F_{ij}(x)+[A_k(x),F_{ij}(x)]\right),F_{ij}(x)\right\rangle=\sum_{i,j}\left\langle F_{ij}(x)+\frac{1}{2}\sum_k x^k\nabla_k F_{ij}(x),F_{ij}(x)\right\rangle,$$

since $\left\langle [A_k,F_{ij}],F_{ij}\right\rangle=\left\langle A_k,[F_{ij},F_{ij}]\right\rangle=0$. On the other hand, an application of the Bianchi identity yields

$$\begin{aligned}&\sum_{i,j}\left\langle F_{ij}(x)-\frac{1}{2}\sum_k x^k\left(\nabla_i F_{jk}+\nabla_j F_{ki}\right)(x),F_{ij}(x)\right\rangle\\ =&\sum_{i,j}\left\langle F_{ij}(x)-\sum_k x^k\nabla_i F_{jk}(x),F_{ij}(x)\right\rangle\\ =&\sum_{i,j}|F_{ij}|^2(x)-\sum_{i,j,k}x^k\left[\partial_i\left\langle F_{jk},F_{ij}\right\rangle-\left\langle F_{jk},\nabla_i F_{ij}\right\rangle\right](x),\end{aligned}$$

where the second line we have used the antisymmetry of F ($F_{ij}=-F_{ji}$) and in the last line the compatibility of ∇ with $\langle\cdot,\cdot\rangle$. On the one hand, A is Yang-Mills so that $\sum_i\nabla_i F_{ij}=0$, whence the last term vanishes. On the other hand, we may integrate what we are left with by parts, so that the integral (2.7) now reads

$$\begin{aligned}&\frac{1}{r^{n-3}}\int_{B_r}\sum_{i,j}|F_{ij}|^2+\sum_{i,j,k}\delta_i^k\left\langle F_{jk},F_{ij}\right\rangle-\frac{1}{r^{n-3}}\int_{\partial B_r}\sum_{i,j,k}x^k\cdot\frac{x^i}{r}\cdot\left\langle F_{jk},F_{ij}\right\rangle\mathrm{d}S_x\\ =&\frac{1}{r^{n-3}}\int_{B_r}\sum_{i,j}|F_{ij}|^2+\left\langle F_{ji},F_{ij}\right\rangle+\frac{1}{r^{n-3}}\int_{\partial B_r}\sum_j\left\langle\sum_k F_{kj}x^k,\sum_i F_{ij}\frac{x^i}{r}\right\rangle\mathrm{d}S_x\\ =&\frac{1}{r^{n-3}}\int_{B_r}\sum_{i,j}|F_{ij}|^2-|F_{ij}|^2+\frac{1}{r^{n-4}}\int_{\partial B_r}\sum_j\left\langle\sum_k F_{kj}\frac{x^k}{r},\sum_i F_{ij}\frac{x^i}{r}\right\rangle\mathrm{d}S_x,\end{aligned}$$

where we have used the antisymmetry of F twice, whence the result follows. □

2.5. The heat flow case. We now provide a local monotonicity formula for the Yang-Mills flow over $\mathbb{R}^n$ along the lines of Ecker's local monotonicity formulæ. Here, as in [20], we prove the formula by scaling. This should be contrasted with the approach taken in Theorem 6.3.2 where we make use of divergence identities.

Let

$$E_r=\left\{(x,t)\in\mathbb{R}^n\times\,]-\infty,0[\,:\Phi(x,t):=\frac{1}{(4\pi(t_0-t))^{\frac{n-4}{2}}}\exp\left(\frac{|x|^2}{4t}\right)\right\}.$$

We first recall an integration formula from [20].

Theorem 2.5.1. *If $X\in C^1(\mathbb{R}^n\times\,]-\infty,0[\,,\mathbb{R}^n)$, then*

$$\iint_{E_r}\operatorname{div}X\mathrm{d}x\mathrm{d}t=-\frac{r}{n-4}\frac{\mathrm{d}}{\mathrm{d}r}\iint_{E_r}X\cdot\frac{x}{2t}\mathrm{d}x\mathrm{d}t$$

whenever these integrals exist.

Proof. See [20, Lemma 1.6] or Theorem 5.4.2. □

Theorem 2.5.2. *Suppose A evolves by the Yang-Mills flow on $\mathbb{R}^n\times\,]-\infty,0[$ and that*

$$\int_{-\frac{r_0^2}{4\pi}}^0\int_{B_{2c_{n,4}r_0}}e(A)\mathrm{d}x\mathrm{d}t<\infty \tag{2.8}$$

for some $r_0 > 0$, where $c_{n,k}$ is as in Theorem 6.3.2. Then the identity

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-4}}\iint_{E_r} e(A)\cdot\frac{n-4}{-2t} - \sum_{j=1}^{n}\left\langle\sum_{i=1}^{n} F_{ij}\cdot\frac{x^i}{2t}, \sum_{i=1}^{n} F_{ij}\cdot\frac{x^i}{2t} + \partial_t A_j\right\rangle \mathrm{d}x\mathrm{d}t\right)$$
$$= \frac{n-4}{r^{n-3}}\iint_{E_r}\sum_{j=1}^{n}|\partial_t A_j + \sum_{i=1}^{n} F_{ij}\cdot\frac{x^i}{2t}|^2\mathrm{d}x\mathrm{d}t. \tag{2.9}$$

holds on $]0, r_0[$.

Remark 2.5.3. This formula should be considered a local analogue of that of Chen and Shen [11] which in this case reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}(e(A)\Phi)(x,t)\mathrm{d}x = -\int_{\mathbb{R}^n}\left(\sum_{j=1}^{n}|\partial_t A_j + \sum_{i=1}^{n} F_{ij}\cdot\frac{x^i}{2t}|^2\Phi\right)(x,t)\mathrm{d}x$$

whenever A decays appropriately at ∞ (cf. Corollary 4.2.4 (i)). In particular, the right hand side of Chen and Shen's formula vanishes precisely when the right hand side of (2.9) does— on scale-invariant solutions (cf. §2.3).

Remark 2.5.4. The condition (2.8) ensures the ensures the summability (on E_{r_0}) of each term occurring in the monotonicity formula, a claim whose proof shall be deferred to §6.3 (cf. Lemma 6.3.1).

Proof of Theorem 2.5.2. We first compute formally, writing $X = (x,t)$ and $Y = (y,s)$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-4}}\iint_{E_r}\frac{1}{4}\sum_{i,j}|F_{ij}(x,t)|^2\cdot\frac{1}{2t}\mathrm{d}X\right)$$
$$\overset{X=P_r(Y)}{=}\frac{\mathrm{d}}{\mathrm{d}r}\left(\iint_{E_1}\frac{1}{4}\sum_{i,j}\left\langle(F_r)_{ij}(y,s),(F_r)_{ij}(y,s)\right\rangle\cdot\frac{1}{2s}\mathrm{d}Y\right) \tag{2.10}$$
$$= \frac{1}{r^{n-3}}\iint_{E_r}\frac{1}{2}\sum_{i,j}\left\langle\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=1}(F_r)_{ij}(x,t),F_{ij}(x,t)\right\rangle\cdot\frac{1}{2t}\mathrm{d}X$$
$$= \frac{1}{r^{n-3}}\iint_{E_r}\sum_{i,j}\left\langle\frac{1}{2t}F_{ij} + \sum_{k=1}^{n}\frac{x^k}{4t}\partial_k F_{ij} + \frac{1}{2}\partial_t F_{ij}, F_{ij}\right\rangle(x,t)\mathrm{d}X, \tag{2.11}$$

As in the static case, we use the identity (2.2) and the Bianchi identity to write the integrand in (2.11) as

$$\sum_{i,j}\left[\frac{1}{2t}|F_{ij}|^2 - \sum_{k=1}^{n}\left\langle\frac{x^k}{2t}\nabla_i F_{jk}, F_{ij}\right\rangle + \frac{1}{2}\left\langle\partial_t F_{ij}, F_{ij}\right\rangle\right].$$

Now, since ∇ is compatible with $\langle\cdot,\cdot\rangle$, we may rewrite the middle term as

$$-\sum_{i,j,k}\left\langle\frac{x^k}{2t}\nabla_i F_{jk}, F_{ij}\right\rangle(x,t) = \sum_{i,j,k} -\partial_i\left\langle\frac{x^k}{2t}F_{jk}, F_{ij}\right\rangle + \frac{\delta_i^k}{2t}\left\langle F_{jk}, F_{ij}\right\rangle + \left\langle\frac{x^k}{2t}F_{jk}, \nabla_i F_{ij}\right\rangle$$
$$= -\sum_{i,j}\frac{1}{2t}|F_{ij}|^2 - \sum_{j,k}\left(\left\langle\frac{x^k}{2t}F_{kj}, \partial_t A_j\right\rangle - \sum_i \partial_i\left\langle\frac{x^k}{2t}F_{kj}, F_{ij}\right\rangle\right),$$

where the antisymmetry of F_{ij} and the Yang-Mills heat equation have been used in the last step. On the other hand,

$$\begin{aligned}\partial_t F_{ij} &= \partial_i \partial_t A_j - \partial_j \partial_t A_i + [\partial_t A_i, A_j] + [A_i, \partial_t A_j] \\ &= (\nabla_i \partial_t A_j - \nabla_j \partial_t A_i),\end{aligned}$$

whence the last term may be rewritten as

$$\begin{aligned}\sum_{i,j} \frac{1}{2} \left\langle \partial_t F_{ij}, F_{ij} \right\rangle &= \frac{1}{2} \sum_{i,j} \left\langle \nabla_i \partial_t A_j, F_{ij} \right\rangle - \left\langle \nabla_j \partial_t A_i, F_{ij} \right\rangle \\ &= \frac{1}{2} \sum_{i,j} \left\langle \nabla_i \partial_t A_j, F_{ij} - F_{ji} \right\rangle \\ &= \sum_{i,j} \left\langle \nabla_i \partial_t A_j, F_{ij} \right\rangle \\ &= \sum_{i,j} \partial_i \left\langle \partial_t A_j, F_{ij} \right\rangle - \left\langle \partial_t A_j, \nabla_i F_{ij} \right\rangle \\ &= \sum_{i,j} \partial_i \left\langle \partial_t A_j, F_{ij} \right\rangle - \sum_{j=1}^{n} |\partial_t A_j|^2,\end{aligned}$$

where the antisymmetry of F and the Yang-Mills heat equation have again been used. Altogether, the integral (2.11) is equal to

$$\frac{1}{r^{n-3}} \iint_{E_r} - \sum_j \left\langle \sum_k \frac{x^k}{2t} F_{kj} + \partial_t A_j, \partial_t A_j \right\rangle + \sum_{i,j} \partial_i \left\langle \partial_t A_j + \sum_k \frac{x^k}{2t} F_{kj}, F_{ij} \right\rangle \mathrm{d}X.$$

Applying Theorem 2.5.1 to the divergence term, we obtain

$$\frac{1}{r^{n-3}} \iint_{E_r} - \sum_j \left\langle \sum_k \frac{x^k}{2t} F_{kj} + \partial_t A_j, \partial_t A_j \right\rangle \mathrm{d}X - \frac{1}{(n-4) r^{n-4}} \frac{\mathrm{d}}{\mathrm{d}r} \left(\iint_{E_r} \sum_{i,j} \left\langle \partial_t A_j + \sum_k \frac{x^k}{2t} F_{kj}, \frac{x^i}{2t} F_{ij} \right\rangle \mathrm{d}X \right).$$

"Completing the derivative," noting that minus the integrand of the latter integral completes the square in the former integrand, we obtain

$$\frac{1}{r^{n-3}} \iint_{E_r} - \sum_j | \sum_k \frac{x^k}{2t} F_{kj} + \partial_t A_j |^2 \mathrm{d}X - \frac{1}{n-4} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{1}{r^{n-4}} \iint_{E_r} \sum_{i,j} \left\langle \partial_t A_j + \sum_k \frac{x^k}{2t} F_{kj}, \frac{x^i}{2t} F_{ij} \right\rangle \mathrm{d}X \right).$$

A careful inspection shows that the preceding steps are valid provided

$$\operatorname{div} \left(\sum_i \left(\sum_j \left\langle F_{ij}, \partial_t A_j + \sum_k \frac{x^k}{2t} F_{kj} \right\rangle \right) e_i \right) \in E_{r_0},$$

which justifies pulling the $\frac{\mathrm{d}}{\mathrm{d}r}$ under the integral sign in (2.10) and applying Theorem 2.5.1, since the condition (2.8) ensures the summability of everything else (cf. Remark 2.5.4). To drop this assumption, we instead compute analogously to before that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{1}{r^{n-4}} \iint_{E_r} \chi_k(-t) \cdot \left(e(A)(X) \cdot \frac{n-4}{-2t} - \sum_{j=1}^{n} \left\langle \sum_{i=1}^{n} F_{ij} \cdot \frac{x^i}{2t}, \sum_{i=1}^{n} F_{ij} \cdot \frac{x^i}{2t} + \partial_t A_j \right\rangle \right) \mathrm{d}X \right) \\ &\quad = \frac{n-4}{r^{n-3}} \iint_{E_r} \chi_k(-t) \sum_{j=1}^{n} |\partial_t A_j + \sum_{i=1}^{n} F_{ij} \cdot \frac{x^i}{2t}|^2 \mathrm{d}X + \frac{n-4}{r^{n-3}} \iint_{E_r} \frac{e(A)}{-t} \cdot (-t) \chi_k'(-t) \mathrm{d}X,\end{aligned}$$

where $\{\chi_k\}_{k \in \mathbb{N}}$ is as in Example A.1. Thus, integrating this expression on $]r_1, r_2[$ with $0 < r_1 < r_2 < r_0$ and using Remark 2.5.4 and the fact that $(-t)\chi_k'(-t) \xrightarrow{k \to \infty} 0$ implies the result. □

In light of the discussion in §2.3, the right hand side of this formula vanishes on scale-invariant solutions. In particular, by [20, Proposition 1.5], we have that

$$\frac{1}{r^{n-4}} \iint\limits_{E_r} \frac{n-4}{-2t} e(A) \mathrm{d}x \mathrm{d}t = \int_{\mathbb{R}^n} e(A)(x,t)\Phi(x,t) \mathrm{d}x$$

for all $t < 0$, $r > 0$ whenever A is scale-invariant. Therefore, on such solutions, this quantity *coïncides* with that in Chen and Shen's formula.

— 3 —

The Static Case: Monotonicity of Energies of p-Dirichlet Type

In this chapter, we investigate the metric properties of an energy functional naturally associated to problems of Dirichlet type in order to obtain identities that, in some sense, state how solutions to problems of Dirichlet type *scale*; these identities then lead to monotonicity identities upon integration. The tensor that naturally arises in these identities is the so-called *energy-momentum tensor* (also known as the *stress-energy tensor*), which is quite well known in the physics literature [46] and was first considered in the context of harmonic maps by Eells and Baird [5]. In [1], Alikakos made use of identities involving this tensor to derive a monotonicity formula for solutions to a certain semilinear elliptic PDE. We take this approach as a starting point for intrinsically deriving similar formulæ for p-Dirichlet-type problems. We note that the energy-momentum tensor is also of independent interest, and has been used to derive conservation laws for and draw conclusions about nonexistence of solutions to certain PDE [5, 42, 9, 2].

3.1. The energy-momentum tensor. Let E be a Riemannian vector bundle with Riemannian connection ∇ over an oriented Riemannian manifold (M^n, g).[1] Consider the Dirichlet-type energy

$$\begin{aligned} F_g : \Gamma(E \otimes \Lambda^k T^*M) &\to \mathbb{R} \\ \psi &\mapsto \int_M e_g(\psi)\mathrm{dvol}_g \end{aligned}$$

where $k \in \mathbb{N}_0$ and $e_g(\psi) := \frac{1}{p}|\psi|^p$ is the **p-Dirichlet energy density** for $p > 1$.[2] Rather than scale the integrand as in the case $M = \mathbb{R}^n$, we vary it with respect to the metric. The following proposition describes the resulting first variation.

Proposition 3.1.1 (Energy-Momentum Tensor). *The unique (symmetric) tensor $T^g_\psi \in \Gamma(T^\star M \otimes T^\star M)$ satisfying*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(t \mapsto e_{g+th}(\psi)(x)\mathrm{dvol}_{g+th}(x)\right) = \left\langle -\frac{1}{2}T^g_\psi(x), h(x)\right\rangle \mathrm{dvol}_g(x)$$

for all symmetric $h \in \Gamma(T^*M \otimes T^*M)$ and $x \in M$ is locally given in a g-ON frame $\{\varepsilon_i\} \leftrightarrow \{\omega^i\}$ by

$$T^g_\psi = |\psi|^{p-2} \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i \otimes \omega^j - e_g(\psi) g.$$

Proof. On the one hand, Proposition 1.7.7 immediately implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(t \mapsto \mathrm{dvol}_{g+th}(x)\right) = \left(\frac{1}{2}\langle g, h\rangle_g \,\mathrm{dvol}_g\right)(x).$$

On the other hand, by Proposition 1.7.8,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \psi, \psi\rangle_{g+th} = \left\langle -\sum_{i,j=1}^n \left(\left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle\right) \omega^i \otimes \omega^j, h\right\rangle_g .$$

The result then follows from

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e_{g+th}(\psi) = \frac{1}{2}|\psi|^{p-2} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \psi, \psi\rangle_{g+th} .$$ □

[1] The metric may depend on a parameter, though the following considerations assume the the parameter is fixed since the dependence of g on its parameter is not relevant to the discussion.

[2] Except in this chapter and the following one, p shall be assumed to be 2.

In [1], Alikakos considered the system[3]

$$\Delta u - \nabla W(u) = 0$$

for $u \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $W \in C^2(\mathbb{R}^n, \mathbb{R}^+)$, which is naturally associated to the energy

$$\int_{\mathbb{R}^n} \frac{1}{2}|\mathrm{d}u|^2 + W(u).$$

There, the energy-momentum tensor is

$$T_{ij} = \partial_i u \cdot \partial_j u - \left(\frac{1}{2}|\mathrm{d}u|^2 + W(u)\right)\delta_{ij},$$

which was shown to enjoy the property div $T = 0$ that ultimately led to a monotonicity formula. This suggests that computing the divergence of T^g_ψ should lead to something useful.

Proposition 3.1.2. *In any local frame as above,*

$$div\, T^g_\psi = -\sum_{j=1}^n \left(\left\langle \delta^\nabla(|\psi|^{p-2}\psi), \iota_{\varepsilon_j}\psi\right\rangle + \left\langle |\psi|^{p-2}\iota_{\varepsilon_j} d^\nabla\psi, \psi\right\rangle\right)\omega^j.$$

Proof. We compute in a local ON frame adapted at x:

$$\begin{aligned}
\mathrm{div}\, T^g_\psi &= \sum_{r=1}^n \iota_{\varepsilon_r}\nabla_{\varepsilon_r} T^g_\psi \\
&= \sum_r \iota_{\varepsilon_r}\Bigg(\partial_{\varepsilon_r}(|\psi|^{p-2})\sum_{i,j}\sum_{J^{k-1}}\left\langle(\psi,\varepsilon_i\wedge\varepsilon_J),(\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle\omega^i\otimes\omega^j \\
&\quad +|\psi|^{p-2}\sum_{i,j}\sum_{J^{k-1}}(\left\langle(\nabla_{\varepsilon_r}\psi,\varepsilon_i\wedge\varepsilon_J),(\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle + \left\langle(\psi,\varepsilon_i\wedge\varepsilon_J),(\nabla_{\varepsilon_r}\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle)\omega^i\otimes\omega^j \\
&\quad - \partial_{\varepsilon_r} e_g(\psi) g\Bigg) \\
&= \sum_j\Bigg(\sum_{J^{k-1}}\underbrace{\sum_i \partial_{\varepsilon_i}(|\psi|^{p-2})\left\langle(\psi,\varepsilon_i\wedge\varepsilon_J),(\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle + |\psi|^{p-2}\left\langle\nabla_{\varepsilon_i}\psi,\varepsilon_i\wedge\varepsilon_J),(\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle}_{=\left\langle\sum_i(\nabla_{\varepsilon_i}(|\psi|^{p-2}\psi),\varepsilon_i\wedge\varepsilon_J),(\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle}\Bigg)\omega^j \\
&\quad + |\psi|^{p-2}\sum_{i,j}\sum_{J^{k-1}}\left(\left\langle(\psi,\varepsilon_i\wedge\varepsilon_J),(\nabla_{\varepsilon_i}\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle - \frac{1}{k}\left\langle(\nabla_{\varepsilon_j}\psi,\varepsilon_i\wedge\varepsilon_J),(\psi,\varepsilon_i\wedge\varepsilon_J)\right\rangle\right)\omega^j \\
&= -\sum_j\left(\sum_{J^{k-1}}\left\langle(\delta^\nabla(|\psi|^{p-2}\psi),\varepsilon_J),(\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle\right)\omega^j \\
&\quad + |\psi|^{p-2}\sum_{i,j}\sum_{J^{k-1}}\left(\left\langle(\psi,\varepsilon_i\wedge\varepsilon_J),(\nabla_{\varepsilon_i}\psi,\varepsilon_j\wedge\varepsilon_J)\right\rangle - \frac{1}{k}\left\langle(\nabla_{\varepsilon_j}\psi,\varepsilon_i\wedge\varepsilon_J),(\psi,\varepsilon_i\wedge\varepsilon_J)\right\rangle\right)\omega^j.
\end{aligned}$$

Now, we note that

$$\left(\mathrm{d}^{\nabla\psi}, \varepsilon_j\wedge\varepsilon_i\wedge\varepsilon_{j_1}\wedge\cdots\wedge\varepsilon_{j_{k-1}}\right)$$

[3] We note here that, apart from the case where Alikakos' equation reduces to a linear one ($W = 0$), there is no overlap between the equations considered here and those considered in [1].

$$
\begin{aligned}
&= \sum_{p=1}^{n} \left(\omega^p \wedge \nabla_{\varepsilon_p}\psi, \varepsilon_j \wedge \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}}\right) \\
&= (\nabla_{\varepsilon_j}\psi, \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}}) - (\nabla_{\varepsilon_i}\psi, \varepsilon_j \wedge \varepsilon_{j_1} \wedge \ldots \varepsilon_{j_{k-1}}) \\
&\quad + \sum_{q=1}^{k-1} (-1)^{q+1} (\nabla_{\varepsilon_{j_q}}\psi, \varepsilon_j \wedge \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \hat{\varepsilon}_{j_q} \wedge \cdots \wedge \varepsilon_{j_{k-1}}),
\end{aligned}
$$

whence

$$
\begin{aligned}
&\sum_j \sum_{J^{k-1}} \left\langle (\nabla_{\varepsilon_j}\psi, \varepsilon_i \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \right\rangle \\
&= \frac{1}{(k-1)!} \sum_{i, j_1, \ldots, j_{k-1}} \left\langle (\nabla_{\varepsilon_j}\psi, \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}}), (\psi, \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}}) \right\rangle \\
&= \sum_i \sum_{J^{k-1}} \left(\left\langle (d^\nabla \psi, \varepsilon_j \wedge \varepsilon_i \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \right\rangle + \left\langle (\nabla_{\varepsilon_i}\psi, \varepsilon_j \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \right\rangle \right) \\
&+ \frac{1}{(k-1)!} \sum_{i, j_1, \ldots, j_{k-1}} \sum_{q=1}^{k-1} (-1)^q \left\langle (\nabla_{\varepsilon_{j_q}}\psi, \varepsilon_j \wedge \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \hat{\varepsilon}_{j_q} \wedge \cdots \wedge \varepsilon_{j_{k-1}}), (\psi, \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}}) \right\rangle .
\end{aligned}
\tag{3.1}
$$

Expanding the sum over q out, keeping track of signs when permuting the basis vectors, we rewrite the last sum (omitting the combinatorial factor) as

$$
\begin{aligned}
&\sum_{i, j_1, \ldots, j_{k-1}} \left\langle (\nabla_{\varepsilon_{j_1}}\psi, \varepsilon_j \wedge \varepsilon_i \wedge \varepsilon_{j_2} \wedge \cdots \wedge \varepsilon_{j_{k-1}}), (\psi, \varepsilon_{j_1} \wedge \varepsilon_i \wedge \varepsilon_{j_2} \wedge \cdots \wedge \varepsilon_{j_{k-1}}) \right\rangle + \ldots \\
&+ \left\langle (\nabla_{\varepsilon_{j_q}}\psi, \varepsilon_j \wedge \varepsilon_{j_1} \wedge \cdots \wedge \underbrace{\varepsilon_i}_{q\text{th entry}} \wedge \cdots \wedge \varepsilon_{j_{k-1}}), (\psi, \varepsilon_{j_q} \wedge \varepsilon_{j_1} \wedge \cdots \wedge \underbrace{\varepsilon_i}_{q\text{th entry}} \wedge \cdots \wedge \varepsilon_{j_{k-1}}) \right\rangle + \ldots \\
&+ \left\langle (\nabla_{\varepsilon_{j_{k-1}}}\psi, \varepsilon_j \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-2}} \wedge \varepsilon_i), (\psi, \varepsilon_{j_{k-1}} \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-2}} \wedge \varepsilon_i) \right\rangle \\
&= (k-1) \sum_{i, j_1, \ldots, j_{k-1}} \left\langle (\nabla_{\varepsilon_i}\psi, \varepsilon_j \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}}), (\psi, \varepsilon_i \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{k-1}}) \right\rangle \\
&= (k-1) \cdot (k-1)! \sum_i \sum_J \left\langle (\nabla_{\varepsilon_i}\psi, \varepsilon_j \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \right\rangle ,
\end{aligned}
$$

where the indices were relabeled in the second last line. Thus (3.1) reduces to

$$
\begin{aligned}
&\sum_i \sum_{J^{k-1}} \left[\left\langle (d^\nabla \psi, \varepsilon_j \wedge \varepsilon_i \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \right\rangle + k \left\langle (\nabla_{\varepsilon_i}\psi, \varepsilon_j \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \right\rangle \right] \\
&= k \left\{ \sum_{L^k} \left\langle (d^\nabla \psi, \varepsilon_j \wedge \varepsilon_L), (\psi, \varepsilon_L) \right\rangle + \sum_i \sum_{J^{k-1}} \left\langle (\nabla_{\varepsilon_i}\psi, \varepsilon_j \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \right\rangle \right\} .
\end{aligned}
$$

The result follows, since the latter term cancels out the unwanted term in the expression for div T^g_ψ above. □

We therefore see that the following conservation law for p-harmonic k-forms may be read off this formula.

Corollary 3.1.3 (Conservation Law). *If ψ is p-harmonic, then* div $T^g_\psi = 0$.

As hinted at earlier, the energy-momentum tensor is thought to contain information about how p-harmonic vector bundle-valued k-forms scale. In [1], for example, the integral of the divergence of T^g_ψ contracted with the radial vector field $x \mapsto \sum_i \frac{x^i}{|x|} \partial_i|_x \in T_x\mathbb{R}^n$ yields an expression that coïncides with what is usually obtained when scaling the Dirichlet integral. In order to make use of this technique more generally, we compute the divergence of the energy-momentum tensor contracted with an arbitrary vector field, henceforth to be interpreted as a 'scaling direction'. The following formula should be compared with [71, Lemma 3.1], where this identity is established for solutions to an inhomogeneous Yang-Mills equation.

Corollary 3.1.4. *Let $U \subset M$ be open and $X \in \Gamma^1(TU)$. Define $Y \in \Gamma^1(TU)$ by*

$$Y := \left(\iota_X T^g_\psi\right)^\sharp = |\psi|^{p-2} \sum_{j=1}^n \left\langle \iota_X \psi, \iota_{\varepsilon_j} \psi \right\rangle \varepsilon_j - e_g(\psi) X.$$

Then

$$\begin{aligned}\text{div } Y = |\psi|^{p-2} \sum_{i=1}^n \left\langle \iota_{\nabla_{\varepsilon_i} X} \psi, \iota_{\varepsilon_i} \psi \right\rangle - e_g(\psi) \text{div } X \\ - \left\langle \delta^\nabla(|\psi|^{p-2}\psi), \iota_X \psi \right\rangle - |\psi|^{p-2} \left\langle \iota_X d^\nabla \psi, \psi \right\rangle.\end{aligned}$$

Proof. We compute in a local ON frame adapted at x:

$$\begin{aligned}\text{div } Y &= \sum_{i=1}^n \left\langle \nabla_{\varepsilon_i} \left(\iota_X T^g_\psi\right)^\sharp, \varepsilon_i \right\rangle \\ &= \sum_{i=1}^n \left\langle \nabla_{\varepsilon_i} \left(\iota_X T^g_\psi\right), \omega^i \right\rangle \\ &= \sum_{i=1}^n \left\langle \iota_{\nabla_{\varepsilon_i} X} T^g_\psi + \iota_X \nabla_{\varepsilon_i} T^g_\psi, \omega^i \right\rangle \\ &= \left\langle T^g_\psi, \nabla X \right\rangle + \iota_X \text{div } T^g_\psi,\end{aligned}$$

where we have used the symmetry of T^g_ψ in the last step. Using proposition 3.1.1, the former term may be written as

$$|\psi|^{p-2} \sum_{i=1}^n \sum_{J^{k-1}} \langle (\psi, \nabla_{\varepsilon_i} X \wedge \varepsilon_J), (\psi, \varepsilon_i \wedge \varepsilon_J) \rangle - e_g(\psi) \text{div } X.$$

On the other hand, using proposition 3.1.2, we may write the latter term as

$$- \left\langle \delta^\nabla(|\psi|^{p-2}\psi), \iota_X \psi \right\rangle - |\psi|^{p-2} \left\langle \iota_X d^\nabla \psi, \psi \right\rangle. \qquad \square$$

3.2. Monotonicity. We now apply the identities derived in §3.1 to the study of Dirichlet type problems on static manifolds. In doing so, we provide a simple intrinsic proof of the monotonicity principle for p-harmonic k-forms with values in vector bundles which in particular includes p-harmonic maps and p-Yang–Mills fields. This should be compared with [1].

Suppose (M, g) is *static*, $x_0 \in M$ is fixed and the locally bounded geometry bounds of Definition 1.7.4 hold and let j_0 be defined by (1.6) of §1.7. Let $k \in \mathbb{N}$ and $p > 1$ be such that $n \geqslant kp + 1$. As remarked earlier, the energy-momentum tensor describes how p-harmonic k-forms scale. Analogously to [1, Theorem 2.1] and [71, Theorem 3.2], we make use of the energy-momentum identities derived earlier to scale k-forms in the radial direction. This is the content of the following lemma.

Lemma 3.2.1. *Let $X \in \Gamma(TU)$ for $U = B_{j_0}(x_0)$ be defined by $X = \nabla(\frac{1}{2}r^2) = r\partial_r$ and suppose Y is as in corollary 3.1.4. Then*

$$\text{div } Y \leqslant (kp - n) e_g(\psi) + \Lambda r e_g(\psi) - J_\psi + |\psi|^{p-2} |\iota_{\partial_r} \psi|^2 \cdot [1 - r f_{\underline{\kappa}}(r)]$$

on U, where $\Lambda = \Lambda(n, k, p, \underline{\kappa}, \bar{\kappa}, inj_{x_0})$ and $J_\psi = \left\langle \delta^\nabla(|\psi|^{p-2}\psi), \iota_X \psi \right\rangle + |\psi|^{p-2} \left\langle \iota_X d^\nabla \psi, \psi \right\rangle$.
If $sec \equiv \kappa \leqslant 0$, then

$$\text{div } Y = (kp - n) e_g(\psi) - J_\psi + \left[(n - kp - 1) e_g(\psi) + |\psi|^{p-2} |\iota_{\partial_r} \psi|^2\right] \cdot (1 - r f_\kappa(r)).$$

Remark 3.2.2. Note that in both cases, the term following J_ψ is nonpositive.[4]

Proof. Let $\{\varepsilon_i\}_{i=1}^n$ be a local orthonormal frame in U with $\varepsilon_1 = \partial_r$. We first compute

$$\begin{aligned}\nabla_{\varepsilon_i}(r\partial_r) &= \langle \varepsilon_i, \partial_r\rangle \partial_r + r\sum_{j=1}^n \left\langle \nabla_{\varepsilon_i}\partial_r, \varepsilon_j\right\rangle \varepsilon_j \\ &= \delta_{i1}\partial_r + r\sum_{j=1}^n (\nabla^2 r, \varepsilon_i\otimes\varepsilon_j)\varepsilon_j,\end{aligned}$$

whence, by lemma B.1,

$$\begin{aligned}\operatorname{div}(r\partial_r) &= 1 + r\cdot\sum_{i=1}^n (\nabla^2 r, \varepsilon_i\otimes\varepsilon_i) \\ &\geqslant 1 + r\sum_{i=2}^n f_{\bar\kappa}(r)\delta_{ii} = 1 + (n-1)rf_{\bar\kappa}(r)\end{aligned}$$

so that

$$\begin{aligned}\operatorname{div} Y &= |\psi|^{p-2}|\iota_{\partial_r}\psi|^2 + |\psi|^{p-2}\sum_{i,j=1}^n r(\nabla^2 r, \varepsilon_i\otimes\varepsilon_j)\left\langle \iota_{\varepsilon_j}\psi, \iota_{\varepsilon_i}\psi\right\rangle - e_g(\psi)\operatorname{div}(r\partial_r) - J_\psi \\ &\leqslant |\psi|^{p-2}|\iota_{\partial_r}\psi|^2 + |\psi|^{p-2}\sum_{i,j=1}^n rf_{\underline{\kappa}}(r)(g_r, \varepsilon_i\otimes\varepsilon_j)\left\langle \iota_{\varepsilon_j}\psi, \iota_{\varepsilon_i}\psi\right\rangle - e_g(\psi)\cdot(1+(n-1)rf_{\bar\kappa}(r)) - J_\psi \\ &= |\psi|^{p-2}|\iota_{\partial_r}\psi|^2 + |\psi|^{p-2}rf_{\underline{\kappa}}(r)\underbrace{\sum_{i=2}^n |\iota_{\varepsilon_i}\psi|^2}_{=kp|\psi|^{2-p}e_g(\psi) - |\iota_{\partial_r}\psi|^2} - e_g(\psi)\cdot(1+(n-1)rf_{\bar\kappa}(r)) - J_\psi \\ &= e_g(\psi)\cdot(kp\cdot rf_{\underline{\kappa}}(r) - 1 - (n-1)rf_{\bar\kappa}(r)) - J_\psi + |\psi|^{p-2}|\iota_{\partial_r}\psi|^2\cdot(1-rf_{\underline{\kappa}}(r)) \\ &= (kp-n)e_g(\psi) + e_g(\psi)\cdot[(rf_{\underline{\kappa}}(r)-1)kp + (n-1)(1-rf_{\bar\kappa}(r))] - J_\psi + |\psi|^{p-2}|\iota_{\partial_r}\psi|^2(1-rf_{\underline{\kappa}}(r)).\end{aligned}$$

Now, in the case $\underline{\kappa} = \bar\kappa = \kappa \leqslant 0$, the above inequalities are actually equalities and

$$\operatorname{div} Y = (kp-n)e_g(\psi) - J_\psi + [(n-kp-1)e_g(\psi) + |\psi|^{p-2}|\iota_{\partial_r}\psi|^2](1-rf_{\underline{\kappa}}(r)).$$

In the general case, we have

$$\begin{aligned}&(rf_{\underline{\kappa}}(r)-1)kp + (n-1)(1-rf_{\bar\kappa}(r)) \\ &\qquad\leqslant \underbrace{\left\{kp\cdot\max_{]0,j_0]}\frac{rf_{\underline{\kappa}}(r)-1}{r} + (n-1)\max_{]0,j_0]}\frac{1-rf_{\bar\kappa}(r)}{r}\right\}}_{=:\Lambda} r,\end{aligned}$$

whence the result follows. □

Exactly as in [1] and [71], the integrated form of this identity leads to a monotonicity identity.

Proposition 3.2.3 (Monotonicity Identity). *The identity*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}R}\left(e^{\Lambda R}R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right) \\ &\qquad\geqslant e^{\Lambda R}R^{kp-n-1}\cdot\left\{\int_{B_R(x_0)} J_\psi + (rf_{\underline{\kappa}}(r)-1)|\psi|^{p-2}|\iota_{\partial_r}\psi|^2\mathrm{dvol}\right.\end{aligned}$$

[4] See Appendix B for the properties of the f_κ, which shall be taken for granted in the following proofs.

$$+R\int_{\partial B_R(x_0)} |\psi|^{p-2}|\iota_{\partial_r}\psi|^2\mathrm{d}S\Bigg\}.$$

holds for $R \in\,]0, j_0[$. If $\sec \equiv \kappa \leqslant 0$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}R}&\left(R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right)\\ &= R^{kp-n-1}\cdot\Bigg\{\int_{B_R(x_0)} J_\psi + (1-rf_\kappa(r))\cdot\left[|\psi|^{p-2}|\iota_{\partial_r}\psi|^2\right.\\ &\qquad \left.+(n-(kp+1))e_g(\psi)\right]\mathrm{dvol}_g + R\int_{\partial B_R(x_0)} |\psi|^{p-2}|\iota_{\partial_r}\psi|^2\mathrm{d}S\Bigg\}\end{aligned}$$

for all $R > 0$.

Proof. By Gauß' theorem,

$$R\int_{\partial B_R(x_0)} |\psi|^{p-2}|\iota_{\partial_r}\psi|^2 - e_g(\psi)\mathrm{d}S = \int_{\partial B_R(x_0)} \langle Y, \partial_r\rangle = \int_{B_R(x_0)} \mathrm{div}\ Y\mathrm{dvol}_g. \tag{3.2}$$

In the case where $\sec \equiv -K^2 \leqslant 0$, the preceding proposition implies that

$$\begin{aligned}\int_{B_R(x_0)} \mathrm{div}\ Y\mathrm{dvol}_g &= (kp-n)\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\\ &\quad -\int_{B_R(x_0)} J_\psi - (1-rf_\kappa(r))\cdot\left[|\psi|^{p-2}|\iota_{\partial_r}\psi|^2 + (n-(kp+1))e_g(\psi)\right]\mathrm{dvol}_g.\end{aligned}$$

Plugging this expression into (3.2) and rearranging a bit, we obtain

$$\begin{aligned}(kp-n)&\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + R\int_{\partial B_R(x_0)} e_g(\psi)\mathrm{d}S\\ &= \int_{B_R(x_0)} J_\psi + (rf_\kappa(r)-1)\left[|\psi|^{p-2}|\iota_{\partial_r}\psi|^2 + (n-(kp+1))e_g(\psi)\right]\mathrm{dvol}_g\\ &\qquad + R\int_{\partial B_R(x_0)} |\psi|^{p-2}|\iota_{\partial_r}\psi|^2\mathrm{d}S,\end{aligned}$$

whence the monotonicity identity follows from the fact that

$$\begin{aligned}(kp-n)&\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + R\int_{\partial B_R(x_0)} e_g(\psi)\mathrm{d}S\\ &= R^{1-kp-n}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right).\end{aligned}$$

In the general case, we have that

$$\begin{aligned}\int_{B_R(x_0)} \mathrm{div}\ Y\ \mathrm{dvol}_g &\leqslant (kp-n)\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + \Lambda R\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\\ &\quad -\int_{B_R(x_0)} J_\psi + (rf_{\underline{\kappa}}(r)-1)|\psi|^{p-2}|\iota_{\partial_r}\psi|^2\mathrm{dvol}_g,\end{aligned}$$

whence an application of (3.2) yields, after a rearranging of terms,

$$(kp-n)\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + \Lambda R\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + R\int_{\partial B_R(x_0)} e_g(\psi)\mathrm{d}S$$

$$\geqslant \int_{B_R(x_0)} J_\psi + (rf_{\underline{K}}(r) - 1)|\psi|^{p-2}|\iota_{\partial_r}\psi|^2 \mathrm{dvol}_g + R\int_{\partial B_R(x_0)} |\psi|^{p-2}|\iota_{\partial_r}\psi|^2 \mathrm{d}S.$$

The monotonicity identity then follows from the fact that

$$\begin{aligned}&(kp-n)\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + \Lambda R\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + R\int_{\partial B_R(x_0)} e_g(\psi)\mathrm{d}S\\ &= R^{1-kp-n}e^{-\Lambda R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^{kp-n}e^{\Lambda R}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right).\end{aligned}$$ □

As with the identities in the preceding chapter, we may immediately read off consequences for p-harmonic k-forms.

Theorem 3.2.4 (Monotonicity Formula). *If ψ is p-harmonic, then*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(e^{\Lambda R}R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right) \geqslant 0$$

on $]0, j_0[$. *If* $\sec \equiv -K^2 \leqslant 0$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right) \geqslant 0$$

on $]0, \infty[$.

Proof. If ψ is p-harmonic, then $J_\psi = 0$. The result then follows immediately from Proposition 3.2.3. □

As corollaries we obtain the following monotonicity formulæ, the former of which is well-known in the case $p = 2$, where it was first established by Price [61], and the latter of which is well known for $p > 1$, having first been established by Schoen and Uhlenbeck [63] and Price [61] in the case $p = 2$ and subsequently generalized by Hardt and Lin [34].

Corollary 3.2.5. *The following hold:*

(i) [61] *Assume the setup of §1.11. If ω is a p-Yang-Mills connection, then*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{e^{\Lambda R}}{R^{n-2p}}\int_{B_R(x_0)} e_g(\underline{\Omega}^\omega)\mathrm{dvol}_g\right) \geqslant 0$$

holds on $]0, j_0[$. *If* $\sec \equiv -K^2 \leqslant 0$, *then the this inequality holds on* $]0, \infty[$ *with* $\Lambda = 0$.

(ii) [63] [34] *Assume the setup of §1.10. If $u : M \to N$ is p-harmonic, then*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{e^{\Lambda R}}{R^{n-p}}\int_{B_R(x_0)} e_g(\mathrm{d}u)\mathrm{dvol}_g\right) \geqslant 0$$

holds on $]0, j_0[$. *If* $\sec_M \equiv -K^2 \leqslant 0$, *then this inequality holds on* $]0, \infty[$ *with* $\Lambda = 0$.

In the same spirit as [65, §4.3], the right bounds on the inhomogeneities $\delta^\nabla(|\psi|^{p-2}\psi)$ and $|\psi|^{p-2}\mathrm{d}^\nabla\psi$ yield similar monotonicity formulæ. An estimate leading us in this direction is the following lemma.[5]

[5] There is less that can be done in this setting owing to the fact that, in a sense, the area functional is an 'L^∞ functional', whereas the p-Dirichlet energy is an 'L^p functional'.

Lemma 3.2.6. *Let $q_\psi = |\delta^\nabla(|\psi|^{p-2}\psi)| + |\psi|^{p-2}|\mathrm{d}^\nabla\psi|$ and p' be such that $\frac{1}{p}+\frac{1}{p'}=1$. Then*

$$J_\psi \geqslant -R\cdot\left(e_g(\psi)+\frac{1}{p'}q_\psi^{p'}\right)$$

on $B_R(x_0)$.

Proof. Using Cauchy-Schwarz and the fact that $r<R$, it is clear that

$$\begin{aligned} J_\psi &= \left\langle \delta^\nabla(|\psi|^{p-2}\psi), \iota_{r\partial_r}\psi\right\rangle + |\psi|^{p-2}\left\langle \iota_{r\partial_r}\mathrm{d}^\nabla\psi,\psi\right\rangle \\ &\geqslant -R\left\{|\delta^\nabla(|\psi|^{p-2}\psi)|\cdot|\iota_{\partial_r}\psi| + |\psi|^{p-2}|\iota_{\partial_r}\mathrm{d}^\nabla\psi|\cdot|\psi|\right\} \\ &\geqslant -R\,|\psi|q_\psi, \end{aligned}$$

whence an application of Young's inequality yields

$$J_\psi \geqslant -R\left(\frac{|\psi|^p}{p}+\frac{q_\psi^{p'}}{p'}\right).$$

□

Thus, using this bound in the monotonicity identity (Proposition 3.2.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(e^{\Lambda R}R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right) \\ + e^{\Lambda R}R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + e^{\Lambda R}R^{kp-n}\int_{B_R(x_0)}\frac{q_\psi^{p'}}{p'}\mathrm{dvol}_g \geqslant 0,$$

or, multiplying through by e^R,

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(e^{(\Lambda+1)r}R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right) + e^{(\Lambda+1)R}R^{kp-n}\int_{B_R(x_0)}\frac{q_\psi^{p'}}{p'}\mathrm{dvol}_g \geqslant 0. \tag{3.3}$$

Therefore, if $R^{kp-n}\int_{B_R(x_0)} q_\psi^{p'}\mathrm{dvol}_g = O(1)$ as $R\to 0$, then we may integrate the latter term on the left-hand side of (3.3) to deduce a monotonicity formula. One case where this may be done is given in the following theorem.

Theorem 3.2.7. *If $q_\psi\big|_{B_{\widetilde{R}}(x_0)} \leqslant \Gamma$ for $0<\widetilde{R}\leqslant j_0$, then*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(e^{(\Lambda+1)R}R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g + \frac{\Gamma^{p'}}{p'}\int_0^R e^{(\Lambda+1)u}u^{kp-n}\,Vol(B_u(x_0))\mathrm{d}u\right)\geqslant 0$$

on $]0,\widetilde{R}[$.

Proof. Applying (3.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(e^{(\Lambda+1)R}R^{kp-n}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_g\right) + \frac{\Gamma^{p'}}{p'}e^{(\Lambda+1)R}R^{kp-n}\mathrm{Vol}(B_R(x_0))\geqslant 0,$$

whence the result follows from the fact that $\mathrm{Vol}(B_R(x_0)) = O(R^n)$ as $R\searrow 0$. □

This implies in particular a monotonicity principle for solutions to an inhomogeneous p-Yang-Mills or p-harmonic map equation with bounded right-hand side, the former of which was established by Uhlenbeck [71] in the case $p=2$.

Corollary 3.2.8. *The following hold:*

1. [71] *Assume the setup of §1.11. If ω is a connection on P such that*

$$\delta^{\nabla}\left(|\underline{\Omega}^{\omega}|^{p-2}\underline{\Omega}^{\omega}\right) = J \in \Gamma\left((P \times_{Ad} \mathfrak{g}) \otimes T^*M\right)$$

and $||J||_{\infty} = \sup_M |J| < \infty$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{e^{(\Lambda+1)R}}{R^{n-2p}}\int_{B_R(x_0)} e_g(\underline{\Omega}^{\omega})\mathrm{dvol}_g + \frac{||J||_{\infty}^{p'}}{p'}\int_0^R e^{(\Lambda+1)u}u^{2p-n}\mathrm{Vol}(B_u(x_0))\mathrm{d}u\right) \geqslant 0$$

on $]0, j_0[$. *If* $\sec \equiv -K^2 \leqslant 0$, *then* $\Lambda = 0$ *and the inequality holds on* $]0, \infty[$.

2. *Assume the setup of §1.10. If $u : M \to N$ is a smooth map such that*

$$\delta^{\nabla}\left(|\mathrm{d}u|^{p-2}\mathrm{d}u\right) = v \in \Gamma(u^{-1}TN)$$

and $||v||_{\infty} = \sup_M |v| < \infty$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{e^{(\Lambda+1)R}}{R^{n-p}}\int_{B_R(x_0)} e_g(\mathrm{d}u)\mathrm{dvol}_g + \frac{||v||_{\infty}^{p'}}{p'}\int_0^R e^{(\Lambda+1)u}u^{2p-n}\mathrm{Vol}(B_u(x_0))\mathrm{d}u\right) \geqslant 0$$

on $]0, j_0[$. *If* $\sec \equiv -K^2 \leqslant 0$, *then* $\Lambda = 0$ *and the inequality holds on* $]0, \infty[$.

Proof. In both cases, the vector bundle-valued differential form ψ is closed, i.e. $\mathrm{d}^{\nabla}\psi = 0$ so that $q_{\psi} = \left|\delta^{\nabla}(|\psi|^{p-2}\psi)\right|$. The result then follows immediately from Theorem 3.2.7. □

Nonlocal Monotonicity of Weighted Energies of Dirichlet Type

After recalling nonlocal monotonicity formulæ for the harmonic map and Yang-Mills flows on static compact manifolds, we apply the identities of Chapter 3 and inequalities of §1.8 to establish nonlocal monotonicity identities for weighted energies of Dirichlet type. As corollaries, we obtain analogues of the aforementioned nonlocal monotonicity formulæ in the evolving manifold setting. Moreover, we use this identity to establish a counterpart of the estimate in [20, Appendix] for vector bundle-valued k-forms satisfying the heat equation in order to later establish that the heat ball integrals we consider are finite.

4.1. Known results. We first review the known nonlocal monotonicity formulæ for the harmonic map and Yang-Mills heat flows.

First suppose that $u : (\mathbb{R}^n, \delta) \times]t_0 - \delta_0, t_0[\to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow, where N is a Riemannian submanifold of $\mathbb{R}^K$ (cf. §1.10). It was shown by Struwe [68] that the monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(4\pi(t_0 - t)\int_{\mathbb{R}^n}\frac{1}{2}|\mathrm{d}u|^2\Phi_{(x_0,t_0)}\mathrm{d}x\right) = -4\pi(t_0-t)\int_{\mathbb{R}^n}\left|\partial_t u + \sum_{i=1}^n \frac{(x-x_0)^i}{2(t-t_0)}\partial_i u\right|^2\Phi_{(x_0,t_0)}\mathrm{d}x \tag{4.1}$$

holds on $]t_0 - \delta_0, t_0[$ whenever u decays appropriately at ∞ (cf. Corollary 4.2.4 (ii)), where

$$\Phi_{(x_0,t_0)}(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}}\exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right)$$

is the Euclidean backward heat kernel. This was subsequently adapted to the case where $u : M \times]t_0 - \delta_0, t_0[\to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow for static compact M and N isometrically embedded in $\mathbb{R}^K$ by Chen and Struwe [12], which takes the form

$$\begin{aligned}(t_0 - t_2)\int_M \frac{1}{2}|\mathrm{d}u|^2(\cdot,t_2)\Phi_{\mathrm{fml}}(\cdot,t_2)\varphi^2\mathrm{dvol}_g \leqslant \frac{e^{c\sqrt{t_0-t_1}}}{e^{c\sqrt{t_0-t_2}}}(t_0-t_1)\int_M\frac{1}{2}|\mathrm{d}u|^2(\cdot,t_1)\Phi_{\mathrm{fml}}(\cdot,t_1)\varphi^2\mathrm{dvol}_g \\ + cE_0\left(\sqrt{t_0-t_1} - \sqrt{t_0-t_2}\right)\end{aligned} \tag{4.2}$$

whenever $t_0 - \delta_0 < t_1 < t_2 < t_0$ and $\int_M \frac{1}{2}|\mathrm{d}u|^2\mathrm{dvol}_g \leqslant E_0$ on $]t_0-\delta_0,t_0[$, where c is a constant depending on the geometries of M and N, Φ_{fml} is the formal backward heat kernel concentrated at (x_0,t_0) (cf. Definition 1.8.3) and $\varphi \in C_0^\infty(M,[0,\infty[)$ is a cut-off function supported in $B_{\mathrm{inj}_{x_0}}(x_0)$.[1] An alternative adaptation of Struwe's formula to this setting has been given by Hamilton [33], taking the form

$$\begin{aligned}(t_0-t_2)\int_M\frac{1}{2}|\mathrm{d}u|^2(\cdot,t_2)\mathrm{P}_{(x_0,t_0)}(\cdot,t_2)\mathrm{dvol}_g \leqslant C_0(t_0-t_1)\int_M\frac{1}{2}|\mathrm{d}u|^2(\cdot,t_1)\mathrm{P}_{(x_0,t_0)}(\cdot,t_1)\mathrm{dvol}_g \\ + C_1(t_2-t_1)E_0\end{aligned} \tag{4.3}$$

whenever $t_0 - \min\{1,\delta_0\} < t_1 < t_2 < t_0$ and $\int_M \frac{1}{2}|\mathrm{d}u|^2\mathrm{dvol}_g \leqslant E_0$ on $]t_0-\delta_0,t_0[$, where C_0 and C_1 are constants depending on the geometry of M with $C_0 = 1$ and $C_1 = 0$ if $\sec_g \geqslant 0$ and $\mathrm{dRic} = 0$, and $\mathrm{P}_{(x_0,t_0)}$ is the canonical backward heat kernel concentrated at (x_0,t_0) (cf. Definition 1.8.1).

Now suppose that $P \to M$ is a principal bundle with semisimple structure group G and $(\omega_t)_{t\in]t_0-\delta_0,t_0[}$ is a one-parameter family of connections on P evolving by the Yang-Mills flow (cf. §1.11). It was shown by Chen and Shen [11] that if M is a static compact Riemannian manifold, that a monotonicity-type formula analogous to that of Chen and Struwe holds, namely

[1] Note that despite the introduction of a cut-off function, this formula is still nonlocal on account of the energy finiteness condition.

$$(t_0-t_2)^2\int_M \frac{1}{2}|\underline{\Omega}^\omega|^2(\cdot,t_2)\Phi_{\mathrm{fml}}(\cdot,t_2)\varphi^2\mathrm{dvol}_g \leqslant \frac{e^{c\sqrt{t_0-t_1}}}{e^{c\sqrt{t_0-t_2}}}(t_0-t_1)^2\int_M \frac{1}{2}|\underline{\Omega}^\omega|^2(\cdot,t_1)\Phi_{\mathrm{fml}}(\cdot,t_1)\varphi^2\mathrm{dvol}_g + cE_0\left(\frac{e^{c\sqrt{t_1-t_0}}}{e^{c\sqrt{t_2-t_0}}}-1\right) \quad (4.4)$$

whenever $t_0-\delta_0 < t_1 < t_2 < t_0$ and $\int_M \frac{1}{2}|\underline{\Omega}^\omega|^2\mathrm{dvol}_g \leqslant E_0$ on $]t_0-\delta_0,t_0[$, where c is a constant depending on the geometry of M, Φ_{fml} is the formal backward heat kernel concentrated at (x_0,t_0) and $\phi \in C_0^\infty(M,[0,\infty[)$ is a cut-off function supported in $B_{\mathrm{inj}_{x_0}}(x_0)$. Hamilton [33] has also established such a formula which takes the form

$$(t_0-t_2)^2\int_M \frac{1}{2}|\underline{\Omega}^\omega|^2(\cdot,t_2)\mathrm{P}_{(x_0,t_0)}(\cdot,t_2)\mathrm{dvol}_g \leqslant C_0(t_0-t_1)^2\int_M \frac{1}{2}|\underline{\Omega}^\omega|^2(\cdot,t_1)\mathrm{P}_{(x_0,t_0)}(\cdot,t_1)\mathrm{dvol}_g + C_1(t_2-t_1)E_0 \quad (4.5)$$

whenever $t_0-\min\{1,\delta_0\} < t_1 < t_2 < t_0$ and $\int_M \frac{1}{2}|\underline{\Omega}^\omega|^2\mathrm{dvol}_g \leqslant E_0$ on $]t_0-\delta_0,t_0[$, where C_0 and C_1 are constants depending on the geometry of M with $C_0 = 1$ and $C_1 = 0$ if $\sec_g \geqslant 0$ and $\mathrm{dRic} = 0$, and $\mathrm{P}_{(x_0,t_0)}$ is the canonical backward heat kernel concentrated at (x_0,t_0).

4.2. Monotonicity identities. We fix $\delta_0 > 0$, let $I =]t_0-\delta_0,t_0[\subset \mathbb{R}$ for some $t_0 \in \mathbb{R}$ and suppose $(M,(g_t)_{t\in I})$ is an evolving manifold with $\partial_t g = h$ and $x_0 \in M$ fixed. We first establish a general monotonicity identity for time-dependent sections of bundles over evolving Riemannian manifolds. The following theorem should be considered the differential form analogue of (1.8) of Remark 1.8.2 or Theorem 1.12.6 (cf. [19]).

Theorem 4.2.1. *Suppose (M,g_t) is complete for each $t \in I$, $f \in C^{2,1}(M\times I,\mathbb{R}^+)$ and $\psi \in \Gamma(E\otimes \Lambda^k T^*M)$ is a time-dependent section over $M\times I$. If*

$$\int_M \left\{\left(1+\left|\frac{\nabla f}{f}\right|+\left|\frac{\nabla f}{f}\right|^2+\left|\frac{\nabla^2 f}{f}\right|\right)|\psi|^2+\left(1+\left|\frac{\nabla f}{f}\right|^2\right)|\mathrm{d}^\nabla\psi|^2+|\delta^\nabla\psi|^2+|\Delta^\nabla\psi|^2\right\} f\mathrm{dvol}_g(\cdot,t) \quad (4.6)$$

is finite for each $t \in I$, then the identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M e_g(\psi)f\mathrm{dvol}_g\right) = \int_M \Big\{&\left\langle\psi,\left(\partial_t+\Delta^\nabla\right)\psi\right\rangle f\\ &+e_g(\psi)\cdot\left(\partial_t+\Delta+\frac{1}{2}tr_g h+\frac{k}{s-t}\right)f\\ &-f\cdot\left[|\mathrm{d}^\nabla\psi|^2+|\iota_{\frac{\nabla f}{f}}\psi-\delta^\nabla\psi|^2\right]\\ &-f\left\langle\nabla^2\log f+\frac{1}{2}h+\frac{1}{2(s-t)}g,\sum_{i,j}\left\langle\iota_{\varepsilon_i}\psi,\iota_{\varepsilon_j}\psi\right\rangle\omega^i\otimes\omega^j\right\rangle\Big\}\mathrm{dvol}_g\end{aligned} \quad (4.7)$$

holds on I for every $s \geqslant t_0$ whenever both integrands are summable over M.

Remark 4.2.2. If $k=0$, i.e. if $\psi \in \Gamma(E)$, then, since $\iota_v\psi \equiv \delta^\nabla\psi \equiv 0$ for any $v \in TM$, the identity (4.7) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2 f\mathrm{dvol}_g\right) = \int_M \left\langle\psi,(\partial_t+\Delta^\nabla)\psi\right\rangle f+\frac{1}{2}|\psi|^2\left(\partial_t+\Delta+\frac{1}{2}\mathrm{tr}_g h\right)f-f|\mathrm{d}^\nabla\psi|^2\mathrm{dvol}_g \quad (4.8)$$

so that, in this case, we obtain a monotonicity identity if ψ is a subsolution to the heat equation in the sense that $\left\langle\psi,\partial_t+\Delta^\nabla\psi\right\rangle \leqslant 0$ and f satisfies $\partial_t f+\Delta f+\frac{1}{2}\mathrm{tr}_g h \leqslant 0$ which holds if f is a solution to the backward heat equation.

Remark 4.2.3. More generally, this identity yields a monotonicity formula provided the following conditions are satisfied:

1. ψ is an appropriate subsolution of the heat equation in the sense that $\left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle \leqslant 0$. This holds with equality if ψ evolves by a Dirichlet-type flow.

2. $f = (s-t)^k\Phi$ where Φ is a subsolution to the backward heat equation, i.e. $(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h)\Phi \leqslant 0$, since then

$$
\begin{aligned}
&(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h + \frac{k}{s-t}) \left((s-t)^k \Phi \right) \\
&\qquad = (s-t)^k (\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h)\Phi - k(s-t)^{k-1}\Phi + k(s-t)^{k-1}\Phi \leqslant 0.
\end{aligned}
$$

This holds with equality if Φ solves the backward heat equation.

3. The matrix Harnack expression

$$
\nabla^2 \log f + \frac{1}{2}h + \frac{1}{2(s-t)}g = \mathcal{H}_s \log f
$$

is nonnegative-definite. To see this, note that letting $\{e_\alpha\}$ be a local frame for $E \otimes \Lambda^{k-1}T^*M$, we may write

$$
\begin{aligned}
\sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j &= \sum_{i,j}\sum_{\alpha} \langle \iota_{\varepsilon_i}\psi, e_\alpha \rangle \left\langle e_\alpha, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \\
&= \sum_{\alpha} \left(\sum_i \langle \iota_{\varepsilon_i}\psi, \varepsilon_\alpha \rangle \omega^i \right) \otimes \left(\sum_i \langle \iota_{\varepsilon_i}\psi, \varepsilon_\alpha \rangle \omega^i \right)
\end{aligned}
$$

so that, if $\mathcal{H}_s \log f \geqslant \lambda g$, we have

$$
\begin{aligned}
\left\langle \mathcal{H}_s \log f, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle &= \sum_\alpha \left\langle \mathcal{H}_s \log f, \left(\sum_i \langle \iota_{\varepsilon_i}\psi, \varepsilon_\alpha \rangle \omega^i \right) \otimes \left(\sum_i \langle \iota_{\varepsilon_i}\psi, \varepsilon_\alpha \rangle \omega^i \right) \right\rangle \\
&\geqslant \lambda \left\langle g, \left(\sum_i \langle \iota_{\varepsilon_i}\psi, \varepsilon_\alpha \rangle \omega^i \right) \otimes \left(\sum_i \langle \iota_{\varepsilon_i}\psi, \varepsilon_\alpha \rangle \omega^i \right) \right\rangle \\
&= \lambda \cdot k|\psi|^2.
\end{aligned}
$$

If g evolves by Ricci flow, i.e. $h = -2\mathrm{Ric}$, then this expression vanishes when g is a *gradient shrinking soliton* (cf. [45, Appendix C]). As may be verified directly, $(M, g_t) \equiv (\mathbb{R}^n, \delta)$ is a special case of this, where f is taken to be the Euclidean backward heat kernel concentrated at $(y, s) \in M \times I$ (cf. Remark 1.8.9).

Proof of Theorem 4.2.1. For convenience, we use the abbreviations

$$
(\partial_t + \Delta^\nabla) = \partial_t + \Delta^\nabla
$$

and

$$
H^* = \partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h + \frac{k}{s-t}.
$$

We first compute the derivative of the integrand:

$$
\partial_t \left(e_g(\psi) f \mathrm{dvol}_g \right) = \partial_t e_g(\psi) \cdot f \mathrm{dvol}_g + e_g(\psi)\partial_t f \cdot \mathrm{dvol}_g + e_g(\psi) f \partial_t \mathrm{dvol}_g
$$

$$
\begin{aligned}
&= \langle \partial_t \psi, \psi \rangle f \mathrm{dvol}_g - \frac{1}{2}\left\langle h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle f \mathrm{dvol}_g \\
&\quad + e_g(\psi)\left(\partial_t f + \frac{1}{2}\mathrm{tr}_g h \cdot f\right) \mathrm{dvol}_g \\
&= \left\langle (\partial_t + \Delta^\nabla)\psi, \psi \right\rangle f \mathrm{dvol}_g - \left\langle \Delta^\nabla \psi, \psi \right\rangle f \mathrm{dvol}_g \\
&\quad - \frac{1}{2}\left\langle h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle f \mathrm{dvol}_g \\
&\quad + e_g(\psi) H^* f \cdot \mathrm{dvol}_g - e_g(\psi)\Delta f \cdot \mathrm{dvol}_g
\end{aligned} \tag{4.9}
$$

where Propositions 1.7.7 and 1.7.8 were used in the second line and the definitions of the respective heat operators were used in line 3. We now 'integrate out' the lone second derivative terms, starting with the latter one. To that end, consider $Y := \left(\iota_{\nabla f} T^g_\psi\right)^\sharp \in \Gamma(TM)$. By Corollary 3.1.4,

$$
\begin{aligned}
e_g(\psi)\Delta f &= \sum_{i=1}^n \left\langle \iota_{\nabla_{\varepsilon_i}\nabla f}\psi, \iota_{\varepsilon_i}\psi \right\rangle - \left\langle \delta^\nabla \psi, \iota_{\nabla f}\psi \right\rangle - \left\langle \iota_{\nabla f} \mathrm{d}^\nabla \psi, \psi \right\rangle - \mathrm{div}\, Y \\
&= \sum_{i,j=1}^n \left\langle \nabla^2 f, \omega^i \otimes \omega^j \right\rangle \left\langle \iota_{\varepsilon_j}\psi, \iota_{\varepsilon_i}\psi \right\rangle - \left\langle \delta^\nabla \psi, \iota_{\nabla f}\psi \right\rangle - \left\langle \iota_{\nabla f} \mathrm{d}^\nabla \psi, \psi \right\rangle - \mathrm{div}\, Y \\
&= \left\langle \nabla^2 f, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle - \left\langle \delta^\nabla \psi, \iota_{\nabla f}\psi \right\rangle - \left\langle \iota_{\nabla f} \mathrm{d}^\nabla \psi, \psi \right\rangle - \mathrm{div}\, Y.
\end{aligned}
$$

On the other hand, by Lemma 1.6.8,

$$
\begin{aligned}
\left\langle \Delta^\nabla \psi, \psi \right\rangle f &= \left\langle \mathrm{d}^\nabla \delta^\nabla \psi, f\psi \right\rangle + \left\langle \delta^\nabla \mathrm{d}^\nabla \psi, f\psi \right\rangle \\
&= \left\langle \delta^\nabla \psi, \delta^\nabla (f\psi) \right\rangle + \mathrm{div}\left(f \sum_{i=1}^n \left\langle \iota_{\varepsilon_i}\psi, \delta^\nabla \psi \right\rangle \omega^i \right) + \left\langle \mathrm{d}^\nabla \psi, \mathrm{d}^\nabla (f\psi) \right\rangle - \mathrm{div}\left(f \sum_{i=1}^n \left\langle \iota_{\varepsilon_i} \mathrm{d}^\nabla \psi, \psi \right\rangle \omega^i \right) \\
&= f|\delta^\nabla \psi|^2 - \left\langle \delta^\nabla \psi, \iota_{\nabla f}\psi \right\rangle + f|\mathrm{d}^\nabla \psi|^2 + \left\langle \mathrm{d}^\nabla \psi, \mathrm{d}f \wedge \psi \right\rangle - \mathrm{div} \underbrace{\left(f \sum_{i=1}^n \left(\left\langle \iota_{\varepsilon_i} \mathrm{d}^\nabla \psi, \psi \right\rangle - \left\langle \iota_{\varepsilon_i}\psi, \delta^\nabla \psi \right\rangle \right) \omega^i \right)}_{:= Z^\flat} \\
&= f|\delta^\nabla \psi|^2 + f|\mathrm{d}^\nabla \psi|^2 - \left\langle \delta^\nabla \psi, \iota_{\nabla f}\psi \right\rangle + \left\langle \iota_{\nabla f} \mathrm{d}^\nabla \psi, \psi \right\rangle - \mathrm{div}\, Z
\end{aligned}
$$

so that the right-hand side of the identity in 4.9 takes the form

$$
\begin{aligned}
&= \left(\left\langle (\partial_t + \Delta^\nabla)\psi, \psi \right\rangle f + e_g(\psi) H^* f \right) \mathrm{dvol}_g + f\left(2\left\langle \delta^\nabla \psi, \iota_{\frac{\nabla f}{f}}\psi \right\rangle - |\delta^\nabla \psi|^2 - |\mathrm{d}^\nabla \psi|^2 \right) \mathrm{dvol}_g \\
&\quad - \left\langle \nabla^2 f + \frac{1}{2} f h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \mathrm{dvol}_g + \mathrm{div}(Y+Z)\mathrm{dvol}_g \\
&= \left(\left\langle (\partial_t + \Delta^\nabla)\psi, \psi \right\rangle f + e_g(\psi) H^* f \right) \mathrm{dvol}_g - f\left(|\iota_{\frac{\nabla f}{f}}\psi - \delta^\nabla \psi|^2 + |\mathrm{d}^\nabla \psi|^2 \right) \mathrm{dvol}_g + \frac{1}{f}|\iota_{\nabla f}\psi|^2 \mathrm{dvol}_g \\
&\quad - f\left\langle \frac{\nabla^2 f}{f} + \frac{1}{2} h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \mathrm{dvol}_g + \mathrm{div}(Y+Z)\mathrm{dvol}_g \\
&= \left(\left\langle (\partial_t + \Delta^\nabla)\psi, \psi \right\rangle f + e_g(\psi) H^* f \right) \mathrm{dvol}_g - f\left(|\iota_{\frac{\nabla f}{f}}\psi - \delta^\nabla \psi|^2 + |\mathrm{d}^\nabla \psi|^2 \right) \mathrm{dvol}_g \\
&\quad - f\left\langle \frac{\nabla^2 f}{f} - \frac{\nabla f \otimes \nabla f}{f^2} + \frac{1}{2} h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \mathrm{dvol}_g + \mathrm{div}(Y+Z)\mathrm{dvol}_g.
\end{aligned} \tag{4.10}
$$

Finally, we compute in an orthonormal frame that

$$
e_g(\psi) \cdot \frac{k}{s-t} = \frac{1}{2} \sum_{J^k} |(\psi, \varepsilon_J)|^2 \cdot \frac{k}{s-t}
$$

$$= \frac{1}{2} \cdot \frac{1}{k} \sum_{j=1}^{n} \sum_{\hat{j}k-1} |(\iota_{\varepsilon_j}\psi, \varepsilon_{\hat{j}})|^2 \cdot \frac{k}{s-t}$$

$$= \frac{1}{2(s-t)} \sum_{j=1}^{n} |\iota_{\varepsilon_j}\psi|^2 = \frac{1}{2(s-t)} \left\langle g, \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \tag{4.11}$$

where the final expression is valid in **any** local frame. Overall, we have the identity

$$\begin{aligned}
&\partial_t \left(e_g(\psi) f \mathrm{dvol}_g\right) \\
&= \left(\left\langle (\partial_t + \Delta^\nabla)\psi, \psi \right\rangle f + e_g(\psi) \left(H^* + \frac{k}{s-t} \right) f \right) \mathrm{dvol}_g - f \left(|\iota_{\frac{\nabla f}{f}}\psi - \delta^\nabla \psi|^2 + |\mathrm{d}^\nabla \psi|^2 \right) \mathrm{dvol}_g \\
&\qquad - f \left\langle \frac{\nabla^2 f}{f} - \frac{\nabla f \otimes \nabla f}{f^2} + \frac{1}{2} h + \frac{1}{2(s-t)} g, \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \mathrm{dvol}_g \\
&\qquad\qquad + \mathrm{div}(Y+Z) \mathrm{dvol}_g.
\end{aligned}$$

Now, if M is compact, we may simply integrate, noting that the integral and t-derivative may be interchanged by standard integration theorems and that the divergences integrate to 0 by Gauß' theorem. For complete M, the interchanging of the derivative and integral is still valid, but Gauß' theorem is only guaranteed to hold provided $|Y+Z|(\cdot,t)$ and $\mathrm{div}(Y+Z)(\cdot,t) \in L^1(M)$ for fixed $t \in I$ [28], but both of these are summable. To see this, we estimate Y, $\mathrm{div}\, Y$, Z and $\mathrm{div}\, Z$ from above. Firstly, note that by Corollary 3.1.4.

$$\begin{aligned}
|Y| &= \left| \sum_{i=1}^{n} f \left\langle \iota_{\frac{\nabla f}{f}}\psi, \iota_{\varepsilon_i}\psi \right\rangle \varepsilon_i - e_g(\psi) \nabla f \right| \\
&\leqslant f \sqrt{\sum_i \left\langle \iota_{\frac{\nabla f}{f}}\psi, \iota_{\varepsilon_i}\psi \right\rangle^2} + e_g(\psi)|\nabla f| \\
&\leqslant f \underbrace{|\iota_{\frac{\nabla f}{f}}\psi|}_{\leqslant \left|\frac{\nabla f}{f}\right| \cdot |\psi|} \underbrace{\sqrt{\sum_i |\iota_{\varepsilon_i}\psi|^2}}_{=\sqrt{k}|\psi|} + \frac{1}{2}|\psi|^2 |\nabla f| \\
&\leqslant \frac{1}{2} f \left| \frac{\nabla f}{f} \right|^2 |\psi|^2 + \frac{k}{2}|\psi|^2 f + \frac{1}{2}|\psi|^2 |\nabla f|
\end{aligned}$$

where the first inequality follows from the triangle inequality and the fact that the $\{\varepsilon_i\}$ are orthonormal, the second from the Cauchy-Schwarz inequality and the third from Young's inequality applied to the terms with underbraces. Secondly, using the same techniques,

$$\begin{aligned}
|\mathrm{div}\, Y| &= \left| \left\langle \nabla^2 f, \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle - \frac{1}{2}|\psi|^2 \Delta f - \left\langle \delta^\nabla \psi, \iota_{\nabla f}\psi \right\rangle - \left\langle \iota_{\nabla f} \mathrm{d}^\nabla \psi, \psi \right\rangle \right| \\
&\leqslant |\nabla^2 f| \left| \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right| + \frac{1}{2}|\psi|^2 |\Delta f| + f|\delta^\nabla \psi| \cdot |\iota_{\frac{\nabla f}{f}}\psi| + f|\psi| \cdot |\iota_{\frac{\nabla f}{f}} \mathrm{d}^\nabla \psi| \\
&\leqslant k|\nabla^2 f| \cdot |\psi|^2 + \frac{1}{2} \left(|\psi|^2 |\Delta f| + |\delta^\nabla \psi|^2 f + \left| \frac{\nabla f}{f} \right|^2 |\psi|^2 f + |\psi|^2 f + \left| \frac{\nabla f}{f} \right|^2 |\mathrm{d}^\nabla \psi|^2 f \right),
\end{aligned}$$

where the last line follows from the fact that

$$\begin{aligned}
\left| \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right| &= \sqrt{\sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle^2} \\
&\leqslant \sqrt{\sum_{i,j} |\iota_{\varepsilon_i}\psi|^2 |\iota_{\varepsilon_j}\psi|^2}
\end{aligned}$$

$$= \sum_i |\iota_{\varepsilon_i}\psi|^2 = k|\psi|^2.$$

Finally, we compute, again using the same techniques, that

$$\begin{aligned}
|Z| &= \left| f \sum_{i=1}^{n} \left(\left\langle \iota_{\varepsilon_i} \mathrm{d}^\nabla \psi, \psi \right\rangle - \left\langle \iota_{\varepsilon_i} \psi, \delta^\nabla \psi \right\rangle \right) \varepsilon_i \right| \\
&\leqslant f \left(\left| \sum_{i=1}^{n} \left\langle \iota_{\varepsilon_i} \mathrm{d}^\nabla \psi, \psi \right\rangle \varepsilon_i \right| + \left| \sum_{i=1}^{n} \left\langle \iota_{\varepsilon_i} \psi, \delta^\nabla \psi \right\rangle \varepsilon_i \right| \right) \\
&= f \left(\sqrt{\sum_{i=1}^{n} \left\langle \iota_{\varepsilon_i} \mathrm{d}^\nabla \psi, \psi \right\rangle^2} + \sqrt{\sum_{i=1}^{n} \langle \iota_{\varepsilon_i} \psi, \delta^\nabla \psi \rangle^2} \right) \\
&\leqslant f \left(|\psi| \sqrt{\sum_{i=1}^{n} |\iota_{\varepsilon_i} \mathrm{d}^\nabla \psi|^2} + |\delta^\nabla \psi| \sqrt{\sum_{i=1}^{n} |\iota_{\varepsilon_i} \psi|^2} \right) \\
&\leqslant \frac{f}{2} \left((1+k)|\psi|^2 + (k+1)|\mathrm{d}^\nabla \psi|^2 + |\delta^\nabla \psi|^2 \right)
\end{aligned}$$

and

$$\begin{aligned}
|\mathrm{div}\, Z| &= \left| \left\langle \iota_{\nabla f} \mathrm{d}^\nabla \psi, \psi \right\rangle - \left\langle \iota_{\nabla f} \psi, \delta^\nabla \psi \right\rangle - f \left\langle \Delta^\nabla \psi, \psi \right\rangle + f |\mathrm{d}^\nabla \psi|^2 + f |\delta^\nabla \psi|^2 \right| \\
&\leqslant f \left(|\iota_{\frac{\nabla f}{f}} \mathrm{d}^\nabla \psi| \cdot |\psi| + |\iota_{\frac{\nabla f}{f}} \psi| \cdot |\delta^\nabla \psi| + |\Delta^\nabla \psi| \cdot |\psi| + |\mathrm{d}^\nabla \psi|^2 + |\delta^\nabla \psi|^2 \right) \\
&\leqslant f \left(\frac{1}{2} \left| \frac{\nabla f}{f} \right|^2 \left(|\mathrm{d}^\nabla \psi|^2 + |\psi|^2 \right) + \frac{3}{2} |\delta^\nabla \psi|^2 + |\psi|^2 + \frac{1}{2} |\Delta^\nabla \psi|^2 + |\mathrm{d}^\nabla \psi|^2 \right).
\end{aligned}$$

By the finiteness of the integral (4.6), all of these expressions are summable, thus allowing us to justify the application of Gauß' theorem. □

Corollary 4.2.4. *Suppose (M, g_t) is complete for each $t \in I$ and $f \in C^{2,1}(M \times I, \mathbb{R}^+)$. Then the following hold:*

(i) Assume the setup of §1.11. If $(\omega_t = \widetilde{\omega} + a(t))_{t \in I}$ is a one-parameter family of connections evolving by the Yang-Mills heat flow and

$$\int_M \left\{ \left(1 + \left| \frac{\nabla f}{f} \right| + \left| \frac{\nabla f}{f} \right|^2 + \left| \frac{\nabla^2 f}{f} \right| \right) |\underline{\Omega}^\omega|^2 + |\delta^\nabla \underline{\Omega}^\omega|^2 + |\partial_t \underline{\Omega}^\omega|^2 \right\} f \mathrm{dvol}_g(\cdot, t)$$

is finite for each $t \in I$, then the identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_M e_g(\underline{\Omega}^\omega) f \mathrm{dvol}_g \right) \\
&= \int_M \Bigg\{ e_g(\underline{\Omega}^\omega) \cdot \left(\partial_t + \Delta + \frac{1}{2} tr_g h + \frac{4}{2(s-t)} \right) f \\
&\qquad - f |\partial_t a + \iota_{\frac{\nabla f}{f}} \underline{\Omega}^\omega|^2 \\
&\qquad\quad - f \left\langle \nabla^2 \log f + \frac{1}{2} h - \frac{1}{2(s-t)} g, \sum_{i,j} \left\langle \iota_{\varepsilon_i} \underline{\Omega}^\omega, \iota_{\varepsilon_j} \underline{\Omega}^\omega \right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg\} \mathrm{dvol}_g
\end{aligned}$$

holds on I whenever both integrands are in $L^1(M)$.

(ii) *Assume the setup of §1.10. If $u : M \times I \to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow, where N is isometrically embedded in $\mathbb{R}^K$, and*

$$\int_M \left\{ \left(1 + \left|\frac{\nabla f}{f}\right| + \left|\frac{\nabla f}{f}\right|^2 + \left|\frac{\nabla^2 f}{f}\right|\right) |\mathrm{d}u|^2 + |\delta^\nabla \mathrm{d}u|^2 + |\partial_t u|^2 + |\partial_t \mathrm{d}u|^2 \right\} f \mathrm{dvol}_g(\cdot, t) \tag{4.12}$$

is finite for each $t \in I$, then the identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M e_g(\mathrm{d}u) f \mathrm{dvol}_g\right) \\
&= \int_M \left\{ e_g(\mathrm{d}u) \cdot \left(\partial_t + \Delta + \frac{1}{2} tr_g h + \frac{2}{2(s-t)}\right) f \right. \\
&\qquad - f|\partial_t u + \iota_{\frac{\nabla f}{f}} \mathrm{d}u|^2 \\
&\qquad\qquad \left. -f \left\langle \nabla^2 \log f + \frac{1}{2} h - \frac{1}{2(s-t)} g, \sum_{i,j} \left\langle \iota_{\varepsilon_i} \mathrm{d}u, \iota_{\varepsilon_j} \mathrm{d}u \right\rangle \omega^i \otimes \omega^j \right\rangle \right\} \mathrm{dvol}_g
\end{aligned}$$

holds on I whenever both integrands are in $L^1(M)$.

Proof. (i) This follows immediately from Theorem 4.2.1 by taking $E = P \times_{\mathrm{Ad}} \mathfrak{g}$, ∇ the covariant derivative induced by ω, $\psi = \underline{\Omega}^\omega$ ($\Rightarrow k = 2$), using Lemma 1.11.4 ($\partial_t \underline{\Omega}^\omega + \Delta^\nabla \underline{\Omega}^\omega = 0$) and the Bianchi identity $\mathrm{d}^\nabla \underline{\Omega}^\omega = 0$ (cf. Proposition 1.5.7), keeping Remark 4.2.3 in mind.

(ii) Take $E = \mathbb{R}^K$, ∇ the flat connection and $\psi = \mathrm{d}u$ ($\Rightarrow k = 1$). Note that by Lemma 1.10.3 (ii) with $X = \frac{\nabla f}{f}$, there holds

$$\left\langle \left(\partial_t + \Delta^\nabla\right) \mathrm{d}u, \mathrm{d}u \right\rangle - \left| \iota_{\frac{\nabla f}{f}} \mathrm{d}u - \delta^\nabla \mathrm{d}u \right|^2 = - \left| \partial_t u + \iota_{\frac{\nabla f}{f}} \mathrm{d}u \right|^2 .$$

Furthermore, by Lemma 1.10.3 (i), $\mathrm{d}^\nabla \mathrm{d}u = 0$. Now, theorem 4.2.1 would then imply the claim if the expression (4.6) were finite. However, the condition (4.12) is not quite the same, since it does not say anything about the summability of $|\Delta^\nabla \mathrm{d}u|^2 f$. We thus go back to the proof of Theorem 4.2.1. All of the steps there are clearly valid in this setting up to the claim that $|\mathrm{div}\, Z|$ may be bounded from above by a summable expression. This is still the case, however, except we must estimate $|\mathrm{div}\, Z|$ from above slightly differently. Note first that, by Lemma 1.10.3 (ii) with $X = 0$ that

$$\left\langle \Delta^\nabla \mathrm{d}u, \mathrm{d}u \right\rangle = - \langle \partial_t \mathrm{d}u, \mathrm{d}u \rangle + |\delta^\nabla \mathrm{d}u|^2 - |\partial_t u|^2,$$

whence we estimate analogously to before that

$$\begin{aligned}
|\mathrm{div}\, Z| &= \left| - \left\langle \iota_{\nabla f} \mathrm{d}u, \delta^\nabla \mathrm{d}u \right\rangle - f \left\langle \Delta^\nabla \mathrm{d}u, \mathrm{d}u \right\rangle + f |\delta^\nabla \mathrm{d}u|^2 \right| \\
&= \left| \left\langle \iota_{\nabla f} \mathrm{d}u, \delta^\nabla \mathrm{d}u \right\rangle + f \langle \partial_t \mathrm{d}u, \mathrm{d}u \rangle - f |\delta^\nabla \mathrm{d}u|^2 + f |\partial_t u|^2 + f |\delta^\nabla \mathrm{d}u|^2 \right| \\
&= \left| \left\langle \iota_{\nabla f} \mathrm{d}u, \delta^\nabla \mathrm{d}u \right\rangle + f \langle \partial_t \mathrm{d}u, \mathrm{d}u \rangle + f |\partial_t u|^2 \right| \\
&\leqslant f \left(\frac{1}{2} \left(1 + \left|\frac{\nabla f}{f}\right|^2\right) |\mathrm{d}u|^2 + \frac{1}{2} |\delta^\nabla \mathrm{d}u|^2 + \frac{1}{2} |\partial_t \mathrm{d}u|^2 + |\partial_t u|^2 \right),
\end{aligned}$$

which is now summable. Proceeding through the rest of the proof of Theorem 4.2.1 then establishes the claim. □

Remark 4.2.5. Taking $M = \mathbb{R}^n$ and f to be $(s-t)^k\Phi$ with Φ the Euclidean backward heat kernel concentrated at $(y,s) \in \mathbb{R}^n \times I$, where $k = 1$ for solutions to (HMHF) (cf. §1.10) and $k = 2$ for solutions to (YMHF) (cf. §1.11), we see that this corollary and Remark 4.2.3 immediately imply Struwe's monotonicity formula (4.1) and a Euclidean analogue of Chen-Shen's formula (4.4).

If either f or ψ is not compactly supported in spacetime, then we can force a *local* monotonicity identity out of the above integrand by introducing a cut-off, as is done e.g. in [11, 19, 39, 12].

Theorem 4.2.6. *Let $\varphi \in C^{2,1}(M \times I, \mathbb{R})$ and $\varphi(\cdot,t) \in C^2_0(M)$ for each $t \in I$, $f \in C^2(\mathcal{D}, \mathbb{R}^+)$ and $\psi \in \Gamma(E \otimes \Lambda^k T^*M)$ a smooth time-dependent section over $\mathcal{D} \subset M \times I$ open with* $\operatorname{supp} \varphi(\cdot,t) \Subset \operatorname{pr}_1(\mathcal{D} \cap (M \times \{t\}))$ *for each $t \in I$. Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M e_g(\psi) f\varphi^2 \mathrm{dvol}_g\right) = \int_M \Bigg\{ & \left[\left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle f + e_g(\psi) \cdot \left(\partial_t + \Delta + \frac{1}{2} tr_g h + \frac{k}{s-t}\right) f\right] \\ & - f \cdot \left[|\mathrm{d}^\nabla \psi|^2 + |\iota_{\frac{\nabla f}{f}} \psi - \delta^\nabla \psi|^2 \right] \\ & - f \left\langle \nabla^2 \log f + \frac{1}{2} h + \frac{1}{2(s-t)} g, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg\} \varphi^2 \\ & + 2 e_g(\psi) f \varphi \left(\partial_t \varphi + \left\langle \nabla\varphi, \frac{\nabla f}{f} \right\rangle \right) \\ & + 2\varphi f \left[\left\langle \iota_{\nabla\varphi}\psi, \delta^\nabla \psi - \iota_{\frac{\nabla f}{f}}\psi \right\rangle - \left\langle \iota_{\nabla\varphi} \mathrm{d}^\nabla \psi, \psi \right\rangle \right] \mathrm{dvol}_g. \qquad (4.13)\end{aligned}$$

on I for $s \geqslant t_0$.

Proof. We first note that

$$\partial_t \left(e_g(\psi) f \varphi^2 \mathrm{dvol}_g \right) = 2 e_g(\psi) f \varphi \partial_t \varphi \mathrm{dvol}_g + \partial_t \left(e_g(\psi) f \mathrm{dvol}_g \right) \varphi^2. \qquad (4.14)$$

Retaining the notation used in the proof of Theorem 4.2.1 and using (4.10) and (4.11), we see that the latter term may be written as

$$\begin{aligned}\Bigg\{ \left[\left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle f + e_g(\psi) \cdot \left(H^* + \frac{k}{s-t} \right) f \right] - f \cdot \left[|\mathrm{d}^\nabla \psi|^2 + |\iota_{\frac{\nabla f}{f}} \psi - \delta^\nabla \psi|^2 \right] \\ - f \left\langle \nabla^2 \log f + \frac{1}{2} h - \frac{1}{2(s-t)} g, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg\} \varphi^2 \mathrm{dvol}_g \\ + \operatorname{div}(Y+Z)\varphi^2 \mathrm{dvol}_g,\end{aligned}$$

but $\operatorname{div}(Y+Z)\varphi^2 = \operatorname{div}(\varphi^2(Y+Z)) - 2\varphi \langle \nabla\varphi, Y+Z \rangle$ and

$$\begin{aligned}\langle \nabla\varphi, Y+Z \rangle &= \left\langle \iota_{\nabla f}\psi, \iota_{\nabla\varphi}\psi \right\rangle - e_g(\psi) \langle \nabla f, \nabla\varphi \rangle + f \left\langle \iota_{\nabla\varphi} \mathrm{d}^\nabla\psi, \psi \right\rangle - f \left\langle \iota_{\nabla\varphi}\psi, \delta^\nabla\psi \right\rangle \\ &= -e_g(\psi) \langle \nabla f, \nabla\varphi \rangle - f \left\langle \iota_{\nabla\varphi}\psi, \delta^\nabla\psi - \iota_{\frac{\nabla f}{f}}\psi \right\rangle + f \left\langle \iota_{\nabla\varphi}\mathrm{d}^\nabla\psi, \psi \right\rangle.\end{aligned}$$

The result then follows from integrating (4.14) after substituting these expressions in and applying Gauß' theorem. □

Let $\mathcal{S}_\psi = \delta^\nabla\psi - \iota_{\frac{\nabla f}{f}}\psi$ and note that the last two terms in the integrand of (4.13) may be estimated as follows:

$$\begin{aligned}& 2e_g(\psi) f\varphi \left(\partial_t\varphi + \left\langle \nabla\varphi, \frac{\nabla f}{f} \right\rangle \right) + 2\varphi f \left(\left\langle \iota_{\nabla\varphi}\psi, \mathcal{S}_\psi \right\rangle - \left\langle \iota_{\nabla\varphi}\mathrm{d}^\nabla\psi, \psi \right\rangle \right) \\ & \leqslant 2e_g(\psi) f\varphi \left(|\partial_t\varphi| + |\nabla\varphi| \cdot \left| \frac{\nabla f}{f} \right| \right) + 2f|\nabla\varphi| \cdot |\psi| \left(|\mathcal{S}_\psi|\varphi + |\mathrm{d}^\nabla\psi|\varphi \right)\end{aligned}$$

$$\leqslant 2e_g(\psi)\varphi\left(|\partial_t\varphi|f + |\nabla\varphi|\cdot|\nabla f|\right) + f\cdot\left(4|\nabla\varphi|^2|\psi|^2 + \frac{1}{2}|\mathcal{S}_\psi|^2\varphi^2 + \frac{1}{2}|\mathrm{d}^\nabla\psi|^2\varphi^2\right)$$
$$\leqslant 2e_g(\psi)\left(\varphi\left(|\partial_t\varphi|f + |\nabla\varphi|\cdot|\nabla f|\right) + 4|\nabla\varphi|^2 f\right) + \frac{1}{2}\left(|\mathcal{S}_\psi|^2 + |\mathrm{d}^\nabla\psi|^2\right)f\varphi^2, \tag{4.15}$$

where the inner products were separated out using the Cauchy-Schwarz inequality and the products treated by Young's inequality. This observation immediately leads to the following monotonicity-type identity:

Lemma 4.2.7. *Let $\varphi \in C_0^{2,1}(M\times I,[0,1])$ be such that*

$$\varphi|_{\mathcal{D}_{r_1,\delta_0}(x_0,t_0)} \equiv \text{const and } \varphi|_{(M\times I)\setminus\mathcal{D}_{r_2,\delta_0}(x_0,t_0)} \equiv 0$$

for $0 < r_1 < r_2 < R$ with $R > 0$ fixed and suppose $f \in C^2(\mathcal{D}_{R,\delta_0}(x_0,t_0),\mathbb{R}^+)$ and $\psi \in \Gamma(E\otimes\Lambda^k T^*M)$ is a smooth time-dependent section over $M\times I$.

If f, $|\nabla f|$ are bounded on $\mathcal{D}_{r_2,\delta_0}(x_0,t_0)\setminus\mathcal{D}_{r_1,\delta_0}(x_0,t_0)$ and the inequalities

$$\left(\partial_t + \Delta + \frac{1}{2}tr_g h + \frac{k}{s-t}\right)f \leqslant a_0 + a_1(t)f \text{ and}$$
$$f\left(\nabla^2\log f + \frac{1}{2}h + \frac{g}{2(t_0-t)}\right) \geqslant (b_0 + b_1(t)f)\,g \tag{4.16}$$

hold on $\mathcal{D}_{r_2,\delta_0}(x_0,t_0)$ for $a_1, b_1 \in C(I)\cap L^1(I)$, $a_0, b_0 \in \mathbb{R}$ and some $s \geqslant t_0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\psi)f\varphi^2\mathrm{dvol}_g\right)$$
$$\leqslant \exp\left(\int_t^{t_0} l\right)\int_M \left\langle\psi,(\partial_t+\Delta^\nabla)\psi\right\rangle f\varphi^2 - \frac{1}{2}f\varphi^2\cdot\left(|\mathrm{d}^\nabla\psi|^2 + \left|\delta^\nabla\psi - \iota_{\frac{\nabla f}{f}}\psi\right|^2\right)\mathrm{dvol}_g$$
$$+ C_0\int_{B^t_{r_2}(x_0)} e_g(\psi)\mathrm{dvol}_g,$$

where $l(t) = a_1(t) - 2kb_1(t)$ and $C_0 = C_0(l,a_0,b_0,f,\varphi,r_1,r_2) > 0$. In particular, if

$$\int_M e_g(\psi)(\cdot,t)\mathrm{dvol}_{g_t} \leqslant E_0.$$

for every $t \in I$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\psi)f\varphi^2\mathrm{dvol}_g + C_0E_0(t_0-t)\right)$$
$$\leqslant \exp\left(\int_t^{t_0} l\right)\int_M \left\langle\psi,(\partial_t+\Delta^\nabla)\psi\right\rangle f\varphi^2 - \frac{1}{2}f\varphi^2\cdot\left(|\mathrm{d}^\nabla\psi|^2 + \left|\delta^\nabla\psi - \iota_{\frac{\nabla f}{f}}\psi\right|^2\right)\mathrm{dvol}_g.$$

Proof. By (4.15), the last two terms in the right-hand integrand of (4.13) may be bounded from above by

$$C_{f,\varphi}e_g(\psi)\chi_{B_{r_2}(x_0)} + \frac{1}{2}\left(|\mathcal{S}_\psi|^2 + |\mathrm{d}^\nabla\psi|^2\right)f\varphi^2,$$

where C_{f,φ,r_1,r_2} is a constant satisfying

$$2\left|\varphi\left(|\partial_t\varphi|f + |\nabla\varphi|\cdot|\nabla f|\right) + 4|\nabla\varphi|^2 f\right| \leqslant C_{f,\varphi,r_1,r_2}\cdot\chi_{\mathcal{D}_{r_2,\delta_0}(x_0,t_0)\setminus\mathcal{D}_{r_1,\delta_0}(x_0,t_0)}; \tag{4.17}$$

such a constant exists in light of hypotheses on f and the fact that $\nabla\varphi$ and $\partial_t\varphi$ are supported in $\mathcal{D}_{r_2,\delta_0}(x_0,t_0)\setminus\mathcal{D}_{r_1,\delta_0}(x_0,t_0)$. With this and the bounds (4.16) in mind, (4.13) implies that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M e_g(\psi) f\varphi^2 \mathrm{dvol}_g\right)\\
&\leqslant \int_M \left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle f\varphi^2 - \frac{1}{2} f\varphi^2 \cdot \left[|\mathrm{d}^\nabla \psi|^2 + |\mathcal{S}_\psi|^2\right] \mathrm{dvol}_g\\
&\qquad + \underbrace{(a_1(t) - 2kb_1(t))}_{=l(t)} \int_M e_g(\psi) f\varphi^2 \mathrm{dvol}_g\\
&\qquad + \left((a_0 - 2kb_0) + C_{f,\varphi,r_1,r_2}\right) \int_{B_{r_2}(x_0)} e_g(\psi)\mathrm{dvol}_g.
\end{aligned}$$

Hence, we immediately see that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right) \int_M e_g(\psi) f\varphi^2 \mathrm{dvol}_g\right)\\
&\leqslant \exp\left(\int_t^{t_0} l\right) \int_M \left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle f - \frac{1}{2} f \cdot \left[|\mathrm{d}^\nabla \psi|^2 + \left|\mathcal{S}_\psi\right|^2\right] \mathrm{dvol}_g\\
&\quad + \underbrace{\left|a_0 - 2kb_0 + C_{f,\varphi,r_1,r_2}\right| \exp \underbrace{\left(\int_t^{t_0} l\right)}_{\leqslant \int_{t_0-\delta}^{t_0} |l|}}_{=:C_0} \int_{B_{r_2}(x_0)} e_g(\psi)\mathrm{dvol}_g, \qquad (4.18)
\end{aligned}$$

whence the result follows. □

4.3. Applications. Assume now that M is of locally bounded geometry about (x_0, t_0) with notation as in Definition 1.7.4 and j_0 as before.

In this section, we let $\varphi : M \times I \to \mathbb{R}^+$ be defined by

$$\varphi(x,t) = \chi\left(\frac{d^t(x,x_0) + r - 2j_0}{2(r - j_0)}\right)$$

with $0 < r < j_0$ and χ as in Example A.1. Note that if $d^t(x,x_0) \leqslant r$ then

$$\frac{d^t(x,x_0) + r - 2j_0}{2(r - j_0)} \geqslant \frac{2r - 2j_0}{2(r - j_0)} = 1$$

so that $\varphi(x,t) = 1$. Similarly, if $d^t(x,x_0) \geqslant j_0$, then $\varphi(x,t) = 0$. Altogether, this implies that $\varphi|_{\mathcal{D}_{r,\delta}(x_0,t_0)} \equiv 1$ and $\varphi|_{(M\times I)\backslash\mathcal{D}_{j_0,\delta}(x_0,t_0)} \equiv 0$. We now use this cut-off function and a suitably weighted version of the formal heat kernel to deduce analogues of the monotonicity formulæ of Chen and Struwe [12] and Chen and Shen [11] from Lemma 4.2.7 in our evolving manifold setting. We first introduce this weighted formal backward heat kernel.

Lemma 4.3.1. *Suppose $k \in \mathbb{N}$ and define $f_k : \mathcal{D}_{j_0,\delta}(x_0,t_0) \to \mathbb{R}^+$ such that*

$$f_k(x,t) = (4\pi(t_0 - t))^k\, \Phi_{\mathrm{fml}}(x,t)$$

with Φ_{fml} as in Definition 1.8.3. Then

$$|f_k| + |\nabla f_k| \leqslant \widetilde{C}$$

on $\mathcal{D}_{j_0,\delta}(x_0,t_0)\backslash\mathcal{D}_{r,\delta}(x_0,t_0)$ with $\widetilde{C}$ depending on n, k, r and j_0, and f_k satisfies the inequalities (4.16) with[2]

[2] The constants C and C_4 are those of Propositions 1.8.7 and 1.8.8 and depend only on the local geometry about (x_0, t_0).

$$\begin{aligned} a_0 &= \max\{\frac{n\mu}{2}, 4C_4\}, \\ a_1(t) &= -\frac{n-2k}{2} a_0 \log(4\pi(t_0 - t)), \\ b_0 &= -\max\{2C, -\lambda_{-\infty}\}, \\ b_1(t) &= \frac{n-2k}{2} b_0 \log(4\pi(t_0 - t)), \end{aligned}$$

and $a_1, b_1 \in C(]t_0 - \delta, t_0[) \cap L^1(]t_0 - \delta, t_0[)$. *Moreover, retaining the notation of Lemma 4.2.7,*

$$\int_t^{t_0} l = \left[\frac{2k-n}{2} a_0 + k(2k-n) b_0\right] \cdot \left(\log(4\pi) \cdot (t_0 - t) + (t_0 - t)\left(\log(t_0 - t) - 1\right)\right),$$

and

$$C_0 \leqslant \left(|a_0 - 2kb_0| + \frac{\widetilde{C}||\chi'||_\infty}{r - j_0} \cdot \left(\frac{\mu}{2} j_0 + 1 + \frac{2||\chi'||_\infty}{r - j_0}\right)\right) \exp(||l||_1).$$

Proof. We first establish boundedness of f_k and ∇f_k on $\mathcal{D} := \mathcal{D}_{j_0,\delta}(x_0, t_0) \backslash \mathcal{D}_{r,\delta}(x_0, t_0)$. To streamling this, we introduce the auxiliary function

$$\eta_m(x,t) := \frac{\exp\left(\frac{d^t(x,x_0)^2}{4(t-t_0)}\right)}{(t_0 - t)^m},$$

where $m \in \mathbb{R}$ is fixed. Note that

$$f_k(x,t) = \frac{\eta_{\frac{n-2k}{2}}(x,t)}{(4\pi)^{\frac{n-2k}{2}}}$$

and, by Proposition 1.8.8,

$$|\nabla f_k|(x,t) = \frac{d^t(x,x_0)}{2 \cdot (4\pi)^{\frac{n-2k}{2}}} \cdot \eta_{\frac{n+2-2k}{2}}(x,t).$$

Thus, since $r < d^t(x,x_0) < j_0$ for $(x,t) \in \mathcal{D}$, it suffices to show that

$$\sup_{\mathcal{D}} \eta_m(x,t) < \infty$$

for $m \in \mathbb{R}$. Now, it is clear that for $(x,t) \in \mathcal{D}$,

$$\eta_m(x,t) \leqslant \frac{\exp\left(\frac{r^2}{4(t-t_0)}\right)}{(t_0 - t)^m}$$

and that the right-hand side is bounded from above provided $t \in]t_0 - \delta, t_0 - \varepsilon[$ for some $\varepsilon \in]0, \delta[$. On the other hand,

$$\lim_{t \nearrow t_0} \frac{\exp\left(\frac{r^2}{4(t-t_0)}\right)}{(t_0 - t)^m} = \frac{4^m}{r^{2m}} \lim_{\tilde{t} \to \infty} \tilde{t}^m \exp(-\tilde{t}) = 0,$$

where the substitution $\tilde{t} = \frac{r^2}{4(t-t_0)}$ was made. This establishes boundedness of η_m on $\mathcal{D}$ for each $m \in \mathbb{R}$ and thus boundedness of f_k and $|\nabla f_k|$ on $\mathcal{D}$.

We now turn our attention to the verification of the inequalities (4.16). Firstly,

$$\begin{aligned}\left(H^* + \frac{k}{t_0 - t}\right) f_k &= (4\pi(t_0-t))^k H^*\Phi_{\text{fml}} + k(4\pi)^k(t_0-t)^{k-1}\Phi_{\text{fml}} - k(4\pi)^k(t_0-t)^{k-1}\Phi_{\text{fml}} \\ &= (t_0-t)^k H^*\Phi_{\text{fml}}\end{aligned}$$

which, together with Proposition 1.8.7, implies that

$$\begin{aligned}\left(H^* + \frac{k}{t_0 - t}\right) f_k \leqslant \left(\frac{n\mu}{2} + \frac{C_4 r^2}{t_0-t}\right) f_k &= \left(\frac{n\mu}{2} + 4C_4 \log\left(\frac{1}{(4\pi(t_0-t))^{(n-2k)/2} f_k}\right)\right) f_k \\ &\leqslant \max\{\frac{n\mu}{2}, 4C_4\}\left(1 + \log\left(\frac{1}{(4\pi(t_0-t))^{(n-2k)/2} f_k}\right)\right) f_k \\ &\leqslant \max\{\frac{n\mu}{2}, 4C_4\}\left(1 + f_k \log\left(\frac{1}{(4\pi(t_0-t))^{(n-2k)/2}}\right)\right)\end{aligned}$$

where we have used Lemma A.3 in the last step. On the other hand, by Proposition 1.8.8,

$$\begin{aligned} f_k \mathcal{H}_{t_0}(\log f_k) &\geqslant -\max\{2C, -\lambda_{-\infty}\} \cdot f_k \left(1 + \frac{r^2}{4(t_0-t)}\right) g \\ &= -\max\{2C, -\lambda_{-\infty}\} \cdot f_k \left(1 + \log\left(\frac{1}{(4\pi(t_0-t))^{(n-2k)/2} f_k}\right)\right) g \\ &\geqslant -\max\{2C, -\lambda_{-\infty}\} \cdot \left(1 + f_k \log\left(\frac{1}{(4\pi(t_0-t))^{(n-2k)/2}}\right)\right) g,\end{aligned}$$

where we have again used Lemma A.3 in the final step. In either case, $t \mapsto \log(t_0 - t)$ is both continuous and summable on $]t_0 - \delta, t_0[$. Hence,

$$\begin{aligned}\int_t^{t_0} l &= \int_t^{t_0} a_1 - 2kb_1 \\ &= \left[\frac{2k-n}{2} a_0 + k(2k-n) b_0\right] \int_t^{t_0} \log\left(4\pi(t_0-u)\right) \mathrm{d}u \\ &= \left[\frac{2k-n}{2} a_0 + k(2k-n) b_0\right] \cdot \left(\log(4\pi)(t_0-t) + (t_0-t)\left(\log(t_0-t) - 1\right)\right).\end{aligned}$$

Finally, we estimate C_0 as follows: we note that on $\mathcal{D}$, the inequalities

$$|\nabla\varphi| \leqslant \frac{||\chi'||_\infty}{2(r-j_0)}$$

and

$$|\partial_t\varphi| \leqslant \frac{||\chi'||_\infty}{2(r-j_0)} \underbrace{|\partial_t d^{\cdot}(x, x_0)|}_{\leqslant \frac{\mu}{2} j_0}$$

so that, by the triangle inequality,

$$2\left|\varphi\left(|\partial_t\varphi| f + |\nabla\varphi| \cdot |\nabla f|\right) + 4|\nabla\varphi|^2 f\right| \leqslant \frac{\widetilde{C}||\chi'||_\infty}{r - j_0}\left(\frac{\mu}{2} j_0 + 1 + 2\frac{||\chi'||_\infty}{r-j_0}\right). \tag{4.19}$$

Using this upper bound[3] in the definition of C_0 ((4.18) in the proof of Lemma 4.2.7), the triangle inequality and the upper bound $|\int_t^{t_0} l| \leqslant ||l||_1$ imply that

$$C_0 \leqslant \left[|a_0 - 2kb_0| + \frac{\widetilde{C}||\chi'||_\infty}{r-j_0}\left(\frac{\mu}{2} j_0 + 1 + 2\frac{||\chi'||_\infty}{r-j_0}\right)\right] ||l||_1. \qquad \square$$

[3]Note that the right-hand side of (4.19) is one such C_{f,φ,r,j_0} satisfying (4.17) in the proof of Lemma 4.2.7.

Corollary 4.3.2. *Let f_k be as in Lemma 4.3.1 and retain the notation of that Lemma. The following hold:*

(i) If ψ is a smooth time-dependent section of $E \otimes \Lambda^k T^*M$ over $M \times]t_0 - \delta, t_0[$ such that $(\partial_t + \Delta^\nabla)\psi = 0$ and

$$\int_M e_g(\psi)(\cdot, t)\mathrm{dvol}_{g_t} \leqslant E_0$$

for every $t \in]t_0 - \delta, t_0[$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\psi) f_k \varphi^2 \mathrm{dvol}_g + C_0 E_0 (t_0 - t)\right) \leqslant -\frac{1}{2}\exp\left(\int_t^{t_0} l\right)\int_M f_k\varphi^2 \cdot \left(|\mathrm{d}^\nabla \psi|^2 + \left|\delta^\nabla \psi - \iota_{\frac{\nabla f_k}{f_k}}\psi\right|^2\right)\mathrm{dvol}_g \leqslant 0$$

holds on $]t_0 - \delta, t_0[$. In particular, assuming the setup of §1.11 and that $(\omega_t = \widetilde{\omega} + a(t))_{t \in]t_0-\delta, t_0[}$ is a one-parameter family of connections evolving by the Yang-Mills flow with

$$\int_M e_g(\underline{\Omega}^\omega)(\cdot, t)\mathrm{dvol}_{g_t} \leqslant E_0$$

for each $t \in]t_0 - \delta, t_0[$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\underline{\Omega}^\omega) f_2 \varphi^2 \mathrm{dvol}_g + C_0 E_0 (t_0 - t)\right) \leqslant -\frac{1}{2}\exp\left(\int_t^{t_0} l\right)\int_M f_2\varphi^2 \left|\partial_t a + \iota_{\frac{\nabla f_2}{f_2}}\underline{\Omega}^\omega\right|^2 \mathrm{dvol}_g \leqslant 0$$

holds on $]t_0 - \delta, t_0[$, where $k = 2$ in the expressions for l and C_0.

(ii) Assume the setup of §1.10. If $u : M \times]t_0 - \delta, t_0[\to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow, where N is isometrically embedded in $\mathbb{R}^K$, and

$$\int_M e_g(\mathrm{d}u)(\cdot, t)\mathrm{dvol}_{g_t} \leqslant E_0$$

for every $t \in]t_0 - \delta, t_0[$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\mathrm{d}u) f_1 \varphi^2 \mathrm{dvol}_g + C_0 E_0 (t_0 - t)\right) \leqslant -\frac{1}{2}\exp\left(\int_t^{t_0} l\right)\int_M f_1\varphi^2 \left|\partial_t u + \iota_{\frac{\nabla f_1}{f_1}}\mathrm{d}u\right|^2 \mathrm{dvol}_g \leqslant 0,$$

holds on $]t_0 - \delta, t_0[$, where $k = 1$ in the expressions for l and C_0.

Remark 4.3.3. Note that, up to the form of the error factor $\int_t^{t_0} l$, integrating these identities from t_1 to t_2 ($t_1, t_2 \in]t_0 - \delta, t_0[$) implies the identities (4.2) and (4.4) in the case where M is static.

Remark 4.3.4. The energy boundedness condition $\int_M e_g(\cdot, t)\mathrm{dvol}_{g_t}$ is satisfied, for example, whenever M is compact and

$$\lim_{t \searrow t_0 - \delta}\int_M e_g(\cdot, t)\mathrm{dvol}_{g_t} < \infty,$$

which is true whenever ψ or u is obtained as the solution of an initial value problem with smooth initial data. This is a consequence of Proposition 1.9.1 and Lemma 1.10.3 (iii) from which more general situations where the energy boundedness condition holds may be gleaned.

Proof. (i) This is a direct consequence of Lemma 4.2.7, where the formula for solutions to (YMHF) follows from exactly the same observations as in the proof of Corollary 4.2.4 (i).

(ii) By Lemma 1.10.3 (ii),

$$\left\langle \mathrm{d}u, (\partial_t + \Delta^\nabla)\mathrm{d}u \right\rangle - \left| \iota_{\frac{\nabla f}{f}} \mathrm{d}u - \delta^\nabla \mathrm{d}u \right|^2 = - \left| \partial_t u + \iota_{\nabla f / f} \mathrm{d}u \right|^2 .$$

On the other hand, $\mathrm{d}^\nabla \mathrm{d}u = 0$, wherefore (4.13) now reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_M e_g(\mathrm{d}u) f \varphi^2 \mathrm{dvol}_g \right) = \int_M \Bigg\{ & e_g(\mathrm{d}u) \cdot \left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h + \frac{1}{t_0 - t} \right) f \\ & - f \cdot \left[|\partial_t u + \iota_{\frac{\nabla f}{f}} \mathrm{d}u|^2 \right] \\ & - f \left\langle \nabla^2 \log f + \frac{1}{2} h + \frac{1}{2(t_0 - t)} g, \sum_{i,j} \left\langle \iota_{\varepsilon_i} \mathrm{d}u, \iota_{\varepsilon_j} \mathrm{d}u \right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg\} \varphi^2 \\ & + 2 e_g(\mathrm{d}u) f \varphi \left(\partial_t \varphi + \left\langle \nabla \varphi, \frac{\nabla f}{f} \right\rangle \right) \\ & - 2 \varphi f \left\langle \iota_{\nabla \varphi} \mathrm{d}u, \partial_t u + \iota_{\frac{\nabla f}{f}} \mathrm{d}u \right\rangle \mathrm{dvol}_g , \end{aligned}$$

where we have used the fact that $\iota_{\nabla\varphi} \mathrm{d}u$ is tangent to N. We may now run through the computation in the proof of Lemma 4.2.7 with $\partial_t u + \iota_{\frac{\nabla f}{f}} \mathrm{d}u$ in place of $\mathcal{S}_\psi$ to yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} & \left(\exp\left(\int_t^{t_0} l \right) \int_M e_g(\mathrm{d}u) f \varphi^2 \mathrm{dvol}_g + C_0 E_0 (t_0 - t) \right) \\ & \leqslant - \frac{1}{2} \exp\left(\int_t^{t_0} l \right) \int_M f \varphi^2 \left| \partial_t u + \iota_{\frac{\nabla f}{f}} \mathrm{d}u \right|^2 \mathrm{dvol}_g , \end{aligned}$$

which is what we sought to prove. □

We may similarly recover Hamilton's formulæ (4.3) and (4.5) in the case where M is compact and static. First, we begin with a lemma.

Lemma 4.3.5. *Suppose $k \in \mathbb{N}$ and define $f_k : M \times]t_0 - 1, t_0[\to \mathbb{R}^+$ such that*

$$f_k(x,t) = (4\pi(t_0 - t))^k \, \mathrm{P}_{(x_0,t_0)}(x,t)$$

with $\mathrm{P}_{(x_0,t_0)}$ as in Definition 1.8.1. Then f_k satisfies the inequalities (4.16) with

$$\begin{aligned} a_0 &= 0 = a_1(t) \\ b_0 &= -F \\ b_1(t) &= -F \log \left(\frac{B}{(4\pi(t_0 - t))^{\frac{n-2k}{2}}} \right), \end{aligned}$$

with B and F as in Theorem 1.8.6, and $a_1, b_1 \in C\left(]t_0 - 1, t_0[\right) \cap L^1\left(]t_0 - 1, t_0[\right)$. Moreover, retaining the notation of Lemma 4.2.7,

$$\int_t^{t_0} l = 2k \left(FB - \frac{(n-2k)F}{2} \left(\log(4\pi) + \log(t_0 - t) - 1 \right) \right) (t_0 - t).$$

Proof. The fact that $a_0 = a_1(t) = 0$ follows from that $\mathrm{P}_{(x_0,t_0)}$ solves the backward heat equation. Moreover, by Theorem 1.8.6 and Lemma A.3,

$$\begin{aligned} f_k \mathcal{H}_{t_0} \log f_k &\geqslant -F\left(1 + f_k \log\left(\frac{B}{(4\pi(t_0-t))^{\frac{n-2k}{2}} f_k}\right)\right) g \\ &\geqslant -F\left(1 + \log\left(\frac{B}{(4\pi(t_0-t))^{\frac{n-2k}{2}}}\right)\right) g, \end{aligned}$$

whence b_0 and b_1 may be read off. It is clear that b_1 is both continuous and summable on $]t_0-1, t_0[$ since $t \mapsto \log(t_0-t)$ is. Finally, we compute

$$\begin{aligned} \int_t^{t_0} l &= -2k \int_t^{t_0} b_1 \\ &= -2k \int_t^{t_0} -FB + F \cdot \left(\frac{n-2k}{2}\right) [\log(4\pi) + \log(t_0-u)]\, \mathrm{d}u \\ &= 2k\left(FB - \frac{(n-2k)F}{2}(\log(4\pi) + \log(t_0-t) - 1)\right)(t_0-t). \end{aligned}$$

□

Corollary 4.3.6. [33] *Let f_k be as in Lemma 4.3.5 and retain the notation of that Lemma. Set $C_0 = 2kF\|l\|_1$. The following hold:*

(i) If ψ is a smooth time-dependent section of $E \otimes \Lambda^k T^*M$ over $M \times]t_0-\delta, t_0[$ such that $(\partial_t + \Delta^\nabla)\psi = 0$ and

$$\int_M e_g(\psi)(\cdot,t)\mathrm{dvol}_{g_t} \leqslant E_0$$

for every $t \in]t_0-\delta, t_0[$, then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\psi) f_k \mathrm{dvol}_g + C_0 E_0 (t_0-t)\right) \\ &\qquad \leqslant -\frac{1}{2}\exp\left(\int_t^{t_0} l\right)\int_M f_k \cdot \left(|\mathrm{d}^\nabla \psi|^2 + \left|\delta^\nabla \psi - \iota_{\frac{\nabla f_k}{f_k}} \psi\right|^2\right)\mathrm{dvol}_g \leqslant 0 \end{aligned}$$

holds on $]t_0-\delta, t_0[$. In particular, assuming the setup of §1.11 and that $(\omega_t = \widetilde{\omega} + a(t))_{t\in]t_0-\delta,t_0[}$ is a one-parameter family of connections evolving by the Yang-Mills flow with

$$\int_M e_g(\underline{\Omega}^\omega)(\cdot,t)\mathrm{dvol}_{g_t} \leqslant E_0$$

for each $t \in]t_0 - \min\{\delta, 1\}, t_0[$, it follows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\underline{\Omega}^\omega) f_2 \mathrm{dvol}_g + C_0 E_0 (t_0-t)\right) \\ &\qquad \leqslant -\frac{1}{2}\exp\left(\int_t^{t_0} l\right)\int_M f_2 \left|\partial_t a + \iota_{\frac{\nabla f_2}{f_2}} \underline{\Omega}^\omega\right|^2 \mathrm{dvol}_g \leqslant 0 \end{aligned}$$

holds on $]t_0 - \min\{\delta, 1\}, t_0[$, where $k = 2$ in the expressions for l and C_0.

(ii) Assume the setup of §1.10. If $u : M \times]t_0-\delta, t_0[\to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow, where N is isometrically embedded in $\mathbb{R}^K$, and

$$\int_M e_g(\mathrm{d}u)(\cdot,t)\mathrm{dvol}_{g_t} \leqslant E_0$$

for every $t \in\,]t_0-\delta, t_0[$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^{t_0} l\right)\int_M e_g(\mathrm{d}u)f_1\mathrm{dvol}_g + C_0E_0(t_0-t)\right)$$
$$\leqslant -\frac{1}{2}\exp\left(\int_t^{t_0} l\right)\int_M f_1\left|\partial_t u + \iota_{\frac{\nabla f_1}{f_1}}\mathrm{d}u\right|^2\mathrm{dvol}_g \leqslant 0,$$

holds on $]t_0-\min\{\delta,1\}, t_0[$, where $k=1$ in the expressions for l and C_0.

Moreover, if $\sec_g \geqslant 0$ and $\mathrm{dRic}\equiv 0$, then $l\equiv 0$ and $C_0 = 0$.

Proof. Since M is compact, there exists an $R>0$ such that $M = B_R(x_0)$. Taking $r_1 = 2R = \frac{1}{2}r_2$ and appealing to Lemma 4.2.7 as in Corollary 4.3.2, noting that $\mathcal{D}_{r_2,\delta}(x_0,t_0)\backslash\mathcal{D}_{r_1,\delta}(x_0,t_0)$ so that we may take $C_{f,\varphi,r_1,r_2} = 0$ (cf. (4.17)), we may proceed exactly as in Corollary 4.3.2, noting that, by (4.18), $C_0 = 2k|b_0|\cdot||l||_1$. That l and C_0 vanish when $\sec_g \geqslant 0$ and $\mathrm{dRic}\equiv 0$ is a consequence of Theorem 1.8.6. □

Integrating these identities and estimating $\int_t^{t_0} l$ accordingly immediately implies Hamilton's formulæ (4.3) and (4.5) (cf. [33]).

4.4. A technical lemma. We shall now make use of Lemma 4.2.7 to derive a technical lemma which shall be made use of in Theorems 6.3.2 and 6.3.6 of Chapter 6 to deduce the finiteness of certain singular intervals. This is the equivalent of Theorem 1.12.7 for Dirichlet-type flows and should be compared with the computation in [20, Appendix].

Lemma 4.4.1. *Let $\psi \in \Gamma\left(E\otimes\Lambda^k T^*M\right)$ be a smooth, time-dependent section over $M\times\,]t_0-\delta,t_0[$ such that either $(\partial_t+\Delta^\nabla)\psi = 0$ or, assuming the setup of §1.10, $\psi = \mathrm{d}u$ ($k=1$) and $u : M\times\,]t_0-\delta,t_0[\,\to N\subset\mathbb{R}^K$ evolves by the harmonic map heat flow, where N is isometrically embedded in $\mathbb{R}^K$. Set*

$$R_r^m(t) = \sqrt{2m(t-t_0)\log\left(\frac{4\pi(t_0-t)}{r^2}\right)}.$$

Then for every $0<r<\min\left\{1, \frac{j_0}{2c_{n,k}}, \sqrt{4\pi\delta}\right\}$ ($c_{n,k} := \sqrt{\frac{n-2k}{2\pi e}}$), the following estimates hold:

1. *For every $t\in\left]t_0 - r^2\exp\left(-\frac{1}{2(n-2k)}\right)/4\pi, t_0\right[$,*

$$\int_{B^t_{R_r^{n-2k}(t)}(x_0)} e_g(\psi)(\cdot,t)\mathrm{dvol}_{g_t}$$
$$\leqslant \widetilde{C}_0 R_r^{n-2k}(t)^{n-2k}\left(\frac{1}{r^{n-2k+2}}\int_{t_0-\frac{r^2}{4\pi}}^{t_0}\int_{B^t_{2c_{n,k}r}(x_0)} e_g(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t\right.$$
$$\left.+\frac{1}{r^{n-2k}}\left(\int_{B^{\cdot}_{2c_{n,k}r}(x_0)} e_g(\psi)\mathrm{dvol}_g\right)(t_0-\frac{r^2}{4\pi})\right).$$

2.

$$\int_{t_0-\frac{r^2}{4\pi}}^{t_0}\int_{B^t_{R_r^{n-2k}(t)}(x_0)} |\mathcal{S}|^2 + |\mathcal{J}|^2\mathrm{dvol}_{g_t}\mathrm{d}t$$

$$\leqslant 2\widetilde{C}_0 r^{n-2k}\left(\frac{1}{r^{n-2k+2}}\int_{t_0-\frac{r^2}{4\pi}}^{t_0}\int_{B^t_{2c_{n,k}r}(x_0)} e_g(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t\right.$$
$$\left.+\frac{1}{r^{n-2k}}\left(\int_{B^\cdot_{2c_{n,k}r}(x_0)} e_g(\psi)\mathrm{dvol}_g\right)(t_0-\frac{r^2}{4\pi})\right).$$

Here $\widetilde{C}_0$ is given by

$$e^{2||l||_1}\max\left\{1,|a_0-2kb_0|,\gamma_{n,k}\max\{||\chi'||_\infty,||\chi'||^2_\infty\}\cdot\left(\frac{\mu}{2}+\frac{1}{2c_{n,k}}+\frac{1}{c^2_{n,k}}\right)\right\}$$

with a_0, b_0 and l as in Lemma 4.3.1 χ as in Example A.1 and $\gamma_{n,k}$ a positive constant such that

$$f\leqslant\frac{\gamma_{n,k}}{r^{n-2k}}$$

and

$$|\nabla f|\leqslant\frac{\gamma_{n,k}}{r^{n-2k+1}}$$

hold on $\mathcal{D}_{2c_{n,k}r,\delta}(x_0,t_0)\backslash\mathcal{D}_{c_{n,k}r,\delta}(x_0,t_0)$, $\mathcal{J}=\mathrm{d}^\nabla\psi$ ($=0$ if $\psi=\mathrm{d}u$ and u solves (HMHF)) and $\mathcal{S}=\delta^\nabla\psi-\iota_{\frac{\nabla f}{f}}\psi$ if $(\partial_t+\Delta^\nabla)\psi=0$ or $\mathcal{S}=\partial_t u+\partial_{\frac{\nabla f}{f}}u$ if $\psi=\mathrm{d}u$ and u solves (HMHF).

Proof. Write e_g for $e_g(\psi)$ and let $f={}^{n-2k}_{\ \ \ s}\Phi$ be as in the proof of Theorem 1.12.7 with $s\geqslant t_0$ to be chosen later. From (1.27) and (1.28) in the proof of Theorem 1.12.7 and the fact that $s-t\geqslant t_0-t$, we see that the bounds (4.16) hold with[4]

$$\begin{aligned}
a_0&=\max\{\frac{n\mu}{2},4C_4\},\\
a_1(t)&=\max\{\frac{n\mu}{2},4C_4\}\log\left(\frac{1}{(4\pi(t_0-t))^{n/2-k}}\right),\\
b_0&=-\max\{2C,-\lambda_{-\infty}\}\text{ and}\\
b_1(t)&=-\max\{2C,-\lambda_{-\infty}\}\log\left(\frac{1}{(4\pi(t_0-t))^{n/2-k}}\right).
\end{aligned}$$

In light of the argument used in the proof of Corollary 4.3.2 (ii), if ψ solves the heat equation or $\psi=\mathrm{d}u$ (with $k=1$) with u solving (HMHF) (cf. §1.10), Lemma 4.2.7 (with $\delta_0=\delta$, $\varphi|_{\mathcal{D}_{r_1,\delta}(x_0,t_0)}$, $r_1=c_{n,k}r$ and $r_2=2r_1$, implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int_t^{t_0}l}\int_M e_g\cdot f\varphi^2\mathrm{dvol}_g\right)\leqslant C_0\int_{B_{2c_{n,k}r}(x_0)}e_g\,\mathrm{dvol}_g-e^{-||l||_1}\int_M\frac{1}{2}f\varphi^2\cdot\left(|\mathcal{J}|^2+|\mathcal{S}|^2\right)\mathrm{dvol}_g,$$

where $e_g:=e_g(\psi)$. Integrating from $t_0-\frac{r^2}{4\pi}$ to $\bar{t}\in\,]t_0-\delta,t_0[$ and using the bound $|\int_I l|\leqslant||l||_1$, we see that

$$e^{-||l||_1}\left(\int_M e_g\cdot f\varphi^2\mathrm{dvol}_g\right)(\bar{t})+\frac{e^{-||l||_1}}{2}\int_{t_0-\frac{r^2}{4\pi}}^{\bar{t}}\int_M f\varphi^2\left(|\mathcal{S}|^2+|\mathcal{J}|^2\right)\mathrm{dvol}_{g_t}\mathrm{d}t$$
$$\leqslant C_0\int_{t_0-\frac{r^2}{4\pi}}^{\bar{t}}\int_{B^t_{2c_{n,k}r}(x_0)}e_{g_t}\mathrm{dvol}_{g_t}\mathrm{d}t+e^{||l||_1}\left(\int_M e_g\cdot f\varphi^2\mathrm{dvol}_g\right)(t_0-\frac{r^2}{4\pi}).$$

[4] This should be compared with Lemma 4.3.1. As in the proof of that Lemma, C and C_4 are constants depending only on the local geometry about (x_0, t_0).

Noting that $\chi_{B^t_{R_r^{n-2k}(t)}(x_0)} \leqslant \chi_{B^t_{c_{n,k}r}(x_0)} \leqslant \varphi(\cdot,t) \leqslant \chi_{B^t_{2c_{n,k}r}(x_0)}$ for every $t \in]t_0 - \frac{r^2}{4\pi}, t_0[$, and

$$f(\cdot,t) \leqslant \frac{1}{(4\pi(s-t))^{\frac{n}{2}-k}} \leqslant \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}-k}}$$

for $t \in I$, as well as multiplying through by $e^{||l||_1}$, we obtain

$$\begin{aligned}\left(\int_{B^t_{R_r^{n-2k}(t)}(x_0)} e_g \cdot f \mathrm{dvol}_g\right)(\bar{t}) &+ \frac{1}{2}\int_{t_0-\frac{r^2}{4\pi}}^{\bar{t}} \int_{B^t_{R_r^{n-2k}(t)}(x_0)} f\left(|\mathcal{S}|^2 + |\mathcal{J}|^2\right) \mathrm{dvol}_{g_t}\mathrm{d}t \\ &\leqslant C_0 e^{||l||_1}\int_{t_0-\frac{r^2}{4\pi}}^{\bar{t}} \int_{B^t_{2c_{n,k}r}(x_0)} e_{g_t}\mathrm{dvol}_{g_t}\mathrm{d}t + \frac{e^{2||l||_1}}{r^{n-2k}}\left(\int_{B^{\cdot}_{2c_{n,k}r}(x_0)} e_g \,\mathrm{dvol}_g\right)(t_0 - \frac{r^2}{4\pi}). \quad (4.20)\end{aligned}$$

To take care of the first term on the left-hand side, we set $s = R_r^{n-2k}(\bar{t})^2 + \bar{t}$ and fix $\bar{t} \in [t_0 - r^2\exp\left(-\frac{1}{2(n-2k)}\right)/4\pi, t_0[$ as in Theorem 1.12.7. As noted there, $s \geqslant t_0$ and, in this case,

$$f(\cdot,\bar{t})\big|_{B^{\bar{t}}_{R_r^{n-2k}(\bar{t})}(x_0)} \geqslant \frac{1}{(4\pi R_r^{n-2k}(\bar{t})^2)^{\frac{n}{2}-k}}\exp(-1/4)$$

so that, discarding the second term on the left-hand side of inequality (4.20),

$$\begin{aligned}\frac{1}{(4\pi)^{\frac{n}{2}-k}\exp(1/4)} \cdot \frac{1}{R_r^{n-2k}(\bar{t})^2}&\left(\int_{B^{\cdot}_{R_r^{n-2k}(\cdot)}(x_0)} e_g \mathrm{dvol}_g\right)(\bar{t}) \\ &\leqslant C_0 e^{||l||_1}\int_{t_0-\frac{r^2}{4\pi}}^{\bar{t}} \int_{B^t_{2c_{n,k}r}(x_0)} e_{g_t}\mathrm{dvol}_{g_t}\mathrm{d}t + \frac{e^{2||l||_1}}{r^{n-2k}}\left(\int_{B^{\cdot}_{2c_{n,k}r}(x_0)} e_g \,\mathrm{dvol}_g\right)(t_0 - \frac{r^2}{4\pi}).\end{aligned}$$

As for the second term, we set $s = \bar{t} = t_0$ and note that, by Example 5.3.1, $t \in \left]t_0 - \frac{r^2}{4\pi}, t_0\right[$ and $d^t(x,x_0) < R_r^{n-2k}(t)$ imply that $f > \frac{1}{r^{n-2k}}$ (cf. introductory remarks in Example 5.3.1) so that, after discarding the first term, we obtain

$$\begin{aligned}\frac{1}{2r^{n-2k}}\int_{t_0-\frac{r^2}{4\pi}}^{t_0}\int_{B^t_{R_r^{n-2k}(t)}(x_0)} &|\mathcal{S}|^2 + |\mathcal{J}|^2 \mathrm{dvol}_{g_t}\mathrm{d}t \\ &\leqslant C_0 e^{||l||_1}\int_{t_0-\frac{r^2}{4\pi}}^{\bar{t}} \int_{B^t_{2c_{n,k}r}(x_0)} e_{g_t}\mathrm{dvol}_{g_t}\mathrm{d}t + \frac{e^{2||l||_1}}{r^{n-2k}}\left(\int_{B^{\cdot}_{2c_{n,k}r}(x_0)} e_g \,\mathrm{dvol}_g\right)(t_0 - \frac{r^2}{4\pi}).\end{aligned}$$

To more explicitly describe C_0, we proceed as in Corollary 4.3.2 and note that

$$\begin{aligned}|\nabla f| &= \frac{1}{(4\pi(s-t))^{\frac{n}{2}-k}}\exp\left(\frac{d^t(x,x_0)^2}{4(t-s)}\right) \cdot \frac{d^t(x,x_0)}{2(s-t)} \\ &\leqslant \frac{c_{n,k}r}{(4\pi)^{\frac{n}{2}-k}(s-t)^{\frac{n}{2}-k+1}}\exp\underbrace{\left(\frac{c_{n,k}^2 r^2}{4(t-s)}\right)}_{=:-u} \\ &= \frac{1}{r^{n-2k+1}} \cdot \frac{4}{\pi^{\frac{n}{2}-k}(c_{n,k})^{n-2k+1}} \cdot \underbrace{u^{\frac{n}{2}-k+1}e^{-u}}_{\text{bounded for } u\in\mathbb{R}^+}\end{aligned}$$

on $\mathcal{D}_{2c_{n,k}r,\delta}(x_0,t_0)\backslash\mathcal{D}_{c_{n,k}r,\delta}(x_0,t_0)$. Similarly, using the same change of variables,

$$f \leqslant \frac{\mathrm{const}(n,k)}{r^{n-2k}}$$

on this set. Finally, we make a choice of cutoff φ:

$$\varphi(x,t) := \chi(-\frac{d^t(x,x_0)}{2c_{n,k}r} + \frac{3}{2}).$$

It is clear that this function satisfies the desired properties[5] and that

$$\begin{aligned}
|\partial_t \varphi(x,t)| &\leqslant \frac{1}{2c_{n,k}r} \cdot \left|\chi'(-\frac{d^t(x,x_0)}{2c_{n,k}r} + \frac{3}{2})\right| \cdot |\partial_t d^{\cdot}(x,x_0)| \\
&\leqslant \frac{||\chi'||_\infty}{2c_{n,k}r} \cdot \frac{\mu}{2} \underbrace{d^t(x,x_0)}_{\leqslant 2c_{n,k}r \text{ on supp } \varphi} \\
&\leqslant \frac{\mu}{2}||\chi'||_\infty,
\end{aligned}$$

whereas

$$\begin{aligned}
|\nabla \varphi(x,t)| &\leqslant \frac{1}{2c_{n,k}r} \cdot \left|\chi'(-\frac{d^t(x,x_0)}{2c_{n,k}r} + \frac{3}{2})\right| \cdot \underbrace{|\nabla d^t(\cdot,x_0)|}_{\leqslant 1} \\
&\leqslant \frac{||\chi'||_\infty}{2c_{n,k}r}.
\end{aligned}$$

Thus, using the definition of C_0 ((4.18) in the proof of Lemma 4.2.7) (cf. proof of Corollary 4.3.2), we see that

$$C_0 \leqslant \left(|a_0 - 2kb_0| + \mathrm{const}(n,k) \cdot \frac{\frac{\mu}{2}||\chi'||_\infty + \frac{||\chi'||_\infty}{2c_{n,k}} + \frac{||\chi'||_\infty^2}{c_{n,k}^2}}{r^{n-2k+2}} \right) e^{||l||_1},$$

where we have used the fact that $r < 1$. Since $r < 1$, the former term may be absorbed into the latter by bounding $|a_0 - 2kb_0|$ and $|\mathrm{const}(n,k,\chi)|$ from above by their maximum, implying the result. □

[5]Since we are within the injectivity radii of $\{g_t\}_{t\in]t_0-\delta,t_0[}$, the only points at which this function might not be smooth are $\{(x_0,t) : t \in]t_0-\delta, t_0[\}$, but it is constant in a cylindrical neighbourhood of these points.

— 5 —

Heat Balls

We introduce the sets over which the integrals appearing in our local monotonicity formulæ take form—heat balls. Heat balls were first introduced by Watson [73] as a generalization of Fulks' 'heat spheres' [27] and subsequently applied variously by Watson [73] and Evans and Gariepy [24] to the study of solutions to the heat equation in Euclidean space and by Ecker [18, 20], Ecker, Knopf, Ni and Topping [22] and Ni [56] to the study of nonlinear evolution equations in more general geometric settings. The presentation here mostly parallels that of Ecker, Knopf, Ni and Topping [22] with a few noticeable differences. In particular, we do not necessarily assume that the "kernel" in question is defined everywhere on the manifold and we assume that the time derivative of its logarithm is summable over its superlevel sets. The latter condition may be dropped in certain applications, but it shall be of use to us in establishing monotonicity formulæ for Dirichlet-type flows. Moreover, we derive integration formulæ analogous to those in [18] in order to simplify computations to be carried out in the following chapters.

5.1. The story so far. Before proceeding to the introduction of heat balls in our general setting, let us first review what is known about heat balls in Euclidean space and those in curved settings.

Let $f : \mathbb{R}^n \times]-\infty, t_0[\to \mathbb{R}^+$ be the usual Euclidean backward heat kernel concentrated at $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, i.e.

$$f(x,t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right),$$

and define for each $r > 0$ the *heat sphere of radius r* by $S_r(x_0, t_0) = \left\{f = \frac{1}{r^n}\right\}$ and the *heat ball of radius r* by $E_r(x_0, t_0) = \left\{f > \frac{1}{r^n}\right\}$. It was first shown by Fulks [27] that if $u \in C^2(\mathcal{D}, \mathbb{R})$ is a solution to $\partial_t u - \Delta u = 0$ in the open domain $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}$, then whenever $(x_0, t_0) \in \mathcal{D}$ and $E_r(x_0, t_0) \Subset \mathcal{D}$,

$$u(x_0, t_0) = \frac{1}{r^n} \int_{S_r(x_0,t_0)} u(x,t) \cdot \frac{|x - x_0|^2}{\sqrt{4|x - x_0|^2(t_0 - t)^2 + \left(|x - x_0|^2 - 2n(t_0 - t)\right)^2}} \mathrm{d}S(x,t),$$

where $\mathrm{d}S$ denotes the usual surface measure in $\mathbb{R}^{n+1}$. This idea was subsequently used by Watson [73] to establish the representation formula

$$u(x_0, t_0) = \frac{1}{r^n} \iint\limits_{E_r(x_0,t_0)} u(x,t) \cdot \frac{|x - x_0|^2}{4(t_0 - t)^2} \mathrm{d}x\mathrm{d}t.$$

The idea is that, since f is bounded outside of any open neighbourhood (x_0, t_0), the $\{E_r(x_0, t_0)\}_{r \in \mathbb{R}^+}$ an increasing[1] one-parameter family of relatively compact sets whose closures contain (x_0, t_0) and, in a certain sense, tend to (x_0, t_0) as $r \searrow 0$. Thus, these representation formulæ are *local* and provide a natural analogue of the usual mean-value formula for solutions to Laplace's equation. They have subsequently been used by Watson [73] and Evans and Gariepy [24] to study solutions to the heat equation. Moreover, in considering an appropriately modified version of the Euclidean backward heat kernel f and different powers of r in the heat ball definition above, Ecker [20, 18] showed that these ideas naturally lead to local monotonicity formulæ for solutions to nonlinear parabolic systems such as the mean curvature flow, the harmonic map heat flow and reaction-diffusion systems. For the latter two systems, the heat balls take the form

$$E_r^\gamma(x_0, t_0) = \{(x,t) \in \mathbb{R}^n \times]-\infty, t_0[\,:\, (4\pi(t_0 - t))^{\frac{\gamma}{2}} f(x,t) > \frac{1}{r^{n-\gamma}}\}$$

for appropriately chosen $\gamma \in]0, n[$ [20].

The heat ball construction was first adapted to a non-Euclidean setting by Ecker [19] where the heat balls take the form

[1] That is, $E_{r_1}(x_0, t_0) \subset E_{r_2}(x_0, t_0)$ for $r_1 < r_2$.

$$\mathcal{M} \cap E_r^{n-m}(x_0, t_0),$$

where $n - m \geqslant 0$ and

$$\mathcal{M} = \bigcup_{t \in]a,b[} F_t(N) \times \{t\}$$

is the space-time track of a one-parameter family $\{F_t : N^m \to \mathbb{R}^n\}_{t \in]a,b[}$ of embeddings evolving by mean curvature flow (cf. §1.12). The construction was subsequently adapted in a different manner by Ecker, Knopf, Ni and Topping [22] to the evolving Riemannian manifold setting, where the heat balls take the form

$$\left\{\Phi > \frac{1}{r^n}\right\} \subset M \times]a,b[\,,$$

where $(M^n, \{g_t\}_{t \in]a,b[})$ is an evolving Riemannian manifold and $\Phi : M \times]a,b[\to \mathbb{R}^+$ is a sufficiently smooth function satisfying properties which are typical of the Euclidean backward heat kernel (cf. §5.2). Finally, the heat sphere construction has been adapted by Ni [56] to the evolving Riemannian manifold setting, where the heat spheres take the form

$$\left\{\mathrm{P}_{(x_0,t_0)} > \frac{1}{r^n}\right\} \subset M \times]a,b[\,,$$

where $(M^n, \{g_t\}_{t \in]a,b[})$ is an evolving Riemannian manifold and $\mathrm{P}_{(x_0,t_0)}$ is the canonical backward heat kernel concentrated at $(x_0, t_0) \in M \times]a,b[$ (cf. Definition 1.8.1).

5.2. The definition. We proceed to define a notion of heat ball in an attempt to unify those of Ecker [19, 18] and Ecker, Knopf, Ni and Topping [22], in particular allowing for different powers of r, whilst also accommodating for kernels which are not globally defined in spacetime.

Fix $t_0 \in \mathbb{R}$, $\delta_0 > 0$, $m \in \mathbb{N}$ and let $\{(M, g_t)\}_{t \in]t_0-\delta_0, t_0[}$ be an evolving manifold. Suppose we are given $\Phi \in C^1(\mathcal{D}, \mathbb{R}^+)$, where $\mathcal{D} \subset M \times]t_0 - \delta_0, t_0[$ is open. Set

$$E_r^m(\Phi) = \left\{\Phi > \frac{1}{r^m}\right\} = \{\log(r^m \Phi) > 0\} \subset \mathcal{D}$$

for $r > 0$ and $0 < m \leqslant \dim M$ and write $\phi = \log(\Phi)$ and $\phi_r^m := \log(r^m \Phi)$.

We assume that there exists an $r_0 \in]0,1[$ such that

(HB1) $E_{r_0}^m(\Phi) \cap \mathrm{pr}_2^{-1}(]t_0 - \delta_0, \tau[) \Subset \mathcal{D}$ for every $\tau \in]t_0 - \delta_0, t_0[$.

(HB2) $|\nabla \phi|^2, \partial_t \phi \in L^1(E_{r_0}^m(\Phi))$, and

(HB3) $\displaystyle\lim_{\tau \nearrow t_0} \int_{\mathrm{pr}_1(E_{r_0}^m(\Phi) \cap (M \times \{\tau\}))} |\phi| \,\mathrm{dvol}_{g_\tau} = 0.$

Definition 5.2.1. With the above definition and assumptions, $E_r^m(\Phi)$ is said to be an (m, Φ)-**heat ball**.

Remark 5.2.2. Since $r_1 < r_2 \Rightarrow E_{r_1}^m(\Phi) \subset E_{r_2}^m(\Phi)$, if r_0 satisfies the above properties then so does $r \in]0, r_0[$.

Remark 5.2.3. In view of (HB1) and (HB3), $\phi \in L^1(E_{r_0}^m(\Phi))$. To see this, note that, by Tonelli's theorem,

$$\iint_{E_{r_0}^m(\Phi)} |\phi| \mathrm{dvol}_{g_t} dt = \iint_{E_{r_0}^m(\Phi) \cap \mathrm{pr}_2^{-1}(]t_0-\delta_0,\tau[)} |\phi| \mathrm{dvol}_{g_t} dt + \int_\tau^{t_0} \int_{\mathrm{pr}_1(E_{r_0}^m(\Phi) \cap (M \times \{t\}))} |\phi| \,\mathrm{dvol}_{g_t} dt.$$

Since the former integral on the right-hand side is over a relatively compact subset of $\mathcal{D}$, i.e. a set on which ϕ is bounded, it is clearly finite. On the other hand, provided τ is close enough to t_0,

$$\int_{\mathrm{pr}_1(E^m_{r_0}(\Phi)\cap(M\times\{t\}))} |\phi|\,\mathrm{dvol}_{g_t} < 1$$

for $t \in]\tau, t_0[$, thus establishing that the latter integral on the right-hand side is also finite.

Remark 5.2.4. Note that by Remark 5.2.2 and (HB3), if $r < r_0 < 1$, then $\phi > -m\log r > 0$ on $E^m_r(\Phi)$ and

$$\begin{aligned}0 = \lim_{\tau\nearrow t_0}\int_{\mathrm{pr}_1(E^m_r(\Phi)\cap(M\times\{\tau\}))}|\phi|\mathrm{dvol}_{g_\tau} &\geqslant \lim_{\tau\nearrow t_0}(-m\log r)\int_{\mathrm{pr}_1(E^m_r(\Phi)\cap(M\times\{\tau\}))}\mathrm{dvol}_{g_\tau}\\ &= (-m\log r)\lim_{\tau\nearrow t_0}\mathrm{Vol}_{g_\tau}(\mathrm{pr}_1(E^m_r(\Phi)\cap(M\times\{\tau\})))\end{aligned}$$

so that

$$\begin{aligned}&\int_{\mathrm{pr}_1(E^m_r(\Phi)\cap(M\times\{\tau\}))}|\phi^m_r|\mathrm{dvol}_{g_\tau}\\ &\quad\leqslant \int_{\mathrm{pr}_1(E^m_r(\Phi)\cap(M\times\{\tau\}))}|\phi|\mathrm{dvol}_{g_\tau} + m|\log r|\cdot\mathrm{Vol}_{g_\tau}(\mathrm{pr}_1(E^m_r(\Phi)\cap(M\times\{\tau\}))) \xrightarrow{\tau\nearrow t_0} 0.\end{aligned}$$

5.3. Examples. We now proceed to give examples of heat balls. In all of the following, let $x_0 \in M$, assume M is of locally bounded geometry about (x_0, t_0) as in Definition 1.7.4 and suppose j_0 is as in (1.6) of §1.7.

The first example is an analogue of the Euclidean heat balls of Watson [73] and Ecker [20], the idea being to mimic their constructions with the formal heat kernel.

Example 5.3.1 (Formal Heat Balls). Suppose M is of locally bounded geometry about (x_0, t_0) and assume the notation in and directly following Definition 1.7.4. We consider

$$\begin{aligned}\Phi = {}^m\Phi_{\mathrm{fml}} &:= \left(\mathcal{D}_{j_0,\delta}(x_0,t_0) \ni (x,t) \mapsto [4\pi(t_0-t)]^{\frac{n-m}{2}}\right)\cdot\Phi_{\mathrm{fml}}\\ &= \left(\mathcal{D}_{j_0,\delta}(x_0,t_0)\ni(x,t)\mapsto \frac{1}{4\pi(t_0-t)^{m/2}}\exp\left(\frac{d^t(x_0,x)^2}{4(t-t_0)}\right)\right)\end{aligned}$$

for fixed $m > 0$.

Note that

$$\begin{aligned}\phi^m_r(x,t) > 0 &\Leftrightarrow \frac{d^t(x,x_0)^2}{4(t-t_0)} - \frac{m\log(4\pi(t_0-t))}{2} > -m\log r\\ &\Leftrightarrow d^t(x,x_0)^2 < 2m(t-t_0)\log\left(\frac{4\pi(t_0-t)}{r^2}\right) =: R^m_r(t)^2.\end{aligned}$$

On the other hand, since $t - t_0 < 0$ in $\mathcal{D}_{j_0,\delta}(x_0,t_0)$, we see that

$$\begin{aligned}R^m_r(t)^2 \geqslant 0 &\Leftrightarrow \log\left(\frac{4\pi(t_0-t)}{r^2}\right) < 0\\ &\Leftrightarrow t > t_0 - \frac{r^2}{4\pi},\end{aligned}$$

whence it is clear that

$$E^m_r(\Phi) = \left(\bigcup_{t\in]t_0-\frac{r^2}{4\pi},t_0[} B^t_{R^m_r(t)}(x_0)\times\{t\}\right)\bigcap\mathcal{D}_{j_0,\delta}(x_0,t_0).$$

Let

$$r_0 = \frac{1}{2}\min\left\{ j_0 \cdot \sqrt{\frac{2\pi e}{m}}, \sqrt{4\pi\delta}, 1 \right\}.$$

We claim that $E_r^m(\Phi)$ is an (m,Φ)-heat ball for $r < r_0$ and now proceed to verify the conditions (HB1)-(HB3).

(HB1) We note that

$$R_r^m \leqslant \sqrt{\frac{m}{2\pi e}}\, r \tag{5.1}$$

wherever R_r^m is defined as is evident from a straightforward computation. Thus, we have that $R_{r_0}^m(t) < \frac{j_0}{2}$ and, from the definition of r_0, $t_0 - \frac{r_0^2}{4\pi} > t_0 - \delta$, whence

$$E_r^m(\Phi) = \bigcup_{t \in]t_0 - \frac{r^2}{4\pi}, t_0[} B^t_{R_r^m(t)}(x_0) \times \{t\} \tag{5.2}$$

and

$$\overline{E_r^m(\Phi) \cap \mathrm{pr}_2^{-1}(]t_0-\delta,\tau[)} = \bigcup_{t \in [t_0 - \frac{r_0^2}{4\pi}, \tau]} \overline{B^t_{R_{r_0}^m(t)}(x_0)} \times \{t\} \subset \mathcal{D}_{j_0,\delta}(x_0,t_0)$$

for every $\tau \in]t_0 - \delta, t_0[$.

(HB2) We use exponential coördinates about x_0 with respect to g_t for some fixed t. Note that since $\sec \geqslant \underline{\kappa}$ in $\mathcal{D}_{j_0,\delta}(x_0,t_0)$, Theorem B.2 and Proposition B.4 imply that

$$\left(\vartheta_{x_0}^* \mathrm{dvol}_{g_t}\right)(x) \leqslant C_{\underline{\kappa}} \mathrm{dvol}_{\mathrm{eucl}}(x), \tag{5.3}$$

for $x \in B_{j_0}(0) \subset \mathbb{R}^n$, where ϑ_{x_0} is as defined in Appendix B and $C_{\underline{\kappa}}$ is some positive constant depending only on $\underline{\kappa}$ and j_0. Now, by Proposition 1.8.8,

$$|\nabla\phi|^2 \circ \vartheta_{x_0} = \frac{|x|^2}{4(t_0-t)^2}$$

and

$$|\partial_t\phi| \circ \vartheta_{x_0} \leqslant \frac{n}{2(t_0-t)} + \frac{|x|^2}{4(t_0-t)^2} + \frac{\mu|x|^2}{4(t_0-t)}$$

hold and, by (5.1), $R_{r_0}^m(t) < j_0$ for every $t \in]t_0 - \frac{r_0^2}{4\pi}, t_0[$ so that (5.3) implies that

$$\begin{aligned}\int_{B^t_{R_{r_0}^m(t)}(x_0)} |\nabla\phi|^2 \mathrm{dvol}_{g_t} &= \int_{B_{R_{r_0}^m(t)}(0)} (|\nabla\phi|^2 \circ \vartheta_{x_0}) \vartheta_{x_0}^* \mathrm{dvol}_{g_t} \\ &\leqslant C_{\underline{\kappa}} \int_{B_{R_{r_0}^m(t)}(0)} \frac{|x|^2}{4(t_0-t)^2} \mathrm{d}x\end{aligned}$$

and

$$\int_{B^t_{R_{r_0}^m(t)}(x_0)} |\partial_t\phi| \mathrm{dvol}_{g_t} = \int_{B_{R_{r_0}^m(t)}(0)} (|\partial_t\phi| \circ \vartheta_{x_0}) \vartheta_{x_0}^* \mathrm{dvol}_{g_t}$$

$$\leqslant C_{\underline{K}} \left\{ \int_{B_{R^m_{r_0}(t)}(0)} \frac{n}{2(t_0-t)} + \frac{|x|^2}{4(t_0-t)^2} + \frac{\mu|x|^2}{4(t_0-t)} \mathrm{d}x \right\}.$$

It therefore suffices to show that

$$\int_{t_0-\frac{r_0^2}{4\pi}}^{t_0} \int_{B_{R^m_{r_0}(t)}(0)} \frac{|x|^2}{(t_0-t)^2} \mathrm{d}x\mathrm{d}t < \infty$$

and

$$\int_{t_0-\frac{r_0^2}{4\pi}}^{t_0} \int_{B_{R^m_{r_0}(t)}(0)} \frac{1}{t_0-t} \mathrm{d}x\mathrm{d}t < \infty.$$

The former integral is equal to

$$\begin{aligned}
&n\omega_n \int_{t_0-\frac{r_0^2}{4\pi}}^{t_0} \frac{1}{(t_0-t)^2} \int_0^{R^m_{r_0}(t)} u^{n+1} \mathrm{d}u\mathrm{d}t \\
&= \frac{n\omega_n (2m)^{1+n/2}}{n+2} \int_{t_0-\frac{r_0^2}{4\pi}}^{t_0} \sqrt{(t-t_0)^{n-2} \log\left(\frac{4\pi(t_0-t)}{r_0^2}\right)}^{\,n+2} \mathrm{d}t \qquad (5.4)\\
&= \left(\frac{r_0^2}{4\pi}\right)^{n/2} \cdot \frac{n\omega_n(2m)^{1+n/2}}{n+2} \int_0^\infty s^{1+n/2} \exp\left(-\frac{n}{2}s\right) \mathrm{d}s,
\end{aligned}$$

which is finite, where in the last line the change of variables $\frac{4\pi(t_0-t)}{r_0^2} =: \exp(-s)$ was made. The latter integral may be evaluated likewise:

$$\begin{aligned}
&\omega_n \int_{t_0-\frac{r_0^2}{4\pi}}^{t_0} \frac{R^m_{r_0}(t)^n}{t_0-t} \mathrm{d}t \\
&= \omega_n (2m)^{n/2} \int_{t_0-\frac{r_0^2}{4\pi}}^{t_0} \sqrt{(t-t_0)^{n-2} \log\left(\frac{4\pi(t_0-t)}{r_0^2}\right)}^{\,n} \mathrm{d}t \qquad (5.5)\\
&= \omega_n \left(\frac{mr_0^2}{2\pi}\right)^{n/2} \int_0^\infty s^{n/2} \exp\left(-\frac{n}{2}s\right) \mathrm{d}s
\end{aligned}$$

and this integral is clearly finite.

(HB3) By the volume comparison argument used in the verification of (HB2) and the fact that $\mathrm{pr}_1\left(E^m_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(\{\tau\})\right) = B_{R^m_{r_0}(\tau)}(p)$, it suffices to show that

$$\lim_{\tau \nearrow t_0} \int_{B_{R^m_{r_0}(\tau)}(0)} \left(|\phi| \circ \vartheta^\tau_{x_0}\right)(x) \mathrm{d}x = 0. \qquad (5.6)$$

Now, for $t_0 - \frac{1}{4\pi} < \tau < t_0$, the integrand is equal to

$$\left| \frac{|x|^2}{4(\tau-t_0)} - \frac{m}{2} \log\left(4\pi(t_0-\tau)\right) \right| \leqslant \frac{|x|^2}{4(t_0-\tau)} + \frac{m}{2}\left(-\log\left(4\pi(t_0-\tau)\right)\right),$$

where the latter term is nonnegative. On the one hand,

$$\begin{aligned}
&\int_{B_{R^m_{r_0}(\tau)}(0)} \frac{|x|^2}{4(t_0-\tau)} \mathrm{d}x \\
&= \frac{n\omega_n}{4(n+2)(t_0-\tau)} R^m_{r_0}(\tau)^{n+2}
\end{aligned}$$

$$= \frac{n\omega_n}{4(n+2)} \sqrt{(2m)^{n+2}(\tau - t_0)^n \log\left(\frac{4\pi(t_0-\tau)}{r_0^2}\right)^{n+2}}.$$

On the other,

$$\begin{aligned}
&\int_{B_{R^m_{r_0}(t)}(0)} \frac{m}{2}\left(-\log(4\pi(t_0-\tau))\right) \mathrm{d}x \\
&= -\frac{m\omega_n}{2} \log(4\pi(t_0-\tau)) R^m_{r_0}(\tau)^n \\
&= \frac{m\omega_n}{2} \sqrt{(2m)^n(\tau - t_0)^n \log\left(\frac{4\pi(t_0-\tau)}{r_0^2}\right)^{n+2}} - \frac{m\omega_n}{2} \log\left(r_0^2\right) R^m_{r_0}(\tau)^n.
\end{aligned}$$

(5.6) now follows from the fact that $R^m_{r_0}(\tau) \xrightarrow{\tau \nearrow t_0} 0$ and

$$\lim_{\tau \nearrow t_0} (\tau - t_0)^n \log\left(\frac{4\pi(t_0-\tau)}{r_0^2}\right)^{n+2} = \left(\frac{r_0^2}{4\pi}\right)^n \lim_{s\to\infty} s^{n+2}\exp(-ns) = 0. \quad \square$$

Remark 5.3.2. If $(M, g_t) \equiv (\mathbb{R}^n, \delta)$, the preceding example reduces to the heat balls of Watson [73] for $m = n$ and to those of Ecker [20] for $m = n - \gamma$ with $\gamma \in \,]0, n[$ fixed.

Following [22], we turn our attention to heat balls constructed from the canonical backward heat kernel on M. However, for later purposes, we shall need appropriate bounds on both $\partial_t \phi$ and $\mathcal{H}_{t_0}\phi$ in heat balls which we only have for the case where M is compact and $h \equiv 0$. For this reason, we now consider heat balls on static compact manifolds. Strictly speaking, these do not generalize the Euclidean heat balls of Watson and Ecker, but they provide an adaptation different from that of Example 5.3.1 in this setting.

Example 5.3.3 (Weighted Heat Balls on Static Compact Manifolds). We suppose that (M, g) is compact and static and let $\mathrm{P}_{(x_0,t_0)}$ denote the canonical backward heat kernel on M centred at $(x_0, t_0) \in M \times \mathbb{R}$. By Theorem 1.8.5, there exists a neighbourhood $\Omega \subset M$ of $x_0 \in M$ and $\tau_0 \in \,]t_0 - 1, t_0[$ such that

$$\frac{1}{2}\Phi_{\text{fml}} - 1 \leqslant \mathrm{P}_{(x_0,t_0)} \leqslant 2\Phi_{\text{fml}} + 1 \tag{5.7}$$

on $\Omega \times [\tau_0, t_0[$.[2] Hence, we set $\mathcal{D} = \Omega \times [\tau_0, t_0[$ and define the map

$$\begin{aligned}
{}^m\mathrm{P}_{(x_0,t_0)} &: \mathcal{D} \to \mathbb{R}^+ \\
(x,t) &\mapsto (4\pi(t_0 - t))^{\frac{n-m}{2}} \mathrm{P}_{(x_0,t_0)}(x,t).
\end{aligned}$$

We claim that $E^m_r({}^m\mathrm{P}_{(x_0,t_0)})$ is a heat ball for

$$r < r_0 := \frac{1}{2}\min\left\{9^{-1/m}, \left(1 + 2\varrho^{-m}\right)^{-1/m}\right\},$$

where $\varrho = \min\left\{\sqrt{4\pi(t_0-\tau_0)}, \sqrt{\frac{2\pi e}{m}}\sup\{y \in \mathbb{R}^+ : B_y(x_0) \Subset \Omega\}, r_0 \text{ of Example 5.3.1}\right\}$.

To simplify notation, we write P for ${}^m\mathrm{P}_{(x_0,t_0)}$ and ρ for $\log \mathrm{P}$. Now, (5.7) by $(t_0 - t)^{\frac{n-m}{2}}$ and noting that $t_0 - t < 1$ for $t \in [\tau_0, t_0[$, it is clear that

$$\frac{1}{2}\Phi - 1 \leqslant \mathrm{P} \leqslant 2\Phi + 1, \tag{5.8}$$

on $\mathcal{D}$, where Φ is as in Example 5.3.1.

[2] This may be seen by first applying Theorem 1.8.5 with $\varepsilon = 1$, then restricting our attention to a smaller neighbourhood of x_0 on which the coefficient of Φ_{fml} lies between $\frac{1}{2}$ and 2.

(HB1) (5.8) immediately implies that

$$E^m_{r_0}(\mathrm{P}) = (\mathrm{P} > \frac{1}{(r_0)^m}) \subset \mathcal{D} \cap (2\Phi + 1 > \frac{1}{(r_0)^m})$$
$$= \mathcal{D} \cap E^m_{\widetilde{r}_0}(\Phi),$$

where $\widetilde{r}_0 = \left(\frac{2}{\frac{1}{(r_0)^m} - 1}\right)^{1/m}$ and, by (5.2),

$$E^m_{\widetilde{r}_0}(\Phi) = \bigcup_{t \in \,]t_0 - \frac{\widetilde{r}_0^2}{4\pi}, t_0[} B^t_{R^m_{\widetilde{r}_0}(t)}(x_0) \times \{t\} \subset B_{\varrho_0}(x_0) \times \,]t_0 - \frac{\widetilde{r}_0^2}{4\pi}, t_0[,$$

where $\varrho_0 = \sqrt{\frac{m}{2\pi e}} \widetilde{r}_0$ (cf. Example 5.3.1). In view of the choice of r_0 above, it is easily verified that $B_{\varrho_0}(x_0) \Subset \Omega$ and $]t_0 - \frac{\widetilde{r}_0^2}{4\pi}, t_0[\subset \,]\tau_0, t_0[$, whence

$$E^m_{r_0}(\mathrm{P}) \subset E^m_{\widetilde{r}_0}(\Phi) \tag{5.9}$$

and

$$E^m_{\widetilde{r}_0}(\mathrm{P}) \cap \mathrm{pr}_2^{-1}(]\tau_0, \tau[) \subset \overline{B_{\varrho_0}(x_0)} \times [t_0 - \frac{\widetilde{r}_0^2}{4\pi}, \tau]$$
$$\subset \Omega \times \,]\tau_0, t_0[\, = \mathcal{D}.$$

(HB2) By Theorem 1.8.6,

$$|\nabla\rho(x,t)|^2 \leqslant \frac{C}{t_0 - t}\left(\log B - \log\left((4\pi(t_0 - t))^{m/2} \cdot \mathrm{P}(x,t)\right)\right)$$
$$\leqslant \frac{C}{t_0 - t}\left(\log B - \frac{m}{2}\log(4\pi(t_0 - t)) + m \log r_0\right)$$
$$\leqslant \frac{C}{t_0 - t}\left(\log B - \frac{m}{2}\log(4\pi(t_0 - t))\right)$$

on $E^m_{r_0}(\mathrm{P})$, where we have used the fact that $\log \mathrm{P}(x,t) \geqslant m \log r_0$ on $E^m_{r_0}(\mathrm{P})$. Likewise, we also have that

$$\partial_t \rho \geqslant -\frac{F}{t_0 - t}\left(1 + \log\left(\frac{B}{(4\pi(t_0 - t))^{m/2} \cdot \mathrm{P}}\right)\right) + \frac{m - n}{2(t_0 - t)}$$
$$\geqslant -\frac{F}{t_0 - t}\left(1 + \log B - \frac{m}{2}\log(4\pi(t_0 - t))\right) + \frac{m - n}{2(t_0 - t)}$$

on $E^m_{r_0}(\mathrm{P})$. Finally, again by Theorem 1.8.6, the upper bound

$$\partial_t \rho \leqslant \frac{n\left(e^{2K(t_0 - t)} - 1\right) + m}{2(t_0 - t)} - e^{-2K(t_0 - t)}|\nabla\rho|^2,$$

holds, where $K > 0$ is such that $\mathrm{Ric} \geqslant -Kg$. Thus, it suffices to show that $(x,t) \mapsto \frac{1}{t_0 - t}$ and $(x,t) \mapsto \frac{\log(4\pi(t_0 - t))}{t_0 - t}$ are in $L^1\left(E^m_{r_0}(\mathrm{P})\right)$. In light of the inclusion (5.9), these functions are summable over $E^m_{r_0}(\mathrm{P})$ if they are summable over $E^m_{\widetilde{r}_0}(\Phi)$.

Now, the former function was already shown to be summable in Example 5.3.1. As for the latter, in light of the volume comparison argument used in Example 5.3.1 (HB2), we may bound the latter integral in modulus from above by a constant depending on a lower sectional curvature bound times

$$\begin{aligned}
&\int_{t_0-\frac{\tilde{r}_0^2}{4\pi}}^{t_0}\int_{B_{R^m_{\tilde{r}_0}(t)}}\frac{|\log\left(4\pi(t_0-t)\right)|}{t_0-t}\mathrm{d}x\mathrm{d}t\\
&\leqslant \omega_n(2m)^{n/2}\int_{t_0-\frac{\tilde{r}_0^2}{4\pi}}^{t_0}\sqrt{(t-t_0)^{n-2}\left[\log\left(\frac{4\pi(t_0-t)}{\tilde{r}_0^2}\right)\right]^{n+2}}\mathrm{d}t\\
&\qquad + 2|\log\tilde{r}_0|\int_{t_0-\frac{\tilde{r}_0^2}{4\pi}}^{t_0}\int_{B_{R^m_{\tilde{r}_0}(t)}}\frac{1}{t_0-t}\mathrm{d}x\mathrm{d}t\\
&=\omega_n\left(\frac{2m\tilde{r}_0^2}{4\pi}\right)^{n/2}\int_0^\infty\left(s^{1+n/2}+2\log\tilde{r}_0\cdot s^{n/2}\right)\exp\left(-\frac{n}{2}s\right)\mathrm{d}s<\infty,
\end{aligned}$$

where a change of variables identical to that in Example 5.3.1 was carried out.

(HB3) Since $r_0 < 9^{-1/m}$ and hence $\tilde{r}_0 < 4^{-1/m}$, it is clear that $\Phi > 4$ on $E^m_{\tilde{r}_0}(\Phi)$, whence

$$1 \leqslant \mathrm{P} \leqslant \frac{5}{4}\Phi \Rightarrow 0 \leqslant \rho \leqslant \log\frac{5}{4} + \log\Phi$$

on $E^m_{r_0}(\mathrm{P})$. Thus, to establish (HB3) it suffices to show that

$$\lim_{\tau\nearrow t_0}\int_{\mathrm{pr}_1(E^m_{\tilde{r}_0}(\Phi)\cap(M\times\{\tau\}))}|\log\Phi|\,\mathrm{dvol}_{g_\tau}=0,$$

but this was established in Example 5.3.4 (HB3). □

We now turn our attention to heat balls obtained by pulling back those of Examples 5.3.1 and 5.3.3 by mean curvature flow (cf. §1.12 for notation and setup) in the appropriate sense. Such heat balls were first considered in the case $(M,g) = (\mathbb{R}^n,\delta)$ by Ecker [18] in a slightly different light. The following example is— in the class of maps considered— a generalization of the heat balls introduced there.

Example 5.3.4 (Formal Heat Balls Pulled Back by MCF). Suppose $F : N^m \times]t_0-\delta_1,t_0[\to M^n$ evolves by mean curvature flow such that the map $(F,\mathrm{pr}_2) : N\times]t_0-\delta_1,t_0[\to M\times]t_0-\delta_1,t_0[$ is proper and fix R_0 as in Theorem 1.12.7. Consider $\underline{\Phi} := {}^m\Phi\circ(F,\mathrm{pr}_2) : (F,\mathrm{pr}_2)^{-1}\left(\mathcal{D}_{j_0,\delta}(x_0,t_0)\right)\to\mathbb{R}^+$ with ${}^m\Phi$ as in Example 5.3.1. We claim that $E^m_r\left(\underline{\Phi_{\mathrm{fml}}}\right) = (F,\mathrm{pr}_2)^{-1}\left(E^m_r\left(\Phi_{\mathrm{fml}}\right)\right)$ is a heat ball for $r < r_0 := \min\{j_0\cdot\sqrt{\frac{2\pi e}{m}}, \sqrt{4\pi\min\{\delta_1,\delta\}}, \frac{1}{2\alpha}R_0, 1\}$. We verify the conditions.

(HB1) By Example 5.3.1 (HB1),

$$\overline{E^m_{r_0}(\Phi)\cap\mathrm{pr}_2^{-1}\left(]t_0-\delta,\tau[\right)}\subset\mathcal{D}_{j_0,\delta}(x_0,t_0),$$

which, when pulled back by (F,pr_2), implies that

$$\begin{aligned}
\overline{E^m_{r_0}(\underline{\Phi})\cap\mathrm{pr}_2^{-1}(]t_0-\delta,\tau[)} &\subset (F,\mathrm{pr}_2)^{-1}\left(\overline{E^m_{r_0}(\Phi)\cap\mathrm{pr}_2^{-1}\left(]t_0-\delta,\tau[\right)}\right)\\
&\subset (F,\mathrm{pr}_2)^{-1}\left(\mathcal{D}_{j_0,\delta}(x_0,t_0)\right).
\end{aligned}$$

On the other hand, $(F,\mathrm{pr}_2)^{-1}\left(\mathcal{D}_{j_0,\delta}(x_0,t_0)\right)$ is relatively compact, whence (HB1) follows.

(HB2) It is clear that

$$E^m_{r_0}(\underline{\Phi}) = (F,\mathrm{pr}_2)^{-1}\left(E^m_{r_0}(\Phi)\right)$$

$$= (F, \mathrm{pr}_2)^{-1}\left(\bigcup_{t\in\left]t_0-\frac{r_0^2}{4\pi},t_0\right[} B^t_{R^m_{r_0}(t)}(x_0)\times\{t\}\right) \tag{5.10}$$
$$= \bigcup_{t\in\left]t_0-\frac{r_0^2}{4\pi},t_0\right[} \underline{B}^t_{R^m_{r_0}(t)}(x_0)\times\{t\}.$$

Now, we note that, by the chain rule, the Cauchy-Schwarz inequality and Young's inequality, the inequality

$$|\partial_t\underline{\phi}| = |\underline{\partial_t\phi} + \left\langle H, \underline{\nabla\phi}\right\rangle| \leqslant |\underline{\partial_t\phi}| + \frac{1}{2}\left(|\underset{\smile}{H}|^2 + \underline{|\nabla\phi|^2}\right)$$

holds. Moreover, since

$$\underset{\smile}{\nabla_g\phi} = \nabla^T\phi + \nabla^\perp\phi,$$

where $\nabla^T\phi(x,t) = ((x,t), \mathrm{d}_x F_t\left(\nabla_{\mathfrak{I}_t}(\underline{\phi})(x,t)\right))$, it is clear that

$$|\nabla_{\mathfrak{I}}(\underline{\phi})| = \sqrt{|\underset{\smile}{\nabla_g\phi}|^2 - |\nabla^\perp\phi|^2} \leqslant |\underset{\smile}{\nabla_g\phi}| = \underline{|\nabla_g\phi|}.$$

Hence, in view of these two inequalities, (5.10) and the gradient and time-derivative bounds in Proposition 1.8.8 (cf. Example 5.3.1 (HB2)), it suffices to show that

$$\int_{t_0-\frac{r_0^2}{4\pi}}^{t_0}\int_{\underline{B}^t_{R^m_{r_0}(t)}(x_0)} \frac{\underline{\mathrm{r}}^2}{(t_0-t)^2}\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t < \infty \tag{5.11}$$

with $\mathrm{r}(x,t) := d^t(x_0, x)$ and

$$\int_{t_0-\frac{r_0^2}{4\pi}}^{t_0}\int_{\underline{B}^t_{R^m_{r_0}(t)}(x_0)} \frac{1}{t_0-t}\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t < \infty \tag{5.12}$$

since, by Theorem 1.12.5,[3]

$$\int_{t_0-\frac{R_0^2}{4\gamma}}^{t_0}\int_{\underline{B}^t_{\sqrt{\frac{m}{2\pi e}}r_0}(x_0)(x_0)} |\underset{\smile}{H}|^2\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t \leqslant 16\exp\left(\frac{m\lambda_\infty R_0^2}{8\gamma}\right)\left(\int_{\underline{B}^{\cdot}_{R_0}(x_0)}\mathrm{dvol}_{\mathfrak{I}_\cdot}\right)(t_0 - \frac{R_0^2}{4\gamma}) < \infty,$$

which establishes that $|\underset{\smile}{H}|^2 \in L^1\left(E^m_{r_0}(\underline{\Phi})\right)$, since

$$E^m_{r_0}(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}(]t_0 - \frac{R_0^2}{4\gamma}, t_0[) \subset \bigcup_{t\in\,]t_0-\frac{R_0^2}{4\gamma},t_0[} \underline{B}^t_{\sqrt{\frac{m}{2\pi e}}r_0}(x_0)\times\{t\}$$

and $|\underset{\smile}{H}|^2 \in L^1\left(E^m_{r_0}(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}(]t_0 - \frac{r_0^2}{4\pi}, t_0 - \frac{R_0^2}{4\gamma}])\right)$, since $E^m_{r_0}(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}(]t_0 - \frac{r_0^2}{4\pi}, t_0 - \frac{R_0^2}{4\gamma}])$ is relatively compact in the domain of F by (HB1).

Now, in light of Theorem 1.12.7 and the fact that $\underline{\mathrm{r}} < R^m_{r_0}(t)$ on $\underline{B}^t_{R^m_{r_0}(t)}(x_0)\times\{t\}$, the estimate

[3] We note here that $\sqrt{\frac{m}{2\pi e}}r_0 \leqslant \sqrt{\frac{m}{2\pi e}}\cdot\frac{1}{2}\cdot\sqrt{\frac{\pi}{2\gamma}}R_0 < \frac{1}{2}R_0$, since $\gamma \geqslant 2m$.

$$\begin{aligned}\int_{\underline{B}^t_{R^m_{r_0}(t)}(x_0)} \frac{\underline{\mathrm{r}}^2}{(t_0-t)^2}\mathrm{dvol}_{\mathfrak{J}_t} &\leqslant \frac{R^m_{r_0}(t)^2}{(t_0-t)^2}\int_{\underline{B}^t_{R^m_{r_0}(t)}(x_0)} \mathrm{dvol}_{\mathfrak{J}_t}\\ &\leqslant \gamma_1 \frac{R^m_{r_0}(t)^{m+2}}{(t_0-t)^2},\end{aligned}$$

holds for $t \in \,]\tau, t_0[$ for $\tau = t_0 - \frac{\exp(-\frac{1}{2m})}{4\pi} r_0^2$, where

$$\gamma_1 = \left(\rho_0 + \frac{\rho_1}{R^m}\right)\left(\int_{\underline{B}^{\cdot}_{R_0}(x_0)} \mathrm{dvol}_{\mathfrak{J}_\cdot}\right)(t_0 - \frac{R_0^2}{4\gamma})$$

with ρ_0 and ρ_1 as in Theorem 1.12.7. Likewise, we have the estimate

$$\int_{\underline{B}^t_{R^m_{r_0}(t)}(x_0)} \frac{1}{t_0-t}\mathrm{dvol}_{\mathfrak{J}_t} \leqslant \gamma_1 \frac{R^m_{r_0}(t)^m}{t_0-t}$$

for $t \in \,]\tau, t_0[$ so that the statements (5.11) and (5.12) are true if

$$\int_\tau^{t_0} \frac{R^m_{r_0}(t)^{m+2}}{(t_0-t)^2}\mathrm{d}t < \infty$$

and

$$\int_\tau^{t_0} \frac{R^m_{r_0}(t)^m}{t_0-t}\mathrm{d}t < \infty,$$

are finite, since finiteness of these two integrals establishes that $|\nabla_{\mathfrak{J}}\underline{\phi}|^2$ and $\partial_t\underline{\phi}$ are in

$$L^1\left(E^m_{r_0}(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}(]\tau, t_0[)\right),$$

and, by (HB1), the relative compactness of $E^m_{r_0}(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}\left(\left]t_0 - \frac{r_0^2}{4\pi}, \tau\right[\right)$ in the domain of F implies summability on the rest of $E^m_{r_0}(\underline{\Phi})$ since $\nabla_{\mathfrak{J}}\underline{\phi}$ and $\partial_t\underline{\phi}$ are smooth.
Now, the former integral is equal to

$$(2m)^{m/2+1}\int_\tau^{t_0} \sqrt{(t-t_0)^{m-2}\left[\log\left(\frac{4\pi(t_0-t)}{r_0^2}\right)\right]^{m+2}}\mathrm{d}t$$

which, in light of the finiteness of the integral (5.4) of Example 5.3.1, is finite. Likewise, the latter integral is equal to

$$(2m)^m\int_\tau^{t_0} \sqrt{(t-t_0)^{m-2}\left[\log\left(\frac{4\pi(t_0-t)}{r_0^2}\right)\right]^{m}}\mathrm{d}t$$

which is also finite in light of the finiteness of the integral (5.5) of Example 5.3.1.

(HB3) In light of (5.10), $\mathrm{pr}_1\left(E^m_{r_0}(\underline{\Phi}) \cap (M \times \{\tau\})\right) = \underline{B}^\tau_{R^m_{r_0}(\tau)}(x_0)$. On the other hand,

$$|\underline{\phi}(\cdot,\tau)| = \left|\frac{\underline{r}^2}{4(\tau-t_0)} - \frac{m}{2}\log\left(4\pi(t_0-\tau)\right)\right| \leqslant \frac{R^m_{r_0}(\tau)^2}{4(t_0-\tau)} + \frac{m}{2}\left(-\log(4\pi(t_0-\tau))\right)$$

on $\underline{B}^{\tau}_{R^m_{r_0}(\tau)}(x_0)$. Therefore, making use of Theorem 1.12.7 again as in (HB2), we see that it suffices to show that

$$\lim_{\tau\nearrow t_0} \frac{R^m_{r_0}(\tau)^{m+2}}{t_0-\tau} = 0$$

and

$$\lim_{\tau\nearrow t_0} R^m_{r_0}(\tau)\log\left(4\pi(t_0-\tau)\right) = 0$$

or, more explicitly,

$$\lim_{\tau\nearrow t_0} \sqrt{(\tau-t_0)^m\left[\log\left(\frac{4\pi(t_0-\tau)}{r_0^2}\right)\right]^{m+2}} = 0$$

and

$$\lim_{\tau\nearrow t_0} \sqrt{(\tau-t_0)^m\left[\log(4\pi(t_0-\tau))\right]^{m+2} - (\tau-t_0)^m\log(r_0^2)} = 0.$$

We know, however, by making the same change of variables as in Example 5.3.1 (HB3) that both of these statements hold true, i.e. by noting that

$$\lim_{\tau\nearrow t_0}(\tau-t_0)^m\left[\log\left(\frac{4\pi(t_0-\tau)}{r_0^2}\right)\right]^{m+2} = \left(\frac{r_0^2}{4\pi}\right)^m \lim_{s\to\infty} s^{m+2}\exp(-ms) = 0. \quad \square$$

Whilst not quite being a generalization of the heat balls introducted by Ecker [18], the following example provides an adaptation of his construction to the case where F evolves by MCF into a static compact manifold.

Example 5.3.5 (Heat Balls on Static Compact Manifolds Pulled Back by MCF). We suppose that (M,g) is compact and static and that F is as in Example 5.3.4. We take $\mathrm{P} : \mathcal{D} \to \mathbb{R}^+$ to be the canonical backward heat kernel on M centred at $(x_0,t_0) \in M \times \,]t_0-\delta_1,t_0[$ as in Example 5.3.3, taking $\tau_0 > t_0-\delta$ if necessary and claim that $E^m_r(\underline{\mathrm{P}})$ is a heat ball for

$$r < r_0 := \min\left\{9^{-1/m}, \left(1+2\varrho^{-m}\right)^{-1/m}, \right\},$$

where $\varrho = \min\left\{\sqrt{4\pi(t_0-\tau_0)}\sqrt{\frac{2\pi e}{m}}\sup\{y\in\mathbb{R}^+ : B_y(x_0)\Subset\Omega\},\ r_0 \text{ of Example 5.3.4}\right\}$.

We note the bound from Example 5.3.3 pulled back to $N\times\,]\tau_0,t_0[$:

$$\frac{1}{2}\underline{\Phi} - 1 \leqslant \underline{\mathrm{P}} \leqslant 2\underline{\Phi} + 1,$$

on $(F,\mathrm{pr}_2)^{-1}(\mathcal{D})$, where $\underline{\Phi}$ is as in Example 5.3.4.

(HB1) The above bound immediately implies that

$$E^m_{r_0}(\underline{\mathrm{P}}) = \left(\underline{\mathrm{P}} > \frac{1}{(r_0)^m}\right) \subset (F,\mathrm{pr}_2)^{-1}(\mathcal{D}) \cap \left(\xi_0\underline{\Phi}+1 > \frac{1}{(r_0)^m}\right)$$
$$= (F,\mathrm{pr}_2)^{-1}(\mathcal{D}) \cap E^m_{\tilde{r}_0}(\underline{\Phi}),$$

where $\widetilde{r}_0 = \left(\frac{2}{\frac{1}{(r_0)^m}-1}\right)^{1/m}$. By Example 5.3.4,

$$E^m_{\widetilde{r}_0}(\underline{\Phi}) = \bigcup_{t\in\,]t_0-\frac{\widetilde{r}_0^2}{4\pi},t_0[} \underline{B}^t_{R^m_{\widetilde{r}_0}(t)}(x_0)\times\{t\} \subset \bigcup_{t\in\,]t_0-\frac{\widetilde{r}_0^2}{4\pi},t_0[} \underline{B}^t_{\varrho_0}(x_0)\times\ \{t\}$$
$$= (F,\mathrm{pr}_2)^{-1}\left(B_{\varrho_0}(x_0)\times\,]t_0-\frac{\widetilde{r}_0^2}{4\pi},t_0[\right),$$

where $\varrho_0 = \sqrt{\frac{m}{2\pi e}}\widetilde{r}_0$ (cf. Example 5.3.1). In view of the choice of r_0 above, it is easily seen that $B_\varrho(x_0)\Subset\Omega$. Thus, we see that

$$E^m_{\widetilde{r}_0}(\underline{\mathrm{P}})\cap \mathrm{pr}_2^{-1}(]\tau_0,\tau[) \subset (F,\mathrm{pr}_2)^{-1}\left(\overline{B_{\varrho_0}(x_0)}\times[t_0-\frac{\widetilde{r}_0^2}{4\pi},\tau_0]\right)$$
$$\subset (F,\mathrm{pr}_2)^{-1}(\Omega\times\,]\tau_0,t_0[) = (F,\mathrm{pr}_2)^{-1}(\mathcal{D}).$$

(HB2) As in Example 5.3.4 (HB2), we note that

$$|\nabla_{\mathfrak{I}}\underline{\rho}| \leqslant \underline{|\nabla_g\rho|} \text{ and}$$
$$|\partial_t\underline{\rho}| \leqslant \underline{|\partial_t\rho|} + \frac{1}{2}\left(|\,\underset{\frown}{\underline{H}}\,|^2 + \underline{|\nabla_g\rho|^2}\right)$$

with $\underline{\rho} = \log\underline{\mathrm{P}}$. Firstly, since $E^m_{r_0}(\underline{\mathrm{P}}) \subset E^m_{\widetilde{r}_0}(\underline{\Phi})$ and r_0 was chosen such that $\widetilde{r}_0$ does not exceed the r_0 of Example 5.3.4, it is clear that $|\,\underset{\frown}{\underline{H}}\,|^2$ is $L^1\left(E^m_{r_0}(\underline{\mathrm{P}})\right)$. Secondly, since $E^m_{r_0}(\underline{\mathrm{P}}) = (F,\mathrm{pr}_2)^{-1}\left(E^m_{r_0}({}^m\mathrm{P}_{(x_0,t_0)})\right)$ with ${}^m\mathrm{P}_{(x_0,t_0)}$ as in Example 5.3.3, it follows from Example 5.3.3 (HB2) that the bounds

$$\underline{|\nabla\rho|^2} \leqslant \frac{C}{t_0-t}\left(\log B - \frac{m}{2}\log(4\pi(t_0-t))\right),$$
$$\underline{\partial_t\rho} \geqslant -\frac{F}{t_0-t}\left(1+\log B-\frac{m}{2}\log(4\pi(t_0-t))\right) + \frac{m-n}{2(t_0-t)}$$

and

$$\underline{\partial_t\rho} \leqslant e^{2K(t_0-t)}\frac{n\left(e^{2K(t_0-t)}-1\right)+m}{2(t_0-t)} - e^{-2K(t_0-t)}\underline{|\nabla\rho|^2}$$

hold on $E^m_{r_0}(\underline{\mathrm{P}})$, where we retain the notation of Example 5.3.3 (HB2). It therefore suffices to show that $(x,t)\mapsto\frac{1}{t_0-t}$ and $(x,t)\mapsto\frac{\log(4\pi(t_0-t))}{t_0-t}$ are in $L^1\left(E^m_{\widetilde{r}_0}(\underline{\Phi})\right)$ and thus in $L^1\left(E^m_{r_0}(\underline{\mathrm{P}})\right)$. An inspection of the computation in Example 5.3.4 (HB2) establishes that $\left((x,t)\mapsto\frac{1}{t_0-t}\right)\in L^1\left(E^m_{\widetilde{r}_0}(\underline{\Phi})\right)$, whereas it suffices to show that $\left((x,t)\mapsto\frac{\log(4\pi(t_0-t))}{t_0-t}\right)\in L^1\left(E^m_{\widetilde{r}_0}(\underline{\Phi})\cap\mathrm{pr}_2^{-1}(]\tau,t_0[)\right)$ for $\tau = t_0-\exp(-\frac{1}{2m})\frac{\widetilde{R}_0^2}{4\gamma}$ as in Example 5.3.4 (HB2) since $E^m_{\widetilde{r}_0}(\underline{\Phi})\cap \mathrm{pr}_2^{-1}(]t_0-\delta,\tau_0[)$ is relatively compact and $(x,t)\mapsto\frac{\log(4\pi(t_0-t))}{t_0-t}$ is bounded on this set. Thus, the integral we are left to establish the finiteness of may, by Theorem 1.12.7, be estimated thus:

$$\int_\tau^{t_0}\int_{B_{R^m_{\widetilde{r}_0}(t)}(x_0)}\frac{\log(4\pi(t_0-t))}{t_0-t}\mathrm{d}x\mathrm{d}t$$
$$\leqslant \gamma_1\left((2m)^{m/2}\int_\tau^{t_0}\sqrt{(t-t_0)^{m-2}\left[\log\left(\frac{4\pi(t_0-t)}{\widetilde{r}_0^2}\right)\right]^{m+2}} + 2\log\widetilde{r}_0\frac{R^m_{\widetilde{r}_0}(t)^m}{t_0-t}\mathrm{d}t\right)$$

$$= \gamma_1 \left(\frac{2m\tilde{r}_0^2}{4\pi}\right)^{m/2} \int_a^\infty \left(s^{1+m/2} + 2\log\tilde{r}_0 \cdot s^{m/2}\right) \exp\left(-\frac{m}{2}s\right) \mathrm{d}s < \infty.$$

Here γ_1 is as in Example 5.3.4 (HB2) and $a = -\log\left(\frac{4\pi(t_0-\tau)}{\tilde{r}_0^2}\right)$.

(HB3) Since $r_0 < 9^{-1/m}$ and hence $\tilde{r}_0 < 4^{-1/m}$, we have that $\underline{\Phi} > 4$ on $E^m_{\tilde{r}_0}(\underline{\Phi})$ so that

$$1 \leqslant \underline{\mathrm{P}} \leqslant \frac{5}{4}\underline{\Phi} \Rightarrow 0 \leqslant \underline{\rho} \leqslant \log\frac{5}{4} + \log\underline{\Phi}$$

on $E^m_{r_0}(\underline{\mathrm{P}})$. Hence, to establish (HB3) it suffices to show that

$$\lim_{\tau \nearrow t_0} \int_{\mathrm{pr}_1(E^m_{\tilde{r}_0}(\underline{\Phi}) \cap (N \times \{\tau\}))} |\log\underline{\Phi}| \mathrm{dvol}_{g_\tau} = 0,$$

but this was shown to hold in Example 5.3.4. □

Remark 5.3.6. Note that the approach taken in Examples 5.3.4 and 5.3.5 is different from that taken by Ecker [18] in that heat balls were considered as subsets of the *parameter space* $N \times]t_0 - \delta_1, t_0[$ as opposed to being subsets of $M \times]t_0 - \delta_1, t_0[$. In our setting, both approaches are equivalent. However, Ecker's approach more readily generalizes to the varifold setting of Brakke [8].

For later purposes (cf. Theorems 7.3.2 and 7.3.4), we shall need to know that if $E^m_r(\Phi)$ is a heat ball for sufficiently small r, then so is $E^m_r(\Phi \cdot \eta)$ provided η is a sufficiently regular function. This motivates the following example.

Example 5.3.7 (Modified Heat Balls). Let $E^m_r(\Phi)$ be any (m,Φ)-heat ball and let $\eta \in L^\infty(E^m_{r_0}(\Phi)) \cap C^1(E^m_{r_0}(\Phi))$ such that $|\nabla\eta|^2$ and $\partial_t\eta \in L^1(E^m_{r_0}(\Phi))$. Set $\tilde{\Phi} := e^\eta \cdot \Phi|_{E^m_{r_0}(\Phi)}$. If we write

$$\eta_{-\infty} \leqslant \eta \leqslant \eta_\infty$$

for $\eta_{\pm\infty} \in \mathbb{R}^\pm$, then

$$e^{\eta_{-\infty}}\Phi \leqslant \tilde{\Phi} \leqslant e^{\eta_\infty}\Phi,$$

whence

$$\left(\Phi > \min\{r, r_0\}^{-m}\right) \subset \left(\tilde{\Phi} > (re^{-\eta_{-\infty}/m})^{-m}\right)$$
$$\text{and } \left(\tilde{\Phi} > r^{-m}\right) \subset \left(\Phi > (\min\{r_0, re^{\eta_\infty/m}\})^{-m}\right),$$

so that

$$E^m_{\min\{r_0, r\exp(\eta_{-\infty}/m)\}}(\Phi) \subset E^m_r(\tilde{\Phi}) \subset E^m_{\min\{r_0, r\exp(\eta_\infty/m)\}}(\Phi), \tag{5.13}$$

which in turn implies that $L^1(E^m_{r\exp(\eta_\infty/m)}(\Phi)) \hookrightarrow L^1(E^m_r(\tilde{\Phi}))$.
Set $\tilde{r}_0 := r_0\exp(-\eta_\infty/m)$ so that $\tilde{r}_0 \in]0, r_0[\subset]0,1[$. We now verify (HB1)-(HB3).

(HB1) (5.13) immediately implies that

$$E^m_{\tilde{r}_0}(\tilde{\Phi}) \cap \mathrm{pr}_2^{-1}(]t_0 - \delta, \tau[) \subset E^m_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(]t_0 - \delta, \tau[) \Subset \mathcal{D}$$

for every $\tau \in]t_0 - \delta, t_0[$.

(HB2) If $\widetilde{\phi} := \log\widetilde{\Phi}$, then $\widetilde{\phi} = \phi + \eta$, whence, in view of (5.13) and the following remark, the assumptions on ϕ and η imply that $\partial_t\widetilde{\phi} = \partial_t\phi + \partial_t\eta \in L^1(E^m_{\widetilde{r}_0}(\widetilde{\Phi}))$ and, since $|\nabla(\phi+\eta)|^2 \leqslant 2\left(|\nabla\phi|^2 + |\nabla\eta|^2\right)$, we also have that $|\nabla(\widetilde{\phi})|^2 \in L^1(E^m_{\widetilde{r}_0}(\widetilde{\Phi}))$.

(HB3) By (5.13), it suffices to show that

$$\lim_{\tau\nearrow t_0}\int_{\mathrm{pr}_1\left(E^m_{r_0}(\Phi)\cap\mathrm{pr}_2^{-1}(\{\tau\})\right)} |\widetilde{\phi}|\mathrm{dvol}_{g_\tau} = 0,$$

but $|\widetilde{\phi}| \leqslant |\phi| + |\eta| \leqslant |\phi| + \underbrace{\max\{\eta_\infty, -\eta_{-\infty}\}}_{=:G}$, whence by Remark 5.2.4,

$$\begin{aligned}&\lim_{\tau\nearrow t_0}\int_{\mathrm{pr}_1\left(E^m_{r_0}(\Phi)\cap\mathrm{pr}_2^{-1}(\{\tau\})\right)} |\widetilde{\phi}|\mathrm{dvol}_{g_\tau}\\ &\leqslant \lim_{\tau\nearrow t_0}\int_{\mathrm{pr}_1\left(E^m_{r_0}(\Phi)\cap\mathrm{pr}_2^{-1}(\{\tau\})\right)} |\phi|\mathrm{dvol}_{g_\tau} + G\lim_{\tau\nearrow t_0}\mathrm{Vol}_{g_\tau}\left(\mathrm{pr}_1\left(E^m_{r_0}(\Phi)\cap(M\times\{\tau\})\right)\right) = 0.\end{aligned}$$

We thus call the $E^m_r(\widetilde{\Phi})$ **η-modified (m,Φ)-heat balls**, or simply **modified heat balls**. □

5.4. Integration formulæ. We now derive integration formulæ for integrals over heat balls in the spirit of [18] and [20]. These shall be used repeatedly in the sequel. To this end, we shall consider the "approximate integrals"

$$J^r_q(f) := \iint f\cdot(\chi_q\circ\phi^m_r)\cdot(\chi_{]t_0-\delta_0,t_0-q^{-1}[}\circ\mathrm{pr}_2)\mathrm{dvol}_{g_t}\mathrm{d}t,$$

where χ_q is as in Example A.1, and analyze them, as well as their derivatives with respect to r, in the limit $q\to\infty$. The idea here is that these approximate the heat ball integrals

$$I^r(f) := \iint_{E^m_r(\Phi)} f\mathrm{dvol}_{g_t}\mathrm{d}t$$

which would, with the right conditions on Φ, yield an integral over $\partial E^m_r(\Phi)$ upon differentiation with respect to r (cf. [18]). However, without additional information about Φ, we wouldn't be able to utilize this technique, which is why we follow the approach of [22].

To streamline the proofs of the integration formulæ to follow, we summarize the relevant properties of these approximate integrals in the

Lemma 5.4.1. *Let $f\in L^1\left(E^m_{r_0}(\Phi)\right)$ and suppose J^r_q and I^r are as above. Then*

1. *Whenever $0<r\leqslant r_0$, we have $|J^r_q(f)|\leqslant I^{r_0}(|f|)$ and $r\mapsto J^r_q(f)$ is smooth.*
2. *For every $r\in]0,r_0]$, $J^r_q(f)\xrightarrow{q\to\infty} I^r(f)$.*
3. *Whenever $0<r_1<r_2<r_0$ and $\int_{r_1}^{r_2}\frac{\mathrm{d}}{\mathrm{d}r}J^{\cdot}_q(f)\xrightarrow{q\to\infty}\int_{r_1}^{r_2}J$ with $J\in L^1(]r_1,r_2[)$, the identity*

$$I^{r_2}(f) - I^{r_1}(f) = \int_{r_1}^{r_2} J.$$

In particular, $\frac{\mathrm{d}}{\mathrm{d}r}J^{\cdot}_q(f) = J$ almost everywhere on $]0,r_0[$.

Proof. 1. By the first inequality in Example A.1, it is clear that

$$|(\chi_q\circ\phi^m_r)(\chi_{]t_0-\delta,t_0-q^{-1}[}\circ\mathrm{pr}_2)| \leqslant \chi_{E^m_r(\Phi)} \leqslant \chi_{E^m_{r_0}(\Phi)}, \tag{5.14}$$

which establishes the inequality.

As for smoothness, we note that $|\frac{\mathrm{d}}{\mathrm{d}r}(\chi_q \circ \phi_r^m)| = \frac{m}{r}|\chi_q' \circ \phi_r^m| \leqslant \frac{\mathrm{const(m,q)}}{r}\chi_q \circ \phi_r^m$, which is summable over $[r_1, r_2]$ as a function of r, thus allowing us to differentiate once under the integral sign by standard theorems from integration theory. Smoothness then follows by taking successive derivatives iterating the same argument.

2. The inequality (5.14) immediately implies that

$$\left|f \cdot (\chi_q \circ \phi_r^m) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2)\right| \leqslant |f| \chi_{E_{r_0}^m(\Phi)}$$

and the convergence properties of χ_q (see Example A.1) that

$$\lim_{q\to\infty} f \cdot (\chi_q \circ \phi_r^m) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2) = f \chi_{E_r^m(\Phi)}.$$

By the dominated convergence theorem, the integral and limit may be interchanged, thus implying the claim.

3. We note that, on the one hand,

$$J_q^{r_2}(f) - J_q^{r_1}(f) = \int_{r_1}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r} J_q^{\cdot}(f).$$

On the other, the left-hand side tends to $I^{r_2}(f) - I^{r_1}(f)$ by the preceding part and the right-hand side tends to $\int_{r_1}^{r_2} J$ by assumption. By the Lebesgue differentiation theorem,

$$\lim_{r_1 \nearrow r} \frac{1}{r - r_1} \int_{r_1}^{r} J = \lim_{r_2 \searrow r} \frac{1}{r_2 - r} \int_{r}^{r_2} J = J(r)$$

for almost every $r \in]0, r_0[$. The equality above then implies the latter claim.

□

Theorem 5.4.2 (Heat ball Gauß). *If $X \in C^1(E_{r_0}^m, TM)$ is a time-dependent section of TM such that* $\operatorname{div} X \in L^1(E_{r_0}^m(\Phi))$ *and $X \in L^2(E_{r_0}^m(\Phi))$, then*

$$\frac{\mathrm{d}}{\mathrm{d}r} \iint\limits_{E_r^m(\Phi)} \langle X, \nabla\phi \rangle \,\mathrm{dvol}_{g_t}\mathrm{d}t = -\frac{m}{r} \iint\limits_{E_r^m(\Phi)} \operatorname{div} X \mathrm{dvol}_{g_t}\mathrm{d}t$$

holds a.e. on $]0, r_0[$.

Proof. Note that

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_r J_q^{\cdot}(\langle X, \nabla\phi\rangle) &= \frac{m}{r} \iint \underbrace{\langle X, \nabla\phi\rangle \cdot (\chi_q' \circ \phi_r^m)}_{=\langle X, \nabla(\chi_q \circ \phi_r^m)\rangle} \cdot (\chi_{]t_0-\delta, t_0-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{dvol}_{g_t}\mathrm{d}t \\
&= -\frac{m}{r} \iint (\operatorname{div} X) \cdot (\chi_q \circ \phi_r^m) \cdot (\chi_{]t_0-\delta, t_0-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{dvol}_{g_t}\mathrm{d}t \\
&= -\frac{m}{r} J_q^r(\operatorname{div} X),
\end{aligned}$$

where the second line follows from the fact that

$$\operatorname{div}(X \cdot (\chi_q \circ \phi_r^m)) = \left\langle X, \nabla(\chi_q \circ \phi_r^m)\right\rangle + (\operatorname{div} X) \cdot (\chi_q \circ \phi_r^m)$$

and an application of Gauß' theorem. Now, since $\operatorname{div} X$ and $\langle X, \nabla\phi\rangle \in L^1(E_{r_0}(\Phi))$, Lemma 5.4.1 implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_r J_q^{\cdot}(\langle X, \nabla\phi\rangle) = -\frac{m}{r}J_q^r(\operatorname{div} X) \xrightarrow{q\to\infty} -\frac{m}{r}I^r(\operatorname{div} X)$$

whence it follows from the fact that

$$|\frac{-m}{r}J_q^r(\operatorname{div} X)| \leqslant \frac{m}{r_1}I^{r_0}(|\operatorname{div} X|) \in L^1(]r_1, r_2[),$$

for $0 < r_1 < r_2 < r_0$ that

$$\int_{r_1}^{r_2} \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_r J_q^{\cdot}(\operatorname{div} X)\mathrm{d}r \xrightarrow{q\to\infty} \int_{r_1}^{r_2} \left(-\frac{m}{r}I^r(\operatorname{div} X)\right) \mathrm{d}r.$$

An application of Lemma 5.4.1 then yields the result. □

Theorem 5.4.3. *If $f \in C^1(E_{r_0}^m(\Phi)) \cap L^\infty(E_{r_0}^m(\Phi))$ and $\partial_t f \in L^1(E_{r_0}^m(\Phi))$, then*

$$\frac{\mathrm{d}}{\mathrm{d}r} \iint\limits_{E_r^m(\Phi)} f \cdot \partial_t\phi \,\mathrm{dvol}_{g_t}\mathrm{d}t = -\frac{m}{r} \iint\limits_{E_r^m(\Phi)} \partial_t f + \frac{1}{2} f\, tr_g h \,\mathrm{dvol}_{g_t}\mathrm{d}t$$

holds a.e. on $]0, r_0[$.

Proof. We compute again:

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_r J_q^{\cdot}(f\cdot\partial_t\phi) &= \frac{m}{r}\iint f\cdot \underbrace{\partial_t\phi\cdot(\chi_q'\circ\phi_r^m)}_{\partial_t(\chi_q\circ\phi_r^m)}\cdot(\chi_{]t_0-\delta_0,t_0-q^{-1}[})\mathrm{dvol}_{g_t}\mathrm{d}t \\
&= \frac{m}{r}\left(\left[\int (f\cdot(\chi_q\circ\phi_r^m))(\cdot,t)\mathrm{dvol}_{g_t}\right]_{t=t_0-\delta_0}^{t=t_0-q^{-1}}\right. \\
&\quad \left. - \iint (\partial_t f + \frac{1}{2}\mathrm{tr}_g h\cdot f)\cdot(\chi_q\circ\phi_r^m)\cdot(\chi_{]t_0-\delta_0,t_0-q^{-1}[}\circ \mathrm{pr}_2)\mathrm{dvol}_{g_t}\mathrm{d}t\right) \\
&= \frac{m}{r}\left(\int (f\cdot(\chi_q\circ\phi_r^m))(\cdot,t_0-q^{-1})\mathrm{dvol}_{g_{t_0-q^{-1}}} - J_q^r(\partial_t f + \frac{1}{2}\mathrm{tr}_g h\cdot f)\right)
\end{aligned}$$

where in the second line we have integrated by parts with respect to t and in the third we made use of the fact that

$$|\chi_q\circ\phi_r^m|(\cdot,t_0-\delta_0) \leqslant \chi_{E_r^m(\Phi)}(\cdot,t_0-\delta_0) = 0.$$

The last equality follows from (HB1), viz. the fact that $E_r^m(\Phi)\cap \mathrm{pr}_2^{-1}(]t_0-\delta_0,\tau[) \Subset \mathcal{D} \subset M \times]t_0-\delta_0, t_0[$ for $r \leqslant r_0$. Now, on the one hand,

$$|f\cdot(\chi_q\circ\phi_r^m)(\cdot,t_0-q^{-1})| \leqslant ||f||_\infty \cdot \chi_{E_r^m(\Phi)\cap \mathrm{pr}_2^{-1}(t_0-q^{-1})},$$

whence

$$\left|\int (f\cdot(\chi_q\circ\phi_r^m))(\cdot,t_0-q^{-1})\mathrm{dvol}_{g_{t_0-q^{-1}}}\right| \leqslant ||f||_\infty\cdot\mu(E_r^m(\Phi)\cap \mathrm{pr}_2^{-1}(t_0-q^{-1})) \xrightarrow{q\to\infty} 0.$$

On the other hand, $\partial_t f + \frac{1}{2}\mathrm{tr}_g h\cdot f \in L^1(E_r^m(\Phi))$ by the assumptions on f in the theorem and since h is smooth. Finally,

$$|J_q^r(\partial_t f + \frac{1}{2}\mathrm{tr}_g h\cdot f)| \leqslant I^{r_0}(|\partial_t f| + \frac{1}{2}||\mathrm{tr}_g h||_\infty ||f||_\infty||).$$

Thus, utilizing these bounds in the same manner as in the Theorem 5.4.2, it follows from an application of Lemma 5.4.1 that

$$\int_{r_1}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_r J_q^{\cdot}(f \cdot \partial_t \phi)\mathrm{d}r = \int_{r_1}^{r_2} \left(-\frac{m}{r} I_r(\partial_t f + \frac{1}{2}\mathrm{tr}_g h \cdot f) \right) \mathrm{d}r$$

whenever $0 < r_1 < r_2 < r_0$. □

Theorem 5.4.4. *If $f, f\phi_r^m \in L^1(E_{r_0}^m(\Phi))$, then*

$$\frac{\mathrm{d}}{\mathrm{d}r} \iint\limits_{E_r^m(\Phi)} f\, \phi_r^m \mathrm{dvol}_{g_t}\, \mathrm{d}t = \frac{m}{r} \iint\limits_{E_r^m(\Phi)} f\, \mathrm{dvol}_{g_t}\, \mathrm{d}t$$

holds a.e. on $]0, r_0[$.

Proof. We compute yet again:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_r J_q^{\cdot}(f \cdot \phi_r^m) &= \frac{m}{r} \iint f \cdot (\chi_q \circ \phi_r^m) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2) \\
&\quad + f \cdot \left(\phi_r^m \cdot \chi_q' \circ \phi_r^m \right) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{dvol}_{g_t} \mathrm{d}t \\
&= \frac{m}{r} J_q^r(f) + \frac{m}{r} \iint f \cdot (\phi_r^m \cdot \chi_q' \circ \phi_r^m) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{dvol}_{g_t} \mathrm{d}t. \quad (5.15)
\end{aligned}$$

Now, $f \in L^1(E_r^m(\Phi))$ so that just as before the first term in (5.15) tends to $\frac{m}{r} I_r(f)$ as $q \to \infty$. On the other hand, by Example A.1,

$$|\phi_r^m \cdot \chi_q' \circ \phi_r^m| \leqslant C \cdot \chi_{(2^{-(q+1)} < \phi_r^m < 2^{-q})} \xrightarrow{q \to \infty} 0$$

and

$$|f \cdot (\phi_r^m \cdot \chi_q' \circ \phi_r^m) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[})| \leqslant C f \chi_{(\phi_{r_0}^m > 0)} \in L^1(\mathcal{D})$$

so that we may proceed as in the preceding proofs. □

Theorem 5.4.5. *If $f \in C^1(E_{r_0}^m(\Phi)) \cap L^\infty(E_{r_0}^m(\Phi))$ and $\partial_t f \cdot \phi_r^m \in L^1(E_{r_0}^m(\Phi))$, then*

$$\iint\limits_{E_r^m(\Phi)} \partial_t \left(f \cdot \phi_r^m \right) \mathrm{dvol}_{g_t}\, \mathrm{d}t = - \iint\limits_{E_r^m(\Phi)} f \cdot \phi_r^m \cdot \frac{1}{2} tr_g h\, \mathrm{dvol}_{g_t}\, \mathrm{d}t$$

for every $r \in]0, r_0[$.

Proof. We consider

$$\begin{aligned}
J_q^r(\partial_t(f \cdot \phi_r^m)) &= \iint \partial_t(f \cdot \phi_r^m) \cdot (\chi_q \circ \phi_r^m) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{dvol}_{g_t} \mathrm{d}t \\
&= - \iint f \cdot \phi_r^m \cdot \chi_q' \circ \phi_r^m \cdot \chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2 \cdot \partial_t \phi \mathrm{dvol}_{g_t} \mathrm{d}t \quad (5.16) \\
&\quad - \iint f \cdot \phi_r^m \cdot \chi_q \circ \phi_r^m \cdot \chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2 \cdot \frac{1}{2} \mathrm{tr}_g h\, \mathrm{dvol}_{g_t} \mathrm{d}t \\
&\quad + \int \left(f \cdot \phi_r^m \cdot (\chi_q \circ \phi_r^m) \right) (\cdot, t_0 - q^{-1}) \mathrm{dvol}_{g_{t_0-q^{-1}}} \\
&\quad - \int \left(f \cdot \phi_r^m \cdot (\chi_q \circ \phi_r^m) \right) (t_0 - \delta_0) \mathrm{dvol}_{g_{t_0-\delta_0}}
\end{aligned}$$

where we have integrated by parts with respect to t in the second line. Now, as in the proof of Theorem 5.4.4, $\phi_r^m \cdot \chi_q' \circ \phi_r^m \xrightarrow{q\to\infty} 0$ and

$$|f \cdot \phi_r^m \cdot \chi_q' \circ \phi_r^m \cdot \partial_t \phi| \leqslant C||f||_\infty |\partial_t \phi| \chi_{E_{r_0}^m(\Phi)} \in L^1(\Omega \times]t_0 - \delta_0, t_0[),$$

whence the first integral in (5.16) tends to 0 as $q \to \infty$. Furthermore, the second integral in (5.16) is equal to $-J_q^r(f \cdot \phi_r^m \cdot \frac{1}{2}\mathrm{tr}_g h)$ which tends to $-I_r(f \cdot \phi_r^m \cdot \frac{1}{2}\mathrm{tr}_g h)$ as $q \to \infty$ in light of the inequality

$$|f \cdot \phi_r^m \cdot \frac{1}{2}\mathrm{tr}_g h| \leqslant \frac{1}{2}||f||_\infty \cdot ||\mathrm{tr}_g h||_\infty \cdot |\phi_r^m| \in L^1(E_r^m(\Phi)).$$

Moreover, the third integral may be handled by estimating as follows:

$$\left|\int \left(f \cdot \phi_r^m \cdot (\chi_q \circ \phi_r^m)\right)(\cdot, t_0 - q^{-1}) \mathrm{dvol}_{g_{t_0 - q^{-1}}}\right| \\ \leqslant ||f||_\infty \int \left(|\phi_r^m| \cdot \chi_{E_r^m(\Phi)}\right)(\cdot, t_0 - q^{-1}) \mathrm{dvol}_{g_{t_0-q^{-1}}} \xrightarrow{q\to\infty} 0.$$

Finally, the fourth integral is equal to 0 since $\chi_q(\phi_r^m(x, t_0 - \delta_0)) \leqslant \chi_{E_{r_0}^m(\Phi)}(x, t_0 - \delta_0) = 0$ by (HB1) (cf. proof of Theorem 5.4.3). □

Theorem 5.4.6. *If $X \in C^1(E_{r_0}^m, TM)$ is a time-dependent section of TM such that* $\operatorname{div} X \in L^\infty(E_{r_0}^m(\Phi))$ *and $X \in L^2(E_{r_0}^m(\Phi))$, then*

$$\iint\limits_{E_r^m(\Phi)} \operatorname{div}(X \cdot \phi_r^m) \mathrm{dvol}_{g_t} \mathrm{d}t = 0$$

for every $r \in]0, r_0[$.

Proof. As in the preceding proof,

$$\begin{aligned} J_q^r(\operatorname{div}(X \cdot \phi_r^m)) &= \iint \operatorname{div}(X \cdot \phi_r^m) \cdot (\chi_q \circ \phi_r^m) \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{dvol}_{g_t} \mathrm{d}t \\ &= -\iint \Big\langle X, \underbrace{\nabla(\chi_q \circ \phi_r^m)}_{=(\chi_q' \circ \phi_r^m)\cdot \nabla\phi} \Big\rangle \cdot \phi_r^m \cdot (\chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{dvol}_{g_t} \mathrm{d}t \\ &= -\iint \langle X, \nabla\phi\rangle \cdot \underbrace{(\phi_r^m \cdot \chi_q' \circ \phi_r^m)}_{\xrightarrow{q\to\infty} 0} \cdot \chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2 \, \mathrm{dvol}_{g_t} \mathrm{d}t, \end{aligned}$$

where the second line is a consequence of Gauß' theorem. Since

$$\left|\langle X, \nabla\phi\rangle \cdot (\phi_r^m \cdot \chi_q' \circ \phi_r^m) \cdot \chi_{]t_0-\delta_0, t_0-q^{-1}[} \circ \mathrm{pr}_2\right| \\ \leqslant C|\langle X, \nabla\phi\rangle \chi_{E_r^m(\Phi)} \in L^1(\Omega \times]t_0 - \delta_0, t_0[),$$

it follows from the dominated convergence theorem that

$$J_q^r(\operatorname{div}(X \cdot \phi_r^m)) \xrightarrow{q\to\infty} 0.$$ □

— 6 —

Monotonicity of Localized Singular Energies of Dirichlet Type

In this chapter we establish local monotonicity identities for time-dependent vector bundle-valued differential forms over heat balls as introduced in Chapter 5 with the help of the formulæ introduced in that chapter. It is then shown that these identities reduce to local monotonicity formulæ when applied to differential forms satisfying the heat equation, in particular establishing monotonicity formulæ for the Yang-Mills and harmonic map heat flows in various cases where M is curved. The formula for the latter flow generalizes that of [20].

6.1. Review. We briefly recall the local monotonicity formula of Ecker [20] which motivated the considerations here.

Set

$$E_r^{\gamma}(x_0,t_0) = \left\{ (x,t) \in \mathbb{R}^n \times]-\infty, t_0[\, : \frac{1}{(4\pi(t_0-t))^{\frac{n-\gamma}{2}}} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) > \frac{1}{r^{n-\gamma}} \right\}$$

as in §5.1. It was shown by Ecker [20] that if $u : \mathbb{R}^n \times]0,T[\to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow, where N is a Riemannian submanifold of $\mathbb{R}^K$, then the local monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-2}} \iint_{E_r^2(x_0,t_0)} \frac{n-2}{2(t_0-t)}|\mathrm{d}u|^2 - \left\langle \sum_{i=1}^n \frac{(x-x_0)^i}{2(t-t_0)}\partial_i u, \partial_t u + \sum_{i=1}^n \frac{(x-x_0)^i}{2(t-t_0)}\partial_i u \right\rangle \mathrm{d}x\mathrm{d}t \right)$$
$$= \frac{n-2}{r^{n-1}} \iint_{E_r^2(x_0,t_0)} \left| \partial_t u + \sum_{i=1}^n \frac{(x-x_0)^i}{2(t-t_0)}\partial_i u \right|^2 \mathrm{d}x\mathrm{d}t \geqslant 0 \quad (6.1)$$

holds whenever $n \geqslant 2$, $E_r^2(x_0,t_0) \subset \mathbb{R}^n \times]0,T[$ and $|\mathrm{d}u|^2 \in L^1(E_r^2(x_0,t_0))$. This formula is a natural local analogue of the monotonicity formula of Struwe [68] (cf. identity (4.1) in §4.1 and, together with a local monotonicity formula for solutions to a certain reaction-difussion equation which was also established in [20], served as motivation for the considerations of Chapter 2.

6.2. Local monotonicity identities. Let $(M, \{g_t\}_{t\in]t_0-\delta_0,t_0[})$ be an evolving Riemannian manifold with $\partial_t g = h$. For a $C^{2,1}$ function $f : \mathcal{D} \subset M \times]t_0-\delta_0, t_0[\to \mathbb{R}$, write $\mathcal{H}_{t_0} f$ for the matrix Harnack expression

$$\nabla^2 f + \frac{1}{2}h + \frac{1}{2(t_0-t)}g.$$

Furthermore, suppose $\Phi \in C^{2,1}(\mathcal{D}, \mathbb{R}^+)$ with $\mathcal{D} \subset M \times I$ open such that $E_r^{n-2k}(\Phi)$ is a heat ball for $r \leqslant r_0$ as defined in §5.2 and let $\phi = \log \Phi$. We begin with a local monotonicity identity which is meant to generalize the identity (6.1) and Theorem 2.5.2 to Dirichlet-type flows in curved settings.

Theorem 6.2.1. *If $n \geqslant 2k$ and $\left(\psi_t \in \Gamma(E \otimes \Lambda^k T^*M)\right)_{t\in]t_0-\delta_0,t_0[}$ is a smooth one-parameter family of sections, then*

$$\left[\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\Phi)} e_g(\psi)\cdot\left(\partial_t\phi + |\nabla\phi|^2\right) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^{\nabla}\psi \right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^{\nabla}\psi, \psi \right\rangle \mathrm{dvol}_{g_t}\, \mathrm{d}t \right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\Phi)} -e_g(\psi)\cdot\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}tr_g h + \frac{k}{t_0-t}\right)\right.$$

$$- \left\langle (\partial_t + \Delta^\nabla)\psi, \psi \right\rangle$$
$$+ |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2$$
$$+ \left(\mathcal{H}_{t_0}\phi, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \varepsilon_i \otimes \varepsilon_j \right) \mathrm{dvol}_{g_t}\, \mathrm{d}t \Bigg) \mathrm{d}r \quad (6.2)$$

holds whenever $0 < r_1 < r_2 < r_0$ *provided both spacetime integrands are in* $L^1\left(E^{n-2k}_{r_0}(\Phi)\right)$. *If*

$$\left((x,t) \mapsto \frac{e_g(\psi)(x,t)}{t_0 - t} \right) \in L^1\left(E^{n-2k}_{r_0}(\Phi)\right), \quad (6.3)$$

then the inequality (6.2) holds with equality.

Remark 6.2.2. If $k = 0$, i.e. $\psi_t \in \Gamma(E)$, the identity (6.2) reads (cf Remark 4.2.2)

$$\left[\frac{1}{r^n} \iint_{E^n_r(\Phi)} \frac{1}{2}|\psi|^2 \left(\partial_t\phi + |\nabla\phi|^2\right) - \left\langle \nabla_{\nabla\phi}\psi, \psi \right\rangle \mathrm{dvol}_{g_t} \mathrm{d}t \right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2} \Bigg(\frac{n}{r^{n+1}} \iint_{E^n_r(\Phi)} -\frac{1}{2}|\psi|^2 \left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h \right)$$
$$- \left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi \right\rangle + |\mathrm{d}^\nabla\psi|^2 \mathrm{dvol}_{g_t} \mathrm{d}t \Bigg) \mathrm{d}r$$

which should be compared with (4.8) of Remark 4.2.2. In particular, this implies a monotonicity formula if $\left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle \leqslant 0$ and $\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h = \frac{\partial_t\Phi + \Delta\Phi + \frac{1}{2}\mathrm{tr}_g h\Phi}{\Phi} \leqslant 0$, the latter of which holds if Φ satisfies the backward heat equation.

Remark 6.2.3. Just as with Theorem 4.2.1, this identity immediately yields a monotonicity formula provided the conditions outlined in Remark 4.2.3 are satisfied with $s = t_0$, i.e. if the following conditions are satisfied:

1. ψ satisfies $\left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle \leqslant 0$, e.g. if ψ evolves by a Dirichlet-type flow.

2. $\Phi = (t_0 - t)^k \mathrm{P}$ where P is a positive subsolution to the backward heat equation, i.e. $(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h)\mathrm{P} \leqslant 0$, since then

$$\begin{aligned} &(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h + \frac{k}{t_0 - t}) \\ &= \frac{(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h + \frac{k}{t_0 - t})\Phi}{\Phi} \\ &= \frac{(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g h)\mathrm{P}}{\mathrm{P}} \leqslant 0. \end{aligned}$$

   This holds with equality if P satisfies the backward heat equation.

3. The matrix Harnack expression

$$\nabla^2\phi + \frac{1}{2}h + \frac{1}{2(t_0 - t)}g$$

   is nonnegative-definite (cf. the corresponding property in Remark 4.2.3). This expression vanishes e.g. if g evolves by Ricci flow ($h = -2\mathrm{Ric}$ and g is a gradient shrinking soliton (cf. [45, Appendix C]). This includes the case where $(M, g_t) \equiv (\mathbb{R}^n, \delta)$ (cf. Remark 1.8.9).

Remark 6.2.4. If Φ is taken to be either the (suitably weighted) formal heat kernel concentrated at (x_0, t_0) (cf. Example 5.3.1) or, if (M, g) is compact and static, the (suitably weighted) canonical backward heat kernel concentrated at (x_0, t_0) (cf. Example 5.3.3), then (6.3) holds for sufficiently small r if $|\psi|^2$ is summable over a certain cylinder (see Lemmata 6.3.1 and 6.3.5 below). Thus, in particular, if $(M, g_t) \equiv (\mathbb{R}^n, \delta)$ and, assuming the setup of §1.10, $\psi = \mathrm{d}u$ for a map $u : M \times]t_0 - \delta, t_0[\to N \subset \mathbb{R}^K$ evolving by the harmonic map heat flow (with $N \subset \mathbb{R}^K$ isometrically embedded and $n \geqslant 2$), the identity (6.1) may be recovered in light of Lemma 1.10.3 (ii) (with $X = \nabla\phi$).

Proof of Theorem 6.2.1. We first assume that $\psi_t \equiv 0$ for $\tau < t < t_0$, whence, by virtue of the smoothness of ψ and ϕ on $E_r^{n-2k}(\Phi) \cap \mathrm{pr}_2^{-1}(]t_0 - \delta, \tau[)$, which is assumed compact, and the fact that each term occuring in the integrals is a product of ψ and something else, each individual term in the identity (6.2) is in $L^1\left(E_{r_0}^{n-2k}(\Phi)\right)$.
Now, let Y be the time-dependent vector field defined by

$$Y = (\iota_{\nabla\phi} T^g_\psi)^\sharp = \sum_j \left\langle \iota_{\nabla\phi}\psi, \iota_{\varepsilon_i}\psi \right\rangle - e_g(\psi)\nabla\phi,$$

where T^g_ψ was defined in Proposition 3.1.1. It is clear that Y is smooth on $E_{r_0}^{n-2k}(\Phi)$ and $|Y| \in L^2(E_{r_0}^{n-2k}(\Phi))$ by virtue of the above remarks. On the other hand,

$$\langle -Y, \nabla\phi \rangle = e_g(\psi)|\nabla\phi|^2 - |\iota_{\nabla\phi}\psi|^2 \in L^1\left(E_{r_0}^{n-2k}(\Phi)\right).$$

Now, by Corollary 3.1.4,

$$\begin{aligned}
\mathrm{div} Y &= \sum_{i=1}^n \left\langle \iota_{\nabla_{\varepsilon_i}\nabla\phi}\psi, \iota_{\varepsilon_i}\psi \right\rangle - e_g(\psi)\Delta\phi - \left\langle \delta^\nabla\psi, \iota_{\nabla\phi}\psi \right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi \right\rangle \\
&= \left\langle \nabla^2\phi, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle - e_g(\psi)\Delta\phi - \left\langle \delta^\nabla\psi, \iota_{\nabla\phi}\psi \right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi \right\rangle \qquad (6.4)
\end{aligned}$$

from which it may be seen that $\mathrm{div} Y \in L^1\left(E_{r_0}^{n-2k}(\Phi)\right)$. Therefore, adopting the notation of §5.4 for "approximate integrals", we note that

$$\begin{aligned}
&\left[\frac{J_q^r\left(e_g(\psi)\cdot(\partial_t\phi + |\nabla\phi|^2) - |\iota_{\nabla\phi}\psi|^2\right)}{r^{n-2k}}\right]_{r=r_1}^{r=r_2} \\
&= \int_{r_1}^{r_2} \frac{2k-n}{r^{n-2k+1}} J_q^r\left(e_g(\psi)\cdot(\partial_t\phi + |\nabla\phi|^2) - |\iota_{\nabla\phi}\psi|^2\right) + \frac{1}{r^{n-2k}}\frac{\mathrm{d}}{\mathrm{d}r} J_q^\cdot\left(e_g(\psi)\cdot(\partial_t\phi + |\nabla\phi|^2) - |\iota_{\nabla\phi}\psi|^2\right)\mathrm{d}r
\end{aligned} \qquad (6.5)$$

so that, on the one hand, by Lemma 5.4.1, the left-hand side tends to

$$\left[\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\Phi)} e_g(\psi)\cdot(\partial_t\phi + |\nabla\phi|^2) - |\iota_{\nabla\phi}\psi|^2 \mathrm{dvol}_{g_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$

as $q \to \infty$ and the first term in the integrand, integrated alone, tends to

$$\int_{r_1}^{r_2} \left(\frac{2k-n}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\Phi)} e_g(\psi)\cdot(\partial_t\phi + |\nabla\phi|^2) - |\iota_{\nabla\phi}\psi|^2 \mathrm{dvol}_{g_t}\mathrm{d}t\right)\mathrm{d}r$$

as $q \to \infty$. On the other, the latter term in the integrand, integrated alone, may be written as

$$\int_{r_1}^{r_2} \frac{1}{r^{n-2k}} \left(\frac{\mathrm{d}}{\mathrm{d}r} J_q(e_g(\psi)\partial_t\phi) - \frac{\mathrm{d}}{\mathrm{d}r} J_q(\langle Y, \nabla\phi\rangle) \right) \mathrm{d}r$$

by virtue of the fact that $e_g(\psi) \cdot (\partial_t\phi + |\nabla\phi|^2) - |\iota_{\nabla\phi}\psi|^2 = e_g(\psi)\partial_t\phi - \langle Y, \nabla\phi\rangle$. By the proofs of Theorems 5.4.2 and 5.4.3, the parenthetical expression may be written as

$$\frac{n-2k}{r} J_q^r(\mathrm{div}Y - \partial_t e_g(\psi) - \frac{1}{2}\mathrm{tr}_g h \cdot e_g(\psi)) + o(1)$$

as $q \to \infty$, where the latter term may be uniformly bounded in terms of ψ and r_0 (cf. proof of Theorem 5.4.3). Using Lemma 5.4.1 and the dominated convergence theorem, the integrability of all of the terms occurring in the approximate integrals immediately implies that the integral of the second term of the integrand in (6.5) tends, as $q \to \infty$, to

$$\int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \mathrm{div}Y - \partial_t e_g(\psi) - \frac{1}{2}\mathrm{tr}_g h \cdot e_g(\psi) \mathrm{dvol}_{g_t} \mathrm{d}t \right) \mathrm{d}r.$$

Altogether, this reads

$$\begin{aligned}
&\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} e_g(\psi) \cdot \left(\partial_t\phi + |\nabla\phi|^2\right) - |\iota_{\nabla\phi}\psi|^2 \mathrm{dvol}_{g_t} \mathrm{d}t \right]_{r=r_1}^{r=r_2} \\
&= \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \langle Y, \nabla\phi\rangle - e_g(\psi) \cdot \partial_t\phi - \partial_t e_g(\psi) - \frac{1}{2} e_g(\psi) \cdot \mathrm{tr}_g h + \mathrm{div}Y \mathrm{dvol}_{g_t} \mathrm{d}t \right) \mathrm{d}r.
\end{aligned} \tag{6.6}$$

By Proposition 1.7.8 and Lemma 1.6.8,

$$\begin{aligned}
\partial_t e_g(\psi) &= \langle \partial_t\psi, \psi\rangle - \left\langle \frac{1}{2}h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \\
&= \left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi \right\rangle - \left\langle \Delta^{\mathrm{d}^\nabla}\psi, \psi \right\rangle - \left\langle \frac{1}{2}h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \\
&= \left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi \right\rangle - |\delta^\nabla\psi|^2 - |\mathrm{d}^\nabla\psi|^2 - \mathrm{div}\left(\sum_{i=1}^n (\left\langle \iota_{\varepsilon_i}\psi, \delta^\nabla\psi \right\rangle - \left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi, \psi \right\rangle)\varepsilon_i \right),
\end{aligned}$$

where $\left(\partial_t + \Delta^\nabla\right) = \partial_t + \Delta^\nabla$, whence

$$\begin{aligned}
&\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} -\partial_t e_g(\psi) \mathrm{dvol}_{g_t} \mathrm{d}t \\
&= \frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} |\delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2 + \left\langle \frac{1}{2}h, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle - \left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi \right\rangle \mathrm{dvol}_{g_t} \mathrm{d}t \\
&\qquad - \frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \mathrm{div}\left(\sum_{i=1}^n (\left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi, \psi \right\rangle - \left\langle \iota_{\varepsilon_i}\psi, \delta^\nabla\psi \right\rangle)\varepsilon_i \right) \mathrm{dvol}_{g_t} \mathrm{d}t.
\end{aligned}$$

To handle the final integral, we proceed as with the approximation (6.5), again making use of the proof of Theorem 5.4.2:

$$\left[\frac{1}{r^{n-2k}}J_q^r\left(\left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\nabla\phi}\psi,\delta^\nabla\psi\right\rangle\right)\right]_{r=r_1}^{r=r_2}$$
$$=\left[\frac{1}{r^{n-2k}}J_q^r\left(\left\langle\sum_{i=1}^n\left(\left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\varepsilon_i}\psi,\delta^\nabla\psi\right\rangle\right)\varepsilon_i,\nabla\phi\right\rangle\right)\right]_{r=r_1}^{r=r_2}$$
$$=\int_{r_1}^{r_2}\frac{n-2k}{r^{n-2k+1}}\left(-J_q^r\left(\left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\nabla\phi}\psi,\delta^\nabla\psi\right\rangle\right)-J_q^r\left(\operatorname{div}\left(\sum_{i=1}^n(\left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\varepsilon_i}\psi,\delta^\nabla\psi\right\rangle)\varepsilon_i\right)\right)\right)\mathrm{d}r$$

whence, taking the limit $q\to\infty$, which is justified exactly as in the previous approximation by means of Lemma 5.4.1, the dominated convergence theorem and the summability of all terms involved, we obtain

$$-\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\operatorname{div}\left(\sum_{i=1}^n(\left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\varepsilon_i}\psi,\delta^\nabla\psi\right\rangle)\varepsilon_i\right)\mathrm{dvol}_{g_t}\mathrm{d}t\right)\mathrm{d}r$$
$$=\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\nabla\phi}\psi,\delta^\nabla\psi\right\rangle\mathrm{dvol}_{g_t}\mathrm{d}t\right)\mathrm{d}r$$
$$+\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)}\left(\left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\nabla\phi}\psi,\delta^\nabla\psi\right\rangle\right)\mathrm{dvol}_{g_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}.$$

Altogether, this implies that

$$\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}-\partial_t e_g(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t\right)\mathrm{d}r$$
$$=\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}|\delta^\nabla\psi|^2+|\mathrm{d}^\nabla\psi|^2+\left\langle\frac{1}{2}h,\sum_{i,j=1}^n\left\langle \iota_{\varepsilon_i}\psi,\iota_{\varepsilon_j}\psi\right\rangle\omega^i\otimes\omega^j\right\rangle\right.$$
$$\left.-\left\langle\left(\partial_t+\Delta^\nabla\right)\psi,\psi\right\rangle+\left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\nabla\phi}\psi,\delta^\nabla\psi\right\rangle\mathrm{dvol}_{g_t}\mathrm{d}t\right)\mathrm{d}r$$
$$+\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)}\left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi,\psi\right\rangle-\left\langle \iota_{\nabla\phi}\psi,\delta^\nabla\psi\right\rangle\mathrm{dvol}_{g_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}.$$

Hence, plugging this expression into (6.6), moving the latter term to the left-hand side, using the expression (6.4) for $\operatorname{div}Y$ and combining like terms, we obtain

$$\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)}\underbrace{e_g(\psi)\cdot\left(\partial_t\phi+|\nabla\phi|^2\right)-\left\langle \iota_{\nabla\phi}\psi,\iota_{\nabla\phi}\psi-\delta^\nabla\psi\right\rangle-\left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi,\psi\right\rangle}_{=:i_1(\psi)}\mathrm{dvol}_{g_t}\ \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$=\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}|\iota_{\nabla\phi}\psi|^2-e_g(\psi)\cdot\left(\partial_t\phi+\Delta\phi+|\nabla\phi|^2+\frac{1}{2}\mathrm{tr}_g h\right)\right.$$

$$+\left\langle \nabla^2\phi + \frac{1}{2}h, \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle - 2\left\langle \delta^\nabla \psi, \iota_{\nabla\phi}\psi \right\rangle$$

$$+ |\delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2 - \left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi \right\rangle \mathrm{dvol}_{g_t}\mathrm{d}t \Bigg) \mathrm{d}r$$

$$= \int_{r_1}^{r_2} \Bigg(\frac{n-2k}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\Phi)} -e_g(\psi)\cdot\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h + \frac{k}{t_0 - t}\right)$$

$$- \left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi \right\rangle + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2$$

$$+ \left\langle \nabla^2\phi + \frac{1}{2}h + \frac{g}{2(t_0-t)}, \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \mathrm{dvol}_{g_t}\mathrm{d}t \Bigg) \mathrm{d}r, \quad (6.7)$$

where in the last line (4.11) was used. This establishes the theorem in the case where ψ vanishes close to t_0.
Now let $\psi \in \Gamma^\infty\left(E \otimes \Lambda^k T^*M\right)$ be an arbitrary smooth time-dependent section and define the smooth family of sections ψ_m by

$$\psi_m(x,t) = \psi_t(x) \cdot \chi_m(t_0 - t)^1.$$

By the properties of χ_m, $\psi_m(\cdot, t) \equiv 0$ for $t > t_0 - 2^{-m-1}$, whence, applying (6.7) to ψ_m,

$$\left[\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\Phi)} \chi_m(t_0 - t)^2 \cdot i_1(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t \right]_{r=r_1}^{r=r_2}$$

$$= \int_{r_1}^{r_2} \Bigg(\frac{n-2k}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\Phi)} \chi_m(t_0 - t)^2 \cdot \Bigg(-e_g(\psi)\cdot\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h + \frac{k}{t_0 - t}\right)$$

$$- \left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi \right\rangle + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2$$

$$+ \left\langle \nabla^2\phi + \frac{1}{2}h + \frac{g}{2(t_0-t)}, \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg) \mathrm{dvol}_{g_t}\mathrm{d}t \Bigg) \mathrm{d}r$$

$$+ \int_{r_1}^{r_2} \Bigg(\frac{n-2k}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\Phi)} 2\chi_m(t_0 - t)\chi_m'(t - t_0)e_g(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t \Bigg) \mathrm{d}r. \quad (6.8)$$

Now, the latter integral on the right-hand side is nonnegative, since $\chi_m' \geqslant 0$. Since $\chi_m(\cdot - t_0) \to \chi_{]-\infty, t_0[}$ pointwise and $\chi_m(\cdot - t_0) \leqslant 1$ on $]t_0 - \delta_0, t_0[$, the result follows from discarding the latter integral on the right-hand side, using the summability of the integrands to apply the dominated convergence theorem and appealing to Lemma 5.4.1.

On the other hand, if $e_g(\psi)\cdot\frac{1}{t_0 - t} \in L^1\left(E_{r_0}^{n-2k}(\Phi)\right)$, it follows from

$$\chi_m(t - t_0)\chi_m'(t - t_0)e_g(\psi) = \chi_m(t_0 - t)\cdot(t_0 - t)\chi_m'(t_0 - t)\cdot\frac{e_g(\psi)}{t_0 - t}$$

and an application of the dominated convergence theorem, noting that $(t_0 - t)\chi_m'(t_0 - t) \xrightarrow{m\to\infty} 0$ (cf. Example A.1) and that $\frac{e_g(\psi)}{t_0 - t}$ is summable, that the integral in the last line of identity (6.8) tends to 0 as $m \to \infty$. □

Corollary 6.2.5. *The following hold:*

1. *Assume the setup of §1.11 and that $n \geqslant 4$. If $(\omega_t = \widetilde{\omega} + a(t))_{t\in]t_0-\delta_0,t_0[}$ is a one-parameter family of connections evolving by the Yang-Mills heat flow, then the identity*

$$\left[\frac{1}{r^{n-4}}\iint\limits_{E_r^{n-4}(\Phi)} e_g(\underline{\Omega}^\omega)\cdot\left(\partial_t\phi+|\nabla\phi|^2\right)-\left\langle\iota_{\nabla\phi}\underline{\Omega}^\omega,\iota_{\nabla\phi}\underline{\Omega}^\omega-\delta^\nabla\underline{\Omega}^\omega\right\rangle\mathrm{dvol}_{g_t}\,\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant\int_{r_1}^{r_2}\left(\frac{n-4}{r^{n-3}}\iint\limits_{E_r^{n-4}(\Phi)}-e_g(\underline{\Omega}^\omega)\cdot\left(\partial_t\phi+\Delta\phi+|\nabla\phi|^2+\frac{1}{2}tr_gh+\frac{4}{2(t_0-t)}\right)\right.$$
$$+|\partial_ta+\iota_{\nabla\phi}\underline{\Omega}^\omega|^2$$
$$\left.+\left(\mathcal{H}_{t_0}\phi,\sum_{i,j}\left\langle\iota_{\varepsilon_i}\underline{\Omega}^\omega,\iota_{\varepsilon_j}\underline{\Omega}^\omega\right\rangle\varepsilon_i\otimes\varepsilon_j\right)\mathrm{dvol}_{g_t}\,\mathrm{d}t\right)\mathrm{d}r$$

holds whenever $0<r_1<r_2<r_0$ provided both integrands are in $L^1\left(E_{r_0}^{n-4}(\Phi)\right)$. If in addition $\left((x,t)\mapsto\frac{e_g(\Omega^\omega)(x,t)}{t_0-t}\right)\in L^1\left(E_{r_0}^{n-4}(\Phi)\right)$, then this identity holds with equality.

2. *Assume the setup of §1.10 and that $n\geqslant 2$. If $u: M\times]t_0-\delta_0,t_0[\to N\subset\mathbb{R}^K$ evolves by the harmonic map heat flow with N isometrically embedded in $\mathbb{R}^K$, then the identity*

$$\left[\frac{1}{r^{n-2}}\iint\limits_{E_r^{n-2}(\Phi)} e_g(\mathrm{d}u)\cdot\left(\partial_t\phi+|\nabla\phi|^2\right)-\left\langle\partial_{\nabla\phi}u,\partial_{\nabla\phi}\mathrm{d}u-\delta^\nabla\mathrm{d}u\right\rangle\mathrm{dvol}_{g_t}\,\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant\int_{r_1}^{r_2}\left(\frac{n-2}{r^{n-1}}\iint\limits_{E_r^{n-2}(\Phi)}-e_g(\mathrm{d}u)\cdot\left(\partial_t\phi+\Delta\phi+|\nabla\phi|^2+\frac{1}{2}tr_gh+\frac{2}{2(t_0-t)}\right)\right.$$
$$+|\partial_tu+\partial_{\nabla\phi}u|^2$$
$$\left.+\left(\mathcal{H}_{t_0}\phi,\sum_{i,j}\left\langle\partial_{\varepsilon_i}u,\partial_{\varepsilon_j}u\right\rangle\varepsilon_i\otimes\varepsilon_j\right)\mathrm{dvol}_{g_t}\,\mathrm{d}t\right)\mathrm{d}r$$

holds whenever $0<r_1<r_2<r_0$ provided both integrands are in $L^1\left(E_{r_0}^{n-2}(\Phi)\right)$. If in addition $\left((x,t)\mapsto\frac{e_g(\mathrm{d}u)(x,t)}{t_0-t}\right)\in L^1\left(E_{r_0}^{n-2}(\Phi)\right)$, then this identity holds with equality.

Proof. For the former claim, apply Theorem 6.2.1 by taking $E=P\times_{\mathrm{Ad}}\mathfrak{g}$, ∇ the covariant derivative induced by ω, $\psi=\underline{\Omega}^\omega$ ($\Rightarrow k=2$) and using the Bianchi identity $\mathrm{d}^\nabla\underline{\Omega}^\omega=0$ (cf. Proposition 1.5.7) and Lemma 1.11.4, keeping Remark 6.2.3 in mind.

The latter claim follows similarly by taking $E=\mathbb{R}^K$, ∇ the flat connection and $\psi=\mathrm{d}u$ ($\Rightarrow k=1$), noting the Bianchi-type identity Lemma 1.10.3 (i). This time, however, we use Lemma 1.10.3 (ii) with $X=\nabla\phi$, which states that

$$\left\langle(\partial_t+\Delta^\nabla)\mathrm{d}u,\mathrm{d}u\right\rangle-\left|\partial_{\nabla\phi}u-\delta^\nabla\mathrm{d}u\right|^2=-\left|\partial_tu+\partial_{\nabla\phi}u\right|^2,$$

which establishes the claim. □

In practice, we do not necessarily know too much about the integrability of the matrix Harnack term. However, the proof of the preceding theorem implies a monotonicity identity nonetheless.

Corollary 6.2.6. *If $n\geqslant 2k$ and $\left(\psi_t\in\Gamma(E\otimes\Lambda^kT^*M)\right)_{t\in]t_0-\delta_0,t_0[}$ is a smooth one-parameter family of sections and*

$$\partial_t\phi+\Delta\phi+|\nabla\phi|^2+\frac{1}{2}tr_gh+\frac{k}{t_0-t}\leqslant a(t)\text{ and}\tag{6.9}$$

$$\mathcal{H}_{t_0}(\phi) \geqslant b(t)g \tag{6.10}$$

on $E_{r_0}^{n-2k}(\Phi)$ with $a, b \in C(]t_0 - \delta_0, t_0[) \cap L^1(]t_0 - \delta_0, t_0[)$, then

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \exp(\xi(t)) \left(e_g(\psi)(\partial_t\phi + |\nabla\phi|^2) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^\nabla\psi \right\rangle - \left\langle \iota_{\nabla\phi} d^\nabla\psi, \psi \right\rangle\right) \mathrm{dvol}_{g_t} \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \exp(\xi(t)) \cdot \left(\left|d^\nabla\psi\right|^2 + \left|\iota_{\nabla\phi}\psi - \delta^\nabla\psi\right|^2 - \left\langle(\partial_t + \Delta^\nabla)\psi, \psi\right\rangle\right) \mathrm{dvol}_{g_t} \mathrm{d}t\right) \mathrm{d}r$$

for $0 < r_1 < r_2 < r_0$ whenever the spacetime integrands are summable over $E_{r_0}^{n-2k}(\Phi)$, where

$$\xi(t) = \int_t^{t_0} a - 2kb.$$

If only the left-hand spacetime integrand is known to be summable over $E_{r_0}^{n-2k}(\Phi)$ and

$$|d^\nabla\psi|^2 + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 - \left\langle\left(\partial_t + \Delta^\nabla\right)\psi, \psi\right\rangle \geqslant 0 \tag{6.11}$$

on $E_{r_0}^{n-2k}(\Phi)$, then

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \exp(\xi(t)) \left(e_g(\psi)(\partial_t\phi + |\nabla\phi|^2) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^\nabla\psi \right\rangle - \left\langle \iota_{\nabla\phi} d^\nabla\psi, \psi \right\rangle\right) \mathrm{dvol}_{g_t} \mathrm{d}t\right]_{r=r_1}^{r=r_2} \geqslant 0$$

for $0 < r_1 < r_2 < r_0$, i.e. the parenthetical quantity is monotone nondecreasing.

Proof. We apply (6.7) to $\psi_m(x,t) := e^{\xi(t)/2}\chi_m(t_0 - t)\psi_t(x)$:

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \chi_m(t_0 - t)^2 e^{\xi(t)} i_1(\psi) \mathrm{dvol}_{g_t} \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\begin{aligned} = \int_{r_1}^{r_2} \Bigg(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} & \exp(\xi(t))\chi_m(t_0 - t)^2 \Bigg(-e_g(\psi) \cdot \left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g h + \frac{k}{t_0 - t}\right) \\ & + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |d^\nabla\psi|^2 - \left\langle(\partial_t + \Delta^\nabla)\psi, \psi\right\rangle \\ & + \left\langle \mathcal{H}_{t_0}(\phi), \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i \otimes \omega^j \right\rangle\Bigg) \\ & + 2\chi_m'(t_0 - t)\chi_m(t_0 - t)e^{\xi(t)}e_g(\psi) - \partial_t\xi \cdot \exp(\xi(t))\chi_m(t_0 - t)^2 e_g(\psi) \mathrm{dvol}_{g_t} \mathrm{d}t\Bigg) \mathrm{d}r. \end{aligned} \tag{6.12}$$

Making use of inequalities (6.9) and (6.10) and noting that $\chi_m'(t_0 - t) \geqslant 0$, $\partial_t\xi = 2kb - a$ and

$$\left\langle g, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i \otimes \omega^j \right\rangle = k|\psi|^2 = 2ke_g(\psi),$$

we may estimate the r-integrand of the right-hand integral of equation (6.12) from below by

$$\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \exp(\xi(t))\chi_m(t_0-t)^2\left(-e_g(\psi)a + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2 \right.$$
$$\left. - \left\langle(\partial_t + \Delta^\nabla)\psi, \psi\right\rangle + 2kbe_g(\psi) - (2kb-a)e_g(\psi)\right)\mathrm{dvol}_{g_t}\mathrm{d}t$$
$$= \frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \exp(\xi(t))\chi_m(t_0-t)^2\cdot\left(\left|\iota_{\nabla\phi}\psi - \delta^\nabla\psi\right|^2 + \left|\mathrm{d}^\nabla\psi\right|^2 - \left\langle(\partial_t + \Delta^\nabla)\psi, \psi\right\rangle\right)\mathrm{dvol}_{g_t}\mathrm{d}t. \tag{6.13}$$

Since $\exp\circ\xi$ is bounded on $]t_0-\delta_0, t_0[$, we may take limits exactly as in the preceding theorem, thus establishing the first claim. For the second, we bound the right-hand side of (6.13) from below by 0 and then take limits. □

Remark 6.2.7. If $(\partial_t + \Delta^\nabla)\psi = 0$, then (6.11) clearly holds. On the other hand, if $\psi = \mathrm{d}u$ and u solves (HMHF) then, by Lemma 1.10.3, the left-hand side of (6.11) is equal to $|\partial_t u + \iota_{\nabla\phi}\mathrm{d}u|^2 \geqslant 0$ so that this condition also holds.

6.3. Applications.

We now proceed to apply Corollary 6.2.6 and Remark 6.2.7 to establish local monotonicity formulæ for forms evolving by Dirichlet-type flows. We first prove a lemma which guarantees the finiteness of certain quantities appearing in the identity (6.3). In all of the following, we assume that:

1. (M^n, g) is an $n \geqslant 2k$-dimensional evolving Riemannian manifold of locally bounded geometry about (x_0, t_0) with bounds as in Definition 1.7.4, and
2. $(\psi_t \in \Gamma(E \otimes \Lambda^k T^*M)_{t\in]t_0-\delta,t_0[}$ is a smooth one-parameter family of sections such that
   a) $(\partial_t + \Delta^\nabla)\psi = 0$, in which case we write $\mathcal{S} = \iota_{\nabla\phi}\psi - \delta^\nabla\psi$ and $\mathcal{J} = \mathrm{d}^\nabla\psi$, or
   b) $\psi = \mathrm{d}u$, where, assuming the setup of §1.10, $u : M \times]t_0-\delta, t_0[\to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow with N isometrically embedded in $\mathbb{R}^K$, in which case we write $\mathcal{S} = \partial_t u + \iota_{\nabla\phi}\mathrm{d}u$ and $\mathcal{J} = 0$.

Lemma 6.3.1. *Let Φ be as in Example 5.3.1 (with $m = n-2k$). If the integral*

$$\int_{t_0-\frac{R^2}{16\pi c_{n,k}^2}}^{t_0} \int_{B_R^t(x_0)} e_g(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t$$

is finite for some $R > 0$, where $c_{n,k} = \sqrt{\frac{n-2k}{2\pi e}}$, then the quantities

$$\left((x,t) \mapsto \frac{e_g(\psi)(x,t)}{t_0 - t}\right),\ e_g(\psi)(\partial_t\phi + |\nabla\phi|^2) - \left\langle\iota_{\nabla\phi}\psi, \mathcal{S}\right\rangle - \left\langle\iota_{\nabla\phi}\mathcal{J}, \psi\right\rangle,\ |\mathcal{J}|^2 \text{ and } |\mathcal{S}|^2$$

are in $L^1(E_r)$ for $r \leqslant r_0 := \min\{\frac{R}{2c_{n,k}}$, r_0 of Example 5.3.1$\}$.

Proof. Firstly, by Lemma 4.4.1,

$$\iint\limits_{E_{r_0}^{n-2k}(\Phi)} |\mathcal{S}|^2 + |\mathcal{J}|^2\mathrm{dvol}_{g_t}\mathrm{d}t$$
$$\leqslant 2\widetilde{C}_0\left(\frac{1}{r_0^2}\int_{t_0-\frac{r_0^2}{4\pi}}^{t_0}\int_{B^t_{2c_{n,k}r_0}(x_0)} e_g(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t + \left(\int_{B^{\cdot}_{2c_{n,k}r_0}(x_0)} e_g(\psi)\mathrm{dvol}_g\right)(t_0 - \frac{r^2}{4\pi})\right),$$

where $\widetilde{C}_0$ is as in Lemma 4.4.1, whence the left-hand side is finite, since $2c_{n,k}r_0 \leqslant R$.

Secondly, again by Lemma 4.4.1,

$$\begin{aligned}&\int_{B^t_{R^{n-2k}_{r_0}(t)}(x_0)} \frac{e_g(\psi)(\cdot,t)}{t_0-t}\mathrm{dvol}_{g_t}\\ &\leqslant \frac{R^{n-2k}_{r_0}(t)^{n-2k}}{t_0-t}\cdot\frac{\widetilde{C}_0}{r_0^{n-2k}}\cdot\left(\frac{1}{r_0^2}\int_{t_0-\frac{r_0^2}{4\pi}}^{t_0}\int_{B^t_{2c_{n,k}r_0}(x_0)} e_g(\psi)\mathrm{dvol}_{g_t}\mathrm{d}t+\left(\int_{B^\cdot_{2c_{n,k}r_0}(x_0)} e_g(\psi)\mathrm{dvol}_g\right)(t_0-\frac{r^2}{4\pi})\right)\end{aligned} \tag{6.14}$$

for each $t\in\left]t_0-\underbrace{r_0^2\exp\left(-\frac{1}{2(n-2k)}\right)/4\pi}_{=:d_{n,k}},t_0\right[\subset\left]t_0-\frac{r_0^2}{4\pi},t_0\right[$. In light of the fact that, by (HB1),

$$\bigcup_{t\in\left]t_0-\frac{r_0^2}{4\pi},t_0-d_{n,k}r_0^2\right[} B^t_{R^{n-2k}_{r_0}(t)}(x_0)\times\{t\}=E^{n-2k}_{r_0}(\Phi)\cap\mathrm{pr}_2^{-1}\left(\left]t_0-\frac{r_0^2}{4\pi},t_0-d_{n,k}r_0^2\right[\right)$$

is relatively compact in $M\times\left]t_0-\delta,t_0\right[$, it suffices to show that

$$\int_{t_0-d_{n,k}r_0^2}^{t_0}\int_{B^t_{R^{n-2k}_{r_0}(t)}(x_0)} \frac{e_g(\psi)(\cdot,t)}{t_0-t}\mathrm{dvol}_{g_t}\mathrm{d}t$$

is finite, which, by (6.14), is guaranteed if

$$\int_{t_0-d_{n,k}r_0^2}^{t_0}\frac{R^{n-2k}_{r_0}(t)^{n-2k}}{t_0-t}\mathrm{d}t$$

is finite. We know from the computation of Example 5.3.1 (HB2) leading to (5.5) that

$$\int_{t_0-\frac{r_0^2}{4\pi}}^{t_0}\frac{R^{n-2k}_{r_0}(t)^{n-2k}}{t_0-t}\mathrm{d}t=\left(\frac{(n-2k)r_0^2}{2\pi}\right)^{n/2-k}\int_0^\infty s^{n/2-k}\exp\left(-\left(\frac{n}{2}-k\right)s\right)\mathrm{d}s,$$

which is finite.

Finally, by the Cauchy-Schwarz inequality and Young's inequality,

$$\begin{aligned}&\left|e_g(\psi)(\partial_t\phi+|\nabla\phi|^2)-\left\langle\iota_{\nabla\phi}\psi,\mathcal{S}\right\rangle-\left\langle\iota_{\nabla\phi}\mathcal{J},\psi\right\rangle\right|\\ &\leqslant e_g(\psi)\left(|\partial_t\phi|+|\nabla\phi|^2\right)+|\nabla\phi|\cdot|\psi|\cdot|\mathcal{S}|+|\nabla\phi|\cdot|\psi|\cdot|\mathcal{J}|\\ &\leqslant e_g(\psi)\left(|\partial_t\phi|+3|\nabla\phi|^2\right)+\frac{1}{2}\left(|\mathcal{S}|^2+|\mathcal{J}|^2\right).\end{aligned}$$

Thus, we need only show that $e_g(\psi)\left(|\partial_t\phi|+3|\nabla\phi|^2\right)$ is in $L^1\left(E^{n-2k}_{r_0}(\Phi)\right)$. Now, by Proposition 1.8.8,

$$\begin{aligned}|\partial_t\phi|+3|\nabla\phi|^2&\leqslant\frac{7d^t(x,x_0)^2}{4(t_0-t)^2}+\frac{n}{2(t_0-t)}+\frac{\mu r^2}{4(t_0-t)}\\ &\leqslant\frac{7}{4(t_0-t)^2}R^{n-2k}_{r_0}(t)^2+\frac{n}{2(t_0-t)}+\frac{\mu}{4(t_0-t)}R^{n-2k}_{r_0}(t)^2\end{aligned}$$

on $E^{n-2k}_{r_0}(\Phi)$ so that, by Lemma 4.4.1,

$$\int_{B^t_{R^{n-2k}_{r_0}(t)}(x_0)} e_g(\psi)(\cdot,t)\cdot\left(|\partial_t\phi|+3|\nabla\phi|^2\right)\mathrm{dvol}_{g_t}$$

$$\leqslant \frac{\widetilde{C}_0}{r^{n-2k}} \cdot c(t) \left(\frac{1}{r_0^2} \int_{t_0 - \frac{r_0^2}{4\pi}}^{t_0} \int_{B^t_{2c_{n,k} r_0}(x_0)} e_g(\psi) \mathrm{dvol}_{g_t} \mathrm{d}t + \left(\int_{B^{\cdot}_{2c_{n,k} r_0}(x_0)} e_g(\psi) \mathrm{dvol}_g \right) (t_0 - \frac{r^2}{4\pi}) \right) \tag{6.15}$$

for each $t \in \left]t_0 - d_{n,k} r_0^2, t_0\right[$, where

$$c(t) = \frac{7}{4(t_0 - t)^2} R_{r_0}^{n-2k}(t)^{n-2k+2} + \frac{n}{2(t_0 - t)} R_{r_0}^{n-2k}(t)^{n-2k} + \frac{\mu}{4(t_0 - t)} R_{r_0}^{n-2k}(t)^{n-2k+2}.$$

As before, it suffices to show that c is summable over $\left]t_0 - d_{n,k} r_0^2, t_0\right[$. Now, the middle term in the expression for c was already shown to be summable on this interval. On the other hand, since $R_{r_0}^{n-2k}(t) \leqslant c_{n,k} r_0$ and

$$\frac{R_{r_0}^{n-2k}(t)^2}{(t_0 - t)^2} = -2m \log \left(\frac{4\pi (t_0 - t)}{r_0^2} \right),$$

which is summable over $]t_0 - \frac{r_0^2}{4\pi}, t_0[$, it is clear that the first and last terms in the expression for $c(t)$ are also summable over this interval. □

Theorem 6.3.2. *Let Φ be as in Example 5.3.1 (with $m = n - 2k$). Suppose that the integral*

$$\int_{t_0 - \frac{R^2}{16\pi c_{n,k}^2}}^{t_0} \int_{B_R^t(x_0)} e_g(\psi) \mathrm{dvol}_{g_t} \mathrm{d}t \tag{6.16}$$

is finite for some $R > 0$, where $c_{n,k} = \sqrt{\frac{n-2k}{2\pi e}}$. Then there exist an $r_0 > 0$ depending on the local geometry of M about (x_0, t_0), δ and R, and a function $\xi \in C(]t_0 - 1, t_0])$ with $\xi(t_0) = 0$ depending on the geometry of M such that for $0 < r_1 < r_2 < r_0$, the identity

$$\left[\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\Phi)} \exp(\xi(t)) \left(e_g(\psi)(\partial_t \phi + |\nabla\phi|^2) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^{\nabla}\psi \right\rangle - \left\langle \iota_{\nabla\phi}\mathcal{J}, \psi \right\rangle \right) \mathrm{dvol}_{g_t} \mathrm{d}t \right]_{r=r_1}^{r=r_2}$$

$$\geqslant \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\Phi)} \exp(\xi(t)) \cdot \left(|\mathcal{J}|^2 + |\mathcal{S}|^2 \right) \mathrm{dvol}_{g_t} \mathrm{d}t \right) \mathrm{d}r$$

holds. In particular, the quantity

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\Phi)} e^{\xi(t)} \left(e_g(\psi) \left(\partial_t \phi + |\nabla\phi|^2 \right) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^{\nabla}\psi \right\rangle - \left\langle \iota_{\nabla\phi}\mathcal{J}, \psi \right\rangle \right) \mathrm{dvol}_{g_t} \mathrm{d}t$$

is monotone nondecreasing in r.

Remark 6.3.3. r_0 is as in Lemma 6.3.1 and ξ is given by (6.17) below.

Remark 6.3.4. Note that

1. If $n \geqslant 4$, $\{\omega_t = \widetilde{\omega} + a(t)\}_{t \in]t_0 - \delta, t_0[}$ is a family of connections on a principal bundle $P \to M^n$ evolving by the Yang-Mills flow according to the setup of §1.11 and the integral (6.16) is finite for $\psi = \underline{\Omega}^{\omega}$, then this identity holds and reads

$$\left[\frac{1}{r^{n-4}}\iint\limits_{E_r^{n-4}(\Phi)} \exp(\xi(t))\left(e_g(\underline{\Omega}^\omega)(\partial_t\phi+|\nabla\phi|^2)-\left\langle \iota_{\nabla\phi}\underline{\Omega}^\omega,\partial_t a+\iota_{\nabla\phi}\underline{\Omega}^\omega\right\rangle\right)\mathrm{dvol}_{g_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2}\left(\frac{n-4}{r^{n-3}}\iint\limits_{E_r^{n-4}(\Phi)}\exp(\xi(t))\left|\partial_t a+\iota_{\nabla\phi}\underline{\Omega}^\omega\right|^2\mathrm{dvol}_{g_t}\mathrm{d}t\right)\mathrm{d}r,$$

and

2. If $n\geqslant 2$, $u : M^n\times]t_0-\delta,t_0[\,\to N\subset\mathbb{R}^K$ evolves by the harmonic map heat flow according to the setup of §1.10, where N is isometrically embedded in $\mathbb{R}^K$, and the integral (6.16) is finite for $\psi=\mathrm{d}u$, then this identity holds and reads

$$\left[\frac{1}{r^{n-2}}\iint\limits_{E_r^{n-2}(\Phi)} \exp(\xi(t))\left(e_g(\mathrm{d}u)(\partial_t\phi+|\nabla\phi|^2)-\left\langle \iota_{\nabla\phi}\mathrm{d}u,\partial_t u+\iota_{\nabla\phi}\mathrm{d}u\right\rangle\right)\mathrm{dvol}_{g_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2}\left(\frac{n-2}{r^{n-1}}\iint\limits_{E_r^{n-2}(\Phi)}\exp(\xi(t))\left|\partial_t u+\iota_{\nabla\phi}\mathrm{d}u\right|^2\mathrm{dvol}_{g_t}\mathrm{d}t\right)\mathrm{d}r.$$

Thus, these identities yield local analogues of the Chen-Struwe and Chen-Shen formulæ(cf. §4.1).

Proof of Theorem 6.3.2. Let r_0 be as in Lemma 6.3.1. We first note that, by Proposition 1.8.7,

$$\begin{aligned}\partial_t\phi+\Delta\phi+|\nabla\phi|^2+\frac{1}{2}\mathrm{tr}_g h+\frac{k}{t_0-t} &\leqslant \left(\frac{n\mu}{2}+\frac{C_4 d^t(x_0,\cdot)^2}{t_0-t}\right)\\ &\leqslant \left(\frac{n\mu}{2}-2(n-2k)C_4\log\left(\frac{4\pi(t_0-t)}{r_0^2}\right)\right) =: a(t)\end{aligned}$$

on $E_{r_0}^{n-2k}(\Phi)$, where C_4 is a constant depending only on the local geometry about (x_0,t_0). Similarly, by Proposition 1.8.8, the inequalities

$$\begin{aligned}\mathcal{H}_{t_0}\phi &\geqslant \left(-\frac{Cd^t(x_0,\cdot)^2}{2(t_0-t)}+\frac{\lambda_{-\infty}}{2}\right)g\\ &\geqslant \left(-(n-2k)C\log\left(\frac{4\pi(t_0-t)}{r_0^2}\right)+\frac{\lambda_{-\infty}}{2}\right)g =: b(t)g\end{aligned}$$

hold. Since a and b are continuous and summable on $]t_0-\delta,t_0[$, we may appeal to Corollary 6.2.6, keeping Remark 6.2.7 in mind and noting that since the integral (6.16) is finite, the integrals of Corollary 6.2.6 all exist by Lemma 6.3.1. To compute ξ, note that

$$\begin{aligned}a(u)-2kb(u) &= \underbrace{\left(\frac{n\mu}{2}-2(n-2k)C_4\log\left(\frac{4\pi}{r_0^2}\right)+2k(n-2k)C\log\left(\frac{4\pi}{r_0^2}\right)-k\lambda_{-\infty}\right)}_{=:\mathfrak{a}}\\ &\quad+\underbrace{2(n-2k)\,(kC-C_4)}_{=:\mathfrak{b}}\log(t_0-u),\end{aligned}$$

whence,

$$\xi(t)=\int_t^{t_0} a-2kb = (\mathfrak{a}-\mathfrak{b})(t_0-t)+\mathfrak{b}(t_0-t)\log(t_0-t), \tag{6.17}$$

whence the result follows. □

We now turn our attention to the case where M is static. We again prove a lemma ensuring finiteness of the integrals we consider.

Lemma 6.3.5. *Let (M,g) be a static compact Riemannian manifold and $\mathrm{P} := {}^{n-2k}\mathrm{P}_{(x_0,t_0)}$ as in Example 5.3.3. If the integral*

$$\int_{t_0-\frac{R^2}{16\pi c_{n,k}^2}}^{t_0}\int_{B_R(x_0)} e_g(\psi)\mathrm{dvol}_{g_t}\,\mathrm{d}t$$

is finite for some $R>0$, where $c_{n,k}=\sqrt{\frac{n-2k}{2\pi e}}$, then the quantities

$$\left((x,t)\mapsto\frac{e_g(\psi)(x,t)}{t_0-t}\right),\ e_g(\psi)\left(\partial_t\rho+|\nabla\rho|^2\right)-\left\langle\iota_{\nabla\rho}\psi,\mathcal{S}\right\rangle-\left\langle\iota_{\nabla\rho}\mathcal{J},\psi\right\rangle,\ |\mathcal{J}|^2\text{ and }|\mathcal{S}|^2$$

are in $L^1(E_r)$ for

$$r\leqslant r_0:=\min\{\frac{1}{2}\left(2(4\pi\delta)^{-\frac{n-2k}{2}}+1\right)^{-1/(n-2k)},\left(\frac{1}{1+\frac{2^{n-2k+1}c_{n,k}^{n-2k}}{R^{n-2k}}}\right)^{1/(n-2k)},\ r_0\text{ of Example 5.3.3}\},$$

where $\rho=\log\mathrm{P}$.

Proof. From Example 5.3.3 (HB1) and Example 5.3.1 (HB1), we know that

$$E_{r_0}^{n-2k}(\mathrm{P})\subset E_{\widetilde{r}_0}^{n-2k}(\Phi)\subset B_{c_{n,k}\cdot\widetilde{r}_0}(x_0)\times]t_0-\frac{\widetilde{r}_0^2}{4\pi},t_0[,\tag{6.18}$$

where

$$\widetilde{r}_0=\left(\frac{2}{\frac{1}{r_0^{n-2k}}-1}\right)^{1/(n-2k)},$$

and Φ is as in Lemma 6.3.1, whence we see that ψ is defined on $E_{r_0}^{n-2k}(\mathrm{P})$, i.e. that $E_{r_0}^{n-2k}(\mathrm{P})\subset B_{c_{n,k}\widetilde{r}_0}(x_0)\times\,]t_0-\delta,t_0[$ provided

$$r_0<\left(2(4\pi\delta)^{-\frac{n-2k}{2}}+1\right)^{-1/(n-2k)},$$

which is guaranteed in view of our choice of r_0. Moreover, since

$$r_0<\left(\frac{1}{1+\frac{2^{n-2k+1}c_{n,k}^{n-2k}}{R^{n-2k}}}\right)^{1/(n-2k)},$$

we have that $c_{n,k}\widetilde{r}_0\leqslant R$. Now, by (6.18), we immediately see that, since $\left((x,t)\mapsto e_g(\psi)(x,t)\cdot\frac{1}{t_0-t}\right)$, $|\mathcal{S}|^2$ and $|\mathcal{J}|^2$ are in $L^1\left(E_{\widetilde{r}_0}^{n-2k}(\Phi)\right)$ and thus also in $L^1\left(E_{r_0}^{n-2k}(\mathrm{P})\right)$. As for the remaining function, we estimate as in Lemma 6.3.1:

$$\begin{aligned}&\left|e_g(\psi)\left(\partial_t\rho+|\nabla\rho|^2\right)-\left\langle\iota_{\nabla\rho}\psi,\mathcal{S}\right\rangle-\left\langle\iota_{\nabla\rho}\mathcal{J},\psi\right\rangle\right|\\&\qquad\leqslant e_g(\psi)\left(|\partial_t\rho|+3|\nabla\rho|^2\right)+\frac{1}{2}\left(|\mathcal{S}|^2+|\mathcal{J}|^2\right).\end{aligned}$$

Now, by the computation in Example 5.3.3 (HB2),

$$|\nabla\rho|^2 \leqslant \frac{\mathrm{const}(\mathrm{geom}, n, k)}{t_0 - t} \cdot \left(1 - \log(\frac{4\pi(t_0 - t)}{r_0^2}\right) \text{ and}$$

$$|\partial_t \rho| \leqslant \mathrm{const}(\mathrm{geom}, n, k) \cdot \left(\frac{1}{t_0 - t}\left(1 + \log\left(\frac{4\pi(t_0 - t)}{r_0^2}\right)\right) + |\nabla\rho|^2\right).$$

We note that

$$-\frac{1}{t_0 - t}\log\left(\frac{4\pi(t_0 - t)}{r_0^2}\right) = \frac{R_{r_0}^{n-2k}(t)^2}{2(n-2k)(t_0 - t)^2}$$

so that it suffices to establish that the integral

$$\iint_{E_{r_0}^{n-2k}(\mathrm{P})} e_g(\psi) \cdot \frac{R_{r_0}^{n-2k}(t)^2}{(t_0 - t)} \mathrm{dvol}_{g_t} \mathrm{d}t$$

is finite, but it was shown in Lemma 6.3.1 that $\left(e_g(\psi) \cdot \frac{R_{r_0}^{n-2k}(t)^2}{(t_0-t)}\right) \in L^1\left(E_{\tilde{r}_0}^{n-2k}(\Phi)\right)$ so that this also follows from (6.18). □

Theorem 6.3.6. *Let (M, g) be a static compact Riemannian manifold and $\mathrm{P} := {}^{n-2k}\mathrm{P}_{(x_0,t_0)}$ as in Example 5.3.3, where $k \in \mathbb{N} \cap [0, \frac{n}{2}[$. Suppose that the integral*

$$\int_{t_0 - \frac{R^2}{16\pi c_{n,k}^2}}^{t_0} \int_{B_R^t(x_0)} e_g(\psi) \mathrm{dvol}_{g_t} \mathrm{d}t \tag{6.19}$$

is finite for some $R > 0$, where $c_{n,k} = \sqrt{\frac{n-2k}{2\pi e}}$. Then there exist an $r_0 > 0$ depending on the local geometry of M about (x_0, t_0), δ and R, and a function $\xi \in C(]t_0 - 1, t_0])$ with $\xi(t_0) = 0$ depending on the geometry of M such that for $0 < r_1 < r_2 < r_0$, the identity

$$\left[\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\mathrm{P})} \exp(\xi(t)) \left(e_g(\psi)(\partial_t\rho + |\nabla\rho|^2) - \left\langle \iota_{\nabla\rho}\psi, \iota_{\nabla\rho}\psi - \delta^\nabla\psi\right\rangle - \left\langle \iota_{\nabla\rho}\mathcal{J}, \psi\right\rangle\right) \mathrm{dvol}_{g_t} \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\mathrm{P})} \exp(\xi(t)) \cdot \left(|\mathcal{J}|^2 + |\mathcal{S}|^2\right) \mathrm{dvol}_{g_t} \mathrm{d}t\right) \mathrm{d}r$$

holds, where $\rho = \log \mathrm{P}$. In particular, the quantity

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\mathrm{P})} e^{\xi(t)} \left(e_g(\psi)\left(\partial_t\rho + |\nabla\rho|^2\right) - \left\langle \iota_{\nabla\rho}\psi, \iota_{\nabla\rho}\psi - \delta^\nabla\psi\right\rangle - \left\langle \iota_{\nabla\rho}\mathcal{J}, \psi\right\rangle\right) \mathrm{dvol}_{g_t} \mathrm{d}t$$

is monotone nondecreasing in r. If $\sec_M \geqslant 0$ and $\mathrm{dRic} \equiv 0$, then $\xi \equiv 0$.

Remark 6.3.7. r_0 is as in Lemma 6.3.5 and ξ is as in (6.20).

Remark 6.3.8. Note that

1. If $n \geqslant 4$, $\{\omega_t = \widetilde{\omega} + a(t)\}_{t\in]t_0-\delta,t_0[}$ is a family of connections on a principal bundle $P \to M^n$ evolving by the Yang-Mills flow according to the setup of §1.11 and the integral (6.19) is finite for $\psi = \underline{\Omega}^\omega$, then this identity holds and reads

$$\left[\frac{1}{r^{n-4}} \iint_{E_r^{n-4}(\mathrm{P})} \exp(\xi(t)) \left(e_g(\underline{\Omega}^\omega)(\partial_t\rho + |\nabla\rho|^2) - \left\langle \iota_{\nabla\rho}\underline{\Omega}^\omega, \partial_t a + \iota_{\nabla\rho}\underline{\Omega}^\omega\right\rangle\right) \mathrm{dvol}_{g_t} \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2} \left(\frac{n-4}{r^{n-3}} \iint_{E_r^{n-4}(\mathrm{P})} \exp(\xi(t)) \left|\partial_t a + \iota_{\nabla\rho}\underline{\Omega}^\omega\right|^2 \mathrm{dvol}_{g_t} \mathrm{d}t\right) \mathrm{d}r,$$

and

2. If $n \geqslant 2$, $u : M^n \times]t_0 - \delta, t_0[\to N \subset \mathbb{R}^K$ evolves by the harmonic map heat flow according to the setup of §1.10, where N is isometrically embedded in $\mathbb{R}^K$, and the integral (6.19) is finite for $\psi = \mathrm{d}u$, then this identity holds and reads

$$\left[\frac{1}{r^{n-2}} \iint_{E_r^{n-2}(\mathrm{P})} \exp(\xi(t)) \left(e_g(\mathrm{d}u)(\partial_t\rho + |\nabla\rho|^2) - \left\langle \iota_{\nabla\rho}\mathrm{d}u, \partial_t u + \iota_{\nabla\rho}\mathrm{d}u\right\rangle\right) \mathrm{dvol}_{g_t} \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$\geqslant \int_{r_1}^{r_2} \left(\frac{n-2}{r^{n-1}} \iint_{E_r^{n-2}(\mathrm{P})} \exp(\xi(t)) \left|\partial_t u + \iota_{\nabla\rho}\mathrm{d}u\right|^2 \mathrm{dvol}_{g_t} \mathrm{d}t\right) \mathrm{d}r.$$

Thus, these identities yield local analogues of Hamilton's formulæ (cf. §4.1).

Proof of Theorem 6.3.6. We work on $E_{r_0}^{n-2k}(\mathrm{P})$ where r_0 is as in Lemma 6.3.5. By Theorem 1.8.6,

$$\begin{aligned}\mathcal{H}_{t_0}\rho &\geqslant -F\left(1 + \log\left(\frac{B}{(4\pi(t_0-t))^{\frac{n}{2}-k}}\right) - \rho\right) g \\ &\geqslant -F\left(1 + \log\left(\frac{B}{(4\pi(t_0-t))^{\frac{n}{2}-k}}\right)\right) g - (F(n-2k)\underbrace{\log r_0}_{\leqslant 0})g\end{aligned}$$

on $E_{r_0}^{n-2k}(\mathrm{P})$ so that the inequalities (6.9) are satisfied by P with

$$a \equiv 0 \text{ and}$$
$$b(t) = -F\left(1 + \log\left(\frac{B}{(4\pi(t_0-t))^{\frac{n}{2}-k}}\right)\right) (= 0 \text{ if } \sec_M \geqslant 0 \text{ and } \mathrm{dRic} = 0)\,.$$

Moreover, it is clear that both a and b define continuous, summable functions on $]t_0 - \delta, t_0[$ and, adopting the notation of Corollary 6.2.6,

$$\begin{aligned}\xi(t) &= 2kF\int_t^{t_0} 1 + \log\left(\frac{B}{4\pi}\right) - \frac{n-2k}{2}\log(t_0-u)\mathrm{d}u \\ &= (t_0 - t)\cdot\left(1 + \log\left(\frac{B}{4\pi}\right) - \frac{n-2k}{2}\left(\log(t_0-t) - 1\right)\right)\end{aligned} \tag{6.20}$$

In light of the finiteness of the finiteness of the integral (6.19), we may appeal to Lemma 6.3.5 to apply Corollary 6.2.6 exactly as in Theorem 6.3.2, which establishes the result. □

— 7 —

Monotonicity of Localized Singular Area of a Submanifold Evolving by Mean Curvature Flow

In this chapter we establish a local monotonicity identity for embeddings evolving by the mean curvature flow using the heat balls of Chapter 5 in a similar manner to Chapter 6. This identity then leads to monotonicity formulæ in various cases where M is curved, thus furnishing a generalization of the local monotonicity formula of Ecker [18]. The various terms arising in the formula should be compared to those arising in the formula of [53] (Theorem 1.12.6).

7.1. Review. We recall the local formula of Ecker [18] which served as motivation for the identityof §7.2.

Let $\{F_t : N^m \to \mathbb{R}^n\}_{t\in]t_0-\delta,t_0[}$ $(t_0 \in \mathbb{R}, \delta_0 > 0)$ be a smooth one-parameter family of embeddings evolving by mean curvature (cf. §1.12) and set

$$E_r(x_0,t_0) = \mathcal{M} \cap \left\{(x,t) \in \mathbb{R}^n \times]-\infty, t_0[\,:\, \Phi(x,t) > \frac{1}{r^m}\right\}$$

for $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}$, where

$$\mathcal{M} = \bigcup_{t\in]t_0-\delta_0,t_0[} F_t(N) \times \{t\}$$

is the spacetime track of $\{F_t\}_{t\in]t_0-\delta_0,t_0[}$ and

$$\Phi(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{m}{2}}} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right).$$

It was shown by Ecker [18] that if F_t is well-defined in a cylinder of the form $B_R(x_0) \times]t_0-\varepsilon,t_0[$ in an appropriate sense,[1] then the identity

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^m}\iint\limits_{E_r(x_0,t_0)} u\left(|(\nabla\phi)^T|^2 + |H|^2\phi_r\right)\mathrm{d}\mathcal{H}^m\mathrm{d}t\right)$$
$$= \frac{n}{r^{n+1}}\iint\limits_{E_r(x_0,t_0)}\left(u\left|H-(\nabla\phi)^\perp\right|^2 - \phi_r\left(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\right)u\right)\mathrm{d}\mathcal{H}^m\,\mathrm{d}t, \quad (7.1)$$

holds for sufficiently small r, where $\mathrm{d}\mathcal{H}^m$ denotes m-dimensional Hausdorff measure, $\phi = \log\Phi$, $\phi_r = \phi + m\log r$ and $u \in C^{2,1}(\mathbb{R}^n \times I, \mathbb{R})$. It is evident that if $u \equiv 1$, then we obtain the monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^m}\iint\limits_{E_r(x_0,t_0)}\left(|(\nabla\phi)^T|^2 + |H|^2\phi_r\right)\mathrm{d}\mathcal{H}^m\mathrm{d}t\right) \geqslant 0,$$

which is a local analogue of Huisken's monotonicity formula (cf. Theorem 1.12.6). Note that all of the quantities appearing here are considered on $\mathcal{M} \subset \mathbb{R}^n \times]t_0-\delta_0,t_0[$, as opposed to being considered on $N \times]t_0-\delta_0,t_0[$.

7.2. Local monotonicity identities. Let $(M,\{g_t\}_{t\in]t_0-\delta_0,t_0[})$ be an evolving Riemannian manifold with $\partial_t g = h$ and $\{F_t = F(\cdot,t) : N^m \to (M^n,g_t)\}_{t\in]t_0-\delta_0,t_0[}$ a family of embeddings

[1] In our case, we assume that $(F,\mathrm{pr}_2) : N \times I \to M \times I$ is proper in place of this.

evolving by mean curvature flow such that $(F, \mathrm{pr}_2) : N \times]t_0 - \delta_0, t_0[\to M \times]t_0 - \delta_0, t_0[$ is proper (cf. §1.12 for notation and setup), and let $\Phi \in C^{2,1}(\mathcal{D}, \to \mathbb{R}^+)$ with $\mathcal{D} \subset M \times I$ be such that $E_r^m(\underline{\Phi})$ is a heat ball[2] for $r < r_0$ (cf. §5.2). As in Chapter 6, write $\mathcal{H}_{t_0} f$ for the matrix Harnack expression

$$\nabla^2 f + \frac{1}{2} h + \frac{g}{2(t_0 - t)}$$

of $f \in C^{2,1}(\mathcal{D})$.

The following theorem should be considered a local analogue of Magni, Mantegazza and Tsatis' generalization [53] of Huisken's monotonicity formula (Theorem 1.12.6).

Theorem 7.2.1. *If $u \in C^{2,1}(E_{r_0}^m(\underline{\Phi}))$ and $\frac{u}{t_0 - t}\underline{\phi} \in L^1\left(E_{r_0}^m(\underline{\Phi})\right)$, then*

$$\left[\frac{1}{r^m} \iint\limits_{E_r^m(\underline{\Phi})} u \left[|\nabla_{\mathfrak{I}_t} \underline{\phi}|^2 + \left(|\underset{\sim}{H}|^2 - \frac{1}{2} tr_{\mathfrak{I}} F_t^* h\right) \underline{\phi_r^n}\right] \mathrm{dvol}_{\mathfrak{I}_t} \mathrm{d}t\right]_{r=r_1}^{r=r_2}$$

$$= \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint\limits_{E_r^m(\underline{\Phi})} -u \cdot \left(\underline{\partial_t \phi + \Delta_{g_t} \phi + |\nabla_{g_t} \phi|^2 + \frac{1}{2} tr_g h} + \frac{n-m}{2(t_0 - t)} \right) \right.$$

$$\left. - \underline{\phi_r^m} \cdot (\partial_t - \Delta_{\mathfrak{I}}) u + u \left|\underset{\sim}{H} - \nabla^\perp \phi\right|^2 + u \cdot tr_g^\perp \mathcal{H}_{t_0}(\phi) \mathrm{dvol}_{\mathfrak{I}_t} \mathrm{d}t \right) \mathrm{d}r \quad (7.2)$$

for $0 < r_1 < r_2 < r_0$ provided both spacetime integrands are in $L^1\left(E_{r_0}^m(\underline{\Phi})\right)$. If $u \geqslant 0$, then the condition $\frac{u}{t_0 - t}\underline{\phi} \in L^1\left(E_{r_0}^m(\underline{\Phi})\right)$ may be lifted so that the the identity (7.2) holds with $\geqslant$ in place of $=$.

Remark 7.2.2. As with Theorem 6.2.1 (cf. Remark 6.2.3), this identity implies a monotonicity formula if $\Phi(\cdot, t) = (t_0 - t)^{\frac{n-m}{2}} \mathrm{P}(\cdot, t)$ for a positive subsolution P of the backward heat equation and if $\mathcal{H}_{t_0} \phi \leqslant 0$, which in particular holds for $(M, g_t) \equiv (\mathbb{R}^n, \delta)$ taken with $\Phi(\cdot, t) = (t_0 - t)^{\frac{n-m}{2}} \mathrm{P}(\cdot, t)$ with P being the standard heat kernel on $\mathbb{R}^n$.

Remark 7.2.3. If u is bounded on $E_{r_0}^m(\underline{\Phi})$ and Φ is either the (suitably weighted) formal backward heat kernel (cf. Example 5.3.4) or, if M is static and compact, the (suitably weighted) canonical backward heat kernel concentrated at (x_0, t_0) (cf. Example 5.3.5), then the estimates of Examples 5.3.4 (HB2) and 5.3.5 (HB2) immediately imply that the integrals of (7.2) (see Theorems 7.3.2 and 7.3.4). In particular, if $(M, g_t) \equiv (\mathbb{R}^n, \delta)$, we recover (7.1) up to the choice of working in $N \times]t_0 - \delta_0, t_0[$ or $M \times]t_0 - \delta_0, t_0[$.

Proof of Theorem 7.2.1. We first assume that $u(\cdot, t) \equiv 0$ for $t \in [\tau_0, t_0[$. Just as in the proof of Theorem 6.2.1, we first approximate:

$$\left[\frac{1}{r^m} J_q^r \left(u \cdot \left[|\nabla_{\mathfrak{I}_t} \underline{\phi}|^2 + \underline{\phi_r^m} \cdot \left(|\underset{\sim}{H}|^2 - \frac{1}{2} \mathrm{tr}_{\mathfrak{I}} F_t^* h\right)\right]\right)\right]_{r=r_1}^{r=r_2}$$

$$= \int_{r_1}^{r_2} \frac{m}{r^{m+1}} J_q^r \left(-u \cdot \left[|\nabla_{\mathfrak{I}_t} \underline{\phi}|^2 + \underline{\phi_r^m} \cdot \left(|\underset{\sim}{H}|^2 - \frac{1}{2} \mathrm{tr}_{\mathfrak{I}} F_t^* h\right)\right]\right) \mathrm{d}r$$

$$+ \int_{r_1}^{r_2} \frac{1}{r^m} \frac{\mathrm{d}}{\mathrm{d}r} J_q^{\cdot} \left(u \cdot \left[|\nabla_{\mathfrak{I}_t} \underline{\phi}|^2 + \underline{\phi_{\cdot}^m} \cdot \left(|\underset{\sim}{H}|^2 - \frac{1}{2} \mathrm{tr}_{\mathfrak{I}} F_t^* h\right)\right]\right) \mathrm{d}r.$$

We note that, since u vanishes near t_0, each individual term in the approximate integrals is summable over $E_{r_0}^m(\underline{\Phi})$,[3] thus allowing us to freely separate these integrals.

To calculate the latter integral, we note that, by Theorem 5.4.2,

[2] Recall that $\underline{\Phi} = \Phi \circ (F, \mathrm{pr}_2) : (F, \mathrm{pr}_2)^{-1}(\mathcal{D}) \to \mathbb{R}^+$.

[3] In particular, each term represents a continuous function supported in $E_{r_0}^m(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}(]t_0 - \delta_0, \frac{1}{2}(t_0 + \tau_0)[)$, a relatively compact set in $N \times]t_0 - \delta_0[$.

$$\frac{\mathrm{d}}{\mathrm{d}r}J_q^{\cdot}\left(\left\langle u\nabla_{\Im_t}\underline{\phi},\nabla_{\Im_t}\underline{\phi}\right\rangle\right)=-\frac{m}{r}J_q^r(\mathrm{div}_{\Im}(u\nabla_{\Im_t}\underline{\phi}))=-\frac{m}{r}J_q^r(\left\langle\nabla_{\Im_t}u,\nabla_{\Im_t}\underline{\phi}\right\rangle+u\Delta_{\Im}\underline{\phi})$$

whilst, by Theorem 5.4.4,

$$\frac{\mathrm{d}}{\mathrm{d}r}J_q^{\cdot}\left(\underline{\phi_{\cdot}^m}\cdot\left(|\underset{\smile}{H}|^2-\frac{1}{2}\mathrm{tr}_{\Im}F_t^*h\right)\right)=\frac{m}{r}J_q^r\left(|\underset{\smile}{H}|^2-\frac{1}{2}\mathrm{tr}_{\Im}F_t^*h\right)+o(1)$$

as $q\to\infty$, where the remainder may be bounded from above uniformly in r. Thus, by Lemma 5.4.1 and the dominated convergence theorem, taking the limit $q\to\infty$ in the above yields

$$\left[\frac{1}{r^m}\iint\limits_{E_r^m(\underline{\Phi})}u\cdot\left[|\nabla_{\Im_t}\underline{\phi}|^2+\underline{\phi_r^m}\cdot\left(|\underset{\smile}{H}|^2-\frac{1}{2}\mathrm{tr}_{\Im}F_t^*h\right)\right]\mathrm{dvol}_{\Im_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$=\int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint\limits_{E_r^m(\underline{\Phi})}-u\cdot\left(|\nabla_{\Im_t}\underline{\phi}|^2+\underline{\phi_r^m}\cdot\left(|\underset{\smile}{H}|^2-\frac{1}{2}\mathrm{tr}_{\Im}F_t^*h\right)\right)-\left\langle\nabla_{\Im_t}u,\nabla_{\Im_t}\underline{\phi}\right\rangle\right.$$
$$\left.-u\Delta_{\Im}\underline{\phi}+u\cdot\left(|\underset{\smile}{H}|^2-\frac{1}{2}\mathrm{tr}_{\Im}F_t^*h\right)\mathrm{dvol}_{\Im_t}\mathrm{d}t\right)\mathrm{d}r,$$

Now, by Proposition 1.12.4

$$\Delta_{\Im}\underline{\phi}=\underline{\Delta_g\phi}-\mathrm{tr}_g^{\perp}\nabla_g^2\phi+\left\langle\underset{\smile}{\nabla_{g_t}\phi},\underset{\smile}{H}\right\rangle.$$

On the other hand, we note that

$$|\nabla_{\Im_t}\underline{\phi}|^2=|\underset{\smile}{\nabla_{g_t}\phi}|^2-|\nabla^{\perp}\phi|^2$$

and

$$\underline{\mathrm{tr}_g h}=\mathrm{tr}_{\Im}F_t^*h+\mathrm{tr}_g^{\perp}h.$$

Proceeding with these identities in mind,

$$=\int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint\limits_{E_r^m(\underline{\Phi})}-u\cdot\left(\underline{\partial_t\phi+\Delta_g\phi+|\nabla_{g_t}\phi|^2+\frac{1}{2}\mathrm{tr}_g h}\right)-u\underline{\phi_r^m}\cdot\left(|\underset{\smile}{H}|^2-\frac{1}{2}\mathrm{tr}_{\Im}F_t^*h\right)\right.$$
$$+u\cdot\underline{\partial_t\phi}-\left\langle\nabla_{\Im_t}u,\nabla_{\Im_t}\underline{\phi}\right\rangle+u\left(|\underset{\smile}{H}|^2+|\nabla^{\perp}\phi|^2-\left\langle\underset{\smile}{\nabla_{g_t}\phi},H\right\rangle\right)$$
$$\left.+\underline{u\mathrm{tr}_g^{\perp}\left(\nabla^2\phi+\frac{1}{2}h\right)}\mathrm{dvol}_{\Im_t}\mathrm{d}t\right)\mathrm{d}r.\quad(7.3)$$

Now, together with Proposition 1.12.2 and the identities

$$\underline{\partial_t\phi}=\partial_t\underline{\phi}-\left\langle\underset{\smile}{\nabla_{g_t}\phi},\underset{\smile}{H}\right\rangle$$

and $\partial_t\underline{\phi}=\partial_t\underline{\phi_r^m}$, an application of Theorem 5.4.5 yields

$$\iint_{E_r^m(\underline{\Phi})} u \cdot \underline{\partial_t \phi} \mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t = \iint_{E_r^m(\underline{\Phi})} u \cdot \partial_t \underline{\phi_r^m} - u \cdot \left\langle \underbrace{\nabla_{g_t} \phi}, \underset{\smile}{H} \right\rangle \mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t$$

$$= \iint_{E_r^m(\underline{\Phi})} -\partial_t u \cdot \underline{\phi_r^m} + u \cdot \underline{\phi_r^m} \left(\left\langle \underset{\smile}{H}, \underset{\smile}{H} \right\rangle - \frac{1}{2} \mathrm{tr}_{\mathfrak{F}} F_t^* h \right) - u \left\langle \underbrace{\nabla_{g_t} \phi}, \underset{\smile}{H} \right\rangle \mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t.$$

On the other hand,

$$\mathrm{tr}_g^{\perp} \frac{g}{2(t_0 - t)} = \frac{n - m}{2(t_0 - t)}.$$

Proceeding from (7.3) with these two identities in mind and completing the square in $\underset{\smile}{H}$ and $\nabla^{\perp}\phi$ and writing

$$L\phi = \partial_t \phi + \Delta_g \phi + |\nabla_{g_t} \phi|^2 + \frac{1}{2} \mathrm{tr}_g h + \frac{n - m}{2(t_0 - t)},$$

we obtain,

$$= \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint_{E_r^m(\underline{\Phi})} -u \cdot \left(\underline{L\phi}\right) - u \cdot \underline{\phi_r^m} \cdot \left(|\, \underset{\smile}{H} \,|^2 - \frac{1}{2} \mathrm{tr}_{\mathfrak{F}} F_t^* h \right) \right.$$

$$- \partial_t u \cdot \underline{\phi_r^m} + u \cdot \underline{\phi_r^m} \cdot \left(\left\langle \underset{\smile}{H}, \underset{\smile}{H} \right\rangle - \frac{1}{2} \mathrm{tr}_{\mathfrak{F}} F_t^{ast} h \right) - u \left\langle \underbrace{\nabla_{g_t} \phi}, \underset{\smile}{H} \right\rangle$$

$$\left. - \left\langle \nabla_{\mathfrak{F}_t} u, \nabla_{\mathfrak{F}_t} \underline{\phi} \right\rangle + u \cdot |\, \underset{\smile}{H} - \nabla^{\perp} \phi|^2 + u \left\langle \underset{\smile}{H}, \nabla^{\perp} \phi \right\rangle + u \mathrm{tr}_g^{\perp} \mathcal{H}_{t_0}(\phi) \mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t \right) \mathrm{d}r$$

$$= \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint_{E_r^m(\underline{\Phi})} -u \cdot \underline{L\phi} + u |\, \underset{\smile}{H} - \nabla^{\perp} \phi|^2 + u \cdot \mathrm{tr}_g^{\perp} \mathcal{H}_{t_0}(\phi) \right.$$

$$\left. - \partial_t u \cdot \underline{\phi_r^m} - \left\langle \nabla_{\mathfrak{F}_t} u, \nabla_{\mathfrak{F}_t} \underline{\phi} \right\rangle \mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t \right) \mathrm{d}r.$$

Now, it is clear that

$$\left\langle \nabla_{\mathfrak{F}_t} u, \nabla_{\mathfrak{F}_t} \underline{\phi} \right\rangle = \mathrm{div}_{\mathfrak{F}} \left(\underline{\phi_r^m} \cdot \nabla_{\mathfrak{F}_t} u \right) - \underline{\phi_r^m} \cdot \nabla_{\mathfrak{F}_t} u$$

which, together with Theorem 5.4.6, implies that

$$\iint_{E_r^m(\underline{\Phi})} - \left\langle \nabla_{\mathfrak{F}_t} u, \nabla_{\mathfrak{F}_t} \underline{\phi} \right\rangle \mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t = \iint_{E_r^m(\underline{\Phi})} \underline{\phi_r^m} \cdot \Delta_{\mathfrak{F}} u \,\mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t$$

which establishes the result in the case where u vanishes close to t_0.

Now, consider $u_l : (F, \mathrm{pr}_2)^{-1}(\mathcal{D}) \to \mathbb{R}$ defined by $u_l(x,t) = \chi_l(t_0 - t) \cdot u(x,t)$. The above implies that

$$\left[\frac{1}{r^m} \iint_{E_r^m(\underline{\Phi})} \chi_l(t_0 - t) \cdot u \left[|\nabla_{\mathfrak{F}_t} \underline{\phi}|^2 + \left(|\, \underset{\smile}{H} \,|^2 - \frac{1}{2} \mathrm{tr}_{\mathfrak{F}} F_t^* h \right) \underline{\phi_r^m} \right] \mathrm{dvol}_{\mathfrak{F}_t} \mathrm{d}t \right]_{r=r_1}^{r=r_2}$$

$$= \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint\limits_{E_r^m(\underline{\Phi})} \chi_l(t_0 - t) \cdot \left(-u \cdot \left(\underline{\partial_t \phi + \Delta_{g_t} \phi + |\nabla_{g_t} \phi|^2 + \frac{1}{2}\mathrm{tr}_g h} + \frac{n-m}{2(t_0-t)} \right) \right.\right.$$

$$\left.\left. - \underline{\phi_r^m} \cdot (\partial_t - \Delta_{\mathfrak{I}}) u + u \left| \underset{\sim}{H} - \nabla^\perp \phi \right|^2 + u \cdot \mathrm{tr}_g^\perp \mathcal{H}_{t_0}(\phi) \right) \mathrm{dvol}_{\mathfrak{I}_t} \mathrm{d}t \right) \mathrm{d}r$$

$$+ \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint\limits_{E_r^m(\underline{\Phi})} u \cdot \frac{1}{t_0 - t} \underline{\phi_r^m} \cdot \chi_l'(t_0 - t) \cdot (t_0 - t) \mathrm{dvol}_{\mathfrak{I}_t} \mathrm{d}t \right) \mathrm{d}r. \quad (7.4)$$

Since $0 \leqslant \chi_l(t_0 - t) \leqslant 1$, the first two integrands may be bounded in absolute value from above by the absolute values of the corresponding integrands occurring in the statement of this theorem, which are assumed summable. Thus, we may pass to the limit $l \to \infty$ in the first two integrals of identity (7.4) as in the proof of Theorem 6.2.1, and if $\frac{1}{t_0 - t}\underline{\phi_r^m} \in L^1\left(E_{r_0}^m(\underline{\Phi})\right)$, then, also as in the proof of Theorem 6.2.1,

$$\left| u \cdot \frac{1}{t_0 - t} \underline{\phi_r^m} \cdot \chi_l'(t_0 - t) \cdot (t_0 - t) \right| \leqslant C \cdot \left| \frac{u}{t_0 - t} \underline{\phi_r^m} \right| \in L^1\left(E_{r_0}^m(\underline{\Phi})\right),$$

allowing us to apply the dominated convergence theorem, which implies that the last integral on the right-hand side vanishes in the limit $l \to \infty$, since $\chi_l'(t_0 - t) \cdot (t_0 - t) \xrightarrow{l \to \infty} 0$. Finally, if $u \geqslant 0$, we may discard the latter integral on the right-hand side by estimating it from below by 0, since $\chi_l' \geqslant 0$, wherefore the aforementioned limits involving the remaining integrals may be taken, thus establishing the result. □

7.3. Applications. As was the case in Chapter 6 with Dirichlet-type flows, we may not know too much about the integrability of the Harnack term. However, we may more carefully go through the steps of the proof of Theorem 7.2.1 in order to derive a monotonicity identity nonetheless.

Theorem 7.3.1. *Suppose $u \in C^{2,1}(E_{r_0}^m(\underline{\Phi}), \mathbb{R}^+)$ and the bounds*

$$\underline{\partial_t \phi + \Delta_{g_t} \phi + |\nabla_{g_t} \phi|^2 + \frac{1}{2} tr_g h + \frac{n-m}{2(t_0 - t)}} \leqslant a(t)$$
$$\underline{\mathcal{H}_{t_0}(\phi)} \geqslant b(t) g \quad (7.5)$$

hold on $E_{r_0}^m(\underline{\Phi})$ with $a, b \in C(]t_0 - \delta_0, t_0[) \cap L^1(]t_0 - \delta_0, t_0[)$ and summable over $E_{r_0}^m(\underline{\Phi})$. Then the inequality

$$\left[\frac{1}{r^m} \iint\limits_{E_r^m(\underline{Z})} u \left[|\nabla_{\mathfrak{I}_t} \underline{\zeta}|^2 + \left(|\underset{\sim}{H}|^2 - \frac{1}{2} tr_{\mathfrak{I}} F_t^* h \right) \underline{\zeta_r^m} \right] \right]_{r=r_1}^{r=r_2}$$

$$\geqslant \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint\limits_{E_r^m(\underline{Z})} -\underline{\zeta_r^m} \cdot (\partial_t - \Delta_{\mathfrak{I}}) u + u \left| \underset{\sim}{H} - \nabla^\perp \zeta \right|^2 \mathrm{dvol}_{\mathfrak{I}_t} \mathrm{d}t \right) \mathrm{d}r$$

holds for $0 < r_1 < r_2 < r_0$ whenever the two spacetime integrands are summable over $E_{r_0}^m(\underline{Z})$, where $Z : E_{r_0}^m(\Phi) \to \mathbb{R}^+$ is defined such that $Z(x,t) = \exp(\xi(t))\Phi(x,t)$ with $\xi(t) = \int_t^{t_0} a - (n-m)b$, $\zeta = \log Z$, $\zeta_r^m = \log(Zr^m)$ and $\tilde{r}_0 = r_0 \exp(-\frac{\sup|\xi|}{m})$.

Proof. Since a and b are summable over $]t_0 - \delta_0, t_0[$, it is clear that ξ is bounded so that, by Example 5.3.7, $E_r^m(\underline{Z})$ is a heat ball for $r < \tilde{r}_0$ and $E_{\tilde{r}_0}^m(\underline{Z}) \subset E_{r_0}^m(\underline{\Phi})$.

Now, we first assume as in the proof of Theorem 7.2.1 that u vanishes close to t_0. Under this assumption, we may apply this theorem to the $E_r^m(\underline{Z})$:

$$\left[\frac{1}{r^m}\iint_{E_r^m(\underline{Z})} u\left[|\nabla_{\mathfrak{I}_t}\underline{\zeta}|^2 + \left(|\underset{\sim}{H}|^2 - \frac{1}{2}\mathrm{tr}_{\mathfrak{I}}F_t^*h\right)\underline{\zeta_r^n}\right]\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$

$$= \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint_{E_r^m(\underline{Z})} -u\cdot\left(\underline{\partial_t\zeta + \Delta\zeta + |\nabla_{\mathfrak{I}_t}\underline{\zeta}|^2 + \frac{1}{2}\mathrm{tr}_g h} + \frac{n-m}{2(t_0-t)}\right)\right.$$

$$\left. - \underline{\zeta_r^m}\cdot(\partial_t - \Delta_{\mathfrak{I}})u + u\left|\underset{\sim}{H} - \nabla^\perp\zeta\right|^2 + u\cdot\mathrm{tr}_g^\perp\underline{\mathcal{H}_{t_0}(\zeta)}\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right)\mathrm{d}r.$$

Applying the bounds for ϕ, noting that $\partial_t\zeta = \partial_t\xi + \partial_t\phi$, $\nabla\zeta = \nabla_{g_t}\phi$ and $\mathcal{H}_{t_0}(\zeta) = \mathcal{H}_{t_0}(\phi)$, we obtain

$$\left[\frac{1}{r^m}\iint_{E_r^m(\underline{Z})} u\left[|\nabla_{\mathfrak{I}_t}\underline{\zeta}|^2 + \left(|\underset{\sim}{H}|^2 - \frac{1}{2}\mathrm{tr}_{\mathfrak{I}}F_t^*h\right)\underline{\zeta_r^n}\right]\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$

$$\geqslant \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint_{E_r^m(\underline{Z})} -u\left(\partial_t\xi + a - (n-m)b\right) - \underline{\zeta_r^m}\cdot(\partial_t - \Delta_{\mathfrak{I}})u + u\left|\underset{\sim}{H} - \nabla^\perp\zeta\right|^2\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right)\mathrm{d}r,$$

but $\partial_t\xi = (n-m)b - a$ so that we obtain the claim whenever u vanishes near t_0. The approximation argument in the proof of Theorem 7.2.1 then establishes the inequality for more general u. □

We may now deduce local monotonicity formulæ from the preceding identity, first starting with the case where (M,g) is evolving and of locally bounded geometry about (x_0,t_0). The following should be considered a generalization of Ecker's local monotonicity formula [18] and ultimately a local analogue of the monotonicity formula that would follow as a consequence of Magni, Mantegazza and Tsatis' formula [53].

Theorem 7.3.2. *Let (M,g) be an evolving Riemannian manifold with locally bounded geometry about (x_0,t_0) and let $\underline{\Phi}$ be as in Example 5.3.4. Suppose $u \in C^{2,1}(E_{r_0}^m(\underline{\Phi}),\mathbb{R}^+) \cap L^\infty(E_{r_0}^m(\underline{\Phi}))$ satisfies*

$$(\partial_t - \Delta_{\mathfrak{I}})u \leqslant 0.$$

Then there exist an $r_0 > 0$ depending on the geometry of M and δ, and a function $\xi \in C(]t_0-1,t_0])$ with $\xi(t_0) = 0$ depending on the geometry of M such that for $0 < r_1 < r_2 < r_0$, the identity

$$\left[\frac{1}{r^m}\iint_{E_r^m(\underline{Z})} u\left[|\nabla_{\mathfrak{I}_t}\underline{\zeta}|^2 + \left(|\underset{\sim}{H}|^2 - \frac{1}{2}tr_{\mathfrak{I}}F_t^*h\right)\underline{\zeta_r^m}\right]\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2} \geqslant \int_{r_1}^{r_2}\left(\iint_{E_r^m(\underline{Z})} u|H - \nabla^\perp\zeta|^2\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right)\mathrm{d}r$$

holds, i.e. the quantity

$$\frac{1}{r^m}\iint_{E_r^m(\underline{Z})} u\left[|\nabla_{\mathfrak{I}_t}\underline{\zeta}|^2 + \left(|\underset{\sim}{H}|^2 - \frac{1}{2}tr_{\mathfrak{I}}F_t^*h\right)\underline{\zeta_r^m}\right]\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t$$

is monotone for $r \in]0,\tilde{r}_0[$, where $\mathrm{Z} : E_{r_0}^m(\underline{\Phi}) \to \mathbb{R}^+$ (r_0 as in Example 5.3.4) is defined such that $\mathrm{Z}(x,t) = \exp(\xi(t))\Phi(x,t)$, $\zeta = \log\mathrm{Z}$ and $\underline{\zeta_r^m} = \log(\mathrm{Z}r^m)$.

Remark 7.3.3. More explicitly, ξ is given by (6.17) and $\tilde{r}_0$ by $r_0\exp(-\frac{\sup|\xi|}{m})$ where r_0 is as in Lemma 6.3.1 (with $k = \frac{n-m}{2}$).

Proof. We proceed as in Theorem 6.3.2 in order to apply the Theorem 7.3.1. To this end, by Lemma 6.3.1, whose proof is valid for $k = \frac{n-m}{2}$, the necessary inequalities for ϕ hold on $E^m_{r_0}(\underline{\Phi})$ (r_0 as in Lemma 6.3.1) with[4]

$$\begin{aligned} a(t) &= \frac{n\mu}{2} - 2mC_4 \log\left(\frac{4\pi(t_0 - t)}{r_0^2}\right) \text{ and} \\ b(t) &= -mC\log\left(\frac{4\pi(t_0-t)}{r_0^2} + \frac{\lambda_{-\infty}}{2}\right), \end{aligned}$$

and these define summable continuous functions on $]t_0 - 1, t_0[$. Moreover, they are summable over $E^m_{r_0}(\underline{\Phi})$, since we may write

$$\iint\limits_{E^m_{r_0}(\underline{\Phi})} a(t)\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t \leqslant \int_{t_0 - \frac{r_0^2}{4\pi}}^{t_0} a(t)\cdot \int_{\underline{B}^t_{R^m_{r_0}(t)}(x_0)} \mathrm{dvol}_{\mathfrak{I}}\mathrm{d}t,$$

and the inner integral is bounded from above by Theorem 1.12.7.

Thus, let ξ and $\widetilde{r}_0$ be as in Theorem 7.3.1. To apply this theorem, it suffices to show, since u is bounded on $E^m_{r_0}(\underline{\Phi}) \supset E^m_{\widetilde{r}_0}(\underline{Z})$, $|\nabla_{\mathfrak{I}_t}\underline{\zeta}| \leqslant |\underset{\smile}{\nabla\zeta}|$ and $|H - \nabla^{\perp}\zeta|^2 \leqslant 2\left(|\underset{\smile}{\nabla\zeta}|^2 + |\underset{\smile}{H}|^2\right)$, that

$$|\underset{\smile}{\nabla\zeta}|^2,\ |\underset{\smile}{H}|^2 \text{ and } \left(|\underset{\smile}{H}|^2 - \frac{1}{2}\mathrm{tr}_{\mathfrak{I}}F_t^* h\right)\underline{\zeta^m_r}$$

are summable over $E^m_{\widetilde{r}_0}(\underline{Z})$. Now, the first function is summable by virtue of the fact that $E^m_{\widetilde{r}_0}(\underline{Z})$. As for the second, this follows from Example 5.3.4 (HB2). Finally, to handle the last function, we go back to the proof of Theorem 5.4.5 and note that, $J_q^{\widetilde{r}_0}$ denoting the approximate integral over $E^m_{\widetilde{r}_0}(\underline{Z})$,

$$J_q^{\widetilde{r}_0}(\partial_t\underline{\zeta}) = J_q^{\widetilde{r}_0}(|\underset{\smile}{H}|^2\underline{\zeta^m_{\widetilde{r}_0}}) - J_q^{\widetilde{r}_0}(\frac{1}{2}\mathrm{tr}_{\mathfrak{I}}F_t^* h\underline{\zeta^m_{\widetilde{r}_0}}) + o(1) \tag{7.6}$$

as $q \to \infty$. Since the geometry of M is locally bounded in a neighbourhood of $(F, \mathrm{pr}_2)\left(E^m_{\widetilde{r}_0}(\underline{Z})\right)$, $\frac{1}{2}\mathrm{tr}_{\mathfrak{I}}F_t^* h$ is bounded on $E^m_{\widetilde{r}_0}(\underline{Z})$ so that, by Remark 5.2.3, the second term on the right-hand side of (7.6) is uniformly bounded in q. On the other hand, by (HB2), the left-hand side of (7.6) is bounded uniformly in q so that, overall, $|J_q^{\widetilde{r}_0}(|\underset{\smile}{H}|^2\underline{\zeta^m_{\widetilde{r}_0}})|$ is uniformly bounded in q. Taking the limit $q \to \infty$, summability follows from the dominated convergence theorem. □

Restricting ourselves to the static compact case, we obtain the following analogue of Ecker's local monotonicity formula, which should be viewed as a localized version of Hamilton's nonlocal formula [33].

Theorem 7.3.4. *Let (M, g) be a static compact Riemannian manifold and let $\underline{\mathrm{P}}$ be as in Example 5.3.5. Suppose $u \in C^{2,1}(E^m_{r_0}(\underline{\Phi}), \mathbb{R}^+) \cap L^{\infty}(E^m_{r_0}(\underline{\Phi}))$ satisfies*

$$(\partial_t - \Delta_{\mathfrak{I}})u \leqslant 0.$$

Then there exist an $r_0 > 0$ depending on the geometry of M and δ, and a function $\xi \in C(]t_0 - 1, t_0])$ with $\xi(t_0) = 0$ depending on the geometry of M such that the identity

$$\left[\frac{1}{r^m}\iint\limits_{E^m_r(\underline{Z})} u\left[|\nabla_{\mathfrak{I}_t}\underline{\zeta}|^2 + \left(|\underset{\smile}{H}|^2 - \frac{1}{2}tr_{\mathfrak{I}}F_t^* h\right)\underline{\zeta^m_r}\right]\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right]_{r=r_1}^{r=r_2} \geqslant \int_{r_1}^{r_2}\left(\iint\limits_{E^m_r(\underline{Z})} u|H - \nabla^{\perp}\zeta|^2\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t\right)\mathrm{d}r \tag{7.7}$$

[4]The constants C and C_4 depend only on the local geometry of M about (x_0, t_0) as in Propositions 1.8.7 and 1.8.8.

holds, i.e. the quantity

$$\frac{1}{r^m}\iint\limits_{E_r^m(\underline{Z})} u\left[|\nabla_{\mathfrak{I}_t}\underline{\zeta}|^2 + \left(|\underset{\sim}{H}|^2 - \frac{1}{2}tr_{\mathfrak{I}}F_t^* h\right)\underline{\zeta_r^m}\right]\mathrm{dvol}_{\mathfrak{I}_t}\mathrm{d}t$$

is monotone for $r \in]0, \widetilde{r_0}[$, *where* $\mathrm{Z} : E_{r_0}^m(\underline{\mathrm{P}}) \to \mathbb{R}^+$ *(*r_0 *as in Example 5.3.5) is defined such that* $\mathrm{Z}(x,t) = \exp(\xi(t))\mathrm{P}(x,t)$, $\zeta = \log \mathrm{Z}$ *and* $\zeta_r^m = \log(\mathrm{Z}r^m)$. *If* $\sec_M \geqslant 0$ *and* $\mathrm{dRic} \equiv 0$, *then* $\xi \equiv 0$.

Remark 7.3.5. More explicitly, ξ is given by (6.20) and $\widetilde{r_0}$ by $r_0 \exp(-\frac{\sup|\xi|}{m})$ where r_0 is as in Lemma 6.3.5 (with $k = \frac{n-m}{2}$).

Proof of Theorem 7.3.4. We proceed as in Theorem 6.3.6 in order to apply Theorem 6.2.6. By Lemma 6.3.5, whose proof is valid for $k = \frac{n-m}{2}$, we have that the inequalities (7.5) hold on $E_{r_0}^m(\underline{\mathrm{P}})$ (r_0 as in Lemma 6.3.5) with

$$a \equiv 0 \text{ and}$$
$$b(t) = -F\left(1 + \log\left(\frac{B}{(4\pi(t_0 - t))^{m/2}}\right)\right)\ (= 0 \text{ if } \sec_M \geqslant 0 \text{ and } \mathrm{dRic} = 0)\,,$$

where the constants are those of Theorem 1.8.6. These define summable continuous functions on $]t_0 - 1, t_0[$ just as in the preceding theorem and they are furthermore summable over $E_{r_0}^m(\underline{\mathrm{P}})$ in light of the argument in the preceding theorem and the inclusion $E_{r_0}^m(\underline{\mathrm{P}}) \subset E_{v_0}^m(\underline{\Phi})$ for $v_0 = \left(\frac{2}{\frac{1}{(r_0)^m} - 1}\right)$ (cf. the inclusion (6.18) in Lemma 6.3.5).

Now, let ξ and $\widetilde{r_0}$ be as in Theorem 7.3.1. Note that, since $E_{\widetilde{r_0}}^m(\underline{\mathrm{Z}}) \subset E_{r_0}^m(\underline{\mathrm{P}}) \subset E_{v_0}^m(\underline{\Phi})$ and the rightmost set is a heat ball, the integrands of (7.7) are summable by the proof of Theorem 7.3.2. Theorem 7.3.1 now applies. ◻

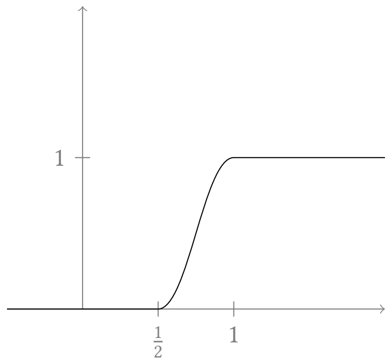
— A —

Analytical Auxiliaries

Auxiliary Functions

In this section we present some auxiliary functions that are used in a few constructions.

Example A.1 (Approximation to $\chi_{]0,\infty[}$). Let $\chi \in C^2(\mathbb{R},[0,1])$ be defined by the following graph:



It is clear that

1. $\chi_{[1,\infty[} \leqslant \chi \leqslant \chi_{[\frac{1}{2},\infty[}$, and
2. $|\chi'| \leqslant C \cdot \chi_{]\frac{1}{2},1[}$ for some $C > 0$.

Now, let

$$\{\chi_m \in C^\infty(\mathbb{R},[0,1])\}_{m\in\mathbb{N}}$$

be defined such that $\chi_m(x) = \chi(2^m x)$. Clearly

$$\chi_m \xrightarrow{m\to\infty} \chi_{]0,\infty[} \quad \text{ptwise}$$

and

$$|x\chi_m'(x)| = |2^m x| \cdot |\chi'(2^m x)| \leqslant C \cdot |2^m x| \cdot \chi_{]\frac{1}{2},1[}(2^m x) \leqslant C\chi_{]2^{-(m+1)},2^{-m}[}(x) \xrightarrow{m\to\infty} 0. \quad \square$$

Example A.2. Define $\eta : \mathbb{R} \to [0,\infty[$ by

$$s \mapsto \eta(s) = (1-s)_+^4 .$$

η is smooth on $\mathbb{R}\backslash\{1\}$ and C^3 on $\mathbb{R}$, whence it may easily be verified that

$$\eta'(s) = -4\,(1-s)_+^3$$

and

$$\eta''(s) = 12\,(1-s)_+^2 . \qquad \square$$

Inequalities of Note

The following inequality from [33] is useful for proving nonlocal monotonicity formulæ:

Lemma A.3. *If $x, y > 0$, then*

$$x\left(1 + \log\left(\frac{y}{x}\right)\right) \leqslant 1 + x\log y$$

— B —

Some Comparison Geometry

Let (M^n, g) be a Riemannian manifold. We collect a few technical lemmata which describe the local behaviour of distance functions on M. To this end, fix $p \in M$ and let $\mathfrak{r} : U_p \to \mathbb{R}^+$ be the **distance function at** p defined by $\mathfrak{r} := \mathrm{dist}(p, \cdot)$, where U_p is $M\backslash\{p\}$ minus the cut locus of p. Set $\partial_\mathfrak{r} := \nabla r \in \Gamma(TM|_{U_p})$ and let $\mathrm{d}\mathfrak{r} \in \Gamma(T^\star M|_{U_p})$ be the element dual to $\partial_\mathfrak{r}$. We write the metric g in polar form, treating $\partial_\mathfrak{r}$ as the radial direction, i.e. such that

$$g = \mathrm{d}\mathfrak{r} \otimes \mathrm{d}\mathfrak{r} + g_\mathfrak{r}, \tag{B.1}$$

where $\iota_{\partial_\mathfrak{r}} g_\mathfrak{r} = 0$.

Theorem B.1 (Hessian Comparison Theorem [57, Theorem 27, p. 175]). *If $k \leqslant sec_g \leqslant K$ in U_p, then*

$$(f_K \circ \mathfrak{r}) \cdot g_\mathfrak{r} \leqslant \nabla^2 \mathfrak{r} \leqslant (f_k \circ \mathfrak{r}) \cdot g_\mathfrak{r} \text{ in } U_p,$$

where $f_s : \mathbb{R}^+ \to \mathbb{R} \cup \{\pm\infty\}$ is defined such that for $r > 0$

$$f_s(r) = \begin{cases} \sqrt{-s}\coth(\sqrt{-s}r), & s < 0 \\ \frac{1}{r}, & s = 0 \\ \sqrt{s}\cot(\sqrt{s}r), & s > 0 \end{cases}.$$

We shall also need to know how volumes may be compared. To this end, we fix an $\mathbb{R}$-vector space isomorphism $A : \mathbb{R}^n \to T_pM$ such that, for every $v, w \in \mathbb{R}^n$, $\langle v, w\rangle = (g_p, Av \otimes Aw)$. The map

$$\vartheta_p := \exp_p \circ A : V_p \subset \mathbb{R}^n \to M$$

then yields exponential coördinates about p, where $V_p = (\exp_p \circ A)^{-1}(U_p)$.

Theorem B.2 (Volume Comparison Theorem [57, Lemma 34, p. 268]). *If $\mathrm{Ric} \geqslant (n-1)k$ in U_p, then*

$$(\vartheta_p^* \mathrm{dvol}_g)(x) \leqslant q_k(|x|)^{n-1} \mathrm{dvol}_{\mathrm{eucl}}(x),$$

for every $x \in V_p$, where $q_k : \mathbb{R}^+ \to \mathbb{R}$ is defined such that for $r > 0$

$$q_k(r) = \begin{cases} \frac{1}{\sqrt{-k}r}\sinh(\sqrt{-k}r), & k < 0 \\ 1, & k = 0 \\ \frac{1}{\sqrt{k}r}\sin(\sqrt{k}r), & k > 0 \end{cases}.$$

For the purposes of estimating certain quantities continually appearing in this thesis, we shall need some very basic properties of the $\{f_s\}_{s\in\mathbb{R}}$ and $\{q_s\}_{s\in\mathbb{R}}$.

Proposition B.3. *If $s > 0$ and $\gamma \in \left]0, \frac{\pi}{\sqrt{s}}\right[$, then there exists a constant $C = C(s, \gamma)$ such that*

$$|1 - rf_s(r)| \leqslant Cr^2$$

on $]0, \gamma]$. If $s < 0$, then there exists an absolute constant $C > 0$ such that

$$1 - rf_s(r) > -Csr^2.$$

Proof. First suppose $s > 0$. Note that

$$\sup_{r\in]0,\gamma]}\left|\frac{1-rf_s(r)}{r^2}\right| = s\cdot\sup_{r\in]0,M]}\left|\frac{1-\sqrt{s}r\cot(\sqrt{s}r)}{(\sqrt{s}r)^2}\right| = s\cdot\sup_{x\in]0,\gamma\sqrt{s}]}\left|\frac{1-x\cot x}{x^2}\right|,$$

whence it suffices to establish the finiteness of the rightmost quantity. Let $q :]0,\pi[\to \mathbb{R}$ $q(x) = x\cot x$. It is clear that q is smooth on its domain. Now, fix $x \in]0,\gamma\sqrt{s}[$ and $\varepsilon \in]0,x[$. By Taylor's theorem with remainder,

$$q(x) = q(\varepsilon) + q'(\varepsilon)(x-\varepsilon) + \frac{q''(t)}{2}(x-\varepsilon)^2$$

for some $t \in [\varepsilon, x]$. Now, note that

$$\lim_{y\searrow 0} q(y) = \lim_{y\searrow 0}\cos y\cdot\frac{y}{\sin y} = 1.$$

On the other hand,

$$q'(y) = \frac{\sin y\cos y - y}{\sin^2 y} = \frac{\frac{\sin(2y)}{2}-\varepsilon}{\sin^2 y} = \frac{o(y^2)}{\sin^2 y}\xrightarrow{y\searrow 0} 0.$$

Finally,

$$\begin{aligned} q''(y) &= \frac{2(y\cos y-\sin y)}{\sin^3 y} = \frac{2(y(1-\frac{y^2}{2}+o(y^3))-y+\frac{y^3}{6}+o(y^4))}{\sin^3 y} \\ &= -\frac{4}{6}\frac{y^3}{\sin^3 y}+\frac{o(y^4)}{\sin^3 y}\xrightarrow{y\searrow 0} -\frac{4}{6}, \end{aligned}$$

whence $C := \frac{1}{2}\sup_{]0,\gamma\sqrt{s}[}|q''| < \infty$. Estimating the above expansion then yields:

$$|1-q(x)| \leqslant |1-q(\varepsilon)| + |q'(\varepsilon)(x-\varepsilon)| + C|x-\varepsilon|^2 \xrightarrow{\varepsilon\searrow 0} C|x|^2.$$

For the case $s < 0$, we proceed in a similar manner, except we show that

$$\sup_{x\in]0,\infty[}\left|\frac{1-x\coth x}{x^2}\right| < \infty$$

and $1 - x\coth x < 0$. We thus consider $q :]0,\infty[\to \mathbb{R}$ defined by $q(x) = x\coth x$ which is again smooth on its domain. We again apply Taylor's theorem. Firstly,

$$\lim_{y\searrow 0} q(y) = \lim_{y\searrow 0}\cosh y\cdot\frac{y}{\sinh y} = 1.$$

Secondly,

$$q'(y) = \frac{\cosh y\sinh y - y}{\sinh^2 y} = \frac{o(y^2)}{\sinh^2 y}\xrightarrow{y\searrow 0} 0.$$

Moreover,

$$q''(y) = \frac{2(y\cosh y-\sinh y)}{\sinh^3 y} = \frac{2}{3}\cdot\frac{x^3}{\sinh^3 y}+\frac{o(y^4)}{\sinh^3 y}\xrightarrow{y\searrow 0}\frac{2}{3},$$

whence it is clear, using the same argument as above, that $|1-q(x)| \leqslant D|x|^2$ for $x \in]0,z]$ for any $z \in \mathbb{R}^+$, where $D = D(z) \in \mathbb{R}^+$. On the other hand,

$$\frac{1-q(y)}{y^2} = \frac{1 - y \cdot \frac{e^{4y}+1}{e^{4y}-1}}{y} = \frac{1}{y} - \frac{e^{4y}+1}{e^{4y}-1} \xrightarrow{y\to\infty} 0,$$

whence it follows that $\frac{1-q(x)}{x^2}$ is uniformly bounded for $x \in]0,\infty[$.

Finally, a quick computation shows that $q' \geqslant 0$ so that $q \geqslant 1$, establishing the result. □

Proposition B.4. *Extend q_s to a function $[0,\infty[\to \mathbb{R}$ by setting $q_s(0) = 1$ for every $s \in \mathbb{R}$. Then q_s is continuous.*

Proof. This is clear, since $\lim_{u\searrow 0} \frac{\sinh u}{u} = \lim_{u \searrow 0} \frac{\sin u}{u} = 1$. □

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