Chapter 3

Convergence and Approximation

3.1 Overview

What do we mean by *convergence*? Consider two smooth Riemann manifolds (M_1, g_1) and $(\tilde{M}_2, \tilde{g}_2)$ and a bi-Lipschitz map

$$\Phi : M_1 \to \tilde{M}_2.$$

The map Φ can be used to pull back the Riemannian metric $\tilde{g}_2$ on $\tilde{M}_2$ to a Riemannian metric g_2 on M_1. Indeed, due to Rademacher's theorem, $\mathrm{d}\Phi$ exists and is regular almost everywhere. Consequently, there exists a positive definite tensor field A on M_1 such that

$$g_2(X,Y) = \tilde{g}_2(\,\mathrm{d}\Phi(X),\, \mathrm{d}\Phi(Y)) = g_1(AX,Y) \quad \text{a.e.}$$

for any vector fields X and Y on M_1. A is (pointwise) symmetric with respect to g_1 (and g_2) and has positive eigenvalues. Henceforth the tensor A will be referred to as the *metric distortion tensor*.

Let (M, g) be a smooth Riemannian manifold. For any g-symmetric tensor field S on M, let $\|S\|_\infty$ denote the essential supremum of the scalar field formed by the eigenvalues of S of pointwise largest absolute value. With this definition one obtains

$$\|A - Id\|_\infty = \sup_{X,Y} \left\| \frac{|g(AX,Y) - g(X,Y)|}{\|X\|_g \cdot \|Y\|_g} \right\|_\infty,$$

where $\|X\|_g$ and $\|Y\|_g$ denote pointwise norms and the supremum is taken over all measurable vector fields X, Y on M.

Definition 3.1.1 (metric convergence)**.** Let (M, g) be a Riemannian manifold. By metric convergence we mean a sequence of Riemannian manifolds $\{\tilde{M}_n, \tilde{g}_n\}$ and a sequence of bi-Lipschitz maps $\{\Phi_n : M \to \tilde{M}_n\}$ such that

$$\|A_n - Id\|_\infty \to 0,$$

i.e. metric distortion vanishes in the limit.

The essential technical tool of this chapter is to express A_n in a geometrically meaningful way. For the case of surfaces in $\mathbb{R}^3$ such an expression is obtained by representing A_n in terms of *pointwise distance* and *normal distance* between surfaces (cf. Theorem 3.2.1).

3.1.1 WHAT WILL BE SHOWN?

It will follow from Theorem 3.2.1 that pointwise convergence of surfaces in $\mathbb{R}^3$ together with the convergence of their normals provides convergence of the metric distortion tensors A_n. This will be used to show convergence of the following objects:

- intrinsic length and area
- Laplace–Beltrami operators
- solutions to the Dirichlet problem
- mean curvature vectors
- geodesics
- Hodge decomposition
- Hodge star operators
- eigenvalues of Laplace–Beltrami operators

Throughout we make a point that the *correct spaces and norms* in which convergence occurs have to be chosen carefully. For example, the Laplace–Beltrami operators converge in their operator norm but there is no hope for pointwise convergence. Likewise, mean curvature vectors will be shown to converge as *functionals* (as elements of the Sobolev space $H^{-1}(M)$), whereas a counterexample to their L^2-convergence will be provided.

3.1.2 WHAT CAN GO WRONG?

If one *requires* pointwise convergence of surfaces, then all that could possibly go wrong for metric convergence is the failure of convergence of normals (Theorem 3.2.1). The most prominent example of this failure is the *lantern of Schwarz* dating back to 1890 (cf. Schwarz [71]).

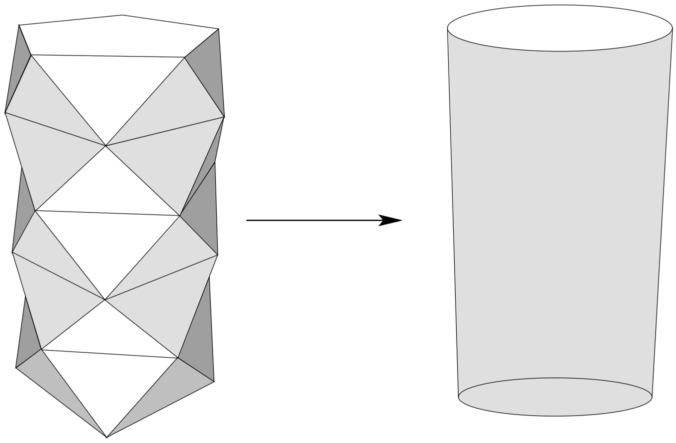

Figure 3.1: The Lantern of Schwarz illustrates pointwise convergence of discrete meshes (all of whose vertices reside on the smooth limit surface) without convergence of normals.

Example (Schwarz lantern)**.** Consider a straight smooth cylinder $C \subset \mathbb{R}^3$ of finite radius and finite height. Divide this cylinder into m evenly spaced circles, and divide each of these circles into n equal segments, thus obtaining a $(m \times n)$ grid of points on C which can be connected by flat triangles. Now twist every other horizontal ring by $\pi/2n$ to generate an alternating grid of points (cf. Figure 3.1). All triangles of the so obtained discrete cylinder $C_{m,n}$ are congruent. Now let m and n approach infinity. The main observation is that the normals of the triangles of the discrete cylinders $C_{m,n}$ will approach the normals of C if and only if

$$\frac{m}{n^2} \longrightarrow 0.$$

The next section will establish that the Schwarz lantern constitutes the most general example of what can go wrong for metric convergence: pointwise convergence of surfaces without convergence of their normals.

3.2 Normal convergence

3.2.1 Shortest distance map

In this section we introduce the *shortest distance map* as an auxiliary tool for comparing a smooth surface to a Euclidean cone surface nearby. Considering this map has been common practice, see e.g. Dziuk [27] and Morvan et al. [57, 58].

Definition 3.2.1 (medial axis, reach)**.** Let $M \subset \mathbb{R}^3$ be a topologically closed subset of $\mathbb{R}^3$. The *medial axis* of M is the set of those points in $\mathbb{R}^3$ which do

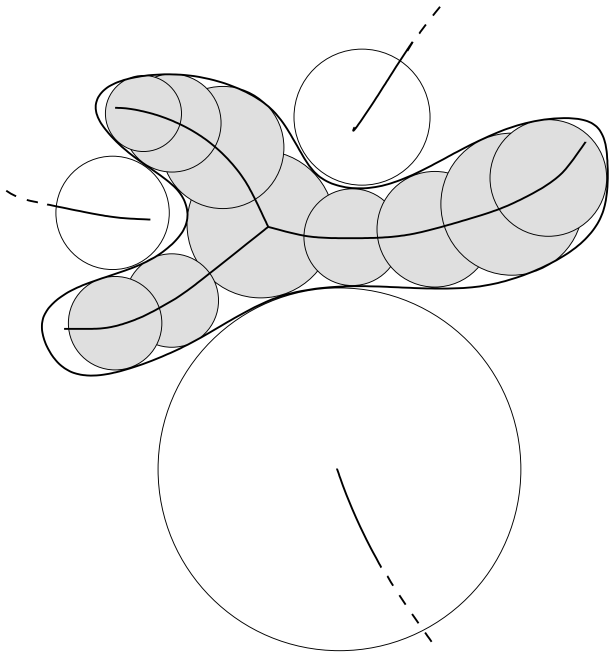

Figure 3.2: Illustration of several branches of the medial axis of a smooth shape.

not have a unique closest neighbor in M. The *reach* of M is the distance of M to its medial axis.

If $M \subset \mathbb{R}^3$ is a smoothly embedded surface, then locally the reach is bounded above by the radii of osculating spheres of M. In other words,

$$\operatorname{reach}(M) \leq \inf_{x \in M} \frac{1}{|\kappa|_{max}(x)}, \tag{3.1}$$

where $|\kappa|_{max}(x)$ denotes the maximal absolute value of the normal curvatures at $x \in M$. Globally, the reach depends on how close (measured in $\mathbb{R}^3$) surface points come to each other. Note that a compact and smoothly embedded surface M always has positive reach (whereas a polyhedron does certainly not). For a general treatment of sets of *positive reach* we refer to Federer [32].

Let M_h is a polyhedral surface within the reach of M. Mapping each point of M_h to its closest point on M is then a well-defined operation.

Definition 3.2.2 (normal graph, shortest distance map)**.** A polyhedral surface M_h is a *normal graph* over a smooth surface M if its distance to M is *strictly less* than the reach of M, and the map $\Phi : M \to M_h$ which takes $p \in M$ to the intersection point $\Phi(p) \in M_h$ of the normal line through p with

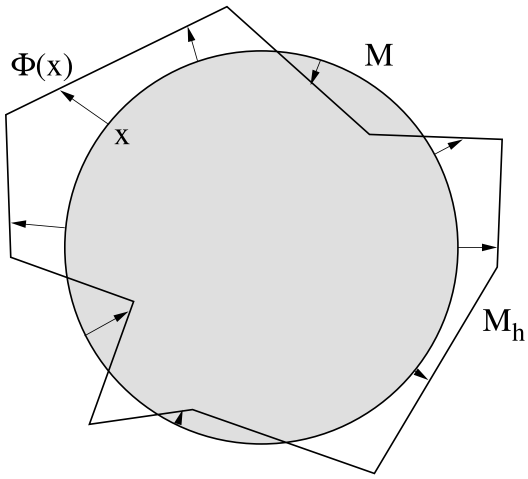

Figure 3.3: M_h is a normal graph over M. At each point $x \in M$, the map Φ takes x to the intersection of the normal line through x with the polyhedral surface M_h.

the polyhedral surface M_h is a *bijection onto the image*[1] up to, and including, the possibly non-empty boundary, ∂M, see Figure 3.3. We sometimes work with the inverse $\Psi = \Phi^{-1}$.

The shortest distance map, Φ, splits into a tangential and a normal component,

$$\Phi(x) = Id_M(x) + \phi(x) \cdot N(x), \tag{3.2}$$

where N is the oriented normal of M, Id_M is the embedding of M into $\mathbb{R}^3$ and ϕ is the (signed) scalar-valued distance function.

The shortest distance map Φ can be used to pull back the metric of the Euclidean cone surface M_h to the smooth surface M. More precisely, one can almost everywhere on M (except for the pre-image of edges and vertices of M_h) define a metric g_A by:

$$g_A(X, Y) := g_{M_h}(\,\mathrm{d}\Phi(X),\, \mathrm{d}\Phi(Y)) = \langle\, \mathrm{d}\Phi(X),\, \mathrm{d}\Phi(Y)\rangle_{\mathbb{R}^3} \quad \text{a.e.}, \tag{3.3}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the standard inner product on $\mathbb{R}^3$. Then there exists a symmetric positive definite 2×2 matrix field $A(x)$, $x \in M$, uniquely defined M-almost everywhere, such that for all vector fields X and Y on M

$$g_A(X, Y) = g(A(X), Y) \quad \text{a.e.}. \tag{3.4}$$

As before, A is referred to as the *metric distortion tensor*.

[1] Specifically, this implies that we require the image of M under Φ to be contained in M_h, i.e., $\Phi(M) \subset M_h$. However, if $\partial M \neq \emptyset$, we do not require that $\Phi(M) = M_h$.

3.2.2 Metric distortion in geometric terms

The metric distortion tensor A is smooth on the pre-image of the interior of triangles of M_h. The next theorem shows that A only depends on the distance between the surfaces M and M_h, the angle between their normals and the curvature of the smooth surface M. A similar result has been obtained by Morvan and Thibert, cf. Remark 3.2.2.

Theorem 3.2.1 (geometric splitting of metric distortion tensor)**.** *Let M_h be a polyhedral surface which is a normal graph over an embedded, smooth surface M. Let N denote the normal field to M, and let N_h denote the pullback under Φ of the (piecewise constant) normal field of M_h to M. Then the metric distortion tensor A satisfies*

$$A = P \circ Q^{-1} \circ P \quad a.e., \tag{3.5}$$

a decomposition into symmetric positive definite matrices P and Q which can be diagonalized (possibly in different ON-frames) to take the form

$$P = \begin{pmatrix} 1 - \phi \cdot \kappa_1 & 0 \\ 0 & 1 - \phi \cdot \kappa_2 \end{pmatrix} \tag{3.6}$$

$$Q = \begin{pmatrix} \langle N, N_h \rangle^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.7}$$

where κ_1 and κ_2 denote the principal curvatures of the smooth surface M and ϕ is the signed scalar distance as in equation (3.2).

Remark 3.2.1. The matrix P is *positive definite* by the assumption that M_h is in the reach of M so that $1 - \phi \cdot \kappa_i > 0$ by inequality (3.1).

Remark 3.2.2. Let $\Psi = \Phi^{-1}$ be the inverse of the shortest distance map. Morvan and Thibert [58] obtain

$$\begin{aligned} d\Psi(X) &= (Id - \phi \mathbf{S})^{-1}(X) && \text{for all } X \text{ parallel to } \mathrm{T}_{\Psi(\cdot)}M \\ d\Psi(N) &= 0 && \text{for all } N \text{ perpendicular to } \mathrm{T}_{\Psi(\cdot)}M \end{aligned}$$

Our splitting of the metric distortion tensor is equivalent to their result.

Proof of Theorem 3.2.1. It suffices to consider a single triangle T_h of M_h. Denote by $\Psi = \Phi^{-1} : M_h \to M$ the inverse of the shortest distance map. For any map $f : M \to \mathbb{R}$ let $f_{T_h} = f \circ \Psi_{|T_h} : T_h \to \mathbb{R}$ denote its pullback to T_h. By equation 3.2, Ψ, restricted to T_h, can be written as

$$\Psi = Id - \phi_{T_h} \cdot N_{T_h}. \tag{3.8}$$

Note that N_{T_h} stands for the pullback of the normal N of M to the triangle T_h, rather than the normal to T_h. Let d denote the outer differential on T_h. Differentiating Ψ yields

$$\mathrm{d}\Psi = Id - N_{T_h} \cdot \mathrm{d}\phi_{T_h} - \phi_{T_h} \cdot \mathrm{d}N_{T_h}. \tag{3.9}$$

Using that $\mathrm{d}N_{T_h} = \mathrm{d}_M N \circ \mathrm{d}\Psi = -\mathbf{S} \circ \mathrm{d}\Psi$, where $\mathbf{S} = -\mathrm{d}_M N$ is the Weingarten operator on M, gives

$$\mathrm{d}\Psi = (Id - \phi \cdot \mathbf{S})^{-1} \circ (Id - N_{T_h} \cdot \mathrm{d}\phi_{T_h}) : \quad \mathrm{T}T_h \to \mathrm{T}M.$$

Setting

$$P := (Id - \phi \cdot \mathbf{S}) : \quad \mathrm{T}M \to \mathrm{T}M, \tag{3.10}$$

$$\tilde{Q} := (Id - N_{T_h} \cdot \mathrm{d}\phi_{T_h}) : \quad \mathrm{T}T_h \to \mathrm{T}M, \tag{3.11}$$

we obtain $\mathrm{d}\Psi = P^{-1} \circ \tilde{Q}$ and hence $\mathrm{d}_M\Phi = \tilde{Q}^{-1} \circ P$. For each $x \in M$ we define a symmetric positive definite operator Q on $\mathrm{T}_x M$ by

$$\langle Q^{-1}(X), Y\rangle_{\mathbb{R}^3} = \langle \tilde{Q}^{-1}(X), \tilde{Q}^{-1}(Y)\rangle_{\mathbb{R}^3}.$$

The definition of the metric distortion tensor, A, and the symmetry of P yield

$$\begin{aligned}\langle A(X), Y\rangle_{\mathbb{R}^3} &= \langle\, \mathrm{d}_M\Phi(X),\, \mathrm{d}_M\Phi(Y)\rangle_{\mathbb{R}^3} \\ &= \langle PQ^{-1}P(X), Y\rangle_{\mathbb{R}^3},\end{aligned}$$

proving the splitting (3.5) of the distortion tensor. Equation (3.6) follows from (3.10). It remains to show (3.7). Let Y be a vector field on the triangle T_h. Taking into account that with respect to ambient Euclidean space, $\mathbb{R}^3$, we have $N_{T_h} \perp \mathrm{im}(\mathrm{d}\Psi)$ and $N_{T_h} \perp \mathrm{im}(\mathrm{d}N_{T_h})$, we obtain from (3.9) that

$$0 = \langle\, \mathrm{d}\Psi(Y), N_{T_h}\rangle = \langle Y, N_{T_h}\rangle - \mathrm{d}\phi_{T_h}(Y). \tag{3.12}$$

By identifying tangent spaces with linear subspaces of $\mathbb{R}^3$, we obtain that $\tilde{Q}$ is the projection operator

$$\tilde{Q}(Y) = Y - N_{T_h} \cdot \langle N_{T_h}, Y\rangle, \tag{3.13}$$

and a straightforward calculation delivers that Q can be diagonalized as claimed. QED

The convergence of distance and normals implies convergence of surface area (cf. [58] for the same result):

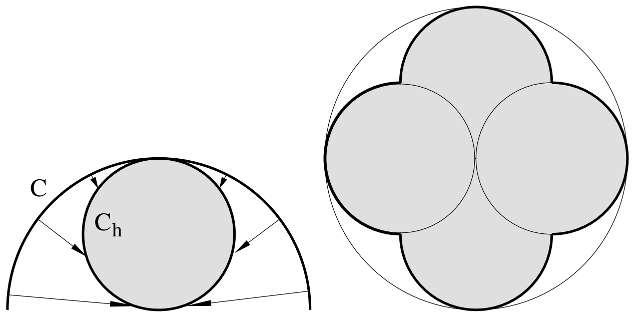

Figure 3.4: The shortest distance map may induce isometrics between non-congruent shapes. Left: The shortest distance map of the half unit circle C induces an isometry to a circle C_h of radius 1/2. Right: By patching together pieces of the left picture one gets isometrics of the unit circle with a *dented circle*.

Corollary 3.2.1 (area distortion). *Under the assumptions of Theorem 3.2.1, the volume elements of M and M_h satisfy*

$$\frac{\mathrm{d}vol_{M_h}}{\mathrm{d}vol_M} = (\det A)^{1/2} = \frac{1 + \phi^2 \cdot \kappa - \phi \cdot \mathrm{H}}{\langle N, N_h \rangle} \quad a.e., \tag{3.14}$$

where κ denotes the Gauss curvature, and H *denotes the scalar mean curvature of M.*

Proof. Equation (3.14) follows immediately from the explicit representation of the distortion tensor A in Theorem 3.2.1, and by using that $\kappa = \kappa_1 \cdot \kappa_2$ as well as $\mathrm{H} = \kappa_1 + \kappa_2$. QED

By bounding the smallest and largest eigenvalue of A we find that:

Corollary 3.2.2 (length distortion). *The infinitesimal distortion of length satisfies*

$$\min_i (1 - \phi \cdot \kappa_i) \leq \frac{\mathrm{d}l_{M_h}}{\mathrm{d}l_M} \leq \frac{\max_i (1 - \phi \cdot \kappa_i)}{\langle N, N_h \rangle} \quad a.e. \tag{3.15}$$

Example (dented circle). Even if the metric distortion induced by the shortest distance map equals the identity (so that the surfaces are isometric) the surfaces need not be congruent. As a consequence, one needs to require pointwise convergence to obtain congruence in the limit.

We give an example of this fact for planar curves; the extension to surfaces is obtained by considering cylinders with cross-sections equal to these planar curves. Consider the half unit circle $C = \{(\cos t, \sin t) : t \in [0, \pi]\}$. Any normal graph C_h over C can be written as

$$C_h = \left\{ ((1 - \phi(t)) \cdot \cos t, (1 - \phi(t)) \cdot \sin t) \in \mathbb{R}^2 : t \in [0, \pi] \right\},$$

where ϕ is the (signed) distance from C to C_h along the unit circle's (inward) normal N. Setting

$$\phi(t) := 1 - \sin t,$$

one readily checks that C_h becomes a circle of radius 1/2 with center $(0, 1/2)$, compare Figure 3.4. The inner product between the normals N of C and N_h of C_h is given by

$$\langle N, N_h \rangle = \sin t = 1 - \phi(t). \tag{3.16}$$

Let $\kappa = 1$ denote the curvature of C. As a special case of (3.5), the metric distortion between the two planar curves C and C_h with respect to the shortest distance map Φ is given by

$$a = \frac{1 - \phi \cdot \kappa}{\langle N, N_h \rangle} = \frac{1 - \phi}{\langle N, N_h \rangle} = 1. \tag{3.17}$$

Hence, in this case, the metric distortion is the identity, although the shapes of C and C_h are clearly not congruent.

3.2.3 Equivalent conditions for convergence

In this section we provide a central result which establishes several equivalent conditions for metric convergence. Before stating our main result, we fix relevant terminology. Specifically, we discuss how the shortest distance map, Φ, can be used to pull back Sobolev spaces on polyhedra to (metrically) equivalent spaces on the smooth reference surface, M. Throughout we write $\|\cdot\|_\infty$ as shorthand for $\|\cdot\|_{L^\infty(M)}$.

Hausdorff distance. Let $M_1, M_2 \subset \mathbb{R}^3$ be non empty subsets. Then the *Hausdorff distance* between M_1 and M_2 is given by

$$\mathrm{d}_H(M_1, M_2) = \inf\{\varepsilon > 0 \,|\, M_1 \subset U_\varepsilon(M_2) \text{ and } M_2 \subset U_\varepsilon(M_1)\},$$

where $U_\varepsilon(M) = \{x \in \mathbb{R}^3 \,|\, \exists y \in M : d(x, y) < \varepsilon\}$.

Normal convergence. Let $\{M_n\}$ be a sequence of normal graphs over M. For each n let $N_n = N_{M_n} \circ \Phi_n$ be the pullback of the normal field of M_n to M. The sequence $\{M_n\}$ is said to converge *normally* to M if $\|N_n - N\|_\infty \to 0$. We talk about *totally normally convergence* if additionally $d_H(M_n, M) \to 0$.

Sobolev norms and spaces. Let M_h be a normal graph over M, so that M_h induces the polyhedral metric g_A on M. In addition to the standard L^2-norm on the smooth reference surface M, the metric g_A yields another norm on $L^2(M)$. These norms are denoted by

$$\|u\|_{L^2}^2 = \int_M u^2 \, \mathrm{d}vol, \tag{3.18}$$

$$\|u\|_{L_A^2}^2 = \int_M u^2 (\det A)^{1/2} \, \mathrm{d}vol, \tag{3.19}$$

respectively, where $\mathrm{d}vol$ is the volume form on M induced by the Riemannian metric g. Similarly, let $H_0^1(M) \subset L^2(M)$ be the space of weakly differentiable functions u on M which either vanish along the (non empty) boundary of M or for which $\int_M u \, \mathrm{d}vol = 0$ if M has no boundary. The space $H_0^1(M)$ can be equipped with the two norms

$$\|u\|_{H_0^1}^2 = \int_M g(\nabla u, \nabla u) \, \mathrm{d}vol, \tag{3.20}$$

$$\|u\|_{H_{0,A}^1}^2 = \int_M g(A^{-1}\nabla u, \nabla u)(\det A)^{1/2} \, \mathrm{d}vol, \tag{3.21}$$

where ∇ denotes the gradient on M induced by the metric g. The last definition is justified by the fact that the perturbed metric, g_A, induces the gradient $\nabla_A = A^{-1}\nabla$. Compactness of M implies that (3.18), (3.19) and (3.20), (3.21) induce pairwise equivalent (but not equal) norms, which, by Theorem 3.2.1, converge to each other under totally normal convergence.

Lemma 3.2.1 (equivalence of norms)**.** *Let* $u \in L^2(M)$*. Setting* $C_1 = \|(\det A)^{-1/2}\|_\infty$ *and* $C_2 = \|(\det A)^{1/2}\|_\infty$ *we obtain*

$$\frac{1}{C_1}\|u\|_{L^2}^2 \le \|u\|_{L_A^2}^2 \le C_2 \|u\|_{L^2}^2.$$

Similarly, for $u \in H_0^1(M)$*, we get*

$$\frac{1}{C_A}\|u\|_{H_0^1}^2 \le \|u\|_{H_{0,A}^1}^2 \le C_A \|u\|_{H_0^1}^2,$$

with $C_A := (\det A)^{1/2}\|A^{-1}\|_\infty$.

Proof. The L^2-estimate is a simple consequence of (3.18) and (3.19). The H_0^1-estimate follows from (3.20) and (3.21) by applying Hölder's inequality and the fact that the 2×2 symmetric tensor field $(\det A)^{1/2}A^{-1}$ has constant determinant 1, so that $\|(\det A)^{1/2}A^{-1}\|_\infty = \|(\det A)^{-1/2}A\|_\infty$. QED

Remark 3.2.3. If the smooth reference surface, M, has empty boundary then $u \in H^1_{0,A}$ implies $0 = \int_M u (\det A)^{1/2} \,\mathrm{d}vol$. However, this does strictly speaking not imply that $u \in H^1_0$ since the mean value, $\int_M u \,\mathrm{d}vol$, does not need to vanish. Nonetheless,

$$[u] = u - \frac{1}{|M|} \int_M u \,\mathrm{d}vol$$

certainly lies in H^1_0. Since the non-standard inner product on H^1_0 vanishes on constants, we are in the sequel silently going to identify u with $[u]$.

Laplace–Beltrami operators. The metrics g and g_A both induce a Laplace–Beltrami operator on M. The *weak form* of these operators is given by

$$\langle \Delta u | v \rangle = - \int_M g(\nabla u, \nabla v) \,\mathrm{d}vol, \tag{3.22}$$

$$\langle \Delta_A u | v \rangle = - \int_M g(A^{-1} \nabla u, \nabla v) (\det A)^{1/2} \,\mathrm{d}vol, \tag{3.23}$$

respectively, where $\langle \cdot | \cdot \rangle$ denotes the pairing between $H^1_0(M)$ and its dual $H^{-1}(M)$. Both Δu and $\Delta_A u$ are elements of $H^{-1}(M)$ and act on $H^1_0(M)$ as bounded linear functionals[2]. Convergence of these operators is understood in the operator norm (denoted by $\| \cdot \|_{\mathrm{op}}$) of linear bounded maps between the spaces $H^1_0(M)$ and $H^{-1}(M)$.

Consistency with Sobolev spaces on polyhedra. The next lemma shows that our definitions of $L^2_A(M)$ and $H^1_{0,A}(M)$, obtained by pulling back the polyhedral metric back to the smooth reference surface, M, *agree* with our previous definitions of $L^2(M_h)$ and $H^1_0(M_h)$ for Euclidean cone surfaces made in Section 2.2. This lemma therefore justifies, a posteriori, our previous definitions of Sobolev spaces on Euclidean cone surfaces.

More precisely, if Φ denotes the shortest distance map from M to M_h then $\Phi(M) \subset M_h$ is a Euclidean cone surface (possibly with piecewise curvilinear boundary)[3], and we can define the spaces $L^2(\Phi(M))$ and $H^1_0(\Phi(M))$ as in Section 2.2. The lemma then shows that $L^2_A(M)$ and $L^2(\Phi(M))$ are *equal*

[2] If $u \in C^\infty(M)$, then certainly $\Delta u \in C^\infty(M)$ as well, but $\Delta_A u$ does not even have to be in $L^2(M)$ since the metric distortion tensor, A, is usually discontinuous; in fact, the distributional components of $\Delta_A u$ (located at the pre-image of the edges of M_h) must not be neglected.

[3] Notice that the presence of curvilinear boundaries (as opposed to piecewise linear ones) does not pose real difficulties. Indeed, the key points of Section 2.2 are to correctly treat cone singularities – necessary adjustments from linear to curvilinear boundaries are straightforward as long as ∂M_h is piecewise smooth.

as metric spaces, and similarly for $H^1_{0,A}(M)$ and $H^1_0(\Phi(M))$. To show this, notice that all we need to prove is *equality of sets*, i.e. that $u \in L^2(\Phi(M))$ if and only if $u \circ \Phi \in L^2(M)$, and $u \in H^1_0(\Phi(M))$ if and only if $u \circ \Phi \in H^1_0(M)$. *Equality of norms* then follows automatically, by construction.

Lemma 3.2.2 (consistency with polyhedral theory)**.** *Let M_h be a polyhedral surface which is a normal graph over the smooth surface M with corresponding shortest distance map, Φ. Then $u \in L^2(\Phi(M))$ if and only if $u \circ \Phi \in L^2(M)$. Similarly, $u \in H^1_0(\Phi(M))$ if and only if $u \circ \Phi \in H^1_0(M)$.*

Proof. Let $u \in L^2(\Phi(M))$. Since Φ is continuous, it follows that $u \circ \Phi$ is measurable on M, and by Corollary 3.2.1 we obtain

$$\|u \circ \Phi\|^2_{L^2(M)} \leq \|(\det A)^{-1/2}\|_\infty \cdot \|u\|^2_{L^2(\Phi(M))}. \tag{3.24}$$

Hence $u \circ \Phi \in L^2(M)$. The opposite direction follows by a similar argument.

Now let $u \in H^1_0(\Phi(M))$. By definition, there exist a sequence $\{u_n\} \subset C^\infty_0(\Phi(M))$ such that $u_n \to u$ in $H^1_0(\Phi(M))$. Each u_n is continuous on $\Phi(M)$ and smooth outside cone singularities (in the sense of Definition 2.2.2). Moreover, Φ^{-1} is continuous on $\Phi(M)$ and smooth on individual triangles. Hence, $u_n \circ \Phi$ is continuous on M and smooth on the pre-image of individual triangles, so that the differential $\mathrm{d}_M(u_n \circ \Phi)$ is well-defined almost everywhere on M. We obtain

$$\|\,\mathrm{d}_M(u_n \circ \Phi)\|_g \leq \|\,\mathrm{d}_M \Phi\| \cdot \|\,\mathrm{d}_{M_h} u_n\|_{g_{M_h}} \quad \text{a.e..} \tag{3.25}$$

Since $\|\,\mathrm{d}_M\Phi\|$ is uniformly bounded, $\|\nabla_M(u_n \circ \Phi)\|_g = \|\,\mathrm{d}_M(u_n \circ \Phi)\|_g$ is square integrable on M. We claim that $u_n \circ \Phi \in H^1_0(M)$. It suffices to show that integration by parts holds for $u_n \circ \Phi$. To show this, let $X \in \mathfrak{X}^\infty(M)$ be a smooth vector field which is compactly supported on M. Let $D_j \subset M$ be small disks around the pre-image of cone singularities. Consider the integral

$$\int_{M\setminus\cup D_j} g(\nabla_M(u_n \circ \Phi), X)\,\mathrm{d}vol + \int_{M\setminus\cup D_j} (u_n \circ \Phi)\,\mathrm{div}\,X\,\mathrm{d}vol. \tag{3.26}$$

We have to show that this integral tends to zero as the disks D_j tend to points. Let $T_i \subset M$ denote the pre-images of the triangles of M_h under Φ. Then (3.26) can be split up as a sum over these T_i. The contribution of each individual triangle can be written as a boundary integral,

$$\oint_{\partial(T_i\setminus\cup D_j)} (u_n \circ \Phi) g(X, \eta)\,\mathrm{d}s,$$

where η is the normal to $\partial(T_i \backslash \cup D_j)$ in M_h. Taking the sum over all triangles, the integrals over inner edges cancel out since $u_n \circ \Phi$ and X are continuous. Because X is compactly supported and u_n is continuous, we obtain that the last integral tends to zero as the disks, D_j, converge to points,

$$\oint_{\partial(M \backslash \cup D_j)} (u_n \circ \Phi) g(X, \eta)\, \mathrm{d}s \longrightarrow 0.$$

Moreover, since $\{u_n\}$ is a Cauchy sequence in $H_0^1(\Phi(M))$, it follows from (3.24) and (3.25) that $\{u_n \circ \Phi\}$ is a Cauchy sequence in $H_0^1(M)$ which converges to some element $v \in H_0^1(M)$. In particular, $\{u_n \circ \Phi\}$ is a Cauchy sequence in $L^2(M)$ which converges to $u \circ \Phi$ by (3.24). Hence $u \circ \Phi = v \in H_0^1(M)$.

The opposite direction i.e., that $u \circ \Phi \in H_0^1(M)$ implies $u \in H_0^1(\Phi(M))$, is proved in a similar fashion. QED

Main result. The following equivalent conditions relate convergence of normals, convergence of Riemannian metrics, and convergence of Laplace–Beltrami operators. This result generalizes a result of Morvan and Thibert [58] who have recently related convergence of normals to convergence of area.

Theorem 3.2.2 (equivalent conditions for convergence)**.** *Let $M \subset \mathbb{R}^3$ be a compact smooth surface, and let $\{M_n\}$ be a sequence of Euclidean cone surfaces which are normal graphs over the smooth surface M and which converge to M in Hausdorff distance. Then the following conditions are equivalent:*

- **i** $\|N_n - N\|_\infty \to 0$ *(normal convergence).*
- **ii** $\|A_n - Id\|_\infty \to 0$ *(metric convergence).*
- **iii** $\|\, \mathrm{d}vol_n - \mathrm{d}vol\|_\infty \to 0$ *(convergence of area).*
- **iv** $\|\Delta_n - \Delta\|_{\mathrm{op}} \to 0$ *(convergence of Laplace–Beltrami operators).*

Proof. The proof is based on translating conditions (ii), (iii) and (iv) into corresponding properties of the metric distortion tensors A_n: convergence of metric tensors by definition means $\|A_n - Id\|_\infty \to 0$, convergence of area measure is equivalent to $\|\det A_n\|_\infty \to 1$, and Lemma 3.2.3 will provide conditions for convergence of Laplace–Beltrami operators. Each single of these conditions can now be shown to be equivalent to the convergence of normals. To see this, let $A_n = P_n \circ Q_n^{-1} \circ P_n$ as in Theorem 3.2.1, and let $\bar{A}_n = (\det A_n)^{1/2} A_n^{-1}$. We claim that

$$\|A_n - Id\|_\infty \to 0 \iff \|\det A_n\|_\infty \to 1 \iff \|\bar{A}_n - Id\|_\infty \to 0$$
$$\iff \|\mathrm{tr}(\bar{A}_n - Id)\|_\infty \to 0$$

are all equivalent conditions to normal convergence. Indeed, by assumption the surfaces converge in Hausdorff distance, so that $\|P_n - Id\|_\infty \to 0$, and from the diagonalization

$$Q_n = \begin{pmatrix} \langle N, N_n\rangle^2 & 0 \\ 0 & 1 \end{pmatrix},$$

one obtains that the above algebraic expressions involving A_n converge if and only if $\langle N, N_n\rangle \to 1$ in L^∞ - which is normal convergence. To complete the proof of the theorem, it remains to show Lemma 3.2.3. QED

Lemma 3.2.3 (convergence of Laplace–Beltrami operators)**.** *Let $M_h \subset \mathbb{R}^3$ be an embedded compact polyhedral surface which is a normal graph over a smooth embedded surface M. Let A be the metric distortion tensor and $\bar{A} := (\det A)^{1/2} A^{-1}$. Then*

$$\frac{1}{2}\|\mathrm{tr}(\bar{A} - Id)\|_\infty \le \|\Delta_A - \Delta\|_{op} \le \|\bar{A} - Id\|_\infty. \tag{3.27}$$

Proof. The upper bound is a straightforward application of definitions (3.22), (3.23), and Hölder's inequality. To prove the lower bound, let $K \subset M$ be the pre-image under the shortest distance map Φ of the 1-skeleton of M_h (its edges and vertices). Then K is a measure zero set. For an arbitrary (but fixed) $x \in M \setminus K$ we will construct a family of functions $\{f_\varepsilon\} \subset H_0^1(M)$ such that

$$\lim_{\varepsilon\to 0} \frac{|\langle(\Delta_A - \Delta)f_\varepsilon | f_\varepsilon\rangle|}{\|f_\varepsilon\|^2_{H_0^1}} = \frac{1}{2}\mathrm{tr}(\bar{A} - Id)(x). \tag{3.28}$$

This will prove the lower bound since it implies

$$\|\Delta_A - \Delta\|_{\mathrm{op}} \ge \frac{1}{2} \sup_{x\in M\setminus K} \mathrm{tr}(\bar{A} - Id)(x).$$

To construct such a family, let $D_\varepsilon(x) \subset M \setminus K$ be a small ε-disk around x, and define in polar coordinates, (r, φ), (induced by the exponential map $\exp_x(r,\varphi) : \mathrm{T}_x M \to M$)

$$f_\varepsilon(r,\varphi) = \begin{cases} \varepsilon - r & \text{for} \quad r < \varepsilon \\ 0 & \text{else.} \end{cases}$$

Then $f_\varepsilon \in H_0^1$ (if M has empty boundary take $f_\varepsilon - \frac{1}{|M|}\int f_\varepsilon$). By the Gauss lemma, $\exp_x$ is a radial isometry so that $g(\nabla f_\varepsilon, \nabla f_\varepsilon) = 1$ on $D_\varepsilon(x) \setminus \{x\}$. By construction, $\nabla f_\varepsilon = 0$ on $M \setminus D_\varepsilon(x)$. It follows that

$$\|f_\varepsilon\|^2_{H_0^1} = \int_M g(\nabla f_\varepsilon, \nabla f_\varepsilon)\,\mathrm{d}vol = |D_\varepsilon(x)|.$$

Moreover,

$$\langle(\Delta_A - \Delta)f_\varepsilon | f_\varepsilon\rangle = -\int_M g((\bar{A} - Id)\nabla f_\varepsilon, \nabla f_\varepsilon)\,\mathrm{d}vol,$$

so that (3.28) is equivalent to

$$\lim_{\varepsilon \to 0} \frac{1}{|D_\varepsilon(x)|} \int_{D_\varepsilon(x)} g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon)\,\mathrm{d}vol = \frac{1}{2}\mathrm{tr}(\bar{A})(x). \tag{3.29}$$

In a first step we are going to prove (3.29) for the case of constant metric and constant $\bar{A}$. In a second step we will deduce the general case.

1. Step. Let $\mathrm{d}vol_x$ denote the volume form on the tangent space T_xM induced by g_x, and let ∂_r denote the unit radial vector field on T_xM. The coefficients of g_x in polar coordinates are given by

$$(g_x)_{12} = 0, \quad (g_x)_{11} = 1, \quad (g_x)_{22} = r^2.$$

The matrix $\bar{A}_x := \bar{A}(x)$ acts as a linear map from T_xM to itself with eigenvalues λ and $1/\lambda$ (since $\det \bar{A}_x = 1$). On the disk of radius ε, $B_\varepsilon(0) \subset \mathrm{T}_xM$, we have

$$\begin{aligned}\int_{B_\varepsilon(0)} g_x(\bar{A}_x\partial_r, \partial_r)\,\mathrm{d}vol_x &= \int_0^\varepsilon \int_0^{2\pi} (\lambda\cos^2\varphi + \frac{1}{\lambda}\sin^2\varphi) r\,\mathrm{d}r\,\mathrm{d}\varphi \\ &= \frac{1}{2}(\lambda + \frac{1}{\lambda}) \cdot |B_\varepsilon(0)| \\ &= \frac{1}{2}\mathrm{tr}\bar{A}_x \cdot |B_\varepsilon(0)|,\end{aligned}$$

proving (3.29) for the case of the constant $g = g_x$ and constant $\bar{A} = \bar{A}_x$.

2. Step. To complete the proof, we show that for $\varepsilon \to 0$ one has

$$\frac{1}{|D_\varepsilon(x)|} \int_{D_\varepsilon(x)} g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon)\,\mathrm{d}vol \longrightarrow \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} g_x(\bar{A}_x\partial_r, \partial_r)\,\mathrm{d}vol_x.$$

Define a 2-form ω_0 on $B_\varepsilon(0) \subset \mathrm{T}_x(M)$ and a 2-form ω_1 on $D_\varepsilon(x)$ by

$$\omega_0 = g_x(\bar{A}_x \partial_r, \partial_r)\, \mathrm{d}vol_x$$
$$\omega_1 = g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon)\, \mathrm{d}vol.$$

Let $\bar{A}^*$ be the pullback of $\bar{A}$, let ω_1^* denote the pullback of ω_1, and let $\mathrm{d}vol^*$ denote the pullback of the volume form $\mathrm{d}vol$ from $D_\varepsilon(x)$ to $B_\varepsilon(0)$. Since $\exp_x$ is a radial isometry (so that $\mathrm{d}\exp_x(\partial_r) = \nabla f_\varepsilon$), it follows that

$$\omega_1^* = g_x(\bar{A}^* \partial_r, \partial_r)\, \mathrm{d}vol^*.$$

From this, and since $\bar{A}$ and the metric are continuous on $D_\varepsilon(x)$, we obtain

$$\|\omega_1^* - \omega_0\|_{\infty, B_\varepsilon(0)} \longrightarrow 0 \quad \text{and} \quad \frac{|B_\varepsilon(0)|}{|D_\varepsilon(x)|} \longrightarrow 1.$$

Hence

$$\begin{aligned}
&\left| \frac{1}{|D_\varepsilon(x)|} \int_{D_\varepsilon(x)} \omega_1 - \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \omega_0 \right| \\
&= \left| \frac{1}{|D_\varepsilon(x)|} \int_{B_\varepsilon(0)} \omega_1^* - \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \omega_0 \right| \\
&\leq \left| \frac{1}{|D_\varepsilon(x)|} - \frac{1}{|B_\varepsilon(0))|} \right| \int_{B_\varepsilon(0)} |\omega_1^*| + \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} |\omega_1^* - \omega_0| \\
&\leq \left| \frac{|B_\varepsilon(0)|}{|D_\varepsilon(x)|} - 1 \right| \|\omega_1^*\|_{\infty, B_\varepsilon(0)} + \|\omega_1^* - \omega_0\|_{\infty, B_\varepsilon(0)} \longrightarrow 0,
\end{aligned}$$

proving our claim. QED

3.3 CONVERGENCE OF METRIC PROPERTIES

In this section, convergence of solutions to the Dirichlet problem, convergence of mean curvature, convergence of discrete minimal surfaces and convergence of geodesics are derived from the results of the previous section. Throughout, M_h is assumed to be a normal graph over the smooth surface M.

3.3.1 DIRICHLET PROBLEM

As before we consider the variational formulation (2.8) of the Dirichlet problem. Using the pullback via the shortest distance map, all of objects are considered to live on the smooth reference surface, M.

Dirichlet problem. Let $V \subset H_0^1(M)$ be a closed linear subspace. If M has boundary, let $V_A = V$. If M has empty boundary, let $V_A = \{v - \bar{v} \mid v \in V\}$, where

$$\bar{v} = \frac{\int_M v(\det A)^{1/2}\, \mathrm{d}vol_M}{\int_M (\det A)^{1/2}\, \mathrm{d}vol_M}.$$

Let $f \in L^2(M)$. The *approximate Dirichlet problem* is to find solutions $u \in V$ and $u_A \in V_A$ such that

$$-\langle \Delta u | v \rangle = (f, v)_{L^2} \quad \forall\, v \in V, \tag{3.30}$$

$$-\langle \Delta_A u_A | v \rangle = (f, v)_{L^2_A} \quad \forall\, v \in V_A. \tag{3.31}$$

Notation. Here we let

$$E : H_0^1(M) \hookrightarrow L^2(M) \tag{3.32}$$

denote the natural embedding, and we let

$$C_E := \sup_{u \in H_0^1} \frac{\|E(u)\|_{L^2}}{\|u\|_{H_0^1}}$$

denote the operator norm of E. Since we use the inner product $(\nabla \cdot, \nabla \cdot)_{L^2}$ on H_0^1, it follows that C_E is the Poincaré constant of M.

The next theorem slightly generalizes a result of Dziuk [27] who considered interpolating sequences of polyhedral surfaces (i.e. all vertices of the approximating sequence reside on the smooth limit surface M).

Theorem 3.3.1 (consistency error of Dirichlet problem)**.** *The solutions to the approximate Dirichlet problems satisfy*

$$\|u - u_A\|_{H_0^1} \leq C_E \cdot \left((C_A - 1) + c_A \cdot C_A \left\|1 - (\det A)^{1/2}\right\|_\infty\right) \cdot \|f\|_{L^2}.$$

Here $C_A = \|(\det A)^{1/2} A^{-1}\|_\infty$*, and* $c_A = 1 + \|\det A^{1/2}\|_\infty \|\det A^{-1/2}\|_\infty$ *if* $\partial M = \emptyset$*, whereas* $c_A = 1$ *if* $\partial M \neq \emptyset$*.*

Proof. Let $\bar{A} := (\det A)^{1/2} A^{-1}$. First assume that $\partial M \neq \emptyset$. By the weak definition of the Dirichlet problems and equations (3.22) and (3.23) we have

$$\int_M g(\nabla u, \nabla v)\, \mathrm{d}vol = \int_M f v\, \mathrm{d}vol \tag{3.33}$$

$$\int_M g(\bar{A} \nabla u_A, \nabla v)\, \mathrm{d}vol = \int_M f v (\det A)^{1/2}\, \mathrm{d}vol, \tag{3.34}$$

for all $v \in V$. Subtracting the last equations from each other and dividing by $\|\nabla v\|_{L^2}$ gives

$$\frac{\big(\nabla u - \bar{A}\nabla u_A, \nabla v\big)_{L^2}}{\|\nabla v\|_{L^2}} \le C_E \|1 - (\det A)^{1/2}\|_\infty \|f\|_{L^2}, \tag{3.35}$$

Writing

$$\nabla u - \nabla u_A = \bar{A}^{-1}(\nabla u - \bar{A}\nabla u_A) + (Id - \bar{A}^{-1})\nabla u,$$

and using (3.35) gives

$$\begin{aligned} \|u - u_A\|_{H_0^1} &= \sup_{v \in V} \frac{(\nabla u - \nabla \hat{u}, \nabla v)_{L^2}}{\|\nabla v\|_{L^2}} \\ &\le C_E \|\bar{A}^{-1}\|_\infty \|1 - (\det A)^{1/2}\|_\infty \|f\|_{L^2} + \|Id - \bar{A}^{-1}\|_\infty \|u\|_{H_0^1}. \end{aligned}$$

Using (3.33), it follows that $\|u\|_{H_0^1} \le C_E \|f\|_{L^2}$. The final estimate for the case $\partial M \neq \emptyset$ follows from the fact that the 2×2 matrix field $\bar{A}$ has pointwise positive eigenvalues and $\det \bar{A} = 1$ so that $C_A = \|\bar{A}\|_\infty = \|\bar{A}^{-1}\|_\infty$ as well as

$$\|Id - \bar{A}^{-1}\|_\infty = \|\bar{A}\|_\infty - 1 = C_A - 1.$$

For the case $\partial M = \emptyset$ equation (3.34) needs to be adjusted as follows:

$$\int_M g(\bar{A}\nabla u_A, \nabla v)\, \mathrm{d}vol = \int_M f \cdot (v - \bar{v}) \cdot (\det A)^{1/2}\, \mathrm{d}vol \tag{3.36}$$

for all $v \in V$ with

$$\bar{v} = \frac{1}{|\Phi(M)|} \int_M v (\det A)^{1/2}\, \mathrm{d}vol.$$

Using Hölder's inequality and the fact that $\int_M v\, \mathrm{d}vol = 0$ gives

$$|\bar{v}| \le \frac{\sqrt{|M|}}{|\Phi(M)|} \|1 - (\det A)^{1/2}\|_\infty \|v\|_{L^2}.$$

This implies

$$\begin{aligned} \left| \int_M f \bar{v} (\det A)^{1/2}\, \mathrm{d}vol \right| &\le |\bar{v}| \sqrt{|M|} \cdot \| \det A^{1/2}\|_\infty \|f\|_{L^2} \\ &\le \frac{|M|}{|\Phi(M)|} \| \det A^{1/2}\|_\infty \|1 - (\det A)^{1/2}\|_\infty \|f\|_{L^2} \|v\|_{L^2}. \end{aligned}$$

Using that $\frac{|M|}{|\Phi(M)|} \leq \|\det A^{-1/2}\|_\infty$ in the last estimate and dividing by $\|\nabla v\|_{L^2}$ implies together with (3.35) that

$$\frac{\big(\nabla u - \bar{A}\nabla u_A, \nabla v\big)_{L^2}}{\|\nabla v\|_{L^2}} \leq C_E \|1 - (\det A)^{1/2}\|_\infty \|f\|_{L^2} \cdot c_A,$$

with $c_A = 1 + \|\det A^{-1/2}\|_\infty \|\det A^{1/2}\|_\infty$. What remains to prove for the case $\partial M = \emptyset$ is identical to the discussion following (3.35) for $\partial M \neq \emptyset$. QED

Corollary 3.3.1 (convergence of Dirichlet problem)**.** *Let $f \in L^2(M)$. If the sequence of polyhedral surfaces $\{M_n\}$ converges totally normally to the smooth surface M, then the solutions to the Dirichlet problems (3.31) on M_n converge in $H^1_0(M)$ to the solution of the Dirichlet problem (3.30) on M.*

Proof. Let $C_A := \|(\det A)^{1/2} A^{-1}\|_\infty$ as in Theorem 3.3.1. Using Theorem 3.2.1, one verifies that totally normal convergence implies $C_A \to 1$ as well as $\|1 - (\det A)^{1/2}\|_\infty \to 0$, so that Theorem 3.3.1 guarantees convergence in $H^1_0(M)$. QED

3.3.2 Finite Element discretization of the Dirichlet problem

In the last section we derived the consistency error of the Dirichlet problem. It remains to estimate the interpolation error using nodal elements. We closely follow the arguments of Dziuk [27] in this section – with the exception that we extend the results of [27] from *interpolating* sequences of polyhedral meshes to *approximating* sequences which converge totally normally.

FE discretization. Using the shortest distance map, $\Phi : M \to M_h$, the finite element space, $S_h \subset H^1(M_h)$, gets mapped to a linear subspace, $\hat{S}_h \subset H^1(M)$[4]. We let $\hat{S}_{h,0} = \hat{S}_h \cap H^1_0(M)$; in particular, any $\hat{u}_h \in \hat{S}_{h,0}$ vanishes along the boundary, ∂M. The objective of this subsection is to compare the solutions $u \in H^1_0(M)$ and $\hat{u} \in \hat{S}_{h,0}$ to the Dirichlet problems

$$-\langle \Delta u | v \rangle = (f, v)_{L^2} \quad \forall\, v \in H^1_0, \tag{3.37}$$

$$-\langle \Delta_A \hat{u}_h | v \rangle = (f, v)_{L^2_A} \quad \forall\, v \in \hat{S}_{h,0}. \tag{3.38}$$

Assumption. We will assume that the solution satisfies $u \in H^2(M)$ (see Definition 3.3.2 for a precise definition of the Sobolev space H^2(M)) together

[4]In the case $\partial M = \emptyset$, the shortest distance map, Φ, induces a bijection form S_h to $\hat{S}_h$. In the case $\partial M \neq \emptyset$, however, where we require only that $\Phi(M) \subset M_h$ (but not necessarily $\Phi(M) = M_h$), the map from S_h to $\hat{S}_h$ is in general not injective.

with an a priori estimate

$$|u|_{H^2(M)} \le c\|f\|_{L^2(M)}. \tag{3.39}$$

This regularity assumption holds for all $f \in L^2(M)$ if $\partial M = \emptyset$ (by classical regularity, cf. [35]), but depends on properties of the boundary if $\partial M \ne \emptyset$.

Overview. In order to provide an upper bound for the difference $(u - \hat{u}_h)$, we take the usual approach of first estimating the interpolation error $(u - I_h u)$, where

$$I_h : H^2(M) \longrightarrow \hat{S}_h, \tag{3.40}$$

is a suitable interpolation operator. We construct $I_h(u)$ as follows. Since we assume that $u \in H^2(M)$, the Sobolev embedding theorem asserts that $u \in C^0(M)$. Hence I_h is well defined by point-sampling u at the vertices obtained by pulling back M_h to M via Φ. Recall from the planar case that

$$\|\nabla u - \nabla I_h u\|_{L^2(T_h)} \le Ch|u|_{H^2(T_h)}, \tag{3.41}$$

where h denotes the diameter of the flat Euclidean triangle T_h, the constant $C > 0$ depends on the *shape regularity* of T_h (see Definition 3.3.1 below), and $|u|_{H^2(T_h)}$ denotes the H^2-seminorm, see [14]. In Theorem 3.3.2, we prove an estimate similar to (3.41) by bounding $\|\nabla u - \nabla I_h u\|_{L^2(T)}$, for the *curved triangle*, $T \subset M$, given by

$$T := \{x \in M \mid \Phi(x) \in T_h \subset M_h\}. \tag{3.42}$$

In order to estimate $\|\nabla u - \nabla I_h u\|_{L^2(T)}$, we make use of (3.41) and bound the seminorm $|u|_{H^2(T_h)}$ on the flat triangle, T_h, in terms of the seminorm $|u|_{H^2(T)}$ on the curved triangle, T, see Lemma 3.3.1. In particular, we will give a precise meaning to the space $H^2(\Omega)$ for any subdomain $\Omega \subset M$. Finally, the bound on $\|\nabla u - \nabla I_h u\|_{L^2(T)}$ will enable us to estimate the difference $(u - \hat{u}_h)$ in $H^1_0(M)$ in Theorem 3.3.3.

Definition 3.3.1 (shape regularity)**.** The shape regularity (aspect ratio) of a planar triangle T_h is the ratio of the radius $R(T_h)$ of its circumcircle to the radius $r(T_h)$ of its incircle. A sequence of triangulations is called shape regular if there exists a constant $\kappa < \infty$ such that $\kappa \ge R(T_h)/r(T_h)$ for all triangles T_h in the sequence.

Theorem 3.3.2 (interpolation error)**.** *Let $u \in H^2(M)$, and let T be a curved triangle, as defined in (3.42), which does not touch the boundary, ∂M. Then*

$$\|\nabla u - \nabla I_h u\|_{L^2(T)} \le Ch\left(|u|_{H^2(T)} + (\|N - N_h\langle N, N_h\rangle\|_{\mathbb{R}^3} + |\phi|)\,\|\nabla u\|_{L^2(T)}\right),$$

where h denotes the diameter of the planar triangle T_h. The constant C only depends on the aspect ratio of T_h, the distance, $|\phi|$, between T and T_h, and the angle between the normals, N of T and N_h of T_h.

Proof. Lemma 3.2.1 together with the estimate (3.41) in the planar case gives

$$\|\nabla u - \nabla I_h u\|_{L^2(T)} \leq C_A \|\nabla u - \nabla I_h u\|_{L^2(T_h)} \leq C_A C h |u|_{H^2(T_h)},$$

where the constant C only depends on the shape regularity of T_h and the constant C_A is given by $C_A := \|A^{-1}(\det A)^{1/2}\|_\infty$. It remains to relate $|u|_{H^2(T_h)}$ to $|u|_{H^2(T)}$, which will be done in Lemma 3.3.1. QED

The Sobolev space $H^2(M)$. We now give a precise meaning to $|u|_{H^2(T)}$ for the curved triangle T. One way to define $H^2(M)$ involves a partition of unity subordinate to a locally finite cover of M by open charts. Two such partitions can be shown to give rise to equivalent norms on $H^2(M)$ provided that M is compact. Here we prefer to give a different, intrinsic, definition of $H^2(M)$ which is not based on charts. Given a Riemannian manifold (M, g), let $\operatorname{Hess_g} u$ denote the Hessian of u. $\operatorname{Hess_g} u$ is a symmetric bilinear form whose action on pairs of vector fields (X, Y) is given by

$$\operatorname{Hess_g} u(X, Y) = g(\nabla_X \nabla u, Y) = X(Yu) - (\nabla_X Y)u. \tag{3.43}$$

Here, as usual, ∇_X denotes the covariant derivative in the direction of X and we write Xu short for the scalar field $\mathrm{d}u(X)$.

Definition 3.3.2 (Hessian and H^2). Let (e_1, e_2) be a smooth $SO(2)$-framing in the tangent bundle of the Riemann surface (M, g). Define the (pointwise) norm of $\operatorname{Hess_g} u$ by

$$|\operatorname{Hess_g} u|^2 = \sum_{i,j} \left(\operatorname{Hess_g} u(e_i, e_j)\right)^2.$$

For any open $\Omega \subset M$, define a seminorm on $H^2(\Omega)$ by

$$|u|^2_{H^2(\Omega)} := \int_\Omega |\operatorname{Hess_g} u|^2 \, \mathrm{d}vol.$$

Finally,

$$\|u\|^2_{H^2(M)} := |u|^2_{H^2(M)} + \|\nabla u\|^2_{L^2(M)} + \|u\|^2_{L^2(M)}$$

gives rise to a norm on $H^2(M)$.

It is straightforward to check that the definition of $|\operatorname{Hess}_g u|$ is independent of any particular $SO(2)$-framing and that $\|\cdot\|_{H^2(M)}$ is a norm which is equivalent to the usual definition involving local charts (provided that M is compact). Clearly, for the planar case, the above definition of $|\cdot|_{H^2}$ coincides with the classical definition in that for standard Euclidean coordinates (x_1, x_2) on a planar domain Ω_h, one has

$$|u|^2_{H^2(\Omega_h)} = \sum_{i,j} \int_{\Omega_h} (\operatorname{Hess} u(\partial_i, \partial_j))^2 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

To complete the proof of Theorem 3.3.2, we need the following H^2-estimate.

Lemma 3.3.1 (H^2-estimate). *Let $u \in H^2(M)$, and let T be a curved triangle, as defined in (3.42), which does not touch the boundary, ∂M. Then*

$$|u|_{H^2(T_h)} \leq C \left(|u|_{H^2(T)} + (\|N - N_h\langle N, N_h\rangle\|_{\mathbb{R}^3} + |\phi|) \|\nabla u\|_{L^2(T)} \right).$$

The constant, $C = C(M, \|A^{-1}\|_\infty, \|P^{-1}\|_\infty, \|\det A\|_\infty) > 0$, only depends on M, the metric distortion tensor, A, and on $P = (Id - \phi\mathbf{S})$ (see Theorem 3.2.1).

Proof. Let (x_1, x_2) be standard Euclidean coordinates on the flat triangle T_h. Then $(x_1 \circ \Phi, x_2 \circ \Phi)$ are coordinates on the curved triangle T. Let (∂_1, ∂_2) denote the framing corresponding to these coordinates on $T \subset M$. From the definition of the metric distortion tensor A (cf. equation (3.4)) it follows that

$$(e_1, e_2) = (A^{1/2}\partial_1, A^{1/2}\partial_2)$$

is an $SO(2)$-framing on the curved triangle T. This gives the pointwise relation

$$\sum_{i,j} (\operatorname{Hess}_g u(\partial_i, \partial_j))^2 \leq \|A^{-1}\|^2_{L^\infty(T)} |\operatorname{Hess}_g u|^2. \tag{3.44}$$

By equation (3.43), the terms under the sum can be written as

$$\operatorname{Hess}_g u(\partial_i, \partial_j) = \operatorname{Hess} u(\partial_i, \partial_j) - (\nabla_{\partial_i}\partial_j)u, \tag{3.45}$$

with $\operatorname{Hess} u(\partial_i, \partial_j) = \partial_i(\partial_j u) = \partial_j(\partial_i u)$. Note that

$$\nabla_{\partial_i}\partial_j = \left(\frac{\partial^2 \Phi^{-1}(x)}{\partial x_i \partial x_j} \right)^{\tan},$$

where $\Phi^{-1} : T_h \to T \subset \mathbb{R}^3$ and $(\cdot)^{\tan}$ denotes the projection to the tangent space $\mathrm{T}_{\Phi^{-1}(x)}M$. Using equations (3.9) and (3.12) from the proof of Theorem 3.2.1, a straightforward calculation shows

$$\nabla_{\partial_i}\partial_j = \mathbf{S}(\partial_i)\langle N, \mathrm{d}\Phi(\partial_j)\rangle_{\mathbb{R}^3} + \mathbf{S}(\partial_j)\langle N, \mathrm{d}\Phi(\partial_i)\rangle_{\mathbb{R}^3} + \phi\nabla_{\partial_i}(\mathbf{S}(\partial_j)),$$

where ϕ denotes the (signed) shortest distance (cf. equation (3.2)), N is the normal field of T, and $\mathbf{S}$ is the shape operator on $T \subset M$. From

$$\nabla_{\partial_i}(\mathbf{S}(\partial_j)) = (\nabla_{\partial_i}\mathbf{S})(\partial_j) + \mathbf{S}(\nabla_{\partial_i}\partial_j),$$

it follows that

$$\nabla_{\partial_i}\partial_j = P^{-1}\left(\mathbf{S}(\partial_i)\langle N, \mathrm{d}\Phi(\partial_j)\rangle_{\mathbb{R}^3} + \mathbf{S}(\partial_j)\langle N, \mathrm{d}\Phi(\partial_i)\rangle_{\mathbb{R}^3} + \phi\cdot(\nabla_{\partial_i}\mathbf{S})(\partial_j)\right),$$

with $P = (Id - \phi\mathbf{S})$ as in Theorem 3.2.1. If g_A denotes the pullback of the flat metric from T_h to T then $\|\partial_i\|_{g_A} = 1$ and hence we have the pointwise relation

$$\|\nabla_{\partial_i}\partial_j\|_{g_A} \leq C\|P^{-1}\|_{L^\infty(T)}\left(\|N - N_h\langle N, N_h\rangle\|_{\mathbb{R}^3} + |\phi|\right),$$

where N_h is the (constant) normal field to T_h and C only depends on properties of M. Applying relation (3.44) and equation (3.45), we find that there exists a constant $C = C(M, \|A^{-1}\|_\infty, \|P^{-1}\|_\infty) > 0$ such that pointwise

$$|\operatorname{Hess} u|^2 \leq C\left(|\operatorname{Hess}_{\mathrm{g}} u|^2 + \left(\|N - N_h\langle N, N_h\rangle\|_{\mathbb{R}^3} + |\phi|\right)^2\|\nabla u\|_g^2\right).$$

The claim follows by integrating the above. QED

Main result. We can now give the main result of this subsection, which compares the solution, u, to the continuous Dirichlet problem (3.37) with the FE solution, $\hat{u}_h$, to the discrete Dirichlet problem (3.38). In the case $\partial M \neq \emptyset$, we assume that the solution, u, to (3.37) satisfies $I_h u \in \hat{S}_{h,0}$, i.e., u is supported sufficiently far away from the boundary[5].

Theorem 3.3.3 (FE error estimate)**.** *Let h denote the mesh size of M_h, and let $\eta > 0$ be a positive constant such that under the shortest distance map*

$$\|\angle(N, N_h)\|_\infty \leq \eta\cdot h \quad \text{and} \quad \|\phi\|_\infty \leq \eta\cdot h. \tag{3.46}$$

[5]This condition is to avoid technical issues when estimating $(u - I_h u)$ for functions u for which $I_h u$ does not vanish along the boundary, ∂M. The condition can be satisfied, for example, if the right hand side, f, is supported sufficiently far away from the boundary.

Then there exists a constant $C > 0$, which only depends on M, η, and the aspect ratio of the triangles of M_h – but not on h itself – such that

$$\|u - \hat{u}_h\|_{L^2(M)} + h\|u - \hat{u}_h\|_{H^1_0(M)} \leq Ch^2\|f\|_{L^2(M)} \quad \text{if } \partial M \neq \emptyset,$$
$$\|u - \hat{u}_h\|_{L^2(M)/\mathbb{R}} + h\|u - \hat{u}_h\|_{H^1_0(M)} \leq Ch^2\|f\|_{L^2(M)} \quad \text{if } \partial M = \emptyset.$$

Here $\|u\|_{L^2(M)/\mathbb{R}}$ is the L^2-norm of the unique representative of $[u] \in L^2/\mathbb{R}$ having zero mean.

Remark. For interpolating meshes the constant η in (3.46) only depends on M and the aspect ratio of the triangles of M_h (cf. Lemma 3.5.1 below). This led Dziuk [27] to the same error estimate as the above for interpolating meshes.

Proof. Let A be metric distortion tensor induced by the shortest distance map, and let $C_A := \|(\det A)^{1/2}A^{-1}\|_\infty$. It follows from assumption (3.46) and the splitting of the metric distortion tensor of Theorem 3.2.1 that

$$C_A - 1 = \mathcal{O}(h) \quad \text{and} \quad (\det A)^{1/2} - 1 = \mathcal{O}(h).$$

Let $u_h \in \hat{S}_{h,0}$ be the FE solution to the Dirichlet problem with respect to Riemannian metric on M, that is

$$-\langle \Delta u_h | v \rangle = (f, v)_{L^2} \quad \forall\, v \in \hat{S}_{h,0}(M).$$

Since $u_h \in \hat{S}_{h,0}(M)$ is the projection of $u \in H^1_0(M)$ with respect to the inner product $(\cdot,\cdot)_{H^1_0(M)}$, it follows that

$$\|u - u_h\|_{H^1_0(M)} \leq \|\nabla u - \nabla I_h u\|_{L^2(M)}.$$

Moreover, Theorem 3.3.1 implies

$$\|\hat{u}_h - u_h\|_{H^1_0(M)} \leq Ch\|f\|_{L^2(M)}.$$

Together with Theorem 3.3.2 and the a priori estimate (3.39) we obtain

$$\begin{aligned}\|u - \hat{u}_h\|_{H^1_0(M)} &\leq \|u - u_h\|_{H^1_0(M)} + \|\hat{u}_h - u_h\|_{H^1_0(M)} \\ &\leq \|\nabla u - \nabla I_h u\|_{L^2(M)} + \|\hat{u}_h - u_h\|_{H^1_0(M)} \\ &\leq Ch\|f\|_{L^2(M)}.\end{aligned}$$

This gives the H^1_0-estimate. Finally, the L^2-estimate on $\|u - \hat{u}_h\|_{L^2(M)}$ follows by employing the Aubin-Nitsche-trick (cf. Dziuk [28]). QED

3.3.3 Mean curvature as a functional

In this section we show that mean curvature vectors converges in the sense of *distributions* or *functionals*, that is, as elements of the Sobolev space $\mathcal{H}^{-1}(M)$ (Theorem 3.3.4). This result is used to show: if a sequence of discrete minimal surfaces converges totally normally to a smooth surface then the limit surface is minimal in the classical sense (Theorem 3.3.5). Finally, we give a counterexample to the convergence of mean curvature vectors in $\mathcal{L}^2$.

As before, let $\Phi : M \to M_h$ be the shortest distance map. Additionally, let $\vec{E} : M \to \mathbb{R}^3$ and $\vec{E}_{M_h} : M_h \to \mathbb{R}^3$ denote the *isometric* embeddings of M and M_h, respectively. Set $\vec{E}_h = \vec{E}_{M_h} \circ \Phi : M \to \mathbb{R}^3$. Recall from Section 2.4.5 that weak mean curvature vectors are functionals, respectively given by

$$\begin{aligned} \vec{H} &= \Delta \vec{E} \in (H^{-1}(M))^3, \\ \vec{H}_A &= \Delta_A \vec{E}_h \in (H^{-1}(M))^3. \end{aligned}$$

In particular, we regard both $\vec{H}$ and $\vec{H}_A$ as $\mathbb{R}^3$-valued functionals on $H_0^1(M)$. We define the norm of these $\mathbb{R}^3$-valued functionals by

$$\|\vec{H}\|_{H^{-1}} = \sup_{0 \neq u \in H_0^1} \frac{\|\langle \vec{H} | u \rangle\|_{\mathbb{R}^3}}{\|u\|_{H_0^1}}.$$

Theorem 3.3.4 (approximation of weak mean curvature)**.** *Let M_h be a normal graph over the smooth surface M. Then*

$$\|\vec{H} - \vec{H}_A\|_{H^{-1}(M)} \leq \sqrt{|M|} \cdot (C_A - 1 + C_A \|Id - \mathrm{d}\Phi\|_\infty), \tag{3.47}$$

where $C_A = \|(\det A)^{1/2} A^{-1}\|_\infty$, $|M|$ is the total area of M, and $\|Id - \mathrm{d}\Phi\|_\infty$ denotes the essential supremum over the pointwise operator norm of the operator $(Id - \mathrm{d}\Phi)(x) : \mathrm{T}_x M \to \mathbb{R}^3$.

Proof. We apply the triangle inequality to

$$\vec{H} - \vec{H}_A = (\Delta \vec{E} - \Delta_A \vec{E}_h) = (\Delta \vec{E} - \Delta_A \vec{E}) + (\Delta_A \vec{E} - \Delta_A \vec{E}_h).$$

For any vector field X on M one has $\langle \nabla \vec{E}, X \rangle_{\mathbb{R}^3} = X$, and $\langle \nabla \vec{E}_h, X \rangle_{\mathbb{R}^3} = \mathrm{d}\Phi(X)$ almost everywhere. Applying (3.22), (3.23), and Hölder's inequality, we obtain

$$\begin{aligned} \left\| \langle \Delta \vec{E} - \Delta_A \vec{E} | u \rangle \right\|_{\mathbb{R}^3} &= \left\| \int_M (A^{-1} (\det A)^{1/2} - Id) \nabla u \, \mathrm{d}vol \right\|_{\mathbb{R}^3} \\ &\leq \sqrt{|M|} \cdot (C_A - 1) \cdot \|u\|_{H_0^1}. \end{aligned}$$

and similarly

$$\begin{aligned}\left\|\langle \Delta_A \vec{E} - \Delta_A \vec{E}_h | u\rangle\right\|_{\mathbb{R}^3} &= \left\| \int_M \left\langle (\nabla \vec{E} - \nabla \vec{E}_h), A^{-1}(\det A)^{1/2} \nabla u \right\rangle \, \mathrm{d}vol \right\|_{\mathbb{R}^3} \\ &= \left\| \int_M (Id - \mathrm{d}\Phi)(A^{-1}(\det A)^{1/2}) \nabla u \, \mathrm{d}vol \right\|_{\mathbb{R}^3} \\ &\leq \sqrt{|M|} \cdot C_A \cdot \|Id - \mathrm{d}\Phi\|_\infty \cdot \|u\|_{H_0^1},\end{aligned}$$

proving the claim. QED

Corollary 3.3.2 (convergence of weak mean curvature)**.** *If a sequence of polyhedral surfaces* $\{M_n\}$ *converges totally normally to the smooth surface* M*, then the corresponding mean curvature functionals converge in* $H^{-1}(M)$.

Proof. Under the assumption of totally normal convergence, we get $C_A \to 1$. It remains to show that $\|Id - \mathrm{d}\Phi\|_\infty \to 0$. Consider a single triangle T_h of M_h. Let $N_{T_h} = N \circ \Phi^{-1}$ denote the pullback of the normal field N on M to the triangle T_h. From the proof of Theorem 3.2.1 we know that $\mathrm{d}\Phi = \tilde{Q}^{-1} \circ P$, where $\tilde{Q}$ is given by $\tilde{Q}(Y) = Y - N_{T_h} \cdot \langle N_{T_h}, Y\rangle$ (cf. equation (3.13)). Totally normal convergence implies $P \to Id$ as well as $\tilde{Q} \to Id$, and hence $\mathrm{d}\Phi \to Id$ almost everywhere. QED

3.3.4 Discrete minimal surfaces

In this section we show that if a sequence of discrete minimal surfaces converges totally normally to a smooth surfaces, then this limit surface must be a minimal surface in the classical sense.

Although existence and regularity of minimal surfaces spanning a given boundary is a well-studied problem, it remains a challenge to explicitly construct minimal surfaces with prescribed boundary data. Pinkall and Polthier [61] suggested an algorithm for numerically approximating area-minimizing polyhedral surfaces by sequentially solving the Dirichlet problem with respect to the metric of the current iterate. For the same purpose, Dziuk [28] used a discretization of the mean curvature flow. Similarly, Ken Brakke's *Surface Evolver* [15] produces numerical approximations of area-minimizing surfaces. Later, various examples of explicitly computable discrete minimal surfaces were discovered [38, 51, 63, 64, 66, 69]. However, it is an open problem whether one can find discrete minimal surfaces arbitrarily close to a given smooth (and possibly unstable) minimal surface. Dziuk and Hutchinson [29, 30] give a positive answer to this problem for the case of minimal surfaces having the topology of a disk, and Pozzi [67] extends their result to annuli[6]. Another related problem is the design a converging refinement scheme of discrete minimal surfaces yielding a smooth minimal surface in the limit. The convergence result of this section provides a step into this direction.

Große-Brauckmann and Polthier [44] constructed examples of compact constant mean curvature (CMC) surfaces of low genus numerically, based on a discrete version of the conjugate surface construction [60]. It is an interesting question whether the convergence results of this section can help to prove that these numerical examples yield smooth CMC surfaces.

Recall from Definition 2.4.1 that discrete minimality implies $\langle \vec{H}_A | u_h \rangle = 0$ for all $u_h \in S_{h,0}$, so that it is a *weaker* condition than $\vec{H}_A = 0$. In view of this fact, the following result is surprising:

Theorem 3.3.5 (convergence of discrete minimal surfaces)**.** *Let $\{M_n\}$ be a sequence of polyhedral minimal surfaces comprised of triangles with uniformly bounded aspect ratio. Assume $\{M_n\}$ converges totally normally to a smooth surface $M \subset \mathbb{R}^3$. Then the smooth limit surface, M, is a minimal surface in the classical sense.*

Proof. Let $\vec{H}$ denote the (smooth) mean curvature vector of the smooth surface M. We are going to show that

$$\langle \vec{H} | u \rangle = \int_M \vec{H} \cdot u \, \mathrm{d}vol = 0$$

for all $u \in C_0^\infty(M)$ which are supported *away from the boundary*, ∂M. As in Section 3.3.2, let $\hat{S}_{n,0} \subset H_0^1(M)$ denote the finite element spaces induced by linear Lagrange elements on the meshes M_n, pulled back under Φ and intersected with $H_0^1(M)$. Let $\hat{u}_n$ be the *orthogonal projection* of u to $\hat{S}_{n,0}$ with respect to the H^1-inner product $(\nabla u, \nabla v)_{L^2(M)}$. We have

$$\left\| \langle \vec{H} | u \rangle \right\|_{\mathbb{R}^3} \leq \left\| \langle \vec{H} | u - \hat{u}_n \rangle \right\|_{\mathbb{R}^3} + \left\| \langle \vec{H} | \hat{u}_n \rangle \right\|_{\mathbb{R}^3}. \tag{3.48}$$

We are going to show that the right hand side of (3.48) tends to zero as the

[6] Recently Bobenko et al. [8] provided a different ('non-linear') view of discrete mean curvature. They show that their approach allows for finding discrete minimal surfaces arbitrarily close to smooth ones.

mesh size of M_n goes to zero. Indeed, since $\vec{H}$ is smooth, it follows that

$$\begin{aligned}\left\|\langle\vec{H}|u-\hat{u}_n\rangle\right\|_{\mathbb{R}^3} &= \left\|\int_M \vec{H}\cdot(u-\hat{u}_n)\,\mathrm{d}vol\right\|_{\mathbb{R}^3}\\ &\le \left\|\vec{H}\right\|_{L^2(M)}\|u-\hat{u}_n\|_{L^2(M)}\\ &\le \left\|\vec{H}\right\|_{L^2(M)}\cdot C_E\cdot\|u-\hat{u}_n\|_{H^1_0(M)}\\ &\le \left\|\vec{H}\right\|_{L^2(M)}\cdot C_E\cdot\|u-I_n u\|_{H^1_0(M)}\,,\end{aligned}$$

where C_E is the Poincaré constant defined of M, and I_n denotes the interpolation operator (cf. Section 3.3.2). The last inequality holds since $\hat{u}_n$ is the *projection* of u. Finally, since $u \in C_0^\infty(M)$ and u can be assumed to be supported sufficiently far away from the boundary, we obtain $I_n u \in \hat{S}_{n,0}$ and hence $\|u - I_n u\|_{H^1_0(M)} \to 0$ by Theorem 3.3.2 (indeed we assume the aspect ratios of the triangles of $\{M_n\}$ to be uniformly bounded).

To estimate the last term in (3.48), let $\vec{H}_n$ denote the weak mean curvature associated with M_n. By assumption, $\vec{H}_n$ vanishes on $\hat{S}_{n,0}$ and hence

$$\begin{aligned}\left\|\langle\vec{H}|\hat{u}_n\rangle\right\|_{\mathbb{R}^3} &= \left\|\langle\vec{H}-\vec{H}_n|\hat{u}_n\rangle\right\|_{\mathbb{R}^3}\\ &\le \left\|\vec{H}-\vec{H}_n\right\|_{H^{-1}(M)}\cdot\|\hat{u}_n\|_{H^1_0(M)}\\ &\le \left\|\vec{H}-\vec{H}_n\right\|_{H^{-1}(M)}\cdot\|u\|_{H^1_0(M)}.\end{aligned}$$

From Corollary 3.3.2 it follows that $\|\vec{H}-\vec{H}_n\|_{H^{-1}(M)} \to 0$. From (3.48) we obtain $\vec{H} = 0$, as asserted. QED

3.3.5 Mean Curvature as a function

Weak mean curvature is a $\mathbb{R}^3$-valued functional. *Discrete mean curvature* is the $\mathbb{R}^3$-valued *piecewise linear function* associated with this functional, in the sense of *discretized functionals* of Section 2.3.4. Corollary 3.3.2 shows that the mean curvature functionals converges in H^{-1}. The objective of this section is to show that the associated discrete mean curvature functions in general fail to converge in L^2. This failure comes to no surprise considering the various recently observed counterexamples to pointwise convergence of discrete differential operators: see e.g. Meek and Walton [53], Borrelli et al. [13], Xu [84, 85], and references therein. The main obstacle seems to be that surface normals of polyhedra approximate normals of smooth surfaces only to order $\mathcal{O}(h)$; whereas, in order to obtain pointwise convergence, one would need $\mathcal{O}(h^2)$-approximation since curvatures correspond to normal derivatives.

Definition 3.3.3 (discrete mean curvature)**.** Discrete mean curvature is the $\mathbb{R}^3$-valued piecewise linear function $\vec{H}_h \in S_h$ defined by

$$\int_{M_h} \vec{H}_h \cdot u_h \, \mathrm{d}vol_{M_h} = \langle \vec{H}_A | u_h \rangle \quad \forall\, u_h \in S_h. \tag{3.49}$$

Note that, only because the dimension of S_h is finite, it is possible to associate a discrete function to the mean curvature functional. There is no infinite-dimensional analogue of this construction. As in Definition 2.3.5, the discrete mean curvature function can be computed *explicitly* on a polyhedral surface M_h:

$$\vec{H}_h = \sum_{p,q} \langle \vec{H}_A | \phi_p \rangle \mathcal{M}^{pq} \phi_q, \tag{3.50}$$

where $\langle \vec{H}_A | \phi_p \rangle$ denotes the evaluation of the mean curvature functional $\vec{H}_A$ on the nodal basis function ϕ_p, and $\mathcal{M}^{pq}$ denotes the inverse of the *mass matrix*, $\mathcal{M}_{pq}$.

Example (counterexample to L^2-convergence)**.** Denote by $\vec{H}$ the smooth mean curvature vector of the smooth surface M, and let $\{\vec{H}_n\}$ denote the sequence of *discrete mean curvature vectors* associated with the sequence of polyhedral surfaces $\{M_n\}$. We show that in general $\|\vec{H}_n - \vec{H}\|_{L^2}$ does not converge to zero. Consider the cylinder M of height 2π and radius 1. We construct a sequence, $\{M_n\}$, of polyhedral surfaces whose vertices lie on this cylinder and which converges to M totally normally. Let the cylinder be parameterized as

$$x = \cos u, \quad y = \sin u, \quad z = v.$$

Let the vertices of M_n be given by

$$\begin{aligned} u &= \frac{i\pi}{n} & i &= 0, ..., 2n-1 \\ v &= \begin{cases} 2j \sin \frac{\pi}{2n} & j = 0, ..., 2n-1 \\ 2\pi & j = 2n \end{cases} \end{aligned}$$

This corresponds (up the uppermost layer) to folding along the vertical lines a regular planar quad-grid of edge length

$$h_n = 2 \sin \frac{\pi}{2n}.$$

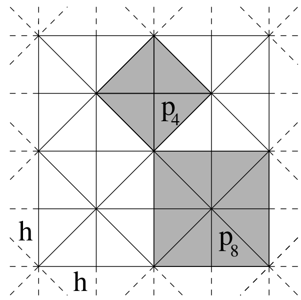

Figure 3.5: Discrete mean curvature does not converge in L^2 for a $4-8$ tessellation of a regular quad grid, because the ratio between the areas of the stencils of p_4 and p_8 does not converge to 1.

In other words, all faces of M_n are rectangular (in fact quadratic except for the uppermost layer). It will now depend on the *tessellation pattern* of this quad-grid whether there is L^2-convergence of discrete mean curvature or not. Indeed, consider the regular $4-8$ tessellation scheme depicted in Figure 3.5. There are two kinds of vertices - those of valence 4 and those of valence 8. Call them p_4 and p_8, respectively. Let ϕ_{p_4} and ϕ_{p_8} denote the corresponding Lagrange basis functions. Using the cotan-formula, it is easy to see that the coefficients of the weak mean curvature satisfy

$$\langle \vec{H} | \phi_{p_4} \rangle = \langle \vec{H} | \phi_{p_8} \rangle = -2\left(1 - \cos\frac{\pi}{n}\right) \cdot \partial_r,$$

where ∂_r denotes the (radial) outward cylinder normal field. By the symmetry of the problem there exist constants $a_n, b_n \in \mathbb{R}$ such that

$$\vec{H}_n = \sum_{p_4} a_n \cdot \phi_{p_4} \cdot \partial_r + \sum_{p_8} b_n \cdot \phi_{p_8} \cdot \partial_r + \text{boundary contributions},$$

where the contributions from the boundary include all vertices one layer away from the upper boundary (as symmetry breaks there). Set

$$\lambda_n := -\left(1 - \cos\frac{\pi}{n}\right).$$

One verifies that

$$a_n = 12 \cdot \frac{\lambda_n}{h_n^2} \cdot \frac{4 + \lambda_n}{8 - \lambda_n^2} \quad \text{and} \quad b_n = 12 \cdot \frac{\lambda_n}{h_n^2} \cdot \frac{\lambda_n}{\lambda_n^2 - 8}.$$

Since $\lim_{n\to\infty}(\lambda_n/h_n^2) = -1/2$, it follows that

$$\lim_{n\to\infty} a_n = -3 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0,$$

so that asymptotically *only the vertices of valence* 4 *but not those of valence* 8 *contribute to discrete mean curvature,*

$$\vec{H}_n \sim -3 \sum_{p_4} \phi_{p_4} \cdot \partial_r + \text{boundary contributions.}$$

Hence, $\vec{H}_n$ is a family of PL-functions oscillating between -3 (at the vertices of valence 4) and 0 (at the vertices of valence 8) with ever growing frequencies. Such a family does not converge in L^2.

3.3.6 Geodesics

Let $\{M_n\}$ be a family of polyhedral surfaces converging totally normally to a compact smooth surface M with associated shortest distance maps Φ_n. By the Hopf-Rinow theorem (cf. Proposition 2.1.1), any two points of M_n can be connected by a minimizing geodesic.

Recall that if U is a locally compact Hausdorff space and V is a metric space then the *compact-open topology* on $C^0(U, V)$ is induced by the uniform metric on $C^0(U, V)$, i.e., the metric given by

$$\text{dist}(f, g) = \sup\{\text{dist}(f(u), g(u)) \,|\, u \in U\}.$$

Theorem 3.3.6 (convergence of geodesics)**.** *Let $p, q \in M$, and let γ_n be a minimizing geodesic connecting $\Phi_n(p)$ to $\Phi_n(q)$ on M_n. Each accumulation point of $\{\gamma_n\}$ in the compact-open topology on $C^0(\mathbb{R}, \mathbb{R}^3)$ is a minimizing geodesic on M. The set of such accumulation points is not empty. In particular, there exists a minimizing geodesic γ on M and a sub-sequence of minimizing geodesics $\{\gamma_{n_i}\}$ on M_{n_i} such that $\gamma_{n_i} \to \gamma$ uniformly.*

Proof. We consider all objects to be defined on the smooth reference surface M by using the pull-backs via Φ_n. In particular, we will (by abuse of notation) refer to γ_n as the minimizing g_n-geodesic between p and q on M. Let A_n denote the metric distortion tensor corresponding to g_n, and let

$$\underline{c}_n := \|A_n^{-1}\|_\infty^{-1/2} \quad \text{and} \quad \overline{c}_n := \|A_n\|_\infty^{1/2}.$$

If β is a Lipschitz curve on M, then the g_n-length $l_n(\beta)$ and the g-length $l(\beta)$ are related by

$$\underline{c}_n \cdot l(\beta) \leq l_n(\beta) \leq \overline{c}_n \cdot l(\beta).$$

The geodesic distance between the points p and q on M equals the infimum over the lengths of all Lipschitz curves connecting these points. The last inequality therefore implies

$$\underline{c}_n \cdot d(p, q) \leq d_n(p, q) \leq \overline{c}_n \cdot d(p, q).$$

Hence, if γ_n is a minimizing geodesic connecting p and q in the g_n-metric, then

$$\underline{c}_n \cdot d(p,q) \leq d_n(p,q) = l_n(\gamma_n) \leq \overline{c}_n \cdot l(\gamma_n),$$
$$\overline{c}_n \cdot d(p,q) \geq d_n(p,q) = l_n(\gamma_n) \geq \underline{c}_n \cdot l(\gamma_n).$$

This implies

$$\frac{\underline{c}_n}{\overline{c}_n} \cdot d(p,q) \leq l(\gamma_n) \leq \frac{\overline{c}_n}{\underline{c}_n} d(p,q).$$

By the assumption of totally normal convergence we have $\underline{c}_n \to 1$ and $\overline{c}_n \to 1$, so that

$$l(\gamma_n) \to d(p,q). \tag{3.51}$$

Now, assume γ is an accumulation point of $\{\gamma_n\}$. Since the length functional $l : C^0(\mathbb{R}, \mathbb{R}^3) \to \mathbb{R}$ is lower semi-continuous, (3.51) implies

$$l(\gamma) \leq \liminf l(\gamma_n) = d(p,q).$$

Hence γ is indeed a minimizing geodesic connecting p to q. It remains to show that the set of such accumulation points is not empty. Note that

$$d(\gamma_n(t), \gamma_n(t')) \leq \frac{1}{\underline{c}_n} \cdot d_n(\gamma_n(t), \gamma_n(t')) = \frac{1}{\underline{c}_n} \cdot |t - t'|,$$

for each t, t' in the domain of γ_n. Hence the family $\{\gamma_n\}$ is equicontinuous. Since $|t - t'|$ is bounded by $\sup_n \operatorname{diam}(M_n) \leq \sup_n \overline{c}_n \cdot \operatorname{diam}(M)$, it follows from the Arzelà-Ascoli theorem that there is an accumulation point in the compact-open topology on $C^0(\mathbb{R}, \mathbb{R}^3)$. QED

3.4 CONVERGENCE OF ALGEBRAIC PROPERTIES

In Section 2.5 we have established a de Rham theory built on discrete differential operators and a corresponding Hodge theory. Our approach was based on mixing conforming and nonconforming finite elements. We derived a discrete Hodge star operator acting on the $2\mathfrak{g}$-dimensional space of harmonic vector fields, and have obtained a corresponding splitting of harmonic fields into holomorphic and anti-holomorphic ones. Here we will show that totally normal convergence provides convergence of all of these objects to their smooth counterparts.

For proving convergence, we draw upon the work by Dodziuk and Patodi from the 1970's, who established a purely discrete theory based on *Whitney* *forms*. Recently Wilson has picked up some loose ends of their work, and has extended it by a discrete Hodge star and a discrete wedge product.

We first review Whitney forms and the convergence results by Dodziuk, Patodi, and Wilson. Then we extend these results to perturbed Riemannian metrics. This will finally allow for proving convergence of Hodge splitting, star operator, and spectral decomposition of the Laplace–Beltrami operator in the FE sense.

3.4.1 Whitney forms I: Overview

Let (M, g) be smooth closed and oriented Riemannian manifold. In his book *Geometric Integration Theory* [81], Whitney constructed a chain map from simplicial cochains to Lipschitz differential forms which is the right-inverse of the de Rham map. To construct this map, Whitney considers finite smooth triangulations of M (e.g. a mesh pulled back to M under the shortest distance map). Any fixed such triangulation gives rise to the spaces of q-chains $C_q(M)$ and q-cochains $C^q(M)$, equipped with the usual boundary, and coboundary operators

$$\partial_q : C_q \to C_{q-1} \quad \text{and} \quad \delta^q : C^q \to C^{q+1}.$$

Whitney constructs a linear chain map from the space of cochains $C^q(M)$ to the space of L^2-forms on M:

$$W : C^q(M) \to L^2\Lambda^q(M).$$

This map is constructed by identifying chains and cochains of the finite complex. Each cochain $c \in C^q$ can be represented as a sum $c = \sum_\tau c_\tau \cdot \tau$ with $c_\tau \in \mathbb{R}$ and τ running over all q-simplices $\tau = [p_0, p_1, ..., p_q]$ (with respect to some fixed ordering of the vertices). For every vertex p let μ_p be the p^{th} barycentric coordinate in the triangulation. The *Whitney map* is defined as

$$W\tau = q! \sum_{i=0}^{q} (-1)^i \mu_{p_i} \, \mathrm{d}\mu_{p_0} \wedge \cdots \wedge \widehat{\mathrm{d}\mu_{p_i}} \wedge \cdots \wedge \mathrm{d}\mu_{p_q}.$$

This map is a chain map, i.e., $\mathrm{d}W = W\delta$, where d denotes the Cartan outer differential. There exists a chain map in the opposite direction from the Whitney map,

$$R : L^2\Lambda^q(M) \to C^q(M),$$

given by integrating a L^2-form ω over chains, i.e.

$$R(\omega)(\tau) = \int_\tau \omega.$$

This map is called the *de Rham map*. It is a chain map by Stokes' theorem. W is the right inverse of the de Rham map R:

$$RW = Id.$$

Conversely, Dodziuk and Patodi show that WR is a good *approximation of the identity* provided that the triangulation behaves well. Let h denote the mesh size of the triangulation (the length of the longest edge) of the Euclidean cone surface M_h. Following Whitney we call

$$\theta = \inf_\tau \frac{area(\tau)}{h^2}$$

the *fullness* of M_h. Fullness is related to *shape regularity* of the triangulation.

Lemma 3.4.1 (shape-regular implies bounded fullness)**.** *A shape-regular sequence $\{M_n\}$ of polyhedral surfaces is of bounded fullness.*

Proof. We work over a single triangle T_h with side lengths $a \geq b \geq c$, and corresponding opposite angles α, β, γ. Recall that shape regularity means that there exists a constant $\kappa < \infty$, independent of T_h, such that

$$\frac{R(T_h)}{r(T_h)} \leq \kappa,$$

where $R(T_h)$ is the radius of the circumcircle, and $r(T_h)$ is the radius of the incircle of T_h. The radius of the incircle is given by

$$r(T_h) = \frac{2\,area(T_h)}{a+b+c}.$$

The radius of the circumcircle is given by

$$R(T_h) = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}.$$

Consequently,

$$0 < \frac{1}{\kappa} \leq \frac{r(T_h)}{R(T_h)} = \frac{4\sin\alpha \cdot area(T_h)}{a(a+b+c)} \leq \frac{4\,area(T_h)}{a(a+b+c)} \leq \frac{4\,area(T_h)}{a^2},$$

which completes the proof. QED

Throughout we are going to assume that the fullness of the meshes M_n is bounded below. Dodziuk and Patodi show:

Theorem 3.4.1 (Whitney-approximation)**.** *There exist a constant C and a positive integer k which only depend on the fullness of the triangulation (but not the triangulation itself) such that*

$$\|\omega - WR(\omega)\|_{L^2} \leq Ch\|(Id + \Delta)^k \omega\|_{L^2}$$

for all smooth q-forms ω on M.

To derive a discrete Hodge splitting on (C^q, δ), Dodziuk and Patodi define an inner product on cochains using the Whitney map and the smooth Riemannian metric on M:

$$(\tau, \sigma)_{C^q} = \int_M g(W\tau, W\sigma)\, \mathrm{d}vol \quad \forall\, \tau, \sigma \in C^q.$$

The adjoint δ^* of δ with respect to this inner product is defined as

$$(\tau, (\delta^q)^* \sigma)_{C^q} = (\delta^q \tau, \sigma)_{C^{q+1}}.$$

Due to the finite dimension of C^q, the operators δ and δ^* give rise to a Hodge decomposition of C^q (cf. Lemma 2.5.2)

$$C^q = \operatorname{im} \delta^{q-1} \oplus \operatorname{im}(\delta^q)^* \oplus \ker \delta^q \cap \ker(\delta^{q-1})^*,$$

which is an orthogonal decomposition with respect to $(\cdot, \cdot)_{C^q}$. Dodziuk and Patodi show that this discrete Hodge decomposition is a good approximation of the smooth Hodge decomposition. In what follows, we are going to omit the superscript q unless this causes confusion.

Theorem 3.4.2 (Whitney-approximation of Hodge decomposition)**.** *Let $\omega \in \Lambda^q$. Then $R\omega \in C^q$. Consider the Hodge decompositions*

$$\begin{aligned} \omega &= \mathrm{d}f + \mathrm{d}^* g + \mathfrak{h} \quad \textit{and} \\ R\omega &= \delta \tilde{f} + \delta^* \tilde{g} + \tilde{\mathfrak{h}}. \end{aligned}$$

Then there exist a constant C and a positive integer k which only depend on the fullness of the triangulation (but not the triangulation itself) such that

$$\begin{aligned} \|\mathrm{d}f - W\delta\tilde{f}\|_{L^2} &\leq Ch\|(Id + \Delta)^k \omega\|_{L^2} \\ \|\mathrm{d}^* g - W\delta^*\tilde{g}\|_{L^2} &\leq Ch\|(Id + \Delta)^k \omega\|_{L^2} \\ \|\mathfrak{h} - W\tilde{\mathfrak{h}}\|_{L^2} &\leq Ch\|(Id + \Delta)^k \omega\|_{L^2}, \end{aligned}$$

where h denotes the mesh size of the triangulation.

Finally, Dodziuk and Patodi use the above approximation theorems to show that the eigenvalues $\{\tilde{\lambda}_i\}$ of the *combinatorial Laplace–Beltrami operator*

$$\tilde{\Delta} = \delta\delta^* + \delta^*\delta$$

converge to the eigenvalues $\{\lambda_i\}$ of the smooth Laplace–Beltrami operator

$$\Delta = \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}.$$

The same result had been obtained by two of the pioneers of the Finite Element method at the same time – see Strang and Fix [76]. The proof relies on the Rayleigh-Ritz method for characterizing eigenvalues (*Min-Max Principle*).

Theorem 3.4.3 (spectral approximation of Laplace–Beltrami)**.** *There exist constants C_1 and C_2 and a positive integer k which only depend on the fullness of the triangulation (but not the triangulation itself) such that*

$$\tilde{\lambda}_i \leq \lambda_i\left(1 + C_1 h(1+\lambda_i)^k\right) \quad \text{if} \quad h(1+\lambda_i)^k \leq C_2$$
$$\lambda_i \leq \tilde{\lambda}_i\left(1 + C_1 h|\log h|(1+(\tilde{\lambda}_i)^{1/2})\right) \quad \text{if} \quad h|\log h|(1+(\tilde{\lambda}_i)^{1/2}) \leq C_2.$$

In particular, for fixed i, $\tilde{\lambda}_i \to \lambda_i$ with the mesh size h tending to 0.

Recently Wilson [82] has extended the theory of Whitney forms to combinatorial wedge product and a combinatorial Hodge star, both of which are shown to converge to their smooth counterparts. The *combinatorial wedge product* $\cup : C^q \otimes C^l \to C^{q+l}$ is defined through

$$\tau \cup \sigma = R(W\tau \wedge W\sigma).$$

This product is graded-commutative ($\tau \cup \sigma = (-1)^{\deg\sigma\deg\tau}\sigma\cup\tau$), and agrees with the usual Alexander-Whitney cup product on cohomology (but not on the cochain level). However, $\cup$ is not associative. Note that δ and $\cup$ satisfy a Leibniz rule ($\delta(\tau\cup\sigma) = \delta\tau\cup\sigma + (-1)^{\deg\tau}\tau\cup\delta\sigma$). The following theorem is due to Wilson.

Theorem 3.4.4 (Whitney-approximation of wedge product)**.** *There exist a constant C and a positive integer k which only depend on the fullness of the triangulation (but not the triangulation itself) such that*

$$\|W(R\omega_1 \cup R\omega_2) - \omega_1\wedge\omega_2\|_{L^2} \leq Ch\lambda(\omega_1,\omega_2),$$

where

$$\lambda(\omega_1,\omega_2) = \|\omega_1\|_\infty \cdot \|(Id+\Delta)^k\omega_2\|_{L^2} + \|\omega_2\|_\infty \cdot \|(Id+\Delta)^k\omega_1\|_{L^2}$$

for all smooth ω_1, ω_2.

From this combinatorial wedge product, Wilson obtains a combinatorial Hodge star. Let $\sigma \in C^{\dim(M)-q}$ and $\tau \in C^q$, then

$$(\bigstar\sigma, \tau)_{C^q} = (\sigma \cup \tau)[M],$$

where $[M]$ is the fundamental class (orientation class) of M. $\bigstar$ is a chain map ($\bigstar\delta = \pm\delta^*\bigstar$), it is (graded) skew-adjoint ($(\bigstar\sigma, \tau) = \pm(\sigma, \bigstar\tau)$), and it induces an isomorphism on cohomology ($\bigstar : H^q \cong H^{\dim(M)-q}$). However, in general $\bigstar^2 \neq \pm Id$ and also $\delta\bigstar \neq \pm\bigstar\delta^*$, which is in contrast to the smooth case.

Theorem 3.4.5 (Whitney-approximation of Hodge star)**.** *Given $\omega \in \Lambda^q$, there exist a constant C and a positive integer k which only depends on the fullness of the triangulation (but not the triangulation itself) such that*

$$\| \star_M \omega - W\bigstar R\omega\|_{L^2} \leq Ch\|(Id + \Delta)^k\omega\|_{L^2}$$

where $\star_M$ is the smooth Hodge star on (M, g) and h denotes the mesh size of the triangulation.

3.4.2 Whitney forms II: Metric perturbation

The results of the previous section show how Whitney forms can be used to impose metric structures on combinatorial data of Riemannian manifolds – in such a way that the smooth structure arises as a limit case. Throughout the previous section we assumed that the Riemannian metric stays *fixed*.

However, a fixed metric does not suffice for our purposes: a sequence of polyhedra converging totally normally to a smooth surface, (M, g), induces *two structures* on M – a sequence of triangulations *and* a sequence of Riemannian metrics. While the previous section covers the case of refining triangulations, it does not cover the case of metric changes. Hence it remains to treat *metric perturbation* in conjunction with Whitney elements.

As before, we let A denote the metric distortion tensor. In order to obtain a *perturbed* discrete Hodge splitting on the space of cochains, (C^q, δ), we first define an inner product on cochains using the Whitney map and the perturbed Riemannian metric:

$$(\tau, \sigma)_A = (W\tau, W\sigma)_{L^2_A} = \int_M g(AW\tau, W\sigma)(\det A)^{1/2}\, \mathrm{d}vol \quad \forall\, \tau, \sigma \in C^q.$$

Recall that the simplicial coboundary operator, δ, is metric independent. Its adjoint operator, δ^*_A, is metric dependent and defined by

$$(\delta\tau, \sigma)_A = (\tau, \delta^*_A\sigma)_A.$$

In the sequel, we shall use

- $(\cdot,\cdot)_{L^2}$ to denote the *unperturbed* inner product on $L^2\Lambda^q$,
- $(\cdot,\cdot)_{L^2_A}$ to denote the *perturbed* inner product on $L^2\Lambda^q$,
- $(\cdot,\cdot)$ to denote the *unperturbed* inner product on C^q,
- $(\cdot,\cdot)_A$ to denote the *perturbed* inner product on C^q.

A straightforward calculation shows the following useful lemma:

Lemma 3.4.2. *Let σ and τ be two q-cochains. Then*

$$\frac{|(\tau,\sigma)_A - (\tau,\sigma)|}{(\tau,\tau)^{1/2}(\sigma,\sigma)^{1/2}} \leq C(A)\cdot|M|,$$

where $C(A) = \|A - Id\|_\infty \|(\det A)^{1/2}\|_\infty + \|1-(\det A)^{1/2}\|_\infty$, and $|M|$ denotes the total area of M.

In the sequel, we use the subscript n simultaneously for both, an index into the *metric independent* sequence of simplicial structures on M, as well as for *metric dependent* operators, such as δ_n^*, associated with the metric distortion tensors, A_n. Throughout we have in mind the picture of polyhedral meshes, M_n, converging to the smooth limit surface, M. Notice that neither the de Rham map R nor the Whitney map W are metric dependent.

Theorem 3.4.6 (convergence of perturbed Hodge decomposition)**.** *Consider the Hodge decomposition of $\omega \in \Lambda^q$ with respect to the unperturbed smooth inner product $(\cdot,\cdot)$ and of $R_n\omega \in C_n^q$ with respect to the perturbed inner products $(\cdot,\cdot)_{A_n}$:*

$$\begin{aligned} \omega &= \mathrm{d}f + \mathrm{d}^*g + \mathfrak{h} \quad \textit{and} \\ R_n\omega &= \delta_n f_n + \delta_n^* g_n + \mathfrak{h}_n. \end{aligned}$$

Assume metric convergence, i.e., $\|A_n - Id\|_\infty \to 0$, and that the mesh size, h, approaches zero. Then

$$\begin{aligned} \|\,\mathrm{d}f - W_n\delta_n f_n\|_{L^2} &\to 0 \\ \|\,\mathrm{d}^*g - W_n\delta_n^* g_n\|_{L^2} &\to 0 \\ \|\mathfrak{h} - W_n\mathfrak{h}_n\|_{L^2} &\to 0, \end{aligned}$$

where $\|\cdot\|_{L^2}$ is the L^2-product with respect to (the smooth metric) g.

Proof. Consider the *unperturbed* Whitney Hodge decomposition of $R_n\omega$ with respect to the smooth metric $(\cdot,\cdot)$ as in Theorem 3.4.2, that is

$$R_n\omega = \delta_n\tilde{f}_n + \tilde{\delta}_n^*\tilde{g}_n + \tilde{\mathfrak{h}}_n.$$

Note carefully that δ does not depend on the metric (but just on the combinatorics), whereas δ^* does, so that indeed $\tilde{\delta}_n^* \neq \delta_n^*$. By Theorem 3.4.2, it suffices to show that

$$\begin{aligned}\|W_n\delta_n\tilde{f}_n - W_n\delta_n f_n\|_{L^2} &\to 0\\ \|W_n\tilde{\delta}_n^*\tilde{g}_n - W_n\delta_n^* g_n\|_{L^2} &\to 0\\ \|W_n\tilde{\mathfrak{h}}_n - W_n\mathfrak{h}_n\|_{L^2} &\to 0.\end{aligned}$$

We start by showing that $\|W_n\delta_n\tilde{f}_n - W_n\delta_n f_n\|_{L^2} \to 0$. Note that

- $\delta_n\tilde{f}_n$ is the projection of $R_n\omega$ to $\operatorname{im}\delta_n$ with respect to $(\cdot,\cdot)$.
- $\delta_n f_n$ is the projection of $R_n\omega$ to $\operatorname{im}\delta_n$ with respect to $(\cdot,\cdot)_{A_n}$.

For better readability, we drop the subscript n in the next calculation.

$$\begin{aligned}\|W\delta\tilde{f} - W\delta f\|_{L^2}^2 &= \Big(W\delta\tilde{f} - W\delta f, W\delta\tilde{f} - W\delta f\Big)_{L^2}\\ &= \Big(\delta\tilde{f} - \delta f, \delta\tilde{f} - \delta f\Big)\\ &= \Big(\delta\tilde{f} - \delta f, (\delta\tilde{f} - R\omega) + (R\omega - \delta f)\Big)\\ &= \Big(\delta\tilde{f} - \delta f, R\omega - \delta f\Big)\\ &\quad (\text{since } (\delta\tilde{f} - R\omega) \perp \operatorname{im}\delta).\end{aligned}$$

Note that $(\delta f - R\omega) \perp_A (\operatorname{im}\delta)$, so that $(\delta\tilde{f} - \delta f, R\omega - \delta f)_A = 0$. Together with Lemma 3.4.2 this implies

$$\begin{aligned}\|W\delta\tilde{f} - W\delta f\|_{L^2}^2 &= \Big(\delta\tilde{f} - \delta f, R\omega - \delta f\Big) - \Big(\delta\tilde{f} - \delta f, R\omega - \delta f\Big)_A\\ &\le C(A)\cdot|M|\cdot\|W\delta\tilde{f} - W\delta f\|_{L^2}\cdot\|WR\omega - W\delta f\|_{L^2}.\end{aligned}$$

Dividing by $\|W\delta\tilde{f} - W\delta f\|_{L^2}$ and taking into account that $C(A)\to 0$ by assumption, it suffices to show that $\|WR\omega - W\delta f\|_{L^2}$ stays bounded: $\|WR\omega\|_{L^2}$ stays bounded by Theorem 3.4.1, and δf is the $(\cdot,\cdot)_A$-*projection* of $R\omega$. Applying Lemma 3.4.2 again shows that $\|W\delta f\|_{L^2}$ stays bounded as well.

We have established that the $(\cdot,\cdot)_{A_n}$-projections of $R_n\omega$ to $(\operatorname{im}\delta_n)$ and the $(\cdot,\cdot)$-projections of $R_n\omega$ to $(\operatorname{im}\delta_n)$ converge to each other. The exact same proof delivers that the projections of $R_n\omega$ to $(\ker\delta_n)$ converge to each other. Consequently, since harmonic forms are the orthogonal complement of $(\operatorname{im}\delta_n)$ in $(\ker\delta_n)$, it follows that

$$\|W_n\tilde{\mathfrak{h}}_n - W_n\mathfrak{h}_n\|_{L^2} \to 0.$$

Since W_n and R_n are metric-independent, it follows for the remaining terms in the Hodge decompositions that

$$\|W_n\tilde{\delta}_n^*\tilde{g}_n - W_n\delta_n^* g_n\|_{L^2} \to 0,$$

which completes the proof. QED

The *perturbed combinatorial Laplace–Beltrami operator* is obtained by

$$\tilde{\Delta}_A = \delta\delta_A^* + \delta_A^*\delta.$$

The next theorem shows that the eigenvalues of this perturbed combinatorial Laplace–Beltrami operator converge to the eigenvalues of the smooth Laplace–Beltrami operator.

As noted by Dodziuk and Patodi, half the spectrum of $\tilde{\Delta}_A$ (resp. Δ), acting on q-forms, is *redundant*: if $E^q(\lambda)$ is the eigenspace of Δ corresponding to an eigenvalue $\lambda \neq 0$ then $E^q(\lambda) = E^q_{\mathrm{d}}(\lambda) \oplus E^q_{\mathrm{d}^*}(\lambda)$, where $E^q_{\mathrm{d}}(\lambda)$ is the λ-eigenspace corresponding to $\mathrm{d}^*\mathrm{d}$ and similarly $E^q_{\mathrm{d}^*}(\lambda)$ is the λ-eigenspace corresponding to $\mathrm{d}\,\mathrm{d}^*$. Moreover, d takes $E^q_{\mathrm{d}}(\lambda)$ *isomorphically* to $E^{q+1}_{\mathrm{d}^*}(\lambda)$ (in particular, on surfaces, the spectrum of the Laplace–Beltrami on 1-forms does not contain any more information than the spectrum of the Laplace–Beltrami on functions). Hence, all spectral information of the full Laplace–Beltrami $\tilde{\Delta}_A$ (resp. Δ) is already encoded in the *half* Laplace–Beltrami $\delta_A^*\delta$ (resp. $\mathrm{d}^*\mathrm{d}$). To study convergence of eigenvalues, it therefore suffices to consider the eigenvalues of $\delta_A^*\delta$ and $\mathrm{d}^*\mathrm{d}$.

The non-zero eigenvalues of the smooth operator $\mathrm{d}^*\mathrm{d}$ acting on Λ^q are enumerated by $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Similarly, the eigenvalues of the perturbed combinatorial operator $\delta_n^*\delta_n$ acting on C^q_n (for henceforth fixed q) are enumerated by $0 < \lambda_1^n \leq \lambda_2^n \leq \cdots$.

Theorem 3.4.7 (spectral conv. of perturbed Laplace–Beltrami)**.** *Let $\{\lambda_i^n\}$ denote the eigenvalues of the perturbed combinatorial operators $\delta_n^*\delta_n$ corresponding to the perturbed Riemannian metrics $g(A_n\cdot,\cdot)$, and let $\{\lambda_i\}$ denote the eigenvalues of the smooth operator* $\mathrm{d}^*\mathrm{d}$. *Then, for fixed i,*

$$\lambda_i^n \to \lambda_i$$

as $h \to 0$ and $\|A_n - Id\|_\infty \to 0$. As a consequence, the spectrum of the perturbed combinatorial Laplacians, $\Delta_n = \delta_n^* \delta_n + \delta_n \delta_n^*$, converges to the spectrum of the smooth Laplace–Beltrami operator, $\Delta = \mathrm{d}^* \mathrm{d} + \mathrm{d}\,\mathrm{d}^*$.

Proof. We use the classical Rayleigh-Ritz method (Min-Max principle) for characterizing eigenvalues. The i^{th} non-zero eigenvalue of $\delta_n^* \delta_n$ is given by

$$\begin{aligned}\lambda_i^n &= \min_{\substack{V_i \subset C_n^q \\ \dim V_i = i}} \max_{0 \neq f \in V_i} \frac{(\delta_n^* \delta_n f, f)_{A_n}}{(f, f)_{A_n}} \\ &= \min_{\substack{V_i \subset C_n^q \\ \dim V_i = i}} \max_{0 \neq f \in V_i} \frac{(\delta_n f, \delta_n f)_{A_n}}{(f, f)_{A_n}}.\end{aligned}$$

As in Theorem 3.4.3, let $\tilde{\lambda}_i^n$ be the i^{th} eigenvalue of the *unperturbed* combinatorial (half) Laplace–Beltrami $\tilde{\delta}_n^* \delta_n$. It suffices to show that

$$|\lambda_i^n - \tilde{\lambda}_i^n| \to 0.$$

Fix i and n. Let V_i denote the space spanned by the eigenvectors corresponding to the first i non-zero eigenvalues of $\tilde{\delta}_n^* \delta_n$ acting on C_n^q. The eigenvalue $\tilde{\lambda}_i^n$ is then given by

$$\tilde{\lambda}_i^n = \sup_{f \in V_i} \frac{(\delta_n f, \delta_n f)}{(f, f)}.$$

Moreover, let λ_i^n denote the i^{th} non-zero eigenvalue of the perturbed operator $\delta_n^* \delta_n$. Then, by the Min-Max principle and Lemma 3.4.2 (assuming $C(A_n)|M| < 1$),

$$\begin{aligned}\lambda_i^n &\leq \sup_{f \in V_i} \frac{(\delta_n f, \delta_n f)_{A_n}}{(f, f)_{A_n}} \\ &\leq \sup_{f \in V_i} \frac{(\delta_n f, \delta_n f)}{(f, f)} \cdot \frac{1 + C(A_n) \cdot |M|}{1 - C(A_n) \cdot |M|} \\ &= \tilde{\lambda}_i^n \cdot \frac{1 + C(A_n) \cdot |M|}{1 - C(A_n) \cdot |M|}.\end{aligned}$$

A similar argument shows that

$$\tilde{\lambda}_i^n \leq \lambda_i^n \cdot \frac{1 + C(A_n) \cdot |M|}{1 - C(A_n) \cdot |M|}.$$

By assumption, $C(A_n) \to 0$, and hence, for fixed i, $|\lambda_i^n - \tilde{\lambda}_i^n| \to 0$. QED

Recall that neither R nor W are metric-dependent, so the combinatorial wedge product $\cup$ is not metric-dependent either. The *perturbed combinatorial Hodge star* is defined by

$$(\bigstar_A \sigma, \tau)_A = (\sigma \cup \tau)[M],$$

where $[M]$ is the fundamental class (orientation class) of M. $\bigstar_A$ is a chain map, it is (graded) skew-adjoint, and it induces an isomorphism on cohomology.

Theorem 3.4.8 (convergence of perturbed Hodge star)**.** *Let $\star_M$ denote the smooth start operator on (M, g), and let $\omega \in \Lambda^q$. Then*

$$\| \star_M \omega - W_n \bigstar_{A_n} R_n \omega \|_{L^2} \to 0,$$

as the mesh size h tends to 0 and the metric distortion approaches the identity, i.e., $\|A_n - Id\|_\infty \to 0$.

Proof. By Theorem 3.4.5, it suffices to show that the perturbed combinatorial Hodge star operators, $\{\bigstar_{A_n}\}$, converge to the unperturbed combinatorial Hodge star operators, $\{\tilde{\bigstar}_n\}$. First, note that the Whitney map commutes with the Hodge star up to a projection:

$$W \bigstar_A = P_A \star_M W,$$

where P_A is the orthogonal projection in Λ^q to the image of C^q under W with respect to $(\cdot, \cdot)_{L^2_A}$ (cf. Wilson [82]). Indeed, let $\sigma \in C^q$ and $\tau \in C^{\dim(M)-q}$, then

$$(W \bigstar_A \sigma, W\tau)_{L^2_A} = (\bigstar_A \sigma, \tau)_A = \int_M W\sigma \wedge W\tau \, \mathrm{d}vol_A = (\star_M W\sigma, W\tau)_{L^2_A}.$$

By the same token, $W\tilde{\bigstar} = \tilde{P} \star_M W$ where $\tilde{P}$ is the orthogonal projection in Λ^q to the image of C^q under W with respect to the unperturbed metric $(\cdot, \cdot)_{L^2}$. Since, by assumption, $\|A_n - Id\|_\infty \to 0$, it follows that $P_{A_n} \to \tilde{P}_n$, and hence

$$W_n \bigstar_{A_n} = P_{A_n} \star_M W_n \longrightarrow \tilde{P}_n \star_M W_n = W_n \tilde{\bigstar}_n,$$

which finishes the proof. QED

3.4.3 Convergence of algebraic FE

We are now in the position to prove convergence of Hodge splitting, star operator and eigenvalues of the Laplace–Beltrami in the FE setting. What remains to be done is twofold: (i) to show that, given totally normal convergence, piecewise constant vector fields, associated with a sequence of Euclidean cone surfaces, approximate arbitrary L^2-vector fields on the limit space M, and (ii) to relate the algebraic-topological FE theory of Section 2.5 to the Whitney setting, so that the convergence results of the last section can be exploited. Throughout this section we assume that the Euclidean cone surfaces, M_n, are normal graphs over the smooth surface $M \subset \mathbb{R}^3$.

1. step. Let $X_n \in \mathfrak{X}_h(M_n)$ be a vector field which is constant on the triangles of M_n. By abuse of notation, we let X_n also denote the pullback to $\mathfrak{X}(M)$ via the shortest distance map.

Lemma 3.4.3 (L^2-density of piecewise constant vector fields)**.** *Let the sequence $\{M_n\}$ converge totally normally to M, and let $X \in \mathfrak{X}(M)$ be a smooth vector field. Let Π_n^c denote the projections $\Pi_n^c : L^2\mathfrak{X}(M) \to \mathfrak{X}_h(M_n)$ to piecewise constant vector fields with respect to the L^2-product $(\cdot,\cdot)_{L^2(M)}$ on M. Then*

$$\|X - \Pi_n^c(X)\|_{L^2(M)} \to 0$$

in the L^2-norm of M.

Proof. The assertion is a consequence of a L^∞-estimate on individual triangles. First, let $T \subset \mathbb{R}^2$ be a triangle equipped with the standard flat metric. Let $X \in \mathfrak{X}(T)$ be a smooth vector field on T. Define $X_h := X(p_0)$ for an arbitrary point $p_0 \in T$. Then

$$\|X - X_h\|_{L^\infty(T)} \le h \cdot \|\,\mathrm{d}X\|_{L^\infty(T)},$$

where h denotes the diameter of T, $\mathrm{d}X(p)$ is the Jacobian matrix of X (thought of as a map from $T \subset \mathbb{R}^2$ to $\mathbb{R}^2$). Extending this construction to all triangles of M_n, we obtain a vector field $X_h \in L^2(M_n)$. Applying Hölder's inequality, it follows that

$$\|X - X_h\|_{L^2(M_n)} \le \sqrt{vol(M_n)} \cdot h \cdot \|\,\mathrm{d}X\|_{L^\infty(M_n)} \longrightarrow 0.$$

Certainly,

$$\|X - \Pi_n^c(X)\|_{L^2(M_n)} \le \|X - X_h\|_{L^2(M_n)}.$$

Moreover, by assumption, M_n converges totally normally to M, which implies that $\|X - \Pi_n^c(X)\|_{L^2(M_n)}$ approaches $\|X - \Pi_n^c(X)\|_{L^2(M)}$ (Lemma 3.2.1). Consequently,

$$\|X - \Pi_n^c(X)\|_{L^2(M)} \longrightarrow 0,$$

which proves the assertion. QED

Remark 3.4.1. Note that the proof of the previous lemma shows that piecewise constant vector fields yield $\mathcal{O}(h)$-convergence on triangles. Interestingly, approximations by *Whitney forms do not give better local convergence rates.* In fact, by [25],

$$\|X^\flat - WRX^\flat\|_{L^\infty(T)} \leq C \cdot h \cdot \|\,\mathrm{d}X\|_{L^\infty(T)},$$

where $X^\flat$ denotes the dual 1-form corresponding to X.

The last lemma completes the first step toward showing convergence of Hodge decomposition, spectrum of the Laplace–Beltrami, and star operator in the FE setting.

2. step. In a second step we shall establish how our algebraic-topological FE theory relates to the Whitney setting. Let the Euclidean cone surface M_h be equipped with the usual (simplicial) co-differentials

$$\delta^q : C^q \to C^{q+1}$$

and their adjoints (induced by the cone metric and the Whitney map),

$$(\delta^q)^* : C^{q+1} \to C^q.$$

The cone metric provides an identification between 1-forms and vector fields in a L^2-sense:

$$\sharp : L^2\Lambda^1(M_h) \to L^2\mathfrak{X}(M_h) \quad \text{such that} \quad \int_{M_h} g_A(\alpha^\sharp, Y) = \int_{M_h} \alpha(Y),$$
$$\flat : L^2\mathfrak{X}(M_h) \to L^2\Lambda^1(M_h) \quad \text{such that} \quad \int_{M_h} X^\flat(Y) = \int_{M_h} g_A(X, Y).$$

We have the following first set of relations:

Lemma 3.4.4 (FE-Whitney relations I). *Let the Whitney map be denoted by $W : C^q(M_h) \to L^2\Lambda^q(M_h)$. Then*

$$\begin{aligned}
\sigma \in \operatorname{im}\delta^0 \quad &\Longleftrightarrow \quad (W\sigma)^\sharp \in \operatorname{im}\nabla_{|S_h},\\
\sigma \in \ker\delta^1 \quad &\Longleftrightarrow \quad (W\sigma)^\sharp \in \ker \mathrm{curl}_\mathrm{h}^*,\\
\sigma \in \ker\delta^1 \cap \ker(\delta^0)^* \quad &\Longleftrightarrow \quad (W\sigma)^\sharp \in \ker \mathrm{curl}_\mathrm{h}^* \cap \ker \mathrm{div}_\mathrm{h}.
\end{aligned}$$

Proof. Let $\sigma = \delta^0 f$ for some $f \in C^0$. By the definition of Whitney forms we can write

$$f = \sum f_i \cdot \mu_i,$$

where $f_i \in \mathbb{R}$ and μ_i is the barycentric coordinate corresponding to the i^{th} vertex. Since W is a chain map, $W\sigma = W\delta f = \mathrm{d}Wf$. Therefore, $W\sigma = \sum f_i \,\mathrm{d}\mu_i$, so that

$$(W\sigma)^\sharp = \sum f_i \nabla \mu_i = \nabla(Wf).$$

This shows $(W\sigma)^\sharp \in \operatorname{im} \nabla_{|S_h}$. Vice versa, let $(W\sigma)^\sharp \in \operatorname{im} \nabla_{|S_h}$ for some $\sigma \in C^1$. By applying the $\flat$ operator, we get $W\sigma \in \operatorname{im} \mathrm{d}_{|S_h}$. Hence there exists $f \in C^0$ such that $W\sigma = \mathrm{d}(Wf) = W\delta^0 f$. Since W is injective (because $RW = Id$), it follows that $\sigma = \delta^0 f$. This shows the first statement.

For the second statement, notice that a simplicial 1-form, $\sigma \in C^1$, is closed if and only if it is locally integrable by an element from C^0. Similarly, a piecewise constant vector field is in $\ker \mathrm{curl}_\mathrm{h}^*$ if and only if it is locally integrable by a function from S_h. The previous argument can now be applied (locally) to get the stated relation between $\ker \delta^1$ and $\ker \mathrm{curl}_\mathrm{h}^*$.

The last statement follows from the previous two because harmonic forms (resp. harmonic vector fields) are the orthogonal complement of $\operatorname{im} \delta^0$ in $\ker \delta^1$ (resp. $\operatorname{im} \nabla_{|S_h}$ in $\ker \mathrm{curl}^*$). Indeed, let $\sigma \in \ker \delta^1 \subset C^1$. Then

$$\begin{aligned}(\sigma, \delta^0 f)_{C^1} &= (W\sigma,\, \mathrm{d}Wf)_{L^2(M_h)} \\ &= \left((W\sigma)^\sharp, (\,\mathrm{d}Wf)^\sharp\right)_{L^2(M_h)} \\ &= \left((W\sigma)^\sharp, \nabla(Wf)\right)_{L^2(M_h)},\end{aligned}$$

which vanishes for every $f \in C^0$ if and only if σ is a harmonic 1-form (resp. $(W\sigma)^\sharp$ is a harmonic vector field). QED

There is a second set of FE-Whitney relations concerning the respective Hodge decompositions. Recall that in Section 2.5.3 we derived that

$$\mathfrak{X}_h(M_h) = \operatorname{im} \nabla_{|S_h} \oplus \operatorname{im} \mathbf{J}\nabla_{|S_h^*} \oplus \ker \mathrm{curl}_\mathrm{h}^* \cap \ker \mathrm{div}_\mathrm{h}\,.$$

The following lemma relates this splitting to the Whitney Hodge decomposition.

Lemma 3.4.5 (FE-Whitney relations II)**.** *Let the Whitney map be denoted by* $W : C^q(M_h) \to L^2\Lambda^q(M_h)$*, and let* Π^c *denote the projection of* $\mathfrak{X}(M)$ *to*

$\mathfrak{X}_h(M_h)$ with respect to the L^2-product on $L^2\mathfrak{X}(M)$. Let $\sigma \in C^1(M_h)$, and consider the Whitney Hodge decomposition

$$\sigma = \delta^0 f + (\delta^1)^* g + \mathfrak{h}.$$

Then the FE Hodge decomposition of $\Pi^c(W\sigma)^\sharp \in L^2\mathfrak{X}_h(M_h)$ is given by

$$\Pi^c(W\sigma)^\sharp = \Pi^c(W\delta^0 f)^\sharp + \Pi^c(W(\delta^1)^* g)^\sharp + \Pi^c(W\mathfrak{h})^\sharp.$$

In other words, the FE Hodge splitting is obtained from the Whitney Hodge splitting by projecting each component separately to the space of piecewise constant vector fields.

Proof. We have to show that

$$\begin{aligned} \Pi^c(W\delta^0 f)^\sharp &\in \operatorname{im} \nabla_{|S_h} \\ \Pi^c(W(\delta^1)^* g)^\sharp &\in \operatorname{im} \mathbf{J}\nabla_{|S_h^*} \\ \Pi^c(W\mathfrak{h})^\sharp &\in \ker \operatorname{curl}^* \cap \ker \operatorname{div}. \end{aligned}$$

The first and last line follow directly from Lemma 3.4.4. It remains to show that $\Pi^c(W(\delta^1)^* g)^\sharp \in \operatorname{im} \mathbf{J}\nabla_{|S_h^*}$. By Lemma 3.4.4,

$$(W\cdot)^\sharp : \operatorname{im}\delta^0 \oplus (\ker\delta^1 \cap \ker(\delta^0)^*) \longrightarrow \operatorname{im}\nabla_{|S_h} \oplus (\ker \operatorname{curl}_\mathrm{h}^* \cap \ker \operatorname{div}_\mathrm{h})$$

is an isomorphism. Now let $k \in \operatorname{im}\delta^0 \oplus (\ker\delta^1 \cap \ker(\delta^0)^*)$. Then

$$\begin{aligned} \left(\Pi^c(W(\delta^1)^* g)^\sharp, (Wk)^\sharp\right)_{L^2(M_h)} &= \left((W(\delta^1)^* g)^\sharp, \Pi^c(Wk)^\sharp\right)_{L^2(M_h)} \\ &= \left((W(\delta^1)^* g)^\sharp, (Wk)^\sharp\right)_{L^2(M_h)} \\ &= \left((\delta^1)^* g, k\right)_{C^1} \\ &= \left(g, \delta^1 k\right)_{C^2} \\ &= 0. \end{aligned}$$

This implies that $\Pi^c(W\delta^* g)^\sharp \perp (\operatorname{im}\nabla_{|S_h} \oplus (\ker \operatorname{curl}_\mathrm{h}^* \cap \ker \operatorname{div}_\mathrm{h}))$, and hence $\Pi^c(W\delta^* g)^\sharp \in \operatorname{im}\mathbf{J}\nabla_{|S_h^*}$. QED

We can now prove convergence of the FE Hodge splittings. For notation, let $\mathbf{J}_M$ denote complex multiplication acting on vector fields on the smooth surface (M, g), and let $\mathbf{J}_n$ denote complex multiplication induced by the cone metrics on the Euclidean cone surfaces (M_n, g_n). We show convergence of the Hodge splitting

$$\mathfrak{X}_h(M_n) = \operatorname{im}\nabla_{n|S_h} \oplus \operatorname{im}\mathbf{J}_n\nabla_{n|S_h^*} \oplus \ker \operatorname{curl}^*{}_n \cap \ker \operatorname{div}_n,$$

for sequences of Euclidean cone surfaces, $M_n \to M$.

Theorem 3.4.9 (FE-convergence of Hodge decomposition). *Assume $M_n \to M$ converges totally normally with mesh size h tending to zero. Let $X \in \mathfrak{X}(M)$ be a smooth vector field. Then the components of the smooth Hodge decomposition*

$$X = \nabla u + \mathbf{J}_M \nabla v + X_{\mathfrak{h}},$$

are approximated by the components of the FE Hodge decomposition,

$$\Pi_n^c X = \nabla_n u_n + \mathbf{J}_n \nabla_n v_n + X_{\mathfrak{h},n},$$

of the piecewise constant vector fields $\Pi_n^c(X) \in \mathfrak{X}_h(M_n)$. Indeed,

$$\begin{aligned} \|\nabla u - \nabla_n u_n\|_{L^2} &\to 0 \\ \| \mathbf{J}_M \nabla v - \mathbf{J}_n \nabla_n v_n\|_{L^2} &\to 0 \\ \|X_{\mathfrak{h}} - X_{\mathfrak{h},n}\|_{L^2} &\to 0 \end{aligned}$$

in the L^2-norm of M.

Proof. Consider the Hodge decompositions of $R_n X^\flat \in C_n^1$ as in Theorem 3.4.6:

$$R_n X^\flat = \delta_n f_n + \delta_n^* g_n + \mathfrak{h}_n.$$

By assumption, $M_n \to M$ converges totally normally, so that the corresponding metric distortion tensors tend to the identity. Theorem 3.4.6 asserts that the Whitney extensions of the components of the above combinatorial Hodge splitting of $R_n X^\flat$ converge to the components of the smooth Hodge splitting of $X^\flat$ on M. Hence, by Lemma 3.4.5, the components of the FE Hodge decomposition of the piecewise constant vector fields

$$X_n = \Pi_n^c \left(W_n R_n X^\flat\right)^\sharp.$$

converge to the components of the smooth Hodge decomposition of X on M. In particular, X_n approaches X. Furthermore, by Lemma 3.4.3, $\Pi_n^c X$ also approaches X. This implies that X_n and $\Pi_n^c X$ tend to one another, and so must their FE Hodge decomposition because each component in this decomposition is obtained by a projection. QED

As a corollary we obtain that the two FE Hodge splittings in Theorem 2.5.2 converge to each other (and to the smooth Hodge splitting on the limit surface).

Corollary 3.4.1. *Assume $M_n \to M$ converges totally normally. Let $X \in \mathfrak{X}(M)$ be a smooth vector field on M, and let $\Pi_n^c(X) \in \mathfrak{X}_h(M_n)$ be the projections to piecewise constant vector fields. Then the components of $\Pi_n^c(X)$ corresponding the two Hodge splittings*

$$\begin{aligned}\mathfrak{X}_h(M_n) &= \operatorname{im} \nabla_{n|_{S_h}} \oplus \operatorname{im} \mathbf{J}_n \nabla_{n|_{S_h^*}} \oplus \ker \operatorname{curl}^*{}_n \cap \ker \operatorname{div}_n \\ &= \operatorname{im} \nabla_{n|_{S_h^*}} \oplus \operatorname{im} \mathbf{J}_n \nabla_{n|_{S_h}} \oplus \ker \operatorname{div}^*{}_n \cap \ker \operatorname{curl}_n\end{aligned}$$

converge to each other (and to the components of the smooth Hodge splitting of X).

Proof. By Theorem 3.4.9, it suffices to show that the components of $\Pi_n^c X$ according to the bottom row Hodge decomposition converge to the components of the smooth Hodge decomposition of X. This is a simple consequence of the fact that the bottom row is the $\mathbf{J}_n$-transformed version of the top row and the fact that metric convergence implies convergence of complex multiplication, i.e., $\mathbf{J}_n \to \mathbf{J}_M$. QED

The preceding theorem can be interpreted as *J-invariance of the FE Hodge decomposition in the limit*. In particular, since $\mathbf{J}_n$ exchanges the two (conforming and nonconforming) harmonic parts of the above splittings,

$$\mathbf{J}_n : \ker \operatorname{curl}^*{}_n \cap \ker \operatorname{div}_n \longrightarrow \ker \operatorname{div}^*{}_n \cap \ker \operatorname{curl}_n,$$

it follows that these two spaces must agree in the limit. This implies that the discrete star operator (see Definition 2.5.3) converges to the smooth Hodge star:

Theorem 3.4.10 (convergence of FE Hodge star)**.** *Assume $M_n \to M$ converges totally normally. Let $\mathfrak{h} \in \ker \operatorname{curl}_M \cap \ker \operatorname{div}_M$ be a smooth harmonic vector field, and let $\mathfrak{h}_n$ be the discrete harmonic part of its L^2-projection, $\Pi_n^c(\mathfrak{h})$, to the space of piecewise constant fields. Let $\star_M = \mathbf{J}_M$ denote the smooth star operator on M, and let $\star_n$ denote the FE Hodge star operators acting on the space of conforming harmonic fields. Then*

$$\| \star_M \mathfrak{h} - \star_n \mathfrak{h}_n \|_{L^2} \longrightarrow 0,$$

in the L^2 metric of M.

As a last point, we show convergence of the spectrum of the FE Laplace–Beltrami operators. In order to be able to speak about eigenvalues of the discrete Laplace–Beltrami operator, this operator needs to be a map from

$S_h(M_h)$ to itself. In other words, we need the *discretized, pointwise* version of the Laplacian in the sense of Definition 2.3.5:

$$\Delta_h = -\operatorname{div_h} \nabla : \; S_h \longrightarrow S_h.$$

The minus sign amounts to requiring the Laplacian to be *positive* semi-definite (as opposed to treating *negative* semi-definite Laplacians as in Section 2.2.5). Computationally, this yields

$$\Delta_h = \mathcal{M}^{-1}\mathfrak{L},$$

with conforming mass matrix $\mathcal{M}$ (cf. Section 2.3.4) and conforming stiffness matrix $\mathfrak{L}$ (cf. Section 2.4.4). Equivalently, the eigenvalue problem can be written in weak form: find all pairs $(\lambda, u_h) \in \mathbb{R} \times S_h$ such that

$$\int_{M_h} g_{M_h}(\nabla u_h, \nabla v_h)\, \mathrm{d}vol_{M_h} = \lambda \int_{M_h} u_h v_h\, \mathrm{d}vol_{M_h} \quad \text{for all } v_h \in S_h.$$

For sequences of polyhedral surfaces $\{M_n\}$, we denote by Δ_h^n the discretized Laplace–Beltrami operators acting on $S_h(M_n)$.

Theorem 3.4.11 (FE-conv. of spectrum of Laplace–Beltrami)**.** *Assume $M_n \to M$ converges totally normally with mesh size h tending to zero. Then for fixed i, the i^{th} non-zero eigenvalues λ_i^n of Δ_h^n converge to the i^{th} non-zero eigenvalues of Δ,*

$$|\lambda_i^n - \lambda_i| \longrightarrow 0,$$

where Δ is the smooth Laplace–Beltrami operator viewed as an unbounded map $\Delta : L^2(M) \to L^2(M)$.

Proof. The Whitney map, W, takes 0-forms to elements of S_h. Since it is a chain map, it follows for $u, v \in C^0$ that

$$\begin{aligned}
(\delta^*\delta u, v)_{C^0} &= (\delta u, \delta v)_{C^1} \\
&= (W(\delta u), W(\delta v))_{L^2} \\
&= (\,\mathrm{d}(Wu),\, \mathrm{d}(Wv))_{L^2} \\
&= (\nabla(Wu), \nabla(Wv))_{L^2} \\
&= (\Delta_h(Wu), Wv)_{L^2}\,.
\end{aligned}$$

Applying Theorem 3.4.7 completes the proof. QED

3.5 Convergence rates for inscribed meshes

A Euclidean cone surface is called *inscribed* or *interpolating* if all of its vertices reside on the smooth limit surface M. Recall that the constants in our approximation estimates have been explicitly stated in terms of the metric distortion tensor A. Indeed, there are three constants, which occur throughout this work:

- $\|A - Id\|_{L^\infty}$: metric distortion,
- $\|(\det A)^{1/2} - 1\|_{L^\infty}$: area distortion,
- $\|(\det A)^{1/2} A^{-1} - Id\|_{L^\infty}$: conformal distortion.

By Theorem 3.2.1, the metric distortion tensor A is the product of two parts, both of which can be expressed entirely in terms of h for inscribed meshes: the first part involves the shortest distance map, and the second part involves the (squared) inner product between the surface normals. The following lemma gives a qualitative estimate for the angle between the surface normals in the case of inscribed meshes, compare Nédélec [59], Amenta et al. [4], and Morvan and Thibert [58].

Lemma 3.5.1 (normal lemma)**.** *Let the polyhedral surface M_h be inscribed into the smooth surface M, and assume that M_h is within the reach of M. Then the angles between the normals N_h (of M_h) and N (of M), compared under the shortest distance map, satisfy*

$$\angle(N, N_h) \leq C \cdot h,$$

where h denotes the mesh size of M and C only depends on the curvature of M and the aspect ratios of the triangles of M_h.

As a consequence of the normal lemma, $\|\phi\|_{L^\infty} \sim \mathcal{O}(h^2)$, where ϕ is the (signed) distance between M_h and M as in equation (3.2). The splitting of A as in Theorem 3.2.1 then shows that:

- $\|A - Id\|_{L^\infty} \sim \mathcal{O}(h^2)$,
- $\|(\det A)^{1/2} - 1\|_{L^\infty} \sim \mathcal{O}(h^2)$,
- $\|(\det A)^{1/2} A^{-1} - Id\|_{L^\infty} \sim \mathcal{O}(h^2)$.

In summary, most of the estimates in this work are *quadratic in mesh size* when it comes to interpolating meshes. For a discussion of *best possible* constants (and in particular, best constants involved in the approximation error of the Dirichlet problem), see Shewchuk [72]. A more thorough discussion of the best possible constants for all of the estimates in this work would require a work of its own.