CHAPTER 3

Poincaré Covariance of the Scattering States

In this chapter we show that the representation U of $\tilde{P}_+^\uparrow$ is fixed on the space of scattering states $\mathcal{H}^{\mathrm{ex}}$ exactly to the degree the intertwiner spaces $F_\sigma^{(n)}$ are known (which in turn are fixed by the statistics operators and the fusion rules) and that the P_1CT operator $U(\tilde{\jmath})$ is fixed up to the S-matrix, and we display the explicit representation of $\tilde{P}_+$ in $\tilde{\mathcal{H}}$ into which they are translated by the Møller operators. Again we will discuss the outgoing and incoming cases simultaneously, notationally distinguished by the signs + and −. Thus, all equations of the form $x(\pm) = y(\pm)$, or $x(\pm) = y(\mp)$, are meant as two equations, condensed into one. The symbol $\mathbf{V}$ will again be used to denote a Cartesian product of mutually disjoint compact subsets V_k of the unit mass shell H_1

3.1. Ray Representation of the Poincaré Group

LEMMA 3.1. *Let ϱ and σ be localized covariant endomorphisms of $\mathcal{A}_{\mathrm{u}}$ with finite statistics. Then every intertwiner $T \in \mathrm{Int}(\sigma|\varrho)$ from ϱ to σ also intertwines the respective representations of the universal covering of the Poincaré group in the sense that*

$$U_\sigma(\tilde{g})\,\pi_0(T) = \pi_0(T)\,U_\varrho(\tilde{g}) \quad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow\,. \tag{3.1}$$

PROOF. We recall the proof from [DHR74, Lemma2.2]. In a first step, let $\varrho = \mathrm{id}$. The linear space $\mathrm{Int}(\pi_0 \circ \sigma|\pi_0)$ of intertwiners from π_0 to $\pi_0 \circ \sigma$ carries the by now familiar scalar product $\langle S, T\rangle\,\mathbf{1} := S^*T \in (\pi_0|\pi_0) = \mathbb{C}\mathbb{1}$. Since σ is assumed to have finite statistics, it contains only finitely many irreducible subrepresentations [DHR71] and hence the above Hilbert space is finite dimensional. Now consider for $\tilde{g} \in \tilde{P}_+^\uparrow$ the operator

$$U_\sigma(\tilde{g})\pi_0(T)U_0(g)^{-1}\,. \tag{3.2}$$

It is easily seen to be also an intertwiner from π_0 to $\pi_0\circ\sigma$, which in addition has the same norm as $\pi_0(T)$. Hence equation (3.2) defines a unitary representation of $\tilde{P}_+^\uparrow$ in $\mathrm{Int}(\pi_0\circ\sigma|\pi_0)$. Since this is a finite dimensional Hilbert space, the representation must be trivial. This implies equation (3.1) for the case $\varrho = \mathrm{id}$. For the general case, we use the above result and the implementation properties (1.3) of U_ϱ and U_σ, to see that $\pi_0(T)\,U_\varrho(\tilde{g})$ coincides with $U_\sigma(\tilde{g})\,\pi_0(T)$ on vectors of the form $(\pi_0\varrho)(A)V\Omega$, where $V \in \mathrm{Int}(\pi_0 \circ \varrho|\pi_0)$ and $A \in \mathcal{A}_{\mathrm{u}}$. Since $\mathcal{H}_0$ is spanned by vectors of this form, this proves equation (3.1). □

The following Lemma has been stated as equation (3.7) in [FGR96].

LEMMA 3.2. *Let $\xi \in X_{\mathbf{V}}^\pm$, and let $\psi_k \in \mathcal{H}^{(1)}_{\alpha_k}(V_k)$. Then for all $\tilde{g} \in \tilde{P}_+^\uparrow$*

$$U(\tilde{g})\,T\psi_n \times \cdots \times \psi_1(\sigma, \boldsymbol{\alpha}, \xi, \pm) = T\,(U_{\alpha_n}(\tilde{g})\psi_n) \times \cdots \times (U_{\alpha_1}(\tilde{g})\psi_1)\,(\sigma, \boldsymbol{\alpha}, \tilde{g}\cdot\xi, \pm)\,. \tag{3.3}$$

PROOF. It suffices to consider (α_k, ψ_k) of the special form $\mathbf{B}_k(f_k, t)\,\boldsymbol{\Omega}$, with $\mathbf{B}_k \in \mathcal{F}(\tilde{I}_k)$ and f_k as in equation (2.7), and with $[\tilde{\mathbf{I}}] = \xi$. Then the left hand side of equation (3.3) reads, due to the above Lemma and the remark after equation (1.23),

$$\mathbf{T} \lim_{t\to\pm\infty} \alpha(\tilde{g})\Big(\mathbf{B}_n(f_n, t)\Big) \cdots \alpha(\tilde{g})\Big(\mathbf{B}_1(f_1, t)\Big)\,\boldsymbol{\Omega}\,. \tag{3.4}$$

Now a short calculation shows that

$$\alpha(\tilde{g})(\mathbf{B}(f, t)) = (\alpha(\tilde{g})\mathbf{B})(g_* f, t) \;\; \text{where } (g_* f)(x) := f(g^{-1}\cdot x)\,. \tag{3.5}$$

Since $\alpha(\tilde{g})\mathbf{B}_k \in \mathcal{F}(\tilde{g}\cdot\tilde{I}_k)$ and further $\alpha(\tilde{g})(\mathbf{B}_k(f_k, t))\boldsymbol{\Omega} = (\alpha_k, U_{\alpha_k}(\tilde{g})\psi_k) \in \mathcal{H}^{(1)}_{\alpha_k}(g \cdot V_k)$, equation (2.8) asserts that the expression (3.4) coincides with the right hand side of equation (3.3). □

There is a canonical representation of $\tilde{P}_+^\uparrow$ in $\tilde{\mathcal{H}}$:

DEFINITION 3.3. Let $\tilde{U}$ be the representation of $\tilde{P}_+^\uparrow$ on $\tilde{\mathcal{H}}$ given by

$$\left(\tilde{U}(x,\tilde{g})\,\psi\right)_{\sigma,\boldsymbol{\alpha}}(\tilde{\boldsymbol{q}}) = \exp i \sum_{k=1}^{n} \{m_{\alpha_k} x \cdot q_k + s_{\alpha_k} \Omega(\tilde{g}; m_{\alpha_k} q_k)\}\, \psi_{\sigma,\boldsymbol{\alpha}}(\tilde{g}^{-1}\cdot\tilde{\boldsymbol{q}}) \tag{3.6}$$

for all $\psi \in L^2_\varepsilon({}^n\widetilde{H}_1; F_\sigma^{(n)})$.

$\tilde{g}\cdot\tilde{\boldsymbol{q}}$ denotes the canonical action of $\tilde{P}_+^\uparrow$ in ${}^n\widetilde{H}_1$, as explained in the Appendix. Note that the above definition coincides for $n=1$ with the earlier definition (2.31).

PROPOSITION 3.4. *Both Møller operators W^+ and W^- intertwine the above representation $\tilde{U}(\tilde{P}_+^\uparrow)$ in $\tilde{\mathcal{H}}$ with the representation $U(\tilde{P}_+^\uparrow)$ of equation (2.28) in $\mathcal{H}^{\mathrm{out,in}}$. More precisely, for all $\tilde{g} \in \tilde{P}_+^\uparrow$:*

$$\begin{aligned} W^+ U(\tilde{g})\,(W^+)^* &= \tilde{U}(\tilde{g}) \quad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow \quad \text{and} \\ W^- U(\tilde{g})\,(W^-)^* &= \tilde{U}(\tilde{g}) \quad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow\,. \end{aligned} \tag{3.7}$$

Consequently, the S-matrix $S := (W^+)^* W^-$ commutes with $U(\tilde{P}_+^\uparrow)$:

$$U(\tilde{g})\,S = S\,U(\tilde{g}) \quad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow\,. \tag{3.8}$$

PROOF. We prove the equation simultaneously for W^+ and W^- and suppress the respective superscripts + and − again. Let $\boldsymbol{\psi} = T\psi_n \times \cdots \times \psi_1(\sigma, \boldsymbol{\alpha}, \xi, \pm)$, with $(\alpha_k, \psi_k) \in \mathcal{H}^{(1)}_{\alpha_k}(V_k)$. Then

$$\begin{aligned} &\big(WU(x,\tilde{g})\boldsymbol{\psi}\big)(\tilde{g}\cdot\tilde{\boldsymbol{q}}) = \big(W\,T\,(U_{\alpha_n}(x,\tilde{g})\psi_n) \times \cdots \times (U_{\alpha_1}(x,\tilde{g})\psi_1)\,(\sigma,\boldsymbol{\alpha},\tilde{g}\cdot\xi,\pm)\big)(\tilde{g}\cdot\tilde{\boldsymbol{q}}) \\ &\quad= \big(\tilde{U}_{\alpha_1}(x,\tilde{g})\tilde{\psi}_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \tilde{U}_{\alpha_n}(x,\tilde{g})\tilde{\psi}_n\big)(g\cdot\boldsymbol{q})\,\big(\sigma\,|\,T\,\varepsilon_{\boldsymbol{\alpha}}\,(\tilde{\varphi}_{g\cdot\mathbf{V},\tilde{g}\cdot\xi}(\tilde{g}\cdot\tilde{\boldsymbol{q}}),\pi^{-1})\,\,|\boldsymbol{\alpha}\pi\,\big) \end{aligned}$$

due to equation (3.3) and the definition (2.66) of the Møller operators. Here we have written $W_{\alpha_k}\psi_k =: \tilde{\psi}_k$ and used $W_\alpha U_\alpha(\tilde{g}) \equiv \tilde{U}_\alpha(\tilde{g})W_\alpha$, see equation (2.31). On the other hand,

$$\begin{aligned} &\big(\tilde{U}(x,\tilde{g})W\boldsymbol{\psi}\big)(\tilde{g}\cdot\tilde{\boldsymbol{q}}) = \exp i \sum_{k=1}^{n} \{m_{\alpha_k} x \cdot q_k + s_{\alpha_k}\Omega(\tilde{g}; m_{\alpha_k} q_k)\}\,(W\boldsymbol{\psi})(\tilde{\boldsymbol{q}}) \\ &= \exp i \sum_{k=1}^{n} \{m_{\alpha_k} x \cdot q_k + s_{\alpha_k}\Omega(\tilde{g}; m_{\alpha_k} q_k)\}\,\big(\tilde{\psi}_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \tilde{\psi}_n\big)(\boldsymbol{q})\,\big(\sigma|\,T\,\varepsilon_{\boldsymbol{\alpha}}\,(\tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}),\pi^{-1})\,\,|\boldsymbol{\alpha}\pi\big) \\ &= \big(\tilde{U}_{\alpha_1}(x,\tilde{g})\tilde{\psi}_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \tilde{U}_{\alpha_n}(x,\tilde{g})\tilde{\psi}_n\big)(\tilde{g}\cdot\boldsymbol{q})\,\big(\sigma|\,T\,\varepsilon_{\boldsymbol{\alpha}}\,(\tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}),\pi^{-1})\,\,|\boldsymbol{\alpha}\pi\,\big)\,. \end{aligned}$$

Here we have used again (2.31). It remains to show

$$\tilde{\varphi}_{g\cdot\mathbf{V},\tilde{g}\cdot\xi}(\tilde{g}\cdot\tilde{\boldsymbol{q}}) = \tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\,. \tag{3.9}$$

We choose a decomposition $\tilde{g} = \tilde{g}_m \cdots \tilde{g}_1$ with paths α_k in $L_+^\uparrow$ representing $\tilde{g}_k$ such that for $k = 1, \ldots, m$

$$\alpha_k(t)\cdot g_{k-1}\cdots g_1\cdot\mathbf{V}\,\cap\, g_{k-1}\cdots g_1\cdot\mathbf{V} \neq \emptyset \;\text{ for all } t \in [0,1]\,.$$

Then by equation (2.55),

$$\tilde{\varphi}_{g\cdot\mathbf{V},\tilde{g}\cdot\xi}(\tilde{g}\cdot\tilde{\boldsymbol{q}}) = b_{g\mathbf{V}\cap g_{m-1}\cdots g_1\mathbf{V}}(\tilde{g}\cdot\xi,\tilde{g}_{m-1}\cdots\tilde{g}_1\cdot\xi)\cdots b_{g_1\mathbf{V}\cap\mathbf{V}}(\tilde{g}_1\cdot\xi,\xi)\,\tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\,. \tag{3.10}$$

All of the b's on the right hand side are of the form $b_{g\mathbf{V}\cap\mathbf{V}}(\tilde{g}\cdot\xi,\xi)$, where in the homotopy class of $\tilde{g}$ there is a path α satisfying $\alpha([0,1])\cdot\mathbf{V}\cap\mathbf{V} \neq \emptyset$. By continuity, they must be trivial. Hence equation (3.9) holds and the proof is complete. □

3.2. Representation of the P$_1$CT-Transformation

From now on we assume our collection Δ of pairwise inequivalent representations[1] chosen such that for each $\sigma \in \Delta$, the modular conjugate representation $\bar{\sigma}$ is also contained in Δ. If in particular σ is self-conjugate, $\sigma \cong \bar{\sigma}$, we thus demand that it in fact coincides with its modular conjugate. As we know from Proposition 1.11, the conjugate of each of our single particle representations $\varrho_\alpha \in \Delta^{(1)}$ [2] is also a single particle representation . Hence our assumption implies that $\Delta^{(1)}$ is stable under modular conjugation: It contains together with each ϱ_α also the modular conjugate $\overline{\varrho_\alpha}$, which we denote by $\varrho_{\bar{\alpha}}$, thus defining a map $\alpha \mapsto \bar{\alpha}$ of the labels satisfying $\bar{\bar{\alpha}} = \alpha$.

If T is an Intertwiner from ϱ to σ, then $\alpha_j^0(T)$ is an intertwiner from $\bar{\varrho}$ to $\bar{\sigma}$ which we will denote by

$$\alpha_{\tilde{j}}\,(\sigma|T|\varrho) := (\bar{\sigma}|\alpha_j^0(T)|\bar{\varrho})\,. \tag{3.11}$$

As a map on $\mathcal{H} = \Delta(S_0) \times \mathcal{H}_0$ this intertwiner coincides with $U(\tilde{j}) \circ \mathbf{T} \circ U(\tilde{j})$, where $U(\tilde{j})$ has been defined in equation (1.76). Now let $\varrho \in \Delta(S_0)$ be irreducible and let V_ϱ be its global self-intertwiner[3]. Then the global self-intertwiner of $\bar{\varrho}$ is given by

$$V_{\bar{\varrho}} = \alpha_j^0(V_\varrho^*)\,. \tag{3.12}$$

To see this, let $(\bar{\varrho}, B) \in \mathcal{F}(\tilde{I})$. Then $(\varrho, \alpha_j^0(B)) \equiv \alpha_{\tilde{j}}\,(\bar{\varrho}, B)$ is in $\mathcal{F}(\tilde{j}\cdot\tilde{I})$ by equation (1.81), and equation (1.36) implies that $(\varrho, V_\varrho^*\,\alpha_j^0(B))$ is in $\mathcal{F}((0,-2\pi)\cdot\tilde{j}\cdot\tilde{I})$. Hence, again by equation (1.81), $(\bar{\varrho}, \alpha_j^0(V_\varrho^*)\,B) \equiv \alpha_{\tilde{j}}\,(\varrho, V_\varrho^*\,\alpha_j^0(B))$ is in $\mathcal{F}(\tilde{j}\cdot(0,-2\pi)\cdot\tilde{j}\cdot\tilde{I})$, which in view of the group relation (1.66) coincides with $\mathcal{F}((0,2\pi)\cdot\tilde{I})$. This proves the claim.

REMARK. The action of $\tilde{j}$ on $\tilde{\mathcal{K}}$, defined after equation (1.81), carries over to equivalence classes of tupels from Definition 2.1, and the resulting action maps $X_{\mathbf{V}}^+$ onto $X_{-j\cdot\mathbf{V}}^-$ and vice versa. This can be seen by a discussion similar to that around equation (2.18): Let ξ be the equivalence class of $(\tilde{I}_1,\dots,\tilde{I}_n)$, and $\mathbf{V} = V_1\times\cdots\times V_n$. $I_k + tV_k$ are mutually spacelike if and only if $j\cdot I_k - t(-j\cdot V_k)$ are. Hence $\xi \in X_{\mathbf{V}}^+$ if and only if $\tilde{j}\cdot\xi \in X_{-j\cdot\mathbf{V}}^-$.

LEMMA 3.5. *$U(\tilde{j})$ maps $\tilde{\mathcal{H}}^{\mathrm{out}}$ onto $\tilde{\mathcal{H}}^{\mathrm{in}}$ and vice versa. Explicitely, let $\xi \in X_{\mathbf{V}}^+$ and $\psi_k \in \mathcal{H}_{\alpha_k}^{(1)}(V_k)$. Then $\tilde{j}\cdot\xi$ is in $X_{-j\cdot\mathbf{V}}^-$ and $U(\tilde{j})\,\psi_k$ is in $\mathcal{H}_{\bar{\alpha}_k}^{(1)}(-j\cdot V_k)$, and*

$$U(\tilde{j})\,T\,\psi_n\times\cdots\times\psi_1(\sigma,\boldsymbol{\alpha},\xi,\pm) = \alpha_j^0(T)\,(U_0(j)\psi_n)\times\cdots\times(U_0(j)\psi_1)\,(\bar{\sigma},\bar{\boldsymbol{\alpha}},\tilde{j}\cdot\xi,\mp)\,. \tag{3.13}$$

PROOF. We proceed in analogy with the proof of Lemma 3.2: Let $(\alpha_k,\psi_k) = \mathbf{B}_k(f_k,t)\,\mathbf{\Omega}$, with $\mathbf{B}_k \in \mathcal{F}(\tilde{I}_k)$ and f_k as in equation (2.7), and with $[\tilde{\mathbf{I}}] = \xi$. Then the left hand side of equation (3.13) reads, exploiting $U(\tilde{j})\,\mathbf{T}\,\mathbf{B}\,\psi = \alpha_{\tilde{j}}(\mathbf{T})\,\alpha_{\tilde{j}}(\mathbf{B})\,U(\tilde{j})\psi$, see equation (1.80),

$$\alpha_{\tilde{j}}(\mathbf{T})\lim_{t\to\pm\infty}\alpha_{\tilde{j}}\Big(\mathbf{B}_n(f_n,t)\Big)\cdots\alpha_{\tilde{j}}\Big(\mathbf{B}_1(f_1,t)\Big)\,\mathbf{\Omega}\,. \tag{3.14}$$

A short calculation shows that

$$\alpha_{\tilde{j}}(\mathbf{B}(f,t)) = (\alpha_{\tilde{j}}\mathbf{B})(f^j,-t) \quad \text{where} \quad \widetilde{(f^j)}(p) := \overline{\tilde{f}(-j\cdot p)}\,. \tag{3.15}$$

Now $\alpha_{\tilde{j}}\mathbf{B}_k$ is in $\mathcal{F}(\tilde{j}\cdot\tilde{I}_k)$ by equation (1.81) and $\alpha_{\tilde{j}}(\mathbf{B}_k(f_k,t))\mathbf{\Omega}$ coincides with $(\bar{\alpha}_k, U_0(j)\psi_k)$, which is contained in $\mathcal{H}_{\bar{\alpha}_k}^{(1)}(-j\cdot V_k)$ due to equation (1.70). Hence, according to the discussion around equation (2.8), the expression (3.14) coincides with the right hand side of equation (3.13). □

Now we define an implementation $\tilde{U}(\tilde{j})$ of $\tilde{j}$ on the reference Hilbert space $\tilde{\mathcal{H}}$, and show in Propostion 3.7 how the Møller operators relate $U(\tilde{j})$ and $\tilde{U}(\tilde{j})$. First we define a conjugation C on the Hilbert space of intertwiners $F = \bigoplus_{\sigma,n} F_\sigma^{(n)}$ by additive extension of

$$\begin{aligned} C\ :\quad & F_{\sigma,\boldsymbol{\alpha}}^{(n)} \to F_{\bar{\sigma},\bar{\boldsymbol{\alpha}}}^{(n)} \\ & (\sigma|T|\boldsymbol{\alpha}) \mapsto c_{\alpha_1}\cdots c_{\alpha_n}\,(\bar{\sigma}|\alpha_j^0(T)|\bar{\boldsymbol{\alpha}})\,. \end{aligned} \tag{3.16}$$

[1] for the definition of the set Δ, see page 33

[2] for the definition of the set $\Delta^{(1)}$, see equation (2.24)

[3] see equation (1.36).

Here c_{α_k} are the complex numbers of unit modulus from equation (1.75), and $\bar{\boldsymbol{\alpha}}$ denotes $(\bar{\alpha}_1, \dots, \bar{\alpha}_n)$, if $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Note that C is indeed antilinear, isometric and satisfies $C^2 = \mathbf{1}$. Then we define a conjugation operator on the reference Hilbert space $\tilde{\mathcal{H}} = \bigoplus_{\sigma,n} L^2_\varepsilon({}^n\widetilde{H}_1, F^{(n)}_\sigma)$ by additive extension of

$$\begin{aligned} \tilde{U}(\tilde{\jmath}) \ : \quad & L^2_\varepsilon({}^n\widetilde{H}_1, F^{(n)}_\sigma) \to L^2_\varepsilon({}^n\widetilde{H}_1, F^{(n)}_{\bar{\sigma}}) \\ & (\tilde{U}(\tilde{\jmath})\,\tilde{\psi})(\tilde{\boldsymbol{q}}) := C \cdot \tilde{\psi}(-\tilde{\jmath}\cdot\tilde{\boldsymbol{q}}) \, . \end{aligned} \tag{3.17}$$

The action of $\tilde{\jmath}$ in ${}^n\widetilde{H}_1$ by $\tilde{\boldsymbol{q}} \mapsto (-\tilde{\jmath})\cdot\tilde{\boldsymbol{q}}$ is explained in Appendix A.1. Note that $\tilde{U}(\tilde{\jmath})\,\tilde{\psi}$ is again equivariant due to the relation

$$(-\tilde{\jmath}) \cdot (\tilde{\boldsymbol{q}}\cdot b) = (-\tilde{\jmath} \cdot \tilde{\boldsymbol{q}})\cdot(-\tilde{\jmath} \cdot b) \, . \tag{3.18}$$

Also $\tilde{U}(\tilde{\jmath})$ is easily seen to be a conjugation operator. Now we show that it indeed implements $\tilde{\jmath}$:

LEMMA 3.6. *$\tilde{U}(\tilde{\jmath})$ extends $\tilde{U}(\tilde{P}^{\uparrow}_{+})$ to a representation of $\tilde{P}_{+}$ in $\tilde{\mathcal{H}}$.*

PROOF. We have to show that

$$(\tilde{U}(\tilde{\jmath})\tilde{U}(\tilde{g})\tilde{U}(\tilde{\jmath})\,\psi)_{\sigma,\boldsymbol{\alpha}}(\tilde{\boldsymbol{q}}) = (\tilde{U}(\tilde{\jmath}\tilde{g}\tilde{\jmath})\,\psi)_{\sigma,\boldsymbol{\alpha}}(\tilde{\boldsymbol{q}}) \, .$$

This follows from the single particle case equation (1.73) and the fact that $(-\tilde{\jmath})\cdot\tilde{g}\cdot(-\tilde{\jmath})\cdot\tilde{\boldsymbol{q}} = (\tilde{\jmath}\tilde{g}\tilde{\jmath})\cdot\tilde{\boldsymbol{q}}$. □

PROPOSITION 3.7. *$\tilde{U}(\tilde{\jmath})$ is related to $U(\tilde{\jmath})$ by*

$$\begin{aligned} & W^- \, U(\tilde{\jmath}) \, (W^+)^* = \tilde{U}(\tilde{\jmath}) \quad \text{and} \\ & W^+ \, U(\tilde{\jmath}) \, (W^-)^* = \tilde{U}(\tilde{\jmath}) \, . \end{aligned} \tag{3.19}$$

Consequently, the P_1CT-operator $U(\tilde{\jmath})$ intertwines the S-matrix $S = (W^+)^* \, W^-$ with its adjoint:

$$U(\tilde{\jmath}) \, S = S^* \, U(\tilde{\jmath}) \, . \tag{3.20}$$

The proof of this proposition will make use of the following Lemma:

LEMMA 3.8. *For all $(b, \pi) \in \widetilde{B}_n(C)$,*

$$\alpha_{\tilde{\jmath}} \left(\varepsilon_{\varrho_1 \cdots \varrho_n}(b, \pi) \right) = \varepsilon_{\bar{\varrho}_1 \cdots \bar{\varrho}_n}(\hat{\jmath}(b), \pi) \, , \tag{3.21}$$

where $\hat{\jmath}$ is the automorphism of B_n defined by

$$\hat{\jmath}(t_k) := t_k^{-1} \quad \text{and} \quad \hat{\jmath}(c_1) := c_1^{-1} \, . \tag{3.22}$$

Note that $\hat{\jmath}$ leaves the subgroup $\mathcal{N}_\varepsilon$ invariant, which has been defined in equation (2.42). In terms of the action of $\tilde{\jmath}$ on ${}^n\widetilde{H}_1$ defined in Appendix A.1, one easily sees that $\hat{\jmath}(b)$ coincides with $-\tilde{\jmath} \cdot b$.

PROOF. To see that $\hat{\jmath}$ determines an automorphism of B_n, we check that it respects the group relations, and hence can be extended to B_n as a homomorphism. It is bijective since it satisfies $\hat{\jmath}^2 = \mathrm{id}$. Since also α_j and $\varepsilon_{\boldsymbol{\varrho}} := \varepsilon_{\varrho_1 \cdots \varrho_n}$ are homomorphisms, we have to prove equation (3.21) only for b in the set of generators. Also, since $\varepsilon_{\boldsymbol{\varrho}}(b, \pi) = \varepsilon_{\boldsymbol{\varrho}\cdot\pi^{-1}}(b, 1)$, we may take $\pi = 1$. We first calculate $\alpha_j(\varepsilon(\varrho_k, \varrho_{k+1}))$. Let U_k, U_{k+1} be intertwiners such that $\mathrm{Ad}U_l \circ \varrho_l \in \Delta(\tilde{I}_l)$ for $l = k, k+1$, where $\tilde{I}_l \in \widetilde{\mathcal{K}}$ are paths with $N(\tilde{I}_k, \tilde{I}_{k+1}) = 0$. According to equation (1.31), $\varepsilon(\varrho_k, \varrho_{k+1}) = (\mathbf{U}^*_{k+1} \times \mathbf{U}^*_k) \circ (\mathbf{U}_k \times \mathbf{U}_{k+1})$. Hence

$$\alpha_j(\varepsilon(\varrho_k, \varrho_{k+1})) = (\alpha_j(\mathbf{U}^*_{k+1}) \times \alpha_j(\mathbf{U}^*_k)) \circ (\alpha_j(\mathbf{U}_k) \times \alpha_j(\mathbf{U}_{k+1})). \tag{3.23}$$

The endomorphism $\mathrm{Ad}(\alpha_j(U_l)) \circ \bar{\varrho}_l$ equals $\alpha_j \circ \mathrm{Ad}U_l \circ \varrho_l \circ \alpha_j$ and is therefore localized in $\tilde{\jmath}\cdot\tilde{I}_l$. Thus by definition (1.31), the right hand side of equation (3.23) equals to $\varepsilon(\bar{\varrho}_k, \bar{\varrho}_{k+1}; \tilde{\jmath}\cdot\tilde{I}_k, \tilde{\jmath}\cdot\tilde{I}_{k+1})$. Since

$N(\tilde{j}{\cdot}\tilde{I}_k, \tilde{j}{\cdot}\tilde{I}_{k+1}) = -N(\tilde{I}_k, \tilde{I}_{k+1}) - 1 = -1$, this intertwiner equals $(\mathbf{V}_{\bar{\varrho}_{k+1}} \times \mathbf{1}_{\bar{\varrho}_k})\, \varepsilon(\bar{\varrho}_k, \bar{\varrho}_{k+1})\, (\mathbf{1}_{\bar{\varrho}_k} \times \mathbf{V}^{-1}_{\bar{\varrho}_{k+1}})$ by virtue of equation (1.38). This result implies

$$\begin{aligned} \alpha_j(\varepsilon_{\boldsymbol{\varrho}}(t_k,1)) &\equiv \mathbf{1}_{\bar{\varrho}_1\cdots\bar{\varrho}_{k-1}} \times \alpha_j(\varepsilon(\varrho_k,\varrho_{k+1})) \times \mathbf{1}_{\bar{\varrho}_{k+2}\cdots\bar{\varrho}_n} \\ &= \mathbf{1}_{\bar{\varrho}_1\cdots\bar{\varrho}_{k-1}} \times (\mathbf{V}_{\bar{\varrho}_{k+1}} \times \mathbf{1}_{\bar{\varrho}_k})\, \varepsilon(\bar{\varrho}_k,\bar{\varrho}_{k+1})\, (\mathbf{1}_{\bar{\varrho}_k} \times \mathbf{V}^{-1}_{\bar{\varrho}_{k+1}}) \times \mathbf{1}_{\bar{\varrho}_{k+2}\cdots\bar{\varrho}_n} \\ &= \varepsilon_{\bar{\boldsymbol{\varrho}}}(c_k,\tau_k) \circ \varepsilon_{\bar{\boldsymbol{\varrho}}}(t_k,1) \circ \varepsilon_{\bar{\boldsymbol{\varrho}}}(c_{k+1}^{-1},1) = \varepsilon_{\bar{\boldsymbol{\varrho}}}(c_k t_k c_{k+1}^{-1},1) = \varepsilon_{\bar{\boldsymbol{\varrho}}}(t_k^{-1},1)\,. \end{aligned}$$

Similarly, equation (3.12) leads to $\alpha_j(\varepsilon_{\boldsymbol{\varrho}}(c_k,1)) \equiv \alpha_j(\mathbf{V}_{\varrho_1}) \times \mathbf{1}_{\bar{\varrho}_2\cdots\bar{\varrho}_n} = \varepsilon_{\bar{\boldsymbol{\varrho}}}(c_k^{-1},1))$. This completes the proof of the Lemma. □

PROOF OF PROPOSITION 3.7. We proceed in analogy with the proof of Lemma 3.4. Let $\boldsymbol{\psi} = T\psi_n \times \cdots \times \psi_1(\sigma,\boldsymbol{\alpha},\xi,+)$, with $\boldsymbol{\psi}_k \in \mathcal{H}^{(1)}_{\alpha_k}(V_k)$, and $\mathbf{T} \in \mathrm{Int}(\sigma|\varrho_{\alpha_1}\cdots\varrho_{\alpha_n})$. Then by virtue of equation (3.13) and the definition (2.66) of the Møller operators, we get

$$\begin{aligned} (W^-U(\tilde{j})\boldsymbol{\psi})(-\tilde{j}{\cdot}\tilde{\boldsymbol{q}}) &= \Big(W^-\ \alpha^0_{\tilde{j}}(T)\,(U_0(\tilde{j})\psi_n) \times \cdots \times (U_0(\tilde{j})\psi_1)\,(\bar{\sigma},\bar{\boldsymbol{\alpha}},\tilde{j}{\cdot}\xi,-)\Big)(-\tilde{j}{\cdot}\tilde{\boldsymbol{q}}) \\ &= (W_{\bar{\alpha}_1}U_0(\tilde{j})\psi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} W_{\bar{\alpha}_n}U_0(\tilde{j})\psi_n)(-j{\cdot}\boldsymbol{q})\,(\bar{\sigma}|\,\alpha^0_{\tilde{j}}(T)\,\varepsilon_{\bar{\boldsymbol{\alpha}}}(\tilde{\varphi}^-_{-j\cdot\mathbf{V},\tilde{j}\cdot\xi}(-\tilde{j}{\cdot}\tilde{\boldsymbol{q}}),\pi^{-1})\,|\bar{\boldsymbol{\alpha}}\pi) \\ &= c_{\alpha_1}\cdots c_{\alpha_n}\,\overline{(\tilde{\psi}_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \tilde{\psi}_n)(\boldsymbol{q})}\,(\bar{\sigma}|\,\alpha^0_{\tilde{j}}(T)\,\varepsilon_{\bar{\boldsymbol{\alpha}}}(\tilde{\varphi}^-_{-j\cdot\mathbf{V},\tilde{j}\cdot\xi}(-\tilde{j}{\cdot}\tilde{\boldsymbol{q}}),\pi^{-1})\,|\bar{\boldsymbol{\alpha}}\pi) \end{aligned} \tag{3.24}$$

Here we have written $W_{\alpha_k}\psi_k =: \tilde{\psi}_k$ and used the intertwining relation on the single particle level $W_{\bar{\alpha}}U_0(\tilde{j}) = \tilde{U}_\alpha(\tilde{j})W_\alpha$ and the explicit form of $\tilde{U}_\alpha(\tilde{j})$, see equation (1.75). On the other hand,

$$\begin{aligned} &(\tilde{U}(\tilde{j})W^+\boldsymbol{\psi})(-\tilde{j}{\cdot}\tilde{\boldsymbol{q}}) = C \cdot W^+\boldsymbol{\psi}(\tilde{\boldsymbol{q}}) \\ &= C\cdot\Big((\tilde{\psi}_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \tilde{\psi}_n)(\boldsymbol{q})\ (\sigma|\,T\,\varepsilon_{\boldsymbol{\alpha}}(\tilde{\varphi}^+_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}),\pi^{-1})\ |\boldsymbol{\alpha}\pi)\Big) \\ &= c_{\alpha_1}\cdots c_{\alpha_n}\,\overline{(\tilde{\psi}_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \tilde{\psi}_n)(\boldsymbol{q})}\,\Big(\bar{\sigma}|\,\alpha^0_{\tilde{j}}(T)\,\alpha^0_{\tilde{j}}\Big(\varepsilon_{\boldsymbol{\alpha}}(\tilde{\varphi}^+_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}),\pi^{-1})\Big)\,|\bar{\boldsymbol{\alpha}}\pi\Big)\,. \end{aligned} \tag{3.25}$$

By virtue of Lemma 3.8, to show that the expression (3.25) coincides with (3.24), it remains to prove

$$\hat{j}\left(\tilde{\varphi}^+_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\right) = \tilde{\varphi}^-_{-j\cdot\mathbf{V},\tilde{j}\cdot\xi}(-\tilde{j}\cdot\tilde{\boldsymbol{q}})\,. \tag{3.26}$$

We will in this proof omit the notation of 'kerε'. Let $\tilde{\varphi}^+_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})$ be given by equation (2.59), with suitable $\gamma, \mathbf{V}_{(\nu)}$ and $\xi_{(\nu)} \in X^+_{\mathbf{V}_{(\nu)}}$. To compute $\tilde{\varphi}^-_{-j\cdot\mathbf{V},\tilde{j}\cdot\xi}(-\tilde{j}\cdot\tilde{\boldsymbol{q}})$, we choose $\gamma'(t) := -j\cdot\gamma(t)$, $\mathbf{V}'_{(\nu)} := -j\cdot\mathbf{V}_{(\nu)}$ and $\xi'_{(\nu)} := \tilde{j}\cdot\xi_{(\nu)} \in X^-_{-j\cdot\mathbf{V}_{(\nu)}}$, where $\xi_{(0)} := \xi_0^+$. Then again by equation (2.59),

$$\begin{aligned} \tilde{\varphi}^-_{-j\mathbf{V},\tilde{j}\xi}(-\tilde{j}\cdot\tilde{\boldsymbol{q}}) =& \varphi_{\tilde{j}\xi\pi}(\pi)\, b^-_{-j(\mathbf{V}\pi\cap\mathbf{V}_{(m-2)})}(\tilde{j}{\cdot}\xi_{(m-1)},\tilde{j}{\cdot}\xi_{(m-2)})\cdots b^-_{-j(\mathbf{V}_{(1)}\cap\mathbf{V}_{(0)})}(\tilde{j}{\cdot}\xi_{(1)},\tilde{j}{\cdot}\xi_{(0)}) \\ & b^-_{-j\mathbf{V}_0\cap\mathbf{V}_0}(\tilde{j}{\cdot}\xi_{(0)},\xi_0^-)\,. \end{aligned} \tag{3.27}$$

We first observe that for all $\pi \in S_n$

$$\varphi_{\tilde{j}\xi}(\pi) = \hat{j}\,(\varphi_\xi(\pi))\ . \tag{3.28}$$

To see this, let $N = N(\xi_k,\xi_{k+1})$. Then $N(\tilde{j}{\cdot}\xi_k,\tilde{j}{\cdot}\xi_{k+1}) = -N-1$ due to the group relation (1.66) and the fact that the action of $\tilde{j}$ inverts the order relation of Definition 1.3. Hence,

$$\begin{aligned} \varphi_{\tilde{j}\xi}(t_k) &= c_k^{N+1} t_k c_{k+1}^{-N-1} = c_k^N t_k^{-1} c_{k+1}^{-N} \\ &= \hat{j}(c_k^{-N} t_k c_{k+1}^N) = \hat{j}\big(\varphi_\xi(t_k)\big)\,, \end{aligned} \tag{3.29}$$

proving equation (3.28). Now we use this result to show that

$$b^-_{-j\mathbf{V}}(\tilde{j}{\cdot}\xi,\tilde{j}{\cdot}\xi') = \hat{j}\big(b^+_{\mathbf{V}}(\xi,\xi')\big)\,. \tag{3.30}$$

In a first step, let ξ' and ξ differ only in the k th entry. Then so do $\tilde{j}{\cdot}\xi$ and $\tilde{j}{\cdot}\xi'$, and according to equation (2.15),

$$b^-_{-j\mathbf{V}}(\tilde{j}{\cdot}\xi,\tilde{j}{\cdot}\xi') = \varphi_{\tilde{j}\xi\pi_k^{-1}}(\pi_k^{-1})\varphi_{\tilde{j}\xi'}(\pi_k) = \hat{j}\big(b^+_{\mathbf{V}}(\xi,\xi')\big)\,, \tag{3.31}$$

due to equation (3.28). This relation carries over to the general case via equation (2.14), which proves (3.30). Plugging equations (3.28) and (3.30) into equation (3.27), we see that in order to prove equation (3.26), it remains to show that the last factor in equation (3.27) is trivial. This is the case if we make a specific choice for ξ_0^+ and S_0, compatible with the restriction on ξ_0^+ specified in the proof of Lemma 2.8 and the requirement $j \cdot S_0 = S_0$. Namely, we take S_0 to be centered around the positive x^2-axis, and ξ_0^+ to be the class of $(\tilde{I}_0^1, \ldots, \tilde{I}_0^n)$, where

$$\tilde{I}_0^k := (0, \omega_k) \cdot S_0 \quad \text{with} \quad \omega_k := \frac{\pi}{2} - \frac{k\pi}{n+1} \in (-\frac{\pi}{2}, \frac{\pi}{2}) . \tag{3.32}$$

(The opening angle of S_0 is assumed to be small enough such that these cones do not overlap.) As chosen before, we have $\xi_0^- = \theta \cdot \xi_0^+$. Denoting $-j \cdot \mathbf{V}_0 \cap \mathbf{V}_0 =: \mathbf{V}$ and exploiting the cocycle condition (2.14), the last factor in equation (3.27) is now given by

$$b_{\mathbf{V}}^-(\tilde{j} \cdot \xi_0^+, \theta \cdot \xi_0^+) = b_{\mathbf{V}}^-(\xi_{(n)}, \xi_{(n-1)}) \cdots b_{\mathbf{V}}^-(\xi_{(1)}, \xi_0^+) , \tag{3.33}$$

where $\xi_{(k)}$ is the class of $(\tilde{j} \cdot \tilde{I}_0^1, \ldots, \tilde{j} \cdot \tilde{I}_0^k, \theta \cdot \tilde{I}_0^{k+1}, \ldots, \theta \cdot \tilde{I}_0^n)$ for $k = 0, \ldots, n$. By equation (2.15), $b_{\mathbf{V}}^-(\xi_{(k)}, \xi_{(k-1)}) = \varphi_{\xi_{(k)}}(\pi_k^{-1}, \pi_k)\varphi_{\xi_{(k-1)}}(\pi_k, 1)$ for $k = 1, \ldots, n$, since $\xi_{(k)}$ and $\xi_{(k-1)}$ differ only in the k th entry where $\xi_{(k-1)}^k = \theta \cdot \tilde{I}_0^k$ is replaced by $\xi_{(k)}^k = \tilde{j} \cdot \tilde{I}_0^k$. From our explicit formula (2.17) we read off that this braid is trivial if $N(\xi_{(k)}^k, \xi_{(k)}^l) = N(\xi_{(k-1)}^k, \xi_{(k)}^l)$ for all $l = 1, \ldots, k-1$. In the case at hand, this means if $N(\tilde{j} \cdot \tilde{I}_0^k, j \cdot \tilde{I}_0^l) = N(\theta \cdot \tilde{I}_0^k, j \cdot \tilde{I}_0^l)$. As we have seen after equation (3.28), $N(\tilde{j} \cdot \tilde{I}_0^k, j \cdot \tilde{I}_0^l) = -N(\tilde{I}_0^k, \tilde{I}_0^l) - 1$, which equals -1 for $l = 1, \ldots, k-1$ with our choice of $\tilde{\mathbf{I}}_0$. On the other hand, $\theta \cdot \tilde{I}_0^k = (0, \omega_k') \cdot S_0$ with $\omega_k' = \pi + \omega_k \in (\frac{\pi}{2}, \frac{3}{2}\pi)$, and $\tilde{j} \cdot \tilde{I}_0^k = (0, \omega_k'') \cdot S_0$ with $\omega_k'' = -\omega_k \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence their relative winding number is $N(\theta \cdot \tilde{I}_0^k, j \cdot \tilde{I}_0^l) = -1$, too. This completes the proof that the last factor in equation (3.27) is trivial, and hence of the first one of equations (3.19) of the Proposition. The second equation works analogously, if one exploits that equation (3.26) is symmetric in the sense that it implies

$$\hat{j}\left(\tilde{\varphi}_{\mathbf{V},\xi}^-(\tilde{\boldsymbol{q}})\right) = \tilde{\varphi}_{-j \cdot \mathbf{V}, \tilde{j} \cdot \xi}^+(-\tilde{j} \cdot \tilde{\boldsymbol{q}}) .$$

□