

## CHAPTER 1

# INTRODUCTION

*Discrete Differential Geometry* (DDG) on *polyhedral surfaces* is concerned with discrete counterparts to the various structural properties of smooth surfaces revealed by classical and modern differential geometry. Despite the youth of the field, a *systematic treatment* of DDG has recently been rapidly maturing – owed to a combined effort of pure and applied mathematicians as well as computer scientists. From the point of view of DDG, the demarcation line between *pure* and *applied* turns blurry – due to a unique combination of rich mathematical structure and direct applicability to geometric algorithms. In a similar vein, this thesis explores the mathematical structure behind some of those loose ends of DDG which have been developed and indeed been applied in many fascinating ways over the past years.

This thesis deals with topological surfaces made up of (finitely) many Euclidean triangles isometrically glued along their boundaries. Interchangeably we call those surfaces *Euclidean cone surfaces*, *triangulated meshes*, *polyhedral surfaces*, or simply *polyhedra*. We consider the following objects:

- *Metric objects*, such as the Laplace–Beltrami operator, gradient, divergence, curl, the mean curvature vector, and geodesics.
- *Algebraic objects*, such as de Rham cohomology, Hodge decomposition, the Hodge star operator, and spectral decomposition of the Laplacian.

We give a *precise meaning* to these objects on polyhedra, and prove *convergence* to their corresponding smooth counterparts.

As a point of departure, we develop the theory of the *Sobolev spaces*  $W^{1,p}$  on polyhedra in Section 2.2. In particular, we outline the theory of weak derivatives, prove a Poincaré lemma, discuss the Dirichlet problem and give an outlook on regularity theory. Although much of the material in Section 2.2 could be deduced from the more general framework of a Sobolev theory for bi-Lipschitz maps (cf. [18, 83, 86]), we hold the view that such a general treatment would obscure the subtle, yet crucial, peculiarities of Euclidean cone structures. Perhaps the most prominent of those peculiarities is the

failure of the *shift theorem* in the presence of cone singularities. Indeed, let  $u$  be the variational solution to the model problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

on some bounded domain  $\Omega \subset \mathbb{R}^2$ , with *smooth* boundary  $\partial\Omega$ . The shift theorem asserts that if  $f \in W^{k,p}$ , with  $1 < p < \infty$ , and  $g$  is a given smooth function, then  $u \in W^{k+2,p}$ . However, in general, this is no longer true for cones: if a cone (of angle smaller than  $2\pi$ ) is cut open and laid out into the plane, the resulting region is *non-convex* – and the shift theorem is in general false for non-convex domains with non-smooth boundary, see [41].

**Discrete differential operators**, and the spaces these operators act on, are introduced in Sections 2.3 and 2.4. In particular, we give a precise meaning to discrete versions of the Laplace–Beltrami operator, divergence, curl, and the mean curvature vector. These operators had been previously employed with much success for both, a discretization of minimal surfaces<sup>1</sup>, as well as for applications in computer graphics<sup>2</sup>. However, in some instances these operators were introduced in a rather ad hoc fashion. The theory of Sobolev spaces on polyhedra allows for a precise treatment. In particular, these operators act between the Sobolev spaces

$$L^2 \text{ or } H^1 \text{ (functions)} \longrightarrow H^{-1} \text{ (functionals)}.$$

We stress the point that the range does no longer consist of *functions* but rather of *functionals*, or *distributions*. The strict discrimination between functions and functionals is important when dealing with convergence.

Akin to the discretization of second order PDEs in the planar case, we discretize Sobolev spaces by piecewise linear functions which are either *conforming* (the degree of freedom being at vertices) or *nonconforming* (the degree of freedom being at edge midpoints),

$$S_h = \{\text{conforming elements}\} \subset \{\text{nonconforming elements}\} = S_h^*.$$

Considering both,  $S_h$  and  $S_h^*$ , turns out to be particularly useful for treating algebraic-topological properties.

<sup>1</sup>For applications to discrete minimal surfaces see e.g. [38, 43, 44, 51, 61, 63, 64, 66, 69].

<sup>2</sup>For applications to computer graphics see e.g. [23, 40, 45, 47, 48, 56, 65, 78].

**Algebraic topology** of polyhedra is developed in Section 2.5. Using discrete differential operators, we derive a discrete de Rham complex whose *cohomology* is isomorphic to singular cohomology. Furthermore, as a consequence of *mixing* conforming and nonconforming elements, we obtain *two distinct versions of a Hodge decomposition* of the space  $\mathfrak{X}_h$  of *piecewise constant* vector fields. For example, the *conforming Hodge decomposition* takes the form

$$\mathfrak{X}_h = \text{im } \nabla|_{S_h} \oplus \text{im } \mathbf{J}\nabla|_{S_h^*} \oplus \ker \text{curl}_h^* \cap \ker \text{div}_h,$$

where  $*$  denotes the nonconforming version of the involved operators,  $\mathbf{J}$  denotes *complex multiplication* and the last summand,  $\ker \text{curl}_h^* \cap \ker \text{div}_h$ , consists of *harmonic vector fields*. On the space of harmonic vector fields we construct a discrete Hodge star operator, an isomorphism

$$\star : \ker \text{curl}_h^* \cap \ker \text{div}_h \xrightarrow{\sim} \mathcal{O},$$

which is shown to converge to the classical Hodge star operator on a smooth limit surface in Section 3.4. The discrete Hodge star operator is obtained by composing complex multiplication with a certain  $L^2$ -projection, and we carefully distinguish between complex multiplication and the Hodge star operator.

The observation that one obtains two distinct Hodge decompositions in the discrete case – a *doubling* which is absent in the smooth setting – is in close analogy to the work of Mercat [54, 55], who considers *two distinct grids* (a primal and a dual one) and obtains Riemann period matrices of *double the dimension* in his work on discrete conformal structures. Our view is also closely linked to that of Desbrun et al. [22] and Glickenstein [36, 37]. Drawing upon the work of Dodziuk and Patodi [24, 25] as well as Wilson [82], we show that the two discrete Hodge decompositions converge to a single one in the smooth limit in Section 3.4.

**Convergence** is treated in Chapter 3. The central result of this chapter may be formulated as follows: *discrete differential operators converge in norm*. Convergence in norm may stand, as in the case of mean curvature vectors, for convergence of *functionals* (distributions) rather than functions. This view is motivated by the observation that the mean curvature vector of an isometrically embedded polyhedral surface has *only distributional components*. Indeed, the mean curvature vector can be written as  $\vec{H} = \Delta \vec{E}$  – the intrinsic surface Laplacian applied to the *piecewise linear* embedding of the polyhedron – and since  $\Delta \vec{E}$  vanishes in the interior of Euclidean triangles, it follows that mean curvature is (distributionally) located at edges.

Let us briefly recall the meaning of convergence in the sense of distributions. Assume we were dealing with  $C^2$  surfaces. Then the mean curvature vectors would be continuous, and their convergence,  $\vec{H}_n \rightarrow \vec{H}$  in  $H^{-1}(M)$ , would imply that

$$\int_M \vec{H}_n \cdot \phi \longrightarrow \int_M \vec{H} \cdot \phi \quad \forall \phi \in C_0^1(M).$$

In other words, one obtains what could be called *convergence of integrated quantities*. This interpretation carries over to Euclidean cone surfaces (which are only of class  $C^{0,1}$ ) in the sense that

$$\langle \vec{H}_n | \phi \rangle \longrightarrow \langle \vec{H} | \phi \rangle \quad \forall \phi \in H_0^1(M),$$

where  $\langle \cdot | \cdot \rangle$  denotes the dual pairing between  $H_0^1$  and  $H^{-1}$ . To show that convergence in the sense of distributions is the best one can generally hope for, we provide a counterexample to  $L^2$ -convergence of mean curvature vectors. This is in accordance with what has been observed in geometric measure theory: in general one cannot expect pointwise convergence of discrete curvatures, but rather convergence in an integrated sense, compare Cheeger et al. [19] and Cohen-Steiner and Morvan [21]. See also the comparative studies of Meek and Walton [53], Borrelli et al. [13], Xu [84, 85], and references therein.

As a first step for establishing convergence we introduce the *shortest distance map* – a bi-Lipschitz map between a smooth surface in  $\mathbb{R}^3$  and a polyhedral surface nearby (in its *reach*). The purpose of this map is to pull back objects from polyhedral surfaces to the smooth reference surface. This technique has been common practice, see e.g. [27, 58, 57]. In particular, for a sequence  $\{M_n\}$  of polyhedra converging to a smooth surface  $M$ , the pulled-back polyhedral metrics  $g_n$  can be expressed in terms of the smooth Riemannian metric  $g$  by

$$g_n(\cdot, \cdot) = g(A_n \cdot, \cdot),$$

where  $A_n$  is a symmetric positive definite  $2 \times 2$  matrix field, called the *metric distortion tensor*. We define *metric convergence* as

$$\|A_n - Id\|_\infty \longrightarrow 0.$$

The main auxiliary result is to split  $A_n$  into a product of two parts: the first one depending only on the pointwise distance between  $M_n$  and  $M$  as well as the shape operator of  $M$ , and the second one depending only on the angle between the normals of  $M_n$  and  $M$  (cf. Theorem 3.2.1). If both of these

parts converge, we speak about *totally normal convergence*. Provided that surfaces converge to each other in Hausdorff distance, Theorem 3.2.2 yields *equivalence of the following conditions*:

- convergence of surface normals
- convergence of volume forms (area)
- convergence of Riemannian metrics
- convergence of Laplace–Beltrami operators.

This equivalence implies that the famous *lantern of Schwarz* constitutes a very general example of what might go wrong – pointwise convergence of surfaces without convergence of their normals. The quantitative relation between normal convergence and convergence of area has recently also been investigated by Morvan and Thibert [58].

The above equivalent conditions for convergence are obtained from corresponding equivalent properties of the metric distortion tensor:

$$\begin{aligned} \|A_n - Id\|_\infty \rightarrow 0 &\iff \|(\det A_n)^{1/2} - 1\|_\infty \rightarrow 0 \iff \\ \|(\det A_n)^{1/2} A_n^{-1} - Id\|_\infty \rightarrow 0 &\iff \|\operatorname{tr}((\det A_n)^{1/2} A_n^{-1} - Id)\|_\infty \rightarrow 0, \end{aligned}$$

which hold as long as the surfaces converge in Hausdorff distance. As a consequence of these equivalent conditions, we obtain *equivalence* of the Sobolev spaces  $W_n^{k,p}$ , with  $k \in \{0, 1\}$ , associated with the sequence. This fact allows for proving (in a straightforward yet somewhat technically involved way) *convergence of metric properties*: Laplace–Beltrami operators, solutions to the Dirichlet problem, mean curvature vectors, and geodesics, see Section 3.3. A slight complication occurs for showing convergence of solutions to the Dirichlet problem in a finite element sense, because a (local)  $H^2$ -estimate is needed. Such an estimate is obtained by extending an argument of Dziuk [27] from *interpolating* meshes to *approximating* meshes. One particularly interesting consequence of totally normal convergence is the fact that a smooth limit surface of a sequence of *discrete minimal surfaces* is a *smooth minimal surface* in the classical sense, see Section 3.3.4.

In Section 3.4, we prove *convergence of algebraic properties* – Hodge decomposition, Hodge star operator, and spectral decomposition of the Laplacian. We obtain convergence of these objects by establishing their link to *Whitney forms* [81]. This relation allows for applying convergence results for Whitney forms obtained by Dodziuk and Patodi in the 1970’s (cf. [24, 25]) and recently by Wilson [82].

## RELATED WORK

Polyhedral surfaces are source to a rich pool of fascinating structures. They bear many more aspects than the (linear) setting explored in this thesis. We list a few related milestone developments – a list which is clearly far from complete.

The rigorous treatment of discrete differential structures has a long and exciting history: Alexandrov [1] and Reshetnyak [68] developed the theory of manifolds of bounded curvature. Thurston [77] and Schramm [70] used circle packings to approximate holomorphic maps and proved a discrete Riemann mapping theorem. Federer [32] and Fu [34] used geometric measure theory, Banchoff [7] studied discrete Morse theory, Stone [75] related global topology of PL-manifolds to their local geometry, Brehm and Kühnel [17] treated approximations of polyhedra by smooth surfaces, Cheeger, Müller and Schrader [19] employed Lipschitz–Killing curvatures, Morvan and Cohen-Steiner [21, 57] studied the normal cycle, and Bobenko, Hoffmann, Mercat, Pinkall, Springborn, and Suris studied discrete integrable systems and discrete conformal structures, see e.g. [8, 9, 11, 12, 55].