Chapter 4

The connection between the Hausdorff and the Fréchet metric

Although the Fréchet metric and the Hausdorff metric do not coincide in general, there is an important special case, namely the case of convex bodies.

First we have to explain how to interpret a convex body $A \in \mathbb{R}^d$ as a parameterized object as we need it for the Fréchet metric. This is not as difficult as you might think. We assume first and in the following that A is of dimension d that means that it has a non empty interior. We concentrate ourself on the boundary of A. This boundary is homeomorphic to the $d-1$ dimensional unit sphere S^{d-1} and, of course, there are in principle two ways to map S^{d-1} homeomorphic onto A, namely orientation preserving or not. In the following whenever we talk about the boundary of A we assume that there is also given an orientation preserving homeomorphism from S^{d-1} to A which will be subsequently called a natural parameterization of ∂A, since all natural parameterizations are derivable from each other by an orientation preserving reparametrization. Then we have the following

Theorem. Let $A, B \subseteq \mathbb{R}^d$ two convex sets with a non empty interior and $d \geqslant 2$. Then $\delta_F(\partial A, \partial B) = \delta_H(\partial A, \partial B) = \delta_H(A, B)$.

The remaining part of this chapter is the proof of this Theorem. The proof we give here is actually not written by us. It can be found in [ABGW90]. Unfortunately the theorem for which the proof is only makes a statement about $d = 2$ and it is necessary to re-read the proof in order to see that the same ideas apply to higher dimensions as well. We take this as an opportunity to give the same proof in a slightly different terminology.

Proof of Theorem. Obviously we have $\delta_F(\partial A, \partial B) \geqslant \delta_H(\partial A, \partial B) \geqslant \delta_H(A, B)$. It remains to show $\delta_F(\partial A, \partial B) \leqslant \delta_H(A, B)$.

To see this we consider first the convex hull of $A \cup B$, let us call this hull C. Let $\varepsilon := \delta_H(A, B)$.

Claim 1. $\delta_H(A, C), \delta_H(B, C) \leqslant \varepsilon$.

Proof. For symmetry reasons we only prove $\delta_H(A, C) \leqslant \varepsilon$. Since $A \subseteq C$ we only have to prove that $\delta_H(A, c) \leqslant \varepsilon$ for any $c \in C$. In the case $c \in A$ there is nothing to show and in the case $c \in B$ this follows from $\delta_H(A, B) \leqslant \varepsilon$. Otherwise we have $c \notin A, B$. In this case there are $a \in A$ and $b \in B$ such that c is on the line segment between a and b.* By $\delta_H(A,B) \leqslant \varepsilon$ and $b \in B$ there is an $a' \in A$ with $\delta(a',b) \leqslant \varepsilon$. Now we consider the triangle $aa'b$. The side $a'b$ is of length $\leqslant \varepsilon$ thus for reasons which have nothing to do with higher dimensional geometry the distance from any point on the side ab to the side aa' is $\leqslant \varepsilon$ as well. Now c *is* on the side ab thus there is a point a'' on the side aa' which $\delta(c,a'') \leqslant \varepsilon$. By convexity follows $a'' \in A$. Thus $\delta_H(A,c) \leqslant \varepsilon$. □

We assume A, B to have smooth boundaries. The result for arbitrary bodies follows by continuity arguments. So for each point $a \in \partial A$ there exists a unique ray emanating from a which is normal to ∂A. Let $n_A(a)$ be the intersection point of r and ∂C. Clearly, a is the point of A closest to $n_A(a)$, so $\delta(a, n_A(a)) \leqslant \varepsilon$ holds for all $a \in \partial A$. Altogether $n_A : \partial A \to \partial C$ is an orientation preserving homeomorphism. Likewise we can construct a mapping $n_B \colon \partial B \to \partial C$.

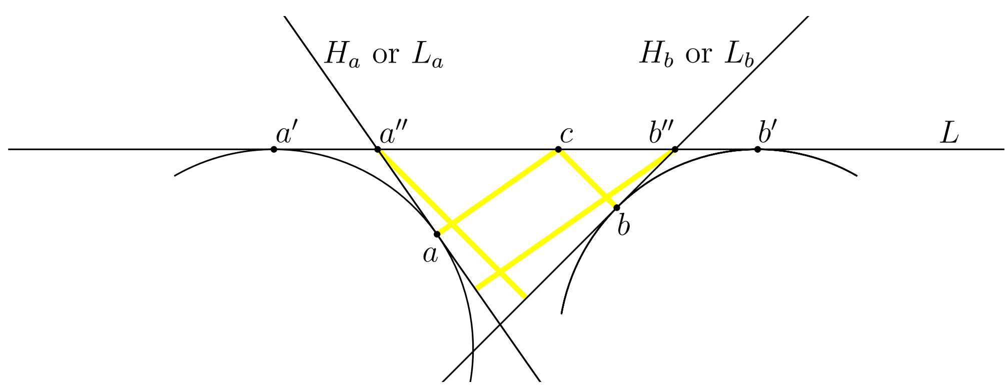
It remained to show that $\delta_F(\partial A, \partial B) \leqslant \varepsilon$ holds. We assume A and B to be given in natural parameterizations thus let $f : S^{d-1} \to \partial A$ be a natural parameterization of ∂A and $g \colon S^{d-1} \to \partial B$ be one of ∂B.

Now let $\sigma := g^{-1} \circ n_B^{-1} \circ n_A \circ f$. Then $\sigma : S^{d-1} \to S^{d-1}$ is an orientation preserving homeomorphism and it remains to show the following

Claim 2. $\delta(f(x), g(\sigma(x))) \leqslant \varepsilon$ holds for all $x \in S^{d-1}$.

Proof. Let therefore $x \in S^{d-1}$. Then $a := f(x)$ as well as $b := g(\sigma(x))$. Then we have to show $\delta(a,b) \leqslant \varepsilon$.

Now let $c := n_A(a)$. Then $c = n_B(b)$ holds as well, by the way. Thus a is the point of A closest to c as well as b is the one of B. By Claim 1 we get $\delta(a,c), \delta(b,c) \leqslant \varepsilon$. In the case $c \in A$ we have $c = a$ and there is nothing to show. Likewise if $c \in B$. Otherwise we have $c \notin A, B$. Then a, b, c are distinct points in $\mathbb{R}^d$ and there is a unique plane going though these points, let us call it the ***base plane***. In this plane we get the situation shown in the picture below.



*By definition c has to be an affine combination of elements of $A \cup B$. By $c \notin A, B$ there has to be elements from A as well as B with non empty coefficients in the affine combination. Then we can rewrite this as as an affine combination of two other affine combinations, namely one of points only of A and one only of points of B. These two combinations give us the a, b we search for.

In other words, the fact that any point $c \in C$ can be written as a convex combination of *two* points out of $A \cup B$ is purely since A and B are convex.

Let H_a be the hyperplane tangential to a and H_b the one for b. Then A lies completely to the left of H_a and B lies completely to the right of H_b. There will be a unique line L trough c in the base plane touching A and B. Let $a' \in L \cap A$ and $b' \in L \cap B$. Since a' and B lie on different sides of H_b the distance of a' to H_b is not greater than the distance of a' to B which is $\leqslant \varepsilon$ by $a' \in A$. Likewise the distance of b' to H_a is $\leqslant \varepsilon$. Now let L_a, L_b be the corresponding intersections of H_a, H_b with the base plane. Since H_a, H_b are perpendicular to the base plane, we also get that the distance of a' to L_b is $\leqslant \varepsilon$ as well as the distance of b' to L_a. Now we are able to forget that we are dealing with $I\!R^d$ since the remaining argumentation in this proof only deals with the base plane.

For any point $x \in L$ let $d(x) := \delta_H(x, L_a) + \delta_H(x, L_b)$. Now let $a'' \in L_a \cap L$ and $b'' \in L_b \cap L$. Then the distance of a'' to L_b may be less than the distance from a' to L_b but not greater. Thus $\delta_H(a'', L_b) \leqslant \varepsilon$. Since $\delta_H(a'', L_a) = 0$ we get $d(a'') \leqslant \varepsilon$. Likewise we get $d(b'') \leqslant \varepsilon$. Since d is affine between a'' and b'' the point $d(c)$ is a convex combination of $d(a'')$ and $d(b'')$ which are both $\leqslant \varepsilon$. It follows

$$\delta(a, b) \leqslant \delta(a, c) + \delta(c, b) = d(c) \leqslant \varepsilon.$$

□

□ **Theorem**