CHAPTER 4

Anyons

In this chapter we are concerned only with Abelian sectors, i.e. sectors which arise from automorphisms of the observable algebra. The set of such sectors has the structure of an Abelian group. Recall that there a composition law $[\varrho_1] \cdot [\varrho_2] := [\varrho_1 \varrho_2]$ which is Abelian due to the statistics operators $\varepsilon(\varrho_1, \varrho_2; N)$ intertwining $\varrho_1\varrho_2$ with $\varrho_2\varrho_1$, and that an Abelian sector has an inverse $[\gamma]^{-1} = [\gamma^{-1}]$.

We will consider a finitely generated subgroup Γ of the set of sectors which satisfies a certain condition, Assumption 2 below. Then there is a choice of representatives of the sectors in Γ which is closed under the group operations of individual composition and inversion, and establishes a monomorphism from Γ into $\Delta(S_0)$ by the assignment $\Gamma \ni \chi \mapsto \gamma_\chi \in \Delta(S_0)$ i.e. satisfies $\gamma_\chi \gamma_{\chi'} = \gamma_{\chi\chi'}$ and $\gamma_1 = \mathrm{id}$.

This circumstance allows one to construct [Reh90] a (charged) *field algebra* $\mathcal{F}_{\mathrm{u}}$ from the observable algebra $\mathcal{A}_{\mathrm{u}}$ and the selected set of sectors Γ, acting in a 'physical Hilbert space' $\bigoplus_{\chi\in\Gamma} \mathcal{H}_\chi$, in the same manner as the field bundle in Chapter 1. One can now translate the assumed properties of the observable algebra, and of the selected set of sectors, into properties of the field algebra. The advantage of this change of pictures lies in the construction of models, where the primary objects are usually charged (unobservable) fields, whereas the observables are the derived objects (consider e.g. the free charged field). We will make use of this in chapter 5.

It turns out that an adapted version of the $\mathrm{P}_1\mathrm{CT}$ operator defined in Chapter 1.4 is directly related to the *modular conjugation* of the field algebra associated to the wedge region W_1. Thus the assumed modular covariance[1] of the observable algebra $\mathcal{A}_{\mathrm{u}}$ lifts to the field algebra $\mathcal{F}_{\mathrm{u}}$. This extends a result of D. Guido and R. Longo [GL95] and of B. Kuckert [Kuc95] for Bose and Fermi fields to the anyonic case. As a consequence, we can derive the strong version of the spin statistics connection, also extending results of the abovementioned articles. Similarly, the geometric significance of the modular *operator* lifts from the observable to the field algebra.

The consideration of automorphisms of the observable algebra only leads to a simple structure of the intertwiner spaces $F^{(n)}_{\sigma,\alpha}$ and of the representation ε of the braid group acting in them. In fact, it is determined by the set of statistics phases of the elementary charges (generators of the group of sectors). Consequently, the reference Hilbert space $\tilde{\mathcal{H}}$ of *scattering states for anyons* is known explicitely and so is the representation of $\tilde{P}_+^\uparrow$ on it, and the $\mathrm{P}_1\mathrm{CT}$ operator is known up to the S-matrix. Taking the modular covariance of the field algebra into account, this means in particular that the modular objects for the field algebras associated to all wedge regions are explicitely given, up to the S-matrix. This opens up new possibilities for the construction of models of anyons. In particular, it will be a basis to discuss the issue of free fields in chapter 5.

4.1. Field Algebra for Anyons

The aim of this section is to describe anyons not in terms of observables and a selected set of representations, but in the more traditional field theoretic picture which goes back to Wick, Wightman and Wigner [WWW52]: One has a *physical Hilbert space* $\mathcal{H}$ which encodes all normal states exhausting all relevant sectors Γ. Vectors corresponding to states in a given sector $\chi \in \Gamma$ are said to carry charge χ. They correspond to pure states and span a subspace $\mathcal{H}_\chi$ of $\mathcal{H}$ which is therefore called coherent subspace or, with a slight abuse of language, "charged sector". Vectors carrying different charges can show no interference and are hence orthogonal.

[1] see Assumption 1, equation 1.62

Thus $\mathcal{H}$ decomposes into the direct sum of charged sectors:

$$\mathcal{H} = \bigoplus_{\chi\in\Gamma} \mathcal{H}_\chi \,. \tag{4.1}$$

$\mathcal{H}_{\chi=1}$ is the vacuum sector and contains the vacuum vector Ω. There is a *field algebra* of operators generating all charged sectors from the vacuum Ω. Field operators mapping Ω into $\mathcal{H}_\chi$ with $\chi \neq 1$ are said to carry charge χ and are unobservable. The observables are operators built up from the fields in such a way such that they leave each charged sector invariant, i.e. they carry no charge. They can be obtained, together with the set of their inequivalent representations, from the field algebra by a symmetry principle: There is a compact Abelian *gauge group* G acting in $\mathcal{H}$. Its character group $\hat{G}$, i.e. the group of its irreducible (hence one dimensional) unitary representations, coincides with the set Γ of sectors, and equation (4.1) is the corresponding decomposition. The field operators carrying charge χ are precisely the ones which transform under G according to the representation χ. In particular, the observables are characterized as the gauge invariant elements. Every unitary field operator Ψ_χ carrying charge χ^{-1} implements an irreducible representation of the observable algebra by

$$A \mapsto \Psi_\chi \, A \, \Psi_\chi^{-1} \,|\, \mathcal{H}_1 \,. \tag{4.2}$$

Different field operators lead to inequivalent representations if and only if they carry different charges χ. Further, the field algebra has a localization structure, and localized fields Ψ_χ lead to representations satisfying the Buchholz-Fredenhagen criterion (0.10).

We make this 'Wick-Wightman -Wigner picture' for anyons precise in the following definition of a field algebra. Then we state our explicite assumptions on the set Γ of selected sectors and arrive at the main result of this section, Proposition 4.3, namely that given an observable algebra $\mathcal{A}$ and a suitable set Γ of sectors there is a field algebra $\mathcal{F}$ with gauge group $G = \hat{\Gamma}$ whose observables coincide with $\mathcal{A}$, and that it is unique within a certain notion of equivalence. Thus there is a one-to-one correspondence

$$\left.\begin{matrix} (\mathcal{A},\Gamma) \\ \text{Algebraic QFT} \end{matrix}\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix} (\mathcal{F}, G=\hat{\Gamma}) \\ \text{`WWW-picture'} \end{matrix}\right.$$

The rest of this section will then be devoted to the construction of $\mathcal{F}$ from $\mathcal{A}$ and Γ (subsection 4.1.2) and to the proof of its uniqueness (subsection 4.1.3). Most of the material in this section is an adaption of the results in [DHR69a] to the anyonic case.

4.1.1. Wick-Wightman-Wigner Picture for Anyons.

DEFINITION 4.1. Let G be a compact Abelian group. An *anyonic field algebra* with *gauge group* G is a family $\mathcal{F} = \mathcal{F}(\tilde{I})_{\tilde{I}\in\tilde{\mathcal{K}}}$ of von Neumann algebras $\mathcal{F}(\tilde{I})$ operating in a Hilbert space $\mathcal{H}$ such that the following holds.

$\mathcal{H}$ carries a unitary representation V of G which contains all inequivalent irreducible representations, i.e. all characters $\chi \in \hat{G}$. It also carries a continuous unitary representation U of the universal covering group $\tilde{P}_+^\uparrow$ of the Poincaré group, which commutes with the action of G, so that

$$\mathcal{H} = \bigoplus_{\chi\in\hat{G}} \mathcal{H}_\chi \,, \quad V(t) = \bigoplus_{\chi\in\hat{G}} \chi(t)\,\mathbb{1} \quad \text{and} \quad U = \bigoplus_{\chi\in\hat{G}} U_\chi \,, \tag{4.3}$$

where the first direct sum denotes the decomposition of $\mathcal{H}$ with respect to $V(G)$. The representation U satisfies the spectrum condition spec$P \subset \bar{V}_+$ and has an invariant unit vector $\mathbf{\Omega} \in \mathcal{H}_{\chi=1}$ which is unique up to a phase. $U_{\chi=1}$ is a vacuum representation of $P_+^\uparrow$. Further, the following properties are satisfied.

0. Inner symmetry: $\mathcal{F}(\tilde{I})$ is mapped onto itself under the action of the global gauge group G. The subalgebra $\mathcal{F}^G(\tilde{I}) := \mathcal{F}(\tilde{I}) \cap V(G)'$ of invariant elements will be interpreted as the *observable subalgebra* of $\mathcal{F}(\tilde{I})$.
1. Isotony: $\mathcal{F}(\tilde{I}) \subset \mathcal{F}(\tilde{J})$ if $\tilde{I} \subset \tilde{J}$ in the sense of (1.22).
2. Locality relative to the observables:

$$\mathcal{F}^G(\tilde{J}) \;\subset\; \mathcal{F}(\tilde{I})' \;\text{ if } J \subset I' \,. \tag{4.4}$$

3. Poincaré covariance: $\mathrm{Ad}U(\tilde{g})\,\mathcal{F}(\tilde{I}) = \mathcal{F}(\tilde{g}\cdot\tilde{I})$ for all $\tilde{g} \in \tilde{P}_+^\uparrow$.
4. Reeh - Schlieder property: The vacuum vector $\boldsymbol{\Omega}$ is cyclic and seperating for each $\mathcal{F}(\tilde{I})$.
5. Irreducibility: $\bigcap_{\tilde{I}\in\tilde{\mathcal{K}}} \mathcal{F}(\tilde{I})' = \mathbb{C}\mathbb{1}$.

REMARK. i) If $\mathcal{F}(\tilde{I}) = \mathcal{F}((0,2\pi)\cdot\tilde{I})$, the field algebra describes bosons or fermions instead of genuine anyons.

ii) That $\boldsymbol{\Omega}$ is cyclic for each $\mathcal{F}(\tilde{I})$ is a consequence of the other assumptions (Reeh-Schlieder theorem). Then one can show that it is also seperating for $\mathcal{F}(\tilde{I})$, i.e. cyclic for $\mathcal{F}(\tilde{I})'$, if one assumes commutation relations for spacelike seperated fields. But the anyonic commutation relations (see equation (4.27) below) do not look very natural, so we want to *derive* them from a simple additional assumption (better, convention, see 2 of Proposition 4.3) rather than assume them.

iii) Two field algebras $\mathcal{F}$ and $\hat{\mathcal{F}}$ with gauge group G and acting in $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively, will be called *unitarily equivalent*, if there is a unitary $W : \hat{\mathcal{H}} \to \mathcal{H}$ which intertwines the representations of G and implements an isomorphism $\mathrm{Ad}W$ from each $\hat{\mathcal{F}}(\tilde{I})$ onto $\mathcal{F}(\tilde{I})$.

4.1.1.1. *Charge Carrying Fields and Spectral Projectors.* There is a unique normalized Haar measure on G, which we denote by $d\lambda(t)$. By definition, it satisfies $\lambda(G) = 1$ and $\lambda(tM) = \lambda(M)$ for any measurable subset M of G and all $t \in G$. It can be used to write the projector P_χ onto the subspace $\mathcal{H}_\chi$ as

$$P_\chi = \int_G d\lambda(t)\,\overline{\chi(t)}V(t)\,,$$

where the integral is understood in the strong operator topology. We will say that an operator in $\mathcal{H}$ *carries charge* χ, or is an irreducible tensor which transforms according to χ, if it maps $\mathcal{H}_{\chi'}$ into $\mathcal{H}_{\chi'\chi}$ for all $\chi' \in \hat{G}$, or equivalently, if $V(t)FV(t)^* = \chi(t)\,F$ for all $t \in G$. These operators can be characterized as the image of the *spectral projectors* E_χ, defined as follows:

$$E_\chi\ : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\,, \quad F \mapsto \int_G d\lambda(t)\,\overline{\chi(t)}\,V(t)FV(t)^*\,. \tag{4.5}$$

The integral is understood in the weak operator topology. It satisfies $E_\chi \circ E_{\chi'} = \delta_{\chi,\chi'}\,E_\chi$, is continuous in the ultraweak topology [DHR69b, Lemma 3.1] and leaves each local field algebra $\mathcal{F}(\tilde{I})$ invariant. Further, the linear span of irreducible tensors in $\mathcal{F}(\tilde{I})$ is weakly dense in $\mathcal{F}(\tilde{I})$:

$$\mathcal{F}(\tilde{I}) = \Big(\bigoplus_{\chi\in\Gamma} E_\chi\,\mathcal{F}(\tilde{I})\Big)^{-w}\,. \tag{4.6}$$

A proof can be found, e.g. in [DR72, Remark 1 after Prop.2.2].

4.1.1.2. *Observable Subalgebra and Decomposition into Sectors.* We denote by $\mathcal{F}^G$ the family $(\mathcal{F}^G(\tilde{I}))_{\tilde{I}\in\tilde{\mathcal{K}}}$ of observable subalgebras or, if the context allows no doubt, their union $\bigcup_{\tilde{I}\in\tilde{\mathcal{K}}} \mathcal{F}^G(\tilde{I})$. [2] The latter leaves each subspace $\mathcal{H}_\chi$ invariant, and hence for each $\chi \in \hat{G}$, a subrepresentation of $\mathcal{F}^G$ is defined by

$$\pi_\chi(A) := A|_{\mathcal{H}_\chi} \quad , A \in \mathcal{F}^G\,.$$

These representations are irreducible and pairwise inequivalent. To see this, we note that for all $\tilde{I}$ the observable algebra $\mathcal{F}^G(\tilde{I})$ coincides with $E_{\chi=1}(\mathcal{F}(\tilde{I}))$. We exploit the continuity of E_1 and the irreducibility of $\mathcal{F}$ to conclude

$$(\mathcal{F}^G)'' = \Big(\bigcup_{\tilde{I}\in\tilde{\mathcal{K}}} E_1(\mathcal{F}(\tilde{I}))\Big)'' = E_1\Big(\big(\bigcup_{\tilde{I}\in\tilde{\mathcal{K}}} \mathcal{F}(\tilde{I})\big)''\Big) = E_1(\mathcal{B}(\mathcal{H})) = V(G)'\,,$$

hence

$$(\mathcal{F}^G)' = V(G)'' \equiv \bigoplus_{\chi\in\hat{G}} \mathbb{C}\mathbb{1}_{\mathcal{H}_\chi}\,. \tag{4.7}$$

This shows that the representations π_χ are irreducible and pairwise inequivalent. It can be shown in analogy to [DHR69b, Thm. 6.1] that they satisfy the Buchholz-Fredenhagen criterion with respect to the vacuum representation $\pi_{\chi=1}$. This implies that they all have the same

[2] We will see shortly that $\mathcal{F}^G(\tilde{I})$ in fact only depends on I, not on $\tilde{I}$.

kernel and are therefore faithful representations. As a consequence, $\mathrm{Ad}U(0,2\pi)$ acts as the identity on $\mathcal{F}^G(\tilde{I})$, since it does so in the vacuum representation π_1. Hence $\mathcal{F}^G(\tilde{I})$ coincides with $\mathcal{F}^G((0,2\pi)\cdot\tilde{I})$ and is thus is independent of the path, i.e. only depends on the endpoint I. We will denote it by $\mathcal{F}^G(I)$. The family $(\mathcal{F}^G(I))_{I\in\mathcal{K}}$ is indeed an observable algebra in the sense of our earlier definition on page 4, save for the property of Haag duality. But now the following result holds.

LEMMA 4.2. *Let Haag duality hold for $\mathcal{F}^G$ in the vacuum representation π_1, i.e. $\pi_1(\mathcal{F}^G(I))' = \pi_1(\mathcal{F}^G(I'))$. Then $\mathcal{F}$ satisfies G-invariant Haag duality, i.e.*

$$\mathcal{F}(\tilde{I})' \cap V(G)' = \mathcal{F}^G(I') \,. \tag{4.8}$$

PROOF. We adapt the proof of [DR90, Lemma 3.8]. The inclusion "$\supset$" follows from relative locality (4.4). To show the other inclusion "$\subset$", let $F \in \mathcal{F}(\tilde{I})' \cap V(G)'$. Then since $\mathcal{F}(\tilde{I})' \cap V(G)' \subset \mathcal{F}^G(I)' \cap V(G)'$,

$$F|\mathcal{H}_{\chi=1} \in \pi_1(\mathcal{F}^G(I))',$$

which coincides with $\pi_1(\mathcal{F}^G(I'))$ by assumption. Since π_1 is faithful, this proves the claim. □

4.1.1.3. *Equivalence with the AQFT Picture.* In the spirit of algebraic quantum field theory, the physical content of the theory should be fixed by the observable algebra $\mathcal{F}^G$ in the vacuum representation $\pi_{\chi=1}$ and the set $\hat{G}$ of representations in consideration. The question arises if these data allow to reconstruct and to a certain extent to fix the field algebra. This has been answered affirmatively for theories with permutation group statistics by Doplicher, Haag and Roberts: in [DHR69a] for the Abelian case, and in [DR90] for the non-Abelian case. In the following we partly transfer these results to the anyonic case and show that for a given observable algebra $\mathcal{A}$ with a given subset Γ of sectors a unique field algebra with gauge group $G = \hat{\Gamma}$ and observable algebra $\mathcal{F}^G = \mathcal{A}$ can be constructed, provided Γ satisfies certain restrictions.

We first state our assumptions on the set of sectors. The first restriction is that we will consider a subgroup Γ of sectors which is generated by a *finite* set

$$\Gamma_{\mathrm{el}} = \{\chi_{(1)}, \dots, \chi_{(R)}\} \tag{4.9}$$

of independent[3] covariant and localizable sectors. They will be called "elementary charges". Note that then all sectors in Γ are covariant and localizable. Also recall that Γ is the direct sum of cyclic groups $\mathbb{Z}_{n_i}$ (where n_i is the order of $\chi_{(i)}$) and copies of $\mathbb{Z}$, and $G \equiv \hat{\Gamma}$ is the direct sum of cyclic groups $\mathbb{Z}_{n_i}$ and copies of $U(1)$. The second restriction is given by the following condition, to which the elementary charges are subject.

ASSUMPTION 2. For every elementary charge $\chi \in \Gamma_{\mathrm{el}}$ satisfying $\chi^n = 1$ for some $n \in \mathbb{N}$ there is a representative γ of χ, localized in S_0, such that $\gamma^n = \mathrm{id}$.

The statistics phase ω_χ of a sector χ with $\chi^n = 1$ satisfies $\omega_\chi^n = \pm 1$. K.-H. Rehren has pointed out in [Reh90] that χ has a representative γ with $\gamma^n = \mathrm{id}$ if and only if $\omega_\chi^n = 1$. Further, in this case γ may be chosen to be localized in any given spacelike cone. We choose R causally disjoint spacelike cones $S_{0,i} \subset S_0$, and for every sector $\chi_{(i)} \in \Gamma_{\mathrm{el}}$ we pick a representative $\gamma_{\chi_{(i)}}$ in accordance with the above assumption, which is localized in $S_{0,i}$. Then, being localized in causally disjoint regions, these representatives commute pairwise, and we can choose a system of representatives $(\gamma_\chi)_{\chi\in\Gamma}$ which is closed under individual multiplication and isomorphic to Γ by the assignment $\Gamma \in \chi \mapsto \gamma_\chi \in \Delta(S_0)$, i.e.

$$\gamma_\chi \gamma_{\hat{\chi}} = \gamma_{\chi\hat{\chi}} \quad \text{and} \quad \gamma_1 = \mathrm{id} \,. \tag{4.10}$$

Namely, we set

$$\chi_{(1)}^{k_1} \cdots \chi_{(R)}^{k_R} = \chi \quad \mapsto \quad \gamma_\chi = \gamma_{\chi_{(1)}}^{k_1} \cdots \gamma_{\chi_{(R)}}^{k_R} \,. \tag{4.11}$$

For any pair of spacelike seperated paths $\tilde{I}, \tilde{J}$ we introduce a unitary 'twist operator' $Z(\tilde{I}, \tilde{J})$ in $\mathcal{H}$. A similar operator playing the same role in the formulation of anyonic commutation

[3] i.e., from $\chi_{(1)}^{k_1} \cdots \chi_{(R)}^{k_R} = 1$ follows $\chi_{(1)}^{k_1} = \ldots = \chi_{(R)}^{k_R} = 1$.

relations has been written down by B. Schroer in [Sch94] for the case $\Gamma = \mathbb{Z}_n$. We denote the statistics phase of the i-th elementary sector $\chi_{(i)}$ by ω_i, i.e.

$$\omega_i \mathbf{1} := \pi_0 \varepsilon(\gamma_{\chi_{(i)}}, \gamma_{\chi_{(i)}}; 0) \quad , i = 1, \ldots, R . \tag{4.12}$$

Then the twist operator is defined by

$$Z(\tilde{I}, \tilde{J}) := \sum_{\chi \in \Gamma} \prod_{i=1}^{R} (\omega_i^{\frac{1}{2}})^{k_i^2 (2N+1)} P_\chi \quad , \text{ where } N := N(\tilde{I}, \tilde{J}) \tag{4.13}$$

and k_i are the coefficients of χ, i.e. $\chi = \prod \chi_{(i)}^{k_i}$. The definition depends on a choice of the square root for every ω_i, but the following results do not.

PROPOSITION 4.3. *Given an observable algebra $\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{K}}$ and a finitely generated subgroup Γ of its simple sectors satisfying Assumption 2, there is a field algebra $\mathcal{F}$ with gauge group $G = \hat{\Gamma}$, unique up to unitary equivalence, such that:*

1. *The observable subalgebra $\mathcal{F}^G$ of $\mathcal{F}$ is the isomorphic image of $\mathcal{A}$ under a faithful representation π, which contains all sectors $\chi \in \Gamma$ exactly once.*
2. *Spacelike separated fields carrying elementary different charges commute.*[4]

Further, this field algebra satisfies twisted Haag duality*: Let $\tilde{I}, \tilde{I}'$ be classes of paths in $\widetilde{\mathcal{K}}$ ending at I and its causal complement I', respectively. Then*

$$Z(\tilde{I}, \tilde{I}')\, \mathcal{F}(\tilde{I}')\, Z(\tilde{I}, \tilde{I}')^* = \mathcal{F}(\tilde{I})' . \tag{4.14}$$

Finally, $\mathcal{F}$ has the intersection property

$$\mathcal{F}(\tilde{I}) = \bigcap_{\tilde{W} \supset \tilde{I}} \mathcal{F}(\tilde{W}) \quad \text{for all } \tilde{I} \in \widetilde{\mathcal{K}} . \tag{4.15}$$

The remainder of this section is devoted to a proof of this proposition. The existence statement is contained in Proposition 4.9 of the next subsection, and uniqueness is shown in Proposition 4.13 of subsection 4.1.3.

4.1.2. Construction of the Field Algebra. We construct the field algebra in such a way that some results of Chapter 1 on the field bundle may be taken over. The subset $\gamma(\Gamma) \times \mathcal{A}_{\mathrm{u}}$ of the field bundle can be completed to a C^* algebra $\mathcal{F}_{\mathrm{u}}$, which in the end is the universal algebra of the family $\mathcal{F}$ to be constructed. In mathematics the construction of $\mathcal{F}_{\mathrm{u}}$ is known as the crossed product of Γ with $\mathcal{A}_{\mathrm{u}}$ via the action $\gamma : \Gamma \to \mathrm{Aut}\mathcal{A}_{\mathrm{u}}$.
Let $\mathcal{F}_{\mathrm{u}}^0$ be defined as the linear space

$$\mathcal{F}_{\mathrm{u}}^0 := \bigoplus_{\chi \in \Gamma} \mathcal{A}_{\mathrm{u}} \qquad \text{(algebraic sum)} . \tag{4.16}$$

By definition, elements are functions from Γ into $\mathcal{A}_{\mathrm{u}}$, denoted as $\mathbf{B} = (B_\chi)_{\chi \in \Gamma}$, with only finitely many components B_χ nonzero. Addition and multiplication with scalars are defined componentwise. Continuing the notation introduced in chapter 1, we define functions with support in one point $\hat{\chi}$ by $(\hat{\chi}, B)_\chi := \delta_{\hat{\chi},\chi} B$, and may write $(B_\chi)_{\chi \in \Gamma} = \sum_{\chi \in \Gamma} (\chi, B_\chi)$. A product is defined on $\mathcal{F}_{\mathrm{u}}^0$ by distributively extending the following analogue of equation (1.11):

$$(\hat{\chi}, \hat{B})(\chi, B) := (\chi\hat{\chi}, \gamma_\chi(\hat{B})B) .$$

It is associative due to the homomorphism property (4.10) of the assignment $\chi \mapsto \gamma_\chi$, thus turning $\mathcal{F}_{\mathrm{u}}^0$ into an algebra. In fact, it is a *-algebra, with the adjoint being defined by antilinear extension of

$$(\chi, B)^* := (\chi^{-1}, \gamma_\chi^{-1}(B^*)) .$$

We introduce a C^*-norm on $\mathcal{F}_{\mathrm{u}}^0$ following a monograph by H. Baumgärtel [Bau95], to which we also refer for the proofs. In a first step, one defines a norm on $\mathcal{F}_{\mathrm{u}}^0$ by $||(B_\chi)_{\chi \in \Gamma}||_2 :=$

[4] This is a convention which replaces that of normal commutation relations for Bose- and Fermi fields.

$\|\sum_{\chi\in\Gamma} B^*_\chi B_\chi\|^{\frac{1}{2}}$. Multiplication in $\mathcal{F}^0_{\rm u}$ is continuous with respect to this norm [Bau95], and hence one can define another norm on $\mathcal{F}^0_{\rm u}$ by

$$\|\mathbf{B}\|_{\mathcal{F}} := \sup_{\boldsymbol{X}\in\mathcal{F}^0_{\rm u}} \frac{\|\mathbf{B}\,\boldsymbol{X}\|_2}{\|\boldsymbol{X}\|_2} < \infty\,,$$

which indeed is a C^*-norm [Bau95]. Now we define the *universal field algebra* $\mathcal{F}_{\rm u}$ as the C^*-algebra obtained by completion of $\mathcal{F}^0_{\rm u}$ with respect to this norm. A *localization* concept on $\mathcal{F}_{\rm u}$, based on paths $\tilde{I} \in \tilde{\mathcal{K}}$, can be defined in analogy to the field bundle in Chapter 1: An element (χ, B) of $\mathcal{F}_{\rm u}$ is said to be localized in a path $\tilde{I} = [(S_0, I_1, \dots, I_n)]$ if there is a chain of charge transporters $U_1, \dots, U_n$ for γ_χ along $\tilde{I}$ such that $U_n \cdots U_1\, B \in \mathcal{A}_{\rm u}(I_n)$. Stated differently, (χ, B) is localized in $\tilde{I}$ if it is of the form $(\chi\,, U^*_{\chi,\tilde{I}} A)$ where $A \in \mathcal{A}_{\rm u}(I)$ and $U_{\chi,\tilde{I}}$ is the product of charge transporters as above.

Lemma and Definition 4.4. *i) The subset $\mathcal{F}^0_{\rm u}(\tilde{I}) \subset \mathcal{F}^0_{\rm u}$ of localized fields defined by*

$$\mathcal{F}^0_{\rm u}(\tilde{I}) := \bigoplus_{\chi\in\Gamma} \{\, (\chi\,, U^*_{\chi,\tilde{I}}\, A) \mid A \in \mathcal{A}_{\rm u}(I)\,, U_{\chi,\tilde{I}} = U_n \cdots U_1 \;\; \textit{as above} \,\} \tag{4.17}$$

*is in fact a *-subalgebra of $\mathcal{F}^0_{\rm u}$. The C^*-algebra obtained by its norm closure in $\mathcal{F}_{\rm u}$ will be denoted by $\mathcal{F}_{\rm u}(\tilde{I})$.*

ii) Let $(\chi_i, B_i) \in \mathcal{F}_{\rm u}(\tilde{I}_i)$, $i = 1, 2$, where $\tilde{I}_1$ and $\tilde{I}_2$ are causally disjoint paths in $\mathcal{K}$ with relative winding number $N := N(\tilde{I}_1, \tilde{I}_2)$. Then

$$(\chi_1, B_1)(\chi_2, B_2) = \varepsilon(\gamma_{\chi_1}, \gamma_{\chi_2}\,; N) \cdot (\chi_2, B_2)(\chi_1, B_1)\,. \tag{4.18}$$

Here, $\varepsilon = (\gamma_{\chi_2}\gamma_{\chi_1}|\varepsilon|\gamma_{\chi_1}\gamma_{\chi_2})$[5] *is understood to act in $\mathcal{F}_{\rm u}$ as $\varepsilon \cdot (\chi_1\chi_2, B) := (\chi_2\chi_1, \varepsilon B)$.*

Proof. i) First note that $\mathcal{F}^0_{\rm u}(\tilde{I})$ is in fact a vector space, because $(\chi, U^*A) + (\chi, V^*B) = (\chi, U^*(A + UV^*B))$ and $A + UV^*B$ is in $\mathcal{A}_{\rm u}(I)$ since UV^* is, according to an argument given on page 14. Next we show that this subspace is stable under multiplication. Let (χ, U^*A) and $(\hat{\chi}, \hat{U}^*\hat{A})$ be in $\mathcal{F}^0_{\rm u}(\tilde{I})$. (The general case follows by distributivity.) Given the chains $U_1, \dots, U_n$ and $\hat{U}_1, \dots, \hat{U}_n$ of charge transporters along $\tilde{I}$ for γ_χ and $\gamma_{\hat{\chi}}$, respectively, we obtain a chain of transporters for $\gamma_{\hat{\chi}\chi}$ by $V_k := \hat{U}_k \times U_k = \hat{U}_k \cdots \hat{U}_1 \gamma_{\hat{\chi}}(U_k) \hat{U}^*_1 \cdots \hat{U}^*_{k-1}$. Now the product can be written as $(\chi, U^*A)\,(\hat{\chi}, \hat{U}^*\hat{A}) = (\hat{\chi}\chi, V^*B)$ with $V := V_n \cdots V_1$ and $B := ({\rm Ad}\hat{U} \circ \gamma_{\hat{\chi}})(A)\hat{A} \in \mathcal{A}_{\rm u}(I)$, hence lies in $\mathcal{F}^0_{\rm u}(\tilde{I})$. To see that $\mathcal{F}^0_{\rm u}(\tilde{I})$ is *-stable, we first note that a chain of charge transporters $U_1, \dots, U_n$ along $\tilde{I}$ for γ_χ yields a chain $V_1, \dots, V_n$ for γ^{-1}_χ by

$$V_k := ({\rm Ad}(U_{k-1} \cdots U_1) \circ \gamma_\chi)^{-1}(U^*_k) = \gamma^{-1}_\chi(U^*_1 \cdots U^*_k U_{k-1} \cdots U_1)\,.$$

Now the adjoint reads $(\chi, U^*A)^* = (\chi^{-1}, V^*B)$ with $V = V_n \cdots V_1$ and $B = ({\rm Ad}U \circ \gamma_\chi)^{-1}(A^*) \in \mathcal{A}_{\rm u}(I)$, and hence is in $\mathcal{F}^0_{\rm u}(\tilde{I})$.

ii) The commutation relations (4.18) can be calculated in complete analogy to Lemma 1.4 of Chapter 1. □

Next we realize the C^*-algebras $\mathcal{F}_{\rm u}(\tilde{I})$ as von Neumann algebras $\mathcal{F}(\tilde{I})$ acting on a physical Hilbert space $\mathcal{H}$. We define $\mathcal{H}$ as the direct sum over all relevant sectors of copies of $\mathcal{H}_0$:

$$\mathcal{H} := \bigoplus_{\chi\in\Gamma} \mathcal{H}_\chi\,,\; \mathcal{H}_\chi := \mathcal{H}_0 \quad \text{(Hilbert sum)}\,. \tag{4.19}$$

By definition, this space consists of functions from Γ into $\mathcal{H}_0$, denoted as $\phi = (\phi_\chi)_{\chi\in\Gamma}$, which have finite norm with respect to the scalar product

$$\langle\, \phi\,, \phi' \,\rangle := \sum_{\chi\in\Gamma} \langle\, \phi_\chi\,, \phi'_\chi \,\rangle_0\,, \tag{4.20}$$

where $\langle\;,\;\rangle_0$ denotes the scalar product in $\mathcal{H}_0$. The subspaces $\mathcal{H}_\chi$ of $\mathcal{H}$ consist of functions with support in one point $\{\chi\}$, denoted (χ, ϕ) and defined by $(\chi, \phi)_{\hat{\chi}} := \delta_{\chi,\hat{\chi}}\, \phi$. In particular, we set $\boldsymbol{\Omega} := (1, \Omega)$. The representations V and U of the gauge group G and of $\tilde{P}^\uparrow_+$ are defined as in equation (4.3), where now U_χ is understood to be the representation U_{γ_χ} which implements $\tilde{P}^\uparrow_+$ in $\pi_0\gamma_\chi$.

[5] For the definition, see equation (1.31).

We define a representation π_{u} of the C^* algebra $\mathcal{F}_{\mathrm{u}}$ in $\mathcal{H}$ by

$$\pi_{\mathrm{u}}((\chi, B))\,(\hat{\chi}, \phi) := (\hat{\chi}\chi, \pi_0\gamma_{\hat{\chi}}(B)\phi)\ . \tag{4.21}$$

(It is in fact the GNS representation of the state $\omega((\chi, B)) := \langle\,\Omega, \pi_0\gamma_\chi(B)\,\Omega\,\rangle_0$.) The family $\mathcal{F} := \mathcal{F}(\tilde{I})_{\tilde{I}\in\tilde{\mathcal{K}}}$ of von Neumann algebras

$$\mathcal{F}(\tilde{I}) := (\pi_{\mathrm{u}}\mathcal{F}_{\mathrm{u}}(\tilde{I}))'' \tag{4.22}$$

will be called the *field algebra for* $\mathcal{A}$, Γ *and* γ. We have to show that $\mathcal{F}$ actually satisfies all requirements of Definition 4.1 for a field algebra and in addition the claims of Proposition 4.3. We first convert the commutation relations into a concise form as a relation in the field algebra $\mathcal{F}$ in terms of the local algebras $\mathcal{F}(\tilde{I})$ and their commutants, which will be called 'twisted locality', see equation (4.27) below. This form is convenient for a derivation of the Reeh-Schlieder property for the algebras $\mathcal{F}(\tilde{I})$, and can be sharpened to twisted Haag duality, which will be used in the derivation of the P_1CT and spin-statistics theorems in Section 4.2. To this end, we first calculate the statistics intertwiners in the vacuum representation from the statistics phases ω_i of the elementary charges $\chi_{(i)}$, which have been defined in equation (4.12). Since our system of representatives γ_χ is chosen such that all of them commute, the statistics operator of any two of them, $\varepsilon(\gamma_\chi, \gamma_{\hat{\chi}}\,; N)$, intertwines $\gamma_\chi\gamma_{\hat{\chi}}$ with itself, hence must be a multiple of unity in the vacuum representation, which we denote by $\varepsilon_0(N; \chi, \hat{\chi})$, i.e.

$$\varepsilon_0(N; \chi, \hat{\chi})\,\mathbf{1}\ := \pi_0\varepsilon(\gamma_\chi, \gamma_{\hat{\chi}}\,; N)\ . \tag{4.23}$$

As the next Lemma shows, these numbers indeed only depend on the equivalence classes χ of the automorphisms γ_χ.

LEMMA AND DEFINITION 4.5. *The map*

$$\begin{aligned} \varepsilon_0(N) :\quad & \Gamma\times\Gamma \to U(1)\ , \\ & (\chi, \hat{\chi}) \mapsto \varepsilon_0(N; \chi, \hat{\chi}) \end{aligned} \tag{4.24}$$

is a symmetric bilinear form. It satisfies $\varepsilon_0(N; \chi, \hat{\chi}) = \varepsilon_0(0; \chi, \hat{\chi})^{2N+1}$ *and is explicitely given by*

$$\varepsilon_0(N; \chi, \hat{\chi}) = \prod_{i=1}^{R} \omega_i^{k_i\hat{k}_i(2N+1)} \qquad \text{if } \chi = \prod \chi_{(i)}^{k_i} \text{ and } \hat{\chi} = \prod \chi_{(i)}^{\hat{k}_i}\ . \tag{4.25}$$

For the proof we need a lemma:

LEMMA 4.6. *Let* ϱ_1, ϱ_2 *and* σ *be localized endomorphisms of the observable algebra. Then*

$\varepsilon(\varrho_1\varrho_2, \sigma\,; N) = \varepsilon(\varrho_1, \sigma\,; N)\ \varrho_1(\varepsilon(\varrho_2, \sigma\,; N)) = (\varepsilon(\varrho_1, \sigma\,; N)\times \mathbf{1}_{\,\varrho_2})\circ(\mathbf{1}_{\,\varrho_1}\times\varepsilon(\varrho_2, \sigma\,; N))$ *and*

$\varepsilon(\sigma, \varrho_1\varrho_2\,; N) = \varrho_1(\varepsilon(\sigma, \varrho_2\,; N))\ \varepsilon(\sigma, \varrho_1\,; N) = (\mathbf{1}_{\,\varrho_1}\times\varepsilon(\sigma, \varrho_2\,; N))\circ(\varepsilon(\sigma, \varrho_1\,; N)\times\mathbf{1}_{\,\varrho_2})$

PROOF. We prove the first of the equations. Let $\tilde{I}, \tilde{J}$ be paths in $\tilde{\mathcal{K}}$ with relative winding number N. Let further for $i = 1, 2$ $U_i = U_i^{(n)}\cdots U_i^{(1)}$ be the product of elements of a chain of charge transporters along $\tilde{I}$ for ϱ_i, and $U = U^{(m)}\cdots U^{(1)}$ be the product of elements of a chain of charge transporters along $\tilde{J}$ for σ. Then $U_1\times U_2 = (U_1^{(n)}\times U_2^{(n)})\circ\cdots\circ(U_1^{(1)}\times U_2^{(1)})$ is the product of the elements of a chain of charge transporters along $\tilde{I}$ for $\varrho_1\varrho_2$. By definition (1.31) of the statistics operator,

$$\begin{aligned} \varepsilon(\varrho_1\varrho_2, \sigma\,; N) &= (U^*\times U_1^*\times U_2^*)\circ(U_1\times U_2\times U) \\ &= (U^*\times U_1^*\times U_2^*)\circ(U_1\times U\times U_2)\circ(U_1^*\times U^*\times U_2^*)\circ(U_1\times U_2\times U) \\ &= \big(((U^*\times U_1^*)\circ(U_1\times U))\times\mathbf{1}_{\,\varrho_2}\big)\ \circ\ \big(\mathbf{1}_{\,\varrho_1}\times((U^*\times U_2^*)\circ(U_2\times U))\big) \\ &= (\varepsilon(\varrho_1, \sigma\,; N)\times\mathbf{1}_{\,\varrho_2})\circ(\mathbf{1}_{\,\varrho_1}\times\varepsilon(\varrho_2, \sigma\,; N))\,. \end{aligned}$$

□

PROOF OF LEMMA 4.5. We show first equation (4.25) for the case $N = 0$. In this case the statistics operators are local intertwiners, hence π_0 can be inverted on them and we can conclude from equation (4.23) that

$$\varepsilon(\gamma_\chi, \gamma_{\hat\chi}; 0) = \varepsilon_0(0;\, \chi, \hat\chi)\, \mathbf{1}_{\mathcal{A}_\mathrm{u}} \,.$$

Therefore the equations of Lemma 4.6 in the present context take the form

$$\begin{aligned}\varepsilon_0(0;\, \chi_1\chi_2, \chi) &= \varepsilon_0(0;\, \chi_1, \chi)\, \varepsilon_0(0;\, \chi_2, \chi) \text{ and} \\ \varepsilon_0(0;\, \chi, \chi_1\chi_2) &= \varepsilon_0(0;\, \chi, \chi_1)\, \varepsilon_0(0;\, \chi, \chi_2) \text{ for all } \chi, \chi_1, \chi_2 \in \Gamma\,,\end{aligned}$$

respectively, rendering $\varepsilon_0(0)$ bilinear. Next we note that

$$\begin{aligned}\varepsilon_0(0;\, \chi_{(i)}, \chi_{(i)}) &= \omega_i \quad \text{by definition (4.12), and} \\ \varepsilon_0(0;\, \chi_{(i)}, \chi_{(j)}) &= 1 \qquad \text{if } i \neq j\,,\end{aligned}$$

and since $\gamma_{\chi_{(i)}}$ and $\gamma_{\chi_{(j)}}$ are localized in causally disjoint regions. Using the last two equations and bilinearity we get the explicit formula (4.25) for $N = 0$. This in turn shows that $\varepsilon_0(0)$ is symmetric. The general dependence of the statistics operators on N is given by equation (1.38), which reads in the vacuum representation

$$\pi_0\varepsilon(\gamma_\chi, \gamma_{\hat\chi}; N) = \pi_0\varepsilon(\gamma_\chi, \gamma_{\hat\chi}; 0) \cdot \pi_0\gamma_\chi(V_{\gamma_{\hat\chi}})^N \,.$$

The last factor can be calculated by applying π_0 to equation (1.48) and exploiting the symmetry of $\varepsilon_0(0)$:

$$\pi_0\gamma_\chi(V_{\gamma_{\hat\chi}}) = \pi_0\varepsilon(\gamma_\chi, \gamma_{\hat\chi}; 0)^2 \,. \tag{4.26}$$

Inserting this into the above equation, we get $\varepsilon_0(N; \chi, \hat\chi) = \varepsilon_0(0; \chi, \hat\chi)^{2N+1}$. But this proves also equation (4.25) for $N \neq 0$. □

LEMMA 4.7. *The commutation relations* (4.18) *in the representation* π_u *are equivalent to a 'twisted locality' in the form*

$$Z(\tilde I_1, \tilde I_2)\, \mathcal{F}(\tilde I_2)\, Z(\tilde I_1, \tilde I_2)^* \subset \mathcal{F}(\tilde I_1)' \quad \textit{for all } \tilde I_1, \tilde I_2 \textit{ with } I_1 \subset I_2' \,. \tag{4.27}$$

PROOF. We rewrite $Z(\tilde I_1, \tilde I_2) = \sum_{\chi\in\Gamma} \varepsilon_0(N; \chi, \chi)^{\frac12}\, P_\chi$, where $N = N(\tilde I_1, \tilde I_2)$. First note that every operator in $\mathcal{H}$ carrying charge $\hat\chi$, i.e. satisfying $F\,P_\chi = P_{\chi\hat\chi}\, F$, also satisfies

$$F\, Z(\tilde I_1, \tilde I_2) = \sum_{\chi\in\Gamma} \varepsilon_0(N; \chi\hat\chi^{-1}, \chi\hat\chi^{-1})^{\frac12}\, P_\chi\, F \,, \quad N = N(\tilde I_1, \tilde I_2) \,.$$

Now let $F_i = \pi_\mathrm{u}((\chi_i, B_i)) \in \mathcal{F}(\tilde I_i)$, and let $Z := Z(\tilde I_1, \tilde I_2)$. Applying the last equation twice and using the properties of bilinearity and symmetry of $\varepsilon_0(N)$ stated in the last Lemma, we get

$$F_1\, ZF_2Z^* = \sum_\chi \frac{\varepsilon_0(\chi\chi_1^{-1}, \chi\chi_1^{-1})^{\frac12}}{\varepsilon_0(\chi\chi_1^{-1}\chi_2^{-1}, \chi\chi_1^{-1}\chi_2^{-1})^{\frac12}} P_\chi\, F_1F_2 = \sum_\chi \frac{1}{\varepsilon_0(\chi_2^{-1}, \chi_2^{-1})^{\frac12}\varepsilon_0(\chi\chi_1^{-1}, \chi_2^{-1})} P_\chi\, F_1F_2$$

and

$$ZF_2Z^*\, F_1 = \sum_\chi \frac{\varepsilon_0(\chi, \chi)^{\frac12}}{\varepsilon_0(\chi\chi_2^{-1}, \chi\chi_2^{-1})^{\frac12}} P_\chi\, F_2F_1 = \sum_\chi \frac{1}{\varepsilon_0(\chi_2^{-1}, \chi_2^{-1})^{\frac12}\varepsilon_0(\chi, \chi_2^{-1})} P_\chi\, F_2F_1 \,,$$

where we have omitted the dependence on the winding number N. These two expressions coincide if and only if

$$F_1F_2 = \sum_\chi \frac{\varepsilon_0(N; \chi\chi_1^{-1}, \chi_2^{-1})}{\varepsilon_0(N; \chi, \chi_2^{-1})} P_\chi\, F_1F_2 = \varepsilon_0(N; \chi_1, \chi_2)\, F_2F_1 \,,$$

i.e. iff equation (4.18) holds for (χ_1, B_1) and (χ_2, B_2) in the representation π_u. This shows in particular that (4.27) implies the commutation relations (4.18) in π_u. On the other hand, the relation $[F_1, ZF_2Z^*] = 0$ extends by distributivity from field operators F_i of the special form as above, $F_i = \pi_\mathrm{u}((\chi_i, B_i))$, to arbitrary field operators in $\pi_\mathrm{u}(\mathcal{F}^0_\mathrm{u}(\tilde I_i))$, and by continuity to all of $\mathcal{F}(\tilde I_i)$. Hence the relations (4.18) imply (4.27). □

LEMMA 4.8. *Twisted locality* (4.27) *can be sharpened to twisted Haag duality, i.e. to equation* (4.14).

PROOF. We can essentially adapt the proof of the corresponding statement in [DR90, Thm. 5.4]. Let $\tilde{I}, \tilde{I}'$ be paths in $\widetilde{\mathcal{K}}$ ending at I and its causal complement I', respectively, and denote $Z := Z(\tilde{I}, \tilde{I}')$. Let F be an operator in $Z^* \mathcal{F}(\tilde{I})' Z$ carrying charge χ. Let further Ψ^* be a unitary in $\mathcal{F}(\tilde{I}')$ carrying the same charge, whose existence we have established before equation (4.30). Then, since $\mathcal{F}(\tilde{I}') \subset Z^* \mathcal{F}(\tilde{I})' Z$ by twisted locality, $F = \Psi^* A$ for some $A \in Z^* \mathcal{F}(\tilde{I})' Z \cap V(G)'$. But this algebra coincides with $\mathcal{F}(\tilde{I})' \cap V(G)'$ and by Haag duality (4.8) with $\mathcal{F}^G(I')$. Hence F is in $\mathcal{F}(\tilde{I}')$. □

Next we give a characterization of $\mathcal{F}(\tilde{I})$ in terms of a representation π of $\mathcal{A}_{\mathrm{u}}$ and certain charge carrying unitary field operators. The observable algebra $\mathcal{A}_{\mathrm{u}}$ acts in $\mathcal{H}$ via the representation

$$\pi := \bigoplus_{\chi \in \Gamma} \pi_0 \gamma_\chi \, .$$

Further, the 'shift' operator

$$\Psi_\chi := \pi_{\mathrm{u}}((\chi^{-1}, \mathbb{1})) : \quad (\hat{\chi}, \phi) \mapsto (\hat{\chi}\chi^{-1}, \phi) \tag{4.28}$$

is a unitary element of $\mathcal{F}(S_0)$ which implements the automorphism γ_χ of $\mathcal{A}_{\mathrm{u}}$ in the representation π of $\mathcal{A}_{\mathrm{u}}$, i.e.

$$\pi(\gamma_\chi(A)) = \Psi_\chi \pi(A) \Psi_\chi^* \, . \tag{4.29}$$

Let $U_1, \ldots, U_n$ be a chain of charge transporters for γ_χ along a path $\tilde{I} \in \widetilde{\mathcal{K}}$, and let $U_{\chi,\tilde{I}} := U_n \cdots U_1$. Then $\Psi_\chi^* \, \pi(U_{\chi,\tilde{I}}^*) \equiv \pi_{\mathrm{u}}((\chi, U^*))$ is in $\mathcal{F}(\tilde{I})$. Furthermore, Lemma 4.4 implies that

$$\mathcal{F}(\tilde{I}) = \Big(\bigoplus_{\chi \in \Gamma} \Psi_\chi^* \, \pi(U_{\chi,\tilde{I}}^*) \, \hat{\pi}(\mathcal{A}_{\mathrm{u}}(I)) \Big)^{-w} \quad \text{(weak closure)} \, . \tag{4.30}$$

As a consequence,

$$E_\chi(\mathcal{F}(\tilde{I})) = \Psi_\chi^* \, \pi(U_{\chi,\tilde{I}}^*) \, \pi(\mathcal{A}_{\mathrm{u}}(I)) \, . \tag{4.31}$$

PROPOSITION 4.9. *$\mathcal{F}$ is a field algebra with gauge group G in the sense of Definition 4.1 and satisfies the claims of Proposition 4.3.*

PROOF. Property 1 of Definition 4.1 (isotony) can be seen from equation (4.30) if one recalls that a chain of charge transporters along $\tilde{I}$ is also a chain along $\tilde{J}$ if $\tilde{I} \subset \tilde{J}$. Property 2 is a special case of twisted locality (4.27). Property 3 (Covariance) can be seen as follows. Analogously to the discussion of the reduced field bundle, see equation (1.24) in Chapter 1, one verifies that

$$\alpha_{\tilde{g}}(\chi, B) := \Big(\chi, \, Y_{\gamma_\chi}(\tilde{g}) \, \alpha_g^0(B)\Big) \tag{4.32}$$

defines a representation of $\tilde{P}_+^\uparrow$ in Aut$\mathcal{F}_{\mathrm{u}}$, which is implemented by U in the representation π_{u}, i.e.

$$\mathrm{Ad}U(\tilde{g}) \circ \pi_{\mathrm{u}} = \pi_{\mathrm{u}} \circ \alpha_{\tilde{g}} \quad \text{ for all } \tilde{g} \in \tilde{P}_+^\uparrow \, . \tag{4.33}$$

The discussion after equation (1.29) in Chapter 1 also shows that $\mathcal{F}_{\mathrm{u}}$ is covariant under α, i.e.

$$\alpha_{\tilde{g}} \; : \; \mathcal{F}_{\mathrm{u}}(\tilde{I}) \to \mathcal{F}_{\mathrm{u}}(\tilde{g} \cdot \tilde{I}) \, . \tag{4.34}$$

Hence $\mathrm{Ad}U(\tilde{g}) \, F(\tilde{I}) \equiv \mathrm{Ad}U(\tilde{g}) \, (\pi_{\mathrm{u}} \, \mathcal{F}_{\mathrm{u}}(\tilde{I}))'' = (\pi_{\mathrm{u}} \alpha_{\tilde{g}} \, \mathcal{F}_{\mathrm{u}}(\tilde{I}))'' \subset (\pi_{\mathrm{u}} \, \mathcal{F}_{\mathrm{u}}(\tilde{g} \cdot \tilde{I}))'' = \mathcal{F}(\tilde{g} \cdot \tilde{I})$, as required. Cyclicity of the vacuum $\mathbf{\Omega}$ for $\mathcal{F}(\tilde{I})$ follows from equation (4.30) and the fact that Ω is cyclic for $\mathcal{A}(I)$. Let $\tilde{J}$ be a path in $\widetilde{\mathcal{K}}$ ending spacelike to I. Then equation (4.27) implies that $\mathcal{F}(\tilde{I})'\mathbf{\Omega}$ contains $Z \, \mathcal{F}(\tilde{J}) \, \mathbf{\Omega}$, where Z is a unitary operator. Hence $\mathbf{\Omega}$ is cyclic for $\mathcal{F}(\tilde{I})'$ and therefore seperating for $\mathcal{F}(\tilde{I})$. Property 5 (irreducibility of $\mathcal{F}$) follows from the irreducibility of $\mathcal{A}$ and the existence of unitary field operators $\Psi_{\chi'\chi^{-1}}^*$ in $\mathcal{F}(S_0)$ connecting any two charged sectors $\mathcal{H}_{\chi'}, \mathcal{H}_\chi$. Property 1 of Proposition 4.3 follows from

$$\mathcal{F}^G(I) = \pi\big(\mathcal{A}_{\mathrm{u}}(I)\big) \, ,$$

which is in turn an immediate consequence of equation (4.31). Twisted Haag duality has been shown in Lemma 4.8. Finally, the intersection property (4.15) follows by simple set theoretic considerations. □

4.1.3. Uniqueness of the Field Algebra. Let $\hat{\mathcal{F}}$ be a field algebra with gauge group G satisfying 1 and 2 of Proposition 4.3 and which acts on a Hilbert space $\hat{\mathcal{H}} = \bigoplus_{\chi\in\Gamma} \hat{\mathcal{H}}_\chi$. We will reveal its structure up to the point where it is clear that it is equivalent to the above field algebra $\mathcal{F}$. We denote by $\hat{\pi}$ both the representation of the family $\mathcal{A}$ in $\hat{\mathcal{H}}$ and its lift to the universal algebra $\mathcal{A}_{\rm u}$. For every $\chi \in \Gamma$ and every localized automorphism γ of $\mathcal{A}_{\rm u}$ in the class of χ we have by assumption a unitary intertwiner from $\pi_0\gamma$ to $\hat{\pi}_\chi := \hat{\pi}|\hat{\mathcal{H}}_\chi$ which maps $\mathcal{H}_0$ isometrically onto $\hat{\mathcal{H}}_\chi$. In particular we fix a unitary intertwiner U_1^* from π_0 to $\hat{\pi}_{\chi=1}$, which we choose such that $\mathbf{\Omega} = U_1^* \Omega$.

The crucial point in the proof of uniqueness is to show that for each $\chi \in \Gamma$ and $\tilde{I} \in \widetilde{\mathcal{K}}$ there are unitary field operators in $\hat{\mathcal{F}}(\tilde{I})$ creating states $\omega_0 \circ \gamma$ with charge χ and localization region I from the vacuum. The following Lemma and the main part of its proof are analogous to [DHR69b, Thm.6.2] and [DR72, Prop.2.1].

Lemma 4.10. *Let $I \in \mathcal{K}, \chi \in \Gamma$ and let γ be an automorphism of $\mathcal{A}_{\rm u}$ in the equivalence class χ, localized in I. Let further $U^* : \mathcal{H}_0 \to \hat{\mathcal{H}}_\chi$ be a unitary intertwiner from $\pi_0\gamma$ to $\hat{\pi}_\chi \subset \hat{\pi}$. Then for any path $\tilde{I} \in \widetilde{\mathcal{K}}$ ending at I there is a unique field operator $\Psi^*_{U,\tilde{I}} \in \hat{\mathcal{F}}(\tilde{I})$ satisfying*

$$\Psi^*_{U,\tilde{I}}\, \mathbf{\Omega} = U^*\, \Omega\,. \tag{4.35}$$

It is unitary, carries charge χ and implements γ in the representation $\hat{\pi}$:

$$\hat{\pi}(\gamma(A)) = \Psi_{U,\tilde{I}}\, \hat{\pi}(A)\, \Psi^*_{U,\tilde{I}} \quad \textit{for all } A \in \mathcal{A}_{\rm u}\,. \tag{4.36}$$

Proof. For short we set $\Psi^* := \Psi^*_{U,\tilde{I}}$. If it exists, it must satisfy

$$\Psi^*\, F\, \mathbf{\Omega} = F\, U^*\, \Omega \quad \text{for all } F \in \mathcal{F}(\tilde{I})'\,. \tag{4.37}$$

Since $\mathbf{\Omega}$ is cyclic for $\mathcal{F}(\tilde{I})$ this defines already an operator. Hence we take the above equation as the definition of the operator Ψ^*. It is densely defined, because $\mathbf{\Omega}$ is also cyclic for $\mathcal{F}(\tilde{I})'$. We show that it is isometric. To this end, note that for any $B \in \mathcal{B}(\hat{\mathcal{H}})$, $P_\chi B P_\chi = P_\chi E_1(B) P_\chi$. Hence

$$\begin{aligned} ||FU^*\Omega||^2 \equiv ||FP_\chi U^*\Omega||^2 &= \langle \Omega, UP_\chi F^*FP_\chi U^*\Omega\rangle_0 = \langle \Omega, UP_\chi E_1(F^*F)P_\chi U^*\Omega\rangle_0 \\ &= \langle \Omega, U_1 F^*FU_1^*\Omega\rangle_0 = ||F\,\mathbf{\Omega}||^2\,, \end{aligned}$$

where we have used that, by Haag duality (4.8), $E_1(F^*F) = \hat{\pi}(A)$ for some A which is in $\mathcal{A}_{\rm u}(I')$ and hence satisfies

$$U\hat{\pi}(A)U^* = \pi_0\gamma(A) = \pi_0(A) = U_1\hat{\pi}(A)U_1^*\,.$$

Thus Ψ^* is isometric and consequently fixed by equation (4.37). Ψ^* carries charge χ, because it maps $\mathbf{\Omega}$ into $\hat{\mathcal{H}}_\chi$ and is in $\hat{\mathcal{F}}(\tilde{I})$, hence the equation $V(t)\Psi^*V(t)^*\,\mathbf{\Omega} = \chi(t)\Psi^*\,\mathbf{\Omega}$ extends to all of $\hat{\mathcal{H}}$ by the Reeh-Schlieder property. To show equation (4.36), consider the isometric map

$$\hat{\pi}(\gamma(A))\mathbf{\Omega} \mapsto \hat{\pi}(A)\, U^*\,\Omega\,, \quad A \in \mathcal{A}_{\rm u}\,.$$

It coincides with Ψ^* on the dense set $\hat{\pi}(\mathcal{A}_{\rm u}(I'))\,\mathbf{\Omega} = U_1^*\,\mathcal{A}(I')\,\Omega$ and hence on all of $\hat{\mathcal{H}}_{\chi=1}$, which means

$$\Psi^*\, \hat{\pi}(\gamma(A))\mathbf{\Omega} = \hat{\pi}(A)\, \Psi^*\, \mathbf{\Omega}\,, \quad A \in \mathcal{A}_{\rm u}\,.$$

This implies that $\Psi^*\, \hat{\pi}(\gamma(A)) = \hat{\pi}(A)\, \Psi^*$ if $A \in \mathcal{A}(\mathcal{O})$ for some $\mathcal{O}$, since then there is some $\tilde{J} \supset \mathcal{O} \cup \tilde{I}$ and $\mathbf{\Omega}$ is cyclic for $\hat{\mathcal{F}}(\tilde{J})'$. By norm continuity, this equation extends to all $A \in \mathcal{A}_{\rm u}$. It remains to show that Ψ^* is onto. Let $\bar{U}$ be an intertwiner from $\pi_0\gamma^{-1}$ to $\hat{\pi}_{\chi^{-1}}$, and let $\bar{\Psi}^*$ be the corresponding operator in $\hat{\mathcal{F}}(\tilde{I})$ intertwining $\hat{\pi}\circ\gamma^{-1}$ with $\hat{\pi}$. Then $\Psi^*\bar{\Psi}^*$ is in $\hat{\pi}(\mathcal{A})' = (\mathcal{F}^G)'$ which coincides with $\bigoplus \mathbb{C}\mathbb{1}_{\mathcal{H}_\chi}$ by equation (4.7). But it is also in $\mathcal{F}^G(I)$ whose restriction to any $\hat{\mathcal{H}}_{\chi'}$ is faithful. Hence $\Psi^*\bar{\Psi}^*$ is a multiple of unity, showing that Ψ^* has a right inverse. □

Lemma 4.11. *Let $(\gamma_\chi)_{\chi\in\Gamma}$ be the collection of automorphisms of $\mathcal{A}_{\rm u}$ defined in equation (4.11). Then there is a collection of unitary intertwiners $U_\chi^* : \mathcal{H}_0 \to \hat{\mathcal{H}}_\chi$ from $\pi_0\gamma_\chi$ to $\hat{\pi}_\chi$ such that*

$$\hat{\Psi}_\chi^*\, U_{\chi'}^* = U_{\chi'\chi}^* \quad \textit{for all } \chi, \chi' \in \Gamma\,, \tag{4.38}$$

where $\hat{\Psi}^*_\chi$ is the unique unitary field operator in $\hat{\mathcal{F}}(S_0)$ satisfying $\hat{\Psi}\chi^*\,\mathbf{\Omega} = U^*_\chi\,\Omega$ and implementing γ_χ. The collection $(\hat{\Psi}_\chi)_{\chi\in\Gamma}$ has the homomorphism properties $\hat{\Psi}_{\mathit{1}} = \mathbb{1}$ and

$$\hat{\Psi}_\chi\,\hat{\Psi}_{\chi'} = \hat{\Psi}_{\chi\chi'} \quad \textit{for all } \chi,\chi' \in \Gamma\,. \tag{4.39}$$

PROOF. The proof partly parallels that of [DHR69a, Thm.5.1]. Let $(U_\chi)_{\chi\in\Gamma}$ be any collection of unitary intertwiners from $\pi_0\gamma_\chi$ to $\hat{\pi}_\chi$ and $\hat{\Psi}^*_\chi$ the corresponding field operators. Then

$$\hat{\Psi}^*_\chi\,\hat{\Psi}^*_{\chi'} = b(\chi,\chi')\,\hat{\Psi}^*_{\chi\chi'} \qquad \text{for all } \chi,\chi' \in \Gamma\,, \tag{4.40}$$

where $b(\chi,\chi')$ is in $\hat{\mathcal{F}}^G(S_0)$ and satisfies, due to equations (4.36) and (4.10), $\mathrm{Ad}\,b(\chi,\chi')\circ\hat{\pi} = \hat{\pi}$ on $\mathcal{A}_\mathrm{u}$. Hence $b(\chi,\chi')$ is in $(\hat{\mathcal{F}}^G)'$ which coincides with $\bigoplus_{\hat{\chi}}\mathbb{C}\mathbb{1}_{\hat{\mathcal{H}}_{\hat{\chi}}}$ by equation (4.7). Since it is also in $\hat{\mathcal{F}}^G(S_0)$, whose restriction to any $\hat{\mathcal{H}}_{\hat{\chi}}$ is faithful, it is a multiple of unity. Then (4.40) and associativity imply the cocycle condition

$$b(\chi,\chi')\,b(\chi\chi',\chi'') = b(\chi',\chi'')\,b(\chi,\chi'\chi'')\,. \tag{4.41}$$

Since we have chosen the automorphisms γ_{χ_i} corresponding to the elementary charges localized in the smaller cones $S_{0,i}\subset S_0$, we can argue as after equation (4.37) to conclude that $\hat{\Psi}^*_{\gamma_i}$ is actually in the smaller algebra $\hat{\mathcal{F}}(S_{0,i})$, so any two of them must commute by property 2 of Proposition 4.3. Hence $b(\chi_i,\chi_j) = b(\chi_j,\chi_i)$. But then the cocycle condition implies $b(\chi,\chi') = b(\chi',\chi)$ for all $\chi,\chi'\in\Gamma$. This can be seen by induction: Let $b(\chi_{i_1}\cdots\chi_{i_n},\chi_k) = b(\chi_k,\chi_{i_1}\cdots\chi_{i_n})$. Then

$$\begin{aligned}
b(\chi_{i_1}\cdots\chi_{i_{n+1}}\,,\,\chi_k) &= b(\chi_{i_1}\cdots\chi_{i_n},\chi_{i_{n+1}})^{-1}\,b(\chi_k,\chi_{i_{n+1}})\,b(\chi_{i_1}\cdots\chi_{i_n},\chi_k\chi_{i_{n+1}})\\
&= b(\chi_{i_1}\cdots\chi_{i_n},\chi_{i_{n+1}})^{-1}\,b(\chi_{i_1}\cdots\chi_{i_n},\chi_k)\,b(\chi_{i_1}\cdots\chi_{i_n}\chi_k,\chi_{i_{n+1}})\\
&= b(\chi_{i_1}\cdots\chi_{i_n},\chi_{i_{n+1}})^{-1}\,b(\chi_k,\chi_{i_1}\cdots\chi_{i_n})\,b(\chi_k\chi_{i_1}\cdots\chi_{i_n},\chi_{i_{n+1}})\\
&= b(\chi_k\,,\,\chi_{i_1}\cdots\chi_{i_n+1})\,.
\end{aligned}$$

This shows that $b(\chi,\chi_k) = b(\chi_k,\chi)$ for all $\chi\in\Gamma$. The general case is shown analogously. But being a symmetric cocycle, b must be a "coboundary", i.e. there is a map c from Γ into the group of complex numbers of modulus 1 such that

$$b(\chi,\chi') = c(\chi)\,c(\chi\chi')^{-1}\,c(\chi').$$

A proof can be found e.g. in [DHR69a, Lemma A.1.2]. If we replace each U_χ by $c(\chi)c(1)^{-1}U_\chi$ and $\hat{\Psi}^*_\chi$ correspondingly, still denoted by the the old symbols, the collection satisfies equation (4.39) instead of (4.40). Now $\hat{\Psi}^*_\chi\,U^*_{\chi'}\,\Omega = U^*_{\chi\chi'}\,\Omega$ follows from the defining equations (4.35) for $\hat{\Psi}^*_{\chi'}$ and $\hat{\Psi}^*_{\chi\chi'}$. Therefore equation (4.38) holds on $\mathcal{A}(S_0')\,\Omega$, and hence on $\mathcal{H}_0$ due to the Reeh-Schlieder property. □

LEMMA 4.12. *Let $\tilde{I}\in\tilde{\mathcal{K}}$, let $U_1,\dots,U_n$ be a chain of charge transporters for γ_χ along $\tilde{I}$, and let $U_{\chi,\tilde{I}} = U_n\cdots U_1$. Then*

$$E_\chi(\hat{\mathcal{F}}(\tilde{I})) = \hat{\Psi}^*_\chi\,\hat{\pi}(U^*_{\chi,\tilde{I}})\,\hat{\pi}(\mathcal{A}_\mathrm{u}(I))\,. \tag{4.42}$$

PROOF. Since $\hat{\Psi}^*_\chi$ is a unitary carrying charge χ, it is sufficient to show that

$$\hat{\Psi}^*_\chi\,\hat{\pi}(U^*_{\chi,\tilde{I}}) \in \hat{\mathcal{F}}(\tilde{I})\,,$$

which we do by induction. Let $(I_0,\dots,I_n)$, be a path in the class $\tilde{I}$, and let $\tilde{I}_k$ be the class of $(I_0,\dots,I_k)$. Let further $\tilde{J}_k$ be the "larger" one of $\tilde{I}_k$ and $\tilde{I}_{k+1}$, i.e. $\tilde{J}_k = \tilde{I}_k$ if $I_k\supset I_{k+1}$ and $\tilde{J}_k = \tilde{I}_{k+1}$ if $I_k\subset I_{k+1}$. We denote $\Psi^*_k := \hat{\Psi}^*_\chi\,\hat{\pi}(U^*_1\cdots U^*_k)$. Suppose Ψ^*_k is in $\hat{\mathcal{F}}(\tilde{I}_k)$. Then Ψ^*_{k+1}, being equal to $\Psi^*_k\,\hat{\pi}(U^*_{k+1})$, is in $\hat{\mathcal{F}}(\tilde{J}_k)$ since $\hat{\pi}(U^*_{k+1})$ is in $\hat{\mathcal{F}}^G(I_k\cup I_{k+1})$. On the other hand , by equation (4.36) and the intertwiner property of U^*_χ,

$$\Psi^*_{k+1}\,\mathbf{\Omega} = U^*_\chi\,\hat{\pi}(U^*_1\cdots U^*_{k+1})\,\Omega\,.$$

The product of unitaries on the right hand side is an intertwiner from $\pi_0\circ\mathrm{Ad}(U_{k+1}\cdots U_1)\circ\gamma_\chi$, which is localized in I_{k+1}, to $\hat{\pi}_\chi$. Then by virtue of Lemma 4.10, there is a unitary $\Psi^*\in\hat{\mathcal{F}}(\tilde{I}_{k+1})$ which coincides with Ψ^*_{k+1} on $\mathbf{\Omega}$. But Ψ^* and Ψ^*_{k+1} are both in $\hat{\mathcal{F}}(\tilde{J}_k)$ and therefore coincide due to the Reeh-Schlieder property. Hence Ψ^*_{k+1} is in $\hat{\mathcal{F}}(\tilde{I}_{k+1})$ and the induction is complete. □

At this point we can exhibit a unitary equivalence from $\hat{\mathcal{F}}$ to $\mathcal{F}$: Let W^* be the direct sum of unitaries U_χ^* of Lemma 4.11, i.e. the unitary operator given by

$$W^* : \bigoplus_\chi \mathcal{H}_0 \overset{\cong}{\to} \hat{\mathcal{H}}, \quad (\chi,\phi) \mapsto U_\chi^* \phi .$$

PROPOSITION 4.13. AdW *establishes a unitary equivalence from* $\hat{\mathcal{F}}$ *to* $\mathcal{F}$.

PROOF. Using equation (4.38) one verifies that $W\,\hat{\Psi}_\chi\,W^* = \Psi_\chi$, and the intertwiner properties of the U_χ^* imply $W\,\hat{\pi}(A)\,W^* = \pi(A)$. In view of equations (4.42) and (4.31), AdW is therefore an isomorphism from $\bigoplus_{\chi\in\Gamma} E_\chi\,\hat{\mathcal{F}}(\tilde{I})$ onto $\bigoplus_{\chi\in\Gamma} E_\chi\,\mathcal{F}(\tilde{I})$. Since these algebras are weakly dense in $\hat{\mathcal{F}}(\tilde{I})$ and $\mathcal{F}(\tilde{I})$ respectively, see equation (4.6), AdW extends to a unitary equivalence from $\hat{\mathcal{F}}(\tilde{I})$ to $\mathcal{F}(\tilde{I})$. □

4.2. Algebraic P_1CT and Spin-Statistics Theorems

From now on we again require Assumption 1, namely that the modular conjugations of the observable algebras associated to wedge regions have geometric significance (see page 23). We show that this property extends to the field algebra leading to a P_1CT theorem (Theorem 4.14), and that a spin-statistics theorem follows from this fact (Theorem 4.15). These results have been obtained for the case of non-Abelian permutation group statistics by D. Guido and R. Longo[6] for compactly localized charges in 4 dimensions [GL95], and by B. Kuckert for charges with localization in spacelike cones in 3 dimensions [Kuc95]. Here they are generalized for the first time to the case of Abelian braid group statistics in 3 dimensions. Also, an expression of the P_1CT operator of the field algebra is given in terms of the modular conjugation of the observable algebra without taking recourse to the modular conjugation of the field algebra (Proposition 4.16).

It is known [Dav95] that for Bose and Fermi fields, and hence in our context at least for the observable algebra, the geometric significance of the modular *conjugations* associated to wedge regions is equivalent to the geometric significance of the modular *operators* associated to them, see Lemma 4.18. We show in Proposition 4.19, that also the latter property extends from the observable algebra to the anyonic field algebra. Hence, both modular conjugations and modular operators of the field algebras associated to wedge regions can be expressed in terms of the representation of the Poincaré group, and this will allow us to calculate them on the space of scattering states in the next section.[7] Let

$$\tilde{W}_1$$

be the homotopy class of the path (S_0, W_1) in $\mathcal{K}$, and let $S(\tilde{W}_1)$ be the Tomita operator for $\mathcal{F}(\tilde{W}_1)$ and $\mathbf{\Omega}$ with polar decomposition $S(\tilde{W}_1) = J\,\Delta^{\frac{1}{2}}$. Again we call J the modular conjugation and Δ the modular operator for $\mathcal{F}(\tilde{W}_1)$ and $\mathbf{\Omega}$.

THEOREM 4.14 (P_1CT - Theorem). *Let the observable algebra satisfy Assumption* 1 *(modular covariance). Then the modular conjugation J for $\mathcal{F}(\tilde{W}_1)$ and $\mathbf{\Omega}$ can be written as*

$$J = Z(\tilde{W}_1, \tilde{j}\cdot\tilde{W}_1)\,\Theta_1 \,, \tag{4.43}$$

where Θ_1 is an antilinear P_1CT operator *which extends the representation $U(\tilde{P}_+^\uparrow)$ to a representation of $\tilde{P}_+$ under which the field algebra is still covariant:*

$$\Theta_1{}^2 = \mathbb{1} \;, \quad \Theta_1\,U(\tilde{g})\,\Theta_1 = U(\tilde{j}\tilde{g}\tilde{j}) \;\; \text{for all } \tilde{g}\in\tilde{P}_+^\uparrow \;, \text{ and} \tag{4.44}$$

$$\Theta_1\,\mathcal{F}(\tilde{I})\,\Theta_1 = \mathcal{F}(\tilde{j}\cdot\tilde{I}) \quad \text{for all } \tilde{I}\in\tilde{\mathcal{K}} \,. \tag{4.45}$$

Further, Θ_1 leaves $\mathbf{\Omega}$ invariant and maps $\mathcal{H}_\chi$ into $\mathcal{H}_{\chi^{-1}}$, so that it can be written $\Theta_1\,(\chi,\phi) =: (\chi^{-1}, \Theta_{1,\chi}\,\phi)$. It implements α_j^0 as follows:

$$\mathrm{Ad}\Theta_{1,\chi}\circ\pi_0\gamma_\chi = \pi_0\gamma_\chi^{-1}\circ\alpha_j^0 \,. \tag{4.46}$$

All these properties, except equation (4.45)*, are also satisfied by J.*

---

[6]Actually, Guido and Longo have assumed only that the net of observables is covariant under the modular groups associated to wedge regions.

[7]as mentioned before, the modular conjugations can be calculated only up to the S-matrix.

Note that the relative winding number $N(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1)$ equals -1, hence $Z(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1)$ is given by

$$Z(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1) = \sum_{\chi\in\Gamma} \varepsilon_0(-1;\chi,\chi)^{\frac{1}{2}}\, P_\chi \,.$$

PROOF. The proof is partly an adaption of B. Kuckert's proof of Theorem 3.4 in [Kuc95]. We show first that J maps $\mathcal{H}_\chi$ into $\mathcal{H}_{\chi^{-1}}$, or equivalently, that

$$J\, P_\chi = P_{\chi^{-1}}\, J \,, \tag{4.47}$$

where $P\chi$ denotes the projector onto $\mathcal{H}_\chi$. Let $V_t := \sum_{\chi\in\Gamma} \chi(t)\, P_\chi$ be the representation of the gauge group G in $\mathcal{H}$. For all $t \in G$, V_t leaves the vacuum vector $\mathbf{\Omega}$ invariant, and $\mathrm{Ad}V_t$ leaves the field algebra $\mathcal{F}(\tilde{W}_1)$ invariant. Consequently, V_t commutes with the modular conjugation J (and also with the modular unitaries Δ^{it}, which we will use later). Writing $P_\chi = \int_G d\lambda(t)\bar{\chi}(t)V_t$ and using the antilinearity of J, this implies equation (4.47). To show equation (4.46), we first consider the restriction of the Tomita operator $S(\tilde{W}_1)$ to the vacuum sector $\mathcal{H}_{\chi=\mathbf{1}}$. As we have seen above, $S(\tilde{W}_1)$ leaves $\mathcal{H}_{\mathbf{1}}$ invariant. Since $\mathcal{F}^G(\tilde{W}_1)\,\mathbf{\Omega}$ is the intersection of $\mathcal{H}_{\mathbf{1}}$ with $\mathcal{F}(\tilde{W}_1)\,\mathbf{\Omega}$, which is a core for $S(\tilde{W}_1)$, $\mathcal{F}^G(\tilde{W}_1)\,\mathbf{\Omega}$ is a core for $S(\tilde{W}_1)|_{\mathcal{H}_{\mathbf{1}}\cap\mathrm{dom}(S)}$. On this core, $S(\tilde{W}_1)$ is easily seen to coincide with $U_1^*\, S_0(W_1)\, U_1$, where U_1^* is the unitary intertwiner from π_0 to $\pi_{\chi=\mathbf{1}}$, mapping ϕ to $(1,\phi)$, and $S_0(W_1)$ is the Tomita operator of $\mathcal{A}(W_1)$, as defined in equation (1.61). But $U_1\, \mathcal{F}^G(\tilde{W}_1)\,\mathbf{\Omega} \equiv \mathcal{A}(W_1)\,\Omega$, which is a core for $S_0(W_1)$, and hence

$$S(\tilde{W}_1)|_{\mathcal{H}_{\mathbf{1}}\cap\mathrm{dom}(S)} = U_1^*\, S_0(W_1)\, U_1 \,.$$

By uniqueness of the polar decomposition, this implies that $J|_{\mathcal{H}_{\mathbf{1}}} = U_1^*\, J_0\, U_1$. Consequently,

$$J\,\pi(A)\,J\,\mathbf{\Omega} = U_1^*\,\pi_0(\alpha_j^0(A))\,U_1\,\mathbf{\Omega} = \pi(\alpha_j^0(A))\,\mathbf{\Omega} \,.$$

If $A \in \mathcal{A}_{\mathrm{u}}(W_1)$, then $J\,\pi(A)\,J$ is in $\mathcal{F}(\tilde{W}_1)'$ by the Tomita-Takesaki theorem, and so is $\pi(\alpha_j^0(A))$ because the fields are local relative to the observables. Since $\mathbf{\Omega}$ is cyclic for $\mathcal{F}(\tilde{W}_1)$, we thus conclude from the above equation

$$J\,\pi(A)\,J = \pi(\alpha_j^0(A)) \tag{4.48}$$

for all $A \in \mathcal{A}_{\mathrm{u}}(W_1)$. Borchers Theorem [Bor92] asserts that the representation of the translation subgroup transforms covariant under J, i.e. $J\,U(x,\mathbb{1}\,)\,J = U(j\cdot x,\mathbb{1}\,)$. This fact extends equation (4.48) to all $A \in \bigcup_{x\in\mathbb{R}^3} \mathcal{A}_{\mathrm{u}}(x+W_1)$. The same equation holds for Θ_1 replacing J, because $Z(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1)$ commutes with gauge invariant operators. This means that the claim (4.46) holds on $\bigcup_{x\in\mathbb{R}^3} \mathcal{A}_{\mathrm{u}}(x+W_1)$. Now let R_χ be an intertwiner from $\bar{\gamma}_\chi = \alpha_j^0 \circ \gamma_\chi \circ \alpha_j^0$ to γ_χ^{-1}. Then the right hand side of equation (4.46) can be rewritten as $\mathrm{Ad}(\pi_0(R_\chi)J_0) \circ \pi_0\gamma_\chi$. Remembering that γ_χ maps $\mathcal{A}_{\mathrm{u}}(x+W_1)$ onto itself, the equation then implies that $\mathrm{Ad}\Theta_{1,\chi} = \mathrm{Ad}(\pi_0(R_\chi)J_0)$ on $\bigcup_{x\in\mathbb{R}^3} \mathcal{A}(x+W_1)$ and hence, by irreducibility, on $\mathcal{B}(\mathcal{H}_0)$. Thus equation (4.46) holds on $\mathcal{A}_{\mathrm{u}}$. The representation property (4.44) is equivalent to $U_{\chi^{-1}}(\tilde{g}) = \Theta_{1,\chi}\, U_\chi(\tilde{\jmath}\tilde{g}\tilde{\jmath})\,\Theta_{1,\chi}^{-1}$ and follows from equation (4.46) by the same argument we have used after equation (1.68). It remains to show equation (4.45), namely that $\mathrm{Ad}\Theta_1$ acts geometrically. Combining the twisted Haag duality (4.14) with the Tomita Takesaki theorem yields

$$J\,\mathcal{F}(\tilde{W}_1)\,J = Z(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1)\,\mathcal{F}(\tilde{\jmath}\cdot\tilde{W}_1)\,Z(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1)^* \,,$$

which is the desired relation for the case $\tilde{I} = \tilde{W}_1$. Since every $\tilde{W} \in \tilde{\mathcal{K}}$ ending at a wedge region is of the form $\tilde{W} = \tilde{g}\cdot\tilde{W}_1$ for some $\tilde{g} \in \tilde{P}_+^\uparrow$, we can exploit the representation property (4.44) to conclude that equation (4.45) holds for all such paths $\tilde{W}$. If $\tilde{I}$ ends at a spacelike cone S_0, the equation still holds due to the intersection property (4.15) of the field algebra. If $\tilde{I}$ ends at the causal complement of a spacelike cone, $\tilde{I} = \tilde{S}'$, we use twisted Haag duality and get

$$\begin{aligned}\Theta_1\,\mathcal{F}(\tilde{S}')\,\Theta_1 &= \Theta_1\,Z(\tilde{S}',\tilde{S})\,\mathcal{F}(\tilde{S})'\,Z(\tilde{S}',\tilde{S})^*\,\Theta_1 = Z(\tilde{S}',\tilde{S})^*\,(\Theta_1\,\mathcal{F}(\tilde{S})\,\Theta_1)'\,Z(\tilde{S}',\tilde{S})\\ &= Z(\tilde{\jmath}\cdot\tilde{S}',\tilde{\jmath}\cdot\tilde{S})\,\mathcal{F}(\tilde{\jmath}\cdot\tilde{S})'\,Z(\tilde{\jmath}\cdot\tilde{S}',\tilde{\jmath}\cdot\tilde{S})^* = \mathcal{F}(\tilde{\jmath}\cdot\tilde{S}') \,.\end{aligned}$$

In the third equation we have used $Z(\tilde{\jmath}\tilde{I},\tilde{\jmath}\tilde{J}) = Z(\tilde{I},\tilde{J})^*$. This completes the proof of equation (4.45) and hence of the theorem. □

As a corollary, we get the 'strong' version of the spin-statistics theorem. We recall that a model independent spin statistics theorem for the case of permutation group statistics has been established in the Wightman framework as early as 1958 [Bur58], and in the algebraic framework in [Bor65].

THEOREM 4.15 (Spin-Statistics Theorem). *Let* $\chi \in \Gamma$. *Its statistics phase* ω_χ, *defined by* $\pi_0\varepsilon(\gamma_\chi, \gamma_\chi, 0) = \omega_\chi \mathbf{1}$, *and its spin phase, defined by* $U_\chi((0, 2\pi)) = \exp(2\pi i s(\chi))\, \mathbb{1}$, *coincide:*

$$\omega_\chi = e^{2\pi i s(\chi)} . \tag{4.49}$$

PROOF. The proof is a straightforword adaption of the proof of [GL95, Thm.2.11] to the present setup. Let $Z := Z(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1)$. Since the relative winding number $N(\tilde{W}_1, \tilde{\jmath}\cdot\tilde{W}_1)$ equals -1, we have by definition of Z and Lemma 4.5

$$Z^2 = \sum_{\chi\in\Gamma} \varepsilon_0(-1;\chi,\chi)\, P_\chi = \sum_{\chi\in\Gamma} \omega_\chi^{-1}\, P_\chi .$$

On the other hand $U((0, -2\pi)) = \sum_\chi \exp(-2\pi i s(\chi))\, P_\chi$, and therefore the claim is equivalent to

$$Z^2 = U((0, -2\pi)). \tag{4.50}$$

Let $\Theta_2 := U((0, \frac{\pi}{2}))\, \Theta_1\, U((0, \frac{\pi}{2}))^*$. Due to the representation property (4.44) of Θ_1, $\Theta_2 = \Theta_1\, U(\tilde{\jmath}(0, \frac{\pi}{2})\tilde{\jmath})\, U((0, -\frac{\pi}{2})) = \Theta_1 U((0, -\pi))$, and we can write

$$U((0, -\pi)) = \Theta_1\, \Theta_2 .$$

Now let $\tilde{W}_2 := (0, \frac{\pi}{2})\cdot\tilde{W}_1$, which is the homotopy class of the path $(S_0, \frac{\pi}{2}\cdot W_1)$. It is invariant under $\tilde{\jmath}$, and therefore $\mathrm{Ad}\Theta_1$, acting geometrically by equation (4.45), leaves $\mathcal{F}(\tilde{W}_2)$ invariant. Hence Θ_1 commutes with the modular conjugation J_2 of $\mathcal{F}(\tilde{W}_2)$. The latter is given by $J_2 = U((0, \frac{\pi}{2}))\, J\, U((0, -\frac{\pi}{2}))$ which coincides with $Z\,\Theta_2$ by virtue of equation (4.43) and of $U(\tilde{g})Z = ZU(\tilde{g})$. Using $\Theta_1 Z^* = Z\,\Theta_1$, we thus have

$$U((0, -2\pi)) = (\Theta_1\, \Theta_2)^2 = \Theta_1 Z^* J_2\; \Theta_1 Z^* J_2 = Z^2\, \Theta_1 J_2 \Theta_1 J_2 = Z^2 .$$

□

We now give an expression for the $\mathrm{P}_1\mathrm{CT}$ -operator Θ_1, respectively for the modular conjugation J of $\mathcal{F}(\tilde{W}_1)$, in terms of the modular conjugation J_0 of the observable algebra analogous to equation (1.76).

PROPOSITION 4.16. *Let* Θ *be an antilinear operator in* $\mathcal{H}$ *which maps* $\mathcal{H}_\chi$ *into* $\mathcal{H}_{\chi^{-1}}$, *allowing for the notation* $\Theta\,(\chi, \phi) =: (\chi^{-1}, \Theta_\chi\, \phi)$.

i) Θ *has all the properties listed for the* P_1CT *operator* Θ_1 *in Theorem* 4.14 *(except the connection* (4.43) *with the modular conjugation) if and only if*

$$\Theta_\chi = \pi_0(R_\chi)\, J_0 , \tag{4.51}$$

where R_χ *is a local intertwiner from* $\bar{\gamma}_\chi$ *to* γ_χ^{-1}, *and the collection of intertwiners satisfies*

$$R_{\chi^{-1}} = \alpha_j^0(R_\chi^*) , \tag{4.52}$$

$$R_1 = \mathbb{1} \quad \text{and}$$

$$R_{\chi\chi'} = R_\chi \times R_{\chi'} . \tag{4.53}$$

ii) A collection of intertwiners satisfying the above requirements is fixed up to a collection $(\lambda_\chi)_\chi$ *of signs which forms a group homomorphism from* Γ *into* $\{\pm 1\}$. *Hence, if* Θ *is defined by equation* (4.51), *it coincides with the the* P_1CT *operator* Θ_1 *up to the unitary* $V = \sum_{\chi\in\Gamma} \lambda_\chi\, P_\chi$.

Note that due to the homomorphism property, $\lambda_\chi = 1$ if $\chi^n = 1$ for some odd integer n.

PROOF. *i)* We first show the 'only if' statement. Let R_χ^0 be the unitary operator in $\mathcal{H}_0$ defined by

$$R_\chi^0 := \Theta_\chi\, J_0 .$$

It is an intertwiner from $\pi_0\bar\gamma_\chi$ to $\pi_0\gamma_\chi^{-1}$, since

$$\mathrm{Ad}R_\chi^0 \circ \pi_0\bar\gamma_\chi = \mathrm{Ad}\Theta_\chi \circ \pi_0\gamma_\chi \circ \alpha_j^0 = \pi_0\gamma_\chi^{-1} \circ \alpha_j^0 \circ \alpha_j^0 = \pi_0\gamma_\chi^{-1} \,.$$

In the second equation we have used equation (4.46) of the P$_1$CT -Theorem. By Haag duality, this implies $R_\chi^0 \in \mathcal{A}(S_0)$, since both $\bar\gamma_\chi$ and γ_χ^{-1} are localized in S_0. Therefore we can define

$$R_\chi := \pi_0^{-1}(R_\chi^0) \,.$$

$R_1 = \mathbf{1}$ follows from the fact that both J_0 and $\Theta_{\chi=1}$ leave the vacuum vector Ω invariant. Equation (4.52) is equivalent to $\mathbf{1} = R^0_{\chi^{-1}} J_0\, R^0_\chi J_0 \equiv \Theta_{\chi^{-1}}\Theta_\chi$ for all χ. But this is equivalent to $\Theta^2 = \mathbf{1}$. Now we show equation (4.53). First note that $\mathrm{Ad}\Theta(\Psi_\chi^*)$ is in $\mathcal{F}(S_0)$ by equation (4.45) of Theorem 4.14, and that it carries charge χ^{-1}. By equation (4.31), this implies that

$$\Theta\Psi_\chi^*\Theta = \Psi^*_{\chi^{-1}}\,\pi(A_\chi) \tag{4.54}$$

for some $A_\chi \in \mathcal{A}_\mathrm{u}(S_0)$. But

$$\begin{aligned} \Theta\Psi_\chi^*\Theta\,(\hat\chi,\phi) &= (\chi^{-1}\hat\chi, R^0_{\hat\chi^{-1}\chi}(R^0_{\hat\chi^{-1}})^*\,\phi) \quad \text{by equ. (4.52), and} \\ \Psi^*_{\chi^{-1}}\,\pi(A_\chi)\,(\hat\chi,\phi) &= (\hat\chi\chi^{-1}, \pi_0\gamma_{\hat\chi}(A_\chi)\,\phi) \quad \text{for all } (\hat\chi,\phi)\in\mathcal{H}_{\hat\chi} \,. \end{aligned} \tag{4.55}$$

Hence equation (4.54) implies, taking again into account that π_0 is injective on the involved local operators, that $R_{\hat\chi^{-1}\chi}R^*_{\hat\chi^{-1}} = \gamma_{\hat\chi}(A_\chi)$. Specializing to $\hat\chi = 1$ and remembering that $R_1 = \mathbf{1}$ this implies $R_\chi = A_\chi$, so that we have

$$R_{\hat\chi^{-1}\chi} = \gamma_{\hat\chi}(R_\chi)\,R_{\hat\chi^{-1}} = R_{\hat\chi^{-1}}\bar\gamma_{\chi^{-1}}(R_\chi) = (R_{\hat\chi^{-1}} \times R_\chi)$$

which is equation (4.53). Now we prove the 'if' statement. The implementation property (4.46) follows from the definitions, and implies the representation property (4.44) in the same manner as in the proof of Theorem 4.14. $\Theta\,\boldsymbol{\Omega} = \boldsymbol{\Omega}$ and $\Theta^2 = 1$ have been treated already in the 'only if' part. Equations (4.52,4.53) imply $\Theta\,\Psi_\chi^*\,\Theta = \Psi^*_{\chi^{-1}}\,\pi(R_\chi)$, which can be seen using equations (4.55). But this transformation property implies the geometric action (4.45) of $\mathrm{Ad}\Theta$: Namely, let $U_1,\ldots U_n$ be a chain of charge transporters for γ_χ along some path $\tilde I$, and let $U = U_n\cdots U_1$. Then by equation (4.31), $E_\chi(\mathcal{F}(\tilde I)) = \Psi_\chi^*\pi(U^*)\,\pi(\mathcal{A}_\mathrm{u}(I))$. Let $A \in \mathcal{A}_\mathrm{u}(I)$. The transformation property of Ψ_χ^* under Θ together with (4.46), which takes here the form $\mathrm{Ad}\Theta\circ\pi = \pi\circ\alpha_j^0$, implies

$$\Theta\,\Psi_\chi^*\pi(U^*\,A)\,\Theta = \Psi^*_{\chi^{-1}}\pi\big(R_\chi\alpha_j^0(U^*)\,\alpha_j^0(A)\big) \,. \tag{4.56}$$

But $\alpha_j^0(U_1)R_\chi^*$, $\alpha_j^0(U_2),\ldots,\alpha_j^0(U_n)$ is a chain of charge transporters for χ^{-1} along $\tilde\jmath\cdot\tilde I$, hence $\Theta\,E_\chi(\mathcal{F}(\tilde I))\,\Theta \subset \mathcal{F}(\tilde\jmath\cdot\tilde I)$.

ii) Being a local intertwiner between two irreducible representations, each R_χ is fixed up to a phase: any other intertwiner is of the form $\hat R_\chi = \lambda_\chi R_\chi$ for some complex number λ_χ of modulus 1. Equation (4.53) being satisfied for both R_χ and $\hat R_\chi$ implies the homomorphism property $\lambda_{\chi\hat\chi} = \lambda_\chi\lambda_{\hat\chi}$. Also, $\lambda_1 = 1$ because both R_1 and $\hat R_1$ are equal to $\mathbf{1}$. Finally, plugging equation (4.52) into (4.53) yields

$$\mathbf{1} \equiv R_{\chi\chi^{-1}} = R_\chi \times R_{\chi^{-1}} = R_\chi \times \alpha_j^0(R_\chi^*)$$

for both $\hat R_\chi$ and R_χ, which implies $\lambda_\chi^2 = 1$. □

As a corollary of the preceding proof, we can show that the P$_1$CT -operator Θ_1 implements an antilinear automorphism $\alpha_{\tilde\jmath}$ of the *universal* field algebra $\mathcal{F}_\mathrm{u}$ as follows. Let

$$\alpha_{\tilde\jmath}\,(\chi,B) := (\chi^{-1}, R_\chi\,\alpha_j^0(B)) \,,$$

where $R_\chi := \pi_0^{-1}(\Theta_1 J_0)$ as in the above proof.

COROLLARY 4.17. *Θ_1 implements the automorphism $\alpha_{\tilde\jmath}$ in the representation π_u. In addition, $\alpha_{\tilde\jmath}$ acts geometrically in the universal field algebra $\mathcal{F}_\mathrm{u}$:*

$$\mathrm{Ad}\Theta_1\circ\pi_\mathrm{u} = \pi_\mathrm{u}\circ\alpha_{\tilde\jmath} \quad \textit{and} \tag{4.57}$$

$$\alpha_{\tilde\jmath} : \mathcal{F}_\mathrm{u}(\tilde I) \to \mathcal{F}_\mathrm{u}(\tilde\jmath\cdot\tilde I) \quad \textit{for all } \tilde I \in \tilde{\mathcal{K}} \,. \tag{4.58}$$

PROOF. Using equation (4.56), we get

$$\Theta_1 \pi_{\mathrm{u}}(\chi, B)\Theta_1 = \Theta_1 \Psi_\chi^* \pi(B)\Theta_1 = \Psi_{\chi^{-1}}^* \pi(R_\chi \alpha_j^0(B)) = \pi_{\mathrm{u}}(\chi^{-1}, R_\chi \alpha_j^0(B)) ,$$

i.e. equation (4.57). To prove equation (4.58), recall that $\mathcal{F}_{\mathrm{u}}(\tilde{I})$ is spanned by elements $(\chi, U^* A)$ with $A \in \mathcal{A}_{\mathrm{u}}(I)$ and $U = U_n \cdots U_1$, where $U_1, \dots, U_n$ is a chain of charge transporters for γ_χ along $\tilde{I}$. On such an element $\alpha_{\tilde{j}}$ acts as

$$\alpha_{\tilde{j}}(\chi, U^* A) = \big(\chi^{-1}, R_\chi\, \alpha_j^0(U^*)\, \alpha_j^0(A)\big) .$$

Now $\alpha_j^0(A)$ is in $\mathcal{A}_{\mathrm{u}}(j \cdot I)$, and $\alpha_j^0(U_1) R_\chi^*$, $\alpha_j^0(U_2), \dots, \alpha_j^0(U_n)$ is a chain of charge transporters for χ^{-1} along $\tilde{j} \cdot \tilde{I}$, hence $\alpha_{\tilde{j}}(\chi, U^* A) \in \mathcal{F}_{\mathrm{u}}(\tilde{j} \cdot \tilde{I})$. □

The geometrical significance not only of the modular conjugations, but also of the modular operators lifts from the observable algebra to the field algebra. In fact, it is known that these two properties are equivalent on the level of the observable algebra. One direction is implicit in the article [GL95] of Guido and Longo, and equivalence has been shown by Davidson in [Dav95].

LEMMA 4.18. *Let J_0 and Δ_0 be the modular conjugation and the modular operator of the observable algebra $\mathcal{A}(W_1)$ associated to the wedge W_1. The following properties are equivalent.*

i) The vacuum represenation U_0 of $P_+^\uparrow$ extends to a representation of P_+ with $U_0(j) = J_0$, under which the observable algebra is covariant:

$$J_0\, U_0(g)\, J_0 = U_0(jgj) \quad \text{and} \quad \mathrm{Ad} J_0 : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(j \cdot \mathcal{O}) . \tag{4.59}$$

ii) The modular unitaries coincide with the representors of the boosts leaving W_1 invariant:

$$\Delta_0^{it} = U_0(\lambda_1(t)) \;, \text{ where } \lambda_1(t) := \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.60}$$

PROOF. We only show how to translate the claim into Theorem 5 of [Dav95]. For any wedge region W, let $J_0(W)$ and $\Delta_0(W)$ be the modular conjugation and the modular operator of $\mathcal{A}(W)$. i) holds if and only if U_0 is the representation generated by the family of modular conjugations $(J_0(W))_W$. The theorem just mentioned and the last equation in its proof assert that this is equivalent to U_0 being the representation generated by the family of modular operators, i.e. to $\Delta_0(W)^{it} = U_0(\lambda_W(t))$, where $\lambda_W(t)$ is the one parameter group of boosts leaving W invariant. This in turn is equivalent to ii). Note that the implication from ii) to i) is contained in Propositions 2.4, 2.8 and 2.9 of [GL95]. □

PROPOSITION 4.19. *Let the modular operator Δ_0 of the observable algebra $\mathcal{A}(W_1)$ have the geometric significance of equation* (4.60)*. Then the analogous statement holds for the field algebra:*

$$\Delta^{it} = U(\tilde{\lambda}_1(t)) , \tag{4.61}$$

where $\tilde{\lambda}_1(t)$ is the unique lift of the one parameter group $\lambda_1(t)$ to $\tilde{P}_+^\uparrow$.

PROOF. We might have proved the claim directly from Theorem 4.14 in analogy to Theorem 5 of [Dav95]. Instead we proceed partly along the lines of the proof of Lemma 3.4 in [GL95]. With the same arguments as in the proof of Theorem 4.14, one can show that Δ^{it} leaves each charge sector $\mathcal{H}_\chi$ invariant, allowing us to write $\Delta^{it} = \bigoplus_{\chi \in \Gamma} \Delta_\chi^{it}$, and that

$$\mathrm{Ad}\Delta_\chi^{it} \circ \pi_0 \gamma_\chi = \pi_0 \gamma_\chi \circ \alpha^0_{\lambda_1(t)} = \mathrm{Ad} U_\chi(\tilde{\lambda}_1(t)) \circ \pi_0 \gamma_\chi$$

on $\bigcup_{x \in \mathbb{R}^3} \mathcal{A}_{\mathrm{u}}(x + W_1)$. Since γ_χ leaves each $\mathcal{A}_{\mathrm{u}}(x + W_1)$ invariant, this implies that

$$c_\chi(t) := \Delta_\chi^{-it}\, U_\chi(\tilde{\lambda}_1(t)) \tag{4.62}$$

is in $\pi_0(\bigcup_{x \in \mathbb{R}^3} \mathcal{A}_{\mathrm{u}}(x + W_1))'$ and hence a multiple of unity. We determine how this equation transforms under a rotation about π, and first show

$$U_\chi((0,\pi))\, \Delta_\chi^{-it}\, U_\chi((0,\pi))^{-1} = \Delta_\chi^{it} . \tag{4.63}$$

To this end we observe that the twisted Haag duality (4.14) implies that the Tomita operator of $\mathcal{F}(\tilde{W}_1)'$ coincides with that of $Z\, \mathcal{F}((0,\pi) \cdot \tilde{W}_1)\, Z^*$, where $Z = Z(\tilde{W}_1, (0,\pi) \cdot \tilde{W}_1)$, which by covariance is given by $Z\,U((0,\pi))\,S(\tilde{W}_1)\,U_\chi((0,\pi))^*\,Z^*$. On the other hand, the Tomita-Takesaki modular theory asserts that the Tomita operator of $\mathcal{F}(\tilde{W}_1)'$ is given by $J\,\Delta^{-\frac{1}{2}}$. We thus have

$$\begin{aligned} J\,\Delta^{-\frac{1}{2}} &= ZU((0,\pi))J\Delta^{\frac{1}{2}}U_\chi((0,\pi))^{-1}Z^{-1} \\ &= U((0,\pi))JZ^{-2}\Delta^{\frac{1}{2}}U_\chi((0,\pi))^{-1} && \text{since } J\,P_\chi = P_{\chi^{-1}}\,J \\ &= JU((0,-\pi))Z^{-2}\Delta^{\frac{1}{2}}U_\chi((0,\pi))^{-1} && \text{by Theorem 4.14} \\ &= J\,U((0,\pi))\Delta^{\frac{1}{2}}U_\chi((0,\pi))^{-1} && \text{by equation (4.50)}\,. \end{aligned}$$

This implies equation (4.63). Further $U((0,\pi))U(\tilde{\lambda}_1(t))U_\chi((0,\pi))^{-1} = U(\tilde{\lambda}_1(-t))$ due to the group relations, and we conclude that equation (4.62) transforms under a π-rotation as

$$\mathrm{Ad}U_\chi((0,\pi))\,(c_\chi(t)) = \Delta_\chi^{it}\,U_\chi(\tilde{\lambda}_1(-t)) \equiv c_\chi(-t)\,.$$

On the other hand, being a multiple of unity, $c_\chi(t)$ is invariant under $\mathrm{Ad}U_\chi((0,\pi))$, hence

$$c_\chi(t) = c_\chi(-t)\,.$$

But $U(\tilde{\lambda}_1(t))$ leaves $\mathbf{\Omega}$ invariant and $\mathrm{Ad}U(\tilde{\lambda}_1(t))$ leaves $\mathcal{F}(\tilde{W}_1)$ invariant, hence $U(\tilde{\lambda}_1(t))$ commutes with $\Delta^{it'}$ for all t, t'. Then $t \mapsto c_\chi(t)$ is a one parameter group, and the last equation implies $c_\chi(t) \equiv 1$, which is the claim. □

4.3. Scattering States: Covariance and Tomita Operators

Considering only automorphisms of the observable algebra leads to trivial fusion rules (4.64), which in turn imply that the fibres $F^{(n)}$ of the n-particle Hilbert spaces are simply isomorphic to $(\mathbb{C}^N)^{\otimes n}$, where N is the number of single particle representations. Further, the representation ε of the braid group in $F^{(n)}$ is explicitely determined by the statistics phases of the elementary charges. Consequently the representation $\tilde{U}$ of $\tilde{P}_+^\uparrow$ from Chapter 3 on the reference Hilbert space of scattering states is explicitely given in the anyonic context, because it depends only on the single particle representations of $\tilde{P}_+^\uparrow$ and the representation ε of the braid group. The same holds then for the representation U of $\tilde{P}_+^\uparrow$ on the space of scattering states, since it is equivalent to $\tilde{U}$ via the Møller operators. Similarly, the $\mathrm{P_1CT}$ -operator is determined up to the S-matrix. As a consequence, the Tomita operators of the field algebras associated to wedge regions are essentially known explicitely, up to the S-matrix.

4.3.1. Structure and Poincaré Covariance of the Space of Scattering States. The construction of scattering states in Chapter 2 is still valid if one substitutes the field bundle $\Delta(S_0) \times \mathcal{A}_\mathrm{u}$ by the universal field algebra $\mathcal{F}_\mathrm{u}$, the set of pairs $\Delta \times \mathcal{H}_0$ by the anyonic Hilbert space $\mathcal{H} = \bigoplus_\Gamma \mathcal{H}_0$, and the action $(\varrho, B) \cdot (\varrho', \psi)$ by the representation $\pi_\mathrm{u}((\chi, B))\,(\chi', \psi)$. In the definition of $\mathcal{F}_\mathrm{u}$ and $\mathcal{H}$ we specialize to a group Γ of sectors which is generated by single particle representations. In detail, let

$$\Gamma^{(1)} = \{\xi_\alpha\,,\ \alpha = 1\ldots, N\}$$

be the set of all sectors of irreducible massive single particle representations corresponding to localizable automorphisms of $\mathcal{A}_\mathrm{u}$ and satisfying Assumption 2, and let Γ be the Abelian group generated by $\Gamma^{(1)}$. Note that $\Gamma^{(1)}$ is in general not linearly independent, since according to Proposition 1.11 it contains together with each ξ_α its inverse ξ_α^{-1}. We will denote the label of ξ_α^{-1} by $\bar{\alpha}$, i.e. $\bar{\alpha}$ is defined by $\xi_\alpha^{-1} = \xi_{\bar{\alpha}}$. Now we choose a maximal linear independent subset of $\Gamma^{(1)}$ as the set of 'elementary charges' Γ_el, c.f. equation (4.9). $\Gamma^{(1)}$ and Γ correspond to the sets $\Delta^{(1)}$ and $\Delta \subset \Delta(S_0)$, respectively, defined in equation (2.24) and before equation (2.27) in Chapter 2. In particular, the space of intertwiners from $\varrho_{\alpha_1} \cdots \varrho_{\alpha_n}$ to $\sigma \in \Delta$ now corresponds to

$$\mathrm{Int}(\pi_0\gamma_\chi\,|\,\pi_0\gamma_{\chi_{\alpha_1}\cdots\chi_{\alpha_n}}) \equiv \begin{cases} \mathbb{C}\mathbb{1} & \text{if } \chi = \chi_{\alpha_1}\cdots\chi_{\alpha_n}\,, \\ \{0\} & \text{else}\,. \end{cases} \tag{4.64}$$

We will identify $\mathbb{C}\mathbb{1}$ with $\mathbb{C}$. The Hilbert space $\mathcal{H}^{\mathrm{ex}} \subset \mathcal{H}$ of scattering states, defined by equation (2.27), can now be written as follows. Let $\mathcal{H}_{\boldsymbol{\alpha}}^{(n)\,\mathrm{out,in}}$ be the closed linear span of all scattering states of the form

$$\psi_n \times \cdots \times \psi_1\,(\boldsymbol{\alpha}, \xi, \pm), \qquad \text{respectively,}$$

and let for $\boldsymbol{\alpha} \in \{1, \dots, N\}^{\times n}$ the *charge* $\chi(\boldsymbol{\alpha})$ *of* $\boldsymbol{\alpha}$ be defined as

$$\chi(\boldsymbol{\alpha}) := \chi_{\alpha_1} \cdots \chi_{\alpha_n} \ .$$

Then

$$\mathcal{H}^{\mathrm{ex}} = \bigoplus_{\substack{n\geq 0\\ \chi\in\Gamma}} \mathcal{H}_\chi^{(n)\,\mathrm{ex}}\,, \quad \text{where } \mathcal{H}_\chi^{(n)\,\mathrm{ex}} := \bigoplus_{\substack{\boldsymbol{\alpha}\in\{1,\dots,N\}^{\times n}\\ \chi(\boldsymbol{\alpha})=\chi}} \mathcal{H}_{\boldsymbol{\alpha}}^{(n)\,\mathrm{ex}} \quad (\mathrm{ex} = \mathrm{out} \text{ or } \mathrm{in})\ .$$

Let the reference Hilbert space $\tilde{\mathcal{H}}$, the Møller operators $W^\pm$ and the representation $\tilde{U}$ of the Poincaré group be defined as in Chapters 2 and 3, see equations (2.64), (2.65), (3.6) and (3.17), respectively. Due to the trivial fusion rules (4.64), they take the following form in the present context. [8] Recall that $N = |\Gamma^{(1)}|$ is the number of single particle sectors. We denote the canonical basis of $\mathbb{C}^N$ by $\{e_\alpha, \alpha = 1, \dots, n\}$, and the induced canonical basis of $(\mathbb{C}^N)^{\otimes n}$ by $\{e_{\boldsymbol{\alpha}} = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}\}$, with the multiindex $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ running through $\{1, \dots, N\}^{\times n}$. Let ε be the representation of the braid group in $(\mathbb{C}^N)^{\otimes n}$ satisfying for $k = 1, \dots, n$

$$\varepsilon(t_k)\, e_{\boldsymbol{\alpha}} = \varepsilon_0(\xi_{\alpha_{k+1}}, \xi_{\alpha_k})^{-1}\, e_{\boldsymbol{\alpha}\tau_k^{-1}} \qquad \text{and} \tag{4.65}$$

$$\varepsilon(c_k)\, e_{\boldsymbol{\alpha}} = \varepsilon_0(\xi_{\alpha_1} \cdots \xi_{\alpha_{k-1}}\,,\, \xi_{\alpha_k})^{-2}\, e_{\boldsymbol{\alpha}}\ , \tag{4.66}$$

where ε_0 is the symmetric bilinear form on Γ with values in $U(1)$ determined by

$$\varepsilon_0(\chi_{(i)}, \chi_{(j)}) = (\omega_i)^{\delta_{i,j}} \quad \text{for all } \chi_{(i)}, \chi_{(j)} \in \Gamma_{\mathrm{el}}\ .$$

Let further $F_\chi^{(n)}$ be the subspace of $(\mathbb{C}^N)^{\otimes n}$ defined by

$$F_\chi^{(n)} := \bigoplus_{\substack{\boldsymbol{\alpha}\in\{1,\dots,N\}^{\times n}\\ \chi(\boldsymbol{\alpha})=\chi}} \mathbb{C} e_{\boldsymbol{\alpha}} \quad \subset (\mathbb{C}^N)^{\otimes n}\ . \tag{4.67}$$

Then

$$\tilde{\mathcal{H}} = \bigoplus_{\substack{n\geq 0\\ \chi\in\Gamma}} \tilde{\mathcal{H}}_\chi^{(n)}\,, \quad \text{where } \tilde{\mathcal{H}}_\chi^{(n)} := L^2_\varepsilon({}^n\tilde{H}_1, F_\chi^{(n)})\ . \tag{4.68}$$

Note that ε leaves each $F_\chi^{(n)}$ invariant, since the set of multiindices $\boldsymbol{\alpha}$ with fixed charge χ is permutation invariant. Elements of $\tilde{\mathcal{H}}$ will be expanded as $\tilde{\psi} = \sum_{n\geq 0, \boldsymbol{\alpha}\in\{1,\dots,N\}^{\times n}} \psi_{\boldsymbol{\alpha}} \otimes e_{\boldsymbol{\alpha}}$. Further, the representation $\tilde{U}$ of $\tilde{P}_+$ on $\tilde{\mathcal{H}}$ is given by

$$\big(\tilde{U}(x,\tilde{g})\,\tilde{\psi}\big)_{\boldsymbol{\alpha}}(\tilde{\boldsymbol{q}}) = \exp i \sum_{k=1}^n \{m_{\alpha_k} x \cdot q_k + s_{\alpha_k} \Omega(\tilde{g}; m_{\alpha_k} q_k)\}\, \tilde{\psi}_{\boldsymbol{\alpha}}(\tilde{g}^{-1} \cdot \tilde{\boldsymbol{q}}) \quad \text{and} \tag{4.69}$$

$$(\tilde{U}(\tilde{j})\,\tilde{\psi})_{\boldsymbol{\alpha}}(\tilde{\boldsymbol{q}}) = c_{\alpha_1} \cdots c_{\alpha_n}\, \overline{\tilde{\psi}_{\bar{\boldsymbol{\alpha}}}(-\tilde{j} \cdot \tilde{\boldsymbol{q}})}\,, \quad \bar{\boldsymbol{\alpha}} = (\bar{\alpha}_1, \dots, \bar{\alpha}_n)\ . \tag{4.70}$$

Again, c_α are the complex numbers of unit modulus from equation (1.75), satisfying $c_\alpha = c_{\bar{\alpha}}$. Note that $\tilde{U}(\tilde{P}_+^\uparrow)$ leaves each $\tilde{\mathcal{H}}_\chi^{(n)}$ invariant, whereas $\tilde{U}(\tilde{j})$ maps $\tilde{\mathcal{H}}_\chi^{(n)}$ onto $\tilde{\mathcal{H}}_{\chi^{-1}}^{(n)}$. Finally, the Møller operators W^+ and W^- from $\mathcal{H}^{\mathrm{out}}$ and $\mathcal{H}^{\mathrm{in}}$, respectively, onto $\tilde{\mathcal{H}}$ take the following form. Let $\boldsymbol{\psi} = \psi_n \times \cdots \times \psi_1(\boldsymbol{\alpha}, \xi, +/-) \in \mathcal{H}_{\boldsymbol{\alpha}}^{(n)\,\mathrm{out/in}}$, respectively. Then

$$(W^\pm \boldsymbol{\psi})(\tilde{\boldsymbol{q}}) = (\psi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \psi_n)(\boldsymbol{q}) \cdot \varepsilon_0\left(\tilde{\varphi}^\pm_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\right)^{-1} \cdot e_{\boldsymbol{\alpha}}\ ,$$

where $\tilde{\varphi}^\pm_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})$ is the braid (modulo $\ker\varepsilon$) defined in equation (2.55), and π is the associated permutation.

[8] This will be shown in the proof of the next proposition.

PROPOSITION 4.20. *The Møller operators W^+ and W^- are unitary operators from $\mathcal{H}^{\rm out}$ and $\mathcal{H}^{\rm in}$ onto $\tilde{\mathcal{H}}$, respectively. They intertwine the representation $\tilde{U}(\tilde{P}_+^\uparrow)$ on $\tilde{\mathcal{H}}$ with the restriction of the representation U on $\mathcal{H}$ to $\mathcal{H}^{\rm out,in}$, respectively, i.e.*

$$W^+ U(\tilde{g})\,(W^+)^* \;=\; \tilde{U}(\tilde{g}) \;=\; W^- U(\tilde{g})\,(W^-)^* \quad \textit{for all } \tilde{g} \in \tilde{P}_+^\uparrow\,. \tag{4.71}$$

Further, the P_1CT -operator Θ_1 maps $\mathcal{H}^{\rm out}$ onto $\mathcal{H}^{\rm in}$ and vice versa and is related to $\tilde{U}(\tilde{j})$ by

$$W^-\,\Theta_1\,(W^+)^* \;=\; \tilde{U}(\tilde{j}) \;=\; W^+\,\Theta_1\,(W^-)^*\,. \tag{4.72}$$

PROOF. Let $F_\sigma^{(n)}$ be *defined* as the intertwinerspace $F_\sigma^{(n)}$ in equation (2.35), with σ replaced by $\pi_0\gamma_\chi$, and ϱ_{α_k} replaced by $\gamma_{\chi_{\alpha_k}}$. By virtue of the fusion rules (4.64), this space can indeed be identified with (4.67). The representation ε of B_n in $F^{(n)}$ has been defined in equation (2.36):

$$\varepsilon(t_k)\,e_{\boldsymbol{\alpha}} := \pi_0\varepsilon_{\boldsymbol{\alpha}}(t_k,1)^{-1}\cdot e_{\boldsymbol{\alpha}\tau_k^{-1}} = \pi_0\varepsilon(\gamma_{\xi_{\alpha_{k+1}}},\gamma_{\xi_{\alpha_k}})^{-1}\cdot e_{\boldsymbol{\alpha}\tau_k^{-1}} = \varepsilon_0(0;\xi_{\alpha_{k+1}},\xi_{\alpha_k})^{-1}\cdot e_{\boldsymbol{\alpha}\tau_k^{-1}}\,.$$

In the second equation we have used Lemma 1.6 and the last equation is the definition (4.23) of $\varepsilon_0(0)$. But $\varepsilon_0(0)$ is just the bilinear form ε_0 defined above, hence we have shown that the definition of ε by equation (4.65) is correct. Equation (4.66) follows from equation (1.43) of Lemma 1.6 and from equation (4.26). Then the definitions of $\tilde{\mathcal{H}}$ and of the Møller operators given above coincide with the earlier definitions (2.64) and (2.65). Finally, the definitions of $\tilde{U}(x,\tilde{g})$ and $\tilde{U}(\tilde{j})$ also coincide with the earlier definitions (3.6) and (3.17).

Now we have identified the above definitions concerning the reference Hilbert space $\tilde{\mathcal{H}}$ with those of Chapters 2 and 3. On the other hand, we have substituted the field bundle with the universal field algebra, which has all the relevant properties used in the derivation of the results of those chapters. These properties are in particular: 1. The representation $U(\tilde{P}_+^\uparrow)$ implements an action under which the field bundle is covariant, i.e. equations (1.25) and (1.29), which have to be replaced by equations (4.33) and (4.34). 2. $U(\tilde{j})$ implements an automorphism $\alpha_{\tilde{j}}$ under which the field bundle is covariant, i.e. equations (1.[illegible]0) and (1.[illegible]1). This corresponds to Θ_1 implementing an automorphism under which the anyonic field algebra is covariant, i.e. equations (4.57) and (4.5[illegible]). Now we may apply Theorem 2.9 of Chapter 2 and Theorems 3.4 and 3.7 of Chapter 3 to prove the claims. ☐

4.3.2. Tomita Operators for Wedge Regions. Recall that, under the assumption of modular covariance, by Theorem 4.14 and Proposition 4.19 the Tomita-operator $S(\tilde{W}_1)$ of $\mathcal{F}(\tilde{W}_1)$ and $\mathbf{\Omega}$ has the polar decomposition

$$S(\tilde{W}_1) = J\Delta^{\frac{1}{2}}\,, \tag{4.73}$$

where the antiunitary part J and the positive part $\Delta^{\frac{1}{2}}$ are determined by

$$\begin{aligned} J &= Z(\tilde{W}_1,\tilde{j}\tilde{W}_1)\,\Theta_1 && \text{and} \\ \Delta^{it} &= U(\tilde{\lambda}_1(t)) && \text{for all } t\in\mathbb{R}\,. \end{aligned}$$

Θ_1 maps $\mathcal{H}^{\rm out}$ onto $\mathcal{H}^{\rm in}$ and vice versa according to Proposition 4.20, while $\Delta^{\frac{1}{2}}$ leaves $\mathcal{H}^{\rm out}$ and $\mathcal{H}^{\rm in}$ invariant since $U(\tilde{P}_+^\uparrow)$ does so (see equation (3.3) in Lemma 3.2). Consequently, the Tomita-operator maps $\mathcal{H}^{\rm out}$ into $\mathcal{H}^{\rm in}$ and vice versa. The same holds for the Tomita operator $S(\tilde{W})$ for any other wedge region $\tilde{W}$, since by covariance of the field algebra

$$S(\tilde{W}) = U(\tilde{g})S(\tilde{W}_1)U(\tilde{g})^{-1}\,, \tag{4.74}$$

where $\tilde{g}$ is any element of $\tilde{P}_+$ such that $\tilde{W} = \tilde{g}\cdot\tilde{W}_1$. Recall further that the S-matrix $S := (W^+)^*\,W^-$ maps $\mathcal{H}^{\rm in}$ isometrically onto $\mathcal{H}^{\rm out}$. Hence the operator

$$S_{\rm in}(\tilde{W}) := S(\tilde{W})\,S$$

leaves $\mathcal{H}^{\rm in}$ invariant. It will be called the *incoming Tomita-operator* for $\mathcal{F}(\tilde{W})$ and $\mathbf{\Omega}$. It is unitarily equivalent to an operator $\tilde{S}(\tilde{W})$ in $\tilde{\mathcal{H}}$ which can be explicitely calculated, and which is defined as follows. Let $\tilde{\Delta}$ be the pendant of Δ in $\tilde{\mathcal{H}}$:

$$\tilde{\Delta}^{it} := \tilde{U}(\tilde{\lambda}_1(t)) \quad \text{for all } t\in\mathbb{R}\,.$$

LEMMA AND DEFINITION 4.21. *Let $\tilde{W}$ be a path in $\mathcal{K}$ ending at a wedge region, and let $\tilde{g}$ be any element of $\tilde{P}_+^\uparrow$ such that $\tilde{W} = \tilde{g}\cdot\tilde{W}_1$. Then*

$$\tilde{S}(\tilde{W}) := \tilde{U}(\tilde{g})\; Z(\tilde{W}_1, \tilde{j}\tilde{W}_1)\; \tilde{U}(\tilde{j})\; \tilde{\Delta}^{\frac{1}{2}}\; \tilde{U}(\tilde{g})^{-1}$$

is independent of $\tilde{g}$ and satisfies $\tilde{S}(\tilde{W})^2 = \mathbf{1}$. Further, this operator is unitarily equivalent to $S_{\rm in}(\tilde{W})$:

$$W^-\, S_{\rm in}(\tilde{W})\, (W^-)^* \;=\; \tilde{S}(\tilde{W})\,. \tag{4.75}$$

PROOF. For short, we denote $Z(\tilde{W}_1, \tilde{j}\tilde{W}_1)$ by Z. We first show equation (4.75) for $\tilde{W} = \tilde{W}_1$. As we have seen in equation (4.71), the S-matrix commutes with the representation U of $\tilde{P}_+^\uparrow$. Hence it also commutes with the modular operator $\Delta^{\frac{1}{2}}$, and consequently $S_{\rm in}(\tilde{W}_1)$ may be written as

$$S_{\rm in}(\tilde{W}_1) = Z\, \Theta_1\, S\, \Delta^{\frac{1}{2}}\,.$$

Now equation (4.75) follows for the case $\tilde{W} = \tilde{W}_1$ with equations (4.71) and (4.72) of Proposition 4.20. Since the S-matrix commutes with $U(\tilde{P}_+^\uparrow)$, the covariance property (4.74) also holds for $S_{\rm in}(\tilde{W})$. Hence by the intertwiner relation (4.71) of W^-,

$$W^-\, S_{\rm in}(\tilde{W})\, (W^-)^* \;=\; \tilde{U}(\tilde{g})\, \tilde{S}(\tilde{W}_1)\, \tilde{U}(\tilde{g})^{-1}$$

for any $\tilde{g}$ with $\tilde{g}\tilde{W}_1 = \tilde{W}$. This shows equation (4.75) and independence of $\tilde{g}$. It is noteworthy that that the latter property can also be shown directly by the group relations. It remains to show $\tilde{S}(\tilde{W}_1)^2 = \mathbf{1}$. From the group relation $\tilde{j}\,\tilde{\lambda}_1(t) = \tilde{\lambda}_1(t)\,\tilde{j}$ for all $t \in \mathbb{R}$ and the fact that $\tilde{U}(\tilde{j})$ is antilinear follows $\tilde{U}(\tilde{j})\,\tilde{\Delta}^{\frac{1}{2}} = \tilde{\Delta}^{-\frac{1}{2}}\,\tilde{U}(\tilde{j})$, which implies the desired relation. □

$\tilde{S}(\tilde{W})$ can in principle be calulated since all involved operators $Z, \tilde{U}(\tilde{P}_+^\uparrow)$ and $\tilde{U}(\tilde{j})$ are explicitelty given. As we have noted after equation (4.70), $\tilde{S}(\tilde{W})$ leaves the subspaces $\tilde{\mathcal{H}}_\chi^{(n)} \oplus \tilde{\mathcal{H}}_{\chi^{-1}}^{(n)}$ (respectively $\tilde{\mathcal{H}}_\chi^{(n)}$ if $\chi^2 = 1$) of $\tilde{\mathcal{H}}$ invariant, hence it is sufficient to consider the restriction of $\tilde{S}(\tilde{W})$ to these subspaces. We have found an explicit formula for $\tilde{W} = \tilde{W}_1$ using a global trivialisation which renders $L^2_\varepsilon({}^n\tilde{H}_1; F)$ isomorphic to $P_+\, L^2(H_1^{\times n}; F)$ and which exhibits the restriction of $\tilde{S}(\tilde{W}_1)$ to the n particle space as the second quantization of its restriction to the single particle space. But the trivialization is adapted to the wedge $\tilde{W}_1$ and it can hardly be controlled how it behaves under Poincaré transformations. There has been more progress on the single particle level, to which we will restrict in the sequel. Note that the physical Tomita operators $S(\tilde{W})$ leave the single particle space $\mathcal{H}^{(1)}$ invariant, which can be seen from the polar decomposition (4.73). Further, on the single particle space the outgoing Møller operator coincides with the incoming one, namely they are the direct sum $W^{(1)}$ of the unitaries W_α which identify the single particle spaces $\mathcal{H}_\alpha^{(1)}$ with $L^2(H_1, \mathrm{d}\mu)$, see equation (2.30). We conclude that the restriction of the physical Tomita operators $S(\tilde{W})$ to $\mathcal{H}^{(1)}$ are unitarily equivalent to the restriction of $\tilde{S}(\tilde{W})$ to $\mathcal{H}^{(1)}$:

$$W^{(1)}\; S(\tilde{W})^{(1)}\; \big(W^{(1)}\big)^* = \tilde{S}(\tilde{W})^{(1)} \quad \text{for all } \tilde{W}\,, \tag{4.76}$$

where we have written $S^{(1)} := S|_{\mathcal{H}^{(1)}\cap \mathrm{dom} S}$. We pick one single particle sector $\xi \in \Gamma^{(1)}$ with mass m and spin s. To be specific, we assume $\xi^2 \neq 1$, and consider the restriction of $\tilde{S}(\tilde{W})$ to $\mathcal{H}_\xi^{(1)} \oplus \mathcal{H}_{\xi^{-1}}^{(1)}$. By $W^{(1)} = W_\xi \oplus W_{\xi^{-1}}$ this Hilbert space is identified with

$$\mathcal{H}_\xi^{(1)} \oplus \mathcal{H}_{\xi^{-1}}^{(1)} \cong L^2(H_m, \mathrm{d}\mu; \mathbb{C}^2)\,.$$

We have replaced the unit mass shell again with H_m. The representation of $\tilde{P}_+$ on this space will just be denoted by $\tilde{U}$ and is, by equations (4.69) and (4.70), given as

$$\big(\tilde{U}(x,\tilde{g})\,\psi\big)_\pm(p) = e^{ix\cdot p}\, e^{is\,\Omega(\tilde{g};p)}\; \psi_\pm(g^{-1}\cdot p) \quad \text{and} \tag{4.77}$$

$$(\tilde{U}(\tilde{j})\,\psi)_\pm(p) = c\,\overline{\psi_\mp(-j\cdot p)}\;,\; |c|^2 = 1\,. \tag{4.78}$$

Note that for different spins s all representations $\tilde{U}$ act in one and the same Hilbert space $L^2(H_m; \mathbb{C}^2)$, and further the representative of $\tilde{j}$ does not depend on s. The spin only enters in the phase factor which represents the Wigner rotation in equation (4.77). According to Definition 4.21 we have

$$\tilde{S}(\tilde{W}_1) = e^{-i\pi s}\,\tilde{U}(\tilde{\jmath})\,\tilde{\Delta}^{\frac{1}{2}} \quad \text{with} \quad \tilde{\Delta} := \exp\Big(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\,\tilde{U}(\tilde{\lambda}_1(t))\Big)\,. \tag{4.79}$$

Here we have taken account of the fact that the restriction of $Z(\tilde{W}_1, \tilde{\jmath}\tilde{W}_1)$ to $\mathcal{H}_\xi \oplus \mathcal{H}_{\xi^{-1}}$ is given by $\exp(-i\pi s)\,\mathbf{1}$, and we have set the complex number c in equation (4.78) to 1. We turn to the calculation of $\tilde{S}(\tilde{W})$. It should be noted, that the interesting point is *not* the calculation of $\tilde{S}(\tilde{W})$ for any *fixed* $\tilde{W}$, but rather how the family of Tomita operators behaves with varying $\tilde{W}$. [9] Ultimately, one would like to know the intersection of the domains of all $\tilde{S}(\tilde{W})$ with $\tilde{W}$ containing a given spacelike cone $\tilde{S}$. We have found a solution only for the family $\tilde{W} = (0,\omega){\cdot}\tilde{W}_1$, with ω varying in $(-\pi, 0\,]$.

Let u be the following function on H_m :

$$u(p) := \Big(\frac{p_0 - p_1}{m}\cdot\frac{p_0 - p_1 + m + ip_2}{p_0 - p_1 + m - ip_2}\Big)^s \quad , \; p_0 := \sqrt{p_1^2 + p_2^2 + m^2}\,. \tag{4.80}$$

Note that for all $p \in H_m$, the real part of $p_0 - p_1 + m \pm ip_2$ is strictly positve, hence the complex number in brackets lies in the cut complex plane $\mathbb{C} \setminus \mathbb{R}_0^-$. Thus the power to $s \in \mathbb{R}$ can be (and will be) defined via the branch of the logarithm with $\ln 1 = 0$. We introduce the following notation. Given a continuous function v on the mass shell and an L^2-function $\phi \in L^2(H_m; \mathbb{C}^2)$, we denote by $v \cdot \phi$ the function $p \mapsto v(p)\cdot\phi(p)$. Further, if f is a Schwartz function on $\mathbb{R}^3$, we denote by $E_m f$ the restriction of its Fourier transform to the mass shell H_m, and for $\vec{f} \in \mathcal{S}(\mathbb{R}^3)\otimes\mathbb{C}^2$, $E_m\vec{f}$ is understood componentwise.

PROPOSITION 4.22. *Let* $\omega \in [0, \pi\,)$. *Then* $\tilde{S}\big((0,-\omega){\cdot}\tilde{W}_1\big)$ *is well defined on the set*

$$\{\, u\cdot E_m\vec{f} \;\;|\, \vec{f} \in \mathcal{S}(\mathbb{R}^3)\otimes\mathbb{C}^2 \;\; \textit{with}\ \mathrm{supp}\vec{f} \subset R(-\omega){\cdot}W_1 \,\}\,,$$

and for $f \in \mathcal{S}\big(R(-\omega){\cdot}\tilde{W}_1\big)$, *the Tomita operator acts as*

$$\tilde{S}\big((0,-\omega){\cdot}\tilde{W}_1\big)\; u\cdot E_m f\otimes e_\pm = e^{2\pi i s}\; u\cdot E_m\bar{f}\otimes e_\mp\,, \qquad \textit{respectively}\,. \tag{4.81}$$

Here, $R(\omega)$ denotes the rotation corresponding to $(0,\omega) \in \tilde{L}_+^\uparrow$.

PROOF. The proof is given on page 83 in Appendix A.3. Here it may be remarked that the action of the modular operator $\tilde{\Delta}^{\frac{1}{2}}\,\phi$ can be calculated by analytic continuation of $\tilde{\Delta}^{it}\,\phi$ in t into the region $\mathbb{R} - i\,[0,\frac{1}{2}]$, but the factor $\exp is\Omega(\tilde{\lambda}_1(t), p)$ has branch points in that region, which can be seen from equation (A.33) in the Appendix. The multiplication operator u is designed such that it 'intertwines' the Tomita-operator $\tilde{S}\big((0,\omega){\cdot}\tilde{W}_1\big)$ for spin s with that for spin 0 simultaneously for all ω in $(-\pi, 0\,]$, thus reducing the problem to the well known case $s = 0$. □

It is noteworthy that $u \cdot E_m f$ is in general *not* in the domain of $\tilde{S}(\tilde{W})$ if $\tilde{W}$ is a generic wedge containing $\tilde{W}_1 \cap (0,\omega){\cdot}\tilde{W}_1$, unless $s = 0$, see Proposition 5.7.

[9] In fact, we have found for fixed $\tilde{W}$ the diagonalization $V_{\tilde{W}} : L^2(H_m, \mathrm{d}\mu; \mathbb{C}^2) \to L^2(\mathbb{R}^2, \mathrm{d}^2\boldsymbol{y}; \mathbb{C}^2)$ such that

$$V_{\tilde{W}}\,\tilde{S}(\tilde{W})\,V_{\tilde{W}}^{-1}\,\psi(y_1, y_2) = e^{-y_1/2}\,\overline{\psi(-y_1, y_2)}\,.$$

But the dependence on $\tilde{W}$ is hardly tractable.