APPENDIX A

Mathematical Supplements

A.1. The (Cylinder) Braid Group

In the Cartesian product $(\mathbb{R}^2)^{\times n}$, let D_n be the set of points $\boldsymbol{x} = (x_1, \dots, x_n)$ $(x_i \in \mathbb{R}^2)$ with $x_i = x_j$ for at least one pair (x_i, x_j) with $i \neq j$. Let S_n be the permutation group of n elements. S_n obviously acts as a transformation group on $(\mathbb{R}^2)^{\times n}$ (on the right), leaving D_n invariant:

$$(\boldsymbol{x}\pi)_i = x_{\pi(i)} \qquad , \pi \in S_n, \boldsymbol{x} \in (\mathbb{R}^2)^{\times n} .$$

We introduce the space

$$^n\mathbb{R}^2 := ((\mathbb{R}^2)^{\times n} \setminus D_n) \,/\, S_n. \tag{A.1}$$

In this space we choose the base point

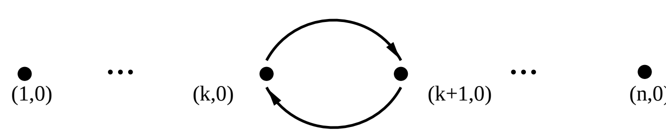
$$\boldsymbol{x}_0 := \boldsymbol{x}_0' \cdot S_n , \quad \text{where} \tag{A.2}$$

$$\boldsymbol{x}_0' := \big((1,0), \dots, (n,0)\big) . \tag{A.3}$$

The fundamental group of $^n\mathbb{R}^2$ with this base point is defined as the set of homotopy classes $[\gamma]$ of closed paths $\gamma : [0,1] \mapsto {}^n\mathbb{R}^2$ starting from $\boldsymbol{x}_0$. The group product is defined by $[\gamma_2]\,[\gamma_1] := [\gamma_2 * \gamma_1]$, where $\gamma_2 * \gamma_1$ is the path which runs first through γ_1 and then through γ_2. This fundamental group is isomorphic to the *braid group* B_n (see e.g. [Bir75]), and will be identified with it in our context:

$$B_n = \pi_1\big({}^n\mathbb{R}^2, \boldsymbol{x}_0\big) .$$

Its elements may be visualized as follows. For a closed path γ at $\boldsymbol{x}_0$ we consider its lift through $\boldsymbol{x}_0'$. It may be pictured as n paths (or strands) in the three-dimensional layer $[0,1] \times \mathbb{R}^2$, which do not intersect. The set of these n strands is called the *geometric braid* associated with γ. Two closed paths at $\boldsymbol{x}_0$ are homotopic if and only if their associated geometric braids can be continuously deformed into one another. Thus the fundamental group $\pi_1\big({}^n\mathbb{R}^2, \boldsymbol{x}_0\big)$ is isomorphic to the set of such topological equivalence classes of geometric braids, where the product corresponds to appending one geometric braid to the other. Algebraically, the braid group B_n has a presentation with generators $t_1, \dots, t_{n-1}$ which satisfy the relations (0.14). The kth elementary braid t_k corresponds to the homotopy class of the S_n-orbit of the following path starting from $\boldsymbol{x}_0'$:



A picture of the corresponding geometric braid can be found on page 8. We finally recall from our introduction that there is a natural homomorphism ν from the braid group onto the permutation group defined by

$$\nu \,: B_n \to S_n \,, \tag{A.4}$$

$$t_k \mapsto \tau_k \quad , \; k = 1, \dots, n-1 \,, \tag{A.5}$$

where τ_k denotes the transposition $(\,k, k+1\,)$. The kernel of this homomorphism is called the *pure braid group.*

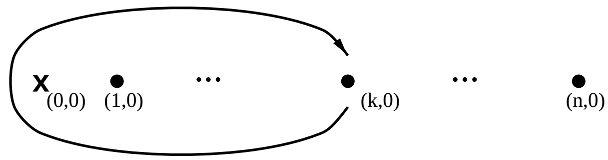
The *cylinder braid group* $B_n(C)$ is the fundamental group of the manifold $((\mathbb{R}^2 - \{0\})^{\times n} \setminus D_n) / S_n$. Algebraically, it arises from B_n by adding one new generator c_1 satisfying the additional relations

$$\begin{aligned} (c_1\, t_1)^2 &= (t_1\, c_1)^2 \quad &&\text{and} \\ c_1\, t_k &= t_k\, c_1 \quad &&\text{if } \ k \geq 2 \,. \end{aligned}$$

For later conveniance, we define for both B_n and $B_n(C)$ the braids

$$c_k := t_{k-1} \cdots t_1\, c_1\, t_1 \cdots t_{k-1} \quad , \quad k = 2, \dots , n \,, \tag{A.6}$$

where c_1 is set to the unit element if c_k is viewed as an element of B_n. As an element of $B_n(C)$, for $k = 1, \dots , n$ c_k corresponds to the homotopy class of the S_n-orbit of the following path:



A.2. The Universal Covering Group of the Poincaré Group

We recall some well known facts in order to establish notation. Elements of the orthochronous Poincaré group $P_+^\uparrow$ of the 3-dimensional Minkowski space $\mathcal{M}_3$ may be written as (a, Λ) with $a \in \mathcal{M}_3$ and $\Lambda \in L_+^\uparrow$, the orthochronous Lorentz group. Group multiplication is given by $(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda')$ with unit element $(0, \mathbf{1}\,)$ such that $P_+^\uparrow$ is the semidirect product of $\mathcal{M}_3$ and $L_+^\uparrow$. A twofold covering of $L_+^\uparrow$ is given as the subgroup of $SL(2, \mathbb{C})$ (conjugate to $SL(2, \mathbb{R})$) consisting of elements of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} , \alpha\bar{\alpha} - \beta\bar{\beta} = 1. \tag{A.7}$$

The corresponding Lorentz transformation $\Lambda = \Lambda(\alpha, \beta) \in L_+^\uparrow$ is given as follows. For $a = (a^0, a^1, a^2) \in \mathcal{M}_3$ we set

$$\underset{\sim}{a} = \begin{pmatrix} a^0 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 \end{pmatrix} \tag{A.8}$$

and define $\Lambda(\alpha, \beta)\, a$ by

$$\underset{\sim}{\Lambda a} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \underset{\sim}{a} \begin{pmatrix} \bar{\alpha} & \beta \\ \bar{\beta} & \alpha \end{pmatrix} . \tag{A.9}$$

In particular for given $p = (p^0, p^1, p^2) \in H_m$ the element of the form

$$\Big(2m(p^0 + m)\Big)^{-\frac{1}{2}} \begin{pmatrix} p^0 + m & p^1 - ip^2 \\ p^1 + ip^2 & p^0 + m \end{pmatrix} \tag{A.10}$$

gives rise to an element in $L_+^\uparrow$ called a boost and is denoted by $\Lambda(p)$. One has
$\Lambda(p)\,(m, 0, 0) = p$. The universal covering group $\tilde{L}_+^\uparrow$ of $L_+^\uparrow$ can be explicitly written as the set

$$\Big\{(\gamma, \omega) \,\Big|\, \gamma \in \mathbb{C},\ |\gamma| < 1,\ \omega \in \mathbb{R}\Big\} \tag{A.11}$$

with the group multiplication $(\gamma, \omega)(\gamma', \omega') = (\gamma'', \omega'')$ being given by

$$\begin{aligned} \gamma'' &= (\gamma' + \gamma e^{-i\omega'})(1 + \gamma\bar{\gamma}' e^{-i\omega'})^{-1} \\ \omega'' &= \omega + \omega' + \frac{1}{i} \log \Big\{ (1 + \gamma\bar{\gamma}' e^{-i\omega'})(1 + \bar{\gamma}\gamma' e^{i\omega'})^{-1} \Big\} \,. \end{aligned} \tag{A.12}$$

Here the logarithm is defined in terms of its power series [Bar47, p.594]. The corresponding element in the twofold covering of $L_+^\uparrow$ described above is then given as

$$(1 - \gamma\bar{\gamma})^{-\frac{1}{2}} \begin{pmatrix} e^{i\frac{\omega}{2}} & \gamma e^{i\frac{\omega}{2}} \\ \bar{\gamma} e^{-i\frac{\omega}{2}} & e^{-i\frac{\omega}{2}} \end{pmatrix} . \tag{A.13}$$

The resulting element in $L_+^\uparrow$ will be denoted by $\Lambda(\gamma,\omega)$. Let

$$\gamma(p) := \frac{p^1 - ip^2}{p^0 + m} \tag{A.14}$$

for given $p \in H_m$. Then the element $h(p) = \big(\gamma(p)\,,\, \omega = 0\big)$ in $\tilde{L}_+^\uparrow$ is such that $\Lambda\big(h(p)\big) = \Lambda(p)$.

The universal covering group $\tilde{\mathcal{P}}_3^\uparrow$ of $P_+^\uparrow$ is now the semidirect product of $\mathcal{M}_3$ with $\tilde{L}_+^\uparrow$, the group multiplication being given by

$$\big(a, (\gamma,\omega)\big)\big(a', (\gamma',\omega')\big) = \big(\, a + \Lambda(\gamma,\omega)a'\,,\, (\gamma,\omega)(\gamma',\omega')\,\big)\,. \tag{A.15}$$

In analogy with the higher dimensional case, $\tilde{\mathcal{P}}_3^\uparrow$ has no nontrivial central extensions.

We will frequently identify elements of $\tilde{P}_+^\uparrow$ and $\tilde{L}_+^\uparrow$ with homotopy classes of paths in $P_+^\uparrow$ and $L_+^\uparrow$, respectively, starting at the unit element and denote them by $\tilde{g}$. The covering projection

$$\tilde{P}_+^\uparrow \to P_+^\uparrow \quad , \quad (a,(\gamma,\omega)) \mapsto (a, \Lambda(\gamma,\omega))$$

will then be written as $\tilde{g} \mapsto g$.

We denote by $\widetilde{U(1)} = \mathbb{R}$ the abelian subgroup of $\tilde{L}_3^\uparrow$ consisting of elements of the form $(0,\omega)$. For arbitrary $p \in H_m$ and $(\gamma,\omega) \in \tilde{L}_3^\uparrow$, the element

$$t\big((\gamma,\omega)\,,\, p\big) = h(p)^{-1}\,(\gamma,\omega)\; h\big(\Lambda(\gamma,\omega)^{-1}p\big) \tag{A.16}$$

is in $\widetilde{U(1)}$ and hence may be written in the form

$$t\big((\gamma,\omega)\,,\, p\big) = \big(0, \Omega((\gamma,\omega)\,,\, p)\big). \tag{A.17}$$

In fact, by (A.16) and (A.12)

$$\begin{aligned}\Omega\big((\gamma,\omega)\,,\, p\big) = \omega \;& + \tfrac{1}{i}\log\Big\{\; \big(1 - \gamma(p)\bar{\gamma}e^{-i\omega}\big)\big(1 - \bar{\gamma}(p)\gamma e^{i\omega}\big)^{-1}\Big\} \\ & + \tfrac{1}{i}\log\Big\{\; \Big(1 + \frac{\gamma - \gamma(p)e^{-i\omega}}{1 - \gamma(p)\bar{\gamma}e^{-i\omega}}\bar{\gamma}(\Lambda(\gamma,\omega)^{-1}p)\Big)\,\cdot \\ & \qquad \Big(1 + \frac{\bar{\gamma} - \bar{\gamma}(p)e^{i\omega}}{1 - \bar{\gamma}(p)\gamma e^{i\omega}}\,\gamma(\Lambda(\gamma,\omega)^{-1}p)\Big)^{-1}\Big\}\,.\end{aligned} \tag{A.18}$$

Note that $\Omega\big((\gamma = 0,\omega)\,,\, p\big) = \omega$ for all ω and p.

Proper Poincaré Group. The proper Poincaré group P_+ can be obtained from the proper orthochroneous Poincaré group by adjoining the reflection j at the x^2-axis (see equation (1.60)) with the appropriate relations:

$$j^2 = 1 \quad , \quad j(x,1)j = (j\cdot x, 1) \quad ,$$

$$j\begin{pmatrix}\lambda_{00} & \lambda_{01} & \lambda_{02}\\ \lambda_{10} & \lambda_{11} & \lambda_{12}\\ \lambda_{20} & \lambda_{21} & \lambda_{22}\end{pmatrix} j \quad = \quad \begin{pmatrix}\lambda_{00} & \lambda_{01} & -\lambda_{02}\\ \lambda_{10} & \lambda_{11} & -\lambda_{12}\\ -\lambda_{20} & -\lambda_{21} & \lambda_{22}\end{pmatrix}\,.$$

Correspondingly, the universal covering group $\tilde{P}_+$ of this (disconnected) group may be defined by adjoining an element $\tilde{j}$ to $\tilde{P}_+^\uparrow$ satisfying the relations

$$\tilde{j}^2 = 1 \quad \text{and} \quad \tilde{j}\;(x,(\gamma,\omega))\;\tilde{j} = (jx, (\bar{\gamma}, -\omega))\;. \tag{A.19}$$

In fact, the map $\tilde{j} \mapsto j$, $(\gamma,\omega) \mapsto \Lambda(\gamma,\omega)$ is a homomorphism and hence a projection.

Action of $\tilde{P}_+$ on ${}^n\widetilde{H}_1$. Let ${}^nH_1$ be defined in analogy to equation (A.1), with $\mathbb{R}^2$ replaced by the unit mass shell H_1. The universal covering space of the latter will be denoted by ${}^n\widetilde{H}_1$. Being diffeomorphic to ${}^n\mathbb{R}^2$, the fundamental group of ${}^nH_1$ can be identified with the braid group B_n. Hence B_n acts canonically on ${}^n\widetilde{H}_1$ from the right. Further, $\tilde{L}_+^\uparrow$ acts as a transformation group on $H_1^{\times n}$ via

$$(\gamma,\omega) : \boldsymbol{q} = (q_1, \dots, q_n) \mapsto \Big(\Lambda(\gamma,\omega)q_1, \dots, \Lambda(\gamma,\omega)q_n\Big)$$

leaving D_n invariant and commuting with the action of S_n. Hence this action of $\tilde{L}_+^\uparrow$ descends to an action on ${}^nH_1$. This action lifts to an action on the universal covering manifold ${}^n\widetilde{H}_1$ written as $(\gamma,\omega) : \tilde{\boldsymbol{q}} \mapsto (\gamma,\omega)\cdot\tilde{\boldsymbol{q}}$ which commutes with the right action of the braid group: $\big((\gamma,\omega)\cdot\tilde{\boldsymbol{q}}\big)\cdot b = (\gamma,\omega)\cdot(\tilde{\boldsymbol{q}}\cdot b)$.

The reflection j does not leave the mass shell H_m invariant, but the action of $\tilde{P}_+^\uparrow$ on H_m can be extended to an action of $\tilde{P}_+$ by the definition

$$\tilde{j} \;:\; p = (p_0, p_1, p_2) \;\mapsto\; -j\cdot p = (p_0, p_1, -p_2)\ .$$

As above, this action descends (for $m = 1$) to an action on ${}^nH_1$, which in turn can be lifted to an action of $\tilde{P}_+$ on ${}^n\widetilde{H}_1$ which extends the above action of $\tilde{P}_+^\uparrow$. Namely, we set

$$\tilde{j} \;:\; \tilde{\boldsymbol{q}} \;\mapsto\; -\tilde{j}\cdot\tilde{\boldsymbol{q}}\ ,$$

where $-\tilde{j}\cdot\tilde{\boldsymbol{q}}$ is defined as follows. If $\tilde{\boldsymbol{q}}$ is the homotopy class of γ, then $-\tilde{j}\cdot\tilde{\boldsymbol{q}}$ is the homoyopy class of the path

$$t \mapsto -j\cdot\gamma(t)\ .$$

This definition presupposes that the base point $\boldsymbol{q}_0$ in ${}^nH_1$ is chosen such that $-j\cdot\boldsymbol{q}_0 = \boldsymbol{q}_0$. We therefore fix $\boldsymbol{q}_0$ as in equation (2.45). Finally, we prove an important cocycle relation of the Wigner rotation (A.17) with respect to the lift $\tilde{j}$ of the reflection.

LEMMA A.1. *For all $\tilde{g} \in \tilde{L}_+^\uparrow$ and $p \in H_m$ the following relation holds.*

$$\Omega\big(\tilde{j}\tilde{g}\tilde{j}\,,\, p\big) = -\Omega(\tilde{g}\,, -j\cdot p)\ . \tag{A.20}$$

PROOF. From the definition of $h(p)$ via equation (A.14) and the group relations (A.19) satisfied by $\tilde{j}$ we get

$$h(-j\cdot p) = \tilde{j}\,h(p)\,\tilde{j}\ . \tag{A.21}$$

This implies $t(\tilde{g}, -j\cdot p) = \tilde{j}\,t(\tilde{j}\tilde{g}\tilde{j}, p)\tilde{j}$ and hence the claim. □

A.3. Calculations Concerning the Tomita Operator

For the proof of Proposition 4.22 we need two Lemmata. Firstly, we use well known results on representions of the additive group of complex numbers by unbounded operators $\tilde{\Delta}^{iz}$ arising from unitary one parameter groups $\tilde{\Delta}^{it} = \tilde{U}(\tilde{\lambda}_1(t))$, see e.g. the article [BW75] of Bisognano and Wichmann:

LEMMA A.2. *If $\phi \in \tilde{\mathcal{H}}$ is in the domain of $\tilde{\Delta}^{\frac{1}{2}}$, then the map*

$$t \mapsto \tilde{U}(\tilde{\lambda}_1(t))\,\phi\ , \quad t \in \mathbb{R}$$

is the boundary value of an analytic $\tilde{\mathcal{H}}$-valued function on the strip $\mathbb{R} - i\,(0, \frac{1}{2})$ which is continuous and bounded on $\mathbb{R} - i\,[0, \frac{1}{2}]$. The analytic function is then $\tilde{\Delta}^{iz}\,\phi$, and the bound is $\max\{\,||\phi||\,,\,||\tilde{\Delta}^{\frac{1}{2}}\,\phi||\,\}$.

In application to $\tilde{\Delta}^{\frac{1}{2}}$, a complication arises from the factor $\exp is\Omega(\tilde{\lambda}_1(t), p)$. Namely, it has branch points in $\mathbb{R} - i\,(0, \frac{1}{2})$, which can be seen from equation (A.33) below. To get rid of this factor, we have designed the function u such that, considered as a multiplication operator, it intertwines $\tilde{S}\big((0,\omega)\cdot W_1\big)$ with $\tilde{S}_0\big((0,\omega)\cdot W_1\big)$ on a certain domain simultaneously for all ω in some intervall contained in $(-\pi, 0]$, thus reducing the problem to the well known case $s = 0$. We first establish the relevant properties of the function u.

DEFINITION AND LEMMA A.3. Let for $p \in H_m$ and (ω, z) in $\mathbb{R}\times\mathbb{R}$ *or* in $[0,\pi)\times \mathbb{R}-i[0,\frac{1}{2}]$

$$u_\omega(z,p) := 2^{-2s}(1+e^{-i\omega})^{2s}\big(1+\frac{\sin\omega}{1+\cos\omega}\frac{p_2+im}{p_0-p_1}e^{2\pi z}\big)^{2s} . \tag{A.22}$$

If $\omega \in [0,\pi)$, then for all $p \in H_m$ the function $z \mapsto u_\omega(z,p)$ is analytic in the open strip $\mathbb{R}-i\,(0,\frac{1}{2})$. Further, it satisfies for all $p \in H_m$ the following equations:

$$u_\omega(t,p) = u_\omega(0,\lambda_1(-t)\cdot p) \qquad \text{for all } \omega \in \mathbb{R}\,, t \in \mathbb{R} \tag{A.23}$$

$$u_\omega(-\frac{i}{2},p) = e^{-2is\omega}\, u_{-\omega}(0,p) \qquad \text{for all } \omega \in [0,\pi)\,. \tag{A.24}$$

LEMMA A.4. *For all $p \in H_m$ the following equations hold.*

$$e^{is\,\Omega(\tilde\lambda_1(t),p)}\,u(\lambda_1(-t)\cdot p) = e^{-2\pi st}u(p) \qquad \textit{for all } t \in \mathbb{R}\,, \tag{A.25}$$

$$\overline{u((-j)\cdot p)} = u(p) \tag{A.26}$$

$$u\big(R(-\omega)\cdot p\big) = u(p)\cdot u_\omega(0,p) \qquad \textit{for all } \omega \in \mathbb{R}\,. \tag{A.27}$$

PROOF. We rewrite $u(p)$ it as follows. Let

$$l(p) := p_0 - p_1 + m + ip_2 \qquad \text{and} \qquad l^s(p) := l(p)^s\,. \tag{A.28}$$

Note that for all $p \in H_m$, the real part of $l(p)$ is strictly positve since $p_0 - p_1 > 0$, hence the function l has values in the cut complex plane $\mathbb{C}\setminus\mathbb{R}_0^-$. Thus its power to $s \in \mathbb{R}$ can be defined via the branch of the logarithm in $\mathbb{C}\setminus\mathbb{R}_0^-$ with $\ln 1 = 0$. In the sequel, all powers to real numbers will be understood this way. Then

$$u(p) = \big(\frac{p_0-p_1}{m}\big)^s\, l^s(p)/\overline{l^s(p)}\,. \tag{A.29}$$

We show equation (A.25) by expanding the denominator and numerator of $e^{i\,\Omega(\tilde\lambda_1(t),p)}$ into powers of $e^{2\pi t}$. Using

$$\big(\lambda_1(-t)p\big)_0 \pm \big(\lambda_1(-t)p\big)_1 = e^{\pm 2\pi t}(p_0 \pm p_1) \tag{A.30}$$

we get for the boost $\gamma\big(\lambda_1(-t)\cdot p\big)$ defined in equation (A.14)

$$\gamma\big(\lambda_1(-t)\cdot p\big) = \frac{e^{-2\pi t}(p_0-p_1)+e^{2\pi t}(p_0+p_1)-2ip_2}{e^{-2\pi t}(p_0-p_1)+e^{2\pi t}(p_0+p_1)+2m}\,. \tag{A.31}$$

Further we can write

$$\tilde\lambda_1(t) = (\,\gamma_t\,,0) \qquad \text{with} \qquad \gamma_t := \tanh(-2\pi t/2) = \frac{e^{-2\pi t}-1}{e^{-2\pi t}+1}\,. \tag{A.32}$$

Putting this into equation (A.18) yields after some calculation

$$\begin{aligned} e^{is\,\Omega(\tilde\lambda_1(t),p)} &= \left(\frac{1-\gamma(p)\gamma_t+\big(\gamma_t-\gamma(p)\big)\,\overline{\gamma\big(\lambda_1(-t)\cdot p\big)}}{1-\bar\gamma(p)\gamma_t+\big(\gamma_t-\bar\gamma(p)\big)\,\gamma\big(\lambda_1(-t)\cdot p\big)}\right)^s \\ &= \left(\frac{(p_0-p_1+m+ip_2)\big(e^{-2\pi t}(p_0-p_1)+m-ip_2\big)}{(p_0-p_1+m-ip_2)\big(e^{-2\pi t}(p_0-p_1)+m+ip_2\big)}\right)^s \\ &= \frac{l^s}{\bar l^s}(p)\cdot\frac{\bar l^s}{l^s}(\lambda_1(-t)p)\,. \end{aligned} \tag{A.33}$$

Using again equation (A.30), this shows equation (A.25). Note that all occuring numerators have stricly positive real part, hence the fractions are complex numbers (of modulus 1) in $\mathbb{C}\setminus\mathbb{R}_0^-$ and the powers to s can again be defined via the branch of the logarithm with $\ln 1 = 0$. Equation (A.26) is easily verified. Now let

$$l_\omega(p) := \frac{1}{2}(1+e^{-i\omega})\big(1+\frac{\sin\omega}{1+\cos\omega}\frac{p_2+im}{p_0-p_1}\big)\,. \tag{A.34}$$

The push-forward of this function under boosts $\lambda_1(t)$ may be read off equation (A.30), and its power to $2s$ is just our function $u_\omega(t,p)$:

$$l_\omega\big(\lambda_1(-t)\cdot p\big)^{2s} = \frac{1}{2^{2s}}(1+e^{-i\omega})^{2s}\big(1+\frac{\sin\omega}{1+\cos\omega}\frac{p_2+im}{p_0-p_1}e^{2\pi t}\big)^{2s} \;, t,\omega\in\mathbb{R}$$
$$= u_\omega(t,p)\;. \tag{A.35}$$

This implies in particular equation (A.23). Let $\omega\in[0,\pi)$. Then the number in brackets in the last factor has strictly positive imaginary part, and we conclude that this number is still in $\mathbb{C}\setminus\mathbb{R}_0^-$ if we let $t\in\mathbb{R}+i(-\frac{1}{2},0]$. The first factor also has strictly positive real part. Hence the powers to $2s$ are well defined via the usual branch of ln in the cut plane $\mathbb{C}\setminus\mathbb{R}_0^-$, and the function $u_\omega(z,p)$ is analytic in the strip. Equation (A.24) is easily verified. Note that for real t, $u_\omega(t,p)$ is well defined for all $\omega\in\mathbb{R}$ due to the relation

$$(1+e^{-i\omega})\frac{\sin\omega}{1+\cos\omega} = -i\,(1-e^{-i\omega})\;.$$

It remains to show equation (A.27). Exploiting the last relation one finds after a lenghty but straightforward calculation that

$$l\big(R(-\omega)p\big) = l(p)\cdot l_\omega(p) \quad \text{for all } \omega\in\mathbb{R}\;,\; p\in H_m\;. \tag{A.36}$$

Using the identity

$$l(p)\cdot\bar{l}(p) \;=\; 2(p_0+m)(p_0-p_1)\;,$$

we can rewrite the function $u(p)$ as

$$u(p) = \big(2m(p_0+m)\big)^{-s}\, l^{2s}(p)\;,$$

and hence equation (A.36) implies that

$$u\big(R(-\omega)p\big) \;\equiv\; \big(2m(p_0+m)\big)^{-s}\cdot l^{2s}(R(-\omega)p) \;=\; u(p)\cdot l_\omega(p)^{2s}\;.$$

Here we have exploited that all three functions occuring in equation (A.36) have values in the cut complex plane, hence $(l\cdot l_\omega)^{2s}$ and $l^{2s}\cdot l_\omega^{2s}$ are defined via the same branch of the logarithm and coincide. Taking account of equation (A.35), this shows equation (A.27). Here the reason becomes apparent why we have to take the unbounded factor p_0-p_1 into the definition of $u(p)$: without this factor, the last equation would read $u(R(-\omega)p) = u(p)\cdot l_\omega(p)^s/\bar{l}_\omega(p)^s$. But the analytic continuation of $\lambda_1(t)_*\bar{l}_\omega$ has zeroes in the strip and cannot serve to define an analytic function $u_\omega(z,p)$. □

PROOF OF PROPOSITION 4.22. We suppress the notation of the basis vectors $e_\pm$ of $\mathbb{C}^2$, which are only switched by the operator $\tilde{U}(\tilde{j})$. The representation $\tilde{U}$ of $\tilde{P}_+^\uparrow$ will be written as $\tilde{U}(\tilde{g})\phi = e^{is\Omega(\tilde{g},\cdot)}\,g_*\,\phi$, and we will use that for $g\in P_+^\uparrow$ the push-forward g_* commutes with the Fourier transformation, i.e. $g_*E_mf = E_mg_*f$. Let f be a Schwartz function with support in $R(-\omega)\cdot W_1$. We first have to show that $\tilde{U}\big((0,\omega)\big)\,u\cdot E_mf$ is in the domain of $\tilde{\Delta}^{\frac{1}{2}}$. To this end, we perform the transformations

$$\begin{aligned}
&\tilde{U}\big(\tilde{\lambda}_1(t)\big)\,\tilde{U}\big((0,\omega)\big)\,u\cdot E_mf && && \text{(A.37)}\\
&= e^{is\,\Omega(\tilde{\lambda}_1(t),\,\cdot\,)}\,e^{is\omega}\;\lambda_1(t)_*R(\omega)_*(u\cdot E_mf) && && \\
&= e^{is\omega}\,e^{is\,\Omega(\tilde{\lambda}_1(t),\,\cdot\,)}\;\lambda_1(t)_*\big(u\cdot u_\omega(0,\cdot)\cdot R(\omega)_*E_mf\big) && \text{by eq. (A.27)} && \\
&= e^{is\omega}\,e^{-2\pi st}\;u\cdot\lambda_1(t)_*u_\omega(0,\,\cdot\,)\cdot\lambda_1(t)_*R(\omega)_*E_mf && \text{by eq. (A.25)} && \\
&= e^{is\omega}\,e^{-2\pi st}\;u\cdot u_\omega(t,\,\cdot\,)\cdot\lambda_1(t)_*E_mR(\omega)_*f && \text{by eq. (A.23)} && \text{(A.38)}\\
&=: e^{is\omega}\,e^{-2\pi st}\;v(t)\cdot\phi(t) && && \text{(A.39)}
\end{aligned}$$

where we have written

$$v(t)(p) := u(p)\cdot u_\omega(t,p) \quad \textit{and} \tag{A.40}$$
$$\phi(t)(p) := \big(\lambda_1(t)_*\,E_m\,R(\omega)_*f\big)(p)$$
$$\equiv (2\pi)^{-3/2}\int_{W_1}\mathrm{d}^3x\;e^{i\,p\cdot\lambda_1(t)\,x}\;(R(\omega)_*\,f)(x)\;. \tag{A.41}$$

We discuss analyticity in t of the last function. For fixed $p \in H_m$, we can extend $\phi(t)(p)$ as a function of t into the strip $\mathbb{R} - i\,[\,0,\frac{1}{2}\,]$ as follows. For any $x \in \mathbb{R}^3$, the vector valued function $t \mapsto \lambda_1(t)x$ can be extended to an entire analytic function on the complex plane, since its dependence on t via $\sinh 2\pi t$ and $\cosh 2\pi t$ is entire analytic. One computes for $t, t' \in \mathbb{R}$

$$\begin{aligned}
\big(\lambda_1(t+it')x\big)^0 &= \cos(2\pi t')\big(\lambda_1(t)x\big)^0 - i\sin(2\pi t')\big(\lambda_1(t)x\big)^1 \,, \\
\big(\lambda_1(t+it')x\big)^1 &= \cos(2\pi t')\big(\lambda_1(t)x\big)^1 - i\sin(2\pi t')\big(\lambda_1(t)x\big)^0 \,, \\
\big(\lambda_1(t+it')x\big)^2 &= x^2 \,.
\end{aligned}$$

This can be rewritten as

$$\lambda_1(t+it')\,x = j_{t'}\,\lambda_1(t)\,x - i\sin(2\pi t')\,\underline{\sigma}\,\lambda_1(t)\,x \,, \tag{A.42}$$

where

$$j_{t'} := \begin{pmatrix} \cos 2\pi t' & 0t & 0 \\ 0 & \cos 2\pi t' & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \underline{\sigma} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{A.43}$$

Note that in particular

$$\lambda_1(-\frac{i}{2})\,x = j\,x \,. \tag{A.44}$$

For $x \in W_1$ the vector $\underline{\sigma}x$ lies in the forward light cone, and hence for $p \in H_m$ the function $p \cdot \underline{\sigma}x$ is bounded below by a function which is linear in $|p_1|$. This produces a damping factor

$$e^{\sin(2\pi t')\;p\cdot\underline{\sigma}\,\lambda_1(t)\,x}$$

in equation (A.41) if we replace t by $t + it' \in \mathbb{R} - i\,(\,0,\frac{1}{2}\,)$, because the support of $R(\omega)_*f$ is in W_1 by assumption. Recall that $\phi(t)$ is for real t a Schwartz function on the mass shell, and the same holds for $it' = -i/2$, since

$$\phi(-\frac{i}{2}) = E_m j_* R(\omega)_* f = \; R(-\omega)_* E_m j_* f \tag{A.45}$$

by equation (A.44) and the group relation $j\,R(\omega) = R(-\omega)\,j.$. Now due to the damping factor, $\phi(z)$ is still a Schwartz function for all z in the strip $\mathbb{R} - i\,[\,0,\frac{1}{2}\,]$ — even for $\mathrm{Im}z = -\frac{1}{4}$ where the transformation $j_{-\frac{1}{4}}$ has a vanishing determinant. The dependence of $\phi(z)$ on z is pointwise analytic, and we denote the pointwise derivative by $\phi'(z)(p)$, i.e.

$$\phi'(z)(p) = (2\pi)^{-3/2} \int_{W_1} \mathrm{d}^3x\; i\,p\cdot\lambda_1'(z)\,x\; e^{i\,p\cdot\lambda_1(z)\,x}\;(R(\omega)_*\,f)(x) \,.$$

Having the same damping factor as above, $\phi'(z)(p)$, is also a Schwartz function for all $z \in \mathbb{R} - i\,[\,0,\frac{1}{2}\,]$. Now $\phi(z)$ is analytic as a Hilbert space valued map in $c \in \mathbb{R} - i(0,\frac{1}{2})$ iff the the difference quotient

$$z \;\mapsto\; \phi_c(z) := \frac{\phi(z) - \phi(c)}{z - c} \;, \quad z, c \in \mathbb{R} - i\,(\,0,\frac{1}{2}\,) \tag{A.46}$$

is continuous in c, converging to $\phi'(c)$. This is in fact the case, since for z in a fixed disc around c the function $p \mapsto |\phi_c(z)(p) - \phi'(c)(p)|^2$ is dominated by an integrable function which we denote by $|\Phi(c)(p)|^2$, and we may therefore apply Lebesgue's theorem on dominated convergence. In fact, $\Phi(c)$ is of the same form as $\phi'(c)$ and is hence also a Schwartz function, which we shall use later.

Next we discuss the large p behaviour of the continuous function $v(z)$ of equation (A.40) for $z \in \mathbb{R} - i[0,\frac{1}{2}]$. We consider only the case $s \geq 0$, the other case working analogously. Then since $|l(p)/\overline{l(p)}| = 1$,

$$|v(z)(p)| = |a_\omega|\;\Big|\frac{p_0 - p_1}{m}\Big|^s\;\Big|1 + b_\omega(z)\,\frac{p_2 + im}{p_0 - p_1}\Big|^{2s} \;,$$

where

$$a_\omega = 2^{-2s}(1+e^{-i\omega})^{2s} \quad ,$$
$$b_\omega(z) = \frac{\sin\omega}{1+\cos\omega}\, e^{2\pi z} \quad .$$

If $p_1 < 0$, then $p_0 - p_1$ is bounded by $m < p_0 - p_1 < 2|p_1| + |p_2| + m$. Hence $|v(z)(p)|$ is bounded for large negative p_1 and large $|p_2|$ by

$$\left| a_\omega \left(\frac{2|p_1|+|p_2|}{m}\right)^n \left(b_\omega(z)\,\frac{p_2}{m}\right)^{2n} \right| ,$$

where n is a natural number greater than s. For $p_1 > 0$, we use the identity

$$\frac{p_2 + im}{p_0 - p_1} = i\,\frac{p_0 + p_1 + m - ip_2}{p_0 - p_1 + m + ip_2} \qquad \text{for all } p \in H_m \ .$$

Since for $p_1 > 0$ we have $m < p_0 + p_1 < 2p_1 + |p_2| + m$ and $p_0 - p_1$ converges to zero as p_1 goes to infinity, $|v(z)(p)|$ is bounded for large positive p_1 and large $|p_2|$ by

$$\left| a_\omega \left(2b_\omega(z)\,\frac{|p_1|+|p_2|}{m}\right)^{2n} \right| .$$

We conclude that $v(z)(p)$ is polynomially bounded for large p. The same holds for its derivative with respect to z for fixed p, which we denote by $v'(c)(p)$, and for the difference quotient $v_c(z) := (z-c)^{-1}(v(z) - v(c))$.

Now the product of each of the functions $v(z), v'(c)$ and $v_c(z)$ with each of the above Schwartz functions $\phi(z), \phi'(c)$ and $\phi_c(z)$ is again a Schwartz function on the mass shell. In order to prove our claim, namely that $z \mapsto v(z)\cdot\phi(z)$ is analytic in the L^2-norm, we have to show that the difference quotient

$$v(c)\cdot\phi_c(z) + v_c(z)\cdot\phi(c) + (z-c)\,v_c(z)\cdot\phi_c(z)$$

is continuous in z, the summands converging to $v(c)\cdot\phi'(c)$, $v'(c)\cdot\phi(c)$ and zero, respectively, as z goes to c. All three summands converge pointwise, and we have to look for integrable dominating functions. To show that $v(c)\cdot(\phi_c(z) - \phi'(c))$ converges to zero, we use the dominating function $|v_c(z)(p)|^2\,|\Phi(c)(p)|^2$, where $\Phi(c)(p)$ is the Schwartz function which was used to show that $\phi(z)$ is analytic. For the second summand, we recall that $v_c(z) - v'(c)$ is a polynomial in p with coefficients depending continuously on z . Hence for z in a given disc around c we can take the maximum of the coefficients of highest order to define a polynomial $V(p)$ such that $|V(p)\cdot\phi(c)(p)|^2$ is an integrable dominating function for $|(v_c(z)(p) - v'(c)(p))\cdot\phi(c)(p)|^2$. Hence the second summand converges to $v'(c)\cdot\phi(c)$. The third summand converges to zero since the norm $v_c(z)\cdot\phi_c(z)$ is bounded for z in a given disc around c by the norm of $V_1\cdot\Phi(c)$, where $V_1(p)$ is a polynomial constructed in analogy to $V(p)$.

Hence $v(z)\cdot\phi(z)$ is analytic in the strip $\mathbb{R} - i\,(0,\frac{1}{2})$ and we conclude that, coming back to equation (A.37), the vector $\tilde{U}\big((0,\omega)\big)\,u\cdot E_m f$ is in the domain of $\tilde{\Delta}^{\frac{1}{2}}$, and that

$$\begin{aligned}
&\tilde{\Delta}^{\frac{1}{2}}\,\tilde{U}\big((0,\omega)\big)\,u\cdot E_m f && \text{by eq. (A.39)}\\
&= e^{is\omega}\,e^{i\pi s}\,u\cdot u_\omega(-\tfrac{i}{2},\cdot)\cdot\phi(-\tfrac{i}{2}) && \\
&= e^{i\pi s}\,e^{-is\omega}\,u\cdot u_{-\omega}(0,\cdot)\cdot R(-\omega)_*\,E_m j_* f && \text{by eq's (A.45) and (A.24)}\\
&= e^{i\pi s}\,e^{-is\omega}\,R(-\omega)_*\big(u\cdot E_m j_* f\big) && \text{by eq. (A.27)}\\
&= e^{i\pi s}\,\tilde{U}\big((0,-\omega)\big)\,(-j)_*\big(\overline{u\cdot E_m f}\big) && \text{by eq. (A.26)}\\
&= e^{i\pi s}\,\tilde{U}(\tilde{j})\,\tilde{U}\big((0,\omega)\big)\,u\cdot E_m\bar{f}\ , &&
\end{aligned}$$

where we have used in the last but one step that $\overline{E_m j_* f} = (-j)_* E_m \bar{f}$. Multiplying the first and last member by $e^{i\pi s}\,\tilde{U}(\tilde{j})$ we get the claimed equation (4.81). □

PROOF OF PROPOSITION 5.7. Again we discuss only the case $s \geq 0$. We identify the restriction of $S(\tilde{W})$ to $\mathcal{H}^{(1)}$ with $\tilde{S}(\tilde{W})^{(1)}$ by equation (4.76). Let $\omega \in (-\pi, 0)$, and let $\tilde{g} := (0, -\frac{\pi}{2})\,\tilde{\lambda}_1(t')\,(0,\omega')$ with $t' \neq 0$ and $\omega' \in (-\pi, 0)$, where the parameters t' and ω' are chosen such that $g\cdot W_1$ contains the intersection $W_1 \cap R(\omega)\cdot W_1$. Let further f be a Schwartz function with support in $W_1 \cap R(\omega)\cdot W_1$. By Lemma A.2, $u \cdot E_m f$ can be in the domain of $S(\tilde{g}\cdot W_1)$ only if the map

$$t \mapsto \tilde{\Delta}^{it}\, \tilde{U}(\tilde{g})^{-1}\, u \cdot E_m f \tag{A.47}$$

is analytic in the strip $\mathbb{R}+i(-\frac{1}{2},0)$. Using the formulae of Lemma A.4 and the cocycle property

$$\Omega\big(\,(0,-\omega')\,\tilde{\lambda}_1(-t')\,(0,\frac{\pi}{2})\,,\,p\big) = -\omega' + \Omega\big(\,\tilde{\lambda}_1(-t')\,,\,R(\omega')p\big) + \frac{\pi}{2}$$

of the Wigner rotation, one calculates for (A.47)

$$\begin{aligned}
&\tilde{U}\big(\tilde{\lambda}_1(t)\big)\,\tilde{U}(\tilde{g}^{-1})\,u\cdot E_m f\\
&= e^{is\,\Omega(\tilde{\lambda}_1(t),\,\cdot\,)}\;\lambda_1(t)_*\,\Big(e^{is\,\Omega(\tilde{g}^{-1},\,\cdot\,)}g_*^{-1}u\cdot g_*^{-1}\cdot E_m f\Big)\\
&= e^{is(\frac{\pi}{2}-\omega')}\;e^{is\,\Omega(\tilde{\lambda}_1(t),\,\cdot\,)}\;\lambda_1(t)_*\,\Big(R(-\omega')_*\big(e^{is\,\Omega(\tilde{\lambda}_1(-t'),\,\cdot\,)}\lambda_1(-t')_*u\big)\cdot R(-\omega')_*\lambda_1(-t')_*l^{2s}_{\frac{\pi}{2}}\cdot E_m\,g_*^{-1}\,f\Big)\\
&= e^{is(\frac{\pi}{2}-\omega')}\;e^{is\,\Omega(\tilde{\lambda}_1(t),\,\cdot\,)}\;\lambda_1(t)_*\,\Big(e^{2\pi st'}\;u\cdot l^{2s}_{-\omega'}\cdot R(-\omega')_*\lambda_1(-t')_*l^{2s}_{\frac{\pi}{2}}\cdot E_m g_*^{-1}f\,\Big)\\
&= e^{is(\frac{\pi}{2}-\omega')}\;e^{-2\pi s(t-t')}\,u\cdot\lambda_1(t)_*\big(\,l^{2s}_{-\omega'}\;\cdot\;R(-\omega')_*\lambda_1(-t')_*l^{2s}_{\frac{\pi}{2}}\,\big)\;\cdot\;\lambda_1(t)_*E_m\,g_*^{-1}\,f\;.
\end{aligned}$$

Here, l_ω is the family of functions on the mass shell defined in equation (A.34). We rewrite this expression as

$$\tilde{U}\big(\tilde{\lambda}_1(t)\big)\,\tilde{U}(\tilde{g})^{-1}\;u\cdot\,E_m f = \;h(t)^{2s}\cdot\psi(t)\qquad\text{for all } t\in\mathbb{R} \tag{A.48}$$

where $h(t)$ and $\psi(t)$ are functions on the right hand side are defined by

$$\begin{aligned}
\psi(t) &:= e^{is(\frac{\pi}{2}-\omega')}e^{-2\pi s(t-t')}\;u\cdot\lambda_1(t)_*E_m\,g_*^{-1}\,f\qquad\text{and}\\
h(t) &:= \lambda_1(t)_*\big(\,l_{-\omega'}\;\cdot\;R(-\omega')_*\lambda_1(-t')_*l_{\frac{\pi}{2}}\,\big)\;.
\end{aligned}$$

Note that g_*f has support in $g\cdot\big(W_1\cap R(\omega)\cdot W_1\big)$ which by assumption is contained in W_1. Hence we know from the proof of Poposition 4.22 that $\psi(t)$ is analytic in the strip (put $\omega=0$ in equation (A.38)). On the other hand, a lengthy but straightforward calculation shows that $h(t)$ is of the form

$$h(t,p) = c\;\frac{e^{-4\pi t}\;a_2 + e^{-2\pi t}\;a_1(p) + a_0(p)}{e^{-2\pi t}\big(e^{-2\pi t} - b(p)\big)}\;,$$

where a_0, a_1, a_2, b and c denote the following constants and functions on the mass shell:

$$\begin{aligned}
a_0(p) &= \frac{(p_0+p_1)}{(p_0-p_1)}\;\big(1-\cos\omega' - e^{2\pi t'}\,\sin\omega'\big)\\
a_1(p) &= 2\;\frac{p_2(e^{2\pi t'}\,\cos\omega' - \sin\omega') + ie^{2\pi t'}m}{(p_0-p_1)}\\
a_2\; &= 1+\cos\omega' + e^{2\pi t'}\,\sin\omega'\\
b(p) &= \frac{\sin\omega'}{1+\cos\omega'}\;\frac{p_2 - im}{p_0-p_1}\\
c\; &= \frac{(1-i)(1+e^{i\omega'})}{1+\cos\omega'}\;.
\end{aligned}$$

For ω' in the specified range, the function b has values in the upper half plane $\mathbb{R}+i\mathbb{R}^+$, and therefore the denominator of $h(t)(p)$ has a zero in the strip $\mathbb{R}+i(-\frac{1}{2},0)$ as a function of t. Having different zeros for generic values of t' (more precisely, for t' in a punctured neighbourhood of 0), the numerator cannot cancel this singularity, and nor can $\psi(t)$ for a generic Schwartz function f. If $s\in\{\frac{1}{2},1,\frac{3}{2},\dots\}$, this produces for every $t\in\mathbb{R}+i(-\frac{1}{2},0)$ a pole of the function $h(t,p)^{2s}$ of the order $|p|^{-2s}$, where $|p| := \sqrt{p_1^2+p_2^2}$, rendering $h(t)^{2s}\cdot\psi(t)$ not square integrable. Hence $\tilde{U}(\tilde{g})^{-1}\,u\cdot E_m f$ cannot be in the domain of $\tilde{\Delta}^{\frac{1}{2}}$. The nature $|p|^{-2s}$ of the pole can be seen by considering the point $e^{-2\pi t_0} = -i\,\frac{\sin\omega'}{1+\cos\omega'}$, where $h(t_0,p)$ has a pole at $p_1=p_2=0$. Its order can be estimated by writing the relevant factor as

$$\frac{1}{e^{-2\pi t_0} - b(p)} = \frac{1+\cos\omega'}{-\sin\omega'}\cdot\frac{p_0-p_1}{p_2 + i(p_0-p_1-m)}$$

and estimating the denominator via the formula $p_0 - m = \frac{|p|^2}{2m} + O(|p|^4) = O(|p|^2)$ with

$$|p_2 + i(p_0 - m - p_1)|^2 = p_2^2 + \big(-p_1 + O(|p|^2)\big)^2 = O(|p|^2) \quad \text{as } |p| \to 0 \,.$$

On the other hand, if $s \in \mathbb{R}\backslash \frac{1}{2}\mathbb{Z}$, then $h(t,p)^{2s}$ has a branch point. This is in contradiction to the analyticity of $\tilde{\Delta}^{it}\, \tilde{U}(\tilde{g})^{-1}\, u \cdot E_m f$ as follows. Let us denote $\tilde{U}(\tilde{g})^{-1}\, u \cdot E_m f =: \phi$. Suppose $\tilde{\Delta}^{it}\, \phi$ is analytic. Then the same holds for $E_V\, \tilde{\Delta}^{it}\, \phi$, where V is a compact subset of the mass shell on which $\tilde{\Delta}^{it}\, \phi$ does not vanish identically, and E_V denotes the projector in $L^2(H_m, \mathrm{d}\mu)$ onto functions with support in V. Let D be the set of complex numbers z in the strip $\mathbb{R} + i(-\frac{1}{2}, 0)$ such that $e^{-2\pi z} - b(p) = 0$ for some $p \in V$. If we analytically continue the L^2-valued map $E_V\, \tilde{\Delta}^{it}\, \phi$ along a closed path in $\mathbb{R} + i(-\frac{1}{2}, 0) \setminus D$ with winding number 1 with respect to D (and winding number 0 with respect to the zeroes of the numerator of $h(t)(p)$), it will pick up a factor $e^{4\pi i s}$ due to the function $h(t,p)^{2s}$ and hence it cannot be analytic in D. We conclude that $u \cdot E_m f$ is not in the domain of $S(\tilde{g}\cdot\tilde{W}_1)$. This implies equation (5.16). □