

Generalized rainbow Turán problems

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Abstract

Alon and Shikhelman [*J. Comb. Theory, B.* **121** (2016)] initiated the systematic study of the following generalized Turán problem: for fixed graphs H and F and an integer n , what is the maximum number of copies of H in an n -vertex F -free graph?

An edge-colored graph is called *rainbow* if all its edges have different colors. The *rainbow Turán number* of F is defined as the maximum number of edges in a properly edge-colored graph on n vertices with no rainbow copy of F . The study of rainbow Turán problems was initiated by Keevash, Mubayi, Sudakov and Verstraëte [*Comb. Probab. Comput.* **16** (2007)].

Motivated by the above problems, we study the following problem: What is the maximum number of copies of F in a properly edge-colored graph on n vertices without a rainbow copy of F ? We establish several results, including when F is a path, cycle or tree.

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1 Introduction

This paper is concerned with variations of the Turán question in extremal graph theory. In the classic setting, given a simple graph F , we are interested in determining the largest possible number of edges in a simple graph G on n vertices without F as a subgraph. This number is called the *Turán number* of F and is denoted by $\text{ex}(n, F)$. In short we will say G is F -free. The prototypical result in the area is Mantel's theorem from 1907 [34]. Mantel showed that if an n -vertex graph does not contain a triangle, it can have at most $\lfloor \frac{n^2}{4} \rfloor$ edges, and this bound is best possible as shown by the balanced complete bipartite graph. Therefore, we have $\text{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$. Turán [35] generalized this to all cliques, and determined $\text{ex}(n, K_k)$ for every k and n . A general result was proven by Erdős and Simonovits [13] as a corollary to a theorem of Erdős and Stone [14]. They proved that for any simple graph F we have $\text{ex}(n, F) = \left(1 - \frac{1}{\chi(F)-1}\right) \binom{n}{2} + o(n^2)$, where $\chi(F)$ is the chromatic number of F . If F is not bipartite, this theorem determines $\text{ex}(n, F)$ asymptotically. However, for bipartite graphs the Erdős-Stone-Simonovits theorem just states that $\text{ex}(n, F)$ is of lower than quadratic order. A general classification of the order of magnitude of bipartite Turán numbers is not known. For paths Erdős and Gallai [12] showed that $\text{ex}(n, P_k) \leq \frac{1}{2}(k-2)n$, where P_k denotes the path on k vertices and equality holds for the graph of disjoint copies of K_{k-1} . Erdős and Sós [10] conjectured that the same should hold for any fixed tree on k vertices. A proof of this conjecture for k large enough was announced by Ajtai, Komlós, Simonovits and Szemerédi. For even cycles Erdős conjectured that $\text{ex}(n, C_{2k}) = \Theta(n^{1+1/k})$. A corresponding upper bound was given by Bondy and Simonovits [8], but so far a matching lower bound has only been found for $k = 2, 3, 5$ ([5, 36]). For more results the interested reader may consult the comprehensive survey on bipartite Turán problems by Füredi and Simonovits [16].

The classical Turán problem has a rich history in combinatorics and several variations and generalizations of it have been studied. Two such variations are rainbow Turán problems and generalized Turán problems. In this paper we will study a natural generalization of these two problems.

The rainbow Turán problem, introduced by Keevash, Mubayi, Sudakov, and Verstraëte [30], is as follows. For a fixed graph F , determine the maximum number of edges in a properly edge-colored graph on n vertices which does not contain a *rainbow* F , i.e., a copy of F all of whose edges have different colors. This maximum is denoted by $\text{ex}^*(n, F)$ and is called the *rainbow Turán number* of F . (We refer the reader to [30] for a discussion on motivations and applications of this problem.) Observe that by definition we always have $\text{ex}^*(n, F) \geq \text{ex}(n, F)$. In relation with the Erdős-Stone-Simonovits theorem in [30] it was shown that if $\chi(F) \geq 3$, then $\text{ex}^*(n, F) = (1 + o(1)) \text{ex}(n, F)$. In the case of the path P_k on $k \geq 4$ vertices it is known that $\frac{k-1}{2}n + O(1) \leq \text{ex}^*(n, P_k) \leq \left(\frac{9k-4}{7}\right)n$, where the lower bound is due to Johnston and Rombach [29], while the upper bound was proven by Ergemlidze, Györi and Methuku [9]. For general trees only some sporadic results are known, for such results see, e.g., [28, 29]. In the case of even cycles Keevash, Mubayi, Sudakov and Verstraëte showed a general lower bound of $\text{ex}^*(n, C_{2k}) = \Omega(n^{1+1/k})$ and

very recently O. Janzer [27] proved the matching upper bound $\text{ex}^*(n, C_{2k}) = O(n^{1+1/k})$. Keevash, Mubayi, Sudakov and Verstraëte also showed that there exists a graph with $\Omega(n \log n)$ edges without a rainbow cycle of any length.

The generalized Turán problem asks the following. Given two graphs H and F , what is the maximum possible number of copies of H in a graph on n vertices that does not contain a copy of F ? This maximum is called the *generalized Turán number* and is denoted by $\text{ex}(n, H, F)$. Note that if $H = K_2$ (an edge), we recover the classical Turán problem.

The first results concerning this function are due to Zykov [37], Erdős [11] and Bollobás [6] who determined $\text{ex}(n, K_r, K_k)$ for every value of n , r and k . Later Bollobás and Györi [7] proved that $\text{ex}(n, C_3, C_5) = \Theta(n^{3/2})$. Györi and Li [24] gave bounds on $\text{ex}(n, C_3, C_{2\ell+1})$. Another well-known result is due to Grzesik [22] and independently to Hatami, Hladký, Král', Norine and Razborov [25], who determined $\text{ex}(n, C_5, C_3)$ exactly. Recently, the systematic study of $\text{ex}(n, H, F)$ was initiated by Alon and Shikhelman [3], and this problem has attracted the interest of a number of researchers; see e.g., [2, 15, 17, 18, 20, 31, 32, 33].

Here we consider a natural generalization of the above two problems and introduce a new variant of the Turán problem. Given two graphs H and F , let $\text{ex}(n, H, \text{rainbow-}F)$ denote the maximum possible number of copies of H in a properly edge-colored graph on n vertices that does not contain a rainbow copy of F . Observe that, by definition, we always have $\text{ex}(n, H, \text{rainbow-}F) \geq \text{ex}(n, H, F)$. In this paper we focus on the case when $H = F$. In other words, we consider the question: How many copies of F can we have in a properly edge-colored graph on n vertices without having a rainbow copy of F ? Our motivation in studying this function comes from attempts to understand the original rainbow Turán problem. To determine $\text{ex}^*(n, F)$ it is important to separate the problem from the classical problem of determining $\text{ex}(n, F)$. In order to do this, one needs to examine properly edge-colored n -vertex graphs that contain more than $\text{ex}(n, F)$ edges without a rainbow copy of F ; such graphs contain many copies of F but no rainbow copy of F . It is then natural to understand how many copies of F can we take before we are forced to have a rainbow copy of F . (See [21] and the discussion in Section 5 for the connection to Ruzsa-Szemerédi-type problems and for further motivation.)

Let us introduce some basic notation that is used throughout this paper. For positive integers k and ℓ , let P_k denote a path on k vertices, and let C_ℓ denote a cycle on ℓ vertices. A *star* is a tree which consists of a vertex that is adjacent to all the other vertices. Let S_p denote the star with p leaves (i.e., a vertex adjacent to p other vertices). A tree is called a *double star* if its longest path has four vertices, or equivalently, if it is not a star and there is an edge uv such that every vertex is adjacent to u or v . If u is adjacent to p leaves and v is adjacent to r leaves, we denote this graph by $S_{p,r}$ and we call both u and v the *centers*. Note that $S_{1,1}$ is just the path P_4 .

In the proofs we will often use the following operation to construct extremal examples. Given a graph G and a vertex v , we delete v from G and replace it by new vertices v_1, \dots, v_b , each of them connected to the neighbors of v . To refer to this operation we will simply say that we *replace* v by b copies of itself. Usually, we will mostly be interested

in the order of magnitude of the function $\text{ex}(n, H, \text{rainbow-}H)$. So when applying this operation (typically to some small graph G), we will not specify the exact value of b , but we will only write $b = \Theta(n)$. By this we will always mean that we choose b to be cn for some appropriate constant c , such that the resulting graph still has at most n vertices.

Main results

First we determine the order of magnitude of the function $\text{ex}(n, P_k, \text{rainbow-}P_k)$ for all k .

Theorem 1. *If $k \geq 5$, then*

$$\text{ex}(n, P_k, \text{rainbow-}P_k) = \Theta(n^{\lfloor \frac{k}{2} \rfloor}).$$

For $k \in \{2, 3\}$ note that we have $\text{ex}(n, P_2, \text{rainbow-}P_2) = \text{ex}(n, P_3, \text{rainbow-}P_3) = 0$ and for $k = 4$ we will show in Proposition 6 that $\text{ex}(n, P_4, \text{rainbow-}P_4) = \Theta(n)$.

Our next result is for cycles.

Theorem 2. *If $k, \ell \geq 2$, then*

$$\text{ex}(n, C_{2k+1}, \text{rainbow-}C_{2k+1}) = \Theta(n^{2k-1})$$

and

$$\Omega(n^{k-1}) = \text{ex}(n, C_{2k}, \text{rainbow-}C_{2k}) = O(n^k).$$

Moreover, if $k \neq \ell$, then

$$\text{ex}(n, C_{2\ell}, \text{rainbow-}C_{2k}) = \Theta(n^\ell).$$

Our final result is about trees. For a tree T , any rainbow- T -free graph G can have at most a linear number of edges: Indeed a graph with sufficiently large (but constant) minimum degree contains a rainbow copy of T . Therefore, by a theorem proved in [1], the maximum possible number of copies of T in G is at most $O(n^{\alpha(T)})$, where $\alpha(T)$ is the size of a maximum independent set in T . This proves that $\text{ex}(n, T, \text{rainbow-}T) = O(n^{\alpha(T)})$ for any tree T . We show that this is close to the truth. Moreover, this implies that $\text{ex}(n, T, \text{rainbow-}T)$ grows fast with the number of vertices of T .

Theorem 3. *Let T be a tree with $\ell \geq 3$ leaves and let T' be the tree resulting from the removal of all leaves from T .*

- (i) *If T' is a single vertex, i.e., T is a star, then $\text{ex}(n, T, \text{rainbow-}T) = 0$.*
- (ii) *If T' is a single edge, i.e., T is a double star, then $\text{ex}(n, T, \text{rainbow-}T) = \Theta(n)$.*
- (iii) *If T' is a star on at least 3 vertices, then $\text{ex}(n, T, \text{rainbow-}T) = \Theta(n^{\ell-d+1})$ where d is the minimum integer such that T' has a leaf adjacent to d leaves in T .*
- (iv) *If T' is not a star, then $\Omega(n^{\alpha(T)-2}) = \text{ex}(n, T, \text{rainbow-}T) = O(n^{\alpha(T)})$.*

We remark that part (iv) of Theorem 3 cannot be improved in the following sense: $\alpha(T)$, $\alpha(T) - 1$ and $\alpha(T) - 2$ are all possible exponents of n . Indeed, $\alpha(T)$ and $\alpha(T) - 1$ occur when T is a path (Theorem 1) and we give a construction (Proposition 16) in Section 4 that shows $\alpha(T) - 2$ is also possible. Note that part (i) of Theorem 3 is immediate as a properly edge-colored star is necessarily rainbow. The remaining parts are proved in Propositions 13, 14 and 15 in Section 4.

Outline of the paper. This paper is organized as follows. In Section 2 we prove some general bounds on the function $\text{ex}(n, F, \text{rainbow-}F)$. Then, in Section 3 we prove our main results Theorem 1 and Theorem 2 (resolving the problem for paths and odd cycles among other results). In Section 4, we prove Theorem 3 about trees, along with corresponding results about some special types of forests. Finally, we make some concluding remarks and present open problems in Section 5.

2 General bounds

Using the graph removal lemma, we show that for any graph H , the number of copies of H in a rainbow- H -free graph is at most $o(n^{|V(H)|-1})$. More precisely:

Proposition 4. *For any graph H on k vertices, we have*

$$(i) \text{ ex}(n, H, \text{rainbow-}H) = O(\text{ex}^*(n, H)n^{k-3}),$$

$$(ii) \text{ ex}(n, H, \text{rainbow-}H) = o(n^{k-1}).$$

Proof. Let G be a properly colored, rainbow- H -free graph on n vertices. To prove (i), let us pick an arbitrary edge of G ; there are at most $\text{ex}^*(n, H)$ ways to do this. Then we pick another edge of the same color; there are less than $n/2$ ways to do this. Next we pick $k - 4$ additional vertices; there are $O(n^{k-4})$ ways to do this. Now note that there are at most $|V(H)|!$ copies of H on each set of $|V(H)|$ vertices picked this way. This way we counted every copy of H at least once, as every copy of H must contain two edges of the same color. This proves (i).

Now let us prove (ii). Observe that (i) immediately gives the upper bound $O(n^{k-1})$, as $\text{ex}^*(n, H) = O(n^2)$. As G contains $o(n^k)$ copies of H , by the graph removal lemma there is a set E_0 of $o(n^2)$ edges such that every copy of H in G contains an edge of E_0 .

We know that every copy of H in G contains two edges of the same color, say e_H and e'_H .

First let us count those copies of H where the edge of E_0 in H shares its color with another edge of H . In this case we can repeat the argument for (i). There are $o(n^2)$ ways to pick an edge e of E_0 and less than $n/2$ ways to pick an edge e' of the same color as e . The remaining $k - 4$ vertices can be picked arbitrarily in $O(n^{k-4})$ ways and in any set of $|V(H)|$ vertices that we picked, we have at most $|V(H)|!$ copies of H . So the total number of such copies of H is $o(n^{k-1})$.

Let us now count those copies of H where the edge of E_0 in H is disjoint from two edges e_H, e'_H of H having the same color. Then these three edges span six vertices. There

are $o(n^2)$ ways to pick the edge of E_0 , $O(n^2)$ ways to pick e_H , $O(n)$ ways to pick e'_H (as it has the same color as e_H) and $O(n^{k-6})$ ways to pick the remaining vertices. So in total there are $o(n^{k-1})$ such copies of H .

Next we count those copies of H where uv is an edge of E_0 in H and $e_H = uv$, $e'_H = xy$ are of the same color and these three edges span five different vertices. We can pick uv in $o(n^2)$ ways, xy in $O(n^2)$ ways, but then uv can be picked in at most one way. The remaining vertices can be picked in $O(n^{k-5})$ ways, so the total number of such copies of H is $o(n^{k-1})$.

Finally, we count those copies of H where uv is an edge of E_0 in H and $e = uv$, $e' = vx$ are of the same color and these three edges span four different vertices. Then we can pick uv in $o(n^2)$ ways, uv in $O(n)$ ways, but then vx can be picked in at most one way. The remaining $k - 4$ vertices can then be picked in $O(n^{k-4})$ ways, so the total number of such copies of H is again $o(n^{k-1})$.

In all four cases we obtained $o(n^{k-1})$ copies of H , which finishes the proof of (ii). \square

It would be interesting to determine if there is a graph H for which the upper bound in Proposition 4 (ii) is sharp. In some special cases, it can be improved. Indeed, for example, we will see later that for k odd, we have $\text{ex}(n, C_k, \text{rainbow-}C_k) = \Theta(n^{k-2})$.

3 Paths and Cycles

We begin with a basic lemma that we will use in the proofs of Theorem 1 and 2.

Lemma 5. *Let U be a set of vertices, A be a set of colors, v_1, \dots, v_k be vertices such that for each $1 \leq i \leq k - 1$, v_i and v_{i+1} have at least $|U| + 2|A| + 6k - 9$ common neighbors. Then there is a rainbow path $v_1u_1v_2u_2 \cdots v_{k-1}u_{k-1}v_k$ that does not use any color in A and $u_i \notin U$ for each $1 \leq i \leq k$.*

Proof. We prove that for every $1 \leq j \leq k$, there is a rainbow path $v_1u_1v_2u_2 \cdots v_{j-1}u_{j-1}v_j$ that does not use any color in A , $u_i \notin U$ and $u_i \neq v_{i'}$ for any $1 \leq i \leq j - 1$ and $1 \leq i' \leq k$. We use induction on j ; the statement is trivial for $j = 1$. Assume we could find such a path $P(j)$. Then we want to find a vertex x connected to both v_j and v_{j+1} such that $x \notin U$, x is not in $P(j)$, x is not any $v_{i'}$ and the colors of xv_j and xv_{j+1} are not on the edges of $P(j)$, nor in A . The number of forbidden vertices is at most $|U| + 2k - 2$ (including v_j and v_{j+1}). The number of forbidden colors is at most $|A| + 2k - 4$. Each of those colors is on at most one edge incident to v_j and at most one edge incident to v_{j+1} , thus it forbids at most two additional vertices. Thus there are at most $|U| + 2|A| + 6k - 10$ forbidden vertices, hence we can find a common neighbor of v_j and v_{j+1} that is not forbidden. \square

Now we are ready to prove Theorem 1, which we restate here for convenience.

Theorem 1. *If $k \geq 5$, then $\text{ex}(n, P_k, \text{rainbow-}P_k) = \Theta(n^{\lfloor \frac{k}{2} \rfloor})$.*

Proof. For the lower bound, we have to construct a graph G with $\Omega(n^{\lfloor \frac{k}{2} \rfloor})$ copies of P_k and give a proper edge coloring of G such that it is rainbow- P_k -free.

Let U_1, U_2, \dots, U_k be disjoint sets of vertices of G defined as follows. Let $U_i = \{u_{i,1}, u_{i,2}, \dots, u_{i,b}\}$ for $i = 1, 3$ and $i = 2j$ for all $j \geq 2$ where $b = \Theta(n)$ is chosen as large as possible so that we still have at most n vertices. For $i = 2$ and $i = 2j + 1$ for all $j \geq 2$, simply let $U_i = \{u_i\}$.

Observe that sets of linear size and sets of size 1 alternate, with one exception, where large sets follow each other. But, as we will see, the important part is between u_2 and u_5 .

The edges of G and the colors of the edges are defined as follows. Let every vertex of U_1 and U_3 be adjacent to u_2 such that the edge $u_2u_{3,i}$ gets color i for each $1 \leq i \leq b$. Let $u_{3,i}$ be adjacent to $u_{4,i}$ for each $1 \leq i \leq b$ such that all these edges get color $b + 1$. Next, let all the vertices of U_4 be adjacent to u_5 such that the edge $u_{4,i}u_5$ gets color i for each $1 \leq i \leq b$. Moreover, for each i with $i \geq 5$, let all of the vertices in U_i be adjacent to U_{i+1} . We extend the coloring already given to an arbitrary proper coloring. It is easy to see that any copy of P_k in G contains either the edges $u_{3,i}u_{4,i}$ and $u_{3,j}u_{4,j}$ for some $i \neq j$, or the edges $u_2u_{3,i}$ and $u_{4,i}u_5$ for some i , which have the same color by definition. Thus G is rainbow- P_k -free and it is easy to see that it contains at least $b^{\lfloor \frac{k}{2} \rfloor} = \Omega(n^{\lfloor \frac{k}{2} \rfloor})$ copies of P_k , as desired.

Now we prove the upper bound. Let G be a properly edge colored rainbow- P_k -free graph. Let us recall first that as it is rainbow- P_k -free, G has $O(n)$ edges. A pair of vertices u, v is called a *thin pair* if they have at most $3k - 4$ common neighbors and a *fat pair* otherwise.

We claim that there is no k -path $v_1v_2 \cdots v_k$ where v_{2i} and v_{2i+2} form a fat pair for every i . Indeed, such a k -path would imply the existence of a rainbow- k -path as follows. When k is even, applying Lemma 5 to fixed vertices v_2, v_4, \dots, v_k and $U = \{v_1\}$, $A = \{\text{color of the edge } v_1v_2\}$ gives a rainbow- k -path. When k is odd, applying Lemma 5 to fixed vertices v_2, v_4, \dots, v_{k-1} and $U = \{v_1, v_k\}$, $A = \{\text{colors of the edges } v_1v_2, v_{k-1}v_k\}$ gives a rainbow path of length $(k - 1)$ ending in vertex v_{k-1} . The vertex v_{k-1} has degree at least $3k - 4$, so we can extend to a rainbow path of length k .

Thus all the k -paths $v_1v_2 \cdots v_k$ have the property that v_{2i} and v_{2i+2} form a thin pair for some $1 \leq i \leq (k - 2)/2$. There are $O(n)$ ways to choose each of the edges $v_1v_2, v_3v_4, \dots, v_{2i-1}v_{2i}$ and each of the edges $v_{2i+2}v_{2i+3}, v_{2i+4}v_{2i+5}, \dots, v_{2l}v_{2l+1}$, where $2l + 1$ is k or $k - 1$ depending on whether k is even or odd. When k is even, there are n further ways to choose v_k . Finally, there are at most $6k - 4$ ways to choose v_{2i+1} . Altogether there are $O(n^{\lfloor k/2 \rfloor})$ k -paths where v_{2i} and v_{2i+2} form a thin pair. As there are constant many ways to choose i , this finishes the proof. \square

It is easy to see that $\text{ex}(n, P_3, \text{rainbow-}P_3) = \text{ex}(n, P_2, \text{rainbow-}P_2) = 0$. We determine $\text{ex}(n, P_4, \text{rainbow-}P_4)$ exactly. In fact we will completely characterize the graphs that have a proper edge-coloring without a rainbow P_4 .

Proposition 6. *A graph G has a proper edge-coloring without a rainbow P_4 if and only if each connected component of G is either a star, or a path, or an even cycle, or has at most four vertices. Consequently, $\text{ex}(n, P_4, \text{rainbow-}P_4) = 12\lfloor n/4 \rfloor$.*

Proof. First we give a proper edge-coloring of G without a rainbow P_4 if its components are as listed. The paths and even cycles are colored with two colors, thus they cannot

contain a rainbow P_4 . Stars do not contain any P_4 , so any proper edge coloring is good. Finally, note that the proper edge-coloring of K_4 with 3 colors does not contain a rainbow P_4 , so any graph with at most four vertices also has a coloring with the same property.

Next let G be a graph with a proper coloring of its edges without a rainbow P_4 , and let us study the components of G . If a component does not contain P_4 , it is a star or a triangle. If a component contains a P_4 , say $abcd$ and there is a fourth edge spanned by these four vertices, we obtain either a C_4 or a triangle with a hanging edge. It is easy to check that no edge can go out to a fifth vertex in either case without creating a rainbow P_4 . Therefore, we may assume that there is no fourth edge spanned by a, b, c and d . Then it is again easy to see that neither b nor c can be connected to a fifth vertex without creating a rainbow P_4 , so if there is another edge, it creates a P_5 . Now suppose we discovered already a P_k for $k \geq 5$. The only further edge that can go between vertices of the path without creating a rainbow P_4 has to go between the endpoints, and so it creates a cycle. Finally, in the same way as before, any edge going to a new vertex from the path must extend the path, in which case we can continue with a P_{k+1} .

Hence the component is either a path, or contains a cycle. It is easy to see that a proper coloring of an odd cycle contains a rainbow P_4 , while an edge added to an even cycle of length at least 6 also creates a rainbow P_4 in any proper coloring. This shows that if the component is not a path, it must be an even cycle, finishing the characterization.

To finish the proof, note that C_k contains k copies, P_k contains $k - 3$ copies, K_4 contains 12 copies, and a star contains no copies of P_4 . Therefore; the number of copies of P_4 is maximized if we take $\lfloor n/4 \rfloor$ disjoint copies of K_4 (and a few isolated vertices). \square

Below we determine the order of magnitude for odd cycles. We restate the corresponding part of Theorem 2 for convenience. Note that obviously $\text{ex}(n, C_3, \text{rainbow-}C_3) = 0$.

Theorem 2 (odd cycles). *If $k \geq 2$, then $\text{ex}(n, C_{2k+1}, \text{rainbow-}C_{2k+1}) = \Theta(n^{2k-1})$.*

Proof. For the lower bound, replace each vertex of a C_{2k+1} with linearly many vertices. We pick two non-adjacent edges of the cycle and for each of these two edges, we introduce a matching between the sets corresponding to the endpoints of the edge. For each other edge, we connect all the vertices between the sets corresponding to its endpoints. Let us color all the edges in the two matchings by the same color and extend this arbitrarily to a proper coloring. Then any rainbow subgraph completely avoids one of the two matchings, hence it is bipartite, and therefore is not a rainbow C_{2k+1} . The number of copies of C_{2k+1} is clearly $\Omega(n^{2k-1})$. Indeed, one has to pick a vertex from each of the classes, except one of the classes incident to the first matching and one of the classes incident to the second matching. There are $\Omega(n^{2k-1})$ ways to pick these vertices and there is a unique C_{2k+1} containing them.

For the upper bound we proceed somewhat similarly to Theorem 1. Again, we use thin pairs: this time a pair u, v is thin if they have at most $6k - 2$ common neighbors. Let us consider a $(2k + 1)$ -cycle $C = v_1v_2 \cdots v_{2k+1}v_1$.

First we claim that for any i , one of the pairs $(v_i, v_{i+2}), (v_{i+2}v_{i+4}), \dots, (v_{i+2k-2}, v_{i+2k})$ is thin (where addition in the subscripts is modulo $2k + 1$). Assume otherwise, and without

loss of generality let the assumption be false for $i = 1$. Then we apply Lemma 5 with fixed vertices $v_1, v_3, \dots, v_{2k+1}$, A containing the color on $v_{2k+1}v_1$ and $U = \emptyset$. This way we obtain a rainbow path from v_1 to v_{2k+1} , with the additional property that the colors on the edges of this path are different from the color on $v_{2k+1}v_1$. This path together with the edge $v_{2k+1}v_1$ forms a rainbow C_{2k+1} ; a contradiction. Thus one of the pairs is thin for $i = 1$; without loss of generality it is v_1v_3 . Now applying the above claim with $i = 3$, we obtain another thin pair. If it is v_{2k+1}, v_2 , then we can apply the above claim again, with $i = 2$, to find a third thin pair. Anyways, this way at the end we find two thin pairs v_i, v_{i+2} and v_j, v_{j+2} in C , such that their clockwise order is $v_i, v_{i+2}, v_j, v_{j+2}$ (note that $i + 2 = j$ or $j + 2 = i$ is possible).

Now consider the following two ways of counting $(2k + 1)$ -cycles. Either pick two disjoint thin pairs in $O(n^4)$ ways and $2k - 5$ other vertices in $O(n^{2k-5})$ ways, or pick two thin pairs sharing a vertex in $O(n^3)$ ways and $2k - 4$ other vertices in $O(n^{2k-4})$ ways. Then order them in a $2k - 1$ -cycle so that vertices in a thin pair are adjacent. This can be done in constantly many ways. Finally we have in both cases constant many choices to put a common neighbor between the vertices of the thin pairs. Clearly every $(2k + 1)$ -cycle is counted at least once in one of the two ways and both cases give $O(n^{2k-1})$ copies of C_{2k+1} , finishing the proof. \square

Let us continue with even cycles. The following theorem was proved in [17].

Theorem 7 (Gerbner, Győri, Methuku, Vizer; [17]). *If $k \neq \ell$, then $\text{ex}(n, C_{2\ell}, C_{2k}) = \Theta(n^\ell)$.*

We will prove a generalization of this theorem in the rainbow setting. We follow the proof of Theorem 7 given in [17], which is based on the proof of the so-called reduction lemma of Győri and Lemons [23]. In [17] the authors also prove a stronger asymptotic bound than that in Theorem 7 for the case when $\ell = 2$. Here we will only determine the order of magnitude in this case, which helps to avoid some difficulties. During the proof we establish some properties of graphs with a proper edge-coloring without a rainbow- C_{2k} , and use these properties to obtain the upper bound $O(n^\ell)$ on the number of copies of $C_{2\ell}$.

Now we are ready to prove Theorem 2 for even cycles. We restate it here for convenience.

Theorem 2 (even cycles). *If $k \geq 2$, then*

$$\Omega(n^{k-1}) = \text{ex}(n, C_{2k}, \text{rainbow-}C_{2k}) = O(n^k).$$

Moreover, if $k \neq \ell$, then

$$\text{ex}(n, C_{2\ell}, \text{rainbow-}C_{2k}) = \Theta(n^\ell).$$

Proof. For $k \neq \ell$, the lower bound follows from Theorem 7. The lower bound $\Omega(n)$ for $\text{ex}(n, C_4, \text{rainbow-}C_4)$ is trivial: we can take $\lfloor n/4 \rfloor$ vertex-disjoint copies of C_4 and color each with two colors properly. The following construction provides the lower bound

$\Omega(n^{k-1})$ for $\text{ex}(n, C_{2k}, \text{rainbow-}C_{2k})$ in case $k \geq 3$. First we take a blow-up of a copy $v_1v_2 \cdots v_{2k}v_1$ of C_{2k} where we replace each of the vertices $v_3, v_6, v_8, v_{10}, \dots, v_{2k-2}, v_{2k}$ with classes of size about $n/(k-1)$ so that the resulting graph has n vertices. We then color the edges v_1v_2 and v_4v_5 with the same color and extend this arbitrarily to a proper coloring. It is easy to see that every copy of C_{2k} contains these distinguished edges, thus it is not rainbow and there are $\Omega(n^{k-1})$ copies of C_{2k} in this graph.

For the upper bound, first we consider the case $k = 2$. Observe that every proper coloring of $K_{2,4}$ contains a rainbow C_4 , hence $\text{ex}(n, C_{2\ell}, \text{rainbow-}C_4) \leq \text{ex}(n, C_{2\ell}, K_{2,4}) = O(n^\ell)$ by a theorem of Gerbner and Palmer [19].

Let us assume now $k > 2$ and start with the case $\ell = 2$. Let G be a graph on n vertices and assume we are given a coloring of G without a rainbow- C_{2k} . For two vertices u and v let $f(u, v)$ denote the number of common neighbors of u and v . We call a pair of vertices (u, v) *fat* if $f(u, v) \geq 6k$ and a copy of C_4 is called *fat* if both opposite pairs in it are fat. First observe that the number of non-fat copies of C_4 is $O(n^2)$, as there are at most $\binom{n}{2}$ non-fat pairs and each such pair is an opposite pair in at most $\binom{6k-1}{2}$ copies of C_4 .

We claim that the number of fat copies of C_4 is also $O(n^2)$. To see this, we go through the fat copies of C_4 one by one in an arbitrary order and pick an edge (from the four edges of the C_4); we always pick the edge which was picked the smallest number of times before (in the case that there is more than one such edge we pick one of them arbitrarily). When this procedure ends, every edge e has been picked a certain number of times. Let us denote this number by $m(e)$ and call it the *multiplicity* of e . Observe that $\sum_{e \in E(G)} m(e)$ is equal to the number of fat copies of C_4 in G . We will show that $m(e) < 128k^3$ for each edge e , thus the number of fat copies of C_4 in G is at most $128k^3 |E(G)| = O(n^2)$.

Let us assume to the contrary that there is an edge $e = ab$ with $m(e) \geq 128k^3$. In this case we will find a rainbow- C_{2k} in G , which will lead to a contradiction that completes the proof. More precisely, we are going to prove the following statement:

Claim 8. *For every $2 \leq t \leq k$ there is a rainbow- C_{2t} in G , that contains an edge e_t with*

$$m(e_t) \geq 16(2k - t)^3.$$

Proof. We proceed by induction on t . For the base step $t = 2$ let us take a fat copy $abcd$ of C_4 containing $e_2 = e = ab$. If the C_4 is not rainbow, we use its fatness to find at least $6k - 2$ other 2-paths between a and c . At most one of those can share a color with the edges ab or bc , thus we can replace the subpath cda with another subpath to obtain a rainbow copy of C_4 containing e_2 .

Let us assume now that we have found a rainbow cycle C of length $2t$, $t \leq k - 1$, containing an edge $e_t = uv$ with multiplicity at least $16(2k - t)^3$. Let us consider the last $16t^2$ times e_t was picked. This way we find a set \mathcal{F}_t of $16t^2$ fat copies of C_4 each containing e_t and containing only edges with multiplicity at least $16(2k - t)^3 - 16t^2 \geq 16(2k - t - 1)^3$.

At most $2\binom{2t-2}{2} < 4t^2$ of the copies of C_4 in \mathcal{F}_t have the other two of their vertices (besides u and v) in C , as we have to pick two other vertices from C and there are at most two 4-cycles containing the edge uv and two other given vertices. Thus, there are more than $12t^2$ fat 4-cycles in \mathcal{F}_t that contain a vertex not in $V(C)$, let \mathcal{F}'_t be their set. Let A

be the set of vertices not in $V(C)$ that are neighbors of u in a 4-cycle in \mathcal{F}'_t , and B be the set of vertices not in $V(C)$ that are neighbors of v in a 4-cycle in \mathcal{F}'_t . We claim that $|A|$ or $|B|$ is more than $2t$. Indeed, suppose the contrary and look at the neighbours of u in the 4-cycles in \mathcal{F}'_t . On the one hand, there are at most $2t$ of them in $V(C)$. Each of them appears in at most $2t$ of the 4-cycles in \mathcal{F}'_t , as the fourth vertices of these cycles are all different and are in B . On the other hand, by assumption there are again at most $2t$ neighbours of u outside $V(C)$ (those in the set A). Each of them is in at most $2t$ 4-cycles in \mathcal{F}'_t where the fourth vertex is in $V(C)$ and, similarly as before, in at most $2t$ 4-cycles in \mathcal{F}'_t where the fourth vertex is outside $V(C)$. This shows that together there can be at most $2t \cdot 2t + 2t \cdot (2t + 2t) = 12t^2$ cycles in \mathcal{F}'_t , a contradiction.

Without loss of generality let u be the vertex which has more than $2t$ neighbors in 4-cycles in \mathcal{F}_t that are not in $V(C)$. Among these more than $2t$ vertices, at least one of them, call it y , has that the color on yu is not used in C . Also, recall that the multiplicity of yu is at least $16(2k - t - 1)^3$. As the 4-cycles are fat, there are at least $6k$ common neighbors of v and y and at least $4k$ of those are not in $V(C)$. There are less than $2k$ colors that are used in C and on the edge yu and each of those colors appears at most twice on edges that connect y and v to the at least $4k$ selected common neighbors. Therefore, there is a common neighbor x such that the colors of the edges vx and yx are neither in C , nor on the edge yu . Then we can replace the edge uv in C with the edges uy , yx , xv to obtain a rainbow cycle of length $2t + 2$, which contains an edge (namely uy) with multiplicity at least $16(2k - t - 1)^3$. \square

This finishes the proof of the case $\ell = 2$. Now we consider the case when $\ell \geq 3$. Note that we have $\frac{1}{2} \sum_{a \neq b, a, b \in V(G)} \binom{f(a,b)}{2} = O(n^2)$ as the left-hand side counts the number of copies of C_4 .

Claim 9. *For every $a \in V(G)$ we have $\sum_{b \in V(G) \setminus \{a\}} f(a, b) \leq cn$ for some $c = c(k)$.*

Note that the left-hand side of the above inequality is the number of paths of length 3 starting at a .

Proof. Let $N_1(a)$ be the set of vertices adjacent to a and $N_2(a)$ be the set of vertices at distance exactly 2 from a . Let E_1 be the set of edges induced by $N_1(a)$ and E_2 be the set of edges uv with $u \in N_1(a)$ and $v \in N_2(a)$. Then clearly $\sum_{b \in V(G) \setminus \{a\}} f(a, b) = 2|E_1| + |E_2|$.

We claim that $E_1 \cup E_2$ does not contain a copy of the $12k$ -ary tree with depth $4k$. Assume it does contain such a copy T and let x_1 be the root of T . In what follows, we will construct a particular path P on $4k$ vertices starting at x_1 . Let the next vertex be an arbitrary child x_2 of x_1 . At later points we always pick the next vertex x_{i+1} to be a child of x_i such that both the color of $x_i x_{i+1}$ and the color of ax_{i+1} (if existent) are different from the colors of ax_j for every $j \leq i$ (if existent) and from the colors used on the path earlier. As there are at most $8k$ forbidden colors, there are at most $8k$ children of x_i that we cannot pick because of the color of $x_i x_{i+1}$ and at most $4k - 1$ children of x_i that we cannot pick because of the color of ax_{i+1} (as the color of ax_j is automatically avoided here). Hence we always have a neighbor to pick and we can really obtain a desired path P in this way. Observe that this path, together with the edges connecting some of its

vertices to a is rainbow. If x_i and x_{i+2k-2} are both in $N_1(a)$ for some i , then they, together with the vertices of P between them and a form a rainbow C_{2k} , a contradiction which finishes the proof. If all the edges of P are in E_2 , then x_1 and x_{2k-1} or x_2 and x_{2k} are both in $N_1(a)$, and the previous case applies. Hence we may assume that the edge $x_i x_{i+1}$ is in E_1 for some i . Without loss of generality we also may assume $i \leq 2k$. Then x_{i+2k-2} and $x_{i+1+2k-2}$ both have to be in $N_2(a)$ (otherwise the earlier case applies), but then the edge between them is not in $E_1 \cup E_2$; a contradiction.

Therefore, we have

$$\sum f(a, b) = 2|E_1| + |E_2| \leq 2|E_1 \cup E_2| \leq 2\text{ex}(n, T) = O(n). \quad \square$$

From now on we follow the proof from [17] more closely, as we have already dealt with the difficulties arising from forbidding only rainbow copies of C_{2k} .

Let us fix vertices v_1, v_2, \dots, v_ℓ and let $g(v_1, v_2, \dots, v_\ell)$ be the number of copies of $C_{2\ell}$ in G of the form $u_1 v_1 u_2 v_2 \cdots u_\ell v_\ell u_1$ for some vertices u_1, u_2, \dots, u_ℓ . Clearly $g(v_1, v_2, \dots, v_\ell) \leq \prod_{j=1}^\ell f(v_j, v_{j+1})$ (where $v_{\ell+1} = v_1$ in the product).

If we add up $g(v_1, v_2, \dots, v_\ell)$ for all possible ℓ -tuples v_1, v_2, \dots, v_ℓ of ℓ distinct vertices in $V(G)$, we count every $C_{2\ell}$ exactly 4ℓ times. Therefore, the number of copies of $C_{2\ell}$ is at most

$$\frac{1}{4\ell} \sum_{(v_1, v_2, \dots, v_\ell)} \prod_{j=1}^\ell f(v_j, v_{j+1}) \leq \frac{1}{4\ell} \sum_{(v_1, v_2, \dots, v_\ell)} \frac{f^2(v_1, v_2) + f^2(v_2, v_3)}{2} \prod_{j=3}^\ell f(v_j, v_{j+1}). \quad (1)$$

Fix two vertices $u, v \in V(G)$ and let us examine what factor $f^2(u, v)$ is multiplied with in (1). It is easy to see that $f^2(u, v)$ appears in (1) whenever $u = v_1, v = v_2$ or $u = v_2, v = v_1$ or $u = v_2, v = v_3$ or $u = v_3, v = v_2$. Let us consider the case $u = v_1$ and $v = v_2$, the other three cases are similar and give only an extra constant factor of 4. In this case $f^2(u, v)$ is multiplied with

$$\frac{1}{8\ell} \left(\prod_{j=3}^{\ell-1} f(v_j, v_{j+1}) \right) f(v_\ell, u) = \frac{1}{8\ell} f(u, v_\ell) \prod_{j=3}^{\ell-1} f(v_j, v_{j+1})$$

for all the choices of $(\ell - 2)$ -tuples v_3, v_4, \dots, v_ℓ of distinct vertices. We claim that

$$\sum_{(v_3, v_4, \dots, v_\ell)} \frac{1}{8\ell} f(u, v_\ell) \prod_{j=3}^{\ell-1} f(v_j, v_{j+1}) \leq \frac{c^{\ell-2} n^{\ell-2}}{8\ell},$$

where $c = c(k)$ is the constant from Claim 9. Indeed, we can rewrite the left-hand side as

$$\frac{1}{8\ell} \left(\sum_{v_\ell \in V(G)} f(u, v_\ell) \left(\sum_{v_{\ell-1} \in V(G)} f(v_\ell, v_{\ell-1}) \cdots \left(\sum_{v_4 \in V(G)} f(v_5, v_4) \left(\sum_{v_3 \in V(G)} f(v_4, v_3) \right) \right) \cdots \right) \right).$$

After repeatedly applying Claim 9 we arrive at the desired upper bound and this finishes the proof (recall that $\sum_{u, v \in V(G)} f^2(u, v) = O(n^2)$). \square

We do not know the order of magnitude of $\text{ex}(n, C_{2k}, \text{rainbow-}C_{2k})$ already for $k = 2, 3$, but we can improve Theorem 2 slightly in these cases.

Proposition 10. *We have $\text{ex}(n, C_4, \text{rainbow-}C_4) = \Omega(n^{3/2})$.*

Proof. Let us take two isomorphic C_4 -free graphs G and G' on $\lfloor n/2 \rfloor$ vertices each with $\Omega(n^{3/2})$ edges. Connect every vertex v in G to its copy v' in G' . Denote the resulting graph by G_0 . Let us color these edges with color 1 and extend this to an arbitrary proper coloring of the edges of G_0 . Every copy of C_4 in G_0 has to use vertices from both G and G' , thus some edge vv' of color 1. The neighbor of v in the C_4 must be a vertex in G and the neighbor of v' must be a vertex in G' . They can only be connected by another edge of color 1, thus the C_4 is not rainbow. On the other hand, for every edge uv of G the 4-cycle $uvv'u'u$ is in G_0 , thus there are $\Omega(n^{3/2})$ copies of C_4 in G_0 . \square

Proposition 11. *We have $\text{ex}(n, C_6, \text{rainbow-}C_6) = O(n^{8/3})$.*

Proof. Let G be a rainbow- C_6 -free graph and $v_1v_2v_3v_4v_5v_6v_1$ be a 6-cycle in it. Note that G has $O(n^{4/3})$ edges. We call a pair of vertices fat if they have at least 14 common neighbors, otherwise we call it thin. If both pairs (v_1, v_3) and (v_3, v_5) are fat, then we can find a rainbow C_6 of the form $v_1uv_3u'v_5v_6v_1$. Indeed, we can apply Lemma 5 for v_1, v_3, v_5 with $U = \{v_6\}$ and A containing the color of v_5v_6 and v_6v_1 . This way we find a rainbow path $v_1uv_3u'v_5$ avoiding v_6 and the colors in A , thus it forms a rainbow 6-cycle with the edges v_5v_6 and v_6v_1 . As a consequence we have that in every 6-cycle there are at most two fat pairs among the pairs (v_i, v_{i+2}) , and hence we can find two thin pairs of the form $(v_i, v_{i+2}), (v_{i+3}, v_{i+5})$ (where addition in the indices is modulo 6).

Therefore, to count 6-cycles, we can do the following. First pick the edges $v_{i+2}v_{i+3}$ and $v_{i+5}v_i$ in $O(n^{8/3})$ ways. Then, as the pairs $(v_i, v_{i+2}), (v_{i+3}, v_{i+5})$ are thin, we are left with only constant many ways to pick v_{i+1} and v_{i+4} . \square

4 Trees and Forests

The aim of this section is to prove Theorem 3 and some additional results about forests. Let us first prove the following proposition that is used later in this section.

Proposition 12. *Let H be a graph that is neither a star nor a triangle and let c be the number of connected components of H . Then $\text{ex}(n, H, \text{rainbow-}H) = \Omega(n^c)$.*

Proof. Let m be the largest edge chromatic number of a component of H . Let us consider a graph G that contains linearly many vertex-disjoint copies of each component. Then G obviously contains $\Omega(n^c)$ copies of H . For each component H' of H , we color each copy of it the same way: using colors from 1 to $\chi'(H')$. This way we properly color G with m colors. Since H has more than m edges, this implies that there is no rainbow copy of H in G , finishing the proof. \square

Next we will determine the order of magnitude for double stars, i.e., part (ii) of Theorem 3.

Proposition 13. *If $p, r \geq 1$, then $\text{ex}(n, S_{p,r}, \text{rainbow-}S_{p,r}) = \Theta(n)$.*

Proof. The lower bound on $\text{ex}(n, S_{p,r}, \text{rainbow-}S_{p,r})$ follows from Proposition 12.

For the upper bound, assume without loss of generality that $p \leq r$ and consider a properly edge-colored graph G on n vertices without a rainbow copy of $S_{p,r}$. We want to bound the number of copies of $S_{p,r}$ in G .

We claim that if a vertex v has degree more than $2p + r$ in G , then it cannot be a center of a copy of $S_{p,r}$. Indeed, if v is a center of some copy, then it has a neighbor u which has at least p neighbors different from v . Let us choose a set A of size p out of these neighbors of u arbitrarily. Then v has at least $p + r$ neighbors not in A and different from u and at least r of them do not have any of the colors that appear on the edges between u and vertices in A . Thus, those r vertices together with u, v and A form a rainbow copy of $S_{p,r}$, which contradicts our assumption.

Now we are ready to count the copies of $S_{p,r}$ in G . We can pick a center of it in at most n ways. Then we can pick one of its neighbors to be the other center in at most $2p + r$ ways and there are at most $\binom{2p+r-1}{r}$ and $\binom{2p+r-1}{p}$ ways to pick the r and p other neighbors of the centers, respectively. Thus, there are $O(n)$ copies of $S_{p,r}$ in G and this finishes the proof. \square

We now prove consider the case when removing all leaves results in a tree, i.e., part (iii) in Theorem 3.

Proposition 14. *Let T be a tree with $\ell \geq 3$ leaves and let T' be the tree resulting from the removal of all leaves from T . If T' is a star on at least 3 vertices, then*

$$\text{ex}(n, T, \text{rainbow-}T) = \Theta(n^{\ell-d+1})$$

where d is the minimum integer such that T' has a leaf adjacent to d leaves in T .

Proof. We begin with the lower bound. Let u be the center of T' and v be a leaf of T' which is adjacent to the fewest number d of leaves in T . We replace v and its leaf neighbors by new edges uv_i and $v_iw_{i,j}$ for $1 \leq i \leq n/|V(T)|$ and $1 \leq j \leq d$. We also replace the leaves of T not incident to v by linearly many copies of themselves. Let H' be the graph obtained this way. Pick an arbitrary edge uv' of T' with $v' \neq v$, then uv' is also an edge of H' . Color the edge uv' and the edges $v_iw_{i,1}$ for every i with the same color. Then we extend this coloring arbitrarily to a proper coloring of H' . Let T_0 be a copy of T in H' let and T'_0 be the star obtained by deleting the leaves of T_0 . The center of T'_0 is u , as u is the only vertex of H' incident to at least two vertices of degree greater than 1. Consider the non-leaf neighbors of u in T_0 . If v_i and v_j are both neighbors of u in T_0 , then they both have at least d leaf neighbors in T_0 . In particular $v_iw_{i,1}$ and $v_jw_{j,1}$ are both in T_0 , thus T_0 is not rainbow. Otherwise, the neighbors of u are its original neighbors and one v_i in T_0 . In particular, uv' and $v_iw_{i,1}$ are in T_0 , thus T_0 is not rainbow. Thus, H' contains no rainbow- T .

We have linearly many choices for each leaf not adjacent to v of T in H' and linearly many choices for the vertex v , so H' contains $\Omega(n^{\ell-d+1})$ copies of T . Therefore, $\text{ex}(n, T, \text{rainbow-}T) = \Omega(n^{\ell-d+1})$.

We now give the upper bound. Observe that for any embedding ϕ of T in G , at least one of the vertices $\phi(w)$ for $w \in V(T') \setminus \{u\}$ must have degree at most $2k$ (otherwise we can find a rainbow embedding of T by modifying ϕ appropriately). So it suffices to show that for any fixed $w \in V(T') \setminus \{u\}$, the number of embeddings ϕ of T in G with the property that the degree of $\phi(w)$ is at most $2k$ is $O(n^{\ell-d+1})$. But we can pick each of the following edges in $O(n)$ ways:

- $\phi(x)\phi(y)$ for $x \in V(T') \setminus \{w\}$ and y a leaf (note $x = u$ is allowed),
- the edge $\phi(u)\phi(w)$.

Once these edges are fixed, there are at most $(2k)^k = O(1)$ ways of picking the vertices corresponding to neighbours of w , and then the embedding ϕ is completely determined. It follows that there are at most $O(n^{\ell-d+1})$ such embeddings, as claimed. \square

We conclude Theorem 3 with a proof of part (iv).

Proposition 15. *Let T be a tree and let T' be the tree resulting from the removal of all leaves from T . If T' is not a star, then*

$$\Omega(n^{\alpha(T)-2}) = \text{ex}(n, T, \text{rainbow-}T) = O(n^{\alpha(T)}).$$

Proof. The upper bound was discussed in introduction, so we prove the lower bound. Let L be the set of leaves in T , $\ell = |L|$ and T' be the tree we obtain by deleting the leaves from T . First we show the bound $\text{ex}(n, T, \text{rainbow-}T) = \Omega(n^\ell)$. As T' is not a star, it has two independent edges. Replace each leaf of T by linearly many copies of itself to obtain a graph H , color the two independent edges by the same color, and extend this coloring arbitrarily to a proper coloring. Clearly, there are no rainbow copies of T in H with this coloring, and H contains $\Omega(n^\ell)$ copies of T .

Now let us show that $\text{ex}(n, T, \text{rainbow-}T) = \Omega(n^{\alpha(T)-2})$. Let A be an independent set in T of order $\alpha(T)$. If $\alpha(T) \leq \ell + 2$, we are done. Otherwise, A contains at least two non-leaf vertices u and v . Let us replace each vertex of A different from u and v by linearly many copies of itself. Let H'' be the resulting graph and A' be the resulting independent set containing u, v and those vertices replacing the remaining vertices of A . The graph H'' clearly contains $\Omega(n^{\alpha(T)-2})$ copies of T . Let us pick edges uu' and vv' such that $u' \neq v'$; this is possible since u and v are not leaves. Color uu' and vv' with the same color and extend this coloring arbitrarily to a proper coloring of H' .

Consider a copy T_0 of T in H'' . A copy T_0 can contain at most $\alpha(T_0) = \alpha(T)$ vertices in the independent set A' , thus T_0 must contain all the vertices in $V(H'') \setminus A' = V(T) \setminus A$. Consider a neighbor u'' of u with $u'' \neq u'$. In T , the only path from u' to u'' is $u'uu''$. Then the same holds in H'' , thus in T_0 . This implies that uu' is in T_0 , and by symmetry we also have vv' in T_0 . Therefore, T_0 is not rainbow which completes the proof. \square

Now let us show that there are trees T such that $\text{ex}(n, T, \text{rainbow-}T) = \Theta(n^{\alpha(T)-2})$.

Proposition 16. *There is an infinite family of trees T such that $\text{ex}(n, T, \text{rainbow-}T) = O(n^{\alpha(T)-2})$.*

Proof. Consider the following tree T on $k + 7$ vertices. Fix a vertex u joined to three vertices x_1, x_2, x_3 . Each x_i has one neighbour y_i apart from u , and each y_i has at least two leaf neighbors and the y_i 's together have k total leaf neighbors. So the leaves and the vertices x_1, x_2, x_3 form a maximum independent set of size $k + 3$.

We claim that if G has no rainbow T then G contains $O(n^{k+1})$ copies of T . First observe that there can be no embedding ϕ of T in G such that $d(\phi(y_i)) > 10k$ for each i and $\phi(u), \phi(y_i)$ have more than $10k$ common neighbours for $i = 1, 2$. Indeed, we can modify such an embedding ϕ on the x_i ($i = 1, 2$) and the leaf edges to obtain a rainbow embedding of T .

Therefore, we have one of the following two cases.

Case 1: The pairs $\phi(u), \phi(y_1)$ and $\phi(u), \phi(y_2)$ have at most $10k$ common neighbours.

We can pick each of the leaf edges (i.e., an edge incident to a leaf) and the edge $\phi(u)\phi(x_3)$ in $O(n)$ ways and then there are at most $100k^2$ ways to pick vertices $\phi(x_1)$ and $\phi(x_2)$, giving $O(n^{k+1})$ such copies.

Case 2: Some y_i has $d(\phi(y_i)) \leq 10k$.

We can pick each of the at most $k - 2$ leaf edges incident to y_j for $j \neq i$ and the edges $\phi(u)\phi(x_j)$ for $j \neq i$ and the edge $x_i y_i$ in $O(n)$ ways, and then there are $O(1)$ ways to pick the leaf edges incident to y_i , giving $O(n^{k+1})$ such copies. \square

In the remainder of this section we prove some sporadic results about special forests.

Proposition 17. *If F is a forest consisting of two stars, then*

$$\text{ex}(n, F, \text{rainbow-}F) = \Theta(n^2).$$

Proof. The lower bound on $\text{ex}(n, F, \text{rainbow-}F)$ follows from Proposition 12.

For the upper bound let us denote the two stars in F by S_p and S_r , $p \leq r$ and let G be a properly edge-colored graph on n vertices without a rainbow copy of F .

Suppose first that G contains a vertex v of degree more than $2p + r$. We claim that this vertex v has to be in every copy of S_p in G . Indeed, assume to the contrary that S is a copy of S_p not containing v . Now at most $p + 1$ neighbors of v are in S and at most p vertices are connected to v using a color from S . Thus we can find r neighbors of v , which together with v form a copy of S_r and with S this gives a rainbow copy of F in G ; a contradiction. So v is contained in every copy of S_p . However F itself contains two disjoint copies of S_p which implies that G is actually F -free.

Therefore, we may assume that every vertex in G has degree at most $2p + r$. Then, when counting the number of copies of F in G , there are at most $\binom{n}{2}$ ways to choose the two centers for the stars and at most $2\binom{2p+r}{p}\binom{2p+r}{r}$ ways to choose the leaves afterwards. Together this shows that the number of copies of F is indeed $O(n^2)$. \square

We remark that if F is made up of more stars we cannot hope for a bound that depends only on the number of stars. To see this, let F consist of three components, two of which are single edges and one that is a star S_r , $r \geq 1$. Let G be a graph with three components, two of which are also only single edges and one that is a star S_{n-5} . Consider

a proper edge-coloring of G where the two edge components have the same color. Then G contains $\binom{n-5}{r}$ copies of F , but no rainbow copy. However, if all the components of F are single edges, we can obtain the following.

Proposition 18. *Let M_k be a matching with $k > 1$ edges. Then*

$$\text{ex}(n, M_k, \text{rainbow-}M_k) = \Theta(n^k).$$

Proof. The lower bound on $\text{ex}(n, M_k, \text{rainbow-}M_k)$ follows from Proposition 12.

For the upper bound consider a properly edge-colored graph G on n vertices without a rainbow copy of M_k . Then according to [28, Theorem 1] G has $O(n)$ edges. To find a copy of M_k we have to pick k edges which can be done in $O(n^k)$ ways. \square

5 Concluding remarks and open problems

In this paper we determined the order of magnitude of $\text{ex}(n, F, \text{rainbow-}F)$ for paths and odd cycles and obtained bounds for even cycles and trees. Several interesting questions remain open. Below we mention a few of them.

- Let K_r denote a clique on r vertices. On the one hand, Proposition 4, part (ii) implies that $\text{ex}(n, K_r, \text{rainbow-}K_r) = o(n^{r-1})$. On the other hand, it is easy to see that $\text{ex}(n, K_r, \text{rainbow-}K_r) = \Omega(n^{r-2})$. Indeed, partition the n vertices into r parts S_1, S_2, \dots, S_r of roughly the same size, and take a matching M_1 between the parts S_1 and S_2 and take another matching M_2 between the parts S_3 and S_4 and the edges of both M_1 and M_2 are colored with the same color. Between every other pair of parts take a complete bipartite graph. It is easy to check that there are $\Omega(n^{r-2})$ copies of K_r in this graph and in any copy of K_r we must have an edge of M_1 and an edge of M_2 , both of which are colored the same. So there is no rainbow copy of K_r . A natural question is to determine the correct order of magnitude of this function for $r \geq 4$.

Postscript. The question above was answered very recently by Gowers and Janzer [21], who showed that the upper bound in Proposition 4 is close to the truth using an interesting geometric construction (similar to a construction of Ruzsa and Szemerédi). More precisely, they showed that $\text{ex}(n, K_r, \text{rainbow-}K_r) = \Omega(n^{r-1-o(1)})$. Furthermore, they also studied the asymmetric setting, and showed that for all $r > s \geq 4$, we have $\text{ex}(n, K_r, \text{rainbow-}K_s) = o(n^{s-1})$. (It is known that when r is large enough compared to s , $\text{ex}(n, K_r, \text{rainbow-}K_s) = 0$; see Alon, Lefmann and Rödl [4].) However, a matching lower bound is not known in many cases. The simplest open case is when $r = 6, s = 4$. In this case they obtained a lower bound of $\text{ex}(n, K_6, \text{rainbow-}K_4) = \Omega(n^{12/5-o(1)})$, while the upper bound is $o(n^3)$ as mentioned above.

- What is the order of magnitude of $\text{ex}(n, C_{2k}, \text{rainbow-}C_{2k})$? Theorem 2 proves some bounds on this function. Proposition 10 and Proposition 11 provide improved bounds in the case when $k = 2, 3$.

Postscript. The question above was answered very recently by Janzer [26], who showed that the lower bound in Theorem 2 is tight for $k \geq 3$ and the upper bound for $k = 2$. In fact, the order of magnitude of $\text{ex}(n, C_s, \text{rainbow-}C_t)$ was determined for every s, t with $s \neq 3$ just like that of $\text{ex}(n, P_\ell, \text{rainbow-}C_{2k})$ for all $k \geq 2$ and $\ell \geq 3$.

- Theorem 3 shows that when T is a tree, then with the exception of stars and double stars, $\text{ex}(n, T, \text{rainbow-}T)$ grows fast with the number of vertices of T . We suspect that a similar phenomenon might be true for general graphs, with some small set of exceptions. One such exception we have encountered was the disjoint union of two stars. Another example is T_p , the triangle with p leaves attached to one of its vertices. For this graph a simple case analysis shows that $\text{ex}(n, T_p, \text{rainbow-}T_p) = O(n)$.
- In this paper we introduced the function $\text{ex}(n, H, \text{rainbow-}F)$ and studied it when $H = F$. It would be interesting to study the case when H and F are different graphs.

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