Applications of Topology, Combinatorics and Algorithms to Discrete Geometry

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Contents

General Introduction

This thesis studies four problems in the playfield of Discrete Geometry, Equivariant Algebraic Topology and Combinatorics. The projects are sorted in chronological order of research done by the author. The goal of the general introduction is to explain how the projects are related and recall some notions.

Mass partition or point partition problems can be approached by equivariant topology. One instance of such a problem is the Grünbaum-Hadwiger-Ramos problem as treated in Chapter 1: A hyperplane seperates $\mathbb{R}^d$ into two halfspaces. Likewise k hyperplanes seperate $\mathbb{R}^d$ into 2^k (possibly empty) orthants. The k hyperplanes equipart a finite Borel measure with continuous density (*mass*) μ if all orthants have equal mass (and in particular no orthant is empty). We wonder, whether it is possible to simultaneously equipart multiple masses: What is the smallest dimension d such that for any j mass on $\mathbb{R}^d$ there exists k affine hyperplanes that equipart all masses simultaneously? The question is wide open and it is already unknown, whether one mass on $\mathbb{R}^4$ can be equiparted by 4 hyperplanes.

Chapter 1 continues an approach to hyperplane mass partition problems based on relative equivariant obstruction theory by Blagojević, Frick, Haase, & Ziegler, see [1].

Definition 0.1. For a finite group G a *topological G-space* is a topological space X with a group action

$$G \times X \to X$$

such that for each g the induced map

$$X \to X, \quad x \mapsto g \cdot x$$

is continuous and such that

$$\mathrm{id} \cdot x = x, \quad (h \cdot g) \cdot x = h \cdot (g \cdot x)$$

for all $x \in X$ and $g, h \in G$.

Morphisms between topological G-spaces are continuous maps $f \colon X \to Y$ between G-spaces with

$$f(g \cdot x) = g \cdot f(x)$$

for all $g \in G$ and $x \in X$.

Hence, when talking about equivariant maps with respect to a group G, we mean that we already equipped the spaces with an appropriate G-action and that this map is a morphism of G-spaces. Other definitions follow from universal properties, i.e. that a G-subspace $A \subseteq X$ is a subspace that is also a G-space such that the G-action on A is the restriction of the G-action on X.

The approach in [1] is the following: Suppose that for some masses $\mu_1, \ldots, \mu_j$ on $\mathbb{R}^d$ no such hyperplanes exist that equipart all masses simultaneously. The authors show that in this case some G-equivariant map $A \to Y$ can be extended to $X \to Y$. If we can show that this map does not exist, the contradiction solves our original problem.

They proceed by constructing a CW-structure on X relative to A that respects the group action. Then the map $X \to Y$ is constructed cell by cell:

By the universal property of the pushout, the map f_n exists if and only if the map h exists, such that the diagram commutes. The map h exists if and only if all the components $S^{n-1} \to Y$ of $f_{n-1} \circ a_{n-1}$

are homotopic to a constant map and represent zero in $\pi_{n-1}(Y)$. Relative equivariant obstruction theory formalizes this: The extension exists if and only if the cochain in

$$\mathcal{C}_G^n\left(X, A; \pi_{n-1}(Y)\right)$$

given by $f_{n-1} \circ a_{n-1}$ is zero.

The crucial observation of obstruction theory is that this cochain is a cocycle and it represents a zero element in cohomology if and only if the map f_{n-2} can be extended to f_n. In our case the space Y is always $(n-2)$-connected and therefore the choice of f_{n-2} is irrelevant and any map $A \to Y$ can be G-equivariantly extended to $A \cup X_n \to Y$ if and only if this obstruction cocycle given by $f_{n-1} \circ a_{n-1}$ vanishes in cohomology.

Computing this obstruction cochain can be difficult in general and in particular its cohomology class. In our case we can find a cell θ such that the boundary of θ consists of pairs related by the G-action. Any G-equivariant cochain that is a coboundary must evaluate on θ as zero modulo 2. Therefore the obstruction cochain does not vanish in cohomology, if it evaluates to one modulo 2 on θ. As the cohomology class of the obstruction cochain is independent of the choice of f_{n-1}, we may proceed to construct any map $f_{n-1} \colon A \cup X_{n-1} \to Y$. The authors of [1] have constructed such a map f_{n-1}, such that the obstruction cochain evaluated on θ modulo 2 is just the number of solutions on θ. In addition, those solutions are in one-to-one correspondence to a combinatorial object, in this case certain 0/1-matrices, see [1]. The enumeration was solved for instances corresponding to two hyperplanes in [1, Lemma 4.5] and for some cases with three and four hyperplanes using a computer, see [1, Remark 4.10].

Counting them with an iterator and not listing them all allows to obtain many more results. In fact, in Chapter 1 we classify those 0/1-matrices and generalize the results from [1]. This also requires a slight generalization of the topological methods developed in [1].

Through insights gained in Chapter 1, the author could contribute to the project in Chapter 2. In this case we consider only the non-trivial orthants (*chambers*) induced by k affine hyperplanes. Those chambers are partitioned in two parts like a chess board: Neighboring chambers belong to different parts. Now, a hyperplane equiparts a mass if the black and the white part of our chess board have the same size with respect to the mass. Again, we wonder whether we can simultaneously equipart multiple masses with some number of hyperplanes in a certain dimension. If this is not possible, some G-equivariant map $A \to Y'$ can be extended to $X \to Y'$, where $A \subseteq X$ are the same G-spaces as in the previous chapter. In particular we can use a similar cell θ with similar boundary computations. The combinatorial object here is slightly easier to classify.

On the other hand, the author expanded and used his knowledge regarding iterators: In Chapter 3 we develop in particular a face iterator for polytopes. This iterator is memory efficient and much faster than previous developments. It can be used to study the faces of large polytopes, where previously this would have been beyond the limits of computations. The faces of polytopes form something that is called a face lattice. It is in particular a graded partially ordered set. Previous algorithms have searched this lattice by a breadth first search such that we visit faces sorted by dimension. However, the number of faces per dimension can grow exponentially with respect to the input and this might need a lot of memory. Also they use mostly global information for each step.

The new algorithm performs a depth first search. For this the lattice is broken into a tree. The disadvantage is that the tree is far from balanced and at each step the first branch might contain up to half of all remaining nodes. However, there are some advantages:

- The algorithm uses only local information that are bounded by the input size. This makes it memory efficient. In addition, the size of the local information decreases at each step, making the algorithm faster in practice.
- The branches of the trees do not depend on each other. This allows easy parallelization.
- How the remainder breaks down into a tree can be modified. This could be useful when searching for faces with particular properties.

We provide several benchmarks to illustrate that the new algorithm is faster. In addition, we continue an appplication developed by Bruns, García-Sánchez, O'Neill and Wilburne [4]: They verified *Wilf's conjecture* for multiplicity ≤ 18. Our algorithm improves this result by verifying multiplicity 19. See Chapter 3 for the details of the conjecture.

Finally, we turn back to a question that was previously approached by equivariant topology: Given $3 \cdot r$ pairwise distinct points in $\mathbb{R}^2$, we can find a partition $S_1, \ldots, S_r$ of pairwise distinct sets of size three

such that the convex hulls intersect:

$$\operatorname{conv}(S_1) \cap \operatorname{conv}(S_2) \cap \cdots \cap \operatorname{conv}(S_r) \neq \emptyset.$$

Instead of triangles in $\mathbb{R}^d$, this can be generalized to $(d+1)r$ points in $\mathbb{R}^d$ and d-simplices that intersect. Tverberg's theorem [8] improves this and shows that even $(d+1)(r-1)+1$ points suffice (not all S_i need to be d-simplices).

This theorem generalizes in many ways. Using methods similar to Chapters 1 and 2, one can show several continuous results: A continuous version of this holds (see Chapter 4), if r is a prime power, see [9], but not otherwise, see [3]. If r is a prime, we can even partition the set of points into color classes of size up to $r-1$ and require the sets S_i to contain at most one point of each color. This optimal colored version even holds in the continuous case, see [2]. Clearly, the optimal colored continuous version cannot hold in the non prime power case. However, it is unkown whether it holds in the case of r being a prime power. Furthermore, the affine version above holds for all r and it is unknown whether there is any r such that an optimal colored affine version would not hold.

Chapter 4 analyzes the first unkown case regarding 10 colored points in $\mathbb{R}^2$. This case is small enough that one can try to do it by hand (for one particular instance of the problem), see [5, Figure 10]. A computer can certainly handle each particular instance, but can it analyze all the instances?

It turns out, that it can: We reduce this into $14,309,547$ graph problems. In each case we construct a k-partite graph, i.e. a graph with vertices $V_1 \sqcup \cdots \sqcup V_k$ such that for $i = 1, \ldots, k$ any two vertices $v, u \in V_i$ are not connected by an edge. If there is a counterexample, then some graph will have a k-clique, which necessarily has a vertex from each part. Finding cliques in a graph is a well-studied problem and one would hope that there already is an implementation that works for us, but that seems not to be the case, as some of the graphs cannot be analyzed by the existing implementations.

There already exist algorithms that treat k-cliques in k-partite graphs, see [6], but their pivot rule did not work for us. Instead we use a pivot rule from a different algorithm, see [7], which has not been applied to k-cliques in k-partite graphs before. The interesting part about this pivot rule is that it is somewhat counter-intuitive: Again we use a depth-first search to find such a k-clique. Instead of trying to create few branches in our search tree, we create branches that contain probably few nodes. This is similar to how many humans would solve a Sudoku: Rule out options that lead to contradictions fast.

Finding a k-clique in a k-partite graph seems to be a strong specification of finding a largest clique in a general graph. However, one strategy for the general graph is to color the graph, such that it becomes a k-partite graph, see [7]. The general case is still a bit more complicated, as it looks for large cliques and not just for k-cliques: In our specialized case we can immediatly backtrack, once a part does not have vertices left.

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Introduction to Chapter 1

A classic reduction of mass/point partition problems to algebraic topology is the Configuration-Space/Test-Map scheme. This scheme can be used to approach a variety of problems that ask for points/hyperplanes that satisfy a certain condition. The classic example of such a reduction is the Borsuk–Ulam theorem. The following proof of this classic theorem aims at introducing the methods used in Chapters 1 and 2. It illustrates how a problem can be reduced to the question whether some equivariant continuous map exists. In some cases, equivariant obstruction theory can then be applied to show that there is an obstruction to such a map.

THEOREM 0.2 (Borsuk–Ulam Theorem). *Given a continuous map f from the n-sphere S^n to $\mathbb{R}^n$. There exist antipodal points $x, -x \in S^d$ that evaluate to the same value, i.e. $f(x) = f(-x)$.*

PROOF. The Configuration-Space/Test-Map scheme usually works as follows: The admissable points/hyperplanes are parametrized by some space. In our case this is just S^n itself, as a point determines its antipodal point as well. This space is called the *configuration space*.

As a next step we set up a *test map*. We measure for each point how far it is from a solution:

$$\tilde{g} \colon S^n \to \mathbb{R}^n, \quad x \mapsto f(x) - f(-x).$$

This map is already $\mathbb{Z}/2$-equivariant with respect to antipodal action. The crucial idea is to assume a contradiction. Suppose such $x, -x$ do not exist with $f(x) = f(-x)$. Then $\tilde{g}(S^n) \subseteq \mathbb{R}^n \setminus \{0\}$ and we can normalize:

$$g \colon S^n \to S^{n-1}, \quad g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||}.$$

However, one can proof that a continuous $\mathbb{Z}/2$-equivariant map $g \colon S^n \to S^{n-1}$ does not exist. This yields a contradiction and contrary to our assumption there is $x \in S^n$ with $f(x) = f(-x)$. To keep the proof consistent with Chapters 1 and 2 we use equivariant obstruction theory as explained from Dieck in [1, Sec. II.3] to show non-existence of such a map:

The spheres can be embedded into each other by considering their equators:

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \ldots$$

Those embeddings induce an inductive CW-structure on S^n. The trivial sphere S^0 has two 0-cells. To the $(n-1)$-skeleton induced by the embedding $S^{n-1} \hookrightarrow S^n$ we attach two n-cells: The upper and lower hemisphere. The antipodal action acts freely on the n-cells.

The map $\mathrm{id} \colon S^{n-1} \to S^{n-1}$ can be $\mathbb{Z}/2$-equivariantly extended to a map $S^n \to S^{n-1}$ if and only if the obstruction cochain

$$\mathfrak{o}(\mathrm{id}) \in \mathcal{C}^n_{\mathbb{Z}/2}\left(S^n; \pi_{n-1}(S^{n-1})\right)$$

is zero. The obstruction cochain assigns to each n-cell the element in $\pi_{n-1}(S^{n-1})$ corresponding to the degree of its gluing map. As the gluing map here is the identity on S^{n-1}, the obstruction cochain is ± 1.

That a specific map $S^{n-1} \to S^{n-1}$ cannot be extended to $S^n \to S^{n-1}$ does not help much. However, the crucial observation of obstruction theory is that the obstruction cochain is more meaningful. The cochain is a cocycle and this cocycle may or may not vanish in cohomology. It vanishes in cohomology if and only if the map on the $(n-2)$-skeleton can be extended to the n-skeleton. In our case the embedding $S^{n-2} \hookrightarrow S^{n-1}$ can be $\mathbb{Z}/2$-equivariantly extended to $S^n \to S^{n-1}$ if and only if the class of the obstruction cocycle

$$[\mathfrak{o}(\mathrm{id})] \in \mathcal{H}^n_{\mathbb{Z}/2}\left(S^n; \pi_{n-1}(S^{n-1})\right)$$

vanishes.

We have already seen that $\mathfrak{o}(\mathrm{id})$ evaluates to ± 1 on the n-cells. Suppose that

$$\mathfrak{h} \in \mathcal{C}^{n-1}_{\mathbb{Z}/2}\left(S^n; \pi_{n-1}(S^{n-1})\right)$$

is a cochain with $\delta\mathfrak{h} = \mathfrak{o}(\mathrm{id})$. Let θ be an n-cell. Its boundary consists of two cells η_1, η_2 that are related as sets by the antipodal action. In particular

$$
\begin{aligned}
\pm 1 = \mathfrak{o}(\mathrm{id})(\theta) \\
= (\delta\mathfrak{h})(\theta) \\
= \mathfrak{h}(\eta_1 + \eta_2)
\end{aligned}
$$

However, we have assumed that $\mathfrak{h}$ is a $\mathbb{Z}/2$-equivariant cochain that must evaluate the same (up to sign) on η_1 and η_2. In each case $\mathfrak{h}(\eta_1 + \eta_2)$ is a multiple of two, which is a contradiction to $\delta\mathfrak{h} = \mathfrak{o}(\mathrm{id})$.

Using obstruction theory, we could now prove that the map $S^{n-2} \hookrightarrow S^{n-1}$ cannot be $\mathbb{Z}/2$-equivariantly extended to a map $S^n \to S^{n-1}$. Finally, as S^{n-1} is $(n-2)$-connected, any two maps $S^{n-2} \to S^{n-1}$ are $\mathbb{Z}/2$-homotopic by [1, Prop. 3.15]. Hence, if one specific map $S^{n-2} \to S^{n-1}$ cannot be $\mathbb{Z}/2$-equivariantly extended to a map $S^n \to S^{n-1}$, then the same holds for any other map. $\qquad\square$

Corollary 0.3 (Ham-Sandwich-theorem)**.** *Given n masses $\mu_1, \ldots, \mu_n$ on $\mathbb{R}^n$ that vanish on affine hyperplanes. There exists an affine hyperplane that equiparts all n masses simultaneously.*

PROOF. To a point $x \in S^n$ we associate a (possibly non-proper) affine halfspace in $\mathbb{R}^n$ by

$$
x = (x_0, \ldots, x_n) \mapsto H(x) := \{ y \in \mathbb{R}^n \colon x_0 + \langle y, (x_1, \ldots, x_n) \rangle \geq 0 \}.
$$

This gives a one-to-one correspondence of affine halfspaces in $\mathbb{R}^n$ and $S^n \setminus \{(\pm 1, 0, \ldots, 0)\}$.

We now consider the continuous map

$$
f \colon S^n \to \mathbb{R}^n, \quad x \mapsto \Big(\mu_1(H(x)), \ldots, \mu_n(H(x)) \Big).
$$

Using the Borsuk–Ulam theorem, there exists $x \in S^n$ with $f(x) = f(-x)$ or

$$
\Big(\mu_1(H(x)), \ldots, \mu_n(H(x)) \Big) = \Big(\mu_1(H(-x)), \ldots, \mu_n(H(-x)) \Big).
$$

This corresponds to a hyperplane that equiparts $\mu_1, \ldots, \mu_n$. W.l.o.g. we may assume that $\mu_1(\mathbb{R}^n) \neq 0$, which implies that this affine hyperplane is proper. $\qquad\square$

In Chapters 1 and 2 we will use the same idea, but the steps will be more involved:

(1) As in Corollary 0.3 we reduce a mass partition problem to some map $S^n \to \mathbb{R}^m$. There exists a group G that acts on S^n and $\mathbb{R}^m$ such that the map is equivariant with respect to G. Suppose that the mass partition problem has no solution for some collection of masses, then the map does not hit the origin and yields a G-equivariant map $S^n \to S^{m-1}$.

(2) With the larger group G comes a different CW-structure on the configuration space S^n. The group G acts no longer freely on the CW-complex. Denote by $A \subset S^n$ the subset of points with non-trivial stabilizer. It turns out that no point in A can correspond to a solution and any collection of masses induces a G-equivariant map $A \to S^{m-1}$. Those maps are shown to be G-homotopic and our goal is to show that they cannot be extended to $S^n \to S^{m-1}$.

Now we use relative obstruction theory, which works very similarly. A G-equivariant extension of those maps $A \to S^{m-1}$ to maps from A union with the $(m-2)$-skeleton of S^n exist and in fact all those extensions are G-homotopic. This means that any extension to A union with the $(m-1)$-skeleton can be used to compute the obstruction cochain. In both Chapters we can find particular masses that have no solution on A and on some $(m-1)$-cell θ allowing us to compute the evaluation of the obstruction cochain on θ.

(3) In the proof of Theorem 0.2, there was essentially only one choice for a cell θ. This is no longer the case and one needs to pick a suitable cell θ.

(4) Computing the boundary of θ is more involved.

(5) The degree of the map $\partial\theta \to S^{m-1}$ is no longer ± 1. In both cases we will use local degrees to compute the parity of the degree, which is enough for our purposes. The parity of the local degree is obtained by counting some combinatorial object.

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CHAPTER 1

Counting Gray codes for an improved upper bound of the Grünbaum–Hadwiger–Ramos problem

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Abstract We give an improved upper bound for the Grünbaum–Hadwiger–Ramos problem: Let $d, n, k \in \mathbb{N}$ such that $d \geq 2^n(1 + 2^{k-1})$. Given 2^{n+1} masses on $\mathbb{R}^d$, there exist k hyperplanes in $\mathbb{R}^d$ that partition it into 2^k sets of equal size with respect to all masses. This is an improvement to the previous bound $d \geq 2^{n+k}$ by Mani-Levitska, Vrećica & Živaljević in 2006.

This is achieved by classifying the number of certain Gray code patterns modulo 2. The reduction was developed by Blagojević, Frick, Haase & Ziegler in 2016. It utilizes the group action of the symmetric group $(\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$ of k oriented hyperplanes. If we restrict to the subgroup $(\mathbb{Z}/2)^k$ as Mani-Levitska et al. we retrieve their bound.

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1. Introduction

Consider the k-dimensional hypercube $[0, 1]^k$. Its graph has several Hamiltonian paths that is, paths of vertices and edges that visit every vertex exactly once. A collection of $0, 1$-vectors representing such a Hamiltonian path is known as k-bit Gray code. The simplest and for our purposes most important such path is the standard k-bit Gray code, which is defined inductively as traversing the front facet by the standard $(k-1)$-bit Gray code and the back facet by the standard $(k-1)$-bit Gray code in reverse order.

The numbers of k-bit Gray codes with marked starting node for $k = 1, \ldots, 5$ are $1, 2, 18, 5712$ and 5859364320[1]. For $k \geq 6$ this number is not known. Invariants of a Gray code are the transition counts $c_1, \ldots, c_k$, where c_i counts the number of edges parallel to the i-th standard base vector for each $i = 1, \ldots, k$. We can classify for which transition counts the number of Gray codes with fixed starting node is odd:

THEOREM 1.1. *The standard k-bit Gray code has transition counts $(2^{k-1}, 2^{k-2}, \ldots, 4, 2, 1)$ and it is the only Gray code (up to choosing a starting node) with those transition counts. Up to permutation those are the only transition counts that are represented by an odd number of Gray codes.*

Given a tuple of $j > 1$ k-bit Gray codes. One can add their transition counts. We try to minimize $\max(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ such that the number of j-tuples of k-bit Gray codes with fixed starting node and sum of transition counts equal to $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ is odd. If $j = 2^n + r$ with $0 \leq r < 2^n$ we find that this value is exactly $2^{k-1}2^n + r$; see Theorem 2.15 (1).

Now the symmetric group of the k-dimensional hypercube acts on Hamiltonian paths not just by changing the starting node, but also by permuting coordinates. It acts on tuples of Gray codes by

[1]https://oeis.org/A003043

diagonal action. Hence, the number of 2-tuples of 4-bit Gray codes with transition counts $(17, 17, 8, 4)$ is even, as those transition counts are invariant with respect to transposing the first two coordinates. We will see however, that this number is not divisible by four; see Proposition 2.26.

We can relax our condition to minimize $\max(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ such that the number of orbits of those j-tuples of Gray codes is odd. Theorem 2.15 establishes that if $j = 2^n + r$ with $0 < r \leq 2^n$, this value is exactly $2^{k-1}2^n + r$; see Theorem 2.15 (2). This differs from the above when $j = 2^n + 2^n$ and this value is $(1 + 2^{k-1})2^n$ instead of $2^k 2^n$. Using methods developed by Blagojević, Frick, Haase, & Ziegler in [3] our combinatorial observations lead to a new bound for the Grünbaum–Hadwiger–Ramos problem:

A *mass* on $\mathbb{R}^d$ is a finite Borel measure that vanishes on every affine hyperplane. Hugo Steinhaus conjectured that any d masses on $\mathbb{R}^d$ can be simultaneously bisected by an appropriate (affine) hyperplane [2]. This is now known as the Ham–Sandwich theorem. Several authors have studied the following generalization:

An arrangment of k hyperplanes in $\mathbb{R}^d$ partitions its complement into 2^k (possibly empty) orthants. A mass μ is equiparted by this arrangment, if each orthant has mass $\frac{1}{2^k}\mu(\mathbb{R}^d)$. The Grünbaum–Hadwiger–Ramos problem is to find the minimal $d = \Delta(j, k)$ such that for any j masses on $\mathbb{R}^d$ there exists an arrangment of k hyperplanes simultaneously equiparting all masses. For a brief history of this problem, we refer the reader to [3].

The lower bound developed by Avis [1] and Ramos [8] is

$$\Delta(j, k) \geq \frac{2^k - 1}{k} j.$$

It is conjectured to be tight and for $k = 1$ this is the case by the Ham–Sandwich theorem. However, even for $k = 2$ exact bounds are elusive: In this case it is shown to be tight for $j = 2^n - 1, 2^n, 2^n + 1$ by Blagojević, Frick, Haase, & Ziegler [3, Thm. 1.5]. Partial results were known before, see [3].

For $k > 2$ the only tight bounds are $\Delta(1, 3) = 3$ by Hadwiger [4] and $\Delta(2, 3) = 5$, $\Delta(4, 3) = 10$ by [3]. The reduction of Hadwiger and Ramos $\Delta(j, k) \leq \Delta(2j, k-1)$ has been used to give (non-optimal) bounds $\Delta(1, 4) \leq 5$ and $\Delta(2, 4) \leq 10$ and $\Delta(1, 5) \leq 10$; see [3].

We complete the list of all previous known bounds with the general upper bound

$$\Delta(j, k) \leq j + (2^{k-1} - 1)2^{\lfloor \log_2 j \rfloor}$$

or equivalently

$$\Delta(2^n + r, k) \leq 2^{n+k-1} + r \quad \text{for } 0 \leq r < 2^n$$

by Mani-Levitska, Vrećica & Živaljević [6]. We will use the methods develeoped in [3] and our combinatorial results briefly described above to generalize this bound:

THEOREM 1.2. *Let $j, k \geq 1$ be integers. It holds that*

$$\Delta(j, k) \leq \left\lceil j + (2^{k-1} - 1)2^{\lfloor \log_2(j - \frac{1}{2}) \rfloor} \right\rceil.$$

Equivalently, for integers $j = 2^n + r$, $n \geq 0$, and $1 \leq r \leq 2^n$ it holds that

$$\Delta(2^n + r, k) \leq 2^{n+k-1} + r \quad and \quad \Delta(1, k) \leq \begin{cases} 2^{k-2} + 1, & \text{if } k \geq 2 \\ 1, & \text{if } k = 1. \end{cases}$$

Note that [3, Thm. 1.5 (iii)] provides a better and tight bound for $\Delta(2^n + 1, 2)$, which cannot be generalized for $k > 2$. Other than this, the above theorem collects all currently known upper bounds. To illustrate the improvements, we give a few examples:

Corollary 1.3.

(1) $\Delta(2^{n+1}, 2) \leq 2^{n+1} + 2^n = 3 \cdot 2^n$, *shown before in [3]*.
(2) $\Delta(2, 3) \leq 5$, *shown before in [3]*.
(3) $\Delta(4, 3) \leq 10$, *shown before in [3]*.
(4) $\Delta(8, 3) \leq 20$, *where the previous best bound is 32; see [6]*.
(5) $\Delta(16, 3) \leq 40$, *where the previous best bound is 64; see [6]*.
(6) $\Delta(1, 4) \leq 5$, *shown before in [3]*.
(7) $\Delta(2, 4) \leq 9$, *where the previous best bound is 10; see [3]*.
(8) $\Delta(4, 4) \leq 18$, *where the previous best bound is 32; see [6]*.
(9) $\Delta(8, 4) \leq 36$, *where the previous best bound is 64; see [6]*.
(10) $\Delta(1, 5) \leq 9$, *where the previous best bound is 10; see [3]*.
(11) $\Delta(2, 5) \leq 17$, *where the previous best bound is 32; see [6]*.

(12) $\Delta(4,5) \le 34$*, where the previous best bound is* 64*; see [6].*
(13) $\Delta(2^{n+1}, k) \le (2^k + 1) \cdot 2^n$*, where the previous best bound is* $2^{k+1} \cdot 2^n$*; see [6].*

If j is not a power of two, our bound coincides with the bound in [6] and we provide an alternative proof of their result. Their bound was derived from a Fadell–Husseini index calculation using the product scheme, parametrizing arrangements of k hyperplanes by $(S^d)^k$. Our proof relies on equivariant obstruction theory with the join scheme, which parametrizes arrangements of k hyperplanes by $(S^d)^{*k}$. (See [3, Section 1.1] for more detail.) Using the join scheme instead of the product scheme, allows to use the full symmetric group $(\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$ of k oriented hyperplanes. If we restrict to the subgroup $(\mathbb{Z}/2)^k$ we retrieve exactly the result by [6].

To conclude the introduction, we remark that for $j = 1$ and $k > 1$ our new bound

$$\Delta(1, k) \le 2^{k-2} + 1$$

is already implied by the reduction of Hadwiger and Ramos $\Delta(1, k) \le \Delta(2, k-1)$ and our new bound

$$\Delta(2, k-1) \le 2^{k-2} + 1.$$

2. Counting Gray codes

Blagojević, Frick, Haase and Ziegler [3] have approached the Grünbaum–Hadwiger–Ramos problem with obstruction theory and developed a reduction of this problem to a combinatorial problem. The topological problem can be solved by counting equivalence classes of certain $0, 1$-matrices modulo 2 with number of rows equal to k. They have classified this parity for the matrices corresponding to $k = 2$. For $j \le 9$, $k = 3$ and $j \le 2$, $k = 4$ they have counted some of them with the help of a computer to obtain new bounds for $\Delta(2,3)$ and $\Delta(4,3)$, see [3, Table 1]. By the reduction of Hadwiger and Ramos they could derive bounds for $\Delta(1,4)$ and $\Delta(2,4)$.

Instead of listing all those matrices, one can use an iterator to count more matrix classes. This helped to understand the underlying structure: For $j > 2$ we provide a full classification, when the number of equivalence classes is odd.

2.1. Gray codes. We will use notation from Knuth [5, pp. 292–294] but start indexing with 1 instead of 0.

Definition 2.1 ([3, Sec. 1.3]). A k-bit Gray code is a $k \times 2^k$ binary matrix containing all column vectors in $\{0,1\}^k$ such that two consecutive vectors differ in only one entry.

A Gray code can be seen as a Hamiltonian path on the edge graph of a hypercube. This is a path visiting each vertex. Knuth restricts to Hamiltonian paths that can be completed to a cycle. We will not restrict to this case.

Definition 2.2. The *standard k-bit Gray code* is a path obtained inductively. The standard 1-bit Gray code is $\big((0), (1)\big)$. For $k > 1$ the standard k-bit Gray code is given by

- each vector of the standard $(k-1)$-bit Gray code with 0 appended,
- in reverse order each vector of the standard $(k-1)$-bit Gray code with 1 appended.

Thus the standard k-bit Gray code traverses the front facet by the standard $(k-1)$-bit Gray code and then in reverse order the back facet.

Example 2.3. The standard 3-bit Gray code is

$$G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We will see later that for $j > 2$ we only need to consider standard k-bit Gray codes.

Definition 2.4. Let $G = (g_1, \ldots, g_{2^k})$ be a k-bit Gray code. Then the *delta sequence of G*

$$\delta(G) := (\delta_1(G), \ldots, \delta_{2^k-1}(G))$$

is defined by $\delta_i(G)$ being the row of the bit change from g_i to g_{i+1} for each $i = 1, \ldots, 2^k - 1$.
A *k-bit delta sequence* is a vector, which is the delta sequence of some k-bit Gray code.

Example 2.5. The standard 3-bit Gray code has delta sequence $(1, 2, 1, 3, 1, 2, 1)$. The standard 4-bit Gray code has delta sequence $(1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1)$. In general, δ_i of the standard k-bit Gray code is equal to one plus the dyadic order of i.

Let D be a k-bit delta seqence. For each choice of a first column $g_1 \in \{0,1\}^k$ there exists exactly one Gray code G with first column g_1 and $\delta(G) = D$.

The symmetric group of the k-dimensional hypercube, $(\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k =: \mathfrak{S}_k^\pm$, acts on the set of all Gray codes by permuting rows and inverting all bits in one row. The group $\mathfrak{S}_k^\pm$ also acts on $[k]^{2^k-1}$ by

$$((\beta_1, \ldots, \beta_k) \rtimes \tau) \cdot (d_1, \ldots, d_{2^k-1}) = (\tau(d_1), \ldots, \tau(d_{2^k-1})).$$

With this action, the map δ from all k-bit Gray codes to $[k]^{2^k-1}$ is $\mathfrak{S}_k^\pm$-equivariant.

Note that $\mathfrak{S}_k^\pm$ acts freely on k-bit Gray codes, that is $\sigma \in \mathfrak{S}_k^\pm$ and G a gray code with $\sigma \cdot G = G$ implies that σ is the unit element. On the other hand, $\mathfrak{S}_k$ acts freely on k-bit delta sequences (but not on $[k]^{2^k-1}$) and $(\mathbb{Z}/2)^k$ acts trivially on k-bit delta sequences. So $(\mathbb{Z}/2)^k$ acts on Gray codes by changing the first column while stabilizing the delta sequence.

Definition 2.6. Let D be a k-bit delta sequence. The *transition counts*

$$c(D) := (c_1(D), \ldots, c_k(D))$$

are defined by letting $c_i(D)$ be the number of i's in D, for every $i = 1, \ldots, k$.

The transition counts of a Gray code are the transition counts of its delta sequence.

Example 2.7. The transition counts of the standard k-bit Gray code are $(2^{k-1}, 2^{k-2}, \ldots, 4, 2, 1)$.

$\mathfrak{S}_k^\pm$ acts on transition counts by trivial action of $(\mathbb{Z}/2)^k$ and by $\mathfrak{S}_k$ permuting coordinates. With this action, taking the transition counts is $\mathfrak{S}_k^\pm$-equivariant.

Lemma 2.8. *Let $D = (d_1, \ldots, d_{2^k-1})$ be a k-bit delta sequence. Suppose for some $0 \leq i < k$ it holds that*

$$c_k(D) = 1, \quad c_{k-1}(D) = 2, \quad \ldots, \quad c_{k-i+1}(D) = 2^{i-1}.$$

Then, $c_{k-i}(D) \geq 2^i$.

PROOF. We induct on i for all k. For $i = 0$ the statement holds as $c_k(D) \geq 1$. Suppose now that $i > 0$ and

$$c_k(D) = 1, \quad c_{k-1}(D) = 2, \quad \ldots, \quad c_{k-i+1}(D) = 2^{i-1}.$$

As $c_k(D) = 1$ it follows that the k-th bit must change at the central position. Then, $\overline{D}_1 := (d_1, \ldots, d_{2^{k-1}-1})$ and $\overline{D}_2 := (d_{2^{k-1}+1}, \ldots, d_{2^k-1})$ are both $(k-1)$-bit delta sequences such that

$$c_{k-1}(\overline{D}_1) + c_{k-1}(\overline{D}_2) = 2, \quad c_{k-2}(\overline{D}_1) + c_{k-2}(\overline{D}_2) = 4, \quad \ldots, \quad c_{k-i+1}(\overline{D}_1) + c_{k-i+1}(\overline{D}_2) = 2^{i-1},$$

which implies by induction that

$$c_{k-1}(\overline{D}_1) = c_{k-1}(\overline{D}_2) = 1, \quad c_{k-2}(\overline{D}_1) = c_{k-2}(\overline{D}_2) = 2, \quad \ldots, \quad c_{k-i+1}(\overline{D}_1) = c_{k-i+1}(\overline{D}_2) = 2^{i-2},$$

and therefore $c_{k-i}(\overline{D}_1) \geq 2^{i-1}$ and $c_{k-i}(\overline{D}_2) \geq 2^{i-1}$. $\qquad\qquad\square$

Definition 2.9. Let D be a k-bit delta sequence. The *shuffle count* of D is the number of delta sequences $\widetilde{D}$ with $c(D) = c(\widetilde{D})$.

We will now proof 1.1 by the following two lemmas:

Lemma 2.10. *Let D be the delta sequence of the standard k-bit Gray code. The shuffle count of D is 1.*

PROOF. For $k = 1$ the statement is clear. Let $\widetilde{D} = (\tilde{d}_1, \ldots, \tilde{d}_{2^k-1})$ be a k-bit delta sequence with $c(\widetilde{D}) = c(D)$. The equality $c_k(\widetilde{D}) = c_k(D) = 1$ implies that the only k-bit change must be at central position and $\tilde{d}_{2^{k-1}} = k$ as desired. By Lemma 2.8 it follows that $\overline{D}_1 := (\tilde{d}_1, \ldots, \tilde{d}_{2^{k-1}-1})$ and $\overline{D}_2 := (\tilde{d}_{2^{k-1}+1}, \ldots, \tilde{d}_{2^k-1})$ are both $(k-1)$-bit delta sequences such that $c(\overline{D}_1) = c(\overline{D}_2) = (2^{k-2}, 2^{k-3}, \ldots, 1)$. By induction this implies that $\overline{D}_1$ and $\overline{D}_2$ are the delta sequence of the standard $(k-1)$-bit Gray code. $\quad\square$

Lemma 2.11. *Let $D = (d_1, \ldots, d_{2^k-1})$ be a k-bit delta sequence. If the shuffle count of D is odd, then D lies in the $\mathfrak{S}_k^\pm$-orbit of the delta sequence of the standard k-bit Gray code.*

PROOF. Denote by D^R the sequence of D in reversed order. If D is a delta sequence, then so is D^R. Hence the parity of the shuffle count is equal to the parity of the number of symmetric delta sequences $\widetilde{D}$ with $c(\widetilde{D}) = c(D)$. If the shuffle count of D is odd, then w.l.o.g. D itself is symmetric. Let G be a Gray code with $\delta(G) = D$. As the statement is invariant under the group action of $\mathfrak{S}_k^\pm$, we may assume that the central entry of D is k.

As D is symmetric, it follows that for each $i = 1, \ldots, 2^{k-1}$ the columns G_i and G_{2^k-i+1} differ exactly in the row k. This implies that $c_k(D) = 1$ and $\overline{D}_1 := (d_1, \ldots, d_{2^{k-1}-1})$ and $\overline{D}_2 := (d_{2^{k-1}+1}, \ldots, d_{2^k-1}) =$

$\overline{D}_1^R$ are $(k-1)$-bit delta sequences. As the shuffle count of D is odd, we may assume that $\overline{D}_1$ itself has odd shuffle count. By induction this implies that $\overline{D}_1$ lies in the $\mathfrak{S}_{k-1}^\pm$-orbit of the standard $(k-1)$-bit Gray code and then D lies in the $\mathfrak{S}_k^\pm$-orbit of the standard k-bit Gray code. $\qquad\qquad\square$

2.2. Equiparting matrices. We generalize [3, Def. 1.1] as follows:

Definition 2.12. Let $\mathfrak{c}_1+\cdots+\mathfrak{c}_k = j(2^k-1)$. A binary matrix $\mathcal{G}$ of size $k\times j2^k$ is a $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrix if

(1) $\mathcal{G} = (G_1,\ldots,G_j)$ for Gray codes $G_1,\ldots,G_j$ with the property that the last column of G_i is equal to the first column of G_{i+1} for $1 \le i < j$; and

(2) the transition counts of $\mathcal{G}$ are given by

$$c(\mathcal{G}) = c(G_1) + \cdots + c(G_j) = (\mathfrak{c}_1,\ldots,\mathfrak{c}_k).$$

If $\mathfrak{c}_1 > \mathfrak{c}_2 = \mathfrak{c}_3 = \cdots = \mathfrak{c}_k$, we have a $(\mathfrak{c}_1 - \mathfrak{c}_2)$-equiparting matrix from [3, Def. 1.1].

Let $\mathfrak{S}_k$ act on $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$ by permutation and let $(\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ denote the stabilizer subgroup. Then $(\mathfrak{S}_k)^\pm_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ acts on $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrices by acting on all Gray codes simultaneously. Note that $(\mathfrak{S}_k)^\pm_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ acts freely on $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrices.

Definition 2.13.

(1) We say that two $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrices $\mathcal{G}$ and $\mathcal{G}'$ are *isomorphic*, if they are in one $(\mathbb{Z}/2)^k$-orbit.

(2) We say that two $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrices $\mathcal{G}$ and $\mathcal{G}'$ are *equivalent*, if they are in one $(\mathfrak{S}_k)^\pm_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$-orbit.

Definition 2.14. Let $j \ge 1$, $k \ge 2$ be integers.

(1) Consider the set $\mathcal{I}(j,k)$ of all tuples of integers $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$ with $\mathfrak{c}_1 + \cdots + \mathfrak{c}_k = j(2^k - 1)$ such that the number of non-isomorphic $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrices is odd.
    Denote by $I(j,k)$ the minimum of $\max(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$ for all $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k) \in \mathcal{I}(j,k)$.

(2) Consider the set $\mathcal{E}(j,k)$ of all tuples of integers $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$ with $\mathfrak{c}_1 + \cdots + \mathfrak{c}_k = j(2^k - 1)$ such that the number of non-equivalent $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrices is odd().
    Denote by $E(j,k)$ the minimum of $\max(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$ for all $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k) \in \mathcal{E}(j,k)$.

Clearly, $E(j,k) \le I(j,k)$. We obtain the following classification of the parity of $(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)$-equiparting matrices:

THEOREM 2.15. *Let $j \ge 1$, $k \ge 2$ be integers.*

(1) Then

$$I(2^n + r, k) = 2^{n+k-1} + r$$

for all $0 \le r < 2^n$ and $n \ge 0$.

(2) Then

$$E(1,k) \le 2^{k-2} + 1$$

and

$$E(2^n + r, k) = 2^{n+k-1} + r$$

for all $1 \le r \le 2^n$ and $n \ge 0$.

Proposition 3.7 establishes that $\Delta(j,k) \le I(j,k)$ and $\Delta(j,k) \le E(j,k)$ and concludes our proof of Theorem 1.2. This reduction has basically been shown in [3] and we generalize it in Section 3. Note that these bounds only differ in the cases

$$E(1,k) \le 2^{k-2} + 1, \quad I(1,k) = 2^{k-1}$$

and

$$E(2^n + 2^n, k) = 2^{n+k-1} + 2^n, \quad I(2^{n+1}, k) = 2^{n+k}.$$

The value of $I(j,k)$ that is obtained by only considering the $(\mathbb{Z}/2)^k$-action is exactly the bound by Mani-Levitska, Vrećica & Živaljević [6]. This seems natural as they also have restricted their attention to the $(\mathbb{Z}/2)^k$-action [6, Proof of Thm. 38]. The value of $E(j,k)$ is the improved bound by considering the full $\mathfrak{S}_k^\pm$-action with the methods developed in [3].

The exact value of $E(1,k)$ remains unknown as we do not understand the precise structure of non-equivalent Gray codes (equiparting matrices for $j = 1$). It has been verified to be tight by computation for $k \le 5$.

2.2.1. *Proof of Theorem 2.15.* The k-bit Gray codes G with transition count $c_k(G) = 1$ are exactly given by composition of two $(k-1)$-bit Gray codes. Let $\mathfrak{c}_1 + \cdots + \mathfrak{c}_{k-1} = 2(2^{k-1} - 1)$ be such that the number of non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_{k-1})$-equiparting matrices is odd. Then it holds that $\mathfrak{c}_1 + \cdots + \mathfrak{c}_{k-1} + 1 = (2^k - 1)$ and the number of non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_{k-1}, 1)$-equiparting matrices is odd, which implies $E(1, k) \leq E(2, k-1)$. $I(1, k) = 2^{k-1}$ is a consequence of Theorem 1.1.

Thus it suffices to verify the values of $E(j, k)$ and $I(j, k)$ for $j > 1$. In order to do so, we use delta sequences and make a few observations:

Definition 2.16. A j-tuple of elements in $[k]^{2^k-1}$

$$\mathcal{D} = (D_1, \ldots, D_j)$$

is a $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequence, if it is a delta sequence of a $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrix

$$\mathcal{G} = (G_1, \ldots, G_j),$$

i.e. for each $i = 1, \ldots, j$ it holds that $\delta(G_i) = D_i$.

The number of non-isomorphic $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices is equal to the number of $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences.

As before $(\mathbb{Z}/2)^k$ acts trivially on $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences and as above $(\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ acts freely by diagonal action. We say that two $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences are equivalent, if they are in one $(\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$-orbit. Thus the number of non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices is equal to the number of non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences.

We denote by $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ the number of non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences modulo 2. Understanding this for all $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ determines $E(j, k)$. As $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ is invariant under the action of $\mathfrak{S}_k$, we may ass well assume that $\mathfrak{c}_1 \geq \mathfrak{c}_2 \geq \cdots \geq \mathfrak{c}_k$.

Understanding $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ for all $\mathfrak{c}_1 > \cdots > \mathfrak{c}_k$ will determine $I(j, k)$:

Lemma 2.17. *If the number of non-isomorphic $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting is odd, then the $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ are pairwise distinct.*

If the $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ are pairwise distinct, then $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices are isomorphic if and only if the are equivalent.

PROOF. As $(\mathfrak{S}_k)^{\pm}_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ acts freely on the set of $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices, the number of $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices is divisible by $|(\mathfrak{S}_k)^{\pm}_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}| = |(\mathbb{Z}/2)^k| \cdot |(\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}|$. Hence the number of non-isomorphic $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices is even unless $|(\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}|$ is odd, which is equivalent to the $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ being pairwise distinct.

The second part is clear. In this case, $(\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ is trivial. □

To determine this parity it suffices to consider only the standard k-bit Gray code:

Lemma 2.18. *Let $j, k \geq 2$ and let $\mathfrak{c}_1 + \cdots + \mathfrak{c}_k = j(2^k - 1)$.*

(1) *Let $(D_1, \ldots, D_j)$ be a $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequence. Let $1 \leq i \leq j$ and let $\widetilde{D}$ be a delta sequence with $c(\widetilde{D}) = c(D_i)$. Then*

$$(D_1, \ldots, D_{i-1}, \widetilde{D}, D_{i+1}, \ldots, D_j)$$

is equivalent to $(D_1, \ldots, D_j)$ if and only if $D_i = \widetilde{D}$.

(2) *The parity $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ is determined by considering only permutations of delta sequences of the standard k-bit Gray code.*

PROOF.   (1) Let $\tau \in (\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ with

$$\tau \cdot (D_1, \ldots, D_j) = (D_1, \ldots, D_{i-1}, \widetilde{D}, D_{i+1}, \ldots, D_j).$$

The group $\mathfrak{S}_k$ acts freely on delta sequences and as $j > 1$ it follows that $\tau = \mathrm{id}$ and $\widetilde{D} = D_i$.

(2) Follows from (1) and Theorem 1.1.

□

We observe another group action: $\mathfrak{S}_j$ acts on $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences by

$$\sigma \cdot (D_1, \ldots, D_j) := (D_{\sigma^{-1}(1)}, \ldots, D_{\sigma^{-1}(j)}).$$

This action commutes with the action of $(\mathfrak{S}_k)_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)}$ and it is the key in determining $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$: If the $\mathfrak{S}_j$-orbit of $(D_1, \ldots, D_j)$ contains an even number of non-equivalent elements, then $(D_1, \ldots, D_j)$ need not be considered for the parity of $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$.

Suppose that $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ is odd. Let $j = 2^n + r$. In the next Proposition we will see that we only need to consider $(D_1, \ldots, D_j)$, where 2^n of them are identical. We have already seen that we only need to consider permutations of the standard k-bit Gray code, which has transition counts $(2^{k-1}, \ldots, 2, 1)$. It follows then that $\mathfrak{c}_1 \geq 2^n 2^{k-1} + r \cdot 1$. Before we proceed, recall some number theory and introduce a term:

The dyadic valuation of an integer n is the largest integer m such that 2^m divides n. To determine whether a fraction is odd, one can compare the dyadic valuation of numerator and denominator.

Lemma 2.19. *The dyadic valuation of* $\binom{j}{a}$ *is*

$$\nu_2\left(\binom{j}{a}\right) = s_2(a) + s_2(j-a) - s_2(j),$$

where $s_2(d)$ is the sum of the dyadic digits of j.

PROOF. By a formula of Legendre, see [7, Theorem 2.6.4], the dyadic valuation of $j!$ is given by

$$\nu_2(j!) = j - s_2(j).$$

$\square$

Lemma 2.20. *The sum of the dyadic digits of positive numbers is subadditive and submultiplicative: Let $n, m \geq 1$ be integers.*

(1)
$$s_2(n) + s_2(m) \geq s_2(n+m)$$
and the inequality is tight, exactly if the dyadic digits of n and m decompose the dyadic digits of $n + m$.

(2)
$$s_2(n)s_2(m) \geq s_2(nm).$$

PROOF. We assume the dyadic digits of n and m to be given as

$$n = \sum_{i=0}^r 2^i n_i, \quad m = \sum_{i=0}^r 2^i m_i.$$

(1) That the inequality is tight, when the dyadic digits are decomposed is clear.

Now by induction on the number of digits of m, it suffices to show that for an integer $j \geq 1$ with $n_j = 1$ it holds that

$$s_2(n) + 1 > s_2(n + 2^j).$$

We induce for fixed n on j from above (note that j is bounded by n). It holds that $s_2(n) = s_2(n - 2^j) + 1$. Then

$$s_2(n + 2^j) = s_2(n - 2^j + 2^{j+1}) \leq s_2(n - 2^j) + s_2(2^{j+1}) = s_2(n) + 1 + 1.$$

If $n_{j+1} = 0$, the last inequality holds tight. If $n_{j+1} = 1$, the last inequality holds by induction hypothesis. The later is never the case for the base case.

(2) With

$$nm = \sum_{k=0}^{2r} \sum_{i+j=k} 2^i 2^j m_i n_i$$

the statment follows from (1).

$\square$

Lemma 2.21. *Let $T \geq 1$ be an integer. It holds that*

$$T - \nu_2(T) \geq s_2(T)$$

and this inequality is tight exactly for $T \in \{1, 2\}$.

PROOF. It is clear that the inequality is tight for $T \in \{1, 2\}$. If T is odd and at least 3 then $T - \nu_2(T) = T > s_2(T)$. The remaining cases follow by induction. If $T - \nu_2(T) \geq s_2(T)$ for some $T \geq 2$, then

$$2T - \nu_2(2T) = 2T - \nu_2(T) - 1$$
$$\geq s_2(T) + T - 1$$
$$= s_2(2T) + T - 1 > s_2(2T).$$

$\square$

Definition 2.22. The *multiplicity* of $(D_1, \ldots, D_j)$ is the multiset formed by:
$$|\{i \in \{1, \ldots, j\} \colon D_i = D\}|$$
for all $D \in \{D_1, \ldots, D_j\}$.

Note that the multiplicity of a $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequence is $(\mathfrak{S}_k)_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)}$-invariant and $\mathfrak{S}_j$-invariant.

Example 2.23. Let D, D', D'' be pairwise distinct.
- (D, D, D', D') has multiplicity $(2, 2)$.
- (D, D, D, D', D') has multiplicity $(2, 3)$.
- (D, D, D', D'', D'') has multiplicity $(1, 2, 2)$.

The multiplicites of those delta sequences we need to consider are limited by j:

Proposition 2.24. *The parity* $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ *is determined by considering only* $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-*delta sequences of multiplicity*
1. $(a_1, \ldots, a_\ell)$, *where the dyadic digits of the* a_i *decompose the dyadic digits of* j.
2. $(\frac{j}{2}, \frac{j}{2})$, *where* j *is a power of two and* $|(\mathfrak{S}_k)_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)}| \geq 2$.

PROOF. Let $\mathcal{D} := (D_1, \ldots, D_j)$ be a $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequence of multiplicity $(a_1, \ldots, a_\ell)$. As $\mathfrak{S}_j$ and $(\mathfrak{S}_k)_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)}$ commute, the number of non-equivalent elements of the $\mathfrak{S}_j$-orbit of $\mathcal{D}$ is equal to the size of the $\mathfrak{S}_j$-orbit divided by the cardinality T of
$$\mathcal{T} := \left\{ \tau \in (\mathfrak{S}_k)_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)} \mid \exists \sigma \in \mathfrak{S}_j \colon \tau\mathcal{D} = \sigma\mathcal{D} \right\}.$$

If $\tau \in \mathcal{T}$ and $\tau(D_{i_1}) = D_{i_2}$, both D_{i_1} and D_{i_2} must have the same cardinality in $(D_1, \ldots, D_j)$. As $(\mathfrak{S}_k)_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)}$ acts freely on delta sequences, $\tau(D_{i_1}) = D_{i_2}$ completely determines τ and the $(a_1, \ldots, a_\ell)$ can be sorted as follows:
$$(a_1, \ldots, a_{\frac{\ell}{T}}, \quad a_1, \ldots, a_{\frac{\ell}{T}}, \quad \ldots, \quad a_1, \ldots, a_{\frac{\ell}{T}}).$$

Note that $T(a_1 + \cdots + a_\ell) = j$ and j must be divisible by T. There are
$$S := \binom{j}{a_1}\binom{j - a_1}{a_2} \cdots \binom{j - a_1 - a_2 - \cdots - a_{\ell-1}}{a_\ell}$$
elements in the $\mathfrak{S}_j$-orbit of $\mathcal{D}$. Hence
$$\nu_2(S) = \sum_{i=1}^{\ell} \Big(s_2(a_i) + s_2(j - a_1 - \cdots - a_i) - s_2(j - a_1 - \cdots - a_{i-1}) \Big)$$
$$= s_2(0) - s_2(j) + \sum_{i=1}^{\ell} s_2(a_i).$$

If the number of non-equivalent elements of the $\mathfrak{S}_j$-orbit of $\mathcal{D}$ is even, $\mathcal{D}$ may be disregarded for the parity of $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$. So suppose that it is odd, it follows that $\nu_2(T) = \nu_2(S)$. However,
$$\nu_2(S) = -s_2(j) + T \sum_{i=1}^{\frac{\ell}{T}} s_2(a_i)$$
$$\geq -s_2(T) s_2\left(\frac{j}{T}\right) + T \sum_{i=1}^{\frac{\ell}{T}} s_2(a_i)$$
$$\overset{(1)}{\geq} -(T - \nu_2(T)) s_2\left(\frac{j}{T}\right) + T \sum_{i=1}^{\frac{\ell}{T}} s_2(a_i) \tag{1}$$
$$= \nu_2(T) s_2\left(\frac{j}{T}\right) + T \left(-s_2\left(\frac{j}{T}\right) + \sum_{i=1}^{\frac{\ell}{T}} s_2(a_i) \right)$$
$$\overset{(2)}{\geq} \nu_2(T) s_2\left(\frac{j}{T}\right) \tag{2}$$
$$\overset{(3)}{\geq} \nu_2(T). \tag{3}$$

Note, that (1) follows from Lemma 2.20 and that (2) follows from Lemma 2.21. So the above inequalities must be tight. This implies:

- $T \in \{1, 2\}$ by (1) and Lemma 2.20.
- The dyadic digits of $a_1, \ldots, a_{\frac{\ell}{T}}$ decompose the dyadic digits of $\frac{j}{T}$ by (2) and Lemma 2.21.
- T is odd or $\frac{j}{T}$ is a power of two by (3).

$\square$

To determine $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ it now suffices to consider those $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences, such that

(1) each D_i is the delta sequence of a permutation of a standard k-bit Gray code by Lemma 2.18 and

(2) the multiplicity $(a_1, \ldots, a_\ell)$ of $(D_1, \ldots, D_j)$ satisfies the conditions of Propostion 2.24.

Suppose that $(D_1, \ldots, D_j)$ is a sequence that satisfies both. Recall that the standard k-bit Gray code has transition counts $(2^{k-1}, 2^{k-2}, \ldots, 1)$. Hence,

$$\max(\mathfrak{c}_1, \ldots, \mathfrak{c}_k) = a_1 \cdot 2^{k-1} + (j - a_1) \cdot 1,$$

as a_1 of the $D_1, \ldots, D_j$ have transition count 2^{k-1} for some row.

If $j = 2^n$ and the $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ are not pairwise distinct, we have that $(a_1, \ldots, a_\ell)$ is one of (2^n) and $(2^{n-1}, 2^{n-1})$. In this case

$$\max(\mathfrak{c}_1, \ldots, \mathfrak{c}_k) \geq 2^n 2^{k-1} + 2^n.$$

Otherwise, if $j = 2^n + r$ for $0 \leq r < 2^n$, we have that a_1 is at least 2^n and we conclude

$$\max(\mathfrak{c}_1, \ldots, \mathfrak{c}_k) \geq 2^n 2^{k-1} + j.$$

The first case implies that

$$E(2^{n-1} + 2^{n-1}, k) \geq 2^{n-1} 2^{k-1} + 2^{n-1}$$

and the second case implies that

$$E(2^n + r, k) \geq 2^n 2^{k-1} + r \leq I(2^n + r, k).$$

This bounds the values of $E(j, k)$ and $I(j, k)$ from below according to Theorem 2.15. To bound the value from above and conclude our proof, we need to find some values for which $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$ is odd.

The case $k = 2$ needs special attention, but was also known before:

Proposition 2.25 (See [3, Section 4.3.1]). *Let $k = 2$ and let $2 \leq j = 2^n + r$ with $0 \leq r \leq 2^n$. Then*

$$N(2^n + 2r, 2^n 2 + r) = 1.$$

Note that $2^n + 2r \neq 2^n 2 + r$ unless $r = 2^n$.

PROOF. There exist exactly two 2-bit delta sequences with transition counts $(1, 2)$ resp. $(2, 1)$. Hence the number of $(2^n + 2r, 2^n 2 + r)$-delta sequences is $\binom{j}{r} = \binom{2^n + r}{r}$. By Lemma 2.19:

$$\nu_2\left(\binom{2^n + r}{r}\right) = s_2(r) + s_2(2^n) - s_2(2^n + r) = s_2(r) + 1 - s_2(2^n + r) = \begin{cases} 0, & \text{if } 0 \leq r < 2^n, \\ 1, & \text{if } r = 2^n. \end{cases}$$

We see that if $r \neq 2^n$, then the number of $(2^n + 2r, 2^n 2 + r)$-delta sequences is odd and $N(2^n + 2r, 2^n 2 + r)$ must be odd as well. If $r = 2^n$, then the number of $(2^n + 2^r, 2^n 2 + r)$-delta sequences is even but not divisible by 4. However, $|\mathfrak{S}_{(2^n + 2r, 2^n 2 + r)}| = 2$, so the delta sequences come in equivalent pairs and $N(2^n + 2r + 2^n 2 + r)$ is also odd in this case. $\square$

The next step is also intuitive. Combining 2^n delta sequences with transition counts $(2, 4, \ldots, 2^{k-1}, 1)$ and 2^n delta sequences with transition counts $(2, 4, \ldots, 2^{k-2}, 1, 2^{k-1})$ will give the only case of

$$2^n \cdot (2 + 2, 4 + 4, \ldots, 2^{k-2} + 2^{k-2}, 2^{k-1} + 1, 2^{k-1} + 1)$$

-equiparting matrices we need to consider and will show that $E(2^{n+1}, k) = 2^{n+k-1} + 2^n$:

Proposition 2.26. *Let $k \geq 2$ and $j = 2^{n+1}$ Then*

$$N(2^{n+2}, 2^{n+3}, \ldots, 2^{n+k-1}, 2^{n+k-1} + 2^n, 2^{n+k-1} + 2^n)$$

is odd.

PROOF. Note that

$$|\mathfrak{S}_{(2^{n+2}, 2^{n+3}, \ldots, 2^{n+k-1}, 2^{n+k-1}+2^n, 2^{n+k-1}+2^n)}| = 2.$$

We only need to consider those delta sequences of permutations of standard k-bit Gray codes of multiplicity (2^{n+1}) or $(2^n, 2^n)$. Such sequences do not exist for multiplicity (2^{n+1}) so we consider the other case.

Suppose $\mathcal{D} = (D_1, \ldots, D_j)$ is such a sequence. As noted above, the transition count $c_i(\mathcal{D}) \geq 2^{k-1}2^n + 2^n$ for two of the i. It immediately follows that 2^n of the $(D_1, \ldots, D_j)$ have transition counts $(2, 4, \ldots, 2^{k-2}, 1, 2^{k-1})$ and the others have transition counts $(2, 4, \ldots, 2^{k-2}, 2^{k-1}, 1)$.

The $\mathfrak{S}_j$-orbit has size $\binom{2^{n+1}}{2^n}$, which is divisible by two, but not by four. Those come in pairs of equivalent permutations, hence the number of non-equivalent elements is odd. $\square$

The other remaining cases are a bit more difficult to see. With the help of a computer we see some examples:

Examples 2.27. Let $k = 4$. The following are all $\mathfrak{c}_1, \ldots, \mathfrak{c}_4$ for different j with $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_4)$ odd and

$$\mathfrak{c}_1 < \mathfrak{c}_2 < \mathfrak{c}_3 < \mathfrak{c}_4$$

such that $\mathfrak{c}_4$ is minimal with this property:

$j = 1$: $(1, 2, 4, 8)$,
$j = 2$: $(2, 4, 8, 16)$,
$j = 3$: $(4, 8, 16, 17)$,
$j = 4$: $(4, 8, 16, 32)$,
$j = 5$: $(6, 12, 24, 33)$, $(6, 16, 20, 33)$, $(8, 10, 24, 33)$, $(10, 12, 20, 33)$, $(8, 16, 18, 33)$,
$j = 6$: $(8, 16, 32, 34)$,
$j = 7$: $(12, 24, 34, 35)$, $(16, 20, 34, 35)$,
$j = 8$: $(8, 16, 32, 64)$,
$j = 9$: $(10, 20, 40, 65)$, $(10, 24, 36, 65)$, $(12, 18, 40, 65)$, $(12, 24, 34, 65)$, $(16, 18, 36, 65)$, $(16, 20, 34, 65)$,
$j = 10$: $(12, 24, 48, 66)$, $(12, 32, 40, 66)$, $(16, 20, 48, 66)$, $(16, 32, 36, 66)$, $(20, 24, 40, 66)$,
$j = 11$: $(14, 28, 56, 67)$, $(14, 32, 52, 67)$, $(14, 36, 48, 67)$, $(14, 40, 44, 67)$, $(16, 26, 56, 67)$, $(16, 32, 50, 67)$,
         $(16, 34, 48, 67)$, $(16, 40, 42, 67)$, $(18, 24, 56, 67)$, $(18, 28, 52, 67)$, $(18, 36, 44, 67)$, $(20, 22, 56, 67)$,
         $(20, 26, 52, 67)$, $(20, 36, 42, 67)$, $(20, 38, 40, 67)$, $(22, 24, 52, 67)$, $(22, 28, 48, 67)$, $(22, 32, 44, 67)$,
         $(24, 26, 48, 67)$, $(24, 32, 42, 67)$, $(24, 34, 40, 67)$, $(24, 36, 38, 67)$, $(26, 28, 44, 67)$,
$j = 12$: $(16, 32, 64, 68)$,
$j = 13$: $(18, 40, 68, 69)$, $(20, 40, 66, 69)$, $(24, 34, 68, 69)$, $(24, 36, 66, 69)$,
$j = 14$: $(24, 48, 68, 70)$, $(32, 40, 68, 70)$
$j = 15$: $(28, 56, 70, 71)$, $(32, 52, 70, 71)$, $(36, 48, 70, 71)$, $(40, 44, 70, 71)$

Apparently there are many values, to verify that the lower bound is achieved. One observation is that there is always a tuple with $2\mathfrak{c}_1 = \mathfrak{c}_2$. This generalizes:

Examples 2.28. Let $k = 5$. The following are all $\mathfrak{c}_1, \ldots, \mathfrak{c}_5$ for different j with $N(\mathfrak{c}_1, \ldots, \mathfrak{c}_5)$ odd and

$$4\mathfrak{c}_1 = 2\mathfrak{c}_2 = \mathfrak{c}_3 < \mathfrak{c}_4 < \mathfrak{c}_5$$

such that $\mathfrak{c}_5$ is minimal with this property:

$j = 1$: $(1, 2, 4, 8, 16)$,
$j = 2$: $(2, 4, 8, 16, 32)$,
$j = 3$: $(4, 8, 16, 32, 33)$,
$j = 4$: $(4, 8, 16, 32, 64)$,
$j = 5$: $(6, 12, 24, 48, 65)$, $(8, 16, 32, 34, 65)$,
$j = 6$: $(8, 16, 32, 64, 66)$
$j = 7$: $(12, 24, 48, 66, 67)$
$j = 8$: $(8, 16, 32, 64, 128)$
$j = 9$: $(10, 20, 40, 80, 129)$, $(12, 24, 48, 66, 129)$
$j = 10$: $(12, 24, 48, 96, 130)$, $(16, 32, 64, 68, 130)$
$j = 11$: $(14, 28, 56, 112, 131)$, $(16, 32, 64, 98, 131)$
$j = 12$: $(16, 32, 64, 128, 132)$
$j = 13$: $(20, 40, 80, 130, 133)$
$j = 14$: $(24, 48, 96, 132, 134)$

Apparently we may start with 2^n times transition counts $(1, 2, 4, 8, 16)$, fill up with $(2, 4, 8, 16, 1)$ until we reach $j = 2^n + 2^{n-1}$ and the remaining onces are $(4, 8, 16, 2, 1)$:

Proposition 2.29. *Let $k > 2$ and $2 \leq j = 2^n + r$ with $0 \leq r < 2^n$. Define integers $w = \min(r, 2^{n-1})$ and $s = r - w$ such that $j = 2^n + w + s$. Then*

$$N(\mathfrak{c}_1, \ldots, \mathfrak{c}_k) = 1,$$

where

$$(\mathfrak{c}_1, \ldots, \mathfrak{c}_k) = 2^n \cdot (1, 2, 4, \ldots, 2^{k-1}) + w \cdot (2, 4, \ldots, 2^{k-1}, 1) + s \cdot (4, 8, \ldots, 2^{k-1}, 2, 1)$$

PROOF. It holds that

$$\mathfrak{c}_{k-1} - \mathfrak{c}_{k-2} = 2^{k-3} 2^n + 2^{k-2} w - (2^{k-1} - 2) s > \begin{cases} 0, & \text{if } s = 0, \\ 2^{k-3} 2w + 2^{k-2} w - 2^{k-1} w = 0, & \text{otherwise,} \end{cases}$$

and

$$\mathfrak{c}_k - \mathfrak{c}_{k-1} = 2^{k-2} 2^n + w + s - 2^{k-1} w - 2s.$$

Hence

$$\mathfrak{c}_1 < \mathfrak{c}_2 < \cdots < \mathfrak{c}_{k-2} < \mathfrak{c}_{k-1} < \mathfrak{c}_k.$$

as $r < 2^n$. This implies that

$$|(\mathfrak{S}_k)_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)}| = 1.$$

By Proposition 2.24 it suffices to consider $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-delta sequences of multiplicity $(a_1, a_2, \ldots, a_\ell)$, where we may assume that $a_1 \geq 2^n$. If $w = 2^{n-1}$, then we may assume that $a_2 \geq 2^{n-1}$ or that $a_1 \geq 2^n + 2^{n-1}$. By Lemma 2.18 it suffices to consider those sequences that consist of permutations of the standard k-bit Gray code.

Suppose $\mathcal{D} = (D_1, \ldots, D_j)$ is such sequence. W.l.o.g. we reorder $\mathcal{D}$ such that $D_1 = \cdots = D_{2^n}$ and such that a is maximal with $D_{2^n+1} = \cdots = D_{2^n+a}$. We will show that $(a_1, \ldots, a_\ell) = (2^n, w, s)$ and that $a = w$ and that

$$c(D_m) = \begin{cases} (1, 2, 4, \ldots, 2^{k-1}), & \text{if } m \leq 2^n, \\ (2, 4, \ldots, 2^{k-1}, 1), & \text{if } 2^n < m \leq 2^n + w, \\ (4, 8, \ldots, 2^{k-1}, 2, 1), & \text{otherwise.} \end{cases}$$

As above 2^n of the $(D_1, \ldots, D_j)$ are equal. Hence at least one of the transition counts is $2^{k-1} 2^n + r = 2^{k-1} 2^n + w + s$. As $\mathfrak{c}_k$ has this value and is strictly the largest, we obtain

$$c_k(D_m) = \begin{cases} 2^{k-1}, & \text{if } m \leq 2^n, \\ 1, & \text{otherwise.} \end{cases}$$

Suppose $s > 0$. This implies $w = 2^{n-1}$ and

$$\mathfrak{c}_{k-1} \equiv 2s \mod 2^n$$

with $2s < 2^n$. However,

$$\nu_2(c_{k-1}(D_1)) = \cdots = \nu_2(c_{k-1}(D_{2^n})) \geq 0$$

and

$$\nu_2(c_{k-1}(D_{2^n+1})) = \cdots = \nu_2(c_{k-1}(D_{2^n+w})) \geq 1.$$

Therefore,

$$c_{k-1}(D_1) + \cdots + c_{k-1}(D_j) \equiv c_{k-1}(D_{2^n+w+1}) + \cdots + c_{k-1}(D_j) \mod 2^n.$$

As they are all at least 2, we conclude that

$$c_{k-1}(D_{2^n+w+1}) = \cdots = c_{k-1}(D_j) = 2.$$

Next we use induction on i to determine the $c_i(D_m)$ for $i \leq k - 2$. By induction we have that

$$c_i(D_m) \geq \begin{cases} 2^{i-1}, & \text{if } m \leq 2^n, \\ 2^i, & \text{if } 2^n < m \leq 2^n + w, \\ 2^{i+1}, & \text{if } 2^n + w < m \leq 2^n + w + s. \end{cases}$$

As $\mathfrak{c}_i = 2^{i-1} 2^n + 2^i w + 2^{i+1} s$, it follows that the inequalities are tight. We conclude the claimed values for $c_1(D_m), \ldots, c_{k-2}(D_m)$ and $c_k(D_m)$ for all m. The values for $c_{k-1}(D_m)$ follow as they are the only values left.

The $\mathfrak{S}_j$-orbit of $\mathcal{D}$ has size $\binom{2^n+w+s}{s} \binom{2^n+w}{w}$, which is not divisible by two. $\qquad \square$

3. Reduction to Equiparting matrices

We will briefly summarize the approach in [3] along with a slight generalization. Via the inclusion

$$\iota \colon \mathbb{R}^d \to \mathbb{R}^{d+1}, \quad (y_1, \ldots, y_d) \mapsto (1, y_1, \ldots, y_d)$$

we parametrize oriented affine hyperplanes in $\mathbb{R}^d$ by S^d with two non-proper hyperplanes that cannot correspond to solutions. A vector $v \in S^d$ corresponds to the oriented hyperplane

$$\left\{ r \in \mathbb{R}^d \colon \langle \iota(r), v \rangle = 0 \right\}$$

with induced positive and negative side.

A collection $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ of masses on $\mathbb{R}^d$ induces a map

$$\psi_{\mathcal{M}} \colon \quad X_{d,k} = (S^d)^{*k} \quad \to \quad W_k \oplus (U_k)^{\oplus j} \cong \mathbb{R}^{k-1} \oplus \left(\mathbb{R}^{(\mathbb{Z}/2)^k} \right)^{\oplus j}$$

$$\lambda_1 v_1 + \cdots + \lambda_k v_k \quad \mapsto \quad \left(\lambda_1 - \frac{1}{k}, \ldots, \lambda_k - \frac{1}{k} \right) \oplus (\lambda_1 \cdots \lambda_k) \cdot \phi_{\mathcal{M}}(v_1, \ldots, v_k),$$

where $\phi_{\mathcal{M}}$ assigns k oriented hyperplanes with normals $(v_1, \ldots, v_k)$ to each mass evaluated on each of the (possibly empty) 2^k regions minus $\frac{1}{2^k}\mu_i(\mathbb{R}^d)$. This map is $\mathfrak{S}_k^{\pm}$-equivariant as explained in [3, Section 2.3]. If 0 is in the image for all collections of masses, then $\Delta(j, k) \leq d$.

If 0 lies not in the image, then $\psi_{\mathcal{M}}$ can be composed with the radial projection $\nu \colon (W_k \oplus U_k^{\oplus j}) \setminus \{0\} \to S(W_k \oplus U_k^{\oplus j})$.

Denote by $X_{d,k}^{>1}$ the subset of those points in $X_{d,k}$ of non-trivial stabilizer. The image of $X_{d,k}^{>1}$ does not contain 0 and in fact any two maps $\psi_{\mathcal{M}}$ and $\psi_{\mathcal{M}'}$ restricted to $X_{d,k}^{>1}$ are $\mathfrak{S}_k^{\pm}$-homotopic [3, Prop 2.2]. This yields:

Proposition 3.1 ([3, Thm. 2.3 (ii)]). *Let $d, k, j \geq 1$ be integers and let $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ be masses on $\mathbb{R}^d$.*

If there is no $\mathfrak{S}_k^{\pm}$-equivariant map

$$X_{d,k} \to S(W_k \oplus U_k^{\oplus j})$$

whose restriction to $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}\mid_{X_{d,k}^{>1}}$, then $\Delta(j, k) \leq d$.

Corollary 3.2. *If there is no $(\mathbb{Z}/2)^k$-equivariant map*

$$X_{d,k} \to S(W_k \oplus U_k^{\oplus j})$$

whose restriction to $X_{d,k}^{>1}$ is $(\mathbb{Z}/2)^k$-homotopic to $\nu \circ \psi_{\mathcal{M}}\mid_{X_{d,k}^{>1}}$, then $\Delta(j, k) \leq d$.

There are different approaches, how to show the non-existent of such equivariant maps. We will use obstruction theory as developed in [3]. As a first step, one can equip $X_{d,k}$ with the CW-structure developed in [3, Section 3]. We assume the reader to be familiar with this CW-structure. We remark that $X_{d,k}^{>1}$ is a subcomplex, which allows us to use relative equivariant obstruction theory. The definition of a (relatively open) cell is as follows:

Definition 3.3 (See [3, Sections 2.2 and 2.3]). Let $(\sigma_1, \ldots, \sigma_k)$ be a permutation of $1, \ldots, k$. Let $(s_1, \ldots, s_k) \in \{+1, -1\}^k$ and let $(i_1, \ldots, i_k) \in \{1, \ldots, d+2\}^k$. Then

$$C_{i_1, \ldots, i_k}^{s_1, \ldots, s_k}(\sigma_1, \ldots, \sigma_k) := \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^{(d+1) \times k} \colon 0 <_{i_1} s_1 x_{\sigma_1} <_{i_2} \cdots <_{i_k} s_k x_{\sigma_k} \right\},$$

where $y <_i y'$ means that y and y' agree in the first $i - 1$ coordinates and $y_i < y_i'$.

This induces a (relatively open) cell

$$D_{i_1, \ldots, i_k}^{s_1, \ldots, s_k}(\sigma_1, \ldots, \sigma_k) := C_{i_1, \ldots, i_k}^{s_1, \ldots, s_k}(\sigma_1, \ldots, \sigma_k) \cap S(\mathbb{R}^{(d+1) \times k}).$$

We proceed as described in [3, Section 2.6]. Let $N_2 = (2^k - 1)j + k - 2$ be the dimension of the sphere $S(W_k \oplus U_k^{\oplus j})$ and let θ be some $(N_2 + 1)$-cell of $X_{d,k}$ and let Z be the union of $X_{d,k}^{>1}$ with the $\mathfrak{S}_k^{\pm}$-orbit of the relative closure of θ. It suffices to show that the map $\nu \circ \psi_{\mathcal{M}}\mid_{X_{d,k}^{>1}}$ cannot be $\mathfrak{S}_k^{\pm}$-equivariantly extended to Z. We may also use Corollary 3.2 and show that this map cannot be $(\mathbb{Z}/2)^k$-equivariantly extended to Z.

Let $\mathcal{S}$ be a subgroup of $\mathfrak{S}_k^{\pm}$. An extension to the N_2-skeleton $Z^{(N_2)}$ can be $\mathcal{S}$-equivariantly extended to Z if and only if the obstruction cocycle

$$\mathfrak{o}(g) \in \mathcal{C}_{\mathcal{S}}^{N_2+1}\left(Z, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j})) \right)$$

is zero. If this cochain is not a coboundary, i.e.

$$0 \neq [\mathfrak{o}(g)] \in \mathcal{H}_{\mathcal{S}}^{N_2+1}\big(Z, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big),$$

then this map cannot be extended to Z independent of the choice on the N_2-skeleton. In general it can be difficult to determine the obstruction cochain and its cohomology class. However, the parity of the cochain can be determined by counting equiparting matrices for a suitable choice of the cell θ and the masses $\mathcal{M}$. Even better, cochains of odd parity do not vanish in this case:

Proposition 3.4. *Let $\mathcal{S}$ be a subgroup of $\mathfrak{S}_k^{\pm}$. Let θ be an $(N_2 + 1)$-cell of $X_{d,k}$ such that the N_2-faces $\eta_1, \ldots, \eta_n, \zeta_1, \ldots, \zeta_n$ in the boundary of θ can be grouped in pairs such that η_i and ζ_i are in one $\mathcal{S}$-orbit for each $i = 1, \ldots, n$.*

Suppose $\mathcal{M}$ is a collection of masses such that 0 is not in the image of $\psi_{\mathcal{M}}$ restricted to the boundary of θ and that the number of 0s in the image of the interior of θ is finite and odd, then $\Delta(j, k) \leq d$.

PROOF. Denote by e_θ the element in the cellular chain group $\mathcal{C}_{N_2+1}(Z, X_{d,k}^{>1})$ corresponding to θ. The image of the boundary of θ under $\psi_{\mathcal{M}}$ does not contain zero. Then by [3, Section 2.6] $\mathfrak{o}(g)(e_\theta)$ is the same as the number of $x \in \operatorname{relint} \theta$ with $\psi_{\mathcal{M}}(x) = 0$ modulo 2.

We proceed as in [3, Proof of Theorem 1.4]. Suppose there exists a cochain

$$\mathfrak{h} \in \mathcal{C}_{\mathcal{S}}^{N_2}\big(Z, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big)$$

such that $\delta\mathfrak{h} = \mathfrak{o}(g)$, then in particular

$$\mathfrak{o}(g)(e_\theta) = (\delta\mathfrak{h})(e_\theta) = \mathfrak{h}(\eta_1 + \cdots + \eta_n + \zeta_1 + \cdots + \zeta_n).$$

However, it holds that $\mathfrak{h}(\eta_i) = \pm\mathfrak{o}(\zeta_i)$ for all $i = 1, \ldots, n$. This is a contradiction and hence the obstruction cocycle is not a coboundary and the map cannot be $\mathcal{S}$-equivariantly extended to Z. By Proposition 3.1 this implies that $\Delta(j, k) \leq d$. $\qquad\square$

Now we will describe a suitable cell θ. Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers such that $dk \geq (2^k - 1)j$. Let further $\ell_1 \geq \cdots \geq \ell_k \geq 0$ be integers with $\ell_1 + \cdots + \ell_k = dk - (2^k - 1)j$.

Consider the cell

$$\theta = D_{\ell_1+1, \ell_2+1, \ldots, \ell_k+1}^{+,+,\ldots,+}(1, 2, \ldots, k).$$

This cell has dimension

$$(d+2)k - k - (\ell_1 + \cdots + \ell_k) - 1 = k + (2^k - 1)j - 1 = N_2 + 1$$

as desired.

By definition, θ parametrizes all arrangements $\mathcal{H} = (H_1, \ldots, H_k)$ of k linear oriented hyperplanes in $\mathbb{R}^{d+1}$ with normal vectors $x_1, \ldots, x_k$ such that x_i has the first ℓ_i coordinates zero and coordinate $\ell_i + 1$ strictly greater than x_{i-1}, where $x_0 := 0$.

Lemma 3.5. *The boundary of θ can be grouped in pairs of $\mathfrak{S}_k^{\pm}$-orbit elements. If further $\ell_1, \ldots, \ell_k$ are pairwise distinct, the boundary can be grouped in pairs of $(\mathbb{Z}/2)^k$-orbit elements.*

PROOF. The cells in the boundary are obtained by introducing one of the following equalities:

$$0 = x_{\ell_1+1, 1}, \quad x_{\ell_2+1, 1} = x_{\ell_2+1, 2}, \quad \ldots, \quad x_{\ell_k+1, k-1} = x_{\ell_k+1, k}.$$

(A) The equality $0 = x_{\ell_1+1, 1}$ induces cells:

$$\eta_1 := D_{\ell_1+2, \ell_2+1, \ldots, \ell_k+1}^{+,+,\ldots,+}(1, 2, \ldots, k), \quad \zeta_1 := D_{\ell_1+2, \ell_2+1, \ldots, \ell_k+1}^{-,+,\ldots,+}(1, 2, \ldots, k)$$

that are related, as sets via $\zeta_1 = \varepsilon_1 \cdot \eta_1$ (ε_1 is the group element flipping the normal of the first hyperplane).

(B) Let $b \in \{2, \ldots, k\}$ and suppose $\ell_{b-1} \geq \ell_b + 1$. Let a be minimal such that $\ell_a \leq \ell_b + 1$ (possibly $a = b$). For the relative interior of θ it holds that

$$0 <_{\ell_b+2} x_a <_{\ell_b+2} x_{a+1} <_{\ell_b+2} \cdots <_{\ell_b+2} x_{b-1} <_{\ell_b+1} x_b$$

and a is minimal with this property. When introducing the equality $x_{\ell_b+1, b} = x_{\ell_b+1, b-1}$ there are $b - a + 1$ possible positions for the b-th hyperplane, each position with both signs. This equality

induces the cells

$$\eta_{b,b} := D^{+,+,\ldots,+}_{\ell_1+1,\ldots,\ell_{b-1}+1,\ell_b+2,\ell_{b+1}+1,\ldots,\ell_k+1}(1,\ldots,k)$$

$$\zeta_{b,b} := \varepsilon_b \cdot \eta_{b,b},$$

$$\eta_{b-1,b} := \tau_{b-1,b} \cdot \eta_{b,b},$$

$$\zeta_{b-1,b} := \varepsilon_{b-1} \cdot \tau_{b-1,b} \cdot \eta_{b,b} = \varepsilon_{b-1} \cdot \eta_{b-1,b},$$

$$\eta_{b-2,b} := \tau_{b-2,b-1} \cdot \tau_{b-1,b} \cdot \eta_{b,b},$$

$$\zeta_{b-2,b} := \varepsilon_{b-2} \cdot \tau_{b-2,b-1} \cdot \tau_{b-1,b} \cdot \eta_{b,b} = \varepsilon_{b-2} \cdot \eta_{b-2,b},$$

$$\cdots$$

$$\eta_{a,b} := \tau_{a,a+1} \cdot \tau_{a+1,a+2} \cdot \cdots \cdot \tau_{b-1,b} \cdot \eta_{b,b},$$

$$\zeta_{a,b} := \varepsilon_a \cdot \tau_{a,a+1} \cdot \tau_{a+1,a+2} \cdot \cdots \cdot \tau_{b-1,b} \cdot \eta_{b,b} = \varepsilon_a \cdot \eta_{a,b}.$$

$\tau_{c,d}$ swaps the hyperplanes c and d.

(C) Let $b \in 2,\ldots,k$ and suppose $\ell_{b-1} = \ell_b$, in particular the $(\ell_1,\ldots,\ell_k)$ are not pairwise distinct. The equality $x_{\ell_b+1,b} = x_{\ell_b+1,b-1}$ induces two cells

$$\eta_{b,b} := D^{+,+,\ldots,+}_{\ell_1+1,\ldots,\ell_{b-1}+1,\ell_b+2,\ell_{b+1}+1,\ldots,\ell_k+1}(1,\ldots,k)$$

$$\zeta_{b-1,b} := \tau_{b-1,b} \cdot \eta_{b,b}.$$

$\qquad\square$

Consider the binomial moment curve

$$\gamma \colon \mathbb{R} \to \mathbb{R}^d, \quad t \mapsto \left(t, \binom{t}{2}, \binom{t}{3}, \ldots, \binom{t}{d}\right)$$

and points

$$q_1 := \gamma(0), \quad q_2 := \gamma(1), \quad \ldots, \quad q_{\ell_1+1} = \gamma(\ell_1).$$

We obtain the following parametrization:

Lemma 3.6. *The relative closure of the cell θ parametrizes those $(v_1,\ldots,v_k)$ in $(S^d)^k$ corresponding to affine oriented hyperplanes $\mathcal{H} = (H_1,\ldots,H_k)$ in $\mathbb{R}^d$ such that*

- *$\{q_1,\ldots,q_{\ell_i}\} \subset H_i$ for $i = 1,\ldots,k$,*
- *if $\ell_i = \ell_{i+1}$ for any $i = 1,\ldots,k-1$, then $v_{\ell_i+1,i} \le v_{\ell_{i+1}+1,i+1}$, where $v_{\ell_i+1,i}$ is the (ℓ_i+1)th-coordinate of the unit normal vector of the hyperplane H_i,*
- *$v_{\ell_i+1,i} \ge 0$ for any $i = 1,\ldots,k$,*

and for relative open cell θ additionally

- *$q_{\ell_i+1} \notin H_i$ for $i = 1,\ldots,k$,*
- *if $\ell_i = \ell_{i+1}$ for any $i = 1,\ldots,k-1$, then $v_{\ell_i+1,i} < v_{\ell_{i+1}+1,i+1}$,*
- *$v_{\ell_i+1,i} > 0$ for any $i = 1,\ldots,k$.*

By parametrize we mean the restriction of $X_{d,k}$ to the embedding of $(S^d)^k$:

$$\left\{(\lambda_1 v_1, \ldots \lambda_k v_k) \in X_{d,k} : \lambda_1 = \cdots = \lambda_k = \frac{1}{k}\right\}.$$

If $\psi_{\mathcal{M}}(x) = 0$, then x lies in the image of the embedding.

PROOF. This proof is analogous to [3, Lem. 3.13]:

By Definition 3.3 the relatively open cell θ corresponds to hyperplanes with normals $v_1,\ldots,v_k$ such that

$$0 <_{\ell_1+1} v_1 <_{\ell_2+1} v_2 <_{\ell_3+1} \cdots <_{\ell_k+1} v_k.$$

As $\ell_1 \ge \ell_2 \ge \cdots \ge \ell_k$, by induction on i we see that this is equivalent to

$$0 = v_{i,1}, \quad 0 = v_{i,2}, \quad \ldots, \quad 0 = v_{i,\ell_i}, \quad 0 \le v_{i-1,\ell_i+1} < v_{i,\ell_i+1}.$$

for each $i = 1,\ldots,k$, where $v_0 := 0$. The relative closure is obtained by allowing the strict inequality to be non-strict.

Observe that a hyperplane with normal v, that is zero in the first ℓ entries contains $q_{\ell+1}$ if and only if the $\ell+1$th entry is zero as well. The characterization of the relatively open cell θ and its closure follow. $\qquad\square$

We are now ready to use Theorem 2.15:

Proposition 3.7.
  (1) $\Delta(j,k) \leq I(j,k)$.
  (2) $\Delta(j,k) \leq E(j,k)$.

Along with Theorem 2.15 this proves Theorem 1.2. The part 1 is strictly weaker, but uses only the $(\mathbb{Z}/2)^k$-action. Its purpose is to explain why $I(j,k)$ coincides with the bound provided in [6].

PROOF. (2) Suppose that $d = E(j,k)$. In this case there are $d = \mathfrak{c}_1 \geq \mathfrak{c}_2 \geq \cdots \geq \mathfrak{c}_k \geq 0$ with

$$\mathfrak{c}_1 + \cdots + \mathfrak{c}_k = j(2^k - 1).$$

such that the number of non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices is odd.

(1) If $d = I(j,k)$ there are such $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ as above which are additionally pairwise distinct. Note that in this case $(\mathfrak{S}_k)_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)}$ is trivial and $(\mathfrak{S}_k)^{\pm}_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)} = (\mathbb{Z}/2)^k$ and non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices are the same as non-isomorphic ones.

Either way, define

$$\ell_1 := d - \mathfrak{c}_1, \quad \ldots, \quad \ell_k := d - \mathfrak{c}_k.$$

Consider the cell

$$\theta = D^{+,+,\ldots,+}_{\ell_1+1,\ell_2+1,\ldots,\ell_k+1}(1,2,\ldots,k)$$
$$= D^{+,+,\ldots,+}_{d-\mathfrak{c}_1+1,d-\mathfrak{c}_2+1,\ldots,d-\mathfrak{c}_k+1}(1,2,\ldots,k).$$

In order to use Proposition 3.4 it remains to find masses $\mathcal{M}$ such that the image of $\psi_{\mathcal{M}}$

  • does not contain 0 restricted to the boundary of θ,
  • contains an odd number of 0s restricted to the relative interior of θ.

We proceed as in [3, Lemma 4.1, Lemma 4.2]. However, in [3] it was omitted to show that the 0s are in the relative open cell θ (and not just in the closure as was done in [3, Thm. 1.3]). Here we provide a proof for completeness.

A mass on $\mathbb{R}$ induces a mass on $\mathbb{R}^d$ via the map $\gamma \colon \mathbb{R} \to \mathbb{R}^d$. For points $p_0 < p_1 < \cdots < p_{2^k}$ we construct a mass $\mu_{p_0,\ldots,p_{2^k}}$ on $\mathbb{R}^d$ induced by μ on $\mathbb{R}$ as follows

  • $\mu([p_i, p_{i+1}]) = 1$ for $i = 0, \ldots, 2^k - 1$ and 1 is uniformly distributed on this interval,
  • $\mu((-\infty, p_0]) = 0$,
  • $\mu([p_{2^k}, \infty)) = 0$.

The mass $\mu_{p_0,\ldots,p_{2^k}}$ can only be equiparted into 2^k pieces by k hyperplanes, if at least $2^k - 1$ points in $\gamma((p_0, p_{2^k}))$ are contained in one of the hyperplanes $H_1, \ldots, H_k$. More than $2^k - 1$ points are needed, unless the hyperplanes contain exactly the points $\gamma(p_1), \ldots, \gamma(p_{2^k-1})$. If we encode the intervals $[p_0, p_1], \ldots, [p_{2^k-1}, p_{2^k}]$ with vectors in $\{0,1\}^k$ according to the region, an equipartition of $\mu_{p_0,\ldots,p_{2^k}}$ by $2^k - 1$ points corresponds exactly to a k-bit Gray code.

Overall we choose points

$$\ell_1 \quad = \quad p_{1,0} < \cdots < p_{1,2^k} \quad = \quad p_{2,0} < \cdots < p_{2,2^k} \quad = \quad \ldots \quad = \quad p_{j,0} < \cdots < p_{j,2^k}$$

and obtain masses

$$\mathcal{M} = \left(\mu_{p_{1,0},\ldots,p_{1,2^k}}, \ldots, \mu_{p_{j,0},\ldots,p_{j,2^k}}\right).$$

An equipartition of all masses simultanously can only be obtained with at least $j(2^k - 1)$ intersection points of the $H_1, \ldots, H_k$ with $\gamma((p_{1,0}, p_{j,2^k}))$. The equipartitions with exactly $j(2^k-1)$ intersection points are encoded by equiparting matrices.

According to Lemma 3.6, for an arrangement $(H_1, \ldots, H_k)$ in the relative closure of θ, the hyperplane H_i contains the points $q_1, \ldots, q_\ell$ on $\gamma(\mathbb{R})$ for each $i = 1, \ldots, k$. Each hyperplane can have at most d intersection points with $\gamma(\mathbb{R})$ and

$$dk = \ell_1 + \cdots + \ell_k + j(2^k - 1).$$

So, hyperplane arrangements in the relative closure of θ have at most $j(2^k - 1)$ intersection points with $\gamma((p_{1,0}, p_{j,2^k}))$. We conclude that arrangments in the $(\mathfrak{S}_k)^{\pm}_{(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)}$-orbit of the relative closure of θ that equipart $\mathcal{M}$ into 2^k pieces are in one-to-one correspondence with $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices.

The number of non-equivalent $(\mathfrak{c}_1, \ldots, \mathfrak{c}_k)$-equiparting matrices is odd by assumption. If each of those arrangements has a representative in the relatively open cell θ and not just in the relative closure, we are done, as $\mathfrak{S}_k^{\pm}$ acts free on the θ-orbit.

Clearly, for $i = 1, \ldots, k$ no hyperplane H_i in such an arrangement may contain the point q_{ℓ_i+1} and we conclude $v_{\ell_i+1,i} \neq 0$. According to Lemma 3.8, we can choose the points $p_{1,0} < \cdots < p_{j,2^k}$ such that $\ell_i = \ell_{i+1}$ for any $i \in \{1, \ldots, k-1\}$ implies $v_{\ell_i+1,i} \neq v_{\ell_{i+1}+1,i+1}$ and then indeed the arrangement lies in the relatively open cell θ by Lemma 3.6. $\square$

Lemma 3.8. *Let $d > \ell_1 > 0$ and $n \geq 2(d - \ell_1)$ be integers. There exist $\ell_1 < \pi_1 < \cdots < \pi_n$ in $\mathbb{R}$ such that for any two oriented hyperplanes H_v, H_w parametrized by $v, w \in S^d$ and $\ell \leq \ell_1$ with the following properties*

    (1) $v_i = w_i = 0$ for $i = 1, \ldots, \ell$, i.e. H_v, H_w contain $q_1, \ldots, q_\ell$,
    (2) $v_{\ell+1} \neq 0 \neq w_{\ell+1}$, i.e. H_v, H_w do not contain $q_{\ell+1}$,
    (3) the sets

$$H_v \cap \{\gamma(\pi_1), \ldots \gamma(\pi_n)\}, \quad H_w \cap \{\gamma(\pi_1), \ldots \gamma(\pi_n)\}$$

    are disjoint and both of cardinality $d - \ell$,

it holds that $v_{\ell+1} \neq w_{\ell+1}$.

PROOF. Let $\pi_1, \ldots, \pi_n$ be algebraically independent with $\ell_1 < \pi_1 < \cdots < \pi_n$. Suppose

$$H_v \cap \{\gamma(\pi_1), \ldots \gamma(\pi_n)\} = \{\gamma(\pi_{I_1}), \ldots \gamma(\pi_{I_{d-\ell}})\}, \quad H_w \cap \{\gamma(\pi_1), \ldots \gamma(\pi_n)\} = \{\gamma(\pi_{J_1}), \ldots \gamma(\pi_{J_{d-\ell}})\}.$$

This means that $\pi_{I_1}, \ldots, \pi_{I_{d-\ell}}$ are roots of the polynomial

$$1v_1 + tv_2 + \binom{t}{2}v_3 + \cdots + \binom{t}{d}v_{d+1}$$

in $\mathbb{Q}(v_1, \ldots, v_{d+1})[t]$. As $v_1 = \cdots = v_\ell = 0$ and $v_1^2 + \cdots + v_{d+1}^2 = 1$, they are algebraic over $\mathbb{Q}(v_{\ell+1}, \ldots, v_d)$.

Likewise, $\pi_{J_1}, \ldots, \pi_{J_{d-\ell}}$ are algebraic over $\mathbb{Q}(w_{\ell+1}, \ldots, w_d)$. By assumptions the transcendence degree of $\mathbb{Q}(\pi_{I_1}, \ldots, \pi_{I_{d-\ell}}, \pi_{J_1}, \ldots \pi_{J_{d-\ell}})$ is $2(d - \ell)$. This implies that $v_{\ell+1}$ and $w_{\ell+1}$ are algebraically independent and in particular distinct. $\square$

Acknowledgements

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Introduction to Chapter 2

With the knowledge aquired in Chapter 1, the author could contribute to the project in Chapter 2, which was published in [3]. The contribution of the author mostly is as follows:

The paper proves that every collection of $2^a(2h+1)+\ell$ masses on $\mathbb{R}^{2^a+\ell}$ can be partitioned in a certain way. The previous version stated the case $\ell = 1$ and the author generalized this case. Along with a number of trivial changes this included reworking Proposition 2.3, Section 3.1.2, and Proposition 3.3.

The contribution of the author of this thesis can also be tracked by comparing [1] with [2].

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More bisections by hyperplane arrangements

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Dedicated to Žarko Mijajlović on the occasion of his 70th birthday

Abstract A union of an arrangement of affine hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$ is the real algebraic variety associated to the principal ideal generated by the polynomial $p_{\mathcal{H}}$ given as the product of the degree one polynomials which define the hyperplanes of the arrangement. A finite Borel measure on $\mathbb{R}^d$ is bisected by the arrangement of affine hyperplanes $\mathcal{H}$ if the measure on the "non-negative side" of the arrangement $\{x \in \mathbb{R}^d : p_{\mathcal{H}}(x) \geq 0\}$ is the same as the measure on the "non-positive" side of the arrangement $\{x \in \mathbb{R}^d : p_{\mathcal{H}}(x) \leq 0\}$.

In 2017 Barba, Pilz & Schnider considered special, as well as modified cases of the following measure partition hypothesis: For a given collection of j finite Borel measures on $\mathbb{R}^d$ there exists a k-element affine hyperplane arrangement that bisects each of the measures into equal halves simultaneously. They showed that there are simultaneous bisections in the case when $d = k = 2$ and $j = 4$. Furthermore, they conjectured that every collection of j measures on $\mathbb{R}^d$ can be simultaneously bisected with a k-element affine hyperplane arrangement provided that $d \geq \lceil j/k \rceil$. The conjecture was confirmed in the case when $d \geq j/k = 2^a$ by Hubard and Karasev in 2018.

In this paper we give a different proof of the Hubard and Karasev result using the framework of Blagojević, Frick, Haase & Ziegler (2016), based on the equivariant relative obstruction theory of tom Dieck, which was developed for handling the Grünbaum–Hadwiger–Ramos hyperplane measure partition problem. Furthermore, this approach allowed us to prove even more, that for every collection of $2^a(2h+1)+\ell$ measures on $\mathbb{R}^{2^a+\ell}$, where $1 \leq \ell \leq 2^a - 1$, there exists a $(2h+1)$-element affine hyperplane arrangement that bisects all of them simultaneously. Our result was extended to the case of spherical arrangements and reproved by alternative methods in a beautiful way by Crabb in 2020.

Contents

1. Introduction and statement of main results

Let $d \geq 1$ be an integer. An *affine hyperplane* in the d-dimensional Euclidean space $\mathbb{R}^d$ is determined by a unit vector $u \in S(\mathbb{R}^d)$ in $\mathbb{R}^d$ and a scalar $a \in \mathbb{R}$ as follows:

$$H_{u,a} := \{x \in \mathbb{R}^d : \langle x, u \rangle = a\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product. In this description the sets $H_{u,a}$ and $H_{-u,-a}$ coincide. An *oriented affine hyperplane* in $\mathbb{R}^d$ determined by a unit vector $u \in S(\mathbb{R}^d)$ and a scalar $a \in \mathbb{R}$ is the triple $H(u,a) := (H_{u,a}, u, a)$. The set of all oriented affine hyperplanes is endowed with a $\mathbb{Z}/2$-action given by the orientation change $H(u,a) \longmapsto H(-u,-a)$. To each oriented affine hyperplane $H(u,a)$ in $\mathbb{R}^d$ we associate the linear polynomial function $p_{u,a} \colon \mathbb{R}^d \longrightarrow \mathbb{R}$ given by $p_{u,a}(x) := \langle x, u \rangle - a$ for $x \in \mathbb{R}^d$. In particular, $H_{u,a} = \{x \in \mathbb{R}^d : p_{u,a}(x) = 0\}$. Furthermore, $p_{u,a}(x) = -p_{-u,-a}(x)$.

Let $k \geq 1$ be an integer. A *k-element affine hyperplane arrangement* $\mathcal{H}$ in $\mathbb{R}^d$ is an ordered k-tuple of oriented affine hyperplanes in $\mathbb{R}^d$. To any k-element affine hyperplane arrangement $\mathcal{H} = (H(u_1, a_1), \ldots, H(u_k, a_k))$ we associate the polynomial function $p_{\mathcal{H}} \colon \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by

$$p_{\mathcal{H}}(x) := \prod_{i=1}^{k} p_{u_i, a_i}(x).$$

The *union* of the arrangement $\mathcal{H}$ in $\mathbb{R}^d$ is the real affine variety

$$\{x \in \mathbb{R}^d : p_{\mathcal{H}}(x) = 0\}.$$

A k-element affine hyperplane arrangement $\mathcal{H} = (H(u_1, a_1), \ldots, H(u_k, a_k))$ in $\mathbb{R}^d$ is *essential* if

$$H(u_r, a_r) \neq H(u_s, a_s) \qquad \text{and} \qquad H(u_r, a_r) \neq H(-u_s, -a_s),$$

for all $1 \leq r < s \leq k$. As expected, a k-element affine hyperplane arrangement is *non-essential* if it is not essential.

Let μ be a *nice* measure on $\mathbb{R}^d$, meaning that μ is a finite Borel measure on $\mathbb{R}^d$ that vanishes on every affine hyperplane in $\mathbb{R}^d$. A k-element arrangement $\mathcal{H}$ *bisects* the family of nice measures $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ if for every $1 \leq r \leq j$:

$$\mu_r\big(\{x \in \mathbb{R}^d : p_{\mathcal{H}}(x) \geq 0\}\big) = \mu_r\big(\{x \in \mathbb{R}^d : p_{\mathcal{H}}(x) \leq 0\}\big) = \frac{\mu_r(\mathbb{R}^d)}{2}.$$

In other words, we are looking for an essential affine hyperplane arrangement and a coloring of the connected components of the complement of its union into two colors with the property that no closures of any two components of the same color share a common facet. This provides a bisection of the space into two parts corresponding to the colors and we ask that this partition bisects every one of the given measures into equal halves.

In this paper, motivated by the recent work of Barba, Pilz & Schnider [3] we study the set $\Lambda \subseteq \mathbb{N}^3$ of all triples (d, j, k) of positive integers such that for every collection of j nice measures in $\mathbb{R}^d$ there exists a k-element affine hyperplane arrangement in $\mathbb{R}^d$ that bisects these measures. It is not hard to observe that the set Λ has the following property:

$$(d, j, k) \in \Lambda \quad \Longrightarrow \quad (d', j, k) \in \Lambda \quad \text{for all } d' \geq d.$$
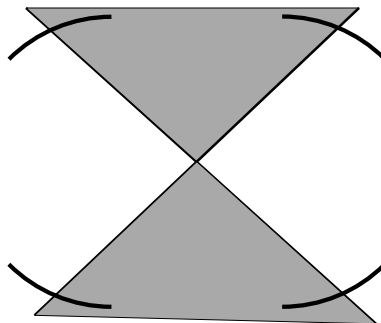
FIGURE 1. Illustration of a black and white bisection of four measures on the plane by an essential 2-element affine hyperplane arrangement.

Furthermore, the ham sandwich theorem is equivalent to the inclusion

$$\{(d, j, 1) : d \geq j \geq 1\} \subseteq \Lambda.$$

The first description of the set Λ follows by considering j pairwise disjoint intervals on a moment curve in $\mathbb{R}^d$ as measures, counting the number of intersection points of a k-element affine hyperplane arrangement with the moment curve (at most dk points) and comparing it with the minimal number of points needed for a bisection of j intervals (at least j points). Consequently, we get

$$(d, j, k) \in \Lambda \quad \Longrightarrow \quad dk \geq j.$$

The idea of considering intervals on a moment curve as measures in the context of the Grünbaum–Hadwiger–Ramos hyperplane measure partition problem originates from the work of Avis [2], and was further used in this context by Ramos [15] and others. For a detailed review of the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem see for example [6] and the references therein. Thus, it is natural to make the following conjecture, see also [3, Conj. 1].

Conjecture 1.1. *Let $d \geq 1$, $j \geq 1$ and $k \geq 1$ be integers. If $d \geq \lceil j/k \rceil$, then $(d, j, k) \in \Lambda$.*

The main result of this paper is derived from the so called "join configuration space / test map scheme" and an application of two different relative equivariant obstruction theories of Bredon [7] and tom Dieck [10]. The join scheme was introduced for the first time in [4], while the relative obstruction theory framework for the study of the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem was developed only in [5]. In particular, in the first part of the theorem we give a different proof of the result by Hubard and Karasev [11, Thm. 1], which in the special case $d = k = 2$ and $j = 4$ is due to Barba, Pilz & Schnider [3, Thm. 2.2].

THEOREM 1.2. *Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers. If*
(a) *$dk = j$ and $d = 2^a$ for some integer $a \geq 0$, or*
(b) *$(d - \ell)k + \ell = j$, k is odd, $d = 2^a + \ell$ for some integers $a \geq 1$, and $1 \leq \ell \leq 2^a - 1$,*
then $(d, j, k) \in \Lambda$.

Thus, Theorem 1.2(a) settles Conjecture 1.1 in the case when $dk - j = 2^a - j = 0$, while Theorem 1.2(b) gives the positive difference $dk - j = \ell k - \ell$, where $k \geq 3$ is odd, and consequently does not settle the conjecture in any additional case. The results of Theorem 1.2 were reproved by Crabb [8] in the broader setting of spherical arrangements by intriguing evaluations of pull-backs of twisted Euler classes.

In order to illustrate the results of Theorem 1.2 we fix the parameter $k = 3$ and consider the set $\Lambda[k = 3] := \{(j, d) \in \mathbb{N}^2 : (d, j, 3) \in \Lambda\}$. In Figure 2 we depicted with a black dot for each j the minimal d such that $(j, d) \in \Lambda[k = 3]$ as Conjecture 1.1 claims. We circled the upper bounds for the dimension d obtained from an application of Theorem 1.2(a). In grey we circled the improved upper bounds on d derived from Theorem 1.2(b).

The main result of this paper, stated in Theorem 1.2, is proven in the following steps:
— The problem regarding the existence of a bisection of a collection of measures in $\mathbb{R}^d$ by a k-element affine hyperplane arrangement is connected to the question about the non-existence of specially constructed $\mathfrak{S}_k^{\pm}$-equivariant maps beween spheres $S^{(d+1)k-1} \longrightarrow S^{j+k-2}$, see Section 2.

— The claim of Theorem 1.2 is obtained as a consequence of the non-existence of $\mathfrak{S}_k^{\pm}$-equivariant maps $S^{(d+1)k-1} \longrightarrow S^{j+k-2}$ with some specific properties, as explained in Theorem 2.5. The non-existence of the relevant map is proved via an application of the equivariant relative obstruction theory of tom Dieck, see Section 3.

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2. From a partition problem to a Borsuk–Ulam type problem

In this section we relate the problem of describing the set $\Lambda \in \mathbb{N}^3$ with a topological problem of the Borsuk–Ulam type. For that we develop both the product and join configuration space / test map scheme even we apply only the join scheme. The join scheme can be efficiently used only in combination with the relative equivariant obstruction theory, as demonstrated in [5].

The space of all oriented affine hyperplanes in $\mathbb{R}^d$ can be identified with the sphere $S^d = S(\mathbb{R}^{d+1})$ of unit vectors in $\mathbb{R}^{d+1}$ where the north pole $e_{d+1} := (0, \dots, 0, 1)$ and the south pole $-e_{d+1} = (0, \dots, 0, -1)$ are interpreted as "extra" oriented affine hyperplanes at infinity. To see this place $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ on "height one", that is via the embedding $(x_1, \dots, x_d) \longmapsto (x_1, \dots, x_d, 1)$. Every oriented affine hyperplane $H(u, a) = (H_{u,a}, u, a)$ in $\mathbb{R}^d$ spans the uniquely oriented linear hyperplane $H(w(u,a), 0) = (H_{w(u,a),0}, w(u,a), 0)$ in $\mathbb{R}^{d+1}$. The corresponding unit normal vector $w(u, a)$ determines a point on the sphere S^d. Furthermore, the associated polynomial function $p_{w(u,a),0} \colon \mathbb{R}^{d+1} \longrightarrow \mathbb{R}$, given by $p_{w(u,a),0}(w) := \langle w, w(u,a) \rangle$ for $w \in \mathbb{R}^{d+1}$, restricts on the embedded $\mathbb{R}^d = \mathbb{R}^d \times \{1\}$ to the polynomial function $p_{u,a}$, that is $p_{u,a} = p_{w(u,a),0}|_{\mathbb{R}^d \times \{1\}}$. The $\mathbb{Z}/2$ action on the space of all oriented affine hyperplanes given by the change of orientation translates into the antipodal action on the sphere, $w \longmapsto -w$ for $w \in S^d$.

Following the presentation in [5, Sec. 2] we consider the following configuration spaces that parameterize all k-element affine hyperplane arrangements in $\mathbb{R}^d$:
— the *join configuration space* $X_{d,k} \cong (S^d)^{*k} \cong S(\mathbb{R}^{(d+1) \times k})$ is a sphere of dimension $dk + k - 1$, (recall the homeomorphism between unit spheres of Euclidan spaces $S(E_1 \oplus E_2) \cong S(E_1) * S(E_2)$), and
— the *product configuration space* $Y_{d,k} \cong (S^d)^k$.

Both configuration spaces are equipped with an action of the group of signed permutations $\mathfrak{S}_k^{\pm} = (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$. To define an action on $X_{d,k}$ we recall that its typical element can be presented as formal ordered convex combinations $\lambda_1 w_1 + \cdots + \lambda_k w_k$, where $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$ and $w_i \in S^d$. Now each copy of $\mathbb{Z}/2$ in $(\mathbb{Z}/2)^k \subseteq \mathfrak{S}_k^{\pm}$ acts antipodally on the appropriate sphere S^d, and the symmetric group $\mathfrak{S}_k \subseteq \mathfrak{S}_k^{\pm}$

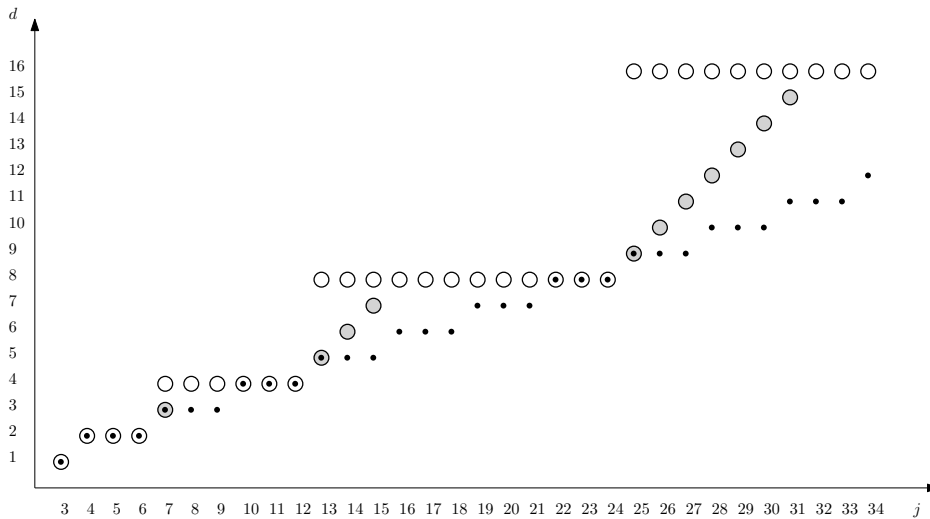FIGURE 2. The shape of the set $\{(j,d) \in \mathbb{N}^2 : (d,j,3) \in \Lambda\}$ as suggested by Conjecture 1.1 (black dot), Theorem 1.2(a) (circled) and Theorem 1.2(b) (circled in grey).

acts by permuting factors in the product. Explicitly, for $((\beta_1, \ldots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ and $\lambda_1 w_1 + \cdots + \lambda_k w_k \in X_{d,k}$ we set:

$$((\beta_1, \ldots, \beta_k) \rtimes \tau) \cdot (\lambda_1 w_1 + \cdots + \lambda_k w_k) :=$$

$$\lambda_{\tau^{-1}(1)}(-1)^{\beta_1} w_{\tau^{-1}(1)} + \cdots + \lambda_{\tau^{-1}(k)}(-1)^{\beta_k} w_{\tau^{-1}(k)}.$$

Alternatively, we can see the join configuration space $X_{d,k}$ as the unit sphere of the real $\mathfrak{S}_k^{\pm}$-representation $\mathbb{R}^{(d+1) \times k}$. The action of $\mathfrak{S}_k^{\pm}$ on $\mathbb{R}^{(d+1) \times k}$ we consider is given by:

$$((\beta_1, \ldots, \beta_k) \rtimes \tau) \cdot (u_1, \ldots, u_k) := \left((-1)^{\beta_1} u_{\tau^{-1}(1)}, \ldots, (-1)^{\beta_k} u_{\tau^{-1}(k)}\right),$$

for $((\beta_1, \ldots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ and $(u_1, \ldots, u_k) \in \mathbb{R}^{(d+1) \times k}$.

The subspace

$$\left\{ \tfrac{1}{k} w_1 + \cdots + \tfrac{1}{k} w_k \in X_{d,k} : (w_1, \ldots, w_k) \in Y_{d,k} \right\}$$

of the join $X_{d,k}$ is homeomorphic to $Y_{d,k}$ and moreover $\mathfrak{S}_k^{\pm}$-invariant. Thus we identify it with $Y_{d,k}$, and the restriction action from $X_{d,k}$ induces an $\mathfrak{S}_k^{\pm}$-action on $Y_{d,k}$. For $k \geq 2$ action of $\mathfrak{S}_k^{\pm}$ on both $X_{d,k}$ and $Y_{d,k}$ is not free. The subspaces of points of $X_{d,k}$ and $Y_{d,k}$ with non-trivial stabilizers with respect to the $\mathfrak{S}_k^{\pm}$-action are

$$X_{d,k}^{>1} := \{\lambda_1 w_1 + \cdots + \lambda_k w_k \in X_{d,k} : \lambda_1 \cdots \lambda_k = 0, \text{ or } \lambda_s = \lambda_r$$
$$\text{with } w_s = \pm w_r \text{ for some } 1 \leq s < r \leq k\},$$

and

$$Y_{d,k}^{>1} := \{(w_1, \ldots, w_k) \in Y_{d,k} : w_s = \pm w_r \text{ for some } 1 \leq s < r \leq k\}.$$

For future use we point out the subspace of $X_{d,k}^{>1}$ given by:

$$(X_{d,k}^{>1})' := \{\lambda_1 w_1 + \cdots + \lambda_k w_k \in X_{d,k}^{>1} : \lambda_1 \cdots \lambda_k = 0\}.$$

Let $V \cong \mathbb{R}$ be the real 1-dimensional $\mathfrak{S}_k^{\pm}$-representation with action defined to be antipodal for every copy of $\mathbb{Z}/2$ in $(\mathbb{Z}/2)^k \subseteq \mathfrak{S}_k^{\pm}$, and trivial for every element of the symmetric group $\mathfrak{S}_k \subseteq \mathfrak{S}_k^{\pm}$. More precisely, when $((\beta_1, \ldots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ and $v \in V$ we have

$$((\beta_1, \ldots, \beta_k) \rtimes \tau) \cdot v := (-1)^{\beta_1} \cdots (-1)^{\beta_k} v.$$

Next consider the vector space $\mathbb{R}^k$ and its vector subspace

$$W_k = \left\{ (y_1, \ldots, y_k) \in \mathbb{R}^k : \sum_{i=1}^{k} y_i = 0 \right\}.$$

The group $\mathfrak{S}_k^{\pm}$ acts on $\mathbb{R}^k$ by permuting coordinates, that is, for $((\beta_1, \ldots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ and $(y_1, \ldots, y_k) \in \mathbb{R}^k$ we get

$$((\beta_1, \ldots, \beta_k) \rtimes \tau) \cdot (y_1, \ldots, y_k) := (y_{\tau^{-1}(1)}, \ldots, y_{\tau^{-1}(k)}). \tag{1}$$

The subspace $W_k \subseteq \mathbb{R}^k$ is $\mathfrak{S}_k^{\pm}$-invariant, and therefore W_k is an $\mathfrak{S}_k^{\pm}$-subrepresentation of $\mathbb{R}^k$.

Now, to an ordered collection $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ of nice measures on $\mathbb{R}^d$ we will associate two continuous $\mathfrak{S}_k^{\pm}$-equivariant maps $\Phi_{\mathcal{M}}$ and $\Psi_{\mathcal{M}}$.

First, we define the continuous map

$$\Phi_{\mathcal{M}} : Y_{d,k} \longrightarrow V^{\oplus j}$$

to be the unique continuous extension of the map $(S^d \setminus \{e_{d+1}, -e_{d+1}\})^k \longrightarrow V^{\oplus j}$ given by

$$\mathcal{H} = (H(u_1, a_1), \ldots, H(u_k, a_k)) = (w(u_1, a_1), \ldots, w(u_k, a_k)) \longmapsto$$

$$\left(\mu_i(\{x \in \mathbb{R}^d : p_{\mathcal{H}}(x) \geq 0\}) - \mu_i(\{x \in \mathbb{R}^d : p_{\mathcal{H}}(x) \leq 0\}) \right)_{i \in \{1, \ldots, j\}}. \tag{2}$$

Indeed, the map (2) is the restriction of the continuous function $(S^d)^k \longrightarrow V^{\oplus j}$ defined by

$$(w_1, \ldots, w_k) \longmapsto \left(\mu_i(\{(x, 1) \in \mathbb{R}^{d+1} : p_{w_1, \ldots, w_k}(x, 1) \geq 0\}) - \right.$$

$$\left. \mu_i(\{(x, 1) \in \mathbb{R}^{d+1} : p_{w_1, \ldots, w_k}(x, 1) \leq 0\}) \right)_{i \in \{1, \ldots, j\}}.$$

Here $p_{w_1, \ldots, w_k} : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}$ is the continuous function $p_{w_1, \ldots, w_k}(w) := \prod_{i=1}^{k} \langle w, w_i \rangle$.

The map $\Phi_{\mathcal{M}}$ is $\mathfrak{S}_k^{\pm}$-equivariant with respect to the already introduced actions on $Y_{d,k}$ and V, assuming the diagonal action on $V^{\oplus j}$.

The key property of the map $\Phi_{\mathcal{M}}$ is that *the k-element affine hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^d$ bisects all the measures from the collection $\mathcal{M}$ if and only if $\Phi_{\mathcal{M}}(\mathcal{H}) = 0 \in V^{\oplus j}$.*

The second continuous map we consider

$$\Psi_{\mathcal{M}} \colon X_{d,k} \longrightarrow W_k \oplus V^{\oplus j}$$

is defined as follows:

$$\lambda_1 w_1 + \cdots + \lambda_k w_k \longmapsto \left(\lambda_1 - \tfrac{1}{k}, \ldots, \lambda_k - \tfrac{1}{k}\right) \oplus \left((\lambda_1 \cdots \lambda_k) \cdot \Phi_{\mathcal{M}}(w_1, \ldots, w_k) \right). \tag{3}$$

It is important to notice that the map we have just defined $\Psi_{\mathcal{M}}$ does not depend on the collection $\mathcal{M}$ when considered on the subspace $(X_{d,k}^{>1})'$. Indeed, if

$$\lambda_1 w_1 + \cdots + \lambda_k w_k \in (X_{d,k}^{>1})',$$

then

$$\Psi_{\mathcal{M}}(\lambda_1 w_1 + \cdots + \lambda_k w_k) = \left(\lambda_1 - \tfrac{1}{k}, \ldots, \lambda_k - \tfrac{1}{k}\right) \oplus 0 \in W_k \oplus V^{\oplus j}.$$

The map $\Psi_{\mathcal{M}}$ is also $\mathfrak{S}_k^{\pm}$-equivariant.

Similarly, *the k-element affine hyperplane arrangement*

$$\mathcal{H} = (H(u_1, a_1), \ldots, H(u_k, a_k)) = (w(u_1, a_1), \ldots, w(u_k, a_k)) = (w_1, \ldots, w_k)$$

in $\mathbb{R}^d$ bisects all the measures from the collection $\mathcal{M}$ if and only if

$$\Psi_{\mathcal{M}}\left(\tfrac{1}{k} w_1 + \cdots + \tfrac{1}{k} w_k\right) = 0 \oplus 0 \in W_k \oplus V^{\oplus j}.$$

From the construction of the $\mathfrak{S}_k^{\pm}$-equivariant maps $\Phi_{\mathcal{M}}$ and $\Psi_{\mathcal{M}}$ we have deduced the following facts; for a similar construction consult [5, Prop. 2.1].

Proposition 2.1. *Let $d \geq 1$, $j \geq 1$ and $k \geq 1$ be integers.*
  (a) *Let $\mathcal{M}$ be a collection of j nice measures on $\mathbb{R}^d$, and let*

$$\Phi_{\mathcal{M}} \colon Y_{d,k} \longrightarrow V^{\oplus j} \quad and \quad \Psi_{\mathcal{M}} \colon X_{d,k} \longrightarrow W_k \oplus V^{\oplus j}$$

  be the $\mathfrak{S}_k^{\pm}$-equivariant maps defined above. If

$$0 \in \operatorname{im} \Phi_{\mathcal{M}} \quad or \quad 0 \in \operatorname{im} \Psi_{\mathcal{M}},$$

  then there exists a k-element affine hyperplane arrangement bisecting all the measures in $\mathcal{M}$.
  (b) *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map of either type*

$$Y_{d,k} \longrightarrow S(V^{\oplus j}) \quad or \quad X_{d,k} \longrightarrow S(W_k \oplus V^{\oplus j}),$$

  then $(d, j, k) \in \Lambda$.

The following essential property of the constructed $\mathfrak{S}_k^{\pm}$-equivariant map $\Psi_{\mathcal{M}}$ needs a modified approach unlike the one used in [5, Prop. 2.2].

Proposition 2.2. *Let $d \geq 2$, $j \geq 1$ and $k \geq 2$ be integers with $j \geq d(k-1) + 2$. Let $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ and $\mathcal{M}' = (\mu_1', \ldots, \mu_j')$ be collections of nice measures on $\mathbb{R}^d$ such that no non-essential k-element affine hyperplane arrangement bisects all of them. Then*
  (a) $0 \notin \operatorname{im} \Psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$,
  (b) $\Psi_{\mathcal{M}}|_{(X_{d,k}^{>1})'} = \Psi_{\mathcal{M}'}|_{(X_{d,k}^{>1})'}$, *and*
  (c) $\Psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ *and* $\Psi_{\mathcal{M}'}|_{X_{d,k}^{>1}}$ *are $\mathfrak{S}_k^{\pm}$-homotopic as maps*

$$X_{d,k}^{>1} \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$$

  which restrict on the subspace $(X_{d,k}^{>1})'$ to the map given by

$$\lambda_1 w_1 + \cdots + \lambda_k w_k \longmapsto \left(\lambda_1 - \tfrac{1}{k}, \ldots, \lambda_k - \tfrac{1}{k}\right) \oplus 0,$$

  where $\lambda_1 w_1 + \cdots + \lambda_k w_k \in (X_{d,k}^{>1})'$ and $\left(\lambda_1 - \tfrac{1}{k}, \ldots, \lambda_k - \tfrac{1}{k}\right) \oplus 0 \in (W_k \oplus V^{\oplus j}) \backslash \{0\}$.

The previous proposition is the special, easy to state, $\ell = 0$ case of a stronger statement which works on invariant subcomplexes of $X_{d,k}$, and therefore on $X_{d,k}$ itself; see Proposition 2.3. Hence, we only prove the more general result.

In the upcoming proposition we use the $\mathfrak{S}_k^{\pm}$-CW structure on the join configuration space $X_{d,k}$ developed in [5, Sec. 3], and reviewed in Section 4 of this paper.

Proposition 2.3. *Let $d \geq 2$, $j \geq 1$ and $k \geq 2$ be integers, and let $1 \leq \ell \leq d-1$ be an integer with $(d-\ell)(k-1) + 2 + \ell \leq j$. Let $Z := \mathfrak{S}_k^{\pm} \cdot \overline{\theta}$ denote the full $\mathfrak{S}_k^{\pm}$-orbit of the closure of the cell $\theta := D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1, 2, \ldots, k)$, and let $Z^{>1} := Z \cap X_{d,k}^{>1}$ and $(Z^{>1})' := Z \cap (X_{d,k}^{>1})'$. Furthermore, let $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ and $\mathcal{M}' = (\mu_1', \ldots, \mu_j')$ be collections of nice measures on $\mathbb{R}^d$ such that no k-element affine hyperplane arrangement parameterized by $Z^{>1}$ bisects them. Then*
  (a) $0 \notin \operatorname{im} \Psi_{\mathcal{M}}|_{Z^{>1}}$, *and*
  (b) $\Psi_{\mathcal{M}}|_{Z^{>1}}$ *and* $\Psi_{\mathcal{M}'}|_{Z^{>1}}$ *are* $\mathfrak{S}_k^{\pm}$-*homotopic as maps*
$$Z^{>1} \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$$
  which restrict on the subspace $(Z^{>1})'$ *to the map given by*
$$\lambda_1 w_1 + \cdots + \lambda_k w_k \longmapsto \left(\lambda_1 - \tfrac{1}{k}, \ldots, \lambda_k - \tfrac{1}{k}\right) \oplus 0, \tag{4}$$
  where $\lambda_1 w_1 + \cdots + \lambda_k w_k \in (Z^{>1})'$ *and* $\left(\lambda_1 - \tfrac{1}{k}, \ldots, \lambda_k - \tfrac{1}{k}\right) \oplus 0 \in (W_k \oplus V^{\oplus j}) \backslash \{0\}$.

PROOF. The first statement follows directly from the assumption that no k-element affine hyperplane arrangement parameterized by $Z^{>1}$ bisects $\mathcal{M}$.

From the assumption on the collections of measures $\mathcal{M}$ and $\mathcal{M}'$ and the first part of the proposition we have that $0 \notin \operatorname{im} \Psi_{\mathcal{M}}|_{Z^{>1}}$ and $0 \notin \operatorname{im} \Psi_{\mathcal{M}'}|_{Z^{>1}}$. Consequently the maps $\Psi_{\mathcal{M}}|_{Z^{>1}}$ and $\Psi_{\mathcal{M}'}|_{Z^{>1}}$ can be considered as $\mathfrak{S}_k^{\pm}$-equivariant maps $Z^{>1} \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$. Furthermore, from the definition of the test map (3) follows that the maps $\Psi_{\mathcal{M}}|_{(Z^{>1})'} = \Psi_{\mathcal{M}'}|_{(Z^{>1})'}$ coincide with the map (4).

In order to prove the second statement we need to construct an $\mathfrak{S}_k^{\pm}$-equivariant homotopy
$$F \colon Z^{>1} \times I \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$$
between the maps $\Psi_{\mathcal{M}}|_{Z^{>1}}$ and $\Psi_{\mathcal{M}'}|_{Z^{>1}}$. Here I denotes the unit interval $[0, 1]$. This will be done using a slight extension of the equivariant obstruction theory of Bredon [7, Ch. II] as presented in [13, Ch. I.5] because the obstruction theory of tom Dieck [10, Sec. II.3] cannot be used in this situation. Indeed, notice that no point in $Z^{>1}$ has a trivial stabilizer.

For simplicity, we denote by
$$K := Z^{>1} \times I \qquad \text{and} \qquad L := Z^{>1} \times \{0\} \cup Z^{>1} \times \{1\} \cup (Z^{>1})' \times I.$$
Define $F_{-1} \colon L \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$ by
$$
\begin{aligned}
F_{-1}|_{Z^{>1} \times \{0\}} &:= \Psi_{\mathcal{M}}|_{Z^{>1}}, \\
F_{-1}|_{Z^{>1} \times \{1\}} &:= \Psi_{\mathcal{M}'}|_{Z^{>1}}, \\
F|_{(Z^{>1})' \times \{t\}} &= \Psi_{\mathcal{M}}|_{(Z^{>1})'} = \Psi_{\mathcal{M}'}|_{(Z^{>1})'}, \quad \text{for all } t \in I.
\end{aligned}
$$
Our aim is to extend the $\mathfrak{S}_k^{\pm}$-equivariant map F_{-1} to an $\mathfrak{S}_k^{\pm}$-equivariant map $F \colon K \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$ extending it one skeleton at a time.

Since $(W_k \oplus V^{\oplus j}) \backslash \{0\}$ is non-empty, and in addition, for every subgroup G of $\mathfrak{S}_k^{\pm}$, the following implication holds:
$$(K \backslash L)^G \neq \emptyset \implies ((W_k \oplus V^{\oplus j}) \backslash \{0\})^G \neq \emptyset$$
we can extend F_{-1} to the 0-skeleton of K obtaining an $\mathfrak{S}_k^{\pm}$-equivariant map F_0. Assume that we have defined a $\mathfrak{S}_k^{\pm}$-equivariant map F_r on the r-th skeleton of K.

The obstructions for the extension of the map F_r to the next skeleton live in the Bredon cohomology group
$$H^{r+1}_{\mathfrak{S}_k^{\pm}}\big(K, L; \widetilde{\omega}_r((W_k \oplus V^{\oplus j}) \backslash \{0\})\big),$$
where $0 \leq r \leq \dim K - 1 = \dim Z^{>1}$. Here $\widetilde{\omega}_r((W_k \oplus V^{\oplus j}) \backslash \{0\}) \colon \mathcal{O}_{\mathfrak{S}_k^{\pm}} \longrightarrow \mathrm{Ab}$ denotes the generic coefficient system. That is a contravariant functor from the category of canonical objects $\mathcal{O}_{\mathfrak{S}_k^{\pm}}$ of the group $\mathfrak{S}_k^{\pm}$ associated to the pair (K, L) into the category of Abelian groups given on objects by
$$\widetilde{\omega}_r((W_k \oplus V^{\oplus j}) \backslash \{0\})(\mathfrak{S}_k^{\pm}/G) = \pi_r(((W_k \oplus V^{\oplus j}) \backslash \{0\})^G, y_0^G),$$

where $(K \backslash L)^G \neq \emptyset$. Here, for every subgroup G of $\mathfrak{S}_k^{\pm}$ with the property that $((W_k \oplus V^{\oplus j}) \backslash \{0\})^G \neq 0$, we chose a G-fixed base point y_0^G in $((W_k \oplus V^{\oplus j}) \backslash \{0\})^G$ such that for every subgroup H_1 contained in the conjugacy class gH_2g^{-1} of another subgroup H_2 holds $gy_0^{H_2} = y_0^{H_1}$. Such a choice can be made beacuse $\mathfrak{S}_k^{\pm}$ is finite. For a detailed account of all relevant notions see [13, Ch. I.5] [7, Ch. I.4].

Let $D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma) \times I$ be an arbitrary $r+1$ cell of $K \setminus L$. The cocycle corresponding to $D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma) \times I$ will have coefficients in $\pi_r(((W_k \oplus V^{\oplus j}) \backslash \{0\})^G, y_0^G)$, where $G \subseteq \mathfrak{S}_k^{\pm}$ is the stabilizer group of the cell $D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma) \times I$.

As $D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma) \times I$ lies in $K \setminus L$, the first $1 \leq i_1 \leq d+1$ and second $k-1$ of the positive integers $i_1, \ldots, i_k$ are less than or equal to $1+\ell$. Furthermore, as G is the stabilizing group of the cell $D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma)$ and $i_1 \leq d+1$, G is not a subgroup of the defining subgroup $(\mathbb{Z}/2)^k$ of the group $\mathfrak{S}_k^{\pm} = (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$. Let $(\beta_1, \ldots, \beta_k) \rtimes \tau \in \mathfrak{S}_k^{\pm}$ be an element that fixes the cell $D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma)$, that is

$$(\beta_1, \ldots, \beta_k) \rtimes \tau \cdot D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma) = D_{i_1,\ldots,i_k}^{(-1)^{\beta_1} s_1,\ldots,(-1)^{\beta_k} s_k}(\tau\sigma) = D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma).$$

Consequently, we have that
— $(-1)^{\beta_q} s_q = s_{\tau^{-1}(q)}$ for all $1 \leq q \leq k$, and
— $i_r = d+2$ for each $\tau(q) < r \leq q$ resp. $q < r \leq \tau(q)$ for all $1 \leq q \leq k$ with $\tau(q) \neq q$.
In particular, $(-1)^{\beta_1} \cdots (-1)^{\beta_k} = 1$ and so $(V^{\oplus j})^G = V^{\oplus j}$.

Next, the dimension of the cell $D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma)$ can be estimated as follows. Let us first introduce $z = z_{(i_1,\ldots,i_k)} := \#\{r : 1 \leq r \leq k \text{ and } i_r = d+2\}$. Notice that $1 \leq z \leq k-1$. Then

$$r = \dim D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma) = (d+1)k - 1 - \sum_{q=1}^{k}(i_q - 1)$$

$$\leq (d+1)k - 1 - \ell(k-1) - z(d+1-\ell).$$

On the other hand,

$$\dim(W_k \oplus V^{\oplus j})^G = k - 1 - z + j.$$

From the assumptions that $1 \leq \ell \leq d-1$ and $(d-\ell)(k-1) + 2 + \ell \leq j$ we get that

$$r + 1 = \dim(D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma) \times I) \leq \dim S((W_k \oplus V^{\oplus j})^G). \tag{5}$$

This conclusion follows from a direct verification of the inequality

$$(d+1)k - \ell(k-1) - z(d+1-\ell) \leq k - 2 - z + j,$$

or more precisely the inequality

$$(d-\ell)(k-1) + 2 + \ell \geq (d+1)k - \ell(k-1) - z(d+1-\ell) - k + 2 + z.$$

Now, the relevant generic coefficient system $\widetilde{\omega}_r((W_k \oplus V^{\oplus j}) \backslash \{0\})(\mathfrak{S}_k^{\pm}/G)$ vanishes. Indeed, from inequality (5) it follows that

$$\widetilde{\omega}_r((W_k \oplus V^{\oplus j}) \backslash \{0\})(\mathfrak{S}_k^{\pm}/G) = \pi_r(((W_k \oplus V^{\oplus j}) \backslash \{0\})^G, y_0^G)$$

$$\cong \pi_r(S((W_k \oplus V^{\oplus j}) \backslash \{0\})^G, \tfrac{1}{\|y_0^G\|} y_0^G) \overset{(5)}{=} 0.$$

Thus, the Bredon cohomology group $H_{\mathfrak{S}_k^{\pm}}^{r+1}(K, L; \widetilde{\omega}_r((W_k \oplus V^{\oplus j}) \backslash \{0\}))$ also vanishes. Consequently all obstructions, in all dimensions $r+1 \leq \dim Z^{>1}$, vanish. Thus, the $\mathfrak{S}_k^{\pm}$-equivariant map $F_{-1} \colon L \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$ extends to an $\mathfrak{S}_k^{\pm}$-equivariant map $F \colon Z^{>1} \times I \longrightarrow (W_k \oplus V^{\oplus j}) \backslash \{0\}$, that is to an $\mathfrak{S}_k^{\pm}$-homotopy between the maps $\Psi_{\mathcal{M}}|_{Z^{>1}}$ and $\Psi_{\mathcal{M}'}|_{Z^{>1}}$. $\square$

Remark 2.4. In general, for a finite group G the category of canonical object $\mathcal{O}_G$ of G consists of all sets of left cosets G/H as objects, where H is a subgroup of G, and all G-equivariant maps $G/H_1 \longrightarrow G/H_2$ between them as morphisms. Here the action of G on the objects is assumed to be induced by the left translations; see [7, Ch. I.3]. A generic coefficient system of group G is any contravariant functor $\omega \colon \mathcal{O}_G \longrightarrow \mathrm{Ab}$ from the category of canonical object into the category of abelian groups; for more details consult [7, Ch. I.4].

Now we combine the criterion stated in Proposition 2.1 (b) and the observations from Proposition 2.2 and Proposition 2.3 into a theorem. In the following, ν denotes the radial $\mathfrak{S}_k^\pm$-equivariant deformation retraction

$$\nu \colon (W_k \oplus V^{\oplus j})\setminus\{0\} \longrightarrow S(W_k \oplus V^{\oplus j}).$$

THEOREM 2.5.
(a) Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers with $d(k-1)+2 \leq j$, and let $\mathcal{M}$ be any collection of j nice measures on $\mathbb{R}^d$ such that no non-essential k-element affine hyperplane arrangement bisects them. If there is no $\mathfrak{S}_k^\pm$-equivariant map

$$X_{d,k} \longrightarrow S(W_k \oplus V^{\oplus j})$$

whose restriction on $X_{d,k}^{>1}$ is $\mathfrak{S}_k^\pm$-homotopic to $\nu \circ \Psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$, then $(d,j,k) \in \Lambda$.
(b) Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers, and let $0 \leq \ell \leq d-1$ be an integer such that $(d-\ell)(k-1)+2+\ell \leq j$. Set $Z := \mathfrak{S}_k^\pm \cdot \overline{\theta}$ to be the $\mathfrak{S}_k^\pm$-orbit of the closure of the cell $\theta := D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1,2,\ldots,k)$, and $Z^{>1} := Z \cap X_{d,k}^{>1}$. If, for a collection $\mathcal{M}$ of j nice measures on $\mathbb{R}^d$ such that $0 \notin \mathrm{im}\,\Psi_{\mathcal{M}}|_{Z^{>1}}$, there is no $\mathfrak{S}_k^\pm$-equivariant map

$$Z \longrightarrow S(W_k \oplus V^{\oplus j})$$

whose restriction on $Z^{>1}$ is $\mathfrak{S}_k^\pm$-homotopic to $\nu \circ \Psi_{\mathcal{M}}|_{Z^{>1}}$, then $(d,j,k) \in \Lambda$.

3. Proof of Theorem 1.2

From this point on we fix an $\mathfrak{S}_k^\pm$-CW structure on the sphere $X_{d,k}$ to be the one introduced and described in [5, Sec. 3] and reviewed in the appendix of this paper, Section 4.

Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers. We want to prove that if one of the conditions (1)-(2) of Theorem 1.2 is satisfied, then for every collection $\mathcal{M}$ of j nice measures in $\mathbb{R}^d$ there exists a k-element affine hyperplane arrangement in $\mathbb{R}^d$ that bisects each of the measures. For this, according to Theorem 2.5(a), in case $j = dk$ it suffices to prove that there is no $\mathfrak{S}_k^\pm$-equivariant map

$$X_{d,k} \longrightarrow S(W_k \oplus V^{\oplus j}),$$

whose restriction on $X_{d,k}^{>1}$ is $\mathfrak{S}_k^\pm$-homotopic to $\nu \circ \Psi_{\mathcal{M}_0}|_{X_{d,k}^{>1}}$, where $\mathcal{M}_0$ is a fixed collection of j nice measures on $\mathbb{R}^d$ such that no non-essential k-element affine hyperplane arrangement bisects them.

Alternatively, according to Theorem 2.5(b), we may consider $Z := \mathfrak{S}_k^\pm \cdot \overline{\theta}$ to be the full $\mathfrak{S}_k^\pm$-orbit of the closure of the cell $\theta := D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1,2,\ldots,k)$ for some $0 \leq \ell \leq d$ such that $(d-\ell)(k-1)+2+\ell \leq j$. If $j = k(d-\ell)+\ell$ this is indeed satisfied. As before, set $Z^{>1} := Z \cap X_{d,k}^{>1}$. Then it suffices to prove that there is no $\mathfrak{S}_k^\pm$-equivariant map

$$Z \longrightarrow S(W_k \oplus V^{\oplus j}),$$

whose restriction on $Z^{>1}$ is $\mathfrak{S}_k^\pm$-homotopic to $\nu \circ \Psi_{\mathcal{M}_0}|_{Z^{>1}}$, where $\mathcal{M}_0$ is a fixed collection of j nice measures on $\mathbb{R}^d$ such that no k-element affine hyperplane arrangement parameterized by $Z^{>1}$ bisects them.

Therefore, our proof of Theorem 1.2 follows directly from the following two propositions. The first proposition gives divisibility criterions for the nonexistence of an $\mathfrak{S}_k^\pm$-equivariant map $Z \longrightarrow S(W_k \oplus V^{\oplus j})$ with required properties.

Proposition 3.1. Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers.
(a) If $dk = j$, and $\frac{1}{k!}\binom{dk}{d,\ldots,d}$ is odd, and $Z := X_{d,k}$, $Z^{>1} := X_{d,k}^{>1}$, or
(b) if there exists an integer ℓ such that $1 \leq \ell \leq d-1$, $(d-\ell)k+\ell = j$ and $\binom{(d-\ell)k+\ell}{d}\frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell}$ is odd, and $Z := \mathfrak{S}_k^\pm \cdot \theta$, $Z^{>1} := Z \cap X_{d,k}^{>1}$ where $\theta := D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1,2,\ldots,k)$,

then there is no $\mathfrak{S}_k^\pm$-equivariant map

$$Z \longrightarrow S(W_k \oplus V^{\oplus j}), \tag{6}$$

whose restriction to $Z^{>1}$ is $\mathfrak{S}_k^\pm$-homotopic to $\nu \circ \Psi_{\mathcal{M}_0}|_{Z^{>1}}$, where $\mathcal{M}_0$ is some fixed collection of j nice measures on $\mathbb{R}^d$ such that that no k-element affine hyperplane arrangement parameterized by $Z^{>1}$ bisects them.

The proof of Proposition 3.1(a) will actually give us more, since by construction of $Z \subseteq X_{d,k}$ the existence of the $\mathfrak{S}_k^\pm$-equivariant map (6) depends only on the primary obstruction. In the case when $Z = X_{d,k}$ we have the following equivalence.

Corollary 3.2. *Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers, and let $dk = j$. Then $\frac{1}{k!}\binom{dk}{d,\ldots,d}$ is even if and only if there exists an $\mathfrak{S}_k^{\pm}$-equivariant map*

$$X_{d,k} \longrightarrow S(W_k \oplus V^{\oplus j})$$

whose restriction on $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \Psi_{\mathcal{M}_0}|_{X_{d,k}^{>1}}$, where $\mathcal{M}_0$ is a certain fixed collection of j nice measures on $\mathbb{R}^d$ such that no non-essential k-element affine hyperplane arrangement bisects them.

The second proposition shows when the divisibility criterions of Proposition 3.1 are satisfied. More precisely, it shows that the assumptions of Theorem 1.2 are equivalent to the divisibility criterions in Proposition 3.1. The first case of the proposition is the content of [11, Lem. 5]. In the second case we restrict ℓ to $2 \leq 2\ell \leq d - 1$ as the case $2\ell \geq d$ does not yield any new bound.

Proposition 3.3. *Let $d \geq 1$, $j \geq 1$ and $k \geq 2$ be integers.*
  (a) *Let $dk = j$. Then $\frac{1}{k!}\binom{dk}{d,\ldots,d}$ is odd if and only if $d = 2^a$ for some integer $a \geq 0$.*
  (b) *Let $(d-\ell)k + \ell = j$ and $2 \leq 2\ell \leq d - 1$. Then $\binom{(d-\ell)k+\ell}{d} \cdot \frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell}$ is odd if and only if k is odd and $d = 2^a + \ell$ for some integer $a \geq 1$.*

The first part of the proposition is the content of [11, Lem. 5]. In the second part of the proposition we restrict to such ℓ where $2 \leq 2\ell \leq d - 1$ as the case $2\ell \geq d$ does not yield any new bounds.

Now, using Theorem 2.5, Propositions 3.1 and Proposition 3.3 we give a proof of the main result of our paper.

PROOF OF THEOREM 1.2. From Theorem 2.5, the claim of Theorem 1.2 holds if we are able to prove that there is no $\mathfrak{S}_k^{\pm}$-equivariant map $Z \longrightarrow S(W_k \oplus V^{\oplus j})$, whose restriction on $Z^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \Psi_{\mathcal{M}_0}|_{Z^{>1}}$, where $\mathcal{M}_0$ is a fixed collection of j nice measures on $\mathbb{R}^d$ such that no k-element affine hyperplane arrangement parameterized by $Z^{>1}$ bisects them. Here $Z := \mathfrak{S}_k^{\pm} \cdot \theta$, $Z^{>1} := Z \cap X_{d,k}^{>1}$ and $\theta := D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1, 2, \ldots, k)$.

An $\mathfrak{S}_k^{\pm}$-equivariant map $Z \longrightarrow S(W_k \oplus V^{\oplus j})$ with described properties, according to Proposition 3.1, does not exist if either,
  — $dk = j$ and $\frac{1}{k!}\binom{dk}{d,\ldots,d}$ is odd, or
  — there exists an integer ℓ such that $1 \leq \ell \leq d - 1$, $(d-\ell)k + \ell = j$ and $\binom{(d-\ell)k+\ell}{d}\frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell}$ is odd.
Next, Proposition 3.3 implies that when:
  — $d = 2^a$ for some integer $a \geq 0$ and $dk = j$, the number $\frac{1}{k!}\binom{dk}{d,\ldots,d}$ is odd, and for
  — $(d-\ell)k + \ell = j$, $2 \leq 2\ell \leq d - 1$, $d = 2^a + \ell$ for some integer $a \geq 1$, the number $\binom{(d-\ell)k+\ell}{d} \cdot \frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell}$ is odd.
Thus we concluded the proof of Theorem 1.2. $\square$

In the next two parts of this section we verify both ingredients of the proof of Theorem 1.2, that is, we prove Proposition 3.1 and Proposition 3.3.

3.1. Proof of Proposition 3.1 and Corollary 3.2. In order to answer the question about the existence of the equivariant map (6) we use the relative equivariant obstruction theory of tom Dieck on an $\mathfrak{S}_k^{\pm}$-invariant subcomplex Z of the sphere $X_{d,k}$ with respect to the group of signed permutations $\mathfrak{S}_k^{\pm}$. For that we follow [5, Sec. 2.6 and Sec. 4] and use the $\mathfrak{S}_k^{\pm}$-CW structure on $(X_{d,k}, X_{d,k}^{>1})$ introduced in [5, Sec. 3] and presented in Section 4. A concise presentation of the relevant equivariant obstruction theory can be found in [10, Sec. II.3].

The study of the existence of the equivariant map (6) is done in three separate steps. First, in Section 3.1.1, we check all the relevant assumptions needed for an application of relative obstruction theory. Furthermore, we identify what is the first obstruction which needs to be calculated and what is the ambient group where this obstruction lives. In the second step, Section 3.1.2, we explain how the obstruction cocycle will be computed using the the binomial moment curve (8) and give the formula (9) for the evaluation of the cocycle on a cell of a corresponding $\mathfrak{S}_k^{\pm}$-CW complex. The third step, proof of the (non-)vanishing of the cohomology class of the obstruction cocycle, is presented in Section 3.1.3 for the case when the primary obstruction is the only obstruction for the existence of the equivariant map (6), and in Section 3.1.4 for the case when there are more obstructions.

3.1.1. *Setting up the obstruction theory.* We consider the problem of the existence of an $\mathfrak{S}_k^{\pm}$-equivariant map

$$Z \longrightarrow S(W_k \oplus V^{\oplus j}), \tag{7}$$

whose restriction to the subcomplex $Z^{>1} := Z \cap X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to the map $\nu \circ \Psi_{\mathcal{M}_0}|_{Z^{>1}}$, where $\mathcal{M}_0$ is a some fixed collection of j nice measures on $\mathbb{R}^d$ such that no k-element affine hyperplane arrangement parameterized by $Z^{>1}$ bisects them.

Let us denote the dimensions of Z and of the sphere $S(W_k \oplus V^{\oplus j})$ as follows

$$M := \dim Z \qquad \text{and} \qquad N := \dim(S(W_k \oplus V^{\oplus j})) = j + k - 2.$$

In the case 3.1(a) we have

$$M := \dim Z = \dim X_{d,k} = \dim D_{1,\ldots,1,1}^{+,\ldots,+,+}(1, 2, \ldots, k) =$$
$$(d+1)k - 1 = j + k - 1 = N + 1$$

because $j = dk$. In the case 3.1(b) we have $j = (d-\ell)k + \ell$ and consequently

$$M := \dim Z = \dim D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+}(1, 2, \ldots, k) =$$
$$(d+1)k - 1 - \ell(k-1) = j + k - 1 = N + 1.$$

In order to apply relative equivariant obstruction theory, as presented by tom Dieck in [10, Sec. II.3], the following requirements need to be satisfied:

— Z is equipped with the structure of a relative $\mathfrak{S}_k^{\pm}$-CW complex $(Z, Z^{>1})$. This is obtained from the relative $\mathfrak{S}_k^{\pm}$-CW structure of $(X_{d,k}, X_{d,k}^{>1})$, as demonstrated in [5, Sec. 3]; see Section 4.

— The N-sphere $S(W_k \oplus V^{\oplus j})$ is path connected and N-simple; for a definition consult for example [1, Def. 5.5.7]. Indeed, we have that $N \geq 1$, and consequently $\pi_N(S(W_k \oplus V^{\oplus j})) \cong \mathbb{Z}$ is abelian for $N = 1$, while $\pi_N(S(W_k \oplus V^{\oplus j})) = 0$ when $N \geq 2$.

— The collection of nice measures $\mathcal{M}_0$ induces the $\mathfrak{S}_k^{\pm}$-equivariant map

$$h \colon Z^{>1} \longrightarrow S(W_k \oplus V^{\oplus j}), \qquad h := \nu \circ \Psi_{\mathcal{M}_0}|_{Z^{>1}},$$

which we want to extend.

The N-sphere $S(W_k \oplus V^{\oplus j})$ is $(N-1)$-connected. Hence, the fixed $\mathfrak{S}_k^{\pm}$-equivariant map $h \colon Z^{>1} \longrightarrow S(W_k \oplus V^{\oplus j})$ can be extended to an $\mathfrak{S}_k^{\pm}$-equivariant map

$$g \colon \mathrm{sk}_N(Z) \cup Z^{>1} \longrightarrow S(W_k \oplus V^{\oplus j}),$$

where $\mathrm{sk}_N(Z)$ denotes the Nth skeleton of Z. Since we have that $M = N + 1$, we now try to extend the map g to the next, final, $(N+1)$th skeleton of Z. The extension of the map g is obstructed by the equivariant cocycle

$$\mathfrak{o}(g) \in \mathcal{C}_{\mathfrak{S}_k^{\pm}}^{N+1}\big(Z, Z^{>1} \, ; \, \pi_N(S(W_k \oplus V^{\oplus j}))\big),$$

while the extension of the map $g|_{\mathrm{sk}_{N-1}(Z) \cup Z^{>1}}$ is obstructed by the cohomology class

$$[\mathfrak{o}(g)] \in \mathcal{H}_{\mathfrak{S}_k^{\pm}}^{N+1}\big(Z, Z^{>1} \, ; \, \pi_N(S(W_k \oplus V^{\oplus j}))\big).$$

The cocycle $\mathfrak{o}(g)$ and the cohomology class $[\mathfrak{o}(g)]$ are called the obstruction cocycle and respectively the obstruction element associated to the map g. Now, the central theorem [10, Thm. II.3.10] tells us that:

— The $\mathfrak{S}_k^{\pm}$-equivariant map $g \colon \mathrm{sk}_N(Z) \cup Z^{>1} \longrightarrow S(W_k \oplus V^{\oplus j})$ extends to the next skeleton $\mathrm{sk}_{N+1}(Z) \cup Z^{>1} = Z$ if and only if the obstruction cocycle vanishes, that is $\mathfrak{o}(g) = 0$.

— The restriction $\mathfrak{S}_k^{\pm}$-equivariant map

$$g|_{\mathrm{sk}_{N-1}(Z) \cup Z^{>1}} \colon \mathrm{sk}_{N-1}(Z) \cup Z^{>1} \longrightarrow S(W_k \oplus V^{\oplus j})$$

extends to $\mathrm{sk}_{N+1}(Z) \cup Z^{>1} = Z$ if and only if the obstruction element vanishes, that is $[\mathfrak{o}(g)] = 0$. Furthermore, since

$$\dim(\mathrm{sk}_N(Z) \cup Z^{>1}) - \dim(S(W_k \oplus V^{\oplus j})) = 1$$

and

$$\mathrm{conn}(S(W_k \oplus V^{\oplus j})) = N - 1$$

according to [10, Prop. II.3.15] any two $\mathfrak{S}_k^{\pm}$-equivariant maps

$$g', g'' \colon \mathrm{sk}_N(Z) \cup Z^{>1} \longrightarrow S(W_k \oplus V^{\oplus j})$$

define cohomologous obstruction cocycles $\mathfrak{o}(g')$ and $\mathfrak{o}(g'')$, or in other words the induced obstruction elements coincide: $[\mathfrak{o}(g')] = [\mathfrak{o}(g'')]$. Thus, *it is enough to compute the obstruction element* $[\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}})]$ *associated to the map* $\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}}$ *induced by a fixed collection of j nice measures* $\mathcal{M}_0$, *which have the property that no k-element affine hyperplane arrangement parameterized by the subcomplex* $Z^{>1}$ *bisects them, or in other words for which* $0 \notin \mathrm{im}(\Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z)})$.

3.1.2. *Evaluation of the obstruction cocycle* $\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}})$. With the fixed cellular structure we assume that an orientation on each cell of the $\mathfrak{S}_k^{\pm}$-CW complex Z is chosen. Furthermore, we choose an orientation on the sphere $S(W_k \oplus V^{\oplus j})$.

Let θ be an arbitrary $(N+1)$-dimensional cell of Z, $f_\theta \colon E^{N+1} \longrightarrow Z$ be the associated characteristic map, and let e_θ denote the corresponding basis element in the cellular chain group $C_{N+1}(Z, Z^{>1})$. Here E^{N+1} denotes the $(N+1)$-dimensional ball. Then by the geometric definition of the obstruction cocycle associated to the map $\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}}$ we have that

$$\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}})(e_\theta) = [\nu \circ \Psi_{\mathcal{M}_0} \circ f_\theta|_{\partial\theta}] \in \pi_N(S(W_k \oplus V^{\oplus j})).$$

For more details of the geometric definition of the obstruction cocycle consult for example [9, Sec. 7.3]. The spheres $\partial\theta$ and $S(W_k \oplus V^{\oplus j})$ have the same dimension and therefore the homotopy class $[\nu \circ \Psi_{\mathcal{M}_0} \circ f_\theta|_{\partial\theta}]$ is completely determined by the degree of the map

$$\partial\theta \xrightarrow{\ f_\theta|_{\partial\theta}\ } \mathrm{sk}_N(Z) \cup Z^{>1} \xrightarrow{\ \nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}}\ } S(W_k \oplus V^{\oplus j}).$$

Recall that the orientations on $\partial\theta$ and $S(W_k \oplus V^{\oplus j})$ are already fixed and so the degree is well defined. For simplicity, let $\kappa := \nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}} \circ f_\theta|_{\partial\theta}$.

Now we want to evaluate degree of the map $\kappa \colon \partial\theta \longrightarrow S(W_k \oplus V^{\oplus j})$. For that we fix $\mathcal{M}_0$ to be the collection of nice measures $(\mu_1, \ldots, \mu_j)$ where μ_r is the measure concentrated on the segment $I_r := \gamma([t_{r1}, t_{r2}])$ of the binomial moment curve in $\mathbb{R}^d$

$$\gamma(t) = \left(\binom{t}{1}, \binom{t}{2}, \binom{t}{3}, \ldots, \binom{t}{d} \right)^T, \tag{8}$$

where

$$\ell < t_{11} < t_{12} < t_{21} < t_{22} < \cdots < t_{j1} < t_{j2}.$$

(Here A^T stands for the transposition of the matrix A.) In the case $Z = X_{d,k}$ we take $\ell = 0$. The intervals $(I_1, \ldots, I_j)$ determined by $t_{r1} < t_{r2}$ can be chosen in such a way that $0 \notin \mathrm{im}(\Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}})$. This requirement will be directly verified for every concrete situation in the next section.

The binomial moment curve is used because the cell $\theta = D_{\ell+1,1,1,\ldots,1}^{+,+,+,\ldots,+}(1,2,3,\ldots,k)$ parametrizes all arrangements $\mathcal{H} = (H_1, \ldots, H_k)$ of k linear hyperplanes in $\mathbb{R}^{d+1}$, where the order and orientation are fixed appropriately, such that
— $\{(1, \gamma(0)), \ldots, (1, \gamma(\ell-1))\} \subseteq H_1$,
— $(1, \gamma(\ell)) \notin H_1$,
— $(1, \gamma(0)) \notin H_2, \ldots, (1, \gamma(0)) \notin H_k$, and
— $H_2, \ldots, H_k$ have unit normal vectors $x_2, \ldots, x_k$ with distinct (positive) first coordinates, that is,

$$\left| \{ \langle x_2, (1, \gamma(0)) \rangle, \langle x_3, (1, \gamma(0)) \rangle, \ldots, \langle x_k, (1, \gamma(0)) \rangle \} \right| = k - 1.$$

For the complete account of these facts see [5, Sec. 3.4].

Next, consider the commutative diagram:

$$
\begin{array}{ccccccc}
\partial\theta & \xrightarrow{\ f_\theta|_{\partial\theta}\ } & \mathrm{sk}_N(Z) \cup Z^{>1} & \xrightarrow{\ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}}\ } & (W_k \oplus V^{\oplus j}) \backslash \{0\} & \xrightarrow{\ \nu\ } & S(W_k \oplus V^{\oplus j}) \\
\downarrow & & \downarrow & & \downarrow & & \\
\theta & \xrightarrow{\ f_\theta\ } & Z & \xrightarrow{\ \Psi_{\mathcal{M}_0}|_Z\ } & W_k \oplus V^{\oplus j}. & &
\end{array}
$$

Here the vertical arrows are inclusions, and the composition of the lower horizontal maps is denoted by $\widehat{\kappa} := \Psi_{\mathcal{M}_0}|_Z \circ f_\theta$. Now, let $E_\varepsilon(0)$ denote the ball with center 0 in the $\mathfrak{S}_k^{\pm}$-representation $W_k \oplus V^{\oplus j}$ of, a sufficiently small, radius $\varepsilon > 0$. Furthermore, let $\widetilde{\theta} := \theta \backslash \widehat{\kappa}^{-1}(E_\varepsilon(0))$. Because of the equality of dimensions $\dim(\theta) = \dim(W_k \oplus V^{\oplus j})$ we can assume that the set of zeros $\widehat{\kappa}^{-1}(0) \subseteq \mathrm{relint}(\theta)$ is finite, say

of cardinality $z \geq 0$. Again finiteness of set of zeroes of the function $\widehat{\kappa}$ is checked in every concrete case independently.

The function $\widehat{\kappa}$ is a restriction of the function $\Psi_{\mathcal{M}_0}$ and therefore *the points in $\widehat{\kappa}^{-1}(0)$ correspond to the k-element affine hyperplane arrangements in* $\operatorname{relint} \theta$ *which bisect $\mathcal{M}_0$*. From the fact that:
— the measures in $\mathcal{M}_0$ are disjoint intervals on a moment curve (8), and that
— each hyperplane cuts the moment curve in at most d distinct points,

it follows that each zero in $\widehat{\kappa}^{-1}(0)$ is isolated and transversal. The boundary of $\widetilde{\theta}$ is composed of the boundary of the cell $\partial \theta$ and in addition z disjoint copies of N-spheres $S_1, \ldots, S_z$, one for each zero of $\widehat{\kappa}$, which are contained in the relative interior of the cell θ. Therefore, the fundamental class of the sphere $\partial \theta$ is equal to the sum (up to a sign) of fundamental classes $\sum [S_i]$ in $H_N(\widetilde{\theta}; \mathbb{Z})$. Keep in mind that the fundamental class of $\partial \theta$ is determined by the cell orientation inherited from the $\mathfrak{S}_k^{\pm}$-CW structure on Z, which we already fixed. Now we define orientation on the spheres $S_1, \ldots, S_z$ in such a way that equality $[\partial \theta] = \sum [S_i]$ is valid. Consequently,

$$\sum (\nu \circ \widehat{\kappa}|_{\widetilde{\theta}})_*([S_i]) = (\nu \circ \widehat{\kappa}|_{\widetilde{\theta}})_*([\partial \theta]) = \kappa_*([\partial \theta]) = \deg(\kappa) \cdot [S(W_k \oplus V^{\oplus j})].$$

Rearranging the left hand side of the equality using the family of continuous maps $\nu \circ \widehat{\kappa}|_{S_i} \colon S_i \longrightarrow S(W_k \oplus V^{\oplus j})$ we get that

$$\sum (\nu \circ \widehat{\kappa}|_{\widetilde{\theta}})_*([S_i]) = \sum (\nu \circ \widehat{\kappa}|_{S_i})_*([S_i]) = \left(\sum \deg(\nu \circ \widehat{\kappa}|_{S_i}) \right) \cdot [S(W_k \oplus V^{\oplus j})].$$

Hence,

$$\deg(\kappa) = \sum \deg(\nu \circ \widehat{\kappa}|_{S_i}).$$

where the sum ranges over all k-element affine hyperplane arrangements in $\operatorname{relint}(\theta)$ which bisect $\mathcal{M}_0$. Thus we have obtained that

$$\begin{aligned} \mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\operatorname{sk}_N(Z) \cup Z^{>1}})(e_\theta) &= [\nu \circ \Psi_{\mathcal{M}_0} \circ f_\theta|_{\partial \theta}] \\ &= [\kappa] \\ &= \deg(\kappa) \cdot \zeta \\ &= \sum \deg(\nu \circ \widehat{\kappa}|_{S_i}) \cdot \zeta. \end{aligned} \qquad (9)$$

Here $\zeta \in \pi_N(S(W_k \oplus V^{\oplus j})) \cong H_N(S(W_k \oplus V^{\oplus j}); \mathbb{Z}) \cong \mathbb{Z}$ is the generator determined by the already fixed orientation on the sphere. The sum (9) ranges over all k-element affine hyperplane arrangements in $\operatorname{relint}(\theta)$ that bisect $\mathcal{M}_0$.

3.1.3. *Evaluation of the obstruction element in the case $Z = X_{d,k}$.* In this section we complete the proof of Proposition 3.1(a) and Corollary 3.2.

Recall that $M = N + 1$ and thus $[\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\operatorname{sk}_N(X_{d,k}) \cup X_{d,k}^{>1}})]$ is the primary obstruction element and also the only obstruction for the existence of the map (7). In particular, this means that an $\mathfrak{S}_k^{\pm}$-equivariant map $X_{d,k} \longrightarrow S(W_k \oplus V^{\oplus j})$, whose restriction on $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \Psi_{\mathcal{M}_0}|_{X_{d,k}^{>1}}$, exists if and only if $[\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\operatorname{sk}_N(X_{d,k}) \cup X_{d,k}^{>1}})] = 0$. We will prove that

$$[\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\operatorname{sk}_N(X_{d,k}) \cup X_{d,k}^{>1}})] = 0 \qquad \Longleftrightarrow \qquad \frac{1}{k!} \binom{dk}{d, \ldots, d} \text{ is even.} \qquad (10)$$

This would conclude the proof of Proposition 3.1(a) and Corollary 3.2.

We have to evaluate the cocycle

$$\mathfrak{o} := \mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\operatorname{sk}_N(X_{d,k}) \cup X_{d,k}^{>1}}) \in \mathcal{C}_{\mathfrak{S}_k^{\pm}}^{N+1}\big(X_{d,k}, X_{d,k}^{>1}; \pi_N(S(W_k \oplus V^{\oplus j}))\big),$$

on the $M(= N+1)$-cells of the M-dimensional sphere $X_{d,k}$. From [5, Thm. 3.11] we know that $X_{d,k}$ has a unique full $\mathfrak{S}_k^{\pm}$-orbit of maximal dimensional cells represented by the cell

$$\theta := D_{1, \ldots, 1}^{+, \ldots, +}(1, 2, \ldots, k).$$

Furthermore, from Theorem 4.1 or [5, Ex. 3.12], we have that θ is given by the inequalities $x_{1,1} < x_{1,2} < \cdots < x_{1,k}$. Thus, having in mind that $\mathfrak{o}$ is an $\mathfrak{S}_k^{\pm}$-equivariant cocycle, it suffices to evaluate $\mathfrak{o}(e_\theta)$.

Consider a collection of j ordered disjoint intervals $\mathcal{M}_0 = (I_1, \ldots, I_j)$ along the moment curve γ, defined in (8), with midpoints $(x_1, \ldots, x_j)$ respectively. Then, according to (9), we have that

$$\mathfrak{o}(e_\theta) = \sum \deg(\nu \circ \widehat{\kappa}|_{S_i}) \cdot \zeta = \big(\sum \pm 1 \big) \cdot \zeta =: a \cdot \zeta, \qquad (11)$$

where the sum ranges over all k-element affine hyperplane arrangements in $\mathrm{relint}(\theta)$ which bisect $\mathcal{M}_0$. We have that:

— $dk = j$,
— any k-element affine hyperplane arrangement in $\mathbb{R}^d$ has at most dk intersection points with the moment curve γ,
— for bisection of a collection of j intervals on γ one needs at least j points, and
— each k-element affine hyperplane arrangement that bisects $\mathcal{M}_0$ is completely determined (up to an orientation of hyperplanes) by a partition of the set of midpoints $\{x_1, \ldots, x_j\}$ of the intervals $(I_1, \ldots, I_j)$ into k subset of cardinality d each, where each of these subset uniquely determines a hyperplane of the k-element affine hyperplane arrangement.

Thus the number of k-element affine hyperplane arrangements that bisect $\mathcal{M}_0$ is $\binom{dk}{d,\ldots,d}2^k$. Using slight perturbations of the intervals $(I_1, \ldots, I_j)$ along the curve γ, without changing their order, we can assume that all the bisecting k-element affine hyperplane arrangements are contained in $\bigcup_{g \in \mathfrak{S}_k^\pm} g \cdot \mathrm{relint}(\theta)$. Thus, the number of k-element affine hyperplane arrangements that bisect $\mathcal{M}_0$ and are contained in $\mathrm{relint}(\theta)$ is $\frac{1}{k!}\binom{dk}{d,\ldots,d}$. This means that the integer a, defined by the equation (11), has the property that

$$a \equiv \frac{1}{k!}\binom{dk}{d,\ldots,d} \mod 2.$$

In the final step let us assume that $[\mathfrak{o}] = 0$, meaning that the cocycle $\mathfrak{o}$ is also a coboundary. Thus there exists a cochain

$$\mathfrak{h} \in \mathcal{C}_{\mathfrak{S}_k^\pm}^N\big(X_{d,k}, X_{d,k}^{>1}\,;\, \pi_N(S(W_k \oplus V^{\oplus j}))\big)$$

such that $\mathfrak{o} = \delta\mathfrak{h}$, where δ denotes the coboundary operator. From (23) or [5, Eq. (11)] we have that

$$\partial e_\theta = (1 + (-1)^d \varepsilon_1) \cdot e_{\gamma_1} + \sum_{i=2}^{k}(1 + (-1)^d \tau_{i-1,i}) \cdot e_{\gamma_{2i-1}}, \tag{12}$$

where the cells $\gamma_1, \ldots, \gamma_{2k}$ are described in Example 4.2, or in [5, p. 755], and $\tau_{i-1,i} \in \mathfrak{S}_k \subseteq \mathfrak{S}_k^\pm$ denotes the transposition that interchanges $i-1$ and i. Thus, $\mathfrak{o} = \delta\mathfrak{h}$ and (12) imply that

$$a \cdot \zeta = \mathfrak{o}(e_\theta) = \delta\mathfrak{h}(e_\theta) = \mathfrak{h}(\partial e_\theta)$$

$$= (1 + (-1)^d \varepsilon_1) \cdot \mathfrak{h}(e_{\gamma_1}) + \sum_{i=2}^{k}(1 + (-1)^d \tau_{i-1,i}) \cdot \mathfrak{h}(e_{\gamma_{2i-1}})$$

$$= (1 + (-1)^{d+j}) \cdot \mathfrak{h}(e_{\gamma_1}) + \sum_{i=2}^{k}(1 + (-1)^{d+1}) \cdot \mathfrak{h}(e_{\gamma_{2i-1}})$$

$$= 2b \cdot \zeta,$$

for some integer b. In this calculation we use the fact that $\mathfrak{h}$ is an equivariant cochain, and that ε_1 and $\tau_{i-1,i}$ act on $V^{\oplus j}$ respectively by multiplication with $(-1)^j$ and trivially. Whereas, ε_1 and $\tau_{i-1,i}$ act on W_k trivially and by multiplication with (-1) respectively. Hence,

$$[\mathfrak{o}] = 0 \iff a \equiv 0 \mod 2 \iff \frac{1}{k!}\binom{dk}{d,\ldots,d} \equiv 0 \mod 2.$$

We verified (10), and concluded a proof of Proposition 3.1(a) and Corollary 3.2.

3.1.4. *Evaluation of the obstruction element in the case* $Z = \mathfrak{S}_k^\pm \cdot \overline{\theta}$ *where* $\theta = D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1, 2, \ldots, k)$. In this section we complete the proof of Proposition 3.1(b).

As before we have that $\dim Z = M = N + 1$ and consequently the obstruction element $[\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}})]$ is the primary obstruction element and the only obstruction to the existence of an $\mathfrak{S}_k^\pm$-equivariant map (7). However, in this case, it is not the only obstruction for the existence of an $\mathfrak{S}_k^\pm$-equivariant map $X_{d,k} \longrightarrow S(W_k \oplus V^{\oplus j})$.

Thus, we prove the implication

$$\binom{(d-\ell)k+\ell}{d} \cdot \frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell} \equiv 1 \mod 2$$

$$\implies \quad [\mathfrak{o}(\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}})] \neq 0. \tag{13}$$

In this way we would prove Proposition 3.1(b) and complete the proof of the proposition.

For that we evaluate the obstruction cocycle

$$\mathfrak{o} := \mathfrak{o}\big(\nu \circ \Psi_{\mathcal{M}_0}|_{\mathrm{sk}_N(Z) \cup Z^{>1}}\big) \in \mathcal{C}^{N+1}_{\mathfrak{S}^\pm_k}\big(Z, Z^{>1}; \pi_N(S(W_k \oplus V^{\oplus j}))\big).$$

on the $M(= N + 1)$-cells of Z. By construction Z is given as the $\mathfrak{S}^\pm_k$-orbit of the cell

$$\theta := D^{+,\ldots,+,+}_{1+\ell,\ldots,1+\ell,1}(1, 2, \ldots, k).$$

and the boundary of its generator e_θ can be written as

$$\partial e_\theta = (1 + (-1)^{d-1}\varepsilon_1)e_{\nu_1} + \sum_{i=2}^{k-1}(1 + (-1)^{d-1}\tau_{i-1,i})e_{\nu_{2i-1}}+ \tag{14}$$

$$\sum_{i=w}^{k}(1 + (-1)^d \varepsilon_i)e_{\mu_{2i}},$$

where $w = \begin{cases} 1, & \ell = 1 \\ k, & \ell \neq 1 \end{cases}$. For more details about the cell θ consult Example 4.3 or [5, pp. 751, 754].

Now we will evaluate $\mathfrak{o}(\theta)$. Since $\mathfrak{o}$ is an $\mathfrak{S}^\pm_k$-equivariant cocycle, in this way we will evaluate the cocycle $\mathfrak{o}$ on all the cells in the orbit of θ.

Consider the moment curve γ defined in (8). We fix a collection of j ordered disjoint intervals $\mathcal{M}_0 = (I_1, \ldots, I_j)$ on γ defined by $I_1 = \gamma([t_{11}, t_{12}]), \ldots, I_j = \gamma([t_{j1}, t_{j2}])$ where

$$\ell < t_{11} < t_{12} < t_{21} < t_{22} < \cdots < t_{j1} < t_{j2}.$$

Then, as in [5, Lem. 3.13], we have that the cell θ parameterizes all k-element affine hyperplane arrangements, where the order and orientation are fixed appropriately, such that the first $k-1$ hyperplanes contain the points $s_1 := \gamma(0), s_2 := \gamma(1), \ldots, s_\ell := \gamma(\ell - 1)$. Thus again, according to (9), we have that

$$\mathfrak{o}(e_\theta) = \sum \deg(\nu \circ \widehat{\kappa}|_{S_i}) \cdot \zeta = \big(\sum \pm 1\big) \cdot \zeta =: a \cdot \zeta, \tag{15}$$

where the sum ranges over all k-element affine hyperplane arrangements in $\mathrm{relint}(\theta)$ which bisect $\mathcal{M}_0$. We have that:

— $(d - \ell)k + \ell = j$,
— any k-element affine hyperplane arrangement in $\mathbb{R}^d$ has at most dk intersection points with the moment curve γ,
— θ parameterizes all k-element affine hyperplane arrangements such that first $k-1$ hyperplane contain the points $s_1, \ldots, s_\ell$, meaning that $(k-1)\ell$ intersection points out of dk cannot be used for interval partitioning,
— for bisection of collection of j intervals on γ one needs at least $j = dk - \ell(k - 1)$ points, and thus
— each k-element affine hyperplane arrangement from θ that bisects $\mathcal{M}_0$ is completely determined (up to an orientation of hyperplanes) by a partition of the set of midpoints $\{x_1, \ldots, x_j\}$ of the intervals $(I_1, \ldots, I_j)$ into $k-1$ subset of cardinality $d-1$ each and one subset of cardinality D, where each of these subsets uniquely determines a hyperplane of the k-element affine hyperplane arrangement.

Consequently, the number of k-element affine hyperplane arrangements from Z which bisect $\mathcal{M}_0$ is $\binom{(d-\ell)k+\ell}{d} \cdot \frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell} \cdot 2^{k-1}$. Again, using slight perturbations of the intervals $(I_1, \ldots, I_j)$ along the curve γ, without changing their order, we can assume that all the bisecting k-element affine hyperplane arrangements are contained in $\bigcup_{g \in \mathfrak{S}^\pm_k} g \cdot \mathrm{relint}(\theta)$. Thus, the number of k-element affine hyperplane arrangements that bisect $\mathcal{M}_0$ and are contained in $\mathrm{relint}(\theta)$ is $\binom{(d-\ell)k+\ell}{d} \cdot \frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell}$. This means that the integer a, defined by equation (15), has the property

$$a \equiv \binom{(d-\ell)k+\ell}{d} \cdot \frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell,\ldots,d-\ell} \quad \mod 2.$$

Next, assume that $[\mathfrak{o}] = 0$, i.e., the cocycle $\mathfrak{o}$ is also a coboundary. Hence there is an N-cochain $\mathfrak{h} \in \mathcal{C}^N_{\mathfrak{S}^\pm_k}\big(Z, Z^{>1}; \pi_N(S(W_k \oplus V^{\oplus j}))\big)$ such that $\mathfrak{o} = \delta\mathfrak{h}$, where δ, as before, is the coboundary operator. Consequently, (14) implies that

$$a \cdot \zeta = \mathfrak{o}(e_\theta) = \delta\mathfrak{h}(e_\theta) = \mathfrak{h}(\partial e_\theta)$$

$$= (1 + (-1)^{d-1}\varepsilon_1) \cdot \mathfrak{h}(e_{\nu_1}) + \sum_{i=2}^{k-1}(1 + (-1)^{d-1}\tau_{i-1,i}) \cdot \mathfrak{h}(e_{\nu_{2i-1}})+$$

$$\sum_{i=w}^{k} (1 + (-1)^d \varepsilon_i) \mathfrak{h}(e_{\mu_{2i}})$$

$$= (1 + (-1)^{d-1+j}) \cdot \mathfrak{h}(e_{\nu_1}) + \sum_{i=2}^{k} (1 + (-1)^d) \cdot \mathfrak{h}(e_{\nu_{2i-1}}) +$$

$$\sum_{i=w}^{k} (1 + (-1)^{d-j}) \mathfrak{h}(e_{\mu_{2i}})$$

$$= 2b \cdot \zeta,$$

for some integer b, where $w = \begin{cases} 1, & \ell = 1 \\ k, & \ell \neq 1 \end{cases}$. Here we use the fact that $\mathfrak{h}$ is an equivariant cochain, and that ε_1 and permutations $\tau_{i-1,i}$ act on $V^{\oplus j}$ respectively by multiplication with $(-1)^j$ and trivially. They act on W_k trivially and by multiplication with (-1) respectively. Therefore, if

$$a \equiv \binom{(d-\ell)k + \ell}{d} \cdot \frac{1}{(k-1)!} \binom{(d-\ell)(k-1)}{d-\ell, \ldots, d-\ell} \not\equiv 0 \mod 2,$$

then $[\mathfrak{o}] \neq 0$, and we concluded the proof of (13) and Proposition 3.1(b).

3.2. Proof of Proposition 3.3. For the proof of the proposition we use the following classical facts going back to Legendre [12], for a modern reference see for example [14, Thm. 2.6.4]. Let p be a prime, $k \geq 1$ be an integer, and let $E_p(k) := \max\{i \in \mathbb{N} \cup \{0\} : p^i \mid k!\}$. There is a unique p-adic presentation of the integer k in the form $k = a_0 + a_1 p + \cdots + a_m p^m$, where $0 \leq a_i \leq p - 1$ for all $0 \leq i \leq m$. Let $\alpha_p(k) := a_0 + a_1 + \cdots + a_m$ denote the sum of coefficients in the p-adic expansion of k. Then

$$E_p(k) = \sum_{j \geq 1} \left\lfloor \frac{k}{p^j} \right\rfloor = \frac{k - \alpha_p(k)}{p - 1}. \tag{16}$$

Furthermore, if $k_1, \ldots, k_t$ are non-negative integers such that $k = k_1 + \cdots + k_t$, then

$$\binom{k}{k_1, \ldots, k_t} \equiv 0 \mod p^r \qquad \Longleftrightarrow \qquad E_p(k) - \sum_{i=1}^{t} E_p(k_i) \geq r. \tag{17}$$

(i) In our proof we assume that $p = 2$, and we also use the inequalities

$$\alpha_2(a + b) \leq \alpha_2(a) + \alpha(b) \qquad \text{and} \qquad \alpha_2(ab) \leq \alpha_2(a)\alpha_2(b), \tag{18}$$

whcih hold for arbitrary integers $a \geq 1$ and $b \geq 1$. Consider the following sequence of equivalences

$$\frac{1}{k!}\binom{dk}{d, \ldots, d} = \frac{(dk)!}{k! d! \cdots d!} \qquad \text{is odd} \iff E_2(dk) = E_2(k) + kE_2(d)$$

$$\iff E_2(k) = kE_2(d) - E_2(dk)$$

$$\overset{(16)}{\iff} k - \alpha_2(k) = k\alpha_2(d) - \alpha_2(dk).$$

Now, if we assume that $\frac{1}{k!}\binom{dk}{d, \ldots, d}$ is odd, then according of the previous equivalences and (18) we have that

$$k - \alpha_2(k) = k\alpha_2(d) - \alpha_2(dk) \geq k\alpha_2(d) - \alpha_2(d)\alpha_2(k) = (k - \alpha_2(k))\alpha_2(d).$$

Since $k \geq 2$ we have that $k - \alpha_2(k) \geq 0$ and consequently $\alpha_2(d) \leq 1$. Thus d must be a power of two.

On the other hand, let us assume that d is a power of two, or in other words $\alpha_2(d) = 1$. Since in this case $\alpha_2(dk) = \alpha_2(k)$ we get the equality

$$k - \alpha_2(k) = k\alpha_2(d) - \alpha_2(dk).$$

Hence, the sequence of equivalences we deduced implies that $\frac{1}{k!}\binom{dk}{d, \ldots, d}$ is odd.

(ii) The product $\binom{(d-\ell)k+\ell}{d} \cdot \frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell, \ldots, d-\ell}$ is odd if and only if both factors are odd. We know that

$$\frac{1}{(k-1)!}\binom{(d-\ell)(k-1)}{d-\ell, \ldots, d-\ell} \text{ is odd} \qquad \Longleftrightarrow \qquad d - \ell = 2^a \text{ for some } a \geq 0.$$

Therefore it remains to discuss when $\binom{(d-\ell)k+\ell}{d}$ is odd, assuming that $d = 2^a + \ell$ where $a \geq 0$ and $2^a > \ell > 0$, which follows from the assumption $d = 2^a + \ell > 2\ell > 0$. Now, the following sequence of equivalences concludes the proof of the second part of the proposition:

$$
\binom{(d-\ell)k+\ell}{d} \text{ is odd} \iff \binom{2^a k + \ell}{2^a + \ell} \text{ is odd}
$$
$$
\iff E_2(2^a k + \ell) = E_2(2^a + \ell) + E_2(2^a(k-\ell))
$$
$$
\stackrel{(16)}{\iff} 2^a k + \ell - \alpha_2(2^a k + \ell) =
$$
$$
2^a + \ell - \alpha_2(2^a + \ell) + 2^a(k-1) - \alpha_2(2^a(k-1))
$$
$$
\iff \alpha_2(2^a k + \ell) = \alpha_2(2^a + \ell) + \alpha_2(2^a(k-1))
$$
$$
\iff \alpha_2(2^a k + \ell) = \alpha_2(2^a + \ell) + \alpha_2(k-1)
$$
$$
\stackrel{2^a \geq \ell}{\iff} \alpha_2(k) + \alpha_2(\ell) = 1 + \alpha_2(\ell) + \alpha_2(k-1)
$$
$$
\iff k \text{ is odd.}
$$

4. Appendix: A $\mathfrak{S}_k^\pm$-CW structure on the join configuration space

In this section, based on the work in [5, Sec. 3], we briefly present a relative $\mathfrak{S}_k^\pm$-CW structure on the join configuration space $X_{d,k}$ which we use in the obstruction theory proof of Theorem 1.2. In particular, this means that the induced $\mathfrak{S}_k^\pm$-CW structure transforms the subspace $X_{d,k}^{>1}$ into an $\mathfrak{S}_k^\pm$-CW subcomplex. As in the original work, the relative $\mathfrak{S}_k^\pm$-CW complex we construct is denoted by $X := (X_{d,k}, X_{d,k}^{>1})$. The construction proceeds in two steps:
— the Euclidean space $\mathbb{R}^{(d+1)\times k}$ is partitioned into a union of (disjoint) relatively open cones, each containing the origin in its closure, on which the $\mathfrak{S}_k^\pm$-action operates by linearly permuting the cones (Section 4.1), and then
— the open cells of a regular $\mathfrak{S}_k^\pm$-CW model are defined as intersections of these relatively open cones with the unit sphere of the Euclidean space $\mathbb{R}^{(d+1)\times k}$, (Section 4.2).

4.1. A stratification of the Euclidean space $\mathbb{R}^{(d+1)\times k}$. First we recall the notion of a stratification of a Euclidean space.

Let E be a Euclidean space. A *stratification of E (by cones)* is a finite collection $\mathcal{C}$ of subsets of E that satisfies the following properties:
— $\mathcal{C}$ consists of finitely many non-empty relatively open polyhedral cones of E,
— $\mathcal{C}$ is a partition of E, i.e., $E = \biguplus_{C \in \mathcal{C}} C$,
— the closure $\overline{C}$ of every cone $C \in \mathcal{C}$ is a union of cones in $\mathcal{C}$.
An element of the family $\mathcal{C}$ is called a *stratum*.

In order to define the desired stratification of the Euclidean space $\mathbb{R}^{(d+1)\times k}$ we first fix following data:
— a permutation $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathfrak{S}_k$,
— a collection of signs $S := (s_1, \ldots, s_k) \in \{+1, -1\}^k$, and
— a collection of integers $I := (i_1, \ldots, i_k) \in \{1, \ldots, d+2\}^k$.
Furthermore, we set x_0 to be the origin of the Euclidean space $\mathbb{R}^{(d+1)\times k}$, $\sigma_0 = 0$ and $s_0 = 1$. Now we define the cone

$$
C_I^S(\sigma) = C_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k) \subseteq \mathbb{R}^{(d+1)\times k}
$$

to be the collection of all points $(x_1, \ldots, x_k) \in \mathbb{R}^{(d+1)\times k}$, $x_i = (x_{1,i}, \ldots, x_{d+1,i})$, such that for each $1 \leq t \leq k$,
— if $1 \leq i_t \leq d+1$, then $s_{t-1} x_{i_t, \sigma_{t-1}} < s_t x_{i_t, \sigma_t}$ with $s_{t-1} x_{i', \sigma_{t-1}} = s_t x_{i', \sigma_t}$ for every $i' < i_t$, and
— if $i_t = d+2$, then $s_{i_{t-1}} x_{\sigma_{t-1}} = s_{i_t} x_{\sigma_t}$.
A triple $(\sigma|I|S) \in \mathfrak{S}_k \times \{1, \ldots, d+2\}^k \times \{+1, -1\}^k$ is called a *symbol*. In the notation of symbols we write instead of the signs $\{+1, -1\}$ just $\{+, -\}$. The set of "inequalities" which define the stratum $C_I^S(\sigma)$ can be shortly denoted by:

$$
C_I^S(\sigma) = C_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k)
$$
$$
= \{(x_1, \ldots, x_k) \in \mathbb{R}^{(d+1)\times k} : 0 <_{i_1} s_1 x_{\sigma_1} <_{i_2} s_2 x_{\sigma_2} <_{i_3} \cdots <_{i_k} s_k x_{\sigma_k}\},
$$

where $y <_i y'$, for $1 \leq i \leq d+1$, means that y and y' agree in the first $i-1$ coordinates and at the i-th coordinate $y_i < y'_i$. The inequality $y <_{d+2} y'$ stands for $y = y'$. Furthermore, each $C_I^S(\sigma)$ equals to the relative interior of a polyhedral cone in $(\mathbb{R}^{d+1})^k$ of codimension $(i_1 - 1) + \cdots + (i_k - 1)$, that means

$$\dim C_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k) = (d+2)k - (i_1 + \cdots + i_k).$$

Let $\mathcal{C}$ denote the family of all strata $C_I^S(\sigma)$ defined by all symbols, that is

$$\mathcal{C} = \left\{ C_I^S(\sigma) : (\sigma|I|S) \in \mathfrak{S}_k \times \{1, \ldots, d+2\}^k \times \{+1, -1\}^k \right\}.$$

Note that different symbols may define the same sets, additionally:

$$C_I^S(\sigma) \cap C_{I'}^{S'}(\sigma) \neq \emptyset \iff C_I^S(\sigma) = C_{I'}^{S'}(\sigma).$$

Since it's not hard to check that $\bigcup \mathcal{C} = \mathbb{R}^{(d+1)\times k}$ we conclude that $\mathcal{C}$ is a stratification of the Euclidean space $\mathbb{R}^{(d+1)\times k}$.

The action of the group $\mathfrak{S}_k^{\pm}$ on the Euclidean space $\mathbb{R}^{(d+1)\times k}$ induces an action on the stratification $\mathcal{C}$ by as follows:

$$\pi \cdot C_I^S(\sigma) = C_I^S(\pi\sigma), \tag{19}$$

$$\varepsilon_t \cdot C_I^S(\sigma) = \varepsilon_t \cdot C_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k)$$
$$= C_{i_1,\ldots,i_k}^{s_1,\ldots,-s_t,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k), \tag{20}$$

where $\pi \in \mathfrak{S}_k$, $1 \leq t \leq k$, and $\varepsilon_1, \ldots, \varepsilon_k$ are the canonical generators of the subgroup $(\mathbb{Z}/2)^k$ of $\mathfrak{S}_k^{\pm}$.

4.2. The $\mathfrak{S}_k^{\pm}$-CW complex induced from the stratification $\mathcal{C}$. The $\mathfrak{S}_k^{\pm}$-CW complex structure on the joint configuration space $X_{d,k} = S(\mathbb{R}^{(d+1)\times k})$ is defined by intersecting each stratum $C_I^S(\sigma)$ of the stratification $\mathcal{C}$ with the unit sphere $S(\mathbb{R}^{(d+1)\times k})$. Since the stratum $C_I^S(\sigma)$ is a relatively open cone which does not contain a line, the intersection

$$D_I^S(\sigma) = D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k) := C_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k) \cap S(\mathbb{R}^{(d+1)\times k})$$

has to be an open cell of dimension $(d+2)k - (i_1 + \cdots + i_k) - 1$. The action of the group $\mathfrak{S}_k^{\pm}$ on the cells $D_I^S(\sigma)$ is induced by from (19) and (20) as follows:

$$\pi \cdot D_I^S(\sigma) = D_I^S(\pi\sigma), \tag{21}$$

$$\varepsilon_t \cdot D_I^S(\sigma) = \varepsilon_t \cdot D_{i_1,\ldots,i_k}^{s_1,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k)$$
$$= D_{i_1,\ldots,i_k}^{s_1,\ldots,-s_t,\ldots,s_k}(\sigma_1, \sigma_2, \ldots, \sigma_k). \tag{22}$$

In this way we have defined a regular $\mathfrak{S}_k^{\pm}$-CW structure on $X_{d,k}$. In particular, the action of the group $\mathfrak{S}_k^{\pm}$ on the Euclidean space $\mathbb{R}^{(d+1)\times k}$ restricts to the cellular action on the model. Thus, we have the following theorem [5, Thm. 3.11].

THEOREM 4.1. *Let $d \geq 1$ and $k \geq 1$ be integers. The family of cells*

$$\left\{ D_I^S(\sigma) : (\sigma|I|S) \neq (\sigma|d+2, \ldots, d+2|S) \right\}$$

forms a finite regular $((d+1)k - 1)$-dimensional $\mathfrak{S}_k^{\pm}$-CW complex $X := (X_{d,k}, X_{d,k}^{>1})$ which models the join configuration space $X_{d,k} = S(\mathbb{R}^{(d+1)\times k})$. It has
  — *one full $\mathfrak{S}_k^{\pm}$-orbit of cells in the maximal dimension $(d+1)k - 1$ induced by the cell $D_{1,\ldots,1,1}^{+,\ldots,+,+}(1, 2, \ldots, k)$, and*
  — *k full $\mathfrak{S}_k^{\pm}$-orbits of cells in dimension $(d+1)k - 2$.*
The (cellular) $\mathfrak{S}_k^{\pm}$-action on $X_{d,k}$ is given by relations (21) and (22). Furthermore the collection of cells

$$\left\{ D_I^S(\sigma) : i_s = d+2 \text{ for some } 1 \leq s \leq k \right\}$$

is a $\mathfrak{S}_k^{\pm}$-CW subcomplex and models $X_{d,k}^{>1}$.

As an illustration of a cell structure we analyze the cells $D_{1,\ldots,1}^{+,\ldots,+}(1, 2, \ldots, k)$ and $D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1, 2, \ldots, k)$, which are used in the proofs of Proposition 3.1(a) and 3.1(b). First we recall [5, Ex. 3.12].

Example 4.2. Let $d \geq 1$ and $k \geq 1$ be integers. Consider the cell
$$\theta := D_{1,1,1,\ldots,1}^{+,+,+,\ldots,+}(1,2,3,\ldots,k)$$
of the $\mathfrak{S}_k^{\pm}$-CW complex $X_{d,k}$. It is determined by the inequalities:
$$0 <_1 x_1 <_1 x_2 <_1 \cdots <_1 x_k.$$

The cells of codimension one in the boundary of θ are obtained by introducing one of the following extra equalities:
$$x_{1,1} = 0, \quad x_{1,1} = x_{1,2}, \quad \ldots \quad x_{1,k-1} = x_{1,k}.$$
Each of these equalities will give two cells, hence there are, in total, $2k$ cells of codimension one in the boundary of the cell θ.

(a) The equality $x_{1,1} = 0$ induces cells:
$$\gamma_1 := D_{2,1,1,\ldots,1}^{+,+,+,\ldots,+}(1,2,3,\ldots,k), \qquad \gamma_2 := D_{2,1,1,\ldots,1}^{-,+,+,\ldots,+}(1,2,3,\ldots,k)$$
which are related, as sets, via $\gamma_2 = \varepsilon_1 \cdot \gamma_1$. Both cells γ_1 and γ_2 belong to the linear subspace
$$V_1 = \left\{(x_1,\ldots,x_k) \in \mathbb{R}^{(d+1) \times k} : x_{1,1} = 0\right\}.$$

(b) The equality $x_{1,r-1} = x_{1,r}$ for $2 \leq r \leq k$ gives cells:
$$\gamma_{2r-1} := D_{1,\ldots,1,2,1,\ldots,1}^{+,+,+,\ldots,+}(1,\ldots,r-1,r,r+1,\ldots,k),$$
$$\gamma_{2r} := D_{1,\ldots,1,2,1,\ldots,1}^{+,+,+,\ldots,+}(1,\ldots,r,r-1,r+1,\ldots,k),$$
satisfying $\gamma_{2r} = \tau_{r-1,r} \cdot \gamma_{2r-1}$. In these cells the index 2 in the subscript $1,\ldots,1,2,1,\ldots,1$ appears at the position r. These cells are contained in the linear subspace
$$V_r = \left\{(x_1,\ldots,x_k) \in \mathbb{R}^{(d+1) \times k} : x_{1,r-1} = x_{1,r}\right\}.$$

Let $e_\theta, e_{\gamma_1}, \ldots, e_{\gamma_{2k}}$ denote a generators in the cellular chain group corresponding to $\theta, \gamma_1, \ldots, \gamma_{2k}$. The boundary of the cell θ is contained in the union of the linear subspaces $V_1, \ldots, V_k$. Therefore we can orient the cells $\gamma_{2i-1}, \gamma_{2i}$ consistently with the orientation of V_i, $1 \leq i \leq k$, given in such a way that
$$\partial e_\theta = (e_{\gamma_1} + e_{\gamma_2}) + (e_{\gamma_3} + e_{\gamma_4}) + \cdots + (e_{\gamma_{2k-1}} + e_{\gamma_{2k}}).$$

Consequently,
$$\partial e_\theta = (1 + (-1)^d \varepsilon_1) \cdot e_{\gamma_1} + \sum_{i=2}^{k}(1 + (-1)^d \tau_{i-1,i}) \cdot e_{\gamma_{2i-1}}. \tag{23}$$

Example 4.3. Let $d \geq 1$, $k \geq 1$, and $1 \leq \ell \leq d-1$. Consider the cell
$$\theta := D_{1+\ell,\ldots,1+\ell,1}^{+,\ldots,+,+}(1,2,\ldots,k)$$
of the $\mathfrak{S}_k^{\pm}$-CW complex $X_{d,k}$, which is given by
$$0 <_{1+\ell} x_1 <_{1+\ell} x_2 <_{1+\ell} \cdots <_{1+\ell} x_{k-1} <_1 x_k.$$
More precisely, it is given by the inequalities
$$0 = x_{1,1} = \cdots = x_{1,k-1} < x_{1,k}, \qquad 0 = x_{r,1} = \cdots = x_{r,k-1}$$
for all $2 \leq r \leq \ell$, and
$$0 < x_{\ell+1,1} < \cdots < x_{\ell+1,k-1}.$$

The cells of codimension one in the boundary of the cell θ are induced by addition of one of the following extra equalities:
$$x_{\ell+1,1} = 0, \quad x_{\ell+1,1} = x_{\ell+1,2}, \quad \ldots, \quad x_{\ell+1,k-2} = x_{\ell+1,k-1}, \quad x_{1,k-1} = x_{1,k}.$$
We have the following cells of codimension 1 in the boundary of θ:

(a) The equality $x_{\ell+1,1} = 0$ gives cells:
$$\nu_1 := D_{\ell+2,\ell+1,\ell+1,\ldots,\ell+1,1}^{+,+,+,\ldots,+,+}(1,2,3,\ldots,k),$$
$$\nu_2 := D_{\ell+2,\ell+1,\ell+1,\ldots,\ell+1,1}^{-,+,+,\ldots,+,+}(1,2,3,\ldots,k),$$
which on the level of sets are related by $\nu_2 = \varepsilon_1 \cdot \nu_1$. Both cells γ_1 and γ_2 belong to the linear subspace
$$V_1 = \left\{(x_1,\ldots,x_k) \in \mathbb{R}^{(d+1) \times k} : x_{1,1} = 0, \ \ldots, \ x_{\ell+1,1} = 0\right\}.$$

(b) The equality $x_{\ell+1,r-1} = x_{\ell+1,r}$ for $2 \leq r \leq k-1$ induces cells:

$$\nu_{2r-1} := D^{+,+,+,\ldots,+,+}_{\ell+1,\ldots,\ell+1,\ell+2,\ell+1,\ldots,\ell+1,1}(1,\ldots,r-1,r,r+1,\ldots,k),$$

$$\nu_{2r} := D^{+,+,+,\ldots,+,+}_{\ell+1,\ldots,\ell+1,\ell+2,\ell+1,\ldots,\ell+1,1}(1,\ldots,r,r-1,r+1,\ldots,k),$$

satisfying $\nu_{2r} = \tau_{r-1,r} \cdot \nu_{2r-1}$. In these cells the index $\ell+2$ in the subscript $\ell+1,\ldots,\ell+1,\ell+2,\ell+1,\ldots,\ell+1,1$ is at the position r. These cells belong to the linear subspace

$$V_r = \left\{(x_1,\ldots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,r-1} = x_{1,r},\ \ldots,\ x_{\ell+1,r-1} = x_{\ell+1,r}\right\}.$$

(c) In the case $\ell = 1$ the last equality $(0 =)x_{1,k-1} = x_{1,k}$ induces $2k$ cells for each $1 \leq r \leq k$ of the form

$$\mu_{2r-1} := D^{+,+,\ldots,+,\ldots,+,+}_{2,2,2,\ldots,2,2}(1,\ldots,r-1,k,r,\ldots,k-1),$$

$$\mu_{2r} := D^{+,+,\ldots,-,\ldots,+,+}_{2,2,2,\ldots,2,2}(1,\ldots,r-1,k,r,\ldots,k-1),$$

satisfying $\mu_{2r} = \varepsilon_r \mu_{2r-1}$. The minus-sign is on the r-th position.

(d) In the case $\ell > 1$ the last equality $(0 =)x_{1,k-1} = x_{1,k}$ induces 2 cells of the form

$$\mu_{2k-1} := D^{+,+,+,\ldots,+,+}_{\ell+1,\ell+1,\ldots,\ell+1,2}(1,2,3,\ldots,k),$$

$$\mu_{2k} := D^{+,+,+,\ldots,+,-}_{\ell+1,\ell+1,\ldots,\ell+1,2}(1,2,3,\ldots,k),$$

satisfying $\mu_{2k} = \varepsilon_k \mu_{2k-1}$. Either way these cells belong to the subspace

$$V_k = \left\{(x_1,\ldots,x_k) \in \mathbb{R}^{(d+1)\times k} : 0 = x_{1,1} = \cdots = x_{1,k}\right\}.$$

Let $e_\theta, e_{\nu_1},\ldots,e_{\nu_{2k-2}}, e_{\mu_1},\ldots,e_{\mu_{2(k-1)}}, e_{\mu_{2k-1}}, e_{\mu_{2k}}$ denote generators in the cellular chain group that correspond to the cells $\theta, \nu_1,\ldots,\nu_{2k-2}, \mu_1,\ldots,\mu_{2(k-1)}, \mu_{2k-1},\mu_{2k}$, respectively. The boundary of the cell θ is a subset of the union of the linear subspaces $V_1,\ldots,V_k$. Hence, we can orient the subspaces and the cells consistently in such a way that for $\ell > 1$ the following equality holds

$$\partial e_\theta = (e_{\nu_1} + e_{\nu_2}) + \cdots + (e_{\nu_{2k-3}} + e_{\nu_{2k-2}}) + (e_{\mu_{2k-1}} + e_{\mu_{2k}}),$$

while for $\ell = 1$ we get

$$\partial e_\theta = (e_{\nu_1} + e_{\nu_2}) + \cdots + (e_{\nu_{2k-3}} + e_{\nu_{2k-2}}) + (e_{\mu_1} + e_{\mu_2}) + \cdots + (e_{\mu_{2k-1}} + e_{\mu_{2k}}).$$

Thus,

$$\partial e_\theta = (1 + (-1)^{d-1}\varepsilon_1)e_{\nu_1} + \sum_{i=2}^{k-1}(1 + (-1)^{d-1}\tau_{i-1,i})e_{\nu_{2i-1}}+ \tag{24}$$

$$\sum_{i=w}^{k}(1 + (-1)^d \varepsilon_i)e_{\mu_{2i}},$$

where $w = \begin{cases} 1, & \ell = 1 \\ k, & \ell \neq 1 \end{cases}$.

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Introduction to Chapter 3

The following chapter is joint work with Christian Stump and was published in [2]. The author of this thesis has developed and implemented the algorithm himself. Providing a formal framework for this implementation was done in collaboration.

As previously mentioned, the results in Chapter 1 were found using an iterator. The next chapter develops an entirely different iterator. It iterates over the faces of a polyhedron. The following theorem establishes the double description of a polytope:

THEOREM 4.4 ([3, Thm. 1.1]). *A subset $P \subseteq \mathbb{R}^d$ is the convex hull of a finite point set (a $\mathcal{V}$-polytope)*

$$P = \operatorname{conv}(V) \quad \text{for some } V \in \mathbb{R}^{d \times n}$$

if and only if it is a bounded intersection of halfspaces (a $\mathcal{H}$-polytope)

$$P = P(A, z) \quad \text{for some } A \in \mathbb{R}^{m \times n}, \ z \in \mathbb{R}^m.$$

The irredundant elements of V are the vertices of a polytope. At least in the full-dimensional case the irreduntant halfspaces induce the facets of the polytope. The unbounded case is slighly more general:

THEOREM 4.5 ([3, Thm. 1.2]). *A subset $P \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points plus a conical combination of vectors (a $\mathcal{V}$-polyhedron)*

$$P = \operatorname{conv}(V) + \operatorname{cone}(Y) \quad \text{for some } V \in \mathbb{R}^{d \times n}, \ Y \in \mathbb{R}^{d \times n'}$$

if and only if it is an intersection of closed halfspaces (a $\mathcal{H}$-polyhedron)

$$P = P(A, z) \quad \text{for some } A \in \mathbb{R}^{m \times n}, \ z \in \mathbb{R}^m.$$

Polytopes and polyhedra are both important objects in discrete geometry and beyond. As we see above each polyhedron is the set of feasible solutions of a system of linear inequalities. Finding an optimal solution corresponds to finding a vertex of a polyhedron that maximes a certain linear function (*linear programming*).

As the similar definitions suggest, polytopes and polyhedra are connected. Via homogenization every polytope and polyhedron in $\mathbb{R}^d$ corresponds to a cone in $\mathbb{R}^{d+1}$ and cones with trivial linear subspace correspond to polytopes. This process is somewhat unique and in particular the combinatorics are preserved. But first we need to describe, what we mean by combinatorics:

Definition 4.6 ([3, Def. 2.1]). Let $P \subseteq \mathbb{R}^d$ be a convex polytope. A linear inequality $\langle c, x \rangle \subseteq c_0$ is *valid* for P if it is satisfied for all points $x \in P$. A *face* of P is any set of the form

$$F = P \cap \{x \in \mathbb{R}^d \colon \langle c, x \rangle = c_0\}$$

where $\langle c, x \rangle \leq c_0$ is a valid inequality for P. The *dimension* of a face is the dimension of its affine hull.

Via the inclusion relation the faces of a polytope form a partially ordered set (*poset*). The intersection of two faces is a face itself and the face poset has a lower bound – the empty set – and an upper bound – the polytope itself. Thus, the poset is a *lattice*. The lattice is graded by dimension. Intervals of this lattice correspond again to face lattices of polytopes. In particular every interval of length 2 contains exactly four elements [3, see Thm. 2.7]:

By homogenization we lift a polytope in $\mathbb{R}^d$ to a cone in $\mathbb{R}^{d+1}$. Faces of a cone are induced by valid linear hyperplanes and an n-dimensional face of the polytope corresponds exactly to a $(n+1)$-dimensional face of the cone.

As for polyhedra the homogenization has some new faces that correspond to faces at infinity of the polyhedron.

In the next chapter we will develop an iterator through the face lattice of a polytope. By the above comments, the reader should have a vague idea that this iterator will also work for polyhedra. It takes as an input the atoms/vertices and coatoms/facets of the face lattice as well as the containment relations (*incidence matrix*). It works by depth-first search and uses an inductive idea: Given a facet F. We know that the interval $[\emptyset, F]$ is again the face lattice of a polytope. We can visit the faces of $[\emptyset, F]$ by calculating the new input for the corresponding

polytope. Besides using local information – which is considerably faster – the algorithm has a clever way of marking faces as visited, which efficiently prevents revisits.

We compare the performance of our implementation to other implementations observing a speed improvement of at least a factor of 10 as well assymptotically better memory usage. The memory usage is now linear to the input times the dimension of the polytope.

We apply our algorithm to a 17-dimensional cone with 162 facets and $11,665,781$ vertices to check for certain faces. Using methods developed by Bruns, García-Sánchez, O'Neill and Wilburne [1] this verifies *Wilf's conjecture* for multiplicity 19.

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CHAPTER 3

A new face iterator for polyhedra and for more general finite locally branched lattices

Jonathan Kliem and Christian Stump

Abstract We discuss a new memory-efficient depth-first algorithm and its implementation that iterates over all elements of a finite locally branched lattice. This algorithm can be applied to face lattices of polyhedra and to various generalizations such as finite polyhedral complexes and subdivisions of manifolds, extended tight spans and closed sets of matroids. Its practical implementation is very fast compared to state-of-the-art implementations of previously considered algorithms. Based on recent work of Bruns, García-Sánchez, O'Neill and Wilburne, we apply this algorithm to prove *Wilf's conjecture* for all numerical semigroups of multiplicity 19 by iterating through the faces of the *Kunz cone* and identifying the possible *bad faces* and then checking that these do not yield counterexamples to Wilf's conjecture.

Contents

1. Introduction

We call a finite lattice $(\mathcal{P}, \leq)$ *locally branched* if all intervals of length two contain at least four elements. We show that such lattices are atomic and coatomic and refer to Section 2 for details.

This paper describes a depth-first algorithm to iterate through the elements in a finite locally branched lattice given its coatoms, see Section 3. It moreover describes variants of this algorithm allowing the iteration over slightly more general posets. Examples of such locally branched lattices (or its mild generalizations) include face posets of

- polytopes and unbounded polyhedra,
- finite polytopal or polyhedral complexes,
- finite polyhedral subdivisions of manifolds,
- extended tight spans, and
- closed sets of matroids.

One may in addition compute all cover relations as discussed in Subsection 4.1. The provided theoretical runtime (without variants) is the same as of the algorithm discussed by V. Kaibel and M. E. Pfetsch in [7], see Section 4.

In practice the chosen data structures and implementation details make the implementation[1] very fast for the iteration and still fast for cover relations in the graded case compared to state-of-the-art implementations of previously considered algorithms, see Section 5.

In Section 6, we apply the presented algorithm to affirmatively settle *Wilf's conjecture* for all numerical semigroups of multiplicity 19 by iterating, up to a certain symmetry of order 18, through all faces of the *Kunz cone* (which is a certain unbounded polyhedron), identifying the *bad faces* which possibly yield counterexamples to Wilf's conjecture, and then checking that these do indeed not yield such counterexamples. This is based on recent work of W. Bruns, P. García-Sánchez, C. O'Neill and D. Wilburne [3] who developed this approach to the conjecture and were able to settle it up to multiplicity 18.

In Section 7, we finally collect detailed runtime comparisons between the implementation of the presented algorithm with the state-of-the-art implementations in `polymake` and in `normaliz`.

2. Formal framework

Let $(\mathcal{P}, \leq)$ be a finite poset and denote by $\prec$ its cover relation[2]. We usually write $\mathcal{P}$ for $(\mathcal{P}, \leq)$ and write $\mathcal{P}^{\mathrm{op}}$ for the opposite poset $(\mathcal{P}^{\mathrm{op}}, \leq_{\mathrm{op}})$ with $b \leq_{\mathrm{op}} a$ if $a \leq b$. For $a, b \in \mathcal{P}$ with $a \leq b$ we denote the interval as $[a, b] = \{p \in \mathcal{P} \mid a \leq p \leq b\}$. If $\mathcal{P}$ has a lower bound $\hat{0}$, its *atoms* are the upper covers of the lower bound,

$$\mathrm{Atoms}(\mathcal{P}) = \{p \in \mathcal{P} \mid \hat{0} \prec p\}$$

and, for $p \in \mathcal{P}$, we write $\mathrm{Atoms}(p) = \{a \in \mathrm{Atoms}(\mathcal{P}) \mid p \geq a\}$ for the atoms below p. Analogously, if $\mathcal{P}$ has an upper bound $\hat{1}$, its *coatoms* are the lower covers of the upper bound, $\mathrm{coAtoms}(\mathcal{P}) = \{p \in \mathcal{P} \mid p \prec \hat{1}\}$. $\mathcal{P}$ is called *graded* if it admits a rank function $r \colon \mathcal{P} \to \mathbb{Z}$ with $p \prec q \Rightarrow r(p) + 1 = r(q)$.

Definition 2.1. $\mathcal{P}$ is *locally branched* if for every chain $a \prec b \prec c$ there exists an element $d \neq b$ with $a < d < c$. If this element is unique, then $\mathcal{P}$ is said to have the *diamond property*.

The diamond property is a well-known property of face lattices of polytopes, see [10, Theorem 2.7 (iii)]. The property of being locally branched has also appeared in the literature in contexts different from the present under the name 2-*thick lattices*, see for example [1] and the references therein.

An obvious example of a locally branched lattice is the Boolean lattice B_n given by all subsets of $\{1, \ldots, n\}$ ordered by containment. We will later see that all locally branched lattices with n atoms are isomorphic to meet semi-sublattices of B_n.

In the following, we assume $\mathcal{P}$ to be a finite lattice with meet operation $\wedge$, join operation $\vee$, lower bound $\hat{0}$ and upper bound $\hat{1}$. We say that
- $\mathcal{P}$ is *atomic* if all elements are joins of atoms,
- $\mathcal{P}$ is *coatomic* if all elements are meets of coatoms,
- $p \in \mathcal{P}$ is *join-irreducible* if p has a unique lower cover $q \prec p$,
- $p \in \mathcal{P}$ is *meet-irreducible* if p has a unique upper cover $p \prec q$.

Atoms are join-irreducible and coatoms are meet-irreducible. The following classification of atomic and coatomic lattices is well-known folklore.

Lemma 2.2. *We have that*
(i) *$\mathcal{P}$ is atomic if and only if the only join-irreducible elements are the atoms,*
(ii) *$\mathcal{P}$ is coatomic if and only if the only meet-irreducible elements are the coatoms.*

PROOF. First observe that for all $p, q \in \mathcal{P}$ we have $p \geq q \Rightarrow \mathrm{Atoms}(p) \supseteq \mathrm{Atoms}(q)$ and $p \geq \bigvee \mathrm{Atoms}(p)$. Moreover, $\mathcal{P}$ is atomic if and only if $p = \bigvee \mathrm{Atoms}(p)$ for all $p \in \mathcal{P}$.

Assume that $\mathcal{P}$ is atomic and let $q \in \mathcal{P}$ join-irreducible and $p \prec q$. Because we have $\mathrm{Atoms}(p) \neq \mathrm{Atoms}(q)$ it follows that $p = \hat{0}$. Next assume that $\mathcal{P}$ is not atomic and let $p \in \mathcal{P}$ minimal such that $p > \bigvee \mathrm{Atoms}(p)$. If $q < p$ then by minimality $q = \bigvee \mathrm{Atoms}(q)$. It follows that $q \leq \bigvee \mathrm{Atoms}(p)$ and p is join-irreducible.

The second equivalence is the first applied to $\mathcal{P}^{\mathrm{op}}$. $\square$

Example 2.3. The face lattice of a polytope has the diamond property, it is atomic and coatomic, and every interval is again the face lattice of a polytope. The face lattice of an (unbounded) polyhedron might neither be atomic nor coatomic as witnessed by the face lattice of the nonnegative orthant in $\mathbb{R}^2$ with five faces. Example 2.11 will explain how to deal with this.

The reason to introduce locally branched posets is the following relation to atomic and coatomic lattices, which has, to the best of our knowledge, not appeared in the literature.

Proposition 2.4. *The following statements are equivalent:*
(i) *$\mathcal{P}$ is locally branched,*
(ii) *every interval of $\mathcal{P}$ is atomic,*

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[1]See `https://trac.sagemath.org/ticket/26887`, merged into `SageMath` version `sage-8.9`.
[2]$a \prec b$ whenever $a < b$ and there does not exist c satisfying $a < c < b$

(iii) every interval of $\mathcal{P}$ is coatomic.

PROOF. $\mathcal{P}$ is locally branched if and only if $\mathcal{P}^{\mathrm{op}}$ is locally branched. Also, $\mathcal{P}$ is atomic if and only if $\mathcal{P}^{\mathrm{op}}$ is coatomic. Hence, it suffices to show i $\Leftrightarrow$ ii. Suppose $\mathcal{P}$ is not locally branched. Then, there exist $p \prec x \prec q$ such that the interval $[p, q]$ contains exactly those three elements. Clearly, $[p, q]$ is not atomic. Now suppose $[p, q] \subseteq \mathcal{P}$ is not atomic. Lemma 2.2 implies that there is x join-irreducible with unique lower cover y with $p < y \prec x$. There exists $z \in [p, q]$ with $z \prec y$ and the interval $[z, x]$ contains exactly those three elements. $\square$

Example 2.5. Figure 1 gives on the left an example of a non-graded locally branched lattice. On the right it gives an example of an atomic, coatomic lattice, which is not locally branched as the interval between the two larger red elements contains only three elements.
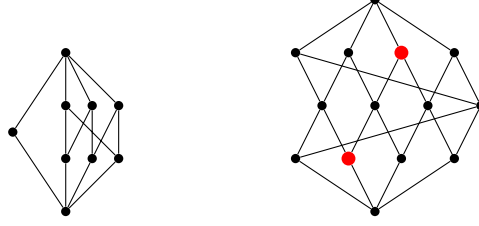
FIGURE 1. a non-graded locally branched lattice and an atomic, coatomic, not locally branched lattice

Let $\mathcal{P}$ be a finite locally branched poset with atoms $\{1, \ldots, n\}$. We have seen that $\mathcal{P}$ is atomic and thus $p = \bigvee \mathrm{Atoms}(p)$ for all $p \in \mathcal{P}$. The following proposition underlines the importance of subset checks and of computing intersections to understanding finite locally branched lattices.

Proposition 2.6. *In a finite locally branched lattice it holds that*

  (i) $p \leq q \Leftrightarrow \mathrm{Atoms}(p) \subseteq \mathrm{Atoms}(q)$.
  (ii) $p \wedge q = \bigvee \left(\mathrm{Atoms}(p) \cap \mathrm{Atoms}(q) \right)$.

PROOF.

  (i) If $p \leq q$ then clearly $\mathrm{Atoms}(p) \subseteq \mathrm{Atoms}(q)$. On the other hand, if $\mathrm{Atoms}(p) \subseteq \mathrm{Atoms}(q)$, then $p = \bigvee \mathrm{Atoms}(p) \leq \bigvee \mathrm{Atoms}(q) = q$, as $\bigvee \mathrm{Atoms}(q)$ is in particular an upper bound for $\mathrm{Atoms}(p)$.
  (ii) By i it holds that $\bigvee \left(\mathrm{Atoms}(p) \cap \mathrm{Atoms}(q) \right)$ is a lower bound of p and q. Also, $\mathrm{Atoms}(p \wedge q) \subseteq \mathrm{Atoms}(p), \mathrm{Atoms}(q)$ and we obtain

$$\mathrm{Atoms}(p \wedge q) \subseteq \mathrm{Atoms}(p) \cap \mathrm{Atoms}(q).$$

$\square$

This proposition provides the following meet semi-lattice [3] embedding [4] of any finite locally branched lattice into a Boolean lattice.

Corollary 2.7. *Let $\mathcal{P}$ be a finite locally branched lattice with $\mathrm{Atoms}(\mathcal{P}) = \{1, \ldots, n\}$. The map $p \mapsto \mathrm{Atoms}(p)$ is a meet semi-lattice embedding of $\mathcal{P}$ into the Boolean lattice B_n.*

Example 2.8. The above embedding does not need to be a join semi-sublattice embedding as witnessed by the face lattice of a square in $\mathbb{R}^2$.

Remark 2.9. Proposition 2.6 shows that checking whether the relation $p \leq q$ holds in $\mathcal{P}$ is algorithmically a subset check $\mathrm{Atoms}(p) \subseteq \mathrm{Atoms}(q)$, while computing the meet is given by computing the intersection $\mathrm{Atoms}(p) \cap \mathrm{Atoms}(q)$.

Justified by Corollary 2.7, we restrict our attention in this paper to meet semi-sublattices of the Boolean lattice.

---

[3]A poset with meet operation.
[4]An poset-embedding preserving meets.

2.1. Variants of this framework and examples. Before presenting in Section 3 the algorithm to iterate over the elements of a finite locally branched lattice together with variants to avoid any element above certain atoms and to avoid any element below certain coatoms (or other elements of B_n), we give the the following main use cases for such an iterator.

Example 2.10 (Polytope)**.** The face lattice of a polytope P has the diamond property and is thus locally branched.

Example 2.11 (Polyhedron)**.** A polyhedron P can be projected onto the orthogonal complement of its linear subspace. The face lattices of those polyhedra are canonically isomorphic. Thus, we can assume that P does not contain an affine line. It is well known (see e.g. [10, Exercise 2.19]) that we may add an extra facet $\overline{F}$ to obtain a polytope $\overline{P}$. The faces of P are exactly the faces of $\overline{P}$ not contained in $\overline{F}$ (together with the empty face). Thus, the iterator visits all non-empty faces of P by visiting all faces of $\overline{P}$ not contained in $\overline{F}$.

Example 2.12 (Polytopal subdivision of manifold)**.** The face poset of a finite polytopal subdivision of a closed manifold (compact manifold without boundary). Adding an artificial upper bound $\hat{1}$, this is a finite locally branched lattice.

Example 2.13 (Extended tight spans)**.** We consider extended tight spans as defined in [6, Section 3] as follows: Let $P \subset \mathbb{R}^d$ be a finite point configuration, and let Σ be a polytopal complex with vertices P, which covers the convex hull of P. We call the maximal cells of Σ facets. We can embedd Σ into a closed d-manifold M: We can add a vertex at infinity and for each face F on the boundary of Σ a face $F \cup \{\infty\}$. In many cases, just adding one facet containing all vertices on the boundary will work as well.

Given a collection Γ of boundary faces of Σ, we can iterate over all elements of Σ, which are not contained in Γ: Iterate over all faces of M, which are not contained in $\Gamma \cup (M \setminus \Sigma)$.

In the case that Γ is the collection of all boundary faces and Σ is therefore the tight span of the polytopal subdivision and if Σ permits to add a single facet F to obtain a closed d-manifold M, we can just iterate over all faces of M not contained in F.

Example 2.14 (Closed sets of a matroid)**.** The MacLane–Steinitz exchange property (see e.g. [9, Lemma 1.4.2]) ensures that the closed sets of a matroid form a locally branched finite lattice.

Example 2.15 (Locally branched lattices with non-trivial intersection)**.** Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be finite locally branched meet semi-sublattices of B_n such that for $p \in \mathcal{P}_i$ and $q \in \mathcal{P}_j$ with $\mathrm{Atoms}(p) \subseteq \mathrm{Atoms}(q)$ it follows that $p \in \mathcal{P}_j$. Then the iterator may iterate through all elements of their union by first iterating through $\mathcal{P}_1$, then through all elements in $\mathcal{P}_2$ not contained in $\mathcal{P}_1$ and so on.

Example 2.16 (Polyhedral complexes)**.** Using the iteration as in the previous example allows to iterate through polytopal or polyhedral complexes.

3. The algorithm

Let $\mathcal{P}$ be a finite locally branched lattice given as a meet semi-sublattice of the Boolean lattice B_n. We assume $\mathrm{Atoms}(\mathcal{P}) = \{1, \ldots, n\}$ and we may identify an element p with $\mathrm{Atoms}(p)$. The following algorithm is a recursively defined depth-first iterator through the elements of $\mathcal{P}$. Given $p \in \mathcal{P}$ and its lower covers $x_1, \ldots, x_k$, the iterator yields p and then computes, one after the other, the lower covers of $x_1, \ldots, x_k$, taking into account those to be ignored, and then recursively proceeds. Being an *iterator* means that the algorithm starts with only assigning the input to the respective variables and then waits in its current state. Whenever an output is requested, it starts from its current state and runs to the point **OUTPUT**, outputs the given output, and again waits.

```
1   DECLARATIONS
2       •  c ,  r ,  v ,  x              – sets of integers
3       •  C, C_new , V, V_new         – duplicate-free lists (of sets of integers)
4       •  D, E                        – lists (of sets of integers), possibly with duplicates
5
6   Algorithm FaceIterator
7
8   INPUT
9       •  C    – all coatoms of P not contained in any of V
10      •  V    – list of subsets of {1,...,n}
11      •  r    – subset of {1,...,n}
12
13  OUTPUT :  Each  c ∈ P  with  c ≠ 1̂  ∧  c ∩ r = ∅  ∧  ∀v ∈ V : c ⊈ v.
14
15  PROCEDURE
16  if  C ≠ [ ] :
17      c := an_element(C)
18      if  c ∩ r = ∅ :
19          OUTPUT c    # continue from here
20
21      D = [ c ∩ x : x ∈ C, x ≠ c ]
22      E = [ x ∈ D : ∀ v ∈ V : x ⊈ v ]
23      C_new = inclusion_maximals(E)
24      V_new = [ v : v ∈ V ]
25
26      # Apply algorithm for sublattice [0, c].
27      FaceIterator(C_new , V_new , r)
28
29      V = V + [ c ∪ r ]              # append V
30      C = [ x ∈ C : x ⊈ c ∪ r ]   # update C accordingly
31      FaceIterator(C, V, r)
```

.

The recursive function calls in line 27 and line 31 can be executed in parallel: **r** can be declared constant. The lists **C** and **V** will be modified, but not their elements.

One should think of **V** as a list of inclusion maximal elements of those already visited.

The algorithm does not visit $\hat{1}$. However, we will still assume that this is the case whenever suitable. This would have to be done, before calling the algorithm.

For polyhedra, a technically elaborated version of this algorithm is implemented in `SageMath`[1]. Before proving the correctness of the algorithm, we provide several detailed examples. In the examples, we do not ignore any atoms and set $r = \emptyset$. Also **V** will be empty if not specified.

Example 3.1. We apply the algorithm to visit faces of a square.
- **INPUT**: $\texttt{C} = [\{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}]$
- $\texttt{c} = \{1,2\}$, **OUTPUT**: $\{1,2\}$
- $\texttt{C}_{\texttt{new}} = [\{1\}, \{2\}]$
- Apply **FaceIterator** to sublattice $[\hat{0}, \{1,2\}]$
  - **INPUT**: $\texttt{C} = [\{1\}, \{2\}]$
  - $\texttt{c} = \{1\}$, **OUTPUT**: $\{1\}$
  - $\texttt{C}_{\texttt{new}} = [\emptyset]$
  - Apply **FaceIterator** to sublattice $[\hat{0}, \{1\}]$
    - **INPUT**: $\texttt{C} = [\emptyset]$
    - $\texttt{c} = \emptyset$, **OUTPUT**: $\emptyset$
    - ($\texttt{C}_{\texttt{new}}$ is empty)
    - Apply **FaceIterator** to sublattice $[\hat{0}, \hat{0}]$ without output
    - Add $\emptyset$ to $\texttt{V}$ (to the copy in this call of `FaceIterator`)
    - Reapply **FaceIterator** to sublattice $[\hat{0}, \{1\}]$
    - **INPUT**: $\texttt{C} = []$, $\texttt{V} = [\emptyset]$
  - $\texttt{V} = [\{1\}]$
  - Reapply **FaceIterator** to sublattice $[\hat{0}, \{1,2\}]$
  - **INPUT**: $\texttt{C} = [\{2\}]$, $\texttt{V} = [\{1\}]$
  - $\texttt{c} = \{2\}$, **OUTPUT**: $\{2\}$
  - Apply **FaceIterator** to sublattice $[\hat{0}, \{2\}]$
    - **INPUT**: $\texttt{C} = []$, $\texttt{V} = [\{1\}]$
  - $\texttt{V} = [\{1\}, \{2\}]$
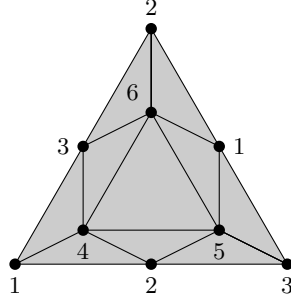
FIGURE 2. Minimal triangulation of $\mathbb{RP}^2$ with vertices $1, \ldots, 6$
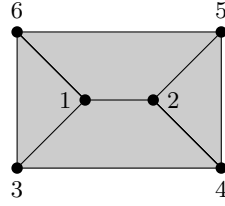
FIGURE 3. Tight span on vertices $1, \ldots, 6$ with interior vertices 1 and 2

- – Reapply **FaceIterator** to sublattice $[\hat{0}, \{1, 2\}]$
  - – **INPUT**: $\mathtt{C} = []$, $\mathtt{V} = [\{1\}, \{2\}]$
- $\mathtt{V} = [\{1, 2\}]$
- Reapply **FaceIterator** to entire lattice
- **INPUT**: $\mathtt{C} = [\{1, 4\}, \{2, 3\}, \{3, 4\}]$, $\mathtt{V} = [\{1, 2\}]$
- $\mathtt{c} = \{1, 4\}$, **OUTPUT**: $\{1, 4\}$
- Apply **FaceIterator** to sublattice $[\hat{0}, \{1, 4\}]$
  - – **INPUT**: $\mathtt{C} = [\{4\}]$, $\mathtt{V} = [\{1, 2\}]$
  - – $\mathtt{c} = \{4\}$, **OUTPUT**: $\{4\}$
  - – Apply **FaceIterator** to sublattice $[\hat{0}, \{4\}]$ without output
- $\mathtt{V} = [\{1, 2\}, \{1, 4\}]$

- ... further outputs: $\{2, 3\}$, $\{3\}$, $\{3, 4\}$

Example 3.2. We apply the algorithm to the minimal triangulation of $\mathbb{RP}^2$ given in Figure 2.
- **INPUT**: $\mathtt{C} = [\{1, 2, 4\}, \ldots, \{4, 5, 6\}]$
- $\mathtt{c} = \{1, 2, 4\}$, **OUTPUT**: $\{1, 2, 4\}$, $\{1, 2\}$, $\{1\}$, $\emptyset$, $\{2\}$, $\{1, 4\}$, $\{4\}$, $\{2, 4\}$
- $\mathtt{c} = \{1, 2, 6\}$, **OUTPUT**: $\{1, 2, 6\}$, $\{1, 6\}$, $\{6\}$, $\{2, 6\}$
- $\mathtt{c} = \{1, 3, 4\}$, **OUTPUT**: $\{1, 3, 4\}$, $\{1, 3\}$, $\{3\}$, $\{3, 4\}$
- $\mathtt{c} = \{1, 3, 5\}$, **OUTPUT**: $\{1, 3, 5\}$, $\{1, 5\}$, $\{5\}$, $\{3, 5\}$
- $\mathtt{c} = \{1, 5, 6\}$, **OUTPUT**: $\{1, 5, 6\}$, $\{5, 6\}$
- $\mathtt{c} = \{2, 3, 5\}$, **OUTPUT**: $\{2, 3, 5\}$, $\{2, 3\}$, $\{2, 5\}$
- $\mathtt{c} = \{2, 3, 6\}$, **OUTPUT**: $\{2, 3, 6\}$, $\{3, 6\}$
- $\mathtt{c} = \{2, 4, 5\}$, **OUTPUT**: $\{2, 4, 5\}$, $\{4, 5\}$
- $\mathtt{c} = \{3, 4, 6\}$, **OUTPUT**: $\{3, 4, 6\}$, $\{4, 6\}$
- $\mathtt{c} = \{4, 5, 6\}$, **OUTPUT**: $\{4, 5, 6\}$

Example 3.3. We apply the algorithm to the tight span given in Figure 3.
- **INPUT**: $\mathtt{C} = [\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{1, 3, 6\}, \{2, 4, 5\}]$,
  $\mathtt{V} = [\{3, 4, 5, 6\}]$
- $\mathtt{c} = \{1, 2, 3, 4\}$, **OUTPUT**: $\{1, 2, 3, 4\}$, $\{1, 2\}$, $\{1\}$, $\{2\}$, $\{1, 3\}$, $\{2, 4\}$
- $\mathtt{c} = \{1, 2, 5, 6\}$, **OUTPUT**: $\{1, 2, 5, 6\}$, $\{1, 6\}$, $\{2, 5\}$
- $\mathtt{c} = \{1, 3, 6\}$, **OUTPUT**: $\{1, 3, 6\}$
- $\mathtt{c} = \{2, 4, 5\}$, **OUTPUT**: $\{2, 4, 5\}$

Example 3.4. Visit all faces of the polyhedral complex given in Figure 4.
- *Incorrect application* by applying to the polyhedra as if they were facets.
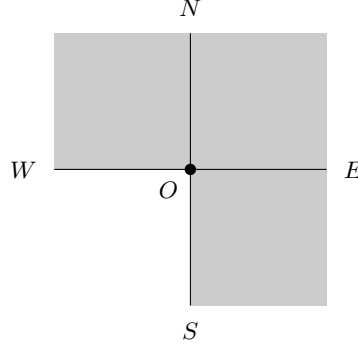
FIGURE 4. Polyhedra complex consisting of three quadrants of the plane with south-west quadrant removed

- – **INPUT**: $\mathtt{C} = [\{W, N, 0\}, \{N, E, 0\}, \{S, E, 0\}]$
- – $\mathtt{c} = \{W, N, 0\}$, **OUTPUT**: $\{W, N, 0\}$
- – $\mathtt{C_{new}} = [\{N, 0\}]$, **OUTPUT**: $\{N, 0\}$
- – $\mathtt{V} = [\{W, N, 0\}]$
- – $\mathtt{c} = \{N, E, 0\}$, **OUTPUT**: $\{N, E, 0\}$
- – $\mathtt{C_{new}} = [\{E, 0\}]$, **OUTPUT**: $\{E, 0\}$
- – $\mathtt{V} = [\{W, N, 0\}, \{N, E, 0\}]$
- – $\mathtt{c} = \{S, E, 0\}$, **OUTPUT**: $\{S, E, 0\}$
- – $\mathtt{C_{new}} = []$
- • *Correct application* by applying successively to all faces of all polyhedra:
  - – Before applying **FaceIterator** to $\{W, N, 0\}$: **OUTPUT**: $\{W, N, 0\}$
  - – Apply algorithm to $\{W, N, 0\}$:
    - ∗ **INPUT**: $\mathtt{C} = [\{W, 0\}, \{N, 0\}]$, $\mathtt{V} = [\{W, N\}]$
    - ∗ **OUTPUT**: $\{W, 0\}$, $\{0\}$, $\{N, 0\}$
  - – Before applying **FaceIterator** to $\{N, E, 0\}$: **OUTPUT**: $\{N, E, 0\}$
  - – Apply algorithm to $\{N, E, 0\}$:
    - ∗ **INPUT**: $\mathtt{C} = [\{E, 0\}]$, $\mathtt{V} = [\{W, N, 0\}, \{N, E\}]$
    - ∗ **OUTPUT**: $\{E, 0\}$
  - – Before applying **FaceIterator** to $\{S, E, 0\}$: **OUTPUT**: $\{S, E, 0\}$
  - – Apply algorithm to $\{S, E, 0\}$:
    - ∗ **INPUT**: $\mathtt{C} = [\{S, 0\}]$, $\mathtt{V} = [\{W, N, 0\}, \{N, E, 0\}, \{S, E\}]$
    - ∗ **OUTPUT**: $\{S, 0\}$

3.1. Correctness of the algorithm. As assumed, let $\mathcal{P}$ be a locally branched meet semi-sublattice of the Boolean lattice B_n. In the following, we see that the algorithm visits each element $p \in \mathcal{P}$ not contained in any of $\mathtt{V}$ and not containing any of $\mathtt{r}$ exactly once. We remark that we could relax the condition on B_n: It suffices for the interval $[p, \hat{1}]$ to be locally branched for p to be visited exactly once under those conditions.

Proposition 3.5. *The algorithm* **FaceIterator** *is well-defined in the following sense: Let* $\mathtt{C}$ *be the list of coatoms of* $\mathcal{P}$ *that are not contained in any of* $\mathtt{V}$.

(i) *Then the call of* **FaceIterator** *in line 27 calls the algorithm for the sublattice* $[\hat{0}, \mathtt{c}]$ *with* $\mathtt{C_{new}}$ *being the list of coatoms of* $[\hat{0}, \mathtt{c}]$ *that are not contained in any of* $\mathtt{V}$.

(ii) *The call of* **FaceIterator** *in line 31 calls the algorithm for* $\mathcal{P}$, *but with* $\mathtt{c} \cup \mathtt{r}$ *appended to* $\mathtt{V}$. *The updated* $\mathtt{C}$ *contains all coatoms of* $\mathcal{P}$ *that are not contained in any of* V.

PROOF.   (i) $\mathtt{C_{new}}$ is a sublist of $\mathtt{E}$, which is a sublist of $\mathtt{D}$. By construction all elements in $\mathtt{D}$ and thus in $\mathtt{C_{new}}$ are strictly below $\mathtt{c}$.

Now, let $\mathtt{d} \prec \mathtt{c} \prec \hat{1}$ in $\mathcal{P}$ and let $\mathtt{d}$ not be contained in any of $\mathtt{V}$. Since $\mathcal{P}$ is locally branched there is an element $\mathtt{x} \neq \mathtt{c}$ with $\mathtt{d} < \mathtt{x} \prec \hat{1}$, implying $\mathtt{d} = \mathtt{c} \cap \mathtt{x}$. If $\mathtt{d}$ is not contained in any of $\mathtt{V}$, then the same must hold for $\mathtt{x}$ as $\mathtt{d} < \mathtt{x}$. This implies that $\mathtt{x}$ is in $\mathtt{C}$ and thus $\mathtt{d}$ is contained in $\mathtt{D}$.

Assume that $\mathtt{d}$ is contained in $\mathtt{D}$. It is contained in $\mathtt{E}$ exactly if it is not contained in any of $\mathtt{V}$ by construction of $\mathtt{E}$ in line 22.

It remains to show that $\mathtt{d}$ in $\mathtt{E}$ is contained in $\mathtt{C_{new}}$ exactly if $\mathtt{d} \prec \mathtt{c}$. As any element in $\mathtt{E}$ is strictly below $\mathtt{c}$, $\mathtt{d} \prec \mathtt{c}$ implies that $\mathtt{d}$ is inclusion maximal. On the other hand, if $\mathtt{d}$ is not inclusion maximal, it lies below a coatom of $[\hat{0}, \mathtt{c}]$. As $\mathtt{d}$ is in $\mathtt{E}$, it cannot be contained in any of $\mathtt{V}$ and the same holds for this coatom. Thus, $\mathtt{d}$ is not inclusion maximal in $\mathtt{E}$.

(ii) Line 30 removes exactly those elements in `C` that are contained in $\mathtt{c} \cup \mathtt{r}$.

$\square$

THEOREM 3.6. *The algorithm* **FaceIterator** *iterates exactly once over all elements in $\mathcal{P}$, which are not contained in any of* `V`*, and do not contain any element in* `r`*.*

PROOF. We argue by induction on the cardinality of `C`. First note that the cardinalities of $\mathtt{C}_{\mathtt{new}}$ and `C` in the two subsequent calls of **FaceIterator** in lines 27 and 31 are both strictly smaller than the cardinality of `C`. If `C` = [], then all elements of $\mathcal{P} \setminus \hat{1}$ are contained in elements of `V`, and the algorithm correctly does not output any element. Suppose that `C` is not empty and let `c` be the element assigned in line 17. Let $p \in \mathcal{P}$. If p is contained in an element of `V`, then it is not contained in the initial `C`. By Proposition 3.5 it will never be contained in `C` in any recursive call and thus cannot be output. On the other hand, if p contains an element in `r`, then it cannot be output by line 18. Otherwise,

- if $p = \mathtt{c}$, then the algorithm outputs p correctly in line 19,
- if $p < \mathtt{c}$, then p is contained in $[\hat{0}, \mathtt{c}]$ and is output by **FaceIterator** in line 27 by induction,
- if $p \not\leq \mathtt{c}$, then $p \not\leq \mathtt{c} \cup \mathtt{r}$ and p is output in the call of **FaceIterator** in line 31 by induction, as it is not contained in any of $\mathtt{V} + [\mathtt{c} \cup \mathtt{r}]$.

$\square$

3.2. Variants of the algorithm. We finish this section with a dualization property followed by explicitly stating the result when applying the algorithm for the variants discussed in Subsection 2.1.

Let $\mathcal{P}$ be a locally branched lattice and `V` be a list of coatoms, and `r` be a list of atoms. Instead of directly applying Theorem 3.6 one can consider $\mathcal{P}^{\mathrm{op}}$, $\mathtt{V}^{\mathrm{op}}$, and $\mathtt{r}^{\mathrm{op}}$. $\mathtt{V}^{\mathrm{op}}$ is now a list of atoms of $\mathcal{P}^{\mathrm{op}}$ (given as indices). $\mathtt{r}^{\mathrm{op}}$ is a list of coatoms of $\mathcal{P}^{\mathrm{op}}$ (each given as list of atoms of $\mathcal{P}^{\mathrm{op}}$).

Corollary 3.7. *The algorithm can be applied to visit all elements of $\mathcal{P}^{\mathrm{op}}$, which are not contained in any of $\mathtt{r}^{\mathrm{op}}$, and do not contain any element in $\mathtt{V}^{\mathrm{op}}$. This is the same as visiting all elements of $\mathcal{P}$, which are not contained in any of* `V`*, and do not contain any element in* `r`*, but that each element is now given as coatom-incidences instead of atom-incidences.*

We later see in Theorem 4.1 that considering $\mathcal{P}^{\mathrm{op}}$ instead of $\mathcal{P}$ might be faster as the runtime depends on the number of coatoms. For example, in Example 3.2 one could apply the algorithm to $\mathcal{P}^{\mathrm{op}}$ to improve runtime as there are 10 facets but only 6 vertices.

Corollary 3.8.   (i) *Let P be a polytope and let $\mathcal{P}$ be its face lattice with coatoms* `C` *given as vertex/atom incidences. The algorithm then outputs every face of P as a list of vertices it contains.*
   (ii) *Let P be an unbounded polyhedron with trivial linear subspace and let $\overline{P}$ be a projectively equivalent polytope with marked face. Provided* `V`*, a list containing just the marked face of $\overline{P}$, and* `C`*, the remaining facets, all given as vertex incidences. The algorithm then outputs all faces of P as vertex/ray incidences.*
  (iii) *Let P be a finite polytopal subdivision of a closed manifold. Let* `C` *be the maximal faces given as vertex incidences. The algorithm then outputs the faces of P as vertex incidences.*
  (iv) *Let Σ be an extended tight span in $\mathbb{R}^d$ as described in Example 2.13. Let Γ be a subset of boundary faces of Σ. As explained in Example 2.13 we can embedd Σ into a (triangulated) manifold M. Given the maximal faces of $M \setminus \Sigma$ and Γ as* `V` *and the remaining maximal faces as* `C` *all as vertex incidences. The algorithm outputs the faces of Σ not contained in any of Γ as vertex incidences.*
   (v) *Let P be a polyhedral complex. Given the atom incidences of the facets of each maximal face. The algorithm can be iteratively applied to output all faces of P:*
        Let F be a maximal face. Given the atom incidences of the facets of F (and possibly the marked far face). As described in (i) and (ii), the algorithm outputs all faces of F. Let $F_1, \dots, F_n$ be some other maximal faces. Append $F_1, \dots, F_n$ (as atom/ray incidences) to `V` *and remove all elements of* `C` *contained in any of $F_1, \dots, F_n$. Then, the algorithm outputs all faces of F not contained in any of $F_1, \dots, F_n$.*

4. Data structures, memory usage, and theoretical runtime

The operations used in the algorithm are `intersetion`, `is_subset` and `union`. It will turn out that the crucial operation for the runtime is the subset check.

For the theoretical runtime we consider representation as (sparse) *sorted-lists-of-atoms*. However, in the implementation we use (dense) *atom-incidence-bit-vectors*. This is theoretically slightly slower, but the crucial operations can all be done using bitwise operations. The improved implementation only considers the significant chunks, which has optimal theoretic runtime again. A chunk contains 64/128/256 bits depending on the architecture. We store for each set, which chunk has set bits. To check whether A is a subset of B, it suffices to loop through the significant chunks of A.

Experiments suggest that for many atoms `RoaringBitmap` described in [8] performs even better. [5]

Observe that a *sorted-lists-of-atoms* needs as much memory as there are incidences. Consider two sets A and B (of integers) of lengths a and b, respectively, and a (possibly unsorted) list C of m sets $C_1, \ldots, C_m$ with $\alpha = |C_1| + \cdots + |C_m|$. Using standard implementations, we assume in the runtime analysis that

- intersection $A \cap B$ and union $A \cup B$ have runtime in $\mathcal{O}(a + b) = \mathcal{O}(\max(a, b))$ and the results can be guaranteed to be sorted,
- a subset check $A \subseteq B$ or $A \subsetneq B$ has runtime in $\mathcal{O}(b)$ and
- to check whether A is subset of any element in C has runtime in $\mathcal{O}(\alpha)$.

Let $d + 1$ be the number of elements in a longest chain in $\mathcal{P}$, let $m = |\mathtt{C}|$, $n = |\operatorname{Atoms}(\mathcal{P})|$, and let

$$\alpha = \sum_{a \in \mathtt{C} \cup \mathtt{V}} |a \cup \mathtt{r}|.$$

(In the case that $\mathtt{V}$ and $\mathtt{r}$ are both empty, the sum of cardinalities of $C_1, \ldots, C_m$ is α. Otherwise it is bounded by α). Let φ be the number of elements in $\mathcal{P}$ that are not contained in any of $\mathtt{V}$. If $\mathtt{r}$ is empty, this is the cardinality of the output.

THEOREM 4.1. *The algorithm has memory consumption $\mathcal{O}(\alpha \cdot d)$ and runtime $\mathcal{O}(\alpha \cdot m \cdot \varphi)$.*

Remark 4.2. We assume constant size of integers as in [7]. To drop this assumption, one needs to multiply our runtime and memory usage by $\log(\max(n, m))$ and likewise for [7].

PROOF. We will assume that recursive calls are not made, when $\mathtt{C}$ resp. $\mathtt{C_{new}}$ are empty. To check whether a list is empty can be performed in constant time.

Then, the number of recursive calls is bounded by φ: Any element assigned to $\mathtt{c}$ is an element from $\mathcal{P}$. Any element in $\mathcal{P}$ is assigned at most once. This follows from the proof of Theorem 3.6. (It follows directly, if $\mathtt{r}$ is empty as then every element assigned to $\mathtt{c}$ is also output.)

Note that for each recursive call of **FaceIterator** the number of elements in $\mathtt{C}$ is bounded by m. The sum of the cardinalities of $\mathtt{C}$, $\mathtt{D}$, $\mathtt{E}$, and $\mathtt{V}$ is bounded by α. So is the cardinality of $\mathtt{r}$.

To prove the claimed runtime, it suffices to show that each call of **FaceIterator** not considering recursive calls has runtime in $\mathcal{O}(m \cdot \alpha)$. With above assumptions, this follows: The check preceeding the output in line 18 can be performed in $\mathcal{O}(\alpha)$. Obtaining $\mathtt{D}$ in line 21 can be done in $\mathcal{O}(n \cdot m)$ (each size is bounded by n and there are at most m intersections to perform). $n \leq \alpha$ as every atom must be contained at least once in a coatom of $\mathtt{C}$, in an element of $\mathtt{V}$ or in $\mathtt{r}$. To check, whether an element is contained in any of V can be done in $\mathcal{O}(\alpha)$. Again there are at most m elements, so the claim holds for line 22. To check wether an element is contained in any of $\mathtt{E}$ can be done in $\mathcal{O}(\alpha)$ and the claim holds for 23. Note, that we can perform a strict subset check for larger indices and a non-strict subset check for smaller indices to remove duplicates as well. Clearly, we can copy V to $\mathtt{V_{new}}$ in this time and append V in line 29. The indivual subset check for each of at most m sets in line 30 is done in $\mathcal{O}(\alpha)$. This proofs the claimed runtime.

A single call of **FaceIterator** has memory usage at most $c \cdot \alpha$ for a global constant c, not taking into account the recursive calls. The call in line 31 does not need extra memory as all old variables can be discarded. The longest chain of the lattice $[0, \mathtt{c}]$ is at most of length $d - 1$. By induction the call of **FaceIterator** in line 27 has total memory consumption at most $(d - 1) \cdot c \cdot \alpha$. The claimed bound follows.   □

Remark 4.3. When searching elements with certain properties, we might observe from $\mathtt{c}$ that all of $[\hat{0}, \mathtt{c}]$ is not of interest. After assigning $\mathtt{c}$ we can skip everything until line 27. This will result in not visiting any further element of $[\hat{0}, \mathtt{c}]$ (some might have been visited earlier).

If $\mathtt{r}$ is empty and the check whether to skip $[\hat{0}, \mathtt{c}]$ can be performed in time $\mathcal{O}(m \cdot |\mathtt{c}|)$, the runtime will reduce linear to the number of elements output:

Appending $\mathtt{V}$ in line 29 and updating $\mathtt{C}$ in line 30 can both be performed in time $\mathcal{O}(m \cdot |\mathtt{c}|)$. This runtime can be accounted for by an upper cover of $\mathtt{c}$, which we must have visited: The sum of the cardinalities of the lower covers of an element is bounded by α. Thus the runtime of skipping elements accounts for runtime in $\mathcal{O}(m \cdot \alpha)$ per element visited.

If we skip some of the $[\hat{0}, \mathtt{c}]$ in this way, the runtime will therefore be in $\mathcal{O}(\alpha \cdot m \cdot \psi)$, where ψ is the cardinality of the output.

4.1. Computing all cover relations. Applying the algorithm to a graded locally branched meet semi-sublattice of B_n while keeping track of the recursion depth allows an a posteriori sorting of the output by the level sets of the grading. The recursion depth is the number of iterative calls using line 27. We obtain the same bound for generating all cover relations as V. Kaibel and M. E. Pfetsch [7]. For a list L of (sorted) subsets of $\{1, \ldots, n\}$ we additionally assume that

- two sets of cardinality a and b resp. can be lexicographically compared in time $\mathcal{O}(\min(a, b))$,

---

[5] `RoaringBitmap` performs better for computing the f-vector of the d-dimensional associahedron for $d \geq 11$. See discussion on `https://github.com/Ezibenroc/PyRoaringBitMap/pull/59`.

- L can be sorted in time $\mathcal{O}(n \cdot |L| \cdot \log |L|)$, and
- if L is sorted, we can look up, whether L contains some set of cardinality a in time $\mathcal{O}(a \cdot \log |L|)$.

Proposition 4.4. *Let $\mathcal{P}$ be a graded meet semi-sublattice of $\mathcal{B}_n$. Assume each level set of $\mathcal{P}$ to be given as sorted-lists-of-atoms, one can generate all cover relations in time $\mathcal{O}(\alpha \cdot \min(m, n) \cdot \varphi)$ with quantities as defined above using the above algorithm.*

Observe that in the situation of this proposition, V and r are both empty and in particular α is the total length of the coatoms. As before φ is the number of elements in $\mathcal{P}$. The level sets are not assumed to be sorted.

PROOF. First, we sort all level-sets. As each element in $\mathcal{P}$ appears exactly once in each level set, all level sets can be sorted in time $\mathcal{O}(n \cdot \varphi \cdot \log \varphi)$.

Then, we intersect each element with each coatom, obtaining its lower covers and possibly other elements. We look up each intersection to determine the lower covers. All such intersections are obtained in time $\mathcal{O}(\varphi \cdot m \cdot n)$. For a fixed element the total length of its intersections with all coatoms is bounded by α. Hence, all lookups are done in time $\mathcal{O}(\varphi \cdot \alpha \cdot \log \varphi)$.

Finally, we note that $m, n \leq \alpha$ and that $\log \varphi \leq \min(m, n)$. □

In the ungraded case, one first sorts all elements in $\mathcal{P}$, and then intersects each element p with all coatoms. The inclusion maximal elements among those strictly below p are lower covers of p. They can be looked up in the list of sorted elements to obtain an index. Observe that all this is done time $\mathcal{O}(\alpha \cdot m \cdot \varphi)$.

4.2. Theoretic comparison. We first compare our approach with the one from V. Kaibel and M. E. Pfetsch [7]. They have written their algorithm in terms of closure operators starting from the vertices. Using the terminology of our paper (applying their algorithm to the dual case), there are some differences:

(i) To translate from *coatom-representation* to *atom-representation* the corresponding coatoms are intersected. Likewise they translate from *atom-representation* to *coatom-representation*.
(ii) They store the *coatom-representation* of c.
(iii) As a first step, they obtain the *atom-representation* of c.
(iv) To obtain a list containing all lower covers, they intersect c with all coatoms not containing c.
(v) For checking, which of the sets is inclusion maximal, they transform them back to *coatom-representation*. The subset check is then trivial.
(vi) They do not store visited faces.

Our runtime is the same as in their approach. They require memory in $\mathcal{O}(\varphi \cdot m)$.

In [7, Section 3.3] however, they mention that one could use lexicographic ordering to avoid storing all the faces and achieve similar memory usage as our approach. To our knowledge, this memory efficient approach has not been implemented.

Advantages of our algorithm to the lexicographic approach are:

(i) The order of output is somewhat flexible. We are free to choose any element c from C (in any recursion step) in line 17 of the algorithm.
(ii) Our order relates to the lattice: By Remark 4.3 we could skip some $[\hat{0}, c]$ and effectively reduce runtime. E.g. we could use the iterator to only visit faces of a polytope, which are not a simplex, in runtime linear to the output.
(iii) Let G be the automorphism group of $\mathcal{P}$. If we sort the elements lexicographically by their *coatom representation*, any first element representative of an orbit, is contained in a first representative of a coatom-orbit. To visit all orbits, it suffices to visit only the first representatives of the facet-orbits and then append each facet in the orbit to V. This will efficiently reduce runtime.

The other algorithm we compare our approach to is described by W. Bruns, P. García-Sánchez, C. O'Neill and D. Wilburne in [3] and was independently developed to our algorithm. It also stores each element in atom-representation. In each step of the algorithm, the atom-representation is computed. Then, each element c is intersected with all coatoms and the inclusion maximal elements are computed just like in our approach via subset checks. Finally the inclusion maximal elements are transformed to atom-representation and, after a lookup, the new ones are stored.

Although there is no theoretic analysis of runtime and memory consumption, it appears that the runtime agrees with [7] (although the implementation by dense *bit-vectors* does not achieve this) and the memory usage is $\mathcal{O}(\varphi \cdot n)$.

They introduced usage of the automorphism group of the atom-coatom-incidences and have first developed a variation that visits the first representative.

5. Performance of the algorithm implemented in SageMath

We present running times for the several computations. An implementation is available through sage-8.9 and later.   This   uses   dense   *bit-vectors*   and   has runtime $\mathcal{O}(n \cdot m \cdot \varphi)$.

The presented algorithm can be parallelized easily as the recursive calls in line 27 and line 31 do not depend on each other. The implementation using bitwise operations allows to use advanced CPU-instructions such as *Advanced Vector Extensions*. Furthermore, *bit-vectors* can be enhanced to account for sparse vectors, obtaining assymptotically optimal runtime. All these improvements are available in `sage-9.4`.

The benchmarks are performed on an Intel® Core™ i7-7700 CPU @ 3.60GHz x86_64-processor with 4 cores and 30 GB of RAM. The computations are done either using

- `polymake 3.3` [5], or
- `normaliz 3.7.2` [2], or
- the presented algorithm in `sage-8.9`, or
- the presented algorithm in `sage-8.9` with additional parallelization, intrinsics, and subsequent improvements as explained above.

Remark 5.1. It appears that there is no difference in performance regarding the f-vector for `polymake 3.3` and `polymake 4.1`. Likewise for `normaliz 3.7.2` and `normaliz 3.8.9` (a computation goal `DualFVector` was added, but we already applied `normaliz` to the dual problem, whenever suitable).
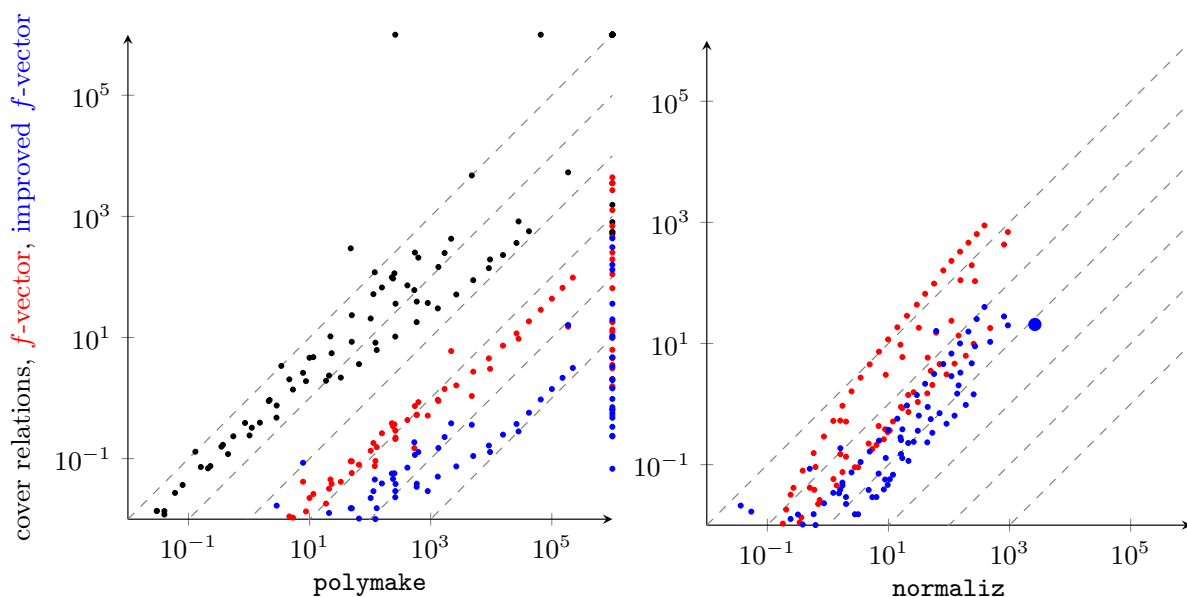
FIGURE 5. Runtime Comparison. Every dot represents one best-of-five computation, and every shifted diagonal is a factor-10 faster runtime. Dots on the boundary represent memory overflows. The left diagram compares `polymake` to three implementations: `SageMath` computing all cover relations (black), `SageMath` computing the f-vector (red) and `SageMath` with aforementioned improvements (blue). The right diagram compares `normaliz` to `SageMath` (without and with improvements) computing the f-vector. E.g. the fat blue dot in the right diagram has coordinates slightly bigger than $(10^3, 10)$ and represents computing the f-vector of the Kunz cone with parameter $m = 15$. It took 2622 seconds with `normaliz` and 21 seconds with `SageMath` with improvements.

We computed:

(i) cover relations and f-vector in `polymake` (x-axis in the left diagram of Figure 5),
(ii) f-vector in `normaliz` *with* parallelization, (x-axis in the right diagram of Figure 5),
(iii) all cover relations with the presented implementation in `SageMath`,
(iv) f-vector with the presented implementation in `SageMath`,
(v) f-vector with the presented implementation in `SageMath` *with* parallelization, intrinsics and additional improvements.

Remark 5.2.   • The computation of the f-vector in i also calculates all cover relations.
- `polymake` also provides a different algorithm to compute the f-vector from the h-vector for simplicial/simple polytopes (providing this additional information sometimes improves the performance in `polymake`).
- `normaliz` does not provide an algorithm to compute the cover relations.

For every algorithm we record the best-of-five computation[6] on

- the simplex of dimension n,
- several instances of the cyclic polytope of dimension 10 and 20,
- the associahedron of dimension n,
- the permutahedron of dimension n embedded in dimension $n + 1$,
- a 20-dimensional counterexample to the Hirsch-conjecture,
- the cross-polytope of dimension n,
- the Birkhoff-polytope of dimension $(n - 1)^2$,
- joins of such polytopes with their duals,
- Lawrence polytopes of such polytopes,
- Kunz cone in dimension $n - 1$ defined in Definition 6.3.

Figure 5 confirms that the implementations behave about the same assymptotically. For computing the cover relations, the implementation in `sage-8.9` is as fast or up to 100 times faster than the implementation in `polymake`. For computing only the f-vector, the implementation in `sage-8.9` is about 1000 times faster than `polymake` and a bit faster than `normaliz`. However, `normaliz` used 4 physical cores (8 threads) for those results, while `sage-8.9` only needed one. With parallelization and other improvements one can gain a factor of about 10 using 4 physical cores.

5.1. Possible reasons for the performance difference. In [3, Remark 5.5] it was mentioned that for one example about 6% of the computation time is needed for converting from *coatom-representation* to *atom-representation* and back and for computing the intersections. Another 4% are required for computing which elemens in `C` are inclusion-maximal. 40% are observed by checking, which elements were seen before. The rest are other operations such as system operations.

Contrary to this, in `sage-8.9` other operations are almost neglectible. About 90% of the time is spend doing subset checks. About 10% of the time is spend computing the intersections. Note, that those times may vary depending on the application. There is no need to do the expensive translation from *coatom-representation* to *atom-representation* and back.

It seems that our algorithm allowed to avoid those 90% that `normaliz` spends with lookups and other operations.

As parallelization is trivial, there is very little overhead even with as much as 40 threads: The overhead is due to the fact that we perform a depth-first search. When the function is called, we can almost immediately dispatch the call at line 31. In this way, there are trivially independent jobs (one per coatom) that can be parallized without overhead. However, the workload is shared badly. In the extreme example of the Boolean lattice, half of the elements visited will be subject to the first job in that way and we should expect this to take half of the time.

Our approach is to have one job per coatom of the coatoms (paralleziation at codimension 2). The jobs are assigned monotonic dynamically. Each thread has independent data structure and recomputes the first `C_new` if necessary. This still has almost no overhead. However, when computing the bad orbits of the Kunz cone with 40 threads, one of them took about a day longer to finish than the others.

Experiments suggest that parallizing at codimension 3 has still reasonable overhead and will pay off with enough threads (depending on the lattice). At level 4 the overhead seems unreasonable.

As for `polymake` the comparison is of course unfair. Their implemenation tries to compute all cover relations in decent time and assymptotically optimal. Of course, one can be much faster, when not storing all cover relations. The implementation in `sage-8.9` to compute the cover relations is usually faster.

We refer to Section 7 for detailed runtimes, which were plotted in Figure 5.

6. Application of the algorithm to Wilf's conjecture

W. Bruns, P. García-Sánchez, C. O'Neill and D. Wilburne provided an algorithm that verifies Wilf's conjecture for a given fixed multiplicity [3]. We give a brief overview of their approach:

Definition 6.1. A *numerical semigroup* is a set $S \subset \mathbb{Z}_{\geq 0}$ containing 0 that is closed under addition and has finite complement.

- Its *conductor* $c(S)$ is the smallest integer c such that $c + \mathbb{Z}_{\geq 0} \subseteq S$.
- Its *sporadic elements* are the elements $a \in S$ with $a < c(S)$ and let $n(S)$ be the number of sporadic elements.
- The *embedding dimension* $e(S) = |S \setminus (S + S)|$ is the number of elements that cannot be written as sum of two elements.
- The *multiplicity* $m(S)$ is the minimal nonzero element in S.

Conjecture 6.2 (Wilf). *For any numerical semigroup S,*

$$c(S) \leq e(S)n(S).$$

---

[6]Benchmarks were taken on a desktop computer. Best-of-five was chosen to account for other processes causing temporary slowdown. All implementations are completely deterministic without randomness.

For fixed mulitplicity m one can analyse certain polyhedra to verify this conjecture.

Definition 6.3 ([3, Def. 3.3]). Fix an integer $m \geq 3$. The *relaxed Kunz polyhedron* is the set P'_m of rational points $(x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}$ satisfying

$$x_i + x_j \geq x_{i+j} \qquad 1 \leq i \leq j \leq m-1, \quad i+j < m,$$
$$x_i + x_j + 1 \geq x_{i+j} \qquad 1 \leq i \leq j \leq m-1, \quad i+j > m,$$

The *Kunz cone* is the set C_m of points $(x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}$ satisfying

$$x_i + x_j \geq x_{i+j} \qquad 1 \leq i \leq j \leq m-1, \quad i+j \neq m.$$

(All indices in this definition are taken modulo m.)

Remark 6.4. Every numerical semigroup S of multiplicity m corresponds to a lattice point in the *relaxed Kunz polyhedron* (not vice versa, thus *relaxed*): x_i is the smallest integer such that $i + mx_i \in S$. The inequalities correspond to $j + mx_j \in S$ implying that $i + j + m(x_i + x_j) \in S$.

Definition 6.5. Let F be a face of P'_m or C_m. Denote by $e(F) - 1$ and $t(F)$ the number of variables not appearing on the right and left hand sides resp. of any defining equations of F.

The Kunz cone is a translation of the relaxed Kunz polyhedron. $e(F)$ and $t(F)$ are invariants of this translation.

Every numerical semigroup S of multiplicty m corresponds to a (all-)positive lattice point in P'_m. If the point corresponding to S lies in the relative interior of some face $F \subseteq P'_m$, then $e(F) = e(S)$ and $t(F) = t(S)$, see [3, Thm. 3.10 & Cor. 3.11]. The following proposition summarizes the approach by which we can check for bad faces:

Proposition 6.6 ([3]). *There exists a numerical semigroup S with multiplicity m that violates Wilf's conjecture if and only if there exists a face F of P'_m with positive integer point $(x_1, \ldots, x_{m-1}) \in F^\circ$ and $f \in [1, m-1]$ such that*

$$mx_i + i \leq mx_f + f \quad \text{for every } i \neq f$$

and

$$mx_f + f - m + 1 > e(F) \cdot \big(mx_f + f - m - (x_1 + \ldots, +x_{m-1}) + 1\big).$$

A face F of P'_m is *Wilf* if no interior point corresponds to a violation of Wilf's conjecture. A face F of C_m is *Wilf*, if the corresponding face in P'_m is Wilf.

Proposition 6.7 ([3] p. 9). *Let F be a face of P'_m or C_m.*
- *If $e(F) > t(F)$, then F is Wilf.*
- *If $2e(F) \geq m$, then F is Wilf.*

Checking Wilf's conjecture for fixed multiplicity m can be done as follows:

(i) For each face F in C_m check if Proposition 6.7 holds.
(ii) If Proposition 6.7 does not hold, check with Proposition 6.6 if the translated face in P'_m contains a point corresponding to a counterexample of the Wilf's conjecture.

We say that a face F of C_m is *bad* if Proposition 6.7 does not hold. The group of units $(\mathbb{Z}/m\mathbb{Z})^\times$ acts on $\mathbb{R}^{m-1}$ by multiplying indices. The advantage of the Kunz cone over the (relaxed) Kunz polyhedron is that it is symmetric with respect to this action. Even more, $e(F)$ and $t(F)$ are invariant under this action. Thus in order to determine the bad faces, it suffices to determine for one representative of its orbit, if it is bad. We say that an orbit is *bad*, if all its faces are bad.

While [3] uses a modified algorithm of `normaliz` to determine all bad orbits, we replace this by the presented algorithm.

We can also apply the symmetry of C_m. As described in Subsection 4.2, we can sort the elements by lexicographic comparison by the *coatom representation*. It suffices to visit the first facet of each orbit to see the first element of each orbit (and possibly more). The concrete implementation is available as a branch of `SageMath`[7].

This implementation worked well enough to apply the presented algorithm to Wilf's conjecture. In Table 1 we compare the runtimes of computing the bad orbits.

These are performed on an Intel® Xeon™ CPU E7-4830 @ 2.20GHz with a total of 1 TB of RAM and 40 cores. We used 40 threads and about 200 GB of RAM. The timings in [3] used only 32 threads and a slightly slower machine.

While testing all bad faces takes a significant amount of time, recent work by S. Eliahou has simplified this task.

THEOREM 6.8 ([4, Thm. 1.1]). *Let S be a numerical semigroup with multiplicity m. If $3e(S) \geq m$ then S satisfies Wilf's conjecture.*

---

[7]See `https://git.sagemath.org/sage.git/tree/?h=u/gh-kliem/KunzConeWriteBadFaces`.

TABLE 1. Runtime comparison for determining the bad orbits

m	# bad orbits	`normaliz`	`SageMath`
15	180,464	3:33 m	7 s
16	399,380	54:39 m	1:14 m
17	3,186,147	19:35 h	16:55 m
18	17,345,725	27:13 d	16:22 h
19	100,904,233		14:22 d

TABLE 2. Number of bad orbits to check and time it took to verify Wilf's conjecture for them

m	# orbits	time
15	193	1 s
16	5,669	11 s
17	7,316	31 s
18	17,233	1:54 m
19	285,684	2:22 h

Checking the remaining orbits can be done quickly (we used an Intel$^{\circledR}$ Core$^{\text{TM}}$ i7-7700 CPU @ 3.60GHz x86_64-processor with 4 cores). For each of the orbits with $3e < m$, we have checked whether the corresponding region is empty analogously to the computation in [3]. See Table 2 for the runtimes.

This computation yields the following proposition.

Proposition 6.9. *Wilf's conjecture holds for $m = 19$.*

7. Detailed runtimes

We give, for each of the five computations, an example of how it is executed for the 2-simplex.

i Compute cover relations and f-vector in `polymake` from vertex-facet-incidences. To our knowledge, this applies the algorithm in [7].

```
polytope> new Polytope(VERTICES_IN_FACETS=>
                       [[0,1],[0,2],[1,2]])->F_VECTOR;
```

.

ii Compute f-vector with `normaliz` (via `pynormaliz`, optional package of `SageMath`). This is the algorithm described in [3].

```
sage: P = polytopes.simplex(2, backend='normaliz')
sage: P._nmz_result(P._normaliz_cone, 'FVector')
```

.

iii Compute cover relations in `SageMath`. This is the algorithm **FaceIterator** with Proposition 4.4.

```
sage: C = CombinatorialPolyhedron([[0,1],[0,2],[1,2]])
sage: C._compute_face_lattice_incidences()  # non-public
```

.

iv & v Compute f-vector in `SageMath` using **FaceIterator**.

```
sage: C = CombinatorialPolyhedron([[0,1],[0,2],[1,2]])
sage: C.f_vector()
```

.

For displaying the runtimes, we use the following notations:

- Δ_d for the d-dimensional simplex,

- $\mathcal{C}_{d,n}$ for the d-dimensional cyclic polytope with n vertices,

- $\mathcal{A}_d$ for the d-dimensional associahedron,

- $\mathcal{P}_d$ for the d-dimensional permutahedron,

- $\mathcal{H}$ for the 20-dimensional counterexample to the Hirsch conjecture,

- $\square_d$ for the d-cube,

- $\mathcal{B}_n$ for the $(n-1)^2$-dimensional Birkhoff polytope,

- P^{op} for the polar dual of a polytope P,

- $L(P)$ for the Lawrence polytope of P,

- K_n the Kunz cone in ambient dimension $n-1$, treated as a inhomogenous polyhedron of dimension $n-2$.

The runtimes of the five best-of-five computations on the various examples are as given in Table 3.

Acknowledgements

TABLE 3. Runtime comparison for obtaining the f-vector in `polymake` i and `normaliz` ii to runtime in `SageMath` for all cover relations iii, f-vector iv and improved algorithm for the f-vector v. Table consists of best-of-five timings rounded to whole seconds and the number of atoms and coatoms, the dimension and the number of elements. "MOF" indicates that the process was killed due to memory overflow, and a dash indicates a runtime of less than half a second.

	Time in s					$\min(n,m)$	$\max(n,m)$	d	$\lvert\varphi\rvert$
	i	ii	iii	iv	v				
$\square_5 \star \square_5^{\mathrm{op}}$	3	—	3	—	—	42	42	11	59,536
$\square_6 \star \square_6^{\mathrm{op}}$	119	2	120	—	—	76	76	13	532,900
$\square_7 \star \square_7^{\mathrm{op}}$	4,771	31	4,744	1	—	142	142	15	4,787,344
$\square_8 \star \square_8^{\mathrm{op}}$	MOF	478	MOF	18	11	272	272	17	43,059,844
$\mathcal{A}_4 \star \mathcal{A}_4^{\mathrm{op}}$	3	—	1	—	—	56	56	9	39,204
$\mathcal{A}_5 \star \mathcal{A}_5^{\mathrm{op}}$	537	2	61	—	—	152	152	11	817,216
$\mathcal{A}_6 \star \mathcal{A}_6^{\mathrm{op}}$	186,000	61	5,335	15	16	456	456	13	18,318,400
$\mathcal{A}_7$	1	—	—	—	—	35	1,430	7	20,794
$\mathcal{A}_8$	21	—	2	—	—	44	4,862	8	103,050
$\mathcal{A}_9$	589	2	39	1	—	54	16,796	9	518,860
$\mathcal{A}_{10}$	28,226	16	826	10	—	65	58,786	10	2,646,724
$\mathcal{A}_{11}$	MOF	237	MOF	196	5	77	208,012	11	13,648,870
$\mathcal{B}_5$	228	9	99	—	—	25	120	16	6,092,722
$\mathcal{B}_6$	MOF	MOF	MOF	2,710	310	36	720	25	19,989,171,034
$\square_{10}$	3	—	—	—	—	20	1,024	10	59,050
$\square_{11}$	19	—	2	—	—	22	2,048	11	177,148
$\square_{12}$	122	1	8	—	—	24	4,096	12	531,442
$\square_{13}$	893	2	37	1	—	26	8,192	13	1,594,324
$\square_{14}$	9,558	9	195	3	—	28	16,384	14	4,782,970
$\square_{15}$	MOF	32	MOF	18	1	30	32,768	15	14,348,908
$\square_{16}$	MOF	154	MOF	111	3	32	65,536	16	43,046,722
$\square_{17}$	MOF	938	MOF	693	20	34	131,072	17	129,140,164
$\square_{18}$	MOF	MOF	MOF	4,410	132	36	262,144	18	387,420,490
$\mathcal{C}_{10,20}$	33	—	2	—	—	20	4,004	10	171,650
$\mathcal{C}_{10,21}$	65	—	4	—	—	21	5,733	10	238,912
$\mathcal{C}_{10,22}$	128	1	6	—	—	22	8,008	10	325,954
$\mathcal{C}_{10,23}$	260	1	10	—	—	23	10,948	10	436,864
$\mathcal{C}_{10,24}$	586	1	18	1	—	24	14,688	10	576,258
$\mathcal{C}_{10,25}$	1,310	2	30	1	—	25	19,380	10	749,312
$\mathcal{C}_{10,26}$	2,647	2	51	2	—	26	25,194	10	961,794
$\mathcal{C}_{10,27}$	5,039	3	88	3	—	27	32,319	10	1,220,096
$\mathcal{C}_{10,28}$	9,128	5	140	5	—	28	40,964	10	1,531,266
$\mathcal{C}_{10,29}$	15,624	7	230	7	—	29	51,359	10	1,903,040
$\mathcal{C}_{10,30}$	26,123	10	365	12	—	30	63,756	10	2,343,874
$\mathcal{C}_{10,31}$	41,903	14	568	19	1	31	78,430	10	2,862,976
$\mathcal{C}_{10,32}$	65,655	20	MOF	29	1	32	95,680	10	3,470,338
$\mathcal{C}_{10,33}$	100,307	29		44	1	33	115,830	10	4,176,768
$\mathcal{C}_{10,34}$	150,241	40		66	2	34	139,230	10	4,993,922
$\mathcal{C}_{10,35}$	222,340	57		98	3	35	166,257	10	5,934,336
$\mathcal{C}_{10,36}$		80		161	5	36	197,316	10	7,011,458
$\mathcal{C}_{10,37}$		110		231	7	37	232,841	10	8,239,680
$\mathcal{C}_{10,38}$		152		329	10	38	273,296	10	9,634,370
$\mathcal{C}_{10,39}$		209		463	16	39	319,176	10	11,211,904
$\mathcal{C}_{10,30}$		287		642	25	40	371,008	10	12,989,698
$\mathcal{C}_{10,41}$		385		889	40	41	429,352	10	14,986,240
$\mathcal{C}_{10,42}$		MOF		1216	67	42	494,802	10	17,221,122
$\mathcal{H}$	MOF		MOF	453		40	36,425	20	353,731,266
$\mathcal{P}_6$	8	1	3	—	—	62	720	5	4,684
$\mathcal{P}_7$	2,179	17	428	6	—	126	5,040	6	47,294
Δ_{16}	2	—	1	—	—	17	17	16	131,072
Δ_{17}	5	—	2	—	—	18	18	17	262,144
Δ_{18}	10	1	5	—	—	19	19	18	524,288
Δ_{19}	22	1	10	—	—	20	20	19	1,048,576
Δ_{20}	50	3	23	—	—	21	21	20	2,097,152
Δ_{21}	113	5	52	—	—	22	22	21	4,194,304
Δ_{22}	252	11	115	—	—	23	23	22	8,388,608
Δ_{23}	548	21	253	1	—	24	24	23	16,777,216
Δ_{24}	MOF	44	553	2	—	25	25	24	33,554,432
Δ_{25}	MOF	91		3	—	26	26	25	67,108,864
Δ_{26}	MOF	189		6	1	27	27	26	134,217,728
Δ_{27}	MOF	MOF		13	2	28	28	27	268,435,456

|	Time in s					min(n,m)	max(n,m)	d	φ
	i	ii	iii	iv	v				
$\mathcal{C}_{20,21}$	48	3	297	—	—	21	21	20	2,097,152
$\mathcal{C}_{20,22}$	260	6	MOF	—	—	22	121	20	4,190,210
$\mathcal{C}_{20,23}$	MOF	12	MOF	1	—	23	506	20	8,341,504
$\mathcal{C}_{20,24}$	MOF	26	MOF	2	—	24	1,716	20	16,474,114
$\mathcal{C}_{20,25}$		44		6	1	25	5,005	20	32,120,832
$\mathcal{C}_{20,26}$		110		24	3	26	13,013	20	61,554,690
$\mathcal{C}_{20,27}$		269		107	9	27	30,888	20	115,546,112
$\mathcal{C}_{20,28}$		811		431	28	28	68,068	20	212,004,866
$\mathcal{C}_{20,29}$		MOF		1,790	98	29	140,998	20	379,838,464
$L(\mathcal{A}_3)$	625	17	209	1	—	28	416	17	10,435,664
$L(\mathcal{C}_{4,8})$	1	—	—	—	—	16	148	12	50,746
$L(\mathcal{C}_{4,9})$	5	—	1	—	—	18	261	13	167,098
$L(\mathcal{C}_{4,10})$	23	1	6	—	—	20	430	14	524,800
$L(\mathcal{C}_{4,11})$	102	2	21	—	—	22	671	15	1,582,332
$L(\mathcal{C}_{4,12})$	412	7	73	—	—	24	1,002	16	4,606,876
$L(\mathcal{C}_{4,13})$	1,696	21	249	1	—	26	1,443	17	13,015,500
$L(\mathcal{C}_{4,14})$	MOF	70	803	5	1	28	2,016	18	35,829,622
$L(\mathcal{C}_{5,8})$	2	—	—	—	—	16	120	13	61,456
$L(\mathcal{C}_{5,9})$	9	—	2	—	—	18	261	14	224,330
$L(\mathcal{C}_{5,10})$	50	1	8	—	—	20	514	15	782,596
$L(\mathcal{C}_{5,11})$	263	5	36	—	—	22	935	16	2,614,020
$L(\mathcal{C}_{5,12})$	1,339	16	146	1	—	24	1,596	17	8,390,656
$L(\mathcal{C}_{5,13})$	MOF	50	554	4	1	26	2,587	18	25,990,044
$L(\mathcal{C}_{5,14})$	MOF	144	MOF	14	2	28	4,018	19	77,999,464
$L(\mathcal{C}_{4,6}^{\mathrm{op}})$	2	—	1	—	—	18	57	13	121,894
$L(\mathcal{C}_{4,7}^{\mathrm{op}})$	MOF	53	510	2	—	28	672	18	24,233,912
$L(\mathcal{C}_{4,8}^{\mathrm{op}})$	MOF	MOF	MOF	1,270	159	40	4,208	24	7,188,015,356
$L(\mathcal{C}_{5,7}^{\mathrm{op}})$	156	8	67	—	—	24	110	17	4,577,866
$L(\mathcal{C}_{5,8}^{\mathrm{op}})$	MOF	MOF	MOF	3,563	434	40	5,928	25	17,364,262,196
$L(\square_5^{\mathrm{op}})$	12	1	5	—	—	20	84	15	479,566
$L(\square_6^{\mathrm{op}})$	239	10	94	—	—	24	152	18	5,909,086
$L(\square_7^{\mathrm{op}})$	MOF	124	1,552	5	1	28	284	21	72,097,678
$L(\square_8^{\mathrm{op}})$	MOF	MOF	MOF	65	10	32	544	24	873,869,950
$L(\square_4)$	MOF	255	3,525	10	1	32	296	20	130,851,046
$\mathcal{K}_{12}$		2			—	60	1,864	10	669,794
$\mathcal{K}_{13}$		16			—	72	7,005	11	4,389,234
$\mathcal{K}_{14}$		137			1	84	15,585	12	21,038,016
$\mathcal{K}_{15}$		2,622			21	98	67,262	13	137,672,474
$\mathcal{K}_{16}$		MOF			241	112	184,025	14	751,497,188

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Introduction to Chapter 4

We return to a point partition problem. In the next chapter, we will show that for ten points in the plane with colors, such that no more than three points have one color, one of the following can be found:

- Three triangles on nine of the ten points, each containing the last point. The triangles are formed each by vertices of three different colors.
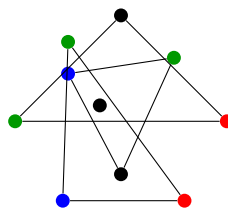
- Two triangles on six of the ten points and two segments on the four remaining points. Each triangle contains the intersection of the segments. Each triangle and each section is formed by vertices of three resp. two different colors.
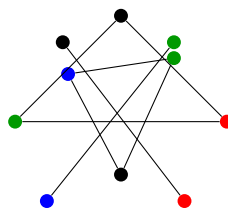
As in Chapters 1 and 2 for a suitable group G we can construct a G-equivariant map $A \to Y$ such that the non-existence of a G-equivariant extension $X \to Y$ shows that such a partition must always exist. However, this approach has worked only if the number of parts is prime. In our case, the number of parts is four and thus the topological methods could not be applied. Instead, we use the help of a computer and a depth first iterator as in Chapter 3 to completely analyze the situation.

We abstract ten points in the plane to an oriented matroid, which allows us to reduce to finitely many cases. Overall there are $14,309,547$ cases that distinguish the orientations of the triangles on each three of the ten points. This does not suffice however, ten points in convex position is exactly one of those cases, but this case does not determine whether the intersection point of two segments is contained in a triangle. We develop a reduction, such that for each of the $14,309,547$ cases we build a k-partite graph – where k depends on the oriented matroid – and if this graph does not have k-clique, one of the above partition can always be found for each choice of color classes.

There exist algorithms to find the maximal clique size (*clique number*) and there even exist algorithms to iterate over k-cliques in a k-partite graph. For some of the graphs such an algorithm can be used, but for some of the graphs the implementations did not terminate in reasonable time, see table 1. Hence, we present a new algorithm that applies the pivot rule of [2] to the k-partite situation as in [1]. The new algorithm can verify that none of the $14,309,547$ graphs has such a clique and therefore there always exists such a colorful partition as above.

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A new k-partite graph k-clique iterator and the optimal colored Tverberg problem for ten colored points

JONATHAN KLIEM

Abstract We provide an algorithm that verifies the optimal colored Tverberg problem for 10 points in the plane: Every 10 points in the plane in color classes of size at most 3 can be partitioned in 4 rainbow pieces such that their convex hulls intersect in a common point.

This is achieved by translating the problem to k-partite graphs and using a new algorithm to verify that those graphs do not have a k-clique.

Contents

1. Introduction

Tverberg's Theorem has many variations. This is a compact presentation of some of them:

Problem 1.1 (Topological/Affine Tverberg)**.** Let $d \geq 1$ and $r \geq 2$ be integers and let $N = (d+1)(r-1)$ and let $f \colon \Delta_N \to \mathbb{R}^d$ be a continous/affine map from the N-simplex to the d-dimensional Euclidean space. Are there pairwise disjoing faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that their images with respect to f intersect?

Optimal Colored: Let further $C_1, \ldots, C_m$ be color classes of the vertex set of size at most $r-1$. Can $\sigma_1, \ldots, \sigma_r$ be chosen rainbow (for each $i = 1, \ldots, r$ vertices of σ_i have pairwise distinct colors)?

The affine Tverberg Problem without colors is solved affirmatively by Tverberg's theorem [14]. For the topological Tverberg Problem the answer is more complicated and depends on r: Bárány, Shlosman and Szücs provided in 1981 that the answer is yes when r is a prime [6]. In 1987, Özaydin extended this for r a prime power [16] in a never published preprint. This first published proof was later provided by Volovikov [15]. Recently, Blagojević, Frick and Ziegler discoverd that the answer is no for all r that are not prime powers [4].

One generalization of the Tverberg Problem is the optimal colored version, which implies other previous generalizations. Blagojević, Matschke and Ziegler [3] showed that the topological optimal colored Tverberg problem holds for primes r. It is unkown whether the optimal colored version holds for prime powers even in the affine case.

We refer the reader to Bárány, Blagojević and Ziegler [7] for the history of the problem. In particular, [7, Figure 10] motivates this paper. We show:

THEOREM 1.2. *Let* $X = \{x_1, \ldots, x_{10}\}$ *be points in* $\mathbb{R}^2$. *Let* $C_1, \ldots, C_m$ *be a partition of* X, *such that* $|C_j| \leq 3$ *for any* $j = 1, \ldots, m$. *There is a partition* $X_1, \ldots, X_4$ *of* X *such that* $|X_i \cap C_j| = 1$ *for any* $i = 1, \ldots, 4$, $j = 1, \ldots, m$ *and such that*

$$\operatorname{conv}(X_1) \cap \operatorname{conv}(X_2) \cap \operatorname{conv}(X_3) \cap \operatorname{conv}(X_4) \neq \emptyset.$$

We obtain this on the level of oriented matroids using a k-clique iterator on k-partite graphs. As other algorithms fail to analyze those graphs, we present a new k-clique iterator for k-partite graphs.

As we will see in Proposition 2.4, we may assume the points $x_1, \ldots, x_{10}$ to be in strong general position, as moving them slightly the new Tverberg partitions will be a subset of the old ones. We may also assume that $|C_1| \geq |C_2| \geq \cdots \geq |C_m|$ and that $|C_m \cup C_{m-1}| > 3$. So $(|C_1|, \ldots, |C_m|)$ is one of $(3,3,3,1)$, $(3,3,2,2)$, $(2,2,2,2,2)$. The cases

$$d = 2, \ r = 4, \ (|C_1|, \ldots, |C_m|) = (3,3,2,2) \quad \text{and} \quad d = 2, \ r = 4, \ (|C_1|, \ldots, |C_m|) = (2,2,2,2,2)$$

can be reduced to

$$d = 3, \ r = 4, \ (|C_1|, \ldots, |C_m|) = (3,3,3,1) \quad \text{and} \quad d = 4, \ r = 4, \ (|C_1|, \ldots, |C_m|) = (3,3,3,3,1)$$

by [3, Reduction of Thm. 2.2 to Thm 2.1]. However, the higher dimensional cases with more points seems much harder to solve.

There are $2800 + 6300 + 945 = 10045$ partitions of $x_1, \ldots, x_{10}$ into color classes of cardinalities $(3,3,3,1)$, $(3,3,2,2)$ and $(2,2,2,2,2)$, respectively. So for each collection of points $x_1, \ldots, x_{10} \in \mathbb{R}^2$ we must check for all those 10045 color partitions, whether there is a rainbow Tverberg partition.

Tverberg partitions of 10 points in general position in $\mathbb{R}^d$ are either of cardinalities $(3,3,3,1)$ or $(3,3,2,2)$. We will call those types $3,3,3,1$ and $3,3,2,2$. For type $3,3,3,1$ we can simply check the oriented matroid on $x_1, \ldots, x_{10}$ induced by the point configuration. For type $3,3,2,2$, this does not suffice: All 10-gons have the same oriented matroid, but their Tverberg partitions are different.

This paper is organized as follows: In Section 2 we reduce the problem to points in strong general position. In Section 3 we will explain how Theorem 1.2 reduces to verifying for each acyclic chirotope on $x_1, \ldots, x_{10}$ of rank 3 that some k-partite graph does not have a k-clique. In Section 4 we will show how a slightly modified graph can be efficiently constructed. Finally, in Section 5 we will introduce a new k-partite k-clique iterator that is able to verify this. We compare its performance with other algorithms on random graphs in Sectoin 6.

2. Reducing the problem to strong general position

To simplify the combinatorics, it would be nice if the points $x_1, \ldots, x_n$ in $\mathbb{R}^d$ are in general position:

Definition 2.1 ([11, Def. 2.1])**.** A finite set $X = \{x_1, \ldots, x_n\}$ in $\mathbb{R}^d$ is said to be in *strong general position* if every subset of X of size $\leq d+1$ is affinely independent and for any collection $\{X_1, \ldots, X_r\}$ of r pairwise disjoint subsets of X we have

$$d - \dim \bigcap_{i=1}^{r} \operatorname{aff}(F_i) = \min \left(d+1, \sum_{i=1}^{r}(d - \dim \operatorname{aff}(F_i)) \right).$$

In the case of points in $\mathbb{R}^2$, this means that no three lines (defined by pairwise distinct points of X) contain a common point.

We will show that we can slightly modify our points without creating new Tverberg partitions to achieve this property:

Lemma 2.2. *The subset of points in strong general position of $(\mathbb{R}^d)^n$ is dense.*

PROOF. Perles and Sigron state this implicitly in [11, Sec. 3] by showing that the points in strong general position can be seen as the subset on which a non-trivial polynomial in all points evaluates non-zero. □

Lemma 2.3. *Let $I_1, \ldots, I_r$ be disjoint subsets of $\{1, \ldots, n\}$. The set $U_{I_1, \ldots, I_r}$ of points $(x_1, \ldots, x_n)$ in $(\mathbb{R}^d)^n$ for which*

$$\emptyset = \bigcap_{i=1}^{r} \operatorname{conv} \{x_j : j \in I_i\}$$

is open.

PROOF. Let D be the minimal distance that can be achieved from the convex sets induced by $I_1, \ldots, I_r$:

$$D = \min_{z \in \mathbb{R}^d} \max_{i=1, \ldots, r} \left(d\left(z, \operatorname{conv}\{x_j : j \in I_i\}\right) \right).$$

This distance D being non-zero is equivalent to

$$\emptyset = \bigcap_{i=1}^{r} \operatorname{conv} \{x_j : j \in I_i\}.$$

Suppose that $(x_1, \ldots, x_n) \in U_{I_1, \ldots, I_r}$, which implies $D > 0$. Let $(y_1, \ldots, y_r) \in (\mathbb{R}^d)^n$ with $d(x_i, y_i) < D$ for all $i = 1, \ldots, n$. Then

$$\min_{z \in \mathbb{R}^d} \max_{i=1, \ldots, r} \left(d\left(z, \operatorname{conv}\{y_j : j \in I_i\}\right) \right)$$

$$> \min_{z \in \mathbb{R}^d} \max_{i=1, \ldots, r} \left(d\left(z, \operatorname{conv}\{x_j : j \in I_i\}\right) \right) - D$$

$$\geq 0.$$

This implies that

$$\emptyset = \bigcap_{i=1}^{r} \operatorname{conv}\{y_j : j \in I_i\}.$$

and therefore $(y_1, \ldots, y_r) \in U_{I_1, \ldots, U_r}$. $\hfill\square$

Proposition 2.4. *Let $X = \{x_1, \ldots, x_n\}$ be a set of points in $\mathbb{R}^d$. There exists a set $Y = \{y_1, \ldots, y_n\}$ in $\mathbb{R}^d$ in strong general position such that for any disjoint subsets $I_1, \ldots, I_r$ of $\{1, \ldots, 10\}$ with*

$$\emptyset = \bigcap_{i=1}^{r} \operatorname{conv}\{x_j : j \in I_i\}.$$

we have

$$\emptyset = \bigcap_{i=1}^{r} \operatorname{conv}\{y_j : j \in I_i\}.$$

PROOF. There exists only finitely many disjoint subsets $I_1, \ldots, I_r$ with

$$\emptyset = \bigcap_{i=1}^{r} \operatorname{conv}(x_j : j \in I_i).$$

For those the sets $U_{I_1, \ldots, I_r}$ are open by Lemma 2.3 and their intersection contains $(x_1, \ldots, x_n)$. This open non-empty intersection must contain some $(y_1, \ldots, y_n)$ in strong general position by Lemma 2.2. $\hfill\square$

3. Reducing the problem to k-partite graphs

To utilize a computer, we use the chirotope axioms of an oriented matroid. For an introduction on oriented matroids we refer the reader to Björner, Las Vergnas, Sturmfels, White and Ziegler [2].

Definition 3.1 ([2, Def. 3.5.3]). A *chirotope* of rank r on a set E is a mapping $\chi \colon E^r \to \{-1, 0, 1\}$, which satisfies the following three properties:

(B0) χ is not identically zero,

(B1) χ is alternating, that is

$$\chi(x_{\sigma_1}, \ldots, x_{\sigma_r}) = \operatorname{sign}(\sigma)\chi(x_1, \ldots, x_r)$$

for all $x_1, \ldots, x_r \in E$ and every permutation σ,

(B2) for all $x_1, \ldots, x_r, y_1, \ldots, y_r \in E$ such that

$$\chi(y_i, x_2, x_3, \ldots, x_r) \cdot \chi(y_1, y_2, \ldots, y_{i-1}, x_1, y_{i+1}, y_{i+2}, \ldots, y_r) \geq 0$$

for $i = 1, \ldots, r$, we have

$$\chi(x_1, \ldots, x_r) \cdot \chi(y_1, \ldots, y_r) \geq 0.$$

Instead of Axiom (B2) will will use an equivalent formulation:

Lemma 3.2 ([2, Lem 3.5.4]). *Let $\chi \colon E^r \to \{-1, 0, 1\}$ be a map satisfying (B0) and (B1). Then (B2) is equivalent to the following: For any $\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r) \neq 0$ there exists $i \in \{1, \ldots, r\}$ such that*

$$\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r) = \chi(y_i, x_2, x_3, \ldots, x_r)\chi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r).$$

As explained on [2, Page 5], every point configuration in $\mathbb{R}^d$ corresponds to an acyclic chirotope of rank $d+1$. We will provide a definition of an acyclic chirotope of rank 3:

Definition 3.3. Let χ be a chirotope on E of rank 3. The chirotope χ is *acyclic* if for every $x_1, \ldots, x_4 \in E$ with $\chi(x_1, x_2, x_3) \neq 0$ one of

$$\chi(x_1, x_2, x_4), \ \chi(x_2, x_3, x_4), \ \chi(x_3, x_1, x_4)$$

is equal to $\chi(x_1, x_2, x_3)$.

Lemma 3.4. *The definition of an acyclic chirotope of rank 3 agrees with the definition [2, Def. 3.4.7]: An oriented matroid is acyclic if it does not contain a positive circuit.*

PROOF. Let C be a circuit with support contained in $x_1, \ldots, x_4$. As x_1, x_2, x_3 is a basis, we may assume that $x_4 \in C^+$.

$x_1 \in \underline{C}$ is equivalent to x_2, x_3, x_4 being a basis.

So either $\chi(x_2, x_3, x_4) = 0$ or by [2, Prop. 3.5.2] we conclude that

$$\chi(x_2, x_3, x_4) = \chi(x_4, x_2, x_3) = -C(x_1)C(x_4) = -C(x_1)\chi(x_1, x_2, x_3).$$

Our statements have been invariant with respect to cyclic permutation of x_1, x_2, x_3. If, C is a positive circuit, then all of

$$\chi(x_1, x_2, x_4), \ \chi(x_2, x_3, x_4), \ \chi(x_3, x_1, x_4)$$

are zero or equal to $-\chi(x_1, x_2, x_3)$. On the other hand, if C is not a positive circuit, then one of them is equal to $\chi(x_1, x_2, x_3)$. $\hfill\square$

Let $x_1, x_2, \ldots, x_{10} \in \mathbb{R}^2$ in strong general position. For any distinct $a, b, c, d \in \{x_1, \ldots, x_{10}\}$ we denote the intersection of the lines ab and cd by $y_{a,b,c,d}$. Note that $y_{a,b,c,d}$ has 8 different notations:

$$y_{a,b,c,d}, \ y_{a,b,d,c}, \ y_{b,a,c,d}, \ y_{b,a,d,c}, \ y_{c,d,a,b}, \ y_{c,d,b,a}, \ y_{d,c,a,b}, \ y_{d,c,b,a}.$$

We define the sets

$$X := \{x_1, \ldots, x_{10}\}, \quad Y := \{y_{a,b,c,d} \colon |\{a, b, c, d\}| = 4, \{a, b, c, d\} \subset \{1, \ldots, 10\}\}.$$

This way the points $x_1, \ldots, x_{10} \in \mathbb{R}^2$ induce an acyclic chirotope of rank 3 on $X \sqcup Y$ with

$$\chi(x, x', x'') \neq 0$$

for all $x, x', x'' \in X$ and

$$\chi(a, b, y_{a,b,c,d}) = 0 = \chi(c, d, y_{a,b,c,d})$$

for all $a, b, c, d \in \{x_1, \ldots, x_{10}\}$ pairwise distinct.

Definition 3.5. The point $y_{a,b,c,d}$ is an *intersection point*, if $y_{a,b,c,d} = \operatorname{conv}(a, b) \cap \operatorname{conv}(c, d)$.

Lemma 3.6. *The point $y_{a,b,c,d}$ is an intersection point, if and only if*

$$\chi(a, b, c)\chi(a, b, d) = -1 = \chi(c, d, a)\chi(c, d, b).$$

PROOF. $y_{a,b,c,d}$ is defined to be the intersection of the lines ab and cd. Thus, it is an intersection point, if and only if $\operatorname{conv}(a, b) \cap \operatorname{conv}(c, d) \neq \emptyset$. As a, b, c, d are in general position there is a cycle C on the ground set (a, b, c, d). By [2, Prop. 3.5.2] we conclude that

$$\chi(a, b, c) = -C(c)C(d)\chi(a, b, d)$$
$$\chi(c, d, a) = -C(a)C(b)\chi(c, d, b).$$

Hence $C = (a, b, -c, -d)$ or $C = (-a, -b, c, d)$ is equivalent to

$$\chi(a, b, c)\chi(a, b, d) = -1 = \chi(c, d, a)\chi(c, d, b).$$

$\square$

As $x_1, \ldots, x_{10}$ are assumed to be in strong general position, we have $\chi(y_{a,b,c,d}, e, f) \neq 0$ unless $\{e, f\} = \{a, b\}$ or $\{e, f\} = \{c, d\}$.

A chirotope on $X \sqcup Y$ determines all Tverberg partitions:

Lemma 3.7. *Let $a, b, c, d, e, f, g \in \{x_1, \ldots, x_{10}\}$ be pairwise distinct and let $y_{a,b,c,d}$ be an intersection point.*

(1) $a \in \operatorname{conv}(e, f, g)$ is equivialent to

$$\chi(e, f, a) = \chi(f, g, a) = \chi(g, e, a).$$

(2)

$$y_{a,b,c,d} \in \operatorname{conv}(e, f, g)$$

is equivalent to

$$\chi(e, f, y_{a,b,c,d}) = \chi(f, g, y_{a,b,c,d}) = \chi(g, e, y_{a,b,c,d}).$$

PROOF. a, e, f, g and $y_{a,b,c,d}, e, f, g$ are in general position by assumptions. Hence, (2) is implied by (1). As a, e, f, g are in general position, there exists a cycle C with ground set a, e, f, g. W.l.o.g. we have $C(g) = 1$. By [2, Prop. 3.5.2] we conclude that

$$\chi(e, f, a) = C(f)C(g)\chi(g, e, a) = C(f)\chi(g, e, a)$$
$$\chi(e, f, a) = C(e)C(g)\chi(f, g, a) = C(e)\chi(f, g, a)$$

$a \in \operatorname{conv}(e, f, g)$ is equivalent to $C = (-a, e, f, g)$, which implies

$$\chi(e, f, a) = \chi(f, g, a) = \chi(g, e, a).$$

For the other direction, this equality implies $C(e) = 1 = C(f)$. As the oriented matroid is acyclic we have $C(a) = -1$. $\square$

This means that restriction of the chirotope to X determines, whether there is a Tverberg partition of type $3, 3, 3, 1$ and the restriction to each $X \cup \{y_{a,b,c,d}\}$ determines whether there is a Tverberg partition of type $3, 3, 2, 2$.

The restrictions of those chirotopes to X are exactly acyclic chirotopes on $x_1, \ldots, x_{10}$ of rank 3. They were previously classified by Aichholzer, Aurenhammer and Krasser [1]. In total there are 14,320,182 such chirotopes and 14,309,547 of them are realizable.

By Lemma 3.7 the chirotope χ determines all Tverberg partitions. The restriction of χ to X, determines all partitions of type $3, 3, 3, 1$, but partitions of type $3, 3, 2, 2$ are not determined other than in special cases. We give a definition to allow us to talk about this:

Definition 3.8. Let $X \subseteq Z \subseteq X \sqcup Y$. The Tverberg partitions on $\chi|_Z$ are those Tverberg partitions that exist by Lemma 3.7:

- Tverberg partitions on $\chi|_Z$ contain all partitions of type $3, 3, 3, 1$ regardless of the choice of Z.

- Tverberg partitions on $\chi\big|_Z$ contain exactly those partitions $(X_1, X_2, \{a, b\}, \{c, d\})$ of type $3, 3, 2, 2$, for which $y_{a,b,c,d} \in Z$.

For the remainder $y := y_{a,b,c,d}$. Depending on $\chi\big|_X$, some of the intersection points can be ignored:

Lemma 3.9. *Let $a, b, c, d, e, f, g \in \{x_1, \ldots, x_{10}\}$ be pairwise distinct such that $y := y_{a,b,c,d}$ is an intersection point.*

(1) If $y \in \mathrm{conv}(e, f, g)$ then
$$\chi(a, b, h) \geq 0$$
for some $h \in \{e, f, g\}$.

(2) If there is a Tverberg partition of type $3, 3, 2, 2$ with $y = y_{a,b,c,d}$, then
$$|\{h \in \{x_1, \ldots, x_{10}\} \setminus \{a, b, c, d\} \colon \chi(a, b, h) = 1\}| \geq 2.$$

PROOF. (1) Suppose that
$$\chi(a, b, f) = \chi(a, b, g) = -1.$$
Then, $y \in \mathrm{conv}(e, f, g)$ implies by Lemma 3.7 that
$$\chi(e, f, y) = \chi(f, g, y) = \chi(g, e, y).$$
We conclude
$$\chi(y, a, b)\chi(e, f, g) = 0,$$
$$\chi(f, a, b)\chi(y, e, g)$$
$$= \chi(g, a, b)\chi(y, f, e) \neq 0.$$
With Lemma 3.2 and $\chi(e, a, b)\chi(y, f, g) \neq 0$ we conclude that
$$\chi(e, a, b)\chi(y, f, g) = \chi(f, a, b)\chi(y, e, g).$$
or
$$\chi(e, a, b) = -\chi(f, a, b).$$

(2) Follows.

$\square$

This means that Tverberg partitions of type $3, 3, 2, 2$ with $y = y_{a,b,c,d}$ can only exist, if at least 2 of the remaining points lie above the line ab. By symmetry also 2 points must lie below and the the same applies for the line cd. We summarize this in the following definition:

Definition 3.10. An intersection point $y = y_{a,b,c,d}$ is *valid* if
$$2 \leq |\{h \in \{x_1, \ldots, x_{10}\} \setminus \{a, b, c, d\} \colon \chi(a, b, h) = 1\}| \leq 4$$
and
$$2 \leq |\{h \in \{x_1, \ldots, x_{10}\} \setminus \{a, b, c, d\} \colon \chi(c, d, h) = 1\}| \leq 4.$$

According to Lemma 3.9 we only need to consider valid intersection points. Depending on $\chi\big|_X$ there are up to 70 valid intersection points. This maximum is attained e.g. for $\chi\big|_X$ being the chirotope of a 10-gon.

Definition 3.11. Let $\chi\big|_X$ be a chirotope on X. We construct a graph $G'(\chi\big|_X)$ as follows. For each valid intersection point $y_{a,b,c,d}$ we have vertices for each extension of $\chi\big|_X$ to $\chi\big|_{X \cup \{y_{a,b,c,d}\}}$. There is an edge between two such extensions, $\chi\big|_{X \cup \{y_{a,b,c,d}\}}$ and $\chi\big|_{X \cup \{y_{a',b',c',d'}\}}$ if they are restrictions of a chirotope on $X \cup \{y_{a,b,c,d}, y_{a',b',c',d'}\}$.

This graph is a $(k-1)$-partite graph, where $k-1$ is the number of valid intersection points.

Any chirotope χ of rank 3 on $X \sqcup Y$ corresponds to a $(k-1)$-clique in $G'(\chi\big|_X)$. According to Lemma 3.7 all the Tverberg partitions are determined by restrictions of the chirotope corresponding to the $(k-1)$-clique.

Definition 3.12. Let $\mathcal{C}$ be the set of all color partitions of cardinalities $(3, 3, 3, 1)$, $(3, 3, 2, 2)$, $(2, 2, 2, 2)$. We construct a graph $G(\chi\big|_X)$ from $G'(\chi\big|_X)$ as follows: We add $\mathcal{C}$ to the vertices. We add the following edges: For each $(C_1, \ldots, C_m) \in \mathcal{C}$ and each $\chi\big|_{X \cup \{y_{a,b,c,d}\}}$ we add an edge if **none** of the Tverberg partitions on $\chi\big|_{X \cup \{y_{a,b,c,d}\}}$ is rainbow with respect to $(C_1, \ldots, C_m)$.

This graph is k-partite, where $k-1$ is the number of valid intersection points. In section 5 we will develop an algorithm, which shows that none of the graphs has a k-clique:

Proposition 3.13. *For each $\chi\big|_X$ the graph $G(\chi\big|_X)$ does not have a k-clique, where $k-1$ is the number of valid intersection points.*

This proposition shows our main Theorem:

PROOF OF THEOREM 1.2. Let $x_1, \ldots, x_{10}$ in $\mathbb{R}^2$ with induced chirotope χ on $X \sqcup Y$ and let $(C_1, \ldots, C_m) \in \mathcal{C}$ be a color partition. Let $y_1, \ldots, y_{k-1}$ be the valid intersection points of $\chi|_X$. The restrictions to the valid intersection points $\chi|_{X \cup \{y_1\}}, \ldots, \chi|_{X \cup \{y_{k-1}\}}$ form a $(k-1)$-clique of $G'(\chi|_X)$.

However, by Proposition 3.13 this $(k-1)$-clique does not extend to a k-clique of $G(\chi|_X)$. In particular $(C_1, \ldots, C_m)$ is not connected to some $\chi|_{X \cup \{y_{a,b,c,d}\}}$. This means that a Tverberg partition on $\chi|_{X \cup \{y_{a,b,c,d}\}}$ is rainbow with respect to $(C_1, \ldots, C_m)$. $\qquad \square$

4. Obtaining the graph

Instead of constructing $G(\chi|_X)$ we will construct a slightly larger graph $H(\chi|_X)$, which is simpler to compute:

Lemma 4.1. *Let G be a graph. If G has a k-clique, then so has any graph that is obtained by*

- *adding vertices,*
- *adding edges,*
- *vertex identification of pairwise non-adjacent vertices.*

PROOF. Trivial. $\qquad \square$

We do not check all axioms when determining all $\chi|_{X \cup \{y_{a,b,c,d}\}}$. Thus we might add some vertices. When checking whether $\chi|_{X \cup \{y_{a,b,c,d}\}}$ and $\chi|_{X \cup \{y_{a',b',c',d'}\}}$ have a common extension, we do not verify all axioms. Thus, we might add some edges. Instead of having a vertex for each $\chi|_{X \cup \{y_{a,b,c,d}\}}$, we will only determine χ for some values that suffice to determine all Tverberg partitions. This operation might identify vertices. All identified vertices are in the same part and are therefore pairwise non-adjacent.

4.1. The vertices of the graph. For the remainder, $a, b, c, d, e, f, g \in \{x_1, \ldots, x_{10}\}$ will be pairwise distinct such that $y := y_{a,b,c,d}$ is a valid intersection point. In addition, $\{i, j, k\}$ is some permutation of $\{e, f, g\}$.

Lemma 4.2. *For any $h \in X \sqcup Y$ it holds that $\chi(a, b, h) = \chi(a, y, h) = \chi(y, b, h)$.*

PROOF. As y is a valid intersection point, we have that $\chi(a, c, d) \neq \chi(c, d, b)$. We inspect
$$\chi(y, c, d)\chi(a, b, g) = 0,$$
$$\chi(b, c, d)\chi(y, a, g)$$
$$= \chi(g, c, d)\chi(y, b, a) \neq 0.$$
Lemma 3.2 implies that
$$\chi(a, c, d)\chi(y, b, g) = \chi(b, c, d)\chi(y, a, g).$$
We conclude that $\chi(y, b, g) = \chi(a, y, g)$. If $\chi(a, b, g) \neq 0$, then one of
$$0 = \chi(a, b, y), \quad \chi(b, g, y) = \chi(g, a, y)$$
must be equal to $\chi(a, b, g)$ as the chirotope is acyclic. If $\chi(a, y, g) \neq 0$, then one of
$$\chi(a, y, b) = 0, \quad \chi(y, g, b) = -\chi(a, y, g), \quad \chi(g, a, b)$$
must be equal to $\chi(a, y, g)$. $\qquad \square$

Lemma 4.3 (See Figure 1). *Suppose that $\chi(a, b, e) = \chi(a, b, f)$ and $\chi(c, d, e) \neq \chi(c, d, f)$ it follows that*
$$\chi(e, f, y) = -\chi(a, b, c)\chi(c, d, f)\chi(a, b, e) = \chi(c, d, a)\chi(c, d, f)\chi(a, b, e).$$

PROOF. By applying Lemma 4.2 we have that
$$\chi(y, y, c)\chi(b, e, f) = 0$$
$$\chi(e, y, c)\chi(y, b, f) = \chi(e, d, c)\chi(a, b, f)$$
$$= \chi(f, y, c)\chi(y, e, b) = \chi(f, d, c)\chi(a, e, b) \neq 0.$$
We conclude with Lemma 3.2 and $\chi(b, y, c)\chi(y, e, f) \neq 0$ that
$$\chi(b, y, c)\chi(y, e, f) = \chi(f, d, c)\chi(a, e, b).$$
The statement follows with Lemma 4.2:
$$\chi(b, y, c) = \chi(b, a, c) = \chi(y, a, c) = \chi(d, a, c).$$
$\qquad \square$

Depending on $\chi(a, b, e), \chi(c, d, e)$ the point e is in one of four regions.

Definition 4.4. If $\chi(a, b, e) = \chi(a, b, f)$ and $\chi(c, d, e) = \chi(c, d, f)$, then e and f are in the *same region* with respect to a, b, c, d. If both equalities are false, then they are in *opposite regions*. If exactly one equality holds, they are in *neighboring regions*.
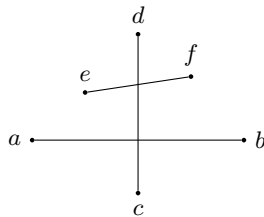
FIGURE 1. Sketch of Lemma 4.3.

Lemma 4.5. *Suppose* $y \in \mathrm{conv}(e, f, g)$. *It follows that there exist* $i, j \in \{e, f, g\}$ *such that* i *and* j *are in opposite regions with respect to* a, b, c, d.

PROOF. Follows from Lemma 3.9 by pigeonhole principle. □

Let i and j be in opposite regions and let $y \in \mathrm{conv}(i, j, k)$. Then k is in the same region as i or j or in a neighboring region to both of them.

By relabeling, we will assume that j and k are in the same region or that $\chi(a, b, i) = \chi(a, b, k)$ and $\chi(c, d, i) \neq \chi(c, d, k)$.

Proposition 4.6 (See Figure 2).

(1) *Suppose that* j *and* k *are in the same region and* i *is in the opposite region to both of them with respect to* a, b, c, d. *The point* y *is contained in* $\mathrm{conv}(i, j, k)$ *if and only if*

$$\chi(i, j, y) \neq \chi(i, k, y).$$

(2) *Suppose that*

$$\chi(a, b, i) = \chi(a, b, k) \neq \chi(a, b, j), \quad \chi(c, d, i) \neq \chi(c, d, j) = \chi(c, d, k).$$

The point y *is contained in* $\mathrm{conv}(i, j, k)$ *if and only if*

$$\chi(i, j, y) = -\chi(a, b, c)\chi(c, d, i)\chi(a, b, i).$$

PROOF. (1) If $y \in \mathrm{conv}(i, j, k)$ then certainly $\chi(i, j, y) \neq \chi(k, i, y)$ by Lemma 3.6. On the other hand, assume that $\chi(i, j, y) \neq \chi(i, k, y)$. We now have

$$\chi(j, c, d)\chi(i, k, y)$$
$$= \chi(k, c, d)\chi(j, i, y) \neq 0,$$
$$\chi(y, c, d)\chi(j, k, i) = 0,$$

which implies by Lemma 3.2 and $\chi(i, c, d)\chi(j, k, y) \neq 0$ that

$$\chi(i, c, d)\chi(j, k, y) = \chi(j, c, d)\chi(i, k, y).$$

This yields $\chi(j, k, y) = -\chi(i, k, y)$ and therefore by Lemma 3.6 we have that $y \in \mathrm{conv}(i, j, k)$.

(2) According to Lemma 4.3 we have that

$$\chi(j, k, y_{c,d,a,b}) = \chi(a, b, c)\chi(a, b, k)\chi(c, d, j)$$

and that

$$\chi(k, i, y) = -\chi(a, b, c)\chi(c, d, i)\chi(a, b, k).$$

As $\chi(c, d, j) = -\chi(c, d, i)$ we conclude

$$\chi(j, k, y) = \chi(k, i, y).$$

Now according to Lemma 3.7, $y \in \mathrm{conv}(i, j, k)$ is equivalent to

$$\chi(i, j, y) = \chi(j, k, y),$$

which is by $\chi(a, b, i) = \chi(a, b, k)$ equivalent to

$$\chi(i, j, y) = -\chi(a, b, c)\chi(c, d, i)\chi(a, b, i).$$

□

We now have made precise what we mean by determining $\chi|_{X \cup \{y\}}$ only for some values: According to Proposition 4.6 it suffices to determine χ for all triples in X and for all tripels (i, j, y), where i, j are in opposite regions with respect to a, b, c, d.

Now, we are not entirely free in choosing such an extension of χ. We will use some obstructions:
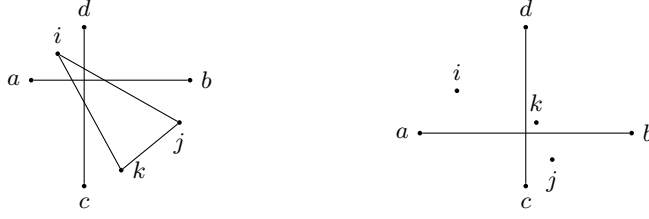
Proposition 4.7 (See Figure 2).
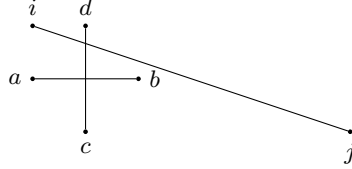
FIGURE 2. Sketch of Propositions 4.6 and 4.7.

FIGURE 3. Sketch of Proposition 4.8.

(1) *Suppose that j and k are in the same region and i is in the opposite region to both of them with respect to a, b, c, d.*

$$\chi(i, j, y) \neq \chi(i, k, y)$$

implies that

$$\chi(i, j, y) = \chi(i, j, k).$$

(2) *Suppose that*

$$\chi(a, b, i) = \chi(a, b, k) \neq \chi(a, b, j), \quad \chi(c, d, i) \neq \chi(c, d, j) = \chi(c, d, k).$$

Now

$$\chi(i, j, y) = -\chi(a, b, c)\chi(c, d, i)\chi(a, b, i).$$

implies

$$\chi(i, j, k) = -\chi(a, b, c)\chi(c, d, i)\chi(a, b, i)$$

PROOF. Both statements follow directly from Proposition 4.6 as the chirotope is acyclic. □

Also $\chi(i, j, y)$ is determined unless $\chi(i, j, a) \neq \chi(i, j, b)$ and $\chi(i, j, c) \neq \chi(i, j, d)$:

Proposition 4.8 (See Figure 3). *Let i and j be in opposite regions. The equality $\chi(i, j, a) = \chi(i, j, b)$ implies $\chi(i, j, a) = \chi(i, j, y)$.*

PROOF. We have by assumptions and Lemma 4.2

$$\chi(a, i, j)\chi(y, b, i) = \chi(a, i, j)\chi(a, b, i)$$
$$= \chi(b, i, j)\chi(a, y, i) = \chi(b, i, j)\chi(a, b, i) \neq 0,$$
$$\chi(i, i, j)\chi(a, b, y) = 0,$$

which implies by Lemma 3.2 and $\chi(y, i, j)\chi(a, b, i) \neq 0$ that

$$\chi(y, i, j)\chi(a, b, i) = \chi(a, i, j)\chi(a, b, i).$$

□

Proposition 4.9 (See Figure 4). *Suppose that i_1, i_2 are in the same region and j_1, j_2 are in the opposite region with respect to a, b, c, d. Also suppose that $\chi(i_1, j_1, a) \neq \chi(i_1, j_1, b)$ and $\chi(i_2, j_2, a) \neq \chi(i_2, j_2, b)$. Now*

$$\chi(i_1, j_1, i_2) = \chi(i_1, j_1, j_2) = -\chi(i_1, j_1, y)$$

implies $\chi(i_2, j_2, y) = \chi(i_1, j_1, y)$.

PROOF. As the chirotope is acyclic one of $\chi(i_1, j_1, b), \chi(j_1, a, b), \chi(a, i_1, b)$ must be equal to $\chi(i_1, j_1, a)$. But $\chi(i_1, j_1, a) \neq \chi(i_1, j_1, b)$ and $\chi(j_1, a, b) = \chi(a, i_1, b)$. Hence it follows that $\chi(i_1, j_1, a) = \chi(a, b, j_1)$. Likewise $\chi(i_2, j_2, a) = \chi(a, b, j_2)$ and in particular $\chi(i_1, j_1, a) = \chi(i_2, j_2, a)$.

We conclude that

$$\chi(y, i_1, j_1)\chi(a, i_2, j_2) = \chi(i_1, j_1, y)\chi(a, b, j_2)$$
$$= \chi(i_2, i_1, j_1)\chi(y, a, j_2) = \chi(j_1, i_1, y)\chi(b, a, j_2)$$
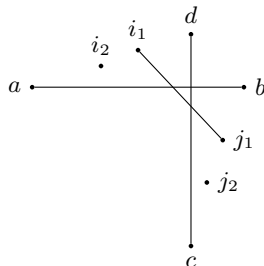
FIGURE 4. Sketch of Proposition 4.9.

$$= \chi(j_2, i_1, j_1)\chi(y, i_2, a) = \chi(j_1, i_1, y)\chi(b, i_2, a) \neq 0,$$

which implies by Lemma 3.2 and $\chi(a, i_1, j_1)\chi(y, i_2, j_2) \neq 0$ that

$$\chi(a, i_1, j_1)\chi(y, i_2, j_2) = \chi(y, i_1, j_1)\chi(a, i_2, j_2).$$

$\square$

4.2. The edges of the graph. Given an acyclic chirotope $\chi|_X$. We are constructing a k-partite graph $H(\chi|_X)$ such that the existence of a k-clique in $G(\chi|_X)$ implies the existence of a k-clique in $H(\chi|_X)$. This graph has the color partitions as vertices in the last part. For each part corresponding to a valid intersection point y it has a vertex v for each collection of orientations of (i, j, y) for i and j in opposite regions that agree with Propositions 4.7, 4.8, and 4.9.

Some vertices v might not be extendable to a chirotope $\chi|_{X \cup \{y\}}$. Also it is possible that some vertices have multiple extensions. However, $\chi|_X$ determines the Tverberg partitions of types $(3, 3, 3, 1)$ by Lemma 3.7. Also the orientations fixed by v determine all Tverberg partitions of type $(3, 3, 2, 2)$ in y by Proposition 4.6.

Hence, v determines the Tverberg partitions on $\chi|_{X \cup \{y\}}$ and v is connected to a color partition if none of those Tverberg partitions are rainbow with respect to this color partition.

Now let w be another vertex corresponding to the valid intersection point $y' := y_{a',b',c',d'} \neq y$. The vertices v and w must be connected by an edge, if the orientations are consistent.

We will always connect v and w unless $\{a, b\} = \{a', b'\}$ or $\{a, b\} = \{c', d'\}$ or $\{c, d\} = \{a', b'\}$ or $\{c, d\} = \{c', d'\}$. By relabeling, those cases are reduced to $a = a'$, $b = b'$, $\chi(a, b, c) = \chi(a, b, c')$, which we will assume for the remainder of this section. For each v and w we will check the following certificate:

Proposition 4.10 (See Figure 5). *Let i, j be distinct from a, b, c', d' such that $\chi(a, b, i) = \chi(a, b, c) = \chi(a, b, c')$ and $\chi(a, b, j) = \chi(a, b, d) = \chi(a, b, d')$. Suppose that $\chi(i, j, y) \neq \chi(c', d', y)$ and let $\chi(i, j, y') \neq 0$. It follows that $\chi(i, j, y') = \chi(c, d, y') = -\chi(c', d', y)$.*

Note that we do not assume that $\chi(i, j, y)$ is non-zero. In particular for $i = c$ and $j = d$ we obtain $\chi(c', d', y) = -\chi(c, d, y')$.

PROOF. For $g \in \{c, c', i\}$ we inspect with Lemma 4.2

$$\chi(a, c, d)\chi(y', y, g) = ?,$$
$$\chi(y, c, d)\chi(a, y', g) = 0,$$
$$\chi(g, c, d)\chi(a, y, y') = 0.$$

As $\chi(y', c, d)\chi(a, y, g) \neq 0$ we conclude with Lemma 3.2 that

$$\chi(y', c, d)\chi(a, y, g) = \chi(a, c, d)\chi(y', y, g).$$

With $\chi(a, y, c) = \chi(a, y, c') = \chi(a, y, i)$ we can therefore conclude that

$$\chi(y', y, c) = \chi(y', y, c') = \chi(y', y, i).$$

Analog

$$\chi(y', y, d) = \chi(y', y, d') = \chi(y', y, j)$$

and in particular by Lemma 4.2

$$\chi(y', y, i) = \chi(y', y, c') = \chi(d', y, c') = -\chi(c', y, d') = -\chi(y', y, d') = -\chi(y', y, j).$$

Hence, $\chi(c', d', y) = -\chi(j, y', y)$. Now by assumptions $\chi(i, j, y) \neq -\chi(j, y', y)$. But as χ is acyclic one of $\chi(i, j, y)$, $\chi(j, y', y) = \chi(y', i, y) \neq 0$ needs to be equal to $\chi(i, j, y') \neq 0$. We conclude that

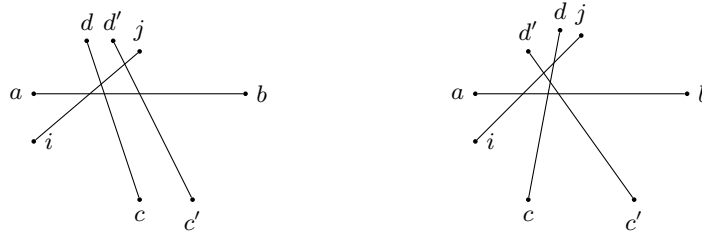$$\chi(i, j, y') = \chi(y', i, y) = \chi(j, y', y) = -\chi(c', d', y).$$

FIGURE 5. Sketch of two cases of Proposition 4.10.

TABLE 1. Runtime in ms of checking for a k-clique for some graphs $H(\chi\big|_X)$.

Graph indexed by [1]	kpkc	FindClique	NetworkX	Cliquer	mcqd
0	17,300	nan	nan	nan	nan
1	17,300	nan	nan	nan	nan
2	17,200	nan	nan	nan	nan
20	5,070	nan	nan	nan	nan
100	17,400	nan	nan	nan	nan
1000	489	21,500	nan	nan	nan
10000	178	3,620	nan	626,000	nan
1000000	1,320	nan	nan	nan	nan
2000000	158	763	nan	9,440	nan
5000000	187	7,950	nan	nan	nan
10000000	22	6	nan	1,240	nan

Also

$$\chi(y', i, y) = \chi(y', c, y) = \chi(y', c, d).$$

$\square$

We will connect v and w unless we find a contradiction to Proposition 4.10.

Given an acyclic chirotope $\chi\big|_X$. The small package `https://github.com/kliem/TenColoredPoints`
constructs the slightly enarged graph $H(\chi\big|_X)$ of $G(\chi\big|_X)$. It then uses the algorithm from Section 5 to verify that $H(\chi\big|_X)$ does not have a k-clique. By Lemma 4.1 this implies that $G(\chi\big|_X)$ also has no k-clique and this chirotope on X satisfies the optimal colored Tverberg problem.

To iterate over all acyclic chirotopes on 10 points of rank 3 one can either use the list by [1][1] or the package
`https://github.com/kliem/pseudo_order_types`.

Note that the list by [1] does not contain the non-realizable acyclic chirotopes on 10 points. It would suffice to prove this instance of the optimal colored Tverberg problem, but we also check the others to show it on the level of chirotopes.

5. k-cliques in a k-partite graph

We provide a new algorithm to iterate over k-cliques in a k-partite graph. It is implemented in `C++`:
`https://github.com/kliem/KPartiteKClique`.

It is based on the depth-first algorithm of Grünert, Irnich, Zimmermann, Schneider and Wulfhorst [8]:

```
def FINDCLIQUE(G):
    parts = G.parts()
    if len(parts) == 0:
        yield []
        return
    P = argmin(part.size() for part in parts)
    for v in P.vertices():
        V1 = v.neighbors()
        G1 = G.induced_subgraph(V1)
        for clique in FINDCLIQUE(G1):
            yield clique + [v]
```

.

---

[1]`http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/ordertypes/`.

This pivot selection is simple and fast for many purposes. However, it did not terminate for some of the graphs encountered during this project.

Consider the graph

```
https://github.com/kliem/PyKPartiteKClique/blob/main/sample_graphs/0.gz.
```

that can be recovered with `kpkc.test.load_tester` from the Python wrapper of `KPartiteKClique`[2]. This is $H(\chi|_X)$, where $\chi|_X$ is the first order type in the enumeration by [1] listed on

```
http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/ordertypes/
```

– 10 points in convex position.

The graph $H(\chi|_X)$ is a 71-partite graph with 10,785 vertices and 6,630,275 edges. The part coresponding to the colors has 10,045 vertices, the other parts have at most 20 vertices.

Such a graph with density 0.11 is unlikely to have a 71-clique. However, judging by the size of the graph, recursion depth 71 is a challenge. This graph has been tested with the following algorithms/implementations and none of them terminated in 24 hours:

- `FindClique` [8],
- `NetworkX` [5] [9],
- `Cliquer` [13],
- `mcqd` [10].

Note that only `FindClique` exploits the given 71-partition, while the other algorithms do note use it. We provide our own implementation of `FindClique`, as the authors have not provided it in [8].

Proving Theorem 1.2 by Proposition 3.13 we need to check for k-cliques in 14,320,182 graphs. Many of them are much simpler than this first one. However, it seems desirable to find an algorithm that terminates for all those graphs in reasonable time.

To reduce the complexity of the graph, we start with vertices with few neighbors as done in [10]:

```
def kpkc(G, prec_depth=5):
    if len(parts) == 0:
        yield []
        return
    if prec_depth:
        G.sort_vertices(key=len_neighbors)
    for v in G.vertices():
        V1 = v.neighbors()
        G1 = G.induced_subgraph(V1)
        for clique in kpkc(G1, prec_depth − 1):
            yield clique + [v]
        G.remove(v)
```

The implementation is a bit more involved. In particular:

- During sorting of the vertices we de facto remove vertices not connected to all parts. If this happend during the first sort, we sort again.
- After removing the second last vertex of a part, we sort again (and de facto select the last vertex).
- After removing the last vertex of a part, we immediatly return.
- The resources for the recursive calls are recycled to avoid memory allocations.
- The induced subgraph only keeps track of the selected vertices and their number of neighbors. The neighbors of a vertex are the intersection of the vertices of the subgraph with the neighbors of the original vertex. The number of neighbors is computed without storing the set of neighbors.

If a vertex has few neighbors, the induced subgraph of the neighbors is likely very easy to handle. This reduces size of the graph vertex by vertex. Table 1 compares the runtime for some graphs we need to analyze for Proposition 3.13. Apparantly, `kpkc` is the only choice suitable to solve our problem.

With `kpkc` we can verify in just 17 seconds that our first graph does not have a 71-clique. We prove Proposition 3.13 by analyzing all 14,320,182 graphs. Building and analyzing all graphs was done in 780 CPU-hours using an Intel® Core™ i7-7700 CPU @ 3.60GHz x86_64-processor.[3] This is an average of 196 ms per instance.

Now, we will inspect benchmarks on random graphs and explain for what type of graphs the new algorithm is suitable.

6. Benchmarks on random graphs

In Tables 2, 3, 4, 5, 6 and 7 we benchmark the implementations on random graphs. Each row represents one graph randomly generated by certain parameters that is tested on all implementations. A timeout after 1000 seconds is marked by "nan". If an implementation is not included in the table, each entry would indicate a timeout.

---

[2]`https://github.com/kliem/PyKPartiteKClique`
[3]Actually, the 14,320,182 graphs were divided to multiple threads on multiple machines with the same specifications.

TABLE 2.   Runtime in ms for sample graphs as constructed by [8]

k	min $\|P_b\|$	max $\|P_b\|$	a	b	kpkc		FindClique		NetworkX		Cliquer	mcqd
					first	all	first	all	first	all	first	first
5	50	50	0.14	0.14	1	1	0	0	11	11	1	1
5	50	50	0.15	0.15	0	1	0	0	4	13	1	1
5	50	50	0.2	0.2	0	1	0	0	2	24	2	2
5	50	50	0.25	0.25	0	2	0	1	1	43	2	2
5	50	50	0.0	0.3	0	1	0	0	1	13	1	1
5	50	50	0.0	0.4	0	1	0	0	1	25	2	2
5	50	50	0.0	0.45	0	3	0	1	1	43	2	2
5	50	50	0.0	0.5	0	4	0	2	1	45	2	2
10	26	37	0.49	0.49	3	40	0	1	259	9,430	46	45
10	26	37	0.5	0.5	0	35	0	1	35	8,260	57	38
10	26	37	0.51	0.51	1	69	0	1	750	14,200	110	66
10	26	37	0.4	0.6	2	47	0	1	217	13,900	72	48
10	26	37	0.3	0.7	1	66	0	4	13	12,700	152	51
10	50	50	0.42	0.42	13	93	0	1	1,090	21,400	107	113
10	50	50	0.43	0.43	4	121	0	2	1,700	27,800	180	131
10	50	50	0.44	0.44	22	151	0	2	316	34,600	214	170
10	50	50	0.46	0.46	5	304	0	4	27	55,200	324	275
10	50	50	0.48	0.48	1	573	0	8	49	83,900	404	395
10	50	50	0.5	0.5	1	997	0	20	16	172,000	1,160	694

TABLE 3.   Runtime in ms for sample graphs as constructed by [8]

k	min $\|P_b\|$	max $\|P_b\|$	a	b	kpkc		FindClique		Cliquer	mcqd
					first	all	first	all	first	first
50	5	15	0.91	0.91	nan	nan	64	64	nan	nan
50	5	15	0.918	0.918	nan	nan	28	10,600	nan	nan
50	5	15	0.92	0.92	nan	nan	8	2,910	nan	nan
20	23	39	0.7	0.7	187,000	320,000	106	193	nan	nan
20	23	39	0.71	0.71	40,700	535,000	52	345	nan	nan
20	23	39	0.72	0.72	5,120	955,000	2	642	nan	nan
20	23	39	0.7	0.73	2,570	nan	4	867	nan	nan
20	23	39	0.65	0.78	1,040	454,000	0	436	nan	nan
30	11	30	0.6	0.6	429	429	0	0	796,000	121,000
30	11	30	0.7	0.7	16,300	16,300	1	1	nan	nan
30	11	30	0.8	0.8	nan	nan	1,160	1,160	nan	nan
30	11	30	0.81	0.81	nan	nan	1,330	1,330	nan	nan
30	11	30	0.82	0.82	nan	nan	436	3,450	nan	nan
30	11	30	0.84	0.84	nan	nan	6	51,300	nan	nan
30	11	30	0.88	0.88	1,440	nan	0	nan	nan	nan
100	10	10	0.7	0.7	52	52	0	0	nan	nan
100	10	10	0.8	0.8	4,760	4,760	0	0	nan	nan
100	10	10	0.85	0.85	221,000	221,000	2	2	nan	nan
100	10	10	0.9	0.9	nan	nan	149	149	nan	nan
100	10	10	0.92	0.92	nan	nan	4,190	4,190	nan	nan
100	10	10	0.94	0.94	nan	nan	nan	nan	nan	nan
100	10	10	0.95	0.95	nan	nan	nan	nan	nan	nan
100	10	10	0.97	0.97	nan	nan	3	nan	nan	nan

We benchmark getting the first k-clique or checking for the existence of a such a k-clique. We also benchmark obtaining all k-cliques, if the implementation has this available. In Tables 6 and 7 we have not included timings for all cliques: Either the graphs have few k-cliques and obtaining all k-cliques is just as fast as obtaining the first k-clique or the graph has many k-cliques and obtaining all of them results in a timout.

6.1. Random graphs as constructed by [8]. We first rerun the benchmarks by [8, Table 2]: A random k-partite graph is generated by parameters

$$(k, \min |P_b|, \max |P_b|, a, b).$$

TABLE 4. Runtime in ms for sample graphs as constructed by [12]

k	min $\|P_b\|$	max $\|P_b\|$	a	b	kpkc		FindClique		NetworkX		Cliquer	mcqd
					first	all	first	all	first	all	first	first
3	100	100	0.1	0.1	0	3	0	2	1	7	1	1
4	100	100	0.15	0.15	0	5	0	2	2	40	4	3
5	100	100	0.2	0.2	0	12	0	2	5	190	9	9
6	100	100	0.25	0.25	0	36	0	3	8	996	23	23
7	50	50	0.35	0.35	0	16	0	1	8	738	13	12
8	50	50	0.4	0.4	0	44	0	2	12	3,890	38	34
9	50	50	0.45	0.45	0	174	0	4	9	20,500	171	125
10	50	50	0.5	0.5	1	925	0	17	143	131,000	1,480	643

TABLE 5. Runtime in ms for rare attraction random graphs

k	max $\|P_b\|$	a	kpkc		FindClique		NetworkX		Cliquer	mcqd
			first	all	first	all	first	all	first	first
5	10	0.1	0	0	0	0	0	0	0	0
5	10	0.2	0	0	0	0	0	1	0	0
5	20	0.05	0	1	0	1	0	4	0	0
5	20	0.1	0	3	0	3	0	6	0	0
5	50	0.01	0	92	0	77	1	155	1	1
5	50	0.02	0	111	0	94	1	166	1	1
10	10	0.4	0	4	0	2	0	22	0	0
10	10	0.6	0	22	0	12	0	51	0	0
10	20	0.3	0	50	0	27	2	790	1	1
10	20	0.5	0	1,160	0	780	1	3,280	1	3
10	50	0.05	0	49	0	66	19	36,800	4	5
10	50	0.1	0	237	0	189	5	47,600	4	5
10	100	0.01	0	5,870	0	7,790	19	nan	17	73
10	100	0.02	0	8,490	0	9,930	14	nan	18	76

Each part has a random number of vertices in $\{\min|P_b|, \min|P_b|+1, \ldots, \max|P_b|\}$ by uniform distribution. To each vertex v we associate a random number p_v, which is uniformly selected from the interval $[a, b]$. Finally, two vertices v, w are connected by an edge with probability $\frac{p_v + p_w}{2}$.

Indeed, FindClique is by far the best choice for those graphs as can be seen in Tables 2 and 3. While [8] only obtains the first 1000 cliques, we try to find all cliques. They used a 100 MHz machine with 32 MB of RAM. Thus already the advance of technology has improved the benchmarks by a factor of at least 36. However, for increasing k our implementation appears to be an improvement. For $k = 100$ we have improved the old timings by a factor of 1000, which is 28-times faster than that factor of 36.

In an intermediate paper Mirghorbani and Krokhmal [12] proposed to improve the data structure of FindClique. They reported that using arrays and bitsets could each gain a factor of about 3. In 4 we have rerun the tests of [12, Table 1], which suggests that our implementation is yet faster by a factor of a bit more than 3 when considering that they only used a 3 GHz machine. If [12] improved FindClique of up to 9 and our implementation is yet an improvement of up to 3, this agrees with the above observation for $k = 100$.

However, the main advantage is that we now have a published implementation of FindClique, which neither [12] nor [8] have provided.

6.2. Rare attraction random graphs. It seems that FindClique is the algorithm of choice for all random graphs as constructed by [8]. The reason seems to be that edges are somewhat equally distributed. Vertices in small parts have about the same expected number of neighbors. This makes the pivot selection of kpkc useless. However, k-partite graphs that correspond to real life problems might behave differently:

Suppose there is only one cement mill in the area, two concrete pumps, twenty conrete mixer trucks, and twenty concrete crews. Nobody can question the quality of the cement mill, because there is no alternative. As there is only two concrete pumps, the truck drivers will usually be willing to work with both of them. Likewise the concrete crews will usually put up with both pump operators. However, it is very much possible that the conrete crews might refuse to work with some truck drivers (always late) or the truck drivers might refuse to work with some crews (always order more trucks than they need).

This problem corresponds to a 4-partite graph. FindClique first selects the cement mill, which is trivial. As a next step it divides the problem in two: 4-cliques containing one pump and 4-cliques containing the other. However, this is only a good choice if the pumps have very different sets of neighbors. The real problem is

TABLE 6.   Runtime in ms for rare attraction random graphs to check for a k-clique

k	max $\lvert P_b\rvert$	a	kpkc	FindClique	Cliquer
50	20	0.5	208	20	nan
50	20	0.6	2,520	422	nan
50	20	0.7	158,000	23,800	nan
50	20	0.71	304,000	248,000	nan
50	20	0.72	nan	712,000	nan
50	20	0.73	nan	761,000	nan
50	20	0.75	nan	nan	nan
50	20	0.76	nan	nan	nan
50	20	0.77	nan	15,600	nan
50	20	0.78	nan	1,500	nan
50	20	0.79	nan	189	nan
50	20	0.8	nan	6	nan
50	50	0.1	36	4,460	nan
50	50	0.2	128	33,300	nan
50	50	0.3	2,090	174,000	nan
50	50	0.4	24,600	nan	nan
50	50	0.5	803,000	nan	nan
50	50	0.6	nan	nan	nan
50	50	0.71	nan	nan	nan
50	50	0.72	nan	29,900	183,000
50	50	0.73	nan	11,700	nan
50	50	0.74	nan	1,550	nan
50	100	0.1	243	nan	nan
50	100	0.2	11,100	nan	nan
50	100	0.3	235,000	nan	nan
50	100	0.4	nan	nan	nan
50	100	0.64	nan	nan	nan
50	100	0.65	nan	nan	119,000
50	100	0.66	nan	nan	nan
50	100	0.67	nan	nan	nan
50	100	0.68	nan	nan	nan
50	100	0.69	nan	22,800	nan
50	100	0.7	nan	2,270	42,900

assigning truck drivers to concrete crews, which `FindClique` solves twice, once for each pump. On the other hand, `kpkc` immediately solves the actual problem..

The graphs $H(\chi\rvert_X)$ seem to behave somewhat like this example. The less choices remain for a valid intersection point y, the less information such a selection will gain. Thus, `FindClique` is likely to divide the problem into two or more very similar problems. `kpkc` instead selects choices that are unlikely to correspond to counter examples and quickly rules them out.

We construct *rare attraction random graphs* parametrized by $(k, \max\lvert P_b\rvert, a)$ as follows: The part i has size

$$1 + \lfloor i\frac{(\max\lvert P_b\rvert - 1)}{k}\rfloor$$

for $i = 1, \dots, k$. Let f be the affine function determined by $f(1) = 1$ and $f(\max\lvert P_b\rvert) = a$. We generate an edge between vertices of different parts of sizes s and t with probability $f(\min(s, t))$. The fewer vertices a part has, the rarer those vertices are. If vertices are rare, other vertices are more likely to be attracted.

Table 5 reveals that for smaller graphs `FindClique` is still the algorithm of choice. In Tables 6 and 7 we see that for $k \geq 50$, $\max\lvert P_b\rvert \geq 50$ and low density `kpkc` is much better in finding k-cliques (or rather verifying their abscence). With high density and probably many k-cliques, `FindClique` is still much faster (at least in finding some k-cliques).

6.3. Conclusion. For finding k-cliques in k-partite graphs it is recommended to use a specialized algorithm. If the graph is large and expected to have few k-cliques and parts with fewer vertices have more neighbors, then `kpkc` is probably the better algorithm. In most other cases, `FindClique` seems better suited. In either case, an algorithm without implementation requires lots of work to be of any use. With

`https://github.com/kliem/KPartiteKClique`

both algorithms are implemented using static polymorphism.

Table 7. Runtime in ms for rare attraction random graphs to check for a k-clique

k	max $\lvert P_b \rvert$	a	kpkc	FindClique
100	20	0.4	48	2
100	20	0.5	435	13
100	20	0.6	4,650	98
100	20	0.7	475,000	9,960
100	20	0.8	nan	nan
100	20	0.89	nan	nan
100	20	0.9	nan	2,430
100	50	0.1	178	7,150
100	50	0.2	292	81,900
100	50	0.3	8,120	496,000
100	50	0.4	46,100	nan
100	50	0.5	nan	nan
100	50	0.87	nan	nan
100	50	0.88	nan	4,240
100	50	0.89	nan	2
100	50	0.9	nan	0
100	100	0.1	892	nan
100	100	0.2	41,600	nan
100	100	0.3	260,000	nan
100	100	0.4	nan	nan
100	100	0.85	nan	nan
100	100	0.86	nan	7,750
100	100	0.87	nan	4
100	100	0.88	nan	0
100	100	0.89	nan	0
100	100	0.9	nan	0

Acknowledgements

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Zum Schluß

Zusammenfassung

In Kapitel 1 zeigen wir obere Schranken des Grünbaum–Hadwiger–Ramos-Problems. Wir geben einen neuen Beweis für fast alle vorhandenen Schranken und zeigen außerdem eine Verbesserung für folgende Fälle: Seien d, n, k natürliche Zahlen mit $d \geq 2^n(1 + 2^{k-1})$ und seien 2^{n+1} Maße in $\mathbb{R}^d$ beliebig gegeben. Es gibt k affine Hyperebenen, die $\mathbb{R}^d$ in 2^k Teile teilen, so dass alle Teile bzgl. aller Maße die gleiche Größe haben.

Im Kapitel 2 benutzen wir ähnliche Methoden für ein ähnliches Problem. Der Raum $\mathbb{R}^d$ wird durch k affine Hyperebenen in bis zu 2^k nicht-leere Kammern geteilt. Wie bei einem Schachbrett gibt es genau zwei Arten die (nicht-leeren) Kammern mit 2 Farben zu färben, so dass Kammern mit gemeinsamer Wand (der Dimension $d-1$) verschiedene Farben haben. Auf diese Art teilen k-Hyperebenen den Raum in 2 Teile: einen schwarzen und einen weißen. Gegeben $2^a(2h + 1) + \ell$ Maße in $\mathbb{R}^{2^a+\ell}$. Wir zeigen, dass es stets $2h + 1$ Hyperebenen gibt, so dass die Aufteilung in schwarz und weiß eine Halbierung bzgl. aller Maße ist.

Im Kapitel 3 entwickeln wir einen schnellen und speichereffizienten Algorithmus zum Iterieren über die Seiten von Polytopen und ähnlichen Objekten. Diesen Algorithmus können wir verwenden um eine zahlentheoretische Vermutung bis zu einem Schwellwert zu überprüfen: Eine numerische Halbgruppe ist eine Teilmenge der nicht-negativen ganzen Zahlen, die 0 enthält, die bzgl. Addition abgeschlossen ist und die alle bis auf endlich viele positiven ganzen Zahlen enthält. Für einige ihrer Invarianten wird eine Ungleichung vermutet (Wilf-Vermutung). Wir weisen diese Vermutung nach, sofern das kleinste nicht-Null Element 19 ist. Dafür iterieren wir über die Seiten eines großen Polyeders.

Im letzten Abschnitt, in Kapitel 4 weisen wir nach, dass es für 10 gefärbte Punkte in der Ebene – mit maximal 3 Punkten je Farbe – stets eine Tverberg Partition gibt, so dass kein Teil zwei Ecken derselben Farbe enthält. Das ist der erste unbekannte Fall der optimal colored Tverberg Conjecture. Dafür reduzieren wir das Problem auf die Suche nach k-Cliquen in k-partiten Graphen (Verallgemeinerung von bipartit) und entwickeln einen neuen Algorithmus, um zeigen zu können, dass unsere Graphen keine solchen Cliquen haben.

Danksagung

Selbständigkeitserklärung

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Mit einer Prüfung meiner Arbeit durch ein Plagiatsprüfungsprogramm erkläre ich mich einverstanden.