Infraparticle quantum fields and the formation of photon clouds

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ABSTRACT: A non-perturbative and exactly solvable quantum field theoretical model for a "dressed Dirac field" is presented, that exhibits all the kinematical features of QED: an appropriate delocalization of the charged field as a prerequisite for the global Gauss Law, superselected photon clouds (asymptotic expectation values of the Maxwell field), infraparticle nature of charged particles that cannot be separated from their photon clouds, broken Lorentz symmetry. The model serves as an intermediate leg on a new roadmap towards full QED, formulated as an off-shell theory, i.e., including a perturbative construction of its interacting charged fields. It also fills a gap in recent discussions of the "Infrared Triangle", and points the way towards a new scattering theory for theories with massless particles of helicity ≥ 1, in which infraparticles can be described at the level of charged fields, rather than just states.

Contents

1   Introduction

1.1   Outline and results

This paper has a two-fold purpose. On the one hand, we shall outline a new roadmap towards the perturbative off-shell construction of QED, including its charged fields, that proceeds in a way that Hilbert space positivity is preserved at every stage. On the other hand, we present and discuss in detail a major intermediate step in this program, which can be performed in an exact non-perturbative way. This "intermediate model" does not yet describe any interaction. It is rather a genuine off-shell construction implementing a highly nontrivial change of the structure of charged free fields. It thus manifestly captures salient infrared features of QED, which are not visible (or not even addressed) in the standard approach. The model is of its own interest because it is the first of its kind in four spacetime dimensions and may serve as an exactly solvable model for developing infraparticle scattering theory.

In section 1.3, we briefly recall the infrared features of QED to be addressed by our exact model. (A more thorough discussion of the conceptual background of these issues can be found in [58, section 1] which is an extended version of the present paper.[1]) Physically, those features are of course due to the long range of the electromagnetic interaction, which is in turn due to the vanishing photon mass. They will be implemented in terms of a "dressed Dirac field" ψ_{qc}, the main new object of our interest, that carries photon clouds along with it.[2] It is heuristically given by (1.3), while its actual construction and analysis is found in section 3. Section 3.4.3 makes contact with the "Infrared Triangle" (section 1.3.3). By computing the lightlike asymptotics of expectation values of the electromagnetic field in states created from the vacuum by the dressed Dirac field, we confirm (and shed some new light) on the views of [71].[3] There emerges a unified picture with the spacelike asymptotics, in which the infraparticle nature of charged particles (see section 1.3.2) becomes visible.

Our approach to QED will be outlined and commented in section 2. It involves a perturbation theory of the charged infrafield of the intermediate model that in this context is regarded as a "free infrafield". First steps in this program in section 4 reveal new structural features of this perturbation theory, including systematic cancellations of non-local terms

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[1]The present paper is an abridged and edited version of the arXiv version 2 of [58].

[2]For comments on the terminology "dressing" see [58, section 1.2].

[3]See also references therein.

illustrated in the important example of the local Gauss Law (section 4.1), a new type of propagator called "cloud propagator" producing new types of IR-divergent loop diagrams (section 4.2), and, perhaps most noteworthy, indications of a new infrared interference mechanism: the diagrams with cloud propagators describe an interference between the infrared divergences responsible for the superselection structure of the intermediate model and the well-known infrared divergences of QED (that separately cause velocity superselection and prevent scattering, respectively [73]). The interference produces a dynamical superselection rule: QED deforms the rigid superselection rule of the intermediate model, and the dressing relaxes velocity superselection of QED so as to admit scattering.

We want to convince the reader that a radical conceptual step of using a new type of quantum fields for infraparticles ("infrafields") such as (1.3) is necessary to get to the roots of the (almost a century old) infrared problems of QED.

The new conceptual mindset towards infrared quantum field theory, and the ensuing need of computational techniques outside the routine repertoire (some of which still remain to be developed) justify the length of our paper. We present many computations in more detail than usual, but we strive not to overburden the main text, by deferring large parts into the appendices.

1.2   String-localized quantum fields

Since the earliest days of Quantum Electrodynamics, one has struggled with the redundancy of the description [23, 47, 52, 70] due to "gauge degrees of freedom", and with the lack of Hilbert space positivity in covariant descriptions [8, 72]. In our manifestly positive reformulation of QED, the redundancy of the description is directly related to the abundance of superselected states of a given electric charge, which are commonly visualized as the "shape of a photon cloud" at spacelike infinity, and quantitatively described by the asymptotic electric flux per solid angle.

Jordan [47] and Dirac [23] were among the first to consider formulating QED with a formally gauge-invariant electrically charged field of the form

$$e^{iq \int_\Gamma dy^\mu A_\mu(y)} \psi(x) \tag{1.1}$$

where $\psi(x)$ is the Dirac field and Γ is a curve extending from x to infinity. Instead of the line-integral along a single curve, smeared curves are also admissible. Similar ideas were pursued by Mandelstam [52], and Steinmann [70]. The question arises how such formal expressions can be defined as quantum fields.

The expression (1.1) is also central to our approach. But we are led to it by a very different, dynamical motivation, that we shall expose in detail in section 2. The guiding principle is Hilbert space positivity, rather than gauge invariance which is no longer an a priori postulate. The latter will rather be an emerging feature following from the perturbative preservation of locality of observables, while the localization of charged fields may (and must) be relaxed as suggested by (1.1). See also the discussion in section 5.

For our purposes, it will be sufficient to be slightly more restrictive than (1.1): we choose straight curves ("strings") of the form $\Gamma_{x,e} = x + \mathbb{R}_+ e$ and average over the directions

e (in a way to be specified). We shall write

$$\phi(x, e) := \int_{\Gamma_{x,e}} dy^\mu A_\mu(y) = \int_0^\infty ds\, A_\mu(x + se)e^\mu, \tag{1.2}$$

and call it the "escort field" because it is evidently not an independent field. We shall write $\phi(x, c)$ for its smearing with a real function $c(e)$ of total weight 1. One may think of the smearing as an "average", although $c(e)$ is not required to be positive. Then we can rewrite the heuristic expression (1.1) in the form

$$\psi_{qc}(x) = e^{iq\phi(x,c)}\psi(x) \qquad \text{(to be properly defined! See section 3.1.1)} \tag{1.3}$$

This formal expression is ill-defined because (1.2) is infrared divergent. We shall provide in section 3.1.1 a non-perturbative definition of the dressed Dirac field ψ_{qc} on a Hilbert space. It serves as an exactly solvable intermediate model on the route to QED. It is a "free infrafield" that does not have a nontrivial interaction, but that implements characteristic kinematical features of QED that quantum fields in the usual axiomatic setting, such as Wightman's [74], fall short of. The desired features of the model are in fact *caused by the cure of the infrared divergence.*

The most prominent of these features is the photon cloud superselection structure. It turns the undesirable gauge redundancy into a tool to "manufacture" charged states with different photon clouds. Further consequences will be discussed below.

We outline in section 2, and elaborate in some more detail in section 4, how full QED should be constructed as a perturbation theory of the free infrafield model. The proposed construction is made possible with a new toolkit of quantum field theory that allows to relax the localization properties of Wightman fields. Notice that the escort field is localized along the string $x + \mathbb{R}_+ e$, and the dressed Dirac field inherits its localization extending to infinity. This departure from the standard axiomatics turns out to be not a defect but a strength of our approach, taylored to implement the said infrared features.

QED becomes the first — and most prominent — instance of a new setup for quantum field theory, originating in various recent conceptual insights gained by axiomatic approaches. A concluding section 5 will place QED in a broader context of axiomatic QFT, and discuss some of the new emerging paradigms for quantum field theory.

Our construction has a long history. It grew out of a simple 2D model by one of the present authors [68], but was made possible in 4D only by the recognition of string-localization as a constructive tool. The need for string-localization was also anticipated abstractly in [17] through an algebraic analysis of localization properties of charged states in massive theories.

String-localization in the sense of causal commutativity was introduced in [61] in the wake of recent conceptual insights concerning the nature of quantum fields, that were imported from the more abstract Haag-Kastler approach ("algebraic QFT" or "Local Quantum Physics" [41]). One of the main motivations was that it can improve ultraviolet divergences of perturbative interactions. The latter reflect the strong vacuum polarization caused by the action of massive higher spin fields on the vacuum, manifest in an increase of their UV scaling dimension. This increase is usually avoided by working in the indefinite

metric state space of gauge theory (Krein space). In contrast, vacuum polarizations can also be alleviated in the physical Hilbert space, by allowing "more room in space" with string-localized quantum fields.

Without interaction, string-localization is rather an artificial possibility for all free fields creating particles of finite spin and helicity, with the benefit that it allows the construction of quantum stress-energy tensors for higher helicity where a point-localized SET does not exist [56]. It is intrinsically necessary only for free fields transforming in Wigner's infinite-spin representations [75], where the increase of scaling dimension with the spin is prohibitive for point-localized fields [61, 64].

The taming of vacuum polarization becomes vital when interactions are turned on. By lowering the UV scaling dimension, naively non-renormalizable theories can become renormalizable. E.g., string-localized "massive QED" with a massive vector boson is power-counting renormalizable on a Hilbert space, whereas the coupling to the Proca field would not be renormalizable, while the coupling to a massive gauge field would not be ghost-free. In the BRST formalism, one looses the charged field [24, 67]. See more on this in section 5.

The present paper emphasizes the mechanism how an auxiliary string-localized free field in the interaction density (we prefer the term "dressing density" [58, footnote 3]) passes its string-localization onto the resulting dressed Dirac field in a new "dynamical" way, substantially different from the artificial construction of string-localized Dirac fields in [61]. In full QED as a perturbation of the dressed Dirac field, the interacting Dirac field remains string-localized. This not only resolves the conflict with the global Gauss Law (section 1.3.1) and restores the local Gauss Law (section 4.1); it also explicitly exhibits the electron as an infraparticle, see section 1.3.2.

1.3 Background

1.3.1 Global Gauss Law and photon clouds

By the Gauss Law, the integral over the charge density can be written as a surface integral over the electric flux at spacelike infinity. In quantum field theory, the flux integral commutes with every (anti-)local field. In particular, if the quantum Gauss Law holds in QED, then the charged field cannot be anti-local, because its commutator with the charge operator is nontrivial.

In standard perturbative QED, the conflict appears in a different guise [8, Chap. 10.B]: the interacting Dirac field on the Krein space *is* anti-local. But the Gauss Law does not hold. Instead, already the free Maxwell field satisfies a modified local (differential) Gauss Law

$$\partial^\mu F^K_{\mu\nu} = -\partial_\nu(\partial A^K) = j^{\text{fict}}_\nu \tag{1.4}$$

with the "fictitious current" (for simplicity in the Feynman gauge) $j^\nu_{\text{fict}} = -\partial^\nu(\partial A^K)$. Because the "fictitious charge operator" $\int d^3x \, j^0_{\text{fict}}(0, \vec{x})$ cannot be written as a surface integral, it has a non-vanishing commutator with the Dirac field with QED interaction, and is the generator of the U(1) symmetry. The fictitious current vanishes in the physical Hilbert space of the free fields, but it contributes a term to the expectation value of the charge operator in an electron state that cancels the total charge, cf. table 1 in section 4.1.

When one passes to the Gupta-Bleuler or BRST quotient space, the charged field ceases to exist. Our approach retains the charged field as an infrafield.

The conflict between locality and the long-range nature of the electromagnetic interaction, manifested in the Gauss Law, is tightly related to two other features: the existence of uncountably many superselection sectors in QED, distinguished by the "profile" of the asymptotic electric flux density at spacelike infinity as a function of the direction, that is often visualized as a "photon cloud" accompanying charged matter sources; and the fact that charged particles cannot have a sharp mass (because they have to carry along their photon cloud): instead they are "infraparticles", see section 1.3.2. Because the superselected photon clouds transform nontrivially under boosts, Lorentz invariance is broken in each irreducible sector.

The string-localization of the dressed Dirac field causes a nontrivial asymptotic flux density (section 3.4). Yet, the total charge is zero in the intermediate model without QED interaction. *With* QED interaction, we verify the local Gauss Law in first perturbative order: the dressing cancels the charge density of fictitious provenience in the Krein space approach (section 4.1, table 1). This result highlights the power of the new approach, insisting on Hilbert space positivity without changing the physical content of QED, while doing full justice to its infrared structure.

1.3.2 Infraparticles

The states generated by the dressed Dirac field are reminiscent of the "dressed states" in the Chung and Faddeev-Kulish approaches to scattering theory [22, 34], see [26, 29] for recent rigorous treatments. These states include infinitely many soft photons in what formally looks like a coherent state, but in fact belongs to a superselection sector orthogonal to the photon Fock space. The shape of the photon cloud depends on the momentum of the charged particle, taylored to cancel the vanishing of perturbative QED scattering amplitudes due to infrared divergences [72, 73]. In contrast, in the dressed states of the intermediate model, the shape of the photon cloud can be freely chosen, but becomes coupled to the momentum when the QED interaction is turned on, see section 4.4.

The infrared superselection sectors are not separately Lorentz covariant, because Lorentz transformations connect photon clouds of different shape. The energy-momentum spectrum of charged states in QED is dissolved above the mass-shell with a sharp lower bound (the mass of the free electron), such that the mass hyberboloid has zero weight [41]. There are no eigenstates of the mass operator (as there are in the tensor product of the Dirac and photon Fock spaces), i.e., isolated electrons separated from their photon cloud do not exist. Such states were called "infraparticle states" [13, 14, 68]. The states created from the vacuum by the dressed Dirac field of our model enjoy these features. In particular, the abstract criterium for the infraparticle nature of charged states [13], formulated in terms of asymptotic properties of expectation values of the electromagnetic field at spacelike infinity, is verified by the analysis in section 3.4.1.

Also the lightlike asymptotics of expectation values of the electromagnetic field in charged states can be computed explicitly, section 3.4.3. Our model therefore provides an underpinning to the Infrared Triangle, section 1.3.3, by way of an exactly solvable

genuinely quantum field theoretical model in which states with the abstract properties discussed in [71] (some of which are imported from classical electrodynamics), are created from the vacuum by an infrafield.

The existence in QED of superselection sectors with broken Lorentz invariance, and the non-existence of charged eigenvectors of the mass operator, hence the necessity for infraparticle states, are known for a long time as necessary consequences of the nontriviality of the asymptotic flux density, by abstract arguments [13, 14, 36]. Although the intermediate model does not warrant the global Gauss Law, it provides the first explicit realization of these features in the context of QED. It also allows to give an example for an infraparticle spectrum in section 3.1.1.

There remains a major challenge: scattering theory for infraparticles. See [58, section 1.2] for more conceptual comments, and section 4.3 for a discussion of the difficulties in the present context.

1.3.3 The infrared triangle

The long-established facts of the previous subsections constitute more facets of the infrared world of QED, that should be added to what was recently called the "Infrared Triangle" [71].

The Infrared Triangle provides a unified view of several issues related to massless particles of helicity 1 and 2 (photons in QED, gravitons in General Relativity). The first corner of the triangle concerns soft photons (resp. gravitons). In QED, the photon clouds accompanying charged particles are responsible for the failure of the formal LSZ prescription as if the electron were a Wigner particle with a sharp mass-shell. The characteristic "shape" of these difficulties in momentum space were captured by soft photon theorems [73]. Prevailing prescriptions to deal with this issue: inclusive cross-sections [7, 76] or dressing factors on the S-matrix [22, 29, 34], do not address the fact that the charged field must be an infrafield.

The second corner of the triangle is related to lightlike infinity ($\mathfrak{I}^\pm$ in Penrose terminology). The r^{-2}-decay of the electric field in spacelike directions is accompanied in lightlike directions by a λ^{-1}-decay of its radiative transversal components and a λ^{-2}-decay of its radial component. The radial and transversal components are not independent of each other: they are related by an asymptotic version (3.29) of the Gauss Law at lightlike infinity [6, 71], see also [58, appendix B.3]. Their behaviour at lightlike infinity provides an analogue to the "memory effect" of gravitational waves, by producing an observable effect on test particles "living on $\mathfrak{I}^\pm$". The effect is computed with the help of (3.29) as a "kick" on the test particle when a pulse of radiation passes by [6].

The third corner of the triangle relates to symmetries that go well beyond charge conservation. The radial component of the electric field on $\mathfrak{I}^+$ has a limiting value at $\mathfrak{I}^+_-$, where $\mathfrak{I}^+$ touches on spacelike infinity $\mathfrak{i}^0$. Remarkably, consistency of scattering amplitudes requires that this limiting value must coincide with the limiting value on $\mathfrak{I}^-_+$, where $\mathfrak{I}^-$ touches on $\mathfrak{i}^0$, i.e., the "interpolation" through $\mathfrak{i}^0$ corresponding to spacelike limits with increasing boost parameter connects to identical values. This matching condition was known earlier in a classical setting, see [43, 45]. Because the limiting values are functions of the direction $\vec{n} \in S^2$, this matching constitutes an infinite number of conservation laws.

There is a second set of conservation laws which manifests itself in a change of sign of the radial component of the magnetic field between $\mathfrak{I}^+_-$ and $\mathfrak{I}^-_+$.

The matching conditions between the electric and magnetic fields at $\mathfrak{I}^+_-$ and at $\mathfrak{I}^-_+$ (which are spheres) are taken as the conservation laws associated with a "new symmetry", (see [42, 49], and [71] for a textbook coverage). When smeared with test functions ε on the sphere, the electromagnetic field operators at $\mathfrak{I}^\pm_\mp$ are interpreted as generators Q_ε of "large gauge transformations" whose gauge parameters do not die off at lightlike infinity. Moreover, they are not globally defined due to a twist entailed by the matching condition. They are considered as the electromagnetic counterparts of BMS super-translations and super-rotations in General Relativity. See also [45] for a critical assessment of some of the interpretations put forward in [71].

Our work contributes to this scenario a genuine quantum model in the bulk (section 3.4.3), with charged states created by charged infrafields. The results confirm (and shed some new light) on the views of [71], connecting them (through the spacelike asymptotics) with the infraparticle nature of charged particles. In this quantum setting, the radiation at lightlike infinity originates from the string-localized dressed Dirac field, and neither massless charges [42] nor the dynamical coupling of massive matter to the electromagnetic field as in full QED must be invoked. The matching condition turns out to be an instance of the PCT theorem. The generators Q_ε transform the charged matter field in the bulk by a complex phase. This is possible because of the string-localization of charged field. The Maxwell field in the bulk is invariant (by locality).[4] While the generators Q_ε preserve the superselection sectors, their expectation values depend on the sector.

The model thus integrates the Infrared Triangle into a much bigger "simplex" of infrared aspects, integrating the most fundamental conceptual issues of Einstein Causality and localization, Hilbert space positivity (i.e., the probability interpretation of quantum theory), PCT, and the very notion of a particle.

2 The roadmap towards QED

The roadmap starts towards QED with a chain of equivalences (2.1), to be briefly explained in the sequel. The chain will be continued with the much less trivial furcation (2.15), where our new approach deploys its power. Schematically, we write

$$\text{QED} \stackrel{1.}{=} \{\psi_0, F\}\big|_{L(c)} \stackrel{2.}{=} \{\psi_0, F^K\}\big|_{L^K(c)} \stackrel{3.}{=} \{\psi_0, F^u\}\big|_{L^u(c)} \tag{2.1}$$

The various string-dependent interaction densities appearing in (2.1) will be explained in Items 1–3 below. $\{\Phi_i\}\big|_L$ is understood as the algebra of the fields $\Phi_i\big|_L$, constructed from the free fields Φ_i with the interaction density L.

Relation 1. is a meaningful definition of QED because the resulting S-matrix and observables are string-independent (see Item 2 below). The equivalences 2. and 3. hold

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[4]Although the interacting Maxwell field is string-independent, our construction provides string-dependent charged states in which its expectation value is string-dependent. Our formulation thus differs from the external field approach in [32] where the Maxwell field itself depends on e, cf. also the external-field example in [57].

because all fields involved have identical correlation functions. The latter determine the fields as operators on a GNS-reconstructed Hilbert space [74]. We are thus addressing the "off-shell" construction of the quantum fields themselves, rather than "on-shell" S-matrices.

We use the framework of causal perturbation theory [33], which provides the interacting fields $\Phi\big|_L$ as power series in integrals over retarded multiple commutators of the free fields $\Phi(x)$ with $L(y_i)$, $i = 1, \ldots, n$, see [27]. E.g., the first order perturbation of a field $\Phi(x)$ with an interaction density L is

$$\Phi^{(1)}(x) = i \int d^4y \, R[\Phi(x), L(y)] := i \int d^4y \, \theta(x^0 - y^0)[\Phi(x), L(y)]. \tag{2.2}$$

By the Wick expansion, retarded commutators turn into retarded propagators times Wick products of fields. The retarded propagators are products of commutator functions with Heaviside functions. Such products are a priori not defined as distributions, and "renormalization" is understood as the process of defining them, possibly with some freedom to be fixed by suitable renormalization conditions, see [33].

Causal perturbation theory assumes the presence of a spacetime cutoff for the coupling constant. The removal of this cutoff is called the "adiabatic limit". Before the adiabatic limit is taken, integration by parts will produce boundary terms whose vanishing has to be controlled in the limit. The adiabatic limit is in fact a delicate issue, especially in massless theories, also in local approaches [25]. We shall, however, ignore these difficulties throughout the present paper; for some case studies see [57].

We now explain the meaning and relevance of the equivalences (2.1).

1. String-localized interaction densities. It is well-known that a local potential for the Maxwell field on a Hilbert space does not exist. Let $F_{\mu\nu}(x)$ be the Maxwell field defined on the Wigner Hilbert space (the Fock space over the unitary representations of helicity ± 1 of the Poincaré group [72, 75]). For any $e \in \mathbb{R}^4 \setminus \{0\}$, the field[5]

$$A_\mu(x, e) := I_e(F_{\mu\nu}e^\nu)(x) \equiv \int_0^\infty ds \, F_{\mu\nu}(x + se)e^\nu \tag{2.3}$$

is a potential for the Maxwell field: $\partial \wedge A(x, e) = F(x)$, as a consequence of the Bianchi identity (homogeneous Maxwell equation) satisfied by $F(x)$. It may be thought of as an axial gauge potential, except that the string direction e is not fixed but transforms along with x under Lorentz transformations. $A(x, e)$ is still a potential for $F(x)$, when it is smeared with a function $c(e)$ of total weight 1: $\partial \wedge A(x, c) = \partial \wedge A(x, e) = F(x)$.

By definition, $A(x, e)$ is localized along the string $x + \mathbb{R}_+ e$, in the sense that two such fields commute whenever their strings $x_i + \mathbb{R}_+ e_i$ are spacelike separated. The study of in- and out-going multi-particle states requires sufficient causal separability of the fields that create these states. This requirement excludes fields localized along timelike strings. In this paper, we shall assume $e^2 < 0$. Because $A(x, e)$ is homogeneous in e, we are free to normalize $e^2 = -1$.

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[5] The short-hand notation I_e will be used throughout. It assumes that the integrand has sufficiently fast decay. In that case, one has $(e\partial)(I_e f)(x) = I_e((e\partial)f)(x) = -f(x)$.

In contrast to the standard gauge-theory approach, $A_\mu(c)$ is not a fundamental field: it is a functional of the free Maxwell field on the Wigner Hilbert space. In particular, it creates from the vacuum only physical states with the two transversal modes of helicity ± 1. The fact that $A_\mu(c)$ is a potential allows a reformulation of QED in which the interaction density

$$L(c) = q\, A_\mu(c) j^\mu. \tag{2.4}$$

is defined on the tensor product of the Wigner Hilbert space and the free Dirac Fock space. Formulating QED with $L(c)$ thus avoids from the outset issues with unphysical states and indefinite metric.

The string direction (or its smearing function c) is an auxiliary quantity, on which observables should not depend. Indeed, the variation of $A_\mu(e)$ with respect to e is a gradient $\partial_{e^\kappa} A_\mu(e) = \partial_\mu (I_e A_\kappa(e))$, so that $\partial_{e^\kappa} L(e) = \partial_\mu (I_e A_\kappa(e) j^\mu)$ is a total derivative, and the c-dependent part of the classical action $\int d^4x\, L(x, c)$ vanishes by Stokes' Theorem. The "lift" of this property to the quantum theory, i.e., the string-independence of $T e^{i \int d^4x\, g(x) L(x,c)}$ in the adiabatic limit $g(x) \to 1$, is nontrivial because derivatives do not commute with time-ordering. It must be imposed as a renormalization condition, see (2.7). When it is fulfilled, then the perturbation of observables with $L(c)$ does not depend on c. This approach (not just for QED) was advocated in [69].

2. Embedding into Krein space. The second equivalence in (2.1) embeds the Maxwell field as $F^K = \partial \wedge A^K$, and along with it the string-localized potential as $A^K(e) = I_e(Fe)$, into the Krein (= indefinite metric) Fock space of the local potential $A^K(x)$. It is an equivalence *before* the IR divergences of QED have been taken care of. It holds, simply because the embedded fields F^K and $A^K(e) = I_e(F^K e)$ have the same correlation functions and propagators as F and $A(e) = I_e(Fe)$.

One finds [57] the relation to the original Krein space potential

$$A^K(x, e) := I_e(F^K_{\mu\nu} e^\nu)(x) = A^K(x) + \partial_\mu \phi(x, e), \tag{2.5}$$

where $\phi(x, e)$ is given by (1.2) (with the Feynman gauge potential $A^K(x)$ in the place of A). This is how the escort field enters the stage in our approach.

The escort field makes the string-independence of the total action manifest: when the interaction density (2.4) is embedded into the Krein space, it splits accordingly into two pieces:

$$L^K(c) = q\, A_\mu^K(c) j^\mu = q\, A_\mu^K j^\mu + q\, \partial_\mu \phi(c) j^\mu. \tag{2.6}$$

The latter, string-dependent part is a total derivative $\partial_\mu V^\mu(c)$. This property ensures the string-independence of the S-matrix: before the adiabatic limit $g(x) \to 1$ is taken, the identity

$$T e^{i \int d^4x\, [L^K(x,c) g(x) + V^\mu(x,c) \partial_\mu g(x)]} = T e^{i \int d^4x\, L^K(x) g(x)} \tag{2.7}$$

(and a similar identity for the interacting observables in causal perturbation theory) holds at tree level in all orders. It holds for loop contributions provided appropriate Ward identities can be fulfilled (confirmed in low orders). It follows that in the adiabatic limit,

where $\partial g = 0$, the renormalized interacting observables are string-independent, and the equivalence 1. in (2.1) holds.

It must be stressed here that $\partial_\mu \phi(e)$ in (2.5) is well defined, while $\phi(e)$ is a logarithmically divergent field that requires an IR cutoff, see section 3.1.1. This divergence is responsible for the superselection structure of QED, see section 3.

3. The physical Maxwell field in Krein space. In the Krein space, the free field strength $F^K = \partial \wedge A^K$ is not source-free due to the fictitious current $-\partial^\nu (\partial A^K)$, see (1.4). The null field (∂A^K) has vanishing self-correlations and correlations with F^K. But its non-vanishing commutator with A^K in the interaction density $q A_\mu^K j^\mu$ is responsible for the failure of the global Gauss Law in Krein space in the standard approach.

F^K creates from the vacuum states with transversal and longitudinal modes. To cure this unphysical feature of F^K, we introduce another potential

$$A_\mu^u(x) := A_\mu^K(x) + u_\mu I_u(\partial A^K)(x) \equiv A_\mu^K(x) + u_\mu \int_0^\infty ds\, (\partial A^K)(x + su) \qquad (2.8)$$

and its field $F^u = \partial \wedge A^u$. Here, u is a future timelike unit vector, for definiteness $u = u_0 \equiv (1, \vec{0})$. The first important difference is that in contrast to (1.4), F^u is the (embedded) *physical* free Maxwell field (isomorphic to F on the Wigner Hilbert space). It does not create unphysical longitudinal null states, and satisfies

$$\partial^\mu F_{\mu\nu}^u = 0. \qquad (2.9)$$

In fact (see section 3.3.1), A_μ^u is equivalent to the Coulomb gauge and $F_{\mu\nu}^u$ are the *transversal* Maxwell fields:

$$\vec{E} := -\vec{\nabla} A_0^u - \partial_0 \vec{A}^u = -\partial_0 (\vec{A}^K - \vec{\nabla}\Delta^{-1}(\vec{\nabla}\vec{A}^K)), \quad \vec{B} := \vec{\nabla} \times \vec{A}^u = \vec{\nabla} \times \vec{A}^K. \qquad (2.10)$$

The second important feature is that A^u and consequently also F^u live on the positive-definite subspace (*not* the Gupta-Bleuler quotient space)

$$\mathcal{H}^u = \left\{ \Phi \in \mathcal{K} : (uA^K)^- \Phi = 0 \right\} \qquad (2.11)$$

of the Krein Fock space. $(uA^K)^-$ stands for the annihilation part. $\mathcal{H}^u$ is the subspace generated by the three spacelike components a_i^* of the creation operators of A_μ^K.

$F^u(x)$ appears to be localized along the timelike string $x + \mathbb{R}_+ u$. But because (∂A^K) is a null field, its self-correlations are the same as those of F^K, hence its commutation relations are local. The situation highlights the "relative" algebraic nature of the concept of localization deduced from commutation relations (Einstein Locality): relative to itself, F^u is not "more non-local" than F. That F^u is non-local relative to A^u, A^K, or ϕ, should rather be blamed on the latter un-observable fields which contain unphysical degrees of freedom.

We define the string-localized potential $A^u(e) := I_e(F^u e)$ associated with F^u. Then it holds

$$A_\mu^u(x, e) = A_\mu^u(x) + \partial_\mu \phi(x, e). \qquad (2.12)$$

For $e \in u^\perp$, the escort field in (2.12) is the same as in (2.5), also living on $\mathcal{H}^u$. By (2.12), the arguments as after (2.6) also apply for the string-independence of the perturbation theory with the interaction density

$$L^u(c) = q\, A^u_\mu(c) j^\mu = q\, A^u_\mu j^\mu + q\, \partial_\mu \phi(c) j^\mu. \tag{2.13}$$

The densities $L^K(c)$ and $L^u(c)$ are defined in terms of observables $A^K(e) = I_e(F^K e)$ and $A^u(e) = I_e(F^u e)$, resp., and differ from each other only by a term involving the null field (∂A^K). Consequently, they give rise to the same perturbative expansions for the interacting observables, as claimed in (2.1).[6]

4. The dressing transformation. While the free fields in the last three entries of (2.1) are point-localized, the interaction densities are string-localized relative to the free fields. In causal perturbation theory, this feature a priori can jeopardize locality of the interacting fields. However, by (2.6) and (2.13), the dangerous string-localization of the interaction density resides entirely in the escort term $q\, \partial_\mu \phi(e) j^\mu$.

Here, the crucial observation (and a main topic of the present paper) comes to bear: *Because the escort term is a total derivative, that part of the interaction can be implemented by an exact construction: the "dressing transformation", in which one can establish locality directly without relying on perturbation theory.*

The idea is thus to exploit the split (2.6) or (2.13), write the interacting Dirac field as[7]

$$\psi|_{L + \partial(q\, \phi(c) j^\mu)} = \left(\psi|_{\partial_\mu(q\, \phi(c) j^\mu)} \right)|_L \tag{2.14}$$

for $L = L^K$ resp. $L = L^u$, and construct $\psi|_{\partial(q\, \phi(c) j^\mu)} = \psi_{qc}$ non-perturbatively. This is how the string-localized dressed Dirac field, formally anticipated in (1.3), enters the stage.

The exact construction in section 3.1.1 of ψ_{qc} as an infrafield defined on a GNS Hilbert space with uncountably many superselection sectors, labelled by smearing functions c, parallels that of the two-dimensional model [1, 30, 68] which for the first time introduced infraparticle fields. The intermediate theory of the dressed Dirac field without the QED interaction is regarded as an autonomous model, referred to as the "dressed model", that already captures essential infrared features of QED in a kinematic way. Several of its interesting properties are studied in section 3.

5. Towards QED. After the dressing transformation, it remains to implement the parts

$$L^K = q\, A^K_\mu j^\mu, \quad \text{resp.} \quad L^u = q\, A^u_\mu j^\mu$$

of the interaction densities (2.6) resp. (2.13) so that

$$\{\psi_0, F^K\}\big|_{L^K(c)} \overset{4.(a)}{=} \{\psi_{qc}, F^K\}\big|_{L^K} \quad \text{resp.} \quad \{\psi_0, F^u\}\big|_{L^u(c)} \overset{4.(b)}{=} \{\psi_{qc}, F^u\}\big|_{L^u}. \tag{2.15}$$

This step requires to extend the algebra of the dressed Dirac field to include the potential A^K_μ or the potential A^u_μ, and use (2.14) also for the respective Maxwell field. The extensions

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[6]The fields A^u and A^K, which would be sensitive to the difference, do not appear in (2.1)!

[7]This formula requires a legitimation. See section 3.1.

are performed in section 3.3. The former extension is represented on an indefinite space containing the vacuum representation of A^K. The subalgebra $\{\psi_{qc}, A_\mu^u\}$ is defined on a proper Hilbert subspace.

The left-hand sides of (2.15) are equivalent by (2.1). The right-hand sides are perturbative expansions around the dressed Dirac field (the "free infrafield"). Both right-hand sides of (2.15) stand for expansions of the same final theory: QED. However, they have complementary benefits and drawbacks. The expansion 4.(a) is only defined on the indefinite Krein space extension of the model $\{\psi_{qc}, F^K\}$, while 4.(b) is defined on the proper Hilbert space of the model $\{\psi_{qc}, F^u\}$. Another charme of 4.(b) is that the unphysical longitudinal photon degree of freedom (which is responsible for the IR properties of charged states, cf. the discussions in [16, 57]) appears only in the dressed charged field, and neither in the Maxwell field nor in the interaction density.

Conversely, 4.(a) has the benefit that its interaction density is point-localized, which secures that the interacting dressed Dirac and Maxwell fields remain (string-)localized. This is not manifest with the non-local interaction density of 4.(b) (involving A^u).

The synthesis is that locality can be controlled in 4.(a), and positivity can be controlled in 4.(b). Because both expansions yield the same correlation functions, QED is perturbatively constructed as a local *and* positive QFT, with string-localized charged fields $\psi_{qc}\big|_{L^K}$, whose perturbative expansion is equivalent to that of $\psi_{qc}\big|_{L^u}$ which is defined on a Hilbert space. For an explicit illustration of the equivalence of perturbative expansions invoked here, see section 4.1.

We call this the "hybrid approach" because of the need of the detour through Krein space in order to assess locality, although the final theory lives on a Hilbert space.

We stress once more that the program is a roadmap. Many technical details remain to be filled in. The equivalences (2.15) hold only in the adiabatic limit, and upon performing the adiabatic limit, boundary terms have to be controlled. Furthermore, it must be established that Ward identities for the conserved Dirac current can be satisfied in every order of perturbation theory also in the present setting. Those issues will not be analyzed in any detail in this paper.

Our focus will be instead on the exact dressing transformation and the analysis of the features of the resulting intermediate models $\{\psi_{qc}, F^K\}$ and $\{\psi_{qc}, F^u\}$.

Comments. The use of string-localized fields is essential in our approach to QED. First of all, it resolves the conflict between (assumed) anti-locality of the Dirac field and the global Gauss Law (section 1.3.1): there is no such conflict with a string-localized field for the simple geometric reason that the string extends to infinity. Second, it allows to formulate QED directly on a Hilbert space. The possibility of a manifest Hilbert space positive perturbative expansion of QED stands behind the celebrated "mystery" why the manifestly non-positive Krein space approach in the end of the day produces a unitary S-matrix. Third, it allows to isolate the infrared sector structure as an effect of the logarithmically divergent escort field.

It has been objected [16] that a formulation of QED based on (a functional of) the Maxwell field such as (2.3) rather than an independent potential cannot be capable of describing charged states, due to the absence of longitudinal photon degrees of freedom in

the Maxwell field. This objection is true when QED is formulated exclusively in terms of observable fields. But our construction includes unobservable charged fields. The model in fact describes a "transfer" of longitudinal photon degrees of freedom onto the Dirac field, where they become responsible for the superselected photon clouds of charged states and the loss of the sharp mass-shell of charged particles. These features survive (with modifications) in the full QED. They were long ago anticipated on the basis of axiomatic considerations (see section 1.3.1, section 1.3.2), and have led to a renormalized construction of non-local charged states satisfying the Gauss Law [53, 54].

The string-localization of the dressed charged field dynamically transferred to it by the coupling to the escort field) is of a very different kind than that of the mere string integration I_e in the string-localized potential or the escort field itself: the latter has no effect on the particle states created by the field. In contrast, the dressed charged field creates infraparticle states that survive in the asymptotic time limit.

The dressed model with fields ψ_{qc} with different c is a string-local theory with infraparticle states, that is a valuable testing ground for infraparticle scattering theory, see section 4.3. An interesting feature of this model is that the string-localization is manifested by commutation relations involving a complex phase depending on the string smearing, see (3.11).

The issue of the timelike string integration I_u in L^u and F^u needs a comment. It makes correlations of the Maxwell field F^u with A^u and with $\phi(e)$ nonlocal. But F^u is a perfectly local field relative to itself. The auxiliary potential and escort fields are no longer part of the model once it has been constructed. Only the unobservable dressed Dirac field is non-local relative to F^u. In this way, the dressed model $\{\psi_{qc}, F^u\}$ becomes a testing ground for the Infrared Triangle. In the full QED the non-locality due to the I_u integration is absent thanks to cancellations as witnessed by the equivalence 3. in (2.1), see also section 4.1 and section 4.2.

The hybrid approach (i.e., the detour through the Krein space by embedding the Maxwell field into the Krein space where the interaction can be split as $L(c) = L^K + \partial_\mu V^\mu(c)$) gains some flexibility for the construction. Unlike in the local BRST formalism based on L^K alone, taking all terms together ensures positivity at every step. It should be mentioned that the strategy applies as well for other models with electrically charged fields, like scalar QED, cf. section 5. Unfortunately, a split like (2.6) is not always possible, e.g., in the case of Yang-Mills [38].

In this setting, the string-localized renormalization theory is expected to become more transparent.[8] It keeps all local observables and the S-matrix string-independent; the only place where the strings make themselves felt is in the charged fields and hence in the states in which the observables are evaluated: notably the expectation values of the Maxwell field in spacelike and lightlike asymptotic directions.

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[8]Emphasizing QED as a perturbation of the dressed model not only constitutes a most efficient reorganization of the perturbative expansion. The use of Weyl formulas rather than expansions of the exponential series also provides an inherent control of infrared structure, see section 4.2 and section 4.4.

3 The dressed model

3.1 The QED dressing transformation

It remained unclear for a long time how infraparticle fields can be realized in dynamical models in $3 + 1$ dimensions. The old two-dimensional model [68] exploited the fact that the massless scalar field φ is formally scale-invariant and as a consequence logarithmically divergent in the infrared. An infrafield was defined as a regularized exponential of φ ("vertex operators" in modern parlance, with conformally invariant correlation functions), tensored with a free Fermi field. The main obstacle in $3+1$ dimensions is the non-existence of local fields of scaling dimension 0 to play the role of φ. An interesting model was presented in [15] with a "Maxwell field of helicity zero". By defining the superselection sectors of this model as algebraic automorphisms, the problem of their implementation by charged fields was circumvented.

The new idea is to construct vertex operators from fields of scaling dimension 0 in $3+1$ dimensions, that arise as string integrals over free fields of dimension 1. In the case of QED, the escort field (1.2) is a natural candidate. The split (2.6) of the QED interaction density suggests to consider the "dynamical" intermediate model whose interaction density is *just* the total derivative

$$L_{\mathrm{dress}}(c) = q\,\partial_\mu\phi(c)j^\mu = q\,\partial_\mu(\phi(c)j^\mu),$$

and to consider the field

$$\psi_{qc}(x) := \psi\big|_{q\,\partial_\mu\phi(c)j^\mu}.$$

Warning. When an interaction density is split into two parts, then the equality (as in (2.14))

$$\Phi\big|_{L+\widetilde{L}} = \left(\Phi\big|_{\widetilde{L}}\right)\big|_L \tag{3.1}$$

does *not* hold in general. However, (3.1) is an identity of unrenormalized perturbative expansions in the adiabatic limit, if $\widetilde{L} = \partial_\mu V^\mu$ is a total derivative, *and* the condition

$$R[\partial_\nu V^\nu(x), X(y)] = \partial_\nu R[V^\nu(x), X(y)] \tag{3.2}$$

is satisfied by $X = L$, $X = V^\mu$ and $X = \partial_\mu V^\mu$. This can be seen by extensive use of the Jacobi identity in the retarded-commutator expansion of both sides of (3.1). The condition is satisfied for $V^\mu = q\,\phi(c)j^\mu$ and $L = L^K$ or $L = L^u$.[9] The renormalization of (3.1) may require further Ward identities, that we have verified only in lower orders.

The potential being replaced by a "pure gauge" $\partial_\mu\phi(c)$, one does not expect an interacting model. The classical Dirac equation in this case has the solution

$$\psi_{qc}(x) = e^{iq\phi(x,c)}\psi_0(x). \tag{3.3}$$

This is our "dynamical" motivation for (1.3), different from Dirac's and others' (cf. section 1.2). It remains to define (3.3) as a quantum field when $\phi(c)$ is the logarithmically divergent escort field.

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[9]There exist weaker conditions for equality. QED is just the simplest instance.

Instead of the formal perturbative construction (see section 3.5), we present below an exact non-perturbative construction, proceeding in close analogy to the strategy applied in [1, 30] for the old two-dimensional model [68]. A main difference is that, because $\phi(c)$ arises from the Krein space decomposition (2.5) of $A^K(c)$, Hilbert space positivity requires the choice of a Lorentz frame (a timelike unit vector $u \in H_1^+$, where H_1^+ is the unit forward hyperboloid).

Let $u = u_0 \equiv (1, \vec{0})$ the standard timelike unit vector. If the string-smearing function c is supported in the sphere $H_1 \cap u_0^\perp = S^2$, where H_1 is the hyperboloid of spacelike unit vectors, and $\widehat{g}(0) = 0$, then $\phi(g, c)$ is well-defined with a positive-definite two-point function. Namely, $A_\mu^K e^\mu$ in (1.2) is positive-definite because vectors $e \in u_0^\perp$ have no zero component. c will be assumed to be real and have unit total weight w.r.t. to the invariant measure on S^2.

A special case may be illustrative. Let $c_0(\vec{e}) = \frac{1}{4\pi}$ the constant smearing function on the sphere S^2. By (A.4), $\phi(c_0) = \Delta^{-1}(\vec{\nabla} \cdot \vec{A}^K)$. Its gradient is the familiar longitudinal vector potential. Thus, if $\rho(\vec{x})$ is a (classical) static charge density distribution with Coulomb electric potential $\phi_{\mathrm{Cb}}(\vec{x}) = -\Delta^{-1}\rho(\vec{x})$, one may rewrite the time-zero Weyl operator $e^{i \int d^3x \, \rho(\vec{x}) \phi(\vec{x}, c_0)}$ as

$$e^{-i \int d^3x \, \phi_{\mathrm{Cb}}(\vec{x}) \, (\vec{\nabla} \cdot \vec{A}^K(\vec{x}))}.$$

This operator creates a "coherent purely longitudinal photon state" with charge distribution ρ, (cf. [50]). It coincides with Dirac's special solution [23, eq. (19)]. As emphasized in [16], the longitudinal photons are essential for the purposes of QED. However, our model admits smearing functions $c(\vec{e})$ without rotational invariance leading to photon clouds with additional transverse degrees of freedom, see section 3.4, and at the same time allows more flexibility of localization.

Here is what will be achieved in the dressed model (the intermediate model regarded as an autonomous theory): because the Dirac current is conserved, $L_{\mathrm{dress}}(c)$ is a total derivative and not expected to generate a nontrivial S-matrix. But the dressed Dirac field exhibits the desired kinematical features: it creates infraparticle states with superselected photon clouds labelled by the smearing function $c(e)$. It inherits the string localization of the escort field, so as to resolve the conflict with the Gauss Law discussed in the Introduction. The photon clouds dissolve the mass-shell of the electron with a sharp lower bound. They break the Lorentz invariance of each sector, as expected [36], but Lorentz invariance of the dressed Dirac field can be restored in a reducible Hilbert space representation, section 3.2.2. They cause expectation values of the Maxwell field to decay asymptotically in spacelike directions like r^{-2}, so as to allow finite flux distributions over the asymptotic sphere, see section 3.4. The detailed features of asymptotic expectation values in lightlike directions fulfill the assumptions expressed in [71] (and the positive electron mass poses no difficulties), section 3.4.3.

3.1.1 Non-perturbative construction

We present the non-perturbative construction of the "dressing factor" $e^{iq\phi(x,c)}$, where c is a real smearing function for the string direction supported in $H_1 \cap u_0^\perp = S^2$ with total weight

1. In section 3.4, we shall see that the smearing function c determines the "shape of the photon cloud".

Instead of admitting only smearing functions $g(x)$ in position space with $\widehat{g}(0) = 0$, we tame the infrared divergence by approximating ϕ by the *massive* escort field ϕ_m, and define a regularized massless limit of its exponential as in [1, 30]. The two-point function of the massive Feynman gauge potential is the massless one with the mass-shell measure $\mu_0(k)$ replaced by $\mu_m(k)$. The massive escort field is defined by the same expression (1.2) in terms of the massive Krein potential. Its two-point function $w_m(x - x'; e, e')$ is given in (A.6) with $\mu_0(k)$ replaced by $\mu_m(k)$.

We choose a real (and rotation symmetric) regulator test function $v(k) > 0$ with $v(0) = 1$. The split $e^{-ikx} = (e^{-ikx} - v(k)) + v(k)$ splits

$$w_m(x; e, e') = w_{m,v}(x; e, e') + d_{m,v}(e, e'), \tag{3.4}$$

where by (A.6) the divergent part (as $m \to 0$) is

$$d_{m,v}(e, e') = -(ee') \int \frac{d\mu_m(k)\, v(k)}{((ke) - i\varepsilon)((ke') + i\varepsilon)}. \tag{3.5}$$

$w_{m,v}(x, e, e')$ has a massless limit $w_v(x, e, e')$ because $e^{-ik(x-x')} - v(k)$ vanishes at $k = 0$. An explicit expression for $w_v(x, e, e')$ can be found in (A.17) and (A.15).

We define the exponential with a normal-ordering w.r.t. the functional defined by $w_{m,v}(x, e, e')$:

$$:e^{i\phi_m(g\otimes c)}:_v = \frac{e^{i\phi_m(g\otimes c)}}{e^{-\frac{1}{2}w_{m,v}(g\otimes c, g\otimes c)}} = e^{-\frac{1}{2}\widehat{g}(0)^2 d_{m,v}(c,c)}\cdot :e^{i\phi_m(g\otimes c)}: . \tag{3.6}$$

It differs from the normal-ordering w.r.t. the massive vacuum state by the displayed divergent factor. Using the Weyl formula, one finds that vacuum correlations of $:e^{i\phi_m(g_i\otimes c_i)}:_v$ involve the total divergent factor $e^{-\frac{1}{2}d_{m,v}(C,C)}$, where $d_{m,v}(C,C)$ is (3.5) smeared with $C(\vec{e}) = \sum_i \widehat{g}_i(0)c_i(\vec{e})$. In the limit $m \to 0$, this factor converges

$$e^{-\frac{1}{2}d_{m,v}(C,C)} \to \delta_{C,0} = \begin{cases} 1 \text{ if } C = 0 \\ 0 \text{ if } C \neq 0 \end{cases} \tag{3.7}$$

because $d_{m,v}(C,C)$ diverges to $+\infty$ unless $C = 0$. In order to see this, we define the integral transform of $c \in C_{\mathbb{R}}^\infty(H^1 \cap u^\perp)$ (cf. appendix A.3)

$$T_c^u(k) := \int_{H^1 \cap u^\perp} d\sigma_u(e)\, \frac{c(e)e}{(ke) + i\varepsilon} \equiv \left\langle \frac{e}{(ke) + i\varepsilon} \right\rangle_c. \tag{3.8}$$

Then $d_{m,v}(C,C) = -\int d\mu_m(k)v(k)(\overline{T_C^{u_0}(k)}T_C^{u_0}(k))$ diverges to $+\infty$ unless $T_C^{u_0}(k) = 0$ for all k on the zero mass-shell. The claim follows, because the integral transform is invertible, see (A.24).

Therefore, we may define $:e^{i\phi(g\otimes c)}:_v$ as the massless limit of (3.6) for $g \in \mathcal{S}(\mathbb{R}^4)$ and $c \in C_{\mathbb{R}}^\infty(S^2)$. Its correlation functions are

$$\left\langle :e^{i\phi(g_1\otimes c_1)}:_v \ldots :e^{i\phi(g_n\otimes c_n)}:_v \right\rangle = \delta_{C,0} \cdot \prod_{i<j} e^{-w_v(g_i\otimes c_i, g_j\otimes c_j)}. \tag{3.9}$$

The state defined by (3.9) satisfies the positivity requirement because the vacuum state on the massive Weyl algebra is positive, and the limit $m \to 0$ preserves positivity. The Kronecker $\delta_{C,0}$ means that the weighted sum $C \in C^\infty_{\mathbb{R}}(S^2)$ over the string smearing functions is superselected. This quantity generalizes the superselected total charge $\sum_i \widehat{g}_i(0)$ of the two-dimensional model, but remains highly uncountable even when $\widehat{g}_i(0)$ will later be restricted to discrete values of electric charges $\pm q$.

Vertex operators $V_{qc}(x)$ are obtained by admitting $g_x(\cdot) = q\delta_x(\cdot) = q\delta(\cdot - x)$ in (3.9), and rescaling $: e^{iq\phi(x,c)} :_v$ by an irrelevant finite c-dependent factor for convenience, see (A.20). Their correlation functions are

$$\langle V_{q_1 c_1}(x_1) \dots V_{q_n c_n}(x_n) \rangle = \delta_{\sum_i q_i c_i, 0} \cdot \prod_{i<j} \left(\frac{-1}{(x_i - x_j)^2_-} \right)^{-\frac{q_i q_j}{8\pi^2} \langle c_i, c_j \rangle} e^{-\frac{q_i q_j}{4\pi^2} \widetilde{H}(x_i - x_j; c_i, c_j)}, \quad (3.10)$$

where $\langle c_i, c_j \rangle$ is a quadratic form on real smearing functions in $C^\infty_{\mathbb{R}}(S^2)$, see (A.19), and $\widetilde{H}(x; e, e')$ is a Lorentz-invariant and in all three variables separately homogeneous distribution, see (A.15). This concludes the non-perturbative construction of vertex operators $V_{qc}(x)$ in four dimensions.

Vertex operators $V_{qc}(x)$ are string-localized in the spacelike cones $x + \bigcup_{e \in \text{supp}(c)} \mathbb{R}_+ e$. In general, they commute up to a phase

$$V_{qc}(x) V_{q'c'}(x') = e^{iqq'\beta(x-x';c,c')} \cdot V_{q'c'}(x') V_{qc}(x), \quad (3.11)$$

where $\beta(x - x'; c, c')$ is given by

$$\beta(x - x'; e, e') = -i[\phi(x, e), \phi(x', e')] = (ee')(I_{-e'} I_e C_0)(x - x'),$$

smeared in e and e'. The commutator function $\beta(x - x'; e, e')$ does not suffer from the IR divergence because the Fourier transform of C_0 vanishes at $k = 0$. It is a geometric quantity that can be characterized in terms the intersection of the wedge $x + \mathbb{R}_+ e - x' - \mathbb{R}_+ e'$ with the null cone [40, 65]. In particular, whenever the (smeared) strings $x_i + \bigcup_{e \in \text{supp}(c_i)} \mathbb{R}_+ e$ are spacelike separated, the phase is zero and the fields commute.[10]

Finally, the dressed Dirac field (1.3) is defined as

$$\psi_{qc}(x) := V_{qc}(x) \otimes \psi_0(x), \qquad \overline{\psi_{qc}}(x) := V_{qc}(x)^* \otimes \overline{\psi}_0(x), \quad (3.12)$$

where q is the unit of electric charge and the real string smearing $c \in C^\infty_{\mathbb{R}}(S^2)$ has unit weight $\int d\sigma(\vec{e}) c(\vec{e}) = 1$. The adjoint vertex operators are $V_{qc}(x)^* = V_{-qc}(x)$. This concludes the non-perturbative construction of the quantum dressed Dirac field $\psi_{qc}(x)$.

Its correlation functions are products of free Dirac and vertex operator correlations. The commutation relations (3.11) modify those of the Dirac field, so that the dressed Dirac field remains anti-local whenever the strings are spacelike separated. The power law decrease of the vertex operator correlations on top of the canonical decrease of the free

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[10]The commutativity up to a phase is not an instance of braid group statistics in the sense of DHR theory [41] because a statistics operator cannot be defined. Namely, by lack of Lorentz covariance of the infrared superselection sectors, one cannot rotate strings into spacelike separated positions.
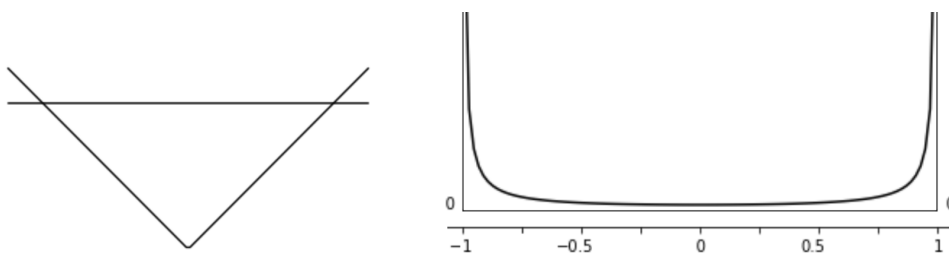
Figure 1. Left: cut at fixed energy through the momentum space forward lightcone. Right: the Fourier transform ρ of the two-point function as a function of momentum (i.e., along the cut) seems to exhibit an inverse power law peak at the maximum value $\frac{|\vec{k}|}{\omega} = 1$. (Qualitative)

Dirac correlations, and details of the direction-dependent function $\widetilde{H}(x; c, c')$ in (3.10) are expected to become important in the future scattering theory of the dressed Dirac field, cf. section 4.3.

The correlation functions (3.10) are translation invariant distributions. Hence the resulting GNS Hilbert space (cf. section 3.2) carries a unitary representation of the translations with positive energy. For the energy-momentum spectrum of infraparticle states, one would like to know the Fourier transform of two-point function of the infrafield (3.12). The c-dependent contribution from the photon cloud arises from the two-point function of the vertex operators. The spectrum of the latter is supported in the interior of V^+. Added to the mass-shell energy-momentum of the free Dirac particle, it describes the dissolution of the mass-shell.

This Fourier transform is impossible to compute for general c. However, because the two-point function drastically simplifies for the constant smearing function $c = c_0$, one can quantify the ensuing dissolution of the mass-shell in this special case.

The two-point function of the vertex operator with constant smearing (see appendix A.2)

$$\langle V_{qc_0}(x_1) V_{-qc_0}(x_2) \rangle = \left[\frac{\left(\frac{x^0 - r - i\varepsilon}{x^0 + r - i\varepsilon} \right)^{\frac{x^0}{r}}}{-(x^2)_-} \right]^{\frac{q^2}{8\pi^2}} \qquad (x = x_1 - x_2, \ r = |\vec{x}|) \tag{3.13}$$

can be written as $(x^0 - i\varepsilon)^{-\frac{\alpha}{\pi}}$ (where $\alpha = \frac{q^2}{4\pi}$ is the fine structure constant) times a power series in $\frac{r^2}{(x^0)^2}$. This structure allows to extract quantitative details of its Fourier transform, and hence of its rotationally invariant energy-momentum distribution $\rho(\omega, \vec{k})$ in the interior of V^+ [65]. By putting $r = 0$, one concludes that the distribution $\rho(\omega) = \int d^3k \, \rho(\omega, \vec{k})$ of energies decays like $\omega^{\frac{\alpha}{\pi} - 1}$. By applying powers of the Laplacian before putting $r = 0$, one can compute averages of powers of $|\vec{k}|^2$ at fixed energy ω. E.g., the average of the invariant masses $\omega^2 - |\vec{k}|^2$ at given energy ω is found to be $\frac{\alpha}{\pi} \cdot \omega^2 + O(\alpha^2)$ with variance $\frac{4}{9} \frac{\alpha}{\pi} \cdot \omega^4 + O(\alpha^2)$. These data are roughly consistent with a power law behaviour $\rho(\omega, k) \sim (\omega^2 - |\vec{k}|^2)^{\frac{\alpha}{\pi} - 1}$ for the dissolution of the mass-shell at fixed energy ω, see figure 1.

3.2 Hilbert space and Lorentz invariance

3.2.1 Fixed Lorentz frame

Until now, we were working in a fixed Lorentz frame, distinguished by the time unit vector $u_0 \in H_1^+$ and restriction of the support of $c(e)$ to the sphere $H_1 \cap u_0^\perp$. We shall now exhibit the structure of the Hilbert space in the fixed Lorentz frame, and then restore Lorentz invariance.

The GNS Hilbert space of the correlation functions (3.9) is the non-separable direct sum

$$\mathcal{H}_{u_0}^{\text{bos}} = \bigoplus_{C \in C_{\mathbb{R}}^\infty(S^2)} \mathcal{H}_{u_0,C}^{\text{bos}}, \tag{3.14}$$

due to the superselection rule $C = 0$ in (3.9). The operators $:e^{i\phi(g \otimes c)}:_v$ in (3.6) take $\mathcal{H}_{u_0,C}^{\text{bos}}$ to $\mathcal{H}_{u_0,C+\widehat{g}(0)c}^{\text{bos}}$. In particular, each sector $\mathcal{H}_{u_0,C}^{\text{bos}}$ carries a representation of the IR-regular Weyl subalgebra generated by $W(g_0,c') = e^{i\phi(g_0 \otimes c')}$ with $\widehat{g}_0(0) = 0$, whose generators are $\partial_\mu \phi(x,c)$. These representations differ from the vacuum representation by automorphisms

$$W(g_0,c') \mapsto e^{[\phi(g \otimes c),\phi(g_0,c')]} \cdot W(g_0,c') \qquad (\widehat{g}(0)c = C),$$

where the commutator is an IR-finite imaginary number because $\widehat{g}_0(0) = 0$.

Consequently, each $\mathcal{H}_{u_0,C}^{\text{bos}}$ in (3.14) is a separable representation space of $\partial_\mu \phi(c)$. Like in the two-dimensional model, the infrared divergence of the escort field infers a rich representation theory for its derivative.

The dressed Dirac field (3.12) is defined on the Hilbert space

$$\mathcal{H}_{u_0}^{\text{dress}} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{u_0,n}^{\text{bos}} \otimes \mathcal{H}_n^{\text{Dirac}}, \tag{3.15}$$

where $\mathcal{H}_n^{\text{Dirac}}$ is the subspace of $\mathcal{H}^{\text{Dirac}}$ of Dirac charge n, and $\mathcal{H}_{u_0,n}^{\text{bos}}$ is the subspace of $\mathcal{H}_{u_0}^{\text{bos}}$ spanned by $\mathcal{H}_{u_0,C}^{\text{bos}}$ with C of total weight n. The cyclic subspace of ψ_{qc} with a fixed c will be some very small subspace (only $C = nc$ occur) of $\mathcal{H}_{u_0}^{\text{dress}} \subset \mathcal{H}_{u_0}^{\text{bos}} \otimes \mathcal{H}^{\text{Dirac}}$.

3.2.2 Lorentz transformations

Because of the restriction $e \in H_1 \cap u_0^\perp$ on the string directions, $\mathcal{H}_{u_0}^{\text{bos}}$ is not Lorentz invariant. But one may repeat the same construction in any other Lorentz frame given by a timelike unit vector $u \in H_1^+$, with strings e restricted to $H_1 \cap u^\perp$. This gives rise to GNS Hilbert spaces $\mathcal{H}_u^{\text{dress}}$. In the Krein space, all massive fields $:e^{i\phi(g \otimes c)}:_{m,v}$ with c smooth smearing functions in $H_1 \cap u^\perp$, are simultaneously defined for all $u \in H_1^+$. For different $u \neq u'$, the inner products $\langle :e^{i\phi(g \otimes c)}:_{m,v} :e^{-i\phi(g' \otimes c')}:_{m,v} \rangle$ where c and c' have equal total weight and $\widehat{g}(0) = \widehat{g}'(0) = q \neq 0$, vanish in the massless limit. The only exception is $c = c' = 0$ which corresponds to the vacuum representation in either frame. The proof of this remarkable claim can be found in appendix A.3.

Therefore, the charged sectors in (3.15) of equal Dirac charge, but constructed in different Lorentz frames are mutually orthogonal. Because of the conservation of the Dirac

charge, the subspaces $\mathcal{H}^{\mathrm{bos}}_{u,n} \otimes \mathcal{H}^{\mathrm{Dirac}}_n$ are mutually orthogonal for different (u,n) with the exception of the common vacuum subspace $\mathcal{H}^{\mathrm{bos}}_{u,C=0} \otimes \mathcal{H}^{\mathrm{Dirac}}_0 \subset \mathcal{H}^{\mathrm{bos}}_{u,0} \otimes \mathcal{H}^{\mathrm{Dirac}}_0$. Let

$$\mathcal{H}^{\mathrm{dress}} := \bigoplus_{u \in H_1^+}{}^{*} \mathcal{H}^{\mathrm{dress}}_u = \bigoplus_{u \in H_1^+}{}^{*} \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{H}^{\mathrm{bos}}_{u,n} \otimes \mathcal{H}^{\mathrm{Dirac}}_n \right), \tag{3.16}$$

where the notation $\bigoplus_u^*$ indicates the identification of the vacuum subspaces ($C = 0$) for all u. Lorentz transformations act on $\mathcal{H}^{\mathrm{dress}}$ by the Wigner representation on the Dirac factor, and by

$$U(\Lambda) : e^{iq\phi(g \otimes c)}:_v \Omega = :e^{iq\phi(g \circ \Lambda^{-1}, c \circ \Lambda^{-1})}:{}_{v \circ \Lambda^{-1}} \Omega,$$

so that $U(\Lambda)\mathcal{H}^{\mathrm{dress}}_u = \mathcal{H}^{\mathrm{dress}}_{\Lambda u}$. (The change of the regularization function $v(k)$ is a unitary equivalence.)

The well-known abstract reason for the breakdown of Lorentz invariance in each sector was already mentioned in section 1.3.1: the "shape of the photon cloud" changes under Lorentz transformations. The expectation values of asymptotic electromagnetic fluxes in states from $\mathcal{H}^{\mathrm{dress}}$ to be computed in section 3.4 show that the latter is given by the string smearing function c. Before we can compute such expectation values, we need to include the Maxwell field into the algebra.

3.3 The dressed Maxwell-Dirac model

The Maxwell field cannot be expressed in terms of the escort field $\phi(x,e)$, so it is not defined in $\mathcal{H}^{\mathrm{bos}}$. We need to add the free Maxwell field to the dressed Dirac model.

3.3.1 Positivity problem

The immediate problem is that (for $m > 0$) $\phi(e)$ and $F_{\mu\nu} = \partial_\mu A^K_\nu - \partial_\nu A^K_\mu$ are not defined on a common positive-definite subspace of the Krein space $\mathcal{K}$. But positivity is essential in section 3.1.1 in order to arrive at a GNS Hilbert space in the first place, and in order to maintain the superselection structure which requires $d_{m,v}(C,C)$ to diverge to $+\infty$ when $m \to 0$.

The following construction holds for every $u \in H_1^+$. For the sake of explicit formulas, we choose again $u = u_0$. For $e \in H_1 \cap u^\perp$, the escort field $\phi(e)$, given by (1.2), is defined on the positive-definite subspace $\mathcal{H}^u \subset \mathcal{K}$, given by (2.11), that is generated by the spatial components of creation operators $a_i^*(k)$ of the Krein potential $\vec{A}^K$. We want to exploit the positivity of this subspace also for the other relevant fields in our construction.

This is achieved with the potential A^u and its field $F^u := \partial \wedge A^u$ already introduced in (2.8). They are also defined[11] on the subspace $\mathcal{H}^u$ characterized by (2.11) because

$$\langle A_0^K(x) A_\nu^u(x') \rangle = -\eta_{0\nu} W_0(x-x') - u_\nu (I_{-u}\partial_0 W_0)(x-x') = 0$$

by virtue of $I_{-u}\partial_0 = I_{-u}(u\partial) = 1$ (see footnote 5) and $u_\nu = \eta_{0\nu}$. They differ only by the null field (∂A^K) from the Feynman gauge fields. Hence, $F_{\mu\nu}^u$ (on $\mathcal{H}^u$) has the same two-point function as $F_{\mu\nu}^K$ (on the Krein space), and the same is true for their string-localized

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[11]Timelike strings don't need a smearing [55] because the denominator (uk) in (A.1) can never vanish except at $k = 0$ — where the singularity is controlled by the infrared regularization.

potentials $A_\mu^K(e)$ and $A_\mu^u(e)$. F^u and F^K are local fields relatively to themselves and to each other. However, $A_\mu^u(x)$ and $\phi(x, e)$ are non-local relatively to $F_{\mu\nu}^u$ (because x cannot be spacelike from $y + \mathbb{R}_+ u$).

The physical reason for the definition (2.8) on $\mathcal{H}^u$ was already mentioned in section 2: F^u is the physical Maxwell field (2.10) creating only the two physical photon states, embedded into the Krein space. Indeed, one may write the potential componentwise as

$$A_0^u = A_0^K + I_u(\partial_0 A^{K0} + (\vec{\nabla}\cdot\vec{A}^K) = -\partial_0\Delta^{-1}(\vec{\nabla}\cdot\vec{A}^K), \quad \vec{A}^u = \vec{A}^K. \tag{3.17}$$

This is gauge equivalent (by $\partial_\mu\Delta^{-1}(\vec{\nabla}\cdot\vec{A}^K)$) to the transverse Coulomb gauge potential

$$A_0^C = 0, \quad \vec{A}^C = \vec{A}^K - \vec{\nabla}\Delta^{-1}(\vec{\nabla}\cdot\vec{A}^K).$$

As a consequence, the equations

$$\partial^\mu A_\mu^u = 0, \qquad \partial^\mu F_{\mu\nu}^u = 0, \tag{3.18}$$

hold as operator identities. In contrast, the Krein space Maxwell field $F^K = \partial\wedge A^K$ contains unphysical longitudinal degrees of freedom visible in its correlations with the unobservable Krein field A^K, and $\partial^\mu F_{\mu\nu}^K = -\partial_\nu(\partial A^K)$ is the fictitious current (1.4). Accordingly, only F^u will enjoy the physical transversality properties in the charged sectors, see footnote 16 below.

3.3.2 Extension to the physical Maxwell field

We extend the construction of section 3.1.1 to include also the physical Maxwell field F^u. By avoiding the field F^K altogether, which carries both physical and unphysical degrees of freedom, the construction neatly separates the observable *field F^u* from the auxiliary escort field responsible for the photon clouds in charged *states* generated by the unobservable charged Dirac field.

To extend the construction of section 3.1.1, it suffices to consider the "multi-component" field

$$\Phi(g \otimes c \oplus f) = \phi(g \otimes c) + A^u(f) = \int dx \left[g(x) \int d\sigma(\vec{e})c(\vec{e})\,\phi(x, e) + f^\mu(x)\,A_\mu^u(x) \right]$$

at mass m. Its two-point function

$$\langle\phi(g \otimes c)\phi(g' \otimes c')\rangle_m + \langle\phi(g \otimes c)A^u(f')\rangle_m + \langle A^u(f)\phi(g' \otimes c')\rangle_m + \langle A^u(f)A^u(f')\rangle_m$$

is positive definite. Only the first term is logarithmically divergent in the limit $m \to 0$, and this term is regularized as in (3.4). Then the massless limit

$$: e^{i\Phi(g\otimes c\oplus f)} :_v \; := \; \lim_{m\to 0} e^{-\frac{1}{2}\widehat{g}(0)^2 d_{m,v}(c,c)} \cdot : e^{i\Phi(g\otimes c\oplus f)} :$$

exists, with correlation functions looking exactly like (3.9) with the obvious additional factors

$$e^{-\langle\phi(g\otimes c)A^u(f')\rangle - \langle A^u(f)\phi(g'\otimes c')\rangle - \langle A^u(f)A^u(f')\rangle} \tag{3.19}$$

As before, the factor $\delta_{\sum_i q_i c_i, 0}$ in (3.9) defines the superselection rule.

Since g and f can be chosen independently, one arrives at an algebra containing the fields $V_{qc}(x)$ and $e^{iA^u(f)}$, represented in a larger GNS Hilbert space

$$\widetilde{\mathcal{H}}^{\mathrm{bos,M}}_{u_0} = \bigoplus_C \widetilde{\mathcal{H}}_{u_0,C}, \tag{3.20}$$

where each $\widetilde{\mathcal{H}}_{u_0,C}$ is a proper extension of $\mathcal{H}^{\mathrm{bos}}_{u_0,C}$.

Since the test functions f^μ of the potential do not contribute to the superselection rule, one can freely vary w.r.t. $f^\mu(x)$. One obtains the fields A^u_μ, preserving each of the subspaces $\widetilde{\mathcal{H}}_{u_0,C}$, i.e., the latter are representation spaces of A^u, and consequently of their exterior derivatives F^u. Thus, each summand in (3.20) carries a representation of the fields $\partial_\mu \phi$ and A^u_μ.

Correlations of vertex operators with $A^u_\mu(x)$ are obtained by variation of the factors (3.19) w.r.t. the test functions $f^\mu(x)$. E.g.,

$$\langle V_{qc}(y_1) : e^{iA^u(f)} : V_{qc}^*(y_2)\rangle = t e^{-q\langle \phi(y_1,c)A^u(f)\rangle + q\langle A^u(f)\phi(y_2,c)\rangle} \cdot \langle V_{qc}(y_1) V_{qc}^*(y_2)\rangle$$

implies

$$\langle V_{qc}(y_1) A^u_\mu(x) V_{qc}^*(y_2)\rangle = iq(\langle \phi(y_1,c) A^u_\mu(x)\rangle - \langle A^u_\mu(x)\phi(y_2,c)\rangle) \cdot \langle V_{qc}(y_1) V_{qc}^*(y_2)\rangle. \tag{3.21}$$

Correlations involving several A^u are obtained similarly. Correlations with $F^u(x)$ are exterior derivatives of correlations with $A^u(x)$.

Now, the dressed Dirac field ψ_{qc} with c supported in $H_1 \cap u_0^\perp$ is defined on the Hilbert space

$$\mathcal{H}^{\mathrm{dress,MD}}_{u_0} = \bigoplus_{n \in \mathbb{Z}} \widetilde{\mathcal{H}}^{\mathrm{bos}}_{u_0,n} \otimes \mathcal{H}^{\mathrm{Dirac}}_n, \tag{3.22}$$

where now $\widetilde{\mathcal{H}}^{\mathrm{bos}}_{u_0,n}$ is the subspace of $\widetilde{\mathcal{H}}^{\mathrm{bos}}_{u_0}$ spanned by $\widetilde{\mathcal{H}}^{\mathrm{bos}}_{u_0,C}$ with C of total weight n.

Lorentz invariance is restored in the same way as in section 3.2.2, by taking a direct sum of $\widetilde{\mathcal{H}}^{\mathrm{dress,MD}}_u$ over all unit time vectors $u \in H_1^+$, $\mathrm{supp}\, c_u \subset H_1 \cap u^\perp$, similar to (3.16).

Remark. The same construction can be done with A^K_μ rather than A^u_μ. This involves Weyl operators of $A^K(g)$ on an indefinite space. Because the superselection structure via the factor $\delta_{C,0}$ in (3.9) requires only positive-definiteness on the escort part of the multi-component correlations, the construction with A^K gives rise to a variant $\mathcal{H}^{K\,\mathrm{dress,MD}}_{u_0}$ of (3.20) with indefinite extensions of the GNS Hilbert spaces H^{bos}_u, where correlations like (3.21), involving A^K rather than A^u, violate positivity. The two variants are needed in order to set up both perturbative expansions in (2.15), whose complementary benefits ensure positivity *and* (string-)localization of the interacting fields at the same time, see section 2 and section 4.

3.4 Expectation values of the Maxwell field

We want to compute the expectation value

$$\langle F^u_{\mu\nu}(x)\rangle_{f,c} := \langle \psi_{qc}(f) F^u_{\mu\nu}(x) \psi_{qc}(f)^*\rangle \big/ \langle \psi_{qc}(f)\psi_{qc}(f)^*\rangle \tag{3.23}$$

of the electromagnetic field strength in charged states from $\mathcal{H}^{\mathrm{dress,M}}$, where

$$\psi_{qc}(f) \equiv \int d^4y\, f(y) V_{qc}(y) \otimes \psi(y).$$

This requires to compute the bosonic factor $\langle V_{qc}(y_1) F^u_{\mu\nu}(x) V_{qc}(y_2)^* \rangle$, multiply with the fermionic factor $\langle \psi_0(y_1)\psi_0^*(y_2)\rangle$, smear with $f(y_1)\overline{f(y_2)}$, and divide by the normalization.

The bosonic factor can be computed from (3.21):

$$\langle V_{qc}(y_1) F^u_{\mu\nu}(x) V^*_{qc}(y_2)\rangle = \tag{3.24}$$
$$= -iq\Big\langle (e \wedge \partial_x)_{\mu\nu} I^x_{-e} + (u \wedge \partial_x)_{\mu\nu} I^x_u (W_0(x - y_2) - W_0(y_1 - x)) \Big\rangle_c \cdot \langle V_{qc}(y_1) V^*_{qc}(y_2)\rangle,$$

where $\langle \cdot \rangle_c$ stands for the smearing of a function of $\vec{e}$ with $c(\vec{e})$, see (B.1).

We are interested in the asymptotic behaviour of $\langle F^u_{\mu\nu}(x)\rangle_{f,c}$ as $x \to \infty$ in various directions. The behaviour of the electric field $\vec{E}(x)$ at $x = (0, r\vec{n})$, $r \to \infty$, in a given state is an abstract criterium for infraparticle states [14]: when the expectation value decays like $r^{-2}\vec{a}(\vec{n})$ with $\vec{a}(\vec{n}) \neq 0$ (and the fluctuations are not too wild), then the state cannot be an eigenvector of the mass operator, and is an infraparticle state instead. Clearly, the Gauss Law would require an r^{-2} decay of the electric flux.

In section 3.4.1, the asymptotic flux will be computed in states (3.23), and the infraparticle criterium will be established for the intermediate model. The dissolution of the mass-shell in the spectrum of the state is manifest because of the contribution of the dressing factor (photon cloud). The point is that the state is created by an "infrafield".

The limiting behaviour in general spacelike and timelike directions will be presented in section 3.4.2. The limiting behaviour in lightlike directions is of particular interest in connection with the Infrared Triangle [71], as briefly outlined in section 1.3.3.

3.4.1 Purely spatial asymptotics

We call "purely spatial" the directions in the hyperplane $(ux) = x^0 = const$.

We are going to show that $\langle V_{qc}(y_1) F^u_{\mu\nu}(x_r) V_{qc}(y_2)^*\rangle$, and consequently also $\langle F^u_{\mu\nu}(x_r)\rangle_{f,c}$ decay like r^{-2} in purely spatial directions $x_r = x_0 + (0, r\vec{n})$, with a nontrivial limit.[12] The string-integrated distributions $I_{-e}W_0(z)$ and $I_u W_0(z)$ appearing in (3.24) are given in (A.7) and (A.8). When $x_r - y_2$ resp. $y_1 - x_r$ are inserted for z, then in the limit $r \to \infty$ the finite values $x_0 - y_i$ may be neglected in (3.24). Namely, the behaviour of $((x_r - y)e)$ and $(x_r - y)^2$ is dominated by $-r(\vec{n} \cdot \vec{e})$ resp. $-r^2$ which are independent of $x_0 - y$. This entails that in (3.23), the norm $\langle \psi_{qc}(f)\psi_{qc}(f)^*\rangle$ also appears in the numerator, and the result is independent of the smearing function f. Moreover, in (3.24) the difference of two-point functions may be replaced by the commutator function. Thus (3.23) simplifies to

$$\lim_{r \to \infty} r^2 \langle F^u_{\mu\nu}(x_r)\rangle_{f,c} = \lim_{r \to \infty} r^2 \langle V_{qc}(0) F^u_{\mu\nu}(x_r) V^*_{qc}(0)\rangle = \tag{3.25}$$
$$= q \lim_{r \to \infty} r^2 \Big\langle \Big[(-e \wedge \partial_z)_{\mu\nu} I^z_{-e} - (u \wedge \partial_z)_{\mu\nu} I^z_u \Big] C_0(z)\big|_{z=(0,r\vec{n})} \Big\rangle_c.$$

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[12]Unless $c = c_0$ is the constant function in which case the limit is identically zero. Namely, $\phi(c_0)$ is longitudinal while F^u is transversal, cf. (A.4) and 3.3.1, so that V_{qc_0} commutes with F^u.

Now, at equal time, $C_0((0, \vec{z})) = 0$ and $\partial_0 C_0((0, \vec{z})) = \delta(\vec{z})$. Because $n^0 = 0$ and $e^0 = 0$, only the electric field $E_i = F_{0i}$ has a non-zero asymptotic expectation value:[13]

$$\lim_{r \to \infty} r^2 \langle E_i(x_r) \rangle_{f,c} = q \lim_{r \to \infty} r^2 \left[\int d\sigma(\vec{e}) \, c(\vec{e}) e_i \int_0^\infty ds \, \delta(r\vec{n} - s\vec{e}) - \frac{1}{\pi} r n_i \int_0^\infty ds \, \delta'(s^2 - \lambda^2) \right]$$
$$= -q \left[c(\vec{n}) - \frac{1}{4\pi} \right] n^i. \tag{3.26}$$

When the flux density is integrated over the "sphere at infinity", the first term gives rise to the charge $-q$ (the charge of the electron). But the first term is the contribution of $F_{\mu\nu}^K$ within F^u, so by Stokes' theorem, its flux arises from the fictitious charge density (1.4), commuted with the escort field. The second term due to the modification (2.8) of the Maxwell field cancels the fictitious total charge.[14]

The vanishing of the total charge is a necessity in the dressing model, because $\partial^\mu F_{\mu\nu}^u = 0$ is an operator identity. *It is physically expected because the model is only designed to implement the infrared features of QED states, notably the non-vanishing asymptotic flux density, without the need to "turn on" the actual QED interaction with the Dirac field, cf. section 2 and section 4.*

In [57], we had done the same computation with $F_{\mu\nu}^K$ rather than $F_{\mu\nu}^u$, and found the same results without the subtraction of the u-terms, and in particular the "correct" physical value $-q$ of the total charge. But the evaluation of F^K in dressed states is a formal prescription, because F^K is not defined on the dressed Hilbert space (3.20), as explained in the remark at the end of section 3.3.2. In particular, dressed expectation values of polynomials in $F_{\mu\nu}^K$ violate positivity in the bulk. On the other hand, the asymptotic flux operators $\lim_{r\to\infty} r^2 F_{\mu\nu}^K(x_r)$ commute with all observables and are therefore multiples of $\mathbf{1}$ in each sector, hence coincide with their expectation values. By the asymptotic absence of fluctuations (which decay like r^{-4}), positivity as a state on the commutative asymptotic algebra is automatic. Therefore, the result in [57] is *justified as a state on the asymptotic flux operators of $F_{\mu\nu}^K$, but not as a state on its full algebra in the bulk.*

In contrast, the present computation is a state also on the bulk algebra of the physical Maxwell field $F_{\mu\nu}^u$. Also its asymptotic flux operators $\lim_{r\to\infty} r^2 F_{\mu\nu}^u(x_r)$ commute with all observables and are multiples of $\mathbf{1}$ in each sector. This classical behaviour of a genuine quantum model is a prerequisite for the existence of a classical limit. It emerges not in some limit sending $\hbar$ to zero, but because in the limit $r \to \infty$ commutators decay faster than expectation values due to long-range photons.

3.4.2 Spacelike and timelike asymptotics

By the same methods as in section 3.4.1, one can compute expectation values of asymptotic fields in other spacetime directions than $x_r = (0, r\vec{e})$. They give functions on the

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[13]The magnetic expectation value will be non-zero when the asymptotic limit is taken in a boosted hyperplane $u'^\perp$ different from the hyperplane $u^\perp$ in which $c(e)$ is supported.

[14]In fact, it subtracts the average over the sphere of the first term. This is not an accident. Namely, by (A.3) the average of $\frac{e}{(ke)}$ equals $\frac{u}{(uk)} - \frac{k}{(uk)^2}$. The second term is annihilated by the exterior derivative, and $\frac{u}{(uk)}$ is the subtracted term (with e replaced by u).

entire spacelike Penrose boundary $\mathfrak{i}^0$ of Minkowski spacetime, whose behaviour adjacent to the lightlike Penrose boundaries $\mathfrak{I}^\pm$ plays a central role in the discussion of asymptotic symmetries in [71]. Our model has the benefit that we can compute these functions in a genuine quantum context, rather than formulating assumptions on their behaviour based on classical considerations.

We shall compute the expectation values of asymptotic fields

$$\lim_{\lambda \to \infty} \lambda^2 \langle F_{\mu\nu}(x_\lambda) \rangle_{f,c}$$

where $x_\lambda = x_0 + \lambda d$ approaches infinity in arbitrary lightlike and timelike directions d. To be specific, we parameterize

$$d_w^\pm = (\pm 1, w\,\vec{n}),$$

where $0 \leq w < 1$ (timelike), $w = 1$ (lightlike), or $w > 1$ (spacelike) is an inverse "velocity parameter". The purely spatial limit corresponds to the limit $w \to \infty$.

The argument as in section 3.4.1 to conclude that in the limit, y_i may be replaced by zero, and the expectation value equals the commutator as in (3.25), applies also for the general spacelike and timelike limits. Consequently, these limits will not depend on the smearing function f of the dressed Dirac field. But the argument does not apply for the lightlike limit, because the leading λ^2-term in $(x_\lambda - y)^2$ is absent, and the dominant terms depends on $x_0 - y$. The lightlike case $w = 1$, which is of particular interest in connection with the Infrared Triangle [71], will be discussed separately in section 3.4.3.

For detailed computations for the spacelike and timelike cases, see appendix B.2. The result is

$$\lim_{\lambda \to \infty} \lambda^2 \langle F_{\mu\nu}^u(x_0 + \lambda d_w^\pm) \rangle_{f,c} = \frac{\pm q}{4\pi} \left\langle (e \wedge d_w^\pm)_{\mu\nu} \frac{\nu}{(1 - w^2 \sin^2 \alpha)^{\frac{3}{2}}} - (u \wedge d_w^\pm)_{\mu\nu} \frac{\nu'}{w^3} \right\rangle_c, \qquad (3.27)$$

where $\alpha = \angle(\vec{n}, \vec{e})$ is the angle of $\vec{e} \in S^2$ relative to $\vec{n} \in S^2$, and the integers ν and ν' are functions of w and α, specified as follows.

- On $\mathfrak{i}^+$ (future timelike), one has $\nu = 1$ and $\nu' = 0$.

- On $\mathfrak{i}^-$ (past timelike), one has $\nu = 1$ and $\nu' = 2$.

- On $\mathfrak{i}^0$ (spacelike, $w > 1$), one has $\nu = 2\theta(\arcsin(w^{-1}) - \alpha)$ and $\nu' = 1$.

The "polar cap" $\alpha < \arcsin(w^{-1}) \in (0, \frac{\pi}{2})$ shrinks to the point $\vec{e} = \vec{n}$ in the limit $w \to \infty$ (purely spatial), in agreement with the result (3.26) for the purely spatial case. The first term in the electric component of (3.27) is parallel to $\vec{e}$, the second term is parallel to $\vec{n}$. Unlike in (3.26), here $\vec{e}$ and $\vec{n}$ are not necessarily parallel. The smearing in $\vec{e}$ will smoothen the step function in α in the spacelike commutator. By choosing the smearing function $c(\vec{e})$, one can "design" asymptotic field expectation values on the spacelike Penrose infinity.

Of special interest in connection with the Infrared Triangle are the limits $w \searrow 1$ where $\mathfrak{i}^0$ is adjacent to lightlike infinity $\mathfrak{I}^\pm$. Here, the "polar cap" becomes the hemisphere $\alpha < \frac{\pi}{2}$, and the expectation values coincide with the limiting values at $\mathfrak{I}_\mp^\pm$, cf. section 3.4.3.

3.4.3 Lightlike asymptotics

In order to make contact with the properties of electromagnetic fields at lightlike infinity $\mathfrak{I}^\pm$ in [71], we study the asymptotic behaviour of expectation values of the Maxwell field F^u in lightlike directions $x_\lambda = x_0 + \lambda \ell^\pm$, where $\ell^\pm = (\pm 1, \vec{n})$. This asymptotics is more subtle than in the spacelike and timelike cases, because the dominating term in z_λ^2 is not only $O(\lambda)$ rather than $O(\lambda^2)$, but it also depends on the initial point $z_0 = x_0 - y_1$ resp. $x_0 - y_2$, where y_i are points in the support of the smearing function f of the dressed Dirac field, as in (B.1). In particular, the limiting behaviour will depend on x_0 and y_i.

In Penrose coordinates $(V, U, \vec{n})$ defined by

$$V = x^0 + |\vec{x}|, \qquad U = x^0 - |\vec{x}|, \qquad \vec{n} = \frac{\vec{x}}{|\vec{x}|},$$

the appropriate asymptotic expansion at $\mathfrak{I}^+$ ($V \to \infty$) is

$$\vec{E}(V, U, \vec{n}) = \frac{2}{V} \vec{E}_\infty(U, \vec{n}) + \frac{4}{V^2} \vec{E}_\infty^{(2)}(U, \vec{n}) + O(V^{-3}). \tag{3.28}$$

For classical electric fields, $\vec{E}_\infty$ is transversal ($(\vec{n} \cdot \vec{E}_\infty) = 0$), while the leading part of the radial field $(\vec{n} \cdot \vec{E})$ appears in $O(V^{-2})$. We shall write $(\vec{n} \cdot \vec{E}_\infty^{(2)}) = E_{r,\infty}$. These leading components at $\mathfrak{I}^+$ fulfill the asymptotic Gauss Law (cf. [6, 71], see also [58, appendix B.3])

$$\partial_U E_{r,\infty}(U, \vec{n}) = (\vec{\nabla}_n - \vec{n}(\vec{n} \cdot \vec{\nabla}_n)) \cdot \vec{E}_\infty(U, \vec{n}). \tag{3.29}$$

We want to establish this classical behaviour also for the asymptotic expectation values in dressed Dirac states. The details of the computation can be found in appendix B.3. For very narrow smearing functions f of the dressed Dirac field around a point y, we find:

$$\lim_{\lambda \to \infty} \lambda \cdot \langle F^u(x_\lambda) \rangle_{f,c} = \frac{q}{4\pi} \cdot \left\langle \frac{\ell \wedge e}{(\ell e)} - \ell \wedge u \right\rangle_c \cdot \delta((\ell x_0) - (\ell y)), \tag{3.30}$$

where again $\langle \cdot \rangle_c$ stands for the smearing of a function of $\vec{e}$ with $c(\vec{e})$. The leading $O(\lambda^{-1})$ contribution to the electric field is[15]

$$\langle \vec{E}_\infty(U, \vec{n}) \rangle_{f,c} = \frac{q}{4\pi} \cdot \left\langle \frac{\vec{e}}{(\vec{n} \cdot \vec{e})} - \vec{n} \right\rangle_c \cdot \delta(U - (\ell y)), \qquad (\ell y) = y^0 - (\vec{n} \cdot \vec{y}). \tag{3.31}$$

Notice the $\vec{n}$-dependent shift in the argument $(\ell y) = y^0 - (\vec{n} \vec{y})$, so that the support $U = (\ell y)$ of the asymptotic electric field along $\mathfrak{I}^+$ depends on $\vec{n}$.

Eq. (3.31) is transversal, in accord with the fact that the radial component $(\vec{n} \cdot \vec{E})$ decays faster than λ^{-1}.[16]

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[15]The expression $\frac{\vec{e}}{(\vec{n} \cdot \vec{e})}$ here and in (3.32), (3.33) and (3.34) should be defined as a distribution in $\vec{e}$. Because (3.33) and (3.34) must average to zero with $c = c_0$ by (3.31), $\frac{1}{(\vec{n} \cdot \vec{e})^p}$ should be interpreted as $\frac{1}{2}\left(\frac{1}{(\vec{n} \cdot \vec{e})_+^p} + \frac{1}{(\vec{n} \cdot \vec{e})_-^p}\right)$.

[16]The subtractions of $\ell \wedge u$ in (3.30) and of $\vec{n}$ in (3.31) would be absent in expectation values of F^K. They secure the transversality of the physical Maxwell tensor F^u, while F^K would fail to be transversal. This is another reason, besides positivity, to adopt F^u as the correct electromagnetic field outside the vacuum sector.

For the constant function $c_0 = \frac{1}{4\pi}$, one has $\langle \frac{\vec{e}}{(\vec{n}\cdot\vec{e})} \rangle_{c_0} = \vec{n}$ (cf. (A.3)). Thus, the term $-\vec{n}$ in (3.31) is the subtraction of the spherical average of the first term, and one may as well write

$$\langle \vec{E}_\infty(U,\vec{n}) \rangle_{f,c} = \frac{q}{4\pi} \cdot \left\langle \frac{\vec{e}}{(\vec{n}\cdot\vec{e})} \right\rangle_{c-c_0} \cdot \delta(U - (\ell y)), \tag{3.32}$$

where $c - c_0$ has weight zero. In particular, for constant smearing $c = c_0$, one has $\langle \vec{E}_\infty \rangle_{f,c_0} = 0$. This physically reflects the fact that the longitudinal field $\phi(c_0)$ cannot generate a transversal electric field, cf. section 3.3.

The field configurations (3.31) in charged states generated by the dressed Dirac field are of a novel, genuinely quantum nature. Their characteristic dependence on $\vec{n}$ shows that they differ from the classical bremsstrahlung of one or several accelerated charged particles in the bulk.

The computation of the subleading radial contribution $\langle E_{r,\infty}(U,\vec{n}) \rangle_{f,c}$ in dressed states is much harder than that of $\langle \vec{E}_\infty(U,\vec{n}) \rangle_{f,c}$. It can be found in appendix B.3. The result is

$$\langle E_{r,\infty}(U,\vec{n}) \rangle_{f,c} = \frac{q}{4\pi} \left\langle \frac{2\theta((\vec{n}\cdot\vec{e})) - \theta(U - (\ell y))}{(\vec{n}\cdot\vec{e})^2} + \theta((\ell y) - U) \right\rangle_c + \langle (\vec{y}\cdot\vec{E}_\infty(U,\vec{n})) \rangle_{f,c}, \tag{3.33}$$

where $(\ell y) = y^0 - (\vec{n}\cdot\vec{y})$. (3.31) and (3.33) fulfill the asymptotic Gauss Law (3.29). The U-derivative of the term $(\vec{y}\cdot\vec{E}_\infty)$ in (3.33) on the left-hand side of (3.29) accounts for the angular derivative of the $\vec{n}$-dependent shift in (3.31) on the right-hand side.

The limiting value of (3.33) as $U \to -\infty$ coincides with the limiting value of the radial electric part of (3.27) as $w \searrow 1$, where $\mathfrak{i}^0$ touches on $\mathfrak{I}^+$ along the sphere $\mathfrak{I}^+_-$:

$$\lim_{U\to-\infty} \langle E_{r,\infty}(U,\vec{n}) \rangle_{f,c} = \frac{q}{4\pi} \left\langle \frac{2\theta((\vec{n}\cdot\vec{e}))}{(\vec{n}\cdot\vec{e})^2} + 1 \right\rangle_c. \tag{3.34}$$

Along the same lines, one obtains the asymptotic expectation values of the magnetic field:

$$\langle \vec{B}_\infty(U,\vec{n}) \rangle_{f,c} = \frac{q}{4\pi} \left\langle \frac{\vec{n}\times\vec{e}}{(\vec{n}\cdot\vec{e})} \right\rangle_c \cdot \delta(U - (\ell y)), \quad \langle B_{r,\infty}(U,\vec{n}) \rangle_{f,c} = \langle (\vec{y}\cdot\vec{B}_\infty(U,\vec{n})) \rangle_{f,c}. \tag{3.35}$$

The asymptotics on $\mathfrak{I}^-$ is related to the result on $\mathfrak{I}^+$ by the anti-unitary time-reversal operator T that exchanges $\mathfrak{I}^+$ and $\mathfrak{I}^-$. In particular, the fulfilment of the matching conditions [71]

$$E_{r,\infty}\big|_{\mathfrak{I}^+_-} = E_{r,\infty}\big|_{\mathfrak{I}^-_+}, \quad B_{r,\infty}\big|_{\mathfrak{I}^+_-} = -B_{r,\infty}\big|_{\mathfrak{I}^-_+} \tag{3.36}$$

has a fundamental origin: *it is a manifestation of the time-reversal (or PCT) invariance of the Dirac and Maxwell fields,* as follows. We have (in standard Dirac conventions)

$$T\psi(y)T^* = \gamma^1\gamma^3\psi(Ty), \quad TA^K_\mu(x)T^* = -(TA^K)_\mu(Tx).$$

The transformation law of A^K_μ along with the fact that $I_u = -I_{-u}$ on all correlations of A^K (see (A.9)), hence $TI_u(\partial A^K)(x)T^* = -I_{-u}(\partial A^K)(Tx) = I_u(\partial A^K)(Tx)$, and trivially $u = -(Tu)$, implies also $TA^u_\mu(x)T^* = -(TA^u)_\mu(Tx)$, hence

$$T\vec{E}(x)T^* = \vec{E}(Tx), \quad T\vec{B}(x)T^* = -\vec{B}(Tx), \quad T\phi(x,e)T^* = -\phi(Tx,e).$$

By the antilinearity of T, $TV_{qc}(x)T^* = V_{qc}(Tx)$. Because the expectation values of E_∞ at $\mathfrak{I}_-^+$ and $\mathfrak{I}_+^-$ vanish, those of $E_{r,\infty}$ are independent of the smearing function $f(y)$ of the Dirac field, see (3.34). Thus, the two expectation values match by virtue of T invariance. Likewise, those of B must differ by a sign. This behaviour is not changed by charge conjugation C, that takes $\vec{E} \to -\vec{E}$, $\vec{B} \to -\vec{B}$, $\phi \to -\phi$, $V_q \to V_{-q}$, nor by parity P, that takes $\vec{E}(x) \to -\vec{E}(Px)$, $\vec{B}(x) \to \vec{B}(Px)$, $\phi(x,e) \to \phi(Px,-e)$, $V_{qc}(x) \to V_{qc\circ P}(Px)$. Both C and P preserve $\mathfrak{I}_-^+$ and $\mathfrak{I}_+^-$, where the latter swaps $\vec{n} \to -\vec{n}$.

Now, the operators in (3.36) commute with all observables and hence are multiples of $\mathbf{1}$ in each superselection sector. They can therefore be identified with their expectation values such as (3.34), and the matching condition for the expectation values entails the matching condition (3.36) for the operators.

The present time-reversal (or PCT) argument pertains to the dressed model. It also applies to full QED, because the QED interaction is time-reversal (and CP) invariant.

The remarkable conservation law (3.36) is discussed at length in [42, 49, 71]. The smeared fields $E_{r,\infty}(\vec{n})$ and $B_{r,\infty}(\vec{n})$ at $\mathfrak{I}_-^+$:

$$Q_\varepsilon^+ = \int_{S^2} d\sigma(\vec{n})\, \varepsilon(\vec{n}) E_{r,\infty}(U,\vec{n})\big|_{U=-\infty}, \quad \widetilde{Q}_{\widetilde{\varepsilon}}^+ = \int_{S^2} d\sigma(\vec{n})\, \widetilde{\varepsilon}(\vec{n}) B_{r,\infty}(U,\vec{n})\big|_{U=-\infty} \quad (3.37)$$

are the electric and magnetic generators of "large gauge transformations" [71] that locally transform the potential by a derivative. When they are written with the help of the asymptotic Gauss Law (3.29) and its magnetic analogue as integrals along $\mathfrak{I}^+$, their commutators with the asymptotic potential on $\mathfrak{I}^+$ can be worked out giving angular derivatives of $\varepsilon(\vec{n})$, i.e., gauge transformations. Being generators of gauge transformations, Q_ε^+ and $\widetilde{Q}_\varepsilon^+$ actually commute with all observables, in particular $\vec{E}_\infty$ and $\vec{B}_\infty$. This is consistent with the commutation of $E_{r,\infty}$ and $B_{r,\infty}$ with $\vec{E}_\infty$ and $\vec{B}_\infty$ on $\mathfrak{I}^+$, cf. [3] and [58, appendix B.4].

By the matching condition, $Q_\varepsilon^+ = Q_\varepsilon^-$, the gauge transformation generated by them can be computed both on $\mathfrak{I}^+$ and on $\mathfrak{I}^-$, and is necessarily globally topologically nontrivial ("large"). This feature is interpreted as an "infinite degeneracy of the vacuum" [71].

The superselection structure of our model is "dual" to this infinite degeneracy in the sense that the large gauge transformations transform the escort field by a shift (which is possible because its localization reaches out to infinity), and hence the sector-creating charged fields by a complex phase. Conversely, the sectors assign expectation values ((3.34) smeared with $\varepsilon(\vec{n})$ for Q_ε^+) to the generators. In full QED, first order corrections to the generators are expected to cancel this phase and render the interacting dressed Dirac field invariant.

3.5 Perturbative dressing transformation

We give a perturbative motivation for the non-perturbative construction of section 3.3. Namely, when the free Dirac field is perturbed with the dressing density $L_{\mathrm{dress}}(c) = q\, \partial_\mu \phi(c) j^\mu$, the tree diagrams can be seen to organize into the Wick-ordered exponential series of $e^{iq\phi(c)} \cdot \psi_0$.

Causal perturbation theory based on Bogoliubov's formula [9] (see also [57] for a discussion of the string-localized case) gives the dressed field $\psi_0\big|_{L_{\mathrm{dress}}(c)}(x)$ as a power series in integrals over retarded multiple commutators of $\psi_0(x)$ with $L_{\mathrm{dress}}(y_i, c)$, see section 2.

In the case at hand, because L_{dress} is a total derivative, an integration by parts turns the retarded integrals for the tree diagrams in each order into $\frac{1}{n!}(iq\phi(x,c))^n\psi_0(x)$, summing up to $e^{iq\phi(x,c)}\psi_0(x)$. E.g., in first order,

$$\psi^{(1)}(x) = i\int d^4y\, R[\psi_0(x), j^\mu(y)]\cdot\partial_\mu\phi(y,c) = -i\int d^4y\,\phi(y,c)\partial_\mu^y R[\psi_0(x), j^\mu(y)]$$
$$= i\phi(x,c)\psi_0(x),$$

thanks to the Ward identity

$$\partial_\mu^y R[\psi_0(x), j^\mu(y)] = -\delta(x-y)\psi_0(y). \tag{3.38}$$

Loop diagrams have to be properly renormalized. This has been achieved up to second order. Apart from UV renormalization, the logarithmical IR divergence in the propagator of the escort field accounts for a multiplicative regularization of the exponential fields as in (3.6).

The characteristic trait of the perturbative construction is the "collapse" of retarded integrals to local expressions in x. Renormalizations must preserve this structure order by order.

By a theorem of Borchers [10], if two (sets of) fields belong to the same Borchers class, then they have the same S-matrix. The Borchers class of a set of fields is the class of all fields on the cyclic Hilbert space of the given fields, that are relatively local w.r.t. the given fields, i.e., they mutually commute at causal separation. In the case at hand, the dressed field belongs to the Borchers class (admitting relative string-locality) of the free fields. Anticipating that the theorem can be extended to the string-localized case, it would imply absence of scattering. Indeed, $S = 1$ is the "classically expected" feature of a quantum theory whose interaction density is a total derivative; it is, however, not automatic in the quantum case, where it must be secured beyond tree level by appropriate renormalization conditions. From this perspective, the above renormalization condition ("preservation of the string-localized Borchers class") is equivalent (or at least not weaker) than the condition that $S = 1$ in each order. Technically, it amounts to requiring that all retarded multiple commutators with L appearing in the perturbative expansion collapse (with the help of integrations by parts) into δ-functions.

One may as well subject the Maxwell field $F_{\mu\nu}^u$ to the dressing transformation with $L_{\text{dress}}(c)$. It turns out that $F^u|_{L_{\text{dress}}(c)} = F^u$ is invariant under this transformation. This is a consequence of the "neutral" Ward identity $\partial_\mu^y R[j^\mu(y), j^\nu(y')] = 0$. E.g., in first and second order,

$$(F^u)^{(1)}(x) = \int d^4y\,\partial_\mu^y R[F^u(x), \phi(y,c)]j^\mu(y) = -\int d^4y\, R[F^u(x), \phi(y,c)]\partial_\mu^y j^\mu(y) = 0,$$
$$(F^u)^{(2)}(x) = -\int d^4y\int d^4y'\, R[F^u(x), \phi(y,c)]\partial_\mu^y R[j^\mu(y), j^\nu(y')]\partial_\nu\phi(y',c) = 0$$

by integration by parts, where we have used that $R[F^u(x), \partial_\mu\phi(y,e)] = \partial_\mu^y R[F^u(x), \phi(y,e)]$ has no freedom of renormalization.

4 Towards QED: perturbation of the dressed Dirac field

The dressing transformation is only the first, "kinematical" step towards the full QED. It produces a free field of a new kind: the dressed Dirac field, that captures essential infrared properties of the actual interacting Dirac field of QED. In order to arrive at QED, the dressed Dirac field has to be subjected to the interaction L^u or L^K, which can only be done perturbatively.

As pointed out in section 2 Item 5, there are two options: to perturb the positive-definite model $\{\psi_{qc}, F^u\}$ with the non-local interaction density L^u, or to perturb the indefinite model $\{\psi_{qc}, F^K\}$ with the local density L^K. Order by order in perturbation theory, the two options give the same result for interacting correlation functions (see section 4.1). Because the former option preserves positivity and has no contributions from the fictitious current, and the latter preserves locality of the observables, the resulting formulation of QED enjoys both properties. This mechanism can work because all u-dependent terms in the former model cancel each other.

The discussion of the local Gauss Law in section 4.1 will illustrate these cancellations in an important instance. In section 4.2 we present systematic considerations concerning the structure of the perturbative expansion of the interacting infrafield. A main message will be that the dressing factor of the free infrafield remains untouched (and along with it the string-localization and the infrared features of the field) in the perturbative expansion. But it causes additional vertices connecting the dressing factor to QED vertices, thus contributing additional terms with novel "cloud propagators" to the perturbative expansion. We discuss the expected difficulties of the future scattering theory for infrafields in section 4.3.

In section 4.4, we illustrate (by way of a nontrivial example) how the diagrams with cloud propagators give rise to an "interference" between the infrared divergencies of QED and those of the free infrafield. Recall that the latter determine the superselection structure of the Hilbert space of the free dressed Dirac-Maxwell theory as in section 3.3. If the observed pattern persists in higher orders, then the dressing divergencies do not simply cancel the QED divergencies. *Rather, the interplay of dressing and QED interaction would "deform" the superselection structure of the dressed model in a momentum-dependent way.*

4.1 The local Gauss Law

We test the validity of the quantum Maxwell equation $\langle \partial_\mu F^{\mu\nu} \rangle = -q \langle j^\nu \rangle$, whose zero component is the local (= differential) Gauss Law, in a charged state created by the interacting charged field, in first order of perturbation theory. In the standard QED approach $\{\psi_0, F^K\}|_{L^K}$, there is a source term that can be attributed to the fictitious current of the free field F^K, which makes the total charge vanish. This is the failure of the global Gauss Law, that cannot be avoided in a local theory (section 1.3.1). In contrast, in the indefinite dressed model $\{\psi_{qc}, F^K\}$ without interaction the global Gauss Law holds, i.e., the total charge is the correct one, but the local charge density can still be identified with the fictitious current density. The most important result is that, when the dressed model is perturbed with the local QED interaction density L^K, the fictitious source term is cancelled and replaced by the Dirac current.

Subsequently, we turn to the perturbation of the positive-definite model $\{\psi_{qc}, F^u\}$ with the non-local interaction density L^u. Here, F^u is sourcefree, as expected for a free theory in a Hilbert space where there is no fictitious current. The main result will be that with QED interaction all non-local contributions arising from $F^u - F^K$ and from $L^u - L^K$ cancel each other, see table 2. This illustrates the equivalence of the two constructions (2.15) and the power of the approach, doing justice to positivity, locality and the infrared structure of QED at the same time.

The upshot is the equality of three expressions

$$\langle\psi_{qc}(y_1)F^K_{\mu\nu}(x)\psi^*_{qc}(y_2)\rangle\big|_{qA^K j} = \langle\psi_{qc}(y_1)F^u_{\mu\nu}(x)\psi^*_{qc}(y_2)\rangle\big|_{qA^u j} = \langle\psi_0(y_1)F_{\mu\nu}(x)\psi^*_0(y_2)\rangle\big|_{qA_{(c)} j},$$

where the second equality holds thanks to section 3.5. The first expression arises formally in the local setting on the GNS Krein space, cf. the remark at the end of section 3.3. The expression in the middle arises in the non-local setting on the extended GNS Hilbert space (3.22). The last expression is perturbatively defined in the positive-definite string-localized setting on the Wigner Hilbert space. It was computed in [57, section 5].

We start by computing the first expression

$$\langle\psi_{qc}(y_1)F^K_{\mu\nu}(x)\psi^*_{qc}(y_2)\rangle\big|^{(1)}_{qA^K_\mu j^\mu} \tag{4.1}$$

of interacting fields with the interaction $L^K = q\, A^K_\mu j^\mu$. In first order in q, it consists of the non-perturbative term as in section 3.4 (with only the contributions from F^K)

$$X_0 = -iq\big(\langle F^K_{\mu\nu}(x)\phi(y_2, c)\rangle - \langle\phi(y_1, c)F^K_{\mu\nu}(x)\rangle\big) \cdot \langle\psi_0(y_1)\psi^*_0(y_2)\rangle$$

plus the perturbative terms

$$X_1 + X_2 + X_3 = \langle\psi_0^{(1)}(y_1)F^K_{\mu\nu}(x)\psi^*_0(y_2)\rangle + \langle\psi_0(y_1)F^K_{\mu\nu}(x)\psi_0^{*(1)}(y_2)\rangle + \langle\psi_0(y_1)F^{K(1)}_{\mu\nu}(x)\psi^*_0(y_2)\rangle,$$

where in this order of q, one may replace the free infrafield by the free Dirac field, and the field perturbations are given by (2.2). X_0 is computed using

$$\langle\phi(y_1, c)F^K(x)\rangle = \partial^x \wedge \langle\phi(y_1)A^K(x)\rangle = \langle e \wedge \partial^x I^{y_1}_e\rangle_c W_0(y_1 - x)$$

etc. $X_1 + X_2$ and X_3 are identical with the expressions computed in [57] *without* the string-dependent parts in that computation. The latter arise when the dressing transformation is implemented perturbatively. As expected from section 3.5, they coincide after an integration by parts and use of (3.38), with the non-perturbative contributions X_0 in the present computation.

Table 1 shows the various contributions as follows.

For the sake of readability, the presentation is quite schematic. In the headline, $\langle Q\rangle$ stands for the charged expectation value of the quantity of interest Q, to which the entries in the respective column contribute according to the indicated interaction. $X_1 + X_2$ resp. X_3 are the contributions due to the perturbations of ψ, ψ^* resp. F; X_0 is the non-perturbative contribution due to the dressing. $\langle\psi\psi^*\rangle$ stands for $\langle\psi(y_1)\psi^*(y_2)\rangle$, and C_0 stands symbolically for the difference of massless two-point functions $i(W_0(x - y_2) - W_0(y_1 - x))$ (which

	$\langle F^K \rangle$	$\partial \langle F^K \rangle$	interaction
$X_1 + X_2$	retarded	$+\partial C_0 \cdot \langle \psi\psi^* \rangle$	$A^K j$
X_3	retarded	$-\langle \psi j \psi^* \rangle$	$A^K j$
X_0	$I_{-e}(\partial \wedge e) C_0 \cdot \langle \psi\psi^* \rangle$	$-\partial C_0 \cdot \langle \psi\psi^* \rangle$	non-pert
$X_1 + X_2$	$I_{-e}(\partial \wedge e) C_0 \cdot \langle \psi\psi^* \rangle$	$-\partial C_0 \cdot \langle \psi\psi^* \rangle$	$\partial\phi(e) j$
X_3	0	0	$\partial\phi(e) j$

Table 1. Contributions and cancellations for the Gauss Law.

asymptotically becomes the commutator function). Every entry has to be smeared with the test functions $f(y_1)\overline{f(y_2)}$ and (where applicable) with $c(e)$. The overall factor q is suppressed.

This is what table 1 tells us.

1. The first two rows give the perturbative contributions due to the interaction L^K, as in standard QED. The "retarded" expressions are the usual retarded integrals in the standard Krein space QED in Feynman gauge. The sum of these two rows is the result of standard perturbation theory. The source entry in row 1 is the "fictitious" contribution from $\partial F^K = j_{\text{fict}} = -\partial(\partial A^K)$ (where $\langle A_\mu^K(y) j_{\text{fict}}^\nu(x) \rangle = -\partial_\mu^y \partial_x^\nu W_0(y-x)$ and $\langle j_{\text{fict}}^\nu(x) A_\mu^K(y) \rangle = -\partial_\mu^y \partial_x^\nu W_0(x-y)$ yield the displayed result after an integration by parts and use of (3.38)). As mentioned in section 1.3.1, its total charge cancels the total charge of the current term in row 2. The presence of this term is the failure of the Gauss Law.

2. The third row is the non-perturbative contribution of the dressing factor, computed as in (3.21). This contribution alone (i.e., the dressed model without QED interaction) yields quantitatively the correct expectation value $-q$ of the total charge; but its origin is still the fictitious current. However, in QED it cancels the source term of fictitious provenience in row 1, so that only the Dirac source term in row 2 remains, i.e., the Gauss Law holds for the interacting fields (in this order).

3. The last two rows give the corresponding contributions when the dressing transformation were done perturbatively. In fact, the last row is identically zero because of the conservation of the Dirac current.

4. The equality of rows 3 and 4 illustrates the equivalence of the perturbative and non-perturbative dressing transformation in this order, see section 3.5, as well as the equivalence (cf. section 2 Item 5) $\{\psi_0, F^K\}|_{L^K(c)} = \{\psi_{qc}, F^K\}|_{L^K}$.

5. The expectation value of F^K is the sum of a retarded term with source $\langle \psi j \psi^* \rangle$ (row 2) and the two entries in rows 1 and 3 that add up to a source-free vacuum solution of the Maxwell equations.

The main message is Item 2: *With the local interaction density L^K, the interacting Maxwell field evaluated in the state created by the interacting infrafield satisfies the local Gauss Law.*

	$\langle F^K \rangle$	$\langle F^u - F^K \rangle$	$\partial \langle F^K \rangle$	$\partial \langle F^u - F^K \rangle$	interaction
$X_1 + X_2$	**	$-I_u(\partial \wedge u)C_0 \cdot \langle \psi\psi^* \rangle$	**	$-\partial C_0 \cdot \langle \psi\psi^* \rangle$	$A^K j$ or $A^u j$
X_3	**	0	**	0	$A^K j$ or $A^u j$
X_0	**	$I_u(\partial \wedge u)C_0 \cdot \langle \psi\psi^* \rangle$	**	$+\partial C_0 \cdot \langle \psi\psi^* \rangle$	non-pert
$X_1 + X_2$	**	$I_u(\partial \wedge u)C_0 \cdot \langle \psi\psi^* \rangle$	**	$+\partial C_0 \cdot \langle \psi\psi^* \rangle$	$\partial \phi(e)\, j$
X_3	0	0	0	0	$\partial \phi(e)\, j$

Table 2. Cancellations of non-local contributions. (Entries indicated by $**$ are identical to table 1.)

In the perturbative approach, this was already one of the main messages of [57]: the string-localized total interaction $q\,A(c)j$ (in Wigner space or in Krein space) yields the correct Gauss Law without a fictitious source term compensating the total charge.

We now compute the expectation value

$$\langle \psi_{qc}(y_1) F^u_{\mu\nu}(x) \psi^*_{qc}(y_2) \rangle \big|^{(1)}_{qA^u_\mu j^\mu} \tag{4.2}$$

of interacting fields with the interaction $L^u = q\,A^u_\mu j^\mu$. Again, there is a non-perturbative term X_0 and three perturbative terms X_1, X_2, X_3. Given the result for (4.1), it is sufficient to consider only the additional contributions due to $F^u - F^K = -I_u(u \wedge \partial)(\partial A^K)$, and concentrate on their cancellation. The contributions due to the modification of the interaction $L^u - L^K = q\,I_u(\partial A^K)(uj)$ neither contributes to $X_1 + X_2$ because the null field (∂A^K) has vanishing correlation with F^K and F^u, nor to X_3 because it has vanishing retarded commutator with F^K and F^u.

The u-contributions in the non-perturbative term X_0 are computed using

$$\langle \phi(y_1, c)(\partial A^K)(x) \rangle = -\langle (e\partial^x)I^{y_1}_e \rangle_c W_0(y_1 - x) = \langle (e\partial^{y_1})I^{y_1}_e \rangle_c W_0(x - y_1) = -W_0(y_1 - x)$$

etc. The u-contributions to X_3 vanish by current conservation. Those to X_1 and X_2 are computed with integration by parts and using again (3.38). We collect the results in another table.

This is what table 2 tells us.

1. The total source term in the dressed model $\{\psi_{qc}, F^u\}$ without interaction (the sum of the contributions in row 3) vanishes.

2. The previous equivalence (Item 4 after table 1) between the non-perturbative (row 3) and perturbative (rows 4 and 5) dressing transformations extends to F^u.

3. All additional non-local terms $I_u(\dots)$ either vanish or cancel each other one-by-one. Therefore, the present a priori non-local expansion is in fact local.

4. Because the present construction is manifestly positive, while the additional terms cancel exactly, the previous model (table 1) is also positive.

5. Items 3 and 4 together illustrate the power of the equivalence (cf. (2.1) and (2.15))

$$\{\psi_{qc}, F^K\}\big|_{L^K} = \{\psi_{qc}, F^u\}\big|_{L^u}$$

(one manifestly local, the other one manifestly positive). The same cancellation also illustrates the equivalence between the two manifestly positive models

$$\{\psi_0, F\}\big|_{L(c)} = \{\psi_{qc}, F^u\}\big|_{L^u}.$$

In higher order, the systematic cancellation of u-dependent terms is harder to see, but it must happen for the abstract reasons given in section 2.

4.2 Systematic considerations

The perturbation theory of the infrafield can be done as a power series expansion in the coupling constant q of L^u, while not at the same time expanding the non-perturbative vertex operator in the free infrafield ψ_{qc}. Instead, the vertex operators are left "intact" in the interacting infrafield, and only their (finite) correlation functions (3.10) (with $q_i = \pm q$) need to be axpanded.[17]

The infrafield ψ_{qc} subjected to the interaction density L^u or L^K will be a power series whose coefficients are retarded multiple commutators of ψ_{qc} with several operators L^u or L^K. The bosonic part contains only a single vertex operator $V_{qc}(x)$ and several potentials $A^u(y_i)$. Such terms require a treatment beyond standard perturbation theory, as follows.

As explained in section 3.3.2, correlations involving one or more potentials A^u are well defined by variation of correlations involving Weyl operators $e^{iA^u(f)}$. This implies that commutators and retarded commutators are of the form

$$[V_{qc}(x), A^u(y)] = V_{qc}(x) \cdot iq[\phi(x,c), A^u(y)] \text{ and } R[V_{qc}(x), A^u(y)] = V_{qc}(x) \cdot iqR[\phi(x,c), A^u(y)].$$

Consequently, the vertex operators associated with the charged fields remain "intact" in the expansion of the interacting infrafield. But they contribute additional string-localized "cloud propagators", with lines in Feynman diagrams connecting vertex operators ("clouds") with interaction vertices (figure 2 in section 4.4). These new propagators are of order 1 in q. They are obtained by string integration over ordinary propagators. "Contractions of vertex operators" need not to be considered since they are already contained in the vertex operator correlations.

There is a potentially important observation: recall that for the construction of vertex operators in section 3.3, we had to restrict smearing functions c to be supported in $u^\perp$ for some $u \in H_1^+$. This was necessary for two reasons: (i) to ensure that the superselected correlation functions of vertex operators satisfy positivity, and (ii) to make sure that the divergent exponent $d_{m,v}(C,C)$ (see section 3.1.1) diverges to $+\infty$, so that $e^{-\frac{q^2}{2}d_{m,v}(C,C)} \to \delta_{C,0}$.

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[17]One might distinguish two (in the beginning independent) couplings q (for the dressing) and q' (for perturbation theory with L^u.) However, the requirement that the resulting S-matrix and observable fields of QED must be string-independent fixes $q' = q$ already in first order.

The observation is that the restriction on the support of smearing functions can be relaxed when vertex operators are tied to the Dirac field as in (3.12) *and* the QED interaction is added! As for (i), the positivity of the perturbative expansion of the free infrafield ψ_{qc} with the QED interaction is indirectly secured by the equivalence (cf. (2.1) and (2.15))

$$\{\psi_0, F\}|_{L(c)} = \{\psi_0, F^K\}|_{L^K(c)} = \{\psi_{qc}, F^K\}|_{L^K}$$

for arbitrary smearing functions; as long as one keeps the strings spacelike so as to maintain sufficient causal separability. Thus, one may admit c_i supported on all of H_1 with unit total weight (and accordingly drop the restriction to $u^\perp$ in (3.8)). As for (ii), charge conservation of the free Dirac field ensures, that in non-vanishing correlations of the dressed Dirac field, $C = \sum_i q_i c_i$ ($q_i = +q$ or $-q$ for each field ψ_{qc} or $\overline{\psi}_{qc}$) has the total weight $\sum_i q_i = 0$. But if C has total weight zero, then $(T_C(k)k) = \sum_i q_i (T_{c_i}(k)k) = 0$ (see (3.8)), hence for $k^2 = 0$, $T_C(k)$ is spacelike or lightlike, and

$$d_{m,v}(C,C) = - \int d\mu_m(k)\, v(k)(\overline{T_C(k)}T_C(k)) \geq 0$$

still diverges to $+\infty$ unless $T_C(k)$ is a multiple of k for all k on the zero mass-shell. Thus, its exponential converges to 1 if C satisfies this condition, and zero otherwise. This suffices for a finite result. When C may be supported on H_1, we do presently not know whether the latter condition implies $C = 0$, giving rise to the "Kronecker delta" $\delta_{C,0}$ as in section 3.1.1; the kernel of the quadratic form $d_{m,v}$ in the limit $m \to 0$ could be larger.

4.3 Scattering theory

Scattering amplitudes of QED are infrared divergent. Bloch and Nordsieck [7] had noticed that the divergence can be cancelled by admitting real soft photons below the observation threshold accompanying charged particles of mass $M > 0$. Later, Weinberg [73] recognized that the real part of the singularities systematically comes with the characteristic logarithmically divergent "soft photon" factors[18]

$$\text{Re}(\alpha B) = \frac{q^2}{2} \lim_{m \to 0} \int d\mu_m(k) \left(\frac{p_{\text{in}}}{(p_{\text{in}}k)} - \frac{p_{\text{out}}}{(p_{\text{out}}k)} \right)^2 = -\infty, \tag{4.3}$$

where p are electron momenta and k virtual soft photon momenta. These factors sum up order by order to yield exponentials of the form $e^{\alpha B} \to 0$ with the consequence that all S-matrix elements in charged states vanish unless $p_{\text{out}} = p_{\text{in}}$. The physical reason is that in- and out-states lie in different superselection sectors ("velocity superselection").

Chung [22] recognized that the same effect as in [7] to cancel the IR divergence (4.3) is achieved by "dressing" the initial and final charged particle states with momentum-dependent coherent real photon states of the form

$$\exp \pm q \lim_{m \to 0} \int d\mu_m(k) \left(\sum_{\ell=1,2} \frac{(p e^\ell(k))}{(pk)} a^{\ell*}(k) - h.c. \right) |0\rangle \tag{4.4}$$

---

[18]$\alpha = \frac{q^2}{4\pi}$ is the fine structure constant.

where $a^{\ell*}(k)$ create states of linear polarization $e^{\ell}(k)$, and $\pm q$ is the charge of the respective particle. (In the limit $m \to 0$, these states again do not lie in the Fock space.) Faddeev and Kulish [34] (see also [26, 29]) proposed to include a similar dressing factor into a redefinition of the S-matrix, stating that their prescription is equivalent to Chung's up to a certain gauge transformation to remove longitudinal photons.[19]

In contrast, in our approach, the x-dependent dressing factors are part of the off-shell charged fields. They are a priori unrelated to the momenta p of the charged particles. But with $p = Mu$ and $c_u(e)$ the constant function on $H_1 \cap u^{\perp}$, i.e., in the rest frame of p, one has

$$\frac{p}{(pk)} = \frac{u}{(uk)} = T^u_{c_u}(k) + \frac{k}{(uk)^2} \tag{4.5}$$

by (A.3) and (3.8). One might therefore expect that with the choice of $c = c_u$ adjusted to the momentum $p = Mu$ of the charged particles, one can achieve with dressed Dirac fields the same cancellation as with dressed states. This is, however, not the case, as will be elaborated in section 4.4. Instead of an exact cancellation of the IR divergence, one finds an "interference" between the superselection rule $c = c'$ for the free infrafield and the velocity superselection $p = p'$ of standard QED into a novel joint superselection rule. In this context, it is crucial that our dressing factors offer more flexibility concerning the choice of c, see section 4.4.

Another distinction (unrelated to scattering theory) is this: in Chung states, the expectation value of the total electric flux at infinity of the free (Wigner) Maxwell field F is zero (in accord with [16]). This is the same finding as with the physical Maxwell field F^u in the positive version of our intermediate model. But while in our approach, the QED correction produces the correct value $\pm q$ (section 4.1), the QED correction in Chung states also vanishes in $O(q)$ when F is embedded into Krein space as F^K or as F^u (Coulomb gauge, cf. section 3.3.1). Thus these states do not resolve the problem with the fictitious charge (which admittedly never was their purpose).

The feature that the off-shell dressing factor change the large-time asymptotic behaviour of the charged field, requires a new (not yet existing) scattering theory. We have not yet succeeded to formulate the correct asymptotic-time limits that would properly define the S-matrix.

The problems are manifold. The LSZ reduction formula is not applicable because it requires a sharp mass-shell; the Haag-Ruelle scattering theory [41] is not applicable because by the absence of a free equation of motion there are no candidates for "asymptotic creation operators" $a_t(f)$ that would create time-independent one-particle states and time-convergent many-particle states. (For theories without an isolated mass-shell or with massless particles, a variety of methods have been developed [2, 12, 18, 19, 44], some of them tested in non-relativistic models [5, 20, 21, 31, 62, 63]; see [58] for more detailed comments). On top of this problem, a major obstacle is the product structure of vertex operator correlations. It seems to indicate that the space of asymptotic "free infraparticle states" may not have the structure of a Fock space.

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[19]Unfortunately, the gauge transformation destroys the necessary condition [34, eq. (16)], so there remains some doubt if this beautiful picture is entirely self-consistent.

We mention here just a specific instance: a generalization of Buchholz' scattering theory of massless waves [12] can be applied to the vertex operator fields of the two-dimensional model of [68] i.e., without the spinor field [30]. This gives a finite result, despite the fact that the assumption of a zero mass-shell is not fulfilled. The S-matrix turns out to be a complex phase proportional to the product of charges. The same strategy applied to the vertex operators in section 3.1.1 in four dimensions seems promising at first sight: the power law factors in (3.10) also produce a complex phase (proportional to the product of charges and the bilinear form $\langle c, c' \rangle$ in (A.19)) — but the homogeneous functions $\widetilde{H}$ in the vertex operator correlations contribute another factor [40]. This factor being possibly of modulus > 1, jeopardizes the interpretation of the computed quantity as a scattering amplitude.

4.4 Dynamical superselection structure

In order to exhibit the cancellation of IR singularities in perturbation theory, one has to think of the factor $\delta_{C,0}$ in vertex operator correlations as the power series expansion of $e^{-\frac{1}{2}d_{m,v}(C,C)}$ (see section 3.1.1) in the limit $m \to 0$, where $C = \sum_i q_i c_i$ with $q_i = \pm q$ is of order q. In lowest order, the IR-divergent term $e^{-\frac{1}{2}d_{m,v}(C,C)}$ combines with IR-divergent contributions of a similar form $e^{-\frac{1}{2}d_{m,v}(C_0,C_0)}$ (with C_0 specified below) from the ordinary Feynman diagrams as in (4.3), and with further IR-divergent contributions from the cloud propagators. If confirmed in higher orders, this leads to an exciting scenario: all these terms conspire to give rise to

$$e^{-\frac{1}{2}d_{m,v}(C+C_0,C+C_0)},$$

where the diagrams with cloud propagators contribute the mixed terms $d_{m,v}(C,C_0) + d_{m,v}(C_0,C)$.

The coupling of the interacting infrafield to an external potential $a_\mu(x)$ is the simplest nontrivial instance which allows a change of momentum, so that the QED result is separately IR-divergent (and hence forbids scattering). For the sake of the illustration, we shall tentatively appeal to the Gell-Mann-Low formula and apply the standard LSZ prescription.[20]

We compute the term linear in the external field a_μ in third order in

$$\langle T[\psi_{qc'}(x')e^{iq\int d^4y\, j^\mu(y)(A_\mu^K+a_\mu)(y)}\overline{\psi}_{qc}(x)]\rangle \tag{4.6}$$

and concentrate on the IR-divergent term, when the LSZ prescription is applied.

In first order, (4.6) equals

$$iq(-i)^2 \int d^4y\, S_F(x'-y)\gamma^\mu S_F(y-x) \cdot a_\mu(y).$$

Inserting the Fourier representations, truncating the Dirac lines with the Klein-Gordon operators $(\square^{(\prime)} + M^2)$, and going on-shell (as in LSZ theory): $p^{(\prime)2} = M^2$, one finds the

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[20]While the GML formula is applicable for local fields also in the presence of IR divergences [26], it may not be properly justified for infrafields. The LSZ formula needs a modification in order to account for the absence of a sharp mass-shell. Yet, it turns out to be good enough to see how the IR-divergent result of QED is modified.
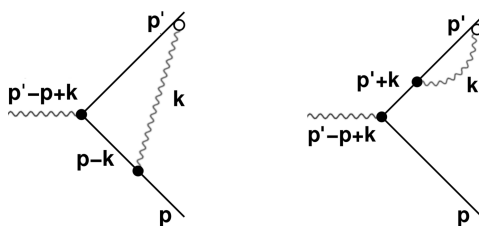
Figure 2. Two diagrams with "cloud propagators" connecting a vertex operator ("cloud", open blob) with a QED vertex. Notice that the open blob is attached to one Dirac line, and does not separate two Dirac lines. The cloud propagator only adds the photon momentum to the outgoing infrafield whose Fourier variable is $p' + k$.

coefficient of $e^{-ip'x'}e^{ipx}$

$$\Gamma^{(1)}(p,p') := -iq\,\widehat{a}_\mu(p'-p) \cdot \frac{M + p\!\!\!/'}{(2\pi)^4}\gamma^\mu\frac{M + p\!\!\!/}{(2\pi)^4}. \tag{4.7}$$

The two-point function of the vertex operators alone (without QED interaction) contribute (in the massive approximation of section 3.1.1) the factor $e^{-\frac{1}{2}d_{m,v}(C,C)}$ with $C = q(c - c')$. This factor gives in third order the IR-divergent term $-\frac{1}{2}d_{m,v}(C,C) \cdot \Gamma^{(1)}(p,p')$. The IR-divergent contribution from the QED diagrams, given in [22, eq. (8)], is of the form (4.3) times $\Gamma^{(1)}(p,p')$. With (4.5), it can be written as $-\frac{1}{2}d_{m,v}(C_0,C_0) \cdot \Gamma^{(1)}(p,p')$, where $C_0 = q(c_u - c_{u'})$. c_u and $c_{u'}$ are the constant smearing functions in the rest frames $u = p/M$ and $u' = p'/M$ of the in- and outgoing Dirac particle, see (A.3). Notice that the two divergent factors cannot cancel each other because (a) they come with the same sign and (b) unlike C, C_0 is momentum-dependent.

In appendix C, we compute the contributions from the four diagrams with cloud propagators. Two of them are depicted in figure 2, the other two have the cloud vertex on the lower line.

They contribute the divergent factor $-\frac{1}{2}(d_{m,v}(C,C_0) + d_{m,v}(C_0,C)) \cdot \Gamma^{(1)}(p,p')$. Thus, they provide the interference terms in $-\frac{1}{2}d_{m,v}(C + C_0, C + C_0)$ with

$$C + C_0 = q \cdot (c - c' + c_u - c_{u'}). \tag{4.8}$$

We conjecture this to be the onset of the perturbative expansion of the factor $e^{-\frac{1}{2}d_{m,v}(C+C_0,C+C_0)}$. At this point the observation at the end of section 4.2 comes to bear: $C + C_0$ is not supported in the intersection of H_1 with any spacelike plane, but because both $T_C(k)$ and $T_{C_0}(k)$ (see (3.8) with the restriction to $u^\perp$ dropped) entering the definition of $d_{m,v}$ are orthogonal to k, the exponent is negative or 0, and the exponential converges either to 0 or to 1. When the smearing functions $c^{(\prime)}$ are suitably supported in H_1, the resulting dynamical selection rule

$$\lim_{m\to 0} e^{-\frac{1}{2}d_{m,v}(C+C_0,C+C_0)} \stackrel{!}{=} 1 \quad \Leftrightarrow \quad T_{C+C_0}(k) \stackrel{!}{=} \alpha(k)k \quad \text{for all } k \text{ with } k^2 = 0$$

can have nontrivial solutions with $p' \neq p$.[21]

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[21]$c^{(\prime)} = -c_{u^{(\prime)}}$ is not a solution because $c^{(\prime)}$ and $c_{u^{(\prime)}}$ all must have unit total weight. Somewhat unnatural solutions with unit total weight would be $c = c_{u'}$, $c' = c_u$ or $c' = c + c_u - c_{u'}$, which all have $\alpha(k) = 0$. We conjecture, however, that solutions with $\alpha(k) \neq 0$ become important.

The upshot is that the cloud contributions do not simply cancel but change the IR singularities $d_{m,v}(C_0, C_0)$ of QED, whose absence requires $u = u'$ (see appendix A.3) and hence forbids scattering as in [22], into a deformed "dynamical" superselection rule that does not require $p = p'$. Conversely, regarded as a perturbation of the free infrafield, the QED interaction modifies the rigid momentum-independent superselection rule $c = c'$ into a momentum-dependent one.

This mechanism of singularity cancellation and the coupling of the cloud-superselection to the momentum appears quite distinct from the one proposed in [7, 76] and [22, 34].

To be sure, the above scenario has been tested only in lowest nontrivial order. Moreover, an LSZ prescription for infrafields that might turn the "suitable choice" of smearing functions c, c' (to comply with the superselection rule) into an asymptotic automatism via some stationary phase mechanism, is still lacking.

5 Emerging new paradigms for QFT and outlook

There are ample new insights arising from string-localized QFT beyond the special case of QED.

Let us start our discussion with what we have learned about QED as a "special instance". We neither claim nor want to construct a "New QED". We rather reformulate off-shell QED in a more conceptual way, avoiding unphysical features wherever possible. The final theory is expected to be equivalent to "ordinary" QED plus off-shell charged fields. Yet, there are many differences.

Our approach distinguishes between observables and states. Unobservable fields are needed to create charged states from the vacuum — simply because observables cannot change a superselection sector. The rich superselection structure of QED (going well beyond electric charge conservation) is clearly addressed in our approach.

The interaction density contains no unphysical quantum degrees of freedom: it couples the Dirac current to a string-localized potential $A(c)$ that is a functional of the Maxwell field strength F — in principle defined in the physical Hilbert space of the Maxwell field with precisely two polarization states. However, it turns out to be far more advantageous to embed the latter into the Krein space and split $A(c)$ into the usual unphysical Krein potential A^K and a derivative of the "escort field". The latter "dresses" the free Dirac field, while the former carries the honest QED interaction.

The embedding of the Maxwell field F sharpens the view upon the Maxwell field in Krein space. The usual Krein space Maxwell field $F^K = \partial \wedge A^K$ involves unphysical photon degrees of freedom that are not visible in its self-correlators but in correlators with the Krein potential. In contrast, the physical Maxwell field is embedded as F^u into Krein space by a nonlocal expression in terms of A^K. But locality is an issue of commutation relations and not of mathematical description and labelling: F^u is a perfectly local field, but the Krein potential is nonlocal relative to F^u. In this way, the picture makes more sense — simply because A^K is unphysical.

In our approach to QED, the Dirac field is coupled to A^u which contains both the escort field and the Krein potential. The escort coupling alone "transfers longitudinal

photon degrees of freedom" to the charged field. The simultaneous coupling has better features than the coupling to A^K alone (which produces vanishing scattering amplitudes due to infrared divergences). The latter are cured by the dressing and intimately connected to the rich superselection structure of charged states. In contrast to the prevailing cures of IR divergences, the cure is included in the field that creates charged states, rather than applied to the states. The highly interesting properties of the resulting charged infrafield (including commutation relations (3.11)) highlight the vanity of attempts to construct off-shell QED within the Wightman axiomatic setting.

The most important message for general QFT is that quantum fields of a new type have to enter the scene when their interactions are mediated by fields of helicity 1 (or more). It seems that the axiomatization of quantum fields pioneered by Wightman and Gårding must be questioned in the case of quantum field theories with long-range interactions and infraparticles. While it is very successful for interactions of fields of spin or helicity below 1 (φ^4 or pion-nucleon interactions), it seems to fall short when particles of spin 1 enter the stage — in particular in the Standard Model. It may be interesting to notice in this respect that the abstract analysis of [17] in the framework of Algebraic QFT can narrow down the localization of charges with a mass gap to spacelike cones, but the general argument can not be sharpened to compact localization.

The traditional axiomatic dichotomy that observable fields must be local while fields creating charged states may be anti-local, is shifted towards a new dichotomy between point-localized and string-localized. To be sure, this does not refer to the string-localized Wigner space potential (2.3) or the Krein space escort field (1.2). Both appear only in the process of the *construction* of the theory: the former allows the formulation of the interaction on a Hilbert space, and the latter serves as a catalyzer to transfer the string-localization of the former onto the charged field. The actual field content of the resulting QED is given by the interacting Maxwell and infrafields. In contrast to the potential and the escort field (no longer present in the final QED), that were just string integrals over local free fields, the string-localization of the charged field is irrevocable. It reflects physical features: photon clouds and the infraparticle nature of charged particles.

The local observables of the final QED: the interacting Maxwell field and current, are the same as in gauge theory. The difference is in the structure of the charged fields, whose string-localization is expected to cure the infrared vanishing of the S-matrix in the local approach.

The axiomatic perspective contemplates the field content of the final theory, not its making. In the orthodox approach, one would *only* consider observables. But with this attitude, one loses direct access to the agents that create charged states from the vacuum. This access was indirectly recovered in one of the great success stories of Algebraic QFT [41]: the Doplicher-Roberts reconstruction of graded-local charged fields from the superselection structure of the observables. But the method works only for global symmetries, and its generalization to local (= gauge) symmetries remained an open problem. Our results do not solve this problem (reconstruction from the superselection structure as a general strategy), but it indicates with an exactly solvable model what a general stategy should be able to envisage. The answer radically departs from the axiomatization of electrically charged

fields in the Wightman setting, and neither complies with the idea of a "graded-local field net" with anti-local Fermi fields.

With an eye on the Standard Model whose main theoretical challenges are posed by massless and massive vector bosons, we make some comments relating to massive QED, Yang-Mills theory, and the Higgs model.

There are two main traditional setups to study QED with massive photons: they describe the free vector bosons by the Proca field, or by a gauge field. The former is defined on a Wigner Hilbert space, but the Proca coupling to the current is power-counting non-renormalizable. The latter allows a renormalizable coupling, but is defined only on an indefinite Krein space.

In both variants, the idea of string-localization comes to rescue: the string-localized Proca potential $A_\mu^P(e) := I_e F_{\mu\nu}^P e^\nu$ (possibly smeared in e) has UV dimension one and makes the coupling renormalizable [61]. As in QED, the interaction density differs from a local density by a total derivative because $A_\mu^P(e) = A_\mu^P + \partial_\mu \phi^P(e)$, where $\phi^P(e) = I_e(A^P e) = -m^{-2}(\partial A^P(e))$ does not have a massless limit [56]. Similarly, the massive Krein potential can be replaced by the potential $A_\mu^K(e) = I_e F_{\mu\nu}^K e^\nu = A_\mu^K + \partial_\mu \phi^K(e)$ that can be defined on a positive-definite subspace of the Krein space. The strategy (2.14) of the present paper can be applied in both approaches:

$$\psi_0\Big|_{L^P(e)} = \left(e^{iq\phi^P(e)}\psi_0\right)\Big|_{L^P}, \qquad \psi_0\Big|_{L^K(e)} = \left(e^{iq\phi^K(e)}\psi_0\right)\Big|_{L^K}.$$

However, in sharp contrast to the massless case, the dressings here do not lead to a dissolution of the mass-shell. The multiplicative contributions of the correlations of massive vertex operators asymptotically tend to 1, and the dressed off-shell fields get "undressed" in the asymptotic time limit of scattering theory.

In the BRST approach [67], one "embeds" the Proca field into the (extended) Krein space with the help of the Stückelberg field Φ, an independent positive-definite scalar field of the same mass: $A^{\mathrm{BRST}} = A^K - m^{-1}\partial\Phi$. An observation of Duch (see [24], implicitly appearing already in [66, section 3]) is the analogue of (2.14) also in this case

$$\psi_0\Big|_{L^{\mathrm{BRST}}} = \left(e^{-iqm^{-1}\Phi}\psi_0\right)\Big|_{L^K}.$$

However, because the Stückelberg field has UV dimension 1, unlike the massless escort field of dimension zero in the present paper, the dressed field is a Jaffe field [46]. Jaffe fields are not polynomially bounded in momentum space, and can therefore only be smeared with a restricted class of test functions of slow decay in position space. This makes them of little use for local QFT. Yet, the dressed field is well-defined, and free of ambiguities. Therefore the right-hand side is renormalizable in the sense of absence of infinitely many undetermined constants, but with poor localization properties.

In general models involving string-localized fields, care must be taken that the quantum S-matrix is string-independent. In first order, this means that the string-dependence of the interaction density must be a total derivative, which clearly restricts the candidate interactions. More excitingly, string-independence as a renormalization condition in higher

orders may lead to further constraints on coupling constants and to the necessity for additional couplings with fixed values of coupling constants. These are particularly interesting for Standard Model physics:

In the case of Yang-Mills theory, a most notable result [39] states that every cubic self-coupling of string-localized massless vector bosons whose string-dependence is a total derivative, must necessarily, in order to ensure string-independence of the quantum S-matrix at second order, come with coefficients that are the structure constants of a compact semisimple Lie algebra. The result and strategy of proof are quite similar to the analogous result in gauge theory [4, 67], with BRST invariance replaced by string-independence. Thus the usual paradigm of non-abelian gauge invariance as a starting point is reverted (in both settings) to become the consequence of a renormalization condition in the service of fundamental physical principles. Moreover, the symmetry is an off-shell property that may not be reflected in the particle spectrum (no physical quarks). An understanding of infrared features of non-abelian theories like confinement, although qualitatively very different, is hardly conceivable without lessons from the abelian case.

String-localized vector potentials can also be used in order to couple *massive* vector bosons to spin-$\frac{1}{2}$ matter without paying with ghosts for power-counting renormalizability. It turns out that string-independence as a renormalization condition in higher orders necessarily requires the presence of a scalar field that must be coupled to the vector potential and to the matter fields like the Higgs field, and that must have the self-coupling of the Higgs field. This field does therefore not "trigger a spontaneous symmetry breakdown mechanism" (note that gauge invariance is not a fundamental principle in the string-localized approach whose breaking needs an explanation, but it rather emerges as a renormalization condition). On the contrary it serves to maintain an invariance property. We shall discuss this in a forthcoming publication [60].

Besides spinor QED, one may also consider scalar QED. With the same coupling $q\,\partial_\mu\phi(c)j^\mu$ where $j^\mu = -i\chi^*\overset{\leftrightarrow}{\partial^\mu}\chi$, one gets again a charged infra field

$$\chi_{qc}(x) = e^{iq\phi(x,c)}\chi_0(x)$$

creating charged states with superselected smearing function c. Again, this field belongs to the free Borchers class (section 3.5) and therefore leads to a trivial S-matrix [10]. But in contrast to the spinor case, string-independence of the theory requires an additional quartic interaction $-\frac{q^2}{2}A(c)^2\chi^*\chi$ that "completes the square" of the covariant derivatives. Thus, gauge symmetry emerges as a renormalization condition. Remarkably, this quartic term can as well be omitted when a different convention (renormalization) for the derivative propagator $\langle T\partial\chi^*\partial\chi\rangle$ is chosen. This scenario was first proven in the local Krein space QFT setting [28] — showing that gauge symmetry of a classical Lagrangian is dispensable when the consistency of the theory can be achieved by a different renormalization condition.

Scalar QED also has an UV divergent "box diagram" (double photon exchange) that requires a counter term of the form $(\chi^*\chi)^2$ with an undetermined coefficient. This UV divergence is not removed in the string-localized approach — simply because the UV behaviour of string-localized scalar QED turns out to be the same as that of Krein space scalar

QED. This may be taken as another indication that string-localized quantum field theory models do not per se describe any "New Physics" — except that they may allow otherwise "power-counting forbidden" interactions. On the other hand, the term $(\chi^*\chi)^2$ should not contribute at Penrose infinity because of its rapid fall-off. This may be an instance where the physics at Penrose infinity cannot capture all features of the bulk theory.

Work in progress by the present authors [59] studies the coupling of massless helicity-2 "gravitons" to a matter stress-energy tensor. The general structure of the theory largely parallels QED with one marked difference: the dressing transformation acts not by an operator-valued phase but like an operator-valued coordinate transformation (of a scalar matter field for simplicity)

$$\chi_{qc}(x) = \chi_0(x - q\beta(x,c)),$$

where $\beta(c)$ is a vector-valued string-localized helicity-2 escort field. As coordinate transformations are the natural analogue of phase transformations with helicity 2, such a formula is less surprising as it may appear at first sight. In momentum space, it is again a phase transformation by $e^{\pm iq(p_\mu \beta^\mu(x,c))}$ that can be defined as a limit of Weyl operators.

In the case of the helicity-2 dressing transformation, the renormalization condition of string-independence (hence triviality) of the S-matrix (or a perturbative version of the Borchers class condition as indicated in section 3.5) seems to require an infinite number of higher-order couplings with unique coefficients, similar to the quartic term $-\frac{q^2}{2}A(c)^2\chi^*\chi$ in the scalar QED case; but in the $h = 2$ case these cannot be absorbed into a renormalization of derivative propagators of the matter field. At least the lowest such terms coincide with the expansion of the coordinate-invariant Lagrangian in general curvilinear coordinates [11]. It thus seems that the classical symmetry (coordinate invariance) of the coupling of matter to helicity-2 massless particles ("gravitons") is again determined by string-independence!

To wrap this up: *gauge symmetry is no longer an a priori postulate.* In the standard view, it is a symmetry principle imposed on the unobservable part of the theory, that constrains the form of the interaction and allows — via BRST — renormalized interactions mediated by particles of spin or helicity 1 with a unitary S-matrix. But in our view, the same structure of the interactions is rather a consequence of the need to implement the fundamental requirements of Hilbert space positivity and renormalizability in the off-shell setting. This can be achieved for spinor and scalar QED, Yang-Mills, the abelian (and perhaps also non-abelian) Higgs model, by means of string-localized interaction densities in such a way that observables remain string-independent and local, while charged fields become string-localized in a way that "looks like a gauge transformation". Ockham would opt for string-localizated charged fields without conflict with the Gauss Law, rather than for Krein with local charged fields in indefinite metric, or BRST without charged fields.

The price for the remarkable advantages of string-localized QFT is that the renormalization theory in position space involving string-localized fields [48] is much more demanding than that of point-localized Wightman fields [33]. Remarkably (at least for a large class of interactions including those of the Standard Model), loop diagrams involving string-smeared string-localized propagators need to be renormalized only at coinciding points (rather than at intersecting strings) [37], so that the benefit of the improved UV scaling dimension [61] fully comes to bear.

We conclude with some speculative remarks.

Theorists are accustomed to employ classical "external" fields to describe a broad range of effects, including anomalous magnetic moments, the Lamb shift, and the process of measurement which invokes "classical" observations. But classical fields do not exist in nature: they are a simplifying idealization while in reality "there is no 'classical world' next to our quantum world" [51].

One may wonder why this idealization yields so stunningly exact results. We have seen in section 3.4.1 that the classical behaviour of the asymptotic electric flux attached to the charged field naturally emerges in the spacelike asymptotic regime, without a limit like $\hbar \to 0$. The long-range photons are responsible for the decoherence that is necessary, e.g., for "classical measurements". The decoherence is thus an intrinsic part of the theory, affiliated with its infrared features, and needs not be accounted on environment effects (the atomic radius is "asymptotically large" compared to the Compton wave length of the electron and of the nucleus). So the above question may be turned around into asking whether, and how, the "built-in" infrared classicality may lead to a new understanding of external field effects. It may even help soothening, in the case of helicity 2, the disconcerting "incompatibility" of classical General Relativity with Quantum Theory.

In the case of self-coupled massless vector fields, there is no known way to separate off a total derivative part from the interaction density that could give rise to a dressing transformation. This is also true for self-coupled helicity-2 fields ("quantum gravity"). Therefore, the infrared features in these cases must make their appearance in a different manner. Confinement comes to one's mind which excludes charged fields altogether from the field content of the interacting theory. The answer must be quite different in the $h = 2$ case.

Acknowledgments

A String-localized correlation functions

Correlation functions and commutators of string-localized fields are string integrals over local correlation functions and commutators. String integrals can be computed in momentum space by the distributional prescription

$$I_e e^{\pm ikx} \equiv \int_0^\infty ds \, e^{\pm ik(x+se)} = \frac{\pm i}{(ke) \pm i\varepsilon} \cdot e^{ikx} \equiv \frac{\pm i}{(ke)_\pm} \cdot e^{\pm ikx}, \qquad (A.1)$$

where the limit $\varepsilon \downarrow 0$ is understood in the sense of distributions.

In view of (A.1), the escort field on the Krein Fock space is

$$\phi(x,e) = \int d\mu_0(k) \left[-i \frac{e^{-ikx}}{(ke)_-}(a^*(k)e) + h.c. \right]. \qquad (A.2)$$

It is a distribution in both x and e. The smearing function in e will be called c. An important special case arises when $c(e)$ is the constant smearing function $c_u(e) = \frac{1}{4\pi}$ on $H_1 \cap u^\perp$, $u \in H_1^+$. Namely, one has

$$\frac{1}{4\pi} \int_{S^2} d\sigma(\vec{e}) \frac{\vec{e}}{(\vec{k} \cdot \vec{e})_{\mp}} = \frac{\vec{k}}{|\vec{k}|^2} \qquad \Leftrightarrow \qquad \frac{1}{4\pi} \int_{H_1 \cap u^\perp} d\sigma(e) \frac{e}{(ke)_{\pm}} = \frac{u}{(uk)} - \frac{k}{(uk)^2}, \quad \text{(A.3)}$$

where the latter is the covariant formulation of the former for $u = u_0$ [56]. In position space, (A.3) can be written as $\frac{1}{4\pi} \int_{S^2} d\sigma(\vec{e}) \, e_\mu I_e = I_u^2 \partial_\mu + u_\mu I_u$. Inserting (A.3) into (A.2), one obtains

$$\phi(x, c_0) = \Delta^{-1}(\vec{\nabla} \cdot \vec{A}^K)(x). \quad \text{(A.4)}$$

A.1 String integrations in position space

We need string integrations over the massless two-point function

$$W_0(x - x') = \int d\mu_0(k) \, e^{-ik(x-x')} = -\frac{1}{(2\pi)^2} \frac{1}{((x-x')^2)_-}, \quad \text{(A.5)}$$

where $\frac{1}{(x^2)_-}$ is the distributional limit $\lim_{\varepsilon \downarrow 0} \frac{1}{(x - i\varepsilon u)^2}$, $u \in H_1^+$ arbitrary. The two-point function $-\eta_{\mu\nu} \cdot W_0(x - x')$ of the Feynman gauge potential yields the two-point function of the escort field

$$\langle \phi(x, e)\phi(x', e') \rangle = -(ee') \cdot (I_{-e'} I_e W_0)(x - x') = -(ee') \int d\mu_0(k) \frac{e^{-ik(x-x')}}{(ke)_-(ke')_+}. \quad \text{(A.6)}$$

It is logarithmically divergent at $k = 0$. Its IR regularization is defined (in section 3.1.1) as a distribution in $x - x'$ and in e, e', see below.

String-integrated correlation functions in position space were computed in [40] for spacelike e. One string integration is easily performed as an elementary integral $s' = s + \frac{(xe)}{e^2} = s - (xe)$)

$$f(x, e) := (2\pi)^2 (I_e W_0)(x) = -\int_0^\infty \frac{ds}{(x + se)^2 - i\varepsilon(x + se)^0} = \frac{\log \frac{-(xe) + \sigma_-}{-(xe) - \sigma_-}}{2\sigma_-}, \quad \text{(A.7)}$$

where $\sigma_- = \sqrt{(xe)^2 + x^2 - i\varepsilon x^0}$. Here and below, log's of fractions are always understood as differences of log's, with phases determined by $i\varepsilon$. This means that it is not allowed to cancel negative factors in the fractions. Note that the branch cut of the square root σ_- resp. ρ is not a singularity of (A.7) and (A.8).

For timelike strings u, $i\varepsilon$ appears in different places. In this case one finds

$$f(x, u) := (2\pi)^2 (I_u W_0)(x) = -\frac{\log \frac{(xu) - i\varepsilon + \rho}{(xu) - i\varepsilon - \rho}}{2\rho}, \qquad (\rho = \sqrt{(xu)^2 - x^2}). \quad \text{(A.8)}$$

Incidentally, one sees by extending the lower integration limit to $-\infty$, that

$$f(x, e) + f(x, -e) = -2\pi i \cdot \frac{\text{sign}(x^0)}{2\sigma_-}, \quad \text{while} \quad f(x, u) + f(x, -u) = 0. \quad \text{(A.9)}$$

For the string-localized commutator, one may use the representation $C_0(z) = \frac{1}{2\pi}\mathrm{sign}(z^0)\delta(z^2)$ of the massless commutator function and evaluate the s-integral over $\delta((z+sa)^2)$ for $a = e \in H_1$ or $a = u \in H_1^+$. When the sign of $z+sa$ is constant, either because $a = e$ with $e^0 = 0$, or because $a = u$ with $z^0 > 0$, then the integral essentially counts the number ν of *positive* zeroes of the polynomial $(z+sa)^2 = a^2 s^2 + 2s(za) + z^2$:

$$\int_0^\infty ds\, \delta(a^2 s^2 + 2s(za) + z^2) = \theta((az)^2 - a^2 z^2) \cdot \frac{\nu}{2\sqrt{(az)^2 - a^2 z^2}}. \tag{A.10}$$

In the purely spatial case $a = e$ with $e^0 = 0$ and $z^0 = 0$, $\vec{z} = r\vec{n}$, the equal-time commutator $C_0(\vec{z}) = 0$ and $\partial_0 C_0(\vec{x}) = \delta(\vec{x})$ is more convenient:

$$(I_e \partial_0 C_0)(\vec{z}) = \int_0^\infty ds\, \delta(r\vec{n} + s\vec{e}) = \frac{\delta(r-s)}{r^2} \cdot \delta_{\vec{e},\vec{n}}, \tag{A.11}$$

where $\delta_{\vec{e},\vec{n}}$ is the δ function on S^2 w.r.t. the normalized invariant measure $d\sigma(\vec{e})$.

The double string-integrated two-point function appearing in (A.6) is regularized by

$$(I_{-e'} I_e W_0)_v(x) := \lim_{\varepsilon \downarrow 0} \lim_{\varepsilon' \downarrow 0} \int d\mu_0(k) \frac{e^{-ikx} - v(k)}{((ke) - i\varepsilon)((ke') + i\varepsilon')} \tag{A.12}$$

as explained in section 3.1.1. Multiplied by $-(ee')$, smeared in e, e' and exponentiated, it appears in correlation functions of the dressed Dirac field. In scattering theory, the causal "separation of wave packets" of the dressed Dirac field at late and early times is controlled by its asymptotic properties, when the smeared strings are spacelike separated.

Eq. (A.12) with two purely spatial strings ($e^0 = e'^0 = 0$) has been computed in [40]. The result is symmetric in $e_1 = e \leftrightarrow e_2 = -e'$. It can be written as

$$(2\pi)^2 (I_{e_2} I_{e_1} W_0)_v(x) = -\frac{1}{2} f(e_1, e_2) \log\left(-\mu_v^2 \cdot (x^2)_-\right) + \frac{H(x; e_1, e_2)}{(e_1 e_2)}, \tag{A.13}$$

where

$$f(e_1, e_2) = \frac{\gamma}{\sin\gamma} \qquad (\gamma = \angle(\vec{e}_1, \vec{e}_2) \in [0, \pi]) \tag{A.14}$$

is the same function as (A.7) with e_1, e_2 substituted for x, e, and $H(x; e_1, e_2)$ is defined for purely spatial e_i as follows. The dimensional factor μ_v^2 depends on the regulator function $v(k)$ in section 3.1.1 and may be a function of γ.

Because the operation I_e is homogeneous of degree -1 in the string e, it is advantageous to relax the normalization $e^2 = -1$, while keeping $e \in u_0^\perp$ spacelike. Then H is homogeneous of degree zero, separately in all three variables. Let

$$\det{}_{x,e_1,e_2} = x^2 e_1^2 e_2^2 - x^2 (e_1 e_2)^2 - e_1^2 (xe_2)^2 - e_2^2 (xe_1)^2 + 2(xe_1)(xe_2)(e_1 e_2)$$

be the Gram determinant of the vectors x, e_1, e_2, which in singular expressions is always understood distributionally as the boundary value from the forward tube $x - i\varepsilon u_0$. For $\{i, j\} = \{1, 2\}$, let $\Lambda_i = (xe_i)(e_i e_j) - e_i^2 (xe_j)$ be the cofactors (signed subdeterminants) of the entries (xe_j) of the Gram matrix. Then it holds

$$\Lambda_i^2 - \det{}_{e_1,e_2} \det{}_{x,e_i} = -e_i^2 \cdot \det{}_{x,e_1,e_2}.$$

Therefore, one can define the homogeneous variables ζ_1, ζ_2 by

$$\pm e^{\pm \zeta_1} = \frac{\Lambda_1 \pm \sqrt{\det_{e_1,e_2} \det_{x,e_1}}}{\sqrt{e_1^2 \det_{x,e_1,e_2}}}, \qquad \pm e^{\pm \zeta_2} = \frac{\Lambda_2 \pm \sqrt{\det_{e_1,e_2} \det_{x,e_2}}}{\sqrt{e_2^2 \det_{x,e_1,e_2}}}.$$

They are real if also x is purely spatial, because in this case all diagonal cofactors are ≥ 0 and $\det_{x,e_1,e_2} \leq 0$. Otherwise, they are defined distributionally as boundary values from the forward tube. It is convenient to define $\gamma = \angle(\vec{e}_1, \vec{e}_2)$ as before, and $D = \frac{\det_{x,e_1,e_2}}{x^2 e_1^2 e_2^2}$. In these variables [40],

$$H(x;e_1,e_2) = \frac{-\cos\gamma}{\sin\gamma}\left[\gamma \log\left(\frac{\sin^2\gamma}{\sqrt{D}}\right) + \frac{\pi}{2}(\zeta_1 + \zeta_2) - \frac{i}{4}\left\{\mathrm{Li}_2\left(e^{i\gamma}e^{\zeta_1}e^{\zeta_2}\right)\begin{array}{l}+(e^{\zeta_1}\leftrightarrow -e^{-\zeta_1})\\+(e^{\zeta_2}\leftrightarrow -e^{-\zeta_2})\\-(e^{i\gamma}\leftrightarrow e^{-i\gamma})\end{array}\right\}\right]. \tag{A.15}$$

The branch cuts of the dilog functions secure that the limit $\gamma \to 0$ (e_1 and e_2 parallel) is regular, while the limit $\gamma \to \pi$ (e_1 and e_2 antiparallel) is singular (as expected because $I_{-e}I_e$ is not defined). The singularity is $\frac{O(\log(\pi-\gamma))}{\pi - \gamma}$ and hence integrable in the string directions w.r.t. the invariant measure on S^2.

Two-point functions of the string-localized potential $A_\mu(e)$ involve only derivatives of (A.13). The derivative is IR regular, and takes a much simpler and highly symmetric form [40]:

$$(2\pi)^2(I_{e_2}I_{e_1}\partial_\mu^x W_0)(x) = -\frac{\left[f(e_1,e_2)\partial_\mu^x + f(x,e_2)\partial_\mu^{e_1} + f(x,e_1)\partial_\mu^{e_2}\right]\det_{x,e_1,e_2}}{2\det_{x,e_1,e_2}}. \tag{A.16}$$

Remarkably [40], despite the fact that in Lorentzian metric $\det_{x,e_1,e_2}$ can vanish in far more configurations than x being linearly dependent of e_1 and e_2 [40], thanks to cancellations of singularities in numerator and denominator, the singular support of this distribution after smearing in e_i, is just $x^2 = 0$, exactly as for the point-local two-point function W_0.

A.2 Vertex operator correlations

From (A.13) we conclude the regularized two-point function of the escort field in section 3.1.1

$$w_v(x,e,e') = \langle \phi(x_1,e)\phi(x_2,e')\rangle_v = \frac{(ee')}{8\pi^2}\widetilde{f}(e,e')\log\left(-\mu_v^2 \cdot (x^2)_-\right) + \frac{\widetilde{H}(x;e,e')}{4\pi^2}, \tag{A.17}$$

where $x = x_1 - x_2$, $\widetilde{f}(e,e') := f(e,-e')$ and $\widetilde{H}(x;e,e') := H(x;e,-e')$ (both symmetric under $e \leftrightarrow -e'$). Because positive and negative powers of $-(x^2)_-$ are well-defined distributions, and the homogeneous distribution $\widetilde{H}$ can be exponentiated, (A.17) can be exponentiated without smearing in x. When $:e^{i\phi(g\otimes c)}:_v$ with $g = q\delta_x$ are inserted into (3.9), the factors become

$$e^{-q_iq_j w_v(x_i-x_j;c_i,c_j)} = e^{\frac{q_iq_j}{8\pi^2}\lambda_v(c_i,c_j)} \cdot \left(\frac{-1}{(x_i-x_j)_-^2}\right)^{-\frac{q_iq_j}{8\pi^2}\langle c_i,c_j\rangle} \cdot e^{-\frac{q_iq_j}{4\pi^2}\widetilde{H}(x_i-x_j;c_i,c_j)}, \tag{A.18}$$

where the quadratic form

$$\langle c, c' \rangle := \int d\sigma(\vec{e})\, c(\vec{e}) \int d\sigma(\vec{e}')c'(\vec{e}')\,(\vec{e}\cdot\vec{e}')f(e,-e') \tag{A.19}$$

determines the power law behaviour. In contrast, the quadratic form

$$\lambda_v(c, c') := \int d\sigma(\vec{e})\, c(\vec{e}) \int d\sigma(\vec{e}')c'(\vec{e}')\,(\vec{e}\cdot\vec{e}')\widetilde{f}(e,e')\log\mu_v(e,-e')$$

just sets a scale and can be eliminated altogether: because it is symmetric, $\sum_i q_i c_i = 0$ implies $\sum_{i<j} \frac{q_i q_j}{8\pi^2}\lambda_v(c_i, c_j) = -\sum_i \frac{q_i^2}{16\pi^2}\lambda_v(c_i, c_i)$, and by conveniently defining

$$V_{qc}(x) := e^{\frac{q^2}{16\pi^2}\lambda_v(c,c)} \cdot {:}\, e^{iq\phi(x,c)} {:}_v, \tag{A.20}$$

one arrives at (3.10), which is independent of the regulator function $v(k)$.

When smeared with the constant function $c_0(e_i) = \frac{1}{4\pi}$, (A.17) simplifies drastically, see [40]:

$$\widetilde{H}(x; c_0, c_0) = \frac{x^0}{2r}\log\frac{x^0 - r - i\varepsilon}{x^0 + r - i\varepsilon} \qquad (r = |\vec{x}|) \tag{A.21}$$

and $\langle c_0, c_0 \rangle = 1$, hence

$$\langle V_{qc_0}(x_1)V_{-qc_0}(x_2) \rangle = \left[\frac{\left(\frac{x^0 - r - i\varepsilon}{x^0 + r - i\varepsilon}\right)^{\frac{x^0}{r}}}{-(x^2)_-} \right]^{\frac{q^2}{8\pi^2}} \qquad (x = x_1 - x_2). \tag{A.22}$$

Among the smearing functions of unit weight, c_0 is a stationary point for $\langle c, c \rangle$. It is presumably a minimum: that is, the power law decay for general $c \neq c_0$ is faster than for c_0.

A.3 Orthogonality of Lorentz transformed sectors

Let $u \neq u'$ forward unit vectors, and c, c' smooth real functions on $H_1 \cap u^\perp$, $H_1 \cap u'^\perp$, respectively, of equal total weight. We claim in section 3.2.2 that, unless $c = c' = 0$, states of the form

$$|f, c\rangle = {:}\, e^{i\phi(f,c)} {:}_v\, \Omega, \quad |f', c'\rangle = {:}\, e^{i\phi(f',c')} {:}_v\, \Omega, \qquad (\widehat{f}(0) = \widehat{f}'(0) = q \neq 0)$$

are mutually orthogonal, because the two-point function $\langle \phi(f, c)\phi(f', c') \rangle$ diverges to $+\infty$.

The argument is as follows. For a momentum four-vector k on the closed forward lightcone, consider $\frac{e}{(ke) + i\varepsilon} = -i\int_0^\infty ds\, e^{is(ke)}$ as a distribution on H_1. For a smooth real function c, supported on $H_1 \cap u^\perp$, consider the integral transform $c(e) \mapsto T_c^u(k)$ as in (3.8). Because $\frac{1}{(ke) + i\varepsilon}$ is rotationally invariant and $c(e)$ is smooth, $T_c^u(k)$ is a smooth function. By definition,

$$(uT_c^u(k)) = 0 \qquad \text{and} \qquad \overline{T_c^u(k)} + T_c^u(P_u k) = 0, \tag{A.23}$$

where $P_u k = 2(uk)u - k$ is the parity reflection in the frame u. The second property is a reality condition reflecting the fact that c is a real function. Moreover, the function $T_c^u(k)$ is homogeneous in k of degree -1, and $(T_c^u(k)k) = \langle 1 \rangle_c$ is the total weight of c.

The smearing function $c(e)$ can be recovered from the restriction of $T_c^u(k)$ to any mass-shell:

$$c(e)e = ir^2 \int d\mu_m(k) \cdot 2(uk) \, e^{-ir(ke)} T_c^u(k), \tag{A.24}$$

where $r > 0$ is arbitrary. We prove (A.24) (without loss of generality in the standard frame $u = u_0$, where $d\mu_m(k) = \frac{d^3k}{(2\pi)^3 2(u_0 k)}$). Let $e = (0, \vec{e})$, and denote by $\langle \cdot \rangle_c$ the smearing with $c(e')$:

$$ir^2 \int d\mu_m(k) \cdot 2(u_0 k) \, e^{-ir(ke)} T_c^{u_0}(k) = \frac{r^2}{(2\pi)^3} \int d^3k \left\langle e \int_0^\infty ds \, e^{ir(\vec{e}\cdot\vec{k}) - is(\vec{e}'\cdot\vec{k})} \right\rangle_c =$$

$$= r^2 \left\langle e \int_0^\infty ds \, \delta(r\vec{e} - s\vec{e}') \right\rangle_c = \left\langle e \int_0^\infty ds \, \delta(r - s)\delta_{\vec{e},\vec{e}'} \right\rangle_c = \left\langle e \, \delta_{\vec{e},\vec{e}'} \right\rangle_c = c(e)e.$$

After these preparations, we turn to the issue at hand. Let c, c' and f, f' as specified above. Write T, T' for $T^u, T^{u'}$. The divergent part of the massive two-point function $\langle \phi(f, c)\phi(f', c') \rangle$ is

$$q^2 d_{m,v}(c - c', c - c') = -q^2 \int d\mu_m(k) \, v(k) |T_c(k) - T'_{c'}(k)|^2. \tag{A.25}$$

For c and c' of equal weight, $T_c(k) - T'_{c'}(k)$ is orthogonal to k, and in the massless limit where $k^2 = 0$, it is either spacelike or a multiple of k. Thus, the integral (A.25) diverges to $+\infty$ unless $T_c(k) - T'_{c'}(k)$ is a multiple of k for all k on the zero mass-shell.

We claim that this is impossible for non-zero real c and c'. Let $T_c(k) - T'_{c'}(k) = \alpha(k) \cdot k$ for all k on the mass-shell. Because $T_c(k) \in u^\perp$ and $T'_{c'}(k) \in u'^\perp$, the coefficient is uniquely fixed:

$$\alpha(k) = \frac{(u'T_c(k))}{(u'k)} = -\frac{(uT'_{c'}(k))}{(uk)}. \tag{A.26}$$

Now assume that the reality condition (the second in (A.23)) holds for both T_c and $T'_{c'}$. Then

$$\frac{(u'T_c(k))}{(u'k)} = -\frac{(uT'_{c'}(k))}{(uk)} = \frac{\overline{(uT'_{c'}(P_{u'}k))}}{(uk)} = -\frac{\overline{(u'T_c(P_{u'}k))}(uP_{u'}k)}{(u'P_{u'}k)(uk)} = \frac{(u'T_c(P_u P_{u'}k))(uP_{u'}k)}{(u'P_{u'}k)(uk)},$$

where (A.26) was used twice. Because $(u'P_{u'}k) = (u'k)$ and $(uP_{u'}k) = (uP_u P_{u'}k)$, this is

$$f(k) = f(P_u P_{u'}k), \quad \text{where} \quad f(k) := (uk)(u'T_c(k)).$$

When $u' = \Lambda u$ with a boost Λ of rapidity $\tau > 0$, then one has $P_u P_{u'} = \Lambda^{-2}$. Because k is arbitrary, this implies that the homogeneous function $f(k)$ must be Λ^2-periodic:

$$f(\Lambda^2 k) = f(k).$$

For the boost Λ there is a (unique up to a positive factor) massless four-momentum k_0 such that $\Lambda k_0 = e^\tau k_0$. Then, for every k, the sequence $e^{-2n\tau}\Lambda^{2n}k$ converges to a multiple of k_0. It follows by continuity and homogeneity that $f(k) = \lim_{n\to\infty} f(e^{-2n\tau}\Lambda^{2n}k) = f(k_0) =: f_0$. Thus, $f(k)$ is constant, hence for all k

$$(u'T_c(k)) = \frac{f_0}{(uk)}.$$

We now insert this into the inversion formula (A.24) (without loss of generality for $u = u_0$), contracted with u'. The integral is a multiple of the massless two-point function $W(re)$, hence a multiple of r^{-2}. It then follows that $c(e)(u'e)$ must be constant. Because $(u'e) = 0$ for $e \in u^\perp \cap u'^\perp$, the constant is zero, and $c(e)$ must be supported on the circle $H_1 \cap u^\perp \cap u'^\perp$. Since c is smooth on $H_1 \cap u^\perp$, this is impossible unless $c = 0$. Similarly for c'.

B Asymptotics

B.1 Dressed expectation values

The expectation value of the electromagnetic field in a dressed Dirac state proceeds by varying w.r.t. $f^{\mu\nu}$ the bosonic factor $\langle V_{qc}(y_1) : e^{iF_{\mu\nu}^u(f^{\mu\nu})} : (x) V_{qc}(y_2)^* \rangle$, as explained in section 3.3. It yields

$$\langle V_{qc}(y_1) F_{\mu\nu}^u(x) V_{qc}(y_2)^* \rangle / \langle V_q(y_1, c) V_q(y_2, c)^* \rangle = -iq(\langle F_{\mu\nu}^u(x) \phi(y_2, c) \rangle - \langle \phi(y_1, c) F_{\mu\nu}^u(x) \rangle).$$
(B.1)

The expectation values in the first factor on the right-hand side are those in the Krein vacuum, and can easily be computed in terms of the massless two-point function W_0:

$$\langle F_{\mu\nu}^u(x) \phi(y, e) \rangle = \langle F_{\mu\nu}(x) \phi(y, e) \rangle - (u_\mu \partial_\nu - u_\nu \partial_\mu) I_u \langle (\partial A^K)(x) \phi(y, e) \rangle =$$
$$= (e \wedge \partial_x)_{\mu\nu} I_{-e}^x W_0(x - y) + (u \wedge \partial_x)_{\mu\nu} I_u^x W_0(x - y).$$
(B.2)

Notice that $\partial^\mu (e \wedge \partial)_{\mu\nu} I_e W_0(x-y) = -\partial_\nu W_0(x-y)$ for every e, so that the total divergence of (B.2) is zero, in accord with the cancellation of the fictitious current in $F_{\mu\nu}^u$, cf. section 3.3.

Denoting by $\langle \, \cdot \, \rangle_c$ the smearing with $c(\vec{e})$ of total weight 1, we obtain from (B.1)

$$\langle V_q(y_1, c) F_{\mu\nu}^u(x) V_q(y_2, c)^* \rangle / \langle V_q(y_1, c) V_q(y_2, c)^* \rangle =$$
$$= -iq \Big\langle (e \wedge \partial_x)_{\mu\nu} I_{-e}^x + (u \wedge \partial_x)_{\mu\nu} I_u^x (W_0(x - y_2) - W_0(y_1 - x)) \Big\rangle_c.$$
(B.3)

B.2 Timelike and spacelike asymptotics

In section 3.4.2, we want to compute the asymptotic behaviour of (3.24) (in the standard frame $u = u_0$) in the spacelike and timelike directions $x_\lambda^\pm = x_0 + \lambda d_w^\pm$, where $d_w^\pm = (\pm 1, w\vec{n})$, $w \neq 1$.

For $w \neq 1$, we are again allowed to ignore x_0 and y_i in (3.24) in the limits $\lambda \to \infty$, and replace both $x_\lambda - y_i = x_0 + \lambda d_w^\pm - y_i$ by $z_\lambda = \lambda \cdot d_w^\pm$. Then, in (B.1), the differences of two-point functions may be replaced by the commutator:

$$-iq[A_\mu^u(x_\lambda), \phi(y, e)] = q(e_\mu I_{-e} + u_\mu I_u) C_0(z_\lambda) = \frac{\pm q}{2\pi} (e_\mu I_{-e} + u_\mu I_u) \delta(z_\lambda^2) + O(\lambda^{-3}), \quad \text{(B.4)}$$

We have used that $C_0(z_\lambda) = \frac{1}{2\pi} \delta(z_\lambda^2) \mathrm{sign}(z_\lambda^0) = \frac{\pm 1}{2\pi} \delta(z_\lambda^2)$ for $z_\lambda = \lambda d_w^\pm$. Hence

$$-iq[F_{\mu\nu}^u(x_\lambda), \phi(y, e)] = \frac{\pm q}{\pi} \Big((z_\lambda \wedge e)_{\mu\nu} I_{-e} + (z_\lambda \wedge u)_{\mu\nu} I_u \Big) \delta'(z_\lambda^2) + O(\lambda^{-3}). \quad \text{(B.5)}$$

The integrals in (B.5) are

$$I_{-e} \delta'(z_\lambda^2) = \frac{\nu}{4} \big((z_\lambda e)^2 + z_\lambda^2 \big)^{-\frac{3}{2}}, \qquad I_u \delta'(z_\lambda^2) = \frac{\nu'}{4} \big((z_\lambda u)^2 - z_\lambda^2 \big)^{-\frac{3}{2}},$$

where ν and ν' are the numbers of *positive* ones among the zeroes $s_\pm$ and $s'_\pm$ of the polynomials $(z_\lambda - se)^2$ and $(z_\lambda + s'u)^2$, respectively, cf. (A.10). ν and ν' are functions of $d_w^\pm$ and e, but independent of λ. One has $s_\pm = \lambda \cdot (-(d_w^\pm e) \pm \sqrt{(d_w^\pm e)^2 + d_w^{\pm 2}})$ and $s'_\pm = \lambda \cdot (-(d_w^\pm)^0 \pm \sqrt{1 - d_w^{\pm 2}})$. Since the zeroes may be positive or negative or complex, several cases have to be distinguished.

- On $\mathfrak{i}^+$ (future timelike, $0 < d_w^{\pm 2} \leq 1$), one has $\nu = 1$ and $\nu' = 0$.
- On $\mathfrak{i}^-$ (past timelike), one has $\nu = 1$ and $\nu' = 2$.
- On $\mathfrak{i}^0$ (spacelike, $w < 1$, $d_w^{\pm 2} < 0$), one has $\nu' = 1$. ν depends on the angle $\alpha = \angle(\vec{n}, \vec{e})$ of $\vec{e} \in S^2$ relative to $\vec{n} \in S^2$. One has $s_\pm = \lambda \cdot (w \cos \alpha \pm \sqrt{1 - w^2 \sin^2 \alpha})$, so that $\nu = 2$ on the "polar cap" $\alpha < \arcsin(w^{-1}) < \frac{\pi}{2}$, and $\nu = 0$ otherwise.

This yields finally, with ν and ν' as specified,

$$\lim_{\lambda \to \infty} \lambda^2 \langle F_{\mu\nu}^u(x_0 + \lambda d_w^\pm) \rangle_{f,c} = \frac{\pm q}{4\pi} \Big\langle (e \wedge d_w^\pm)_{\mu\nu} \frac{\nu}{\sqrt{1 - w^2 \sin^2 \alpha}^3} - (u \wedge d_w^\pm)_{\mu\nu} \frac{\nu'}{w^3} \Big\rangle_c. \quad \text{(B.6)}$$

B.3 Lightlike asymptotics

We want to compute the asymptotic behaviour of (3.24) (in the standard frame $u = u_0$) in the lightlike direction $x_\lambda^\pm = x_0 + \lambda \ell^\pm$, where $\ell^\pm = (\pm 1, \vec{n})$. The trajectories $x_\lambda^\pm$ reach the lightlike infinity $\mathfrak{I}^\pm$ at the Penrose points $(U, \vec{n})$ with $U = (\ell^+ x_0)$, resp. $(V, \vec{n})$ with $V = -(\ell^- x_0)$.

In the lightlike limit, the dependence of (3.24) on the "initial points" $x_0 - y_i$ ($i = 1, 2$) does not drop out, giving functions of $U - (\ell^+ y_i)$ on $\mathfrak{I}^+$ resp. $V + (\ell^- y_i)$ on $\mathfrak{I}^-$. Therefore, one should compute the differences of two-point functions in (3.24) rather than just commutators as in section B.2. We first investigate the expexted deviation, by inspection of (A.7) and (A.8).

The differences in (3.24) would require to consider expressions like

$$\log \frac{(z_1 e) + \sigma_{1,-}}{(z_2 e) + \sigma_{2,+}} - \log \frac{(z_1 e) - \sigma_{1,-}}{(z_2 e) - \sigma_{2,+}} \quad \text{and} \quad \log \frac{z_1^0 - i\varepsilon + \rho_1}{z_2^0 + i\varepsilon + \rho_2} - \log \frac{z_1^0 - i\varepsilon - \rho_1}{z_2^0 + i\varepsilon - \rho_2},$$

evaluated at $z_i = x_0 + \lambda \ell^\pm - y_i$, in the limit $\lambda \to \infty$. In the limit, the numerators and denominators of either one of the log's are small, so that the behaviour is dictated by the $i\varepsilon$ prescriptions. For $y_1 = y_2$, the log's would just be multiples of $2\pi i$ times Heaviside functions. The corrections for $y_1 \neq y_2$ can in principle be worked out from these expressions. We want to spare that labour, by just noting that a narrow smearing in y would essentially smoothen the step functions.

Keeping this in mind, we shall from now on assume a narrow smearing and set $y_1 = y_2$, so that we can compute the complex phases by returning to the commutators. However, in contrast to section B.2, we have to keep the dependence on $z_0 = x_0 - y$ in $z_\lambda^\pm = x_0 + \lambda \ell^\pm - y$.

For $x_\lambda = x_0 + \lambda \ell^+$, the Penrose coordinate is $V_\lambda = x_\lambda^0 + |\vec{x}_\lambda| \approx 2\lambda$. In order to compute the leading order $\vec{E}_1$ in (3.28), one may as well expand in λ^{-1} (which simplifies the computation). The difference will become effective only in the computation of $\vec{E}_2$. Similar for $x_\lambda = x_0 + \lambda \ell^-$.

Because $\mathrm{sign}((z_\lambda^\pm)^0)$ in the commutator function C_0 is constant $= \pm 1$ for sufficiently large λ,

$$f(z_\lambda^\pm, -e) - f(-z_\lambda^\pm, e) = -i(2\pi)^2 I_{-e} C_0(z_\lambda^\pm) = \mp 2\pi i \int_0^\infty ds\, \delta((z_\lambda^\pm - se)^2).$$

As in appendix B.2,

$$f(z_\lambda^\pm, -e) - f(-z_\lambda^\pm, e) = \mp 2\pi i \cdot \frac{\nu^\pm}{2\sqrt{(z_\lambda^\pm e)^2 + z_\lambda^{\pm 2}}},$$

where $\nu^\pm$ is the number of positive zeroes of $(z_\lambda^\pm - se)^2 = -s^2 - 2s(z_\lambda^\pm e) + z_\lambda^{\pm 2}$. For sufficiently large λ, $\nu^\pm = 2\theta(-(z_\lambda^\pm e)) + \mathrm{sign}((z_\lambda^\pm e))\theta(z_\lambda^{\pm 2})$. Thus,

$$f(z_\lambda^\pm, -e) - f(-z_\lambda^\pm, e) = \mp i\pi \cdot \frac{-2\theta(-(z_\lambda^\pm e)) + \theta(z_\lambda^{\pm 2})}{(z_\lambda^\pm e)\sqrt{1 + w_\lambda^\pm}}, \qquad \left(w_\lambda^\pm \equiv \frac{z_\lambda^{\pm 2}}{(z_\lambda^\pm e)^2} \right). \qquad (B.7)$$

Similarly, for sufficiently large λ,

$$f(z_\lambda^\pm, u) - f(-z_\lambda^\pm, -u) = \mp i\pi \cdot \frac{2\theta(-(z_\lambda^\pm)^0) + \theta(-z_\lambda^{\pm 2})}{(z_\lambda^\pm)^0\sqrt{1 - v_\lambda^\pm}}, \qquad \left(v_\lambda^\pm \equiv \frac{z_\lambda^2}{(z_\lambda^0)^2} \right). \qquad (B.8)$$

For the asymptotic treatment in leading order, notice that $(z_\lambda^\pm e) \approx \lambda(\ell^\pm e)$ and $(z_\lambda^\pm)^0 \approx \pm\lambda$ and $z_\lambda^{\pm 2} \approx 2(\ell^\pm z_0)$ are $O(\lambda)$, hence $w_\lambda^\pm$ and $v_\lambda^\pm$ are $O(\lambda^{-1})$. Thus, in the limit,

$$f(z_\lambda^\pm, -e) - f(-z_\lambda^\pm, e) \approx \frac{\pm i\pi}{\lambda} \cdot \frac{2\theta(-(\ell^\pm e)) - \theta((\ell^\pm z_0))}{(\ell^\pm e)} + O(\lambda^{-2}). \qquad (B.9)$$

Similarly

$$f(z_\lambda^\pm, u) - f(-z_\lambda^\pm, -u) \approx \frac{\mp i\pi}{\lambda} \cdot \left(2\theta(\mp 1) + \theta(-(\ell^\pm z_0)) \right) + O(\lambda^{-2}). \qquad (B.10)$$

We now compute the asymptotic expectation values of the electromagnetic field on $\mathfrak{I}^+$, $\ell \equiv \ell^+$, $x_\lambda = x_0 + \lambda\ell$, $z_\lambda = x_\lambda - y$. (B.9) and (B.10) yield the leading behaviour (order λ^{-1})

$$\langle F_\infty^u(U, \vec{n}) \rangle_{f,c} := \lim_{\lambda \to \infty} \lambda \cdot \langle F^u(x_\lambda) \rangle_{f,c} = \frac{q}{4\pi} \left\langle \frac{(\ell \wedge e)}{(\ell e)} - (\ell \wedge u) \right\rangle_c \delta(U - (\ell y)), \qquad (B.11)$$

where $U = (\ell x_0)$ and $\vec{n}$ are the corresponding coordinates on $\mathfrak{I}^+$. One reads off the expectation value (3.31) of the electric field. It is transverse (orthogonal to $\vec{n}$) as expected, see footnote 16.

The radial electric field decays like λ^{-2}. Its asymptotic expectation value $\langle E_{r,\infty}(U, \vec{n}) \rangle_{f,c} \equiv \langle (\vec{n} \cdot \vec{E}_\infty^{(2)}(U, \vec{n})) \rangle_{f,c}$ according to (3.28) is computed as

$$\langle E_{r,\infty}(U, \vec{n}) \rangle_{f,c} = \lim_{\lambda \to \infty} \frac{V_\lambda^2}{4} \cdot \langle (\vec{n}_\lambda \cdot \vec{E}(V_\lambda, U_\lambda, \vec{n}_\lambda)) \rangle_{f,c} \qquad \text{where} \quad \vec{n}_\lambda = \vec{n} + \lambda^{-1}(\vec{x} - (\vec{n} \cdot \vec{x})\vec{n}).$$

We have to study the subleading orders of (B.7) and (B.8). It is convenient to write $(\vec{n}_\lambda \cdot \vec{E})$ as $(u \wedge \ell_\lambda).F^u$, where $(a \wedge b).(c \wedge d) \equiv (ac)(bd) - (ad)(bc)$, and $\ell_\lambda = (1, \vec{n}_\lambda)$. This yields

$$\langle E_{r,\infty}(U,\vec{n})\rangle_{f,c} = \frac{-iq}{4\pi^2} \lim_{\lambda \to \infty} \frac{V_\lambda^2}{4} \cdot (u \wedge \ell_\lambda)$$
$$\cdot \Big\langle (e \wedge \partial)(f(z_\lambda, -e) - f(-z_\lambda, e)) + (u \wedge \partial)(f(z_\lambda, u) - f(-z_\lambda, -u)) \Big\rangle_c.$$

Inserting (B.7) and (B.8),

$$\langle E_{r,\infty}(U,\vec{n})\rangle_{f,c} = \frac{q}{4\pi} \lim_{\lambda \to \infty} \frac{V_\lambda^2}{4} \cdot (u \wedge \ell_\lambda) . \Big\langle (e \wedge \partial)\Big(\frac{2\theta(-(z_\lambda e)) - \theta(z_\lambda^2)}{(z_\lambda e)\sqrt{1 + \frac{z_\lambda^2}{(z_\lambda e)^2}}}\Big) + (u \wedge \partial)\Big(\frac{-\theta(-z_\lambda^2)}{z_\lambda^0 \sqrt{1 - \frac{z_\lambda^2}{z_\lambda^{0\,2}}}}\Big)\Big\rangle_c,$$

where $\frac{V_\lambda}{2} = \lambda + \frac{1}{2}(x_0^0 + (\vec{n}\vec{x}_0))$. The derivatives "see" only z_λ^2. Acting on the denominators of the two terms inside $\langle \ldots \rangle$, they are separately $O(\lambda^{-2})$. Their contribution to $\langle E_{r,\infty}\rangle_{f,c}$ is

$$\frac{q}{4\pi}\Big\langle \frac{2\theta((\vec{n}\cdot\vec{e})) - \theta(U - (y^0 - (\vec{n}\cdot\vec{y})))}{(\vec{n}\cdot\vec{e})^2} + \theta((y^0 - (\vec{n}\cdot\vec{y})) - U)\Big\rangle_c.$$

The contributions from the derivatives acting on the numerators are proportional to $\delta(z_\lambda^2)$ which is $O(\lambda^{-1})$. The coefficients are $O(1)$, but the two terms in $\langle \ldots \rangle_c$ cancel each other. The subleading terms contribute

$$\frac{q}{4\pi}\Big\langle \frac{(\vec{e}\cdot\vec{y})}{(\vec{n}\cdot\vec{e})} - (\vec{n}\cdot\vec{y})\Big\rangle_c \delta(U - (y^0 - (\vec{n}\cdot\vec{y}))) = \langle (\vec{y}\cdot\vec{E}_\infty(U,\vec{n}))\rangle_{f,c}.$$

Thus, with $(\ell y) = y^0 - (\vec{n}\cdot\vec{y})$, we obtain (3.33).

C Cloud propagator contributions

For the argument in section 4.4, we still have to compute the contributions to (4.6) due to the four diagrams with cloud propagators, two of which are depicted in figure 2. We compute the first depicted diagram. It equals

$$2 \cdot \frac{(iq)^2}{2}\langle T\psi_0(x')j^\mu(y_1)j^\nu(y_2)\overline{\psi}_0(x)\rangle \cdot a_\mu(y_1) \cdot iq\langle T\phi(x',c')A_\nu^K(y_2)\rangle,$$

where the brackets stand for $\langle T\psi(\cdot)\overline{\psi}(\cdot)\rangle = -iS_F(\cdot)$ and S_F is the Dirac Feynman propagator. The last factor is the first-order contribution to $\langle TV_{qc'}(x')A_\nu^K(y_2)\rangle$, i.e., the cloud propagator $-q\langle e'_\nu I_{e'} G_{0,F}(x' - y_2)\rangle_{c'}$ where $G_{0,F}$ is the massless scalar Feynman propagator. Inserting the Fourier representations, we get

$$(iq)^2 \cdot \int d^4y_1\, d^4y_2\, a_\mu(y_1)\int \frac{d^4p'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} e^{-ip'(x'-y_1)}e^{-iq(y_1-y_2)}e^{-ip(y_2-x)}.$$

$$\cdot (-i)^3 \frac{M + \slashed{p}'}{M^2 - p'^2 - i\varepsilon}\gamma^\mu \frac{M + \slashed{q}}{M^2 - q^2 - i\varepsilon}\gamma^\nu \frac{M + \slashed{p}}{M^2 - p^2 - i\varepsilon} \cdot iq \int \frac{d^4k}{(2\pi)^4}\Big\langle \frac{e'_\nu}{(ke')_-}\Big\rangle_{c'}\frac{e^{-ik(x'-y_2)}}{m^2 - k^2 - i\varepsilon}.$$

The y-integrations yield $(2\pi)^4\delta(q - p + k) \cdot \widehat{a}_\mu(p' - q)$. The remaining exponential factors $e^{-i(p'+k)x'}e^{ipx}$ become $e^{-ip'x'}e^{ipx}$ after a change of the integration variable p'. Now, we are

interested in the singular behaviour near $k = 0$ when p and p' are on-shell. Truncating with $(\Box^{(\prime)} + M^2)$ according to the LSZ prescription, cancels the denominator of the last Dirac propagator. The denominator of the first Dirac propagator (with the cloud vertex attached) is cancelled only up to $O(k)$. The (unknown) infraparticle truncation, that properly accounts for the absence of a sharp mass-shell, should justify to let $k \to 0$ in this term before going on-shell with p'. Then we get the coefficient of $e^{-ip'x'}e^{ipx}$:

$$q^3 \cdot \widehat{a}_\mu(p'-p) \int \frac{d^4k}{(2\pi)^4} \cdot \frac{M+\slashed{p}'-\slashed{k}}{(2\pi)^4} \gamma^\mu \frac{M+\slashed{p}-\slashed{k}}{M^2-(p-k)^2-i\varepsilon} \gamma^\nu \frac{M+\slashed{p}}{(2\pi)^4} \cdot \left\langle \frac{e'_\nu}{(ke')-i\varepsilon} \right\rangle_{c'} \cdot \frac{1}{m^2-k^2-i\varepsilon}.$$

With $\gamma^\nu(M + \slashed{p}) = 2p^\nu + (M - \slashed{p})\gamma^\nu$, the first factor under the integral becomes $\approx \frac{M+\slashed{p}'}{(2\pi)^4}\gamma^\mu \frac{2p^\nu}{2(pk)} \frac{M+\slashed{p}}{(2\pi)^4}$ plus finite terms $O(k^0)$. This allows to factor out the first-order vertex amplitude $\Gamma^{(1)}(p,p')$ of (4.7). The divergent coefficient has the real part

$$\text{Re}\left[q^2 \int \frac{d^4k}{(2\pi)^4} \frac{p^\nu}{(pk)_-} \left\langle \frac{e'_\nu}{(ke')_-} \right\rangle_{c'} \cdot \frac{i}{m^2-k^2-i\varepsilon}\right] = -\frac{q^2}{2}\text{Re}\left[\int \frac{d^4k}{(2\pi)^3} \frac{p^\nu}{(pk)} \left\langle \frac{e'_\nu}{(ke')_-} \right\rangle_{c'} \cdot \delta(k^2-m^2)\right].$$

(For this equality, we have used the symmetry of the propagator under $k \to -k$). As we think of this as the onset of an exponential whose phase does not matter, we ignore the imaginary part. Furthermore, we may replace $\frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2)$ by $2d\mu_m(k)$.

Proceeding similarly with the other three diagrams, we obtain the total coefficient

$$q^2\text{Re}\left[\int d\mu_m(k) \left(\frac{p^\nu}{(pk)} - \frac{p'^\nu}{(p'k)}\right)\left(\left\langle \frac{e_\nu}{(ke)_-} \right\rangle_c - \left\langle \frac{e'_\nu}{(ke')_-} \right\rangle_{c'}\right)\right] = -\frac{1}{2}(d_{m,v}(C,C_0)+d_{m,v}(C_0,C))$$
$$+\text{finite}$$

with $C = q(c - c')$ and $C_0 = q(c_u - c_{u'})$, $u = p/M$, $u' = p'/M$. This is the interference term between $-\frac{1}{2}d_{m,v}(C,C)$ and $-\frac{1}{2}d_{m,v}(C_0,C_0)$, anticipated in section 4.4.

References

[1] E. Abdalla, M.B. Abdalla and D. Rothe, *Non-perturbative methods in two-dimensional quantum field theory*, World Scientific, Singapore (1991).

[2] S. Alazzawi and W. Dybalski, *Compton scattering in the Buchholz-Roberts framework of relativistic QED*, *Lett. Math. Phys.* **107** (2017) 81 [arXiv:1509.03997] [INSPIRE].

[3] A. Ashtekar, *Asymptotic quantization*, Monographs and textbooks in physical science **2**, Bibliopolis, Naples, Italy (1987).

[4] A.W. Aste and G. Scharf, *Non-Abelian gauge theories as a consequence of perturbative quantum gauge invariance*, *Int. J. Mod. Phys. A* **14** (1999) 3421 [hep-th/9803011] [INSPIRE].

[5] V. Béaud, W. Dybalski and G.M. Graf, *Infraparticle states in the massless Nelson model — revisited*, arXiv:2105.05723 [INSPIRE].

[6] L. Bieri and D. Garfinkle, *An electromagnetic analogue of gravitational wave memory*, *Class. Quant. Grav.* **30** (2013) 195009 [arXiv:1307.5098] [INSPIRE].

[7] F. Bloch and A. Nordsieck, *Note on the radiation field of the electron*, *Phys. Rev.* **52** (1937) 54 [INSPIRE].

[8] N.N. Bogoliubov, A.A. Logunov, A.I. Oksak and I.T. Todorov, *General principles of quantum field theory*, Kluwer, Dordrecht, The Netherlands (1990).

[9] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the theory of quantized fields*, Wiley, New York, NY, U.S.A. (1959).

[10] H.-J. Borchers, *Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix* (in German), *Nuovo Cim.* **15** (1960) 784.

[11] T. Brüers, *Perturbative analysis of infrared dressing transformation of helicity* 2, master thesis, Göttingen University, Göttingen, Germany (2021).

[12] D. Buchholz, *Collision theory for massless bosons*, *Commun. Math. Phys.* **52** (1977) 147 [INSPIRE].

[13] D. Buchholz, *The physical state space of quantum electrodynamics*, *Commun. Math. Phys.* **85** (1982) 49 [INSPIRE].

[14] D. Buchholz, *Gauss' law and the infraparticle problem*, *Phys. Lett. B* **174** (1986) 331 [INSPIRE].

[15] D. Buchholz, S. Doplicher, G. Morchio, J.E. Roberts and F. Strocchi, *A model for charges of electromagnetic type*, hep-th/9705089 [INSPIRE].

[16] D. Buchholz, F. Ciolli, G. Ruzzi and E. Vasselli, *On string-localized potentials and gauge fields*, *Lett. Math. Phys.* **109** (2019) 2601 [arXiv:1904.10055] [INSPIRE].

[17] D. Buchholz and K. Fredenhagen, *Locality and the structure of particle states*, *Commun. Math. Phys.* **84** (1982) 1 [INSPIRE].

[18] D. Buchholz, M. Porrmann and U. Stein, *Dirac versus Wigner: towards a universal particle concept in local quantum field theory*, *Phys. Lett. B* **267** (1991) 377 [INSPIRE].

[19] D. Buchholz and J.E. Roberts, *New light on infrared problems: sectors, statistics, symmetries and spectrum*, *Commun. Math. Phys.* **330** (2014) 935 [arXiv:1304.2794] [INSPIRE].

[20] T. Chen, J. Fröhlich and A. Pizzo, *Infraparticle scattering states in non-relativistic QED. I. The Bloch-Nordsieck paradigm*, *Commun. Math. Phys.* **294** (2010) 761 [arXiv:0709.2493] [INSPIRE].

[21] T. Chen, J. Fröhlich and A. Pizzo, *Infraparticle scattering states in nonrelativistic quantum electrodynamics. II. Mass shell properties*, *J. Math. Phys.* **50** (2009) 012103 [arXiv:0709.2812].

[22] V. Chung, *Infrared divergence in quantum electrodynamics*, *Phys. Rev.* **140** (1965) B1110 [INSPIRE].

[23] P.A.M. Dirac, *Gauge-invariant formulation of quantum electrodynamics*, *Can. J. Phys.* **33** (1955) 650 [INSPIRE].

[24] P. Duch, *Massive QED*, unpublished notes, (2018).

[25] P. Duch, *Weak adiabatic limit in quantum field theories with massless particles*, *Annales Henri Poincaré* **19** (2018) 875 [arXiv:1801.10147] [INSPIRE].

[26] P. Duch, *Infrared problem in perturbative quantum field theory*, *Rev. Math. Phys.* **33** (2021) 2150032 [`arXiv:1906.00940`] [ɪɴSPIRE].

[27] M. Dütsch and K. Fredenhagen, *A local (perturbative) construction of observables in gauge theories: the example of QED*, *Commun. Math. Phys.* **203** (1999) 71 [`hep-th/9807078`] [ɪɴSPIRE].

[28] M. Dütsch, F. Krahe and G. Scharf, *Scalar QED revisited*, *Nuovo Cim. A* **106** (1993) 277 [ɪɴSPIRE].

[29] W. Dybalski, *From Faddeev-Kulish to LSZ. Towards a non-perturbative description of colliding electrons*, *Nucl. Phys. B* **925** (2017) 455 [`arXiv:1706.09057`] [ɪɴSPIRE].

[30] W. Dybalski and J. Mund, *Interacting massless infraparticles in* $1 + 1$ *dimensions*, `arXiv:2109.02128` [ɪɴSPIRE].

[31] W. Dybalski and A. Pizzo, *Coulomb scattering in the massless Nelson model IV. Atom-electron scattering*, `arXiv:1902.08799` [ɪɴSPIRE].

[32] W. Dybalski and B. Wegener, *Asymptotic charges, large gauge transformations and inequivalence of different gauges in external current QED*, *JHEP* **11** (2019) 126 [`arXiv:1907.06750`] [ɪɴSPIRE].

[33] H. Epstein and V. Glaser, *The role of locality in perturbation theory*, *Ann. Inst. H. Poincaré* **A 19** (1973) 211.

[34] P.P. Kulish and L.D. Faddeev, *Asymptotic conditions and infrared divergences in quantum electrodynamics*, *Theor. Math. Phys.* **4** (1970) 745 [*Teor. Mat. Fiz.* **4** (1970) 153] [ɪɴSPIRE].

[35] R. Ferrari, L.E. Picasso and F. Strocchi, *Some remarks on local operators in quantum electrodynamics*, *Commun. Math. Phys.* **35** (1974) 25 [ɪɴSPIRE].

[36] J. Fröhlich, G. Morchio and F. Strocchi, *Charged sectors and scattering states in quantum electrodynamics*, *Ann. Phys.* **119** (1979) 241.

[37] C. Gaß, *Renormalization in string-localized field theories: a microlocal analysis*, to appear in *Annales Henri Poincaré* (2022) [`arXiv:2107.12834`] [ɪɴSPIRE].

[38] C. Gaß, *Constructive aspects of string-localized quantum field theory*, Ph.D. thesis, Göttingen University, Göttingen, Germany (2022).

[39] C. Gaß, J.M. Gracia-Bondía and J. Mund, *Revisiting the Okubo-Marshak argument*, *Symmetry* **13** (2021) 1645.

[40] C. Gaß, K.-H. Rehren and F. Tippner, *On the spacetime structure of infrared divergencies in QED*, *Lett. Math. Phys.* **112** (2022) 37 [`arXiv:2109.10148`] [ɪɴSPIRE].

[41] R. Haag, *Local quantum physics*, Springer, Berlin, Heidelberg, Germany (1996).

[42] T. He, P. Mitra, A.P. Porfyriadis and A. Strominger, *New symmetries of massless QED*, *JHEP* **10** (2014) 112 [`arXiv:1407.3789`] [ɪɴSPIRE].

[43] A. Herdegen, *Long range effects in asymptotic fields and angular momentum of classical field electrodynamics*, *J. Math. Phys.* **36** (1995) 4044 [ɪɴSPIRE].

[44] A. Herdegen, *Infraparticle problem, asymptotic fields and Haag-Ruelle theory*, *Annales Henri Poincaré* **15** (2014) 345 [`arXiv:1210.1731`] [ɪɴSPIRE].

[45] A. Herdegen, *Asymptotic structure of electrodynamics revisited*, *Lett. Math. Phys.* **107** (2017) 1439 [`arXiv:1604.04170`] [ɪɴSPIRE].

[46] A.M. Jaffe, *High-energy behavior in quantum field theory. I. Strictly localizable fields*, *Phys. Rev.* **158** (1967) 1454 [InSPIRE].

[47] P. Jordan, *Zur Quantenelektrodynamik. III. Eichinvariante Quantelung und Diracsche Magnetpole* (in German), *Z. Phys.* **97** (1935) 535.

[48] J.M. Gracia-Bondía, J. Mund and J.C. Várilly, *The chirality theorem*, *Annales Henri Poincaré* **19** (2018) 843 [arXiv:1702.03383] [InSPIRE].

[49] D. Kapec, M. Pate and A. Strominger, *New symmetries of QED*, *Adv. Theor. Math. Phys.* **21** (2017) 1769 [arXiv:1506.02906] [InSPIRE].

[50] B.S. Kay, *Quantum electrostatics, Gauss's law, and a product picture for quantum electrodynamics; or, the temporal gauge revised*, *Found. Phys.* **52** (2022) 6 [arXiv:2003.07473] [InSPIRE].

[51] N.P. Landsman, *Algebraic theory of superselection sectors and the measurement problem in quantum mechanics*, *Int. J. Mod. Phys. A* **6** (1991) 5349 [InSPIRE].

[52] S. Mandelstam, *Quantum electrodynamics without potentials*, *Annals Phys.* **19** (1962) 1 [InSPIRE].

[53] G. Morchio and F. Strocchi, *A nonperturbative approach to the infrared problem in QED: construction of charged states*, *Nucl. Phys. B* **211** (1983) 471 [*Erratum ibid.* **232** (1984) 547] [InSPIRE].

[54] G. Morchio and F. Strocchi, *Charge density and electric charge in quantum electrodynamics*, *J. Math. Phys.* **44** (2003) 5569 [hep-th/0301111] [InSPIRE].

[55] J. Mund and E.T. Oliveira, *String-localized free vector and tensor potentials for massive particles with any spin: I. Bosons*, *Commun. Math. Phys.* **355** (2017) 1243 [arXiv:1609.01667] [InSPIRE].

[56] J. Mund, K.-H. Rehren and B. Schroer, *Relations between positivity, localization and degrees of freedom: the Weinberg-Witten theorem and the van Dam-Veltman-Zakharov discontinuity*, *Phys. Lett. B* **773** (2017) 625 [arXiv:1703.04408] [InSPIRE].

[57] J. Mund, K.-H. Rehren and B. Schroer, *Gauss' law and string-localized quantum field theory*, *JHEP* **01** (2020) 001 [arXiv:1906.09596] [InSPIRE].

[58] J. Mund, K.-H. Rehren and B. Schroer, *Infraparticle quantum fields and the formation of photon clouds*, arXiv:2109.10342 [InSPIRE].

[59] J. Mund, K.-H. Rehren and B. Schroer, *Infraparticle structure of matter coupled to gravity*, in preparation.

[60] J. Mund, K.-H. Rehren and B. Schroer, *How the Higgs potential got its shape*, in preparation.

[61] J. Mund, B. Schroer and J. Yngvason, *String-localized quantum fields and modular localization*, *Commun. Math. Phys.* **268** (2006) 621 [math-ph/0511042] [InSPIRE].

[62] A. Pizzo, *One-particle (improper) states in Nelson's massless model*, *Annales Henri Poincaré* **4** (2003) 439 [InSPIRE].

[63] A. Pizzo, *Scattering of an infraparticle: the one-particle sector in Nelson's massless model*, *Annales Henri Poincaré* **6** (2005) 553 [InSPIRE].

[64] K.-H. Rehren, *Pauli-Lubanski limit and stress-energy tensor for infinite-spin fields*, *JHEP* **11** (2017) 130 [arXiv:1709.04858] [InSPIRE].

[65] S. Ried, *Eigenschaften des Vertexoperators in der string-lokalisierten Formulierung der QED* (in German), bachelor thesis, Göttingen University, Göttingen, Germany (2021).

[66] H. Ruegg and M. Ruiz-Altaba, *The Stückelberg field*, *Int. J. Mod. Phys. A* **19** (2004) 3265 [`hep-th/0304245`] [INSPIRE].

[67] G. Scharf, *Quantum gauge theories: a true ghost story*, Wiley, New York, NY, U.S.A. (2001).

[68] B. Schroer, *Infraparticles in quantum field theory*, *Fortsch. Phys.* **11** (1963) 1 [INSPIRE].

[69] B. Schroer, *The role of positivity and causality in interactions involving higher spin*, *Nucl. Phys. B* **941** (2019) 91 [`arXiv:1712.02346`] [INSPIRE].

[70] O. Steinmann, *Perturbative QED in terms of gauge invariant fields*, *Ann. Phys.* **157** (1984) 232.

[71] A. Strominger, *Lectures on the infrared structure of gravity and gauge theory*, Princeton University Press, Princeton, NJ, U.S.A. (2018).

[72] S. Weinberg, *The quantum theory of fields*, Cambridge University Press, Cambridge, U.K. (1995).

[73] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516 [INSPIRE].

[74] R.F. Streater and A.S. Wightman, *PCT, spin and statistics and all that*, Benjamin, New York, NY, U.S.A. (1964).

[75] E.P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, *Annals Math.* **40** (1939) 149 [INSPIRE].

[76] D.R. Yennie, S.C. Frautschi and H. Suura, *The infrared divergence phenomena and high-energy processes*, *Annals Phys.* **13** (1961) 379 [INSPIRE].