CHAPTER 5

Can there be Free Fields for Anyons?

What is a "free field"? – In $3+1$ dimensions this question has a quite definite answer: It is the *free Fock space field*, which associates to a given particle type a quantum field (see equation (5.2) below). For anyons in $2+1$ dimensions, in contrast, free fields satisfying the requirements from relativistic quantum field theory have not been constructed yet[1]. In fact, the original yet unreached aim of the present thesis was to construct such a model. Instead, we were able to show that such a model *cannot* exist for anyons without giving up certain ideas one has about "free fields".

Let's recall what the free field is in $3+1$ dimensions. An elementary particle, alone in the world, should be quantum mechanically described by an irreducible ray representation of the Poincaré group. This idea is due to E.P. Wigner, who also largely solved the mathematical problem of classifying these [Wig39], i.e. of *deducing* what particle types may occur – a highlight in the interplay between mathematics and phsics. Yet, the corresponding single particle theories did not seem to support a satisfactory localization concept[2], and also do not offer a principle to incorporate interaction in an unambiguous way, or even to take account of annihilation and creation of particles encountered in high energy physics. These issues have been solved by the so-called second quantization, where the single particle wave function is replaced by an operator valued local quantum field acting in the bosonic or fermionic Fock space over the one particle space. It satisfies the Wightman axioms and describes a system of arbitrarily many noninteracting identical particles.

In $2+1$ dimensions, however, the situation is far from being so satisfying. As we have seen in Section 1.3, in $d=2+1$ the classification of elementary particles may be carried through as well, the massive particles being labelled by their mass $m>0$ and spin $s\in\mathbb{R}$, which labels the representation of the universal covering $\mathbb{R}$ of the rotation subgroup $SO(2)$. But a straightforward definition of a second quantized field as in equation (5.2) fails for anyons (i.e. particles with non semiinteger spin) on account of two apparent complications: The Hilbert space $\mathcal{H}$ does not have a canonical Fock space structure, and there are no so-called u- and v-spinors, which in $d=3+1$ serve to make the free fields transform under a finite dimensional (non-unitary) representation of the Lorentz group, see equation (5.1).

In fact, a large part of the time this Ph.D. thesis took the author has been used up in the unsuccessful attempt to overcome these two difficulties and construct a direct substitute of the free Fock space field. A (non-canonical) Fock space structure may be imposed on $\mathcal{H}$ by a trivialization of the vector bundle in terms of which the anyonic Hilbert space is described [Mun92, MS95]. The problem of the u- and v- spinors may be either avoided by the use of the modular localization concept to be introduced below, or settled by using an infinite dimensional representation of the universal covering group of the Lorentz group, which is in $d=2+1$ the universal covering of $SL(2,\mathbb{R})$[3].

After constructive attempts at a direct substitute of the free Fock space field have failed, a more general characterization of a "free field" was desireable. We advocate the viewpoint

[1] D.R.Grigore has constructed free fields in $d=2+1$ for any spin [Gri94] in a bosonic Hilbert space, but in contradiction to the weak spin statistics connection mentioned in the introduction [Fre89, FM89], all of these fields have bosonic statistics. Presumably, this is due to the fields having infinitely many components.

[2] see, however, the remarks at the beginning of Section 5.2 on (modular) localization in Hilbert space.

[3] This route has been proposed by R. Schrader and the author in [MS95], where the ribbon braid group is considered instead of the braid group and the so-called principal series representation of $\widetilde{SL(2,\mathbb{R})}$ plays the role of the abovementioned representation D. – D. R. Grigore has used a similar solution in the mentioned article [Gri94]

that the essential feature of a free field is that it establishes a "second quantization functor" assigning to each single particle space $\mathcal{H}^{(m,s)}$ a field algebra, which is thus altogether with its localization concept (namely, the net structure, microcausality and Poincaré covariance) completely determined by the single particle space. Note that this implies that already on the single particle level there is a notion of localization – this is the concept of modular localization, which we introduce in Section 5.2. To be specific, we assume that the local algebras of a free theory are generated by basic fields which create only single particle states out of the vacuum (Section 5.1). But under slightly stronger assumptions, we can establish two no-go results: In Section 5.2 we assume that the basic fields are determined by the single particle vectors which they create from the vacuum, thus incorporating the idea that the field algebra is fixed by the single particle space and its localization concept. We show that in the case of non-zero spin this assumption, together with a certain intersection property of the field algebra (equation (5.12)) which is usually satisfied if the field algebras are generated by local quantum fields, is in conflict with the assumption of modular covariance. This is achieved by using our results on the explicit form of the Tomita operator for wedge regions in Section 4.3 and Appendix A.3. The conflict is due to the representation of the Wigner rotation and is peculiar to $2+1$ dimensions. In Section 5.3 we show that the basic fields must violate a certain mild regularity condition, reminiscent of the Wightman axioms. Otherwise we can establish, via a theorem à la Jost-Schroer, commutation relations of the fields which are not compatible with anyonic statistics. The basic idea which has lead the author to this no-go theorem is due to B. Schroer, and the result has been published in [Mun98].

5.1. What is a "Free Field"?

We briefly review the definition of free Fock space fields in $3+1$ dimensions. The physically relevant irreducible ray representations of the Poincaré group (hence particle types) are labelled by the mass parameter $m \geq 0$ and the spin (respectively helicity for $m = 0$) $s \in \{0, \frac{1}{2}, 1, \dots\}$ which determines a unitary irreducible representation V_s of the rotation subgroup $SU(2)$ in $\mathbb{C}^{2s+1}$ (respectively the twofold covering of $SO(2)$). So-called u- and v-spinors intertwine V_s with a (non-unitary) finite dimensional representation[4] D of the covering group $SL(2,\mathbb{C})$ of the Lorentz group in the sense that

$$\underline{u}(p)\, V_s(R(A,p)) = D(A)\, \underline{u}(A^{-1}\cdot p) \qquad \text{for all } A \in SL(2,\mathbb{C}),\ p \in H_m\,, \tag{5.1}$$

where $R(A,p) \in SU(2)$ is the Wigner rotation, and $\underline{v}(p)$ is defined by $\underline{v}(p) := \underline{u}(p)\, V_s(\sigma_2)$. Then the (neutral) *free Fock space field* for the particle type (m,s), transforming according to D, is the operator valued distribution in the bosonic or fermionic Fock space over the one particle space $\mathcal{H}^{(m,s)}$ defined by

$$\Phi_\nu(x) = (2\pi)^{-\frac{3}{2}} \int_{H_m} \mathrm{d}\mu(p) \sum_{k=1}^{2s+1} \left(e^{-ip\cdot x} u_{\nu k}(p)\, a_k(p) + e^{ip\cdot x} v_{\nu k}(p)\, a_k^*(p) \right) ,\nu = 1, \dots, L. \tag{5.2}$$

The u- and v-spinors are responsible that the covariance properties of the creation and annihilation operators $a^*(p), a(p)$ are carried over from p-space to x-space by the Fourier transformation. The net $(\mathcal{F}(\mathcal{O}))_\mathcal{O}$ of von Neumann algebras generated by the smeared Fock space fields via

$$\mathcal{F}(\mathcal{O}) = \left\{ \exp i(\Phi(f) + \Phi(f)^*) \mid f \in \mathcal{S}(\mathcal{O};\mathbb{C}^L) \right\}'' \tag{5.3}$$

contains the physically relevant information and does not depend on the representation $D(SL(2,\mathbb{C}))$ used in the definition of the free Fock space field, i.e. it is completely determined by the particle type (m,s). In fact, (5.3) may be reformulated without taking recourse to the u- and v-spinors and the representation D. Recall that for every single particle vector ψ the so-called Segal-operator $\varphi(\psi)$ is a self-adjoint operator in Fock space which creates ψ out of the vacuum, i.e. $\varphi(\psi)\,\mathbf{\Omega} = \psi$, and is uniquely determined by ψ on a certain core independent of ψ[5]. Now $\Phi(f) + \Phi(f)^*$ may be rewritten as $\varphi(\hat{f})$ where $\hat{f}$ is the single particle vector $\hat{f} = \underline{v}^t \cdot E_m f + \underline{u}^* \cdot E_m \bar{f}$. Here $E_m f$ denotes the restriction of the Fourier transform of f to the

[4]In fact, for given s there are several representations D possible.

[5]see [RS75]

mass hyperboloid, and we have used matrix notation. Further, the set of single particle vectors $\hat{f}$ arising from functions f with support in a given space time region $\mathcal{O}$ may be characterized without taking recourse to Fourier transformation, namely via modular localization, which we introduce in Section 5.2. Denoting this set by $K(\mathcal{O})$, we thus have rewritten $\mathcal{F}(\mathcal{O})$ as the algebra generated by the Segal operators $\varphi(\psi)$ with $\psi \in K(\mathcal{O})$.

Summing up, the free field construction (5.2) establishes a "second quantization functor", assigning to each single particle state space $\mathcal{H}^{(m,s)}$ a local net $(\mathcal{F}(\mathcal{O}))_{\mathcal{O}}$ of field algebras. We advocate the viewpoint that this is the essential feature of a free field, and are led to the following definition.

DEFINITION 5.1. A field algebra $\mathcal{F} = (\mathcal{F}(\tilde{I}))_{\tilde{I}\in\tilde{\mathcal{K}}}$ is called a *free field algebra*, if each $\mathcal{F}(\tilde{I})$ is generated by a star stable linear set $\Phi(\tilde{I})$ of closed operators in $\mathcal{H}$ (the *free fields localized in* $\tilde{I}$) creating only one particle states out of the vacuum:

$$\varphi\,\mathbf{\Omega} \in \mathcal{H}^{(1)} \quad \text{for all } \varphi \in \Phi(\tilde{I})\,, \tag{5.4}$$

and the set of these one particle states is dense in $\mathcal{H}^{(1)}$.

To appreciate this definition, recall that in $3+1$ dimensions the Jost-Schroer theorem[6] asserts that if the free fields are assumed to be Wightman fields, the above condition already characterizes them as the free Fock space fields (5.2). (In fact, this line of argumentation leads for anyons to our no-go result in Section 5.3.)

For notational conveniance, we will consider in this chapter only one elementary charge $\chi_{\rm e}$ generating the group Γ of sectors, i.e. $\Gamma = \{\chi_{\rm e}^q \,|\, q \in \mathbb{Z}\,\}$. We denote by s the spin of $\chi_{\rm e}$, by $\omega \equiv \exp 2\pi i s$ its statistics phase, and by m its mass. The set $\Gamma^{(1)}$ of single particle sectors contains $\chi_{\rm e}$ and $\chi_{\rm e}^{-1}$ and may or may not contain other sectors.

5.2. Free Fields and Modular Localization

We first introduce the concept of modular localization in Hilbert space, which is not tight to the free field case (see, e.g. [Sch97]).

Modular Localization in Hilbert Space. Localization of observables is one of the fundamental concepts of algebraic quantum field theory. We have seen that it entails a notion of localization also for the (unobservable) field algebra. For states, however, the notion of localization has turned out less appropriate. Newton and Wigner have proposed a definition of localization for single particle state vectors of massive elementary particles [NW94], but this concept of localization of a particle is meaningful only for accuracies above the order of a compton wave length. Yet, as mentioned in the introduction to this chapter, the localization concept of of the field algebra – to be specific: the 'net' structure, microcausality and Poincaré covariance – can be carried over to the Hilbert space $\mathcal{H}$ on which the field algebra acts. The idea is to consider states in $\mathcal{F}(\tilde{I})\,\mathbf{\Omega}$ as localized in $\tilde{I}$ (relative to the vacuum $\mathbf{\Omega}$), and find a topology such that the closure of this set in $\mathcal{H}$ may be viewed as the subspace of vectors localized in $\tilde{I}$. It must be finer than the norm topology, which destroys all information about localization, since by the Reeh-Schlieder property $\mathcal{F}(\tilde{I})\,\mathbf{\Omega}$ is norm-dense in $\mathcal{H}$ for any $\tilde{I}$. The suitable topology is the graph norm topology of the Tomita-operator for $\mathcal{F}(\tilde{I})$ and $\mathbf{\Omega}$. (The closure is then just the domain of the Tomita-operator.) We will work out this concept in the present subsection and end up with a family of closed subspaces of $\mathcal{H}$ which reflect the defining properties of a field algebra, see Definition 4.1. Thereby this family encodes in particular a notion of localization, which we call *modular localization in* $\mathcal{H}$. The relevant properties are listed in Proposition 5.3.

The Tomita-operator for $\mathcal{F}(\tilde{I})$ and $\mathbf{\Omega}$ will be denoted by $S(\tilde{I})$. It is an unbounded antilinear operator satisfying $S(\tilde{I})^2 = \mathbb{1}$. By virtue of the latter property, any vector in the domain of $S(\tilde{I})$ may be uniquely written as $\psi = \psi_+ + \psi_-$ with $S(\tilde{I})\,\psi_\pm = \pm\,\psi_\pm$, namely $\psi_\pm := \frac{1}{2}(\psi \pm S(\tilde{I})\psi)$. We denote by $K(\tilde{I})$ the eigenspace to $+1$:

$$K(\tilde{I}) = \{\,\phi \in \mathrm{dom}S(\tilde{I})\,|\,S(\tilde{I})\,\phi = \phi\,\}\,.$$

[6] This theorem is due to B. Schroer [Sch58] and has been elaborated by R. Jost [Jos61] and further by K. Pohlmeyer [Poh69]. For a didactic account, see [SW64, Thm. 4-15]. O. Steinmann has extended it to string-localized fields satisfying modified Wightman assumptions and Bose or Fermi statistics [Ste82].

It is a real linear subspace of $\mathcal{H}$, which carries the norm induced by the norm on $\mathcal{H}$ and is closed in the corresponding topology.[7] Since $S(\tilde{I})$ is antilinear, multiplication with the imaginary unit i maps its eigenspace to $+1$ onto the eigenspace to -1 and vice versa, and hence the above discussion implies that

$$K(\tilde{I}) + iK(\tilde{I}) = \mathrm{dom}S(\tilde{I}) \quad \text{and} \quad K(\tilde{I}) \cap iK(\tilde{I}) = \{0\}\,. \tag{5.5}$$

Due to these properties, the operator $S(\tilde{I})$ can be recovered from $K(\tilde{I})$ by setting

$$S(\tilde{I})(\psi + i\phi) := \psi - i\phi \quad \text{for all } \psi, \phi \in K(\tilde{I})\,,$$

and is thus completely determined by the real space $K(\tilde{I})$. For any subset M of $\mathcal{H}$ we define the *symplectic complement* M' of M as the set of vectors $\psi \in \mathcal{H}$ satisfying $\mathrm{Im}\langle\psi, \phi\rangle = 0$ for all $\phi \in M$. It is a closed real linear space [LRT78].

LEMMA 5.2. *$K(\tilde{I})$ and its symplectic complement may be directly obtained from $\mathcal{F}(\tilde{I})$ through the relations*

$$K(\tilde{I}) = \overline{\mathcal{F}(\tilde{I})^{\mathrm{sa}}\,\mathbf{\Omega}} \qquad \textit{and} \tag{5.6}$$

$$K(\tilde{I})' = \overline{(\mathcal{F}(\tilde{I})')^{\mathrm{sa}}\,\mathbf{\Omega}}\,. \tag{5.7}$$

Here the closures are understood in the norm topology, and $\mathcal{F}(\tilde{I})^{\mathrm{sa}}$ denotes the set of self adjoint elements in $\mathcal{F}(\tilde{I})$.

PROOF. A vector ϕ is in $K(\tilde{I})$ if and only if it is in the domain of $S(\tilde{I})$ and $S(\tilde{I})\,\phi = \phi$, i.e. if and only if there is a sequence of operators B_n in $\mathcal{F}(\tilde{I})$ such that both $B_n\,\mathbf{\Omega}$ and $B_n^*\,\mathbf{\Omega}$ converge to ϕ. Setting $A_n := \frac{1}{2}(B_n + B_n^*)$, this is equivalent to the existence of a sequence A_n of self adjoint operators in $\mathcal{F}(\tilde{I})$ such that $A_n\,\mathbf{\Omega}$ converges to ϕ. This shows the first equation. Now let $\psi \in K(\tilde{I})'$. Then for all $\phi_1, \phi_2 \in K(\tilde{I})$

$$\langle\,\psi\,,\,\phi_1 + i\phi_2\,\rangle = \langle\,\phi_1 - i\phi_2\,,\,\psi\,\rangle \equiv \langle\,S(\tilde{I})(\phi_1 + i\phi_2)\,,\,\psi\,\rangle\,.$$

Hence[8] ψ is in the domain of $S(\tilde{I})^*$ and $S(\tilde{I})^*\psi = \psi$. Conversely, if ψ satisfies the latter condition, it is clearly in $K(\tilde{I})'$. Hence $K(\tilde{I})'$ is the eigenspace of $S(\tilde{I})^*$ to the eigenvalue $+1$. By virtue of Tomita-Takesaki theory, $S(\tilde{I})^*$ is just the Tomita-operator for $\mathcal{F}(\tilde{I})'$ and $\mathbf{\Omega}$. Hence the second equation may be proved with the same argumentation as the first one. □

PROPOSITION 5.3. *The family $(K(\tilde{I}))_{\tilde{I}\in\widetilde{\mathcal{K}}}$ satisfies the following properties, which correspond to those of Definition* 4.1*:*

1. Isotony: *$K(\tilde{I}) \subset K(\tilde{J})$ if $\tilde{I} \subset \tilde{J}$. in the sense of (1.22).*
2. Twisted Haag duality: *Let $\tilde{I}, \tilde{I}'$ be classes of paths in $\widetilde{\mathcal{K}}$ ending at I and its causal complement I', respectively. Then*

$$Z(\tilde{I}, \tilde{I}')\,K(\tilde{I}') \;=\; K(\tilde{I})'\,. \tag{5.8}$$

3. Poincaré covariance: *For all $\tilde{I} \in \widetilde{\mathcal{K}}$,*

$$U(\tilde{g})\,K(\tilde{I}) \;=\; K(\tilde{g}\cdot\tilde{I}) \quad \textit{for all } \tilde{g} \in \tilde{P}_+^\uparrow \textit{ and}$$
$$\Theta_1\,K(\tilde{I}) \;=\; K(\tilde{j}\cdot\tilde{I})\,.$$

4. Property of being standard: *$K(\tilde{I}) + iK(\tilde{I})$ is dense in $\mathcal{H}$, and $K(\tilde{I}) \cap iK(\tilde{I}) = \{0\}$.*

PROOF. Properties 1 and 3 follow directly from the corresponding properties of $\mathcal{F}(\tilde{I})$ via equation (5.6). To prove property 2, we also exploit equation (5.6) to rewrite the left hand side of equation (5.8) as $\overline{(Z(\tilde{I}, \tilde{I}')\,\mathcal{F}(\tilde{I}')\,Z(\tilde{I}, \tilde{I}')^*)^{\mathrm{sa}}\,\mathbf{\Omega}}$, which by twisted Haag duality (4.14) coincides with $\overline{(\mathcal{F}(\tilde{I})')^{\mathrm{sa}}\,\mathbf{\Omega}}$, and hence by equation (5.7) of the last Lemma with the right hand side of equation (5.8). Property 4 follows from equation (5.5). □

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[7]Note that the restriction of the scalar product to the vector space $K(\tilde{I})$ is not real valued and hence does not turn it into real Hilbert space.

[8]Note that the adjoint of an antilinear operator S is characterized by $\langle S^*\psi, \phi\rangle = \langle S\phi, \psi\rangle$ for all $\phi \in \mathrm{dom}S$.

Characterization of Free Fields via Modular Localization. In a free theory we expect the properties of the last Proposition to be satisfied already on the one particle level, i.e. for $\mathcal{H}$ replaced by $\mathcal{H}^{(1)}$ and $K(\tilde{I})$ replaced by

$$K(\tilde{I})^{(1)} := K(\tilde{I}) \cap \mathcal{H}^{(1)} ,$$

which we then consider as the space of one particle vectors localized in $\tilde{I}$. Further we wish the free fields to establish a 'second quantization functor' which assigns to each such local one particle space $K(\tilde{I})^{(1)}$ the local field algebra $\mathcal{F}(\tilde{I})$:

$$\left.\begin{array}{c}\mathcal{H}^{(1)} \\ \text{localization: } K(\tilde{I})^{(1)}\end{array}\right\} \quad \longmapsto \quad \left\{\begin{array}{c}\mathcal{F} \\ \text{localization: } \mathcal{F}(\tilde{I})\end{array}\right. \tag{5.9}$$

Such a theory will be called a modular free field algebra (Definition 5.4). This picture is only slightly more restrictive than the earlier Definition 5.1 of a free field algebra: It follows from Definition 5.1 under the additional assumption that the set $\Phi(\tilde{I})\mathbf{\Omega}$, assumed to be dense in $\mathcal{H}^{(1)}$, is already a core for the restriction of $S(\tilde{I})$ to $\mathcal{H}^{(1)}$ (Lemma 5.5).

DEFINITION 5.4. A field algebra $\mathcal{F} = (\mathcal{F}(\tilde{I}))_{\tilde{I}\in\tilde{\mathcal{K}}}$ is called a *modular free field algebra*, if for every $\tilde{I} \in \tilde{\mathcal{K}}$ the following holds. $S(\tilde{I})$ leaves $\mathcal{H}^{(1)}$ invariant, i.e. $S(\tilde{I})\left(\mathcal{H}^{(1)} \cap \mathrm{dom} S(\tilde{I})\right) \subset \mathcal{H}^{(1)}$, and $\mathcal{F}(\tilde{I})$ is generated by a star stable linear set $\Phi(\tilde{I})$ of closed operators in $\mathcal{H}$ creating only one particle states out of the vacuum, and for every $\psi \in K(\tilde{I})^{(1)}$ there is a closed symmetric operator $\varphi(\psi) \in \Phi(\tilde{I})$ which satsfies $\varphi(\psi)\,\mathbf{\Omega} = \psi$.

LEMMA 5.5. *Let $\mathcal{F}$ be a free field algebra in the sense of Definition 5.1 and let $\Phi(\tilde{I})\,\mathbf{\Omega}$ be a core for $S(\tilde{I})|_{\mathcal{H}^{(1)}\cap \mathrm{dom} S(\tilde{I})}$ for each $\tilde{I} \in \tilde{\mathcal{K}}$. Then $\mathcal{F}$ is a modular free field algebra. Conversely, let $\mathcal{F}$ be modular free. Then it is a free field algebra, and further the properties 1 to 4 of Proposition 5.3 hold on the one particle level, i.e. for $\mathcal{H}$ replaced by $\mathcal{H}^{(1)}$ and $K(\tilde{I})$ replaced by $K(\tilde{I})^{(1)}$.*

PROOF. Let $\mathcal{F}$ be a free field algebra with $\mathcal{F}(\tilde{I})$ generated by $\Phi(\tilde{I})$. Since $S(\tilde{I})$ is a closed operator, one easily verifies that $S(\tilde{I})$ is well defined on $\varphi\,\mathbf{\Omega}$ for every $\varphi \in \Phi(\tilde{I})$, and that

$$S(\tilde{I})\,\varphi\,\mathbf{\Omega} = \varphi^*\,\mathbf{\Omega} \,. \tag{5.10}$$

Hence $S(\tilde{I})\left(\Phi(\tilde{I})\,\mathbf{\Omega}\right) \subset \mathcal{H}^{(1)}$. Using again that $S(\tilde{I})$ is closed and that $\Phi(\tilde{I})\,\mathbf{\Omega}$ is assumed to be a core, one concludes that $P_{\mathcal{H}^{(1)}}\,S(\tilde{I}) \subseteq S(\tilde{I})\,P_{\mathcal{H}^{(1)}}$, i.e. that $S(\tilde{I})$ leaves $\mathcal{H}^{(1)}$ invariant. Note that for every $\psi \in K(\tilde{I})$ there is a closed symmetric operator $\varphi(\psi)$ affiliated with $\mathcal{F}(\tilde{I})$ and satisfying $\varphi(\psi)\,\mathbf{\Omega} = \psi$, namely the closure of the operator

$$F'\,\mathbf{\Omega} \mapsto F'\,\psi \quad \text{for all } F' \in \mathcal{F}(\tilde{I})' \,. \tag{5.11}$$

Since these operators are affiliated with $\mathcal{F}(\tilde{I})$, we may add all $\varphi(\psi)$ with $\psi \in K(\tilde{I})^{(1)}$ to the set $\Phi(\tilde{I})$ which generates the algebra $\mathcal{F}(\tilde{I})$ without enlarging the latter. This shows that $\mathcal{F}$ is modular free. Conversely, let $\mathcal{F}$ be modular free. Then it is clearly also free. Using that by assumption $\mathcal{H}^{(1)} \cap \mathrm{dom} S(\tilde{I})$ coincides with the dense set $P_{\mathcal{H}^{(1)}}\,\mathrm{dom} S(\tilde{I})$, one easily verifies that the properties 1 to 4 of Proposition 5.3 still hold if one replaces $\mathcal{H}$ by $\mathcal{H}^{(1)}$ and $K(\tilde{I})$ by $K(\tilde{I})^{(1)}$. □

'No-Go' for Modular Free and Modular Covariant Non-Scalar Fields. We assume that $\mathcal{F}$ is a modular free field algebra satisfying modular covariance, i.e. the modular objects have the geometric significance as in Theorem 4.14 and Proposition 4.19. We require that in addition for every $\psi \in K(\tilde{I})$ the closed symmetric operator φ satisfying $\varphi\,\mathbf{\Omega} = \psi$ (which by Definition 5.4 is contained in $\Phi(\tilde{I})$) is uniquely determined by ψ. This could be implemented either as in the case of the Segal operator: one has a common core and a common prescription $\psi \mapsto \varphi(\psi)$ for all φ and all $\tilde{I}$. One could also require φ to be determined by the conditions that it creates ψ from the vacuum and that it is affiliated with $\mathcal{F}(\tilde{I})$, where $\tilde{I}$ is the localization region of ψ. Being affiliated with $\mathcal{F}(\tilde{I})$, φ must satisfy the prescription (5.11), and then φ is uniquely determined (in the latter sense) by ψ if and only if it is the closure of the operator (5.11), i.e. if and only if $\mathcal{F}(\tilde{I})'\,\mathbf{\Omega}$ is a *core* for φ. But ψ is also localized in any $\tilde{J}$ which contains

$\tilde{I}$, and hence, if φ is to be fixed by ψ in the above sense, already the smaller set $\mathcal{F}(\tilde{J})'\,\mathbf{\Omega}$ must be a core for φ.

In either case, we may denote φ it by $\varphi(\psi)$, and we demand in particular that φ does not carry any ψ-independent information about the localization region.

ASSUMPTION 3. Let $\psi \in K(\tilde{I})$. Then there is a closed symmetric operator $\varphi(\psi)$ affiliated with $\mathcal{F}(\tilde{I})$ satisfying $\varphi(\psi)\,\mathbf{\Omega} = \psi$ and which is uniquely determined by ψ in the abovementioned sense.

We further assume the field algebra to satisfy the following intersection property:

ASSUMPTION 4. $\mathcal{F}$ satisfies the following intersection property: Let $\tilde{I}_1 \cap \tilde{I}_2 \neq \emptyset$ in the sense of definition (1.22). Then

$$\mathcal{F}(\tilde{I}_1 \cap \tilde{I}_2) = \mathcal{F}(\tilde{I}_1) \cap \mathcal{F}(\tilde{I}_2)\,. \tag{5.12}$$

Note that this assumption is independent of all axioms and assumptions made so far. But it may be viewed to encode that the algebra is generated by a local quantum field. In the case of a modular free field algebra satisfying Assumption 3 one can then conclude that the same intersection property holds for the real one particle spaces:

LEMMA 5.6. *Let $\mathcal{F}$ be a modular free field algebra satisfying Assumptions 3 and 4, and let $\tilde{I}_1, \tilde{I}_2 \in \tilde{\mathcal{K}}$ be such that their union $\tilde{I}_1 \cup \tilde{I}_2$ is contained in some $\tilde{I} \in \tilde{\mathcal{K}}$. Then*

$$K(\tilde{I}_1 \cap \tilde{I}_2)^{(1)} = K(\tilde{I}_1)^{(1)} \cap K(\tilde{I}_2)^{(1)}\,. \tag{5.13}$$

Note that for any field algebra satisfying the intersection property (5.12) one has, due to the Reeh-Schlieder property,

$$\mathcal{F}(\tilde{I}_1 \cap \tilde{I}_2)^{\mathrm{sa}}\mathbf{\Omega} = \mathcal{F}(\tilde{I}_1)^{\mathrm{sa}}\mathbf{\Omega} \,\cap\, \mathcal{F}(\tilde{I}_2)^{\mathrm{sa}}\mathbf{\Omega}\,.$$

But this implies, via equation (5.6), only the inclusion "$\subset$" of the claimed equation and not "$\supset$", because in general

$$\big(\mathcal{F}(\tilde{I}_1)^{\mathrm{sa}}\mathbf{\Omega} \,\cap\, \mathcal{F}(\tilde{I}_2)^{\mathrm{sa}}\mathbf{\Omega}\big)^{-} \subsetneqq \mathcal{F}(\tilde{I}_1)^{\mathrm{sa}}\mathbf{\Omega}^{-} \cap \mathcal{F}(\tilde{I}_2)^{\mathrm{sa}}\mathbf{\Omega}^{-}\,.$$

PROOF. To show "$\supset$", let $\psi \in K(\tilde{I}_1)^{(1)} \cap K(\tilde{I}_2)^{(1)}$. Then by Definition 5.4 there are symmetric operators $\varphi_i(\psi)$ affiliated with $\mathcal{F}(\tilde{I}_i)$ (i=1,2) and satisfying $\varphi_1\,\mathbf{\Omega} = \psi = \varphi_2\,\mathbf{\Omega}$. Assumption 3 demands that $\varphi_1 = \varphi_2 = \varphi(\psi)$. Due to Assumption 4, $\varphi(\psi)$ is affiliated with $\mathcal{F}(\tilde{I}_1 \cap \tilde{I}_2)$. By the same argument leading to equation (5.10), one concludes that

$$S(\tilde{I}_1 \cap \tilde{I}_2)\,\psi \equiv S(\tilde{I}_1 \cap \tilde{I}_2)\,\varphi(\psi)\,\mathbf{\Omega} = \varphi(\psi)^*\,\mathbf{\Omega} \equiv \varphi(\psi)\,\mathbf{\Omega}\,,$$

hence $\psi \in K(\tilde{I}_1 \cap \tilde{I}_2)^{(1)}$. □

Now we will show that if the field algebra satisfies modular covariance, our explicit results on the Tomita-operator for wedge regions in Proposition 4.22 imply that equation (5.13) is in fact *violated* in the case of non-zero spin. We only discuss the case $s \geq 0$. In order not to burden notation, we restrict to one massive single particle sector χ satisfying $\chi^2 = 1$. Let $\omega \in (-\pi, 0)$, and let

$$K_\omega := \{\, u \cdot E_m f \mid f \in \mathcal{S}_{\mathbb{R}}\big(W_1 \cap (0,\omega)W_1\big)\,\}\,,$$

where u is the function on the mass shell defined in equation (4.80), and $\mathcal{S}_{\mathbb{R}}(M)$ denotes the space of real valued Schwartz functions with support in the spacetime region M, and $E_m f$ is the restriction of the Fourier transform of f to the mass shell. Then by Proposition 4.22,

$$K_\omega \subset K(\tilde{W}_1)^{(1)} \cap K\big((0,\omega)\cdot\tilde{W}_1\big)^{(1)}\,. \tag{5.14}$$

On the other hand, we have the following result:

PROPOSITION 5.7. *Let $\tilde{g}$ be an element of $\tilde{P}_+^\uparrow$ of the form $\tilde{g} = (0, -\frac{\pi}{2})\,\lambda_1(t')\,(0,\omega')$ with $t' \neq 0$ and $\omega' < 0$, where the parameters t' and ω' are chosen such that $\tilde{g}\cdot\tilde{W}_1$ contains the intersection $\tilde{W}_1 \cap (0,\omega)\cdot\tilde{W}_1$. Then*

$$K_\omega \subset K(\tilde{g}\cdot\tilde{W}_1)^{(1)} \quad \textit{if } s = 0\,, \quad \textit{whereas} \tag{5.15}$$

$$K_\omega \not\subset K(\tilde{g}\cdot\tilde{W}_1)^{(1)} \quad \textit{if } s > 0\,. \tag{5.16}$$

(In fact, we conjecture that for $s > 0$ the intersection $K_\omega \cap K(\tilde{g}\cdot\tilde{W}_1)^{(1)}$ is trivial.) Note that such a choice of $\tilde{g}$ exists: If one takes ω close to $-\pi$, and t' and ω' close to zero, then $\tilde{W}_1 \cap (0,\omega)\cdot\tilde{W}_1$ is a spacelike cone of small opening angle in the negative y-direction, and $\tilde{g}\cdot\tilde{W}_1$ is a wedge in approximately the same direction.

PROOF. A proof of the proposition is given on page 85 in Appendix A.3. □

But by virtue of isotony, the space $K\big(\tilde{W}_1 \cap (0,\omega)\cdot\tilde{W}_1\big)^{(1)}$ is contained in $K(\tilde{g}\cdot\tilde{W}_1)^{(1)}$, and hence Proposition 5.7 shows that K_ω is not contained in $K\big(\tilde{W}_1 \cap (0,\omega)\cdot\tilde{W}_1\big)^{(1)}$ if $s > 0$. Consequently, in this case

$$K\big(\tilde{W}_1 \cap (0,\omega)\cdot\tilde{W}_1\big)^{(1)} \subsetneq K(\tilde{W}_1)^{(1)} \cap K\big((0,\omega)\cdot\tilde{W}_1\big)^{(1)} ,$$

in contradiction to equation (5.13). We conclude: *Let $\mathcal{F}$ be a modular free field algebra satisfying modular covariance and the intersection property* (5.12)*, and let the spin of the generating elementary charge χ_e be non-zero. Then the closed symmetric operators, which by Definition 5.4 are affiliated with $\mathcal{F}(\tilde{I})$ and map the vacuum vector into $K(\tilde{I})^{(1)}$, are* not *uniquely determined by the single particle states which they create from the vacuum.* This shows that under the stated assumptions, the basic fields φ which generate the local algebras cannot be as canonical objects as the Segal operators used in 'second quantization'. Viewed differently, we conclude that the 'functor' (5.9) is not injective:

$$\begin{array}{ccc} K(\tilde{I})^{(1)} \cap K(\tilde{J})^{(1)} & \longmapsto & \mathcal{F}(\tilde{I}) \cap \mathcal{F}(\tilde{J}) \\ \not\parallel & & \parallel \\ K(\tilde{I} \cap \tilde{J})^{(1)} & \longmapsto & \mathcal{F}(\tilde{I} \cap \tilde{J}) \ . \end{array}$$

This is a surprising result, and it shall be discussed in the near future how the free fields which have been constructed in $d = 3 + 1$, e.g. the spin $\frac{1}{2}$ field, behave in this respect if they are restricted to $2 + 1$ dimensions.

5.3. No-Go via Jost-Schroer Theorem

Here we skip the Assumptions 3 and 4 of the last section. Instead, we strengthen our definition of a free field algebra into a direction which is reminiscent of the Wightman axioms. Namely, we assume that for two fields φ_1, φ_2 with spacelike separated localization regions, the norm of $\varphi_1 U(x) \varphi_2 \boldsymbol{\Omega}$ is polynomially bounded in x (Assumption 5). We then arrive at the no-go result in two steps: If the asymptotic *directions* of the localization regions of two fields φ_1 and φ_2 are spacelike separated, their "twisted commutator" is a c-number function, even if the localization regions overlap (Proposition 5.10). This is completely analogous to the (first part of the) well known Jost-Schroer theorem. On the other hand, these commutation relations are consistent only in the case of permutation group statistics (Lemma 5.11).

ASSUMPTION 5 (Regularity condition). Let $\varphi_1 \in \Phi(\tilde{I}_1)$ and $\varphi_2 \in \Phi(\tilde{I}_2)$ with $I_1 \subset I_2'$. Then $U(x)\varphi_2 \boldsymbol{\Omega}$ is in the domain of φ_1 for all $x \in \mathbb{R}^3$, and the function

$$x \mapsto \| \varphi_1 U(x) \varphi_2 \boldsymbol{\Omega} \| \tag{5.17}$$

is polynomially bounded for large x and locally integrable. Further, the free fields carry definite charges, i.e. for each $\varphi \in \Phi(\tilde{I})$ there is some $\chi \in \Gamma^{(1)}$ such that $V_t \varphi V_t^* = \chi(t) \cdot \varphi$ for all $t \in \mathbb{R}$. In addition, there exist nontrivial free fields in $\Phi(\tilde{I})$ carrying the particular charges $\chi = \chi_e$ and $\chi = \chi_e^{-1}$.

Note that the regularity condition (5.17) can be violated only if the fields are unbounded operators.

REMARK. Alternatively, we could have made the following stronger assumption.

ASSUMPTION 5'. Let $\varphi \in \Phi(\tilde{I})$. Then φ is uniquely determined by the single particle vector $\varphi \boldsymbol{\Omega}$ which it creates from the vacuum. Further, the domain of φ is invariant under translations, and for all $\psi \in \operatorname{dom}\varphi$ the function

$$x \mapsto \| \varphi U(x) \psi \|$$

is polynomially bounded for large x and locally integrable. Finally, the domain of φ contains the set $\Phi(\tilde{I}_2)\,\mathbf{\Omega}$ if $I_2 \subset I'$.

These conditions *imply* Assumption 5 in the following sense. By assumption, each $\varphi \in \Phi(\tilde{I})$ is the closure of the operator (5.11). On its core $\mathcal{F}(\tilde{I})'\,\mathbf{\Omega}$ it may be decomposed into the finite sum $\varphi = \sum_{\chi\in\Gamma^{(1)}} \varphi_\chi$, where φ_χ is defined, in analogy to equation (4.5), by

$$\varphi_\chi\, F'\,\mathbf{\Omega} := F'\,P_\chi\,\varphi\,\mathbf{\Omega} \quad \text{for all } F' \in \mathcal{F}(\tilde{I})' .$$

φ_χ clearly carries charge χ and is affiliated with $\mathcal{F}(\tilde{I})$. Replacing $\Phi(\tilde{I})$ by the set of all φ_χ with $\varphi \in \Phi(\tilde{I})$, Assumption 5 is satisfied.

We have chosen Assumption 5 so that the free fields φ can be decomposed into creation and annihilation parts at least on vectors of the form $\varphi'\,\mathbf{\Omega}$ (Lemma 5.8). This will suffice to establish Proposition 5.10, which is the analogon of the first part of the Jost-Schroer theorem.

For $\varphi \in \Phi(\tilde{I})$, let $\varphi(x) := U(x)\,\varphi\,U(x)^{-1}$ be the translated field, which is affiliated with $\mathcal{F}(\tilde{I}+x)$,[9] and let $H_m^\pm := \pm H_m$ be the positive and negative mass shell, respectively.

LEMMA 5.8. *Let φ_1 and φ_2 be spacelike seperated fields as in Assumption 5. Then the $\mathcal{H}$-valued function $\varphi_1(x)\,\varphi_2(y)\,\mathbf{\Omega}$ is a tempered distribution, whose Fourier transform has support contained in $(H_m^- \cup H_m^+) \times H_m^+$. Let F^+ and F^- be defined by the corresponding decomposition, i.e.*

$$\varphi_1(x)\,\varphi_2(y)\,\mathbf{\Omega} = F^+(x,y) + F^-(x,y)\,, \qquad \textit{with} \quad \operatorname{supp}\widetilde{F^\pm} \subset H_m^\pm \times H_m^+ . \tag{5.18}$$

Then

$$F^-(x,y) = \langle\,\mathbf{\Omega}\,,\,\varphi_1(x)\,\varphi_2(y)\,\mathbf{\Omega}\,\rangle\,\mathbf{\Omega}\,, \tag{5.19}$$

and

$$\mathrm{sp}_P F^+(x,y) \subseteq H_m^+ + H_m^+ . \tag{5.20}$$

Here $\mathrm{sp}_P\psi$ denotes the spectral support of ψ w.r.t. the energy momentum operators.

PROOF. Let $f,g \in \mathcal{S}(\mathbb{R}^3)$, the space of Schwartz functions. Due to the temperedness condition in Assumption 5, Riesz' theorem asserts the existence of a unique vector $F(f,g) \in \mathcal{H}$ satisfying

$$\langle\,\phi, F(f,g)\,\rangle \;=\; \int dxdy\, f(x)\, g(y)\,\langle\,\phi\,,\,\varphi_1(x)\,\varphi_2(y)\,\mathbf{\Omega}\,\rangle \;\text{ for all } \phi \in \mathcal{H}\,,$$

and whose norm can be estimated by

$$\|F(f,g)\| \le \int dxdy\, |f(x)g(y)|\; \|\varphi_1(x)\varphi_2(y)\mathbf{\Omega}\|\,.$$

By Assumption 5, this shows that the linear map $F\,:\,(f,g) \mapsto F(f,g)$ is continuous in both entries w.r.t. the usual locally convex topology on Schwartz space, i.e. F is a vector valued tempered distribution.

To prove the statement on the support of its Fourier transform, let f_1 be in $C_0^\infty(\mathbb{R}^3)$, i.e. a smooth function with compact support, and $g \in \mathcal{S}(\mathbb{R}^3)$. For all $A \in \mathcal{F}(\tilde{I}_1 + \mathrm{supp} f_1)'$ one verifies that

$$\langle\, A\,\mathbf{\Omega},\, F(f_1,g)\,\rangle \;=\; \langle\, A\,\tilde{\tilde{f}}_1(P)\,\varphi_1^*\,\mathbf{\Omega}\,,\,\tilde{g}(P)\,\varphi_2\,\mathbf{\Omega}\,\rangle\,.$$

Since $\varphi_2\,\mathbf{\Omega}$ and $\varphi_1^*\,\mathbf{\Omega}$ are in $\mathcal{H}^{(1)}$, the scalar product vanishes if $\mathrm{supp}\tilde{g} \cap H_m^+ = \emptyset$, or if f_1 is of the form $f_1 = (\Box + m^2)f$ for some function $f \in C_0^\infty(\mathbb{R}^3)$. Taking into account the fact that $\mathcal{F}(\tilde{I}_1 + \mathrm{supp} f_1)'\,\mathbf{\Omega}$ is dense in $\mathcal{H}$, we conclude that

$$\begin{aligned} F\big((\Box + m^2)f, g\big) &= 0 \text{ for all } f \in C_0^\infty(\mathbb{R}^3),\; g \in \mathcal{S}(\mathbb{R}^3)\,, \quad \text{and} \\ F(f,g) &= 0 \text{ for all } f \in C_0^\infty(\mathbb{R}^3) \text{ and } g \in \mathcal{S}(\mathbb{R}^3) \text{ with } \mathrm{supp}\tilde{g} \cap H_m^+ = \emptyset\,. \end{aligned}$$

By continuity of F, these two properties extend to all $f \in \mathcal{S}(\mathbb{R}^3)$. Now we can proceed as in the case of Wightman fields [SW64]:

[9](and *not* localized at x)

The support of the Fourier transform of F consists of two disjoint sets contained in $H_m^+ \times H_m^+$ and $H_m^- \times H_m^+$, respectively, thus defining the decomposition (5.18). To analyse the energy momentum supports, we extend F via the Schwartz nuclear theorem to a continuous linear map from $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ into $\mathcal{H}$, and note that for all $f, g \in \mathcal{S}(\mathbb{R}^3)$ we have $e^{ix\cdot P}\, F(f \otimes g) = \widetilde{F}(\, e^{ix\cdot(p_1+p_2)}\, \tilde{f} \otimes \tilde{g}\,)$. By linearity and continuity this yields

$$h(P)\, F(f \otimes g) \;=\; \widetilde{F}\,\Big(h(p_1+p_2)\cdot \tilde{f} \otimes \tilde{g}\Big) \quad \text{for all } h \in \mathcal{S}(\mathbb{R}^3). \tag{5.21}$$

From this equation and from the support properties of $\tilde{F}$ we conclude that $h(P)\, F^{\pm}(f,g) \;=\; 0$ if $\operatorname{supp} h \cap (H_m^{\pm} + H_m^{+}) \;=\; \emptyset$, respectively. This shows that $\operatorname{sp}_P F^{\pm}(f,g) \subset H_m^{\pm} + H_m^{+}$. Since $H_m^- + H_m^+$ intersects the energy momentum spectrum only in $\{0\}$, the vector $F^-(f,g)$ must be a multiple of the vacuum vector $\mathbf{\Omega}$, the factor being $\langle \mathbf{\Omega}\,,\, F^-(f,g)\,\rangle \;=\; \langle \mathbf{\Omega}\,,\, F(f,g)\,\rangle$. This shows that $F^{\pm}(x,y)$ are, like $F(x,y)$, well defined as *functions*, and have the properties (5.19) and (5.20). □

Our subsequent arguments will involve spacelike commutation relations of the free (possibly unbounded) fields φ. On the vacuum vector they can be derived from those of the field algebra elements:

LEMMA 5.9. *Let $I_2 \subset I_1'$, and let $\varphi_1 \in \Phi(\tilde{I}_1)$ and $\varphi_2 \in \Phi(\tilde{I}_2)$ be field operators carrying charges $\chi_{\mathrm{e}}^{q_1}$ and $\chi_{\mathrm{e}}^{q_2}$, respectively. Then*

$$\varphi_1\, \varphi_2\, \mathbf{\Omega} \;=\; \omega^{q_1 q_2 (2N+1)}\, \varphi_2\, \varphi_1\, \mathbf{\Omega}\,, \qquad \textit{where } N := N(\tilde{I}_1, \tilde{I}_2)\,. \tag{5.22}$$

PROOF. One approximates the positive part of the polar decomposition of φ_1 by operators in $\mathcal{F}(\tilde{I}_1)$ and uses equation (4.27) to show

$$\varphi_1\, Z(\tilde{I}_1, \tilde{I}_2)\, \varphi_2\, \mathbf{\Omega} = Z(\tilde{I}_1, \tilde{I}_2)\, \varphi_2\, Z(\tilde{I}_1, \tilde{I}_2)^{-1}\, \varphi_1\, \mathbf{\Omega}\,.$$

This implies equation (5.22) by the arguments used in the proof of Lemma 4.7, if one takes account of the fact that $\varepsilon_0(N;\, \chi_{\mathrm{e}}^{q_1}, \chi_{\mathrm{e}}^{q_2}) = \omega^{q_1 q_2 (2N+1)}$ by equation (4.25). □

Now we are ready to establish an analogon to the (first part of the) Jost-Schroer theorem: If in the situation of Lemma 5.9, we translate the localization regions such that they are not spacelike separated any more, only a multiple of $\mathbf{\Omega}$ is added to the right hand side of the commutation relation (5.22). More precisely:

PROPOSITION 5.10. *Let $I_2 \subset I_1'$, and let $\varphi_1 \in \Phi(\tilde{I}_1)$ and $\varphi_2 \in \Phi(\tilde{I}_2)$ be field operators carrying charges $\chi_{\mathrm{e}}^{q_1}$ and $\chi_{\mathrm{e}}^{q_2}$, respectively. Then the fields satisfy the commutation relations*

$$\varphi_1(x)\, \varphi_2\, \mathbf{\Omega} - \omega^{q_1 q_2 (2N+1)}\, \varphi_2\, \varphi_1(x)\, \mathbf{\Omega} \;=\; c_{\varphi_1, \varphi_2}(x)\, \mathbf{\Omega} \quad \textit{for all } x \in \mathbb{R}^3\,, \tag{5.23}$$

Here $N = N(\tilde{I}_1, \tilde{I}_2)$ and $c_{\varphi_1,\varphi_2}(x)$ is the scalar product of the vacuum with the left hand side of equation (5.23).

Note that these commutation relations extend from $\mathbf{\Omega}$ to $\bigcap_{i=1,2} \mathcal{F}(\tilde{I}_i + x_i)'\, \mathbf{\Omega}$, which is dense in $\mathcal{H}$ if $\tilde{I}_1$ and $\tilde{I}_2$ have "equal winding numbers", see equation (5.25).

PROOF. After having established Lemma 5.8, the proof of this proposition is a straightforward adaption of the proof of theorem 4-15 in [SW64] to the present anyonic case: Let $F_{1,2}^{+}(x,y)$ be the component of $\varphi_1(x)\varphi_2(y)\, \mathbf{\Omega}$ whose Fourier transform has support in $H_m^+ + H_m^+$ according to Lemma 5.8, and $F_{2,1}^{+}(x,y)$ that of $\varphi_2(x)\varphi_1(y)\, \mathbf{\Omega}$. Let further $\omega_{12} := \omega^{q_1 q_2 (2N(\tilde{I}_1, \tilde{I}_2)+1)}$, see equation (5.22). Lemma 5.8 asserts that

$$\varphi_1(x)\varphi_2\, \mathbf{\Omega} - \omega_{12}\, \varphi_2\varphi_1(x)\, \mathbf{\Omega} = c_{\varphi_1,\varphi_2}(x)\, \mathbf{\Omega} + F_{1,2}^{+}(x,0) - \omega_{12}\, F_{2,1}^{+}(0,x). \tag{5.24}$$

We have to show that the last two terms add up to zero. For all $\psi \in \mathcal{H}$, the distribution $F_\psi(x) \;=\; \big\langle\, \psi\,,\, F_{1,2}^{+}(x,0) - \omega_{12}\, F_{2,1}^{+}(0,x)\,\big\rangle$ is the boundary value of an analytic function, since its Fourier transform has support in the cone V_+ according to Lemma 5.8 [RS75, Thm. IX.16]. Further, equations (5.24) and (5.22) imply that F_ψ vanishes on the real open set of points satisfying $I_1 + x \subset I_2'$. Due to the edge of the wedge theorem, this forces F_ψ to vanish identically as a distribution [SW64, Thm. 2-17], and hence as a function. □

LEMMA 5.11. *Assume, some of the commutator functions* $c_{\varphi_1,\varphi_2}(x)$ *appearing in equation* (5.23) *of Proposition* 5.10 *do* not *vanish identically in* x. *Then the commutation relations* (5.23) *are consistent only if the statistics phase* ω *is* 1 *or* -1, *i.e. only in the case of permutation group statistics.*

REMARK. The additional assumption of the proposition does not seem to be a severe restriction: if it were violated, we only needed a criterion allowing us to deduce commutation relations for the field algebra elements from those of the fields (like an energy bound satisfied by the fields), to conclude that the local observable algebras are commutative – in contradiction to our general framework given in Chapter 4.

PROOF. We choose two spacelike seperated spacelike cones $I_1, I_2 \in \mathcal{K}$, two corresponding paths $\tilde{I}_1, \tilde{I}_2 \in \widetilde{\mathcal{K}}$ s.t. $\tilde{I}_1 < \tilde{I}_2 < (0, 2\pi) \cdot \tilde{I}_1$, and two fields $\varphi_1 \in \Phi(\tilde{I}_1)$, $\varphi_2 \in \Phi(\tilde{I}_2)$ together with a translation vector $x \in \mathbb{R}^3$ s.t. $c_{\varphi_1,\varphi_2}(x) \neq 0$. This presupposes that the charges $\chi_{\mathrm{e}}^{q_1}$ and $\chi_{\mathrm{e}}^{q_2}$ of the fields satisfy $\chi_{\mathrm{e}}^{q_1+q_2} = 1$, since the left hand side of equation (5.23) is in $\mathcal{H}_{q_1+q_2}$ while the right hand side is in $\mathcal{H}_1$. We choose $q_1 = -1$ and $q_2 = 1$. Next we pick a cone $I_3 \in \mathcal{K}$ spacelike[10] to $I_1 + x$ and I_2 and such that their union $I_1 + x \cup I_2 \cup I_3$ is contained in some $I_x \in \mathcal{K}$. Now we choose $\tilde{I}_3 \in \widetilde{\mathcal{K}}$ ending at I_3 with $\tilde{I}_1 < \tilde{I}_3 < \tilde{I}_2$ (to be definite) and a non-zero field $\varphi_3 \in \Phi(\tilde{I}_3)$ carrying charge χ_{e}. Due to the localization properties of $\varphi_1(x)$ and φ_3, the commutator $c_{\varphi_1,\varphi_3}(x)$ vanishes. We have chosen the localization regions so that there is a path $\tilde{I}_x \in \widetilde{\mathcal{K}}$ containing $\tilde{I}_1 + x, \tilde{I}_2$ and $\tilde{I}_3$ in the sense of (1.22). Using isotony of the field algebra, this implies that the subspace

$$D := \mathcal{F}(\tilde{I}_1 + x)' \, \boldsymbol{\Omega} \cap \bigcap_{i=2,3} \mathcal{F}(\tilde{I}_i)' \, \boldsymbol{\Omega} \tag{5.25}$$

contains $\mathcal{F}(\tilde{I}_x)' \, \boldsymbol{\Omega}$, which is dense in $\mathcal{H}$ due to the Reeh-Schlieder property, see Property 4 of Definition 4.1. Thus, D is a dense subspace on which equation (5.23) holds. Now let $\psi \in D$. Denoting $\omega_{ij} := \omega^{q_i q_j (2N(\tilde{I}_i, \tilde{I}_j)+1)}$ and $c_{12} := c_{\varphi_1,\varphi_2}(x)$, we get from Proposition 5.10 and from equation (5.22)

$$\begin{aligned} \langle \varphi_1(x)^* \, \psi \, , \, \varphi_2 \, \varphi_3 \, \boldsymbol{\Omega} \rangle \; = & \\ = \langle \psi \, , c_{12} \, \varphi_3 \, \boldsymbol{\Omega} \rangle + \omega_{12}\omega_{13}\omega_{23} \; \langle \varphi_3^* \, \psi \, , \varphi_2 \, \varphi_1(x) \, \boldsymbol{\Omega} \rangle & \qquad (5.26) \\ = \langle \psi \, , \omega_{23}\omega_{13} c_{12} \, \varphi_3 \, \boldsymbol{\Omega} \rangle + \omega_{12}\omega_{13}\omega_{23} \; \langle \varphi_3^* \, \psi \, , \varphi_2 \, \varphi_1(x) \, \boldsymbol{\Omega} \rangle \, . & \qquad (5.27) \end{aligned}$$

In equation (5.26) we have first commuted φ_2^* with $\varphi_1(x)^*$ and then $\varphi_1(x)$ with φ_3, and in (5.27) first φ_3 with φ_2, and then φ_3^* with $\varphi_1(x)^*$. Note that all of the twofold products $\varphi_2^* \, \varphi_1(x)^*$ etc. are well defined on D. Since D is dense, we conclude that

$$c_{12} \, \varphi_3 \, \boldsymbol{\Omega} = \omega_{23}\omega_{13} \, c_{12} \, \varphi_3 \, \boldsymbol{\Omega} \, .$$

By assumption, $c_{12} \, \varphi_3 \, \boldsymbol{\Omega} \neq 0$, so that $\omega_{23} \, \omega_{13} = 1$ follows. On the other hand, we have chosen the localization regions such that $N(\tilde{I}_1, \tilde{I}_3) = -1$ and $N(\tilde{I}_2, \tilde{I}_3) = 0$, which implies that $\omega_{13} = \omega^{-q_1 q_3} = \omega$ and $\omega_{23} = \omega^{q_2 q_3} = \omega$. Hence $\omega^2 = 1$. ☐

Proposition 5.10 and Lemma 5.11 imply our no-go theorem for free anyons:

THEOREM 5.12. *If the statistics phase of the elementary charge is not* 1 *or* -1, *i.e. in the case of anyons, there can be no free field algebra satisfying Assumption 5 and for which some of the commutator functions* $c_{\varphi_1,\varphi_2}(x)$ *appearing in equation* (5.23) *of Proposition* 5.10 *do not vanish identically in* x.

[10] It is a necessary condition for the proposition that this geometric situation can be achieved. This is not the case, e.g. if the "free fields" are not localizable in regions smaller than wedge regions.