CHAPTER 2

Structure of the Space of Scattering States

2.1. Construction and Properties of Particle States

Experimental data of high energy physics typically stem from collision processes, where incoming particle configurations evolve into outgoing ones. Hence, in order to relate our theoretical framework of algebraic quantum field theory to physical phenomena, we have to identify the states in the theory which can be interpreted as particle configurations. Phenomenologically, the notion of particle states can be made precise as follows [Haa92]. An n-particle coincidence experiment at time t is an arrangement of n particle detectors which are far separated from each other spatially, and switched on during some time interval around t. By a state with n *incoming or outgoing particles* one means a state which for all times t earlier or later, respectively, than some t_0 gives a signal in at least one n-particle coincidence experiment at t but in no m-particle coincidence experiment at t if $m > n$. The theoretical notion corresponding to such a state within the framework of algebraic quantum field theory is defined by the same words, now viewed as theoretical notions: A 'detector' is a positive element of the observable algebra with vanishing vacuum expectation value, which is essentially localized in some bounded region of space-time; A 'coincidence arrangement' of detectors is modelled by their product; Finally, a state is said to 'give a signal' in such an arrangement if it assigns an expectation value significantly different from zero to the corresponding product. Note that the detector cannot be a strictly localized observable due to the Reeh-Schlieder theorem, according to which no local observable (other than 0) can annihilate the vacuum.

We now address the question of the existence and construction of particle states in the theory, in the sense of vectors in the Hilbert space $\mathcal{H}$. The question of single particle states ($n = 1$) has to be distinguished from that of multiparticle states ($n > 1$). For the existence of single particle states, there are no model independent reasons. However, if the spectrum of the mass operator P_ϱ^2 in a given representation ϱ of the observable algebra contains an eigenvalue m^2 (as required from the representations we are interested in, see equation (0.9)), the corresponding eigenspace $\mathcal{H}_\varrho^{(1)}$ indeed satisfies the above criterion for 1-particle states. This is implied as a special case by the following discussion. Conversely, it has been shown by Enss in [Ens75], that under certain assumptions on the theory the eigenvectors of the mass operator are the only 1-particle states . If in addition the eigenvalue m^2 is isolated, as is the case for our massive single particle representations (see again equation (0.9)), the construction of multiparticle states from single particle states is possible in a model independent way. This construction is known as Haag-Ruelle scattering theory. It has first been carried out by R. Haag [Haa58], and improved by D. Ruelle who showed [Rue62] that certain assumptions of Haag can be derived from the Wightman axioms. It has been adapted to the case of particles with braid group statistics by K. Fredenhagen, M. Gaberdiel and S. Rüger in [FGR96].

It is noteworthy that Wigner's definition of elementary particles, relying on an eigenvalue of the mass operator P^2, is too restrictive for certain models, even for quantum electrodynamics. For example, a single electron state cannot belong to an eigenspace of P^2, as has been pointed out by B. Schroer [Sch63], who coined the name 'infraparticles' for such generalized particles. The theoretical concept describing both infraparticles and Wigner particles, is that of 'particle weights' introduced by D. Buchholz [Buc87]. It still fits into the phenomenological characterization of particle configurations given at the beginning of this section.

We briefly sketch the construction of Haag-Ruelle scattering states in the present framework as developped in [FGR96]. Given n single particle vectors, the aim is to construct a corresponding in- or outgoing n-particle vector. The first step is to construct one particle creation operators which are almost localized and create single particle states with a prescribed compact momentum support from the vacuum. To this end, let $\mathbf{B} = (\varrho, B) \in \mathcal{F}(\tilde{I})$ such that $\mathrm{sp}_{P_\varrho} B\Omega$ has nonvanishing intersection with the mass hyperboloid H_m, where m is the mass of a particle described by ϱ in the sense of equation (0.9). Further, let $f \in \mathcal{S}(\mathcal{M}^3)$ be a test function whose Fourier transform

$$\tilde{f}(p) := (2\pi)^{-\frac{2}{3}} \int \mathrm{d}^3x \, e^{ip\cdot x} f(x) \tag{2.1}$$

has a compact support contained in the open forward light cone V_+ and satisfying $\mathrm{supp}\tilde{f} \cap \mathrm{spec}P_\varrho \subset H_m$. For all $t \in \mathbb{R}$, let f_t be defined by $\tilde{f}_t(p) := \exp(i\frac{p^2-m^2}{2m}t)\tilde{f}(p)$. For large $|t|$, its support is essentially contained in the region $t\,V(f)$, where $V(f)$ is the velocity support of f,

$$V(f) := \frac{1}{m}\,\mathrm{supp}\tilde{f}\,. \tag{2.2}$$

More precisely [FGR96, Lemma 3.1], for all $\varepsilon > 0$, $N \in \mathbb{N}$, there is a constant $c > 0$ such that

$$|f_t(x)| < c\,\left|\mathrm{dist}\left(\frac{x}{t}\,, V(f)\right) + 1\right|^{-N} \quad \text{for all } x \notin t\,V(f)^\varepsilon\,. \tag{2.3}$$

Here, dist is a euclidean metric on $\mathcal{M}^3$, and we have used the notation

$$V^\varepsilon := \{v \in \mathcal{M}^3\,,\ \mathrm{dist}(v, V) < \varepsilon\} \tag{2.4}$$

for a compact set $V \subset \mathcal{M}^3$ and $\varepsilon > 0$. Now the announced one particle creation operator is defined by

$$\mathbf{B}(f,t) := \int \mathrm{d}^3x\, f_t(x)\,\alpha(x, \mathbb{1}\,)(\mathbf{B}) \quad \in \mathcal{F}\,. \tag{2.5}$$

For large $|t|$, it is essentially localized in $\tilde{I} + t\,V(f)^\varepsilon$. This means, that $\mathbf{B}(f,t)$ can be approximated for any $\varepsilon > 0$ by field operators $\mathbf{B}^\varepsilon(f,t)$ localized in $\tilde{I} + t\,V(f)^\varepsilon$, such that $||\mathbf{B}(f,t) - \mathbf{B}^\varepsilon(f,t)||\ |t|^N \to 0$ for all $N \in \mathbb{N}$. Explicitely, $\mathbf{B}^\varepsilon(f,t)$ is defined like $\mathbf{B}(f,t)$ in equation (2.5), but with the integral only performed over the region $t\,V(f)^\varepsilon$. The vector

$$\mathbf{B}(f,t)\,\mathbf{\Omega} \quad \in \mathcal{H}_\varrho^{(1)} \tag{2.6}$$

coincides with $(\varrho, (2\pi)^{\frac{2}{3}}\tilde{f}(P_\varrho)\,B\,\Omega)$, hence it is independent of t and its spectral support is contained in $\mathrm{supp}\tilde{f} \cap H_m$. In particular, it is a single particle vector.

Now we consider n such one particle creation operators essentially localized in regions which become mutually spacelike for large t: We choose n localizations $\tilde{I}_1, \dots, \tilde{I}_n \in \tilde{\mathcal{K}}$ and n compact subsets $V_1, \dots, V_n$ of the unit mass shell H_1^+, such that for a suitable $\varepsilon > 0$ the regions $I_k + t\,V_k^\varepsilon$ are mutually spacelike for large t. For $k = 1, \dots, n$, let $\mathbf{B}_k = (\varrho_k, B_k) \in \mathcal{F}(\tilde{I}_k)$, where ϱ_k is a single particle representation with mass m_k, and let f_k be a testfunction such that $\mathrm{supp}\tilde{f}_k$ is contained in $m_k\,V_k^\varepsilon$ and intersects the energy momentum spectrum $\mathrm{spec}P_{\varrho_k}$ only in $H_{m_k}^+$. Here m_k is the mass of the particle described by ϱ_k. Then the vector valued function

$$t \mapsto \mathbf{B}_n(f_n,t)\cdots\mathbf{B}_1(f_1,t)\,\mathbf{\Omega} \qquad \subset \quad \mathcal{H}\varrho_1\cdots\varrho_n \tag{2.7}$$

converges for $t \to \infty$, and the limit vector depends, for fixed localizations $\tilde{I}_k$, only on the single particle vectors $\psi_k := \mathbf{B}_k(f_k,t)\,\mathbf{\Omega}$. It is interpreted as an *outgoing state* of n particles with state vectors ψ_k and 'velocity supports' V_k. A state vector of n *incoming particles* may be defined analogously as the limit for $t \to -\infty$ of the vectors (2.7), provided the localizations $\tilde{I}_k$ and velocity supports V_k are chosen such that the regions $I_k + t\,V_k^\varepsilon$ are mutually spacelike for all t *smaller* than some $t_0 < 0$. Since the outgoing and incoming n-particle vectors only depend on the single particle vectors $\psi_1, \dots, \psi_n$ and the localizations $\tilde{\mathbf{I}} = (\tilde{I}_1, \dots, \tilde{I}_n)$, we may denote them by

$$\lim_{t\to\pm\infty} \mathbf{B}_n(f_n,t)\cdots\mathbf{B}_1(f_1,t) \ =: \psi_n \times \cdots \times \psi_1(\tilde{\mathbf{I}}, \pm)\,, \tag{2.8}$$

respectively. The order of factors in these states can be permuted with the commutation relation (1.53), which survives in the limit $t \to \pm\infty$, respectively:

$$\begin{aligned} &\psi_{(\pi'\pi)^{-1}(n)} \times \cdots \times \psi_{(\pi'\pi)^{-1}(1)} \left(\tilde{\mathbf{I}} \cdot (\pi'\pi)^{-1}, \pm\right) \\ &= \left(\varepsilon_{\boldsymbol{\varrho}} \circ \varphi_{\tilde{\mathbf{I}}}\right)(\pi', \pi)\, \psi_{\pi^{-1}(n)} \times \cdots \times \psi_{\pi^{-1}(1)} \left(\tilde{\mathbf{I}} \cdot \pi^{-1}, \pm\right) . \end{aligned} \tag{2.9}$$

Here, $\tilde{\mathbf{I}} \cdot \pi$ denotes the natural right action of the permutation group on tupels as defined before equation (A.1). Let us now examine the dependence of the n-particle vectors on the localizations $\tilde{I}_k$ of the creation operators. As a first step, we exchange one of the localizations: Assume, that the jth single particle vector can be created also by an operator essentially localized in a different region, i.e. $\psi = \mathbf{D}(g,t)\Omega$ for some $\mathbf{D} \in \mathcal{F}(\tilde{J})$, where $\tilde{J} + t\,V(f_j)^\varepsilon$ is causally disjoint from $\tilde{I}_k + t\,V(f_k)^\varepsilon, k \neq j$, for large t (or small t, respectively), and $\mathrm{supp}\tilde{g} \subset \mathrm{supp}\tilde{f}_j$. We exploit equation (2.9) to commute $\mathbf{B}_j(f_j, t)$ to the right where it is applied to the vacuum and can be replaced by $\mathbf{D}(g,t)$, which we then commute back to the jth position. As a result,

$$\begin{aligned} &\psi_n \times \cdots \times \psi_1(\tilde{\mathbf{I}}, \pm) \\ &= \varepsilon_{\boldsymbol{\varrho}}\left(\varphi_{\tilde{\mathbf{I}}}(\pi_j^{-1}, \pi_j)\, \varphi_{\tilde{\mathbf{I}}^{(j)}}(\pi_j, 1)\right)\, \psi_n \times \cdots \times \psi_1(\tilde{\mathbf{I}}^{(j)}, \pm) \\ &\equiv \varepsilon_{\boldsymbol{\varrho}}\left(\left(\varphi_{\tilde{\mathbf{I}}}(\pi_j, 1)\right)^{-1} \varphi_{\tilde{\mathbf{I}}^{(j)}}(\pi_j, 1)\right)\, \psi_n \times \cdots \times \psi_1(\tilde{\mathbf{I}}^{(j)}, \pm) . \end{aligned} \tag{2.10}$$

Here, $\tilde{\mathbf{I}}^{(j)} \in \tilde{\mathcal{K}}^{\times n}$ denotes the tupel arising from $\tilde{\mathbf{I}} = (\tilde{I}_1, \ldots, \tilde{I}_n)$ if $\tilde{I}_j$ is replaced by $\tilde{J}$, and π_j is any permutation mapping j to 1. In particular, the multiparticle vector does not change at all if $\tilde{J}$ has the same relative winding numbers as $\tilde{I}_k$ with $\tilde{I}_1, \ldots, \tilde{I}_{k-1}, \tilde{I}_{k+1}, \ldots, \tilde{I}_n$, since $\varphi_{\tilde{\mathbf{I}}}$ only depends on the relative winding numbers. We are led to the following definition.

Definition 2.1. Let $\mathbf{V} := (V_1, \ldots, V_n)$ be a tupel of mutually disjoint compact subsets of the unit mass shell H_1^+. Let $\tilde{\mathcal{K}}_{\mathbf{V}}^+$ and $\tilde{\mathcal{K}}_{\mathbf{V}}^-$ denote the set of tupels $\tilde{\mathbf{I}} = (\tilde{I}_1, \ldots, \tilde{I}_n)$ in $\tilde{\mathcal{K}}$ such that the regions $I_k + t\,V_k$ are mutually spacelike for all t larger or smaller, respectively, than some $t_0 \in \mathbb{R}$. Two tupels $\tilde{\mathbf{I}}$ and $\tilde{\boldsymbol{J}}$ in $\tilde{\mathcal{K}}_{\mathbf{V}}^+$ will be considered equivalent, if all of their relative winding numbers coincide, i.e $N(\tilde{I}_k, \tilde{I}_l) = N(\tilde{J}_k, \tilde{J}_l)$ for $k \neq l$. The same equivalence relation is defined on $\tilde{\mathcal{K}}_{\mathbf{V}}^-$. The corresponding sets of equivalence classes will be denoted by

$$X_{\mathbf{V}}^+ \quad \text{and} \quad X_{\mathbf{V}}^- ,$$

respectively. Elements of these sets will be denoted by ξ. Also, we will write $_\xi$ instead of $_{\tilde{I}}$.

As the above discussion shows, we may write

$$\psi_n \times \cdots \times \psi_1(\tilde{\mathbf{I}}, \pm) =: \psi_n \times \cdots \times \psi_1(\xi, \pm) , \tag{2.11}$$

where $\xi \in X_{\mathbf{V}}^\pm$ is the equivalence class of $\tilde{\mathbf{I}}$. Working out further the dependence of this vector on the localizations along the lines of equation (2.10), we are led to the following result. Let

$$\mathcal{N} := \bigcap_{\boldsymbol{\varrho}} \ker\, \pi_0 \varepsilon_{\boldsymbol{\varrho}} \cap PB_n , \tag{2.12}$$

where $\boldsymbol{\varrho}$ runs through all n-fold products of irreducible massive single particle representations of $\mathcal{A}_{\mathrm{u}}$. Here $\ker\, \pi_0\varepsilon_{\boldsymbol{\varrho}}$ denotes the normal subgroup of all $b \in B_n$ such that $\pi_0\varepsilon_{\boldsymbol{\varrho}}(b, 1) = \mathbf{1}$, and PB_n denotes the pure braid group, i.e. the kernel of the natural homomorphism ν from B_n onto S_n. Note that the cylinder braid c_1 is in the kernel of all representations $\pi_0\varepsilon_{\boldsymbol{\varrho}}$, hence we may view them as representations of B_n instead of $B_n(C)$.

Lemma and Definition 2.2. *Given a Cartesian product* $\mathbf{V}$ *of* n *compact mutually disjoint subsets of* H_1^+, *there are maps* $b_{\mathbf{V}}^+$ *and* $b_{\mathbf{V}}^-$ *with*

$$b_{\mathbf{V}}^\pm : X_{\mathbf{V}}^\pm \times X_{\mathbf{V}}^\pm \to B_n/\mathcal{N}$$

satisfying the cocycle conditions

$$\textit{i)}\ b_{\mathbf{V}}^\pm(\xi, \xi')\, b_{\mathbf{V}}^\pm(\xi', \xi) = 1 \cdot \mathcal{N} \qquad \textit{for all } \xi, \xi' \in X_{\mathbf{V}}^\pm \tag{2.13}$$

$$\textit{ii)}\ b_{\mathbf{V}}^\pm(\xi, \xi')\, b_{\mathbf{V}}^\pm(\xi', \xi'') = b_{\mathbf{V}}(\xi, \xi'') \qquad \textit{for all } \xi, \xi', \xi'' \in X_{\mathbf{V}}^\pm \tag{2.14}$$

and uniquely characterized by the further condition

$$b^{\pm}_{\mathbf{V}}([\tilde{\mathbf{I}}],[\tilde{\mathbf{I}}^{(j)}]) = \varphi_{[\tilde{\mathbf{I}}]}(\pi_j^{-1},\pi_j)\varphi_{[\tilde{\mathbf{I}}^{(j)}]}(\pi_j,1)\cdot\mathcal{N}\;, \tag{2.15}$$

if $\tilde{\mathbf{I}}^{(j)}$ *differs from* $\tilde{\mathbf{I}}$ *only in the* j *th entry. Here,* $\pi_j \in S_n$ *denotes the transposition of* 1 *and* j. *The dependence of outgoing or incoming multiparticle vectors on the localizations is effected by* $b^+_{\mathbf{V}}$ *or* $b^-_{\mathbf{V}}$, *respectively, as follows. Let* $\psi_k \in \mathcal{H}^{(1)}_{\varrho_k}$, *with velocity support in* $\mathbf{V}$. *Then*

$$\psi_n \times \cdots \times \psi_1(\xi,\pm) = \varepsilon_{\boldsymbol{\varrho}}\left(b^{\pm}_{\mathbf{V}}(\xi,\xi')\right)\,\psi_n\times\cdots\times\psi_1(\xi',\pm)\ \ \textit{for all}\ \xi,\xi'\in X^{\pm}_{\mathbf{V}}\,. \tag{2.16}$$

In equation (2.16), as well as in the following, the embedding $B_n \hookrightarrow \widetilde{Bn},\ b\mapsto(b,1)$ is implicitely understood.

Remark. The claim follows by iterating the procedure (2.10). Note that only pure braids can occur in this process. Hence $b^{\pm}_{\mathbf{V}}$ are actually maps into the pure braid group PB_n modulo the intersection of all $\ker\pi_0\varepsilon_{\boldsymbol{\varrho}}$. This is equivalent to our formulation (2.12), which we have chosen for later conveniance. Equation (2.15) can be put into a more explicit form. Suppose $\tilde{\mathbf{I}}^{(j)}$ arises from $\tilde{\mathbf{I}}$ as described after equation (2.10). According to Definition 1.7, we choose paths γ_k corresponding to $\tilde{I}_k$, and a path $\hat{\gamma}$ corresponding to $\tilde{J}$. They start from r_0 (corresponding to S_0), and the respective endpoints will be denoted by r_k and $\hat{r}$. Then

$$\varphi_{[\tilde{\mathbf{I}}]}(\pi_j^{-1},\pi_j)\varphi_{[\tilde{\mathbf{I}}^{(j)}]}(\pi_j,1)\ =\ (b,1)\;,$$

where $b\in PB_n$ is the homotopy class of the S_n-orbit of the path $(\beta_1,\dots,\beta_n)$ in $(\mathbb{R}^2\setminus\{0\})^{\times n}\setminus\Delta_n$ which is defined by

$$\begin{array}{llccl}\beta_i & =(i,\gamma_i^{-1})\,* & (i,r_i) & *\,(i,\gamma_i) & \text{for } 1\le i<j\\ \beta_j & =(j,\gamma_j^{-1})\,* & (\tfrac12\mapsto j,r_j)*(\tfrac12,\gamma_j*\hat{\gamma}^{-1})*(j\mapsto\tfrac12,\hat{r}) & *\,(j,\hat{\gamma}) & \\ \beta_k & = & (k,r_0) & & \text{for } j<k\le n\,.\end{array} \tag{2.17}$$

We note that it would be more constructive and hence satisfying to have a direct proof that a $B_n(C)$-valued map $b_{\mathbf{V}^{\pm}}$ can be defined with the stated properties, without taking recourse to the existence of massive single particle representations. We conjecture that this is possible, but have only shown the case $n=2$. Fredenhagen, Gaberdiel and Rüger have solved this problem for $\mathbf{V}$ and ξ in a certain class [FGR96, Lemma 3.3].

There is a one-to-one correspondence between $X^+_{\mathbf{V}}$ and $X^-_{\mathbf{V}}$ and a relation between $b^+_{\mathbf{V}}$ and $b^-_{\mathbf{V}}$. It relies on the observation that I_k+tV_k are mutually spacelike for large *negative* t if and only if $-I_k+tV_k$ are mutually spacelike for large *positive* t. Hence, if θ is a map from $\widetilde{\mathcal{K}}$ into itself such that $\theta\cdot\tilde{I}$ ends at $-I$ if $\tilde{I}$ ends at I, and is compatible with relative winding numbers, then θ maps $X^+_{\mathbf{V}}$ into $X^-_{\mathbf{V}}$ and vice versa. A natural choice is as follows. Let T denote time reversal in $\mathcal{M}^3$, $Tx := x-2(e_0\cdot x)e_0$, where e_0 is the positive timelike unit vector which is stable under rotations $(0,\omega)\in L^{\uparrow}_{+}$. In coordinates adapted to e_0 this reads $T\,(x^0,\vec{x})=(-x^0,\vec{x})$. T acts on paths in $\mathcal{K}$ via $T\cdot(I_0,\dots,I_m):=(TI_0,\dots,TI_m)$ if our reference cone $S_0=I_0$ satisfies $TS_0=S_0$. This will be assumed in the sequel. T clearly respects homotopy of paths in $\mathcal{K}$, and hence is well defined on $\widetilde{\mathcal{K}}$. Now we define

$$\theta\cdot\tilde{I} := (0,\pi)\cdot(T{\cdot}\tilde{I})\,. \tag{2.18}$$

This choice for θ has the profitable property of preserving relative winding numbers[1]. Hence it can be carried over to eqivalence classes of tupels from Definition 2.1: $\theta\cdot[(\tilde{I}_1,\cdots,\tilde{I}_n)] := [(\theta{\cdot}\tilde{I}_1,\cdots,\theta{\cdot}\tilde{I}_n)]$. It is also easily seen to be invertible, hence, in view of our above observation, it maps $X^+_{\mathbf{V}}$ *onto* $X^-_{\mathbf{V}}$. Since θ preserves relative winding numbers, the groupoid homomorphism φ_ξ, which is defined in equation (1.49) via the relative winding numbers of $\xi_1,\dots,\xi_n$, coincides with $\varphi_{\theta\xi}$. This implies, in view of the characterization of the maps $b^+_{\mathbf{V}}$ and $b^-_{\mathbf{V}}$ by equation (2.15), that $b^-_{\mathbf{V}}(\theta{\cdot}\xi,\theta{\cdot}\xi')=b^+_{\mathbf{V}}(\xi,\xi')$. We sum up this discussion in the

[1] defined in equation (1.34).

COROLLARY 2.3. *The map θ, defined in equation (2.18), maps $X^+_{\mathbf{V}}$ onto $X^-_{\mathbf{V}}$ and vice versa, and satisfies*

$$\varphi_{\theta\xi}(\pi,\pi') = \varphi_\xi(\pi,\pi') \quad \textit{for all } (\pi,\pi') \in \widetilde{S}_n \quad \textit{and} \tag{2.19}$$

$$b^-_{\mathbf{V}}(\theta\cdot\xi,\theta\cdot\xi') = b^+_{\mathbf{V}}(\xi,\xi') \quad \textit{for all } \xi,\xi' \in X^+_{\mathbf{V}}\,. \tag{2.20}$$

The scalar product of two multiparticle vectors with the same localizations has been calculated by Fredenhagen, Gaberdiel and Rüger [FGR96, Theorem 3.2], exploiting a version of the cluster theorem [Rue62, DHR69a], which has been adapted to the present plektonic setup in [FGR96, Lemma 2.2]. We state the result in the following theorem.

THEOREM 2.4. *Let $V_1,\dots,V_n$ be compact subsets of the unit mass shell H_1^+ and $\tilde I_1,\dots,\tilde I_n \in \widetilde{\mathcal{K}}$ be localizations such that for suitable $\varepsilon > 0$, the regions $I_k + t V_k^\varepsilon$ are mutually spacelike for all t larger (or smaller, respectively) than some $t_0 \in \mathbb{R}$. Let further $\mathbf{B}_k = (\varrho_k, B_k) \in \mathcal{F}(\tilde I_k)$, $\hat{\mathbf{B}}_k = (\hat\varrho_k, \hat B_k) \in \mathcal{F}(\tilde I_k)$, where ϱ_k and $\hat\varrho_k$ are irreducible massive single particle representations with mass m_k, and let f_k, $\hat f_k$ be test functions whose Fourier transforms have support in $m_k V_k^\varepsilon$ and intersect* specP_{ϱ_k} *only in $H^+_{m_k}$. Finally, let $\mathbf{T}$ be a local intertwiner from $\varrho_1\cdots\varrho_n$ to $\hat\varrho_1\cdots\hat\varrho_n$. Then the scalar product of the multiparticle states arising from the two sets of single particle states $\psi_k = \mathbf{B}_k(f_k,t)\Omega$ and $\hat\psi_k = \hat{\mathbf{B}}_k(\hat f_k,t)\,\Omega$ is given by*

1. *If $\hat\varrho_k = \varrho_k$ for $k = 1,\dots,n$, then*

$$\left\langle \hat\psi_n \times\cdots\times \hat\psi_k\,(\tilde{\mathbf{I}},\pm)\,,\, \mathbf{T}\,\psi_n \times\cdots\times \psi_1\,(\tilde{\mathbf{I}},\pm) \right\rangle = \phi_1\cdots\phi_n(T) \prod_{k=1}^n \left\langle \hat\psi_k, \psi_k \right\rangle\,. \tag{2.21}$$

2. *If, on the other hand, $\hat\varrho_k$ is inequivalent to ϱ_k for some $k \in \{1,\dots,n\}$, then the above scalar product on the left hand side is zero.*

In equation (2.21), ϕ_k denotes the unique left inverse of ϱ_k, and $\phi_1\cdots\phi_n(T) \in \mathbb{C}\mathbb{1}$ has been identified with a complex number.

2.2. Structure of the Space of Scattering States

From the last theorem it is clear that *'locally'*, i.e. for a given compact velocity support[2] $\mathbf{V} \subset H_1^{\times n}$, the n-particle vectors can be viewed as tensor products of single particle vectors, which in turn can be identified with functions on $\mathbf{V}$. (This is made precise in Corollary 2.6.) The aim of this section is to reveal the *global* structure of the space of scattering vectors, i.e. of sums of vectors of the above type with different velocity supports $\mathbf{V}$. This has been already achieved in principle in [FGR96]. Here, we refine their results and prove them using concepts which are adapted to the structure of the space of scattering vectors in a natural way such that it becomes transparent how the Poincaré group and the $\mathrm{P}_1\mathrm{CT}$ operator act in it. Also, we obtain Møller operators which map the outgoing and the incoming scattering states canonically into one and the same reference Hilbert space, thus opening up the possibility to define in a natural way the S matrix, which was obscured before by the absence of a canonical comparison map between the outgoing and incoming reference Hilbert spaces. The following structure emerges. Let ${}^nH_1$ be the manifold of n noncoinciding velocities,

$${}^nH_1 := \big(H_1^{\times n} \setminus D_n\big)/S_n\,. \tag{2.22}$$

Here D_n denotes the set of points $(q_1,\dots,q_n) \in H_1^{\times n}$ with $q_i = q_j$ for at least one pair (q_i,q_j) with $i \neq j$ (see Appendix A.1). Since H_1 is diffeomorphic to $\mathbb{R}^2$, the braid group B_n is the fundamental group of ${}^nH_1$ and hence acts naturally on its universal covering manifold ${}^n\widetilde{H}_1$. [3] It turns out that the space of n-particle vectors is isomorphic to the space of functions on ${}^n\widetilde{H}_1$ with values in a direct sum of intertwinerspaces, which satisfy the so-called equivariance property

$$\tilde\psi(\tilde{\boldsymbol{q}}) = \varepsilon(b)\,\tilde\psi(\tilde{\boldsymbol{q}}\cdot b) \quad \text{for all } b \in B_n\,. \tag{2.23}$$

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[2]We put the word 'local' in quotation marks here to get no confusion with localization in spacetime

[3]The explicit identification of B_n with the fundamental group of ${}^nH_1$ is given before equation (2.47), and the action of the fundamental group in the univeral covering space of a manifold is explained in the Appendix.

Here, ε is a representation of B_n which is fixed by the homomorphisms $\boldsymbol{\varepsilon}_{\boldsymbol{\varrho}}$ defined in Definition 1.6 and the 'fusion rules' governing the decomposition of products of the considered massive single particle representations ϱ_i into irreducible representations. Note that this space is isomorphic to the space of square integrable sections in the vector bundle which is associated to ${}^n\widetilde{H}_1$, viewed as a principal fibre bundle, via ε. It must be taken aware that this function space has no canonical tensor product structure. The space of all scattering vectors is then the direct sum over all n-particle spaces. This shall be worked out in detail now. A remark on the notation is in order: The analysis for the outgoing case $t \to \infty$ is independent from that for the incoming case. Nonetheless, we will discuss both cases simultaneously throughout this section, and use superskripts "out,in" or $\pm$ to distinguish them.

2.2.1. Definition of the Hilbert Space $\mathcal{H}^{\mathrm{ex}}$ of Scattering Vectors. We consider a finite number of sectors corresponding to irreducible massive single particle representations of $\mathcal{A}_{\mathrm{u}}$, labelled by $\alpha = 1, \ldots , N$. Out of each sector we pick a localized endomorphism $\varrho_\alpha \in \Delta(S_0)$ and collect these (pairwise inequivalent) representations in the set

$$\Delta^{(1)} := \{\varrho_\alpha \, , \, \alpha = 1, \ldots , N\} \, . \tag{2.24}$$

Not to burden notation, we will replace 'ϱ_α' and '$\varrho_{\alpha_1} \cdots \varrho_{\alpha_n}$' by '$\alpha$' and '$\boldsymbol{\alpha}$,' respectively, whenever possible, e.g. we will write $(\alpha, \psi) \in \mathcal{H}_\alpha$, P_α, $\boldsymbol{\varepsilon}_{\boldsymbol{\alpha}}$ instead of $(\varrho_\alpha, \psi) \in \mathcal{H}_{\varrho_\alpha}$, P_{ϱ_α}, $\boldsymbol{\varepsilon}_{\varrho_{\alpha_1} \cdots \varrho_{\alpha_n}}$. In particular, the mass of the particle species described by ϱ_α will be denoted by m_α. We have seen that n-fold products $\pi_0 \varrho_{\alpha_1} \cdots \varrho_{\alpha_n}$ are representations of $\mathcal{A}_{\mathrm{u}}$ which contain n-particle scattering states

$$(\varrho_{\alpha_n}, \psi_n) \times \cdots \times (\varrho_{\alpha_1}, \psi_1)\,(\xi, \pm) \quad \subset \quad \mathcal{H}_{\varrho_{\alpha_1} \cdots \varrho_{\alpha_n}} \, .$$

In general, the representation $\pi_0 \varrho_{\alpha_1} \cdots \varrho_{\alpha_n}$ is reducible, and hence vectors of the above form correspond to mixed states on $\mathcal{A}_{\mathrm{u}}$ according to the interpretation (1.8). In order to describe pure n-particle states, we have to map such vectors onto a fibre $\mathcal{H}_\sigma$ over an irreducible representation σ, using an intertwiner $\mathbf{T}$ from $\varrho_{\alpha_1} \cdots \varrho_{\alpha_n}$ to σ. The resulting vector will be denoted by

$$T\psi_n \times \cdots \times \psi_1(\sigma, \boldsymbol{\alpha}, \xi, \pm) := (\sigma | T | \varrho_{\alpha_1} \cdots \varrho_{\alpha_n})\,(\varrho_{\alpha_n}, \psi_n) \times \cdots \times (\varrho_{\alpha_1}, \psi_1)\,(\xi, \pm) \, , \tag{2.25}$$

and corresponds to a pure state in the sector σ describing n particles of types $\alpha_1, \ldots , \alpha_n$. We define the space of all n particle vectors in $\mathcal{H}_\sigma$ as follows. Let $\mathbf{V} = V_1 \times \cdots \times V_n$ be the Cartesian product of mutually disjoint compact subsets V_i of the unit mass shell H_1. Then we define

$$\mathcal{H}^{(n)}_\sigma(\mathbf{V})^{\mathrm{out}} \quad \text{and} \quad \mathcal{H}^{(n)}_\sigma(\mathbf{V})^{\mathrm{in}}$$

as the closed subspaces of $\mathcal{H}_\sigma$ spanned by all outgoing or incoming, respectively, n-particle vectors with velocity support in $\mathbf{V}$ of the form (2.25), where $\boldsymbol{\alpha} \in \{1, \ldots , N\}^{\times n}$, $\mathbf{T}$ is an an intertwiner in $\mathrm{Int}(\sigma | \varrho_{\alpha_1} \cdots \varrho_{\alpha_n})$ such that $\pi_0(T) \in \mathcal{A}(S_0)$, $\xi \in X^{\pm}_{\mathbf{V}}$, and (α_i, ψ_i) is of the form $\mathbf{B}_i(f_i, t)\,\boldsymbol{\Omega}$ with $\mathrm{supp}\tilde{f} \cap \mathrm{spec}P_{\alpha_i} \subset V_i$ and $\mathbf{B}_i = (\alpha_i, B_i) \in F(\tilde{I}_i)$ for a suitable $\tilde{I}_i$ chosen such that $(\tilde{I}_1, \ldots , \tilde{I}_n)$ is in the class ξ.[4] The space of all outgoing or incoming, respectively, n-particle vectors in $\mathcal{H}_\sigma$, is then defined as

$$\mathcal{H}^{(n)\,\mathrm{out,in}}_\sigma := \left(\sum_{\mathbf{V}} \mathcal{H}^{(n)}_\sigma(\mathbf{V})^{\mathrm{out,in}} \right)^{-} \, , \text{ respectively.} \tag{2.26}$$

Here the sum and closure are understood within $\mathcal{H}_\sigma$. The sum of Hilbert spaces is meant in the algebraic sense, i.e. it consists of finite linear combinations $\psi = \sum \psi_{\mathbf{V}}$. It runs over all Cartesian products $\mathbf{V} = V_1 \times \cdots \times V_n \subset H_1^{\times n}$ of mutually disjoint compact subsets V_i of the unit mass shell. Note that the union of all these sets $\mathbf{V}$ is a covering of $H_1^{\times n} \setminus D_n$, where D_n denotes the set defined after equation (2.22). In order to describe the set of all scattering states on $\mathcal{A}_{\mathrm{u}}$ in terms of rays in a Hilbert space, we have to consider all particle numbers $n \geq 0$ and all sectors $[\sigma]$ which have n-particle states. These are exactly the sectors contained as a subrepresentation in $[\varrho_{\alpha_1} \cdots \varrho_{\alpha_n}]$ for some $n > 0$ and $\boldsymbol{\alpha} \in \{1, \ldots , N\}^{\times n}$. We only consider covariant sectors with finite statistics. Out of every such sector we pick a reference morphism localized in S_0. For sectors of single particle representations we make the same choice as in (2.24) for $\Delta^{(1)}$. The collection of these pairwise inequivalent irreducible representations will

[4] See Definition 2.1.

be denoted by Δ. It is countable [Reh90] since we started with a finite set $\Delta^{(1)}$. Vectors in different fibres $\mathcal{H}_\sigma, \mathcal{H}_{\sigma'}$ (with $\sigma \ncong \sigma'$) correspond to states carrying different charges and can therefore show no interference. Hence they must be described by orthogonal vectors. We shall have occasion to discuss on this issue in Section 4.1. Thus we define the Hilbert space of outgoing (incoming, respectively) scattering states as the direct sum over all relevant sectors $\sigma \in \Delta$ and all particle numbers $n \geq 0$ of the n-particle spaces in $\mathcal{H}_\sigma$:

$$\mathcal{H}^{\text{out,in}} := \bigoplus_{n\geq 0, \sigma\in\Delta} \mathcal{H}_\sigma^{(n)\text{out,in}} . \tag{2.27}$$

Here we understand $\mathcal{H}_\sigma^{(0)\text{out,in}} = \delta_{\sigma,\text{id}}\, \mathbb{C}\Omega$. $\mathcal{H}^{\text{out}}$ and $\mathcal{H}^{\text{in}}$ will be viewed as subsets of the (redundant) bundle $\hat{\mathcal{H}} = \Delta(S_0) \times \mathcal{H}_0$, and also as subspaces of the Hilbert space $\oplus_{\sigma\in\Delta}\mathcal{H}_0$. Continuing the earlier notation, the corresponding decomposition of elements in $\mathcal{H}^{\text{out,in}}$ will be written as $\psi = \sum_{\sigma\in\Delta} (\sigma, \psi_\sigma)$, with $\psi_\sigma \in \mathcal{H}_0$. The observable algebra is represented in $\mathcal{H}^{\text{out,in}}$ by the direct sum of representations $\pi_0\sigma$, and the Poincaré group as in $\hat{\mathcal{H}}$, c. f. equations (1.23) and (1.76):

$$U(\tilde{g})\,(\sigma, \psi) = (\sigma, U_\sigma(\tilde{g})\,\psi) \tag{2.28}$$

$$U(\tilde{j})\,(\sigma, \psi) = (\bar{\sigma}, U_0(j)\,\psi) . \tag{2.29}$$

2.2.2. Reference Hilbert Space, Møller Operators and S-Matrix. We first need to make a few remarks on the single particle spaces. We will assume the degeneracies g_α of the representations of the Poincaré group on the single particle spaces as $g_\alpha = 1$, c.f. equation (1.58). In other words, we assume that $\mathcal{H}_\alpha^{(1)}$ is irreducible and hence describes an elementary particle in the sense of Wigner.[5] Since we discuss scattering theory with particles of different masses m_α, it is advisible to go over from the momenta $p \in H_m$ to the velocities $q = p/m_\alpha \in H_1$. Hence we will replace $L^2(H_m, \mathrm{d}\mu)$ by $L^2(H_1, \mathrm{d}\mu_1)$, and each $\tilde{U}_{\varrho_\alpha}$ by the equivalent representation living on H_1 which we denote by $\tilde{U}_\alpha$. Explicitely, we then have

$$\begin{aligned} W_\alpha\, U(x, \tilde{g})\,|_{\mathcal{H}_\alpha^{(1)}} &= \tilde{U}_\alpha(x, \tilde{g})\, W_\alpha \quad \text{for all } (x, \tilde{g}) \in \tilde{P}_+^\uparrow \quad \text{and} \\ W_{\bar{\alpha}}\, U(\tilde{j})\,|_{\mathcal{H}_\alpha^{(1)}} &= \tilde{U}_\alpha(\tilde{j})\, W_\alpha , \end{aligned} \tag{2.30}$$

where $\tilde{U}_\alpha$ acts on $L^2(H_1, \mathrm{d}\mu_1)$ by

$$(\tilde{U}_\alpha(x, \tilde{g})\,\psi)(q) = e^{i m_\alpha x\cdot q + i s_\alpha \Omega(\tilde{g}; m_\alpha q)}\, \psi(\tilde{g}^{-1}\cdot q) \qquad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow \tag{2.31}$$

$$(\tilde{U}_\alpha(\tilde{j})\,\psi)(q) = c_\alpha\, \overline{\psi(-j\cdot q)} \quad . \tag{2.32}$$

Here c_α is a complex number of unit modulus, and $c_{\bar{\alpha}} = c_\alpha$. For a given compact subset V of H_1, we denote the closed subspace of vectors in $\mathcal{H}_\alpha$ with spectral support in $m_\alpha V$ by

$$\mathcal{H}_\alpha^{(1)}(V) := \{(\alpha, \psi)\, , \, \text{sp} P_\alpha \psi \subset m_\alpha V\} . \tag{2.33}$$

Note that this space is identified with $\{\alpha\} \times L^2(V, \mathrm{d}\mu_1)$, the closed span of functions with support in V, by the intertwiner W_α.

LEMMA 2.5. *Given any fixed* $\tilde{I} \in \tilde{\mathcal{K}}$, $\mathcal{H}_\alpha^{(1)}(V)$ *is spanned by vectors of the form* $\mathbf{B}(f, t)\,\mathbf{\Omega}$ *as in equation* (2.6), *where* $\text{supp}\tilde{f} \cap \text{spec}P_\alpha \subset V$ *and* $\mathbf{B} = (\alpha, B) \in \mathcal{F}(\tilde{I})$.

PROOF. Assume $\psi \in \mathcal{H}_\alpha^{(1)}(V)$ is orthogonal to all vectors of the above form. Since $\mathbf{B}(f, t)\,\mathbf{\Omega} = (\alpha, (2\pi)^{\frac{2}{3}} \tilde{f}(P_\alpha)\, B\,\Omega)$, this would imply that $g(P_\alpha)\,\psi$ is orthogonal to $B\,\Omega$ for all $g \in C_0^\infty(m_\alpha V)$ and $B \in \mathcal{F}(\tilde{I})$. A version of the Reeh-Schlieder theorem [SW64] adapted to the present setup implies that then $g(P_\alpha)\,\psi = 0$ for all $g \in C_0^\infty(m_\alpha V)$, hence $\text{sp}_{P_\alpha}\psi \cap m_\alpha V = \emptyset$. Since on the other hand the vector ψ is in $\mathcal{H}_\alpha^{(1)}(V)$ by assumption, it must be zero. □

We come to the case $n > 1$. Given $\sigma \in \Delta$ and an n-tupel $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ of charge labels $\alpha_i \in \{1, \ldots, N\}$, the space of local intertwiners $\text{Int}(\pi_0\sigma|\pi_0\varrho_{\alpha_1}\cdots\varrho_{\alpha_n})(S_0)$ has the structure of a Hilbert space, the scalar product being given by

$$\langle T, S\rangle\,\mathbf{1} = \phi_{\alpha_n}\cdots\phi_{\alpha_1}\big(\pi_0^{-1}(T^*S)\big)\, , \text{ for } T, S \in \text{Int}(\pi_0\sigma|\pi_0\varrho_{\alpha_1}\cdots\varrho_{\alpha_n})(S_0) . \tag{2.34}$$

[5]Otherwise, the subsequent formulas would get more complicated without changing the essence of the analysis.

Note that $\phi_{\alpha_n} \cdots \phi_{\alpha_1}$ is a left inverse of $\varrho_{\alpha_1} \cdots \varrho_{\alpha_n}$, so that the right hand side is indeed a multiple of unity as we have remarked after Definition 1.9. Let $F_\sigma^{(n)}$ be the Hilbert space consisting of the direct sum of all these intertwiner spaces:

$$F_\sigma^{(n)} := \bigoplus_{\boldsymbol{\alpha} \in \{1,\dots,N\}^{\times n}} \mathrm{Int}(\pi_0\sigma | \pi_0 \varrho_{\alpha_1} \cdots \varrho_{\alpha_n})(S_0) . \tag{2.35}$$

It is known to be finite dimensional [Reh90]. We denote its elements by $(\sigma|T|\boldsymbol{\alpha})$. The groupoid homomorphism $\varepsilon_{\boldsymbol{\alpha}} = \varepsilon_{(\varrho_{\alpha_1},\dots,\varrho_{\alpha_n})}$ defined in Lemma and Definition 1.6 induces a unitary representation of B_n in $F_\sigma^{(n)}$, which we denote by ε :

$$\begin{aligned} \varepsilon(b)\,(\sigma|T|\boldsymbol{\alpha}) &:= \big(\sigma|\,T\,\pi_0\varepsilon_{\boldsymbol{\alpha}}(b^{-1},\pi)\,|\boldsymbol{\alpha}\cdot\pi^{-1})\big) , \\ &\equiv \big(\sigma|\,T\,\pi_0\varepsilon_{\boldsymbol{\alpha}\cdot\pi^{-1}}(b^{-1},1)\,|\boldsymbol{\alpha}\cdot\pi^{-1})\big) , \end{aligned} \tag{2.36}$$

where π is the permutation associated to b, i.e. $\pi = \nu(b)$. To verify that ε is indeed a representation, one needs equation (1.44). Note that $\pi_0\varepsilon(b,\pi)$ is in $\mathcal{A}(S_0)$, and that ε in fact represents B_n, since the additional generator c_1 of $B_n(C)$ is in its kernel. Now we can formulate the 'local' structure of the n-particle spaces, which is an immediate consequence of Theorem 2.4 and Lemma 2.5:

COROLLARY 2.6. *Let $\mathbf{V} = V_1 \times \cdots \times V_n$ be the Cartesian product of mutually disjoint compact subsets V_i of the unit mass shell H_1, and let $\xi \in X_{\mathbf{V}}^+$ or $X_{\mathbf{V}}^-$. Then there is an isometric isomorphism $i_{\mathbf{V},\xi}^+$ or $i_{\mathbf{V},\xi}^-$, respectively, with*

$$i_{\mathbf{V},\xi}^{\pm} : \mathcal{H}_\sigma^{(n)}(\mathbf{V})^{\mathrm{out,in}} \;\rightarrow\; L^2(\mathbf{V}, \mathrm{d}\tilde{\mu}; F_\sigma^{(n)}) , \tag{2.37}$$

$$T\psi_n \times \cdots \times \psi_1(\sigma,\boldsymbol{\alpha},\xi,\pm) \;\mapsto\; W_{\alpha_1}\psi_1 \otimes \cdots \otimes W_{\alpha_n}\psi_n \otimes (\sigma|\pi_0(T)|\boldsymbol{\alpha}) , \tag{2.38}$$

if $\psi_k \in \mathcal{H}_{\alpha_k}^{(1)}(V_k)$. Here, $\mathrm{d}\tilde{\mu}$ denotes the (Lorentz invariant) product measure $\mathrm{d}\mu_1^{\times n}$ on $H_1^{\times n}$.

REMARK. n-particle vectors $T\psi_n \times \cdots \times \psi_1(\sigma,\boldsymbol{\alpha},\xi,\pm)$ have so far been defined only if the single particle vectors are of the form $(\alpha_i,\psi_i) = \mathbf{B}_i(f_i,t)\boldsymbol{\Omega}$, see equation (2.25), but due to the Corollary, they can now be defined for all $\psi_i \in \mathcal{H}_{\alpha_i}^{(1)}(V_i)$ via $(i_{\mathbf{V},\xi}^{\pm})^{-1}$.

We now analyze the 'global' structure, i.e. to the structure of $\mathcal{H}_\sigma^{(n)\,\mathrm{out,in}}$. If no confusion can arise, we will frequently omit the sub- and superscripts σ and (n) of the various Hilbert spaces. If the outgoing and incoming cases don't have to be treated differently, we will also frequently suppress the notation of 'out, in' and '$\pm$', or replace them by 'ex'. To analyze the structure of $\mathcal{H}_\sigma^{(n)\,\mathrm{ex}}$, we first observe that $\mathcal{H}(\mathbf{V}\cdot\pi)$ coincides with $\mathcal{H}(\mathbf{V})$ for all $\pi \in S_n$ due to equation (2.9). This is the case even though $\mathbf{V} \cap \mathbf{V}\cdot\pi = \emptyset$ for $\pi \neq 1$, the sets V_k being mutually disjoint. Thus it is more natural to label $\mathcal{H}(\mathbf{V})$ not by the region $\mathbf{V}$, but by the (disjoint) union of all $\mathbf{V}\cdot\pi$ with $\pi \in S_n$, which in turn will be identified with the subset $\mathbf{V}/S_n := \{\boldsymbol{q}\cdot S_n \mid \boldsymbol{q} \in \mathbf{V}\}$ of ${}^nH_1$. Hence we define

$$\mathcal{H}_\sigma^{(n)}(\mathbf{V}/S_n)^{\mathrm{ex}} := \mathcal{H}_\sigma^{(n)}(\mathbf{V})^{\mathrm{ex}} . \tag{2.39}$$

Note that the sum in equation (2.26) runs effectively over the subsets $\mathbf{V}/S_n$ of ${}^nH_1$, where $\mathbf{V}$ is of the admitted type, and that the union of these sets exhausts ${}^nH_1$.

On the other hand, we can also identify $L^2(\mathbf{V}, \mathrm{d}\tilde{\mu}; F)$ canonically with $L^2(\mathbf{V}/S_n, \mathrm{d}\mu; F)$, where $\mathrm{d}\mu$ denotes the canonical measure on ${}^nH_1$ inherited from $\mathrm{d}\mu_1^{\times n}$, i.e., $\mu(\mathbf{V}/S_n) = \tilde{\mu}(\mathbf{V})$. To this end, we exploit the fact that $\mathbf{V} \cap \mathbf{V}\cdot\pi = \emptyset$ for $\pi \neq 1$ to identify $L^2(\mathbf{V}; F_\sigma^{(n)})$[6] with the space of permutation invariant functions on the disjoint union $\dot{\cup}_{\pi\in S_n}\mathbf{V}\cdot\pi$, that is the space $P_+\,L^2(\dot{\cup}_{\pi\in S_n}\mathbf{V}\cdot\pi\,;\,F_\sigma^{(n)})$, where

$$\begin{aligned} P_+(\tilde{\psi}_1 \otimes \cdots \otimes \tilde{\psi}_n \otimes \mathbf{T}) &:= (\tilde{\psi}_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \tilde{\psi}_n) \otimes \mathbf{T} \\ &:= (n!)^{-1} \sum_{\pi\in S_n} \tilde{\psi}_{\pi(1)} \otimes \cdots \otimes \tilde{\psi}_{\pi(n)} \otimes \mathbf{T} . \end{aligned} \tag{2.40}$$

[6] In the sequel, we suppress the notation of the measures if no confusion can arise.

This space will in turn be viewed as $L^2(\mathbf{V}/S_n; F_\sigma^{(n)})$. After this identification, the maps $i_{\mathbf{V},\xi}$ of Corollary 2.6 are isometries from $\mathcal{H}(\mathbf{V}/S_n)$ onto $L^2(\mathbf{V}/S_n; F_\sigma^{(n)})$, and read

$$i_{\mathbf{V},\xi}\, T\psi_n \times \cdots \times \psi_1(\sigma, \boldsymbol{\alpha}, \xi) = \tilde{\psi}_1 \otimes_s \cdots \otimes_s \tilde{\psi}_n \otimes (\sigma|\pi_0(T)|\boldsymbol{\alpha}) \ , \text{ if } \boldsymbol{\psi}_k \in \mathcal{H}^{(1)}_{\alpha_k}(V_k)\,, \qquad (2.41)$$

where we have written $\tilde{\psi}_k := W_{\alpha_k}\psi_k$. Note that this prescription implies that the product $\psi_{\pi(n)} \times \cdots \times \psi_{\pi(1)}$ has first to be permuted into the unique order such that $\boldsymbol{\psi}_k \in \mathcal{H}^{(1)}_{\alpha_k}(V_k)$. Hence for the prescription (2.41) to be well-defined, it is essential that the isometries $i_{\mathbf{V},\xi}$ are still labelled by $\mathbf{V}$, not by $\mathbf{V}/S_n$. In order to reveal the structure of $\mathcal{H}_\sigma^{(n)\,\mathrm{ex}}$, one has to compute how the different isometries on the intersections of 'local' scattering spaces are related: Given $\xi \in X_\mathbf{V}$ and $\xi' \in X_{\mathbf{V}'}$, we compare $i_{\mathbf{V},\xi}$ and $i_{\mathbf{V}',\xi'}$ on $\mathcal{H}(\mathbf{V}/S_n) \cap \mathcal{H}(\mathbf{V}'/S_n)$. This will involve the maps $b^{\pm}_{\mathbf{V}}(\xi,\xi')$ with values in $B_n/_{\mathcal{N}}$ which have been defined in Lemma 2.2. Obviously, Lemma 2.2 still holds if we replace $\mathcal{N}$ by the normal subgroup of B_n defined by

$$\mathcal{N}_\varepsilon := \ker\varepsilon \cap PB_n \ . \qquad (2.42)$$

LEMMA 2.7. *Let $\mathbf{V}$ and $\mathbf{V}'$ be the Cartesian products of mutually disjoint compact subsets V_i and V_i', respectively, of the unit mass shell H_1 such that $\mathbf{V}/S_n \cap \mathbf{V}'/S_n$ is nonempty, and let ξ and ξ' be in $X^{\pm}_{\mathbf{V}}$ and $X^{\pm}_{\mathbf{V}'}$, respectively. Then for all $\tilde{\psi} \in L^2(\mathbf{V}/S_n \cap \mathbf{V}'/S_n; F_\sigma^{(n)})$*

$$\left(i^{\pm}_{\mathbf{V}',\xi'}\,(i^{\pm}_{\mathbf{V},\xi})^{-1}\,\tilde{\psi}\right)(\boldsymbol{q}) = \varepsilon\left(\tilde{g}^{\pm}_{\mathbf{V}',\xi';\mathbf{V},\xi}(\boldsymbol{q})\right)\cdot\tilde{\psi}(\boldsymbol{q})\ . \qquad (2.43)$$

Here $\tilde{g}^{\pm}_{\mathbf{V}',\xi';\mathbf{V},\xi}$ is the locally constant $B_n(C)/_{\mathcal{N}_\varepsilon}$-valued function on $\mathbf{V}/S_n \cap \mathbf{V}'/S_n$ defined as follows. $\mathbf{V}/S_n \cap \mathbf{V}'/S_n$ is the disjoint union of the sets $(\mathbf{V}\cdot\pi \cap \mathbf{V}')/S_n, \pi \in S_n$, and on each such set $\tilde{g}^{\pm}_{\mathbf{V}',\xi';\mathbf{V},\xi}$ takes the constant value

$$\tilde{g}^{\pm}_{\mathbf{V}',\xi';\mathbf{V},\xi}(\boldsymbol{q}) = b_{\mathbf{V}'\cap\mathbf{V}\pi}(\xi',\xi\cdot\pi)\ \left(\varphi_\xi(\pi^{-1})\cdot\mathcal{N}_\varepsilon\right)\ \textit{for all } \boldsymbol{q} \in (\mathbf{V}\cdot\pi\cap\mathbf{V}')/S_n\ . \qquad (2.44)$$

PROOF. Let $\tilde{\psi} = \tilde{\psi}_1 \otimes_s \cdots \otimes_s \tilde{\psi}_n \otimes (\sigma|\pi_0(T)|\boldsymbol{\alpha})$, where $\tilde{\psi}_k \in L^2(V_k \cap V'_{\pi^{-1}(k)}; F)$, and let $\psi_k := W_{\alpha_k}^{-1}\tilde{\psi}_k$. Using equations (2.9) and (2.16), we get

$$\begin{aligned}
i_{\mathbf{V}',\xi'}\, i^{-1}_{\mathbf{V},\xi}\,\tilde{\psi} &= i_{\mathbf{V}',\xi'}\, T\psi_n \times \cdots \times \psi_1(\sigma,\boldsymbol{\alpha},\xi) \\
&= i_{\mathbf{V}',\xi'}\, T\,\varepsilon_{\boldsymbol{\alpha}}\left(\varphi_\xi(\pi,\pi^{-1})\right)\,\varepsilon_{\boldsymbol{\alpha}\pi}\left(b_{\mathbf{V}\pi\cap\mathbf{V}'}(\xi\cdot\pi,\xi')\right)\,\psi_{\pi(n)}\times\cdots\times\psi_{\pi(1)}(\sigma,\boldsymbol{\alpha}\cdot\pi,\xi') \\
&= \tilde{\psi}_{\pi(1)}\otimes_s\cdots\otimes_s\tilde{\psi}_{\pi(n)}\left(\sigma\right|T\,\pi_0\varepsilon_{\boldsymbol{\alpha}}\left(\varphi_\xi(\pi,\pi^{-1})\right)\,\varepsilon_{\boldsymbol{\alpha}\pi}\left(b_{\mathbf{V}\pi\cap\mathbf{V}'}(\xi\cdot\pi,\xi')\right)\,\left|\boldsymbol{\alpha}\cdot\pi\right) \\
&= \mathbb{1}_{L^2}\otimes\varepsilon\left(b_{\mathbf{V}'\cap\mathbf{V}\pi}(\xi',\xi\cdot\pi)\,\varphi_\xi(\pi^{-1})\right)\ \tilde{\psi}\ .
\end{aligned}$$

In the last equation we have used

$$\varphi_\xi(\pi,\pi^{-1}) = (\varphi_{\xi\pi}(\pi),\pi^{-1}) = (\varphi_\xi(\pi^{-1})^{-1},\pi^{-1})$$

and $b(\xi\cdot\pi,\xi') = b(\xi',\xi\cdot\pi)^{-1}$. By linearity this equation extends to $L^2(\mathbf{V}\cdot\pi\cap\mathbf{V}')/S_n\,; F_\sigma^{(n)})$. $\mathbf{V}/S_n\cap\mathbf{V}'/S_n$ is easily seen to be the disjoint union of the sets $(\mathbf{V}\cdot\pi\cap\mathbf{V}')/S_n, \pi\in S_n$. Hence the claim follows by decomposing an arbitrary function in $L^2(\mathbf{V}/S_n\cap\mathbf{V}'/S_n\,; F_\sigma^{(n)})$ into the sum of $n!$ functions with the disjoint supports $(\mathbf{V}\cdot\pi\cap\mathbf{V}')/S_n, \pi\in S_n$. □

Now we claim that the maps $\tilde{g}^{\pm}_{\mathbf{V}',\xi';\mathbf{V},\xi}$ are the transition functions of a sub-bundle of ${}^n\widetilde{H}_1$ in a certain local trivialization. We identify the universal covering manifold manifold ${}^n\widetilde{H}_1$ of ${}^nH_1$ with the set of homotopy classes $\tilde{\boldsymbol{q}} = [\gamma]$ of paths γ starting from a base point $\boldsymbol{q}_0 \in {}^nH_1$. We choose the base point

$$\boldsymbol{q}_0 := \boldsymbol{q}_0'\cdot S_n \quad \text{with} \qquad (2.45)$$

$$\boldsymbol{q}_0' := \left((\sqrt{1+1},1,0),\ldots,(\sqrt{n^2+1},n,0)\right)\ . \qquad (2.46)$$

Recall that B_n is the fundamental group of $({}^n\mathbb{R}^2,\boldsymbol{x}_0)$ with the base point $\boldsymbol{x}_0 = \left((1,0),\ldots,(n,0)\right)\cdot S_n$. Hence the projection $(\sqrt{q_1^2+q_2^2+1},q_1,q_2)\mapsto(q_1,q_2)$ fixes a diffeomorphism of the respective universal covering manifolds, and an identification the fundamental groups:

$$\pi_1({}^nH_1,\boldsymbol{q}_0)\equiv B_n\ . \qquad (2.47)$$

Now ${}^n\widetilde{H}_1$ is a principal fibre bundle over the base space ${}^nH_1$ with structure group B_n, the right action of B_n being given by

$$\tilde{\boldsymbol{q}}\cdot b = [\gamma * \beta] \quad \text{if } \tilde{\boldsymbol{q}} = [\gamma]\,,\; b = [\beta]\,.$$

Here, $\gamma * \beta$ denotes the path which runs first through β and then through γ. Consequently,

$$ {}^n\widetilde{H}_1^\varepsilon := {}^n\widetilde{H}_1/\mathcal{N}_\varepsilon \tag{2.48}$$

is a principal fibre bundle over ${}^nH_1$ with structure group $B_n/\mathcal{N}_\varepsilon$, a so-called reduction of the bundle ${}^n\widetilde{H}_1$ [KN63]. It may be viewed as the set of equivalence classes, denoted $[\gamma]_\varepsilon$, of paths γ with respect to the equivalence relation

$$\gamma \sim_\varepsilon \gamma' \quad \text{iff} \quad [\gamma^{-1} * \gamma'] \in \mathcal{N}_\varepsilon\,.$$

For $M \subset {}^nH_1$, we denote by ${}^n\widetilde{H}_1^\varepsilon|_M$ the restriction of the bundle ${}^n\widetilde{H}_1^\varepsilon$ to M, i.e. the set of equivalence classes $[\gamma]_\varepsilon$ of paths ending in M. If no confusion can arise, we will denote points in ${}^n\widetilde{H}_1^\varepsilon$ also by $\tilde{\boldsymbol{q}}$.

LEMMA 2.8. *${}^n\widetilde{H}_1^\varepsilon$, considered as a principal fibre bundle, has a local trivialization whose transition functions are given by $\tilde{g}^+_{\mathbf{V}',\xi';\mathbf{V},\xi}$. More precisely, for every Cartesian product $\mathbf{V}$ of mutually disjoint compact subsets of H_1 and every $\xi \in X^+_{\mathbf{V}}$, there is a locally constant map*

$$\tilde{\varphi}^+_{\mathbf{V},\xi} : {}^n\widetilde{H}_1^\varepsilon|_{\mathbf{V}/S_n} \quad \to \quad B_n/\mathcal{N}_\varepsilon\,, \tag{2.49}$$

whose restriction to every fibre ${}^n\widetilde{H}_1^\varepsilon|_{\{\boldsymbol{q}\}}$ is bijective, and which satisfies

$$\tilde{\varphi}^+_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}\cdot b) = \tilde{\varphi}^+_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\cdot b \quad \text{for all } b \in B_n/\mathcal{N}_\varepsilon\,,\; \tilde{\boldsymbol{q}} \in {}^n\widetilde{H}_1^\varepsilon|_{\mathbf{V}/S_n}\,. \tag{2.50}$$

If $\mathbf{V}'$ is a second subset of $H_1^{\times n}$ of the above kind and $\xi' \in X^+_{\mathbf{V}'}$, then the two corresponding maps are related by

$$\tilde{\varphi}^+_{\mathbf{V}',\xi'}(\tilde{\boldsymbol{q}})\, \tilde{\varphi}^+_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})^{-1} = \tilde{g}^+_{\mathbf{V}',\xi';\mathbf{V},\xi}(\boldsymbol{q}) \quad \text{for all } \boldsymbol{q} \in \mathbf{V}/S_n \cap \mathbf{V}'/S_n\,. \tag{2.51}$$

The same statement holds with all '+' signs replaced by '-' signs.

Note that the maps $\tilde{\varphi}^\pm_{\mathbf{V},\xi}$ define local trivializations ${}^n\widetilde{H}_1^\varepsilon|_{\mathbf{V}/S_n} \cong \mathbf{V}/S_n \times B_n/\mathcal{N}_\varepsilon$ by $\tilde{\boldsymbol{q}} \mapsto (\boldsymbol{q}, \tilde{\varphi}^\pm_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}))$.

PROOF. Sticking to our notational convention, we suppress the superscripts $\pm$ frequently. We first check a necessary condition which the functions $\tilde{g}_{\mathbf{V}',\xi';\mathbf{V},\xi}$ have to satisfy in order to be the transition functions in any local trivialization, namely the cocycle conditions

$$\begin{aligned} \tilde{g}_{\mathbf{V},\xi;\mathbf{V}',\xi'}(\boldsymbol{q})\, \tilde{g}_{\mathbf{V}',\xi';\mathbf{V},\xi}(\boldsymbol{q}) &= 1\cdot\mathcal{N}_\varepsilon \;\text{ for all } \boldsymbol{q} \in \mathbf{V}/_{S_n} \cap \mathbf{V}'/_{S_n} \quad \text{and} \\ \tilde{g}_{\mathbf{V}'',\xi'';\mathbf{V}',\xi'}(\boldsymbol{q})\, \tilde{g}_{\mathbf{V}',\xi';\mathbf{V},\xi}(\boldsymbol{q}) &= \tilde{g}_{\mathbf{V}'',\xi'';\mathbf{V},\xi}(\boldsymbol{q}) \;\text{ for all } \boldsymbol{q} \in \mathbf{V}/_{S_n} \cap \mathbf{V}'/_{S_n} \cap \mathbf{V}''/_{S_n}\,. \end{aligned} \tag{2.52}$$

These equations follow from the cocycle properties of the maps $b_{\mathbf{V}}(\xi,\xi')$, see Lemma 2.2, and the homomorphism property of $\varphi_\xi(\pi,\pi')$. The maps $\tilde{\varphi}^\pm_{\mathbf{V},\xi}$ to be constructed must be constant on every connected component of ${}^n\widetilde{H}_1^\varepsilon|_{\mathbf{V}/S_n}$. Now two points $[\gamma]_\varepsilon$ and $[\gamma']_\varepsilon$ are in the same connected component of ${}^n\widetilde{H}_1^\varepsilon|_{\mathbf{V}/S_n}$ if there is a path α in $\mathbf{V}/S_n$ such that $[\gamma^{-1} * \alpha * \gamma'] \in \mathcal{N}_\varepsilon$. Hence the above requirement implies that for for all $[\gamma]_\varepsilon \in {}^n\widetilde{H}_1^\varepsilon$ with $\gamma(1) \in \mathbf{V}/S_n$,

$$\tilde{\varphi}_{\mathbf{V},\xi}([\alpha * \gamma]_\varepsilon) = \tilde{\varphi}_{\mathbf{V},\xi}([\gamma]_\varepsilon) \quad \text{if } \alpha \text{ is a path in } \mathbf{V}/S_n\,. \tag{2.53}$$

Let $\boldsymbol{q}_0'$ be the base point of $H_1^{\times n} \setminus D_n$ defined in equation (2.46). We fix a neighbourhood $\mathbf{V}_0$ of $\boldsymbol{q}_0'$ and choose $\xi_0^\pm \in X^\pm_{\mathbf{V}_0}$ which will be specified later, and define

$$\tilde{\varphi}_{\mathbf{V}_0,\xi_0}([\alpha]_\varepsilon) := 1\cdot\mathcal{N}_\varepsilon\,, \quad \text{if } \alpha \text{ is a path in } \mathbf{V}_0/S_n\,. \tag{2.54}$$

Now we will see, how the conditions (2.51) and (2.53) *determine* $\tilde{\varphi}_{\mathbf{V},\xi}$ up to an overall constant which is fixed by equation (2.54). Given $\tilde{\boldsymbol{q}}$, we pick a path γ representing $\tilde{\boldsymbol{q}}$ and choose $m+1$ sets $\mathbf{V}_{(\nu)}, \nu = 0,\dots,m$ such that the sets $\mathbf{V}_{(\nu)}/S_n$ cover γ, with $\mathbf{V}_{(0)} = \mathbf{V}_0$. Let further $\xi_{(\nu)} = \xi^\pm_{(\nu)} \in X^\pm_{\mathbf{V}_{(\nu)}}, \nu = 1,\dots,m$, and $\xi_{(0)} := \xi_0^\pm$. Now γ can be written $\gamma = \gamma_m * \cdots * \gamma_0$, where γ_ν is a path in $\mathbf{V}_{(\nu)}/S_n$. Iterated application of equations (2.51) and (2.53) yields, together with equation (2.54),

$$\tilde{\varphi}_{\mathbf{V}_{(m)},\xi_{(m)}}(\tilde{\boldsymbol{q}}) = \tilde{g}_{\mathbf{V}_{(m)},\xi_{(m)};\mathbf{V}_{(m-1)},\xi_{(m-1)}}(\gamma_{m-1}(1)) \cdots \tilde{g}_{\mathbf{V}_{(1)},\xi_{(1)};\mathbf{V}_{(0)},\xi_{(0)}}(\gamma_0(1))\,. \tag{2.55}$$

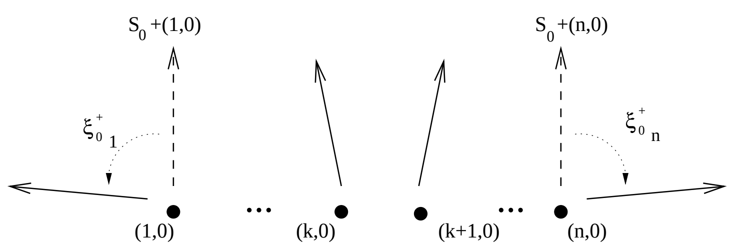
Now we *define* $\tilde{\varphi}_{\mathbf{V}_{(m)},\xi_{(m)}}$ by this expression. It is well-defined due to the cocycle conditions (2.52), and independent of the path representing $\tilde{\boldsymbol{q}}$. It is easily seen to be locally constant and hence smooth, and by construction any two such maps satisfy equation (2.51). From the definition follows immediately equation (2.51) and that

$$\tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}\cdot b) = \tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\, \tilde{\varphi}_{\mathbf{V}_0,\xi_0}(b) \quad \text{for all } b \in B_n/\mathcal{N}_\varepsilon\ . \tag{2.56}$$

Now we consider the restriction of $\tilde{\varphi}_{\mathbf{V},\xi}$ to the fibre in ${}^n\widetilde{H}_1^\varepsilon$ over $\boldsymbol{q} \in \mathbf{V}/S_n$. This fibre consists of the set $\tilde{\boldsymbol{q}}\cdot b$ for some fixed $\tilde{\boldsymbol{q}}$, where b runs through $B_n/\mathcal{N}_\varepsilon$. Hence, the last equation shows that the restriction of $\tilde{\varphi}_{\mathbf{V},\xi}$ is a bijective map onto $B_n/\mathcal{N}_\varepsilon$ if and only if the map

$$i^\pm : B_n/\mathcal{N}_\varepsilon \ \to\ B_n/\mathcal{N}_\varepsilon\ , \tag{2.57}$$
$$b \ \mapsto\ \tilde{\varphi}^\pm_{\mathbf{V}_0,\xi_0^\pm}(b) \tag{2.58}$$

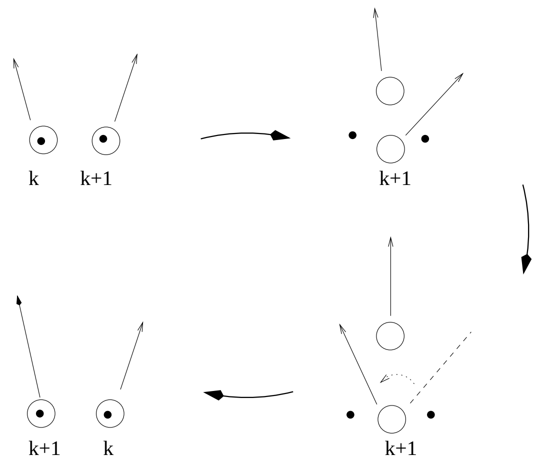
is one to one. If this is the case, $i^\pm$ is also a group homomorphism due to equation (2.56), and then the claimed equation (2.50) also holds, with b replaced by $i^\pm(b)$ on its right hand side. Note that i^+ and i^- depend (only) on the choices of ξ_0^+ and ξ_0^-. In order to simplify the formulae, we fix a suitable choice for $\xi_0^+ \in X^+_{\mathbf{V}_0}$ in the following. Namely, we choose ξ_0^+ such that $(\xi_0^+)_n < \ldots < (\xi_0^+)_1 < (0,2\pi)\cdot(\xi_0^+)_n$, which is equivalent to the condition that $N\left((\xi_0^+)_i, (\xi_0^+)_j\right) = 0$ if $i < j$, Thus ξ_0^+ is fixed up to a common rotation about multiples of 2π of all $(\xi_0^+)_k$. Projected onto the plane as after equation (2.46), the situation looks as follows.



Here we have idealized the spacelike cones by rays. It remains to prove that $i^\pm$ is bijective. This will be achieved by direct calculation. To this end, we first evaluate equation (2.55) further. Let $\tilde{\boldsymbol{q}} = [\gamma]_\varepsilon$ and let $\mathbf{V}$ be a neighbourhood such that $\boldsymbol{q} \in \mathbf{V}/S_n$. Let further $\bar{\gamma}$ be the lift of γ to $H_1 \setminus D_n$ through $\boldsymbol{q}_0'$. We choose $\mathbf{V}_{(\nu)}$ to be a covering of $\bar{\gamma}, \nu = 0, \ldots, m-1$, such that $\mathbf{V}_{(m-1)}/S_n = \mathbf{V}/S_n$. Then $\mathbf{V}_{(m-1)} = \mathbf{V}\cdot\pi$ for some unique permutation π, and we can calculate $\tilde{\varphi}_{\mathbf{V},\xi}$ using equation (2.44):

$$\tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}) = \left(\varphi_{\xi\pi}(\pi)\cdot\mathcal{N}_\varepsilon\right) b_{\mathbf{V}\pi\cap\mathbf{V}_{(m-2)}}(\xi_{(m-1)}, \xi_{(m-2)}) \cdots b_{\mathbf{V}_{(1)}\cap\mathbf{V}_{(0)}}(\xi_{(1)}, \xi_{(0)})\ . \tag{2.59}$$

Now we consider the particular case that $\tilde{\boldsymbol{q}}$ is the $\mathcal{N}_\varepsilon$-orbit of one of the generators $t_k = [\beta_k]$ of $\pi_1({}^nH_1, \boldsymbol{q}_0)$ and that $(\mathbf{V},\xi) = (\mathbf{V}_0, \xi_0^+)$, and compute $i^+([\beta_k]_\varepsilon)$. The lift of the representant β_k of t_k to $H_1^{\times n} \setminus D_n$ is pictured in the figure on page 78. We cover this path by a sequence $\mathbf{V}_{(0)} = \mathbf{V}_0, \ldots, \mathbf{V}_{(3)}$ such that $\mathbf{V}_0 \subset \mathbf{V}_{(1)} \supset \mathbf{V}_{(2)} \subset \mathbf{V}_{(3)} \supset \mathbf{V}_0$, and choose appropriate $\xi_{(m)} \in X^+_{\mathbf{V}_m}$, such that the situation looks as follows (we have pictured only every second step, in particular only the smaller regions $\mathbf{V}_{(0)}, \mathbf{V}_{(2)}, \mathbf{V}_{(0)}$):



One computes for the second of the above steps $b^+_{\mathbf{V}_{(2)}}(\xi_{(2)}, \xi_{(3)}) = t_k^2 \cdot \mathcal{N}_\varepsilon$, while all other steps yield trivial b's. Further, one has $\varphi_{\xi_0^+ \cdot \tau_k}(\tau_k) = c_{k+1}^{-1} t_k c_k$, and hence equation (2.59) yields

$$\begin{aligned} i^+([\beta_k]_\varepsilon) &= \left(\varphi_{\xi_0^+ \cdot \tau_k}(\tau_k) \cdot \mathcal{N}_\varepsilon\right) b^+_{\mathbf{V}_{(2)}}(\xi_{(2)}, \xi_{(3)}) = c_{k+1}^{-1} t_k c_k \, t_k^2 \cdot \mathcal{N}_\varepsilon \\ &= c_k t_k c_{k+1}^{-1} \, t_k^2 \cdot \mathcal{N}_\varepsilon \\ &= t_k \cdot \mathcal{N}_\varepsilon \quad , \text{ i.e.} \end{aligned}$$

for our choice of ξ_0^+, i^+ is the identity on $B_n/_{\mathcal{N}_\varepsilon}$. To compute i^-, we recall Corollary 2.3, which immediately implies that

$$\tilde{g}^-_{\mathbf{V}',\theta\xi';\mathbf{V},\theta\xi}(\boldsymbol{q}) = \tilde{g}^+_{\mathbf{V}',\xi';\mathbf{V},\xi}(\boldsymbol{q}) \quad \text{for all } \xi \in X^+_{\mathbf{V}}, \xi' \in X^+_{\mathbf{V}'} \ . \tag{2.60}$$

Hence, we fix the choice $\xi_0^- = \theta \cdot \xi_0^+$, and we choose $\xi^-_{(\nu)} := \theta \cdot \xi^+_{(\nu)}$ in the definition (2.55) of $\tilde{\varphi}^-_{\mathbf{V}_{(m)}, \xi^-_{(m)}}$. Then we have

$$\tilde{\varphi}^-_{\mathbf{V},\theta\xi} = \tilde{\varphi}^+_{\mathbf{V},\xi} \quad \text{for all } \xi \in X^+_{\mathbf{V}} \ . \tag{2.61}$$

In particular, we then have $i^- = i^+$, and both automorphisms are the identity on $B_n/_{\mathcal{N}_\varepsilon}$ as demanded in equation (2.50), or more precisely they are just the identifications of $\pi_1({}^n H_1, \boldsymbol{q}_0)/_{\mathcal{N}_\varepsilon}$ with $B_n/_{\mathcal{N}_\varepsilon}$ of equation (2.47). □

Equation (2.43) implies that $\mathcal{H}_\sigma^{(n)\,\mathrm{ex}}$ is canonically isomorphic to the Hilbert space of square integrable sections in the vector bundle which is associated via ε to the principle fibre bundle characterized by the transition functions $\tilde{g}^\pm_{\mathbf{V}',\xi';\mathbf{V},\xi}$, i.e. in view of Lemma 2.8, to ${}^n\widetilde{H}_1^\varepsilon$:

$${}^n\widetilde{H}_1^\varepsilon \times_\varepsilon F_\sigma^{(n)} \equiv {}^n\widetilde{H}_1 \times_\varepsilon F_\sigma^{(n)} \ .$$

This Hilbert space in turn is in a natural way isomorphic to the space of equivariant functions on ${}^n\widetilde{H}_1$, which we denote by $L^2_\varepsilon({}^n\widetilde{H}_1, F_\sigma^{(n)})$. The proof of these remarks is standard in fibre bundle theory, see e.g [KN63]; but instead we will directly define the resulting maps W^{ex} : $\mathcal{H}_\sigma^{(n)\,\mathrm{ex}} \to L^2_\varepsilon({}^n\widetilde{H}_1, F_\sigma^{(n)})$ (i.e. the Møller operators) and prove them to be unitaries. First we define the reference Hilbert space $L^2_\varepsilon({}^n\widetilde{H}_1, F_\sigma^{(n)})$. We recall that a function $\tilde{\psi}$ on ${}^n\widetilde{H}_1$ into $F_\sigma^{(n)}$ is called *equivariant* with respect to ε, if it satisfies

$$\tilde{\psi}(\tilde{\boldsymbol{q}}) = \varepsilon(b)\, \tilde{\psi}(\tilde{\boldsymbol{q}} \cdot b) \quad \text{for all } b \in \pi_1({}^n H_1, \boldsymbol{q}_0) \ . \tag{2.62}$$

By definition, $L^2_\varepsilon({}^n\widetilde{H}_1 \, , \, F_\sigma^{(n)})$ is the Hilbert space completion of the functions on ${}^n\widetilde{H}_1$ into $F_\sigma^{(n)}$ which are equivariant with respect to ε and have finite norm with respect to the scalar product

$$\left\langle \tilde{\psi}, \tilde{\psi}' \right\rangle := \int_{{}^n H_1} \left\langle \tilde{\psi}(\tilde{\boldsymbol{q}}), \tilde{\psi}'(\tilde{\boldsymbol{q}}) \right\rangle_F \, \mathrm{d}\mu(\boldsymbol{q}) \ , \tag{2.63}$$

where $<\ ,\ >_F$ is the scalar product in $F_\sigma^{(n)}$. Finally, we define the refernce Hilbert space $\tilde{\mathcal{H}}$ as the direct sum over all particle numbers n and sectors $[\sigma]$:

$$\tilde{\mathcal{H}} := \mathbb{C} \oplus \bigoplus_{n>0,\sigma\in\Delta} L^2_\varepsilon({}^n\widetilde{H}_1\ ,\ F_\sigma^{(n)})\ . \tag{2.64}$$

We will denote the elements of this Hilbert space as follows. An equivariant function in $L^2_\varepsilon({}^n\widetilde{H}_1\ ;\ F_\sigma^{(n)})$ can be written as a direct sum of 2^N functions $\tilde{\psi}_{\boldsymbol{\alpha}}$ with values in the intertwiner space $F^{(n)}_{\sigma,\boldsymbol{\alpha}}$. Correspondingly, we decompose a vector $\tilde{\psi}\in\tilde{\mathcal{H}}$ as $\tilde{\psi}=\sum_{\sigma,n,\boldsymbol{\alpha}}\tilde{\psi}_{\sigma,\boldsymbol{\alpha}}$.

THEOREM 2.9. *$\mathcal{H}^{\rm out}$ and $\mathcal{H}^{\rm in}$ are in a natural way isomorphic to $\tilde{\mathcal{H}}$. The unitaries effecting these equivalences are the so-called Møller operators W^+ and W^-, which are defined as the linear and isometric extensions of the following operators. Let $\mathbf{V}$ be the Cartesian product of mutually disjoint compact subsets of H_1, and let $\xi\in X^\pm_{\mathbf{V}}$. Then for all $\boldsymbol{\psi}\in\mathcal{H}_\sigma^{(n)}(\mathbf{V}/S_n)^{\rm out,in}$,*

$$(W^\pm\boldsymbol{\psi})(\tilde{\boldsymbol{q}}) := \begin{cases}\varepsilon\big(\tilde{\varphi}^\pm_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\big)^{-1}\ (i^\pm_{\mathbf{V},\xi}\boldsymbol{\psi})(\boldsymbol{q})\ , & \text{if } \boldsymbol{q}\in\mathbf{V}/S_n\ ,\\ 0\ , & \text{else.}\end{cases} \tag{2.65}$$

In particular, if $\boldsymbol{\psi}=T\psi_n\times\cdots\times\psi_1(\sigma,\boldsymbol{\alpha},\xi,\pm)$ with $\boldsymbol{\psi}_k\in\mathcal{H}^{(1)}_{\alpha_k}(V_k)$, then

$$(W^\pm\boldsymbol{\psi})(\tilde{\boldsymbol{q}}) = (\tilde{\psi}_1\otimes_{\rm s}\cdots\otimes_{\rm s}\tilde{\psi}_n)(\boldsymbol{q})\ \big(\sigma|\,T\,\varepsilon_{\boldsymbol{\alpha}}(\tilde{\varphi}^\pm_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}}),\pi^{-1})\ |\boldsymbol{\alpha}\pi\big)\ , \tag{2.66}$$

where π is the permutation associated to $\tilde{\varphi}^\pm_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})$, see (A.4), *and where $\tilde{\psi}_k := W_{\alpha_k}\psi_k$.*

PROOF. We omit the superscript $\pm$. To see that W is well defined on $\mathcal{H}_\sigma^{(n)}(\mathbf{V}/S_n)^{\rm out,in}$ by equation (2.65), let $\mathbf{V}'$ be another Cartesian product of sets, $\xi'\in X^\pm_{\mathbf{V}'}$, and $\boldsymbol{\psi}\in\mathcal{H}(\mathbf{V}/S_n\cap\mathcal{H}_{\mathbf{V}'/S_n})$. This intersection is mapped into $L^2(\mathbf{V}/S_n\cap\mathbf{V}'/S_n;F)$ by both $i^\pm_{\mathbf{V},\xi}$ and $i^\pm_{\mathbf{V}',\xi'}$. For $\boldsymbol{q}\in\mathbf{V}/S_n\cap\mathbf{V}'/S_n$, equations (2.51) and (2.43) imply

$$\begin{aligned}\varepsilon\left(\tilde{\varphi}_{\mathbf{V},\xi}(\tilde{\boldsymbol{q}})\right)^{-1}\ (i_{\mathbf{V},\xi}\boldsymbol{\psi})\,(\boldsymbol{q}) &= \varepsilon\left(\tilde{\varphi}_{\mathbf{V}',\xi'}(\tilde{\boldsymbol{q}})\right)^{-1}\ \varepsilon\big(g_{\mathbf{V}',\xi';\mathbf{V},\xi}(\boldsymbol{q})\big)\ (i_{\mathbf{V},\xi}\boldsymbol{\psi})\,(\boldsymbol{q})\\ &= \varepsilon\left(\tilde{\varphi}_{\mathbf{V}',\xi'}(\tilde{\boldsymbol{q}})\right)^{-1}\ (i_{\mathbf{V}',\xi'}\boldsymbol{\psi})\,(\boldsymbol{q})\ .\end{aligned}$$

This also shows that the expression is independent of the leaf $\mathbf{V}$ over $\mathbf{V}/S_n$ which one takes in evaluating equation (2.65) (note that $\boldsymbol{\psi}$ only carries the information $\mathbf{V}/S_n$), and that it is indedpendent of $\xi\in X_{\mathbf{V}}$. Hence W is well defined on $\mathcal{H}_\sigma^{(n)}(\mathbf{V}/S_n)^{\rm out,in}$. $W\boldsymbol{\psi}$ is equivariant with respect to the representation ε of $\pi_1({}^nH_1,\boldsymbol{q}_0)$ on account of equation (2.50). We now show that W is isometric. Locally, this follows from $i_{\mathbf{V},\xi}\boldsymbol{\psi}$ being isometric. Namely, for all $\boldsymbol{\psi}\in\mathcal{H}(\mathbf{V}/S_n)^{\rm ex}$ we have

$$\begin{aligned}||W\boldsymbol{\psi}\,||^2 &= \int_{{}^nH_1}|\,(W\boldsymbol{\psi})(\tilde{\boldsymbol{q}})\,|^2_F\ {\rm d}\mu(\boldsymbol{q}) = \int_{{}^nH_1}|\,(i_{\mathbf{V},\xi}\boldsymbol{\psi})(\boldsymbol{q})\,|^2_F\ {\rm d}\mu(\boldsymbol{q})\\ &= ||i_{\mathbf{V},\xi}\boldsymbol{\psi}||^2 = ||\boldsymbol{\psi}||^2\ .\end{aligned}$$

${}^nH_1$ can be exhausted by a family of disjoint subsets $\mathbf{V}_i/S_n$, where each $\mathbf{V}_i$ is the Cartesian product of compact and pairwise disjoint subsets of H_1. Hence the finite sums $\boldsymbol{\psi}=\sum_{\rm finite}\boldsymbol{\psi}_i$ of vectors $\boldsymbol{\psi}_i\in\mathcal{H}_\sigma^{(n)}(\mathbf{V}_i/S_n)^{\rm ex}$ with $\mathbf{V}_i/S_n\cap\mathbf{V}_j/S_n=\emptyset$ for $i\neq j$ already span $\mathcal{H}_\sigma^{(n)\,\rm ex}$, and it suffices to show that $W^\pm$ is isometric on such vectors. From the cluster theorem [FGR96, Lemma 2.2] one can show in analogy to the derivation of [FGR96, Theorem 3.2], that the spaces $\mathcal{H}_\sigma^{(n)}(\mathbf{V}_i/S_n)^{\rm ex}$ are pairwise orthogonal if the sets $\mathbf{V}_i/S_n$ are pairwise disjoint. Hence for $\boldsymbol{\psi}$ of the above type we have

$$||\sum_{\rm finite}\boldsymbol{\psi}_i|| = \sum||\boldsymbol{\psi}_i|| = \sum||W^\pm\boldsymbol{\psi}_i|| = ||W^\pm\sum\boldsymbol{\psi}_i||\ ,$$

where we have used in the last step that $W^\pm$ maps $\mathcal{H}_{\mathbf{V}_i/S_n}$ into $L^2_\varepsilon({}^n\widetilde{H}_1|_{\mathbf{V}_i/S_n};F)$, and that these subspaces of $L^2_\varepsilon({}^n\widetilde{H}_1;F)$ are also pairwise orthogonal. Finally, one easily checks that for any $\mathbf{V}$, $W^\pm$ maps $\mathcal{H}_\sigma^{(n)}\mathbf{V}/S_{n\,\rm ex}$ *onto* $L^2_\varepsilon({}^n\widetilde{H}_1|_{\mathbf{V}/S_n};F)$, and since these subspaces span $L^2_\varepsilon({}^n\widetilde{H}_1;F)$, $W^\pm$ is surjective. □

REMARK. Anticipating results from the next chapter, some remarks are in order at this point. 1.) The Møller operators intertwine the dynamics in $\mathcal{H}^{\mathrm{ex}}$ with the 'free' dynamics, namely the natural implementation of time translations in $L^2_\varepsilon({}^n\widetilde{H}_1, F^{(n)}_\sigma)$. This is a special case of Proposition 3.4 in Chapter 3, where this intertwining property is shown for the whole Poincaré group. 2.) We define the S-matrix as the isometric operator

$$S := (W^+)^* W^- : \mathcal{H}^{\mathrm{in}} \to \mathcal{H}^{\mathrm{out}} . \tag{2.67}$$

Due to equation (2.61), we have explicitely

$$S\,T\psi_n \times \cdots \times \psi_1(\sigma, \boldsymbol{\alpha}, \theta\cdot\xi, -) = T\psi_n \times \cdots \times \psi_1(\sigma, \boldsymbol{\alpha}, \xi, +) . \tag{2.68}$$

It will be shown in the next chapter, that it satisfies the familiar commutation relations with $U(\tilde{P}_+)$:

$$U(\tilde{g})\,S = S\,U(\tilde{g}) \quad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow \quad \text{and}$$
$$U(\tilde{j})\,S = S^*\,U(\tilde{j}) .$$

3.) For the case $N = 1$, i.e. considering only one particle type, the structure coincides with the one proposed in [MS95]. The same holds for the ray representation of the Poincaré group in the next chapter.