Spiral Waves in Circular and Spherical Geometries

The Ginzburg-Landau Paradigm

Dissertation

zur Erlangung des Grades eines
Doktors der Naturwissenschaften

eingereicht am
Fachbereich Mathematik und Informatik
der Freien Universität Berlin

vorgelegt von

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Berlin, May 2017

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Datum der Disputation:
13. Juli 2017

Glory to God

耶穌對她說：復活在我，生命也在我。

約翰福音第十一章二十五節

Jesus said unto her, I am the resurrection, and the life.

John 11:25

Jesus sprach zu ihr: Ich bin die Auferstehung und das Leben.

Johannes 11:25

Acknowledgments

God loves me. He comforts me when I am lonely, confused, and struggling. He saves my whole family from the valley of the shadow of death.

I am deeply grateful to my advisor Prof. Bernold Fiedler. He reveals a beautiful world of mathematics. He believes in my ability to explore this world. He treats me as a father does, demanding and warm.

I am very grateful to Prof. Chen Chiun-Chuan (陳俊全教授) for giving me the honor of reading and evaluating my thesis. I am also grateful to Prof. Lin Shao-Shiung (林紹雄教授) for letting me know my advisor.

Many thanks go to all members of the Nonlinear Dynamics group: to my teachers Pavel Gurevich, Stefan Liebscher, and Martin Väth for sharpening my mathematical skills and broadening my academic horizon; to Bernhard Brehm for suggesting me study spiral waves on spheres; to Mark Curran, Phillipo Lappicy, and Isabelle Schneider for reading the manuscript and many suggestions on this thesis; to Mattias Bosewitz, Anna Karnauhova, Alejandro López, and Hannes Stuke for exchanging lots of ideas in mathematics. I am deeply indebted to Ulrike Geiger for assisting me with all bureaucratic issues and helping my family.

My research is funded by the Ministry of Education of Taiwan, the German Academic Exchange Service, the Collaborative Research Center 910 Project A4, and the Berlin Mathematical School.

During my research, I am supported by all memories in my homeland Taiwan, especially by the mountains and ocean. I would like to express profound gratitude to my parents and sister. I am extremely lucky that my beloved wife and daughter accompany me all the way of exploration. Glory to God!

Abstract

In this thesis we prove the existence of m-armed spiral wave solutions for the complex Ginzburg-Landau equation in the circular and spherical geometries. Instead of applying the shooting method in the literature, we establish a functional approach and generalize the known results of existence for rigidly-rotating spiral waves. Moreover, we prove the existence of two new patterns: frozen spirals in the circular and spherical geometries, and 2-tip spirals in the spherical geometry.

Contents

Chapter 1

Introduction

1.1 The paradigm of Ginzburg-Landau spiral waves

Pattern formation of spiral waves on two-dimensional manifolds has been observed in physiological, chemical, and biological models; see the pioneering investigations [WiRo46], [Be51], [ZhRo64] and [Wi72]. In physiology, spiral waves of electricity appear on heart tissues during cardiac arrhythmia and life-threatening fibrillation. In chemistry, the diffusive Belousov-Zhabotinsky reaction triggers intricate spiral waves on a petri dish; see the survey [Be&al97]. In biology, spiral waves arise during aggregation of slime mold via chemotactic movement; see [FaLe98].

From a mathematical point of view, spiral waves can be triggered by a mechanism composed of the *Turing instability* and *symmetry breaking*; see [Tu52], [GoSt03], and [Mu03]. The Turing instability is induced by a diffusion process, and mostly together with an *excitable* or *oscillatory* reaction kinetic. Hence *reaction-diffusion systems* are ideal mathematical models for studying spiral waves. The symmetry of a reaction-diffusion system is best described as *equivariance* with respect to certain group actions. Equivariance of the underlying reaction-diffusion system should be carefully distinguished from spatio-temporal symmetries of its solutions. For many more details see the surveys [FiSc03], [GoSt03], and [Fi&al07].

Concerning the mechanism that triggers spiral waves, however, few rigorous results are available. One of the popular models is the following *cubic supercritical Ginzburg-Landau equation* on $\mathbb{R}^2$:

$$\partial_t \Psi = \Delta_{\mathbb{R}^2} \Psi + (1 - |\Psi|^2 - i\,\beta\,|\Psi|^2)\,\Psi. \tag{1.1}$$

Here $\beta \in \mathbb{R}$ is a prescribed *kinetics parameter*. The significance of (1.1) is threefold. First, it plays a central role in the theory of nonlinear hydrodynamics and condensed matter physics; see [Pi06]. Second, it is a *normal form* for general parameter-dependent PDEs near the Hopf instability in reaction kinetics; see [Sc98] and [Mi02]. Third, the solutions reveal a beautiful world of spatio-temporal patterns; see [ArKr02].

One important feature of (1.1) is the *global S^1-equivariance*: Ψ is a solution of (1.1) if and only if $e^{i\vartheta}\,\Psi$ is also a solution, for each $\vartheta \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. This *global gauge symmetry* is closely related to the appearance of *rotating wave solutions*. More specifically, it allows one to pursue the following *m-armed spiral Ansatz*:

$$\Psi(t,s,\varphi) = e^{-i\Omega t}\Big(A(s)\,e^{ip(s)}\Big)e^{im\varphi}. \tag{1.2}$$

Here $m \in \mathbb{N}$ is fixed and $(s,\varphi) \in [0,\infty) \times [0,2\pi)$ denotes polar coordinates on $\mathbb{R}^2$; see [Co&al78], [Gr80], and [KoHo81]. Once a nontrivial solution of (1.1) in the form of (1.2) exists, its associated *spiral pattern* consists of a *tip* and a spiral-like *shape*. The tip is a jump discontinuity of the phase field $P(t,s,\varphi) := -\Omega t + p(s) + m\varphi$ and resides at $s = 0$. The shape is defined as the zero contour of the phase field on $\mathbb{R}^2$,

$$\big\{\big(s\cos(\varphi(t,s)), s\sin(\varphi(t,s))\big) : P(t,s,\varphi(t,s)) = 0\big\},$$

and it takes the form of a *rigidly-rotating spiral* for our Ansatz (1.2).

Let us substitute the spiral Ansatz (1.2) into (1.1). This results in an ODE system for the amplitude function $A(s)$ and the phase derivative $p'(s)$:

$$A'' + \frac{1}{s}A' - \frac{m^2}{s^2}A - A\,(p')^2 + (1-A^2)\,A = 0, \tag{1.3}$$

$$A\,p'' + 2\,A'\,p' + \frac{1}{s}A\,p' + (\Omega - \beta\,A^2)\,A = 0. \tag{1.4}$$

The rotation frequency $\Omega \in \mathbb{R}$ is an unknown parameter that we have to determine.

The approach developed in [Gr80] and [KoHo81] for solving the resulting ODE system (1.3–1.4) involves two steps that can be sketched as follows.

- ***Step 1: shooting method***

  Begin with the case $\beta = 0$. It follows that necessarily $\Omega = 0$ and $p'(s)$ is identically zero. Hence it suffices to solve a second-order ODE for $A(s)$:

  $$A'' + \frac{1}{s}A' - \frac{m^2}{s^2}A + (1-A^2)\,A = 0. \tag{1.5}$$

  Notice that the location $s = 0$ of the tip is a *singularity* of (1.5). By analyticity, any bounded nontrivial solution $A(s)$ of (1.5) satisfies the power series expansion

  $$A(s) = a_m s^m + O(s^{m+1}) \quad \text{as } s \to 0.$$

  The *shooting method* continues this solution globally, with $a_m \neq 0$ as the shooting parameter.

- ***Step 2: perturbation arguments***
  The cases $0 < |\beta| \ll 1$ can be treated by careful phase portrait analysis. In particular, it is shown that the shooting manifold and the center-unstable manifold of the trivial solution intersect transversely; see [KoHo81] Theorem 3.1. Therefore, their intersection, which produces m-armed spirals persists for small complex perturbations $-i\,\beta\,|\Psi|^2\,\Psi$ on the reaction kinetic of (1.1).

Motivated by the existence of spiral waves on $\mathbb{R}^2$, in [Pa&al94] the authors considered (1.1) on the unit disk $\mathcal{B}^2$ equipped with *Neumann boundary conditions*

$$\partial_t \Psi = \frac{1}{b}\Delta_{\mathcal{B}^2}\Psi + (1 - |\Psi|^2 - i\,\beta\,|\Psi|^2)\,\Psi,$$

and added a *diffusion parameter* $b > 0$. Based on numerical evidences, they conjectured that m-armed spiral waves exist for all

$$b > j_{0,m}^2.$$

Here $j_{0,m}$ is the first positive zero of the derivative of the Bessel function J_m. This conjecture was confirmed rigorously by the shooting method; see [Ts10].

We have now arrived at the entrance of the thesis. Let us raise the following questions.

- ***Question 1: other geometries of spatial domains?***
  Clearly, the spiral Ansatz (1.2) is based on polar coordinates. So, besides disks, do m-armed spiral waves exist when the spatial domain $\mathcal{M}$ is *any* two-dimensional compact manifold that admits polar coordinates?

- ***Question 2: other topologies?***
  The unit 2-sphere admits polar coordinates but it differs from disks, topologically, and by absence of any boundary. So, does topological structure affect the shape of spiral patterns?

- ***Question 3: other boundary conditions?***
  Besides Neumann boundary conditions, do m-armed spiral waves exist when $\mathcal{M}$ is equipped with Dirichlet boundary conditions or Robin boundary conditions? Do choices of boundary conditions affect the shape of spiral patterns near the boundary of $\mathcal{M}$?

- ***Question 4: complex diffusion parameter?***
  Let us introduce the *complex diffusion parameter* $\eta \in \mathbb{R}$ and consider

$$\partial_t \Psi = \frac{1}{b}(1 + i\,\eta)\Delta_{\mathcal{M}}\Psi + (1 - |\Psi|^2 - i\,\beta\,|\Psi|^2)\,\Psi. \tag{1.6}$$

  Do spiral wave solutions of (1.6) exist for all small parameters $0 < |\eta|,\ |\beta| \ll 1$? How does the rotation frequency Ω depend on choices of η and β?

- ***Question 5: justification of the m-armed spiral Ansatz?***
  Is the spiral Ansatz (1.2) just a smart guess? Or can one, instead, derive the spiral Ansatz from systematic considerations based on *equivariance* of (1.6)?

- ***Question 6: nodal solutions?***
  In the literature, so far, only positive solutions $A(s) > 0$ for $s > 0$ of (1.5) have been discussed; see [Gr80] and [KoHo81]. This restriction seems unnatural. So, can we find *nodal solutions*, that is, nontrivial solutions $A(s)$ that have zeros besides the location of tips?

1.2 Main goal: New functional approach

This thesis is devoted to establishing a new functional approach for pattern formation of Ginzburg-Landau spiral waves.

We accomplish this main goal and answer Questions 1–6. Each question, *except Question 6*, possesses a definite complete answer together with rigorous treatments. We formulate Question 6 as a conjecture and provide a plausible way to solve it.

We now give brief answers to each question. In this thesis we consider

$$\partial_t \Psi = \frac{1}{b}(1 + i\,\eta)\Delta_{\mathcal{M},(\alpha)}\Psi + (1 - |\Psi|^2 - i\,\beta\,|\Psi|^2)\,\Psi. \tag{1.7}$$

Here $\Delta_{\mathcal{M},(\alpha)}$ is the *Laplace-Beltrami operator* defined on an *admissible surface of revolution* $\mathcal{M}$:

$$\mathcal{M} = \big\{ \big(a(s)\cos(\varphi), a(s)\sin(\varphi), \tilde{a}(s)\big) : s \in [0, s_*],\ \varphi \in [0, 2\pi) \big\}.$$

Prototypical examples of $\mathcal{M}$ are the unit disk (for $a(s) = s$ and $\tilde{a}(s) = 0$) and the unit 2-sphere (for $a(s) = \sin(s)$ and $\tilde{a}(s) = \cos(s)$). If the boundary of $\mathcal{M}$, denoted by $\partial\mathcal{M}$, is nonempty, then we consider *Robin boundary conditions with the ratio* $\alpha \in [-\infty, 0]$, that is, $\partial_{\mathbf{n}}\Psi = \alpha\,\Psi$ on $\partial\mathcal{M}$. Note that $\alpha = 0$ stands for Neumann boundary conditions and $\alpha = -\infty$ stands for Dirichlet boundary conditions, symbolically. The ratio α is required to be nonpositive so that solutions do not grow at $\partial\mathcal{M}$.

For a nontrivial solution $\Psi = A\, e^{iP}$ of (1.7) in the form of the spiral Ansatz (1.2), a tip of Ψ is a *phase singularity*, that is, a jump discontinuity of the phase field P. Each tip is characterized by a nonzero *winding number* of Ψ. For $\mathcal{M}$ without boundary, we prove that generically the sum of winding numbers is zero. Hence the geometry of $\mathcal{M}$ affects the global shape of spiral patterns. For instance, the spherical geometry supports *2-tip spirals*.

Based on our symmetry perspective, we describe equivariance of (1.7) with respect to the following group action of the 2-torus $S^1 \times S^1$:

$$((\vartheta,\gamma)\cdot v)(s,\varphi) := e^{-i\vartheta} v(s,\varphi-\gamma) \tag{1.8}$$

for all $(\vartheta,\gamma) \in S^1 \times S^1$, $s \in [0, s_*]$, and $\varphi \in [0, 2\pi)$. A *relative equilibrium* of (1.7) is a time orbit that is contained in its group orbit under equivariance (1.8). We interpret an *Ansatz* as the form of solutions associated with a relative equilibrium. Then we justify the m-armed spiral Ansatz (1.2) by the variational structure possessed by (1.7) with $\eta = \beta = 0$. Moreover, due to the special form of the spiral Ansatz, we show that every spiral pattern hits boundary points in normal direction, regardless of which Robin boundary conditions are chosen.

Our functional approach views the spiral Ansatz (1.2) differently:

$$\Psi(t,s,\varphi) = e^{-i\Omega t}\,\psi(s,\varphi), \quad \psi(s,\varphi) := u(s)\, e^{im\varphi}. \tag{1.9}$$

Here the radial part $u(s)$ of ψ is complex valued. We adopt the L^2-functional setting for the Laplace-Beltrami operator:

$$\Delta_{\mathcal{M},(\alpha)} : H^2_{(\alpha)}(\mathcal{M},\mathbb{C}) \subset L^2(\mathcal{M},\mathbb{C}) \to L^2(\mathcal{M},\mathbb{C}),$$

with $H^2_{(\alpha)}(\mathcal{M},\mathbb{C})$ as its domain. The significance of considering (1.9) is that the closed L^2-subspace

$$L^2_m := \left\{\psi \in L^2(\mathcal{M},\mathbb{C}) : \psi(s,\varphi) = u(s)\, e^{im\varphi}\right\}$$

is *invariant* under $\Delta_{\mathcal{M},(\alpha)}$ and also under the nonlinearity of (1.7) due to the global S^1-equivariance. Hence the following restriction is well defined:

$$\Delta_{m,(\alpha)} := \Delta_{\mathcal{M},(\alpha)}\Big|_{H^2_{m,(\alpha)}} : H^2_{m,(\alpha)} \subset L^2_m \to L^2_m.$$

Here $H^2_{m,(\alpha)} := H^2_{(\alpha)} \cap L^2_m$ is a closed H^2-subspace. Let us substitute the spiral Ansatz (1.9) into (1.7). Then we obtain the following *elliptic equation* for ψ:

$$(1+i\,\eta)\,\Delta_{m,(\alpha)}\psi + i\,b\,\Omega\,\psi + b\,(1-|\psi|^2 - i\,\beta\,|\psi|^2)\,\psi = 0. \tag{1.10}$$

Here parameters $\eta \in \mathbb{R}, b > 0$, and $\beta \in \mathbb{R}$ are given and we have to determine the unknown rotation frequency $\Omega \in \mathbb{R}$. Moreover, once a solution pair (Ω, ψ) of (1.10) exists for some prescribed parameters η, b, and β, the following *frequency-parameter relation* holds:

$$\int_{\mathcal{M}} \Big(\Omega - \eta + (\eta - \beta)\, |\psi|^2 \Big) |\psi|^2 = 0. \tag{1.11}$$

This relation is easily derived by multiplication of (1.10) with the complex conjugate $\overline{\psi}$, integrating over $\mathcal{M}$, and sorting out the resulting real part and imaginary part. It is remarkable that we use the relation (1.11) to characterize parameter subregimes of spiral patterns.

Compared to the ODEs (1.3–1.4) for $A(s)$ and $p'(s)$ treated by the shooting method, ***our functional approach treats the rather standard elliptic equation*** (1.10). An immediate advantage is that once a *weak solution* $\psi \in H^2_{m,(\alpha)}$ of (1.10) exists, it becomes smooth by the well-known embedding theorems and the Schauder elliptic regularity theory; see [He81] and [GiTr83].

We expect nontrivial weak solutions of (1.10) that *bifurcate from the trivial solution* $\psi \equiv 0$. Why do we expect so? The reason is that $\Delta_{m,(\alpha)}$ shares its spectral property and nodal property of eigenfunctions with regular Sturm-Liouville operators: The spectrum of $\Delta_{m,(\alpha)}$ consists of *simple eigenvalues*,

$$0 > \mu_0^{m,(\alpha)} > \mu_1^{m,(\alpha)} > \ldots > \mu_n^{m,(\alpha)} > \ldots, \qquad \lim_{n \to \infty} \mu_n^{m,(\alpha)} = -\infty,$$

and each associated eigenfunction $\mathbf{e}_n^{m,(\alpha)}(s, \varphi) = u_n^{m,(\alpha)}(s)\, e^{im\varphi}$ satisfies a *nodal property*, that is, its radial part $u_n^{m,(\alpha)}(s)$ possesses exactly n simple zeros in $(0, s_*)$.

The proof based on our functional approach consists of three new steps.

- ***New Step 1: global bifurcation analysis***

  Begin with the case $\eta = \beta = 0$. The frequency-parameter relation (1.11) implies $\Omega = 0$, and thus it suffices to solve the equation

  $$\Delta_{m,(\alpha)}\psi + b\,(1 - |\psi|^2)\,\psi = 0. \tag{1.12}$$

  The unknown $\psi \in H^2_{m,(\alpha)}$ is of the form $\psi(s, \varphi) = u_{\mathcal{R}}(s)\, e^{im\varphi}$ where the radial part $u_{\mathcal{R}}$ is *real valued*. Because the spectrum of $\Delta_{m,(\alpha)}$ consists of simple eigenvalues, nontrivial solutions of (1.12) near the bifurcation point $b = -\mu_n^{m,(\alpha)} > 0$ form a unique local bifurcation curve $\mathcal{C}_n^{m,(\alpha)}$, for each $n \in \mathbb{N}_0$; see [CrRa71]. Moreover, the radial part of any nontrivial solution of $\mathcal{C}_n^{m,(\alpha)}$ possesses exactly n simple zeros. This *nodal structure* of bifurcation curves is very crucial to show that $\mathcal{C}_n^{m,(\alpha)}$ is possibly *global*, in the sense that it exists for all diffusion parameters $b > -\mu_n^{m,(\alpha)}$.

However, in this thesis we can only prove that the *principal bifurcation curve* $\mathcal{C}_0^{m,(\alpha)}$ is global in the above sense. It remains unsolved whether other bifurcation curves $\mathcal{C}_n^{m,(\alpha)}$ for $n \in \mathbb{N}$, which contain nodal solutions of (1.12), are global.

- ***New Step 2: perturbation arguments***
  We prove that the principal bifurcation curve $\mathcal{C}_0^{m,(\alpha)}$ persists for all small parameters $0 < |\eta|, |\beta| \ll 1$ by the equivariant implicit function theorem of [RePe98].

- ***New Step 3: determination on the types of pattern***
  For each solution proved in Step 2, we determine whether it exhibits a spiral pattern by the frequency-parameter relation (1.11).

We briefly state our main theorem, which generalizes the results in [Ts10] for Ginzburg-Landau spiral waves.

Theorem. *For all diffusion parameters $b > -\mu_0^{m,(\alpha)}$, the cubic supercritical Ginzburg-Landau equation* (1.7) *possesses m-armed frozen or rigidly-rotating spiral wave solutions. Moreover, in the spherical geometry every spiral wave exhibits a 2-tip spiral.*

It is worth noting that the existence of *frozen spirals* and *2-tip spirals* is new. The rotation frequency Ω can be zero because it depends on *both* parameters η and β; see the frequency-parameter relation (1.11). This explains why the literature could not find frozen spirals because the complex diffusion parameter η was not introduced.

1.3 Outline

The essence of the thesis is already contained in the sections above. Readers who wish to see more details about the main results or understand why our functional approach works can hop to Chapter 4 directly. Readers who like a taste of establishing our whole mathematical stage are welcome to read the thesis from beginning to end.

In Chapter 2 we establish the basic mathematical setting. Although we consider the general Ginzburg-Landau equation on admissible surfaces of revolution, readers may keep in mind that the cubic supercritical case (1.7) on a disk or on a 2-sphere is the main example that captures the essence of our analysis. The key result is that $\Delta_{m,(\alpha)}$ shares its spectral property and nodal property of eigenfunctions with regular Sturm-Liouville operators. We also prove the existence and uniqueness of solutions for the initial value problem. We are interested, however, in pattern-forming solutions. For example, we show that the real Ginzburg-Landau equation possesses a variational structure, which is crucial to describe the m-armed spirals.

In Chapter 3 we define spiral patterns and we design spiral Ansätze. Our first aim is to derive the m-armed spiral Ansatz (1.2) based on our symmetry perspective. Then we establish a functional setting for solving the resulting elliptic equation. We next obtain several consequences, once a nontrivial solution exists. For instance, zeros of the solution neither accumulate at tips nor at boundary points; its phase derivative possesses a continuous extension to tips and simple zeros; its associated pattern hits boundary points in normal direction. In the end we obtain a simple criterion that determines whether a solution exhibits a spiral pattern.

In Chapter 4 we follow the three steps of Section 1.2 to prove our main results: the existence of Ginzburg-Landau spiral waves. We also provide a plausible way for finding nodal solutions.

In Chapter 5 we conclude with an overview and we discuss several directions for future research.

Chapter 2

Basic mathematical setting

In this chapter we establish the basic mathematical setting. In Section 2.1 we introduce the Ginzburg-Landau equation and propose two assumptions on its nonlinearity for the existence of spiral wave solutions. In Section 2.2 we conduct linear analysis based on the L^2-spectral decomposition of the Laplace-Beltrami operator. In Section 2.3 we study the Laplace-Beltrami operator on surfaces of revolution that intersect the axis of rotation, which are spatial domains for our spirals. In Section 2.4 we solve the initial value problem of the Ginzburg-Landau equation. In Section 2.5 we show that the real Ginzburg-Landau equation possesses a variational structure.

2.1 Global S^1-equivariance

2.1.1 Ginzburg-Landau equation

Consider the ***Ginzburg-Landau equation*** (***GLe*** for abbreviation):

$$\partial_t \Psi = \frac{1}{b}(1 + i\,\eta)\,\Delta_{\mathcal{M},(\alpha)}\Psi + f(|\Psi|^2;\beta)\,\Psi, \tag{2.1}$$

where $\Psi(t,x) \in \mathbb{C}$ for $(t,x) \in [0,\infty) \times \mathcal{M}$. (2.1) is a complex-valued scalar semilinear parabolic equation and possesses ***global*** S^1***-equivariance***: Ψ is a solution if and only if the phase shift $e^{i\vartheta}\,\Psi$ is a solution, for each $\vartheta \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. We specify the spatial domain $\mathcal{M}$ of (2.1) as an admissible manifold and keep in mind that disks and 2-spheres are main examples.

Definition. *A two-dimensional real analytic manifold $\mathcal{M}$ is called an **admissible manifold** if it is compact, connected, and oriented. Its boundary $\partial\mathcal{M}$, if being nonempty, is a one-dimensional real analytic submanifold.*

Remark. We require real analyticity of $\mathcal{M}$ in the proof of Lemma 2.4 (ii) as we apply the Frobenius series, only.

In fact, due to the Whitney embedding theorem, an admissible manifold $\mathcal{M}$ is a hypersurface in $\mathbb{R}^3$ and thus a Riemannian manifold. A volume form dVol of $\mathcal{M}$ is globally defined as an orientation is chosen.

We explain objects involved in (2.1).

- $b > 0$ is a manipulable ***bifurcation parameter***. We use the reciprocal of b for simplicity of notation in the proof of the main theorems.

- $\eta \in \mathbb{R}$ is a given ***complex diffusion parameter***.

- $\Delta_{\mathcal{M},(\alpha)}$ is the ***Laplace-Beltrami operator*** on $\mathcal{M}$. If $\partial\mathcal{M}$ is nonempty, then we consider Robin boundary conditions with the ratio $\alpha \in [-\infty, 0]$, that is, $\partial_{\mathbf{n}}\Psi = \alpha\,\Psi$ on $\partial\mathcal{M}$.

- $f \in C^3([0,\infty)\times\mathbb{R}^d, \mathbb{C})$ with $d \in \mathbb{N}$ is a complex-valued nonlinearity that depends on given ***kinetics parameters*** $\beta = (\beta_j)_{j=1}^d \in \mathbb{R}^d$. Note that C^3-smoothness of f is required to determine the shape of local bifurcation curves.

Remark ($\lambda - \omega$ system)**.** For a more general nonlinearity $f = f(|\Psi|;\beta)$, which still possesses the global S^1-equivariance, (2.1) is called the $\lambda - \omega$ system; see [KoHo73]. However, we tend not to solve the $\lambda - \omega$ system because the nonlinearity is not real Fréchet differentiable at $\Psi = 0$.

2.1.2 Two assumptions on nonlinearity

We seek spiral wave solutions that bifurcate from the zero equilibrium $\Psi = 0$. For this purpose, we require two assumptions on the complex-valued nonlinearity $f = f_{\mathcal{R}} + i\, f_{\mathcal{I}}$, where $f_{\mathcal{R}}$ and $f_{\mathcal{I}}$ are real valued.

(A1) $f_{\mathcal{R}}(0;\mathbf{0}) = 1$, and there exists a constant $C(f_{\mathcal{R}}) > 0$ that only depends on choices of $f_{\mathcal{R}}(\cdot\,;\mathbf{0})$ such that

$$f_{\mathcal{R}}(y;\mathbf{0}) \begin{cases} = 0, & y = C(f_{\mathcal{R}}), \\ < 0, & y > C(f_{\mathcal{R}}). \end{cases}$$

Moreover, we assume

$$f_{\mathcal{I}}(y;\mathbf{0}) = 0 \quad \text{for all } y \geq 0. \tag{2.2}$$

(A2) $\partial_y f_{\mathcal{R}}(0;\mathbf{0}) < 0$ and $\partial_y f_{\mathcal{R}}(y;\mathbf{0}) \leq 0$ for all $y \in (0, C(f_{\mathcal{R}}))$.

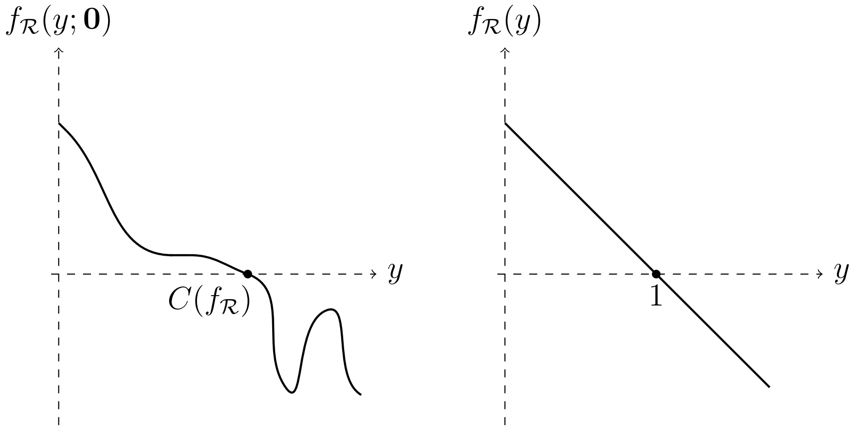

Figure: The graph of $f_{\mathcal{R}}(y;\mathbf{0})$ that satisfies (A1–A2). *Left*: a typical choice of $f_{\mathcal{R}}(y;\mathbf{0})$ for the general GLe. *Right*: $f_{\mathcal{R}}(y) = 1 - y$ for the cubic supercritical GLe.

We explain the meaning of the assumptions. First, although $f_{\mathcal{R}}(0;\mathbf{0}) > 0$ is sufficient to yield linear instability of the zero equilibrium, we can assume $f_{\mathcal{R}}(0;\mathbf{0}) = 1$ without loss of generality, possibly after a rescaling of time and a transformation of the unknown Ψ. Second, if $\eta = 0$ and $\beta = \mathbf{0}$, then the GLe possesses real coefficients by (2.2). Hence we classify the GLe as follows.

Definition. *Under* (A1) *the Ginzburg-Landau equation* (2.1) *is called to be* ***real*** *if* $\eta = 0$ *and* $\beta = \mathbf{0}$*, or* ***complex*** *if* $\eta \neq 0$ *or* $\beta \neq \mathbf{0}$*.*

The real GLe reads

$$\partial_t \Psi = \frac{1}{b}\Delta_{\mathcal{M},(\alpha)}\Psi + f_{\mathcal{R}}(|\Psi|^2;\mathbf{0})\,\Psi. \tag{2.3}$$

Notice that the unknown Ψ is still complex valued. Third, as we consider the diffusion-free ODE of (2.3),

$$D_t\Phi = f_{\mathcal{R}}(|\Phi|^2;\mathbf{0})\,\Phi, \tag{2.4}$$

and apply the polar form $\Phi(t) = R(t)\,e^{i\Theta(t)}$, nontrivial solutions satisfy the following *decoupled* ODE system:

$$D_t R = f_{\mathcal{R}}(R^2;\mathbf{0})\,R,$$
$$D_t\Theta = 0.$$

Thus we interpret that the GLe is *weakly coupled* for all parameters $0 < |\eta|,\ |\beta| \ll 1$. Last, (A1–A2) admit a stable limit cycle of (2.4) that connects the unstable zero equilibrium and the stable circle of equilibria $\{z \in \mathbb{C} : |z| = \sqrt{C(f_{\mathcal{R}})}\}$.

Remark. We see in Chapter 4 that (A1) yields a C^0-bound for solutions of the real GLe. This bound is crucial to obtain *global* bifurcation curves of nontrivial solutions. We require (A2) for our perturbation arguments to solve the complex GLe.

Example. The main example is the ***cubic supercritical GLe*** with the nonlinearity

$$f(y;\beta) = 1 - y - i\,\beta\, y, \quad \beta \in \mathbb{R}.$$

Thus $f_{\mathcal{R}}(y) = 1 - y$, and the minus sign stands for supercriticity. Clearly, $C(f_{\mathcal{R}}) = 1$ and (A1–A2) are fulfilled.

2.2 Linear analysis

We conduct linear analysis on the GLe by studying the *Laplace-Beltrami operator*. Due to compactness of admissible manifolds, the Laplace-Beltrami operator yields an L^2*-spectral decomposition*, which is significant for our bifurcation analysis.

2.2.1 Laplace-Beltrami operator

To define the Laplace-Beltrami operator, the coordinates-free approach constructs the Hodge star operator (see [Ro97] Chapter 1), while the coordinates-dependent approach relies on choices of local coordinates (see [Ec04] Appendix A). Both approaches define the Laplace-Beltrami operator as composition of the divergence operator and the gradient operator. We adopt the coordinates-dependent approach because we need polar coordinates to solve differential equations related to spiral wave solutions.

Let $\mathcal{M}$ be an admissible manifold locally parametrized by a smooth mapping

$$E : D \to \mathcal{M}, \quad x = E(p) = E(p_1, p_2).$$

Here D is a subset of $\mathbb{R}^2$ with bounded interior. The vectors $\partial_{p_1} E(p)$ and $\partial_{p_2} E(p)$ form a basis of the tangent space of $\mathcal{M}$ at $x = E(p)$, denoted by $T_x\mathcal{M}$. The Riemannian metric g of $\mathcal{M}$ is given by

$$g(p) := (g_{jk}(p)) = \Big(\big\langle \partial_{p_j} E(p), \partial_{p_k} E(p) \big\rangle_{\mathbb{R}^3} \Big), \quad j,k = 1,2,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ is the standard dot product of $\mathbb{R}^3$. The volume of the parallelepiped spanned by the tangent vectors is equal to $\sqrt{\det(g(p))}$. Thus the volume form of $\mathcal{M}$ induces the inner product

$$\langle v_1, v_2 \rangle_g := \int_D v_1(E(p))\, \overline{v_2(E(p))}\, \sqrt{\det(g(p))}\, \mathrm{d}p. \tag{2.5}$$

Let $T_x^*\mathcal{M}$ denote the dual space of $T_x\mathcal{M}$, and $\Lambda^1(T_x^*\mathcal{M})$ denote the space of smooth 1-forms on $T_x^*\mathcal{M}$. The gradient operator is by definition the exterior derivative

$\nabla_g : C^\infty(\mathcal{M}, \mathbb{C}) \to \Lambda^1(T_x^*\mathcal{M})$. Since $T_x^*\mathcal{M}$ is isomorphic to $T_x\mathcal{M}$ as vector spaces, as we express a cotangent vector in $T_x^*\mathcal{M}$ with respect to the basis of $T_x\mathcal{M}$, the inverse matrix $(g^{jk}(p))$ of $g(p)$ appears. It follows that the gradient operator is given by

$$\nabla_g v(p) = \sum_{j,k=1}^{2} g^{jk}(p)\,\partial_{p_j} v(E(p))\,\partial_{p_k} E(p).$$

The divergence operator is the adjoint of the gradient operator with respect to (2.5). For a cotangent vector $w(p) = \sum_{j=1}^{2} w_j(p)\, g^{jk}(p)\,\partial_{p_k} E(p)$, the divergence operator is given by

$$\mathrm{div}_g w(p) = \frac{1}{\sqrt{\det(g(p))}} \sum_{j,k=1}^{2} \partial_{p_k}\left(\sqrt{\det(g(p))}\, g^{jk}(p)\, w_j(p)\right).$$

It can be verified that ∇_g and div_g are independent of choices of parametrizations. We now define the ***Laplace-Beltrami operator*** on $\mathcal{M}$ by

$$\begin{aligned}\Delta_{\mathcal{M}} v(p) &:= \mathrm{div}_g(\nabla_g v)(p)\\ &= \frac{1}{\sqrt{\det(g(p))}} \sum_{j,k=1}^{2} \partial_{p_k}\left(\sqrt{\det(g(p))}\, g^{jk}(p)\,\partial_{p_j} v(p)\right).\end{aligned} \tag{2.6}$$

2.2.2 Embedding theorems

We define Sobolev spaces $H^q(\mathcal{M}, \mathbb{C})$ for $q \geq 0$ and collect the embedding theorems, which reduce the problem of proving smooth solutions to finding weak solutions.

Sobolev spaces and standard (e.g., Sobolev, Rellich-Kondarachov) embedding theorems have been established for the Euclidean space; see [He81] and [GiTr83]. Since regularity of a function is a local property, and locally a smooth manifold is diffeomorphic to a bounded set in the Euclidean space, the embedding theorems can be generalized for admissible manifolds by using a partition of unity.

Remark (argument using a partition of unity)**.** Since $\mathcal{M}$ is compact, it is covered by finitely many coordinates charts $\{D_j, \zeta_j\}_{j=1}^{\ell}$. Here $\zeta_j : D_j \to \mathcal{M}$ and D_j is a subset of $\mathbb{R}^2$ with bounded interior. Let $\{P_k\}_{k\in K}$ be a partition of unity subordinated to the covering $\{D_j\}_{j=1}^{\ell}$. Then any statement regarding regularity of a function $v : \mathcal{M} \to \mathbb{C}$ is reduced to regularity of the function

$$\sum_{k\in K} P_k(\cdot)\, v(\zeta_j(\cdot)) : D_j \to \mathbb{C},$$

for each $j \in \{1, 2, ..., \ell\}$. Thus we can apply the known embedding theorems for the Euclidean space.

It is known that the Sobolev space $H^q(\mathbb{R}^2, \mathbb{C})$ is completion of smooth functions with compact support under the norm

$$|v|^2_{H^q(\mathbb{R}^2)} := \int_{\mathbb{R}^2} |\hat{v}(\sigma)|^2 \, (1 + |\sigma|^2)^q \, \mathrm{d}\sigma.$$

Here $\hat{v}(\sigma)$ is the Fourier transform defined by

$$\hat{v}(\sigma) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\cdot\sigma} \, v(x) \, \mathrm{d}x.$$

To define the Sobolev space $H^q(\mathcal{M}, \mathbb{C})$ on $\mathcal{M}$, using a partition of unity it suffices to consider $\mathcal{M}$ as a subset D of $\mathbb{R}^2$ with bounded interior. It is known that the Sobolev space on D is defined as

$$H^q(D, \mathbb{C}) := \big\{ v : D \to \mathbb{C} : \text{ there exists a } \tilde{v} \in H^q(\mathbb{R}^2, \mathbb{C}) \text{ such that } \tilde{v}\big|_D = v \big\}$$

equipped with the norm

$$|v|_{H^q(D)} := \inf \big\{ |\tilde{v}|_{H^q(\mathbb{R}^2)} : \tilde{v} \in H^q(\mathbb{R}^2, \mathbb{C}), \, \tilde{v}\big|_D = v \big\}.$$

It can be verified that the above norm is independent of choices of partitions of unity.

We collect embedding theorems after introducing the following concepts.

Definition. *Let X and Y be Banach spaces and $\mathcal{F} : Y \to X$ be a (typically nonlinear) mapping. $\mathcal{F}$ is called to be **bounded** (or **compact**) if $\mathcal{F}$ maps each bounded set in Y into a bounded (or precompact) set in X.*

Definition. *Let X and Y be Banach spaces. Y is called to be **continuously** (or **compactly**) **embedded** into X if $Y \subset X$ and the inclusion operator $\mathbb{E} : Y \to X$, $\mathbb{E}w = w$, is bounded (or compact).*

Lemma 2.1 (embedding theorems). *Let $\mathcal{M}$ be an admissible manifold. Then the following statements hold:*

(i) *$H^{q_1}(\mathcal{M}, \mathbb{C})$ is continuously embedded into $H^{q_2}(\mathcal{M}, \mathbb{C})$ if $q_1 \geq q_2 \geq 0$. The embedding is compact if $q_1 > q_2 \geq 0$.*

(ii) *If $q - 1 > k + \nu$ for some $k \in \mathbb{N}_0$ and $\nu \in (0, 1]$, then $H^q(\mathcal{M}, \mathbb{C})$ is continuously embedded into $C^{k,\nu}(\mathcal{M}, \mathbb{C})$. In particular, $H^2(\mathcal{M}, \mathbb{C})$ is continuously embedded into $C^{0,\nu}(\mathcal{M}, \mathbb{C})$ for any $\nu \in (0, 1)$.*

Proof. See [Ro97] Theorem 1.20 and Theorem 1.22. $\bowtie$

2.2.3 L^2-spectral decomposition

Let $L^2(\mathcal{M},\mathbb{C}) := H^0(\mathcal{M},\mathbb{C})$. Consider the following L^2-functional setting for the Laplace-Beltrami operator:

$$\Delta_{\mathcal{M},(\alpha)} : \mathcal{D}(\Delta_{\mathcal{M},(\alpha)}) \subset L^2(\mathcal{M},\mathbb{C}) \to L^2(\mathcal{M},\mathbb{C}).$$

Here $\mathcal{D}(\Delta_{\mathcal{M},(\alpha)})$ denotes the domain of $\Delta_{\mathcal{M},(\alpha)}$, which is

- $H^2(\mathcal{M},\mathbb{C})$ if $\partial\mathcal{M}$ is empty;

or if $\partial\mathcal{M}$ is nonempty, then we consider *Robin boundary conditions with the ratio* α:

- $H^2_\alpha(\mathcal{M},\mathbb{C}) := \{v \in H^2(\mathcal{M},\mathbb{C}) : \partial_{\mathbf{n}} v = \alpha\, v \text{ on } \partial\mathcal{M}\}$.

Here $\mathbf{n}$ is the unit outer normal vector field on $\partial\mathcal{M}$; the normal derivative is defined by $\partial_{\mathbf{n}} v = \nabla_g v \cdot \mathbf{n}$. Throughout this thesis we assume

$$\alpha \in [-\infty, 0].$$

Note that Robin boundary conditions become Neumann boundary conditions if $\alpha = 0$.

Notation. Symbolically $\alpha = -\infty$ stands for Dirichlet boundary conditions.

Notation. We use the notations $\Delta_{\mathcal{M},(\alpha)}$ and $H^2_{(\alpha)}(\mathcal{M},\mathbb{C})$ for the two cases:

- $\Delta_{\mathcal{M},(\alpha)} = \Delta_{\mathcal{M}}$ and $H^2_{(\alpha)}(\mathcal{M},\mathbb{C}) = H^2(\mathcal{M},\mathbb{C})$ if $\partial\mathcal{M}$ is empty;
- $\Delta_{\mathcal{M},(\alpha)} = \Delta_{\mathcal{M},\alpha}$ and $H^2_{(\alpha)}(\mathcal{M},\mathbb{C}) = H^2_\alpha(\mathcal{M},\mathbb{C})$ if we consider Robin boundary conditions with the ratio $\alpha \in [-\infty, 0]$.

We need the following concepts to illustrate the L^2-spectral structure of $\Delta_{\mathcal{M},(\alpha)}$.

Definition. *Let X be a Banach space and $\mathcal{L} : X \to X$ be a linear operator.*

- *The **resolvent set of** $\mathcal{L}$ is defined as the set*

$$\rho(\mathcal{L}) := \big\{\lambda \in \mathbb{C} : \mathcal{L} - \lambda \text{ is bijective and its inverse is bounded}\big\}.$$

- *The **spectrum of** $\mathcal{L}$ is defined as $\sigma(\mathcal{L}) := \mathbb{C} \setminus \rho(\mathcal{L})$.*

- *The **point spectrum of** $\mathcal{L}$ is defined as the set*

$$\sigma_p(\mathcal{L}) := \{\lambda \in \mathbb{C} : \mathcal{L} - \lambda \text{ is not injective}\}.$$

*Elements of $\sigma_p(\mathcal{L})$ are called **eigenvalues**. The **geometric multiplicity** of an eigenvalue λ is defined by the dimension of* $\ker(\mathcal{L} - \lambda)$. *The **algebraic multiplicity** of λ is defined by*

$$\dim\left(\bigcup_{j=1}^{\infty} \ker(\mathcal{L} - \lambda)^j\right).$$

*λ is called to be **simple** if its algebraic multiplicity is one.*

- *The **essential spectrum of** $\mathcal{L}$ is defined as $\sigma(\mathcal{L}) \setminus \sigma_p(\mathcal{L})$.*

Definition. *Let X be a Banach space and $\mathcal{L} : X \to X$ be a linear operator. $\mathcal{L}$ is called to **have compact resolvent** if for all $\lambda \in \rho(\mathcal{L})$, the resolvent operator $(\mathcal{L} - \lambda)^{-1} : X \to X$ is compact.*

We now collect the well-known L^2-spectral decomposition of the Laplace-Beltrami operator. The motivation is to solve the Schrödinger equation on 2-spheres, which models motion of an electron in the hydrogen atom; see [Ki08] Section 4.9.

Lemma 2.2 (L^2-spectral decomposition)**.** *Let $\mathcal{M}$ be an admissible manifold. Then the following statements hold:*

(i) $\Delta_{\mathcal{M},(\alpha)}$ *is self-adjoint on* $L^2(\mathcal{M}, \mathbb{C})$ *and has compact resolvent.*

(ii) *The spectrum of $\Delta_{\mathcal{M},(\alpha)}$ consists of eigenvalues of finite multiplicity. Counting multiplicity all eigenvalues can be listed as*

$$0 \geq \lambda_0^{(\alpha)} > \lambda_1^{(\alpha)} \geq \lambda_2^{(\alpha)} \geq ... \geq \lambda_n^{(\alpha)} \geq ..., \quad \lim_{n\to\infty} \lambda_n^{(\alpha)} = -\infty.$$

In particular, the essential spectrum of $\Delta_{\mathcal{M},(\alpha)}$ is empty.

Remark.

(i) Since $\Delta_{\mathcal{M},(\alpha)}$ is self-adjoint, the geometric multiplicity and the algebraic multiplicity of each eigenvalue are equal. Thus *multiplicity* results in no ambiguity.

(ii) The L^2-spectral decomposition may not exist if $\mathcal{M}$ is not compact. For instance, the spectrum of the Laplace operator on $\mathbb{R}^2$ consists of essential spectrum, only. We emphasize that absence of the essential spectrum of $\Delta_{\mathcal{M},(\alpha)}$ plays a significant role in our bifurcation analysis.

Proof. See [Ro97] Theorem 1.29. $\bowtie$

2.3 Laplace-Beltrami operator on admissible surfaces of revolution

In this section we analyze the Laplace-Beltrami operator on a surface of revolution $\mathcal{M}$ that is *admissible*, in the sense that it intersects the axis of rotation. The rotational symmetry of $\mathcal{M}$ admits the L^2-subspace L^2_m that consists of radially symmetric functions multiplied by a fixed Fourier mode $e^{im\varphi}$ of the azimuthal angle. The key observation is that L^2_m is *invariant* under the Laplace-Beltrami operator. Thus we study the Laplace-Beltrami operator restricted on L^2_m, for which we call the *projected operator* $\Delta_{m,(\alpha)}$. We show that $\Delta_{m,(\alpha)}$ shares its spectral property and nodal property of eigenfunctions with regular Sturm-Liouville operators. In the end we collect the spectral information of $\Delta_{m,(\alpha)}$ on the unit disk and on the unit 2-sphere.

2.3.1 Admissible surfaces of revolution

Let $\mathcal{M}$ be symmetric with respect to all rotations around a fixed axis. Then $\mathcal{M}$ is a surface of revolution. Since the Laplace-Beltrami operator commutes with rotations, reflections, and translations, we assume without loss of generality that the x_3-axis in $(x_1, x_2, x_3) \in \mathbb{R}^3$ is the axis of rotation. Hence $\mathcal{M}$ admits the following parametrization by ***polar coordinates*** (s, φ):

$$\mathcal{M} = \big\{ \big(a(s)\cos(\varphi), a(s)\sin(\varphi), \tilde{a}(s)\big) : s \in [0, s_*],\ \varphi \in [0, 2\pi) \big\}, \tag{2.7}$$

where $a(s) \geq 0$ for all $s \in [0, s_*]$. Since $\mathcal{M}$ is real analytic, $a(s)$ and $\tilde{a}(s)$ are real analytic functions. We call $\mathcal{M}$ an ***admissible surface of revolution*** if it intersects the axis of rotation.

Lemma 2.3. *Let $\mathcal{M}$ be an admissible surface of revolution. Then $a(s)$ and $\tilde{a}(s)$ satisfy the following conditions:*

(i) $(a'(s))^2 + (\tilde{a}'(s))^2 = 1$ *for all* $s \in [0, s_*]$.

(ii) $a(0) = 0$, $a'(0) = 1$, *and* $a(s) > 0$ *for all* $s \in (0, s_*)$.

(iii) *If $\partial\mathcal{M}$ is nonempty, then $a(s_*) > 0$; if $\partial\mathcal{M}$ is empty, then $a(s_*) = 0$ and $a'(s_*) = -1$.*

Proof: For (i), for the generatrix $J = (a(s), 0, \tilde{a}(s))$ of $\mathcal{M}$ we choose s to be the arclength parameter. For (ii) and (iii), since $\mathcal{M}$ intersects the axis of rotation and is connected, there are at least one, but at most two distinct points of intersection between J and the x_3-axis. Such points are the end points of J. We can always let $a(0) = 0$. If $\partial\mathcal{M}$ is empty, then also $a(s_*) = 0$. Smoothness of $\mathcal{M}$ implies $a'(0) = 1$, and $a'(s_*) = -1$ if in addition $a(s_*) = 0$. The proof is complete. $\bowtie$

Disks and 2-spheres are main examples of admissible surfaces of revolution.

Example (unit disk). Polar coordinates for the unit disk are given by $a(s) = s$ and $\tilde{a}(s) = 0$ for $s \in [0, 1]$. Indeed every admissible surface of revolution with boundary is diffeomorphic to the unit disk.

Example (unit 2-sphere). Polar coordinates for the unit 2-sphere are given by $a(s) = \sin(s)$ and $\tilde{a}(s) = \cos(s)$ for $s \in [0, \pi]$. Indeed every admissible surface of revolution without boundary is diffeomorphic to the unit 2-sphere.

Observe that a 2-sphere is also symmetric with respect to the reflection of its equatorial plane. This leads to the following definition.

Definition. *An admissible surface of revolution without boundary is called to* ***possess the reflection symmetry*** *if* $a(s) = a(s_* - s)$ *for all* $s \in [0, s_*]$.

2.3.2 Projected operator $\Delta_{m,(\alpha)}$

Let $\mathcal{M}$ be an admissible surface of revolution. By (2.6) the Laplace-Beltrami operator on $\mathcal{M}$ is given by

$$\Delta_{\mathcal{M}} = \partial_{ss} + \frac{a'(s)}{a(s)}\,\partial_s + \frac{1}{a^2(s)}\,\partial_{\varphi\varphi}.$$

For each fixed $m \in \mathbb{Z} \setminus \{0\}$, we define the following closed L^2-subspace:

$$L^2_m := \big\{\psi \in L^2(\mathcal{M}, \mathbb{C}) : \psi(s, \varphi) = u(s)\, e^{im\varphi}\big\}.$$

Note that each function $\psi \in L^2_m$ is multiplication of a radially symmetric function $u(s)$ by the fixed Fourier mode $e^{im\varphi}$ of the azimuthal angle. The Laplace-Beltrami operator restricted on L^2_m is given by

$$\Delta_{\mathcal{M}}\psi = \left(u'' + \frac{a'}{a}u' - \frac{m^2}{a^2}u\right)e^{im\varphi}, \quad \psi(s, \varphi) = u(s)\, e^{im\varphi},$$

and thus L^2_m is *invariant* under $\Delta_{\mathcal{M}}$. The domain of $\Delta_{\mathcal{M},(\alpha)}$ restricted on L^2_m is

- $H^2_m := H^2(\mathcal{M}, \mathbb{C}) \cap L^2_m$ equipped with the H^2-norm if $\partial\mathcal{M}$ is empty;
- $H^2_{m,\alpha} := H^2_\alpha(\mathcal{M}, \mathbb{C}) \cap L^2_m$ equipped with the H^2-norm if we consider Robin boundary conditions with the ratio $\alpha \in [-\infty, 0]$.

Definition. *The **projected operator** $\Delta_{m,(\alpha)}$ is defined by*

$$\Delta_{m,(\alpha)} := \Delta_{\mathcal{M},(\alpha)}\Big|_{H^2_{m,(\alpha)}} : H^2_{m,(\alpha)} \subset L^2_m \to L^2_m. \tag{2.8}$$

Note that due to the volume form of $\mathcal{M}$, $\psi \in L^2_m$ if and only if

$$|\psi|^2_{L^2_m} := \int_0^{s_*} |u(s)|^2\, a(s)\, \mathrm{d}s < \infty. \tag{2.9}$$

2.3.3 Spectral property of $\Delta_{m,(\alpha)}$

The projected operator $\Delta_{m,(\alpha)}$ is a singular Sturm-Liouville operator because $a(0) = 0$ (and also $a(s_*) = 0$ if $\partial\mathcal{M}$ is empty). However, it is singular merely due to polar coordinates. We show that indeed $\Delta_{m,(\alpha)}$ shares its spectral property with regular Sturm-Liouville operators (see [Har02] Chapter XI).

Lemma 2.4 (spectral property)**.** *Let $\mathcal{M}$ be an admissible surface of revolution and $m \in \mathbb{Z} \setminus \{0\}$ be fixed. Then the following statements hold:*

(i) $\Delta_{m,(\alpha)}$ *is self-adjoint on L^2_m and has compact resolvent.*

(ii) *The spectrum of $\Delta_{m,(\alpha)}$ consists of simple eigenvalues, which can be listed as*

$$0 > \mu_0^{m,(\alpha)} > \mu_1^{m,(\alpha)} > \mu_2^{m,(\alpha)} > \ldots > \mu_n^{m,(\alpha)} > \ldots, \quad \lim_{n\to\infty} \mu_n^{m,(\alpha)} = -\infty.$$

In particular, $\Delta_{m,(\alpha)} : H^2_{m,(\alpha)} \to L^2_m$ is a linear homeomorphism.

Proof: For (i), $\Delta_{m,(\alpha)}$ is self-adjoint on L^2_m, since it is restriction of the Laplace-Beltrami operator on the closed invariant L^2-subspace L^2_m. Furthermore, $\Delta_{m,(\alpha)}$ has compact resolvent due to the L^2-spectral decomposition; see Lemma 2.2. This proves the item (i).

By the item (i), the spectrum of $\Delta_{m,(\alpha)}$ consists of real eigenvalues. The restriction (2.8) implies

$$\sigma(\Delta_{m,(\alpha)}) \subset \sigma(\Delta_{\mathcal{M},(\alpha)}),$$

so counting the multiplicity, eigenvalues of $\Delta_{m,(\alpha)}$ can be listed as

$$0 \geq \mu_0^{m,(\alpha)} \geq \mu_1^{m,(\alpha)} > \mu_2^{m,(\alpha)} \geq \ldots \geq \mu_n^{m,(\alpha)} \geq \ldots, \quad \lim_{n\to\infty} \mu_n^{m,(\alpha)} = -\infty.$$

Since eigenvalues of $\Delta_{m,(\alpha)}$ are real, as we consider the eigenvalue problem

$$\Delta_{m,(\alpha)}\psi = \mu\,\psi, \quad \psi(s,\varphi) = u(s)\, e^{im\varphi},$$

it suffices to assume that u is real valued.

For (ii), we first show that zero is not an eigenvalue of $\Delta_{m,(\alpha)}$. Suppose the contrary that $\psi \in H^2_{m,(\alpha)}$ were nontrivial and would satisfy $\Delta_{m,(\alpha)}\psi = 0$. Then ψ would also satisfy $\Delta_{\mathcal{M},(\alpha)}\psi = 0$. Hence zero is the principal eigenvalue of $\Delta_{\mathcal{M},(\alpha)}$, which implies that $\psi(s,\varphi)$ cannot have sign changes in $(0, s_*) \times [0, 2\pi)$; see [GiTr83] Theorem 8.38. However, since $\psi(s,\varphi) = u(s)\, e^{im\varphi}$ and $m \neq 0$, $\psi(s,\varphi)$ must have a sign change, which is a contradiction.

We next prove that all eigenvalues are simple. The eigenvalue problem is equivalent to the second-order ODE

$$u'' + \frac{a'}{a}u' - \frac{m^2}{a^2}u = \mu\, u. \tag{2.10}$$

We have four observations. First, since $\mathcal{M}$ is real analytic, it is well known that L^2-eigenfunctions of $\Delta_{m,(\alpha)}$ are real analytic. Second, (2.10) is a linear second-order ODE, so there are two linearly independent solutions. Third, since $\Delta_{m,(\alpha)}$ is self-adjoint, an eigenvalue is simple if and only if its geometric multiplicity is one. Thus it suffices to show that one of the two linearly independent solutions is unbounded in L^2_m. Four, from the weighted inner product (2.9), a solution $u(s)$ of (2.10) is unbounded in L^2_m if there is an $r_- \leq -1$ such that $u(s) = u_0\, s^{r_-} + o(s^{r_-})$ as $s \to 0$, where $u_0 \neq 0$ is some constant.

Since $a(s)$ is real analytic and $a(s) = s + o(s)$ as $s \to 0$, $s = 0$ is a regular-singular point of (2.10). Thus there exists a radius of convergence $\delta > 0$ such that every solution $u(s)$ of (2.10) can be expanded by the Frobenius series

$$u(s) = \sum_{j=0}^{\infty} u_j\, s^{j+r}, \quad u_0 \neq 0,$$

for all $s \in (0, \delta)$; see [Har02] Corollary 11.1. Substituting the series into (2.10) and equating the coefficients, we obtain the leading exponents

$$r_+ = |m|, \quad r_- = -|m|.$$

Therefore, as $s \to 0$, the two linearly independent solutions of (2.10) admit the following expansion:

$$u_1(s) = u_0\, s^{|m|} + o(s^{|m|}), \quad u_2(s) = u_0\, s^{-|m|} + o(s^{-|m|}).$$

Hence $u_2(s)$ is unbounded in L^2_m. The proof is complete. $\bowtie$

2.3.4 Nodal property of eigenfunctions of $\Delta_{m,(\alpha)}$

We show that the projected operator $\Delta_{m,(\alpha)}$ also shares its nodal property of eigenfunctions with regular Sturm-Liouville operators (see [Har02] Chapter XI).

Definition. *Let $u : [0, s_*] \to \mathbb{C}$ be a smooth function and $s_0 \in [0, s_*]$ be a zero of u, that is, $u(s_0) = 0$. We call s_0 to be **simple** if $u'(s_0) \neq 0$, or **multiple** if $u'(s_0) = 0$.*

Lemma 2.5. *Let $\mathbf{e}_n^{m,(\alpha)}(s, \varphi) = u_n^{m,(\alpha)}(s)\, e^{im\varphi}$ be the L^2_m-normalized eigenfunction associated with the eigenvalue $\mu_n^{m,(\alpha)}$ of the projected operator $\Delta_{m,(\alpha)}$. Then the following statements hold:*

(i) (***nodal property***) *$u_n^{m,(\alpha)}(s)$ possesses exactly n simple zeros in $(0, s_*)$.*

(ii) *If $\mathcal{M}$ possesses the reflection symmetry, then*

$$u_n^{m,(\alpha)}(s) = (-1)^n\, u_n^{m,(\alpha)}(s_* - s)$$

for all $n \in \mathbb{N}_0$ and $s \in [0, s_*]$. Consequently, $u_n^{m,(\alpha)}(s)$ satisfies a Neumann boundary condition (or a Dirichlet boundary condition) at $s = \frac{s_*}{2}$ if $n \in \mathbb{N}_0$ is even (or odd).

Proof: To treat the singularity $s = 0$ in the eigenvalue problem (2.10), the weighted inner product (2.9) hints the Euler multiplier

$$\left(\frac{ds}{d\tau} =\right)\ \dot{s} = a(s). \tag{2.11}$$

Let us apply (2.11) on (2.10). Then we obtain the extended ODE system

$$\begin{aligned} \dot{u}(\tau) &= v(\tau), \\ \dot{v}(\tau) &= m^2\, u(\tau) + \mu\, a^2(s(\tau))\, u(\tau), \\ \dot{s}(\tau) &= a(s(\tau)), \end{aligned} \tag{2.12}$$

for all $\tau \in (-\infty, \tau_*)$. Note that we can recover the original variable $s \in [0, s_*]$ via the mapping $\tau = \tau(s)$ such that $\tau'(s) = \frac{1}{a(s)}$ and the following conditions hold:

- $\lim_{s\to 0} \tau(s) = -\infty$.
- If $\partial\mathcal{M}$ is nonempty, that is, $a(s_*) > 0$, then $\tau_* = \tau(s_*) \in \mathbb{R}$.
- If $\partial\mathcal{M}$ is empty, that is, $a(s_*) = 0$, then $\tau_* = \lim_{s\to s_*} \tau(s) = \infty$.

For (i), we first show that all zeros of a nontrivial solution $u(\tau)$ of (2.12) in $(-\infty, \tau_*)$ are simple. Suppose the contrary that there were a nontrivial solution $u(\tau)$ of (2.12) that possesses a multiple zero $\tilde{\tau} \in (-\infty, \tau_*)$. Then $(0, 0, a(s(\tilde{\tau})))$ is evaluation of the vector field of (2.12) at $\tau = \tilde{\tau}$. Since $(u, v, s)(\tau) = (0, 0, s(\tau))$ for all $\tau \in (-\infty, \tau_*)$ is also a solution that satisfies $(\dot{u}, \dot{v}, \dot{s})(\tilde{\tau}) = (0, 0, a(s(\tilde{\tau})))$, the uniqueness of ODE initial value problems implies that $u(\tau)$ is identically zero, which is a contradiction.

We next prove that $u_n^{m,(\alpha)}(\tau)$ possesses exactly n zeros in $(-\infty, \tau_*)$. Since there are no multiple zeros, the Prüfer transformation

$$\begin{aligned} u(\tau) &= R(\tau)\cos(\theta(\tau)), \\ v(\tau) &= R(\tau)\sin(\theta(\tau)), \end{aligned}$$

is well defined for all $\tau \in (-\infty, \tau_*)$. Note that zeros of $u(\tau)$ are points of intersection between the phase portrait of (2.12) and the v-axis. We differentiate the following angle function with respect to τ:

$$\theta(\tau) := \arctan\left(\frac{v(\tau)}{u(\tau)}\right),$$

and use (2.12). Then we obtain the following first-order ODE system:

$$\begin{aligned} \big(\dot{\theta} = \big)\ \dot{\theta}_\mu &= -\sin^2(\theta) + \big(m^2 + \mu\, a^2(s)\big)\cos^2(\theta), \\ \dot{s} &= a(s). \end{aligned} \tag{2.13}$$

Notice that (2.13) is *decoupled with* $R(\tau)$ and we have two crucial properties.

(P1) The angle function $\theta_\mu(\tau)$ is strictly decreasing at points where the phase portrait intersects the v-axis, since $\dot{\theta}_\mu(\tilde{\tau}) < 0$ holds for any $\tilde{\tau} \in (-\infty, \tau_*)$ such that $\cos^2(\theta_\mu(\tilde{\tau})) = 0$.

(P2) If $\mu_1 < \mu_2$ and $\lim_{\tau\to-\infty} \theta_{\mu_1}(\tau) = \lim_{\tau\to-\infty} \theta_{\mu_2}(\tau)$, that is, θ_{μ_1} and θ_{μ_2} share the same initial angle at $\tau = -\infty$, then $\theta_{\mu_1}(\tau) < \theta_{\mu_2}(\tau)$ for all $\tau \in (-\infty, \tau_*)$.

We now study the behavior of an eigenfunction $\psi(\tau, \varphi) = u(\tau)\, e^{im\varphi}$ near $\tau = -\infty$ where $u(\tau)$ is a nontrivial solution of (2.12). We claim

$$\lim_{\tau\to-\infty} u(\tau) = 0, \quad \lim_{\tau\to-\infty} v(\tau) = 0. \tag{2.14}$$

To prove the claim, note that $\tau = -\infty$ corresponds to $s = 0$ by the Euler multiplier (2.11). Continuity of eigenfunctions at $s = 0$ implies $\lim_{s\to 0} u(s) = 0$, and hence $\lim_{\tau\to-\infty} u(\tau) = 0$. On the other hand, by the chain rule $v(\tau) = u'(s)\, a(s)$, where $s = s(\tau)$ is solved by (2.11), it is equivalent to show $\lim_{s\to 0} u'(s)\, a(s) = 0$. Smoothness of eigenfunctions, and in particular $|\nabla_g \psi|_{C^0} < \infty$ implies that $u'(0)$ exists. Hence $\lim_{s\to 0} u'(s)\, a(s) = 0$ because $a(0) = 0$.

As a result of the claim, we can without loss of generality impose the asymptotic conditions (2.14) on the ODE system (2.12). Then $\lim_{\tau\to-\infty}(u, v, s)(\tau) = (0, 0, 0)$ for every solution of (2.12), that is, the phase portrait of (2.12) converges backwards to the equilibrium $(0, 0, 0)$ as $\tau \to -\infty$. Calculating the Jacobian shows that the equilibrium is hyperbolic and $(1, |m|)$ is the (u, v)-component of the expanding direction at $\tau = -\infty$. Hence the initial angle satisfies

$$\lim_{\tau\to-\infty} \tan(\theta_\mu(\tau)) = |m|, \tag{2.15}$$

which is *fixed for all* $\mu \in \mathbb{R}$ and depends on the choice of $m \neq 0$, only.

We deal with two cases: Either $\partial\mathcal{M}$ is empty or not.

Suppose that $\partial\mathcal{M}$ is nonempty. The radial part $u(\tau)$ of an eigenfunction is given by a point of intersection between the phase portrait of (2.12) at $\tau_* \in \mathbb{R}$ and the line $L_\alpha := \{(u, v, s_*) : v = \alpha\, a(s_*)\, u\}$, due to Robin boundary conditions. Since the slope of L_α is nonpositive, as $\tau \to -\infty$ the initial angle (2.15) implies that the (u, v)-component of the phase portrait stays in the interior of the first or the third quadrant of the (u, v)-phase-plane. Moreover, by (2.13) we see that if $\mu \geq 0$, then the (u, v)-component of the phase portrait gets trapped in the interior of the first or the third quadrant for all $\tau \in (-\infty, \tau_*)$. Using (P1) and (P2), since the slope of L_α is nonpositive, we can tune $\mu \searrow -\infty$ and assure that $u_n^{m,(\alpha)}(\tau)$ possesses exactly n zeros in $\tau \in (-\infty, \tau_*)$.

Suppose that $\partial\mathcal{M}$ is empty. By similar arguments, the phase portrait of (2.12) converges forwards to the hyperbolic equilibrium $(u, v, s) = (0, 0, s_*)$ as $\tau \to \infty$. At $\tau = \infty$, the (u, v)-component of the contracting direction is given by $(1, -|m|)$, whose slope is negative. Hence we use (P1) and (P2) and tune $\mu \searrow -\infty$ to imply that $u_n^{m,(\alpha)}(\tau)$ possesses exactly n zeros in $(-\infty, \tau_*)$. This proves the item (i).

For (ii), the reflection symmetry assures that (2.10) is unchanged as we apply the new variable $s \mapsto s_* - s$. Since all eigenvalues are simple (see Lemma 2.4 (ii)), either $u_n^{m,(\alpha)}(s) = u_n^{m,(\alpha)}(s - s_*)$ or $u_n^{m,(\alpha)}(s) = -u_n^{m,(\alpha)}(s - s_*)$ for all $s \in [0, s_*]$. Since $u_n^{m,(\alpha)}(s)$ possesses exactly n zeros in $(0, s_*)$ by the item (i), we see that $n \in \mathbb{N}_0$ is even if and only if $\frac{s_*}{2}$ is not a zero of $u_n^{m,(\alpha)}(s)$, and thus if and only if $u_n^{m,(\alpha)}(s) = u_n^{m,(\alpha)}(s_* - s)$ for all $s \in [0, s_*]$. The completes the proof. $\bowtie$

2.3.5 Example: Disks and associated Bessel functions

Let $\mathcal{M}$ be the unit disk. The eigenvalues of the projected operator $\Delta_{m,(\alpha)}$ are

$$\mu_n^{m,(\alpha)} = -j_{n,m,(\alpha)}^2,$$

where $j_{n,m,(\alpha)}$ is the $(n+1)$th positive zero of the equation

$$(J_{|m|})'(s) = \alpha\, J_{|m|}(s).$$

Here $J_{|m|}$ is the Bessel function of the first kind of index $|m|$ and can be expressed by

$$J_{|m|}(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+|m|}\, k!\, (k+|m|)!}\, s^{2k+|m|};$$

see [Be68] Section 4.1. The eigenfunctions of $\Delta_{m,(\alpha)}$ are

$$\mathbf{e}_n^{m,(\alpha)}(s,\varphi) = J_{|m|}(j_{n,m,(\alpha)}\, s)\, e^{im\varphi},$$

where $J_{|m|}(j_{n,m,(\alpha)}\, s)$ is the ***associated Bessel function***.

2.3.6 Example: Spheres and associated Legendre functions

Let $\mathcal{M}$ be the unit 2-sphere. The eigenvalues and eigenfunctions of the projected operator Δ_m possess the following closed form:

$$\mu_n^m = -(|m|+n)(|m|+n+1), \quad \mathbf{e}_n^m(s,\varphi) = P_{|m|+n}^m(\cos(s))\, e^{im\varphi}.$$

Here $P_{|m|+n}^m$ is the ***associated Legendre function***. The Rodrigues' Formula writes $P_{|m|+n}^m$ explicitly:

$$P_{|m|+n}^m(\sigma) = \frac{(1-\sigma^2)^{\frac{m}{2}}}{2^{|m|+n}\,(|m|+n)!}\, \frac{d^{m+|m|+n}}{d\sigma^{m+|m|+n}}\, (\sigma^2-1)^{|m|+n}$$

for $\sigma := \cos(s) \in [-1,1]$; see [Be68] Section 3.8.

2.4 Initial value problem of Ginzburg-Landau equation

In this section we prove that the initial value problem of the GLe possesses unique smooth local (in time) solutions. This result of existence and uniqueness is a prerequisite for seeking pattern-forming solutions.

We adopt the analytic semigroup theory. Because the Laplace-Beltrami operator generates an analytic semigroup, the problem of finding strong solutions is reduced to studying regularity of the underlying nonlinearity; see [He81]. We then gain smoothness of strong solutions by the embedding theorems and the Schauder elliptic regularity theory.

Let $X = L^2(\mathcal{M}, \mathbb{C})$. It is well known that the fractional space of X with an exponent $\zeta \in (0, 1)$ is

$$X^\zeta = H^{2\zeta}(\mathcal{M}, \mathbb{C}).$$

The nonliearity $\mathcal{N}_\beta : X^\zeta \to X$ of the GLe is a superposition mapping defined by

$$\mathcal{N}_\beta(v)(x) := f(|v(x)|^2; \beta)\, v(x). \tag{2.16}$$

We determine exponents $\zeta \in (0, 1)$ such that $\mathcal{N}_\beta$ is locally Lipschitz continuous, because it is a sufficient condition for the existence of strong solutions.

Notation. The dependence on $\beta \in \mathbb{R}^d$ is irrelevant to our analysis in this section, so we denote by $f(y; \beta) = f(y)$ and also $\mathcal{N}_\beta = \mathcal{N}$.

Lemma 2.6. *Let $X = L^2(\mathcal{M}, \mathbb{C})$ and $\zeta > \frac{1}{2}$ be fixed. Then $\mathcal{N} : X^\zeta \to X$ defined in* (2.16) *is locally Lipschitz continuous.*

Proof. The mapping $\widetilde{\mathcal{N}} : \mathbb{C} \to \mathbb{C}$ defined by $\widetilde{\mathcal{N}}(z) := f(|z|^2)\, z$ is locally Lipschitz continuous because f is C^1. Hence there is a continuous componentwise increasing function $B : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$|\widetilde{\mathcal{N}}(z_1) - \widetilde{\mathcal{N}}(z_2)| \le B(|z_1|, |z_2|)\, |z_1 - z_2|$$

for all $z_1, z_2 \in \mathbb{C}$.

Since $\zeta > \frac{1}{2}$, X^ζ is continuously embedded into $C^0(\mathcal{M}, \mathbb{C})$ (see Lemma 2.1 (ii)), that is, there is a constant $C_1 > 0$ such that $|v|_{C^0} \le C_1\, |v|_{X^\zeta}$ for all $v \in X^\zeta$.

Let $C_2 > 0$ be a fixed constant. Given any $v_1, v_2 \in X^\zeta$ with $|v_1|_{X^\zeta} \le C_2$ and $|v_2|_{X^\zeta} \le C_2$, then $|v_1|_{C^0} \le C_1 C_2$ and $|v_2|_{C^0} \le C_1 C_2$, and so

$$\begin{aligned} |\mathcal{N}(v_1) - \mathcal{N}(v_2)|_X &\le \sqrt{\mathrm{Vol}(\mathcal{M})}\, B(|v_1|_{C^0}, |v_2|_{C^0})\, |v_1 - v_2|_{C^0} \\ &\le \sqrt{\mathrm{Vol}(\mathcal{M})}\, B(C_1 C_2, C_1 C_2)\, C_1\, |v_1 - v_2|_{X^\zeta} \\ &=: C_3\, |v_1 - v_2|_{X^\zeta}, \end{aligned}$$

where C_3 depends only on $\mathcal{M}$, C_1, and C_2, which is the desired conclusion. $\bowtie$

Lemma 2.7. *Let $X = L^2(\mathcal{M}, \mathbb{C})$ and $\zeta \in (\frac{1}{2}, 1)$ be fixed. Then the Ginzburg-Landau equation* (2.1) *possesses a unique classical local solution for each initial condition $\Psi_0 \in X^\zeta$.*

Proof. For each $b > 0$ and $\eta \in \mathbb{R}$, we define the infinitesimal generator

$$\Delta_{b,\eta} := \frac{1}{b}(1 + i\,\eta)\,\Delta_{\mathcal{M},(\alpha)}$$

with the domain $\mathcal{D}(\Delta_{b,\eta}) := \mathcal{D}(\Delta_{\mathcal{M},(\alpha)})$. Since $\Delta_{\mathcal{M},(\alpha)}$ is sectorial (see [Em09] Lemma 3.2) and $b > 0$, $\Delta_{b,\eta}$ is also sectorial. Since $\zeta > \frac{1}{2}$, by Lemma 2.6 the nonlinearity $\mathcal{N} : X^\zeta \to X$ of the GLe is locally Lipschitz continuous. Therefore, for each initial condition $\Psi_0 \in X^\zeta$, there exists a unique strong local solution $\Psi \in C^0([0,T], X^\zeta)$ such that $\partial_t\Psi \in C^0((0,T), X^\zeta)$, where $T = T(\Psi_0) > 0$ is the maximal time interval of existence; see [He81] Theorem 3.3.3 and Theorem 3.5.2.

We show that the strong solution is smooth. Since X^ζ is continuously embedded into $C^{0,\kappa}(\mathcal{M},\mathbb{C})$ for any $\kappa \in (0, 2\zeta - 1)$ (see Lemma 2.1 (ii)), for each fixed $t \in (0,T)$ we have

$$\Delta_{b,\eta}\Psi(t,\cdot) = \partial_t\Psi(t,\cdot) - \mathcal{N}(\Psi(t,\cdot))(\cdot) \in C^{0,\kappa}(\mathcal{M},\mathbb{C}).$$

Using a partition of unity, the Schauder elliptic regularity theory yields a classical local solution $\Psi \in C^0((0,T), C^{2,\kappa}(\mathcal{M},\mathbb{C}))$ such that $\partial_t\Psi \in C^0((0,T), C^{0,\kappa}(\mathcal{M},\mathbb{C}))$; see [GiTr83] Theorem 6.14 for the case $\alpha = -\infty$ and Theorem 6.31 for the other cases $\alpha \in (-\infty, 0]$. The proof is complete. $\bowtie$

2.5 Variational structure of real Ginzburg-Landau equation

An evolution PDE system is said to possess a *variational structure* if it can be derived by variation of a *Lyapunov functional*, and thus its global dynamics are fairly simple: All bounded trajectories converge to local extreme points of the Lyapunov functional. In this section we obtain a variational structure for the real GLe, which is a crucial ingredient to design our spiral Ansatz.

Let us introduce the following concepts in dynamical systems.

Definition. *Let Y be a Banach space and $Z \subset [0,\infty) \times Y$. A mapping $\mathcal{S} : Z \to Y$ is called a* ***local semiflow*** *on Y if the following conditions hold.*

- *Z is open in $[0,\infty) \times Y$ and the projection of Z onto the second component is equal to Y.*
- *For each $v \in Y$, there exists a* ***maximal time interval of existence*** *$T(v) > 0$ such that $\mathcal{S}(t,v) \in Z$ for all $t \in [0, T(v))$.*

- (*semigroup property*) *For each* $v \in Y$, $\mathcal{S}(0, v) = v$ *and*

$$\mathcal{S}(t_1 + t_2, v) = \mathcal{S}(t_1, \mathcal{S}(t_2, v)) = \mathcal{S}(t_2, \mathcal{S}(t_1, v))$$

for all $t_1 + t_2 \in [0, T(v))$ *such that* $t_1 \in [0, T(\mathcal{S}(t_2, v)))$ *and* $t_2 \in [0, T(\mathcal{S}(t_1, v)))$.

- (*strong continuity*) *The mapping*

$$(t, v) \in [0, T(v)) \times Y \quad \mapsto \quad \mathcal{S}(t, v) \in Y$$

is continuous.

Definition. *Let* $\mathcal{S}$ *be a local semiflow on a Banach space* Y *and* $v \in Y$. *Then we define the following:*

- *The* ***forward time orbit of*** v *is defined as* $\mathcal{O}^+(v) := \{\mathcal{S}(t, v) : t \in [0, T(v))\}$.
- v *is called an* ***equilibrium*** *if* $\mathcal{O}^+(v) = \{v\}$.
- $\mathcal{O}^+(v)$ *is called a* ***nontrivial periodic orbit*** *if* v *is not an equilibrium and there exists a period* $t_p > 0$ *such that* $\mathcal{S}(t_p, v) = v$.

Definition. *Let* $\mathcal{S}$ *be a local semiflow on a Banach space* Y. *A continuous mapping* $\mathcal{E} : Y \to \mathbb{R}$ *is called a* ***strict Lyapunov functional*** *if it is bounded from below, and the mapping* $t \in [0, T(v)) \mapsto \mathcal{E}[\mathcal{S}(t, v)] \in \mathbb{R}$ *is strictly decreasing for every* $v \in Y$, *except at equilibria.*

We now study the GLe (2.1) for every fixed $b > 0$ and $\alpha \in [-\infty, 0]$.

First, for each $\eta \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$, the GLe generates a local semiflow $\mathcal{S}_{(\eta,\beta)}$ on $Y := X^\zeta = H^{2\zeta}(\mathcal{M}, \mathbb{C})$ with $\zeta \in (\frac{1}{2}, 1)$, that is,

$$\mathcal{S}_{(\eta,\beta)}(t, \Psi_0) = \Psi(t, \cdot)$$

for all $\Psi_0 \in X^\zeta$ and $t \in [0, T(\Psi_0))$; see Lemma 2.7.

Second, the GLe can be associated with the following *complex-valued functional*:

$$\mathcal{E}_{(\eta,\beta)}[v] := \frac{1 + i\,\eta}{b} \int_{\mathcal{M}} |\nabla_g v|^2 - \int_{\mathcal{M}} F(|v|^2; \beta) - \frac{1 + i\,\eta}{b\,\alpha} \int_{\partial\mathcal{M}} |v|^2, \tag{2.17}$$

where $F(y; \beta) = \int_0^y f(\sigma; \beta)\, \mathrm{d}\sigma$ is a primitive function of f. The functional (2.17) is well defined and continuous on X^ζ, since X^ζ is continuously embedded into $C^0(\mathcal{M}, \mathbb{C})$; see Lemma 2.1 (ii). Notice that the boundary integral vanishes if $\partial\mathcal{M}$ is empty, or if we consider either Neumann boundary conditions ($\alpha = 0$) or Dirichlet boundary conditions ($\alpha = \infty$ in symbol).

Third, every solution Ψ is smooth, and due to the Euler-Lagrange equation the GLe is equivalent to the variational equation

$$\partial_t \Psi = -\frac{\delta \mathcal{E}_{(\eta,\beta)}}{\delta \overline{\Psi}}.$$

Here $\overline{\Psi}$ denotes the complex conjugate of Ψ. Moreover, applying the divergence theorem it is straightforward to obtain

$$\frac{d}{dt}\, \mathcal{E}_{(\eta,\beta)}[\Psi(t,\cdot)] = -2 \int_{\mathcal{M}} \left|(\partial_t \Psi)(t,\cdot)\right|^2. \tag{2.18}$$

However, the set of complex numbers is not an ordered field, so the range of $\mathcal{E}_{(\eta,\beta)}$, as long as it is complex valued, can possibly be extremely complicated along some forward time orbits. This observation reflects the difficulty to analyzing the complex GLe, but explains why it yields abundant and intricate spatio-temporal patterns; for instance see [ChMa96] and [ArKr02] for phase diagrams.

Lemma 2.8 (variational structure)**.** *Suppose that* (A1) *holds. Then the real Ginzburg-Landau equation* (2.3) *is associated with a strict Lyapunov functional. Consequently, there is no nontrivial periodic orbit.*

Proof: Since $\eta = 0$ and $\beta = \mathbf{0}$, the functional $\mathcal{E} := \mathcal{E}_{(0,\mathbf{0})} : X^\zeta \to \mathbb{R}$ defined in (2.17) is real valued. Thus by (2.18), $\mathcal{E}$ is strictly decreasing along every forward time orbit, except at equilibria. Moreover, $\mathcal{E}$ is bounded from below due to (A1) and $\alpha \in [-\infty, 0]$. This proves that $\mathcal{E}$ is a strict Lyapunov functional.

To prove the consequence, suppose the contrary that $v \in Y$ were not an equilibrium and $\mathcal{S}_{(0,\mathbf{0})}(t_p, v) = v$ for some period $t_p > 0$. Since $\mathcal{E}$ is strictly decreasing in time and bounded from below, we have $-\infty < \mathcal{E}[v] = \mathcal{E}[\mathcal{S}_{(0,\mathbf{0})}(t_p, v)] < \mathcal{E}[v]$, which is a contradiction. This completes the proof. $\bowtie$

2.6 Summary

We have set up the mathematical stage: the Ginzburg-Landau equation as the main equation; the two assumptions for the existence of spiral wave solutions. We have shown that the projected operator shares its spectral property and nodal property of eigenfunctions with regular Sturm-Liouville operators, due to the L^2-spectral decomposition of the Laplace-Beltrami operator. Having solved the initial value problem of the Ginzburg-Landau equation, we go further and seek pattern-forming solutions. Moreover, in Chapter 3 we derive our spiral Ansatz by the variational structure of the real Ginzburg-Landau equation.

Chapter 3

Spiral patterns and spiral Ansätze

In this chapter we design spiral patterns and their associated spiral Ansätze. In Section 3.1 we define tips as phase singularities, which are characterized by nonzero winding numbers. We prove that generically the sum of winding numbers is zero for admissible manifolds without boundary. In Section 3.2 we define spatio-temporal patterns. Based on equivariance, in Section 3.3 we interpret an Ansatz as the form of solutions associated with a relative equilibrium. In Section 3.4 we show that relative equilibria of the Ginzburg-Landau equation are quasi-periodic with at most two frequencies. Then we derive a relation between frequencies and parameters of the Ginzburg-Landau equation. In Section 3.5 we seek rigidly-rotating spiral patterns. We justify the popular m-armed spiral Ansatz by the variational structure of the real Ginzburg-Landau equation. In the end we derive a simple criterion for parameters of the Ginzburg-Landau equation that support spiral patterns.

3.1 Tips

A spiral pattern is composed of tips and a spiral-like shape. In this section we propose two equivalent definitions for tips. The study of tips plays a key role in the dynamics of nonlinear (e.g., fluid, optical, electromagnetic) fields. In different contexts tips are also called phase singularities, co-dimension two topological defects, wave dislocations, or quantized vortices; see [Pi99], [De01], [ArKr02] and references therein.

3.1.1 Jump discontinuities of phase field

Let $\Psi : [0, T) \times \mathcal{M} \to \mathbb{C}$ be a smooth solution of the initial value problem of the GLe, where $T > 0$ is the maximal time interval of existence; see Lemma 2.7. We express $\Psi = \Psi_{\mathcal{R}} + i\,\Psi_{\mathcal{R}}$ where $\Psi_{\mathcal{R}}$ and $\Psi_{\mathcal{I}}$ are real valued.

Let $t \in [0, T)$ be fixed and $\mathcal{Z}[\Psi](t)$ denote the set of zeros of $\Psi(t, \cdot)$. Let $x_0 \in \mathcal{Z}[\Psi](t)$. We call x_0 an *isolated zero* if there is an open subset $U \subset \mathcal{M}$ such that $\mathcal{Z}[\Psi](t) \cap U = \{x_0\}$. All isolated zeros of $\Psi(t, \cdot)$ form a set $\mathcal{Z}_{iso}[\Psi](t)$. For each $x_0 \in \mathcal{Z}_{iso}[\Psi](t)$, we can apply the ***polar form***

$$\Psi(t, x) = A(t, x)\, e^{iP(t,x)} \tag{3.1}$$

pointwisely for all $x \in U \setminus \{x_0\}$. Here $A = \sqrt{\Psi_{\mathcal{R}}^2 + \Psi_{\mathcal{I}}^2} > 0$ is the ***amplitude function*** and P is the real-valued ***phase field***. Note that $A(t, \cdot)$ and $P(t, \cdot)$ inherit the same smoothness of $\Psi(t, \cdot)$ in $U \setminus \{x_0\}$; see [Bo&al00] Theorem 3. Clearly, the phase field $P(t, \cdot)$ is indeterminate at x_0, so we classify $\mathcal{Z}_{iso}[\Psi](t)$ as follows:

- isolated zeros at which $P(t, \cdot)$ is *continuously extendable*, that is, there exists a continuous function $\widetilde{P}(t, \cdot) : U \to \mathbb{R}$ such that $\widetilde{P}(t, x) = P(t, x)$ holds for all $x \in U \setminus \{x_0\}$.
- isolated zeros at which $P(t, \cdot)$ undergoes a *jump discontinuity*. We call such zeros ***tips***.

3.1.2 Nonzero winding numbers

We characterize tips by a topological invariant: the winding number. For each isolated zero $x_0 \in \mathcal{Z}_{iso}[\Psi](t)$, there is a smooth positively-oriented loop $L \subset \mathcal{M}$ around x_0, which does not pass through any other zeros of $\Psi(t, \cdot)$. Hence in the polar form (3.1) the phase field $P(t, \cdot)$ is smooth in an open neighborhood of L in $\mathcal{M}$. We call such loops L *permissible*.

It follows that the net change of $P(t, \cdot)$ along any permissible loop yields the same integer multiple of 2π. Therefore, the ***winding number*** of $\Psi(t, \cdot)$ at x_0 is well defined and derived by the formula

$$\mathrm{w}(\Psi(t, \cdot), x_0) := \frac{1}{2\pi} \oint_L \mathrm{d}P(t, \cdot) = \frac{1}{2\pi} \oint_L \nabla_g P(t, \cdot) \cdot dl, \tag{3.2}$$

where dl is the arclength element. Note that the winding number is also called the *topological charge*; see [ArKr02].

It is well known that the winding number coincides with the topological degree $\mathrm{Deg}(\cdot)$ for continuous self-mappings on the unit circle $\partial\mathcal{D}$. Here $\mathcal{D}$ is the open unit disk; see [Hat02] and [OuRu09] IV.. Homotopy invariance implies that the winding number does not change as permissible loops smoothly shrink to the point x_0. Therefore, from the formula (3.2) we expect $\mathrm{w}(\Psi(t, \cdot), x_0) \neq 0$ if and only if $\nabla_g P(t, \cdot)$

diverges as permissible loops L shrink to x_0, and thus $P(t, \cdot)$ undergoes a jump discontinuity at x_0. This observation results in the following definition.

Definition. ***The set of tips*** *of* $\Psi(t, \cdot)$ *at time* $t \in [0, T)$ *is given by*

$$\mathcal{T}[\Psi](t) := \{x \in \mathcal{Z}_{iso}[\Psi](t) : \mathrm{w}(\Psi(t, \cdot), x) \neq 0\}.$$

Let us justify the definition. We call a continuous self-mapping $h : \partial\mathcal{D} \to \partial\mathcal{D}$ to be ***continuously extendable to*** $H : \mathcal{D} \cup \partial\mathcal{D} \to \partial\mathcal{D}$ if H is a continuous function and $H\big|_{\partial\mathcal{D}} = h$.

Lemma 3.1. *The following statements hold:*

(i) *A continuous self-mapping* $h : \partial\mathcal{D} \to \partial\mathcal{D}$ *is continuously extendable to* $H : \mathcal{D} \cup \partial\mathcal{D} \to \partial\mathcal{D}$ *if and only if* $\mathrm{Deg}(h) = 0$.

(ii) *Let* $x_0 \in \mathcal{Z}_{iso}[\Psi](t)$*. Then the phase field* $P(t, \cdot)$ *of the solution* $\Psi(t, \cdot)$ *undergoes a jump discontinuity at* x_0 *if and only if* $\mathrm{w}(\Psi(t, \cdot), x_0) \neq 0$.

Proof. For (i), suppose that $H : \mathcal{D} \cup \partial\mathcal{D} \to \partial\mathcal{D}$ is a continuous extension of $h : \partial\mathcal{D} \to \partial\mathcal{D}$. Then h is homotopic to the constant function $H(0)$ by the homotopy $\widetilde{H}_1 : [0, 1] \times \partial\mathcal{D} \to \partial\mathcal{D}$ defined by $\widetilde{H}_1(\sigma, x) := H(\sigma x)$. Hence $\mathrm{Deg}(h) = 0$.

Conversely, suppose $\mathrm{Deg}(h) = 0$. By the Hopf degree theorem (see [Hat02] Corollary 4.25), h is homotopic to a constant function h_0, so there is a homotopy $\widetilde{H}_2 : [0, 1] \times \partial\mathcal{D} \to \partial\mathcal{D}$ such that $\widetilde{H}_2(0, x) = h(x)$ and $\widetilde{H}_2(1, x) = h_0$. Since

$$\mathcal{D} \cup \partial\mathcal{D} = \{(\sigma, x) \in [0, 1] \times \partial\mathcal{D} : |x| = \sigma\},$$

$\widetilde{H}_2$ is a continuous extension of h. This proves the item (i).

For (ii), observe that the formula (3.2) involves $P(t, \cdot)$ only, and each permissible loop L is diffeomorphic to $\partial\mathcal{D}$. Thus to calculate the winding number, it suffices to consider $\widetilde{\Psi}(t, \cdot) := e^{iP(t, \cdot)}$ as a self-mapping on $\partial\mathcal{D}$. Since the winding number coincides with the topological degree, by the item (i), $\mathrm{w}(\Psi(t, \cdot), x_0) \neq 0$ if and only if $\widetilde{\Psi}(t, \cdot)$ is not continuously extendable to $\mathcal{D} \cup \partial\mathcal{D}$, or equivalently, $P(t, \cdot)$ undergoes a jump discontinuity at x_0. The proof is complete. $\bowtie$

Remark. Time $t \geq 0$ is a homotopy parameter, so the winding number is a conserved quantity before an annihilation of tips occurs.

3.1.3 Zero sum constraint

In this subsection we restrict our attention to admissible manifolds *without boundary*. We prove that generically the sum of winding numbers is zero.

Our starting point is that the *Brouwer degree* is well defined for $\Psi(t,\cdot):\mathcal{M}\to\mathbb{C}$ because both $\mathcal{M}$ and $\mathbb{C}$ are real two-dimensional and without boundary. Moreover, it shares all properties with the standard Brouwer degree on open sets in the Euclidean space. In particular, if $\mathbf{0}\in\mathbb{C}$ is a regular value, then the Brouwer degree is equal to the sum of winding numbers; see [Vä12] Chapter 9.

Lemma 3.2 (zero sum constraint)**.** *Let $\mathcal{M}$ be an admissible manifold without boundary and $t\geq 0$ be fixed. Assume $\mathcal{Z}[\Psi](t)=\mathcal{T}[\Psi](t)=\{x_1,...,x_\ell\}$. Then*

$$\sum_{j=1}^{\ell}\mathrm{w}(\Psi(t,\cdot),x_j)=0.$$

In particular, $\mathcal{T}[\Psi](t)$ contains at least two tips if $\mathcal{T}[\Psi](t)$ is nonempty.

Remark.

(i) The assumption is a generic condition for smooth functions on $\mathcal{M}$ due to the Morse-Sard Theorem.

(ii) The sum of winding numbers is independent of the genus of $\mathcal{M}$. Nevertheless, Lemma 3.2 does not contradict the Poincaré-Hopf Theorem because $\Psi(t,\cdot)$ is not a tangent vector field on $\mathcal{M}$.

(iii) $\mathcal{M}$ needs to be compact and without boundary because there are patterns with only one tip on $\mathbb{R}^2$ or on disks.

Proof. Since $\mathcal{M}$ is compact, $\Psi(t,\cdot):\mathcal{M}\to\mathbb{C}$ is homotopic to a constant function h_0 via the homotopy $H:[0,1]\times\mathcal{M}\to\mathbb{C}$ defined by $H(\sigma,x):=\sigma\,\Psi(t,x)+(1-\sigma)\,h_0$; see [Vä12] Definition 9.36. Hence the Brouwer degree of $\Psi(t,\cdot)$ is zero. Since all zeros of $\Psi(t,\cdot)$ are tips by assumption, the sum of their winding numbers is zero. ⋈

One can use triangulation on $\mathcal{M}$ to prove the zero sum constraint; see [Da&al04]. Due to this constraint, an annihilation of tips must involve anti-rotating spirals, that is, spirals whose tips are associated with equal winding numbers of opposite signs.

3.2 Spatio-temporal patterns

We see patterns in the physical space of experiments or numerical simulations. In this section we define spatio-temporal patterns, following [GoSt03] Section 6.6.

Let $\Psi : [0,T) \times \mathcal{M} \to \mathbb{C}$ be a smooth solution of the GLe. Let us interpret $\mathcal{M}$ as an observable region and seek a real-valued observation function

$$\Psi_{obs} : [0,T) \times \mathcal{M} \to \mathbb{R}$$

such that the mapping $\Psi(t,x) \mapsto \Psi_{obs}(t,x)$ is well defined, continuous, and respects equivariance of the GLe (see Section 3.3.1 for more details). For each $t \in [0,T)$, a *spatial pattern* is defined as the level set

$$\mathcal{P}_{spa}[\Psi](t) := \big\{x \in \mathcal{M} : \Psi_{obs}(t,x) = \ell_0\big\}.$$

Here $\ell_0 \in \mathbb{R}$ is a level of measurements. Choices of observable functions are not unique. We choose

$$\Psi_{obs}(t,x) := \mathrm{Im}(\Psi(t,x))$$

and $\ell_0 = 0$ in order to see the location of tips. One may choose $\Psi_{obs} := \mathrm{Re}(\Psi)$, however, such a choice yields a constant phase-shift of spatial patterns, only.

Definition. *The* ***spatial pattern of*** $\Psi(t,\cdot)$ ***at time*** $t \geq 0$ *is defined as*

$$\mathcal{P}_{spa}[\Psi](t) := \big\{x \in \mathcal{M} : \mathrm{Im}(\Psi(t,x)) = 0\big\}.$$

The ***spatio-temporal pattern*** *of* Ψ *is defined as*

$$\mathcal{P}_{spa-tem}[\Psi] := \big\{\mathcal{P}_{spa}[\Psi](t) : t \in [0,T)\big\}.$$

In the polar form $\Psi = A\,e^{iP} = A\cos(P) + i\,A\sin(P)$, the relation $\mathrm{Im}(\Psi) = 0$ results in the equivalent definition

$$\mathcal{P}_{spa}[\Psi](t) = \mathcal{Z}[\Psi](t) \cup \big\{x \in \mathcal{M} : P(t,x) = 0 \pmod{\pi}\big\}. \tag{3.3}$$

Thus the spatial pattern of $\Psi(t,\cdot)$ consists of zeros of $\Psi(t,\cdot)$ and zero curves (modulo π) of its phase field $P(t,\cdot)$. We are particularly interested in ***rigidly-rotating patterns***, that is,

$$\mathcal{P}_{spa}[\Psi](t) = R_{\Omega t}\Big(\mathcal{P}_{spa}[\Psi](0)\Big)$$

for all $t \geq 0$. Here $R_{\Omega t}$ is a rotation through the angle Ωt around a fixed center and $\Omega \neq 0$ is the rotating frequency.

3.3 Ansätze: Symmetry perspective

An Ansatz is a desired form of special solutions. To design Ansätze based on systematic considerations, we adopt *symmetry perspective* and interpret an Ansatz as the form of solutions associated with a *relative equilibrium*, that is, a group orbit that is invariant under the semiflow of the underlying evolution PDE. Hence our design of Ansätze shall rely on *equivariance*. In this section we review necessary background knowledge from the theory of equivariant dynamical systems; see [ChLa00], [FiSc03], and [GoSt03]. We then obtain equivariance of the GLe.

3.3.1 Equivariance, relative equilibria, and Ansätze

In this subsection our aim is to reach the definition of relative equilibria and their associated Ansätze. Moreover, we prove that every relative equilibrium is group isomorphic to a torus.

Continuous symmetries can be described by Lie groups. Because admissible manifolds are compact, we restrict our attention to compact Lie groups.

Definition. *A **compact Lie group** is a group that is also a smooth compact manifold such that the group operations of multiplication and inversion are smooth.*

Let X and Y be Banach spaces and $Y \subset X$. Typically Y is equipped with a stronger topology than the topology of X. Let $(\Gamma, \cdot)$ be a compact Lie group, and $GL(X)$ be the group of bounded linear invertible operators on X with composition as the group operation.

Definition. *We define the following:*

- *An **action of** Γ **on** X is a group homomorphism $\rho_X : \Gamma \to GL(X)$, that is,*

  $$\rho_X(\gamma_1 \cdot \gamma_2) = \rho_X(\gamma_1)\, \rho_X(\gamma_2)$$

  for all $\gamma_1, \gamma_2 \in \Gamma$.

- *The pair (X, ρ_X) is called a **representation of** Γ **on** X.*

- *An action ρ_X is called to be **strongly continuous** if the mapping*

  $$(\gamma, v) \in \Gamma \times X \quad \mapsto \quad \rho_X(\gamma)v \in X$$

  is continuous.

Let (X, ρ_X) and (Y, ρ_Y) be two representations of Γ on X and Y, respectively. Due to topology considerations we always assume

$$\rho_X\Big|_Y(\gamma) = \rho_Y(\gamma) \tag{3.4}$$

for all $\gamma \in \Gamma$.

Definition. *A mapping $\mathcal{Q} : Y \to X$ is called to be Γ-**equivariant** if*

$$\mathcal{Q}(\rho_Y(\gamma)v) = \rho_X(\gamma)\mathcal{Q}(v)$$

for all $\gamma \in \Gamma$ and $v \in Y$.

Example (Γ-equivariant evolution equation)**.** Consider the Γ-equivariant evolution equation on Y:

$$\partial_t v(t) = \mathcal{Q}(v(t)), \quad v(0) = v_0 \in Y. \tag{3.5}$$

Here $\mathcal{Q} : Y \to X$ is a Γ-equivariant mapping. Assume the existence of a local semiflow $\mathcal{S}$ on Y, that is, $S(t, v_0) = v(t)$ for all $t \in [0, T(v_0))$. From Γ-equivariance and the topological condition (3.4), we see that $v(t)$ is a solution of (3.5) if and only if $\rho_Y(\gamma)v(t)$ is also a solution, for each $\gamma \in \Gamma$ and $t \in [0, T(v))$.

We specify the meaning of *symmetry* in two slightly different ways.

Definition. *Let (X, ρ_X) be a representation of Γ on X and $v \in X$. Then we define the following:*

- *The **isotropy subgroup of** v is defined as*

$$\Sigma_v := \big\{\gamma \in \Gamma : \rho_X(\gamma)v = v\big\}.$$

- *Given a subgroup H of Γ, the H**-fixed subspace of** X is defined as*

$$\mathrm{Fix}_X(H) := \big\{v \in X : \rho_X(\gamma)v = v \text{ for all } \gamma \in H\big\}.$$

Evidently, Σ_v assembles all symmetries of the element v, and the H-fixed subspace collects all elements that satisfy the symmetry property H. One important feature of the H-fixed subspace is the *invariance* under any Γ-equivariant mapping.

Lemma 3.3. *Let $\mathcal{Q} : Y \to X$ be Γ-equivariant and H be a subgroup of Γ. Then $\mathcal{Q}(\mathrm{Fix}_Y(H)) \subset \mathrm{Fix}_X(H)$.*

Proof. [see [ChLa00] Chapter 2] Given any $v \in \mathrm{Fix}_Y(H)$ and $\gamma \in H$, by Γ-equivariance we have $\rho_X(\gamma)\mathcal{Q}(v) = \mathcal{Q}(\rho_Y(\gamma)v) = \mathcal{Q}(v)$, so $\mathcal{Q}(v) \in \mathrm{Fix}_X(H)$. $\bowtie$

From now on we consider the Γ-equivariant evolution equation (3.5). We are interested in those group orbits that are invariant under the local semiflow $\mathcal{S}$.

Definition. *Consider the Γ-equivariant evolution equation* (3.5)*. Let (Y, ρ_Y) be a representation of Γ on Y. Then we define the following:*

- *The **group orbit** of $v_0 \in Y$ is defined as $\Gamma v_0 := \{\rho_Y(\gamma)v_0 : \gamma \in \Gamma\}$.*
- *A group orbit Γv_0 is called a **relative equilibrium** if $T(v_0) = \infty$ and it contains the time orbit of v_0, that is,*

  $$\big\{\mathcal{S}(t, v_0) : t \geq 0\big\} \subset \Gamma v_0.$$

  Hence there exists a one-parameter family in time $\{\gamma^t \in \Gamma : t \geq 0\}$ such that

  $$\Gamma v_0 = \big\{\rho_Y(\gamma^t)v_0 : t \geq 0,\, \gamma^t \in \Gamma\big\}.$$

- *The **Ansatz** of a relative equilibrium Γv_0 is the form of solutions*

  $$\mathcal{S}(t, v_0) := \rho_Y(\gamma^t)v_0, \quad t \geq 0.$$

It follows by definition that a relative equilibrium Γv_0 is invariant under the semiflow, that is, $\mathcal{S}(\tilde{t}, \Gamma v_0) \subset \Gamma v_0$ for all $\tilde{t} \geq 0$. Typical examples of relative equilibria are equilibria when Γ consists of the group identity only, traveling waves when Γ consists of translations along a line, and rotating waves when Γ consists of rotations around a center.

The one-parameter family in time $\{\gamma^t \in \Gamma : t \geq 0\}$ realizes the Ansatz. Therefore, the key to design Ansätze is to answer the question: *What is time*?

Our answer is based on the semigroup property of semiflows: Time is a continuous (typically nonlinear) action of the abelian semigroup $(\mathbb{R}_{\geq 0}, +)$ on the space Y. So, we expect that the one-parameter family in time, and thus its associated relative equilibrium, is group isomorphic to an abelian subgroup of Γ.

We justify our observation with the following terminology.

Definition. *We define the following:*

- *A Lie group is called a **torus** it is compact, connected, and abelian* (see [ChLa00] Theorem 7.2.2 that justifies this definition).

Let Γ be a compact Lie group.

- *A torus of Γ is called to be **maximal** if it is maximal among all subgroups of Γ under set-theoretic inclusion.*

Since it is well known that all maximal tori lie in a single conjugacy class, we define:

- *The **rank of** Γ is the dimension of its maximal tori.*

Lemma 3.4. *Let Γ be a compact Lie group and $\Gamma v_0 \subset Y$ be a relative equilibrium of the Γ-equivariant evolution equation* (3.5)*. Then the closure of Γv_0 in Γ is group isomorphic to a torus.*

Definition. *A relative equilibrium is called to be **quasi-periodic with** n **frequencies** if it is group isomorphic to a torus of dimension n.*

Proof. [see [ChLa00] Theorem 7.2.4] Let $\Gamma v_0 = \{\rho_Y(\gamma^t)u_0 : t \geq 0,\, \gamma^t \in \Gamma\}$. We identify Γv_0 as a subset of Γ via the mapping $\rho_Y(\gamma^t)v_0 \in \Gamma v_0 \mapsto \gamma^t \in \Gamma$. It suffices to prove that $H := \{\gamma^t \in \Gamma : t \geq 0\}$ is a torus of Γ. First, since Γv_0 is a relative equilibrium, the semigroup property implies that H is an abelian subgroup of Γ. Second, strong continuity of the semiflow yields connectedness of H. Last, since Γ is compact, the closure of H in Γ is also compact. The proof is complete. $\bowtie$

3.3.2 Application to Ginzburg-Landau equation

In this subsection we obtain a standard strongly continuous representation and show equivariance of the GLe

$$\partial_t \Psi = \mathcal{Q}(b|\, \Psi;\, \eta, \beta) := \frac{1}{b}(1 + i\,\eta)\Delta_{\mathcal{M},(\alpha)}\Psi + f(|\Psi|^2; \beta)\,\Psi.$$

Notation. Since $b > 0$, $\eta \in \mathbb{R}$, $\alpha \in [-\infty, 0]$, and $\beta \in \mathbb{R}^d$ are irrelevant to our analysis in this subsection, we consider without loss of generality

$$\partial_t \Psi = \mathcal{Q}(\Psi) := \Delta_{\mathcal{M}}\Psi + f(|\Psi|^2)\,\Psi. \tag{3.6}$$

Let $X = L^2(\mathcal{M}, \mathbb{C})$. Recall that (3.6) generates a local semiflow on any fractional space $Y = X^\zeta$ with $\zeta \in (\frac{1}{2}, 1)$; see Lemma 2.7.

Let $\Gamma_{\mathcal{M}}$ be a Lie group that assembles all symmetries of $\mathcal{M}$, that is, $\gamma\mathcal{M} = \mathcal{M}$ for all $\gamma \in \Gamma_{\mathcal{M}}$. Then $\Gamma_{\mathcal{M}}$ is a Lie subgroup of the real orthogonal group $\mathbf{O}(3, \mathbb{R})$, which consists of all 3×3 real matrices γ such that $\gamma^{-1} = \gamma^T$. The group action on $\mathcal{M}$,

$$(\gamma, x) \in \Gamma_{\mathcal{M}} \times \mathcal{M} \quad \mapsto \quad \gamma^{-1}x \in \mathcal{M},$$

induces the following action of $\Gamma_{\mathcal{M}}$ on X:

$$(\rho_X(\gamma)v)(x) := v(\gamma^{-1}x)$$

for all $\gamma \in \Gamma_{\mathcal{M}}$, $v \in X$, and $x \in \mathcal{M}$. Note that the inverse γ^{-1} appears so that ρ_X becomes an action. Together with the global S^1-equivariance, the symmetry group of the GLe is

$$\Gamma := S^1 \times \Gamma_{\mathcal{M}}.$$

Note that the inverse of $(\vartheta, \gamma) \in \Gamma$ is given by $(-\vartheta, \gamma^{-1})$. We now consider the following action ρ_X of Γ on X:

$$\big(\rho_X((\vartheta, \gamma))v\big)(x) := e^{-i\vartheta}\, v(\gamma^{-1}x) \tag{3.7}$$

for all $(\vartheta, \gamma) \in \Gamma$, $v \in X$, and $x \in \mathcal{M}$. Accordingly, we define the action ρ_Y of Γ on Y also by (3.7), so the topological condition (3.4) is trivially fulfilled.

Lemma 3.5. *The following statements hold:*

(i) *The action* (3.7) *of* Γ *on* X *and* Y *is strongly continuous.*

(ii) *The Ginzburg-Landau equation* (3.6) *is* Γ*-equivariant under the action* (3.7).

Remark. Lemma 3.5 (i) does not necessarily hold if Γ is not a compact Lie group; for instance, if Γ contains translations.

Before the proof, we denote by $L(Y, X)$ the space of bounded linear operators that map Y into X equipped with the operator norm $\|\cdot\|$.

Proof. For (i), it suffices to prove the case for ρ_X because the proof of the other case ρ_Y is the same, due to the topological condition (3.4). Since $\|\rho_X(\vartheta, \gamma)\|_{L(X,X)} = 1$ for all $(\vartheta, \gamma) \in \Gamma$, and

$$\big|\rho_X(\vartheta_1, \gamma_1)v_1 - \rho_X(\vartheta_2, \gamma_2)v_2|_X \le |v_1 - v_2|_X + |\rho_X((\vartheta_2, \gamma_2)^{-1} \cdot (\vartheta_1, \gamma_1))v_2 - v_2|_X,$$

it remains to prove that $\lim_{(\vartheta,\gamma) \to (0,\mathrm{id})} |\rho_X(\vartheta, \gamma)v - v|_X = 0$ holds for each fixed $v \in X$. Here $(0, \mathrm{id}) \in \Gamma$ is the group identity.

Given $v \in X$ and $\epsilon > 0$, since $C^1(\mathcal{M}, \mathbb{C})$ is dense in X, there is a smooth function w such that $|v - w|_X < \epsilon$. Since $\mathcal{M}$ is compact, w is uniformly continuous, so there is a $\delta > 0$ such that $|w(x) - w(\tilde{x})| < \epsilon$ whenever x, $\tilde{x} \in \mathcal{M}$ satisfy $|x - \tilde{x}| < \delta$. Hence, as $(\vartheta, \gamma) \to (0, \mathrm{id})$ in Γ, we have $|(\vartheta, \gamma)^{-1}x - x| < \delta$ for all $x \in \mathcal{M}$. Therefore,

$$\begin{aligned} |\rho_X(\vartheta, \gamma)v - v|_X &\le \|\rho_X(\vartheta, \gamma)\|_{L(X,X)}\, |v - w|_X + |\rho_X(\vartheta, \gamma)w - w|_X + |w - v|_X \\ &\le C\epsilon, \end{aligned}$$

where $C > 0$ is a constant that depends on $\mathcal{M}$ and δ, only. This proves the item (i).

For (ii), observe that $\mathcal{Q}$ defined in (3.6) is S^1-equivariant and the Laplace-Beltrami operator as well as the nonlinearity defined as a superposition mapping commute with rotations, reflections, and translations. The proof is complete. $\bowtie$

3.4 General Ansatz for Ginzburg-Landau equation

In this section we show that all relative equilibria of the GLe are quasi-periodic with at most two frequencies. Moreover, we derive a relation between frequencies and parameters of the GLe.

3.4.1 General Ansatz and surfaces of revolution

Consider the GLe under the action (3.7). Since every relative equilibrium is group isomorphic to a torus of a compact Lie group Γ, the rank of Γ determines the maximal number of frequencies.

For the maximality, it suffices to consider that $\mathcal{M}$ possesses the full symmetries in $\mathbb{R}^3$, that is, $\Gamma_{\mathcal{M}} = \mathbf{O}(3, \mathbb{R})$ and thus $\Gamma = S^1 \times \mathbf{O}(3, \mathbb{R})$. It follows that the rank of Γ is two, since S^1 is a one-dimensional torus and $\mathbf{SO}(2, \mathbb{R}) \cong S^1$ is a maximal torus of $\mathbf{O}(3, \mathbb{R})$; see [ChLa00] Example 7.23. Hence $S^1 \times \mathbf{SO}(2, \mathbb{R})$ is a maximal torus of Γ. As a result, all relative equilibria are quasi-periodic with at most two frequencies. We call an Ansatz to be ***general*** if its associated relative equilibrium is quasi-periodic with exactly two frequencies.

To classify all maximal tori of Γ, we have two observations. First, it is well known that maximal tori of a compact Lie group form a single conjugacy class. Second, by Euler's rotation theorem, every element of $\mathbf{O}(3, \mathbb{R})$ is a rotation around some axis. Therefore, up to a rotation, $S^1 \times \mathbf{SO}(2, \mathbb{R})$ is the *unique* maximal torus of Γ. Hence we can assume without loss of generality that the action of $\mathbf{SO}(2, \mathbb{R})$ is described by all rotations around the x_3-axis in $(x_1, x_2, x_3) \in \mathbb{R}^3$. Consequently, the general Ansatz requires $\mathcal{M}$ being a *surface of revolution*. Moreover, by the action (3.7) the general Ansatz is defined by

$$\Psi(t, s, \varphi) = e^{-i\Omega t}\, \psi(s, \varphi - ct). \tag{3.8}$$

Here (s, φ) denotes polar coordinates on $\mathcal{M}$; see (2.7).

We immediately see that in the co-rotating frame with the ***rotation frequency*** $\Omega \in \mathbb{R}$, (3.8) is the form of a periodic traveling wave solutions for the complex-valued ***profile*** ψ with the ***wave speed*** $c \in \mathbb{R}$. Notice that Ω and c are *unknown frequencies.*

3.4.2 Frequency-parameter relation

The general Ansatz (3.8) yields a relation between the unknown frequencies and parameters of the GLe. To see the derivation, we first introduce the co-moving frame

$$\xi := \varphi - ct$$

and substitute (3.8) into the GLe (2.1). The global S^1-equivariance yields the following *elliptic equation* for ψ:

$$-\,i\,\Omega\,\psi - c\,\partial_\xi\psi = \frac{1}{b}(1+i\,\eta)\,\Delta_{\mathcal{M},(\alpha)}\psi + f(|\psi|^2;\beta)\,\psi. \tag{3.9}$$

Second, we multiply (3.9) by the complex conjugate $\overline{\psi}$, integrate over $\mathcal{M}$, and use the divergence theorem and Robin boundary conditions. Then we obtain

$$i\,\Omega\int_{\mathcal{M}}|\psi|^2+c\int_{\mathcal{M}}(\partial_\xi\psi)\,\overline{\psi} = \frac{1}{b}(1+i\,\eta)\Big(\int_{\mathcal{M}}|\nabla_g\psi|^2-\alpha\int_{\partial\mathcal{M}}|\psi|^2\Big)-\int_{\mathcal{M}}f(|\psi|^2;\beta)\,|\psi|^2. \tag{3.10}$$

Note that all integrals are taken over the volume form with respect to polar coordinates. Third, since $\psi(s,\xi+2\pi)=\psi(s,\xi)$ for all $s\in[0,s_*]$ and $\xi\in\mathbb{R}$, we have

$$\int_{\mathcal{M}}\mathrm{Re}\big((\partial_\xi\psi)\,\overline{\psi}\big) = \frac{1}{2}\int_{\mathcal{M}}\partial_\xi(|\psi|^2)=0$$

and thus $\int_{\mathcal{M}}(\partial_\xi\psi)\,\overline{\psi}$ is purely imaginary. We now sort out the real part and imaginary part in (3.10):

$$\frac{1}{b}\Big(\int_{\mathcal{M}}|\nabla_g\psi|^2-\alpha\int_{\partial\mathcal{M}}|\psi|^2\Big) = \int_{\mathcal{M}}f_{\mathcal{R}}(|\psi|^2;\beta)\,|\psi|^2, \tag{3.11}$$

$$\Omega\int_{\mathcal{M}}|\psi|^2+c\int_{\mathcal{M}}\mathrm{Im}\big((\partial_\xi\psi)\,\overline{\psi}\big) = \frac{\eta}{b}\Big(\int_{\mathcal{M}}|\nabla_g\psi|^2-\alpha\int_{\partial\mathcal{M}}|\psi|^2\Big)-\int_{\mathcal{M}}f_{\mathcal{I}}(|\psi|^2;\beta)\,|\psi|^2. \tag{3.12}$$

Then (3.11–3.12) yield the following ***frequency-parameter relation***:

$$\int_{\mathcal{M}}\Big(\Omega-\eta\,f_{\mathcal{R}}(|\psi|^2;\beta)+f_{\mathcal{I}}(|\psi|^2;\beta)\Big)|\psi|^2+c\int_{\mathcal{M}}\mathrm{Im}\big((\partial_\xi\psi)\,\overline{\psi}\big)=0. \tag{3.13}$$

3.5 Spiral Ansatz for Ginzburg-Landau equation

In this section we consider $\mathcal{M}$ as an *admissible surface of revolution* because of our *tip assumption for rigidly-rotating spirals.* We justify the popular *m-armed spiral Ansatz* by the variational structure of the real GLe. We then establish a *functional setting* to solve the resulting elliptic equation. We next obtain several consequences, once a nontrivial solution exists. For instance, zeros of the solution neither accumulate at tips nor at boundary points; its phase derivative possesses a continuous extension to tips and simple zeros; its associated pattern hits boundary points in normal direction. In the end we obtain a criterion that determines whether a nontrivial solution exhibits a spiral pattern.

3.5.1 Tip assumption: Rigidly-rotating spirals

Let Ψ be a solution of the GLe on an admissible surface of revolution $\mathcal{M}$; see (2.7). We propose an assumption on tips so that Ψ can exhibit rigidly-rotating spirals.

Tip Assumption (rigidly-rotating spirals)**.** We assume that tips of Ψ reside on the x_3-axis, that is,

- there is only one tip $(0, 0, \tilde{a}(0))$ if $\partial\mathcal{M}$ is nonempty;
- there are two distinct tips $(0, 0, \tilde{a}(0))$ and $(0, 0, \tilde{a}(s_*))$ if $\partial\mathcal{M}$ is empty.

Moreover, tips are pinned for all time.

Since each tip is associated with a nonzero winding number, we assign $m \in \mathbb{Z}\setminus\{0\}$ as the winding number of Ψ at the tip $(0, 0, \tilde{a}(0))$. Up to a change of directions $\varphi \mapsto -\varphi$, we assume without loss of generality

$$m \in \mathbb{N}.$$

In the general Ansatz (3.8), since by definition a tip is an isolated zero of Ψ, there is an $\hat{s} \in (0, s_*]$ such that ψ admits the polar form,

$$\psi(s, \xi) = A(s, \xi)\, e^{iP(s,\xi)}, \quad \xi = \varphi - ct,$$

for all $s \in (0, \hat{s})$ and $\xi \in \mathbb{R}$. Thus the formula of winding numbers (3.2) yields

$$2\pi m = \oint_{C_s} \nabla_g P \cdot dl = \int_0^{2\pi} (\partial_\xi P)(s, \sigma + \xi_0)\, \mathrm{d}\sigma, \tag{3.14}$$

where C_s is the positively-oriented circle given by the intersection of $\mathcal{M}$ and the plane $\{(x_1, x_2, x_3) : x_3 = \tilde{a}(s),\ s \in (0, \hat{s})\}$ and $\xi_0 \in \mathbb{R}$ is any reference point. From (3.14) we have

$$P(s, \xi_0 + 2\pi) = P(s, \xi_0) + 2\pi m \tag{3.15}$$

for all $s \in (0, \hat{s})$ and $\xi_0 \in \mathbb{R}$.

3.5.2 The m-armed spiral Ansatz

In this subsection we assume that only the location of tips are zeros of Ψ. Hence we can apply *one polar form* in the general Ansatz (3.8), that is, $\hat{s} = \tilde{s}$ and thus

$$\Psi(t, s, \varphi) = e^{-i\Omega t} A(s, \xi)\, e^{iP(s,\xi)}, \quad \xi = \varphi - ct, \tag{3.16}$$

for all $s \in (0, s_*)$ and $\xi \in \mathbb{R}$. We now consider the real GLe (2.3). Since $\eta = 0$ and $\beta = \mathbf{0}$, the frequency-parameter relation (3.13) reads

$$\Omega \int_{\mathcal{M}} |\psi|^2 + c \int_{\mathcal{M}} \mathrm{Im}\big((\partial_\xi \psi)\, \overline{\psi}\big) = 0.$$

We are interested in nontrivial solutions, so $c = 0$ implies $\Omega = 0$. In this case (3.16) transforms the real GLe into its equilibrium equation, only. Thus we consider $c \neq 0$.

Lemma 3.6 (m-armed spiral Ansatz)**.** *Consider the real Ginzburg-Landau equation* (2.3) *on an admissible surface of revolution $\mathcal{M}$. Suppose that Ψ is a smooth nontrivial solution in the form of* (3.16)*. Assume $c \neq 0$ and Ω and c are rationally dependent. Then the following statements hold:*

(i) $\Omega + mc = 0$.

(ii) *$A(s, \xi) = A(s)$, that is, the amplitude function A is radially symmetric.*

(iii) *$P(s, \xi) = p(s) + m\xi$. Hence the phase field $P(s, \cdot)$ is a linear circle mapping with $m \in \mathbb{N}$ as its rotation number, for each fixed $s \in (0, s_*)$.*

Proof: Since $c \neq 0$, (3.15) implies

$$\begin{aligned} \Psi\left(t + k\frac{2\pi}{c}, s, \varphi\right) &= e^{-ik(\frac{\Omega}{c}+m)2\pi}\, \Psi(t, s, \varphi) \\ &= e^{-i2k\frac{\Omega}{c}\pi}\, \Psi(t, s, \varphi) \end{aligned} \tag{3.17}$$

for all $t \geq 0$, $s \in (0, s_*)$, and $\varphi \in [0, 2\pi)$. Since t is nonnegative, we choose $k \in \mathbb{N}$ (or $k \in -\mathbb{N}$) if $c > 0$ (or $c < 0$). We assume $c > 0$ without loss of generality.

Since Ω and c are rationally dependent, $\Psi(t, \cdot)$ is periodic in time by (3.17). The variational structure of the real GLe implies that $\Psi(t, \cdot)$ is independent of time; see Lemma 2.8. Thus (3.16) reads

$$e^{-i\Omega t} A(s, \varphi - ct)\, e^{iP(s,\varphi - ct)} = A(s, \varphi)\, e^{iP(s,\varphi)}. \tag{3.18}$$

Hence $A(s, \varphi - ct) = A(s, \varphi)$ for all $t \geq 0$, $s \in (0, s_*)$, and $\varphi \in [0, 2\pi)$. Since $c \neq 0$, continuity of A implies $A(s, \varphi) = A(s)$. This proves the item (ii).

Now (3.18) reads

$$A(s)\, e^{i\left(-\Omega t + P(s,\varphi - ct)\right)} = A(s)\, e^{iP(s,\varphi)}.$$

Since $A(s) > 0$ for all $s \in (0, s_*)$, continuity of P implies

$$-\Omega t + P(s, \varphi - ct) = P(s, \varphi). \tag{3.19}$$

Let us plug $t = \frac{2\pi}{c} > 0$ into (3.19) and use (3.15). Then we obtain $\Omega + mc = 0$. This proves the item (i).

For (iii), we differentiate (3.19) with respect to t, which yields

$$-\Omega - c\,\partial_\xi P(s, \varphi - ct) = 0.$$

Thus $\partial_\xi P(s, \varphi - ct) = -\frac{\Omega}{c} = m$, by the item (i). Hence $P(s, \varphi - ct) = p(s) + m(\varphi - ct)$. This completes the proof. ⋈

In view of Lemma 3.6, we define the ***m-armed spiral Ansatz*** by

$$\Psi(t, s, \varphi) = e^{-i\hat{\Omega} t}\, \psi(s, \varphi), \quad \psi(s, \varphi) = u(s)\, e^{im\varphi},$$

where $\hat{\Omega} := \Omega + mc$ and u is complex valued.

Notation. Since $\hat{\Omega}$ is the only unknown frequency in the m-armed spiral Ansatz, from now on we omit the hat, so we consider

$$\Psi(t, s, \varphi) = e^{-i\Omega t}\, \psi(s, \varphi), \quad \psi(s, \varphi) = u(s)\, e^{im\varphi}. \tag{3.20}$$

Accordingly, the frequency-parameter relation (3.13) reads

$$\int_{\mathcal{M}} \left(\Omega - \eta\, f_{\mathcal{R}}(|\psi|^2; \beta) + f_{\mathcal{I}}(|\psi|^2; \beta) \right) |\psi|^2 = 0. \tag{3.21}$$

Remark. Lemma 3.6 justifies that the m-armed spiral Ansatz (3.20) is the *unique* rigidly-rotating spiral Ansatz for the real GLe, if the unknown frequencies Ω and c are rationally dependent. However, it is unclear whether other spiral Ansätze are possible if Ω and c are *irrationally dependent.* To treat this case by our proof, the difficulty is that the set of equilibria is not discrete due to the global S^1-equivariance. Thus the relation (3.17) is fruitless, even all ω-limit sets are contained in the set of equilibria by the LaSalle's invariance principle; see [Sa76] Theorem 3.1.

3.5.3 Functional setting

In the m-armed spiral Ansatz (3.20), observe that the profile ψ is an element of the closed L^2-subspace L^2_m. Thus the resulting elliptic ***full equation*** for ψ reads

$$\mathcal{F}(b|\,\Omega,\psi;\,\eta,\beta) := (1+i\,\eta)\,\Delta_{m,(\alpha)}\psi + i\,b\,\Omega\,\psi + b\,f(|\psi|^2;\beta)\,\psi = 0; \tag{3.22}$$

see (3.9) with $c=0$. Here $\Delta_{m,(\alpha)} : H^2_{m,(\alpha)} \subset L^2_m \to L^2_m$ is the projected operator; see Section 2.3.2. Notice that we have to determine the unknown frequency $\Omega \in \mathbb{R}$.

Notation. We use the notation $(b|\,\Omega,\psi;\,\eta,\beta)$ to distinguish $b>0$ as the bifurcation parameter; $\Omega\in\mathbb{R}$ and $\psi \in H^2_{m,(\alpha)}$ as the unknowns of the full equation (3.22); $\eta\in\mathbb{R}$ and $\beta\in\mathbb{R}^d$ as the parameters of the GLe.

We establish our functional setting based on the following simple fact.

Lemma 3.7 (invariance)**.** *The range of $\mathcal{F}$ in* (3.22) *is contained in L^2_m, that is, $\mathcal{F}:(0,\infty)\times\mathbb{R}\times H^2_{m,(\alpha)}\times\mathbb{R}\times\mathbb{R}^d \to L^2_m$ is well defined.*

Proof: Let $\psi\in H^2_{m,(\alpha)}$ and $\psi(s,\varphi)=u(s)\,e^{im\varphi}$. Then ψ is continuous, because $H^2_{m,(\alpha)}$ is a H^2-closed subspace and $H^2(\mathcal{M},\mathbb{C})$ is continuously embedded into $C^0(\mathcal{M},\mathbb{C})$; see Lemma 2.1 (ii). By the global S^1-equivariance, we have

$$f(|\psi(s,\varphi)|^2;\beta)\,\psi(s,\varphi) = \big(f(|u(s)|^2;\beta)\,u(s)\big)\,e^{im\varphi}.$$

Since f is continuous, we see $f(|\psi|^2;\beta)\,\psi\in L^2_m$. ⋈

3.5.4 A priori smoothness

In our functional setting, once a *weak solution* $\psi\in H^2_{m,(\alpha)}$ of the full equation (3.22) exists, it becomes smooth by the embedding theorem and the Schauder elliptic regularity theory.

Lemma 3.8 (a priori smoothness)**.** *Let $\psi \in H^2_{m,(\alpha)}$ be a weak solution of the full equation* (3.22)*. Then $\psi\in C^{2,\nu}(\mathcal{M},\mathbb{C})$ for any $\nu\in(0,1)$.*

Proof: Since $H^2(\mathcal{M},\mathbb{C})$ is continuously embedded into $C^{0,\nu}(\mathcal{M},\mathbb{C})$ for any $\nu\in(0,1)$, we have $\psi\in C^{0,\nu}(\mathcal{M},\mathbb{C})$; see Lemma 2.1 (ii). Since $f\in C^1$, from the full equation we see $\Delta_{\mathcal{M},(\alpha)}\,\psi\in C^{0,\nu}(\mathcal{M},\mathbb{C})$. Hence $\psi\in C^{2,\nu}(\mathcal{M},\mathbb{C})$ by using a partition of unity and the Schauder elliptic regularity theory; see [GiTr83] Theorem 6.14 for the case $\alpha=-\infty$, and Theorem 6.31 for the other cases $\alpha\in(-\infty,0]$. ⋈

We often use a by-product of smoothness: If $\psi\in C^{2,\nu}(\mathcal{M},\mathbb{C})$, then in particular

$$|\nabla_g\psi|_{C^0} = \sup_{s\in[0,s_*]}\left(|u'(s)|^2 + \frac{m^2}{a^2(s)}\,|u|^2\right) < \infty. \tag{3.23}$$

3.5.5 Dynamics near tips and boundary

Lemma 3.9. *Suppose that $\psi \in C^{2,\nu}(\mathcal{M},\mathbb{C})$ and $\psi(s,\varphi) = u(s)\, e^{im\varphi}$ is a nontrivial solution of the full equation (3.22). Let $u(s) = u_{\mathcal{R}}(s) + i\, u_{\mathcal{I}}(s)$ where $u_{\mathcal{R}}$ and $u_{\mathcal{I}}$ are real valued. Then the following statements hold:*

(i) (*near the tips*) *$s = 0$ is an isolated zero of $u(s)$. If in addition $\partial\mathcal{M}$ is empty, then $s = s_*$ is also an isolated zero of $u(s)$.*

(ii) (*near the boundary*) *Let $\partial\mathcal{M}$ be nonempty. Then $s = s_*$ cannot be a multiple zero of $u(s)$.*

Proof: Let $\psi(s,\varphi) = \psi_{\mathcal{R}}(s,\varphi) + i\,\psi_{\mathcal{I}}(s,\varphi)$ where $\psi_{\mathcal{R}}(s,\varphi) = u_{\mathcal{R}}(s)\, e^{im\varphi}$ and $\psi_{\mathcal{I}}(s,\varphi) = u_{\mathcal{I}}(s)\, e^{im\varphi}$. We show that $s = 0$ is an isolated zero of both $u_{\mathcal{R}}(s)$ and $u_{\mathcal{I}}(s)$.

The full equation is equivalent to the following system:

$$\begin{aligned} \Delta_{m,(\alpha)}\psi_{\mathcal{R}} - \eta\, \Delta_{m,(\alpha)}\psi_{\mathcal{I}} + b\,\big(-\Omega\,\psi_{\mathcal{I}} + f_{\mathcal{R}}(|\psi|^2;\beta)\,\psi_{\mathcal{R}} - f_{\mathcal{I}}(|\psi|^2;\beta)\,\psi_{\mathcal{I}}\big) = 0, \\ \eta\,\Delta_{m,(\alpha)}\psi_{\mathcal{R}} + \Delta_{m,(\alpha)}\psi_{\mathcal{I}} + b\,\big(\Omega\,\psi_{\mathcal{R}} + f_{\mathcal{I}}(|\psi|^2;\beta)\,\psi_{\mathcal{R}} + f_{\mathcal{R}}(|\psi|^2;\beta)\,\psi_{\mathcal{I}}\big) = 0, \end{aligned} \tag{3.24}$$

where $|\psi|^2 = u_{\mathcal{R}}^2 + u_{\mathcal{I}}^2$. Using the Euler multiplier

$$\left(\frac{ds}{d\tau} =\right)\ \dot{s} = a(s),$$

it follows that (3.24) is equivalent to the extended ODE system:

$$\begin{aligned} \dot{u_{\mathcal{R}}} &= v_{\mathcal{R}}, \\ \dot{v_{\mathcal{R}}} &= m^2\, u_{\mathcal{R}} - b\, a^2(s)\big(G_1 + \eta\, G_2\big), \\ \dot{u_{\mathcal{I}}} &= v_{\mathcal{I}}, \\ \dot{v_{\mathcal{I}}} &= m^2\, u_{\mathcal{I}} - b\, a^2(s)\big(-\eta\, G_1 + G_2\big), \\ \dot{s} &= a(s). \end{aligned} \tag{3.25}$$

Here G_1 and G_2 are defined as follows:

$$\begin{aligned} G_1 = G_1(u_{\mathcal{R}}, u_{\mathcal{I}};\, \Omega,\eta,\beta) &:= \frac{1}{1+\eta^2}\big(-\Omega\, u_{\mathcal{I}} + f_{\mathcal{R}}(|\psi|^2;\beta)\, u_{\mathcal{R}} - f_{\mathcal{I}}(|\psi|^2;\beta)\, u_{\mathcal{I}}\big), \\ G_2 = G_2(u_{\mathcal{R}}, u_{\mathcal{I}};\, \Omega,\eta,\beta) &:= \frac{1}{1+\eta^2}\big(\Omega\, u_{\mathcal{R}} + f_{\mathcal{I}}(|\psi|^2;\beta)\, u_{\mathcal{R}} + f_{\mathcal{R}}(|\psi|^2;\beta)\, u_{\mathcal{I}}\big). \end{aligned}$$

For (i), note that $\tau = -\infty$ corresponds to $s = 0$. Continuity of $\psi_{\mathcal{R}}(\tau)$ and $\psi_{\mathcal{I}}(\tau)$ at $\tau = -\infty$ implies $\lim_{\tau\to-\infty} u_{\mathcal{R}}(\tau) = \lim_{\tau\to-\infty} u_{\mathcal{I}}(\tau) = 0$.

We next show $\lim_{\tau\to-\infty} v_{\mathcal{R}}(\tau) = 0$ and the proof for $\lim_{\tau\to-\infty} v_{\mathcal{I}}(\tau) = 0$ is analogous. By the chain rule $\dot{u_{\mathcal{R}}}(\tau) = u_{\mathcal{R}}'(s)\, a(s)$. Since $\psi \in C^{2,\nu}(\mathcal{M},\mathbb{C})$, by (3.23) we see that $u'_{\mathcal{R}}(0)$ exists. Hence $\lim_{\tau\to-\infty} \dot{u_{\mathcal{R}}}(\tau) = \lim_{s\to 0} u'_{\mathcal{R}}(s)\, a(s) = 0$ because $a(0) = 0$.

Observe $G_j(0,0;\,\Omega,\eta,\beta) = 0$ for all $j = 1,2$, $\Omega \in \mathbb{R}$, $\eta \in \mathbb{R}$, and $\beta \in \mathbb{R}^d$. Thus $\lim_{\tau\to-\infty} u_{\mathcal{R}}(\tau) = \lim_{\tau\to-\infty} u_{\mathcal{I}}(\tau) = 0$ and (3.25) imply that as $\tau \to -\infty$, the phase portrait of (3.25) converges to the equilibrium $(u_{\mathcal{R}}, v_{\mathcal{R}}, u_{\mathcal{I}}, v_{\mathcal{I}}, s) = (0,0,0,0,0)$. Since the equilibrium is hyperbolic and the expanding direction at $\tau = -\infty$ is given by $(1,m,1,m,1)$, we see that $\tau = -\infty$ is an isolated zero of both $u_{\mathcal{R}}(\tau)$ and $u_{\mathcal{I}}(\tau)$, or equivalently, $s = 0$ is an isolated zero of both $u_{\mathcal{R}}(s)$ and $u_{\mathcal{I}}(s)$.

If $\partial\mathcal{M}$ is empty, then similar arguments show that the phase portrait of (3.25) converges to the hyperbolic equilibrium $(u_{\mathcal{R}}, v_{\mathcal{R}}, u_{\mathcal{I}}, v_{\mathcal{I}}, s) = (0,0,0,0,s_*)$ as $\tau \to \infty$, and follows its contracting direction $(1,-m,1,-m,-1)$. This proves the item (i).

For (ii), suppose the contrary that $s = s_*$ were a multiple zero of $u(s)$, that is, $u(s_*) = u'(s_*) = 0$. Then $u(\tau_*) = \dot{u}(\tau_*) = 0$, where $\tau_* = \tau(s_*) \in \mathbb{R}$ is solved by the Euler multiplier. Since $a(s_*) > 0$, and thus $s = s_*$ is not a singularity of (3.25), the uniqueness of ODE initial value problems implies that $u(s)$ is identically zero, which is a contradiction. The proof is complete. ⋈

3.5.6 Continuous extension of phase derivative

Let $\psi \in C^{2,\nu}(\mathcal{M},\mathbb{C})$ be a nontrivial solution of the full equation (3.22) and $\psi(s,\varphi) = u(s)\, e^{im\varphi}$. Suppose that $\tilde{s} \in [0, s_*]$ is an isolated zero of $u(s)$. Then there is a $\delta > 0$ such that ψ admits the polar form

$$\psi(s,\varphi) = \Big(A(s)\, e^{ip(s)}\Big) e^{im\varphi}, \quad s \in (\tilde{s}, \tilde{s}+\delta)$$

(or consider $(\tilde{s}-\delta, \tilde{s})$ if $\tilde{s} = s_*$). We substitute the polar form into (3.22) and then obtain the following ODE system:

$$\Big(A'' + \frac{a'}{a}A' - \frac{m^2}{a^2}A - A\,(p')^2\Big) - \eta\Big(A\,p'' + 2\,A'\,p' + \frac{a'}{a}A\,p'\Big) + b\, f_{\mathcal{R}}(|A|^2;\beta)\, A = 0, \tag{3.26}$$

$$\Big(A\,p'' + 2\,A'\,p' + \frac{a'}{a}A\,p'\Big) + \eta\Big(A'' + \frac{a'}{a}A' - \frac{m^2}{a^2}A - A\,(p')^2\Big) + b\,\Big(\Omega + f_{\mathcal{I}}(|A|^2;\beta)\Big)A = 0. \tag{3.27}$$

Observe that the ***phase derivative*** $p'(s)$ is an independent variable in (3.26–3.27). We ask if $p'(s)$ admits a unique continuous extension to the isolated zero $s = \tilde{s}$.

To answer it, we combine (3.26) and (3.27) and then obtain

$$(1+\eta^2)\Big(A\,p'' + 2\,A'\,p' + \frac{a'}{a}A\,p'\Big) = -b\Big(\Omega - \eta\, f_{\mathcal{R}}(|A|^2;\beta) + f_{\mathcal{I}}(|A|^2;\beta)\Big)A. \tag{3.28}$$

The trick is to apply the following identity:

$$(a\,A^2\,p')' = a\,A\biggl(A\,p'' + 2\,A'\,p' + \frac{a'}{a}A\,p'\biggr).$$

Let us multiply (3.28) by $a(s)A(s)$ and use the above identity. Then we obtain

$$(a\,A^2\,p')' = \frac{-b}{1+\eta^2}\,a\,A^2\biggl(\Omega - \eta\,f_{\mathcal{R}}(|A|^2;\beta) + f_{\mathcal{I}}(|A|^2;\beta)\biggr). \tag{3.29}$$

To integrate (3.29) over $(\tilde{s}, s)$ with $s \in (\tilde{s}, \tilde{s}+\delta)$, we need to show that the one-sided limit $\lim_{s\searrow\tilde{s}} a(s)A^2(s)\,p'(s)$ exists. Since $u'(s) = (A'(s) + i\,A(s)\,p'(s))\,e^{ip(s)}$ for all $s \in (\tilde{s}, \tilde{s}+\delta)$, and $|u'(s)| < \infty$ for all $s \in [0, s_*]$ by (3.23), we have

$$|u'(s)|^2 = (A'(s))^2 + (A(s)\,p'(s))^2 \tag{3.30}$$

and so $\lim_{s\searrow\tilde{s}} A(s)\,p'(s)$ exists. Therefore, $\lim_{s\searrow\tilde{s}} A(s) = 0$ and (3.30) imply

$$|u'(\tilde{s})| = \lim_{s\searrow\tilde{s}} |A'(s)|, \tag{3.31}$$

$$\lim_{s\searrow\tilde{s}} a(s)\,A^2(s)\,p'(s) = 0. \tag{3.32}$$

Consequently, we can integrate (3.29) over $(\tilde{s}, s)$ for any $s \in (\tilde{s}, \tilde{s}+\delta)$, which yields

$$p'(s) = \frac{-b}{(1+\eta^2)\,a(s)A^2(s)} \int_{\tilde{s}}^{s} a(\sigma)A^2(\sigma)\biggl(\Omega - \eta\,f_{\mathcal{R}}(|A(\sigma)|^2;\beta) + f_{\mathcal{I}}(|A(\sigma)|^2;\beta)\biggr)\mathrm{d}\sigma. \tag{3.33}$$

Lemma 3.10. *Suppose that $\psi \in C^{2,\nu}(\mathcal{M}, \mathbb{C})$ and $\psi(s,\varphi) = u(s)\,e^{im\varphi}$ is a nontrivial solution of the full equation (3.22). Let $\tilde{s} \in [0, s_*]$ be either the location of a tip or a simple zero of $u(s)$, then in the polar form of ψ the phase derivative $p'(s)$ is continuously extendable to $s = \tilde{s}$ by the one-sided limit $\lim_{s\to\tilde{s}} p'(s) = 0$.*

Proof: Let $\tilde{s}$ be the location of a tip, that is, $a(\tilde{s}) = 0$. We apply L'Hôpital's rule on (3.33). It follows that $\lim_{s\to\tilde{s}} p'(s) = 0$ if the one-sided limit

$$\lim_{s\to\tilde{s}} \frac{a(s)}{a'(s) + 2\,a(s)\frac{A'(s)}{A(s)}} \tag{3.34}$$

is also zero. Suppose $\tilde{s} = 0$. Then $a'(0) = 1$, and by definition $\lim_{s\searrow 0} A'(s) \geq 0$. Thus the denominator in the limit (3.34) is positive as $s \searrow 0$. Since $a(0) = 0$, this limit (3.34) exists and is equal to zero. Our proof is also valid for the case $\tilde{s} = s_*$, since $a'(s_*) = -1$ and $\lim_{s\nearrow s_*} A'(s) \leq 0$.

On the other hand, let $\tilde{s} \in (0, s_*)$ be a simple zero of $u(s)$, that is, $u(\tilde{s}) = 0$

and $u'(\tilde{s}) \neq 0$. Then $\lim_{s\to\tilde{s}} A(s) = 0$, and $\lim_{s\to\tilde{s}} A'(s) \neq 0$ by (3.31). We apply L'Hôpital's rule on (3.33). Then $\lim_{s\to\tilde{s}} p'(s) = 0$ because $a(\tilde{s}) > 0$ and thus

$$\lim_{s\to\tilde{s}} \frac{a(s)A(s)}{a'(s)A(s) + 2\,a(s)A'(s)} = 0.$$

This completes the proof. $\bowtie$

3.5.7 Types of pattern

In this subsection we classify spatio-temporal patterns associated with the m-armed spiral Ansatz (3.20). Consider a nontrivial solution Ψ of the GLe in the polar form

$$\Psi(t,s,\varphi) = e^{-i\Omega t}\Big(A(s)\,e^{ip(s)}\Big)e^{im\varphi}.$$

The spatio-temporal pattern exhibited by Ψ is determined by the zero contour of the phase field, that is,

$$-\,\Omega t + p(s) + m\varphi = 0 \pmod{\pi} \tag{3.35}$$

for $t \geq 0$, $s \in [0, s_*]$, and $\varphi \in [0, 2\pi)$; see (3.3). Since $\varphi \in [0, 2\pi)$, (3.35) is equivalent to the following relation:

$$\varphi = \varphi(t,s) = \frac{\Omega t - p(s) + k\pi}{m}, \quad k = 0, 1, ..., 2m-1. \tag{3.36}$$

We obtain the spatio-temporal pattern via polar coordinates:

$$(t,s) \quad \mapsto \quad \big(a(s)\cos(\varphi(t,s)), a(s)\sin(\varphi(t,s)), \tilde{a}(s)\big) \in \mathcal{M}.$$

From (3.36) we readily see that $m \in \mathbb{N}$ is the ***number of arms*** and the phase function $p(s)$ determines the shape of the spatio-temporal pattern. Evidently, a pattern looks straight if $p(s)$ is a constant function, or equivalently, the phase derivative $p'(s)$ is identically zero. This leads to the following definition.

Definition. *Consider a spatio-temporal pattern associated with the m-armed spiral Ansatz* (3.20)*. Then we define the following:*

- *Such a pattern is called to be **rotating** if $\Omega \neq 0$, or **frozen** if $\Omega = 0$.*
- *Such a pattern is called a **spiral pattern** if $p'(s)$ is not identically zero, or a **meridian pattern** if $p'(s)$ is identically zero.*

For instance, if $\mathcal{M}$ is a 2-sphere, then a meridian pattern is seen as a union of usual meridians; if $\mathcal{M}$ is a disk, then it is seen as a pinwheel centered at the origin.

3.5.8 Shape of pattern near boundary

In this subsection we focus on an admissible surface of revolution $\mathcal{M}$ with boundary. Given a pattern associated with the m-armed spiral Ansatz (3.20), we study the angle between the tangent vector along the pattern and the tangent vector of the boundary, at a common boundary point.

First, the unit tangent vector field on $\partial\mathcal{M}$ is given by

$$\mathbf{t}(\varphi) = \big(-a(s_*)\sin(\varphi), a(s_*)\cos(\varphi), 0\big),$$

where $a(s_*) > 0$ because $\partial\mathcal{M}$ is nonempty. Second, a pattern is parametrized by

$$\mathbf{r}(t,s) := \big(a(s)\cos(\varphi(t,s)), a(s)\sin(\varphi(t,s)), \tilde{a}(s)\big),$$

where by (3.36)

$$\varphi(t,s) = \frac{\Omega t - p(s) + k\pi}{m}.$$

The boundary angle is determined by the dot product

$$\partial_s\mathbf{r}(t,s_*)\cdot\mathbf{t}(\varphi(t,s_*)) = -\frac{a^2(s_*)\,p'(s_*)}{m}. \tag{3.37}$$

Lemma 3.11. *Let $\partial\mathcal{M}$ be nonempty. Suppose that $\psi \in C^{2,\nu}(\mathcal{M},\mathbb{C})$ and $\psi(s,\varphi) = u(s)\,e^{im\varphi}$ is a nontrivial solution of the full equation* (3.22)*. Then in the polar form of ψ it follows $p'(s_*) = 0$. Consequently, the pattern of ψ hits boundary points in normal direction, by* (3.37)*.*

Proof: Suppose $\alpha \in (-\infty, 0)$. Then $u(s_*) = 0$ if and only if $u'(s_*) = 0$. Since ψ is nontrivial, by Lemma 3.9 (ii), $s = s_*$ cannot be a multiple zero of $u(s)$. Hence $u(s_*) \neq 0$ and so we can translate Robin boundary conditions in the polar form: $A'(s_*) = \alpha\,A(s_*)$ and $A(s_*)\,p'(s_*) = 0$. Since $A(s_*) > 0$, we see $p'(s_*) = 0$.

Suppose $\alpha = 0$. Then $u(s_*) \neq 0$ by Lemma 3.9 (ii) and we apply the previous arguments.

Suppose $\alpha = -\infty$, that is, $u(s_*) = 0$. Then $s = s_*$ is a simple zero of $u(s)$ by Lemma 3.9 (ii). Hence $p'(s_*) = 0$ by Lemma 3.10. The proof is complete. $\bowtie$

3.5.9 Decoupling effect for real Ginzburg-Landau equation

We show that the real GLe yields frozen meridian patterns, only. Consequently, to solve the real GLe, we consider without loss of generality the simpler Ansatz

$$\psi(s,\varphi) = u_{\mathcal{R}}(s)\,e^{im\varphi},$$

where $u_{\mathcal{R}}(s)$ is *real valued.* We call this result a *decoupling effect.*

Lemma 3.12 (decoupling effect). *Consider the full equation* (3.22) *with* $\eta = 0$ *and* $\beta = \mathbf{0}$. *Suppose that* $\psi \in C^{2,\nu}(\mathcal{M}, \mathbb{C})$ *and* $\psi(s,\varphi) = u(s)\, e^{im\varphi}$ *is a nontrivial solution. Then* $\Omega = 0$, *and in the polar form of* ψ *it follows that* $p'(s)$ *is identically zero.*

Proof: Since $\eta = 0$ and $\beta = \mathbf{0}$, $f_{\mathcal{I}}$ is identically zero (see the assumption (2.2)), and thus $\Omega = 0$ by the frequency-parameter relation (3.21). Then $p'(s)$ is identically zero by (3.33) and Lemma 3.10. $\bowtie$

3.5.10 Criterion for spiral patterns

For each fixed $b > 0$, the type of pattern exhibited by a nontrivial solution of the full equation (3.22) depends on parameters $\eta \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$. We derive a criterion that characterizes a parameter subregime of spiral patterns.

Assume that nontrivial solutions of the full equation are parametrized in an open subset $\mathcal{U}$ of the parameter regime $(\eta, \beta) \in \mathbb{R} \times \mathbb{R}^d$, that is,

$$\Omega = \widetilde{\Omega}(\eta, \beta) \in \mathbb{R}, \quad \psi = \widetilde{\psi}(\eta, \beta) \in C^{2,\nu}(\mathcal{M}, \mathbb{C})$$

for all $(\eta, \beta) \in \mathcal{U}$. Let $\widetilde{p}'(\eta, \beta)$ be the phase derivative of $\widetilde{\psi}(\eta, \beta)$ in the polar form. The parameter regime of spiral patterns is defined as

$$\mathcal{U}_{spiral} := \big\{(\eta, \beta) \in \mathcal{U} : \widetilde{p}'(\eta, \beta) \text{ is not identically zero}\big\}.$$

Lemma 3.13. *Suppose that* $(\widetilde{\Omega}(\eta, \beta), \widetilde{\psi}(\eta, \beta)) \in \mathbb{R} \times C^{2,\nu}(\mathcal{M}, \mathbb{C})$ *is a solution of the full equation* (3.22). *If in the polar form of* $\widetilde{\psi}(\eta, \beta)$ *the phase derivative* $\widetilde{p}'(\eta, \beta)$ *is identically zero, then*

$$\widetilde{\Omega}(\eta, \beta) - \eta\, f_{\mathcal{R}}(0; \beta) + f_{\mathcal{I}}(0; \beta) = 0. \tag{3.38}$$

Consequently, $\{(\eta, \beta) \in \mathcal{U} : (3.38) \text{ is violated}\}$ *is a parameter subregime of spiral patterns.*

Proof: Since $\widetilde{p}'(\eta, \beta)$ is identically zero, $s = 0$ is an isolated zero of $\widetilde{A}(s) := |\widetilde{\psi}(\eta, \beta)(s)|$, and $a(s) > 0$ for $s \in (0, s_*)$, by (3.29) there is a $\delta > 0$ such that

$$\widetilde{\Omega}(\eta, \beta) - \eta\, f_{\mathcal{R}}(|\widetilde{A}(s)|^2; \beta) + f_{\mathcal{I}}(|\widetilde{A}(s)|^2; \beta) = 0 \tag{3.39}$$

for all $s \in (0, \delta)$. Since $\lim_{s \searrow 0} \widetilde{A}(s) = 0$, by continuity of $\widetilde{A}$, $f_{\mathcal{R}}$, and $f_{\mathcal{I}}$, we let $s \searrow 0$ in (3.39) and then obtain (3.38). $\bowtie$

3.6 Summary

We have defined the constituents of spiral patterns: tips by nonzero winding numbers; spiral-like shapes by spatio-temporal patterns. As we seek rigidly-rotating spiral patterns, we have justified the popular m-armed spiral Ansatz by equivariance and the variational structure of the real Ginzburg-Landau equation. Moreover, we have derived the frequency-parameter relation and a criterion for spiral patterns; they are crucial to prove the existence of spiral patterns in the next chapter.

Chapter 4

Existence of Ginzburg-Landau spiral waves

In this chapter we prove the existence of Ginzburg-Landau spiral waves. In Section 4.1 we combine all necessary ingredients from the previous chapters and establish a *functional approach* to prove our main theorems. In Section 4.2 we solve the real Ginzburg-Landau equation by global bifurcation analysis. In Section 4.3 we solve the complex Ginzburg-Landau equation by the equivariant implicit function theorem. Then we determine parameter subregimes of spiral patterns by the frequency-parameter relation. In Section 4.4 we provide a plausible way to solve the conjecture on the existence of nodal solutions.

4.1 Functional approach

4.1.1 Preliminaries

Consider the *Ginzburg-Landau equation*

$$\partial_t \Psi = \frac{1}{b}(1 + i\,\eta)\Delta_{\mathcal{M},(\alpha)}\Psi + f(|\Psi|^2;\beta)\,\Psi. \tag{4.1}$$

- $b > 0$ is a manipulable *bifurcation parameter*. We use the reciprocal of b for simplicity of notation in the proof of the main theorems.
- $\eta \in \mathbb{R}$ is a given *complex diffusion parameter* and $\beta \in \mathbb{R}^d$ are given *kinetics parameters*.

The *Laplace-Beltrami operator* $\Delta_{\mathcal{M},(\alpha)}$ is defined on an *admissible surface of revolution* $\mathcal{M}$. If $\partial\mathcal{M}$ is nonempty, then we consider *Robin boundary conditions with the ratio* $\alpha \in [-\infty, 0]$, that is, $\partial_{\mathbf{n}}\Psi = \alpha\,\Psi$ on $\partial\mathcal{M}$.

Notation. In the functional setting $\Delta_{\mathcal{M},(\alpha)} : H^2_{(\alpha)}(\mathcal{M},\mathbb{C}) \subset L^2(\mathcal{M},\mathbb{C}) \to L^2(\mathcal{M},\mathbb{C})$:

- $\Delta_{\mathcal{M},(\alpha)} = \Delta_{\mathcal{M}}$ and $H^2_{(\alpha)}(\mathcal{M},\mathbb{C}) = H^2(\mathcal{M},\mathbb{C})$ if $\partial\mathcal{M}$ is empty;
- $\Delta_{\mathcal{M},(\alpha)} = \Delta_{\mathcal{M},\alpha}$ and $H^2_{(\alpha)}(\mathcal{M},\mathbb{C}) = H^2_{\alpha}(\mathcal{M},\mathbb{C})$ if we consider Robin boundary conditions with the ratio $\alpha \in [-\infty, 0]$.

More precisely, $\mathcal{M}$ admits the parametrization by polar coordinates

$$\mathcal{M} = \big\{ \big(a(s)\cos(\varphi), a(s)\sin(\varphi), \tilde{a}(s)\big) : s \in [0, s_*],\ \varphi \in [0, 2\pi) \big\},$$

where $a(s)$ and $\tilde{a}(s)$ are real analytic functions, $a(s) \geq 0$ for all $s \in [0, s_*]$, and moreover, the following conditions hold (see Lemma 2.3):

- $(a'(s))^2 + (\tilde{a}(s))^2 = 1$ for all $s \in [0, s_*]$.
- $a(0) = 0$, $a'(0) = 1$, and $a(s) > 0$ for all $s \in (0, s_*)$.
- If $\partial\mathcal{M}$ is nonempty, then $a(s_*) > 0$; if $\partial\mathcal{M}$ is empty, then $a(s_*) = 0$ and $a'(s_*) = -1$.

For each fixed $m \in \mathbb{N}$ and $\alpha \in [-\infty, 0]$, we substitute the following *m-armed spiral Ansatz* into the GLe (4.1):

$$\Psi(t, s, \varphi) = e^{-i\Omega t}\, \psi(s, \varphi), \quad \psi(s, \varphi) := u(s)\, e^{im\varphi},$$

where $u(s)$ is *complex valued*. Then we obtain the elliptic ***full equation*** for ψ:

$$\mathcal{F}(b|\, \Omega, \psi;\, \eta, \beta) := (1 + i\,\eta)\, \Delta_{m,(\alpha)}\psi + i\, b\, \Omega\, \psi + b\, f(|\psi|^2; \beta)\, \psi = 0, \qquad (4.2)$$

and the *rotation frequency* $\Omega \in \mathbb{R}$ is an unknown parameter that we have to determine. Here $\Delta_{m,(\alpha)} : H^2_{m,(\alpha)} \subset L^2_m \to L^2_m$ is the *projected operator*; see Section 2.3.2. Note that $L^2_m := \{\psi \in L^2(\mathcal{M},\mathbb{C}) : \psi(s,\varphi) = u(s)\, e^{im\varphi}\}$ is a closed L^2-subspace and $H^2_{m,(\alpha)} := H^2_{(\alpha)}(\mathcal{M},\mathbb{C}) \cap L^2_m$ is a closed H^2-subspace. The domain and the range of $\mathcal{F}$ are given by

$$\mathcal{F} : (0, \infty) \times \mathbb{R} \times H^2_{m,(\alpha)} \times \mathbb{R} \times \mathbb{R}^d \to L^2_m.$$

Notation. We use the notation $(b|\, \Omega, \psi;\, \eta, \beta)$ to distinguish $b > 0$ as the bifurcation parameter; $\Omega \in \mathbb{R}$ and $\psi \in H^2_{m,(\alpha)}$ as the unknowns of the full equation (4.2); $\eta \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$ as the parameters of the GLe.

We consider the complex-valued nonlinearity $f \in C^3([0,\infty)\times\mathbb{R}^d,\mathbb{C})$ and express $f = f_{\mathcal{R}} + i\, f_{\mathcal{I}}$ where $f_{\mathcal{R}}$ and $f_{\mathcal{I}}$ are real valued. Then the following *frequency-parameter relation* holds (see Section 3.4.2):

$$\int_{\mathcal{M}} \Big(\Omega - \eta\, f_{\mathcal{R}}(|\psi|^2;\beta) + f_{\mathcal{I}}(|\psi|^2;\beta)\Big)|\psi|^2 = 0. \tag{4.3}$$

We require two assumptions on the nonlinearity f.

(A1) $f_{\mathcal{R}}(0;\mathbf{0}) = 1$, and there exists a constant $C(f_{\mathcal{R}}) > 0$ such that

$$f_{\mathcal{R}}(y;\mathbf{0}) \begin{cases} = 0, & y = C(f_{\mathcal{R}}), \\ < 0, & y > C(f_{\mathcal{R}}). \end{cases}$$

Moreover, we assume

$$f_{\mathcal{I}}(y;\mathbf{0}) = 0 \quad \text{for all } y \geq 0. \tag{4.4}$$

(A2) $\partial_y f_{\mathcal{R}}(0;\mathbf{0}) < 0$ and $\partial_y f_{\mathcal{R}}(y;\mathbf{0}) \leq 0$ for all $y \in (0, C(f_{\mathcal{R}}))$.

4.1.2 Main theorems: Existence of spiral patterns

A solution pair (Ω,ψ) of the full equation (4.2) is called to be ***nontrivial*** if ψ is not identically zero. A nontrivial solution pair ***exhibits a spiral pattern*** if in the polar form of ψ, that is,

$$\psi(s,\varphi) = \Big(A(s)\, e^{ip(s)}\Big)\, e^{im\varphi},$$

the phase derivative $p'(s)$ is not identically zero. Otherwise it ***exhibits a meridian pattern***. Moreover, a pattern is called to be ***rotating*** if the rotation frequency $\Omega \neq 0$, or ***frozen*** if $\Omega = 0$.

Theorem I (Ginzburg-Landau equation)**.** *Let $\mu_0^{m,(\alpha)} < 0$ be the principal eigenvalue of $\Delta_{m,(\alpha)}$. Then for each $b > -\mu_0^{m,(\alpha)}$, there exists an $\epsilon > 0$ such that the full equation (4.2) possesses nontrivial solution pairs $(\Omega(\eta,\beta),\psi(\eta,\beta))$ parametrized by all $(\eta,\beta)\in\mathbb{R}\times\mathbb{R}^d$ and $0 \leq |\eta|, |\beta| < \epsilon$. Moreover, the following statements hold:*

(i) $(\Omega(\eta,\beta),\psi(\eta,\beta))$ *exhibits a* ***rotating spiral pattern*** *if $(\eta,\beta)\neq(0,\mathbf{0})$ lies in a small cone with $(0,\mathbf{0})$ as the vertex and $\{(\eta,\mathbf{0}) : 0 \leq |\eta| < \epsilon\}$ as the axis.*

(ii) *Suppose $d = 1$. Assume $\partial_\beta f_{\mathcal{I}}(y;0) \neq 0$ for all $y \in (0, C(f_{\mathcal{R}}))$ and $f_{\mathcal{I}}(0;\beta) = 0$ for all $\beta\in\mathbb{R}$. Then $(\Omega(\eta,\beta),\psi(\eta,\beta))$ exhibits a* ***rotating spiral pattern*** *if $(\eta,\beta)\neq(0,0)$ lies in a small cone with $(0,0)$ as the vertex and $\{(0,\beta) : 0 \leq |\beta| < \epsilon\}$ as the axis.*

Remark.

(i) Theorem 4.1.2 (ii) generalizes Theorem 2 in [Ts10] for the GLe nonlinearity, in the sense that we allow $\mathcal{M}$ to be without boundary or equipped with Robin boundary conditions, and more importantly, we also introduce the complex diffusion parameter η.

(ii) The constant $\epsilon > 0$ depends on choices of admissible surfaces of revolution $\mathcal{M}$, $\alpha \in [-\infty, 0]$, $f \in C^3([0,\infty) \times \mathbb{R}^d, \mathbb{C})$ satisfying (A1–A2), $m \in \mathbb{N}$, and $b > -\mu_0^{m,(\alpha)}$. One may derive a *lower bound* for this constant; see [Ts10] for $\mathcal{M}$ being a disk equipped with Neumann boundary conditions.

Proof: See the proof of Theorem 4.11 and Theorem 4.12. $\bowtie$

Theorem II (cubic supercritical Ginzburg-Landau equation)**.** *Suppose* $d = 1$ *and* $f(y;\beta) = 1 - y - i\,\beta\,y$. *Then there exists a smooth strictly decreasing function* $\widetilde{\eta}$: $(-\epsilon, \epsilon) \to \mathbb{R}$ *with* $\widetilde{\eta}(0) = 0$ *such that* $(\Omega(\eta,\beta), \psi(\eta,\beta))$ *exhibits a* ***rotating spiral pattern*** *if* $\eta \neq \beta$ *and* $\eta \neq \widetilde{\eta}(\beta)$*, or a* ***frozen spiral pattern*** *if* $\eta \neq \beta$ *and* $\eta = \widetilde{\eta}(\beta)$.

Proof: See the proof of Theorem 4.13. $\bowtie$

4.1.3 Global bifurcation analysis

Our proof consists of three steps.

- ***Step 1: global bifurcation analysis***

  We first solve the real GLe, that is, $\eta = 0$ and $\beta = \mathbf{0}$. Then $\Omega = 0$ due to the frequency-parameter relation (4.3) and the assumption (4.4). Thus the full equation (4.2) becomes the ***reduced equation***

$$\mathcal{F}(b|\,0,\psi;\,0,\mathbf{0}) = \Delta_{m,(\alpha)}\psi + b\,f_{\mathcal{R}}(|\psi|^2;\mathbf{0})\,\psi = 0. \tag{4.5}$$

  The unknown ψ is of the form $\psi(s,\varphi) = u_{\mathcal{R}}(s)\,e^{im\varphi}$ where the radial part $u_{\mathcal{R}}$ is *real valued*; see Lemma 3.12. Thus ψ exhibits a frozen meridian pattern.

  We seek nontrivial solutions of (4.5) that bifurcate from the trivial solution. Since the spectrum of $\Delta_{m,(\alpha)}$ consists of *simple eigenvalues* $\mu_n^{m,(\alpha)} < 0$ for each $n \in \mathbb{N}_0$, local bifurcation curves $\mathcal{C}_n^{m,(\alpha)}$ of nontrivial solutions exist. Moreover, $u_{\mathcal{R}}(s)$ solves a second-order ODE, so we can characterize each local bifurcation curve by the nodal property of its associated eigenfunction. We then adopt *open-closed-arguments* to prove that the *principal bifurcation curve* $\mathcal{C}_0^{m,(\alpha)}$ is *global* and *undergoes no secondary bifurcations.*

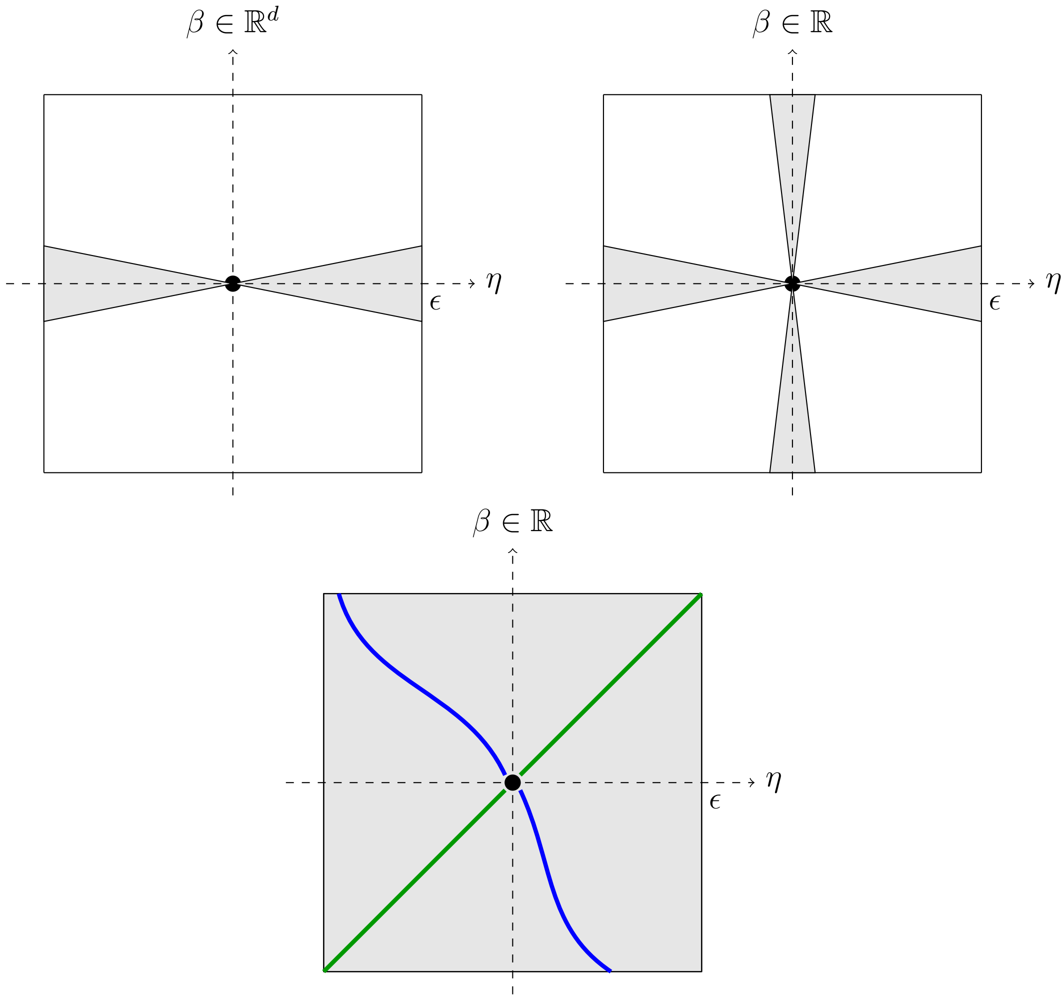

Figure: Types of pattern in the (η, β)-parameter space. The origin $(\eta, \beta) = (0, \mathbf{0})$ always supports frozen spiral patterns. Each parameter in the shaded gray region supports rotating spiral patterns. *Upper left*: Theorem I (i), for which we only assume (A1–A2). *Upper right*: Theorem I (ii), for which we need the additional assumptions on $f_{\mathcal{I}}$. *Lower*: Theorem II. We can completely classify the types of pattern for the cubic supercritical GLe. Each parameter on the diagonal green line $\eta = \beta$ supports frozen meridian patterns. Each parameter on the blue curve $\eta = \widetilde{\eta}(\beta)$ supports frozen spiral patterns.

- ***Step 2: perturbation arguments***
  We solve the complex GLe and prove that $\mathcal{C}_0^{m,(\alpha)}$ *persists* under all small parameters $0 < |\eta|$, $|\beta| \ll 1$ by the equivariant implicit function theorem in [RePe98].

- ***Step 3: determination on the type of pattern***
  For each nontrivial solution pair proved in Step 2, we determine whether it exhibits a frozen or rotating spiral pattern by the frequency-parameter relation (4.3) and the criterion (3.38) of spiral patterns.

4.1.4 Linearization

The spirit of bifurcation analysis is a reduction of a nonlinear problem to its linear one. Thus we study properties of the linearization of $\mathcal{F}$ in the full equation (4.2).

The first concern is whether $\mathcal{F}$ is Fréchet differentiable. Notice that the mapping $\psi \in H^2_{m,(\alpha)} \mapsto |\psi|^2 \in L^2_m$ is *not* complex Fréchet differentiable. Nevertheless, the complex differentiability is not required in bifurcation analysis, so we identify $\mathbb{C}$ as a real vector space; the inner product of L^2_m, now as a real vector space, is defined by

$$\langle \psi_1, \psi_2 \rangle_{L^2_m} := \mathrm{Re}\left(\int_0^{s_*} u_1(s)\, \overline{u_2(s)}\, a(s)\, \mathrm{d}s \right),$$

where $\psi_j(s,\varphi) = u_j(s)\, e^{im\varphi}$ for $j = 1, 2$. Thus a Fréchet derivative is a bounded linear operator over $\mathbb{R}$.

Notation. We do not identify $\mathbb{C}$ as $\mathbb{R}^2$ because multiplication by complex numbers yields simplicity of notations.

Lemma 4.1. *$\mathcal{F}$ in the full equation* (4.2) *is C^3 real Fréchet differentiable.*

Proof: It suffices to show that all k-times ($k \leq 3$) real Gâteaux derivatives of $\mathcal{F}$ exist and are bounded; see [ChHa82] Theorem 1.3. Since $f \in C^3$, it remains to show that the nonlinear part, $\psi \in H^2_{m,(\alpha)} \mapsto f(|\psi|^2; \beta)\, \psi \in L^2_m$, is k-times real Gâteaux differentiable with respect to ψ and β. The proof is straightforward by induction and the continuous embedding of $H^2(\mathcal{M}, \mathbb{C})$ into $C^0(\mathcal{M}, \mathbb{C})$; see Lemma 2.1 (ii). $\bowtie$

We study the real Gâteaux derivative

$$\mathcal{L}_{(b|\,\psi)} := D_\psi \mathcal{F}(b|\, 0, \psi;\, 0, \mathbf{0}) : H^2_{m,(\alpha)} \subset L^2_m \to L^2_m.$$

Definition. *Let X and Y be Banach spaces. A bounded linear operator $\mathcal{L} : Y \to X$ is called to be* ***Fredholm*** *if* $\dim \ker \mathcal{L}$ *and* $\dim \operatorname{coker} \mathcal{L}$ *are finite. The* ***index*** *of $\mathcal{L}$ is defined by* $\dim \ker \mathcal{L} - \dim \operatorname{coker} \mathcal{L}$.

Lemma 4.2. *$\mathcal{L}_{(b|\psi)}$ is self-adjoint on L^2_m and Fredholm of index zero.*

Proof: $\mathcal{L}_{(b|\psi)}$ is self-adjoint on L^2_m because $\mathcal{F}$ does not possess first-order differential operators. Being a linear elliptic operator, $\mathcal{L}_{(b|\psi)}$ is Fredholm; see [Ag&al97] Theorem 2.4.1. Its index is zero due to self-adjointness. $\bowtie$

4.2 Frozen meridian patterns for real Ginzburg-Landau equation

In this section we prove the existence of *frozen meridian patterns* for the *real* GLe. We first obtain local bifurcation curves by the bifurcation results from simple eigenvalues. We next label each bifurcation curve by the nodal property of its associated eigenfunction. Then we show that the *principal bifurcation curve* is *global.*

4.2.1 Local bifurcation from simple eigenvalues

To solve the reduced equation (4.5), note that the unknown ψ is of the form $\psi(s,\varphi) = u_{\mathcal{R}}(s)\, e^{im\varphi}$ where the radial part $u_{\mathcal{R}}$ is *real valued*; see Lemma 3.12.

Notation. For simplicity of notation, in the subsequent analysis we simply write $\psi(s,\varphi) = u(s)\, e^{im\varphi}$ where u is real valued. Let us introduce the closed L^2_m-subspace

$$L^2_{m,\mathcal{R}} := \left\{\psi \in L^2_m : \psi(s,\varphi) = u(s)\, e^{im\varphi} \text{ where } u \text{ is real valued}\right\}$$

and $H^2_{m,(\alpha),\mathcal{R}} := H^2_{m,(\alpha)} \cap L^2_{m,\mathcal{R}}$.

It is straightforward to obtain the linearization $\mathcal{L}_{(b|\psi)} := D_\psi \mathcal{F}(b|\, 0, \psi;\, 0, \mathbf{0})$:

$$\mathcal{L}_{(b|\psi)} U = \Delta_{m,(\alpha)} U + b\, f_{\mathcal{R}}(|\psi|^2; \mathbf{0})\, U + 2\, b\, \partial_y f_{\mathcal{R}}(|\psi|^2; \mathbf{0})\, |\psi|^2\, U, \tag{4.6}$$

where $U \in H^2_{m,(\alpha),\mathcal{R}}$. We now obtain nontrivial solutions of (4.5) that bifurcate from the trivial solution.

Lemma 4.3 (local bifurcation curves)**.** *For each $n \in \mathbb{N}_0$, nontrivial solutions of the reduced equation* (4.5) *near* $(-\mu_n^{m,(\alpha)}|\, 0)$ *form a **unique C^2 local bifurcation curve***

$$\mathcal{C}_n^{m,(\alpha)} := \left\{(b_n(\sigma)|\, \sigma\, \mathbf{e}_n^{m,(\alpha)} + v_n(\sigma)) : 0 \le |\sigma| \ll 1\right\} \subset (0,\infty) \times H^2_{m,(\alpha),\mathcal{R}}.$$

Here $\mathbf{e}_n^{m,(\alpha)}$ is the L^2_m-normalized eigenfunction associated with the eigenvalue $\mu_n^{m,(\alpha)}$ of $\Delta_{m,(\alpha)}$. Moreover, the following statements hold:

(i) *The shape of the local bifurcation curve is a supercritical pitchfork, because*

$$b_n(0) = -\mu_n^{m,(\alpha)}, \quad D_\sigma b_n(0) = 0, \quad D_\sigma^2 b_n(0) > 0.$$

(ii) $v_n(\sigma)$ *satisfies*

$$\left\langle v_n(\sigma), \mathbf{e}_n^{m,(\alpha)} \right\rangle_{L^2_m} = 0, \tag{4.7}$$

$v_n(0) = 0$*, and* $D_\sigma v_n(0) = 0$*. In particular, the local bifurcation curve intersects* $\mathbb{R} \times \{0\}$ *only at the* ***bifurcation point*** $(-\mu_n^{m,(\alpha)} | \, 0)$*.*

(iii) $((\mathbb{Z}_2 \times \mathbb{Z}_2)$*-equivariance) Suppose that* $\mathcal{M}$ *is without boundary and possesses the reflection symmetry, that is,* $a(s) = a(s_* - s)$ *for all* $s \in [0, s_*]$*. Then*

$$\psi_n(\sigma)(s, \varphi) = (-1)^n \, \psi_n(\sigma)(s_* - s, \varphi)$$

for all $0 \leq |\sigma| \ll 1$*,* $s \in [0, s_*]$*, and* $\varphi \in [0, 2\pi)$*.*

Proof: Except the proof of $D_\sigma b_n(0) = 0$ and $D_\sigma^2 b_n(0) > 0$, the item (i) and (ii) follow from the well-known local bifurcation results from simple eigenvalues. To see why, note $f_{\mathcal{R}}(0; \mathbf{0}) = 1$ by (A1), so by (4.6) the linearization at the trivial solution is $\mathcal{L}_{(b|\,0)} U = \Delta_{m,(\alpha)} U + b\, U$. Hence local bifurcation curves exist by [CrRa71] Theorem 1.7, since the spectrum of $\Delta_{m,(\alpha)}$ consists of *simple* eigenvalues $\mu_n^{m,(\alpha)} < 0$ for each $n \in \mathbb{N}_0$; see Lemma 2.4 (ii). These bifurcation curves are C^2 because $\mathcal{F}$ is C^3 real Fréchet differentiable; see Lemma 4.1.

To calculate $D_\sigma b_n(0)$ and $D_\sigma^2 b_n(0)$, we substitute $\psi = \psi(\sigma) = \sigma \, \mathbf{e}_n^{m,(\alpha)} + v_n(\sigma)$ into (4.5), differentiate (4.5) with respect to σ, use self-adjointness of $\Delta_{m,(\alpha)}$, and use (4.7). Then it follows

$$D_\sigma b_n(0) = -2 \left\langle D_\sigma v_n(\sigma), \mathbf{e}_n^{m,(\alpha)} \right\rangle_{L^2_m} = 0,$$

$$D_\sigma^2 b_n(0) = 2 \, \mu_n^{m,(\alpha)} \, \partial_y f_{\mathcal{R}}(0; \mathbf{0}) \int_0^{s_*} \left| u_n^{m,(\alpha)}(s) \right|^4 a(s) \, \mathrm{d}s > 0,$$

since $\partial_y f_{\mathcal{R}}(0; \mathbf{0}) < 0$ by (A2). Here $u_n^{m,(\alpha)}(s)$ is the radial part of the eigenfunction $\mathbf{e}_n^{m,(\alpha)}$. This proves the item (i–ii).

For (iii), with the reflection symmetry, (4.5) is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-equivariant under the following actions:

$$(\rho_{L^2_{m,\mathcal{R}}}(\gamma_N) \psi)(s, \varphi) := \psi(s_* - s, \varphi), \quad (\rho_{L^2_{m,\mathcal{R}}}(\gamma_D) \psi)(s, \varphi) := -\psi(s_* - s, \varphi).$$

Here γ_N (or γ_D) generates $\Gamma^N := \{\mathrm{id}, \gamma_N\} \cong \mathbb{Z}_2$ (or $\Gamma^D := \{\mathrm{id}, \gamma_D\} \cong \mathbb{Z}_2$). We consider the action γ_N only, because the proof for the other action γ_D is analogous.

Let us denote by $\mathcal{F}_0^N$ the restriction of $\mathcal{F}(b|\,0,\psi;0,\mathbf{0})$ on the Γ^N-fixed subspace. By Lemma 3.3, we see

$$\mathcal{F}_0^N : (0,\infty)\times \mathrm{Fix}_{H^2_{m,(\alpha),\mathcal{R}}}(\Gamma^N) \to \mathrm{Fix}_{L^2_{m,\mathcal{R}}}(\Gamma^N).$$

If $n \in \mathbb{N}_0$ is even, then $(-\mu_n^{m,(\alpha)}|\,0)$ is a bifurcation point of the restricted equation $\mathcal{F}_0^N(b|\,\psi) = 0$; see Lemma 2.5 (ii). Hence $\mathcal{C}_n^{m,(\alpha)}$ is contained in $(0,\infty)\times\mathrm{Fix}_{H^2_{m,(\alpha),\mathcal{R}}}(\Gamma^N)$ due to the uniqueness of local bifurcation curves. The proof is complete. $\bowtie$

For each $n \in \mathbb{N}_0$, since the local bifurcation curve $\mathcal{C}_n^{m,(\alpha)}$ intersects $\mathbb{R}\times\{0\}$ only at the bifurcation point $(-\mu_n^{m,(\alpha)}|\,0)$, the following decomposition holds:

$$\mathcal{C}_n^{m,(\alpha)} = \mathcal{C}_{n,+}^{m,(\alpha)} \cup \mathcal{C}_{n,-}^{m,(\alpha)} \cup \big\{(-\mu_n^{m,(\alpha)}|\,0)\big\}.$$

Here $\mathcal{C}_{n,+}^{m,(\alpha)}$ is the subset of $\mathcal{C}_n^{m,(\alpha)}$ that collects all $\sigma > 0$, and similarly $\mathcal{C}_{n,-}^{m,(\alpha)}$ collects all $\sigma < 0$.

Notation. The notation ι always applies for both cases $+$ and $-$.

Since the shape of $\mathcal{C}_n^{m,(\alpha)}$ is a supercritical pitchfork, taking $|\sigma|$ sufficiently small if necessary, we have the following characterization for $\mathcal{C}_{n,\iota}^{m,(\alpha)}$:

- ***monotone parametrization in*** $b \in (-\mu_n^{m,(\alpha)}, -\mu_n^{m,(\alpha)} + \delta_n]$**.**
  There exists a $\delta_n > 0$ and a smooth function

  $$\hat{\psi}_{n,\iota} : (-\mu_n^{m,(\alpha)}, -\mu_n^{m,(\alpha)} + \delta_n] \to H^2_{m,(\alpha),\mathcal{R}}$$

  such that $(b|\,\psi) \in \mathcal{C}_{n,\iota}^{m,(\alpha)}$ if and only if $b \in (-\mu_n^{m,(\alpha)}, -\mu_n^{m,(\alpha)}+\delta_n]$ and $\psi = \hat{\psi}_{n,\iota}(b)$.

Our main task is to prove that δ_n *extends to infinity*, and thus $\mathcal{C}_n^{m,(\alpha)}$ is ***global*** and undergoes no secondary bifurcations in $(0,\infty)\times H^2_{m,(\alpha),\mathcal{R}}$.

The main idea is to show that the set of $\delta > 0$ such that $\mathcal{C}_{n,\iota}^{m,(\alpha)}$ admits a monotone parametrization in $b \in (-\mu_n^{m,(\alpha)}, -\mu_n^{m,(\alpha)} + \delta]$ is open and closed in $(0,\infty)$. For openness, we apply the implicit function theorem. However, we can only prove openness for the ***principal bifurcation curve*** $\mathcal{C}_0^{m,(\alpha)}$. Nevertheless, we can prove closedness for *every* bifurcation curve.

4.2.2 Nodal structure of bifurcation curves

In this subsection we label each bifurcation curve by the number of simple zeros of its associated eigenfunction, as long as it admits a monotone parametrization.

Lemma 4.4 (nodal structure)**.** *Suppose that* $\mathcal{C}_{n,\iota}^{m,(\alpha)}$ *admits a monotone parametrization in* $b \in (-\mu_n^{m,(\alpha)}, -\mu_n^{m,(\alpha)} + \delta]$ *for some* $\delta > 0$. *Then for each* $(b|\,\psi) \in \mathcal{C}_{n,\iota}^{m,(\alpha)}$ *and* $\psi(s,\varphi) = u(s)\,e^{im\varphi}$, $u(s)$ *possesses exactly* n *simple zeros in* $(0, s_*)$.

Proof. By the Schauder elliptic regularity theory (see Lemma 3.8), $\psi \in C^{2,\nu}(\mathcal{M}, \mathbb{C})$ for any $\nu \in (0,1)$ and $u(s)$ is a nontrivial solution of the following second-order ODE:

$$u'' + \frac{a'}{a}u' - \frac{m^2}{a^2}u + b\,f_{\mathcal{R}}(u^2; \mathbf{0})\,u = 0. \tag{4.8}$$

Our proof is based on the following facts.

- Since identically zero is a solution of (4.8), the uniqueness of ODE initial value problems implies that all zeros of $u(s)$ in $(0, s_*)$ are simple; see the proof of Lemma 2.5 (i).

- Simple zeros of $u(s)$ neither accumulate at $s = 0$ nor at $s = s_*$; see Lemma 3.9. Hence there are *finitely many* simple zeros of $u(s)$ in $(0, s_*)$.

- Since $\mathcal{C}_{n,\iota}^{m,(\alpha)}$ admits a monotone parametrization, the mapping $b \mapsto u = u(\cdot; b)$ is well defined for all $b \in (-\mu_n^{m,(\alpha)}, -\mu_n^{m,(\alpha)} + \delta]$ and smooth. Thus we have the ODE smooth dependence of solutions on the parameter b.

The above three facts imply that all nontrivial solutions $u(s; b)$ of (4.8) with $b \in (-\mu_n^{m,(\alpha)} - \mu_n^{m,(\alpha)} + \delta]$ possess the same number of simple zeros in $(0, s_*)$. Recall that the radial part $u_n^{m,(\alpha)}(s)$ of the eigenfunction $\mathbf{e}_n^{m,(\alpha)}$ possesses exactly n simple zeros in $(0, s_*)$; see Lemma 2.5 (i). Therefore, nontrivial solutions $u(s; b)$ with b sufficiently near $-\mu_n^{m,(\alpha)}$ possess exactly n simple zeros in $(0, s_*)$; see Lemma 4.3 (ii). This completes the proof. $\bowtie$

4.2.3 C^0-bound

We obtain an important C^0-bound for solutions of the reduced equation.

Lemma 4.5 (C^0-bound)**.** *Let* $(b|\,\psi) \in \mathcal{C}_n^{m,(\alpha)}$ *and* $\psi(s,\varphi) = u(s)\,e^{im\varphi}$. *Then*

$$|\psi|_{C^0} = \sup_{s \in [0, s_*]} |u(s)| \le \sqrt{C(f_{\mathcal{R}})}. \tag{4.9}$$

Proof. By the Schauder elliptic regularity theory (see Lemma 3.8), $\psi \in C^{2,\nu}(\mathcal{M}, \mathbb{C})$ and $u(s)$ satisfies the ODE

$$u'' + \frac{a'}{a}u' - \frac{m^2}{a^2}u + b\, f_{\mathcal{R}}(u^2; \mathbf{0})\, u = 0. \tag{4.10}$$

Suppose the contrary that $|\psi|_{C^0} > \sqrt{C(f_{\mathcal{R}})}$. Then there would be an $s_M \in [0, s_*]$ such that $|u(s_M)| > \sqrt{C(f_{\mathcal{R}})}$. Since $[0, s_*]$ is compact, we assume without loss of generality that s_M is a global extreme point: $|u(s_M)| \geq |u(s)|$ for all $s \in [0, s_*]$. We consider the case $u(s_M) > \sqrt{C(f_{\mathcal{R}})}$ because the proof for the other case $u(s_M) < -\sqrt{C(f_{\mathcal{R}})}$ is analogous.

We show $s_M \in (0, s_*)$ by dealing with the following four cases:

- Case 1: $\partial\mathcal{M}$ is empty.
  $u(0) = u(s_*) = 0$ by Lemma 3.9 (i), so $s_M \in (0, s_*)$.

If $\partial\mathcal{M}$ is nonempty, then $u(0) = 0$ by Lemma 3.9 (i).

- Case 2: Dirichlet boundary conditions.
  In this case $u(s_*) = 0$, so $s_M \in (0, s_*)$.

- Case 3: Robin boundary conditions with the ratio $\alpha \in (-\infty, 0)$.
  Suppose the contrary $s_M = s_*$. Since $\alpha < 0$, Robin boundary conditions imply $u'(s_*) < 0$, and thus $s = s_*$ cannot be a global extreme point of $u(s)$, which is a contradiction. Hence $s_M \in (0, s_*)$.

- Case 4: Neumann boundary conditions.
  Suppose the contrary $s_M = s_*$. We aim at a contradiction from the equation (4.10). Since $u(s_*) > \sqrt{C(f_{\mathcal{R}})}$ and $u'(s_*) = 0$, by continuity there would exist a $\delta > 0$ such that $u(s) > \sqrt{C(f_{\mathcal{R}})}$ and

  $$|u'(s)| < \frac{m^2}{\max_{s\in[0,s_*]} a(s)} \sqrt{C(f_{\mathcal{R}})} \tag{4.11}$$

  hold for all $s \in (s_* - \delta, s_*)$. Thus (A1) implies $f_{\mathcal{R}}(|u(s)|^2; \mathbf{0}) < 0$ for all $s \in (s_* - \delta, s_*)$. Since $b > 0$, from (4.10) the following inequalities hold for all $s \in (s_* - \delta, s_*)$:

  $$\begin{aligned} u''(s) &\geq \frac{m^2}{a^2(s)}\, u(s) - \frac{|a'(s)|}{a(s)}\, |u'(s)| \\ &\geq \frac{m^2}{a^2(s)} \sqrt{C(f_{\mathcal{R}})} - \frac{1}{a(s)}\, |u'(s)| \\ &= \frac{1}{a^2(s)} \left(m^2 \sqrt{C(f_{\mathcal{R}})} - a(s)\, |u'(s)| \right) \\ &> 0. \end{aligned}$$

We have used $\sup_{s\in[0,s_*]}|a'(s)|\le 1$ (see Lemma 2.3 (i)) for the second inequality, and (4.11) for the last inequality. Since $u'(s_*)=0$ and $u''(s)>0$ for all $s\in(s_*-\delta,s_*)$, we see $u'(s)<0$ for all $s\in(s_*-\delta,s_*)$. Thus $s=s_*$ cannot be a global extreme point of $u(s)$, which is a contradiction. Hence $s_M\in(0,s_*)$.

Therefore, $s_M\in(0,s_*)$, that is, s_M lies in the interior of $[0,s_*]$, so $u'(s_M)=0$ and $u''(s_M)\le 0$, yielding a contradiction as we plug $s=s_M$ into (4.10). ⋈

4.2.4 Hyperbolicity: Comparison of principal eigenvalues

We aim at proving openness of a monotone parametrization. More precisely, if $\mathcal{C}_{n,\iota}^{m,(\alpha)}$ admits a monotone parametrization in $b\in(-\mu_n^{m,(\alpha)},-\mu_n^{m,(\alpha)}+\delta]$ for some $\delta>0$, then we need to show that it extends to $b\in(-\mu_n^{m,(\alpha)},-\mu_n^{m,(\alpha)}+\widetilde{\delta})$ for some $\widetilde{\delta}>\delta$.

We obtain such an extension by the implicit function theorem. Thus for each $(b|\,\psi)\in\mathcal{C}_{n,\iota}^{m,(\alpha)}$, we need to show that the linearization $\mathcal{L}_{(b|\,\psi)}$ defined in (4.6) is a linear homeomorphism. Because $\mathcal{L}_{(b|\,\psi)}$ is Fredholm of index zero (see Lemma 4.2), it suffices to show that $\ker\mathcal{L}_{(b|\,\psi)}$ is trivial, or equivalently, zero is not an eigenvalue of $\mathcal{L}_{(b|\,\psi)}$. In this case ψ is called to be ***hyperbolic*** because it is a hyperbolic equilibrium of the real GLe restricted on $H^2_{m,(\alpha),\mathcal{R}}$:

$$\partial_t\Psi=\Delta_{m,(\alpha)}\Psi+b\,f_{\mathcal{R}}(|\Psi|^2;\mathbf{0})\,\Psi. \tag{4.12}$$

However, we can only prove openness for the principal bifurcation curve $\mathcal{C}_0^{m,(\alpha)}$. We leave the question of openness for other bifurcation curves as a conjecture and provide a plausible way to solve it; see Section 4.4.

Lemma 4.6. *Suppose that there is a $\delta>0$ such that $\mathcal{C}_{0,\iota}^{m,(\alpha)}$ admits a monotone parametrization in $b\in(-\mu_n^{m,(\alpha)},-\mu_n^{m,(\alpha)}+\delta]$. Then there exists a $\widetilde{\delta}>\delta$ such that the monotone parametrization extends to $b\in(-\mu_n^{m,(\alpha)},-\mu_n^{m,(\alpha)}+\widetilde{\delta})$.*

Proof: We denote by $b_0:=-\mu_n^{m,(\alpha)}+\delta$ and $\psi_0:=\hat{\psi}_{n,\iota}(b_0)$ where $\hat{\psi}_{n,\iota}$ is the monotone parametrization of $\mathcal{C}_{0,\iota}^{m,(\alpha)}$. Once we prove hyperbolicity of $(b_0|\,\psi_0)$, this lemma follows from the the implicit function theorem; see [ChHa82] Theorem 2.3.

By (4.6) the linear equation $\mathcal{L}_{(b_0|\,\psi_0)}U=0$ for $U\in H^2_{m,(\alpha),\mathcal{R}}$ reads

$$\Big(\Delta_{m,(\alpha)}+b_0\,f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})+2\,b_0\,\partial_y f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\,|\psi_0|^2\Big)U=0.$$

To show that U is identically zero, we compare the principal eigenvalues of two different but related systems. To see it, let μ_1^* be the principal eigenvalue of the following eigenvalue problem:

$$\Big(\Delta_{m,(\alpha)} + b_0\, f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\Big) V_1 = \mu_1\, V_1. \tag{4.13}$$

Since $\psi_0 \in H^2_{m,(\alpha),\mathcal{R}}$ is a nontrivial solution of (4.13) for $\mu_1 = 0$, and by Lemma 4.4 its radial part $u_0(s)$ does not possess zeros in $(0, s_*)$, we see $\mu_1^* = 0$ and $\ker\mathcal{L}_{(b_0|\,\psi_0)} = \mathrm{span}_{\mathbb{R}}\langle\psi_0\rangle$; see [GiTr83] Theorem 8.38.

Compare μ_1^* with the principal eigenvalue μ_2^* of the following eigenvalue problem:

$$\Big(\Delta_{m,(\alpha)} + b_0\, f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0}) + 2\, b_0\, \partial_y f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\, |\psi_0|^2\Big) V_2 = \mu_2\, V_2. \tag{4.14}$$

Since ψ_0 satisfies the C^0-bound (4.9), by (A2), we have $\partial_y f_{\mathcal{R}}(|\psi_0(s,\varphi)|^2;\mathbf{0}) \le 0$ for all $s \in (0, s_*]$ and $\varphi \in [0, 2\pi)$. Moreover, $s = 0$ is an isolated zero of any nontrivial solution $V_2(s,\varphi)$ of (4.14); see the proof in Lemma 3.9 (i). Since $\partial_y f_{\mathcal{R}}(|\psi_0(0,\varphi)|^2;\mathbf{0}) = \partial_y f_{\mathcal{R}}(0;\mathbf{0}) < 0$ by (A2), we have

$$\int_{\mathcal{M}} \partial_y f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\, |\psi_0|^2\, |V_2|^2 < 0.$$

Hence the Rayleigh quotient for principal eigenvalues (see [CaCo03] Theorem 2.1) implies $\mu_2^* < \mu_1^* = 0$ and so U is identically zero. $\bowtie$

Remark (stability)**.** The proof in Lemma 4.6 tells us more than hyperbolicity: The principal eigenvalue of (4.14) is *negative* for every $(b|\,\psi) \in \mathcal{C}^{m,(\alpha)}_{0,\iota}$. Since the real GLe (4.12) generates a local semiflow on $H^{2\zeta}_{m,(\alpha),\mathcal{R}}$ for any $\zeta \in (\frac{1}{2}, 1)$ (see Lemma 2.7), ψ is a locally asymptotically stable equilibrium of (4.12) under $H^{2\zeta}_{m,(\alpha),\mathcal{R}}$-perturbations. It is interesting to determine the stability of ψ under full $H^{2\zeta}_{(\alpha)}$-perturbations.

4.2.5 Monotone extension of bifurcation curves

We prove closedness of a monotone parametrization for *every* bifurcation curve. As a result, a bifurcation curve is global if we can prove openness of its monotone parametrization.

Lemma 4.7. *Suppose that there is a $\widetilde{\delta} > 0$ such that $\mathcal{C}^{m,(\alpha)}_{n,\iota}$ admits a monotone parametrization in $b \in (-\mu_n^{m,(\alpha)}, -\mu_n^{m,(\alpha)} + \widetilde{\delta})$. Then the monotone parametrization extends to $b = -\mu_n^{m,(\alpha)} + \widetilde{\delta}$.*

Proof: Let $b_n := -\mu_n^{m,(\alpha)} + \widetilde{\delta}$, and $(b^j|\,\psi^j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{C}^{m,(\alpha)}_{n,\iota}$ such that $b^j \nearrow b_n$. We show that there exists a $\psi_n \in H^2_{m,(\alpha),\mathcal{R}}$ such that $\lim_{j\to\infty} \psi^j = \psi_n$

holds in $H^2_{m,(\alpha),\mathcal{R}}$ and $(b_n|\,\psi_n) \in \mathcal{C}^{m,(\alpha)}_{n,\iota}$ and the monotone parametrization extends to $b = b_n$.

The key of proof is the C^0-bound (4.9), that is, $|\psi^j|_{C^0} \leq \sqrt{C(f_\mathcal{R})}$ for all $j \in \mathbb{N}$. First, we have the uniform L^2-bound

$$|\psi^j|^2_{L^2} = \int_\mathcal{M} |\psi^j|^2 \leq C(f_\mathcal{R})\,\mathrm{Vol}(\mathcal{M}).$$

Second, let us multiply (4.5) by the complex conjugate $\overline{\psi}$, integrate over $\mathcal{M}$, apply the divergence theorem, and use Robin boundary conditions. Then

$$\begin{aligned}\int_\mathcal{M} |\nabla_g \psi^j|^2 &= \alpha \int_{\partial\mathcal{M}} |\psi^j|^2 + b^j \int_\mathcal{M} f_\mathcal{R}(|\psi^j|^2;\mathbf{0})\,|\psi^j|^2 \\ &\leq |\alpha|\,C(f_\mathcal{R})\,\mathrm{Vol}(\partial\mathcal{M}) + b^j\,C(f_\mathcal{R})\,\mathrm{Vol}(\mathcal{M}).\end{aligned}$$

In the inequality we have used $\sup_{y\in[0,C(f_\mathcal{R})]} |f_\mathcal{R}(y;\mathbf{0})| = 1$ due to (A1–A2). Thus we have a uniform H^1-bound on every compact b-subinterval. Since $H^1(\mathcal{M},\mathbb{C})$ is compactly embedded into $L^2(\mathcal{M},\mathbb{C})$ (see Lemma 2.1 (i)), passing to a subsequence if necessary, there exists a $\psi_n \in L^2_{m,\mathcal{R}}$ such that $\lim_{j\to\infty} \psi^j = \psi_n$ holds in $L^2_{m,\mathcal{R}}$

Third, since $f_\mathcal{R}$ is C^1, it follows by the triangle inequality and the C^0-bound (4.9) that $\lim_{j\to\infty} b^j\, f_\mathcal{R}(|\psi^j|^2;\mathbf{0})\,\psi^j = b_n\, f_\mathcal{R}(|\psi_n|^2;\mathbf{0})\,\psi_n$ holds in $L^2_{m,\mathcal{R}}$. By Lemma 2.4 (ii), $\Delta_{m,(\alpha)} : H^2_{m,(\alpha),\mathcal{R}} \to L^2_{m,\mathcal{R}}$ is a linear homeomorphism, so

$$\lim_{j\to\infty} \Delta^{-1}_{m,(\alpha)}\big(b^j\, f_\mathcal{R}(|\psi^j|^2;\mathbf{0})\,\psi^j\big) = \Delta^{-1}_{m,(\alpha)}\big(b_n\, f_\mathcal{R}(|\psi_n|^2;\mathbf{0})\,\psi_n\big)$$

holds in $H^2_{m,(\alpha),\mathcal{R}}$. Since every $(b^j|\,\psi^j)$ is a solution of (4.5), $\lim_{j\to\infty}\psi^j = \psi_n$ holds in $H^2_{m,(\alpha),\mathcal{R}}$ and $(b_n|\,\psi_n)$ is also a solution of (4.5).

Last, near each bifurcation point, nontrivial solutions of (4.5) form a *unique* local bifurcation curve whose shape is a supercritical pitchfork; see Lemma 4.3. Thus ψ_n is not identically zero by the implicit function theorem. Hence $(b_n|\,\psi_n) \in \mathcal{C}^{m,(\alpha)}_{n,\iota}$ by Lemma 4.4, and the monotone parametrization extends to $b = b_n$ by Lemma 4.6. The proof is complete. ⋈

Lemma 4.8. *The principal bifurcation curve* $\mathcal{C}^{m,(\alpha)}_0$ *is global and undergoes no secondary bifurcations in* $(0,\infty) \times H^2_{m,(\alpha),\mathcal{R}}$.

Proof: The set of $\delta > 0$ such that $\mathcal{C}^{m,(\alpha)}_{0,\iota}$ admits a monotone parametrization in $b \in (-\mu^{m,(\alpha)}_n, -\mu^{m,(\alpha)}_n + \delta]$ is nonempty by Lemma 4.3, open in $(0,\infty)$ by Lemma 4.6, and closed in $(0,\infty)$ by Lemma 4.7. ⋈

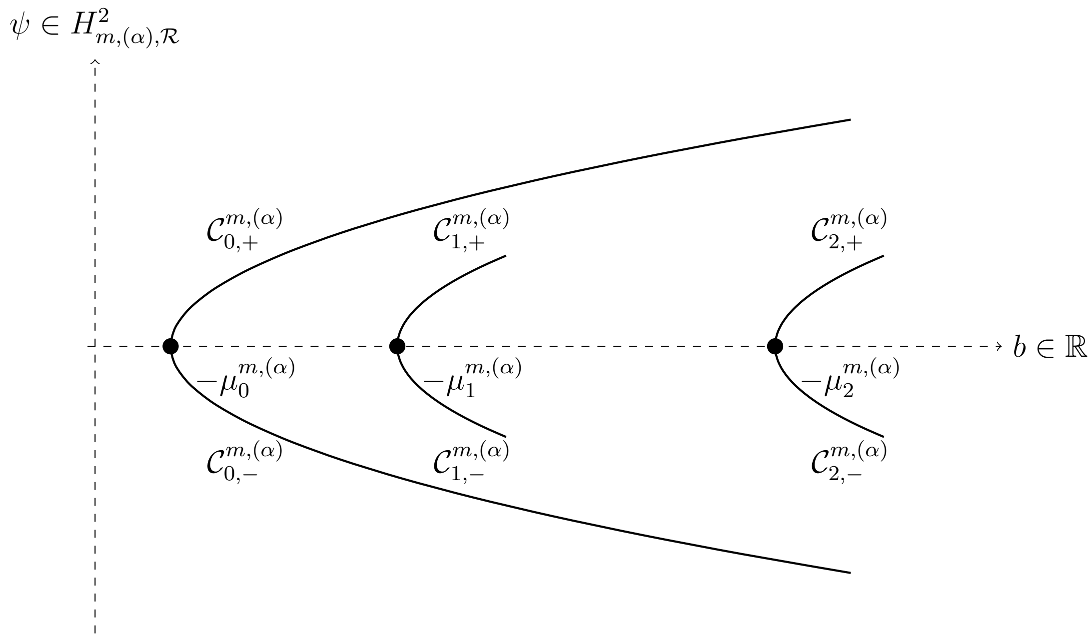

Figure: Bifurcation diagram $(b|\,\psi) \in \mathbb{R} \times H^2_{m,(\alpha),\mathcal{R}}$ of the reduced equation (4.5). Notice that the ψ-component of each bifurcation curve may be unbounded, since the H^2-bound of solutions is not guaranteed by the C^0-bound (4.9).

4.3 Spiral patterns for complex Ginzburg-Landau equation

In this section we prove the existence of *genuine spiral patterns* for the *complex* GLe. We adopt *perturbation arguments* and use the *equivariant implicit function theorem.* We then determine parameter subregimes of spiral patterns.

4.3.1 Perturbation arguments

We solve the unknowns $\Omega \in \mathbb{R}$ and $\psi \in H^2_{m,(\alpha)}$ of the full equation (4.2):

$$\mathcal{F}(b|\,\Omega, \psi;\, \eta, \beta) = 0.$$

Let $(b_n|\,\psi_n) \in \mathcal{C}^{m,(\alpha)}_{n,\iota}$ be a solution of the reduced equation (4.5):

$$\mathcal{F}(b_n|\,0, \psi_n;\, 0, \mathbf{0}) = 0.$$

We need to parametrize the unknowns $\Omega \in \mathbb{R}$ and $\psi \in H^2_{m,(\alpha)}$ by parameters $b > 0$, $\eta \in \mathbb{R}$, and $\beta \in \mathbb{R}^d$ in the vicinity of $(b_n|\,0, \mathbf{0})$. Such perturbation arguments lead us to study the real Gâteaux derivative

$$\mathcal{L}_{(b_n|\,\psi_n)} := D_\psi \mathcal{F}(b_n|\,0, \psi_n; 0, \mathbf{0}) : H^2_{m,(\alpha)} \subset L^2_m \to L^2_m.$$

Since $\psi_n(s,\varphi) = u_n(s)\, e^{im\varphi}$ where u_n is real valued, it is straightforward to derive

$$\mathcal{L}_{(b_n|\,\psi_n)}U = \Delta_{m,(\alpha)}U + b_n\, f_{\mathcal{R}}(|\psi_n|^2;\mathbf{0})\, U + 2\, b_n\, \partial_y f_{\mathcal{R}}(|\psi_n|^2;\mathbf{0})\, |\psi_n|^2\, U_R\, e^{im\varphi}. \quad (4.15)$$

Here $U = (U_R + i\, U_I)\, e^{im\varphi} \in H^2_{m,(\alpha)}$ where $U_{\mathcal{R}}$ and $U_{\mathcal{I}}$ are real valued. Recall that $\mathcal{L}_{(b_n|\,\psi_n)}$ is self-adjoint on L^2_m and Fredholm of index zero; see Lemma 4.2.

To study $\ker\mathcal{L}_{(b_n|\,\psi_n)}$, we recall the $(S^1\times S^1)$-equivariance of $\mathcal{F}$: $\psi(s,\varphi) = u(s)\, e^{im\varphi}$ is a solution of the full equation (4.2) if and only if $e^{-i\vartheta}\,\psi(s,\varphi-\gamma)$ is also a solution, for each $(\vartheta,\gamma)\in S^1\times S^1$. Since $e^{-i\vartheta}\,\psi(s,\varphi-\gamma) = e^{-i(\vartheta+m\gamma)}\,\psi(s,\varphi)$, we identify without loss of generality that $\mathcal{F}$ possesses S^1-equivariance. This S^1-equivariance implies that $\ker\mathcal{L}_{(b_n|\,\psi_n)}$ contains a nontrivial subspace $T_{\psi_n}(S^1\psi_n)$, that is, the tangent space along the group orbit of ψ_n. It follows $T_{\psi_n}(S^1\psi_n) = \mathrm{span}_{\mathbb{R}}\langle i\,\psi_n\rangle$ and hence

$$\mathrm{span}_{\mathbb{R}}\langle i\,\psi_n\rangle \subset \ker\mathcal{L}_{(b_n|\,\psi_n)}.$$

Therefore, the standard implicit function theorem is not applicable. This situation, however, is amendable to the following *equivariant implicit function theorem*, because $\Omega\in\mathbb{R}$ is a *real one-dimensional* unknown.

Lemma 4.9 (equivariant implicit function theorem)**.** *Let X, Y, and Λ be Banach spaces. Suppose that $\mathcal{H}\in C^k(Y\times\Lambda, X)$ for $k\ge 2$ and $\mathcal{H}(v_0,0)=0$. Let Γ be a compact Lie group and ρ_X (resp. ρ_Y) be an action of Γ on X (resp. on Y). Suppose that $\mathcal{H}(\cdot,\lambda)$ is Γ-equivariant, that is, $\mathcal{H}(\rho_Y(\gamma)v,\lambda) = \rho_X(\gamma)\mathcal{H}(v,\lambda)$ for all $\gamma\in\Gamma$, $v\in Y$, and $\lambda\in\Lambda$. Assume the following:*

- *$\mathcal{L} := D_v\mathcal{H}(v_0,0)$ is Fredholm of index zero.*
- *The isotropy subgroup is trivial, that is, $\Sigma_{v_0} = \{\mathrm{id}\}$.*
- *Assume $\Lambda = \Lambda_1\times\Lambda_2$ such that*

$$\dim\Lambda_1 = \dim\ker\mathcal{L} \quad (4.16)$$

and

$$X = \mathcal{L}Y \oplus D_\lambda\mathcal{H}(v_0,0)\Lambda_1. \quad (4.17)$$

Then there exist neighbourhoods $W\subset Y$ of the group orbit Γv_0 and $W_j\subset\Lambda_j$ of $\lambda_j=0$ for $j=1,2$, and C^k-mappings $\widetilde{v}: W_2\to Y_0$ and $\widetilde{\lambda}_1 : W_2\to\Lambda_1$ with $\widetilde{v}(0)=v_0$ and $\widetilde{\lambda}_1(0)=0$ such that for each $(v,\lambda_1,\lambda_2)\in W\times W_1\times W_2$, $\mathcal{H}(v,\lambda_1,\lambda_2)=0$ if and only if $v=\rho_Y(\gamma)\widetilde{v}(\lambda_2)$ for some $\gamma\in\Gamma$ and $\lambda_1=\widetilde{\lambda}_1(\lambda_2)$.

Proof: See [RePe98] Theorem 3.1. ⋈

To apply Lemma 4.9, we choose

$$\mathcal{H} := \mathcal{F}, \quad X := L^2_m, \quad Y := H^2_{m,(\alpha)}, \quad \Omega \in \Lambda_1 := \mathbb{R}, \quad (b|\,\eta,\beta) \in \Lambda_2 := \mathbb{R}^{d+2}$$

Moreover, we take

$$v_0 := \psi_n, \quad \Gamma := S^1, \quad \mathcal{L} := \mathcal{L}_{(b_n|\,\psi_n)}.$$

Then it follows $\Sigma_{\psi_n} = \{0\}$. Let us denote by $B^{d+1}_{\epsilon_2}(\mathbf{0})$ the $(d+1)$-dimensional ball with radius $\epsilon_2 > 0$ centered at the origin.

Lemma 4.10 (perturbation arguments)**.** *Let $(b_0|\,\psi_0) \in \mathcal{C}^{m,(\alpha)}_{0,\iota}$. Then there exists an $\epsilon_1 > 0$ and smooth functions*

$$\widetilde{\psi} : (b_0 - \epsilon_1, b_0 + \epsilon_1) \times B^{d+1}_{\epsilon_1}(\mathbf{0}) \to H^2_{m,(\alpha)}, \quad \widetilde{\psi}(b_0|\,0,\mathbf{0}) = \psi_0,$$
$$\widetilde{\Omega} : (b_0 - \epsilon_1, b_0 + \epsilon_1) \times B^{d+1}_{\epsilon_1}(\mathbf{0}) \to \mathbb{R}, \qquad \widetilde{\Omega}(b_0|\,0,\mathbf{0}) = 0,$$

such that for each $(b|\,\eta,\beta) \in (b_0 - \epsilon_1, b_0 + \epsilon_1) \times B^{d+1}_{\epsilon_1}(\mathbf{0})$, $\mathcal{F}(b|\,\Omega,\psi;\,\eta,\beta) = 0$ if and only if $\Omega = \widetilde{\Omega}(b|\,\eta,\beta)$ and $\psi = \widetilde{\psi}(b|\,\eta,\beta)$.

Proof: We apply Lemma 4.9 and it suffices to verify (4.16) and (4.17).

To verify (4.16), since $\Lambda_1 = \mathbb{R}$ is real one-dimensional and $\mathrm{span}_{\mathbb{R}}\langle i\,\psi_0\rangle \subset \ker\mathcal{L}_{(b_0|\,\psi_0)}$, we need to show $\mathrm{span}_{\mathbb{R}}\langle i\,\psi_0\rangle = \ker\mathcal{L}_{(b_0|\,\psi_0)}$.

By (4.15), the linear equation $\mathcal{L}_{(b_0|\,\psi_0)}U = 0$ for $U = (U_R + i\,U_I)\,e^{im\varphi} \in H^2_{m,(\alpha)}$ is equivalent to the following decoupled system:

$$\begin{aligned}
\Big(\Delta_{m,(\alpha)} + b_0\,f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0}) + 2\,b_0\,\partial_y f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\,|\psi_0|^2\Big)U_R\,e^{im\varphi} &= 0,\\
\Big(\Delta_{m,(\alpha)} + b_0\,f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\Big)U_I\,e^{im\varphi} &= 0.
\end{aligned}$$

Since $(b_0|\,\psi_0) \in \mathcal{C}^{m,(\alpha)}_{0,\iota}$, the same proof in Lemma 4.6 shows that $U_{\mathcal{R}}\,e^{im\varphi}$ is identically zero and $U_{\mathcal{I}}\,e^{im\varphi} = \psi_0$. Hence $U = i\,\psi_0$ and so $\ker\mathcal{L}_{(b_0|\,\psi_0)} \subset \mathrm{span}_{\mathbb{R}}\langle i\,\psi_0\rangle$.

We next verify (4.17). Let us calculate the Gâteaux derivative

$$D_{(\Omega;\,\eta,\beta)}\mathcal{F}(b_0|\,0,\psi_0;\,0,\mathbf{0})(\Omega_*;\,0,\mathbf{0}) = i\,b_0\,\Omega_*\,\psi_0.$$

Suppose that $V = (V_R + i\,V_I)\,e^{im\varphi} \in H^2_{m,(\alpha)}$ solves the equation $\mathcal{L}_{(b_0|\,\psi_0)}V = i\,b_0\,\Omega_*\,\psi_0$. Then the equation for $V_I\,e^{im\varphi}$,

$$\Big(\Delta_{m,(\alpha)} + b_0\,f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\Big)V_I\,e^{im\varphi} = b_0\,\Omega_*\,\psi_0,$$

implies $b_0\,\Omega_*\langle\psi_0,\psi_0\rangle_{L^2_m} = 0$, due to self-adjointness. Hence $\Omega_* = 0$ and so (4.17) is verified. The proof is complete. ⋈

4.3.2 Determination on spiral patterns

We prove that nontrivial solution pairs obtained by our perturbation arguments can exhibit spiral patterns.

Theorem 4.11. *Let $b_0 > -\mu_0^{m,(\alpha)}$ be fixed. By Lemma 4.10, let $(\widetilde{\Omega}(\eta,\beta), \widetilde{\psi}(\eta,\beta))$ be nontrivial solution pairs of the full equation (4.2). Then there exists an $\epsilon > 0$ such that the following statements hold as $(\eta,\beta) \in B_\epsilon^{d+1}(\mathbf{0})$:*

(i) *There exists a smooth function $\widetilde{\eta} : B_\epsilon^d(\mathbf{0}) \to \mathbb{R}$, $\widetilde{\eta}(\mathbf{0}) = 0$, such that $\widetilde{\Omega}(\eta,\beta) = 0$ if and only if $\eta = \widetilde{\eta}(\beta)$.*

(ii) *There exists a smooth function $\kappa_1 : (-\epsilon,\epsilon) \to [0,\infty)$ such that $\kappa_1(\eta) > 0$ if $\eta \neq 0$, and $(\widetilde{\Omega}(\eta,\beta), \widetilde{\psi}(\eta,\beta))$ exhibits a rotating spiral pattern for all $\eta \neq 0$ and $0 \leq |\beta| \leq \kappa_1(\eta)$.*

Proof: For (i), the main idea is to use the frequency-parameter relation (4.3)

$$\mathcal{J}(\widetilde{\Omega}(\eta,\beta), \widetilde{\psi}(\eta,\beta), \eta, \beta) := \int_{\mathcal{M}} \Big(\widetilde{\Omega} - \eta\, f_{\mathcal{R}}(|\widetilde{\psi}|^2;\beta) + f_{\mathcal{I}}(|\widetilde{\psi}|^2;\beta)\Big) |\widetilde{\psi}|^2 = 0. \quad (4.18)$$

Using $\widetilde{\Omega}(0,\mathbf{0}) = 0$, $\widetilde{\psi}(0,\mathbf{0}) = \psi_0$, and the assumption (4.4): $f_{\mathcal{I}}(y;\mathbf{0}) = 0$ for all $y \geq 0$, we differentiate $\mathcal{J}$ with respect to η and evaluate at $(\eta,\beta) = (0,\mathbf{0})$. Then we obtain

$$\partial_\eta \widetilde{\Omega}(0,\mathbf{0}) = \frac{\int_{\mathcal{M}} f_{\mathcal{R}}(|\psi_0|^2;\mathbf{0})\,|\psi_0|^2}{\int_{\mathcal{M}} |\psi_0|^2}. \quad (4.19)$$

Since ψ_0 satisfies the C^0-bound (4.9): $|\psi_0|_{C^0} \leq \sqrt{C(f_{\mathcal{R}})}$, (A1–A2) and (4.19) imply

$$0 < \partial_\eta \widetilde{\Omega}(0,\mathbf{0}) < 1. \quad (4.20)$$

Since $\widetilde{\Omega}(0,\mathbf{0}) = 0$ and $0 < \partial_\eta \widetilde{\Omega}(0,\mathbf{0})$, the implicit function theorem yields the existence of $\widetilde{\eta}$. This proves the item (i)

For (ii), recall the criterion (3.38) for spiral patterns, that is, $(\widetilde{\Omega}(\eta,\beta), \widetilde{\psi}(\eta,\beta))$ exhibits a spiral pattern if

$$\widetilde{\Omega}(\eta,\beta) - \eta\, f_{\mathcal{R}}(0;\beta) + f_{\mathcal{I}}(0;\beta) \neq 0. \quad (4.21)$$

Since $f_{\mathcal{R}}(0;\mathbf{0}) = 1$ by (A1), we have $\partial_\eta \widetilde{\Omega}(0,\mathbf{0}) < f_{\mathcal{R}}(0;\mathbf{0})$ by (4.20). Since $\widetilde{\Omega}(0,\mathbf{0}) = 0$, by smoothness of $\widetilde{\Omega}$, we see $\widetilde{\Omega}(\eta,\mathbf{0}) \neq \eta\, f_{\mathcal{R}}(0;\mathbf{0})$ for all $\eta \neq 0$ sufficiently near zero. By the assumption (4.4) and smoothness of $f_{\mathcal{R}}$ and $f_{\mathcal{I}}$, there exists a smooth function $\kappa_1 = \kappa_1(\eta)$ such that (4.21) holds for all $\eta \neq 0$ sufficiently near zero and $0 \leq |\beta| \leq \kappa_1(\eta)$. The proof is complete. $\bowtie$

Notice that Theorem 4.11 (ii) does not assure the existence of spiral patterns if η is almost zero, but $\beta \neq \mathbf{0}$ is large. Such a result of existence requires more assumptions on $f_{\mathcal{I}}$. For instance, suppose $d = 1$, that is, $\beta \in \mathbb{R}$. Then a prototype considered in [KoHo81] and [Ts10] is $f_{\mathcal{I}}(y;\beta) = \beta\,\omega(y)$, where $\omega(0) = 0$ and $\omega(y) \neq 0$ for all $y \in (0, C(f_{\mathcal{R}}))$. Evidently, the cubic supercritical GLe nonlinearity $f_{\mathcal{I}}(y;\beta) = -\beta\, y$ is the simplest example of the prototype. Based on the prototype we impose the following two assumptions:

$$\partial_\beta f_{\mathcal{I}}(y;0) \neq 0 \quad \text{for all } y \in (0, C(f_{\mathcal{R}})); \tag{4.22}$$

$$f_{\mathcal{I}}(0;\beta) = 0 \quad \text{for all } \beta \in \mathbb{R}. \tag{4.23}$$

Theorem 4.12. *Suppose $d = 1$ and under the same assumptions of Theorem 4.11. Then the following statements hold:*

(i) *Assume (4.22). Then $\widetilde{\eta}$ proved in Theorem 4.11 (ii) is invertible.*

(ii) *Assume (4.22) and (4.23). Then there exists a smooth function $\kappa_2 : (-\epsilon, \epsilon) \to [0, \infty)$ such that $\kappa_2(\beta) > 0$ if $\beta \neq 0$, and $(\widetilde{\Omega}(\eta, \beta), \widetilde{\psi}(\eta, \beta))$ exhibits a rotating spiral pattern for all $\beta \neq 0$ and $0 \leq |\eta| \leq \kappa_2(\beta)$.*

Proof: For (i), we differentiate $\mathcal{J}$ in (4.18) with respect to β and evaluate at $(\eta, \beta) = (0, 0)$. Then (4.22) implies

$$\partial_\beta \widetilde{\Omega}(0,0) = \frac{-\int_{\mathcal{M}} \partial_\beta f_{\mathcal{I}}(|\psi_0|^2; 0)\, |\psi_0|^2}{\int_{\mathcal{M}} |\psi_0|^2} \neq 0. \tag{4.24}$$

Thus $\widetilde{\eta}$ is invertible. This proves the item (i).

For (ii), by (4.23) the criterion (4.21) now reads

$$\widetilde{\Omega}(\eta, \beta) - \eta\, f_{\mathcal{R}}(0;\beta) \neq 0. \tag{4.25}$$

By $\widetilde{\Omega}(0,0) = 0$ and (4.24), we have $\widetilde{\Omega}(0,\beta) \neq 0$ for all $\beta \neq 0$ sufficiently near zero. By smoothness of $f_{\mathcal{R}}$, there exists a smooth function $\kappa_2 = \kappa_2(\beta)$ such that (4.25) holds for all $\beta \neq 0$ sufficiently near zero and $0 \leq |\eta| \leq \kappa_2(\beta)$. This completes the proof. $\bowtie$

Although nontrivial solution pairs of the full equation are parametrized by all parameters (η, β) in $B_\epsilon^{d+1}(\mathbf{0})$, the main question is to determine the type of pattern for *each* (η, β) in $B_\epsilon^{d+1}(\mathbf{0})$. For general GLe, the local information of $(\eta, \beta) = (0, \mathbf{0})$ is not sufficient for the criterion (4.21) to give a definite answer. But still, we have a definite answer for the cubic supercritical GLe.

Theorem 4.13. *Under the same assumptions of Theorem* 4.11*, consider the cubic supercritical Ginzburg-Landau equation, that is,* $d = 1$ *and* $f(y;\beta) = 1 - y - i\,\beta\, y$*. Then the following statements hold as* $(\eta,\beta) \in B^2_\epsilon(\mathbf{0})$*:*

(i) $\widetilde{p}'(\eta,\beta)(s)$ *is identically zero if and only if* $\eta = \beta$*.*

(ii) *The smooth function* $\widetilde{\eta} = \widetilde{\eta}(\beta)$ *proved in Theorem* 4.11 (ii) *is strictly decreasing.*

Consequently, $(\widetilde{\Omega}(\eta,\beta), \widetilde{\psi}(\eta,\beta))$ *exhibits a rotating spiral pattern if* $\eta \neq \beta$ *and* $\eta \neq \widetilde{\eta}(\beta)$*; it exhibits a frozen spiral pattern if* $\eta \neq \beta$ *and* $\eta = \widetilde{\eta}(\beta)$*.*

Proof: For (i), suppose that $\widetilde{p}'(\eta,\beta)$ is identically zero. The criterion (3.38) for spiral patterns and the specific form of nonlinearity imply $\widetilde{\Omega}(\eta,\beta) = \eta$. Therefore, (4.18) reads $\int_{\mathcal{M}} (\eta - \beta)\,|\widetilde{\psi}|^4 = 0$. Since $\widetilde{\psi}$ is not identically zero, we see $\eta = \beta$.

Conversely, suppose $\eta = \beta$. Then (4.18) reads $\int_{\mathcal{M}} (\widetilde{\Omega}(\eta,\eta) - \eta)\,|\widetilde{\psi}|^2 = 0$ and so $\widetilde{\Omega}(\eta,\eta) = \eta$. Thus the m-armed spiral Ansatz is of the form $\Psi(t,s,\varphi) = e^{-i\eta t}\,\widetilde{\psi}(s,\varphi)$, where Ψ solves the equation

$$\partial_t \Psi = \frac{1}{b_0}(1 + i\,\eta)\Delta_{\mathcal{M},(\alpha)}\,\Psi + (1 - |\Psi|^2 - i\,\eta\,|\Psi|^2)\,\Psi.$$

It is easy to verify that $\widetilde{\psi}$ solves the real GLe:

$$\Delta_{m,(\alpha)}\widetilde{\psi} + b_0\,(1 - |\widetilde{\psi}|^2)\,\widetilde{\psi} = 0.$$

Therefore, $\widetilde{p}'(\eta,\eta)(s)$ is identically zero due to the decoupling effect; see Lemma 3.12. This proves the item (i).

For (ii), since $f_{\mathcal{I}}(y;\beta) = -\beta\, y$ fulfills the assumption (4.22), Theorem 4.12 (i) implies that $\widetilde{\eta}$ is invertible. It remains to show $D_\beta\widetilde{\eta}(0) < 0$. We differentiate the relation $\widetilde{\Omega}(\widetilde{\eta}(\beta),\beta) = 0$ with respect to β and evaluate at $\beta = 0$. Then we obtain

$$D_\beta\widetilde{\eta}(0) = -\frac{\partial_\beta\widetilde{\Omega}(0,0)}{\partial_\eta\widetilde{\Omega}(0,0)}.$$

By (4.20) we have $\partial_\eta\widetilde{\Omega}(0,0) > 0$. Moreover, since $\partial_\beta f_{\mathcal{I}}(y;0) = -y^2$, by (4.24) we see $\partial_\beta\widetilde{\Omega}(0,0) > 0$. Hence $D_\beta\widetilde{\eta}(0) < 0$. The proof is complete. ⋈

4.4 Nodal solutions

We propose a conjecture that all other bifurcation curves are also global and provide a plausible way to solve it. In the end we present a partial result of the conjecture.

4.4.1 Conjecture

Consider $(b_n|\,\psi_n) \in \mathcal{C}_{n,\iota}^{m,(\alpha)}$ for $n \in \mathbb{N}$ and $\psi_n(s,\varphi) = u_n(s)\, e^{im\varphi}$ where the radial part $u_n(s)$ is real valued and possesses n simple zeros in $(0, s_*)$; see Lemma 4.4. Recall the linear equation $\mathcal{L}_{(b_n|\,\psi_n)}U = 0$ for $U = (U_\mathcal{R} + i\, U_\mathcal{I})\, e^{im\varphi} \in H^2_{m,(\alpha)}$, which is equivalent to the following decoupled system:

$$\Big(\Delta_{m,(\alpha)} + b_n\, f_\mathcal{R}(|\psi_n|^2; \mathbf{0}) + 2\, b_n\, \partial_y f_\mathcal{R}(|\psi_n|^2; \mathbf{0})\, |\psi_n|^2\Big) U_R\, e^{im\varphi} = 0, \tag{4.26}$$

$$\Big(\Delta_{m,(\alpha)} + b_n\, f_\mathcal{R}(|\psi_n|^2; \mathbf{0})\Big) U_I\, e^{im\varphi} = 0. \tag{4.27}$$

Let us review the *logic chain* of proof for the existence of spiral wave solutions.

- $\mathcal{C}_{n,\iota}^{m,(\alpha)}$ is global and undergoes no secondary bifurcations if $U_\mathcal{R}\, e^{im\varphi}$ is identically zero for *every* $(b_n|\,\psi_n) \in \mathcal{C}_{n,\iota}^{m,(\alpha)}$; see Lemma 4.6, 4.7, and 4.8.
- Spiral wave solutions exist for the bifurcation parameter $b = b_n$ if $U_\mathcal{R}\, e^{im\varphi}$ is identically zero and ψ_n is the unique nontrivial solution of (4.27) up to multiplication by real numbers; see Lemma 4.10 and notice that verification of (4.17) requires self-adjointness, only.
- Determination on the types of pattern is a direct consequence of perturbation arguments; see Lemma 4.10.

Indeed (4.26–4.27) are equivalent to the following second-order linear ODEs:

$$U_\mathcal{R}'' + \frac{a'}{a}U_\mathcal{R}' - \frac{m^2}{a^2}U_\mathcal{R} + b_n\, f_\mathcal{R}(u_n^2; \mathbf{0})\, U_\mathcal{R} + 2\, b_n\, \partial_y f_\mathcal{R}(u_n^2; \mathbf{0})\, u_n^2\, U_R = 0, \tag{4.28}$$

$$U_\mathcal{I}'' + \frac{a'}{a}U_\mathcal{I}' - \frac{m^2}{a^2}U_\mathcal{I} + b_n\, f_\mathcal{R}(u_n^2; \mathbf{0})\, U_\mathcal{I} = 0. \tag{4.29}$$

If we assume that $f_\mathcal{R}$ is real analytic, then by the Frobenius series it follows that both (4.28) and (4.29) possess *at most one* bounded nontrivial solution; see the proof in Lemma 2.4 (ii). Since u_n already solves (4.29), it remains to show that U_R is identically zero.

Conjecture. For each $(b_n|\,\psi_n) \in \mathcal{C}_{n,\iota}^{m,(\alpha)}$ with $n \in \mathbb{N}$, if $U_\mathcal{R}\, e^{im\varphi} \in H^2_{m,(\alpha)}$ is a solution of (4.26), then $U_\mathcal{R}$ is identically zero.

Remark.

(i) Once the conjecture is solved, we obtain $2(n+1)$ different spiral wave solutions for each $b > -\mu_n^{m,(\alpha)}$ and suitable choices of parameters η and β.

(ii) Comparison of principal eigenvalues used in the proof of Lemma 4.6 is fruitless, since $u_n(s)$ possesses simple zeros in $(0, s_*)$, and thus $\mu_1^* > 0$.

(iii) For $m = 0$, the conjecture has been solved if $\partial\mathcal{M}$ is either equipped with Dirichlet boundary conditions or empty; see [Na83] and [La17].

Here is a logic for solving the conjecture: Suppose that there were a nontrivial solution $U_{\mathcal{R}}\, e^{im\varphi} \in H^2_{m,(\alpha)}$ of (4.26). Then $U_{\mathcal{R}}(s)$ would be a bounded nontrivial solution of (4.28), and thus $s = 0$ is an isolated zero of $U_{\mathcal{R}}(s)$; see Lemma 3.9 (i). We would reach a contradiction if we could show that *any* bounded nontrivial solution v of (4.28) satisfies

- either $v'(s_*) \neq \alpha\, v(s_*)$ if $u_n'(s_*) = \alpha\, u_n(s_*)$, that is, when $\partial\mathcal{M}$ is equipped with Robin boundary conditions with the ratio $\alpha \in [-\infty, 0]$;
- or $v(s_*) \neq 0$ if $u_n(s_*) = 0$, that is, when $\partial\mathcal{M}$ is empty;
- or that v violates certain symmetries of u_n.

One plausible way to solve the conjecture is *phase portrait analysis* and the idea of proof is *comparison of angle functions of solutions.* Let v be a bounded nontrivial solution of (4.28). We define the associated angle functions of solutions by

$$\theta_n(s) := \arctan\left(\frac{u_n'(s)}{u_n(s)}\right), \quad \theta_v(s) := \arctan\left(\frac{v'(s)}{v(s)}\right).$$

Note that θ_n and θ_v are well defined because all zeros of nontrivial solutions of (4.28) or (4.29) are simple.

Since $s = 0$ is an isolated zero of both $u_n(s)$ and $v(s)$, by linearity we consider without loss of generality $u_n(s) > 0$ and $v(s) > 0$ for all $s > 0$ sufficiently near zero. Then by analyticity the power series expansion holds

$$u_n(s) = C_1\, s^m + O(s^{m+1}), \quad v(s) = C_2\, s^m + O(s^{m+1}), \quad \text{as } s \to 0, \tag{4.30}$$

for some $C_1, C_2 > 0$. Hence the initial angles are equal and satisfy

$$\lim_{s\to 0} \theta_n(s) = \lim_{s\to 0} \theta_v(s) = \frac{\pi}{2}.$$

Since $\partial_y f_{\mathcal{R}}(0; \mathbf{0}) < 0$ and $\partial_y f_{\mathcal{R}}(u_n^2(s); \mathbf{0}) \leq 0$ for all $s \in (0, s_*]$ by (A2), from (4.28) and (4.29) we obtain the following comparison:

$$\theta_n(s) < \theta_v(s) \quad \text{for all } s \in (0, s_*]. \tag{4.31}$$

In other words, the phase portrait $(u_n(s), u_n'(s))$ *rotates faster* than $(v(s), v(s))$.

We have three observations. First, $\theta_n(s)$ (and $\theta_v(s)$) is strictly decreasing at the u_n'-axis (and the v'-axis) in the phase plane. Second, if $v\, e^{im\varphi} \in H^2_{m,(\alpha)}$ is a nontrivial solution of (4.26), then by (4.31) there exists a $k \in \mathbb{N}$ such that the difference of angle functions at $s = s_*$ satisfies

$$\theta_v(s_*) - \theta_n(s_*) = k\pi. \tag{4.32}$$

Third, since $u_n(s)$ possesses exactly n simple zeros in $(0, s_*)$, we have

$$-\frac{\pi}{2} - n\pi \le \theta_n(s_*) < \frac{\pi}{2} - n\pi,$$

and thus by (4.31),

$$-\frac{\pi}{2} - n\pi < \theta_v(s_*) < \frac{\pi}{2}.$$

We readily see that if $n = 0$, then $0 < \theta_v(s_*) - \theta_0(s_*) < \pi$ and hence (4.32) cannot hold. Therefore, for $n = 0$ the conjecture is true because $v\, e^{im\varphi} \notin H^2_{m,(\alpha)}$. For general $n \in \mathbb{N}$, however, (4.32) can still possibly hold.

To violate (4.32), one way is to show that $v(s)$ also possesses exactly n simple zeros in $(0, s_*)$. Then $-\frac{\pi}{2} - n\pi < \theta_v(s_*) < \frac{\pi}{2} - n\pi$, and so $0 < \theta_v(s_*) - \theta_n(s_*) < \pi$. Therefore, the conjecture is true if we can solve the following problem.

Problem. For $(b_n|\, \psi_n) \in \mathcal{C}^{m,(\alpha)}_{n,\iota}$, show that every bounded nontrivial solution $v(s)$ of (4.28) possesses exactly n zeros in $(0, s_*)$.

If $u_n(s_*) = 0$, then the problem has been solved in [Na83] for $m = 0$, and in [Sc97] for $m \neq 0$ but *subcritical nonlinearity*. For $m \neq 0$ and supercritical nonlinearity, the problem remains open. The main difficulty is lack of a *correct sign* that allows us to apply the maximum principles on suitable auxiliary functions for comparison.

4.4.2 Partial result

Partial results of the conjecture are available, say, when the full equation (4.2) possesses certain equivariance so that *solutions obey certain symmetries*. For instance, consider that $n = 1$ and $\mathcal{M}$ is without boundary and possesses the reflection symmetry. Then $s = \frac{s_*}{2}$ is the *unique* zero of $u_1(s)$, and also a zero of $U_{\mathcal{R}}(s)$ if $U_{\mathcal{R}}(s)\, e^{im\varphi} \in H^2_{m,(\alpha)}$ is a nontrivial solution of (4.26); see Lemma 4.3 (iii). This symmetry of solutions yields a contradiction, because from (4.31) we already know $v(\frac{s_*}{2}) > 0$ if v is any bounded nontrivial solution of (4.28). Let us give a proof.

Theorem 4.14. *The conjecture is true if $n = 1$ and $\mathcal{M}$ is without boundary and possesses the reflection symmetry.*

Proof: Let v be any bounded nontrivial solution of (4.28) and $v(s) > 0$ for all $s > 0$ sufficiently near zero. By the $\mathbb{Z}_2$-equivariance induced by the reflection symmetry, $u_1(s) > 0$ for all $s \in (0, \frac{s_*}{2})$ and $u_1(\frac{s_*}{2}) = 0$. It suffices to show $v(\frac{s_*}{2}) \neq 0$. Indeed we prove more: $v(s) > 0$ for all $s \in (0, \frac{s_*}{2}]$.

Suppose the contrary that there were a *first zero* $\tilde{s} \in (0, \frac{s_*}{2}]$ of $v(s)$, that is, $v(s) > 0$ for all $s \in (0, \tilde{s})$ and $v(\tilde{s}) = 0$. Then by definition

$$v'(\tilde{s}) \leq 0. \tag{4.33}$$

To compare the angle functions of u_1 and v, it is natural to consider the *auxiliary function of Riccati type*

$$W := \frac{v}{u_1}.$$

Note that $W(s)$ is bounded in $(0, \tilde{s}]$ due to the power series expansion (4.30) as $s \to 0$, and the fact that all zeros of $u_1(s)$ or $v(s)$ in $(0, s_*)$ are simple.

It is straightforward to verify that $W(s)$ satisfies the following ODE for $s \in (0, \tilde{s})$:

$$\left(u_1\, W'' + 2\, u_1'\, W' + \frac{a'}{a} u_1\, W'\right) = -2\, b_n\, \partial_y f_{\mathcal{R}}(u_1^2; \mathbf{0})\, u_1^3\, W. \tag{4.34}$$

Multiplying (4.34) by $a(s)u_1(s)$, we obtain

$$(a\, u_1^2\, W')' = -2\, b_n\, a\, \partial_y f_{\mathcal{R}}(u_1^2; \mathbf{0})\, u_1^4\, W. \tag{4.35}$$

Using $a(0) = 0$, $a(s) > 0$ for all $s \in (0, \tilde{s})$, (A2), and $W(s) > 0$ for all $s \in (0, \tilde{s})$, we integrate (4.35) over $(0, \tilde{s})$ and obtain

$$\lim_{s \to \tilde{s}} u_1^2(s) W'(s) > 0.$$

Since $u_1^2(s)W'(s) = u_1(s)v'(s) - u_1'(s)v(s)$, using $u_1(\tilde{s}) \geq 0$, (4.33), and $v(\tilde{s}) = 0$, we reach a contradiction: $\lim_{s \to \tilde{s}} u_1^2(s)W'(s) \leq 0$. The proof is complete. $\bowtie$

Chapter 5

Conclusion

In this chapter we present a brief overview of this thesis. Then we discuss an issue related to our analysis and indicate four directions for future research.

5.1 Overview

In this thesis we establish a *functional approach* to prove the *existence of Ginzburg-Landau spiral waves*. Based on systematic considerations, we justify the popular *m-armed spiral Ansatz* by *equivariance* and the *variational structure* of the real Ginzburg-Landau equation. This spiral Ansatz transforms the Ginzburg-Landau equation into an *elliptic equation*. To solve this elliptic equation by our functional approach, we adopt *global bifurcation analysis* and the result of existence is essentially a consequence of *compactness*.

The advantage of our functional approach is threefold. First, it avoids smart, but tricky, estimates used in the shooting method. Second, it works for more general underlying spatial domains, not only in the *circular geometry*, but also in the *spherical geometry*. Third, it permits the occurrence of a *mixed diffusion process* when a complex diffusion parameter is introduced. Thus our result of existence of rigidly-rotating spiral waves greatly generalizes those in the literature; see [KoHo81] and [Ts10]. Moreover, we prove the existence of two new patterns: *frozen spirals* in circular and spherical geometries, and *2-tip spirals* in the spherical geometry.

5.2 Discussion and outlook

The Ginzburg-Landau spiral waves we have proved typically possess two features: *slowly rotating* and possibly *slightly twisting*, due to perturbation arguments; see

Lemma 4.10. Such spiral waves are very different from those observed in the Belousov-Zhabotinsky equation; see [Be&al97]. These two features are rooted in the global S^1-equivariance, and also in the assumption (4.4) by which the Ginzburg-Landau equation is weakly coupled for all small parameters $0 < |\eta|, |\beta| \ll 1$. To seek fast rotating or greatly twisting spiral waves, we need to either consider sufficiently large $|\eta|, |\beta|$ (see the heuristic attempts in [Ha82]) or replace (4.4).

Finally, we indicate four directions for future research.

- ***Direction 1: nodal solutions***
  We can ask whether Theorem I and II hold for $n \in \mathbb{N}$; see the discussion in Section 4.4.

- ***Direction 2: Ginzburg-Landau scroll waves***
  Pattern formation of scroll waves requires three-dimensional spatial domains; see the survey [FiSc03]. We can ask whether Ginzburg-Landau scroll waves exist, for instance, on the three-dimensional unit ball

  $$\mathcal{B} = \big\{ \big(r\sin(s)\cos(\varphi), r\sin(s)\sin(\varphi), r\cos(s)\big) : r \in [0,1], s \in [0,\pi], \varphi \in [0,2\pi) \big\}.$$

  Analogously, consider the *m-armed scroll wave Ansatz*:

  $$\Psi(t,r,s,\varphi) = e^{-\Omega t}\,\psi(r,s,\varphi), \quad \psi(r,s,\varphi) = u(r,s)\,e^{im\varphi}.$$

  The major difference is that the resulting elliptic equation is *not* equivalent to a second-order ODE, because $u(r,s)$ depends on both r and s. In particular, not all eigenvalues of $\Delta_{m,(\alpha)}$ are simple. But still, the *principal eigenvalue of* $\Delta_{m,(\alpha)}$ *is simple*. Thus for the case $\eta = 0$ and $\beta = \mathbf{0}$, the principal bifurcation curve $\mathcal{C}^{m,(\alpha)}_{0,\mathcal{B}}$ always exists, at least locally. This curve would be global if we could prove that it possesses the nodal structure; see Lemma 4.4.

- ***Direction 3: stability of Ginzburg-Landau spiral waves***
  The Ginzburg-Landau spiral waves we have proved is locally asymptotically stable under $H^{2\zeta}_{m,(\alpha)}$-perturbations for any $\zeta > \frac{1}{2}$. We can ask their stability under the full $H^{2\zeta}_{(\alpha)}$-perturbations. Note that formal asymptotic expansions and numerical evidences suggest that the one-armed spiral waves be stable, while multi-armed ones be unstable; see [Ha82] and [Ts10].

- ***Direction 4: spatio-temporal feedback (de-)stabilization***
  Once the stability of a spiral wave solution $\widetilde{\Psi}$ is known, we can study whether

the stability changes when *noninvasive spatio-temporal feedback control* is introduced, that is, when we consider the *delay Ginzburg-Landau equation*:

$$\partial_t \Psi = \frac{1}{b}(1+i\,\eta)\Delta_{\mathcal{M},(\alpha)}\Psi + f(|\Psi|^2;\beta) + k_g\left(\Psi - e^{ik_0}\,\Psi(t-k_t, s, \varphi - k_\varphi)\right). \quad (5.1)$$

Here $k_g \in \mathbb{C}$ is the feedback gain, $k_o \in S^1$ is the transformation of the output, $k_t > 0$ is the time delay, and $k_\varphi \in S^1$ is the spatial delay on the azimuthal angle; see the *method of control triple* established recently in [Sch16]. It is worth noting that the framework of feedback control is indeed model-independent. The main task is to prove the existence of parameters (k_o, k_t, k_φ) such that (5.1) is noninvasive, that is, the introduced control term vanishes if $\Psi = \widetilde{\Psi}$, and more importantly, there exist a $k_g \in \mathbb{C}$ such that $\widetilde{\Psi}$ changes its stability under the dynamics of (5.1).

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Deutsche Zusammenfassung

In dieser Arbeit etablieren wir eine *funktionalanalytische Methode*, um die *Existenz von Ginzburg-Landau Spiralwellen* zu beweisen. Auf der Grundlage von systematischen Erwägungen rechtfertigen wir den beliebten *m-armigen Spiralansatz* mit Hilfe von *Äquivarianz* und der *variationellen Struktur* der reellen Ginzburg-Landau Gleichung. Dieser Spiralansatz verwandelt die Ginzburg-Landau-Gleichung in eine *elliptische Gleichung*. Um diese elliptische Gleichung mit unserer funktionalanalytischen Methode zu lösen, führen wir eine *globale Bifurkationsanalyse* durch, und das Ergebnis der Existenz ist im Wesentlichen eine Folge der *Kompaktheit*.

Aus unserer funktionalanalytischen Methode ergeben sich drei Vorteile: Erstens vermeidet sie die raffinierten, aber heiklen Abschätzungen der *shooting-Methode*. Zweitens funktioniert sie für allgemeinere zugrunde liegende räumliche Bereiche, und dies nicht nur in der *Kreisgeometrie*, sondern auch in der *sphärischen Geometrie*. Drittens ermöglicht sie das Auftreten eines *gemischten Diffusionsprozesses*, wenn ein komplexer Diffusionsparameter eingeführt wird. In diesem Sinne ist unser Ergebnis eine große Verallgemeinerung der Existenzresultate in der Literatur. Insbesondere beweisen wir die Existenz von zwei neuen Mustern; den *gefrorenen Spiralwellen* in der Kreisgeometrie und der sphärischen Geometrie, sowie den *2-Spitzen Spiralen* in der sphärischen Geometrie.

Selbstständigkeitserklärung

Hiermit bestätige ich, DAI, Jia-Yuan, dass ich die vorliegende Dissertation mit dem Thema

Spiral waves in circular and spherical geometries
The Ginzburg-Landau paradigm

selbstständig angefertigt und nur die genannten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

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Curriculum Vitae

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Academic Experience

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