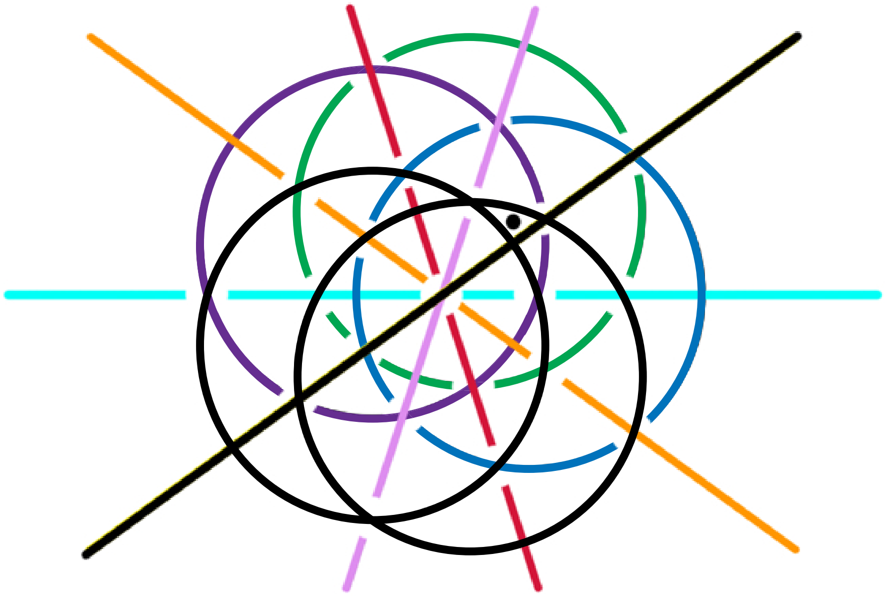

On Three Ehrhart Theories & Simplicial Hyperplane Arrangements

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von

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Selbständigkeitserklärung
Ich versichere hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Zudem wurde die Arbeit auch noch nicht in einem früheren Promotionsverfahren eingereicht.

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Dedicated to my parents, Christopher and Lorelei.

"I say so strange a dreaminess did there then reign over the ship and all over the sea, only broken by the intermitting dull sound of the sword, that it seemed as if this were the Loom of Time, and I myself were a shuttle mechanically weaving and weaving away at the Fates. There lay the fixed threads of the warp subject to but one single, ever returning, unchanging vibration, and that vibration merely enough to admit of the crosswise interblending of other threads with its own. This warp seemed necessity; and here, thought I, with my own hand I ply my own shuttle and weave my own destiny into these unalterable threads."

\- Herman Melville, *Moby-Dick*

SUMMARY

This dissertation presents recent contributions to two major topics in discrete geometry. The first topic is *Ehrhart theory*, which is the study of the discrete volume of convex polytopes. Ehrhart theory is named for Eugène Ehrhart, who showed that for a positive integer k, the number of lattice points in the k-th dilate of a lattice polytope is given by a polynomial in k [51]. The first three chapters of this dissertation each handle a different type of Ehrhart theory.

In Chapter 1, which is part of joint work in progress with Donghyun Kim and Mariel Supina, we investigate *equivariant* Ehrhart theory, which unites Ehrhart theory with representation theory to study the properties of convex polytopes and their symmetries. Let P be a lattice polytope invariant under the linear action of a finite group. Stapledon introduced a formal power series, called the equivariant Ehrhart series $\mathrm{EE}(P;t)$, which encodes simultaneously, for all group elements, the number of fixed lattice points in each dilate [99]:

$$\mathrm{EE}(P;t) = \sum_{m\geq 0} \chi_{mP} t^m = \frac{\mathrm{H}^*(P;t)}{\det(I - t\cdot\rho)} = \frac{\tilde{\mathrm{H}}(P;t)}{(1-t^N)^{d+1}}.$$

We prove basic results about this series, including proofs that it has the above rational generating functions. We also provide two original methods for calculating the equivariant Ehrhart series using invariant triangulations. Finally, we present `Sagemath` code to calculate the equivariant Ehrhart series. It could provide the means for the interested mathematician to prove or disprove the following conjecture:

Conjecture (Stapledon: [99, Conjecture 12.2])**.** *If the* H^**-series is polynomial, then it has nonnegative, integral coefficients, i.e., it is effective.*

In Chapter 2, we study *rational* Ehrhart theory. This chapter is joint work with Matthias Beck and Sophie Rehberg [11]. For polytopes with rational vertices, the Ehrhart counting function is given by a quasipolynomial. We extend the Ehrhart counting function to count the number of lattice points in rational and real dilates of rational polytopes. This continues work of Linke, who showed that the rational and real Ehrhart counting functions of rational polytopes are quasipolynomial [77]. Our goal is to add a generating function perspective to this work. We associate two rational generating functions to a rational polytope that completely describe its rational and real Ehrhart counting functions. We provide structural theorems about these generating functions: rationality, nonnegativity theorems, connections to the h^*-polynomial in classical Ehrhart theory, and combinatorial reciprocity theorems. We also extend the notion of Gorenstein polytopes to the rational setting.

In Chapter 3, we perform a computational investigation of the *multivariate* Ehrhart theory of tropical polytopes using methods from toric geometry. This chapter is joint work with Marie-Charlotte Brandenburg and Leon Zhang [20]. The tropical convex hull of a finite set of points is not necessarily a classically convex polytope. When it is, we call the resulting convex hull a *polytrope*. Polytropes appear classically as al-

cove polytopes of type A, which are of much interest in their own right. For instance, it is not known if alcove polytopes of type A have unimodal h*-vectors, which are vectors associated with the Ehrhart polynomial. In this work, we compute multivariate volume, Ehrhart, and h*-polynomials for all polytropes up to dimension 4, in which there are 27 248 types of maximal polytropes. We also provide a combinatorial description of the coefficients of the volume polynomials in dimension 3 in terms of regular central subdivisions of the fundamental polytope FP_3.

In Chapter 4, which comprises the second part of this dissertation, we study *simplicial hyperplane arrangements*. Chapter 4 is joint work with Jean-Philippe Labbé and Michael Cuntz published in Annals of Combinatorics [34]. A finite, central, real hyperplane arrangement is called *simplicial* if the bounding hyperplanes to each region have linearly independent normal vectors. Our work is particularly motivated by the following open question:

Open Question. What is the number of isomorphism classes of rank three simplicial hyperplane arrangements?

In 1971, Grünbaum published a catalogue of rank-3 simplicial arrangements with three infinite families and 90 sporadic arrangements [61]. Since then, 5 new arrangements have been found [33]. In [34], we provide the most up to date catalogue of simplicial hyperplane arrangements of rank 3, giving normals and invariants. In Chapter 4, we add structure to the catalogue by classifying the arrangements according to whether their associated lattices of regions are always, sometimes, or never *congruence normal* depending on the choice of base region. Our novel methods for checking congruence normality work in any dimension and make use of the oriented matroid associated to a hyperplane arrangement. We prove that finite Weyl groupoids of any dimension have congruence normal lattices of regions.

ZUSAMMENFASSUNG

Diese Dissertation präsentiert neue Beiträge zur zwei wichtigen Themen in diskreter Geometrie. Das erste Thema ist *Ehrhart Theorie*, welche das diskrete Volumen konvexer Polytope studiert. Ehrhart Theorie ist nach Eugène Ehrhart benannt. Ehrhart zeigte 1962, dass die Anzahl an Gitterpunkten in der k-ten Streckung eines Gitterpolytopes mit einem Polynom in k übereinstimmt, wobei k eine natürliche Zahl ist [51]. Die ersten drei Kapiteln dieser Dissertation behandeln verschiedene Verallgemeinerungen von Ehrhart Theorie.

In Kapitel 1, eine gemeinsame Arbeit mit Donghyun Kim und Mariel Supina, untersuchen wir *Äquivariante* Ehrhart Theorie, diese vereint Ehrhart Theorie mit Darstellungstheorie um die Eigenschaften von konvexen Polytopen und ihren Symmetrien zu untersuchen. Sei P ein ganzzahliges Polytop das invariant unter der linearen Aktion einer endlichen Gruppe ist. Stapledon hat eine formale Potenzreihe eingeführt, die äquivariante Ehrhart Reihe $\mathrm{EE}(P;t)$ genannt wird. Diese kodiert gleichzeitig, für jedes Gruppenelement die Anzahl der Gitterpunkte, welche Fixpunkte unter der Aktion des Gruppenelementes sind, in jeder Streckung des Polytopes:

$$\mathrm{EE}(P;t) = \sum_{m\geq 0} \chi_{mP} t^m = \frac{\mathrm{H}^*(P;t)}{\det(I - t\cdot\rho)} = \frac{\tilde{\mathrm{H}}(P;t)}{(1-t^N)^{d+1}}.$$

Wir zeigen grundlegende Eigenschaften für diese Reihe, unter anderem beweisen wir die obige Darstellung als rationale erzeugenden Funktion. Wir stellen auch zwei neue Methoden vor um mit Hilfe von invarianten Triangulierungen die äquivariante Ehrhart Reihe zu berechnen. Schließlich präsentieren wir `Sagemath` Code um die H^*-Reihe zu berechnen. Es könnte für die interessierte Mathematiker/innen ein Mittel bereitstellen um die folgende Vermutung zu beweisen oder zu widerlegen:

Vermutung (Stapledon: [99, Conjecture 12.2])**.** Wenn die H^*-reihe ein Polynom ist, dann hat diese nicht-negative ganzzahlige Koeffizienten.

In Kapitel 2, studieren wir *rationale* Ehrhart Theorie. Dieses Kapitel ist gemeinsame Arbeit mit Matthias Beck und Sophie Rehberg [11]. Für rationale Polytopes ist die Ehrhart-Gitterpunktzählfunktion für natürliche Zahlen durch ein Quasipolynom gegeben. Wir erweitern die Ehrhart-Gitterpunktzählfunktion um die Anzahl von Gitterpunkten in rationalen und reellen Streckungen von rationalen Polytopen zu zählen. Das erweitert Arbeit von Linke, die gezeigt hat das die rationale und reelle Ehrhart Gitterpunktzählfunktionen von rationalen Polytopen quasipolynomiell sind [77]. Unser Ziel ist es eine rationale erzeugenden Funktion zur diese Arbeit hinzuzufügen. Wir definieren für jedes Polytop zwei rationale erzeugenden Funktionen, die die rationalen und reellen Gitterpunktzählfunktionen vollständig beschreiben. Wir geben strukturelle Theoreme über diese erzeugenden Funktionen: Rationalität und Nicht-Negativität, Verbindungen zu dem h^*-Polynom in klassischer Ehrhart Theorie, und Theoreme zur kombinatorischen Reziprozität. Wir erweitern auch den Begriff von Gorenstein Polytopen auf das rationale Szenario.

In Kapitel 3 machen wir eine computergestützte Untersuchung der *multivariaten* Ehrhart Theorie von tropischen Polytopen mit Hilfe von Methoden aus der torischen Geometrie. Dieses Kapitel ist gemeinsame Arbeit mit Marie-Charlotte Brandenburg und Leon Zhang. Die tropische konvexe Hülle einer endlichen Punktmenge ist nicht notwendigerweise konvex im klassischen Sinne. Ist dies jedoch der Fall, so nennen wir die tropische konvexe Hülle ein *Polytrop*. Polytrope tauchen auch als alkove Polytopen von Typ A auf, die von großem Interesse sind. Zum Beispiel ist nicht bekannt, ob alkove Polytope von Typ A unimodale h*-Vektoren haben. In diesem Kapitel berechnen wir multivariate Volumina, Ehrhartpolynome und h*-Polynome für alle Polytrope bis Dimension 4, wo es 27 248 Arten von maximalen Polytropen gibt. Wir geben auch eine kombinatorische Beschreibung der Koeffizienten des Volumenpolynomen in Dimension 3 im Hinblick auf die reguläre und zentrale Unterteilungen des Fundamentalpolytopes FP_3.

In Kapitel 4 studieren wir *simpliziale Hyperebenenarrangements*, das zweite Hauptthema dieser Dissertation. Kapitel 4 ist gemeinsame Arbeit mit Michael Cuntz und Jean-Philippe Labbé, die bereits in den Annals of Combinatorics publiziert wurde [34]. Ein endliches, zentrales, reelles Hyperebenenarrangement heißt *simplizial* wenn die Stützehyperebenen zu jeder Region linear unabhängige Normalenvektoren haben. Unser Arbeit wurde besonders durch die folgende offene Frage motiviert:

Offene Frage. Wie viele Isomorphismusklassen von simplizialen Hyperebenenarrangements mit Rang drei gibt es?

1971 hat Grünbaum einen Katalog von simplizialen Arrangements mit Rang drei veröffentlicht. Dieser enthält drei unendliche Familien und 90 sporadischen Arrangements [61]. Seitdem sind 5 neue Arrangements gefunden wurden [33]. In [34] präsentieren wir den aktuellsten Katalog von simplizial Hyperebenenarrangements mit Rang drei, mit Normalvektoren und Invarianten. Außerdem geben wir dem Katalog Struktur durch eine Klassifizierung der Arrangements bezüglich der Frage, ob der Regionenverband immer, manchmal, oder nie *kongruenz normal* ist, abhängig von der Wahl der Basisregion. Unsere neuen Methoden zur Prüfung ob eine Arrangement kongruenz normal ist funktionieren in jeder Dimension und benutzen das zugehörige orientierte Matroid zu einem Hyperebenenarrangement. Wir zeigen auch, dass die endlichen Weyl-Gruppoide in jeder Dimension kongruenz normale Regionenverbände haben.

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CONTENTS

INTRODUCTION

As a child, perhaps you ran to the candy store with 70 cents in your hand and had to figure out how many tootsie rolls (10¢) and jawbreakers (25¢) you could buy, given that you had to buy your best friend a jawbreaker. Systems of linear equations and inequalities lie at the heart of polyhedral geometry and are abundant in life. The mathematics that has emerged from this simple setup is bountiful, with rich problems and applications. As in the case of the jawbreakers and tootsie rolls, often, the only solutions of importance are the integral ones, since in the real world, it's not so easy to split things up (without chipping a tooth). *Ehrhart theory* is the study of integral points in polytopes and their dilates, for instance representing the different buying scenarios above; in other words, it is the study of the discrete volume of polytopes. The *Ehrhart counting function* evaluated at a positive integer k returns the number of lattice points in the k-th dilate of a polytope. Ehrhart's theorem states that for a lattice polytope, i.e. a polytope whose vertices have integral coordinates, this counting function is given by a polynomial of degree equal to the dimension of the polytope [51]. More generally, for a rational polytope, the Ehrhart counting function is given by a quasipolynomial [51], and for a vector-dilated polytope by a set of quasipolynomials, see [65] or [103]. So, for example, Ehrhart theory can be used to model: problems in enumerative combinatorics [4] such as knapsack-type problems, chromatic polynomials of graphs [14] and hypergraphs [89], solutions in computer programming and compiler optimization [103]; and has deep ties to toric geometry [55, Section 5.3]. Open questions in Ehrhart theory include: What polynomials can be Ehrhart polynomials of lattice polytopes? It is known that the leading coefficient of the Ehrhart polynomial encodes the volume of the polytope, and the second coefficient is equal to $\frac{1}{2}$ the sum of the relative volumes of the facets [12, Sections 5.3 and 5.4], but what do the other coefficients encode geometrically? Which lattice polytopes have positive Ehrhart polynomial coefficients [78]? The coefficients of the Ehrhart polynomial may be transformed to other bases to create the so called h*- and f*-*vectors* [22]. Much study has gone into the characterization of these vectors, and it is open to determine inequalities among their coefficients, for instance when they are unimodal [21]. For example, Bruns and Römer showed in 2005 that Gorenstein polytopes with a regular unimodular triangulation have unimodal h*-vectors [24].

Changing tack, we can ignore the inequalities in our linear system and study the equations themselves. Each linear equation gives rise to a *hyperplane*, a $(d-1)$-dimensional subset of d-dimensional space. A set of hyperplanes forms a *hyperplane arrangement*. It is often useful to study hyperplane arrangements combinatorially. For example one can use Möbius inversion to calculate the number of regions of the arrangement [106]. One can also create various posets from a hyperplane arrangement, such as the *poset of regions* [50] or *facial weak order* [46]. Björner, Edelman, and Ziegler showed that the poset of regions is a lattice if the arrangement is *simplicial* [17]. Much attention has lately been given to discerning which arrangements are the normal fans of polytopes, mostly focusing on Coxeter cases, see [82] and [81]. For simplicial arrangements an open problem is to count isomorphism classes of arrangements, see [34], [33], or [62].

The rest of this introduction outlines the contents of this dissertation. The investigation of Ehrhart theory within this dissertation is threefold. Chapter 1 discusses *equivariant Ehrhart theory*, a generalization of Ehrhart theory that takes the symmetry of the polytope into account. Suppose we have a lattice polytope and a group of its symmetries. For any one of these symmetries, we can ask, How many lattice points in the polytope are fixed by the symmetry? What about for any integral dilate of the polytope? In 2010, Alan Stapledon introduced equivariant Ehrhart theory [99] and with it the equivariant Ehrhart series, which is an analogue of the usual Ehrhart series and encodes simultaneously for all group elements the number of fixed lattice points in each dilate of a polytope invariant under a linear group action. This chapter focuses on the study of the equivariant Ehrhart series. Our first major contribution is a sequence of structural results on the equivariant Ehrhart series that provide more detail to Stapledon's results and extend them. Stapledon showed that the coefficients of the series are quasipolynomial, and he studied a generating function for the series with numerator $\mathrm{H}^*(P;t)$ that is an equivariant analogue of the h*-polynomial. We reprove quasipolynomiality results and additionally show that the equivariant Ehrhart series has another rational generating function with the classical denominator $(1-t^N)^{d+1}$ where N and d are positive integers. Our second contribution in this chapter is implemented open-source `Sagemath` code for the computation of the numerator $\mathrm{H}^*(P;t)$. This allows anyone to compute the H*-series as long as they have a current version of `Sagemath`. As it opens experimental approaches, it could be useful to answer some of the open questions about equivariant Ehrhart theory: When is the H*-series polynomial and effective? What is the H*-series for (nearly any chosen) family of polytope and group action? Finally, the main theorems of this chapter, Theorem 1.4.1 and Theorem 1.4.6, provide two new ways to compute the equivariant Ehrhart series by utilizing the symmetry of the polytope and creating symmetric triangulations.

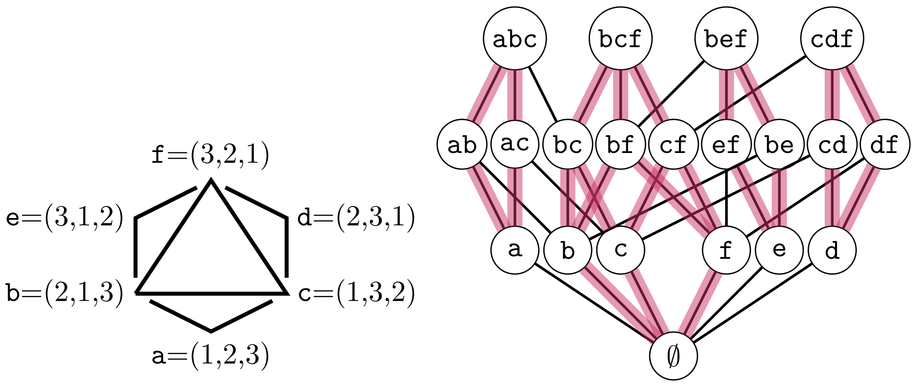


Figure 1: A half-open decomposition of the permutahedron Π_3 determined by a triangulation (left image) and a partition of the face poset of the triangulation into intervals shown via its Hasse diagram (right image)

In Chapter 2, we continue to play with the power of rational generating functions. For for both lattice and rational polytopes, the Ehrhart series have rational generating functions. However, the usual Ehrhart series only encodes the number of lattice points in integral dilates of a polytope. If we want to perform this count for rational dilates or even real dilates, things get much more subtle (and rewarding). For example, for the line segment [1,2] and a nonnegative dilation factor λ,

$$\begin{aligned} |\ \lambda[1,2] \cap \mathbb{Z}\ | &= \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 \\ &= \begin{cases} n+1 & \text{if } \lambda = n & \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n & \text{if } n < \lambda < n + \frac{1}{2} & \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n+1 & \text{if } n + \frac{1}{2} \leq \lambda < n+1 & \text{for some } n \in \mathbb{Z}_{>0}\,. \end{cases} \end{aligned}$$

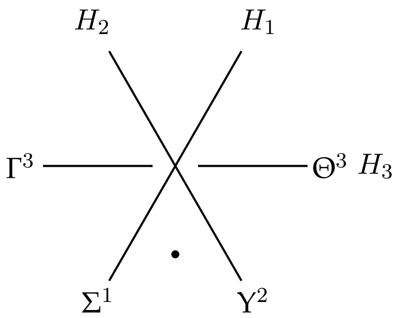
Continuing work of Linke, Stapledon, and Henk among others (see [77],[100], and [65]), we look at rational and real dilates of rational polytopes. Our main contribution is a discretization of the problem: we create a rational generating function that describes the *real and rational Ehrhart theory* of any rational polytope. This viewpoint is inspired by [100]. Additionally, we define γ-*rational* Gorenstein polytopes, which are the rational analogue of lattice Gorenstein polytopes. We show that γ-rational Gorenstein polytopes enjoy many of the appealing equivalent properties that characterize lattice Gorenstein polytopes, for example, the numerators of their rational generating functions are symmetric. We reprove Betke and McMullen's theorem that states that the h*-polynomial of a lattice polytope may be decomposed into a sum of two symmetric polynomials with nonnegative coefficients [15] and offer many examples, including examples related to period collapse.

How much can you tell from the normal fan of a polytope? Namely, if I hand you a normal fan, can you tell me, for any polytope P with this normal fan, how many lattice points are in P? This is the subject of *multivariate Ehrhart theory* and the study of Chapter 3. We describe methods and algorithms from toric geometry [43] for computing multivariate versions of volume, Ehrhart and h*-polynomials of lattice *polytropes*, which are both tropically and classically convex. Polytropes are also known as alcoved polytopes of type A, which are beloved in their own right (see [75, 76]). These algorithms are applied to all polytropes of dimensions $2, 3$ and 4, yielding a large class of integer polynomials. For example, in the case of 2-dimensional polytropes, with facet normals $\mathbf{e}_i - \mathbf{e}_j \leq a_{ij}$ for $i \neq j \in [3]$ (and seen for example in the chart $x_3 = 0$), the multivariate Ehrhart function is

$$\frac{1}{2} \sum_{i \neq j \in [3]} \left(a_{ij} - a_{ij}^2 \right) + a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32} + 1,$$

as may also be determined by Pick's theorem ([12, Section 2.6]). We give a complete combinatorial description of the coefficients of volume polynomials of 3-dimensional polytropes in terms of regular central subdivisions of the *fundamental polytope*. Finally, we provide a partial characterization of the analogous coefficients in dimension 4.

We continue to study normal fans of polytopes in the second part of this dissertation. Namely, we concern ourselves with the normal fans of simple zonotopes, otherwise known as *simplicial hyperplane arrangements*. A catalogue of simplicial hyperplane arrangements was first given by Grünbaum in 1971, see [61] or [62]. These arrangements naturally generalize finite Coxeter arrangements and also the weak order through the poset of regions. The weak order is known to be a *congruence normal* lattice, and congruence normality of lattices of regions of simplicial arrangements can be determined using polyhedral cones called *shards* [88]. In Chapter 4, we add structure to Grünbaum's catalogue by determining which arrangements *always* or *sometimes* or *never* lead to congruence normal lattices of regions. To this end, we use oriented matroids to recast shards as restricted covectors with entries $(0, +, -, *)$, which we call *shard covectors*. For example, the $*$ in coordinate 2 of the shard covector σ^1 in Figure 2 indicates that the points in shard Σ^1 lie on both sides of hyperplane H_2; the position of hyperplane H_2 is irrelevant for describing the points in the shard Σ^1.



	1	2	3
σ^1	0	$*$	$*$
υ^2	$*$	0	$*$
γ^3	+	-	0
θ^3	-	+	0

Figure 2: The 4 shards of the Coxeter arrangement A_2 with respect to the chosen region marked with a dot, and their corresponding shard covectors.

We show that shards are in bijection with shard covectors, and use an intersection operation on shard covectors to determine congruence noramlity on the level of the oriented matroid. We also show that lattices of regions coming from finite Weyl groupoids of any rank are always congruence normal.

The following notation is used throughout. $\mathbb{N} = \{0, 1, 2, \dots\}$, $d, m, n \in \mathbb{N} \setminus \{0\}$, and $[m] := \{1, 2, \dots, m\}$. We use bold faced $\mathbf{n}, \mathbf{p}, \mathbf{x}$, etc. to denote vectors in the real Euclidean space $\mathbb{R}^d$ equipped with the usual dot product $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. The linear span of a finite, ordered set of vectors $\mathbf{P}$ is denoted $\operatorname{span}(\mathbf{P})$, its affine hull by $\operatorname{aff}(\mathbf{P})$, and its convex hull by $\operatorname{conv}(\mathbf{P})$. To ease reading, we often abuse notation and write for instance $\operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)$ instead of $\operatorname{span}(\{\mathbf{x}_1, \mathbf{x}_2\})$. The orthogonal complement of a linear subspace $A \subseteq \mathbb{R}^d$ is denoted $A^\top$. The relative interior of a subset $\mathbf{P}$ of $\mathbb{R}^d$ is denoted by $\mathbf{P}^\circ$.

In this section, we give select background on polyhedral geometry and Ehrhart theory that is used in this dissertation. We adopt the vocabulary of polyhedral geometry as in [108], and freely use notions of polyhedra and their faces.

A **(real) hyperplane** H is a codimension-1 affine subspace in $\mathbb{R}^d$:

$$H := \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{n} \cdot \mathbf{x} = a \text{ for some nonzero } \mathbf{n} \in \mathbb{R}^d \text{ and } a \in \mathbb{R} \right\}.$$

The vector $\mathbf{n}$ is called the **normal** of H. A **finite hyperplane arrangement** $\mathscr{A}$ is a finite non-empty set of m hyperplanes. If $a = 0$ for all hyperplanes in $\mathscr{A}$, then the hyperplane arrangement is called **central**. In this case, the hyperplanes are completely determined by their normals.

A **polytope** $P \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points in $\mathbb{R}^d$. Equivalently, polytopes are determined by hyperplane arrangements as the bounded intersections of finitely many closed half-spaces: $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}$ for some $A \in \mathbb{R}^{m \times d}$ and $\mathbf{b} \in \mathbb{R}^m$. A polytope $P \subseteq \mathbb{R}^d$ is called **full-dimensional** if its dimension, $\dim(P)$, is equal to d. Occasionally, we write d-polytope to refer to a d-dimensional polytope. A polytope is a **lattice polytope** if its vertices are in a lattice $\Lambda \subseteq \mathbb{R}^d$, which we normally take to be $\mathbb{Z}^d$. Let $P \subseteq \mathbb{R}^d$ be a polytope with vertices $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. The **cone over** P, denoted $\operatorname{Cone}(P)$, is the following set of points in $\mathbb{R}^{d+1}$:

$$\operatorname{Cone}(P) = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} \ \middle| \ \mathbf{x} = \sum_{i=1}^{n} \lambda_i (\mathbf{v}_i, 1) \text{ where } \lambda_i \geq 0 \text{ for all } i \in [n] \right\}.$$

Definition 0.0.1 ([108, Definition 5.1])**.** A **polyhedral complex** $\mathcal{C}$ is a finite collection of polyhedra in $\mathbb{R}^d$ such that:

1. the empty polyhedron is in $\mathcal{C}$,
2. if $P \in \mathcal{C}$, then all faces of P are also in $\mathcal{C}$,
3. the intersection $P \cap Q$ of two polyhedra $P, Q \in \mathcal{C}$ is a face of both P and of Q.

A **polytopal subdivision** of a polytope P is a polytopal complex $\mathcal{C}$ with underlying space $|\mathcal{C}| = P$. The subdivision is called a **triangulation** if all polytopes in $\mathcal{C}$ are simplices. We think of a triangulation $\mathcal{T}$ as a simplicial complex and identify faces with sets of vertices when convenient. A triangulation of a polytope into lattice simplices is called a **lattice triangulation**.

The main reference we follow for Ehrhart theory is [12]. The **Ehrhart counting function** of P, written $\operatorname{ehr}(P;m)$, gives the number of lattice points in the m-th dilate of P for $m \in \mathbb{Z}_{\geq 1}$:

$$\operatorname{ehr}(P;m) = |mP \cap \mathbb{Z}^n| = |\{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} \leq m\mathbf{b}\}|.$$

Ehrhart's theorem [51] says that for positive integers, $\operatorname{ehr}(P;m)$ agrees with a polynomial in m of degree equal to the dimension of P. Furthermore, the constant term of this polynomial is equal to 1 and the coefficient of the leading term is equal to the Euclidean volume of P within its affine span. The interpretation of other coefficients of the Ehrhart polynomial is an active direction of research, see for example [53]. Ehrhart-Macdonald reciprocity says that $\operatorname{ehr}(P,-m) = (-1)^{\dim(P)}|(mP)^\circ \cap \mathbb{Z}^n|$, see [13, Section 4.6].

Generating functions play a central role in Ehrhart theory. The **Ehrhart series** $\operatorname{Ehr}(P;t)$ of a polytope P is the formal power series given by

$$\operatorname{Ehr}(P;t) = 1 + \sum_{m \geq 1} \operatorname{ehr}(P;m)t^m.$$

For a d-dimensional lattice polytope, the Ehrhart series has the rational generating function

$$\operatorname{Ehr}(P;t) = 1 + \sum_{m \geq 1} \operatorname{ehr}(P;m)t^m = \frac{\mathrm{h}^*(P;t)}{(1-t)^{d+1}},$$

where $\mathrm{h}^*(P;t) = \sum_{i=0}^{d} \mathrm{h}_\mathrm{i}^*\, t^i$ is a polynomial in t of degree at most d, called the **h^*-polynomial**. Furthermore, each h_i^* is a non-negative integer [95]. The coefficients of the h^*-polynomial form the **h^*-vector**: $(\mathrm{h}_0^*, \mathrm{h}_1^*, \ldots, \mathrm{h}_\mathrm{d}^*)$.

The **normalized volume** of P is defined as $\operatorname{Vol}(P) = \dim(P)!\operatorname{vol}(P)$, where $\operatorname{vol}(P)$ is the Euclidean volume of P within its affine span. The normalized volume $\operatorname{Vol}(P)$ measures the volume of P with respect to unimodular simplices, which have normalized volume 1. The Euclidean volume $\operatorname{vol}(P)$ measures the volume of P with respect to unit $\dim(P)$-dimensional cubes, which have Euclidean volume 1. The normalized volume of P is equal to the sum of the coefficients of the h^*-polynomial. The Ehrhart polynomial may be recovered from the h^*-vector through the transformation

$$\operatorname{ehr}_P(m) = \sum_{i=0}^{d} \mathrm{h}_\mathrm{i}^* \binom{m+d-i}{d}. \tag{1}$$

Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope with **denominator** k, i.e., k is the smallest positive integer such that kP is a lattice polytope. Then $\operatorname{ehr}(P;m)$ is a **quasipolynomial** with period dividing k, i.e., of the form $\operatorname{ehr}(P;m) = c_d(m)m^d + \cdots + c_1(m)m + c_0(m)$ where $c_0(m), c_1(m), \ldots, c_d(m)$ are periodic functions. In this case, the Ehrhart series has the rational generating function

$$\operatorname{Ehr}(P;t) \; := \sum_{m \in \mathbb{Z}_{\geq 0}} \operatorname{ehr}(P;m)\, t^m \; = \; \frac{\mathrm{h}^*(P;t)}{(1-t^k)^{d+1}} \tag{2}$$

where $\mathrm{h}^*(P;t) \in \mathbb{Z}[t]$ has degree $< k(d+1)$.

Part I

EHRHART THEORY

"Och så ska man ju ha några stunder att bara sitta och glo också!"

- Astrid Lindgren, Diary 1964

1 EQUIVARIANT EHRHART THEORY THROUGH SYMMETRIC TRIANGULATIONS

How can we bring the symmetries of a polytope in to bear on its Ehrhart theory? Alan Stapledon's work from 2011 [99] gives one approach to this question. Suppose we have a lattice polytope and a group of its symmetries. For any one of the symmetries, we can count the number of lattice points in the polytope that are fixed by that symmetry. We can do the same for any integral dilate of the polytope. Stapledon introduced a formal power series, which is an analogue of the usual Ehrhart series, and encodes *simultaneously* for all group elements the number of fixed lattice points in each integral dilate. Let P be a lattice polytope invariant under the linear action of a group G. The mentioned analogue of the Ehrhart series is the **equivariant Ehrhart series**:

$$\mathrm{EE}(P;t) = \sum_{m\geq 0} \chi_{mP} t^m,$$

where χ_{mP} is the character of the permutation representation of G on the lattice points in the m-th dilate of P. There is much left to explore in this novel, natural, and beautiful generalization of Ehrhart theory. In this chapter, we set out to do just that, and provide three main contributions. First, we give fundamental structural results on the equivariant Ehrhart series and add details to the proofs that are not explained in Stapledon's original work. Second, we provide new, implemented `Sagemath` [91] methods for the computation of the equivariant Ehrhart series. Third, we prove two theorems, Theorem 1.4.1 and Theorem 1.4.6, for computing the equivariant Ehrhart series using symmetric triangulations.

The chapter is structured as follows. The symmetric triangulations that we use are referred to as G-invariant half-open decompositions and are introduced in Section 1.1. They are closely related to partitionability of posets. Also in Section 1.1, we provide the necessary background on the representation theory of finite groups.

In Section 1.2, we give the mentioned structural results for the equivariant Ehrhart series. We show that χ_{mP} is a quasipolynomial in m and that $\mathrm{EE}(P;t)$ has the following generating functions:

$$\mathrm{EE}(P;t) = \sum_{m\geq 0} \chi_{mP} t^m = \frac{\mathrm{H}^*(P;t)}{\det(I - t\cdot\rho)} = \frac{\tilde{\mathrm{H}}(P;t)}{(1-t^N)^{d+1}}.$$

The first generating function $\frac{\mathrm{H}^*(P;t)}{\det(I-t\cdot\rho)}$ is studied in [99], and the numerator $\mathrm{H}^*(P;t)$ is a priori a formal power series in t with coefficients in the character ring of the acting group. In the denominator, I is the identity matrix, ρ is the matrix representation of the acting group, and for $g \in G$, $t\cdot\rho[g]$ is the matrix representation of g where each entry is multiplied by the variable t. In [99], $\mathrm{H}^*(P;t)$ is referred to as $\phi[t]$. This notation was updated in [3] to reflect that $\mathrm{H}^*(P;t)$ is a generalization of the usual $\mathrm{h}^*(P;t)$ polynomial, and we adopt $\mathrm{H}^*(P;t)$ here.

The second rational generating function, $\frac{\tilde{\mathrm{H}}(P;t)}{(1-t^N)^{d+1}}$, is a typical rational expression for a formal power series with coefficients given by a quasipolynomial [97, Chapter 4], and we do not believe it has been looked at before in the setting of equivariant Ehrhart theory. In the denominator, N is the exponent of G and d is the dimension of the polytope. The numerator, $\tilde{\mathrm{H}}(P;t)$, is necessarily a polynomial in t. One benefit of this rational generating function is that it is easy to transform from $\tilde{\mathrm{H}}(P;t)$ to the quasipolynomial χ_{mP}, see Example 1.4.9 and Example 1.4.10.

Section 1.2.1 discusses the author's implementation of the H^*-series in `Sagemath`, which is the second major contribution of this chapter. The new methods are open-source and included with current `Sagemath` versions. This code will allow the mathematical community to compute examples and explore the topic, and was already used by the authors of [3] for verification of their computations on the equivariant Ehrhart theory of the permutahedron under a symmetric group action. We explain how to use the code and provide examples. One motivation for considering the H^*-series is its connection to toric geometry, see [99, Sections 7 and 8]. A lattice polytope $P \subseteq \mathbb{R}^d$ defines a projective toric variety X_P. A hypersurface in X_P is given by the vanishing set of $f = \sum_{\mathbf{v}\in P\cap Z^d} a(\mathbf{v})x^{\mathbf{v}}$, where $a(\mathbf{v}) \in \mathbb{C}$. This hypersurface is said to be **G-invariant** if $a(\mathbf{v}) = a(\mathbf{w})$ for all lattice points $\mathbf{v}, \mathbf{w} \in P$ in the same orbit. The hypersurface is **smooth** if the gradient vector $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is never zero when $x_1, \ldots, x_n \in \mathbb{C}^*$. The hypersurface is **nondegenerate** if $f|_F = \sum_{\mathbf{v}\in F\subseteq P\cap\mathbb{Z}^d} a(\mathbf{v})x^{\mathbf{v}}$ is smooth for all faces F of P. The H^*-series is **effective** if the coefficient of t^i is an **effective character** for all i, where an effective character is a nonnegative integral sum of the irreducible characters of the group. Stapledon conjectured [99, Conjecture 12.2] that the following are equivalent:

1. There exists a G-invariant nondegenerate hypersurface for P.
2. The H^*-series is effective.
3. The H^*-series is polynomial.

He showed that $1 \Rightarrow 2 \Rightarrow 3$. It is known that 2 does not imply 1, but the equivalence of 2 and 3 is still open. Stapledon also showed that when a polytope's vertices coincide with the primitive ray generators of its G-invariant face fan, then $\mathrm{H}^*(P;t)$ coincides with the G-representation on the cohomology of the associated toric variety [99, Proposition 8.1].

In Section 1.3, we review a theorem of Stapledon which gives the equivariant Ehrhart series of a simplex; it allows one to compute $\mathrm{H}^*(P;t)$ by counting fixed lattice points in the fundamental parallelepiped. In Theorem 1.4.1, we generalize this theorem, allowing geometric computation of the H^*-series for polytopes with G-invariant half-open decompositions. In Theorem 1.4.6 we give a second method for computing the equivariant Ehrhart series using G-invariant half-open decompositions, and show that in this case χ_{mP} is polynomial in m. Theorems 1.4.1 and 1.4.6 make up the third major contribution of this chapter.

Finally, we provide two additional sections, Section 1.5 and Section 1.6, with more exploratory results. In Section 1.5, we turn our attention to restricted representations. We show that if the H^*-series of P under the action of G (admits a G-invariant nondegenerate hypersurface / is effective / is polynomial), then the H^*-series of P under the action of G restricted to a subgroup will have the same property. However, we show in Example 1.5.4 that even if $\mathrm{H}^*(P;t)$ is not polynomial for a G-action, it may be polynomial for the action of a subgroup. Likewise, an irreducible representation of a group G does not necessarily restrict to an irreducible representation of a subgroup, making it worthwhile to study the equivariant Ehrhart theory of subgroup actions.

In light of these observations, in Section 1.6, we consider the permutahedron under the action of cyclic groups. The equivariant Ehrhart theory of the permutahedron under the action of the symmetric group was studied by Ardila, Supina, and Vindas-Meléndez in [3]. They find combinatorial expressions for the fixed subsets of the permutahedron for each conjugacy class of the symmetric group. They also show that Stapledon's conjecture [99, Conjecture 12.2] holds for the permutahedron under the symmetric group action. In Theorem 1.6.5, we give an explicit expression for the H^*-series of the permutahedron under the action of prime cyclic groups and show that it is effective as a corollary. In Theorem 1.6.2, we give a result on the usual Ehrhart theory of the permutahedron: the coefficients of the h^*-polynomial of the permutahedron Π_p for prime p are equivalent to 1 modulo p.

This chapter is part of joint work in progress with Donghyun Kim and Mariel Supina.

1.1 HALF-OPEN DECOMPOSITIONS

In this section, we define the half-open decompositions of polytopes that are used to calculate the equivariant Ehrhart series in Section 1.4. For background definitions on discrete geometry see Section 2.

Definition 1.1.1. A **half-open decomposition** of a lattice polytope P with lattice triangulation $\mathcal{T}$ is a partition of the face poset of $\mathcal{T}$ into intervals such that the empty set is not in its own class. We use $\mathcal{T}^{[)}$ to refer to the half-open decomposition, i.e., the triangulation together with the partition of the face poset.

Definition 1.1.1 is closely related to that of partitionability; the face poset of a pure simplicial complex is **partitionable** if it can be partitioned into disjoint intervals such that each maximal element is a facet of the complex, see [96, Chapter 2]. In this case, it is easy to read off the h-vector of the simplicial complex by recording the heights of the minimal simplices in the intervals. It was conjectured by Stanley that Cohen-Macaulay simplicial-complexes are partitionable, but a counterexample was found in 2016, see [48]. The following two examples are used as running examples throughout the chapter.

Example 1.1.2. The two-dimensional permutahedron $\Pi_3 \subseteq \mathbb{R}^3$ obtained as the convex hull of the permutations of the coordinates $(1,2,3)$ admits a half-open decomposition. Figure 3 shows a triangulation of Π_3 and a partition of the face poset of the triangulation. The broken edges in the triangulation indicate which faces are open and which are closed in the cone over Π_3 according to the partition.

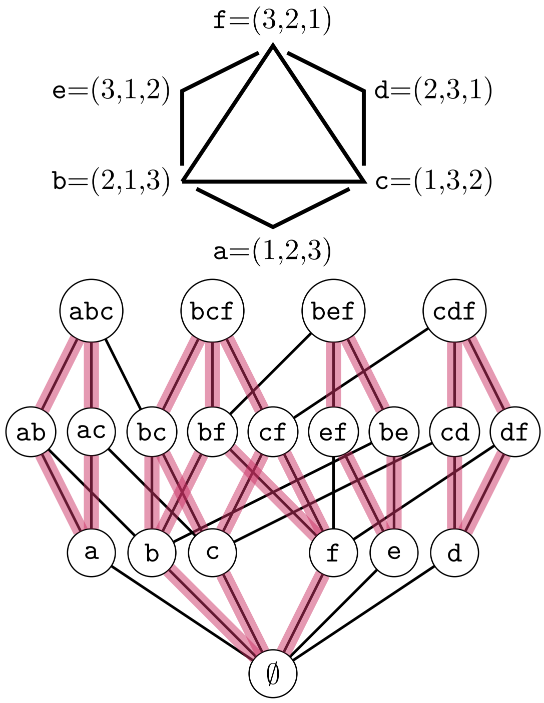


Figure 3: A half-open decomposition of the permutahedron Π_3 determined by a triangulation (top image) and a partition of the face poset of the triangulation into intervals shown via its Hasse diagram (bottom image)

Example 1.1.3. The 0/1 square admits a half-open decomposition. Triangulate the square into two triangles: $\Delta(abc) = \mathrm{conv}(\{(0,1),(0,0),(1,1)\})$ and $\Delta(bcd) = \mathrm{conv}(\{(0,0),(1,1),(1,0)\})$, as shown on the left in Figure 4. Partition the face poset into three intervals: $[a, abc]$, $[\emptyset, bc]$, $[d, bcd]$ to create the half-open decomposition, as shown on the right in Figure 4. The broken edges in the triangulation shown in Figure 4 indicate which faces in the cone over the triangulated square are open according to the partition of the face poset. The maximal simplex bc in the interval $[\emptyset, bc]$ is not full-dimensional, which is a flexibility offered by these half-open decompositions.

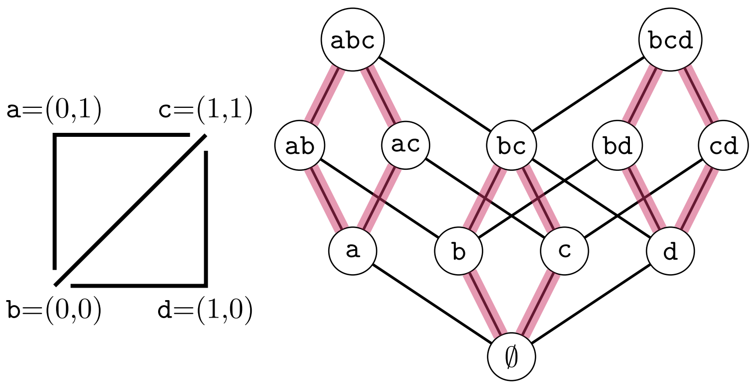


Figure 4: A half-open decomposition of the 0/1 square determined by a triangulation (left image) and a partition of the face poset of the triangulation into intervals shown via its Hasse diagram (right image)

Definition 1.1.4. Let $P \subseteq \mathbb{R}^d$ be a lattice polytope, and let $I = [\underline{S}, \overline{S}]$ be an interval in a half-open decomposition $\mathcal{T}^{[)}$ of P. The **half-open cone over the interval** I, denoted $\mathrm{Cone}(I)$, is the set of points $\mathbf{x} \in \mathbb{R}^{d+1}$ such that:

$$\mathbf{x} = \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \mu_j(\mathbf{v}_j, 1), \text{ where } \lambda_i \in \mathbb{R}_{>0} \text{ and } \mu_j \in \mathbb{R}_{\geq 0}.$$

Proposition 1.1.5. *The cone over a lattice polytope P with half-open decomposition $\mathcal{T}^{[)}$ is equal to the disjoint union of half-open cones over the intervals of $\mathcal{T}^{[)}$:*

$$\mathrm{Cone}(P) = \bigsqcup_{I \in \mathcal{T}^{[)}} \mathrm{Cone}(I).$$

Proof. The inclusion $\mathrm{Cone}(P) \supseteq \bigsqcup_{I \in \mathcal{T}^{[)}} \mathrm{Cone}(I)$ is clear. The origin is contained in the half-open cone, $\mathrm{Cone}(I)$, for the unique interval $I = [\emptyset, \overline{S}]$ that contains the empty set. Let $\mathbf{z} \in \mathrm{Cone}(P)$, $\mathbf{z} \neq 0$. There exists a minimal simplex $S = \mathrm{conv}(\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathcal{T}^{[)}$ (with respect to inclusion) such that $\mathbf{z} \in \mathrm{Cone}(S)$ and $\mathbf{z} = \sum_{i=1}^n \lambda_i(\mathbf{v}_i, 1)$, $\lambda_i \in \mathbb{R}_{>0}$ for all $i \in [n]$. The simplex S is contained in a unique interval $I = [\underline{S}, \overline{S}]$ of $\mathcal{T}^{[)}$. Thus,

$$\mathbf{z} = \sum_{\mathbf{v}_i \in S} \lambda_i(\mathbf{v}_i, 1) = \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_i \in S \setminus \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v} \in \overline{S} \setminus S} 0(\mathbf{v}, 1).$$

This implies that all vertices of $\underline{S}$ have a coefficient in $\mathbb{R}_{>0}$, and all vertices of $\mathbf{v} \in \overline{S} \setminus \underline{S}$ have a coefficient in $\mathbb{R}_{\geq 0}$. Thus, $\mathbf{z} \in \mathrm{Cone}(I)$.

It remains to show that the union $\bigsqcup_{I \in \mathcal{T}^{[)}} \mathrm{Cone}(I)$ is disjoint. Suppose $\mathbf{z} \neq 0$, $\mathbf{z} \in \mathrm{Cone}(I)$, and $\mathbf{z} \in \mathrm{Cone}(I')$ for two intervals $I = [\underline{S}, \overline{S}]$, $I' = [\underline{S}', \overline{S}']$ of $\mathcal{T}^{[)}$. Then there is a unique face of the simplicial complex $T = \mathrm{conv}(\mathbf{v}_1, \dots, \mathbf{v}_n) \subseteq \overline{S} \cap \overline{S}'$ such that $\mathbf{z} = \sum_{i=1}^n \lambda_i(\mathbf{v}_i, 1)$, where $\lambda_i \in \mathbb{R}_{>0}$ for all $i \in [n]$. As $\mathbf{z}$ is contained in $\mathrm{Cone}(I)$ and $\mathrm{Cone}(I')$, $\underline{S} \subseteq T \subseteq \overline{S}$ and $\underline{S}' \subseteq T \subseteq \overline{S}'$. Thus T is contained in two intervals in the half-open decomposition, and they must be the same. $\square$

Definition 1.1.6. Let $P \subseteq \mathbb{R}^d$ be a lattice polytope, and let $I = [\underline{S}, \overline{S}]$ be an interval in a half-open decomposition $\mathcal{T}^{[)}$ of P. The **half-open fundamental parallelepiped of the interval** I, denoted $\Pi_I^{[)}$, is the set of points $\mathbf{x} \in \mathbb{R}^{d+1}$ such that:

$$\mathbf{x} = \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \mu_j(\mathbf{v}_j, 1), \text{ where } \lambda_i \in (0, 1] \text{ and } \mu_j \in [0, 1).$$

For in an interval I of a half-open decomposition, $\mathrm{Box}(I)$ is defined to be the set of lattice points in the half-open fundamental parallelepiped $\Pi_I^{[)}$:

$$\mathrm{Box}(I) = \Pi_I^{[)} \cap \mathbb{Z}^{d+1}.$$

Throughout, we use $\mathrm{Box}(I)_k$ to denote the set of lattice points in the half-open fundamental parallelepiped of I at height k, i.e. with last coordinate equal to k.

Proposition 1.1.7. *Let P be a lattice polytope with half-open decomposition $\mathcal{T}^{[)}$. Every lattice point $\mathbf{x} \in \mathrm{Cone}(P)$ can be expressed uniquely as $\mathbf{x} = \mathbf{w} + \mathbf{z}$ for* *some* $\mathbf{w} = \sum_{\mathbf{v}_i \in \overline{S}} \lambda_i(\mathbf{v}_i, 1)$, *with* $\lambda_i \in \mathbb{Z}_{\geq 0}$ *and* $\mathbf{z} \in \mathrm{Box}(I)$ *for some interval* $I = [\underline{S}, \overline{S}] \in \mathcal{T}^{[)}$.

Proof. Let $\mathbf{x} \in \mathrm{Cone}(P)$. Then by Proposition 1.1.5, $\mathbf{x} \in \mathrm{Cone}(I)$ for a unique interval $I = [\underline{S}, \overline{S}]$ of $\mathcal{T}^{[)}$, and

$$
\begin{aligned}
\mathbf{x} &= \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \mu_j(\mathbf{v}_j, 1), \quad \text{where } \lambda_i \in \mathbb{R}_{>0} \text{ and } \mu_j \in \mathbb{R}_{\geq 0}, \\
&= \sum_{\mathbf{v}_i \in \underline{S}} \Big(\lceil \lambda_i \rceil - 1 + \lambda_i - (\lceil \lambda_i \rceil - 1) \Big) (\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \left(\lfloor \mu \rfloor + (\mu - \lfloor \mu \rfloor) \right) (\mathbf{v}_j, 1) \\
&= \left(\sum_{\mathbf{v}_i \in \underline{S}} \left(\lceil \lambda_i \rceil - 1 \right) (\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \lfloor \mu \rfloor (\mathbf{v}_j, 1) \right) \\
&\quad + \left(\sum_{\mathbf{v}_i \in \underline{S}} \left(1 - \lceil \lambda_i \rceil + \lambda_i \right) (\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \left(\mu_j - \lfloor \mu_j \rfloor \right) (\mathbf{v}_j, 1) \right). \quad \square
\end{aligned}
$$

Definition 1.1.8. Let P be a lattice polytope invariant under the action of a group G. A half-open decomposition $\mathcal{T}^{[)}$ of P is called G**-invariant** if

1. Simplices are sent to simplices (the triangulation is G-invariant).
2. The action of G induces an automorphism of the face poset of $\mathcal{T}^{[)}$ such that intervals of $\mathcal{T}^{[)}$ are sent to intervals.

In broad strokes, another way to create half-open decompositions of (the cone over) a triangulated polytope is to use a generic vector in the interior of one simplex and take the "sunny-side" of each face; one asks if it is possible to remain in a face of the triangulation when walking in the direction of the vector, see [13, Secton 5.3]. This creates a shelling of the simplicial complex, which is a stronger notion than that of a partition. As we show in the next example, this method can fail to create symmetric half-open decompositions in the case of non-regular triangulations. For equivariant Ehrhart theory this matters, as we may like to exploit the symmetry that can appear in non-regular triangulations to make the computations of the equivariant Ehrhart series easier.

Example 1.1.9. In Figure 5, the top image shows a famously non-regular triangulation of a 2-dimensional simplex, sometimes called the "Mother of all Examples" [44]. One can check that no sunny-side decomposition will yield a G-invariant half-open decomposition. This triangulation is nevertheless invariant under a rotation by 120° about the center. Figure 5 also shows two different $\mathbb{Z}/3\mathbb{Z}$-invariant partitions of the face poset of the triangulation. The first relates to a shelling that is not obtainable using "sunny-side" decompositions. The second does not correspond to a shelling, but it still produces an invariant half-open decomposition.

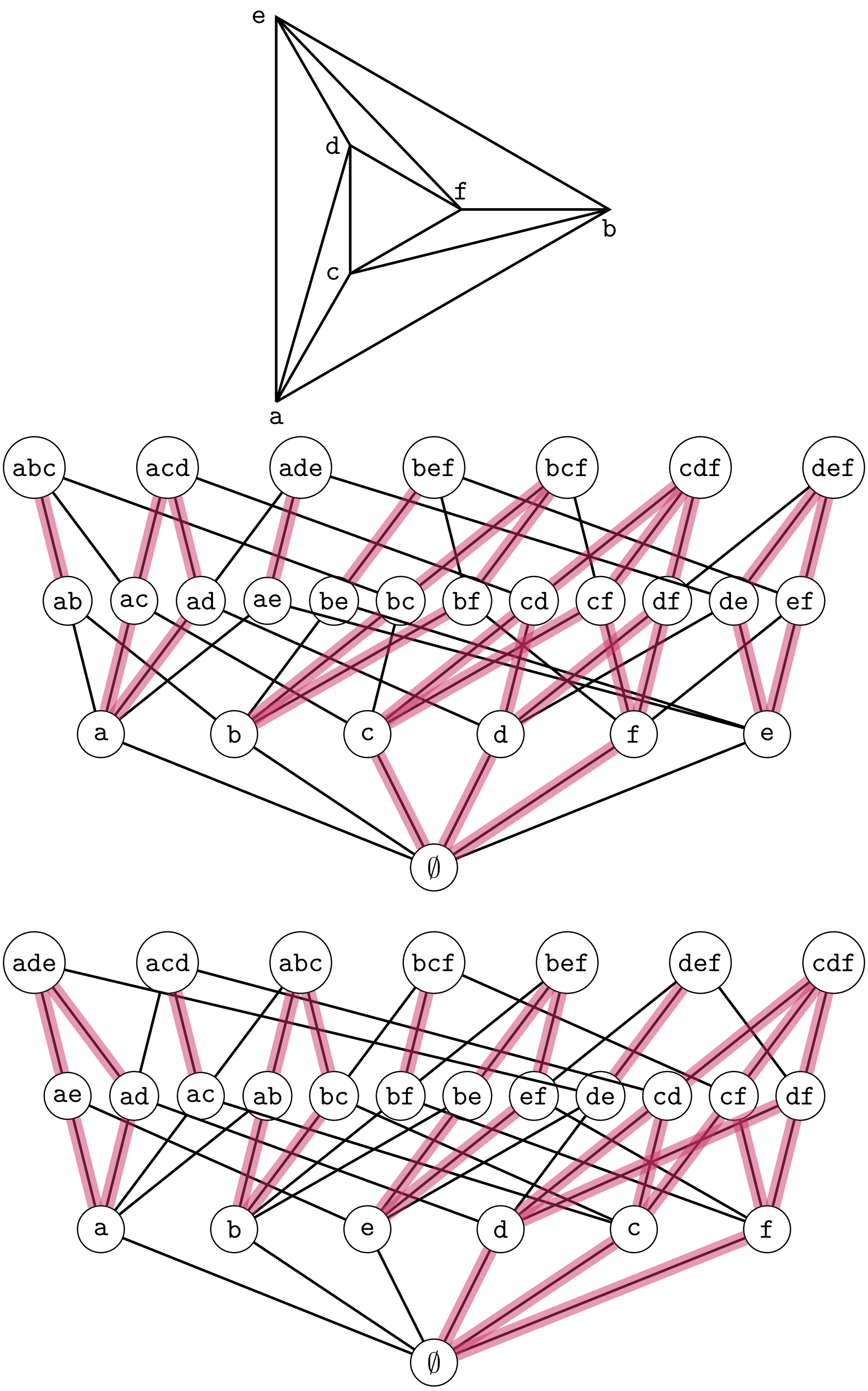


Figure 5: A non-regular, symmetric triangulation of a triangle and two $\mathbb{Z}/3\mathbb{Z}$-invariant partitions of the face lattice shown via Hasse diagram

1.1.1 *Representation Theory of Finite Groups*

For a nice introduction to representation theory of finite groups, see Sagan's book [90] or Serre's book [93]. Let V be an n-dimensional vector space over a field K of characteristic 0. A **representation** ρ of a group G on V is a group homomorphism from G to the group of invertible K-linear transformations of V, $\rho : G \to GL(V)$. The vector space V is also referred to as a G-**module**. A **G-module isomorphism** $f : V \to W$ between two G-modules is a vector space isomorphism such that $g(f(v)) = f(g(v))$ for all $g \in G$ and $v \in V$. If V is a finite, n-dimensional vector space, choosing a basis for V allows us to equivalently write ρ as a group homomorphism from G to the group of invertible $(n \times n)$-matrices with entries in K, $\rho : G \to GL_n(V)$. This is called a **matrix representation** of G on V. We always consider G to be a finite group, and work interchangeably with representations and matrix representations. We usually take $K = \mathbb{C}$, and take V to be a finite-dimensional vector space unless otherwise stated.

A subspace $W \subseteq V$ is called G-**invariant** if $W = G(W)$ as a set. A representation is **irreducible** if there are no nontrivial, proper invariant subspaces of V under the action of G. There are notions of both addition and multiplication on representations. Given G-modules V and W with corresponding representations ρ and ψ respectively, the sum $\rho + \psi$ is a representation on $V \bigoplus W$ and the product $\rho \otimes \psi$ is a representation on $V \bigotimes W$, see, for example, [90, Chapter 1.11]. The ring spanned by all formal sums and products of representations of a group G is called the **representation ring** and denoted $R[G]$. A finite group G always has a one-dimensional irreducible representation, the trivial representation. The trivial representation sends each group element to the matrix [1]. If the group G is the symmetric group S_n, with $n > 1$, then there is an additional one-dimensional representation, the sign representation. The sign representation sends each group element $\sigma \in S_n$ to the matrix [sign(σ)].

Let $\rho : G \to GL_n(V)$ be a representation of G. The **character** of ρ, written χ_ρ, is the function $G \to \mathbb{C}$ such that $\chi_\rho(g) := \text{trace}(\rho(g))$; the trace of a matrix is the sum of its diagonal entries. The characters of irreducible representations are referred to as **irreducible characters**. For a group G, the characters of the trivial representation and sign representation are denoted by χ_{triv} and χ_{sign} throughout.

Characters are **class functions**, functions from the group to the complex numbers that take the same value on every conjugacy class. In fact, a class function is a character of a representation if and only if it can be written as a nonnegative integral linear combination of the irreducible characters of G [93, Chapter 9]. Following the notation of [93, Chapter 9], we write $R^+(G)$ for the set of the class functions that are characters and refer to them as **effective characters**. The group generated by $R^+(G)$ is denoted $R(G)$ and called the **character ring**; it consists of all class functions that can be written as linear integral combinations of the irreducible characters. Elements of $R(G)$ are referred to as **virtual characters**. As the product of two characters is the character of the tensor product of the corresponding G-modules, $R(G)$ is really a ring, as its name suggests. It is a subring of the $\mathbb{C}$-vector space $F_\mathbb{C}(G)$ of class functions on G with values in $\mathbb{C}$, called the **ring of class functions**. There is an inner product on $F_\mathbb{C}(G)$ such that for two class functions ϕ, χ of G, $\langle \phi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\chi(g)}$, where $\bar{\cdot}$ denotes the complex conjugate. The characters of the irreducible representations form an orthonormal basis for $F_\mathbb{C}(G)$ with respect to this inner product. The dimension of $F_\mathbb{C}(G)$ (and therefore the number of irreducible representations G) is equal to the number of conjugacy classes of G.

Let k be the number of conjugacy classes of G, and let $\{\chi_1, \dots, \chi_k\}$ be the irreducible characters of G in some order. The **character table** of G is a $k \times k$ matrix T such that $[T_{ij}]$ is the value of the character of the i-th irreducible representation of G at the j-th conjugacy class. Let G act by permutation on a finite (ordered) set S. The **permutation representation** of G on S is the group homomorphism that sends every element of G to its permutation matrix.

We work with the tensor powers of V and the related symmetric and exterior powers. The k-th tensor power of V is denoted $\bigotimes^k(V)$. For intuition and helpful exercises, see [93, Sections 1.5 and 2.1, Exercise 9.3]. We omit definitions of tensor powers here, but recall some details of the symmetric and exterior powers. The following definitions are from [56, Appendix B].

Definition 1.1.10. The ***k*-th symmetric power**, $\mathrm{Sym}^k(V)$ of a vector space V is the quotient of the tensor product $\bigotimes^k(V)$ by the subspace generated by all tensor elements $\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k - \mathbf{v}_{\sigma(1)} \otimes \cdots \otimes \mathbf{v}_{\sigma(k)}$, for $\sigma \in S_k$. We denote the simple tensors in the quotient space by $\mathbf{v}_1 \odot \cdots \odot \mathbf{v}_k$.

Definition 1.1.11. The ***k*-th exterior power**, $\Lambda^k(V)$ of a vector space V is the quotient of $\bigotimes^k(V)$ by the subspace generated by all $\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k$ such that two of the vectors are equal. We denote the simple tensors in the quotient space by $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$.

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for V. The set $\{\mathbf{e}_{i_1} \odot \cdots \odot \mathbf{e}_{i_k} \mid i_\ell \leq i_{\ell+1}\}$ is a basis for $\mathrm{Sym}^k(V)$. A basis for $\Lambda^k(V)$ is the set $\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \mid i_\ell < i_{\ell+1}\}$. If ρ is a representation of G on V, then G also acts (diagonally) on $\bigotimes^k V$ by

$$g(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k) = g\mathbf{v}_1 \otimes \cdots \otimes g\mathbf{v}_k.$$

The group G acts diagonally on $\mathrm{Sym}^k(V)$ and $\Lambda^k(V)$ in the same manner.

It is useful to look at the characters of these actions. For $g \in G$, let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal eigenbasis of the action of g on V, with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. This is possible as $\rho(g)$ can be written as a unitary matrix, see [93, Chapter 1]. Then

$$g(\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_k}) = \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_k} \mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_k},$$

where $i_1 \leq i_2 \leq \cdots \leq i_k$. This shows that an eigenvector of $\mathrm{Sym}^k(V)$ has an eigenvalue that is a monomial in the eigenvalues of $\rho(g)$ of degree k. So the sum of the eigenvalues of g acting on $\mathrm{Sym}^k(V)$ is the sum of all monomials in the eigenvalues of degree k. With respect to the eigenbasis of the action of g on V,

$$\det(I - t\rho(g)) = \det\left(\begin{bmatrix} 1 - t\lambda_1 & & \\ & \ddots & \\ & & 1 - t\lambda_n \end{bmatrix}\right) = \Pi_{i=1}^n (1 - t\lambda_i).$$

In fact, $\det(I - t\rho(g))$ is independent of the choice of basis to express ρ since

$$\det\left(B^{-1}(I - t\rho(g))B\right) = \det\left(I - t\rho(g)\right),$$

for a change of basis matrix B, and

$$B^{-1}\left(I - t\rho(g)\right)B = \left(B^{-1}IB\right) - t\left(B^{-1}\rho(g)B\right) = I - t\left(B^{-1}\rho(g)B\right).$$

Thus the generating function of the series of characters of the symmetric powers has the rational form:

$$\sum_{k\geq 0}\chi_{\mathrm{Sym}^k(V)}t^k = \frac{1}{\det(I - t\cdot\rho)}. \tag{3}$$

For the k-th exterior power, $g(\mathbf{e}_{i_1}\wedge\mathbf{e}_{i_2}\wedge\cdots\wedge\mathbf{e}_{i_k}) = \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}\mathbf{e}_{i_1}\wedge\mathbf{e}_{i_2}\wedge\cdots\wedge\mathbf{e}_{i_k}$, where $i_1 < i_2 < \cdots < i_k$. The character of the k-th exterior product evaluated at g is the sum of all square-free homogeneous monomials in λ_i of degree k. As such, we can rewrite

$$\det(I - t\cdot\rho) = \sum_{i=0}^{n}(-1)^i\chi_{\Lambda^i(V)}t^i. \tag{4}$$

We have now recovered Lemma 3.1 in [99], which states:

Lemma 1.1.12 ([99, Lemma 3.1])**.** *Let G be a finite group and let V be an r-dimensional representation. Then*

$$\sum_{m\geq 0}\mathrm{Sym}^m\,Vt^m = \frac{1}{1 - Vt + \Lambda^2Vt^2 - \cdots + (-1)^r\Lambda^rVt^r}. \tag{5}$$

Moreover, if an element $g\in G$ acts on V via a matrix A, and if I denotes the identity $r\times r$ matrix, then both sides equal $\frac{1}{\det(I-tA)}$ when the associated characters are evaluated at g.

Remark 1.1.13. Stapledon works interchangeably with characters and their representations in [99]. For example, Stapledon writes $R(G)$ both for the representation ring of G and the character ring. Both (5) and its character analogue are said to be elements of $R(G)[[t]]$. The author has kept the notions separate here.

1.2 SETUP AND QUASIPOLYNOMIALITY

For a $\mathbb{Z}$-module M, we write $M_{\mathbb{R}}$ for $M\otimes_{\mathbb{Z}}\mathbb{R}$ and $M_{\mathbb{C}}$ for $M\otimes_{\mathbb{Z}}\mathbb{C}$. We use the following setup throughout the chapter.

Setup 1.2.1. *Let G be a finite group acting linearly on a lattice $M'\cong M\times\mathbb{Z}$ of rank $d+1$ such that the $\mathbb{Z}$-coordinate of the lattice points in M' is preserved under the action of G. Let $P\subseteq M'_{\mathbb{R}}$ be a d-dimensional G-invariant polytope with vertices in $M\times\{1\}$, where G-invariant means that as a set, $G(P) = P$.*

We assume $M = \mathbb{Z}^d$ when convenient. Let id_G denote the identity element of the group G. The **exponent** of a finite group G is the smallest positive integer N such that $g^N = \mathrm{id}_G$ for all $g\in G$. For any $g\in G$, the subset of P fixed by G, denoted P^g and called the **fixed subpolytope**, is the convex hull of the barycenters of orbits of vertices of P [99, Lemma 5.4].

This implies that any fixed subpolytope of P for $g\in G$ is a rational polytope with denominator dividing N. Let χ_{mP} be the character of the complex permutation representation induced by the action of G on the lattice points in $mP\cap M'$.

Definition 1.2.2 (Equivariant Ehrhart Series)**.** The equivariant Ehrhart series, $\mathrm{EE}(P;t)$, is the formal power series in $R(G)[[t]]$, such that the coefficient of t^m for $m \in \mathbb{Z}_{\geq 0}$ is the character χ_{mP}:

$$\mathrm{EE}(P;t) = \sum_{m \geq 0} \chi_{mP} t^m.$$

Evaluating the equivariant Ehrhart series at $g \in G$, $\mathrm{EE}(P;t)[g]$, gives the Ehrhart series of P^g.

Theorem 1.2.3 ([99, Theorem 5.7])**.** *Let P be a lattice d-polytope invariant under the action of a group G as in the Setup 1.2.1 with k conjugacy classes and exponent $N \geq 1$. There exist polynomials $f_{j,0}m^0 + f_{j,1}m^1 + \cdots + f_{j,d}m^d \in F_{\mathbb{C}}(G)[m]$ for $j \in N$, such that*

$$\chi_{mP} = f_{j,0}m^0 + f_{j,1}m^1 + \cdots + f_{j,d}m^d,$$

when $m \equiv j \mod N$. Thus, viewing χ_{mP} as a function on m, we call χ_{mP} a quasipolynomial.

Proof. Evaluating the equivariant Ehrhart series at $g \in G$ yields the Ehrhart series of the fixed subpolytope P^g. Furthermore, for all $g \in G$, $\mathrm{ehr}(P^g;t)$ can be expressed as a quasipolynomial of period N and degree equal to $\dim(P^g) \leq d$. Therefore,

$$\begin{aligned}
\mathrm{EE}(P;t)[g] = \sum_{m \geq 0} \chi_{mP}[g] t^m &= \sum_{a \geq 0} \sum_{j=0}^{N-1} \chi_{(aN+j)P}[g] t^{aN+j} \\
&= \sum_{a \geq 0} \sum_{j=0}^{N-1} \mathrm{ehr}(P^g; aN+j) t^{aN+j} \\
&= \sum_{a \geq 0} \sum_{j=0}^{N-1} \left(\sum_{i=0}^{d} c^g_{j,i} (aN+j)^i \right) t^{aN+j},
\end{aligned}$$

where $c^g_{j,i} \in \mathbb{Q}$ is the coefficient of the degree i term in the j-th constituent of the Ehrhart quasipolynomial of P^g. Let $\{g_1, \ldots, g_k\}$ be conjugacy class representatives of G. Define class functions $f_{j,i}[g_\ell] := c^{g_\ell}_{j,i}$ for all $j \in [N]$, $\ell \in [k]$, and $i \in \{0, 1, \ldots, d\}$. Then for $j \in N$, and $m \equiv j \mod N$, $\chi_{mP} = f_{j,0}m^0 + \ldots f_{j,d}m^d$. $\square$

Corollary 1.2.4. *Let $\{\chi_1, \chi_2, \ldots, \chi_k\}$ be the irreducible characters of G. There exist polynomials $\lambda_{j,i}(m) \in \mathbb{Q}[m]$ for each $j \in N$ and $i \in [k]$ of degree $\leq d$ such that*

$$\chi_{mP} = \sum_{i=1}^{k} \lambda_{j,i}(m) \chi_i,$$

for $m \equiv j \mod N$.

Proof. Continuing from the proof of Theorem 1.2.3, as each $f_{j,i}$ is a class function, it may be written as a $\mathbb{C}$-linear combination of irreducible characters. For $m \equiv j \mod N$, we have

$$\begin{aligned}\chi_{mP} &= f_{j,0}m^0 + f_{j,1}m^1 + \cdots + f_{j,d}m^d \\ &= \sum_{\ell=0}^{d}\left(\sum_{i=1}^{k} z_{\ell,i}\chi_i\right) m^\ell,\end{aligned}$$

where a priori $z_{\ell,i} \in \mathbb{C}$ for all $\ell \in \{0,\dots,d\}$ and $i \in \{1,\dots,k\}$. As χ_{mP} is an effective character, for fixed m we also have $\chi_{mP} = \sum_{i=1}^{k} \mu_{m,i}\chi_i$, where $\mu_{m,i} \in \mathbb{Z}_{\geq 0}$. Combining and comparing coefficients of χ_i yields equations of the form

$$\mu_{m,i} = \sum_{\ell=0}^{d} z_{\ell,i} m^\ell,$$

for each $m \in \mathbb{Z}_{\geq 0}$ such that $m \equiv j \mod N$. This implies that the $z_{\ell,i}$s are rational (for example through use of the Vandermonde-matrix). □

Corollary 1.2.5. *Let P be a rational polytope with denominator $S \in \mathbb{Z}_{>0}$ such that P is invariant under the linear action of a group G. Then χ_{mP} is quasipolynomial in m with period dividing SN.*

Theorem 1.2.6. *Let P be a G-invariant lattice polytope of dimension d as in the Setup 1.2.1, where G has exponent N and irreducible characters $\{\chi_1,\dots,\chi_k\}$. The equivariant Ehrhart series has the following rational expression:*

$$\mathrm{EE}(P;t) = \sum_{m\geq 0} \chi_{mP} t^m = \frac{\tilde{\mathrm{H}}(P;t)}{(1-t^N)^{d+1}},$$

where $\tilde{\mathrm{H}}(P;t)$ is a polynomial with coefficients in $R(G)\otimes_{\mathbb{Z}}\mathbb{Q}$ of degree $\leq N(d+1)-1$.

Proof. By Corollary 1.2.4, there exists polynomials $\lambda_{j,i}(m) \in \mathbb{Q}[m]$ for each $j \in \{0,\dots,N-1\}$ and $i \in [k]$ of degree $\leq d$ such that for $m \equiv j \mod N$, $\chi_{mP} = \sum_{i=1}^{k} \lambda_{j,i}(m)\chi_i$. Therefore,

$$\begin{aligned}\mathrm{EE}(P;t) &= \sum_{a\geq 0}\sum_{j=0}^{N-1}\sum_{i=1}^{k} \lambda_{j,i}(aN+j)\chi_i t^{aN+j} \\ &= \sum_{j=0}^{N-1} t^j \sum_{i=1}^{k}\chi_i \sum_{a\geq 0}\lambda_{j,i}(aN+j)t^{aN},\end{aligned}$$

where $\lambda_{j,i}(aN+j)$ is a polynomial in a of degree $d_{j,i} \leq d$ for all $i \in \{1,\dots,k\}$ and $j \in \{0,\dots,N-1\}$. Then,

$$\mathrm{EE}(P;t) = \sum_{j=0}^{N-1} t^j \sum_{i=1}^{k} \chi_i \frac{g_{j,i}(t^N)}{(1-t^N)^{d_{j,i}+1}},$$

where $g_{j,i}(t) \in \mathbb{Q}[t]$ is a polynomial of degree $\leq d_{j,i} \leq d$ for all $i \in \{1, \ldots, k\}$ and $j \in \{0, \ldots, N\}$. Thus,

$$\mathrm{EE}(P;t) = \frac{\sum_{j=0}^{N-1} t^j \sum_{i=1}^{k} \chi_i (1-t^N)^{d-d_{j,i}} g_{j,i}(t^N)}{(1-t^N)^{d+1}}. \qquad \square$$

Stapledon considers a different denominator for the equivariant Ehrhart series. We can rewrite $\mathrm{EE}(P;t) = \sum_{m\geq 0} \chi_{mP} t^m = \frac{\mathrm{H}^*(P;t)}{\det(I - t\cdot\rho)}$, where $\mathrm{H}^*(P;t)$ is a priori a power series in t with coefficients in $R(G)$. The denominator, $\det(I - t\cdot\rho)$, is a polynomial in t of degree $d+1$ with coefficients in the character ring, see Equation (4). Here, ρ is the matrix representation of G on the ambient vector space, and $t\cdot\rho(g)$ is the matrix representation of g with each entry multiplied by t.

Corollary 1.2.7. *The* H^**-series can be represented as a rational function in* t *with coefficients in* $R(G)$.

Proof. By Theorem 1.2.6, the equivariant Ehrhart series has a rational generating function in t with coefficients in $R(G)\otimes_{\mathbb{Z}}\mathbb{Q}$. Likewise, by Equation (4), $\det(I - t\cdot\rho)$ is a polynomial in t with coefficients in $R(G)$. The product

$$\mathrm{H}^*(P;t) = \det(I - t\cdot\rho)\frac{\tilde{\mathrm{H}}(P;t)}{(1-t^N)^{d+1}}$$

is also a rational function in t with coefficients in $R(G)\otimes_{\mathbb{Z}}\mathbb{Q}$. As $(1-t^N)^{d+1}\,\mathrm{H}^*(P;t)$ is an element of $R(G)[t]$, $\det(1 - t\cdot\rho)\tilde{\mathrm{H}}(P;t) \in R(G)[t]$. $\square$

Recall that the H^*-series is effective if the coefficient of t^i is an effective character, i.e., it decomposes as a nonnegative integral combination of the irreducible characters.

Lemma 1.2.8. *If the* H^**-series is effective, then it is polynomial.*

Proof. The H^*-series is a priori an infinite formal sum: $\mathrm{H}^*(P;t) = \sum_{i\geq 0}\mathrm{H}^*_{\mathrm{i}}\, t^i$. On the level of series, we have:

$$\frac{\mathrm{H}^*(P;t)[\mathrm{id}_G]}{\det(I - t\cdot\rho(\mathrm{id}_G))} = EE(P;t)[\mathrm{id}_G] = Ehr(P;t) = \frac{\mathrm{h}^*(P;t)}{(1-t)^{d+1}}.$$

As $\det(I - t\cdot\rho[\mathrm{id}_G]) = (1-t)^{d+1}$, $\mathrm{H}^*(P;t)[\mathrm{id}_G] = \mathrm{h}^*(P;t)$. Let $\{\chi_1, \ldots, \chi_k\}$ be the irreducible characters of G. Since $\mathrm{H}^*(P;t)$ is effective, $\mathrm{H}^*_{\mathrm{i}} = \sum_{j=1}^{k} z_{i,j}\chi_j$, with $z_{i,j} \in \mathbb{Z}_{\geq 0}$ for all i,j. Since $\chi_j[\mathrm{id}_G] > 0$ for all j, no cancellation can occur and $\mathrm{H}^*(P;t)$ must be polynomial. $\square$

It is an open question to determine if $\mathrm{H}^*(P;t)$ is effective if and only if it is polynomial [99, Conjecture 12.1]. At the time of writing, the equivariant Ehrhart series of only a few families of polytopes has been completely described, including centrally symmetric polytopes under a $\mathbb{Z}/2\mathbb{Z}$ action [99, Section 11], hypercubes under symmetric group actions [99, Section 9], reflexive polytopes [99, Section 8], and permutahedra under symmetric group actions [3]. The open-source implementation of code for calculating the H^*-series discussed next may be helpful in answering the effectiveness question and describing the equivariant Ehrhart series of more families of polytopes.

1.2.1 *Implementation of* $\mathrm{H}^*(P;t)$ *in* `Sagemath`

The author has implemented methods in `Sagemath` for the computation of the H^*-series, see https://trac.sagemath.org/ticket/27637. The code is open-source and available for public use with current `Sagemath` versions. It works via the following algorithm:

Algorithm 1.2.9 (Computing $\mathrm{H}^*(P;t)$)**.**
INPUT: A rational polytope P and a group G that acts as in the Setup 1.2.1.
OUTPUT: The H^*-series: a rational function in t with coefficients in $R(G)$.

1: Fix conjugacy class representatives $\{g_1, \ldots, g_k\}$ of G and the corresponding character table T of G.
2: For each representative g_i compute:
- A_i = the Ehrhart series of the fixed subpolytope,
- $B_i = \det(I - t\cdot\rho(g_i))$.

3: Solve the system of equations $\mathbf{x}T = [A_iB_i, \ldots, A_kB_k]$, for $\mathbf{x}$.
4: Output the rational function $x_1\chi_1 + \cdots + x_k\chi_k$.

Some explanation is needed for the input of Algorithm 1.2.9. The usage of the code is demonstrated in the coming examples. The polytope must have backend `normaliz`. The acting group must be obtained as a subgroup of the polytope's `restricted_automorphism_group`, which is a method of polytopes in `Sagemath`. The restricted automorphism group is the group of linear transformations mapping the polytope to itself and such that d-dimensional faces are mapped to d-dimensional faces. Furthermore, the output of the `restricted_automorphism_group` must be set to `permutation`. This means that every group element is expressed as a permutation of the vertices of the polytope. The restricted automorphism group also has the option to represent group elements as matrices. Full-dimensional polytopes are lifted to height 1 for this representation, and for lower-dimensional polytopes, the matrices act as the identity on the orthogonal space to the affine span of polytope. The trickiest part about using the `Hstar_function` is creating the acting group one is interested in as a subgroup of the restricted automorphism group. To compute the Ehrhart series of the fixed subpolytopes in Step 2 of Algorithm 1.2.9, we implemented the method `fixed_subpolytopes` which returns the fixed subpolytope for each conjugacy class representative, and may be useful to some researchers in its own right.

Example 1.2.10. The H^*-polynomial of the 2-dimensional permutahedron $\Pi_3 = \operatorname{conv}\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ in $\mathbb{R}^3$ is invariant under the action of the symmetric group S_3 permuting the standard basis vectors of $\mathbb{R}^3$. Actually, S_3 acts linearly on the lattice

$$\left\{\mathbf{x} \in \mathbb{Z}^3 \,:\, \sum_{i=1}^{3} x_i = 0\right\} \times \; (2,2,2)\mathbb{Z},$$

and Π_3 is an invariant lattice polytope of height one with respect to this sublattice of $\mathbb{Z}^3$. As shown in [3], the H*-series of Π_3 under the action of S_3, where $\chi_{triv}, \chi_{sign}, \chi_\Delta$ denote the irreducible representations, is

$$\mathrm{H}^*(\Pi_3; t) = \chi_{triv} + (\chi_{triv} + \chi_{sign} + \chi_\Delta)\, t + \chi_{triv} t^2.$$

We now make this computation in `Sagemath`. To do so, we first create the permutahedron `P3` and is restricted automorphism group `G`.

```
sage: P3 = polytopes.permutahedron(3, backend='normaliz')
sage: G = P3.restricted_automorphism_group(output='
   permutation')
sage: G.gens()
[(1,2)(3,4), (0,1)(2,3)(4,5), (0,5)(1,3)(2,4)]
sage: P3.vertices()
(A vertex at (1, 2, 3),
 A vertex at (1, 3, 2),
 A vertex at (2, 1, 3),
 A vertex at (2, 3, 1),
 A vertex at (3, 1, 2),
 A vertex at (3, 2, 1))
```

Inspection shows that the generator $(0,1)(2,3)(4,5)$ of G corresponds to the reflection across the $(x_1 = x_2)$-hyperplane, and the generator $(0,5)(1,3)(2,4)$ corresponds to reflection across the $(x_1 = x_3)$-hyperplane. Together, these two elements generate S_3. We now create S_3 as a subgroup of G and then use the `Hstar_function` to compute $\mathrm{H}^*(\texttt{P3}; t)$. We set the output to `complete` for a more verbose output.

```
sage: H = G.subgroup(gens=[G.gens()[1], G.gens()[2]]); H
Subgroup generated by [(0,1)(2,3)(4,5), (0,5)(1,3)(2,4)] of
(Permutation Group with generators
[(1,2)(3,4), (0,1)(2,3)(4,5), (0,5)(1,3)(2,4)])
sage: H.order()
6
sage: P3.Hstar_function(acting_group=H, output='complete')
{'Hstar': chi_0*t^2 + (chi_0 + chi_1 + chi_2)*t + chi_0,
 'Hstar_as_lin_comb': (t^2 + t + 1, t, t),
 'conjugacy_class_reps': [(), (0,1)(2,3)(4,5), (0,3,4)
    (1,5,2)],
 'character_table': [ 1  1  1]
 [ 1 -1  1]
 [ 2  0 -1],
 'is_effective': True}
```

The permutahedron Π_3 is also invariant under the action of the cyclic group $\mathbb{Z}/3\mathbb{Z}$ permuting the standard basis vectors. We can likewise compute this H*-series, where χ_0 is the trivial representation, and χ_1, χ_2 are the other one-dimensional irreducible characters of $\mathbb{Z}/3\mathbb{Z}$. We again create a subgroup of the restricted automorphism group G, this time to obtain a $\mathbb{Z}/3\mathbb{Z}$ action.

```
sage: g = G[1]; g
(0,4,3)(1,2,5)
sage: H = G.subgroup(gens=[g]); H
Subgroup generated by [(0,4,3)(1,2,5)] of (Permutation
   Group
```

```
with generators [(1,2)(3,4), (0,1)(2,3)(4,5), (0,5)(1,3)
   (2,4)])
sage: P3.Hstar_function(acting_group=H)
chi_0*t^2 + (2*chi_0 + chi_1 + chi_2)*t + chi_0
```

The representation of $\mathbb{Z}/3\mathbb{Z}$ is a restriction of the representation of S_3 on Π_3. Although knowing the quasipolynomial χ_{mP} for the symmetric group action implicitly determines the quasipolynomial for the cyclic group action, there is in general no easy way to transfer between the expressions in terms of irreducible characters without using the inner product on characters. This is because an irreducible representation of a group G does not necessarily restrict to an irreducible representation of a subgroup.

Example 1.2.11 ([99, Example 7.6])**.** This example shows that $\mathrm{H}^*(P;t)$ is not always a polynomial. Let P be the polytope with vertices $\pm(0,0,1)$, $\pm(1,0,1)$,$\pm(0,1,1)$, $\pm(1,1,1)$ and let $G = \mathbb{Z}/2\mathbb{Z}$ act on P as follows:

```
sage: P = Polyhedron(vertices
   =[[0,0,1],[0,0,-1],[1,0,1],[-1,0,-1],[0,1,1],
[0,-1,-1],[1,1,1],[-1,-1,-1]], backend='normaliz')
sage: K = P.restricted_automorphism_group(output='
   permutation')
sage: G = K.subgroup(gens=[K[6]]); G
Subgroup generated by [(0,2)(1,3)(4,6)(5,7)] of (
   Permutation Group with generators [(2,4)(3,5), (1,2)
   (5,6), (0,1)(2,3)(4,5)(6,7), (0,7)(1,3)(2,5)(4,6)])
```

To check what matrix the vertex permutation $(0,2)(1,3)(4,6)(5,7)$ corresponds to, we use the `match_permutations_to_matrices` function. This function requires the conjugacy class representatives of G as an input. The result agrees with the matrix given in [99, Example 7.6].

```
sage: conj_reps = G.conjugacy_classes_representatives()
sage: Dict = P.match_permutations_to_matrices(conj_reps,
   acting_group = G)
sage: list(Dict.keys())[0]
(0,2)(1,3)(4,6)(5,7)
sage: list(Dict.values())[0]
[-1  0  1  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
sage: len(G)
2
sage: G.character_table()
[ 1  1]
[ 1 -1]
```

Then we calculate the rational function $\mathrm{H}^*(P;t)$:

```
sage: Hst = P.Hstar_function(G); Hst
(chi_0*t^4 + (3*chi_0 + 3*chi_1)*t^3 + (8*chi_0 + 2*chi_1)*
   t^2 + (3*chi_0 + 3*chi_1)*t + chi_0)/(t + 1)
```

We can format the output as `Hstar_as_lin_comb` to see it written exactly as in [99, Example 7.6]. The first coordinate is the coefficient of the trivial character; the second is the coefficient of the sign character:

```
sage: P.Hstar_function(G,output='Hstar_as_lin_comb')
((t^4 + 3*t^3 + 8*t^2 + 3*t + 1)/(t + 1),
(3*t^3 + 2*t^2 + 3*t)/(t + 1))
```

To see the documentation of the `Hstar_function`, or of the related supporting methods, `fixed_subpolytopes`, `permutations_to_matrices`, or indeed of any function in `Sagemath`, one can type `?` after the function:

```
sage: P = polytopes.cube(backend='normaliz')
sage: P.fixed_subpolytopes?
```

To see the both source code and the documentation simultaneously, type `??` after the function:

```
sage: P = polytopes.cube(backend='normaliz')
sage: P.Hstar_function??
```

1.3 SIMPLICES

In this section we recall some results of Stapledon [99, Secton 6] on the equivariant Ehrhart series of simplices and elaborate on the proofs. Of particular focus is [99, Proposition 6.1] which gives an expression for the H^*-series of a G-invariant simplex by counting fixed lattice points in the fundamental parallelepiped. These proof ideas are generalized in Theorem 1.4.1 to obtain a similar theorem for general polytopes.

As in the Setup 1.2.1, let $M' = M \oplus \mathbb{Z}$ be a $(d+1)$-dimensional lattice. Let S be a d-dimensional G-invariant simplex with vertices $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_d\} \in M \times \{1\}$. Since S is a simplex, we can choose to have a single interval $[\emptyset, S]$ in our half-open decomposition, see Section 1.1 for clarification. Then, $\mathrm{Box}([\emptyset, S])$ is the set of lattice points in the half-open parallelepiped spanned by the vertices of S:

$$\mathrm{Box}([\emptyset, S]) = \left\{ \mathbf{v} \in M' \mid \mathbf{v} = \sum_{i=0}^{d} a_i \mathbf{v}_i \text{ with } 0 \leq a_i < 1 \text{ for all } i \in \{0, \dots, d\} \right\}.$$

Furthermore $\mathrm{Box}([\emptyset, S])_k$ denotes the set of lattice points in the half-open parallelepiped of S at height k, i.e. with $\mathbb{Z}$-coordinate equal to k. Recall that for an action of a group G on a set X, X^g denotes the subset of X fixed by the element $g \in G$. Recall also, that the equivariant Ehrhart series has the generating function $\frac{\mathrm{H}^*(P;t)}{\det(I - t\cdot\rho)}$. If $\mathrm{H}^*(P;t)$ is polynomial in t, then we denote the coefficient of t^i in $R(G)$ by $\mathrm{H}^*_{\mathrm{i}}$.

Proposition 1.3.1 ([99, Proposition 6.1])**.** *If S is a G-invariant lattice simplex as in the Setup 1.2.1 with vertices $\{\mathbf{v}_0, \mathbf{v}_1, \dots \mathbf{v}_d\}$ in $M \times \{1\}$, then $\mathrm{H}^*_{\mathrm{i}}$ is the permutation representation induced by the action of G on* $\mathrm{Box}([\emptyset, S])_i$*. In particular,*

$$\sum_{m \geq 0} \mathrm{ehr}(S^g; m) t^m = \frac{\sum_{i=0}^{d} |\mathrm{Box}([\emptyset, S])_i^g| \, t^i}{\det(I - t \cdot \rho(g))}.$$

Proof. We consider 2 infinite-dimensional vector spaces over $\mathbb{C}$ and 1 finite-dimensional vector space over $\mathbb{C}$, all described by their respective bases:

$$\begin{aligned}\mathbb{C}\left[\mathrm{Cone}(S)\cap M'\right] \text{ with basis: } & \{\mathbf{x}\in \mathrm{Cone}(S)\cap M'\} \\ \mathbb{C}\left[\mathbb{Z}_{\geq 0}\,\mathrm{vert}(S)\right] \text{ with basis: } & \left\{\mathbf{x}=\sum_{i=0}^{d} z_i\mathbf{v}_i \middle| z_i\in\mathbb{Z}_{\geq 0} \text{ for all } i\in\{0,\dots,d\}\right\} \\ \mathbb{C}\left[\mathrm{Box}([\emptyset,S])\right] \text{ with basis: } & \{\mathbf{x}\in \mathrm{Box}([\emptyset,S])\}\,.\end{aligned}$$

We first show that there is an isomorphism of graded G-modules:

$$\mathbb{C}[\mathrm{Cone}(S)\cap M'] \cong \mathbb{C}[\mathbb{Z}_{\geq 0}\,\mathrm{vert}(S)] \bigotimes_{\mathbb{C}} \mathbb{C}[\mathrm{Box}([\emptyset,S])].$$

The basis of $\mathbb{C}[\mathrm{Cone}(S)\cap M']$ is the set of all lattice points in the cone over the simplex. The basis of $\mathbb{C}[\mathbb{Z}_{\geq 0}\,\mathrm{vert}(S)] \bigotimes_{\mathbb{C}} \mathbb{C}[\mathrm{Box}([\emptyset,S])]$ is the set of all simple tensors $\mathbf{w}\otimes\mathbf{z}$ where $\mathbf{w}\in\mathbb{Z}_{\geq 0}\,\mathrm{vert}(S)$ and $\mathbf{z}\in\mathrm{Box}([\emptyset,S])$. Every lattice point $\mathbf{x}$ in $\mathrm{Cone}(S)$ has a unique expression $\mathbf{x}=\mathbf{w}+\mathbf{z}$ where $\mathbf{w}\in\mathbb{Z}_{\geq 0}\,\mathrm{vert}(S)$ and $\mathbf{z}\in\mathrm{Box}([\emptyset,S])$. The map sending $\mathbf{x}\mapsto\mathbf{w}\otimes\mathbf{z}$ is then a bijection between bases. The action of G preserves the height of lattice points in the cone by assumption, and the isomorphism is also a G-module isomorphism by linearity and the diagonality of the action. We now identify $\mathbb{C}[\mathbb{Z}_{\geq 0}\,\mathrm{vert}(S)]$ with the $\mathbb{C}$-vector space $\sum_{m\geq 0}\mathrm{Sym}^m(M'\bigotimes_{\mathbb{Z}}\mathbb{C})$, where the symmetric product is taken over $\mathbb{C}$. These two vector spaces are isomorphic as graded G-modules, so they have the same characters. Let $\mathbb{Z}^m_{\geq 0}\,\mathrm{vert}(S)$ be the set:

$$\left\{\mathbf{x}=\sum_{i=0}^{d} z_i\mathbf{v}_i \quad\middle|\quad z_i\in\mathbb{Z}_{\geq 0} \text{ for all } i\in\{0,\dots,d\} \text{ and } \sum_{i=0}^{d} z_i = m\right\}.$$

Then,

$$\chi_{\mathbb{C}[\mathbb{Z}^m_{\geq 0}\,\mathrm{vert}(S)]} = \chi_{\mathrm{Sym}^m(M'\bigotimes\mathbb{C})} \text{ for all } m\geq 0.$$

When a group acts on a tensor product of vector spaces, the characters multiply:

$$\begin{aligned}\sum_{m\geq 0}\chi_{mS}t^m &= \sum_{j\geq 0}\chi_{\mathrm{Sym}^j(M'\otimes\mathbb{C})}t^j\sum_{i=0}^{d}\chi_{\mathrm{Box}([\emptyset,S])_i}t^i \\ &= \frac{\sum_{i=0}^{d}\chi_{\mathrm{Box}([\emptyset,S])_i}t^i}{\det(I-t\cdot\rho)},\end{aligned}$$

where the last equality holds by Equation (3). Evaluating at a specific $g\in G$ yields

$$\sum_{m\geq 0}\mathrm{ehr}(S^g;m)t^m = \frac{\sum_{i=0}^{d}|\mathrm{Box}([\emptyset,S])_i^g|\,t^i}{\det(I-t\cdot\rho(g))}.$$

□

Corollary 1.3.2 ([99, Proposition 6.1])**.** *The multiplicity of the trivial representation in* $\mathrm{H}^*_{\mathrm{k}}$ *equals the number of G-orbits of points in* $\mathrm{Box}([\emptyset,S])_k$*.*

Proof. The vector space $\mathbb{C}[\mathrm{Box}([\emptyset,S])_k]$ is a G-module; it carries the induced permutation representation of G acting on the lattice points in $\mathrm{Box}([\emptyset,S])_k$. Call the character of this representation Φ. To find the multiplicity of the trivial representation, we take the inner product of Φ with χ_{triv}.

$$\begin{aligned}\langle \Phi, \chi_{triv} \rangle &= \frac{1}{|G|} \sum_{g \in G} \Phi(g) \overline{\chi_{triv}(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} |\operatorname{Box}([\emptyset, S])_k^g|.\end{aligned}$$

By the Cauchy–Frobenius Lemma, this is exactly the number of orbits of lattice points in $\operatorname{Box}([\emptyset, S])_k$. □

The next corollary is in the same spirit, and uses Frobenius reciprocity, see [90, Chapter 1]. We write $\operatorname{Ind}_H^G \rho$ for the induced representation of G from the representation ρ of a subgroup $H \subseteq G$. We write $\operatorname{Res}_H^G \rho$ for the restricted representation on a subgroup $H \subseteq G$ from the representation ρ of G. Given a representation ρ of G, we can construct a new irreducible, one-dimensional representation (and class function), $\det(\rho(g))$, which sends every element of G to the determinant of its corresponding matrix.

Corollary 1.3.3 ([99, Proposition 6.1])**.** *Let S be a G-invariant simplex as in Proposition 1.3.1. The multiplicity of* $\det(\rho)$ *in* H_k^* *is equal to the number of orbits $\mathcal{O}$ of points in* $\operatorname{Box}([\emptyset, S])_k$ *such that for any $\mathbf{x} \in \mathcal{O}$,* $\det(\rho(h)) = 1$ *for all $h \in$* $\operatorname{Stab}(\mathbf{x})$.

Proof. Let $\mathcal{O}$ be an orbit of points in $\operatorname{Box}([\emptyset, S])_k$, and let $\mathbf{x} \in \mathcal{O}$. Let H be the isotropy subgroup of $\mathbf{x}$ in G, i.e. $H = \{g \in G \,|\, g\mathbf{x} = \mathbf{x}\}$. Let $\mathbf{1}$ be the trivial representation of H on the one-dimensional vector space $\mathbb{C}[\mathbf{x}]$.

Claim 1.3.4. $\operatorname{Ind}_H^G \mathbf{1}$ is G-module isomorphic to $\mathrm{H}_k^* \big|_{\mathcal{O}}$.

Proof of Claim. Let $\{a_i \,|\, i = 1, \ldots, m\}$ be a set of coset representatives of H in G. Then the vector space V_{Ind} of the induced representation is an m-dimensional vector space over $\mathbb{C}$ with one basis vector $a_i\mathbf{x}$ for each coset representative: $V_{\mathrm{Ind}} = \mathbb{C}[\{a_i\mathbf{x}\}]$. In an induced representation, the action of H on V_{Ind} is $a_i \cdot (a_j\mathbf{x}) := (a_k)\mathbf{x}$, where $a_i a_j \in a_k H$ for a coset representative a_k. The action of G on V_{Ind} is then given as follows. Let $g \in G$ so that $g = a_i \cdot h$ for some coset representative a_i and some element $h \in H$. Then $g \cdot a_j\mathbf{x} = (a_i \cdot h) \cdot a_j\mathbf{x} = (a_i \cdot h \cdot a_j) \cdot \mathbf{x} = a_k\mathbf{x}$, for some coset representative a_k. The action of G on V_{Ind} is then given through linear extension of its action on the basis. The G-module isomorphism easily follows. □

By Claim 1.3.4 and Frobenius reciprocity, $\langle\, \mathrm{H}_k^* \big|_{\mathcal{O}}, \det(\rho) \,\rangle = \langle\, \operatorname{Ind}_H^G \mathbf{1}, \det(\rho) \,\rangle = \langle\, \mathbf{1}, \operatorname{Res}_H^G \det(\rho) \,\rangle$. As both $\mathbf{1}$ and the restriction of $\det(\rho)$ are irreducible representations, $\langle\, \mathbf{1}, \operatorname{Res}_H^G \det(\rho) \,\rangle$ is either equal to one or zero, and it is equal to one exactly if the determinant of each element in H is 1. □

1.4 POLYTOPES WITH SYMMETRIC HALF-OPEN DECOMPOSITIONS

In this section, we give two new methods for computing the equivariant Ehrhart series of a polytope using symmetric half-open decompositions. The first method is presented in Theorem 1.4.1 and generalizes Stapledon's proposition on the equivariant Ehrhart series for simplices (Proposition 1.3.1) to lattice polytopes that admit symmetric half-open decompositions and satisfy a cardinality condition on orbits. In this case, the H^*-series is polynomial and effective (Corollary 1.4.2). The second method is given in Theorem 1.4.6 and allows us to compute a rational generating function for the equivariant Ehrhart series with the denominator $(1-t)^{\dim(P)+1}$. This theorem also uses G-invariant half-open decompositions, and further requires a condition on the box points in the intervals. In this case, χ_{mP} is polynomial as a function in m.

The criteria of these theorems may appear restrictive, but the upshot is that the equivariant Ehrhart theory is especially well-behaved and has a geometric interpretation, as we illustrate in the examples following each theorem.

As in the Setup 1.2.1, let P be a lattice polytope with vertices in $M \times 1$, invariant under the linear action of a finite group G. For a half-open decomposition $\mathcal{T}^{[)}$ of P, let $\mathrm{Box}(\mathcal{T}^{[)})_i$ denote the union of lattice points with $\mathbb{Z}$-coordinate equal to i in the half-open parallelepipeds of the intervals of $\mathcal{T}^{[)}$ (see Section 1.1 for details on half-open decompositions).

Theorem 1.4.1. *Let $\mathcal{T}^{[)}$ be a G-invariant half-open decomposition of a d-dimensional polytope P as in the Setup 1.2.1 such that $\dim(\overline{S}) = d$ for all intervals $[\underline{S}, \overline{S}]$ of $T^{[)}$ and such that all orbits of intervals of $T^{[)}$ have order $|G|$ except for a unique G-invariant interval. Then the H^*-polynomial is the permutation representation on the union of box points in $\mathcal{T}^{[)}$, graded by height:*

$$\mathrm{H}^*(P;t) = \sum_{i=0}^{d} \chi_{\mathrm{Box}(\mathcal{T}^{[)})_i} t^i.$$

Proof. Let n be the number of orbits of intervals of $\mathcal{T}^{[)}$. There necessarily exists a G-invariant simplex $\overline{S_0}$ contained in an interval $[\underline{S_0}, \overline{S_0}]$ of $\mathcal{T}^{[)}$ that contains the G-fixed barycenter of the vertices of P. Label its vertices by $\{\mathbf{v}_{0,0}, \ldots, \mathbf{v}_{0,d}\}$ and label the orbit of the interval $[\underline{S_0}, \overline{S_0}]$ by $\mathcal{O}_0$. Order the other orbits $\mathcal{O}_1, \ldots, \mathcal{O}_{n-1}$ of intervals of $\mathcal{T}^{[)}$. For each $i \in [n-1]$, label representative simplices $\underline{S_i}$ and $\overline{S_i}$ contained in an interval $[\underline{S_i}, \overline{S_i}]$ of $\mathcal{O}_i$ and label the vertices of $\overline{S_i}$ as $\mathbf{v}_{i,0}, \ldots, \mathbf{v}_{i,d}$. As in the proof of Theorem 1.3.1, identify the G-module $\mathbb{C}[\mathbb{Z}_{\geq 0} \operatorname{vert}(\overline{S_0})]$ with the polynomial ring and graded G-module, $\mathbb{C}[X_0, \ldots, X_d]$ where $g(X_i) := X_j$ if $g(\mathbf{v}_{0,i}) = \mathbf{v}_{0,j}$. Viewing $g \in G$ as a permutation on $\{0, 1, \ldots, d\}$, we write, $g(X_0^{c_0} \cdots X_d^{c_d}) = X_{g(0)}^{c_0} \cdots X_{g(d)}^{d}$. We define a map f from

$$\mathbb{C}\left[X_0, \ldots, X_d\right] \bigotimes_{\mathbb{C}} \left(\mathbb{C}\left[\mathrm{Box}\left(\left[\underline{S_0}, \overline{S_0}\right]\right)\right] \bigoplus_{\substack{i:|\mathcal{O}_i|=|G| \\ g \in G}} \mathbb{C}\left[\mathrm{Box}\left(\left[\underline{gS_i}, \overline{gS_i}\right]\right)\right] \right) \tag{6}$$

to $\mathbb{C}[\mathrm{Cone}(P) \cap M']$ by defining f on a basis and extending bilinearly. To shorten notation, for a lattice point $\mathbf{z}_0$ in $\mathrm{Box}([\underline{S_0}, \overline{S_0}])$ or $g\mathbf{z}_i$ in $\mathrm{Box}([\underline{gS_i}, \overline{gS_i}])$, we denote the corresponding basis vector of

$$\left(\mathbb{C}\left[\mathrm{Box}\left(\left[\underline{S_0}, \overline{S_0}\right]\right)\right] \bigoplus_{\substack{i:|\mathcal{O}_i|=|G| \\ g\in G}} \mathbb{C}\left[\mathrm{Box}\left(\left[\underline{gS_i}, \overline{gS_i}\right]\right)\right] \right) \tag{7}$$

as $\mathbf{z}_0$ or $g\mathbf{z}_i$ respectively. The direct sum (7) is also a graded G-module. A basis for the tensor product (6) is the set

$$\left\{ \begin{array}{r} (X_0^{c_0} \cdots X_d^{c_d}) \otimes \mathbf{z}_0 : \{c_0, \dots, c_d\} \in \mathbb{Z}_{\geq 0}, \mathbf{z}_0 \in \mathrm{Box}\left(\left[\underline{S_0}, \overline{S_0}\right]\right), \\ g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{z}_i : \{c_0, \dots, c_d\} \in \mathbb{Z}_{\geq 0}, \mathbf{z}_i \in \mathrm{Box}\left(\left[\underline{S_i}, \overline{S_i}\right]\right), \ g \in G, \ |\mathcal{O}_i| = |G| \end{array} \right\}.$$

Define

$$f(g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{z}_i) = f((X_{g(0)}^{c_0} \cdots X_{g(d)}^{c_d}) \otimes g\mathbf{z}_i) := \sum_{j=0}^{d} c_j g(\mathbf{v}_{i,j}) + g\mathbf{z}_i.$$

Suppose $g\mathbf{z}_0 = \mathbf{w}_0$ for box points $\mathbf{z}_0$ and $\mathbf{w}_0$ in $\mathrm{Box}([\underline{S_0}, \overline{S_0}])$ and therefore

$$X_{g(0)}^{c_0} \cdots X_{g(d)}^{c_d} \otimes g\mathbf{z}_0 = X_0^{c_{g^{-1}(0)}} \cdots X_d^{c_{g^{-1}(d)}} \otimes \mathbf{w}_0.$$

To check f is well-defined, we compute:

$$\begin{aligned} f\left(X_{g(0)}^{c_0} \cdots X_{g(d)}^{c_d} \otimes g\mathbf{z}_0\right) &= \sum_{j=0}^{d} c_j g(\mathbf{v}_{i,j}) + g(\mathbf{z}_0) \\ &= \sum_{j=0}^{d} c_j \mathbf{v}_{i,g(j)} + \mathbf{w}_0 \\ &= \sum_{\gamma=0}^{d} c_{g^{-1}(\gamma)} \mathbf{v}_{i,\gamma} + \mathbf{w}_0 \\ &= f(X_0^{c_{g^{-1}(0)}} \cdots X_d^{c_{g^{-1}(d)}} \otimes \mathbf{w}_0). \end{aligned}$$

The map f is a G-module isomorphism, as we verify on the basis. For $h \in G$,

$$\begin{aligned} f\left(h\left(g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{z}_i\right)\right) &= f\left((X_{hg(0)}^{c_0} \cdots X_{hg(d)}^{c_d}) \otimes hg\mathbf{z}_i\right) \\ &= \sum_j c_j hg(\mathbf{v}_{i,j}) + hg\mathbf{z}_i \\ &= h\left(\sum_j c_j g(\mathbf{v}_{i,j}) + g\mathbf{z}_i\right) \\ &= h(f(g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{z}_i)). \end{aligned}$$

The G-module isomorphism also respects the grading and yields an equality among characters.

$$\begin{aligned}\sum_{m\geq 0} \chi_{mP} t^m &= \chi_{\mathbb{C}[\mathbb{Z}_{\geq 0}\operatorname{vert}(S_0)]} \sum_{i=0}^{d} \chi_{\operatorname{Box}(\mathcal{T}^{[)})_i} t^i \\ &= \frac{1}{\det(I - t\cdot\rho)} \sum_{i=0}^{d} \chi_{\operatorname{Box}(\mathcal{T}^{[)})_i} t^i,\end{aligned}$$

where the second equality holds through the identification of $\mathbb{C}[\mathbb{Z}_{\geq 0}\operatorname{vert}(\overline{S_0})]$ and $\sum_{m\geq 0}\operatorname{Sym}^m(M' \otimes_{\mathbb{Z}} \mathbb{C})$ as G-modules. □

Stapledon conjectured [99, Conjecture 12.2] that if $\mathrm{H}^*(P;t)$ is effective, then $\mathrm{H}^*(1)$ is a permutation representation.

Corollary 1.4.2. *The* H^**-series is polynomial, effective, and a permutation representation. Thus* $\mathrm{H}^*(P;1)$ *is also a permutation representation, and Stapledon's Conjecture 12.2 in [99] is satisfied.*

The next corollaries are proved analogously to Corollary 1.3.2 and Corollary 1.3.3.

Corollary 1.4.3. *The multiplicity of the trivial representation in* H^*_k *equals the number of* G*-orbits in* $\operatorname{Box}(\mathcal{T}^{[)})_k$*.*

Corollary 1.4.4. *The multiplicity of* $\det(\rho)$ *in* H^*_k *is equal to the number of orbits* $\mathcal{O}$ *of points in* $\operatorname{Box}(\mathcal{T}^{[)})_k$ *such that for any* $\mathbf{x} \in \mathcal{O}$, $\det(\rho(h)) = 1$ *for all* $h \in \operatorname{Stab}(\mathbf{x})$.

Example 1.4.5. [Examples 1.1.2 and 1.2.10 continued] The two-dimensional permutahedron $\Pi_3 \subseteq \mathbb{R}^3$ under the action of the group $\mathbb{Z}/3\mathbb{Z}$ cyclically permuting the standard basis vectors admits a G-invariant half-open decomposition as dictated by Theorem 1.4.1. This half-open decomposition is described in Figure 3 through a triangulation $\mathcal{T}$ of Π_3 and a partition of the face poset of $\mathcal{T}$ into intervals. In this partition, each maximal simplex in an interval is a triangle and every orbit of triangles has order 1 or 3. With this half-open decomposition, the lattice points in the union of the fundamental parallelepipeds are

$$\begin{aligned}&\{(0,0,0)\} && \text{at height } 0,\\ &\{(1,2,3),(2,3,1),(3,1,2),(2,2,2)\} && \text{at height } 1,\\ &\{(4,4,4)\} && \text{at height } 2.\end{aligned}$$

At height 0 and 2, there is a unique $\mathbb{Z}/3\mathbb{Z}$-invariant box point, each giving a copy of the trivial representation. At height 1, the character of the permutation representation on the 4 lattice points evaluates to 4 at the identity element, and 1 at the other two group elements. Taking the inner product to express this representation in the basis of irreducible representations for $\mathbb{Z}/3\mathbb{Z}$: $\chi_{triv}, \chi_\zeta, \chi_{\zeta^2}$, yields the H^*-polynomial

$$\mathrm{H}^*(\Pi_3;t) = \chi_{triv} + (2\chi_{triv} + \chi_\zeta + \chi_{\zeta^2})t + \chi_{triv}t^2.$$

This calculation agrees with the `Sagemath` computation from Example 1.2.10.

The next theorem introduces a second way to compute the equivariant Ehrhart series using G-invariant half-open decompositions. The conditions required by the theorem ensure that the quasipolynomial χ_{mP} is actually polynomial in m. For a G-invariant half-open decomposition $T^{[)}$ of a polytope, let $\mathcal{O}$ denote an orbit of intervals of $\mathcal{T}^{[)}$, $\dim(\mathcal{O}) := \dim(\overline{S})$ for any interval $I = [\underline{S}, \overline{S}] \in \mathcal{O}$, and $\mathrm{Box}(\mathcal{O})_i := \bigcup_{I \in \mathcal{O}} \mathrm{Box}(I)_i$. Let $\chi_{\mathrm{Box}(\mathcal{O})_i}$ denote the permutation character on the lattice points in $\mathrm{Box}(\mathcal{O})_i$.

Theorem 1.4.6. *Let P be a G-invariant d-polytope as in the Setup 1.2.1 with a G-invariant half-open decomposition $\mathcal{T}^{[)}$ such that no d-face of the triangulation forms an interval. Furthermore, suppose that for each interval $I = [\underline{S}, \overline{S}]$ of $\mathcal{T}^{[)}$ and each $g \in G$, if a box point $\mathbf{z} \in \mathrm{Box}(I)$ is fixed by g, then the simplex $\overline{S}$ is fixed by g.*

The equivariant Ehrhart series has the following rational generating function:

$$\sum_{m \geq 0} \chi_{mP} t^m = \frac{\sum_{\mathcal{O} \in \mathcal{T}^{[)}} (1-t)^{d-\dim(\mathcal{O})} \sum_{i=0}^{\dim(\mathcal{O})} \chi_{\mathrm{Box}(\mathcal{O})_i} t^i}{(1-t)^{d+1}}.$$

In this case, χ_{mP} is a polynomial in m with coefficients in $R(G)$.

Proof. We break $\mathrm{Cone}(P)$ into orbits and describe the permutation representation on each piece. Let $\mathcal{O}$ be an orbit of intervals of $\mathcal{T}^{[)}$, and let $g \in G$. Furthermore, let $\mathrm{Cone}(\mathcal{O}) := \bigcup_{I \in \mathcal{O}} \mathrm{Cone}(I)$.

Claim 1.4.7.

$$\sum_{\mathbf{u} \in \mathrm{Cone}(\mathcal{O})^g \cap \mathbb{Z}^{d+1}} t^{\mathbf{u}} = \sum_{I = [\underline{S}, \overline{S}] \in \mathcal{O}} \frac{\sum_{\mathbf{m} \in \mathrm{Box}(I)^g} t^{\mathbf{m}}}{\prod_{\mathbf{s} \in \overline{S}} (1 - t^{\mathbf{s}})} \tag{8}$$

Proof of Claim. We first show that the left hand side of Equation (8) is a subset of the right in terms of the lattice points appearing in the exponents of the series expansions. Suppose $\mathbf{u} \in \mathrm{Cone}(\mathcal{O}^g) \cap \mathbb{Z}^{d+1}$. By Proposition 1.1.5, there exists a unique interval $I = [\underline{S}, \overline{S}] \in \mathcal{O}$ such that $\mathbf{u} \in \mathrm{Cone}(I)$. Let $\{\mathbf{s}_1, \ldots, \mathbf{s}_{n+1}\}$ be the vertices of $\overline{S}$. Then we may write $\mathbf{u}$ uniquely as $\mathbf{u} = \sum_{i=1}^{n+1} (c_i \mathbf{s}_i) + \mathbf{z}$, with $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n+1]$, and $\mathbf{z} \in \mathrm{Box}(I)$ by Proposition 1.1.7. As the expression of $\mathbf{u}$ is unique, $\mathbf{u} = g(\mathbf{u}) = \sum_{i=1}^{n+1} (c_i g(\mathbf{s}_i)) + g(\mathbf{z})$ implies that $\mathbf{z} = g(\mathbf{z})$. This implies that the coefficient of $t^{\mathbf{u}}$ in the series expansion of the right side of Equation (8) is 1.

We now show the right hand side of Equation (8) is a subset of the left. By Proposition 1.1.7, every lattice point $\mathbf{u}$ in the series expansion of the right side of Equation (8) as $t^{\mathbf{u}}$ has coefficient one. Furthermore, there exists a unique interval $I = [\underline{S}, \overline{S}] \in \mathcal{O}$ with $\mathrm{vert}(\overline{S}) = \{\mathbf{s}_1, \ldots \mathbf{s}_{n+1}\}$ such that $\mathbf{u} = \sum_{i=1}^{n+1} (c_i \mathbf{s}_i) + \mathbf{z}$, with $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n+1]$ and $\mathbf{z} \in \mathrm{Box}(I)^g$. Then,

$$\begin{aligned} g\left(\sum_{i=1}^{n+1} c_i \mathbf{s}_i + \mathbf{z}\right) &= \sum_{i=1}^{n+1} (c_i g(\mathbf{s}_i)) + g(\mathbf{z}) \\ &= \sum_{i=1}^{n+1} (c_i g(\mathbf{s}_i)) + \mathbf{z} \\ &= \sum_{i=1}^{n+1} (c_i \mathbf{s}_i) + \mathbf{z} \quad \text{(by assumption)}. \end{aligned}$$

Thus $\mathbf{u} \in (\mathrm{Cone}(\mathcal{O})^g)$ and Equation (8) holds. $\square$

Homogenize, sending $t \to (\mathbf{1}, t_{d+1})$. Let $n = \dim(\mathcal{O})$. Then Equation (8) becomes

$$\sum_{m \geq 0} \left| \mathrm{Cone}(\mathcal{O})^g \cap (\mathbb{Z}^d, m) \right| t^m = \frac{\sum_{k=0}^{n} \left| Box(\mathcal{O})_k^g \right| t^k}{(1-t)^{n+1}}.$$

In terms of characters this is:

$$\sum_{m \geq 0} \chi_{\mathcal{O}_m} t^m = \frac{\sum_{k=0}^{n} \chi_{\mathrm{Box}(\mathcal{O})_k} t^k}{(1-t)^{n+1}},$$

where $\chi_{\mathcal{O}_m}$ denotes the permutation representation on $\mathrm{Cone}(\mathcal{O}) \cap (\mathbb{Z}^d, m)$. Summing over all the orbits yields,

$$\begin{aligned} \sum_{m \geq 0} \chi_{mP} t^m &= \sum_{\mathcal{O} \in T^{[)}} \frac{\sum_{k=0}^{\dim(\mathcal{O})} \chi_{\mathrm{Box}(\mathcal{O})_k} t^k}{(1-t)^{\dim(\mathcal{O})+1}} \\ &= \frac{\sum_{\mathcal{O} \in T^{[)}} (1-t)^{d-\dim(\mathcal{O})} \chi_{\mathrm{Box}(\mathcal{O})_k} t^k}{(1-t)^{d+1}}. \end{aligned}$$

$\square$

Remark 1.4.8. Allowing triangulations into simplices with rational vertices should allow one to generalize Theorem 1.4.6 so that it works for many more G-invariant lattice polytopes as in the setup. The denominator would have to change, and χ_{mP} would no longer necessarily be polynomial.

Example 1.4.9 (Example 1.1.3 continued)**.** Let P be the unit 0/1-square lifted to height 1 in $\mathbb{R}^3$ (i.e. with last coordinate 1), and let $\mathbb{Z}/2\mathbb{Z}$ act on P by reflection across the hyperplane $x_1 = x_2$. Let $T^{[)}$ be the half-open decomposition of P given in Example 1.1.3. Then $T^{[)}$ is a G-invariant half-open decomposition of P. We have the following box points:

$$\mathrm{Box}([\emptyset, bc]) = \{(0,0,0)\} \quad \mathrm{Box}([a, abc]) = \{(0,1,1)\} \quad \mathrm{Box}([d, bcd]) = \{(1,0,1)\}$$

The half-open decomposition $T^{[)}$ satisfies the additional property that for all intervals $I = [\underline{S}, \overline{S}] \in T^{[)}$ and all $g \in G$, if there exists a g-fixed box point $\mathbf{z} \in \mathrm{Box}(I)$, then $g(\mathbf{s}) = \mathbf{s}$ for all vertices $\mathbf{s} \in \mathrm{vert}(\overline{S})$. Under the action of $\mathbb{Z}/2\mathbb{Z}$, $\mathrm{Box}([\emptyset, bc])$ forms an orbit, and $\mathrm{Box}([a, abc]) \cup \mathrm{Box}([d, bcd])$ forms an orbit. Denote the corresponding characters of permutation representations on the orbits by $\chi_{/}$ and $\chi_{\gtrless}$. Applying Theorem 1.4.6 yields:

$$\sum_{m \geq 0} \chi_{mP} t^m = \frac{(1-t)(\chi_{/}) + (\chi_{\gtrless})t}{(1-t)^{d+1}}.$$

We can rewrite $\chi_{/}$ and $\chi_{\gtrless}$ in the basis of irreducible characters of $\mathbb{Z}/2\mathbb{Z}$. Let e be the identity element and g be the non-identity element of $\mathbb{Z}/2\mathbb{Z}$. The group's character table is as follows.

$\mathbb{Z}/2\mathbb{Z}$	e	g
χ_{triv}	1	1
χ_{sign}	1	-1

As the box point $(0,0,0) \in \mathrm{Box}([\emptyset, bc])$ is $\mathbb{Z}/2\mathbb{Z}$-fixed, $\chi_{\not} = \chi_{triv}$. For $\chi_{\gtrless}$, we have

$$\begin{aligned}\langle \chi_{\gtrless}, \chi_{triv}\rangle &= \frac{1}{|\mathbb{Z}/2\mathbb{Z}|}\sum_{h\in\mathbb{Z}/2\mathbb{Z}} \chi_{\gtrless}(h)\overline{\chi_{triv}(h)} \\ &= \frac{1}{2}(2+0) = 1.\end{aligned}$$

In the same fashion, we compute $\langle \chi_{\gtrless}, \chi_{sign}\rangle = 1$. This allows us to rewrite the equivariant Ehrhart series as

$$\sum_{m\geq 0} \chi_{mP} t^m = \frac{(1-t)\chi_{triv} + (\chi_{triv}+\chi_{sign})t}{(1-t)^3} = \frac{\chi_{triv}+\chi_{sign}t}{(1-t)^3}.$$

Evaluating at e and g yields:

$$\begin{aligned}\sum_{m\geq 0} \chi_{mP}[e] t^m &= \frac{\chi_{triv}[e] + \chi_{sign}[e]t}{(1-t)^3} = \frac{1+t}{(1-t)^3}, \\ \sum_{m\geq 0} \chi_{mP}[g] t^m &= \frac{\chi_{triv}[g] + \chi_{sign}[g]t}{(1-t)^3} = \frac{1}{(1-t)^2}.\end{aligned}$$

These are the Ehrhart series of P and the fixed line segment $[(0,0,1),(1,1,1)]$ respectively. We compute the polynomial χ_{mP} using the usual transformation from the h*-polynomial to the Ehrhart polynomial, see Equation (1) in Section 2:

$$\begin{aligned}\chi_{mP} &= \chi_{triv}\binom{t+2}{2} + \chi_{sign}\binom{t+1}{2} \\ &= \frac{1}{2}(\chi_{triv}+\chi_{sign})t^2 + \frac{1}{2}(3\chi_{triv}+\chi_{sign})t + \chi_{triv}\end{aligned}$$

Evaluating at the identity and non-identity elements yields the Ehrhart polynomials $(1+t)^2$ and $1+t$ respectively.

Example 1.4.10 (Example 1.4.5 continued)**.** Again consider the two dimensional permutahedron $\Pi_3 \subseteq \mathbb{R}3$ under the action of $\mathbb{Z}/3\mathbb{Z}$ cyclically permuting the standard basis vectors. It is necessary to use a different half-open decomposition from that in Example 1.4.5 to compute the equivariant Ehrhart series.

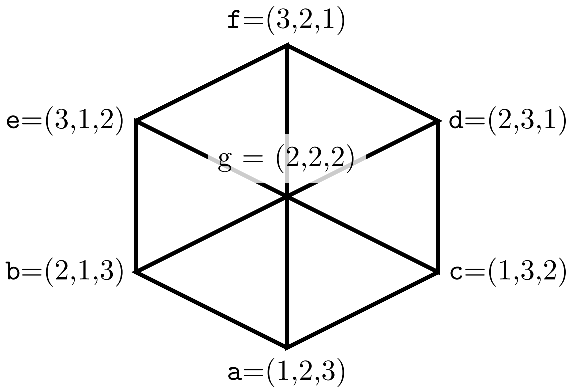


Figure 6: A triangulation of the permutahedron Π_3 into six triangles.

As shown in Figure 6, we triangulate Π_3 using a barycentric subdivision. The intervals we use for a G-invariant half-open decomposition are:

$$[\emptyset, g], [a, abg], [b, beg], [c, acg], [d, cdg], [e, efg], [f, dfg].$$

The only lattice points that can be box points must have a coordinate sum that is a multiple of 6. The box points for each of the respective intervals are:

$$\{(0,0,0)\}, \{(1,2,3)\}, \{(2,1,3)\}, \{(1,3,2)\}, \{(2,3,1)\}, \{(3,1,2)\}, \{(3,2,1)\}$$

There are 3 orbits of box points. The character table of $\mathbb{Z}/3\mathbb{Z}$ is as follows, with ζ a cube root of unity.

$\mathbb{Z}/3\mathbb{Z}$	id	g	g^2
χ_{triv}	1	1	1
χ_g	1	ζ	ζ^2
χ_{g^2}	1	ζ^2	ζ

We calculate the following rational generating function for the equivariant Ehrhart series:

$$\sum_{m \geq 0} \chi_{m\Pi_3} t^m = \frac{\chi_{triv} + 2(\chi_g + \chi_{g^2})t + \chi_{triv} t^2}{(1-t)^3}$$

Evaluating at the group elements yields the Ehrhart series of the fixed subpolytopes. Transforming to χ_{mP} using the usual transformation gives:

$$\begin{aligned} \chi_{m\Pi_3} &= \chi_{triv} \binom{t+2}{2} + 2(\chi_g + \chi_{g^2}) \binom{t+1}{2} + \chi_{triv} \binom{t}{2} \\ &= (\chi_{triv} + \chi_g + \chi_{g^2}) t^2 + (\chi_{triv} + \chi_g + \chi_{g^2}) t + \chi_{triv}. \end{aligned}$$

Evaluating at the identity and g yields the Ehrhart polynomials of the fixed subpolytopes, $3t^2 + 3t + 1$ and 1 respectively.

Theorem 1.4.6 cannot be applied to Π_3 under the action of S_3 permuting the standard basis vectors, because a S_3-invariant half-open decomposition satisfying the requirements of the theorem cannot be found. Studying the equivariant Ehrhart theory of the permutahedron lead to the following questions.

Question 1.4.11. For which n is there a G-invariant triangulation as in Theorem 1.4.1 or Theorem 1.4.6 of the permutahedron $\Pi_n \in \mathbb{R}^n$ under the cyclic group action permuting the standard basis vectors? Can Theorem 1.4.6 be extended to compute equivariant Ehrhart theory using symmetric zonotopal tilings?

1.5 RESTRICTED REPRESENTATIONS

Let P be a polytope invariant under the action of G as in the Setup 1.2.1, and let $\mathcal{H}$ be a subgroup of G. The action of G on P induces an action of $\mathcal{H}$ on P by restriction. Let $\mathrm{H}^*{}_G(P;t)$ and $\mathrm{H}^*{}_{\mathcal{H}}(P;t)$ be their respective equivariant H^*-series.

Lemma 1.5.1. *If there exists a G-invariant non-degenerate hypersurface, then the same hypersurface is a non-degenerate $\mathcal{H}$-invariant hypersurface.*

Proof. Let $\sum_{\mathbf{v}\in P\cap\mathbb{Z}^d} a(\mathbf{v})\mathbf{x}^{\mathbf{v}} = 0$ be a G-invariant non-degenerate hypersurface. Every $\mathcal{H}$-orbit of lattice points in P is contained in a G-orbit of lattice points. Thus $a(\mathbf{v}) = a(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w}$ in an orbit of $\mathcal{H}$, and the hypersurface is $\mathcal{H}$-invariant. The non-degeneracy condition is independent of the group. □

Lemma 1.5.2. *If $\mathrm{H}^*{}_G(P;t)$ is effective, then $\mathrm{H}^*{}_{\mathcal{H}}(P;t)$ is effective.*

Proof. Each coefficient of t^i in $\mathrm{H}^*{}_G(P;t)$ is a nonnegative integral linear combination of irreducible characters of G. Thus, for each t^i there exists a vector space which is a direct sum of irreducible representations of G. Restricting to the action of $\mathcal{H}$ decomposes each summand further into a finite number of irreducible representations of $\mathcal{H}$. □

Lemma 1.5.3. *If $\mathrm{H}^*{}_G(P;t)$ is polynomial, then $\mathrm{H}^*{}_{\mathcal{H}}(P;t)$ is polynomial.*

Proof. Let $\{\chi_1, \ldots, \chi_m\}$ be the irreducible representations of G, and let $\{\mu_1, \ldots, \mu_k\}$ be the irreducible representations of $\mathcal{H}$. $\mathrm{H}^*{}_G(P;t)$ can be seen as a linear function in $R(G)$ with coefficients in the field of rational functions in t. Suppose $\mathrm{H}^*{}_G(P;t)$ is polynomial so that $c_{i,0}t^0 + c_{i,1}t^1 + \cdots + c_{i,n}t^n$ is the coefficient of χ_i and $c_{i,j} \in \mathbb{C}$ for all $j \in \{0, \ldots, n\}$, and $i \in [m]$. Then the coefficient of μ_k in $\mathrm{H}^*{}_{\mathcal{H}}(P;t)$ is

$$\sum_{i=0}^{m} (c_{i,0}t^0 + \cdots + c_{i,n}t^n)\langle \chi_i|_{\mathcal{H}}, \mu_k\rangle_{\mathcal{H}},$$

where $\langle \Phi, \Psi\rangle_{\mathcal{H}} \in \mathbb{C}$ denotes the inner product between characters Φ, Ψ of $\mathcal{H}$, and $\chi_i|_{\mathcal{H}}$ is the representation χ_i restricted to $\mathcal{H}$. □

Example 1.5.4. The cyclic group $\mathbb{Z}/n\mathbb{Z} \subseteq S_n$ acting on the permutahedron gives an interesting example of the failure of the converse of Lemma 1.5.3. Namely, $\mathrm{H}^*{}_{S_n}(\Pi_n;t)$ is not polynomial for $n \geq 4$, but $\mathrm{H}^*{}_{\mathbb{Z}/n\mathbb{Z}}(\Pi_n;t)$ is polynomial for all n, see Corollary 1.6.1. Here we show how the rational function collapses to a polynomial for the case $n = 4$.

For the S_4 action on Π_4,

$$\begin{aligned}\mathrm{H^*}_{S_4}(\Pi_4;t) = \; & \frac{(\chi_2+\chi_3+\chi_4)t^4}{t+1} + \frac{(\chi_0+5\chi_1+6\chi_2+9\chi_3+6\chi_4)t^3}{t+1} \\ & + \frac{(\chi_0+7\chi_1+8\chi_2+14\chi_3+9\chi_4)t^2}{t+1} + \frac{(\chi_0+3\chi_1+3\chi_2+5\chi_3+4\chi_4)t}{t+1} \\ & + \frac{\chi_4}{t+1},\end{aligned}$$

where the character tables are as follows.

S_4	()	(12)	(12)(34)	(123)	(1234)
χ_0	1	-1	1	1	-1
χ_1	3	-1	-1	0	1
χ_2	2	0	2	-1	0
χ_3	3	1	-1	0	-1
χ_4	1	1	1	1	1

$\mathbb{Z}/4\mathbb{Z}$	()	(1234)	(12)(34)	(1432)
μ_0	1	1	1	1
μ_1	1	-1	1	-1
μ_2	1	i	-1	-i
μ_3	1	-i	-1	i

We compute $\langle \chi_i|_{\mathcal{H}}, \mu_j\rangle_{\mathcal{H}}$ for all i, j. The following table summarizes these products.

$\langle \chi, \mu\rangle_{\mathcal{H}}$	μ_0	μ_1	μ_2	μ_3
χ_0	0	1	0	0
χ_1	1	0	1	1
χ_2	1	1	0	0
χ_3	0	1	1	1
χ_4	1	0	0	0

Restricting $\mathrm{H^*}_{S_4}(\Pi_4;t)$ to $\mathrm{H^*}_{\mathbb{Z}/4\mathbb{Z}}(\Pi_4;t)$ yields the rational function:

$$\begin{aligned}&\frac{(2\mu_0+2\mu_1+\mu_2+\mu_3)t^4+(17\mu_0+16\mu_1+14\mu_2+14\mu_3)t^3}{1+t} \\ &+ \frac{(24\mu_0+23\mu_1+21\mu_2+21\mu_3)t^2}{1+t} + \frac{(10\mu_0+9\mu_1+8\mu_2+8\mu_3)t}{1+t} + \frac{\mu_0}{1+t}\end{aligned}$$

The numerator is divisible by (1+t), and the resulting polynomial is

$$\begin{aligned}\mathrm{H^*}_{\mathbb{Z}/4\mathbb{Z}}(\Pi_4;t) = \; & (2\mu_0+2\mu_1+\mu_2+\mu_3)\,t^3 + (15\mu_0+14\mu_1+13\mu_2+13\mu_3)\,t^2 \\ & + (9\mu_0+9\mu_1+8\mu_2+8\mu_3)\,t+\mu_0.\end{aligned}$$

The H^*-series of Π_4 with respect to the cyclic group action can be verified in `Sagemath` as follows.

```
sage: P4 = polytopes.permutahedron(4, backend='normaliz')
sage: G = P4.restricted_automorphism_group(output='
    permutation')
sage: G[41]
(0,9,16,18)(1,11,22,12)(2,15,10,19)(3,17,20,6)(4,21,8,13)
    (5,23,14,7)
```

```
sage: verts = P4.vertices_list()
sage: print(verts[0],verts[9],verts[16],verts[18])
[1, 2, 3, 4] [2, 3, 4, 1] [3, 4, 1, 2] [4, 1, 2, 3]
sage: Z4 = G.subgroup(gens= [G[41]])
sage: P4.Hstar_function(acting_group=Z4,output='complete')
{'Hstar': (2*chi_0 + 2*chi_1 + chi_2 + chi_3)*t^3 + (15*
   chi_0 + 14*chi_1 + 13*chi_2 + 13*chi_3)*t^2 + (9*chi_0 +
    9*chi_1 + 8*chi_2 + 8*chi_3)*t + chi_0,
'Hstar_as_lin_comb': (2*t^3 + 15*t^2 + 9*t + 1, 2*t^3 + 14*
   t^2 + 9*t, t^3 + 13*t^2 + 8*t, t^3 + 13*t^2 + 8*t),
'conjugacy_class_reps': [(),
 (0,9,16,18)(1,11,22,12)(2,15,10,19)(3,17,20,6)(4,21,8,13)
    (5,23,14,7),
 (0,16)(1,22)(2,10)(3,20)(4,8)(5,14)(6,17)(7,23)(9,18)
    (11,12)(13,21)(15,19),
 (0,18,16,9)(1,12,22,11)(2,19,10,15)(3,6,20,17)(4,13,8,21)
    (5,7,14,23)],
 'character_table': [      1       1       1       1]
 [      1      -1       1      -1]
 [      1 -zeta4      -1   zeta4]
 [      1  zeta4      -1 -zeta4],
 'is_effective': True}
```

1.6 PERMUTAHEDRA UNDER CYCLIC GROUP ACTIONS

In [3], Ardila, Supina, and Vindas-Meléndez describe the equivariant Ehrhart theory of the permutahedron under the action of the symmetric group. The symmetric group S_n acts by permuting the standard basis vectors of $\mathbb{R}^n$, and the $(n-1)$-dimensional permutahedron Π_n is invariant with respect to this action. As discussed in Section 1.5, the equivariant Ehrhart theory can change significantly when restricting to subrepresentations. For example, Example 1.5.4 shows how $\mathrm{H}^*(\Pi_4; t)$ is not polynomial for the action of S_n but is polynomial for the action of $\mathbb{Z}/4\mathbb{Z}$ also given by permuting the standard basis vectors. Because of this subtlety, in this section, we study the equivariant Ehrhart theory of the standard permutahedra under the action of the cyclic group.

Corollary 1.6.1. *The* H^**-series of the permutahedron* Π_n *under the action of* $\mathbb{Z}/n\mathbb{Z}$ *permuting the standard basis vectors is polynomial.*

Proof. By Lemma 5.2 of [3], for $\sigma \in S_n$ with cycle type $\lambda = (\ell_1, \ldots, \ell_m)$, $\mathrm{H}^*(\Pi_n; t)[\sigma]$ is polynomial if and only if the number of even parts in λ is 0, m, or $m-1$. For any element σ of $\mathbb{Z}/n\mathbb{Z}$, all cycles of σ have the same length, so $\mathrm{H}^*(\Pi_n; t)[\sigma]$ is polynomial, and thus $\mathrm{H}^*(\Pi_n; t)$ itself is polynomial. □

We consider the case where $\mathbb{Z}/p\mathbb{Z}$ acts on Π_p for some prime p. First, we show a general result on the usual h^*-polynomial of Π_p.

Theorem 1.6.2. *The* h^**-polynomial of the permutahedron* Π_p *for prime* p *has coefficients equivalent to* 1 *modulo* p*.*

Proof. If $p = 2$, $\mathrm{h}^*(\Pi_2; t) = 1$. Let $p > 2$. Let $\mathcal{I}$ be the set of all nonempty linearly independent subsets of $\{e_j - e_k : 1 \le j < k \le p\}$, and let S be a linearly independent subset in $\mathcal{I}$. The elements of $\mathcal{I}$ correspond to forests on p labeled vertices (excluding the graph with no edges) [12, Lemma 9.6]. Associate to each S the half-open parallelepiped given by the Minkowski sum $\sum_{e_j - e_k \in S} (0, e_j - e_k]$. As discussed in [12, Chapter 9], the $(p-1)$-dimensional permutahedron Π_p is equal to the disjoint union of translates of the parallelotopes:

$$\Pi_p = \{0\} \cup \bigcup_{S \in I} \left(\sum_{e_j - e_k \in S} (0, e_j - e_k] \right).$$

Let $g = (1, 2, \dots, p)$ be a generator of $\mathbb{Z}/p\mathbb{Z}$. The group $\mathbb{Z}/p\mathbb{Z}$ acts on the independent sets $\mathcal{I}$ by cyclically permuting the forest vertex labels. Each orbit has size p, as follows. For $1 \le k < p$, suppose for contradiction $g^k S = S$ for some $S \in \mathcal{I}$. Then for each edge (i, j) in the forest corresponding to S, the edge $(|i + k|_p, |j + k|_p)$ is also in the forest. As k is coprime to p, the forest thus contains p edges and therefore a cycle, which is a contradiction. Thus g^k does not fix any $S \in \mathcal{I}$, and $|\mathbb{Z}/p\mathbb{Z}S| = p$. Let $\mathcal{T} \subset \mathcal{I}$ be a transversal containing one representative from each orbit in $\mathcal{I}$. Each half-open parallelepiped in an orbit has the same Ehrhart series, and is translated by an integral vector. Thus,

$$\begin{aligned}
\mathrm{Ehr}(\Pi_p; t) &= \frac{1}{1-t} + p \sum_{S \in \mathcal{T}} \mathrm{Ehr}(S; t) \\
&= \frac{(1-t)^{p-1}}{(1-t)^p} + \sum_{S \in \mathcal{T}} \frac{p \cdot \mathrm{h}^*(S; t) \cdot (1-t)^{p-|S|-1}}{(1-t)^p}.
\end{aligned}$$

Claim 1.6.3. The coefficients of $(1-t)^{p-1}$ are equivalent to one mod p.

Proof of Claim. The coefficient of t^i is $\binom{p-1}{i}(-1)^i$, and

$$\begin{aligned}
\binom{p-1}{i} = \frac{(p-1)\cdots(p-i)}{i!} &\equiv \frac{(-1)\cdots(-i)}{i!} \mod p \\
&\equiv \frac{(-1)^i i!}{i!} \mod p \equiv (-1)^i \mod p.
\end{aligned}$$

The coefficient of t^i in the sum $(1-t)^{p-1} + \sum_{S \in \mathcal{T}} p \cdot \mathrm{h}^*(S; t) \cdot (1-t)^{p-|S|-1}$ is then equivalent to one mod p. □

□

Example 1.6.4. For the permutahedron Π_3, a transversal of the orbits of linearly independent subsets of $\{e_j - e_k \mid 1 \le j < k \le 3\}$ under the action of $\mathbb{Z}/3\mathbb{Z}$ is $\mathcal{T} = \{\{e_1 - e_2\}, \{e_1 - e_2, e_2 - e_3\}\}$. Thus

$$\mathrm{Ehr}_{\Pi_3}(t) = \frac{1 - 2t + t^2}{(1-t)^3} + \frac{3t(1-t)}{(1-t)^3} + \frac{3(t+t^2)}{(1-t)^3} = \frac{1 + 4t + t^2}{(1-t)^3}.$$

For prime p, we can explicitly construct the H^*-series of Π_p under the cyclic group action $\mathbb{Z}/p\mathbb{Z}$.

Theorem 1.6.5. *For a prime number $p \neq 2$, the H^*-series of Π_p under the action of $\mathbb{Z}/p\mathbb{Z}$ is*

$$\mathrm{H}^*(\Pi_p; t) = \sum_{i=0}^{p-1} \left(\chi_0 + \frac{\mathrm{h}_\mathrm{i}^* - 1}{p} \sum_{j=0}^{p-1} \chi_j \right) t^i,$$

where h_i^* is the coefficient of t^i in the h^*-polynomial of Π_p.

Proof. It is enough to show that our formula for $\mathrm{H}^*(\Pi_p; t)$ specializes to the fixed Ehrhart series for each conjugacy class. Let ζ be an n-th root of unity. Then

$$\sum_{i=0}^{n-1} \zeta^i = \frac{1-\zeta^n}{1-\zeta} = \begin{cases} n \text{ if } \zeta = 1 \text{ (L'Hospitals)}, \\ 0 \text{ else.} \end{cases}$$

For the identity element, we have

$$\mathrm{H}^*(\Pi_p; t)[\mathrm{id}] = \sum_{i=0}^{p-1} \left(1 + \frac{\mathrm{h}_\mathrm{i}^* - 1}{p} p \right) t^i = \sum_{i=0}^{n-1} \mathrm{h}_\mathrm{i}^* \, t^i.$$

For $g \neq \mathrm{id}$, our formula gives $\mathrm{H}^*(\Pi_p; t)[g] = \sum_{i=0}^{p-1}(1+0)t^i$. Combining this with $\det(I - t \cdot \rho(g)) = (1 - t^p)$ gives

$$\frac{\mathrm{H}^*(\Pi_p; t)[g]}{\det(I - t \cdot \rho(g))} = \frac{1 + t + \cdots + t^{p-1}}{1 - t^p} = \frac{1}{1-t}. \qquad \square$$

Corollary 1.6.6. *Let p be prime. For the action of $\mathbb{Z}/p\mathbb{Z}$ on Π_p, the H^*-series is effective.*

Proof. By Theorem 1.6.2, h_i^* is equivalent to one mod p for all i. Thus, $\frac{\mathrm{h}_\mathrm{i}^* - 1}{p}$ is a nonnegative integer, and by Theorem 1.6.5, the coefficient of t^i in the polynomial $\mathrm{H}^*(\Pi_p; t)$ is an effective character. $\square$

Theorem 1.6.5 generalizes to any polytope under a $\mathbb{Z}/n\mathbb{Z}$ action with the property that only one lattice point is fixed for any non-trivial group element.

Corollary 1.6.7. *Let $P \subseteq \mathbb{R}^n$ be a $(n-1)$-dimensional lattice polytope invariant under the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ permuting the coordinates of $\mathbb{R}^n$ as in the Setup 1.2.1. Furthermore, suppose that for each non-identity element $\zeta \in \mathbb{Z}/n\mathbb{Z}$, P^ζ is a single lattice point. Then,*

$$\mathrm{H}^*(P; t) = \sum_{i=0}^{n-1} \left(\chi_0 + \frac{\mathrm{h}_\mathrm{i}^* - 1}{n} \sum_{j=0}^{n-1} \chi_j \right) t^i,$$

where h_i^* is the coefficient of z^i in the h^*-polynomial of P, and $\{\chi_0, \ldots, \chi_{n-1}\}$ are the irreducible representations of $\mathbb{Z}/n\mathbb{Z}$.

Proof. The proof is analogous to the proof of Theorem 1.6.5. $\square$

2 RATIONAL EHRHART THEORY THROUGH GENERATING FUNCTIONS

Let $P \subseteq \mathbb{R}^d$ be a rational d-dimensional polytope with **denominator** k, i.e., k is the smallest positive integer such that kP is a lattice polytope. Recall from the Background Section on page 7, that in this case, $\operatorname{ehr}(P;n)$ is a **quasipolynomial**, i.e., of the form $\operatorname{ehr}(P;n) = c_d(n)n^d + \cdots + c_1(n)n + c_0(n)$ where $c_0(n), c_1(n), \ldots, c_d(n)$ are periodic functions. The least common period of $c_0(n), c_1(n), \ldots, c_d(n)$ is the **period** of $\operatorname{ehr}(P;n)$; this period divides the denominator k of P; again this goes back to Ehrhart [51]. Equivalently,

$$\operatorname{Ehr}(P;t) \; := 1 + \sum_{n \in \mathbb{Z}_{>0}} \operatorname{ehr}(P;n)\, t^n \; = \; \frac{\mathrm{h}^*(P;t)}{(1-t^k)^{d+1}}, \tag{9}$$

where $\mathrm{h}^*(P;t) \in \mathbb{Z}[t]$ has degree $< k\,(d+1)$.

Because polytopes can be described by a system of linear equalities and inequalities, they appear in a wealth of areas; likewise Ehrhart quasipolynomials have applications in number theory, combinatorics, computational geometry, commutative algebra, representation theory, and many other areas, see, e.g., [12].

This chapter studies Ehrhart counting functions with rational and real dilation parameters. We define the **rational Ehrhart counting function**

$$\operatorname{rehr}(P;\lambda) := \left| \lambda P \cap \mathbb{Z}^d \right| \,,$$

where $\lambda \in \mathbb{Q}$, and the **real Ehrhart counting function**

$$\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda) := \left| \lambda P \cap \mathbb{Z}^d \right|,$$

for $\lambda \in \mathbb{R}$. Naturally, we have $\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda) = \operatorname{rehr}(P;\lambda)$ when $\lambda \in \mathbb{Q}$, and so strictly speaking there is no need for separate notations. However, as P is a rational polytope, it suffices to compute $\operatorname{rehr}(P;\lambda)$ at certain rational arguments to fully understand $\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda)$; we quantify and make this statement precise shortly. To the best of our knowledge, Linke [77] initiated the study of $\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda)$ from the Ehrhart viewpoint. She proved several fundamental results starting with the fact that $\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda)$ is a **quasipolynomial** in the real variable λ, that is,

$$\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda) \; = \; c_d(\lambda)\, \lambda^d + c_{d-1}(\lambda)\, \lambda^{d-1} + \cdots + c_0(\lambda)$$

where $c_0(\lambda), c_1(\lambda), \dots, c_d(\lambda)$ are periodic functions from $\mathbb{R}$ to $\mathbb{R}$. Here is a first example, which we will revisit below:

$$\begin{aligned}\overline{\mathrm{r}}\mathrm{ehr}([1,2];\lambda) &= \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 \\ &= \begin{cases} n+1 & \text{if } \lambda = n & \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n & \text{if } n < \lambda < n + \frac{1}{2} & \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n+1 & \text{if } n + \frac{1}{2} \le \lambda < n+1 & \text{for some } n \in \mathbb{Z}_{>0}\,. \end{cases}\end{aligned}$$

Linke views the coefficient functions as piecewise-defined polynomials, which allows her, among many other things, to establish differential equations relating the coefficient functions. Essentially concurrently, Baldoni–Berline–Köppe–Vergne [5] developed an algorithmic theory of **intermediate sums** for polyhedra, which includes $\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda)$ as a special case.

Our main contribution is to add a generating-function viewpoint to the study of rational and real Ehrhart theory presented in [5] and [77]. To set it up, we need the following definition. Suppose the rational polytope P is given by the irredundant halfspace description

$$P \;=\; \left\{ \mathbf{x} \in \mathbb{R}^d : \, \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}, \tag{10}$$

where $\mathbf{A} \in \mathbb{Z}^{n \times d}$ and $\mathbf{b} \in \mathbb{Z}^n$ such that the greatest common divisor of b_i and the entries in the ith row of $\mathbf{A}$ equals 1, for every $i \in [n]$. We define the **codenominator** r of P to be the least common multiple of the nonzero entries of $\mathbf{b}$:

$$r := \mathrm{lcm}(b_1, b_2, \dots, b_n).$$

As we assume that P is full dimensional, the codenominator is well-defined. Our nomenclature arises from determining r using duality, as follows. Let P° denote the relative interior of P, and let $(\mathbb{R}^d)^\vee$ be the dual vector space. If $P \subseteq \mathbb{R}^d$ is a rational polytope such that $\mathbf{0} \in P^\circ$, the **polar dual polytope** is

$$P^\vee := \{\mathbf{x} \in (\mathbb{R}^d)^\vee : \, \langle \mathbf{x}, \mathbf{y} \rangle \ge -1 \text{ for all } \mathbf{y} \in P\},$$

and $r = \min\{q \in \mathbb{Z}_{>0} : \, q\,P^\vee \text{ is a lattice polytope}\}$.

We will see in Section 2.1 that $\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda)$ is fully determined by evaluations at rational numbers with denominator $2r$ (see Corollary 2.1.6 below for details); if $\mathbf{0} \in P$ then we actually need to know only evaluations at rational numbers with denominator r. Thus we associate two generating series to the rational Ehrhart counting function, the **rational Ehrhart series**, to a full-dimensional rational polytope P with codenominator r:

$$\mathrm{REhr}(P;t) := 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{rehr}\left(P; \frac{n}{r}\right) t^{\frac{n}{r}}, \tag{11}$$

and the **refined rational Ehrhart series**

$$\mathrm{RREhr}(P;t) := 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{rehr}\left(P; \frac{n}{2r}\right) t^{\frac{n}{2r}}\,.$$

Continuing our comment above, we typically study $\mathrm{REhr}(P;t)$ for polytopes such that $\mathbf{0} \in P$, and $\mathrm{RREhr}(P;t)$ for polytopes such that $\mathbf{0} \notin P$.

Section 2.1 also contains, as a first set of main results, structural theorems about these generating functions: rationality and its consequences for the quasipolynomial $\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda)$ (Theorem 2.1.7 and Theorem 2.1.13), nonnegativity theorems (Lemma 2.1.12), connections to the h*-polynomial (Corollary 2.1.15), and combinatorial reciprocity theorems (Corollary 2.1.17 and Corollary 2.1.18).

One can find a precursor of sorts to our generating functions $\mathrm{REhr}(P;t)$ and $\mathrm{RREhr}(P;t)$ in work by Stapledon [100], and in fact this work was our initial motivation to look for and study rational Ehrhart generating functions. We explain the connection of [100] to our work in Section 2.2. In particular, we deduce that in the case $\mathbf{0} \in P^\circ$ the generating function $\mathrm{REhr}(P;t)$ exhibits additional symmetry.

A d-dimensional, pointed, rational cone $C \subseteq \mathbb{R}^d$ is called **Gorenstein** if there exists a point $\mathbf{p} \in C \cap \mathbb{Z}^d$ such that $C^\circ \cap \mathbb{Z}^d = \mathbf{p} + C \cap \mathbb{Z}^d$ (see, e.g., [9, 94]). The point $\mathbf{p}$ is called the **Gorenstein point** of the cone. Recall that the cone over a polytope, $\mathrm{Cone}(P)$, of a rational polytope $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ is the following set of points in $\mathbb{R}^{d+1}$:

$$\mathrm{Cone}(P) := \left\{(x_0, \mathbf{x}) \in \mathbb{R}^{d+1} \ : \ \mathbf{A}\mathbf{x} \le x_0 \mathbf{b}\,,\ x_0 \ge 0\right\}.$$

For a cone $C \subseteq \mathbb{R}^d$, the **dual cone** $C^\vee \subseteq (\mathbb{R}^d)^\vee$ is

$$C^\vee := \{\mathbf{y} \in (\mathbb{R}^d)^\vee \ : \ \langle \mathbf{y}, \mathbf{x}\rangle \ge 0 \text{ for all } \mathbf{x} \in C\}.$$

A lattice polytope $P \subset \mathbb{R}^d$ is **Gorenstein** if $\mathrm{Cone}(P)$ is Gorenstein; in the special case where the Gorenstein point of that cone is $(1, \mathbf{q})$ for some $\mathbf{q} \in \mathbb{Z}^d$, we call P **reflexive** [8, 66]. Reflexive polytopes can alternatively be characterized as those lattice polytopes (containing the origin) whose polar duals are also lattice polytopes, i.e., they have codenominator 1. This definition has a natural extension to rational polytopes [54]. Gorenstein and reflexive polytopes play an important role in Ehrhart theory, as they have palindromic h*-polynomials. In Section 2.3, we define **γ-rational Gorenstein polytopes** and show that they satisfy many of the characterizing properties of lattice Gorenstein polytopes, such as having symmetric numerators of their generating functions. Furthermore, the definition does not depend on the existence of a polar dual polytope. Interestingly, there are many r-rational Gorenstein polytopes; any rational polytope containing the origin in its interior is r-rational Gorenstein, see Corollary 2.3.2.

We mention the recent notion of an **l-reflexive polytope** P ("reflexive of higher index") [74]. A lattice point $\mathbf{x} \in \mathbb{Z}^d$ is **primitive** if the gcd of its coordinates is equal to one. The l-reflexive polytopes are precisely the lattice polytopes of the form (10) with $\mathbf{b} = (l, l, \ldots, l)$ and primitive vertices; this means P has codenominator l and $\frac{1}{l}P$ has denominator l.

We conclude with two short sections further connecting our work to the existing literature. Section 2.4 exhibits how one can deduce a theorem of Betke–McMullen [15] (and also its rational analogue given in [10]) from rational Ehrhart theory.

Ehrhart's theorem gives an upper bound for the period of the quasipolynomial $\mathrm{ehr}(P;n)$, namely, the denominator of P. When the period of $\mathrm{ehr}(P;n)$ is smaller than the denominator of P, we speak of **period collapse**. One can witness this phenomenon most easily in the Ehrhart series, as period collapse means that the rational function (9) factors in such a way that there are no kth roots of unity that are poles. It is an interesting question whether or how much period collapse happens in rational Ehrhart theory, and how it compares to the classical scenario. In Section 2.5, we offer data points that show that each of the four combinations for whether a polytope's rational and classical Ehrhart quasipolynomials exhibit period collapse occur.

This chapter is joint work with Matthias Beck and Sophie Rehberg [11].

2.1 RATIONAL EHRHART DILATIONS

We call a d-dimensional polytope in $\mathbb{R}^n$ a *d-polytope*, and we assume throughout this chapter that all polytopes are full dimensional. Consequently, we could write the rational generating function $\mathrm{Ehr}(P;t)$ with denominator $(1-t)(1-t^k)^d$; in other words, $\mathrm{h}^*(P;t)$ always has a factor $(1+t+\cdots+t^{k-1})$.

Example 2.1.1. We feature the following intervals as running examples.

- $P_1 := \left[-1, \frac{2}{3}\right]$, codenominator $r = 2$

$$\begin{aligned}
\overline{\mathrm{r}}\mathrm{ehr}(P_1;\lambda) &= \lceil \lambda \rceil + \left\lceil \tfrac{2}{3}\lambda \right\rceil + 1 \\
&= \begin{cases}
\frac{5}{3}n+1 & \text{if } n \le \lambda < n+\frac{1}{2} \quad \text{for some } n \in 3\mathbb{Z}_{>0}, \\
\frac{5}{3}n+1 & \text{if } n+\frac{1}{2} \le \lambda < n+1 \quad \text{for some } n \in 3\mathbb{Z}_{>0}, \\
\frac{5}{3}n+2 & \text{if } n+1 \le \lambda < n+\frac{3}{2} \quad \text{for some } n \in 3\mathbb{Z}_{>0}, \\
\frac{5}{3}n+3 & \text{if } n+\frac{3}{2} \le \lambda < n+2 \quad \text{for some } n \in 3\mathbb{Z}_{>0}, \\
\frac{5}{3}n+4 & \text{if } n+2 \le \lambda < n+\frac{5}{2} \quad \text{for some } n \in 3\mathbb{Z}_{>0}, \\
\frac{5}{3}n+4 & \text{if } n+\frac{5}{2} \le \lambda < n+3 \quad \text{for some } n \in 3\mathbb{Z}_{>0}.
\end{cases}
\end{aligned}$$

- $P_2 := \left[0, \frac{2}{3}\right]$, codenominator $r = 2$

$$\begin{aligned}
\overline{\mathrm{r}}\mathrm{ehr}(P_2;\lambda) &= \left\lfloor \tfrac{2}{3}\lambda \right\rfloor + 1 \\
&= \tfrac{2}{3}n+1 \quad \text{if } n \le \lambda < n+\tfrac{3}{2} \quad \text{for some } n \in \tfrac{3}{2}\mathbb{Z}_{>0}.
\end{aligned}$$

- $P_3 := [1, 2]$, codenominator $r = 2$

$$\begin{aligned}
\overline{\mathrm{r}}\mathrm{ehr}(P_3;\lambda) &= \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 \\
&= \begin{cases}
n+1 & \text{if } \lambda = n \quad \text{for some } n \in \mathbb{Z}_{>0}, \\
n & \text{if } n < \lambda < n+\frac{1}{2} \quad \text{for some } n \in \mathbb{Z}_{>0}, \\
n+1 & \text{if } n+\frac{1}{2} \le \lambda < n+1 \quad \text{for some } n \in \mathbb{Z}_{>0}.
\end{cases}
\end{aligned}$$

The real Ehrhart counting function $\overline{\mathrm{rehr}}(P_3;\lambda)$ is not monotone. For example, $\overline{\mathrm{rehr}}(P_3;0)=1$, $\overline{\mathrm{rehr}}(P_3;\frac{1}{4})=0$, and $\overline{\mathrm{rehr}}(P_3;\frac{1}{2})=1$.

- $P_4 := 2P_3 = [2,4]$, codenominator $r=4$.

$$\begin{aligned}\overline{\mathrm{rehr}}(P_4;\lambda) &= \lfloor 4\lambda\rfloor - \lceil 2\lambda\rceil + 1\\ &= \begin{cases} 2n+1 & \text{if } \lambda = n \qquad\qquad \text{for some } n\in\frac{1}{2}\mathbb{Z}_{>0},\\ 2n & \text{if } n<\lambda<n+\frac{1}{4} \quad \text{for some } n\in\frac{1}{2}\mathbb{Z}_{>0},\\ 2n+1 & \text{if } n+\frac{1}{4}\le\lambda<n+\frac{1}{2} \quad \text{for some } n\in\frac{1}{2}\mathbb{Z}_{>0}.\end{cases}\end{aligned}$$

We can see in these examples (and prove below in general terms) that $\overline{\mathrm{rehr}}(P;\lambda)$ is a quasipolynomial in the real variable λ.

Remark 2.1.2. If P is a lattice polytope, then the denominator of $\frac{1}{r}P$ divides r. However, the denominator of $\frac{1}{r}P$ need not equal r, as in the case of P_4 above.

Remark 2.1.3. If $\frac{1}{r}P$ is a lattice polytope, its Ehrhart polynomial is invariant under lattice translations. Unfortunately, this does not clearly translate to invariance of $\mathrm{rehr}(P;\lambda)$. Consider the line segment $[-1,1]$ and its translation $P_4=[2,4]$. For any $\lambda\in(0,\frac{1}{4})$, $\mathrm{rehr}([-1,1],\lambda)=1$ and $\mathrm{rehr}(P_4,\lambda)=0$. This observation raises two related questions: 1) Is there an example of a polytope and a translate with the same codenominator? We expect not in dimension one. 2) Given a rational polytope P, for which r and $\tilde{P}$ could $P=\frac{1}{r}\tilde{P}$?

Lemma 2.1.4. *Let $P\subseteq\mathbb{R}^d$ be rational d-polytope. If $\mathbf{0}\in P$, then $\mathrm{rehr}(\lambda)$ is monotone for $\lambda\in\mathbb{Q}_{\ge0}$.*

Proof. Let $\lambda<\omega$ be positive rationals. Suppose $\mathbf{x}\in\mathbb{R}^d$ and $\mathbf{x}\in\lambda P$. Then $\mathbf{x}$ satisfies all n facet-defining inequalities of λP: $\langle\mathbf{a}_i,\mathbf{x}\rangle\le\lambda b_i$ for all $i\in[n]$. If $b_i=0$, then $\langle\mathbf{a}_i,\mathbf{x}\rangle\le\lambda\cdot 0=\omega\cdot 0$. Otherwise, $b_i>0$, and $\langle\mathbf{a}_i,\mathbf{x}\rangle\le\lambda b_i<\omega b_i$. So $\mathbf{x}\in\omega P$. ☐

Proposition 2.1.5. *Let $P\subseteq\mathbb{R}^d$ be a rational d-polytope with codenominator r.*

(i) *The number of lattice points in λP is constant for $\lambda\in(\frac{n}{r},\frac{n+1}{r})$, $n\in\mathbb{Z}_{\ge0}$.*

(ii) *If $\mathbf{0}\in P$, then the number of lattice points in λP is constant for $\lambda\in[\frac{n}{r},\frac{n+1}{r})$, $n\in\mathbb{Z}_{\ge0}$.*

Proof. (i). Suppose there exist two rationals λ and ω such that $\frac{n}{r}<\lambda<\omega<\frac{n+1}{r}$, and $\mathrm{rehr}(\lambda)\ne\mathrm{rehr}(\omega)$. Then there exists $\mathbf{x}\in\mathbb{Z}^d$ such that either ($\mathbf{x}\in\omega P$ and $\mathbf{x}\notin\lambda P$) or ($\mathbf{x}\in\lambda P$ and $\mathbf{x}\notin\omega P$). Suppose ($\mathbf{x}\in\omega P$ and $\mathbf{x}\notin\lambda P$). Then there exists a facet F with integral, reduced inequality $\langle\mathbf{a},\mathbf{v}\rangle\le b$ of P such that

$$\langle\mathbf{a},\mathbf{x}\rangle\le\omega b,\quad \langle\mathbf{a},\mathbf{x}\rangle>\lambda b,\ \text{and}\quad \langle\mathbf{a},\mathbf{x}\rangle\in\mathbb{Z}.$$

As $\lambda<\omega$, this implies $b>0$. We have,

$$b\frac{n}{r}<\lambda b<\langle\mathbf{a},\mathbf{x}\rangle\le\omega b<\frac{n+1}{r}b.$$

As $r = bk$, with $k \in \mathbb{Z}_{>0}$, this is equivalent to

$$n < \lambda r < k\langle \mathbf{a}, \mathbf{x}\rangle \leq \omega r < n+1.$$

This is a contradiction because $k\langle \mathbf{a}, \mathbf{x}\rangle$ is an integer. The second case is proved analogously: Assume ($\mathbf{x} \notin \omega P$ and $\mathbf{x} \in \lambda P$). Then there exists again a facet F with integral, reduced inequality $\langle \mathbf{a}, \mathbf{v}\rangle \leq b$ of P such that

$$\langle \mathbf{a}, \mathbf{x}\rangle > \omega b, \quad \langle \mathbf{a}, \mathbf{x}\rangle \leq \lambda b, \text{ and } \quad \langle \mathbf{a}, \mathbf{x}\rangle \in \mathbb{Z}.$$

As $\lambda < \omega$, this implies $b < 0$. We have,

$$\frac{n+1}{r}|b| > \omega|b| > -\langle \mathbf{a}, \mathbf{x}\rangle \geq \lambda|b| > \frac{n}{r}|b|\,.$$

As $\frac{r}{|b|} \in \mathbb{Z}_{>0}$, this is equivalent to

$$n+1 > \omega r > -\frac{r}{|b|}\langle \mathbf{a}, \mathbf{x}\rangle \geq \lambda r > n\,. \tag{12}$$

This leads to the same contradiction.

(ii) If $\mathbf{0} \in P$ we know that $\mathbf{b} \geq \mathbf{0}$. So in the proof above only the first case applies. (This can also be seen as a consequence of Lemma 2.1.4.) Allowing $\frac{n}{r} \leq \lambda$ leads, with the same computations, to the following weakened version of (2.1):

$$n \leq \lambda r < k\langle \mathbf{a}, \mathbf{x}\rangle \leq \omega r < n+1\,,$$

which is still strong enough for the contradiction. This is not the case in (12). □

It follows that we can compute the real Ehrhart function $\overline{\mathrm{r}}\mathrm{ehr}$ from the rational Ehrhart function:

Corollary 2.1.6. *Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r. Then*

$$\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda) \;=\; \begin{cases} \mathrm{rehr}(P;\lambda) & \text{if } \lambda \in \frac{1}{r}\mathbb{Z}_{\geq 0}\,,\\ \mathrm{rehr}(P;\lfloor\lambda\rceil) & \text{if } \lambda \notin \frac{1}{r}\mathbb{Z}_{\geq 0}\,, \end{cases} \tag{13}$$

where

$$\lfloor\lambda\rceil := \frac{2j+1}{2r} \quad \text{for} \quad \left|\lambda - \frac{2j+1}{2r}\right| < \frac{1}{2r} \quad \text{and} \quad j \in \mathbb{Z}\,.$$

In words, $\lfloor\lambda\rceil$ is the element in $\frac{1}{2r}\mathbb{Z}$ with odd numerator that has the smallest Euclidean distance to λ on the real line. Furthermore, if $\mathbf{0} \in P$, then

$$\overline{\mathrm{r}}\mathrm{ehr}(P;\lambda) \;=\; \mathrm{rehr}\left(P;\frac{\lfloor r\lambda\rfloor}{r}\right).$$

Theorem 2.1.7. *If $P \subseteq \mathbb{R}^d$ is a rational d-polytope with codenominator r, and $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}P$ is a lattice polytope, then*

$$\operatorname{REhr}(P;t) \;:=\; \sum_{n \in \mathbb{Z}_{\geq 0}} \operatorname{rehr}\left(P;\frac{n}{r}\right) t^{\frac{n}{r}} \;=\; \frac{\operatorname{rh}^*(P;t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}},$$

where $\operatorname{rh}^*(P;t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative integral coefficients. Consequently, $\operatorname{rehr}(P;\lambda)$ and $\overline{\operatorname{rehr}}(P;\lambda)$ are quasipolynomials.

Remark 2.1.8. Our implicit definition of $\operatorname{rh}^*(P;t)$ depends on m. We will sometimes use the notation $\operatorname{rh}^*_m(P;t)$ to make this dependency explicit. Naturally, one often tries to choose m minimal, which gives a canonical definition of $\operatorname{rh}^*(P;t)$, but sometimes it pays to be flexible.

Remark 2.1.9. By usual generating function theory the degree of $\operatorname{rh}^*_m(P;t)$ is less than or equal to $m(d+1)-1$ as a polynomial in $t^{\frac{1}{r}}$.

Proof of Theorem 2.1.7. Our conditions imply that $\frac{1}{r}P$ is a rational polytope with denominator dividing m. Thus by standard Ehrhart theory,

$$\operatorname{REhr}(P;t) \;=\; \operatorname{Ehr}\left(\tfrac{1}{r}P;t^{\frac{1}{r}}\right) \;=\; \frac{\mathrm{h}^*\left(\frac{1}{r}P;t^{\frac{1}{r}}\right)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}},$$

and $\mathrm{h}^*(\frac{1}{r}P;t)$ has nonnegative integral coefficients. □

Corollary 2.1.10. *The period of the quasipolynomial* $\operatorname{rehr}(P;\lambda)$ *divides $\frac{m}{r}$, i.e., this period is of the form $\frac{j}{r}$ with $j \mid m$.*

Corollary 2.1.11. *The period of the quasipolynomial* $\operatorname{ehr}(P;\lambda)$ *divides $\frac{m}{\gcd(m,r)}$.*

Proof. Viewed as a function of the integer parameter n, the function $\operatorname{rehr}(P;\frac{n}{r})$ has period dividing m. Thus $\operatorname{ehr}(P;n) = \operatorname{rehr}(P;n)$ has period dividing $\frac{m}{\gcd(m,r)}$. □

Corollary 2.1.12.

$$\operatorname{REhr}(P;t) \;=\; \frac{\operatorname{rh}^*_r(P;t)}{(1-t)^{d+1}},$$

where $\operatorname{rh}^*_r(P;t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative coefficients.

For polytopes that don't contain the origin, the following variant of Theorem 2.1.7 is useful.

Theorem 2.1.13. *If $P \subseteq \mathbb{R}^d$ is a rational d-polytope with codenominator r, and $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{2r}P$ is a lattice polytope, then*

$$\operatorname{RREhr}(P;t) \;:=\; 1 + \sum_{n \in \mathbb{Z}_{>0}} \operatorname{rehr}\left(P;\frac{n}{2r}\right) t^{\frac{n}{2r}} \;=\; \frac{\operatorname{rrh}^*(P;t)}{\left(1-t^{\frac{m}{2r}}\right)^{d+1}},$$

where $\operatorname{rrh}^*(P;t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{2r}}]$ with nonnegative coefficients.

The proof of Theorem 2.1.13 is virtually identical to that of Theorem 2.1.7. Many of the following assertions come in two versions, one for REhr and one for RREhr. We typically write an explicit proof for only one version, as the other is analogous.

Corollary 2.1.14. *Let $P \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r. The real and rational Ehrhart functions, $\overline{\mathrm{r}}\mathrm{ehr}(P,\lambda)$ and $\mathrm{rehr}(P,\lambda)$, are given by quasipolynomials of period 1.*

Corollary 2.1.15. *If $\frac{m}{r}$ (resp. $\frac{m}{2r}$) in Theorem 2.1.7 (resp. Theorem 2.1.13) is integral we can retrieve the h^*-polynomial from the rh^*-polynomial (resp. rrh^*-polynomial) by applying the operator Int that extracts from a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ the terms with integer powers of t: $\mathrm{h}^*(P;t) = \mathrm{Int}(\mathrm{rh}^*(P;t))$ (resp. $\mathrm{h}^*(P;t) = \mathrm{Int}(\mathrm{rrh}^*(P;t))$).*

Example 2.1.16 (Example 2.1.1 continued)**.** Here are the (refined) rational Ehrhart series of the running examples. Recall that the rational Ehrhart series of P in the variable t can be computed as the Ehrhart series of $\frac{1}{r}P$ in the variable $t^{\frac{1}{r}}$ (resp. the refined rational Ehrhart as the Ehrhart series of $\frac{1}{2r}P$ in the variable $t^{\frac{1}{2r}}$).

- $P_1 := [-1, \frac{2}{3}]$, $r = 2$, $m = 6$

$$\begin{aligned}\mathrm{REhr}(P_1;t) &= \frac{1 + t^{\frac{1}{2}} + t + t^{\frac{3}{2}} + t^2}{(1-t)\left(1 - t^{\frac{3}{2}}\right)} \\ &= \frac{1 + t^{\frac{1}{2}} + 2t + 3t^{\frac{3}{2}} + 4t^2 + 4t^{\frac{5}{2}} + 4t^3 + 4t^{\frac{7}{2}} + 3t^4 + 2t^{\frac{9}{2}} + t^5 + t^{\frac{11}{2}}}{(1-t^3)^2}\end{aligned}$$

- $P_2 := [0, \frac{2}{3}]$, $r = 2$, $m = 3$

$$\mathrm{REhr}(P_2;t) = \frac{1}{\left(1 - t^{\frac{1}{2}}\right)\left(1 - t^{\frac{3}{2}}\right)} = \frac{1 + t^{\frac{1}{2}} + t}{\left(1 - t^{\frac{3}{2}}\right)^2}$$

- $P_3 := [1, 2]$, $r = 2$. $\frac{1}{4}P_3 = [\frac{1}{4}, \frac{1}{2}]$ and $m = 4$, so $\frac{m}{2r} = 1$. See Figure 7.

$$\mathrm{RREhr}(P_3, t) = \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1-t)^2} = \frac{\left(1 + t^{\frac{3}{4}}\right)\left(1 + t^{\frac{1}{2}}\right)}{(1-t)^2}$$

- $P_4 := [2, 4]$, $r = 4$. Then $\frac{1}{8}P_4 = [\frac{1}{4}, \frac{1}{2}]$ and $m = 4$, so $\frac{m}{2r} = \frac{1}{2}$. See Figure 8.

$$\begin{aligned}\mathrm{RREhr}(P_4;t) &= \frac{1 + t^{\frac{1}{4}} + t^{\frac{3}{8}} + 2t^{\frac{1}{2}} + t^{\frac{5}{8}} + 2t^{\frac{3}{4}} + 2t^{\frac{7}{8}} + t + 2t^{\frac{9}{8}} + t^{\frac{5}{4}} + t^{\frac{11}{8}} + t^{\frac{13}{8}}}{(1-t)^2} \\ &= \frac{1 + t^{\frac{1}{4}} + t^{\frac{3}{8}} + t^{\frac{5}{8}}}{(1 - t^{\frac{1}{2}})^2}\end{aligned}$$

Choosing m to be minimal means $\mathrm{rrh}^*_4(P_4;t) = (1 + t^{\frac{3}{8}})(1 + t^{\frac{1}{4}}) = 1 + t^{\frac{1}{4}} + t^{\frac{3}{8}} + t^{\frac{5}{8}} = \mathrm{rrh}^*_4(P_3; t^{\frac{1}{2}})$ The real Ehrhart quasipolynomial $\overline{\mathrm{r}}\mathrm{ehr}$ has period $\frac{1}{2}$. The rational Ehrhart counting function agrees with a quasipolynomial for $\lambda \in \frac{1}{2r}\mathbb{Z}$.

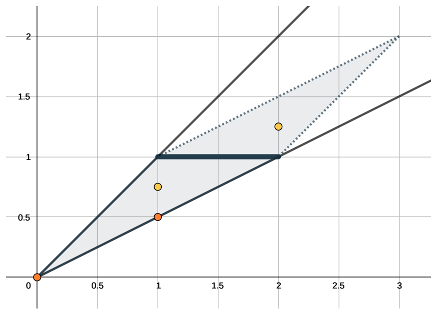


Figure 7: The cone $\mathrm{Cone}(P_3)$ over $P_3 = [1,2]$. The lattice points in the fundamental parallelepiped with respect to the lattice $\frac{1}{4}\mathbb{Z} \times \mathbb{Z}$ are $(0,0)$, $(\frac{1}{2},1)$, $(\frac{3}{4},1)$, $(\frac{5}{4},2)$.

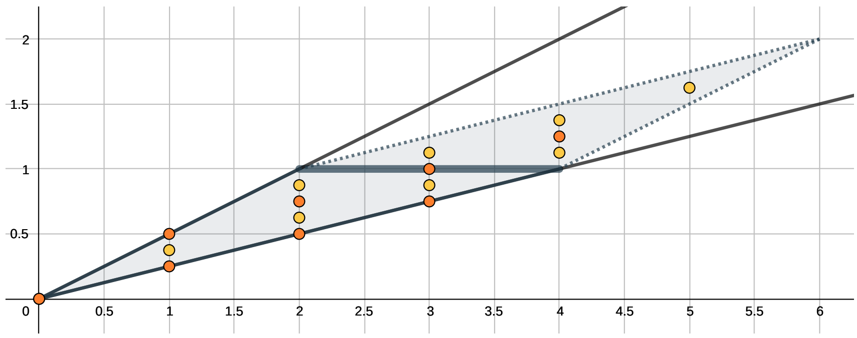


Figure 8: The cone $\mathrm{Cone}(P_4)$ over $P_4 = [2,4]$. The lattice points in the fundamental parallelepiped with respect to the lattice $\frac{1}{8}\mathbb{Z} \times \mathbb{Z}$ are shown in the figure.

From the (refined) rational Ehrhart series of these examples, we can recompute the quasipolynomials found earlier. For example, for P_3:

$$\begin{aligned}
\mathrm{RREhr}(P_3, t) &= \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1-t)^2} \\
&= \left(1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}\right) \sum_{j \geq 0} (j+1)\, t^j \\
&= \sum_{j\geq 0} (j+1)\, t^j + \sum_{j\geq 0} (j+1)\, t^{j+\frac{1}{2}} + \sum_{j\geq 0} (j+1)\, t^{j+\frac{3}{4}} + \sum_{j\geq 0} (j+1)\, t^{j+\frac{5}{4}}
\end{aligned}$$

With a changes of variables we compute for $\lambda \in \frac{1}{4}\mathbb{Z}$

$$\mathrm{rehr}(\lambda) = \begin{cases} \lambda + 1 & \text{if } \lambda \in \mathbb{Z}, \\ \lambda - \frac{1}{4} & \text{if } \lambda \equiv \frac{1}{4} \mod 1, \\ \lambda + \frac{1}{2} & \text{if } \lambda \equiv \frac{1}{2} \mod 1, \\ \lambda + \frac{1}{4} & \text{if } \lambda \equiv \frac{3}{4} \mod 1. \end{cases}$$

We recover the reciprocity result for the rational Ehrhart function of rational polytopes proved by Linke [77, Corollary 1.5].

Corollary 2.1.17. *Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope. Then $(-1)^d\,\overline{\mathrm{rehr}}(P;-\lambda)$ equals the number of interior lattice points in λP, for any $\lambda > 0$.*

Proof. Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r. The fact that both $\mathrm{rehr}(P;\lambda)$ and $\overline{\mathrm{rehr}}(P;\lambda)$ are quasipolynomials allows us to extend (13) to the negative (and therefore all) real numbers via

$$\overline{\mathrm{rehr}}(P;\lambda) \;=\; \begin{cases} \mathrm{rehr}(P;\lambda) & \text{if } \lambda \in \frac{1}{r}\mathbb{Z}\,, \\ \mathrm{rehr}(P;\lfloor\lambda\rceil) & \text{if } \lambda \notin \frac{1}{r}\mathbb{Z}\,. \end{cases}$$

By standard Ehrhart–Macdonald Reciprocity, $(-1)^d\,\mathrm{rehr}(P;-\frac{n}{2r}) = \mathrm{ehr}(\frac{1}{2r}P;-n)$ equals the number of lattice points in the interior of $\frac{n}{2r}P$. The result now follows from $\lfloor -\lambda\rceil = -\lfloor\lambda\rceil$. $\square$

Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope, and let $\mathrm{rehr}(P^\circ;\lambda) := |\lambda P^\circ \cap \mathbb{Z}^d|$. We define the (refined) rational Ehrhart series of the interior of a polytope as follows:

$$\mathrm{REhr}(P^\circ;t) := \sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}(P^\circ;\lambda)t^\lambda$$
$$\mathrm{RREhr}(P^\circ;t) := \sum_{\lambda\in\frac{1}{2r}\mathbb{Z}_{>0}} \mathrm{rehr}(P^\circ;\lambda)t^\lambda\,,$$

where r as usual denotes the codenominator of P.

Corollary 2.1.18. *Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope.*

(i) *The (refined) rational Ehrhart series of the open polytope P° have the rational expressions*

$$\mathrm{REhr}(P^\circ;t) = \frac{\mathrm{rh}^*_m(P^\circ;t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}} \quad \text{and} \quad \mathrm{RREhr}(P^\circ;t) = \frac{\mathrm{rrh}^*_m(P^\circ;t)}{\left(1-t^{\frac{m}{2r}}\right)^{d+1}}\,,$$

where $\mathrm{rh}^*_m(P^\circ;t)$ and $\mathrm{rrh}^*_m(P^\circ;t)$ are polynomials in $\mathbb{Z}[t^{\frac{1}{r}}]$ and $\mathbb{Z}[t^{\frac{1}{2r}}]$, respectively.

(ii) *The (refined) rational Ehrhart series fulfill the reciprocity relations*

$$\mathrm{REhr}(P^\circ;t) = (-1)^{d+1}\,\mathrm{REhr}\left(P;\frac{1}{t}\right),$$
$$\mathrm{RREhr}(P^\circ;t) = (-1)^{d+1}\,\mathrm{RREhr}\left(P;\frac{1}{t}\right).$$

(iii) *The rh^*- and rrh^*-polynomials of the polytope P and its interior P° are related by*

$$\mathrm{rh}^*_m(P^\circ;t) = \left(t^{\frac{m}{r}}\right)^{d+1}\mathrm{rh}^*_m\left(P;\frac{1}{t}\right),$$
$$\mathrm{rrh}^*_m(P^\circ;t) = \left(t^{\frac{m}{2r}}\right)^{d+1}\mathrm{rrh}^*_m\left(P;\frac{1}{t}\right).$$

Proof. Identity (i) follows from Ehrhart-Macdonald reciprocity (see, e.g., [12, Theorem 4.4]) and Remark 2.1.9:

$$\begin{aligned}\mathrm{REhr}(P^\circ;t) &= \sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}(P^\circ;\lambda)t^\lambda = \sum_{n\in\mathbb{Z}_{>0}} \mathrm{ehr}\left(\frac{1}{r}P^\circ;n\right)t^{\frac{n}{r}} \\ &= \mathrm{Ehr}\left(\frac{1}{r}P^\circ;t^{\frac{1}{r}}\right) = (-1)^{d+1}\,\mathrm{Ehr}\left(\frac{1}{r}P;t^{-\frac{1}{r}}\right) \\ &= (-1)^{d+1}\frac{\mathrm{h}^*\left(\frac{1}{r}P;t^{-\frac{1}{r}}\right)}{\left(1-t^{-\frac{m}{r}}\right)^{d+1}} = \frac{\left(t^{\frac{m}{r}}\right)^{d+1}\mathrm{h}^*\left(\frac{1}{r}P;t^{-\frac{1}{r}}\right)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}.\end{aligned}$$

For identities (ii) and (iii) we again apply Ehrhart-Macdonald reciprocity:

$$\begin{aligned}\frac{\left(t^{\frac{m}{r}}\right)^{d+1}\mathrm{rh}^*_m\left(P;\frac{1}{t}\right)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}} &= \frac{(-1)^{d+1}\,\mathrm{rh}^*_m\left(P;\frac{1}{t}\right)}{\left(1-\left(\frac{1}{t}\right)^{\frac{m}{r}}\right)^{d+1}} = (-1)^{d+1}\,\mathrm{REhr}\left(P;\frac{1}{t}\right) \\ &= (-1)^{d+1}\,\mathrm{Ehr}\left(\frac{1}{r}P;\frac{1}{t^{\frac{1}{r}}}\right) = \mathrm{Ehr}\left(\frac{1}{r}P^\circ;t^{\frac{1}{r}}\right) \\ &= \sum_{\lambda\in\mathbb{Z}_{>0}} \mathrm{ehr}\left(\frac{1}{r}P^\circ;\lambda\right)t^{\frac{\lambda}{r}} = \sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}\left(P^\circ;\frac{\lambda}{r}\right)t^{\frac{\lambda}{r}} \\ &= \mathrm{REhr}\left(P^\circ;t\right) = \frac{\mathrm{rh}^*_m(P^\circ;t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}.\end{aligned}$$

□

2.2 HOW TO COUNT LATTICE POINTS USING THE BOUNDARY

We recall the setup from [100]. Let $P \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r and $\mathbf{0} \in P$. Let $\partial_{\neq 0}(P)$ denote the union of facets of P that do not contain the origin. In order to study all rational dilates of the boundary of P, Stapledon introduces the generating function

$$\mathrm{WEhr}(P;t) \;:= 1 + \sum_{\lambda\in\mathbb{Q}_{>0}} \left|\partial_{\neq 0}(\lambda P)\cap\mathbb{Z}^d\right| t^\lambda \;=\; \frac{\widetilde{\mathrm{h}}(P;t)}{(1-t)^d}\,, \tag{14}$$

where $\widetilde{\mathrm{h}}(P;t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with fractional exponents. The generating function WEhr is closely related to the (rational) Ehrhart series: the truncated sum $1+\sum_{\lambda\in\mathbb{Q}>0}^{\omega} |\partial_{\neq 0}(\lambda P)\cap\mathbb{Z}^d|$ equals the number of lattice points in ωP. Proposition 2.1.5 allows us to discretize this sum:

Corollary 2.2.1. *Let $P \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r and $\mathbf{0} \in P$. The number of lattice points in λP equals $1+\sum_{\omega\in\frac{1}{r}\mathbb{Z}_{>0},\,\omega<\lambda} |\partial_{\neq 0}(\omega P)\cap\mathbb{Z}^d|$.*

Proof. As $\mathbf{0} \in P$, every nonzero lattice point in λP occurs in $\partial_{\neq 0}(\omega P)$ for some unique $\omega \in \mathbb{Q}$ where $0 < \omega \leq \lambda$. Using Lemma 2.1.4, we have $\lambda P\cap\mathbb{Z}^d = \mathbf{0}\cup\bigsqcup_{\omega\in\mathbb{Q}_{>0}}^{\lambda}(\partial_{\neq 0}(\omega P)\cap\mathbb{Z}^d)$. By Proposition 2.1.5, the union $\bigsqcup_{\omega\in\mathbb{Q}_{>0}}^{\lambda}(\partial_{\neq 0}(\omega P)\cap\mathbb{Z}^d)$ is discrete and disjoint. □

Similarly, $\widetilde{\mathrm{h}}(P;t)$ is related to $\mathrm{h}^*(\frac{1}{r}P;t^{\frac{1}{r}})$ and to $\mathrm{rh}^*_m(P;t)$, as we show in Lemma 2.2.2 and Corollary 2.2.5. Recall that we use $\mathrm{rh}^*_m(P;t)$ to keep track of the denominator of $\mathrm{REhr}(P;t) = \frac{\mathrm{rh}^*_m(P;t)}{(1-t^{\frac{m}{r}})^{d+1}}$.

Lemma 2.2.2. *Let $P \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r such that $\mathbf{0} \in P$, and let k be the denominator of $\frac{1}{r}P$. Then*

$$\mathrm{h}^*\left(\frac{1}{r}P;t^{\frac{1}{r}}\right) = \frac{\left(1-t^{\frac{k}{r}}\right)^{d+1}}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(P;t)\,.$$

Proof. Applying classical Ehrhart theory and Proposition 2.1.5 we compute

$$\begin{aligned}
\frac{\mathrm{h}^*\left(\frac{1}{r}P;t^{\frac{1}{r}}\right)}{\left(1-t^{\frac{k}{r}}\right)^{d+1}} &= \mathrm{Ehr}\left(\frac{1}{r}P;t^{\frac{1}{r}}\right)\\
&= 1+\sum_{n\in\mathbb{Z}_{>0}} \mathrm{ehr}\left(\frac{1}{r}P;n\right)t^{\frac{n}{r}}\\
&\underset{\text{Cor. 2.2.1}}{=} 1+\sum_{n\in\mathbb{Z}_{>0}}\left(1+\sum_{j=1}^{n}\left|\partial_{\neq 0}\left(\frac{j}{r}P\right)\cap\mathbb{Z}^d\right|\right)t^{\frac{n}{r}}\\
&= 1+\sum_{n\in\mathbb{Z}_{>0}} t^{\frac{n}{r}}+\sum_{j>0}\sum_{n\geq j}\left|\partial_{\neq 0}\left(\frac{j}{r}P\right)\cap\mathbb{Z}^d\right|t^{\frac{n}{r}}\\
&= 1+\frac{t^{\frac{1}{r}}}{1-t^{\frac{1}{r}}}+\sum_{j>0}\left|\partial_{\neq 0}\left(\frac{j}{r}P\right)\cap\mathbb{Z}^d\right|\sum_{n\geq j}t^{\frac{n}{r}}\\
&= \frac{1-t^{\frac{1}{r}}+t^{\frac{1}{r}}+\sum_{j>0}\left|\partial_{\neq 0}\left(\frac{j}{r}P\right)\cap\mathbb{Z}^d\right|t^{\frac{j}{r}}}{1-t^{\frac{1}{r}}}\\
&= \frac{\mathrm{WEhr}(P;t)}{1-t^{\frac{1}{r}}}\\
&= \frac{\widetilde{\mathrm{h}}(P;t)}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}.
\end{aligned}$$

□

Remark 2.2.3. The factor multiplying $\widetilde{\mathrm{h}}(P;t)$ in Lemma 2.2.2 can be rewritten in terms of finite geometric series. Let the codenominator $r = ks$ for some $s \in \mathbb{Z}_{\geq 1}$. Rewriting yields

$$\begin{aligned}
\frac{\left(1-t^{\frac{k}{r}}\right)^{d+1}}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d} = \frac{\left(1-t^{\frac{k}{r}}\right)}{\left(1-t^{\frac{1}{r}}\right)}\cdot\left(\frac{\left(1-t^{\frac{k}{r}}\right)}{(1-t)}\right)^d &= \frac{\left(1-t^{\frac{1}{s}}\right)}{\left(1-t^{\frac{1}{ks}}\right)}\cdot\left(\frac{1}{1+t^{\frac{1}{s}}+\cdots+t^{\frac{s-1}{s}}}\right)^d\\
&= \frac{1+t^{\frac{1}{r}}+\cdots+t^{\frac{k-1}{r}}}{\left(1+t^{\frac{1}{s}}+\cdots+t^{\frac{s-1}{s}}\right)^d}.
\end{aligned}$$

If $k = r$, this simplifies to $(1+t^{\frac{1}{r}}+\cdots+t^{\frac{r-1}{r}})$.

Remark 2.2.4. Lemma 2.2.2 corrects [100, Remark 3], which was missing the factor between $\mathrm{h}^*(\frac{1}{r}P; t^{\frac{1}{r}})$ and $\widetilde{\mathrm{h}}(P;t)$.

Corollary 2.2.5. *Let $P \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r such that $\mathbf{0} \in P$. Let k be the denominator of $\frac{1}{r}P$. Then*

$$\mathrm{rh}^*_k(P;t) \;=\; \mathrm{h}^*\left(\tfrac{1}{r}P; t^{\frac{1}{r}}\right) \;=\; \frac{\left(1-t^{\frac{k}{r}}\right)^{d+1}}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(P,t)\,.$$

Remark 2.2.6. In [98, Equation (14)] and [100, Equation (6)], Stapledon shows that $\mathrm{h}^*(P;t) = \Psi(\widetilde{\mathrm{h}}(P;t))$, where $\Psi\colon \bigcup_{r\in\mathbb{Z}_{>0}} \mathbb{R}[t^{\frac{1}{r}}] \to \mathbb{R}[t]$ is defined by $\Psi(t^\lambda) = t^{\lceil\lambda\rceil}$. In the case of a lattice polytope with $\frac{m}{r} \in \mathbb{Z}$ we give a different construction to recover the h^*-polynomial from the rrh^*- and rh^*-polynomial by applying the operator Int (see Corollary 2.1.15). Corollary 2.2.5 shows that, after a bit of computation, these two constructions are equivalent.

Remark 2.2.7. For a lattice d-polytope $P \subseteq \mathbb{R}^d$ with codenominator r, $\mathbf{0} \in P$, and denominator of $\frac{1}{2r}P = k$, we can relate $\mathrm{rrh}^*(P;t)$ and $\mathrm{h}^*(\frac{1}{2r}P; t^{\frac{1}{2r}})$ in a similar way. We again write $\mathrm{rrh}^*_k(P;t)$ to emphasize that it is the numerator of $\frac{\mathrm{rrh}^*_k(P;t)}{(1-t^{\frac{k}{2r}})^{d+1}}$. Then

$$\mathrm{rrh}^*_k(P;t) = \mathrm{h}^*\left(\frac{1}{2r}P; t^{\frac{1}{2r}}\right) = \frac{\left(1-t^{\frac{k}{2r}}\right)^{d+1}}{\left(1-t^{\frac{1}{2r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(P;t)\,.$$

Corollary 2.2.8. *Let $P \subseteq \mathbb{R}^d$ be a lattice d-polytope with $\mathbf{0} \in P^\circ$. Let r be the codenominator of P and k be the denominator of $\frac{1}{r}P$. Then $\mathrm{rh}^*_k(P;t)$ is palindromic.*

Proof. From [98, Corollary 2.12] we know that $\widetilde{\mathrm{h}}(P;t)$ is palindromic if $\mathbf{0} \in P^\circ$. We compute using Corollary 2.2.5.

$$\begin{aligned}
\mathrm{rh}^*_k\left(P;t^{-1}\right) &= \frac{\left(1-t^{\frac{-k}{r}}\right)^{d+1}}{\left(1-t^{\frac{-1}{r}}\right)(1-t^{-1})^d}\,\widetilde{\mathrm{h}}\left(P;t^{-1}\right)\\
&= \frac{t^{\frac{-(d+1)k}{r}}}{t^{\frac{-1}{r}}}\,\frac{\left(1-t^{\frac{k}{r}}\right)^{d+1}}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(P;t)\\
&= \frac{1}{t^{\frac{k(d+1)-1}{r}}}\,\mathrm{rh}^*_k(P;t)
\end{aligned}$$

This implies, since the constant term of $\mathrm{rh}^*_k(P;t)$ is 1, that the degree of $\mathrm{rh}^*(P;t)$ (measured as a polynomial in $t^{\frac{1}{r}}$) equals $k(d+1)-1$. □

This suggests that there is a 3-step hierarchy for rational dilations: $\mathbf{0} \in P^\circ$ comes with extra symmetry, $\mathbf{0} \in P$ comes with Proposition 2.1.5(ii) and so we "only" have to compute $\mathrm{rh}^*(P;t) \in \mathbb{Z}[t^{\frac{1}{r}}]$, and $\mathbf{0} \notin P$ means we have to compute $\mathrm{rrh}^*(P;t) \in \mathbb{Z}[t^{\frac{1}{2r}}]$. Corollary 2.2.8 is related to Gorenstein properties of rational polytopes, which we consider in the next section.

2.3 GORENSTEIN MUSINGS

Our main goal in this section is to extend the notion of Gorenstein polytopes to the rational case. A rational d-polytope $P \subseteq \mathbb{R}^d$ is **γ-rational Gorenstein** if $\mathrm{Cone}(\frac{1}{\gamma}P)$ is a Gorenstein cone. In this paper we explore this definition for parameters $\gamma = r$ and $\gamma = 2r$, other parameters are still to be investigated. The archetypal r-rational Gorenstein polytope is a rational polytope that contains the origin in its interior, see Corollary 2.3.2. The definition of γ-rational Gorenstein does not require that the origin is contained in the polytope, hence, it does not require the existence of a polar dual. A lattice polytope P is 1-rational Gorenstein if and only if it is a Gorenstein polytope in the classical sense.

Analogous to the lattice case, the following theorem shows that a polytope containing the origin is r-rational Gorenstein if and only if it has a palindromic rh^*-polynomial. Let $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope. We may assume that for some index $i \in [n]$, $b_j = 0$ for $j = 1, \dots, i$ and $b_j \neq 0$ for $j = i+1, \dots, n$; thus we can write P as follows:

$$P = \begin{Bmatrix} \mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}_j, \mathbf{x} \rangle \le 0 & \text{for } j = 1, \dots, i \\ \langle \mathbf{a}_j, \mathbf{x} \rangle \le b_j & \text{for } j = i+1, \dots, n \end{Bmatrix}, \tag{15}$$

where $\mathbf{a}_j$ are the rows of $\mathbf{A}$.

Theorem 2.3.1. *Let $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope with codenominator r and $\mathbf{0} \in P$, as in (10) and (15). Then the following are equivalent for $g, m \in \mathbb{Z}_{\ge 1}$ and $\frac{m}{r}P$ a lattice polytope:*

(i) *P is r-rational Gorenstein with Gorenstein point $(g, \mathbf{y}) \in \mathrm{Cone}(\frac{1}{r}P)$.*

(ii) *There exists a (necessarily unique) integer solution $(g, \mathbf{y})$*

$$\begin{aligned} -\langle \mathbf{a}_j, \mathbf{y} \rangle &= 1 && \textit{for } j = 1, \dots, i \\ b_j\, g - r\, \langle \mathbf{a}_j, \mathbf{y} \rangle &= b_j && \textit{for } j = i+1, \dots, n\,. \end{aligned}$$

(iii) *$\mathrm{rh}^*(P; t)$ is palindromic:*

$$t^{(d+1)\frac{m}{r} - \frac{g}{r}}\, \mathrm{rh}^*_m\!\left(P; \frac{1}{t}\right) \;=\; \mathrm{rh}^*_m(P; t)\,.$$

(iv) *$(-1)^{d+1} t^{\frac{g}{r}}\, \mathrm{REhr}(P; t) = \mathrm{REhr}(P; \frac{1}{t})$.*

(v) *$\mathrm{rehr}(P; \frac{n}{r}) = \mathrm{rehr}(P^\circ; \frac{n+g}{r})$ for all $n \in \mathbb{Z}_{\ge 0}$.*

(vi) *$\mathrm{Cone}(\frac{1}{r}P)^\vee$ is the cone over a lattice polytope, i.e., there exists a lattice point $(g, \mathbf{y}) \in \mathrm{Cone}(\frac{1}{r}P)^\circ \cap \mathbb{Z}^{d+1}$ such that for every primitive ray generator $(v_0, \mathbf{v})$ of $\mathrm{Cone}(\frac{1}{r}P)^\vee$.*

$$\langle (g, \mathbf{y})\,, (v_0, \mathbf{v}) \rangle = 1\,.$$

The equivalence of (i) and (vi) is well known (see, e.g., [7, Definition 1.8] or [23, Exercises 2.13, 2.14]); for the sake of completeness we include a proof below.

Corollary 2.3.2. *Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r. If $\mathbf{0} \in P^\circ$, then P is r-rational Gorenstein with Gorenstein point $(1, 0, \dots, 0)$ and $\mathrm{rh}^*(P;t)$ is palindromic.*

Example 2.3.3 (continued)**.** We check the Gorenstein criterion for the running examples such that $\mathbf{0} \in P$.

- Let $P_1 := \left[-1, \frac{2}{3}\right]$ so that $r = 2$. Let $m = 6$. Then,

$$\mathrm{rh}_6^*(P_1;t) = 1 + t^{\frac{1}{2}} + 2t + 3t^{\frac{3}{2}} + 4t^2 + 4t^{\frac{5}{2}} + 4t^3 + 4t^{\frac{7}{2}} + 3t^4 + 2t^{\frac{9}{2}} + t^5 + t^{\frac{11}{2}}.$$

  The polynomial $\mathrm{rh}_6^*(P_1;t)$ is palindromic and therefore (by Theorem 2.3.1), P_1 is 2-rational Gorenstein. This is to be expected; as $\mathbf{0} \in P^\circ$, Lemma 2.2.2 shows that $\mathrm{rh}^*(P_1;t)$ must be palindromic.

- Let $P_2 := \left[0, \frac{2}{3}\right]$ so that $r = 2$. Let $m = 3$. Then,

$$\mathrm{rh}_3^*(P_2;t) = 1 + t^{\frac{1}{2}} + t.$$

  The polynomial $\mathrm{rh}_3^*(P_2;t)$ is palindromic and P_2 is 2-rational Gorenstein with Gorenstein point $(g, \mathbf{y}) = (4, 1) \in \mathrm{Cone}(\frac{1}{2}P_2)$.

Example 2.3.4 (Haasenlieblingsdreieck)**.** The triangle $\Delta := \mathrm{conv}\{(0,0),(2,0),(0,2)\}$ is not Gorenstein in the classic (integral) setting, but it is 2-rational Gorenstein: we compute

$$\mathrm{REhr}(P,t) \;=\; \frac{1}{\left(1 - t^{\frac{1}{2}}\right)^3} \;=\; \frac{1 + 3t^{\frac{1}{2}} + 3t + t^{\frac{3}{2}}}{(1-t)^3}\,.$$

Example 2.3.5. The triangle $\nabla := \mathrm{conv}\{(0,0),(0,1),(3,1)\}$ has codenominator 1. It is not 1-rational Gorenstein as $|\nabla^\circ \cap \mathbb{Z}^2| = 0$ and $|(2\nabla)^\circ \cap \mathbb{Z}^2| = 2$.

Proof of Theorem 2.3.1. (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) We compute using reciprocity (Corollary 2.1.18)

$$\begin{aligned} 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}(P;\lambda) t^\lambda &= \frac{\mathrm{rh}_m^*(P;t)}{\left(1 - t^{\frac{m}{r}}\right)^{(d+1)}} = \frac{t^{(d+1)\frac{m}{r} - \frac{g}{r}}\, \mathrm{rh}_m^*\left(P; \frac{1}{t}\right)}{\left(1 - t^{\frac{m}{r}}\right)^{(d+1)}} \\ &= t^{-\frac{g}{r}} \frac{\mathrm{rh}_m^*(P^\circ;t)}{\left(1 - t^{\frac{m}{r}}\right)^{(d+1)}} = t^{-\frac{g}{r}} \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}(P^\circ;\lambda) t^\lambda\,. \end{aligned}$$

That is equivalent to

$$\begin{aligned} t^{\frac{g}{r}}\, \mathrm{REhr}(P,t) = t^{\frac{g}{r}} \left(1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}(P;\lambda) t^\lambda \right) &= \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}(P^\circ;\lambda) t^\lambda \\ = \mathrm{REhr}(P^\circ,t) = (-1)^{d+1}\, \mathrm{REhr}\left(P, \frac{1}{t}\right). \end{aligned}$$

Comparing coefficients gives the third equivalence:

$$\operatorname{rehr}\left(P; \frac{n}{r}\right) = \operatorname{rehr}\left(P; \frac{n+g}{r}\right) \quad \text{for } n \in \mathbb{Z}_{\geq 0}\,.$$

(v) $\Rightarrow$ (i) Since

$$\operatorname{rehr}\left(P; \frac{n}{r}\right) = \operatorname{rehr}\left(P; \frac{n+g}{r}\right) \quad \text{for } n \in \mathbb{Z}_{\geq 0}$$

it suffices to show one inclusion:

$$\operatorname{Cone}\left(\frac{1}{r}P\right)^{\circ} \cap \mathbb{Z}^{d+1} \supseteq \left((g,\mathbf{y}) + \operatorname{Cone}\left(\frac{1}{r}P\right)\right) \cap \mathbb{Z}^{d+1}\,,$$

where $\mathbf{y}$ is the unique interior lattice point in $\frac{g}{r}P^{\circ}$. Indeed, since $(g,\mathbf{y}) \in \operatorname{Cone}(\frac{1}{r}P)^{\circ} \cap \mathbb{Z}^{d+1}$, $(g,\mathbf{y}) + \mathbf{z} \in \operatorname{Cone}(\frac{1}{r}P)^{\circ} \cap \mathbb{Z}^{d+1}$ for all $\mathbf{z} \in \operatorname{Cone}(\frac{1}{r}P) \cap \mathbb{Z}^{d+1}$.

(i) $\Rightarrow$ (iii) By the definition of P being r-rational Gorenstein we have

$$\operatorname{Cone}\left(\frac{1}{r}P\right)^{\circ} \cap \mathbb{Z}^{d+1} = (g,\mathbf{y}) + \operatorname{Cone}\left(\frac{1}{r}P\right) \cap \mathbb{Z}^{d+1}.$$

Computing integer point transforms gives:

$$\sigma_{\operatorname{Cone}\left(\frac{1}{r}P\right)^{\circ}}(\mathbf{z}) = \mathbf{z}^{(g,\mathbf{y})}\sigma_{\operatorname{Cone}\left(\frac{1}{r}P\right)}(\mathbf{z})\,.$$

Applying reciprocity (see, e.g., [12, Theorem 4.3]) yields

$$\sigma_{\operatorname{Cone}\left(\frac{1}{r}P\right)^{\circ}}(\mathbf{z}) = (-1)^{d+1}\sigma_{\operatorname{Cone}\left(\frac{1}{r}P\right)}\left(\frac{1}{\mathbf{z}}\right) = \mathbf{z}^{(g,\mathbf{y})}\sigma_{\operatorname{Cone}\left(\frac{1}{r}P\right)}(\mathbf{z})\,. \tag{16}$$

By specializing $\mathbf{z} = (t^{\frac{1}{r}}, 1, \dots, 1)$ in (16) we obtain the following relation between Ehrhart series for $\frac{1}{r}P$ in the variable $t^{\frac{1}{r}}$ and $t^{-\frac{1}{r}}$:

$$(-1)^{d+1}\operatorname{Ehr}\left(\frac{1}{r}P, \frac{1}{t^{\frac{1}{r}}}\right) = t^{\frac{g}{r}}\operatorname{Ehr}\left(\frac{1}{r}P, t^{\frac{1}{r}}\right). \tag{17}$$

From (the proof of) Theorem 2.1.7 we know that

$$\operatorname{Ehr}\left(\frac{1}{r}P, t^{\frac{1}{r}}\right) = \operatorname{REhr}(P,t) = \frac{\operatorname{rh}^*_m(P;t)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}},$$

where m is an integer such that $\frac{1}{r}P$ is a lattice polytope. Substituting this into (17) yields

$$\left(t^{\frac{m}{r}}\right)^{d+1}\frac{\operatorname{rh}^*_m\left(P;\frac{1}{t}\right)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}} = (-1)^{d+1}\frac{\operatorname{rh}^*_m\left(P;\frac{1}{t}\right)}{\left(1 - \frac{1}{t^{\frac{m}{r}}}\right)^{d+1}} = t^{\frac{g}{r}}\frac{\operatorname{rh}^*_m(P;t)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}$$

and therefore
$$t^{\frac{(d+1)m}{r}-\frac{g}{r}}\,\mathrm{rh}^*_m\left(P;\frac{1}{t}\right)=\mathrm{rh}^*_m(P;t)\,.$$

(ii) $\Leftrightarrow$ (vi) The primitive ray generators of $\mathrm{Cone}(\frac{1}{r}P)^\vee$ are the primitive facet normals of $\mathrm{Cone}(\frac{1}{r}P)$, that is,
$$(0,-\mathbf{a}_j)\;\text{ for } j=1,\ldots,i \quad\text{ and }\quad \left(1,-\frac{r}{b_j}\mathbf{a}_j\right)\;\text{ for } j=i+1,\ldots,n\,.$$
Since $\mathbf{0}\in P$, $b_j\geq 0$ for all $j=1,\ldots,n$. The statement follows.

(vi) $\Rightarrow$ (i) Since $(g,\mathbf{y})\in\mathrm{Cone}(\frac{1}{r}P)^\circ\cap\mathbb{Z}^{d+1}$ is an interior point, $(g,\mathbf{y})+\mathrm{Cone}(\frac{1}{r}P)\subseteq\mathrm{Cone}(\frac{1}{r}P)^\circ$ follows directly. Let $(x_0,\mathbf{x})\in\mathrm{Cone}(\frac{1}{r}P)^\circ$, then for any primitive ray generator $(v_0,\mathbf{v})$ of $\mathrm{Cone}(\frac{1}{r}P)^\vee$ (being the primitive facet normals of $\mathrm{Cone}(\frac{1}{r}P)$) we have
$$\langle(x_0,\mathbf{x})-(g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle=\underbrace{\langle(x_0,\mathbf{x})\,,(v_0,\mathbf{v})\rangle}_{>0}-\underbrace{\langle(g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle}_{=1}\geq 0\,.$$
Hence, $(x_0,\mathbf{x})-(g,\mathbf{y})\in\mathrm{Cone}(\frac{1}{r}P)$ and $(x_0,\mathbf{x})\in(g,\mathbf{y})+\mathrm{Cone}(\frac{1}{r}P)$.

(i) $\Rightarrow$ (vi) From the definition of Gorenstein point we know that $(g,\mathbf{y})\in\mathrm{Cone}(\frac{1}{r}P)^\circ$ and hence
$$\langle(g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle>0$$
for all primitive facet normals $(v_0,\mathbf{v})$ of $\mathrm{Cone}(\frac{1}{r}P)$. Since the facet normals $(v_0,\mathbf{v})$ are primitive, i.e., $\gcd((v_0,\mathbf{v}))=1$, there exists an integer point in the shifted hyperplane H defined by
$$H=\left\{(x_0,\mathbf{x})\in\mathbb{R}^{d+1}\;:\;\langle(v_0,\mathbf{v}),(x_0,\mathbf{x})\rangle=1\right\},$$
and hence H contains a d-dimensional sublattice. As $H\cap\mathrm{Cone}(\frac{1}{r}P)^\circ$ contains a pointed cone (e.g., the shifted recession cone), it contains a lattice point $(z_0,\mathbf{z})\in\mathrm{Cone}(\frac{1}{r}P)^\circ$.

So, for any facet of $\mathrm{Cone}\left(\frac{1}{r}P\right)$ there exists a lattice point $(z_0,\mathbf{z})$ in the interior of $\mathrm{Cone}(\frac{1}{r}P)$ at lattice distance one from the facet. Since $(g,\mathbf{y})+\mathrm{Cone}(\frac{1}{r}P)=\mathrm{Cone}(\frac{1}{r}P)^\circ$, there exists a point $(r_0,\mathbf{r})\in\mathrm{Cone}(\frac{1}{r}P)$ such that
$$(g,\mathbf{y})+(r_0,\mathbf{r})=(z_0,\mathbf{z})\,.$$
Then,
$$1=\langle(z_0,\mathbf{z})\,,(v_0,\mathbf{v})\rangle=\underbrace{\langle(g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle}_{>0}+\underbrace{\langle(r_0,\mathbf{r})\,,(v_0,\mathbf{v})\rangle}_{\geq 0}$$
and $\langle(g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle=1$.

□

As usual we state a version of Theorem 2.3.1 for the refined rational Ehrhart series and the rrh*-polynomial. Here, the polytopes under consideration are not required to contain the origin. This means that in the description (15) of the polytope the vector $\mathbf{b} \in \mathbb{Z}^n$ might have negative entries and we use absolute values when multiplying inequalities or facet normals with entries of $\mathbf{b}$. Except for this small difference, the proof is the same as that of Theorem 2.3.1 so we omit it.

Theorem 2.3.6. *Let $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope with codenominator r, as in* (10) *and* (15)*. Then the following are equivalent for $g, m \in \mathbb{Z}_{\ge 1}$ and $\frac{m}{2r}P$ a lattice polytope:*

(i) *P is $2r$-rational Gorenstein with Gorenstein point $(g, \mathbf{y}) \in \operatorname{Cone}(\frac{1}{2r}P)$.*

(ii) *There exists a (necessarily unique) integer solution $(g, \mathbf{y})$*

$$\begin{aligned} -\langle \mathbf{a}_j, \mathbf{y}\rangle &= 1 \qquad \text{for } j = 1, \dots, i \\ b_j\, g - 2r\,\langle \mathbf{a}_j, \mathbf{y}\rangle &= |b_j| \qquad \text{for } j = i+1, \dots, n\,. \end{aligned} \tag{18}$$

(iii) *$\operatorname{rrh}^*(P; t)$ is palindromic:*

$$t^{(d+1)\frac{m}{2r} - \frac{g}{2r}}\ \operatorname{rrh}^*_m\left(P; \frac{1}{t}\right) \ = \ \operatorname{rrh}^*_m(P; t)\,.$$

(iv) $(-1)^{d+1} t^{\frac{g}{2r}} \operatorname{RREhr}(P; t) = \operatorname{RREhr}(P; \frac{1}{t})$.

(v) $\operatorname{rehr}(P; \frac{n}{2r}) = \operatorname{rehr}(P^\circ; \frac{n+g}{2r})$ *for all $n \in \mathbb{Z}_{\ge 0}$.*

(vi) $\operatorname{Cone}(\frac{1}{2r}P)^\vee$ *is the cone over a lattice polytope, i.e., there exists a lattice point $(g, \mathbf{y}) \in \operatorname{Cone}(\frac{1}{2r}P)^\circ \cap \mathbb{Z}^{d+1}$ such that for every primitive ray generator $(v_0, \mathbf{v})$ of $\operatorname{Cone}(\frac{1}{2r}P)^\vee$.*

$$\langle (g, \mathbf{y})\,, (v_0, \mathbf{v})\rangle = 1\,.$$

Theorem 2.3.6 could be generalized to ℓr-rational Gorenstein polytopes for $\ell \in \mathbb{Z}_{>0}$. However it's not clear that computationally this would provide any new insights to the (rational) Ehrhart theory of the polytopes.

Corollary 2.3.7.

(i) *If $\mathbf{0} \in P^\circ$, then P is also $2r$-rational Gorenstein with the same Gorenstein point $(1, 0 \dots, 0)$ (see Corollary 2.3.2).*

(ii) *If $\mathbf{0} \in P$ and P is r-rational Gorenstein, then P is also $2r$-rational Gorenstein.*

(iii) *If P is $2r$-rational Gorenstein and the first coordinate g of the Gorenstein point $(g, \mathbf{y})$ is even, then P is also r-rational Gorenstein.*

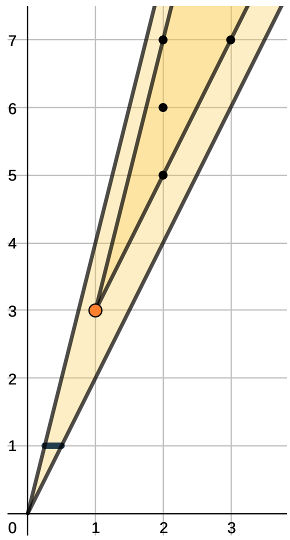


Figure 9: The cone $\mathrm{Cone}(\frac{1}{4}P_3) = \mathrm{Cone}(\frac{1}{8}P_4)$ with Gorenstein point $(3,1)$ highlighted in orange. The other lattice points $\mathrm{Cone}(\frac{1}{4}P_3)^\circ \cap \mathbb{Z}^2$ are marked in black. Observe that $(3,1) + \mathrm{Cone}(\frac{1}{4}P_3) \cap \mathbb{Z}^2 = \mathrm{Cone}(\frac{1}{4}P_3)^\circ \cap \mathbb{Z}^2$.

Proof of (ii). Since $\mathbf{0} \in P$ we know that rehr is constant on $[\frac{n}{r}, \frac{n+1}{r})$ and we compute

$$\begin{aligned}
\mathrm{RREhr}(P;t) &= 1 + \sum_{n\in\mathbb{Z}_{>0}} \mathrm{rehr}\left(P; \frac{n}{2r}\right) t^{\frac{n}{2r}} \\
&= 1 + \mathrm{rehr}\left(P, \frac{1}{2r}\right) t^{\frac{1}{2r}} + \sum_{n\in\mathbb{Z}_{>0}} \left(\mathrm{rehr}\left(P; \frac{2n}{2r}\right) t^{\frac{2n}{2r}} + \underbrace{\mathrm{rehr}\left(P; \frac{2n+1}{2r}\right)}_{=\mathrm{rehr}\left(P; \frac{n}{r}\right)} t^{\frac{2n+1}{2r}} \right) \\
&= 1 + t^{\frac{1}{2r}} + \sum_{n\in\mathbb{Z}_{>0}} \mathrm{rehr}\left(P; \frac{n}{r}\right) t^{\frac{n}{r}} \left(1 + t^{\frac{1}{2r}}\right) \\
&= \left(1 + t^{\frac{1}{2r}}\right) \mathrm{REhr}(P;t)\,,
\end{aligned}$$

where we also use that $\mathrm{rehr}(P,0) = \mathrm{rehr}\left(P, \frac{1}{2r}\right) = 1$.

$\square$

Example 2.3.8. (continued) We check the Gorenstein criterion for the running examples such that $\mathbf{0} \notin P$.

- Let $P_3 := [1,2]$ so that $r = 2$. Let $m = 4$. Then, $\mathrm{rrh}_4^*(P_3;t) = 1 + t^{\frac{2}{4}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}$.
- Let $P_4 := [2,4]$ so that $r = 4$. Let $m = 4$. Then, $\mathrm{rrh}_4^*(P_4;t) = 1 + t^{\frac{1}{4}} + t^{\frac{3}{8}} + t^{\frac{5}{8}}$.

Both polynomials $\mathrm{rrh}_4^*(P_4;t)$ and $\mathrm{rrh}_4^*(P_3;t)$ are palindromic and therefore P_3 is 4-rational Gorenstein and P_4 is 8-rational Gorenstein. In fact, $\frac{1}{4}P_3 = \frac{1}{8}P_4$ and so $\mathrm{Cone}(\frac{1}{4}P_3) = \mathrm{Cone}(\frac{1}{8}P_4)$. The Gorenstein point is $(g, \mathbf{y}) = (3,1)$. See Figure 9.

Example 2.3.9 (A polytope that is not $2r$-rational Gorenstein)**.** Let $P_5 = [1,4]$. Then $r = 4$ and $2r = 8$, so $\frac{1}{2r}P_5 = [\frac{1}{8}, \frac{1}{2}]$. The first lattice point in the interior of $\mathrm{Cone}(\frac{1}{8}P_5)$ is $(g,\mathbf{y}) = (3,1)$. However, $(3,1)$ does not satisfy Condition (ii) from Theorem 2.3.1; it is at lattice distance 5 from one of the facets of $\mathrm{Cone}(\frac{1}{8}P_5)$.

Remark 2.3.10. The **codegree** of a lattice polytope is defined as $\dim(P) + 1 - \deg(\mathrm{h}^*(t))$. Analogously, in the rational case, one can define the **rational codegree**

of $\mathrm{rh}_m^*(P;t)$ to be $\frac{m}{r}(\dim(P)+1) - \deg(\mathrm{rh}_m^*(P;t))$, where the degree of $\mathrm{rh}_m^*(P;t)$ is its (possibly fractional) degree as a polynomial in t. Likewise, the **rational codegree of** $\mathrm{rrh}_m^*(t)$ is defined as $\frac{m}{2r}(\dim(P)+1) - \deg(\mathrm{rrh}_m^*(P;t))$. It holds that the rational codegree of $\mathrm{rh}^*(P;t)$ is the smallest integral dilate of $\frac{1}{r}P$ containing interior lattice points. The proof requires no new insights and we omit it here.

2.4 SYMMETRIC DECOMPOSITIONS

We now use the stipulations of the last section to give a new proof of the following theorem. As we will see, our proof will also yield a rational version (Theorem 2.4.3 below).

Theorem 2.4.1 (Betke–McMullen [15]). *Let $P \subseteq \mathbb{R}^d$ be a lattice d-polytope that contains a lattice point in its interior. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that*

$$\mathrm{h}^*(P;t) = a(t) + t\,b(t)\,, \quad t^d\,a\left(\tfrac{1}{t}\right) = a(t)\,, \quad t^{d-1}\,b\left(\tfrac{1}{t}\right) = b(t)\,.$$

Proof. Suppose P is a lattice d-polytope with codenominator r. If P contains a lattice point in its interior, we may assume it is the origin as the h^*-polynomial is invariant under lattice translations. Corollary 2.3.2 says

$$t^{d+1-\frac{1}{r}}\,\mathrm{rh}_r^*\left(P;\frac{1}{t}\right) \;=\; \mathrm{rh}_r^*(P;t)\,. \tag{19}$$

On the other hand, as we noted in the beginning of Section 2.1, the h^*-polynomial of a rational d-polytope always has a factor, that carries over (by the proof of Theorem 2.1.7) to

$$\mathrm{rh}_r^*(P;t) \;=\; \left(1+t^{\frac{1}{r}}+\cdots+t^{\frac{r-1}{r}}\right)\widetilde{\mathrm{h}}(P;t)$$

for some $\widetilde{\mathrm{h}}(P;t) \in \mathbb{Z}[t^{1/r}]$ (which is, of course, very much related to Section 2.2). Moreover, by (19) this polynomial satisfies $t^d\,\widetilde{\mathrm{h}}(P;\frac{1}{t}) = \widetilde{\mathrm{h}}(P;t)$. Note that

$$\mathrm{REhr}(P;t) \;=\; \frac{\left(1+t^{\frac{1}{r}}+\cdots+t^{\frac{r-1}{r}}\right)\widetilde{\mathrm{h}}(P;t)}{(1-t)^{d+1}} \;=\; \frac{\widetilde{\mathrm{h}}(P;t)}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d} \tag{20}$$

and the Gorenstein property of $\frac{1}{r}P$ imply that $\widetilde{\mathrm{h}}(P;t)$ equals the h^*-polynomial (in the variable $t^{\frac{1}{r}}$) of the boundary of $\frac{1}{r}P$. Indeed, the rational Ehrhart series of ∂P is

$$\mathrm{REhr}(P;t) - \mathrm{REhr}(P^\circ;t) = \frac{\mathrm{rh}_r^*(P;t)}{(1-t)^{d+1}} - \frac{t^{d+1}\,\mathrm{rh}_r^*\left(P;\frac{1}{t}\right)}{(1-t)^{d+1}} = \frac{\mathrm{rh}_r^*(P;t)}{(1-t)^{d+1}} - \frac{t^{\frac{1}{r}}\,\mathrm{rh}_r^*(P;t)}{(1-t)^{d+1}} \tag{21}$$

$$= \frac{(1-t^{\frac{1}{r}})\,\mathrm{rh}_r^*(P;t)}{(1-t)^{d+1}} = \frac{\widetilde{\mathrm{h}}(P;t)}{(1-t)^d}\,. \tag{22}$$

The (triangulated) boundary of a polytope is shellable [108, Chapter 8], and this shelling gives a half-open decomposition of the boundary, which yields nonnegativity of the h*-vector. Hence, $\widetilde{\mathrm{h}}(P;t)$ has nonnegative coefficients.

Let Int be the operator that extracts from a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ the terms with integer powers of t. Thus

$$a(t) := \mathrm{Int}\Big(\widetilde{\mathrm{h}}(P;t)\Big)$$

is a polynomial in $\mathbb{Z}[t]$ with nonnegative coefficients satisfying $t^d\, a(\frac{1}{t}) = a(t)$. The polynomial $a(t)$ can be interpreted as the h*-polynomial of the boundary of P. With (20), we compute

$$\begin{aligned} \mathrm{h}^*(P;t) &= \mathrm{Int}\Big(\Big(1 + t^{\frac{1}{r}} + \cdots + t^{\frac{r-1}{r}}\Big)\,\widetilde{\mathrm{h}}(P;t)\Big) \\ &= a(t) + \mathrm{Int}\Big(\Big(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\Big)\,\widetilde{\mathrm{h}}(P;t)\Big). \end{aligned}$$

Since $\beta(t) := \Big(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\Big)\,\widetilde{\mathrm{h}}(P;t)$ satisfies $t^{d+1}\,\beta\Big(\frac{1}{t}\Big) = \beta(t)$, the polynomial

$$b(t) := \frac{1}{t}\,\mathrm{Int}\Big(\Big(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\Big)\,\widetilde{\mathrm{h}}(P;t)\Big)$$

satisfies $t^{d-1}\,b\Big(\frac{1}{t}\Big) = b(t)$, and $\mathrm{h}^*(P;t) = a(t) + t\,b(t)$ by construction. □

Remark 2.4.2. We could have started this proof with (14) and then used Stapledon's results [100] that $\widetilde{\mathrm{h}}(P;t)$ is palindromic and nonnegative.

The rational version of this theorem is a special case of [10, Theorem 4.7].

Theorem 2.4.3. *Let $Q \subseteq \mathbb{R}^d$ be a rational d-polytope with denominator k that contains a lattice point in its interior. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that*

$$\mathrm{h}^*(Q;t) = a(t) + t\,b(t)\,, \quad t^{k(d+1)-1}\,a\Big(\frac{1}{t}\Big) = a\,(t)\;, \quad t^{k(d+1)-2}\,b\Big(\frac{1}{t}\Big) = b(t)\,.$$

Proof. We repeat our proof of Theorem 2.4.1 for $P := k\,Q$, except that instead of the operator Int, we use the operator Rat_k which extracts the terms with powers that are multiples of $\frac{1}{k}$. So now $a(t) := \mathrm{Rat}_k(\widetilde{\mathrm{h}}(P;t))$,

$$b(t) := \frac{1}{t^{\frac{1}{k}}}\,\mathrm{Rat}_k\Big(\Big(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\Big)\,\widetilde{\mathrm{h}}(P;t)\Big)$$

and $\mathrm{h}^*(P;t) = a(t^k) + t\,b(t^k)$. □

2.5 PERIOD COLLAPSE

One of the classic instances of period collapse in integral Ehrhart theory is the triangle

$$\Delta := \mathrm{conv}\{(0,0), (1, \tfrac{p-1}{p}), (p,0)\} \tag{23}$$

where $p \geq 2$ is an integer. Here

$$\mathrm{Ehr}(\Delta; t) \; = \; \frac{1 + (p-2)\, t}{(1-t)^3},$$

and so, while the denominator of Δ equals p, the period of $\mathrm{ehr}(\Delta; n)$ collapses to 1: the quasipolynomial $\mathrm{ehr}(\Delta; n) = \frac{p-1}{2}\, n^2 + \frac{p+1}{2}\, n + 1$ is a polynomial.

As mentioned in the Introduction, we offer data points towards the question of whether or how much period collapse happens in rational Ehrhart theory, and how it compares to the classical scenario.

Example 2.5.1. We consider the triangle Δ defined in (23) with $p = 3$. Both the denominator and the codenominator of Δ equal 3. We compute

$$\mathrm{REhr}(\Delta; t) \; = \; \frac{1 + t^{\frac{5}{3}}}{\left(1 - t^{\frac{1}{3}}\right)^2 (1 - t^3)} \, .$$

The accompanying rational Ehrhart quasipolynomial $\mathrm{ehr}(P; \lambda)$ thus has period 3. We can retrieve the integral Ehrhart series from the rational by rewriting

$$\mathrm{REhr}(\Delta; t) \; = \; \frac{\left(1 + t^{\frac{5}{3}}\right)\left(1 + t^{\frac{1}{3}} + t^{\frac{2}{3}}\right)^2}{(1-t)^2 (1 - t^3)} \; = \; \frac{\left(1 + t^{\frac{5}{3}}\right)\left(1 + 2t^{\frac{1}{3}} + 3t^{\frac{2}{3}} + 2t + t^{\frac{4}{3}}\right)}{(1-t)^2 (1 - t^3)}$$

and then disregarding the fractional powers in the numerator, which gives

$$\mathrm{Ehr}(\Delta; t) \; = \; \frac{1 + 2t + 2t^2 + t^3}{(1-t)^2 (1 - t^3)} \; = \; \frac{1 + t}{(1-t)^3} \, .$$

Hence the classical Ehrhart quasipolynomial exhibits period collapse while the rational does not.

Example 2.5.2. The recent paper [52] describes certain families of polytopes arising from graphs, which exhibit period collapse. One example is the pyramid

$$P \; := \; \mathrm{conv}\left\{(0,0,0)\,, \left(\tfrac{1}{2}, 0, 0\right), \left(0, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\right)\right\} .$$

which has denominator 4 and codenominator 1. In particular, its rational Ehrhart series equals the standard Ehrhart series, and

$$\mathrm{REhr}(P; t) \; = \; \mathrm{Ehr}(P; t) \; = \; \frac{1 + t^3}{(1-t)\,(1-t^2)^3}$$

shows that $\mathrm{ehr}(P; n)$ and $\mathrm{ehr}(P; \lambda)$ both have period 2, i.e., they both exhibit period collapse.

Example 2.5.3. Recall the running examples $P_1 = [-1, \frac{2}{3}]$ and $P_2 = [0, \frac{2}{3}]$. Restricting the real Ehrhart quasipolynomial from page 46 to positive integers we retrieve the Ehrhart quasipolynomials:

$$\operatorname{ehr}(P_1; n) = \begin{cases} \frac{5}{3}n + 1 & \text{if } n \equiv 0 \bmod 3, \\ \frac{5}{3}n + \frac{1}{3} & \text{if } n \equiv 1 \bmod 3, \\ \frac{5}{3}n + \frac{2}{3} & \text{if } n \equiv 2 \bmod 3, \end{cases} \quad \operatorname{ehr}(P_2; n) = \begin{cases} \frac{2}{3}n + 1 & \text{if } n \equiv 0 \bmod 3, \\ \frac{2}{3}n + \frac{1}{3} & \text{if } n \equiv 1 \bmod 3, \\ \frac{2}{3}n + \frac{2}{3} & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

We can observe the period 3 here. Recall the rational Ehrhart series from page 50:

$$\operatorname{REhr}(P_1; t) = \frac{1 + t^{\frac{1}{2}} + t + t^{\frac{3}{2}} + t^2}{(1-t)\left(1 - t^{\frac{3}{2}}\right)}, \quad \operatorname{REhr}(P_2; t) = \frac{1}{\left(1 - t^{\frac{1}{2}}\right)\left(1 - t^{\frac{3}{2}}\right)} = \frac{1 + t^{\frac{1}{2}} + t}{\left(1 - t^{\frac{3}{2}}\right)^2}.$$

We can read off from the series that $\frac{3}{2}$ is a rational period of both $\operatorname{rehr}(P_1; \lambda)$ and $\operatorname{rehr}(P_2; \lambda)$. Hence these are examples of polytopes with period collapse in their rational Ehrhart quasipolynomials but not in their integral Ehrhart quasipolynomials.

Example 2.5.4. Consider the line segment $P_6 := [0, \frac{1}{2}]$. We easily compute the Ehrhart quasipolynomial:

$$\operatorname{ehr}(P_6, n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 0 \bmod 2, \\ \frac{n}{2} + \frac{1}{2} & \text{if } n \equiv 1 \bmod 2, \end{cases}$$

with period 2. As the codenominator r of P_6 is 1, the rational Ehrhart series equals the classical Ehrhart series, which we compute as

$$\operatorname{REhr}(P_6; t) = \operatorname{Ehr}(P_6, t) = \frac{1}{(1-t)(1-t^2)} = \frac{1+t}{(1-t^2)^2}.$$

In this case neither the rational Ehrhart quasipolynomial nor the integral Ehrhart quasipolynomial exhibit period collapse.

The question about possible period collapse of an Ehrhart quasipolynomial is only one of many one can ask for a given rational polytope. To mention just one further example, there are many interesting questions and conjectures on when the h*-polynomial is unimodal. One can, naturally, extend any such question to rational Ehrhart series.

3 MULTIVARIATE EHRHART THEORY OF POLYTROPES

Polytropes are a fundamental class of polytopes, which masquerade in the literature as alcoved polytopes of type A [75], [76]. They include order polytopes, some associahedra and matroid polytopes, hypersimplices, and Lipschitz polytopes. **Polytropes** are tropical polytopes which are classically convex [71]. They are closely related to the notion of Kleene stars and the problem of finding shortest paths in weighted graphs [102], [72]. Polytropes also arise in a range of algorithmic applications to other fields, including phylogenetics [105], mechanism design [30], and building theory [73].

It is well known that computing and approximating the volume of a polytope is "difficult" [6]. More specifically, there is no polynomial-time algorithm for the exact computation of the volume of a polytope [49], even when restricting to the class of polytopes defined by a totally unimodular matrix. However, viewing polytropes as the "building blocks" of tropical polytopes, understanding their volumes is a step towards understanding the volume of tropical polytopes. Determining whether the volume of such a tropical polytope is zero is equivalent to deciding whether a mean payoff game is winning [2]. The volume of a tropical polytope can hence serve as a measurement of how far a game is from being winning [58].

Unimodular triangulations of polytropes were studied in the language of affine Coxeter arrangements in [75], producing a volume formula and non-negativity of the h-vector corresponding to the triangulation. Motivated by a novel possibility for combining algebraic methods with enumerative results from tropical geometry, we continue to study the volume of polytropes, both continuously and discretely. A measure of discrete volume for a polytope is the number of lattice points it contains. The Ehrhart counting function encodes the discrete volume by counting the number of lattice points in any positive integral dilate of a polytope. For lattice polytopes, this counting function is given by a univariate polynomial, the Ehrhart polynomial, with leading term equal to the Euclidean volume of the polytope. Rewriting the Ehrhart polynomial in the basis of binomial coefficients determines the h^*-polynomial and reveals additional beautiful connections between the coefficients and the geometry of the polytope. It is an area of active research to determine the relations between the h^*-coefficients of alcoved polytopes [92, Question 1]; for example, it is conjectured that the h^*-vectors of alcoved polytopes of type A are unimodal.

In recent work, Loho and Schymura [79] developed a separate notion of volume for tropical polytopes driven by a tropical version of dilation, which yields an Ehrhart theory for a new class of tropical lattices. This notion of volume is intrinsically tropical and exhibits many natural properties of a volume measure, such as being monotonic and rotation-invariant. Nevertheless, the discrete and classical volume can be more relevant for certain applications; for example, the irreducible components of a Mustafin variety correspond to the lattice points of a certain tropical polytope [25], [107].

We pass from univariate polynomials to multivariate polynomials to push the connections between the combinatorics of the polynomials and the geometry even further. Combinatorial types of polytropes have been classified up to dimension 4 [102], [72]. Each polytrope of the same type has the same normal fan. Given a normal fan, we create multivariate polynomial functions in terms of the rays that yield the (discrete) volume and h*-evaluation for any polytrope of that type.

We first use algebraic methods to compute the multivariate volume polynomials, following the algorithm in [43]. We then transform these polynomials into multivariate Ehrhart polynomials, which are highly related to vector partition functions, using the Todd operator. Finally we perform the change of basis to recover the h*-polynomials.

Result 3.0.1. *We compute the multivariate volume, Ehrhart, and* h**-polynomials for all types of polytropes of dimension* ≤ 4*.*

Furthermore, these methods could be extended to higher dimensions with increased computation power. Our code and the resulting polynomials are publicly available on a Github repository[1].

Each combinatorial type of polytrope of dimension $n-1$ corresponds to a certain triangulation of the **fundamental polytope** FP_n, the polytope with vertices $e_i - e_j$ for $i, j \in [n]$ [72]. Our computations show that the volume polynomials of polytropes of dimension 3 have integer coefficients with a strong combinatorial meaning:

Theorem 3.0.2. *The coefficients of the volume polynomials of maximal* 3*-dimensional polytropes reflect the combinatorics of the corresponding regular central subdivision of* FP_3*.*

For example, each coefficient of a monomial of the form $a_{ij}a_{kl}a_{st}$ is either 6 or 0. This reflects whether the the vertices $e_i - e_j, e_k - e_l$ and $e_s - e_t$ form a face in the triangulation of FP_4 or not. Similarly, the coefficient of the monomial $a_{ij}^2 a_{kl}$ is -3 if the vertex $e_k - e_l$ is incident to a triangulating edge of a square facet of FP_3 and 0 otherwise. These intriguing observations naturally lead to a question of generalization.

Question 3.0.3. How do the coefficients of the volume polynomials of maximal $(n-1)$-dimensional polytropes reflect the combinatorics of the corresponding regular central subdivision of FP_n?

To emphasize this question, we show that our data of volume polynomials of dimension 4 is highly structured:

Theorem 3.0.4. *In the 8855-dimensional space of homogeneous polynomials of degree 4, the 27248 normalized volume polynomials of 4-dimensional polytropes span a 70-dimensional affine subspace.*

Finally, we present a partial characterization of the coefficients of these polynomials. For example, the coefficient of a monomial of the form $a_{ij}a_{ik}$ is always either 0 or 6, and the sum of all coefficients of this form is always 300, in each of the 27248 polynomials.

1 https://github.com/mariebrandenburg/polynomials-of-polytropes

Overview

In this chapter we describe methods for computing the multivariate volume, Ehrhart, and h*-polynomials for all polytropes. We begin by describing the Ehrhart theory, tropical geometry, and algebraic geometry necessary for these methods in 3.1. In 3.2, we describe our methods and apply them to 2-dimensional polytropes. In Section 3.3, we apply these methods to compute the volume, Ehrhart, and h*-polynomials of polytropes of dimension 3 and 4. We give a complete description of the coefficients of volume polynomials of 3-dimensional polytropes in terms of regular central subdivisions of the fundamental polytope, and give a partial characterization of these coefficients in dimension 4.

This chapter is joint work with Marie-Charlotte Brandenburg and Leon Zhang [20].

3.1 BACKGROUND ON POLYTROPES AND TORIC GEOMETRY

In this section we give a brief overview of the background material we use for our results. Note that throughout this chapter, we assume that P is a lattice polytope unless stated otherwise.

For a lattice polytope $P = \{x \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ with $A \in \mathbb{Z}^{m\times n}$, $\mathbf{b} \in \mathbb{Z}^m$, the **multivariate Ehrhart counting function** of P, $\mathrm{ehr}_P(\mathbf{a}) : \mathbb{Z}^m \to \mathbb{Z}$, gives the number of lattice points in the vector dilated polytope:

$$\mathrm{ehr}_P(\mathbf{a}) = |\{x \in \mathbb{Z}^n : A\mathbf{x} \leq \mathbf{a}\}|.$$

This counting function is closely related to vector partition functions, which can be used to show that $\mathrm{ehr}_P(\mathbf{a})$ is piecewise-polynomial [40]. Vector partition functions and the related Ehrhart theory have been widely studied, see, for example [101],[65].

3.1.1 *Tropical convexity*

In this subsection we review some basics of tropical arithmetic and tropical convexity. We refer readers to [47] or [70] for a more detailed exposition.

Over the **min-plus tropical semiring** $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ we define for $a, b \in \mathbb{T}$ the operations of addition $a \oplus b$ and multiplication $a \odot b$ by

$$a \oplus b = \min(a, b), \quad a \odot b = a + b.$$

We can similarly define vector addition and scalar multiplication: for any scalars $a, b \in \mathbb{T}$ and for any vectors $\mathbf{v} = (v_1, \dots, v_n), \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{T}^n$, we define

$$a \odot \mathbf{v} = (a + v_1, a + v_2, \dots, a + v_n),$$

$$a \odot \mathbf{v} \oplus b \odot \mathbf{w} = (\min(a + v_1, b + w_1), \dots, \min(a + v_n, b + w_n)).$$

Let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathbb{R}^n$ be a finite set of points. The **tropical convex hull** of V is given by the set of all tropical linear combinations

$$\operatorname{tconv}(V) = \{a_1 \odot \mathbf{v}_1 \oplus \cdots \oplus a_r \odot \mathbf{v}_r |\ a_1, \ldots, a_r \in \mathbb{R}\}.$$

A tropically convex set in $\mathbb{R}^n$ is closed under tropical scalar multiplication. As a consequence, we can identify a tropically convex set P contained in $\mathbb{R}^n$ with its image in the **tropical projective torus** $\mathbb{TP}^{n-1} = \mathbb{R}^n/(\mathbb{R} \odot (1, \ldots, 1))$. A **tropical polytope** is the tropical convex hull of a finite set V in $\mathbb{TP}^{n-1}$. A **tropical lattice polytope** is a tropical polytope whose spanning points are all contained in $\mathbb{Z}^n$. Let $P = \operatorname{tconv}(V) \subseteq \mathbb{TP}^{n-1}$ be a tropical polytope. The **(tropical) type** of a point $\mathbf{x}$ in $\mathbb{TP}^{n-1}$ with respect to V is the collection of sets $S = (S_1, \ldots, S_n)$, where an index i is contained in S_j if

$$(\mathbf{v}_i)_j - x_j = \min((\mathbf{v}_i)_1 - x_1, \ldots, (\mathbf{v}_i)_n - x_n).$$

Geometrically, we can view the type of $\mathbf{x}$ as follows: a **max-tropical hyperplane** $H_{\mathbf{a}} \subseteq \mathbb{TP}^{n-1}$ with **apex** at $\mathbf{a} \in \mathbb{TP}^{n-1}$ is the set of points $\mathbf{y} \in \mathbb{TP}^{n-1}$ such that the maximum of $\{a_i + y_i : i \in [n]\}$ is attained at least twice. The max-tropical hyperplane $H_{\mathbf{0}}$ induces a complete polyhedral fan $\mathcal{F}_{\mathbf{0}}$ in $\mathbb{TP}^{n-1}$. For each $i \in [r]$, let H_i be a max-tropical hyperplane with apex $\mathbf{v}_i$. Each of these hyperplanes determines a translate $\mathcal{F}_{H_i}$ of $\mathcal{F}_{\mathbf{0}}$. Two points $\mathbf{x}, \mathbf{y} \in \mathbb{TP}^{n-1}$ lie in the same face of $\mathcal{F}_{H_i}$ if and only if $\mathbf{v}_i - \mathbf{x}$ and $\mathbf{v}_i - \mathbf{y}$ achieve their minima in the same set of coordinates. For a point $\mathbf{x} \in \mathbb{TP}^{n-1}$ with type $S = (S_1, \ldots, S_n)$, the set S_j records for which hyperplanes H_i the point $\mathbf{x}$ lies in a face of $\mathcal{F}_{H_i}$ such that $\mathbf{v}_i - \mathbf{x}$ is minimal in coordinate j. 10 shows when i is contained in S_j based on the position of $\mathbf{x}$ in $\mathbb{TP}^2$.

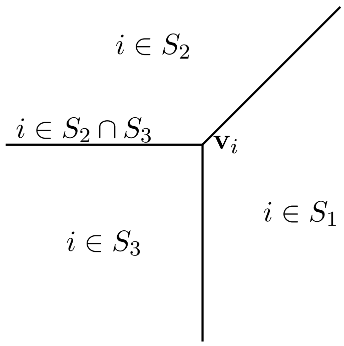


Figure 10: The max-tropical hyperplane $H_i \subseteq \mathbb{TP}^2$ in the chart where the third coordinate is 0, with faces labeled for type identification.

The tropical polytope P consists of all points $\mathbf{x}$ whose type $S = (S_1, \ldots, S_n)$ has all S_i nonempty. These are precisely the bounded regions of the subdivision of $\mathbb{TP}^{n-1}$ induced by the max-tropical hyperplanes $H_1, \ldots H_r$, as illustrated in 11. Each collection of points with the same type is called a **cell**. Each cell with all S_i nonempty is a **polytrope**: a tropical polytope that is classically convex [71]. In this way all tropical polytopes have a decomposition into polytropes. A tropical polytope P has a unique minimal set of points V such that $P = \operatorname{tconv}(V)$ [47, Prop. 21]. If P is itself a polytrope, then there is a unique maximal cell whose type with respect to

V is said to be the **type** of the polytrope P, a labeled refinement of the unlabeled combinatorial type of P as a polytope.

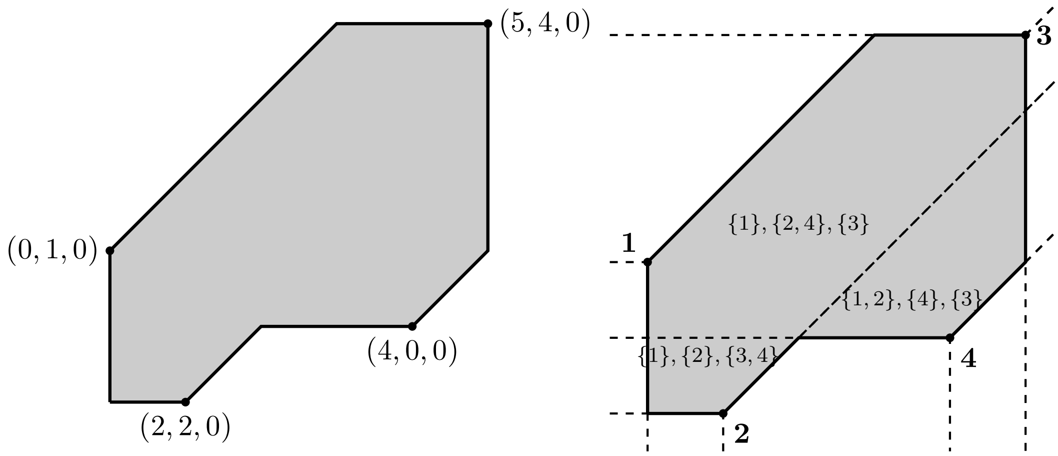


Figure 11: A 2-dimensional tropical polytope in $\mathbb{TP}^2$ spanned by four vertices, pictured in the chart where the last coordinate is 0, and its decomposition into three polytropes, labeled with their types.

3.1.2 *Polytropes*

We now delve deeper into a discussion of polytropes, reviewing certain results of [102] and [72].

Let $\mathbf{c}$ be a vector in $\mathbb{R}^{n^2-n}$. We can identify $\mathbf{c}$ with an $n \times n$ matrix having zeros along the diagonal. Under this identification, $\mathbf{c}$ describes weights on the edges of a complete directed graph with n vertices. The entry c_{ij} represents the weight of the edge going from vertex v_i to vertex v_j.

Example 3.1.1. Let $n = 3$ and consider the vector $\mathbf{c} = (3, 2, 3, 4, 5, 6) \in \mathbb{R}^6$. We view $\mathbf{c}$ as the 3×3 matrix

$$\mathbf{c} = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix},$$

where each off-diagonal entry represents the weight of an edge in a complete directed graph on 3 vertices.

We define $\mathcal{R}_n \subseteq \mathbb{R}^{n^2-n}$ to be the set of all vectors $\mathbf{c}$ with no negative cycles in the corresponding weighted graph. The **Kleene star** $\mathbf{c}^* \in \mathbb{R}^{n\times n}$ of $\mathbf{c}$ is the matrix such that $\mathbf{c}^*_{ij}$ is the weight of the lowest-weight path from i to j. It can be computed as the $(n-1)$th tropical power $\mathbf{c}^{\odot(n-1)}$. Since $\mathbf{c}$ has no negative cycles, $\mathbf{c}^*$ is zero along the diagonal, and we can again identify $\mathbf{c}^*$ with a vector in $\mathbb{R}^{n^2-n}$. The **polytrope region** $\mathcal{P}ol_n \subseteq \mathcal{R}_n \subseteq \mathbb{R}^{n^2-n}$ is the closed cone given by

$$\mathcal{P}ol_n = \{\mathbf{c} \in \mathcal{R}_n : \mathbf{c} = \mathbf{c}^*\}.$$

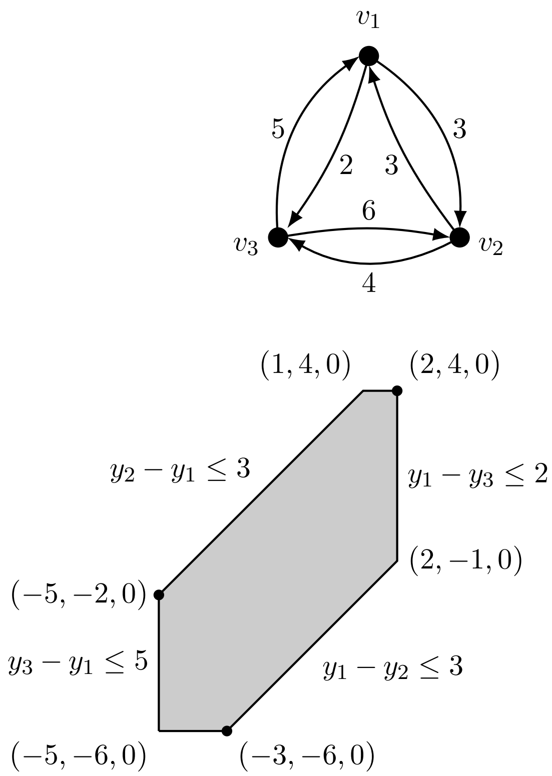


Figure 12: The complete directed graph and polytrope Q corresponding to the Kleene star $\mathbf{c} = (3,2,3,4,5,6)$. The polytrope Q is pictured in the chart where the last coordinate is zero.

Points in the polytrope region correspond to weighted graphs whose edges satisfy the triangle inequality. As the name suggests, the polytrope region parametrizes the set of all polytropes:

Proposition 3.1.2 ([84, Th. 1], [102, Prop. 13])**.** *Let $P \subseteq \mathbb{TP}^{n-1}$ be a non-empty set. The following statements are equivalent:*

1. *P is a polytrope.*
2. *There is a matrix $\mathbf{c} \in \mathcal{P}ol_n$ such that $P = tconv(\mathbf{c})$, where the columns of the matrix $\mathbf{c}$ are taken as a set of n points in $\mathbb{TP}^{n-1}$.*
3. *There is a matrix $\mathbf{c} \in \mathcal{P}ol_n$ such that*

$$P = \{y \in \mathbb{R}^n \mid y_i - y_j \leq c_{ij}, y_n = 0\}.$$

Furthermore, the $\mathbf{c}$'s in the last two statements are equal, and are uniquely determined by P.

Note in particular that polytropes in $\mathbb{TP}^{n-1}$ are tropical simplices, i.e. the tropical convex hull of exactly n points. A polytrope of dimension $n-1$ is **maximal** if it has $\binom{2n-2}{n-1}$ vertices as an ordinary polytope. To see why this is indeed the maximal number of classical vertices, we note that a polytrope is dual to a regular subdivision of the product of simplices $\Delta_{n-1} \times \Delta_{n-1}$. The normalized volume of this polytope is $\binom{2n-2}{n-1}$, bounding the number of maximal cells in the regular subdivision and hence the number of vertices of the polytrope. This bound is attained in every dimension [47, Proposition 19].

Let R be the polynomial ring $R = \mathbb{R}[x_{ij} \mid (i,j) \in [n]^2,\ i \neq j]$. Given a vector $\mathbf{v} = (v_{12}, \ldots, v_{n(n-1)})$ contained in $\mathbb{N}^{n^2-n}$, we write $\mathbf{x}^{\mathbf{v}}$ for the monomial $\prod x_{ij}^{v_{ij}}$. A vector $\mathbf{c} \in \mathbb{R}^{n^2-n}$ determines a partial ordering $>_{\mathbf{c}}$ on the monomials of R, where monomials are compared using the dot product of their exponent vector with $\mathbf{c}$, i.e. $\mathbf{x}^{\mathbf{u}} >_{\mathbf{c}} \mathbf{x}^{\mathbf{v}}$ if $\mathbf{u} \cdot \mathbf{c} > \mathbf{v} \cdot \mathbf{c}$. Given a polynomial $f = \sum_{\mathbf{v}} \alpha_{\mathbf{v}} \mathbf{x}^{\mathbf{v}}$, some of its monomial terms will be maximal with respect to this partial ordering. We define the initial term $in_{\mathbf{c}}(f)$ to be the sum of all such maximal terms of f. The initial ideal $in_{\mathbf{c}}(I)$ of an ideal $I \subseteq R$ is generated by all initial terms $in_{\mathbf{c}}(f)$ for $f \in I$. In general, a generating set $\{f_i\}$ for the ideal I need not satisfy $\langle in_{\mathbf{c}}(f_i)\rangle = in_{\mathbf{c}}(I)$. If $\langle in_{\mathbf{c}}(f_i)\rangle = in_{\mathbf{c}}(I)$ does hold, we call the generating set $\{f_i\}$ a **Gröbner basis** of I with respect to the weight vector $\mathbf{c}$. For a more detailed introduction to Gröbner bases, see [28].

We consider the toric ideal

$$I = \langle x_{ij}x_{ji} - 1, x_{ij}x_{jk} - x_{ik} \mid i,j,k \in [n] \text{ pairwise distinct}\rangle,$$

which appears in [102] as the toric ideal associated with the all-pairs shortest path program. Let $\mathbf{c} \in \mathbb{R}^{n^2-n}$. The **Gröbner cone** $\mathcal{C}_{\mathbf{c}}(I)$ is given by

$$\mathcal{C}_{\mathbf{c}}(I) = \{\mathbf{c}' \in \mathbb{R}^{n^2-n} : in_{\mathbf{c}'}(I) = in_{\mathbf{c}}(I)\}.$$

This is a closed, convex polyhedral cone. The collection of all such cones is a polyhedral fan, the **Gröbner fan** $\mathcal{GF}_n$ of the ideal I. Let $\mathcal{GF}_n|_{\mathcal{P}ol_n}$ be the restriction of the Gröbner fan of I to the polytrope region $\mathcal{P}ol_n$. This polyhedral fan captures the tropical types of polytropes:

Theorem 3.1.3 ([102, Th. 17 - 18])**.** *Cones of $\mathcal{GF}_n|_{\mathcal{P}ol_n}$ are in bijection with tropical types of polytropes in $\mathbb{TP}^{n-1}$. Open cones of $\mathcal{GF}_n|_{\mathcal{P}ol_n}$ are in bijection with types of maximal polytropes in $\mathbb{TP}^{n-1}$.*

Up to the action of the symmetric group S_3 on the labels of the vertices, in dimension 2 there is precisely one maximal tropical type of polytrope, namely the hexagon. In dimension 3 there are 6 distinct maximal tropical types up to symmetry, see [71] or [69]. Using 3.1.3, [102] showed that in dimension 4 there are 27248 distinct types up to the symmetric group action. In higher dimensions, this number is unknown. These tropical type counts were independently confirmed in [72] using the following identification:

Proposition 3.1.4 ([47, Theorem 1, Lemma 7])**.** *Let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathbb{TP}^{n-1}$. There is a piecewise-linear isomorphism between the tropical polytope $tconv(V)$ and the polyhedral complex of bounded faces of the unbounded polyhedron*

$$\mathcal{P}_V = \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{r+n}/(1, \ldots, 1, -1, \ldots, -1) \mid y_i + z_i \leq v_{ij} \text{ for all } i \in [r], j \in [n]\}.$$

The boundary complex of $\mathcal{P}_V$ is polar to the regular subdivision of the products of simplices $\Delta_{r-1} \times \Delta_{n-1}$ defined by the weights v_{ij}.

In particular, the bounded faces of $\mathcal{P}_V$ are dual to the interior faces of the regular subdivision of $\Delta_{r-1} \times \Delta_{n-1}$, i.e. the faces not completely contained in the boundary of $\Delta_{r-1} \times \Delta_{n-1}$. If $\mathrm{tconv}(V)$ is a polytrope, then 3.1.2 implies that $r = n$. Even more, $\mathrm{tconv}(V)$ is a polytrope if and only if the bounded region of $\mathcal{P}_V$ consists of a single bounded face [47, Th. 15], and hence all maximal cells in the dual subdivision of $\Delta_{n-1} \times \Delta_{n-1}$ share some vertex.

By the Cayley trick [68], this is identical to studying mixed subdivisions of the dilated simplex $n \cdot \Delta_{n-1}$. Regular subdivisions of products of simplices can thus be related to certain regular subdivisions of the **fundamental polytope** FP_n, a subpolytope of $n \cdot \Delta_{n-1}$ introduced by Vershik [104] and further studied by Delucchi and Hoessly [45]:

$$FP_n = \mathrm{conv}\{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j \in [n]\}.$$

The fundamental polytope FP_4 is pictured in 13. A **regular central subdivision**

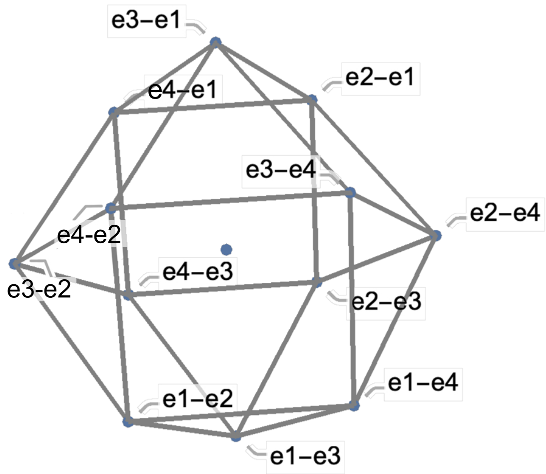


Figure 13: The fundamental polytope FP_4 with unique interior lattice point $\mathbf{0}$, whose regular central subdivisions correspond to tropical types of 3-dimensional polytropes.

of FP_n is a regular subdivision in which the unique relative interior lattice point $\mathbf{0}$ of FP_n is lifted to height 0 and is a vertex of each maximal cell. The number of tropical types can be enumerated using the following theorem:

Theorem 3.1.5 ([72, Th. 22])**.** *The tropical types of full-dimensional polytropes in* $\mathbb{TP}^{n-1}$ *are in bijection with the regular central subdivisions of* FP_n*.*

We connect our computational results to regular central subdivisions of the fundamental polytope in 3.3.3, 3.3.4.

3.1.3 *Toric geometry*

In order to compute multivariate volume polynomials of polytropes we use methods from toric geometry. We now give a brief summary of the toric geometry needed in our computation. For further details, the reader may consult [29, Ch. 12.4, 13.4] or [55, Ch. 5.3].

Let P be a lattice polytope with m facets, so that P is given by

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{u}_i \rangle + c_i \geq 0 \text{ for } i \in [m]\},$$

where $\mathbf{u}_i$ is the primitive facet normal of the facet F_i. We denote by Σ the normal fan of P, and by X the toric variety defined by the fan Σ. We assume that X is smooth, so Σ is a simplicial fan and P a simple polytope.

Let $d = \dim(X)$. A torus-invariant prime divisor D_i of X is a subvariety of X of codimension 1, which is in bijection with a ray of Σ and hence also with a facet F_i of P. Given the polytope P, we can define the divisor D_P as the linear combination $D_P = \sum_{i=1}^m c_i D_i$. At the same time, as D_i is an irreducible subvariety of X of codimension 1, it gives rise to a cohomology class $[D_i] \in H^2(X, \mathbb{Q})$ and $[D_P] = [\sum_{i=1}^m c_i D_i] = \sum_{i=1}^m c_i [D_i] \in H^2(X, \mathbb{Q})$.

Given irreducible subvarieties $V, W \subseteq X$ with $\dim(V) = k_1, \dim(W) = k_2$, we can consider the cup product $[V] \smile [W] \in H^{2(k_1+k_2)}(X, \mathbb{Q})$. If $k_1 + k_2 = d$, then by Poincare-duality $[V] \smile [W] \in H^{2d}(X, \mathbb{Q}) \cong H_0(X, \mathbb{Q}) \cong \mathbb{Q}$ and thus we can define the **integral (or intersection product)** $\int_X ([V] \smile [W]) \in \mathbb{Q}$. We use $[V]^2$ as shorthand notation for $([V] \smile [V])$.

Theorem 3.1.6 ([29, Theorem 13.4.1])**.** *The normalized volume of P is given by*

$$\operatorname{Vol}(P) = \int_X \left[\sum_{i=1}^m c_i D_i \right]^d .$$

In order to be able to compute these polynomials systematically, we make use of an identification of the cohomology ring as a polynomial ring. Let K be a field of characteristic 0 and S be a simplicial complex on m vertices. The **Stanley-Reisner ideal** M in the polynomial ring $R = K[x_1, \ldots, x_m]$ is the ideal generated by the (inclsuion-minimal) non-faces of S, i.e.

$$M = \langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \text{ is not a face of S } \rangle.$$

Let $\mathcal{B}$ be a basis of $\mathbb{Z}^n$. Since Σ is simplicial, we can consider the **Stanley-Reisner ideal** M of Σ, i.e. the Stanley-Reisner ideal of the boundary complex ∂P° of the polar of P. The cohomology ring $H^*(X, \mathbb{Q})$ is isomorphic to the quotient ring $R/(L + M)$, where L is the ideal

$$L = \left\langle \sum_{i=1}^m \langle \mathbf{b}, \mathbf{u}_i \rangle x_i \,\middle|\, \mathbf{b} \in \mathcal{B} \right\rangle .$$

The variable x_i in $R/(L + M)$ corresponds to $[D_i]$, the cohomology class of a torus-invariant prime-divisor, and hence to a facet of P. Therefore, the expression in 3.1.6 translates to a polynomial

$$\left(\sum_{i=1}^m c_i x_i \right)^d \in R/(L + M).$$

The top cohomology group is a one-dimensional vector space. A canonical choice of a basis vector in $R/(L + M)$ is any square-free monomial $\mathbf{x}^{\boldsymbol{\alpha}}$ which indexes a vertex of P. The expression $(\sum_{i=1}^m c_i x_i)^{\dim(X)}$ has a representation $\delta \cdot \mathbf{x}^{\boldsymbol{\alpha}}$ in $R/(L + M)$.

The volume of P will be given by the coefficient δ, up to a correcting factor that solely depends on the choice of the basis $\mathbf{x}^{\boldsymbol{\alpha}}$ [43, Algorithm 1].

Replacing the values $c_1, \dots, c_m$ defining the facets of P by indeterminates $a_1, \dots, a_m$ we obtain a polynomial which gives the volume $\operatorname{Vol}(P)$ of the polytope P when evaluated at $c_1, \dots, c_m$. We hence refer to such a polynomial as a **volume polynomial** of P. This polynomial depends only on the normal fan of P, and so polytopes with the same normal fan determine the same volume polynomial. 3.2.4 describes how to compute the integral of a cohomology class of X.

3.2 COMPUTING MULTIVARIATE POLYNOMIALS

In this section we introduce our multivariate polynomials of interest and describe methods for computing these functions for polytropes, motivated by the methods in [102] and [43].

3.2.1 *Computing multivariate volume polynomials*

We seek to compute a **multivariate volume polynomial** for each tropical type of polytrope as discussed in 3.1.3: that is, a polynomial in variables a_{ij} for each tropical type which evaluates to the volume of a polytrope $P(\mathbf{c})$ of the appropriate type when given the respective Kleene star $\mathbf{c}$. For each tropical type, our computation of such a polynomial will depend on a fixed Kleene star $\mathbf{c}$ of the appropriate type.

Consider the "indeterminate polytrope"

$$P(\mathbf{a}) = \{\mathbf{y} \in \mathbb{R}^n \mid y_i - y_j \le a_{ij}, y_n = 0\},$$

defined by indeterminates a_{ij}. By 3.1.2, a_{ij} is the weight of the shortest path in a weighted complete digraph. As $P(\mathbf{a})$ is contained in the linear space given by $y_n = 0$, we can project onto the first $n-1$ coordinates, which yields

$$P(\mathbf{a}) = \{\mathbf{y} \in \mathbb{R}^{n-1} \mid B\mathbf{y} \le \mathbf{a}\}$$

for a suitable matrix $B \in \mathbb{Z}^{(n^2-n)\times(n-1)}$ with rows B_{ij} indexed by $ij \in [n]^2, i \ne j$. This is a representation of $P(\mathbf{a})$ given by $n^2 - n$ inequalities $B_{ij}\mathbf{y} \le a_{ij}$ and $\mathbf{y} = (y_1, \dots, y_{n-1})$, as in 3.2.1 below. Note that for each $j \in [n-1]$ there is an inequality $-y_j \le a_{nj}$. We introduce a nonnegative slack-variable y_{ij} for each inequality and replace the inequality by the equation $y_i - y_j + y_{ij} = a_{ij}$. This gives a representation as $(B \mid Id_n)\mathbf{y} = \mathbf{a}$ with $\mathbf{y} = (y_1, \dots, y_{n-1}, y_{12}, \dots, y_{n-1,n})$.

In particular, we have the equation $-y_j + y_{nj} = a_{nj}$. We can thus substitute the variable $y_j, j \in [n-1]$ by $y_{nj} - a_{nj}$, which leaves us with a system of equations of the form

$$\begin{aligned} y_{ij} + y_{ni} - y_{nj} &= a_{ij} + a_{ni} - a_{nj} & (1)\\ y_{jn} + y_{nj} &= a_{jn} + a_{nj}. & (2) \end{aligned}$$

Adding these equations gives $y_{ij} + y_{ni} + y_{jn} = a_{ij} + a_{ni} + a_{jn}$ (1'). The set of solutions to the system with equations (1) and (2) is equal to the set of solutions to the system with (1') and (2), yielding a matrix A such that

$$P(\mathbf{a}) = \{\mathbf{y} \in \mathbb{R}^{n^2-n}_{\geq 0} \mid A\mathbf{y} = A\mathbf{a}\}$$

and $\ker(A) \cap \mathbb{R}^{n^2-n}_{\geq 0} = 0$, thus fulfilling the general assumptions in [43]. The expressions $a_{ij} + a_{ni} + a_{jn}$ and $a_{in} + a_{ni}$ have a nice interpretation in terms shortest paths of the complete digraph: these are the weights of the shortest cycle passing through i and n and the shortest directed cycle passing through i, j and n respectively. A matrix is called **totally unimodular** if every minor equals $-1, 0$, or 1. A full-dimensional lattice polytope in $\mathbb{R}^n$ is called **unimodular** if each of its vertex cones is generated by a basis of $\mathbb{Z}^n$. This condition is sometimes referred to as **smooth** or **Delzant**. It is well-known that our constraint matrix A is totally unimodular [102, Section 2.3.3], and that maximal polytropes are unimodular and simple [60, Section 7.3].

Example 3.2.1. Any 2-dimensional polytrope $P(\mathbf{a})$ has an H-description as

$$P(\mathbf{a}) = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{array}{r} y_1 - y_2 \leq a_{12}, y_2 - y_1 \leq a_{21} \\ y_1 \leq a_{13}, y_2 \leq a_{23} \\ y_1 \geq -a_{31}, y_2 \geq -a_{32} \end{array} \right\},$$

when $\mathbf{a}$ is contained in the polytrope region $\mathcal{P}ol_3$. We want to compute the constraint matrix A by turning the above description of a polytrope into one involving only equalities. We begin by translating the above to a matrix description of $P(\mathbf{a})$:

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} a_{12} \\ a_{13} \\ a_{21} \\ a_{23} \\ a_{31} \\ a_{32} \end{pmatrix}$$

Introducing slack variables y_{ij}, we get the representation

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{23} \\ y_{31} \\ y_{32} \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{13} \\ a_{21} \\ a_{23} \\ a_{31} \\ a_{32} \end{pmatrix}$$

Substituting $y_1 = y_{31} - a_{31}, y_2 = y_{32} - a_{32}$ and deleting zero-columns and zero-rows gives us

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{12} \\ y_{13} \\ y_{21} \\ y_{23} \\ y_{31} \\ y_{32} \end{pmatrix} = \begin{pmatrix} a_{12} + a_{31} - a_{32} \\ a_{13} + a_{31} \\ a_{21} - a_{31} + a_{32} \\ a_{23} + a_{32} \end{pmatrix}$$

This is equivalent to

$$A\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{12} \\ y_{13} \\ y_{21} \\ y_{23} \\ y_{31} \\ y_{32} \end{pmatrix} = \begin{pmatrix} a_{12} + a_{23} + a_{31} \\ a_{13} + a_{31} \\ a_{21} + a_{13} + a_{32} \\ a_{23} + a_{32} \end{pmatrix} = A\mathbf{a}.$$

This gives us the desired representation $P(\mathbf{a}) = \{\mathbf{y} \in \mathbb{R}_{\geq 0}^{n^2-n} \mid A\mathbf{y} = A\mathbf{a}\}$.

Let $K = \mathbb{Q}(a_{ij} \mid (i,j) \in [n]^2,\ i \neq j)$, for a_{ij} indeterminate variables, and consider the polynomial ring $R = K[x_{ij} \mid (i,j) \in [n]^2,\ i \neq j]$. Following [43], we consider the toric ideal seen previously in 3.1.2:

$$I = \langle \mathbf{x}^{\mathbf{r}} - 1 \mid \mathbf{r} \text{ is a row of } A\rangle = \langle x_{in}x_{ni} - 1, x_{ij}x_{jn}x_{ni} - 1\rangle = \langle x_{ij}x_{ji} - 1, x_{ij}x_{jk} - x_{ik}\rangle,$$

where $(i,j,k) \in [n]^3$ are pairwise distinct.

Fix a tropical type and Kleene star $\mathbf{c}$ corresponding to a polytrope $P(\mathbf{c})$ of that type. We write $M = in_{\mathbf{c}}(I)$ for the initial ideal of I with respect to the weight vector $\mathbf{c}$.

Proposition 3.2.2 ([43, Prop. 2.3])**.** *The initial ideal M is the Stanley-Reisner ideal of the normal fan Σ of the simple polytope $P(\mathbf{c})$.*

In order to compute the volume polynomial, we need to know the minimal primes of M.

Proposition 3.2.3 ([19, Lemmas 5 and 6], [43])**.** *The facets of $P(\mathbf{c})$ are in bijection with variables x_{ij}. The vertices of $P(\mathbf{c})$ are in bijection with minimal primes of M.*

In the above bijection, the facet F_{ij} given by the inequality $y_i - y_j = c_{ij}$ is identified with the variable x_{ij}. A vertex v of the polytrope can be identified with the minimal prime $\langle x_{ij} \mid ij \notin \mathcal{I}_v\rangle$, where $\mathcal{I}_v = \{ij \mid F_{ij} \text{ contains } v\}$. Thus, a minimal prime is generated by variables which correspond to facets that do not contain a given vertex v.

Let X be the smooth toric variety defined by the normal fan Σ of the unimodular polytope $P(\mathbf{c})$. Let $\mathbf{u}_{ij}$ denote the primitive ray generators of the normal fan of $P(\mathbf{c})$, i.e. $\mathbf{u}_{ij} = \mathbf{e}_j - \mathbf{e}_i$ for $i \neq j \in [n-1] \times [n-1]$ and $\mathbf{u}_{ni} = \mathbf{e}_i,\ \mathbf{u}_{in} = -\mathbf{e}_i$

for $i \in [n-1]$. Let further $\mathcal{B}$ be a basis for $\mathbb{Z}^n$. Recall that the cohomology ring $H^*(X, \mathbb{Q})$ is isomorphic to the quotient ring $R/(L+M)$, where M is the initial ideal from above and L is the ideal

$$L = \left\langle \sum_{\substack{ij \in [n]\times[n] \\ i \neq j}} \langle \mathbf{b}, \mathbf{u}_{ij} \rangle x_{ij} \;\middle|\; \mathbf{b} \in \mathcal{B} \right\rangle.$$

Choosing $\mathcal{B}$ to be the standard basis for $\mathbb{Z}^n$, for a given vector $\mathbf{b} = \mathbf{e}_k$ we get

$$\sum_{\substack{ij \in [n]\times[n] \\ i \neq j}} \langle \mathbf{e}_k, \mathbf{u}_{ij} \rangle x_{ij} = \sum_{j \in [n]} x_{kj} - \sum_{j \in [n]} x_{jk}$$

and so the ideal is equal to

$$L = \left\langle \sum_{j \in [n]} x_{kj} - \sum_{j \in [n]} x_{jk} \;\middle|\; k \in [n] \right\rangle.$$

Considering the complete directed graph K_n on n vertices, this ideal can be viewed as generated by the cuts of K_n that isolate a single vertex.

Let D be the divisor on X corresponding to the polytrope $P(\mathbf{a})$ given by the indeterminates a_{ij}, i.e. $P(\mathbf{a}) = \{\mathbf{y} \in \mathbb{R}^n \mid y_i - y_j \leq a_{ij}, y_n = 0\}$. We can write D as

$$D = \sum_{\substack{ij \in [n]\times[n] \\ i \neq j}} a_{ij} D_{ij},$$

where D_{ij} is the prime divisor corresponding to the ray of Σ spanned by $\mathbf{u}_{ij}$. Let

$$q = \sum_{\substack{ij \in [n]\times[n] \\ i \neq j}} a_{ij} x_{ij}$$

be the polynomial in R representing the divisor D.

In the following we present an algorithm to compute the integral of a top cohomology class of X. As the dimension of a polytrope $P(\mathbf{c})$ defined by a Kleene star $\mathbf{c} \in \mathcal{GF}|_{\mathcal{P}ol_n}$ is $n-1$, we can compute the volume polynomial restricted to an open maximal cone of $\mathcal{GF}|_{\mathcal{P}ol_n}$ by

$$\mathrm{Vol}_{P(\mathbf{a})}(\mathbf{a}) = \int_X [D]^{n-1}.$$

Note that the integral $\int_X [D]^{n-1}$ is a constant in R and thus a polynomial with variables a_{ij}, as discussed in 3.1.3. If the input of 3.2.4 is given by the polynomial $p = q^{n-1}$, the output is a multivariate volume polynomial.

Algorithm 3.2.4 (Computing the integral of a cohomology class of X [43, Alg. 1]).

INPUT: A polynomial $p(x)$ with coefficients in a field $k \supset \mathbb{Q}$.

OUTPUT: The integral $\int_X p$ of the corresponding cohomology class on X.

1: Compute a Gröbner basis $\mathcal{G}$ for the ideal $M + L$.
2: Find a minimal prime $\langle x_j \mid x_j \notin \mathcal{I}_v \rangle$ of M, and compute the normal form of $\prod_{i \in \mathcal{I}_v} x_i$ modulo the Gröbner basis $\mathcal{G}$. It looks like $\gamma \cdot \mathbf{x}^{\boldsymbol{\alpha}}$, where γ is a non-zero element of k and $\mathbf{x}^{\boldsymbol{\alpha}}$ is the unique standard monomial of degree $n-1$.
3: Compute the normal form of p modulo $\mathcal{G}$ and let $\delta \in k$ be the coefficient of $\mathbf{x}^{\boldsymbol{\alpha}}$ in that normal form.
4: Output the scalar $\delta/\gamma \in k$.

Recall that $\mathcal{GF}_n|_{\mathcal{P}ol_n}$ is the restriction of the Gröbner fan of I to the polytrope region $\mathcal{P}ol_n$. By 3.1.3, open cones of $\mathcal{GF}_n|_{\mathcal{P}ol_n}$ are in bijection with types of maximal polytropes. Since 3.2.4 only depends on $in_{\mathbf{c}}(I)$ and not the choice of $\mathbf{c}$ itself, the multivariate volume polynomial is constant along an open cone of $\mathcal{GF}_n|_{\mathcal{P}ol_n}$. This reflects the fact that polytropes of the same tropical type have the same normal fan. Therefore, maximal polytropes of the same type have the same multivariate volume polynomial, and it suffices to compute the polynomial for only one representative $\mathbf{c}$ for each maximal cone. Furthermore, the polynomials agree on the intersection of the closure of two of these cones [101]. Thus, given a Kleene star $\mathbf{c}$ corresponding to a non-maximal polytrope $P(\mathbf{c})$, we can choose any of the maximal closed cones that contain $\mathbf{c}$ and evaluate the corresponding multivariate volume polynomial at $\mathbf{c}$ to compute the volume of $P(\mathbf{c})$.

Example 3.2.5. We apply the above discussion to compute the multivariate volume polynomial for 2-dimensional polytropes. Note that the volume, Ehrhart- and h*-polynomial of the hexagon can be derived by more elementary methods as, for example, counting unimodular simplices in an alcoved triangulation and Pick's formula. However, as the presentation is less clear in dimensions 3 and 4, we showcase the algebraic machinery on this example. The toric ideal I is

$$I = \langle x_{12}x_{23}x_{31} - 1,\ x_{13}x_{31} - 1,\ x_{21}x_{13}x_{32} - 1,\ x_{23}x_{32} - 1 \rangle.$$

We also have L as

$$L = \langle x_{12} + x_{13} - x_{21} - x_{31},\ x_{21} + x_{23} - x_{12} - x_{32},\ x_{31} + x_{32} - x_{13} - x_{23} \rangle.$$

Let $\mathbf{c} = (3, 2, 3, 4, 5, 6)$. Then the corresponding polytrope $Q(\mathbf{c})$ is the hexagon displayed in 14, with facets labeled according to 3.2.3.

The initial ideal M of I with respect to the weight vector $\mathbf{c}$ is

$$M = \langle x_{12}x_{21},\ x_{13}x_{21},\ x_{12}x_{23},\ x_{12}x_{31},\ x_{13}x_{31},\ x_{23}x_{31},\ x_{13}x_{32},\ x_{21}x_{32},\ x_{23}x_{32} \rangle.$$

A Gröbner basis for $M + L$ is given by

$$\begin{aligned}\mathcal{G} = \langle & x_{31} - x_{12} + x_{21} - x_{13},\ x_{13}x_{21},\ x_{12}x_{13} + x_{13}^2,\ x_{32} - x_{23} + x_{12} - x_{21}, \\ & x_{13}x_{23} + x_{13}^2,\ x_{21}^2 - x_{13}^2,\ x_{12}x_{21},\ x_{12}^2 - x_{13}^2,\ x_{13}^3,\ x_{21}x_{23} + x_{13}^2,\ x_{12}x_{23},\ x_{23}^2 - x_{13}^2 \rangle.\end{aligned}$$

Any vertex gives us a minimal prime. We choose the vertex v incident to the facets labeled by x_{31} and x_{32}, giving us the minimal prime $\langle x_{ij} \mid ij \notin \mathcal{I}_v \rangle = \langle x_{12}, x_{13}, x_{21}, x_{23} \rangle$

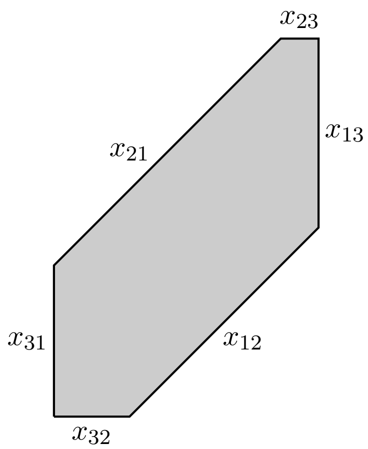


Figure 14: The polytrope Q corresponding to the vector $\mathbf{c} = (3, 2, 3, 4, 5, 6)$.

and the monomial $\prod_{ij \in \mathcal{I}_v} x_{ij} = x_{31}x_{32}$. Modulo the Gröbner basis $\mathcal{G}$, this gives us $\gamma \cdot \mathbf{x}^{\boldsymbol{\alpha}} = (-1)x_{13}^2$, so $\gamma = -1$ and $\mathbf{x}^{\boldsymbol{\alpha}} = x_{13}^2$.

Let $q = \sum_{\substack{ij \in [n]\times[n] \\ i \neq j}} a_{ij}x_{ij}$. This is the polynomial in $R/(L+M)$ corresponding to the divisor described in 3.2.1. We want to compute the volume of the polytrope $Q(\mathbf{c})$. This can be done by applying 3.2.4 to $p = q^2$.

The polynomial q^2 modulo $\mathcal{G}$ is

$$\begin{gathered}(a_{12}^2 - 2a_{12}a_{13} + a_{13}^2 + a_{21}^2 - 2a_{13}a_{23} - 2a_{21}a_{23} \\ +a_{23}^2 - 2a_{21}a_{31} + a_{31}^2 - 2a_{12}a_{32} - 2a_{31}a_{32} + a_{32}^2)x_{13}^2,\end{gathered}$$

so the coefficient δ of $\mathbf{x}^{\boldsymbol{\alpha}}$ gives us the volume polynomial for the normalized volume

$$\begin{aligned}\mathrm{Vol}_{Q(\mathbf{a})}(\mathbf{a}) = \frac{\delta}{\gamma} &= -(a_{12}^2 - 2a_{12}a_{13} + a_{13}^2 + a_{21}^2 - 2a_{13}a_{23} - 2a_{21}a_{23} \\ &\quad +a_{23}^2 - 2a_{21}a_{31} + a_{31}^2 - 2a_{12}a_{32} - 2a_{31}a_{32} + a_{32}^2).\end{aligned}$$

Evaluating at the original vector $\mathbf{c}$ gives 79, which is the normalized volume of the original polytope. The volume polynomial $\mathrm{vol}_{Q(\mathbf{a})}(\mathbf{a})$ for the Euclidean volume of $Q(\mathbf{a})$ is given as

$$\mathrm{vol}_{Q(\mathbf{a})}(\mathbf{a}) = \frac{1}{2}\mathrm{Vol}_{Q(\mathbf{a})}(\mathbf{a}).$$

Remark 3.2.6. Polytropes have also appeared in the literature as **alcoved polytopes of type** A. The volumes of alcoved polytopes of type A were studied in [75, Theorem 3.2] and extended to general root systems in [76, Theorem 8.2], where the normalized volume of an alcoved polytope is described as a sum of discrete volumes of related alcoved subpolytopes. More specifically, given a fixed alcoved polytope of type A, the normalized volume of the respective polytope P can be computed as

$$\mathrm{Vol}(P) = \sum_{\omega \in S_{n-1}} \left| P_\omega \cap \mathbb{Z}^{n-1} \right|,$$

where $P_\omega = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid \mathbf{x} + \Delta_\omega \subseteq P\}$ and $\Delta_\omega = \{\mathbf{y} \in \mathbb{R}^{n-1} \mid 0 \leq y_{\omega(1)} \leq \cdots \leq y_{\omega(n-1)} \leq 1\}$. While this is a formula that yields a value for the normalized volume for polytropes of any dimension, it does not allow a parametrized approach resulting in multivariate polynomials.

3.2.2 *Computing multivariate Ehrhart polynomials*

We use the Todd operator to pass from the multivariate volume polynomials to the multivariate Ehrhart polynomials of polytropes. We begin by defining single and multivariate versions of the Todd operator and then explain the method we used for computations. Finally, we compute the multivariate and univariate Ehrhart polynomials of our running example. For more thorough background information on the Todd operator, see [12, Chapter 12] and [29, Chapter 13.5].

The Todd operator is related to the Bernoulli numbers, a sequence of rational numbers B_k for $k \in \mathbb{Z}_{\geq 0}$ whose first few terms are $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0$. They are defined through the following generating function:

$$\frac{z}{\exp(z) - 1} = \sum_{k\geq 0} \frac{B_k}{k!} z^k.$$

Definition 3.2.7. The **Todd operator** is the differential operator

$$\mathrm{Todd}_h = 1 + \sum_{k\geq 1} (-1)^k \frac{B_k}{k!} \left(\frac{d}{dh}\right)^k.$$

Note that for a polynomial $f(h)$ of degree d, the function $\mathrm{Todd}_h(f)$ is a polynomial: since $(\frac{df}{dh})^k = 0$ for any $k > d$, we get the finite expression

$$\mathrm{Todd}_h(f) = 1 + \sum_{k=1}^{d} (-1)^k \frac{B_k}{k!} \left(\frac{df}{dh}\right)^k.$$

The Todd operator can be succinctly expressed in shorthand as

$$\mathrm{Todd}_h = \frac{\frac{d}{dh}}{1 - \exp\left(-\frac{d}{dh}\right)}.$$

In order to compute the multivariate Ehrhart polynomials, we use a multivariate version of the Todd operator. For $\mathbf{h} = (h_1, h_2, \ldots, h_m)$, we write

$$\mathrm{Todd}_{\mathbf{h}} = \prod_{j=1}^{m} \left(\frac{\frac{\partial}{\partial h_j}}{1 - \exp(-\frac{\partial}{\partial h_j})} \right).$$

The Todd operator allows one to pass from a continuous measure of volume on a polytope to a discrete measure: a lattice point count. Let $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$, $\mathbf{b} \in \mathbb{R}^m$. For $\mathbf{h} \in \mathbb{R}^m$, the **shifted polytope** $P_{\mathbf{h}}$ is defined as

$$P_{\mathbf{h}} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b} + \mathbf{h}\}.$$

Theorem 3.2.8 (Khovanskii-Pukhlikov, [12, Ch. 12.4])**.** *Let $P \subseteq \mathbb{R}^n$ be a unimodular d-polytope. Then*

$$\#(P \cap \mathbb{Z}^n) = \operatorname{Todd}_{\mathbf{h}} \operatorname{vol}(P_{\mathbf{h}})|_{\mathbf{h}=0}.$$

In words, the number of lattice points of P equals the evaluation of the Todd operator at $\mathbf{h} = 0$ on the relative Euclidean volume of the shifted polytope $P_{\mathbf{h}}$.

In 3.2.8, one applies the Todd operator to the volume of a shifted version $P_{\mathbf{h}}$ of the polytope P. In our setting of multivariate volume polynomials that are constant on fixed cones of the polytrope region in the Gröbner fan, a nice simplification occurs that allows us to ignore this shift. As discussed in Section 3.1.2, a polytrope P can be described as

$$P = \{\mathbf{x} \in \mathbb{R}^{n-1} : x_i - x_j \leq c_{ij},\ -c_{ni} \leq x_i \leq c_{in}\},$$

where $i, j \in [n-1], i \neq j$, for some $\mathbf{c} \in \mathbb{R}^{n^2-n}$. Its volume is given by evaluating the multivariate volume polynomial $\operatorname{vol}_P(\mathbf{a})$ at $\mathbf{c}$. The shifted polytrope $P_{\mathbf{h}}$ has the description

$$P_{\mathbf{h}} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : x_i - x_j \leq c_{ij} + h_{ij},\ -(c_{ni} + h_{ni}) \leq x_i \leq c_{in} + h_{in}\right\},$$

for any $\mathbf{h} \in \mathbb{R}^{n^2-n}$. As long as $\mathbf{h}$ is small enough, the shifted polytrope remains in the same cone and its volume polynomial is given by evaluating the multivariate volume polynomial $\operatorname{vol}_P(\mathbf{a} + \mathbf{h})$ at $\mathbf{c}$. As $\operatorname{vol}_P(\mathbf{a})$ is a polynomial,

$$\left(\prod_{i \neq j \in [n]} \frac{\partial}{\partial h_{ij}}\right) \operatorname{vol}_P(\mathbf{a} + \mathbf{h})\Big|_{\mathbf{h}=0} = \left(\prod_{i \neq j \in [n]} \frac{\partial}{\partial a_{ij}}\right) \operatorname{vol}_P(\mathbf{a}).$$

Example 3.2.9. We now apply the Todd operator to the multivariate volume polynomial of the tropical hexagon $Q(\mathbf{a})$ in our running example. As in the previous example, this 2-dimensional example can be computed with more elementary methods, such as Pick's formula. However, this example generalizes to higher dimensions, and we use it to present our methods in a managable size. Recall from 3.2.5 that the volume polynomial is

$$\operatorname{vol}_{Q(\mathbf{a})}(\mathbf{a}) = \sum_{i \neq j \in [3]} -\frac{1}{2} a_{ij}^2 + \sum_{i \neq j \neq k \in [3]} (a_{ij} a_{ik} + a_{ji} a_{ki}).$$

Evaluating the polynomial at a specific Kleene star $\mathbf{c}$ returns the volume of the corresponding polytrope. Applying the multivariate Todd operator to this volume polynomial, we compute that $\mathrm{Todd}_{\mathbf{h}} \operatorname{vol}_{Q(\mathbf{a})}(\mathbf{a}+\mathbf{h})\Big|_{\mathbf{h}=0}$ is:

$$\left(\frac{\frac{\partial}{\partial h_{32}}}{1-\exp(-\frac{\partial}{\partial h_{32}})}\right)\cdots\left(\frac{\frac{\partial}{\partial h_{13}}}{1-\exp(-\frac{\partial}{\partial h_{13}})}\right)\left(1+\sum_{k\geq 1}(-1)^k\frac{B_k}{k!}\Big(\frac{\partial}{\partial h_{12}}\Big)^k\right)\operatorname{vol}_{Q(\mathbf{a})}(\mathbf{a}+\mathbf{h})\Big|_{\mathbf{h}=0}$$

$$= \left(\frac{\frac{\partial}{\partial a_{32}}}{1-\exp(-\frac{\partial}{\partial a_{32}})}\right)\cdots\left(\frac{\frac{\partial}{\partial a_{13}}}{1-\exp(-\frac{\partial}{\partial a_{13}})}\right)\left(1+\sum_{k\geq 1}(-1)^k\frac{B_k}{k!}\Big(\frac{\partial}{\partial a_{12}}\Big)^k\right)\operatorname{vol}_{Q(\mathbf{a})}(\mathbf{a})$$

$$= \left(\frac{\frac{\partial}{\partial a_{32}}}{1-\exp(-\frac{\partial}{\partial a_{32}})}\right)\cdots\left(\frac{\frac{\partial}{\partial a_{13}}}{1-\exp(-\frac{\partial}{\partial a_{13}})}\right)\left(\operatorname{vol}_{Q(\mathbf{a})}(\mathbf{a})+\frac{1}{2}[-a_{12}+a_{13}+a_{32}]-\frac{1}{12}\right)$$

$$= -\frac{1}{2}a_{12}^2+a_{12}a_{13}-\frac{1}{2}a_{13}^2-\frac{1}{2}a_{21}^2+a_{13}a_{23}+a_{21}a_{23}-\frac{1}{2}a_{23}^2+a_{21}a_{31}-\frac{1}{2}a_{31}^2$$
$$+a_{12}a_{32}+a_{31}a_{32}-\frac{1}{2}a_{32}^2+\frac{1}{2}a_{12}+\frac{1}{2}a_{13}+\frac{1}{2}a_{21}+\frac{1}{2}a_{23}+\frac{1}{2}a_{31}+\frac{1}{2}a_{32}+1$$
$$= \operatorname{vol}_{Q(\mathbf{a})}(\mathbf{a})+\sum_{i\neq j\in[3]}\frac{a_{ij}}{2}+1.$$

Hence, for integral Kleene stars $\mathbf{c}\in\mathbb{Z}^6$ (i.e. whenever $Q(\mathbf{c})$ is unimodular), we get that

$$\#(Q(\mathbf{c})\cap\mathbb{Z}^2)=\operatorname{vol}_{Q(\mathbf{c})}(\mathbf{c})+\sum_{i\neq j\in[3]}\frac{c_{ij}}{2}+1.$$

Note that this implies that $\sum_{i\neq j\in[3]}\frac{c_{ij}}{2}$ is the number of lattice points on the boundary of $Q(\mathbf{c})$. Evaluating this polynomial at the weight vector $\mathbf{c}=(3,2,3,4,5,6)$ gives 52, which is the number of lattice points in the polytrope. Evaluating at $t\mathbf{c}=(3t,2t,3t,4t,5t,6t)$ gives the univariate Ehrhart polynomial of the polytrope $Q(\mathbf{c})$:

$$\operatorname{ehr}_{Q(\mathbf{c})}(t)=\frac{79}{2}t^2+\frac{23}{2}t+1.$$

3.2.3 *Computing multivariate* h*-*polynomials*

Finally, we can also compute a multivariate h*-polynomial from the multivariate Ehrhart polynomial corresponding to each tropical type. We explain the method here. The interested reader can also consult [12] for further details.

As discussed in 2, the coefficients $\{\mathrm{h}^*_{\mathrm{i}}\}$ of the h*-polynomial $\mathrm{h}^*(t)=\mathrm{h}^*\,0+\mathrm{h}^*_1\,t+\cdots+\mathrm{h}^*_{\mathrm{d}}\,t^d$ are the coefficients of the Ehrhart polynomial expressed in the basis $\left\{\binom{t+d-i}{d}\mid i\in\{0,1,\ldots,d\}\right\}$ of the vector space of polynomials in t of degree at most d. To transform the Ehrhart polynomial to the h*-polynomial, we perform a change of basis. The Eulerian polynomials play a central role in this transformation.

The Eulerian polynomials $A_d(t)$ are defined through the generating function:

$$\sum_{j\geq 0} j^d t^j=\frac{A_d(t)}{(1-t)^{d+1}}.$$

Explicitly, we can write the Eulerian polynomials as

$$A_d(t) = \sum_{m=1}^{d} A(d, m-1)t^m,$$

where $A(d,m)$ is the Eulerian number that counts the number of permutations of $[d]$ with exactly m ascents. The first few Eulerian polynomials are $A_0(t) = 1, A_1(t) = t$, and $A_2(t) = t^2 + t$. Recall the Ehrhart series of a d-dimensional polytope:

$$\mathrm{Ehr}_P(t) = \sum_{k\geq 0} \mathrm{ehr}_P(k)t^k = \sum_{k\geq 0}(\lambda_0 + \lambda_1 k + \cdots + \lambda_d k^d)t^k = \sum_{i=0}^{d} \frac{\lambda_i A_i(t)}{(1-t)^{i+1}}.$$

On the other hand, we have

$$\mathrm{Ehr}_P(t) = \frac{\mathrm{h^*}_P(t)}{(1-t)^{d+1}}.$$

Comparing yields an expression for the h*-polynomial in terms of the coefficients of the Ehrhart polynomial:

$$\mathrm{h^*}_P(t) = \sum_{i=0}^{d} \lambda_i A_i(t)(1-t)^{d-i}.$$

To compute the multivariate h*-polynomials, we collect the terms of each degree in the Ehrhart polynomials and apply the transformation.

Example 3.2.10. We compute the multivariate h*-polynomial of the hexagon $Q(\mathbf{a})$ from the Ehrhart polynomial $\mathrm{ehr}_{Q(\mathbf{a})}(t\mathbf{a}) = \lambda_2 t^2 + \lambda_1 t + 1$ from 3.2.9. With these coefficients we can compute

$$\begin{aligned}
\lambda_2 A_2(t)(1-t)^0 =& \Bigg(\sum_{i\neq j\in[3]} -\frac{1}{2}a_{ij}^2 + \sum_{i\neq j\neq k\in[3]} [a_{ij}a_{ik} + a_{ji}a_{ki}]\Bigg)(t^2+t)\\
\lambda_1 A_1(t)(1-t)^1 =& \Big(\sum_{i\neq j\in[3]} \frac{1}{2}a_{ij}\Big)(-t^2+t)\\
\lambda_0 A_0(t)(1-t)^2 =& t^2 - 2t + 1.
\end{aligned}$$

The sum of these three polynomials gives the multivariate h*-polynomial of the hexagon:

$$\begin{aligned}
\mathrm{h^*}_{Q(\mathbf{a})}(\mathbf{a}, t) =& \Bigg(\sum_{i\neq j\in[3]} -\frac{1}{2}[a_{ij}^2 + a_{ij}] + \sum_{i\neq j\neq k\in[3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] + 1\Bigg)t^2\\
&+ \Bigg(\sum_{i\neq j\in[3]} \frac{1}{2}[a_{ij} - a_{ij}^2] + \sum_{i\neq j\neq k\in[3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] - 2\Bigg)t + 1.
\end{aligned}$$

Evaluating $h^*_{Q(\mathbf{a})}(\mathbf{a},t)$ at $(\mathbf{c},t) = (3,2,3,4,5,6,t)$ yields the univariate h*-polynomial of the hexagon $Q(\mathbf{c})$ from 3.2.5:

$$\mathrm{h^*}_{Q(\mathbf{c})}(\mathbf{c},t) = 29t^2 + 49t + 1.$$

The coefficients of $\mathrm{h^*}_{Q(\mathbf{c})}(\mathbf{c},t)$ sum to 79, which equals the normalized volume of $Q(\mathbf{c})$ observed previously in 3.2.1.

3.3 EXPERIMENTS AND OBSERVATIONS

In this section we describe the results of our application of Section 3.2 for maximal polytropes of dimension at most 4. Since the Ehrhart and h*-polynomials are computed from the volume polynomials, we mainly focus our investigation on the volume polynomials. All scripts and results of our computations can be found at

https://github.com/mariebrandenburg/polynomials-of-polytropes

and will be made accessible in `Polymake` through the polytope database

https://polydb.org.

3.3.1 *Data and computation*

In the computation that is described in this section, we used data from [72] containing the vertices of one polytrope for each maximal tropical type of dimension 3 and 4 up to the action of the symmetric group. The vertices of each polytrope were arranged to form a Kleene star and corresponding weight vector $\mathbf{c}$. The methods described in Section 3.2 were then applied to obtain multivariate volume, Ehrhart, and h*-polynomials for the corresponding tropical type. Our computations were performed on a desktop computer with a 3.6 GHz quad-core processor. On average, the running time was about 5 minutes for each 4-dimensional volume polynomial, 0.15 seconds for each Ehrhart polynomial, and 0.73 seconds for each h*-polynomial. Parallelization is possible as the computations are independent for each tropical type.

In order to verify our computational results, we independently computed the univariate volume and Ehrhart polynomials with respect to our input data and compared them with our multivariate results, as explained in 3.2.5, 3.2.9, 3.2.10. To check the h*-polynomial of a representative polytrope, we attempted to compute its h*-polynomial by computing its Ehrhart series with `Normaliz` and compared this with our multivariate h*-polynomial evaluated at the corresponding weight vector. We attempted to perform this check on a cluster, capping the `Normaliz` computation of each polytrope's Ehrhart series at 10 minutes. We checked 1459 polytropes. For 670 of them, the `Normaliz` computation finished in under 10 minutes, and the h*-polynomials matched. Checking the `Normaliz` computation for individual polytropes revealed that the Ehrhart series computation could take as long as 12 hours, in comparison to the 5 minutes required by our methods.

3.3.2 *2-dimensional polytropes*

First we consider 2-dimensional polytropes. As noted in 3.1.2, there is a unique class of maximal polytropes up to permutation of vertex labels. The volume, Ehrhart, and h*-polynomials are computed in Examples 3.2.5, 3.2.9, and 3.2.10 respectively. We note that the volume, Ehrhart, and h*-polynomials are all symmetric with respect to the S_3 action, as expected.

3.3.3 *3-dimensional polytropes*

In the case of maximal 3-dimensional polytropes, up to the symmetric group action there are 6 types of maximal polytropes. We applied the algorithms in 3.2 to nonnegative points in maximal cones corresponding to these 6 types, yielding the volume, Ehrhart, and h*-polynomials of their corresponding tropical types.

Example 3.3.1. One of the six volume polynomials is

$$\begin{aligned}
&2a_{12}^3 - 3a_{12}^2a_{13} + a_{13}^3 - 3a_{12}^2a_{14} + 6a_{12}a_{13}a_{14} - 3a_{13}^2a_{14} + a_{21}^3 - 3a_{13}^2a_{23} + 6a_{13}a_{14}a_{23} - 3a_{14}^2a_{23} \\
&- 3a_{14}a_{23}^2 - 3a_{21}a_{23}^2 + a_{23}^3 - 3a_{21}^2a_{24} + 6a_{14}a_{23}a_{24} + 6a_{21}a_{23}a_{24} - 3a_{14}a_{24}^2 - 3a_{23}a_{24}^2 + a_{24}^3 \\
&- 3a_{21}^2a_{31} + 6a_{21}a_{24}a_{31} - 3a_{24}^2a_{31} - 3a_{24}a_{31}^2 + a_{31}^3 - 3a_{12}^2a_{32} + 6a_{12}a_{14}a_{32} - 3a_{14}^2a_{32} - 3a_{31}^2a_{32} \\
&- 3a_{14}a_{32}^2 + 6a_{14}a_{24}a_{34} + 6a_{24}a_{31}a_{34} + 6a_{14}a_{32}a_{34} + 6a_{31}a_{32}a_{34} - 3a_{14}a_{34}^2 - 3a_{24}a_{34}^2 - 3a_{31}a_{34}^2 \\
&- 3a_{32}a_{34}^2 + 2a_{34}^3 + 6a_{21}a_{31}a_{41} - 3a_{31}^2a_{41} + 6a_{31}a_{32}a_{41} - 3a_{32}^2a_{41} - 3a_{21}a_{41}^2 - 3a_{32}a_{41}^2 + a_{41}^3 \\
&- 3a_{12}^2a_{42} + 6a_{12}a_{13}a_{42} - 3a_{13}^2a_{42} + 6a_{12}a_{32}a_{42} + 6a_{32}a_{41}a_{42} - 3a_{13}a_{42}^2 - 3a_{32}a_{42}^2 - 3a_{41}a_{42}^2 \\
&+ a_{42}^3 - 3a_{21}^2a_{43} + 6a_{13}a_{23}a_{43} + 6a_{21}a_{23}a_{43} - 3a_{23}^2a_{43} + 6a_{21}a_{41}a_{43} - 3a_{41}^2a_{43} + 6a_{13}a_{42}a_{43} \\
&+ 6a_{41}a_{42}a_{43} - 3a_{13}a_{43}^2 - 3a_{21}a_{43}^2 - 3a_{42}a_{43}^2 + a_{43}^3.
\end{aligned}$$

We devote the remainder of this subsection to an analysis of the coefficients of the normalized volume polynomials, which we write as follows:

$$\mathrm{Vol}(\{\mathbf{x} \in \mathbb{R}^4 \mid x_i - x_j \leq a_{ij}, x_4 = 0\}) = \sum_{\mathbf{v}} \alpha_{\mathbf{v}} \mathbf{a}^{\mathbf{v}},$$

where $\mathbf{v} \in \mathbb{N}^{12}$ has coordinates summing to 3. Note that there is a natural decomposition of the set of all possible exponent vectors $\mathbf{v}$ into three different disjoint subsets T_{111}, T_{21}, and T_3, one for each partition of 3.

Recall that the 6 types of maximal 3-dimensional polytropes correspond to different regular central triangulations of the fundamental polytope FP_4, as discussed in 3.1.2. A regular central triangulation is determined by a choice of triangulating edge in each of the six square facets of FP_4. The coefficients of the volume polynomials encode the data of these six facet triangulations as follows:

- Let $\mathbf{v} \in T_{111}$, so that the monomial $\mathbf{a}^{\mathbf{v}}$ is $a_{ij}a_{kl}a_{st}$ for some $i \neq j, k \neq l, s \neq t$ and $(i,j) \neq (k,l) \neq (s,t)$. The coefficients $\alpha_{\mathbf{v}}$ in this case are determined directly by the triangulation of FP_4:

$$\alpha_{\mathbf{v}} = \begin{cases} 6 & \text{if } \mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_k - \mathbf{e}_l, \mathbf{e}_s - \mathbf{e}_t \text{ form a 2-dimensional simplex in the} \\ & \text{corresponding regular central triangulation,} \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\mathbf{v} \in T_{21}$, so that the monomial $\mathbf{a}^{\mathbf{v}}$ is $a_{ij}^2 a_{kl}$ for some $i \neq j$, $k \neq l$, and $(i,j) \neq (k,l)$. The coefficient $\alpha_{\mathbf{v}}$ is nonzero only if $\mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{e}_k - \mathbf{e}_l$ are adjacent vertices of FP_4. In that case, it is determined by the square facet S of FP_4 containing $\mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{e}_k - \mathbf{e}_l$:

$$\alpha_{\mathbf{v}} = \begin{cases} -3 & \text{if } \mathbf{e}_k - \mathbf{e}_l \text{ incident to triangulating edge of } S \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\mathbf{v} \in T_3$, so that the monomial $\mathbf{a}^{\mathbf{v}}$ is a_{ij}^3 for some $i \neq j$. The coefficient $\alpha_{\mathbf{v}}$ is given by

$$\alpha_{\mathbf{v}} = 7 - \deg(\mathbf{e}_i - \mathbf{e}_j),$$

  where $\deg(\mathbf{e}_i - \mathbf{e}_j)$ is the number of edges incident to the vertex $\mathbf{e}_i - \mathbf{e}_j$ in the regular central subdivision of FP_4.

We note that the above descriptions of the coefficients of the volume polynomial imply that the sums of coefficients corresponding to each partition of 3 are the same for all six volume polynomials:

$$\sum_{\mathbf{v} \in T_3} \alpha_{\mathbf{v}} = 12, \quad \sum_{\mathbf{v} \in T_{21}} \alpha_{\mathbf{v}} = -108, \quad \sum_{\mathbf{v} \in T_{111}} \alpha_{\mathbf{v}} = 120.$$

Example 3.3.2. Consider the polytrope P with facet coefficients c_{ij} given by the matrix

$$\begin{pmatrix} 0 & 11 & 20 & 29 \\ 21 & 0 & 19 & 20 \\ 20 & 29 & 0 & 11 \\ 19 & 20 & 21 & 0 \end{pmatrix}.$$

Assigning the weight c_{ij} to the vertex $\mathbf{e}_i - \mathbf{e}_j$ of the fundamental polytope FP_4, and weight 0 to the central vertex at the origin, produces the regular central triangulation in 15. The volume polynomial corresponding to this polytrope is the polynomial displayed in 3.3.1. We see that the coefficients corresponding to $a_{12}^3, a_{12}^2 a_{14}, a_{32}^2 a_{42}$, and $a_{31}a_{32}a_{41}$ are equal to $2, -3, 0$, and 6 respectively, as summarized by the discussion above.

3.3.4 *4-dimensional polytropes*

Finally we consider 4-dimensional polytropes. In this case, up to the action of the symmetric group S_5 there are 27248 types of maximal polytropes. We applied the methods of 3.2 to obtain multivariate volume, Ehrhart, and h*-polynomials for these polytropes.

We can embed the 27248 normalized volume polynomials using the canonical basis in the vector space of homogeneous polynomials of degree 4, having dimension $\binom{23}{4} = 8855$. The affine span of these volume polynomials has dimension 70, implying that there is much structure in their coefficients. We note that this equals the number of facets in a regular central triangulation of FP_5.

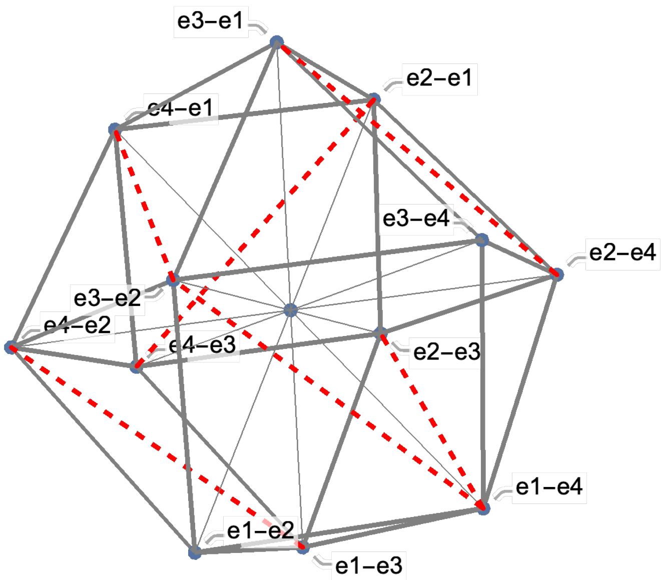


Figure 15: The regular central triangulation of FP_4 corresponding to the polytrope in 3.3.2, with triangulating edges of square facets of FP_4 colored red.

We were able to experimentally verify the facts collected in 1. For example, all coefficients for monomials corresponding to the partition $2+2=4$ lie in the set $\{0,6\}$, and the sum of all such coefficients is 300. Furthermore, the S_5-orbit of the monomials $a_{12}a_{13}a_{14}a_{15}$ and $a_{21}a_{31}a_{41}a_{51}$ always appears in the volume polynomial with coefficient 24. Finally, the coefficient -4 always appears exactly twice as often as the coefficient 12.

Partition	Example monomial	Possible coefficients	Coefficient sum
4	a_{12}^4	$-6,-3,-2,-1,0,1,2,3$	-20
3 + 1	$a_{12}^3a_{13}$	$-4,0,4,8$	320
2 + 2	$a_{12}^2a_{13}^2$	$0,6$	300
2+1+1	$a_{12}a_{13}a_{14}^2$	$-12,0,12$	-2160
1+1+1+1	$a_{12}a_{13}a_{14}a_{15}$	$0,24$	1680

Table 1: Summary statistics for coefficients of 4-dimensional volume polynomials.

As in the 3-dimensional case, a monomial corresponding to the partition $1+1+1+1=4$ had coefficient 24 if and only if it appeared as a face in the corresponding triangulation. Beyond these observations, we were unable to detail the exact relationship between the volume polynomials and their corresponding regular central triangulations.

Question 3.3.3. How do the coefficients of the volume polynomials of maximal $(n-1)$-dimensional polytropes reflect the combinatorics of the corresponding regular central subdivision of FP_n?

A natural first step would be to prove that, for $\mathbf{v}$ with partition $1+1+\cdots+1 = n-1$, the coefficient $\alpha_{\mathbf{v}}$ is nonzero if and only if it corresponds to a face in the regular central triangulation.

Part II

SIMPLICIAL HYPERPLANE ARRANGEMENTS

O the grey dull day! It seemed a limbo of painless patient consciousness through which souls of mathematicians might wander, projecting long slender fabrics from plane to plane of ever rarer and paler twilight, radiating swift eddies to the last verges of a universe ever vaster, farther and more impalpable.

- James Joyce, *A Portrait of the Artist as a Young Man*

The first part of this dissertation focuses on the bounded, convex regions, i.e. polytopes, determined by a finite set of real hyperplanes. We have seen what happens to the discrete volumes of these polytopes when we shift the hyperplanes, either uniformly by a rational or real factor in Chapter 2, or independently in Chapter 3. Now, we zoom out and study the entire hyperplane arrangement instead of a specific bounded region. In order to facilitate our study, we choose to look at *central, simplicial arrangements*, which were introduced by Melchior in 1940, [80]. The systematic study of simplicial hyperplane arrangements began with one of the great geometers of the 20th century, Branko Grünbaum, who enumerated them along with regular polytopes, tilings, and patterns. His catalogue [61] of simplicial arrangements of rank 3 (arrangements of 2-dimensional planes in 3-dimensional space), appeared in 1971. It included three infinite families and 90 sporadic arrangements.

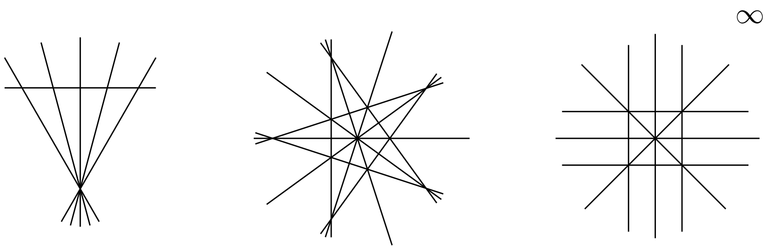


Figure 16: Arrangements from the three infinite families of simplicial arrangements of rank 3 drawn in the projective plane

Since then, the list has grown to include 95 sporadic arrangements (see [62], [34]), and it is still open to determine if the list is complete, or indeed, whether there is a finite list at all. In order to answer these questions, it is natural to search for geometric, combinatorial, and algebraic structures lurking behind the list. Luckily, this search has a history, and there are already many tools at our disposal for studying hyperplane arrangements.

Tools from Combinatorics: One method for extracting combinatorial information from a hyperplane arrangement is to associate to it a partially ordered set, or poset. For example, this was done with great success by Zaslavsky in his doctoral dissertation in 1974 [106]; he used the intersection poset of a hyperplane arrangement and Möbius inversion to show that the number of regions of an arrangement $\mathcal{A}$ is $(-1)^n \chi_{\mathcal{A}}(-1)$, where $\chi_{\mathcal{A}}$ is the characteristic polynomial of the arrangement. Skipping ahead to 1984, Paul Edelman introduced a different partial order on hyperplane arrangements called the *poset of regions* and gave an expression for its Möbius function [50]. Björner, Edelman, and Ziegler showed in 1990 that if the poset of regions is a lattice, then the base region must be simplicial, but the converse does not hold in general [17]. They also showed that the poset of regions of any simplicial arrangement is a lattice. Nathan Reading later showed that simpliciality can be weakened to *tightness*—which is a connectivity condition on facets of regions—in order to obtain lattices [88, Chapter 9]. Today, posets continue to be useful tools for gaining new insights on hyperplane arrangements. In [46], Dermenjian, Hohlweg, and Pilaud introduce the facial weak order and show that if the poset of regions is a lattice, then the facial weak order is a lattice. One of their definitions for the facial weak order depends on another useful combinatorial object associated to a hyperplane arrangement, its *oriented matroid*. In Chapter 4, we continue to use posets of regions and oriented matroids to study simplicial hyperplane arrangements.

Tools from Algebra: It turns out that finite Weyl groupoids provide an algebraic justification for around half of the sporadic arrangements from Grünbaum's list of rank 3 simplicial arrangements. Finite Weyl groupoids are algebraic structures generalizing Weyl groups that were introduced by Heckenberger in 2005 [63] to better understand the symmetries of Nichols algebras and related Hopf algebras. They were further studied by Heckenberger and Volkmar Welker [64] and classified by Heckenberger and Michael Cuntz in the 2000s-2010s [35–37]. Each Weyl groupoid originates from the data of a "Cartan graph", leading to a so-called "root system". In turn, these root systems generalize the usual notion of root system of a Weyl group. Notably, they form the set of normals of certain simplicial hyperplane arrangements. Finite Weyl groupoids of rank 3 account for 53 simplicial arrangements [31]. Cuntz also proved that the list is complete for arrangements with up to 27 lines [32] and found a new arrangement with 35 hyperplanes in 2020 using a greedy algorithm [33]. In Chapter 4, we show that the simplicial arrangements coming from finite Weyl groupoids have a nice combinatorial property: their posets of regions are always congruence normal. It is possible that this theorem could lead to new algebraic insights on Weyl groupoids and classification results for simplicial hyperplane arrangements.

4 CONGRUENCE NORMALITY OF SIMPLICIAL HYPERPLANE ARRANGEMENTS

In this chapter, we continue to search for combinatorial structure regulating the known list of simplicial hyperplane arrangements of rank 3. Simplicial arrangements through their lattices of regions provide generalizations of the weak order of finite Coxeter groups. Unlike in the Coxeter case, a simplicial arrangement may lead to several non-isomorphic lattices of regions. Apart from being lattices, much less is known about the poset of regions of simplicial arrangements. We investigate which posets of regions of simplicial arrangements possess the property of congruence normality.

Our motivation for looking at congruence normality stems from the study of lattice congruences. Lattice congruences of the weak order of Coxeter arrangements generate several objects of interest. For example, the permutahedron is perhaps the most studied example of a simple zonotope, that comes from the braid arrangement, or Coxeter arrangement of type A. The corresponding poset of regions is the weak order of the symmetric group and is a lattice. Moreover, Tamari and Cambrian lattices, generalized permutahedra, and associahedra are all related to lattice congruences of the weak order [67, 83, 87]. In particular, in type A and B, every lattice congruence leads to a polytope [81, 82]. To which extent do these constructions extend to general simplicial arrangements? We focus here on two important properties used to study lattice congruences and shard polytopes: *congruence normality* and *congruence uniformity*. Coxeter arrangements are congruence normal and uniform [26]. Congruence uniform lattices admit a bijection between their join-irreducible elements and the join-irreducible elements in the lattice of lattice congruences. Congruence uniform lattices are thus particularly nice lattices as they allow one to more easily study the lattice of congruences. Reading characterized congruence uniformity of posets of regions using *tightness* and *shards* (i.e. pieces of hyperplanes) [88, Corollary 9-7.22]. Reading also showed that supersolvable hyperplane arrangements have congruence uniform posets of regions for some canonical choice of base region [85]. Congruence uniform lattices admit a combinatorial construction whose geometric aspects in this context have yet to be explored in detail.

In this chapter, we determine congruence uniformity and normality of posets of regions of simplicial hyperplane arrangements of rank 3 and draw several conclusions. To do so, we approach posets of regions through the oriented matroids naturally associated to the normals of the hyperplane arrangements. We use covectors and the *intersection* operation as our main tools to elevate Reading's characterization of congruence uniformity to the level of oriented matroids (see Theorem 4.2.18 and Corollary 4.2.19). Namely, we introduce *shard covectors*—which are covectors with some "$*$" entries—and show they are in bijection with shards (see Theorem 4.2.12).

This approach led to the following results. The posets of regions of hyperplane arrangements coming from finite Weyl groupoids are always congruence normal and congruence uniform (see Theorem 4.3.2). This result provides a new proof that finite

Coxeter arrangements are obtainable through a finite sequence of interval doublings (i.e. congruence uniform) [26, Theorem 6]. We further classify the known rank-3 simplicial arrangements according to whether their posets of regions are always or sometimes or never congruence normal depending on the base region (see Table 2). The approach through covectors gives a way to determine congruence normality of posets of regions without the data of the poset or resorting to polyhedral objects (i.e. shards). Notably, this classification could not have been carried out through the computation of the posets of regions due to their large size. Hence, this framework provides an oriented matroid approach to study congruence normality and uniformity for large posets of regions. As an interesting outcome of this classification, five arrangements have exceptional behavior. Two of the five arrangements are always congruence normal: the non-crystallographic arrangement corresponding to the Coxeter group H_3 and its point-line dual arrangement which has 31 hyperplanes. The three other arrangements are never congruence normal: they have yet to show any connection to other known structures. Furthermore, we provide instructive examples which give deeper insight into congruence uniformity for posets of regions. We verified that within supersolvable simplicial arrangements (by [38, Theorem 1.2] these are the arrangements in 2 of the 3 infinite families) only four are always congruence normal and all others are only sometimes congruence normal, see Theorems 4.3.5 and 4.3.6. The algorithms used to carry out the verifications and the data to construct known simplicial hyperplane arrangements are available as a `Sagemath`-package [27]. The construction of this package is another contribution of this chapter, as it allows the mathematical community to easily access the current known list of simplicial hyperplane arrangements of rank 3 in a form that that is ready to use for computations. This chapter is part of the published article [34] which also includes a list of normal vectors to the known simplicial arrangements of rank 3 along with invariants and wiring diagrams. The gathering of this list of normals and the computation of the invariants and wiring diagrams is the work of Michael Cuntz. For the sake of brevity the lists of normal, invariants, and wiring diagrams are not included in this chapter as the data comprises many pages.

The chapter is structured as follows. In Section 4.1, we present the necessary background notions of congruence normality and uniformity and the theory of shards. In Section 4.2, we recast shards and the forcing relation using covectors. In Section 4.3, we present the result of the application of the approach of Section 4.2 to the known rank-3 simplicial hyperplane arrangements.

This chapter is part of joint work with Jean-Philippe Labbé and Michael Cuntz published in Annals of Combinatorics [34].

4.1 PRELIMINARIES ON LATTICES AND THEIR CONGRUENCES

In Section 4.1.1, we review the notion of a lattice congruence. In Section 4.1.2, we define hyperplane arrangements and posets of regions. In Sections 4.1.3 and 4.1.4, we discuss the notions of congruence normality and uniformity. Finally, in Section 4.1.5, we describe Reading's characterization of congruence uniformity for tight hyperplane arrangements using *shards*. The material presented in this section is mostly based on material treated in the book chapter [88, Chapter 9].

4.1.1 *Lattice congruences*

For ease of reading, in this chapter, the relative interior of a subset $\mathbf{P}$ of $\mathbb{R}^d$ is denoted by $\mathrm{int}(\mathbf{P})$. Let $L = (P; \wedge, \vee)$ be a finite lattice, where P is a poset $(P, \leq)$. An element $j \in L$ is **join-irreducible** if j covers a unique element $j_\bullet \in L$. Similarly, an element $m \in L$ is **meet-irreducible** if m is covered by a unique element $m^\bullet \in L$. We denote the subposet of join-irreducible elements of a lattice L by $L_\vee$ and the subposet of meet-irreducible elements by $L_\wedge$. An **order ideal** of a poset P is a subposet $Q \subseteq P$ that satisfies $x \in Q$ and $y \leq x \Rightarrow y \in Q$. The order ideals of a poset P can be ordered by containment to get the **poset of order ideals** denoted $\mathcal{O}(P)$. When L is self-dual, join- and meet-irreducible elements are canonically in bijection. The dual map therefore allows one to refine statements involving L and its irreducible elements. Join-irreducible elements (and dually meet-irreducible elements) and posets of order ideals are very useful to understand finite distributive lattices.

Lemma 4.1.1 ([16, Theorem 17.3])**.** *Let L be a lattice, $L_\vee$ be its subposet of join-irreducible elements, and $\mathcal{O}(L_\vee)$ be the poset of order ideals of $L_\vee$. If L is finite and distributive, then L is isomorphic to $\mathcal{O}(L_\vee)$.*

Recall that cosets of a normal subgroup $N \trianglelefteq G$ determine a *congruence* relation, and lead to a quotient group G/N, which is the image of the map sending an element to its coset. Analogously, in lattice theory, intervals play the role of cosets, and under certain conditions, they form a *quotient lattice*. In this case, the equivalence relation is called a lattice congruence. For a thorough discussion on congruences and quotient lattices, we refer the reader to [88, Chapter 9-5 and 9-10] and the references therein.

Definition 4.1.2 (Lattice congruence)**.** An equivalence relation $\equiv$ on the elements of a lattice L is a **lattice congruence** if $x_1 \equiv x_2$ and $y_1 \equiv y_2$ implies $x_1 \wedge y_1 \equiv x_2 \wedge y_2$ and $x_1 \vee y_1 \equiv x_2 \vee y_2$ for any elements $x_1, x_2, y_1, y_2 \in L$.

Lemma 4.1.3 (see e.g. [88, Proposition 9-5.2])**.** *An equivalence relation on a lattice is a lattice congruence if and only if the following three conditions are satisfied:*

1. *Every equivalence class is an interval.*
2. *The map $\pi_\downarrow$ sending each element to the minimal element in its equivalence class is order-preserving.*
3. *The map $\pi_\uparrow$ sending each element to the maximal element in its equivalence class is order-preserving.*

Given a lattice congruence, the images of $\pi_\downarrow$ and $\pi_\uparrow$ are sublattices, i.e. the join and meet operations are preserved on the equivalence classes, and they are referred to as **quotient lattices***.*

Our main motivation for studying the property of congruence normality comes from the study of lattice congruences. Lattice congruences of a lattice can be numerous and the relations between them may be challenging to describe. In spite of that, the set of lattice congruences on a lattice L may be partially ordered by refinement. The equivalence relation with singleton classes is the smallest lattice congruence and its associated quotient lattice is the lattice itself. Furthermore, the equivalence relation with a unique class is the coarsest lattice congruence whose associated quotient lattice has exactly one element. It turns out that under this partial order by refinement, the set of lattice congruences forms a distributive lattice which is called the **lattice of congruences** and is denoted by $\mathrm{Con}(L)$ [57]. The lattices of congruences we consider here are finite and therefore complete. Consequently, given any set of relations, there is a smallest lattice congruence which contains these relations [88, Proposition 9-5.13]. This makes it possible to define two important congruences related to join- and meet-irreducible elements. Consider a join-irreducible element $j \in L_\vee$, then there is a smallest lattice congruence $\mathrm{con}_\vee(j)$ such that j and $j_\bullet$ are equivalent. Similarly, for a meet-irreducible element m, there is a smallest lattice congruence $\mathrm{con}_\wedge(m)$ such that m and the unique element $m^\bullet$ that covers it are equivalent. In this case, we say that the congruence $\mathrm{con}_\vee$ **contracts** j, and that $\mathrm{con}_\wedge$ contracts m. As $\mathrm{Con}(L)$ is finite and distributive, we may use Lemma 4.1.1 to obtain that $\mathrm{Con}(L)$ is isomorphic to $\mathcal{O}(\mathrm{Con}(L)_\vee)$. A congruence is determined by an order ideal of join-irreducible congruences, i.e., by the join-irreducibles it contracts [88, Corollary 9-5.15].

Definition 4.1.4. Let $\mathrm{con}_\vee : L_\vee \to \mathrm{Con}(L)$ be the map that sends a join-irreducible element $j \in L_\vee$ to the smallest lattice congruence in $\mathrm{Con}(L)$ such that $j \equiv j_\bullet$. Dually, the map $\mathrm{con}_\wedge$ is similarly defined for meet-irreducible elements.

The image of the map $\mathrm{con}_\vee$ is $\mathrm{Con}(L)_\vee$, i.e., the congruence $\mathrm{con}_\vee(j)$ is join-irreducible in $\mathrm{Con}(L)$ and for every join-irreducible congruence α in $\mathrm{Con}(L)$, there exists a join-irreducible $j \in L_\vee$ such that $\mathrm{con}_\vee(j) = \alpha$ [88, Proposition 9-5.14]. It may happen that two distinct join-irreducibles give rise to the same congruence, i.e. that $\mathrm{con}_\vee$ is not injective, leading to an equivalence relation on join-irreducible elements in $L_\vee$. Through the map $\mathrm{con}_\vee$, these equivalence classes of join-irreducible elements in L are in bijection with join-irreducible congruences of L.

4.1.2 *Poset of regions of a real hyperplane arrangement*

Let $\mathcal{A}$ be a real, finite, central hyperplane arrangement. We denote the hyperplanes in $\mathcal{A}$ by $H_1, \dots, H_m$ and often reuse their indices to refer to objects canonically related to them. The **rank** of $\mathcal{A}$ is the dimension of the linear span of the normal vectors of the hyperplanes in $\mathcal{A}$. The complement of the arrangement in the ambient space $(\mathbb{R}^d \setminus \bigcup_{i\in[m]} H_i)$ is disconnected, and the closures of the connected components are the **regions** of the arrangement. The set of regions of $\mathcal{A}$ is denoted by $\mathcal{R}(\mathcal{A})$. A region is called **simplicial** if the normal vectors of its facet-defining hyperplanes are linearly independent. A hyperplane arrangement is **simplicial** if every region in its complement is simplicial. Throughout, we use the notation $\mathcal{A}(m, r)_i$ to denote the i-th simplicial hyperplane arrangement with m hyperplanes and r regions from our catalogue of simplicial arrangements, which is given in its totality in [34]. Additionally, we denote by $\mathcal{F}_i(m)$ the arrangement in the i-th infinite family with m hyperplanes, see Section 4.3.3.

To proceed further, a **base region** B of $\mathcal{A}$ is chosen. For each hyperplane $H_i \in \mathcal{A}$, we fix a normal vector $\mathbf{n}_i \in \mathbb{R}^d$ such that $\mathbf{n}_i \cdot \mathbf{x} < 0$, for all $\mathbf{x} \in B$. Given a region R

of $\mathcal{A}$, the **separating set** $\mathrm{Sep}_B(R)$ of R is the set of hyperplanes $H_i \in \mathcal{A}$ such that $\mathbf{n}_i \cdot \mathbf{x} > 0$, for all $\mathbf{x} \in R$. The separating set of a region is the set of hyperplanes that separate it from the base region B.

Definition 4.1.5 (Poset of regions, $P_B(\mathcal{A})$)**.** Let $\mathcal{A}$ be a hyperplane arrangement with base region B. The **poset of regions** $P_B(\mathcal{A})$ of $\mathcal{A}$ with base region B is the partially ordered set $(\mathcal{R}(\mathcal{A}), \leq)$ such that

$$R_1 \leq R_2 \text{ if and only if } \mathrm{Sep}_B(R_1) \subseteq \mathrm{Sep}_B(R_2),$$

for all $R_1, R_2 \in \mathcal{R}(\mathcal{A})$.

An **upper facet** of a region $R \in \mathcal{R}(\mathcal{A})$ is a facet of R which corresponds to a cover relation of R in $P_B(\mathcal{A})$. A hyperplane arrangement is **tight** with respect to B when the upper facets of every region intersect pairwise along a codimension-2 face, i.e. they are neighbors in the facet-adjacency graph. When a hyperplane arrangement $\mathcal{A}$ is tight with respect to every base region, we say that $\mathcal{A}$ is tight. For convenience, when a hyperplane arrangement is tight, we also call the corresponding posets of regions tight. The usual definition of tightness also requires the dual statement to hold. As posets of regions are self-dual, we have restricted the statement to upper facets. The following lemma is a refinement of [17, Theorem 3.4].

Lemma 4.1.6. *Let $\mathcal{A}$ be a finite, central hyperplane arrangement with base region B.*

1. *If $\mathcal{A}$ is tight with respect to B, then $P_B(\mathcal{A})$ is a lattice. [88, Theorem 9-3.2]*
2. *If $\mathcal{A}$ is simplicial, then $\mathcal{A}$ is tight. [88, Proposition 9-3.3]*

Reading developed an approach to study congruences of lattices of regions that is thoroughly described in [88, Chapter 9]. In particular, for posets of regions, tightness is equivalent to *semidistributivity* [88, Theorem 9-3.8] (see Section 4.1.4 for the definition of semidistributivity). Furthermore, in order to describe the interplay between join-irreducible elements, the combinatorial notion of "polygonality" of a lattice is used; in the case of posets of regions, this notion is equivalent to the notion of tightness [88, Theorem 9-6.10]. Using the polygonality property, it is possible to describe which join-irreducibles *force* other ones to be contracted. This forcing relation can then be read off from the hyperplane arrangement using pieces of hyperplanes called *shards* (see Definition 4.1.15 in Section 4.1.5). The interest in the notion of tightness lies in the fact that being tight and having *acyclicity* on shards characterizes congruence uniformity, see Theorem 4.1.19 in Section 4.1.5.

Throughout this chapter, we restrict our study to finite, central, and tight hyperplane arrangements, so that the posets of regions are guaranteed to be complete lattices regardless of the choice of base regions. We refer the reader to [88, Chapter 9-3, 9–6] for further details on tightness and polygonality.

4.1.3 *Congruence normality*

Definition 4.1.7 (Congruence normality, [42, Section 1, p.400])**.** Let L be a lattice, $L_\vee \subseteq L$ be the subposet of join-irreducible elements of L, and $L_\wedge$ be the subposet of meet-irreducible elements of L. The lattice L is **congruence normal** if

$$j \leq m \quad \text{implies} \quad \operatorname{con}_\vee(j) \neq \operatorname{con}_\wedge(m),$$

for all $j \in L_\vee$, and $m \in L_\wedge$. A hyperplane arrangement is called **congruence normal** if its lattices of regions are congruence normal for every choice of base region.

Equivalently, finite congruence normal lattices are exactly the lattices obtained from a one-element lattice by a sequence of *doublings* of *convex sets* [42, Section 3], see also [1, Theorem 3-2.39] and [59]. The following example illustrates a local condition showing how a lattice may fail to be congruence normal.

Example 4.1.8. Consider the lattice L_3 with the Hasse diagram illustrated in Figure 17. The element c is join-irreducible, and the smallest congruence $\operatorname{con}_\vee(c)$ such that $b \equiv c$ is illustrated on the right-hand side.

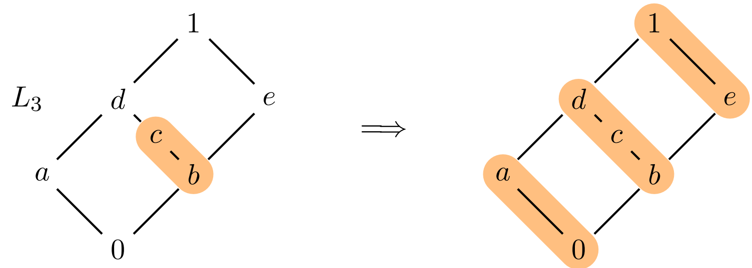


Figure 17: The Hasse diagram of the lattice L_3 which is not congruence normal and the equivalence classes of $\operatorname{con}_\vee(c) = \operatorname{con}_\wedge(c)$.

Following Definition 4.1.2, setting $b \equiv c$ forces the lattice to project onto a three-element chain. By order-reversing symmetry, the smallest congruence such that $c \equiv d$ is the same as the smallest congruence such that $b \equiv c$. Since c is also meet-irreducible, we get $\operatorname{con}_\wedge(c) = \operatorname{con}_\vee(c)$. Since $c \leq c$, the lattice L_3 is not congruence normal.

This example complements Reading's example of forcing of polygons nicely, see e.g. [88, Example 9-6.6] and the exercise on congruence normality of polygonal lattices [88, Exercice 9.55]. The intervals $[0, d]$ and $[b, 1]$ intersect on more than one cover and removing c from L_3 makes it congruence normal. Unfortunately, such local obstructions may not be used on lattices of regions of a hyperplane arrangement. The corresponding Hasse diagrams are isomorphic to the 1-skeleta of the associated zonotopes, and two polygons as in the example may not intersect along more than one cover relation for convexity reasons. As we shall see in Example 4.1.13, there are non-congruence normal lattices of regions.

4.1.4 *Congruence uniformity*

A lattice is **join-semidistributive** if for $x, y, z \in L$,

$$x \vee y = x \vee z \text{ implies } x \vee (y \wedge z) = x \vee y.$$

It is **meet-semidistributive** if

$$x \wedge y = x \wedge z \text{ implies } x \wedge (y \vee z) = x \wedge y.$$

A lattice that is both join-semidistributive and meet-semidistributive is called **semidistributive**.

Definition 4.1.9 (Congruence uniformity, [41, Definition 4.1])**.** Let L be a finite lattice. If the maps $\mathrm{con}_\vee$ and $\mathrm{con}_\wedge$ are injective, then L is called **congruence uniform**.

Congruence uniformity describes the lattice of congruences of the involved lattice through the map $\mathrm{con}_\vee$. If L is a finite congruence uniform lattice, then the map $\mathrm{con}_\vee$ gives a order-preserving bijection between $L_\vee$ and $\mathrm{Con}(L)_\vee$. Lemma 4.1.1 then permits one to study the whole of $\mathrm{Con}(L)$. Congruence uniformity is a stronger condition than congruence normality in that it should be obtained from a one-element lattice by a sequence of *doublings of intervals* [41, Theorem 5.1].

Theorem 4.1.10 ([42, Section 2])**.** *A finite lattice is congruence uniform if and only if it is both congruence normal and semidistributive.*

Corollary 4.1.11. *A tight poset of regions $P_B(\mathcal{A})$ is congruence normal if and only if it is congruence uniform.*

Proof. By Lemma 4.1.6, the poset of regions $P_B(\mathcal{A})$ of a $\mathcal{A}$ is a finite lattice, independent of the choice of base region B. Furthermore, $\mathcal{A}$ is tight with respect to B if and only if $P_B(\mathcal{A})$ is semidistributive [88, Theorem 9-3.8]. ☐

Remark 4.1.12.

1. Since lattices of regions are self-dual, it suffices to verify the injectivity of $\mathrm{con}_\vee$ to determine whether they are congruence uniform.

2. Semidistributivity can be described using sublattice avoidance [1, Theorem 3-1.4]. The six sublattices obstructing semidistributivity are illustrated in Figure 17 and 18. Four out of the six non-semidistributive lattices are not congruence normal (L_3, L_4, L_5, and M_3) and share the property that two polygons share more than 1 cover. Nevertheless, semidistributivity is neither necessary nor sufficient to obtain congruence normality: L_1 and L_2 are congruence normal but not semidistributive and Example 4.1.13 gives a poset of regions which is semidistributive but not congruence normal.

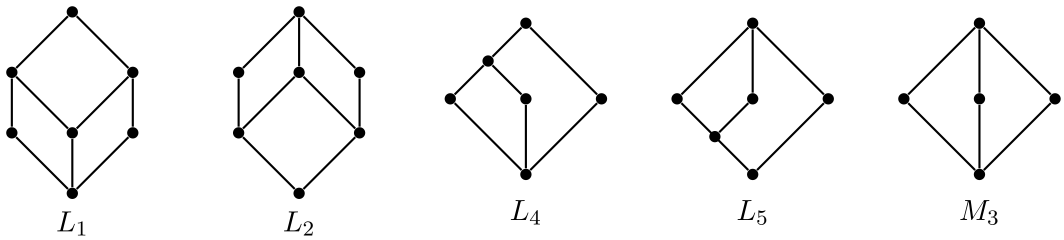


Figure 18: Five of the six sublattices that obstruct semidistributivity, the sixth is L_3 illustrated in Figure 17

3. While considering two polygons in a polygonal lattice, and verifying congruence normality as in Example 4.1.8, one realizes that M_3, L_3, L_4, and L_5 should be avoided. For poset of regions, this comes as no surprise as polygonality, tightness and semidistributivity are equivalent [88, Theorem 9-3.8 and 9-6.10]. In general, one might be tempted to ask, what is the relationship between polygonal and semidistributive lattices?

Example 4.1.13 ([85, Figure 5] and [88, Exercise 9.69])**.** Figure 19 illustrates the stereographic projection of the simplicial hyperplane arrangement $\mathscr{A}(10,60)_3$ in $\mathbb{R}^3$ with 10 hyperplanes through the intersection of 5 hyperplanes which are mapped to lines. This arrangement is $\mathcal{A}(10,1)$ in Grünbaum's list [62, p.2-3], see Section 4.3.

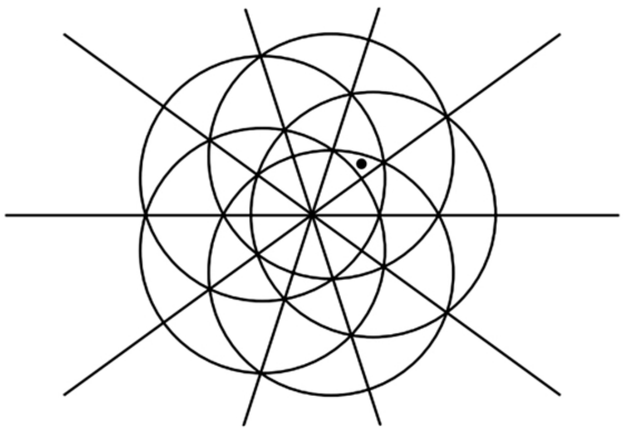


Figure 19: The simplicial hyperplane arrangement $\mathscr{A}(10,60)_3 = \mathscr{F}_2(10)$ whose lattice of regions with the marked base region is not congruence normal

The lattice of regions with respect to the base region marked by a black dot is semidistributive as the arrangement is simplicial. In Example 4.1.20, we use shards to demonstrate that this arrangement is not congruence normal, hence not uniform by Corollary 4.1.11. It is the smallest known simplicial hyperplane arrangement of rank three with that property.

Examples 4.1.8 and 4.1.13 illustrate failures to be congruence normal. Example 4.1.13 is particularly interesting in that it does not fail to be congruence normal because of forbidden sublattices blocking semidistributivity.

4.1.5 *Congruence normality of simplicial hyperplane arrangements through shards*

Reading characterized congruence uniformity of posets of regions via two conditions, the first one is tightness and the second is phrased using pieces of hyperplanes called *shards*. When the arrangement is central, these pieces are polyhedral cones defined through certain subarrangements.

Definition 4.1.14 (Rank-2 subarrangements and their basic hyperplanes, see [88, Definition 9-7.1])**.** Let $\mathcal{A}$ be a hyperplane arrangement with base region B, and let $1 \leq i < j \leq m$. The set

$$\mathcal{A}|_{i,j} := \{H \in \mathcal{A} \ : \ H \supset (H_i \cap H_j)\}$$

is called a **rank-2 subarrangement** of $\mathcal{A}$. The two facet-defining hyperplanes of the region of $\mathcal{A}|_{i,j}$ that contains B are called the **basic** hyperplanes of $\mathcal{A}|_{i,j}$.

Definition 4.1.15 (Shards, see [88, Definition 9-7.2])**.** Let $H_i \in \mathcal{A}$ and set

$$\mathrm{pre}(H_i) := \{H_k \in \mathcal{A} \ : \ H_k \text{ is basic in } \mathcal{A}|_{i,k} \quad \text{and} \quad H_i \text{ is not basic in } \mathcal{A}|_{i,k}\}\,.$$

The intersection of the hyperplanes in $\mathrm{pre}(H_i)$ with the hyperplane H_i breaks H_i into closed regions called **shards**. We denote shards by capital Greek letters such as Σ, Θ, Υ, etc. The hyperplane of $\mathcal{A}$ that contains a shard Σ is denoted by H_Σ. We write Σ^i to indicate that it is contained in H_i. Hyperplanes in $\mathrm{pre}(H_i)$ are said to **cut** the hyperplane H_i.

Example 4.1.16 (Example 4.1.13 continued)**.** Figure 20 illustrates the 29 shards obtained from the base region marked with a dot.

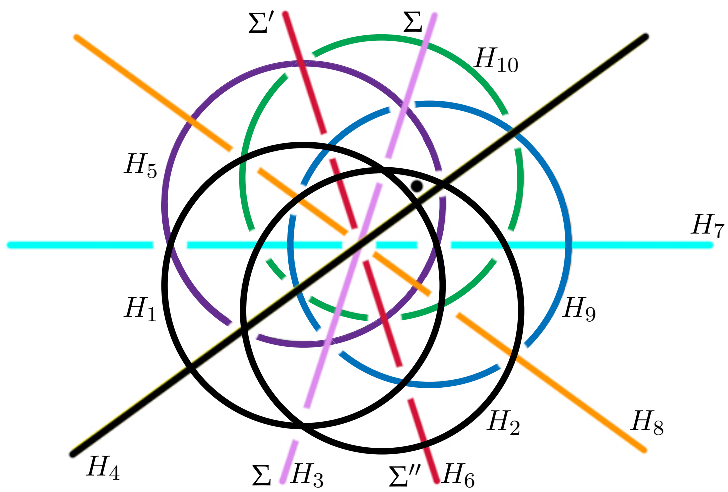


Figure 20: The shards of the simplicial hyperplane arrangement $\mathcal{A}(10,60)_3 = \mathcal{F}_2(10)$ with respect to the dotted region

On one hand, due to the particular choice of projection, it is necessary to distinguish whether two unbounded straight line-segments lying on a common line form 1 or 2 shards. For example, the unbounded line-segments on the line labeled H_3 form one shard Σ, and H_4 is basic and therefore is one shard itself. The unbounded line-segments on the line labeled H_6 form 2 distinct shards Σ' and Σ''. Similarly, the hyperplanes H_7 and H_8 also split like H_6. On the other hand, it is possible to solve this by changing the projection to obtain only circles, though simultaneously losing symmetry.

The following directed graph records the cutting relation among hyperplanes.

Definition 4.1.17 (Directed graph $\mathcal{H}_B(\mathcal{A})$ [86, Section 3])**.** Let $\mathcal{H}_B(\mathcal{A})$ be the directed graph whose vertices are the hyperplanes of the arrangement $\mathcal{A}$, and whose oriented edges are such that

$$H_i \to H_j \qquad \text{if and only if} \qquad H_i \in \operatorname{pre}(H_j).$$

The following directed graph keeps track of the cutting relation along with the "geometric proximity" between shards.

Definition 4.1.18 (Shard digraph, see [86, Section 3][88, Definition 9.7.16])**.** Let $\mathcal{S}\!h_B(\mathcal{A})$ be the directed graph on the shards of $\mathcal{A}$ such that

$$\Sigma^i \to \Sigma^j \quad \text{if and only if} \quad \begin{cases} \bullet\ H_{\Sigma^i} \to H_{\Sigma^j} \text{ in } \mathcal{H}_B(\mathcal{A}) \text{ and} \\ \bullet\ \Sigma^i \cap \Sigma^j \text{ has dimension } d-2. \end{cases}$$

The following theorem gives a characterization of congruence uniformity in terms of the directed graph on shards.

Theorem 4.1.19 ([88, Corollary 9-7.22])**.** *Let $\mathcal{A}$ be a hyperplane arrangement with a base region B. The poset of regions $P_B(\mathcal{A})$ is a congruence uniform lattice if and only if $\mathcal{A}$ is tight with respect to B and $\mathcal{S}\!h_B(\mathcal{A})$ is acyclic. In this case, $\mathcal{S}\!h_B(\mathcal{A})$ is isomorphic to the Hasse diagram of* $\operatorname{Con}(P_B(\mathcal{A}))_\vee$.

By Corollary 4.1.11, the theorem implies that acyclicity of the directed graph on shards $\mathcal{S}\!h_B(\mathcal{A})$ characterizes the normality and uniformity of tight posets of regions $P_B(\mathcal{A})$.

Example 4.1.20 (Example 4.1.16 continued)**.** Let $\Sigma^6, \Theta^{10}, \Upsilon^8$, and Ξ^9 be the shards illustrated in Figure 21. The directed graph on shards contains the cycle $\Sigma^6 \to \Theta^{10} \to \Upsilon^8 \to \Xi^9 \to \Sigma^6$. Thus, for this choice of base region, the lattice of regions is not congruence normal.

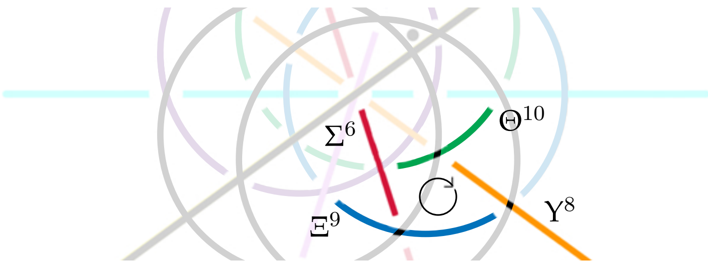


Figure 21: A cycle in the shards of the simplicial hyperplane arrangement from Example 4.1.13.

4.2 CONGRUENCE NORMALITY THROUGH RESTRICTED COVECTORS

In this section, we recast shards as certain restricted covectors—which we call *shard covectors*—in the point configuration dual to the arrangement $\mathcal{A}$. We then describe how to detect cycles in $\mathcal{S}h_B(\mathcal{A})$ using shard covectors. This reduces the verification of congruence normality for tight posets of regions to its simplest combinatorial expression, one that does not require the entire poset nor the usage of polyhedral objects. Furthermore, it is possible to express an obstruction to congruence normality for tight hyperplane arrangements.

In Section 4.2.1, we introduce restricted covectors and the intersection operation. In Section 4.2.2, we define affine point configurations and their lines. In Section 4.2.3, we interpret shards as covectors. In Section 4.2.4, we translate the forcing relation on shards into the language of covectors. Finally, in Section 4.2.5 we describe examples of obstructions to congruence normality in terms of restricted covectors.

4.2.1 *Restricted covectors and the intersection operation*

For standard references on covectors and oriented matroids, we refer the reader to the books [18, 44].

Definition 4.2.1 (Covector and restricted covector)**.** Let $\mathbf{P} = \{\mathbf{p}_i\}_{i\in[m]}$ be an ordered set of vectors in $\mathbb{R}^d$. A **covector** on $\mathbf{P}$ is a vector of signs $(c_i)_{i\in[m]} \in \{0,+,-\}^m$ defined as

$$c := (\operatorname{sign}(\mathbf{c}\cdot\mathbf{p}_i + a))_{i\in[m]},$$

where $\mathbf{c} \in \mathbb{R}^d$ and $a \in \mathbb{R}$. Given a subset $\mathbf{U} \subseteq \mathbf{P}$ and a covector c on $\mathbf{P}$, the **restricted covector** $c|_{\mathbf{U}}$ with respect to $\mathbf{U}$ is equal to c on the entries $\{j \,:\, \mathbf{p}_j \in \mathbf{U}\}$ and contains a "$*$" symbol in every other entry.

Intuitively, a restricted covector "forgets" about certain hyperplanes while keeping them encoded. Similarly, reversing the roles of $\mathbf{c}$ and $\mathbf{p}_i$ above, covectors may be thought of as *sign evaluations* of a certain vector $\mathbf{x}$ with respect to a set of vectors:

Definition 4.2.2 (Sign evaluation of a vector)**.** Let $\mathbf{P} = \{\mathbf{p}_i\}_{i\in[m]}$ be an ordered set of vectors in $\mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$. The **sign evaluation** of $\mathbf{x}$ with respect to $\mathbf{P}$ is the covector

$$c_{\mathbf{P}}(\mathbf{x}) := (\operatorname{sign}(\mathbf{p}_i\cdot\mathbf{x}))_{i\in[m]}\,.$$

Inspired by the composition operation on *vectors* (i.e. affine dependences) of oriented matroids [18, Chapter 3], we define an intersection operation on restricted covectors.

Definition 4.2.3 (Intersection of restricted covectors)**.** The commutative **intersection operation** $\cap$ from $\{0,+,-,*\}\times\{0,+,-,*\}$ to $\{0,+,-,*\}$ is defined as

$$+\cap+ := +, \qquad +\cap- = -\cap+ := 0, \qquad -\cap- := -,$$

$$0\cap\varepsilon = \varepsilon\cap 0 := 0, \qquad *\cap\varepsilon = \varepsilon\cap * := \varepsilon,$$

where $\varepsilon \in \{0,+,-,*\}$. Let $c, d \in \{0,+,-,*\}^m$ be two restricted covectors, then their **intersection** $c \cap d$ is the vector of signs $(c_i \cap d_i)_{i\in[m]}$.

The vector of signs $(c_i \cap d_i)_{i\in[m]}$ is not necessarily a covector, though it nevertheless records the information of the sign evaluation of points in an intersection. It is possible to interpret this intersection operation using subsets of the real numbers. That is, if one replaces the four symbols $0, +, -, *$ respectively by the sets $\{0\}, \mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}, \mathbb{R}$, and consider their intersections, we get exactly the same results. The associativity of this operation then follows easily.

4.2.2 *Affine point configurations and lines*

We use duality to pass from a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$ with a base region B to an acyclic point configuration $\mathcal{A}_B^*$, see [18, Section 1.2] for more detail. Indeed, the normals $\{\mathbf{n}_i\}_{i\in[m]}$ are oriented so that the linear hyperplane orthogonal to $\mathbf{v}_B \in B^\circ$ separates them from the base region B, i.e. $\mathbf{v}_B \cdot \mathbf{n}_i < 0$, for all $i \in [m]$, making the set $\{\mathbf{n}_i\}_{i\in[m]}$ acyclic.

Definition 4.2.4 (Affine point configuration relative to a base region)**.** Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$, $B \in \mathcal{R}(\mathcal{A})$, and $\mathbf{v}_B \in \mathrm{int}(B)$. Let

$$\mathbb{A}_B := \left\{\mathbf{x} \in \mathbb{R}^d \;:\; \mathbf{v}_B \cdot \mathbf{x} = -1\right\},$$

and associate the point $\mathbf{p}_i := -\frac{1}{\mathbf{v}_B \cdot \mathbf{n}_i} \cdot \mathbf{n}_i \in \mathbb{A}_B \subset \mathbb{R}^d$ to the normal $\mathbf{n}_i$. The ordered set of vectors $\{\mathbf{p}_i\}_{i\in[m]}$ is the **affine point configuration of $\mathcal{A}$ relative to the base region** B and is denoted $\mathcal{A}_B^*$.

Choosing a different normal vector $\mathbf{v}_B \in \mathrm{int}(B)$ yields an affine point configuration which is projectively equivalent to $\mathcal{A}_B^*$. Hence, up to projective transformation, this construction does not depend on the choice of $\mathbf{v}_B$.

Definition 4.2.5 (Lines of a point configuration, $\mathcal{L}(\mathbf{P})$)**.** Let $\mathbf{P} = \{\mathbf{p}_i\}_{i\in[m]}$ be an ordered set of vectors in $\mathbb{R}^d$. A subset of $\mathbf{P}$ consisting of all the points that lie on the affine hull of two distinct points of $\mathbf{P}$ is called a **line**. The set of lines of $\mathbf{P}$ is denoted by $\mathcal{L}(\mathbf{P})$.

Lemma 4.2.6. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$ with base region B, $\ell \in \mathcal{L}(\mathcal{A}_B^*)$, and $\mathbf{p}_i$ and $\mathbf{p}_j$ be the two vertices of the segment* $\mathrm{conv}(\ell)$*.*

i) The lines in $\mathcal{L}(\mathcal{A}_B^*)$ are in bijection with the rank-2 subarrangements of $\mathcal{A}$.

ii) The hyperplanes H_i and H_j are the basic hyperplanes of the rank-2 subarrangement corresponding to ℓ.

Proof. i) Let $\mathcal{A}' := \{H_i \;:\; i \in \mathcal{I}\}$, for some $\mathcal{I} \subseteq [m]$. The subarrangement $\mathcal{A}'$ is a rank-2 subarrangement if and only if

$$\dim\left(\bigcap_{i\in\mathcal{I}} H_i\right) = d-2 \quad \text{and} \quad \dim\left(\bigcap_{i\in\mathcal{I}\cup\{j\}} H_i\right) < d-2, \text{ for every } j \notin \mathcal{I}.$$

Equivalently,

$$\begin{aligned}&\dim\left(\mathrm{span}(\mathbf{n}_i \;:\; i \in \mathcal{I})\right) = 2 \quad \text{and}\\ &\dim\left(\mathrm{span}(\{\mathbf{n}_j\} \cup \{\mathbf{n}_i \;:\; i \in \mathcal{I}\})\right) > 2, \text{ for every } j \notin \mathcal{I}.\end{aligned}$$

By passing to the affine point configuration in the affine space $\mathbb{A}_B$, the above statement is equivalent to $\{\mathbf{p}_i \ : \ i \in \mathcal{I}\} \in \mathcal{L}(\mathcal{A}_B^*)$. Thus the map sending a rank-2 subarrangement $\mathcal{A}'$ to the line $\{\mathbf{p}_i \ : \ i \in \mathcal{I}\}$ is a bijection.

ii) Let $B|_{i,j}$ be the region of $\mathcal{A}|_{i,j}$ that contains B:

$$B|_{i,j} = \left\{\mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{p}_k \cdot \mathbf{x} \leq 0, \text{ for all } \mathbf{p}_k \in \ell\right\},$$

by part *i)*. Let $\mathbf{p}_k$ be the normal of a facet F of $B|_{i,j}$ and $\mathbf{x}$ be contained in the relative interior of F so that $\mathbf{p}_k \cdot \mathbf{x} = 0$. Since $\mathbf{p}_i$ and $\mathbf{p}_j$ are the vertices of $\operatorname{conv}(\ell)$, we have $\mathbf{p}_k = \lambda_k \mathbf{p}_i + (1-\lambda_k)\mathbf{p}_j$, for some $0 \leq \lambda_k \leq 1$. Then

$$0 = \mathbf{x} \cdot \mathbf{p}_k = \lambda_k(\mathbf{x} \cdot \mathbf{p}_i) + (1-\lambda_k)(\mathbf{x} \cdot \mathbf{p}_j).$$

As $\mathbf{p}_i \cdot \mathbf{x} \leq 0$ and $\mathbf{p}_j \cdot \mathbf{x} \leq 0$, the above equality implies that $\mathbf{p}_k$ must be $\mathbf{p}_i$ or $\mathbf{p}_j$. $\square$

4.2.3 *Shards as restricted covectors*

Let $\mathcal{A}$ be a tight hyperplane arrangement with respect to a base region B and $\mathcal{A}_B^*$ be its associated affine point configuration. Every shard Σ of $\mathcal{A}$ has a corresponding unique join-irreducible region J_Σ [88, Proposition 9-7.8]. In the lattice of regions, J_Σ is the meet of all regions R such that

$$H_\Sigma \in \operatorname{Sep}(R) \quad \text{and} \quad R \cap \Sigma \text{ has dimension } d-1.$$

The next lemma shows how $\operatorname{pre}(H_\Sigma)$ and $\operatorname{Sep}(J_\Sigma)$ yield a description of the shard as the intersection of half-spaces. It is originally stated for simplicial arrangements, though the same holds true for tight hyperplane arrangements.

Lemma 4.2.7 (see [86, Lemma 3.7])**.** *A shard Σ has the following description:*

$$\Sigma = \left\{\mathbf{x} \in H_\Sigma \ \middle| \ \begin{array}{l} \mathbf{n}_i \cdot \mathbf{x} \geq 0 \text{ if } H_i \in \operatorname{pre}(H_\Sigma) \cap \operatorname{Sep}(J_\Sigma) \\ \mathbf{n}_i \cdot \mathbf{x} \leq 0 \text{ if } H_i \in \operatorname{pre}(H_\Sigma) \setminus \operatorname{Sep}(J_\Sigma) \end{array} \right\}.$$

To interpret shards on a hyperplane H_i as covectors, we focus on a certain subconfiguration containing $\mathbf{p}_i$.

Definition 4.2.8 (Subconfiguration localized at a point)**.** Let $\mathbf{p}_i \in \mathcal{A}_B^*$. The **subconfiguration $\mathcal{A}_{B,i}^*$ of $\mathcal{A}_B^*$ localized at $\mathbf{p}_i$** contains $\mathbf{p}_i$ and the vertices of the convex hulls of lines of $\mathcal{A}_B^*$ that contain $\mathbf{p}_i$ in their interior.

Lemma 4.2.6 ii) and Definition 4.1.17 imply the following lemma.

Lemma 4.2.9. *The subconfiguration $\mathcal{A}_{B,i}^*$ satisfies*

$$\mathcal{A}_{B,i}^* = \{\mathbf{p}_i\} \cup \{\mathbf{p}_j \ : \ H_j \in \operatorname{pre}(H_i)\}.$$

Definition 4.2.10 (Shard covectors of a point)**.** Let $\mathbf{p}_i \in \mathcal{A}_B^*$. A **shard covector of $\mathbf{p}_i$** is a restricted covector $\sigma^i = c|_{\mathcal{A}_{B,i}^*}$ with respect to $\mathcal{A}_{B,i}^*$ such that

- $\sigma_j^i = *$ if and only if $\mathbf{p}_j \notin \mathcal{A}_{B,i}^*$, and

- the restriction of σ^i to the subconfiguration $\mathcal{A}^*_{B,i}$ is a covector with exactly one zero in position "i".

Example 4.2.11. In Figure 22, the left image illustrates the affine point configuration $\mathcal{A}(6,24)^*$ for the rank-3 braid arrangement with 6 hyperplanes. The right image illustrates the subconfiguration of $\mathcal{A}(6,24)^*$ localized at $\mathbf{p}_6$, $\mathcal{A}(6,24)^*_6$.

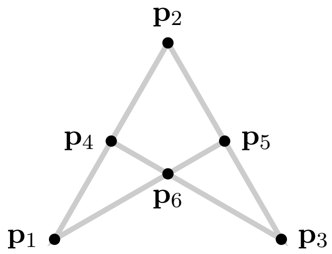


Figure 22: The point configuration $\mathcal{A}(6,24)^*$ for the rank-3 braid arrangement and the subconfiguration $\mathcal{A}(6,24)^*_6$ localized at $\mathbf{p}_6$

There are two pairs of oppositely signed shard covectors of $\mathbf{p}_6$:

$$\sigma^{6,+} = (+,*,+,-,-,0), \qquad \theta^{6,+} = (+,*,-,+,-,0),$$
$$\sigma^{6,-} = (-,*,-,+,+,0), \qquad \theta^{6,-} = (-,*,+,-,+,0).$$

It is possible to obtain these shard covectors by drawing a line through $\mathbf{p}_6$ in $\mathcal{A}(6,24)^*_6$, and choosing a positive and a negative side. Rotating the line about $\mathbf{p}_6$ in all possible directions, and recording the sign evaluations of the points in $\mathcal{A}(6,24)^*_6$ relative to the line exhausts all possibilities.

We now associate a restricted covector to each shard using the sign evaluation of vectors. Let Σ^i be a shard contained in hyperplane H_i, and let $\mathbf{x} \in \operatorname{int}(\Sigma^i)$. Using Lemma 4.2.7 and 4.2.9, we get

$$c_{\mathcal{A}^*_{B,i}}(\mathbf{x}) := \left(c_{\mathcal{A}^*_{B,i}}(\mathbf{x})_j = \begin{cases} 0 & \text{if } j = i, \\ + & \text{if } H_j \in \operatorname{pre}(H_i) \cap \operatorname{Sep}(J_\Sigma) \\ - & \text{if } H_j \in \operatorname{pre}(H_i) \setminus \operatorname{Sep}(J_\Sigma) \end{cases} \right)_{j \in [m] \text{ and } \mathbf{p}_j \in \mathcal{A}^*_{B,i}} .$$

Completing this sign evaluation to the configuration $\mathcal{A}^*_B$, we get the restricted covector

$$\sigma^i := \left(\sigma^i_j = \begin{cases} c_{\mathcal{A}^*_{B,i}}(\mathbf{x})_j & \text{if } H_j \in \operatorname{pre}(H_i) \cup \{H_i\} \\ * & \text{if } H_j \notin \operatorname{pre}(H_i) \cup \{H_i\} \end{cases} \right)_{j \in [m]} .$$

This restricted covector is independent of the choice of vector $\mathbf{x} \in \operatorname{int}(\Sigma^i)$, thanks to Lemma 4.2.7, and only depends on the choice of base region B.

Theorem 4.2.12. *Let $\mathcal{A}$ be a tight hyperplane arrangement with respect to a base region B. The map sending a shard Σ^i to the shard covector σ^i gives a bijection between the shards of $\mathcal{A}$ with base region B and the shard covectors of $\mathcal{A}^*_B$.*

Proof. Injectivity. Suppose $\sigma^i = \theta^i$ for two shards Σ^i and Θ^i, for some $i \in [m]$. By the definition of σ^i and θ^i, the shard covectors are obtained from some points $\mathbf{x} \in \mathrm{int}(\Sigma^i)$ and $\mathbf{y} \in \mathrm{int}(\Theta^i)$ and

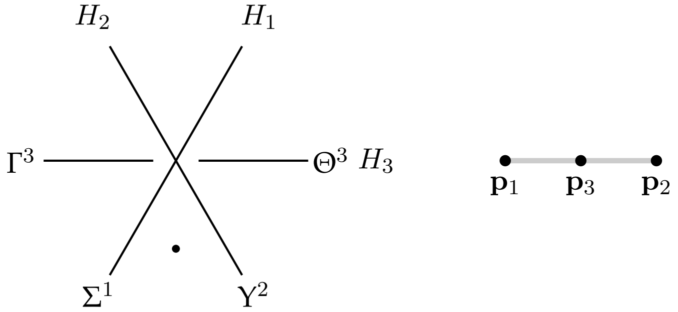
$$\mathrm{sign}(\mathbf{n}_j \cdot \mathbf{x}) = \mathrm{sign}(\mathbf{n}_j \cdot \mathbf{y}) \quad \text{for every } j \text{ such that } H_j \in \mathrm{pre}(H_i) \cup \{H_i\}.$$

By Lemma 4.2.7, an H-description of the shard is given by the sign evaluation of any of its points in the relative interior with respect to the hyperplanes in $\mathrm{pre}(H_i)$. Because Σ^i and Θ^i are both shards on hyperplane H_i, and the sign evaluation of $\mathbf{x}$ and $\mathbf{y}$ agree on all normals in $\mathrm{pre}(H_i)$, Σ^i and Θ^i must be the same.

Surjectivity. Let c be a shard covector with a unique zero at position $i \in [m]$. Considered as a sign evaluation, there is an $\mathbf{x} \in \mathbb{R}^d$ such that $c = c_{\mathcal{A}_B^*}(\mathbf{x})|_{\mathcal{A}_{B,i}^*}$. The linear hyperplane with normal $\mathbf{x}$ separates the normal vectors in $\mathrm{pre}(H_i)$ as c dictates. Thus $\mathbf{x}$ is a point in the relative interior of a shard Σ^i of H_i such that $\sigma^i = c$. $\square$

By Theorem 4.2.12, there is a unique shard covector associated to every shard. We therefore use lowercase Greek letters σ^i to denote the unique shard covector corresponding to a shard Σ^i.

Example 4.2.13. Consider the Coxeter arrangement A_2 with three hyperplanes H_1, H_2, H_3, and choose a base region B between H_1 and H_2, as in Figure 23. Hyperplanes H_1 and H_2 are basic, giving shards Σ^1 and Υ^2, and hyperplane H_3 splits into two shards Γ^3 and Θ^3. The affine point configuration $\mathcal{A}_B^*$ has three points on a line $\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2$, and admits exactly four shard covectors. The table shows the bijection between shards and shard covectors using the map of Theorem 4.2.12.



	1	2	3
σ^1	0	$*$	$*$
υ^2	$*$	0	$*$
γ^3	+	-	0
θ^3	-	+	0

Figure 23: The shards of the Coxeter arrangement A_2, and their corresponding shard covectors.

4.2.4 *Forcing relation on covectors*

In this section, we use Theorem 4.2.12 and interpret the shard digraph $\mathcal{S}h_B(\mathcal{A})$ using shard covectors of $\mathcal{A}_B^*$. In Definition 4.1.18, the first condition to get an edge $\Sigma^i \to \Sigma^j$ translates to the shard covectors of $\mathbf{p}_j$ having a $+$ or $-$ at position "i". The second condition requires one to interpret the dimension of intersection of two shards using shard covectors. To do so, we define **line covectors** of two hyperplanes.

Definition 4.2.14 (Line covector)**.** Let $\ell \in \mathcal{L}(\mathcal{A}_B^*)$. A **line covector** of ℓ is a covector $\hbar^\ell$ on $\mathcal{A}_B^*$ such that

$$\hbar_k^\ell = 0 \quad \text{if and only if} \quad \mathbf{p}_k \in \ell.$$

Line covectors record possible sign evaluations of non-zero points in the intersection of two hyperplanes with respect to $\mathcal{A}_B^*$. They come in oppositely signed pairs which we denote by $\hbar^{\ell,+}$ and $\hbar^{\ell,-}$. In the case of rank-3 hyperplane arrangements, these covectors are actually cocircuits of the oriented matroid. For higher-rank hyperplane arrangements, the set of 0-indices of a line covector gives a flat of rank 2 in the underlying matroid.

Example 4.2.15 (Example 4.2.11 continued)**.** Let $\ell = \{\mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_6\}$. Since $\mathbb{A}_B$ has dimension 2, the line ℓ has exactly two line covectors. From Figure 22, we deduce that the line covectors of ℓ are:

$$\hbar^{\ell,+} = (0, +, -, +, 0, 0),$$
$$\hbar^{\ell,-} = (0, -, +, -, 0, 0).$$

Lemma 4.2.16. *Let $\mathcal{A}_B^* = \{\mathbf{p}_i\}_{i\in[m]}$ be an affine point configuration, $1 \le i < j \le m$, ℓ be the line spanned by $\mathbf{p}_i$ and $\mathbf{p}_j$, and $\hbar^\ell$ be a line covector of ℓ. The set*

$$\left\{ \mathbf{x} \in (H_i \cap H_j) \ : \ c_{\mathcal{A}_B^*}(\mathbf{x}) = \hbar^\ell \right\}$$

has dimension $d-2$.

Proof. Let $\mathbf{x} \in H_i \cap H_j$ with $c_{\mathcal{A}_B^*}(\mathbf{x}) = \hbar^\ell$. For any $\mathbf{v} \in \operatorname{span}(\mathbf{n}_i, \mathbf{n}_j)^\perp$ and $\varepsilon > 0$, the k-th entry of $c_{\mathcal{A}_B^*}(\mathbf{x} + \varepsilon\mathbf{v})$ is equal to

$$\begin{aligned} c_{\mathcal{A}_B^*}(\mathbf{x} + \varepsilon\mathbf{v})_k &= \operatorname{sign}\left(\mathbf{x}\cdot\mathbf{n}_k + \varepsilon(\mathbf{v}\cdot\mathbf{n}_k)\right) \\ &= \begin{cases} 0 & \text{if } k \in \{i,j\}, \\ \operatorname{sign}(\mathbf{x}\cdot\mathbf{n}_k + \varepsilon(\mathbf{v}\cdot\mathbf{n}_k)) & \text{if } k \notin \{i,j\}. \end{cases} \end{aligned}$$

When ε is chosen small enough, then

$$c_{\mathcal{A}_B^*}(\mathbf{x} + \varepsilon\mathbf{v})_k = c_{\mathcal{A}_B^*}(\mathbf{x})_k = \hbar_k^\ell.$$

Thus $\dim(\{\mathbf{x} \in (H_i \cap H_j) \ : \ c_{\mathcal{A}_B^*}(\mathbf{x}) = \hbar^\ell\}) = \dim(\operatorname{span}(\mathbf{n}_i, \mathbf{n}_j)^\perp) = d-2$. □

Example 4.2.17 (Example 4.2.11 continued)**.** Figure 24 shows a stereographic projection of $\mathcal{A}(6, 24)$ broken into shards. The shards $\Theta^{6,+}$ and $\Sigma^1 = H_1$ are thickened and one sees that H_1 cuts H_6. The shards Σ^1 and $\Theta^{6,+}$ intersect at a point so there is an oriented edge $\Sigma^1 \to \Theta^{6,+}$ in the shard digraph.

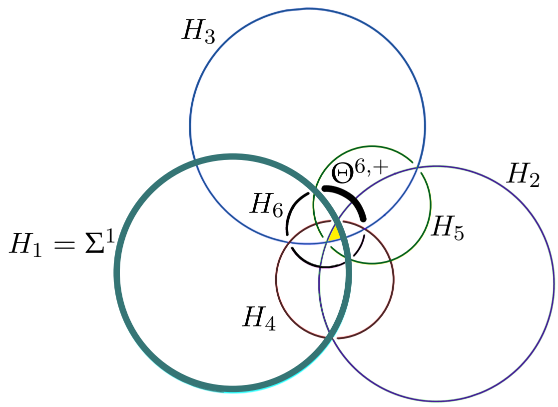


Figure 24: The shards of the arrangement $\mathcal{A}(6,24)$ shown via stereographic projection

This fact translates to a property of the corresponding shards covectors $\sigma^1 = (0,*,*,*,*,*)$ and $\theta^{6,+} = (+,*,-,+,-,0)$. Consider the line $\ell = \{\mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_6\}$ and the line covector $\hbar^{\ell,+} = (0,+,-,+,0,0)$. Then $\hbar^{\ell,+} \cap \theta^{6,+} \cap \sigma^1 = (0,+,-,+,0,0)$. In comparison, $\hbar^{\ell,-} = (0,-,+,-,0,0)$, and $\hbar^{\ell,-} \cap \theta^{6,+} \cap \sigma^1 = (0,-,0,0,0,0) \neq \hbar^{\ell,-}$. Figure 25 illustrates the affine point configuration $\mathcal{A}(6,24)^*$ along with the three oriented lines describing the involved covectors.

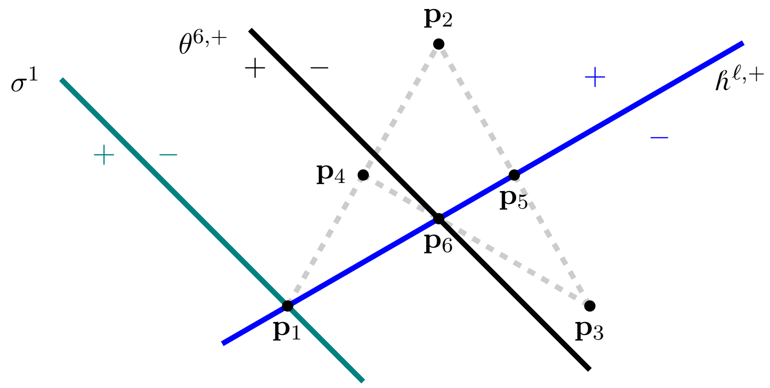


Figure 25: The point configuration $\mathcal{A}(6,24)^*$ and hyperplanes describing the covectors σ^1, $\theta^{6,+}$, and $\hbar^{\ell,+}$, where $\ell = \{\mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_6\}$

It is possible to interpret the fact that the two shards intersect at a point as follows. Apply a clockwise rotation to the line labeled $\theta^{6,+}$ about the point $\mathbf{p}_6$ until it collides with the line corresponding to $\hbar^{\ell,+}$. During the rotation, the line did not cross any points in $\mathcal{A}^*_{B,6}$. Similarly, applying the same with the line labeled σ^1 about $\mathbf{p}_1$ does not cross any points in $\mathcal{A}^*_{B,1} = \varnothing$.

The theorem below shows that the above equality is exactly the necessary and sufficient condition for the two involved shards to have an intersection of dimension $d-2$.

Theorem 4.2.18. *Let $\mathcal{A}^*_B = \{\mathbf{p}_i\}_{i\in[m]}$ be an affine point configuration, $1 \le i < j \le m$, and let ℓ be the line spanned by $\mathbf{p}_i$ and $\mathbf{p}_j$. Furthermore, let $\Sigma^i \subseteq H_i$ and*

$\Theta^j \subseteq H_j$ be two shards. The intersection $\Sigma^i \cap \Theta^j$ has dimension $d-2$ if and only if there exists a line covector $\hbar^\ell$ such that $\hbar^\ell \cap \sigma^i \cap \theta^j = \hbar^\ell$.

Proof. Assume $\dim(\Sigma^i \cap \Theta^j) = d-2$. Hence there exists $\mathbf{x} \in \Sigma^i \cap \Theta^j$ such that the sign evaluation $c_{\mathcal{A}^*_B}(\mathbf{x})_k$ equals zero if and only if $\mathbf{p}_k \in \ell$. Therefore $c_{\mathcal{A}^*_B}(\mathbf{x})$ is a line covector of ℓ. If $\mathbf{x}$ is in the boundary of Σ^i, then for $\mathbf{z} \in \operatorname{int}(\Sigma^i), \mathbf{p}_k \in \mathcal{A}^*_{B,i}$, either $\operatorname{sign}(\mathbf{x}\cdot\mathbf{p}_k) = \operatorname{sign}(\mathbf{z}\cdot\mathbf{p}_k)$ or $\operatorname{sign}(\mathbf{x}\cdot\mathbf{p}_k) = 0$. As $c_{\mathcal{A}^*_B}(\mathbf{x})_k$ equals zero if and only if $\mathbf{p}_k \in \ell$, $\sigma^i_k = c_{\mathcal{A}^*_B}(\mathbf{x})_k$ for all k such that $\mathbf{p}_k \in (\mathcal{A}^*_{B,i} \setminus \ell)$. Likewise, $\theta^j_k = c_{\mathcal{A}^*_B}(\mathbf{x})_k$ for all k such that $\mathbf{p}_k \in (\mathcal{A}^*_{B,j} \setminus \ell)$. Thus,

$$c_{\mathcal{A}^*_B}(\mathbf{x}) \cap \sigma^i \cap \theta^j = c_{\mathcal{A}^*_B}(\mathbf{x}).$$

Assume now that there exists a line covector $\hbar^\ell$ such that $\hbar^\ell \cap \sigma^i \cap \theta^j = \hbar^\ell$. Let $S = \{\mathbf{x} \in (H_i \cap H_j) \ : c_{\mathcal{A}^*_B}(\mathbf{x}) = \hbar^\ell\}$. By Lemma 4.2.16, $\dim(S) = d-2$. Let $\mathbf{x} \in S$, $\mathbf{y} \in \operatorname{int}(\Sigma^i)$, and $\mathbf{z} \in \operatorname{int}(\Theta^j)$. As $\hbar^\ell \cap \sigma^i = \hbar^\ell$, $\operatorname{sign}(\mathbf{x}\cdot\mathbf{p}_k) = \operatorname{sign}(\mathbf{y}\cdot\mathbf{p}_k)$ for all $\mathbf{p}_k \in \mathcal{A}^*_{B,i} \setminus \ell$. For $\mathbf{p}_k \in \ell$, we have $\mathbf{x}\cdot\mathbf{p}_k = 0$. For $0 \le \lambda \le 1$, let $\mathbf{m}_\lambda = (1-\lambda)\mathbf{y} + \lambda\mathbf{x}$. Then $c_{\mathcal{A}^*_{B,i}}(\mathbf{m}_\lambda)_k = \operatorname{sign}(\mathbf{y}\cdot\mathbf{p}_k)$ for all k such that $\mathbf{p}_k \in \mathcal{A}^*_{B,i}$, and $\lambda \in [0,1)$. This shows that $\mathbf{m}_\lambda \in \Sigma^i$ for all $\lambda \in [0,1)$, and thus $\mathbf{x}$ is contained in Σ^i. A similar argument with $\mathbf{z}$ shows that $\mathbf{x}$ is in Θ^j. □

Corollary 4.2.19. *There is a directed arrow $\Sigma^i \to \Theta^j$ in $\mathcal{S}\hbar_B(\mathcal{A})$ if and only if $\theta^j_i \in \{-,+\}$ and there exists a line covector $\hbar^\ell$ such that $\hbar^\ell \cap \sigma^i \cap \theta^j = \hbar^\ell$.*

4.2.5 *Examples of obstruction to congruence normality*

Example 4.2.20 (Example 4.1.13 continued)**.** The normal vectors $\{\mathbf{n}_i\}_{i\in[10]}$ for this configuration can be chosen as follows. Let $\tau = \frac{1+\sqrt{5}}{2}$ and $\mathbf{n}_1 = (0,1,0), \mathbf{n}_2 = (1,0,0), \mathbf{n}_3 = (1,1,0), \mathbf{n}_4 = (1,1,1), \mathbf{n}_5 = (\tau+1,\tau,\tau), \mathbf{n}_6 = (\tau+1,\tau+1,1), \mathbf{n}_7 = (\tau+1,\tau+1,\tau), \mathbf{n}_8 = (2\tau,2\tau,\tau), \mathbf{n}_9 = (2\tau+1,2\tau,\tau), \mathbf{n}_{10} = (2\tau+2,2\tau+1,\tau+1)$. Let B be the base region containing the vector $\mathbf{v} = (-1,-1,-2)$. Figure 26 illustrates $\mathcal{A}(10,60)^*_{3,B}$ along with four lines describing the shard covectors

$$\sigma^6 = (+,*,-,+,*,0,*,*,-,*), \quad \theta^8 = (-,*,-,+,*,*,*,0,*,+),$$
$$\upsilon^9 = (*,-,-,*,+,*,*,+,0,*), \quad \xi^{10} = (*,-,*,+,+,-,+,*,-,0).$$

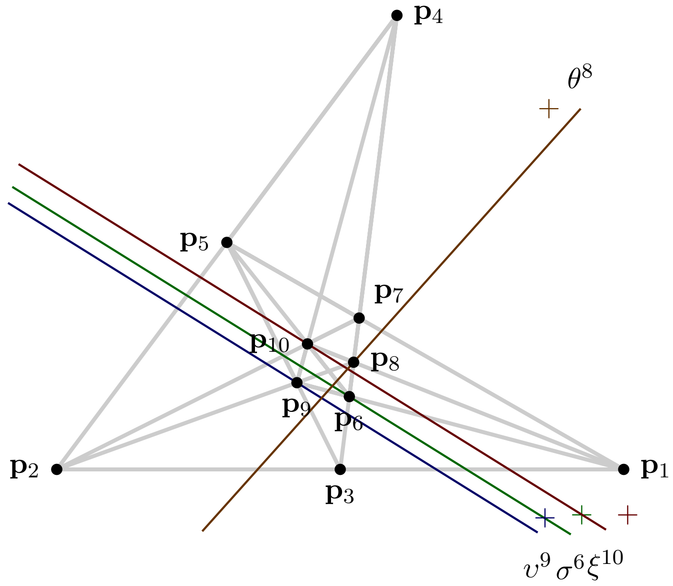


Figure 26: The point configuration $\mathcal{A}(10,60)^*_{3,B}$

Let $\ell_1 = \{\mathbf{p}_2, \mathbf{p}_8, \mathbf{p}_9\}, \ell_2 = \{\mathbf{p}_1, \mathbf{p}_6, \mathbf{p}_9\}, \ell_3 = \{\mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_{10}\}$, and $\ell_4 = \{\mathbf{p}_1, \mathbf{p}_8, \mathbf{p}_{10}\}$ and consider the four line covectors

$$\hbar^{\ell_1} = (-,0,-,+,+,-,+,0,0,+), \quad \hbar^{\ell_2} = (0,-,-,+,+,0,+,+,0,+),$$
$$\hbar^{\ell_3} = (+,-,-,+,0,0,+,+,-,0), \quad \hbar^{\ell_4} = (0,-,-,+,+,-,+,0,-,0).$$

As υ^9 has a "+" in position 8, H_8 cuts H_9. Furthermore, one computes that $\hbar^{\ell_1} \cap \theta^8 \cap \upsilon^9 = \hbar^{\ell_1}$. By Corollary 4.2.19, there is a directed arrow $\Theta^9 \to \Upsilon^9$ in $\mathcal{S}\hbar_B(\mathcal{A})$. Similar computations reveal that $\theta^8 \to \upsilon^9 \to \sigma^6 \to \xi^{10} \to \theta^8$ is a cycle in $\mathcal{S}\hbar_B(\mathcal{A})$. Thus, the poset of regions of $\mathcal{A}(10,60)_3$ with respect to the base region B is not congruence normal.

Example 4.2.21. Removing the hyperplane H_4 from the arrangement $\mathcal{A}(10,60)_3$ and taking the base region that contains the vector $\mathbf{v} = (-1,-1,2)$, one obtains a non-simplicial, tight (hence semidistributive) poset of regions with 52 regions that is not congruence normal as the cycle $\theta^8 \to \upsilon^9 \to \sigma^6 \to \xi^{10} \to \theta^8$ still occurs in the shard digraph. Figure 27 illustrates the resulting affine point configuration. Is there a tight poset of regions which is not congruence normal with at most 8 hyperplanes?

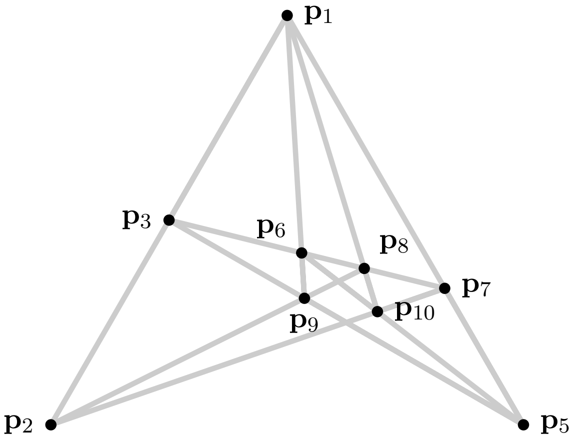


Figure 27: An affine point configuration leading to a tight, non-congruence normal hyperplane arrangement

Example 4.2.22. It is possible to have cycles in $\mathscr{H}_B(\mathscr{A})$ while $\mathscr{Sh}_B(\mathscr{A})$ is acyclic, settling the question raised in [85, p. 203]. Figure 28 shows the affine point configuration of arrangement $\mathscr{A}(14,116)$ with respect to the base region that contains the vector $\approx (0.38, 2.85, -7.85)$. There is a cycle $H_1 \to H_4 \to H_7 \to H_1$ in $\mathscr{H}_B(\mathscr{A})$. However, this cycle does not lead to any cycle among shards included in these three hyperplanes as $\mathscr{Sh}_B(\mathscr{A})$ was computed to be acyclic in this case. This can be seen geometrically as follows. Apply a rotation to the line spanned by the points 1 and 2 about the point 4 until it collides with the line spanned by the points 4 and 14. During the rotation (be it clockwise or counter-clockwise) the line crossed points in $\mathscr{A}^*_{B,4}$. This means that a shard on hyperplane 4 intersecting with a shard on hyperplane 1 along a face of dimension $d-2$ can not have an intersection with a shard on hyperplane 7 that has dimension $d-2$.

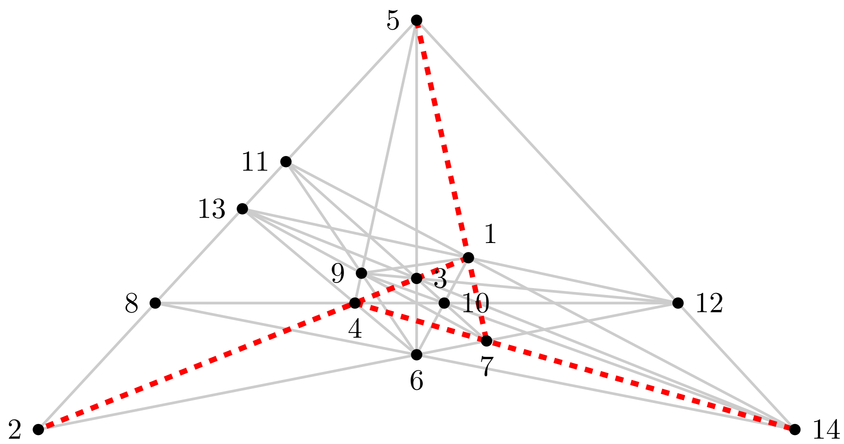


Figure 28: The point configuration of arrangement $\mathscr{A}(14,116)$ with respect to the base region containing $\approx (0.38, 2.85, -7.85)$

4.3 CONGRUENCE NORMALITY CLASSIFICATION OF SIMPLICIAL ARRANGEMENTS

As the number of regions of a rank-three hyperplane arrangement grows quadratically with the number of hyperplanes, its poset of regions becomes costly to construct in practice when the number of hyperplanes gets large. Consequently, checking if a given poset of regions is obtainable through doublings of convex sets becomes impractical. If one uses shards as geometric objects to determine congruence normality, then one needs to determine polyhedral cones contained in each hyperplane and the dimensions of intersection for pairs of shards. In contrast, the combinatorial methods developed in Section 4.2 make the determination of congruence normality for posets of regions of rank-3 hyperplane arrangements tractable and could be extended to higher dimensions given a method for determining the covectors of the oriented matroid. Additionally, the oriented matroid approach makes it possible and natural to check congruence normality for non-realizable oriented matroids.

One of the motivations for studying congruence normality is to better understand simplicial hyperplane arrangements. In rank 3, the number of simplicial hyperplane arrangements is unknown [33, 62]. So far, three infinite families and 95 sporadic arrangements have been found. It is conjectured that there are only finitely many sporadic arrangements. The largest sporadic arrangement found so far has 37 hyperplanes. In this section, we apply our reformulation of shards as shard covectors to classify which of the known simplicial hyperplane arrangements of rank 3 are congruence normal. This verification was carried out using `Sage` [91]. The computations took around 18 hours on 8 Intel Cores (i7-7700 @3.60Hz). The verification for each poset of regions was computed independently, for example the cocircuits were recomputed for each reorientation of the set of normals, but the computations of intersections on covectors were cached. The computation could be further improved by applying the reorientation on cocircuits directly in order to avoid recomputing them.

Our results are summarized in Table 2. We use the following notation: $\mathcal{A}(m,r)_i$ denotes the i-th hyperplane arrangement with m hyperplanes and r regions. The hyperplane arrangement in the i-th infinite family with m hyperplanes is denoted $\mathcal{F}_i(m)$, see Section 4.3.3. We refer to congruence normality using the acronym **CN** and use **NCN** for non-congruence normality. The normals of the 119 arrangements from the known sporadic arrangements and two of the infinite families as well as the corresponding wiring diagrams are listed in Appendix of [34]. The list includes the sporadic arrangements and the arrangements from the infinite families with at most 37 hyperplanes.

$P_B(\mathcal{A})$ always **CN**	$P_B(\mathcal{A})$ sometimes **CN**	$P_B(\mathcal{A})$ never **CN**
Rank-3 Finite Weyl Groupoids	$\mathcal{F}_2(m)$ $(m \geq 10)$	$\mathcal{A}(22, 288)$
(including $\mathcal{F}_2(m)$ $(m \leq 8)$	$\mathcal{F}_3(m)$ $(m \geq 17)$	$\mathcal{A}(25, 360)$
and $\mathcal{F}_3(m)$ $(m \leq 13)$)	41 arrangements	$\mathcal{A}(35, 680)$
$\mathcal{A}(15, 120)$		
$\mathcal{A}(31, 480)$		
$\mathcal{F}_1(m)$		
55 arrangements	61 arrangements	3 arrangements
see Section 4.3.1	see Sections 4.3.2 and 4.3.3	see Section 4.3.4
and Table 3	and Table 4	and Table 5

Table 2: Classification of rank-3 simplicial hyperplane arrangements with at most 37 hyperplanes according to the congruence normality of their posets of regions

Table 2 provides material to check the veracity of [81, Conjecture 145], which postulates the existence of certain "shard" polytopes for tight congruence normal arrangements. Section 4.3.1 looks at the arrangements that are always **CN**, Section 4.3.2 at the arrangements that are sometimes **CN**, and Section 4.3.4 at the arrangements that are never **CN**. In Section 4.3.5 we finish by discussing these results and compiling related questions.

4.3.1 *Always **CN** simplicial arrangements*

Fifty-five of the 119 arrangements are congruence normal, that is, for any choice of base region, the poset of regions is congruence normal, see Table 3.

Finite Weyl Groupoids					
$\mathcal{F}_2(6) = \mathcal{A}(6, 24)$	$\mathcal{A}(13, 96)_2$	$\mathcal{A}(18, 180)_1$	$\mathcal{A}(20, 220)_2$	$\mathcal{A}(26, 364)_1$	$\mathcal{A}(29, 448)_1$
$\mathcal{A}(7, 32)$	$\mathcal{A}(13, 96)_3$	$\mathcal{A}(18, 180)_2$	$\mathcal{A}(21, 240)_1$	$\mathcal{A}(26, 364)_2$	$\mathcal{A}(29, 448)_2$
$\mathcal{F}_2(8) = \mathcal{A}(8, 40)$	$\mathcal{A}(14, 112)_1$	$\mathcal{A}(19, 192)_1$	$\mathcal{A}(21, 240)_2$	$\mathcal{A}(27, 392)_1$	$\mathcal{A}(29, 448)_3$
$\mathcal{F}_3(9) = \mathcal{A}(9, 48)$	$\mathcal{A}(15, 128)_1$	$\mathcal{A}(19, 192)_2$	$\mathcal{A}(21, 240)_3$	$\mathcal{A}(27, 392)_2$	$\mathcal{A}(30, 476)$
$\mathcal{A}(10, 60)_1$	$\mathcal{A}(16, 144)_1$	$\mathcal{A}(19, 200)_1$	$\mathcal{A}(22, 264)_1$	$\mathcal{A}(27, 392)_3$	$\mathcal{A}(31, 504)_1$
$\mathcal{A}(10, 60)_2$	$\mathcal{A}(16, 144)_2$	$\mathcal{A}(19, 200)_2$	$\mathcal{A}(25, 336)_1$	$\mathcal{A}(28, 420)_1$	$\mathcal{A}(31, 504)_2$
$\mathcal{A}(11, 72)$	$\mathcal{A}(17, 160)_1$	$\mathcal{A}(19, 200)_3$	$\mathcal{A}(25, 336)_2$	$\mathcal{A}(28, 420)_2$	$\mathcal{A}(34, 612)_1$
$\mathcal{A}(12, 84)_1$	$\mathcal{A}(17, 160)_2$	$\mathcal{A}(20, 216)$	$\mathcal{A}(25, 336)_3$	$\mathcal{A}(28, 420)_3$	$\mathcal{A}(37, 720)_1$
$\mathcal{A}(12, 84)_2$	$\mathcal{A}(17, 160)_3$	$\mathcal{A}(20, 220)_1$	$\mathcal{A}(25, 336)_4$		
$\mathcal{F}_3(13) = \mathcal{A}(13, 96)_1$					

Others	
$H_3 = \mathcal{A}(15, 120)$	$H_3^* = \mathcal{A}(31, 480)$

Table 3: List of congruence normal rank-3 simplicial arrangements

Fifty-three of these arrangements come from finite Weyl groupoids of rank 3 [35]. Finite Weyl groupoids correspond to (generalized) *crystallographic* root systems. In the present context, affine point configurations $\mathcal{A}_B^*$ play the role of these root systems. A root system is **crystallographic** if there exists a choice of normals $\{\mathbf{n}_i\}_{i\in[m]}$ for the hyperplanes such that for any base region, all normals are integral linear combinations of normals to the basic hyperplanes [31, Section 1]. Given a base region B, denote the set of rays of $\text{span}^+(\mathcal{A}_B^*)$ by Δ and call the elements of $\mathcal{A}_B^*$ the **positive roots**. A positive root $\mathbf{p}_i \in \mathcal{A}_B^*$ is **constructible** if

$$\mathbf{n}_i \in \Delta \quad \text{or} \quad \mathbf{n}_i = \mathbf{n}_\alpha + \mathbf{n}_\beta,$$

where $\alpha, \beta \in \mathcal{A}_B^*$. We call $\mathcal{A}_B^*$ **additive** if every positive root in $\mathcal{A}_B^*$ is constructible. If $\mathcal{A}_B^*$ is additive, then it is possible to define the **root poset** $(\mathcal{A}_B^*, \leq)$ by

$$\mathbf{p}_i \leq \mathbf{p}_j \quad \iff \quad \mathbf{n}_j - \mathbf{n}_i \in \mathbb{N}\Delta.$$

The following is a fundamental result about finite Weyl groupoids.

Theorem 4.3.1 ([31, Corollary 5.6] and [35, Theorem 2.10])**.** *A simplicial arrangement $\mathcal{A}$ corresponds to a finite Weyl groupoid if and only if $\mathcal{A}_B^*$ is additive for every choice of base region B.*

Theorem 4.3.1 leads directly to the following theorem, which provides a new proof that finite Coxeter arrangements are congruence normal [26, Theorem 6].

Theorem 4.3.2. *Let $\mathcal{A}$ be the hyperplane arrangement of a finite Weyl groupoid $\mathcal{W}$. For any choice of base region B, the lattice of regions $P_B(\mathcal{A})$ is congruence normal.*

Proof. Via the contrapositive statement, having a cycle in the graph $\mathcal{H}_B(\mathcal{A})$ is a necessary condition for $P_B(\mathcal{A})$ *not* to be congruence normal. By Corollary 2.5 in [35], such a cycle between hyperplanes yields a cycle in the order defining the root poset of $\mathcal{A}_B^*$. Hence, when $P_B(\mathcal{A})$ is not congruence normal, the positive roots $\mathcal{A}_B^*$ do not lead to a root poset. Thus $\mathcal{A}_B^*$ can not be additive. □

Remark 4.3.3. There are arrangements such that $\mathcal{A}_B^*$ is additive, but there is no relation between $\mathbf{p}_j$ and $\mathbf{p}_i$ in the root poset for two positive roots $\mathbf{p}_j$ and $\mathbf{p}_i$ with $H_i \in \text{pre}(H_j)$. The additional assumptions that the arrangement is simplicial and $\mathcal{A}_B^*$ is additive with respect to every base region ensure the relation exists.

There are two additional **CN** arrangements that do not stem from finite Weyl groupoids. Arrangement $\mathcal{A}(15, 120)$ is the Coxeter arrangement for the Coxeter group H_3 and arrangement $\mathcal{A}(31, 480)$ is its point-line dual. As discussed in [39], there is a root poset for H_3 supporting the fact that its arrangement is always congruence normal. The dual arrangement $\mathcal{A}(31, 480)$ is also always congruence normal, as we verified directly. Is there a proof of congruence normality for $\mathcal{A}(31, 480)$ using duality with H_3?

4.3.2 *Simplicial arrangements that are sometimes congruence normal*

Sixty-one of the 119 arrangements are congruence normal for some base regions and not congruence normal for others, see Table 4. Among them is the arrangement $\mathcal{A}(10, 60)_3$ which appeared in Example 4.1.13.

Name	**CN**	**NCN**
$\mathcal{F}_2(10) = \mathcal{A}(10,60)_3$	40	20
$\mathcal{F}_2(12) = \mathcal{A}(12,84)_3$	36	48
$\mathcal{A}(13,104)$	24	80
$\mathcal{F}_2(14) = \mathcal{A}(14,112)_2$	28	84
$\mathcal{A}(14,112)_3$	72	40
$\mathcal{A}(14,116)$	40	76
$\mathcal{A}(15,128)_2$	72	56
$\mathcal{A}(15,132)_1$	60	72
$\mathcal{A}(15,132)_2$	48	84
$\mathcal{A}(16,140)$	120	20
$\mathcal{F}_2(16) = \mathcal{A}(16,144)_3$	32	112
$\mathcal{A}(16,144)_4$	84	60
$\mathcal{A}(16,144)_5$	108	36
$\mathcal{A}(16,148)$	52	96
$\mathcal{F}_3(17) = \mathcal{A}(17,160)_4$	96	64
$\mathcal{A}(17,160)_5$	120	40
$\mathcal{A}(17,164)$	76	88
$\mathcal{A}(17,168)_1$	48	120
$\mathcal{A}(17,168)_2$	48	120
$\mathcal{F}_2(18) = \mathcal{A}(18,180)_3$	36	144
$\mathcal{A}(18,180)_4$	84	96
$\mathcal{A}(18,180)_5$	120	60
$\mathcal{A}(18,180)_6$	120	60
$\mathcal{A}(18,184)_1$	100	84
$\mathcal{A}(18,184)_2$	72	112
$\mathcal{A}(19,200)_4$	120	80
$\mathcal{A}(19,204)$	72	132
$\mathcal{F}_2(20) = \mathcal{A}(20,220)_3$	40	180
$\mathcal{A}(20,220)_4$	120	100
$\mathcal{F}_3(21) = \mathcal{A}(21,240)_4$	80	160
$\mathcal{A}(21,240)_5$	120	120
$\mathcal{A}(21,248)$	88	160
$\mathcal{A}(21,252)$	36	216
$\mathcal{F}_2(22) = \mathcal{A}(22,264)_2$	44	220
$\mathcal{A}(22,264)_3$	168	96
$\mathcal{A}(22,276)$	60	216
$\mathcal{A}(23,296)$	112	184
$\mathcal{A}(23,304)$	8	296
$\mathcal{A}(24,304)$	112	192
$\mathcal{F}_2(24) = \mathcal{A}(24,312)$	48	264
$\mathcal{A}(24,316)$	184	132
$\mathcal{A}(24,320)$	24	296
$\mathcal{A}(25,320)$	288	32
$\mathcal{F}_3(25) = \mathcal{A}(25,336)_5$	48	288
$\mathcal{A}(25,336)_6$	48	288
$\mathcal{F}_2(26) = \mathcal{A}(26,364)_3$	52	312
$\mathcal{A}(26,380)$	20	360
$\mathcal{A}(27,400)$	48	352
$\mathcal{F}_2(28) = \mathcal{A}(28,420)_4$	56	364
$\mathcal{A}(28,420)_5$	84	336
$\mathcal{A}(28,420)_6$	84	336
$\mathcal{A}(29,440)$	136	304
$\mathcal{F}_3(29) = \mathcal{A}(29,448)_4$	56	392
$\mathcal{A}(30,460)$	240	220
$\mathcal{F}_2(30) = \mathcal{A}(30,480)$	60	420
$\mathcal{F}_2(32) = \mathcal{A}(32,544)$	64	480
$\mathcal{F}_3(33) = \mathcal{A}(33,576)$	64	512
$\mathcal{F}_2(34) = \mathcal{A}(34,612)_2$	68	544
$\mathcal{F}_2(36) = \mathcal{A}(36,684)$	72	612
$\mathcal{F}_3(37) = \mathcal{A}(37,720)_2$	72	648
$\mathcal{A}(37,720)_3$	96	624

Table 4: Simplicial arrangements that are sometimes congruence normal

Reading proved that the poset of regions of a supersolvable hyperplane arrangements is congruence normal with respect to a canonical base region [85, Theorem 1]. In rank 3, the infinite families are exactly the irreducible supersolvable ones [38, Theorem 1.2]. However, we show below that $\mathcal{F}_2(m)$ with $m \geq 10$ and $\mathcal{F}_3(m)$ with $m \geq 17$ always have a base region for which the associated lattice of regions is not congruence normal. For $\mathcal{F}_2(m)$ with $m \leq 8$ and $\mathcal{F}_3(m)$ with $m \leq 13$, the posets of regions are always congruence normal, see Section 4.3.1.

4.3.3 *Congruence normality for the infinite families*

There are three infinite families of rank-3 simplicial hyperplane arrangements [61]. The first family, $\mathcal{F}_1(m)$ with $m \geq 3$ is the family of **near-pencils** in the projective plane with m hyperplanes. The second family, $\mathcal{F}_2(m)$, for even $m \geq 6$ consists of the hyperplanes defined by the edges of the regular $\frac{m}{2}$-gon and each of its $\frac{m}{2}$ lines of symmetry. The third family, $\mathcal{F}_3(m)$, for $m = 4k+1$, $k \geq 2$, is obtained from $\mathcal{F}_2(m-1)$ by adding the line at infinity. Examples of these families are illustrated in Figure 29.

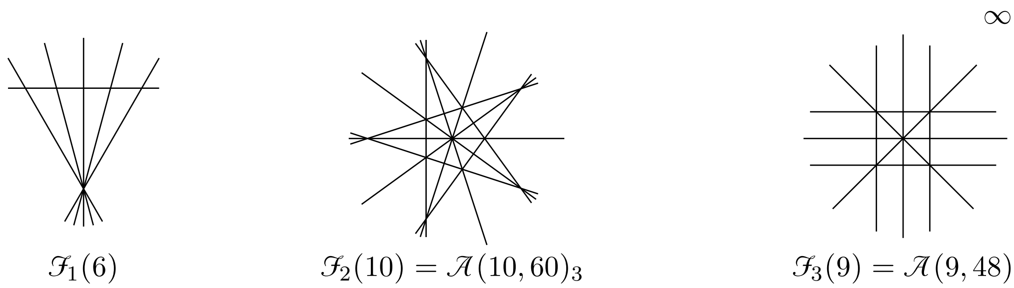


Figure 29: Arrangements from the three infinite families of simplicial arrangements of rank 3 drawn in the projective plane

Theorem 4.3.4. *The near-pencil arrangements of* $\mathcal{F}_1(m)$ *are congruence normal.*

Proof. There is exactly one rank-2 subarrangement with at least three hyperplanes. Thus, for any choice of base region, the length of any path in the directed graph on shards is at most one, so there are no cycles. □

Theorem 4.3.5. *The second family* $\mathcal{F}_2(m)$ *is sometimes congruence normal for* $m \geq 10$*.*

Proof. In rank 3, the infinite families are exactly the irreducible supersolvable ones, thus there exists a canonical choice of base region such that the poset of regions is congruence normal [38, Theorem 1.2]. On the other hand, with respect to a certain choice of base region, there is a guaranteed four-cycle in the shards as demonstrated in Figure 30. The figure shows the arrangement on two projective planes and how some of the hyperplanes intersect at infinity. To represent the central, three-dimensional hyperplane arrangement, we intersect it with the unit sphere at the origin, and use two centrally symmetric planar charts, giving the left and the right sides of the image, which are glued together in the middle by the hyperplane (a dotted line in this case) at infinity. Let the base region be bounded by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{r}_2$. At point 1, the hyperplane $\mathbf{e}_5$ is cut by $\mathbf{r}_5$. At point 2, the hyperplane $\mathbf{r}_6$ is cut by $\mathbf{e}_5$. At point 3, the hyperplane $\mathbf{e}_4$ is cut by $\mathbf{r}_6$. At point 4, the hyperplane $\mathbf{r}_5$ is cut by $\mathbf{e}_4$. Thus there is a cycle in the shard digraph. Adapting this procedure when $m \geq 14$ similarly provides a 4-cycle for every member of $\mathcal{F}_2(m)$. □

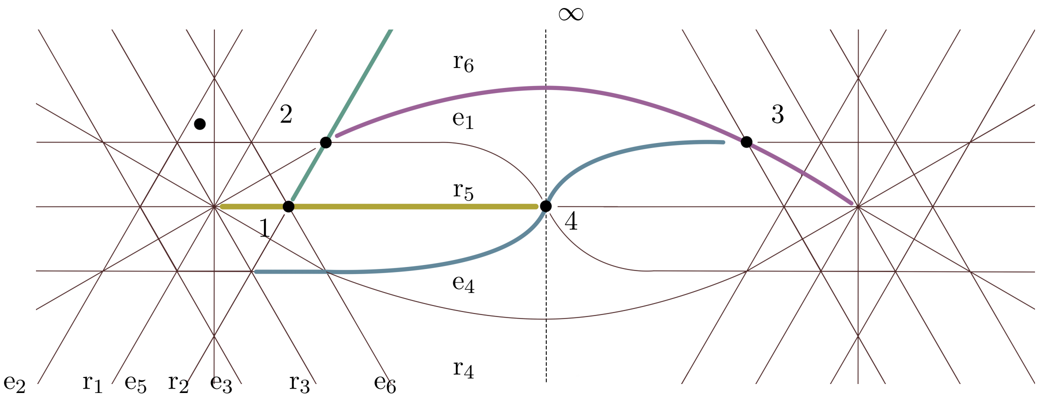


Figure 30: The simplicial hyperplane arrangement $\mathcal{A}(12,84)_3$ from $\mathcal{F}_2$ whose lattice of regions with the marked base region is not congruence normal

Theorem 4.3.6. *The third family $\mathcal{F}_3(m)$ is sometimes congruence normal for $m \geq 17$.*

Proof. The proof is similar to that of Theorem 4.3.5. For $m \geq 17$, a four-cycle among shards still occurs, and its location relative to the base region is illustrated in Figure 31 for $m = 17$. The line at infinity is included in these arrangements, and one of the intersection points in the cycle occurs in a rank-2 subarrangement that includes the hyperplane at infinity. Relative to the plane graph, the cycle involves the same description as a embedded cycle for the family $\mathcal{F}_2$. □

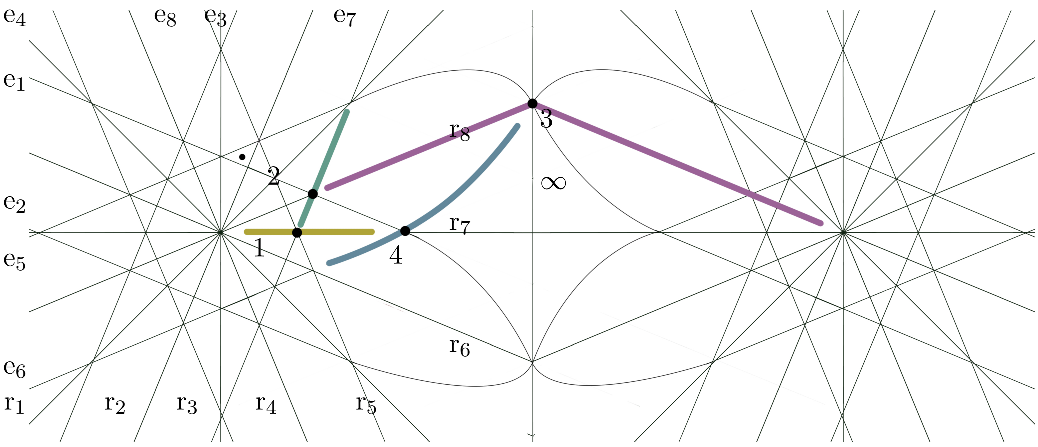


Figure 31: The simplicial hyperplane arrangement $\mathcal{A}(17,160)_4$ from $\mathcal{F}_3$ whose lattice of regions with the marked base region is not congruence normal

4.3.4 *Never **CN** simplicial arrangements*

Three of the known simplicial arrangements of rank 3 are never congruence normal, see Table 5. That is, there is no choice of base region such that the lattice of regions is congruence normal. The first arrangement is an arrangement with 22 hyperplanes with normals related to $\sqrt{5}$, see Figure 32. The second arrangement has 25 hyperplanes with normals related to $\sqrt{5}$ and is shown in Figure 33. The third arrangement is the new sporadic arrangement found in [33]. It is the only known arrangement with 35 hyperplanes and is illustrated in Figure 34. We are not aware of any geometric explanation for the provenance of these arrangements and why they are never congruence normal.

Table 5: Simplicial arrangements that are never congruence normal

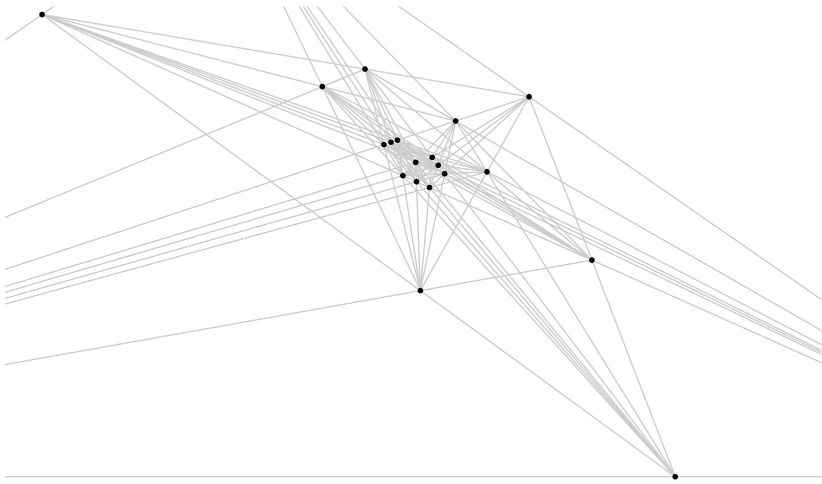


Figure 32: The point configuration $\mathcal{A}(22, 288)^*$. Three points are not shown and can be obtained by continuing the line segments.

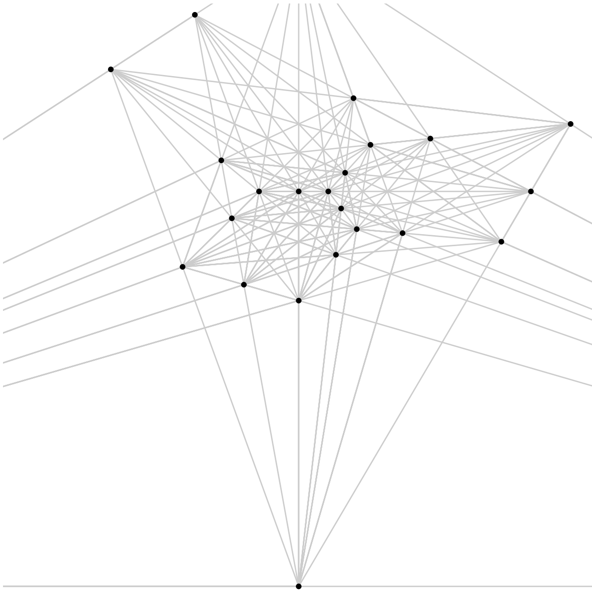


Figure 33: The point configuration $\mathscr{A}(25,360)^*$. Three points are not shown and can be obtained by continuing the line segments.

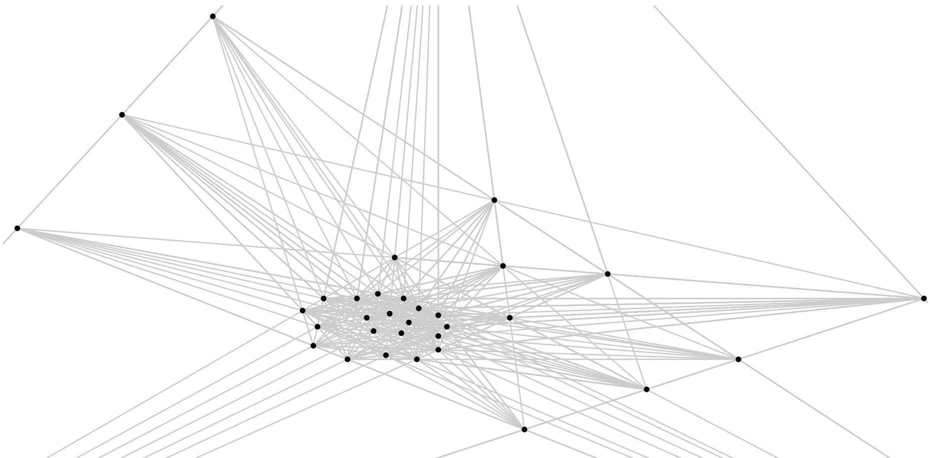


Figure 34: The point configuration $\mathscr{A}(35,680)^*$. Three points are not shown and can be obtained by continuing the line segments.

4.3.5 *Observations and consequences*

We make a few remarks on the verification and its implications. The number of shards do not depend on the choice of base region: indeed, [81, Lemma 146] says that in a simplicial arrangement, the number of shards is the number of rays in the arrangement minus the dimension. So, computing the number of shards leads to the number of facets of the corresponding simple zonotope. For rank-3 simplicial arrangements, the number of shards is one less than half the number of regions. For example, the arrangements $\mathscr{A}(30, 480)$ and $\mathscr{A}(31, 480)$ have different numbers of hyperplanes but the same number of shards and regions.

Finally, we end with questions that arose from this investigation.

QUESTION 1 What is the relationship between polygonal and semidistributive lattices?

QUESTION 2 Is there a hyperplane arrangement with at most 8 hyperplanes that yields a tight poset of regions which is not congruence normal?

QUESTION 3 Is there a proof of congruence normality for $\mathscr{A}(31, 480)$ using dualty with H_3?

QUESTION 4 Is there a geometric explanation for the provenance of the three arrangements that are never congruence normal? Are the posets of regions all isomorphic?

QUESTION 5 Reading used "signed subsets" to describe when an edge occurs between two shards in type A and B [86]. Can shard covectors be used in conjunction with positive roots to describe forcing on shards?

QUESTION 6 Apart from being dual to 2-neighborly, what can be said about the combinatorial types of the regions in a tight hyperplane arrangement?

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