CHAPTER 1

Plektons in Algebraic QFT

1.1. The Field Bundle

We further develop some of the concepts of local quantum physics mentioned in the introduction.

Universal Algebra and Localized Endomorphisms. An elegant way to discuss the composition and also the statistics of sectors in this setting is to embed the family $\mathcal{A}(I)_{I\in\mathcal{K}}$ into an abstract C*-algebra, the *universal algebra* $\mathcal{A}_{\mathrm{u}}$ associated to $\mathcal{A}(I)_{I\in\mathcal{K}}$, such that the representations $\{\pi\}$ extend to endomorphisms of that algebra. This construction has been proposed by Fredenhagen in [Fre90a]. $\mathcal{A}_{\mathrm{u}}$ is characterized by the following conditions:

1. There are unital embeddings $i^I : \mathcal{A}(I) \to \mathcal{A}_{\mathrm{u}}$ such that for all $I, J \in \mathcal{K}$
$$i^J\,|_{\mathcal{A}(I)} = i^I \text{ if } I \subset J.$$
2. For every family of representations $\{\pi\}$ which is consistent in the sense of (0.6) there is a unique representation π of $\mathcal{A}_{\mathrm{u}}$ in $\mathcal{H}_\pi$ such that $\pi \circ i^I = \pi^I$ for all $I \in \mathcal{K}$.

The particular representation π_0 resulting from the defining (identical) representation, $\pi_0 \circ i^I = \mathrm{id}_{\mathcal{A}(I)}$, will from now on be called the *vacuum representation* of $\mathcal{A}_{\mathrm{u}}$. It has the peculiar property, that it is in general not faithful, due to the existence of global intertwiners (see below).

To each representation $\{\pi\}$ of the family $\mathcal{A}(I)_I$ localized in S an endomorphism ϱ of $\mathcal{A}_{\mathrm{u}}$ can be associated which satisfies $\pi = \pi_0 \circ \varrho$, where π is the extension of $\{\pi\}$ to $\mathcal{A}_{\mathrm{u}}$.

> We sketch the construction of ϱ following [FRS92]. In view of the characterizing properties of $\mathcal{A}_{\mathrm{u}}$, it suffices to construct a consistent family of homomorphisms $\varrho^I : \mathcal{A}(I) \to \mathcal{A}_{\mathrm{u}}$ satisfying $\pi^I = \pi_0 \circ \varrho^I$. These will determine ϱ via $\varrho^I = \varrho \circ i^I$. Given $I \in \mathcal{K}$, we choose a spacelike cone $S_1 \in I'$ and a region I_{01} containing both S and S_1. (Such a region exists, e.g. take $I_{01} := S_2'$, where S_2 is a spacelike cone in $S' \cap S_1'$.) Then there is a unitary $U \in \mathcal{A}(I_{01})$ such that
> $$\mathrm{Ad}U \circ \pi^{S_1'} = \mathrm{id}_{\mathcal{A}(S_1')}. \tag{1.1}$$
> Namely, $U := V_{S_1}^* V_S$, with V_S and V_{S_1} as in equation (0.10), satisfies equation (1.1). This equation implies, remembering that $\pi^{S'} = \mathrm{id}_{\mathcal{A}(S')}$ and $I_{01}' \subset S' \cap S_1'$, that $\mathrm{Ad}U\,|_{\mathcal{A}(I_{01}')} = \mathrm{id}_{\mathcal{A}(I_{01}')}$. Hence $U \in \mathcal{A}(I_{01})$ by Haag duality. Now we define $\varrho^I := \mathrm{Ad}\left(i^{I_{01}}(U^*)\right) \circ i^I$, and convince ourselves that this definition is independent of the choice of S_1, I_{01} and U.

Endomorphisms of the universal algebra arising that way will be called *localized* in S, and the set of such endomorphisms will be denoted by $\Delta(S)$. A characterization of this set without reference to the family $\mathcal{A}(I)_I$ can be found in [FGR96].

The notion of superselection sectors translates into the present setup as follows. Let $\{\pi\}$ and $\{\hat\pi\}$ be two representations of the family $\mathcal{A}(I)_{I\in\mathcal{K}}$ in $\mathcal{H}_0$, both localized in S, and let $\varrho, \hat\varrho \in \Delta(S)$ be the corresponding endomorphisms of $\mathcal{A}_{\mathrm{u}}$. Then $\{\pi\}$ and $\{\hat\pi\}$ are equivalent if and only if ϱ and $\hat\varrho$ are inner equivalent, i.e. if there is a unitary element $U \in \mathcal{A}_{\mathrm{u}}$ of the universal algebra such that $\hat\varrho = \mathrm{Ad}U \circ \varrho$ (see [Fre93]). Consequently, we call the inner equivalence class of ϱ its sector and denote it by $[\varrho]$.

Poincaré Covariance of Endomorphisms. The vacuum representation U_0 of $P_+^\uparrow$ on $\mathcal{H}_0$ determines uniquely a representation α^0 of $P_+^\uparrow$ by automorphisms of $\mathcal{A}_{\mathrm{u}}$ which is implemented by U_0, i.e.

$$\mathrm{Ad}U_0(g) \circ \pi_0 = \pi_0 \circ \alpha_g^0 \quad \text{for all } g \in P_+^\uparrow . \tag{1.2}$$

Let ϱ be an endomorphism of $\mathcal{A}_{\mathrm{u}}$ corresponding to the representation $\{\pi\}$ of $\mathcal{A}(I)_{I\in\mathcal{K}}$ and let $U_\varrho := U_\pi$ be the unitary representation of $\tilde{P}_+^\uparrow$ satisfying equation (0.7). Then U_ϱ implements α^0 in the representation $\pi_0\varrho$ of $\mathcal{A}_{\mathrm{u}}$:

$$\mathrm{Ad}U_\varrho(\tilde{g}) \circ \pi_0\varrho = \pi_0\varrho \circ \alpha_g^0 \text{ for all } \tilde{g} \in \tilde{P}_+^\uparrow . \tag{1.3}$$

Equations (1.2) and (1.3) imply

$$\mathrm{Ad}\big(U_0(g)U_\varrho(\tilde{g}^{-1})\big) \circ \pi_0\varrho = \pi_0 \circ \alpha_g^0 \circ \varrho \circ \alpha_{g^{-1}}^0 \quad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow . \tag{1.4}$$

The right hand side of this equation is a representation of $\mathcal{A}_{\mathrm{u}}$ localized in $g\cdot S$, hence we can conclude as after equation (1.1): if $\tilde{g}$ is in a small enough neighbourhood of the identity such that $S \cup g\cdot S \subset I_g$ for some $I_g \in \mathcal{K}$, then Haag duality implies that $U_0(g)U_\varrho(\tilde{g}^{-1}) \in \mathcal{A}(I_g)$. In this case we define

$$Y_\varrho^*(\tilde{g}) := \pi_0^{-1}\left(U_0(g)U_\varrho(\tilde{g}^{-1})\right) \tag{1.5}$$

and $Y_\varrho(\tilde{g}) := Y_\varrho^*(\tilde{g})^*$. The map $\tilde{g} \mapsto Y_\varrho^*(\tilde{g})$ has the cocycle property $Y_\varrho^*(\tilde{g}_2\tilde{g}_1) = \alpha_{g_2}^0\big(Y_\varrho^*(\tilde{g}_1)\big)Y_\varrho^*(\tilde{g}_2)$, which allows to extend it continuously to arbitrary elements $\tilde{g} \in \tilde{P}_+^\uparrow$. It lifts equation (1.4) from $\mathcal{B}(\mathcal{H}_0)$ to $\mathcal{A}_{\mathrm{u}}$:

$$\mathrm{Ad}Y_\varrho^*(\tilde{g}) \circ \varrho = \alpha_g^0 \circ \varrho \circ \alpha_{g^{-1}}^0 \quad \text{for all } \tilde{g} \in \tilde{P}_+^\uparrow . \tag{1.6}$$

Also, it can be used to implement α_0 in the product representation $\pi_0 \circ \varrho_2\varrho_1$ by a representation $U_{\varrho_2\varrho_1}$ of $\tilde{P}_+^\uparrow$, defined as follows:

$$U_{\varrho_2\varrho_1}(\tilde{g}) := (\pi_0\varrho_2)\left(Y_{\varrho_1}(\tilde{g})\right) U_{\varrho_2}(\tilde{g}) . \tag{1.7}$$

That $U_{\varrho_2\varrho_1}$ has the claimed property (1.3), can be seen by using equation (1.4).

Field Bundle. The semantic relevance of a vector in Hilbert space is that it defines a *state* in the sense of expectation values on the algebra of observables. Since we have arranged all relevant representations $\pi_0 \circ \varrho$ of the observable algebra to act on the same Hilbert space $\mathcal{H}_0$, the representation has to be specified along with the vector in order to determine a state

$$A \mapsto \langle\psi, \pi_0\varrho(A)\psi\rangle,\ A \in \mathcal{A}_{\mathrm{u}} , \tag{1.8}$$

and we are led to consider pairs

$$\mathcal{H} := \Delta(S_0) \times \mathcal{H}_0 \quad \supset \quad \mathcal{H}_\varrho := \{\varrho\} \times \mathcal{H}_0 . \tag{1.9}$$

We consider here a fixed spacelike cone S_0 to reduce the redundancy. On every fibre $\mathcal{H}_\varrho$ a scalar product is declared by that of $\mathcal{H}_0$. The *field bundle* is defined by

$$\mathcal{F} := \Delta(S_0) \times \mathcal{A}_u . \tag{1.10}$$

We recall that a field algebra has not yet been established in the case of nontrivial and non-Abelian braid group statistics. The field bundle is a simple substitute, which mimicks the essential feature of charged fields as operations on the set of states, capable of creating charged states (ϱ, ψ) from the vacuum (id, Ω). The structure is rich enough to carry a localization concept and to discuss Haag-Ruelle scattering theory. Its elements are called *generalized field operators.* A multiplication in $\mathcal{F}$ is defined by

$$(\varrho_2, B_2)\,(\varrho_1, B_1) := (\varrho_1\varrho_2, \varrho_1(B_2)\,B_1) , \tag{1.11}$$

and the norm of a generalized field operator by $||(\varrho, B)|| := ||B||$. Finally, $\mathcal{F}$ acts in $\mathcal{H}$ in a way which is consistent with the multiplication by

$$(\varrho_2, B)\cdot(\varrho_1, \psi) := (\varrho_1\varrho_2, \pi_0 \circ \varrho_1(B)\,\psi). \tag{1.12}$$

Intertwiners. An element $T \in \mathcal{A}_{\mathrm{u}}$ satisfying

$$\hat{\varrho}(A)T = T\varrho(A) \quad \text{for all } A \in \mathcal{A}_{\mathrm{u}} \tag{1.13}$$

is called an *intertwiner* with source ϱ and range $\hat{\varrho}$ (or from ϱ to $\hat{\varrho}$), and the set of such intertwiners will be denoted by $\mathrm{Int}(\hat{\varrho}|\varrho)$, or by $\mathrm{Int}(\hat{\varrho}|\varrho)(S_0)$ if they are in addition required to be in $\mathcal{A}_{\mathrm{u}}(S_0)$. We will denote the source and range of an intertwiner $T \in \mathrm{Int}(\hat{\varrho}|\varrho)$ together with T in the form $\mathbf{T} := (\hat{\varrho}|T|\varrho)$. Intertwiners act on $\mathcal{H}$ and $\mathcal{F}$ via

$$(\hat{\varrho}|T|\varrho)\,(\varrho,\psi) := (\hat{\varrho},\pi_0(T)\psi) \quad \text{and} \quad (\hat{\varrho}|T|\varrho)\,(\varrho,B) := (\hat{\varrho},TB), \tag{1.14}$$

respectively. The adjoint of an intertwiner and the composition of two intertwiners with matching source and range will be written as

$$(\hat{\varrho}|T|\varrho)^* \;:=\; (\varrho|T^*|\hat{\varrho}) \quad \text{and} \tag{1.15}$$

$$(\varrho_2|T_2|\varrho)\circ(\varrho|T_1|\varrho_1) \;:=\; (\varrho_2|T_2\,T_1|\varrho_1), \tag{1.16}$$

respectively. Further, if $T_i \in \mathrm{Int}(\hat{\varrho}_i|\varrho_i)\,, i = 1,2$, one easily verifies that $\hat{\varrho}_2(T_1)T_2 \equiv T_2\varrho_2(T_1)$ is an intertwiner from $\varrho_2\varrho_1$ to $\hat{\varrho}_2\hat{\varrho}_1$ which will be denoted

$$(\hat{\varrho}_2|T_2|\varrho_2)\times(\hat{\varrho}_1|T_1|\varrho_1) := (\hat{\varrho}_2\hat{\varrho}_1|T_2\varrho_2(T_1)|\varrho_2\varrho_1). \tag{1.17}$$

Note that the last observation implies that the composition of endomorphisms respects the division into equivalence classes, justifying the definition of charge composition as in equation (0.11). The two products $\circ$ and $\times$ are associative and satisfy the following identities, which will be useful in the sequel:

$$(\mathbf{T}_1 \times \mathbf{T}_2)^* \;=\; \mathbf{T}_1^* \times \mathbf{T}_2^* \tag{1.18}$$

$$\mathbf{T}_2\mathbf{B}_2 \cdot \mathbf{T}_1\mathbf{B}_1 \;=\; (\mathbf{T}_1 \times \mathbf{T}_2)\cdot\mathbf{B}_2\mathbf{B}_1\,. \tag{1.19}$$

$$(\mathbf{T}_1 \circ \mathbf{S}_1)\times(\mathbf{T}_2 \circ \mathbf{S}_2) \;=\; (\mathbf{T}_1 \times \mathbf{T}_2)\circ(\mathbf{S}_1 \times \mathbf{S}_2)\,, \tag{1.20}$$

whenever the left hand side is defined. In intermediate steps of calculations we will frequently use the notations (ϱ, B) and $(\hat{\varrho}, |T|\varrho)$ and the definitions and identities (1.11) to (1.19) also for endomorphisms localized in other regions than S_0.

Localization of the 'Fields'. The localization concept for $\mathcal{F}$ should satisfy the requirement that the fields are relatively local w.r.t. the observables: if (ϱ, B) is localized in $I \in \mathcal{K}$, it should commute with (id, A) for all $A \in \mathcal{A}_{\mathrm{u}}(I')$. But this condition is not sufficient to imply commutation relations between spacelike seperated fields. As a way out, we first observe that in the vacuum representation π_0, relative locality is equivalent to the following condition: If $U \in \mathcal{B}(\mathcal{H}_0)$ is a unitary intertwiner such that $\mathrm{Ad}U \circ \pi_0 \circ \varrho$ is localized in I, then $U\pi_0(B) \in \mathcal{A}(I)$. From this, commutation relations can be derived in π_0. In order to lift them (and the localization concept) from $\mathcal{B}(\mathcal{H}_0)$ to $\mathcal{A}_{\mathrm{u}}$, one has to lift $U \in \mathcal{B}(\mathcal{H}_0)$ to $\mathcal{A}_{\mathrm{u}}$. If $I \cup S_0$ are contained in some $J \in \mathcal{K}$, then $U \in \mathcal{A}(J)$ by Haag duality, and one might replace U by $i^J(U) \in \mathcal{A}_{\mathrm{u}}(J)$ in the above condition. But the resulting localization condition depends on the choice of J. Hence, similar to the situation of multivalued analytic functions, one is led to either introduce a "cut", i.e. here, to exclude a fixed spacelike direction from the set of allowed localization regions [BF82], or to consider the generalized field operators as living on paths in $\mathcal{K}$. Such a localization concept has been introduced by Fredenhagen, Gaberdiel and Rüger in [FGR96]. We recall (from page 6) that a *path* in $\mathcal{K}$ is a finite sequence of regions $I_k \in \mathcal{K}, k = 0,\dots,n$ with $I_0 = S_0$, where either $I_{k-1} \subset I_k$ or $I_{k-1} \supset I_k$ for $k = 1,\dots,n$. The path is said to end at I_n. For each k there is a unitary $U_k \in \mathcal{A}_{\mathrm{u}}(I_{k-1} \cup I_k)$ such that $\mathrm{Ad}(U_k \cdots U_1) \circ \varrho$ is localized in I_k. Then (ϱ, B) is called *localized* in the path $(I_0,\dots,I_n)$ if $U_n \cdots U_1\,B \in \mathcal{A}_{\mathrm{u}}(I_n)$.

To see that this definition is independent of the choice of the intertwiners U_k, let $\hat{U}_1,\dots,\hat{U}_n$ be another such choice. Then $\hat{U}_1U_1^{-1}$ is an intertwiner between two representations localized in I_1, hence its image under π_0 is in $\mathcal{A}(I_1)$ by Haag duality. Since $\hat{U}_1U_1^{-1}$ itself is local, namely $\in \mathcal{A}_{\mathrm{u}}(I_0 \cup I_1)$, we can invert π_0 and get $\hat{U}_1U_1^{-1} \in \mathcal{A}_{\mathrm{u}}(I_1)$. Iterating this argument we conclude $V := \hat{U}_n \cdots \hat{U}_1U_1^{-1} \cdots U_n^{-1} \in \mathcal{A}_{\mathrm{u}}(I_n)$. Hence, $\hat{U}_n \cdots \hat{U}_1\,B \equiv V\,U_n \cdots U_1\,B$ is in $\mathcal{A}_{\mathrm{u}}(I_n)$ if $U_n \cdots U_1\,B$ is.

As proposed in [FRS92], we call two paths in $\mathcal{K}$ *homotopic*, if they end at the same region $I \in \mathcal{K}$ and result from each other by a finite number of operations of the following type:

1. insertion of a region $\hat{I}_k$ between I_{k-1} and I_k, if the resulting sequence is still a path ending at I, and
2. omission of one of the regions I_k, if the resulting sequence is still a path ending at I.

By an argument analogous to the above one it is easily seen that the property of (ϱ, B) of being localized in a path $(I_0, \dots, I_n)$ is stable under such operations and hence depends only on the homotopy class of the path. We will denote the homotopy class of a path ending at I by $\tilde{I}$, and the set of these classes by $\widetilde{\mathcal{K}}$. Further, the set of generalized field operators localized in $\tilde{I}$ will be denoted by

$$\mathcal{F}(\tilde{I})\ .$$

We say that $\tilde{I}$ and $\tilde{J} \in \widetilde{\mathcal{K}}$ are causally disjoint, if they end at regions I and J which are causally disjoint. Homotopy of paths in $\mathcal{K}$ can be equivalently characterized by homotopy of paths in the set of spacelike directions,

$$H_i := \{p \in \mathbb{R}^3 \,/\, p^2 = -1\}\ , \tag{1.21}$$

as follows. We fix a point r_0 in H_i corresponding to S_0, i.e. $r_0 \in H_i \cap S_0 - \mathrm{a}(S_0)$, where $\mathrm{a}(S_0)$ denotes the apex of S_0. Let γ be a path in H_i and $(I_0, \dots, I_n)$ a path in $\mathcal{K}$. We say that γ *corresponds to* $(I_0, \dots, I_n)$ if it starts from r_0 and has the decomposition $\gamma = \gamma_n * \cdots * \gamma_0$, where γ_k is a path in $I_k - \mathrm{a}(I_k)$. (In particular, γ ends in $I_n - \mathrm{a}(I_n)$.) We will also say that the homotopy class $\tilde{r} \in \widetilde{H_i}$ of γ corresponds to the homotopy class $\tilde{I} \in \widetilde{\mathcal{K}}$ of $(I_0, \dots, I_n)$. One easily verifies:

COROLLARY 1.1. *Let γ and $\hat{\gamma}$ be paths in H_i corresponding to two paths in $\mathcal{K}$ which end at the same region $I \in \mathcal{K}$. Then the paths in $\mathcal{K}$ are homotopic if and only if $\hat{\gamma} * \gamma^{-1}$ is homotopic to a path in $I - \mathrm{a}(I)$.*

The equivalence class of paths $\hat{\gamma}$ satisfying this last condition with rwespect to γ will be denoted by $[\gamma]_I$. In view of the Corollary, we can identify $\tilde{I}$ with the tupel $(I, [\gamma]_I)$. We write

$$(I, [\gamma]_I) \subset (J, [\hat{\gamma}]_J) \tag{1.22}$$

if $I \subset J$ and $[\gamma]_I = [\hat{\gamma}]_I$.

Poincaré Covariance of the Field Bundle. Consider the representation U of $\tilde{P}_+^\uparrow$ in $\mathcal{H}$ defined by

$$U(\tilde{g})(\varrho, \psi) := (\varrho, U_\varrho(\tilde{g})\psi)\ , \tag{1.23}$$

and the action α in the field bundle $\mathcal{F}$ given by

$$\alpha_{\tilde{g}}(\varrho, B) := \Big(\varrho, Y_\varrho(\tilde{g})\, \alpha_g^0(B)\Big)\ , \tag{1.24}$$

where $Y_\varrho(\tilde{g})$ is the cocycle defined in equation (1.5). U implements α in the sense that

$$U(\tilde{g})\ (\varrho, B) \cdot U(\tilde{g}^{-1})\, \psi = (\alpha_{\tilde{g}}(\varrho, B)) \cdot \psi \quad \text{for all } \psi \in \mathcal{H}\ . \tag{1.25}$$

Further, $\tilde{P}_+^\uparrow$ has a natural action $\tilde{g} : \tilde{I} \mapsto \tilde{g} \cdot \tilde{I}$ on $\widetilde{\mathcal{K}}$: Let $\tilde{I}$ be the class of $(I_0, \dots, I_n)$. We choose a decomposition $\tilde{g} = \tilde{g}_N \cdots \tilde{g}_1$ with representants $\alpha_k \in \tilde{g}_k$ and regions $J_k \in \widetilde{\mathcal{K}}$ such that for $k = 1, \dots, N$

$$\alpha_k(t) g_{k-1} \cdots g_1 {\cdot} S_0 \cup g_{k-1} \cdots g_1 {\cdot} S_0 \subset J_k \quad \text{for all } t \in [0,1]\ . \tag{1.26}$$

Then $\tilde{g} \cdot \tilde{I}$ is the class of the path

$$(S_0, J_1, g_1{\cdot}S_0, J_2, g_2 g_1{\cdot}S_0, \dots, J_N, g{\cdot}S_0, g{\cdot}I_1, \dots, g{\cdot}I_n)\ . \tag{1.27}$$

It can easily be shown to be independent of the representants of $\tilde{I}$ and $\tilde{g}$. In terms of the definition after the last corollary, this action reads as follows. Let $\tilde{I} = (I, [\gamma]_I)$, and $\tilde{g} = [\alpha] \in \tilde{P}_+^\uparrow$ with $\alpha(1) = g \in P_+^\uparrow$. Then

$$\tilde{g} \cdot \tilde{I} := (g \cdot I, [\gamma_{\tilde{g}}]_{g \cdot I}), \qquad \text{with } \gamma_{\tilde{g}}(t) := \alpha(t) \cdot \gamma(t)\ . \tag{1.28}$$

LEMMA 1.2. *The field bundle is covariant under the action α of $\tilde{P}_+^\uparrow$:*

$$\alpha(\tilde{g}) :\ \mathcal{F}(\tilde{I}) \to \mathcal{F}(\tilde{g} \cdot \tilde{I})\ . \tag{1.29}$$

Proof. Let $(\varrho, U^*A) \in \mathcal{F}(\tilde{I})$, i.e. A is in $\mathcal{A}_{\rm u}(I)$ and $U = U_n \cdots U_1$, where $(U_1, \dots, U_n)$ is a chain of charge transporters for ϱ along $(I_0, \dots, I_n)$. By equation (1.6) $Y_\varrho^*(\tilde{g}_k \cdots \tilde{g}_1)\, Y_\varrho(\tilde{g}_{k-1} \cdots \tilde{g}_1)$ is an intertwiner from $\alpha^0_{g_{k-1}\cdots g_1} \circ \varrho \circ \alpha^0_{(g_{k-1}\cdots g_1)^{-1}}$ to $\alpha^0_{g_k \cdots g_1} \circ \varrho \circ \alpha^0_{(g_k\cdots g_1)^{-1}}$, which is in $\mathcal{A}_{\rm u}(J_k)$ due to condition (1.26). Therefore

$$(\mathbf{1}\,, Y_\varrho^*(\tilde{g}_1)\,, \mathbf{1}\;, \dots, \; Y_\varrho^*(\tilde{g})\, Y_\varrho(\tilde{g}_{N-1} \cdots \tilde{g}_1)\,, \alpha^0_{\tilde{g}}(U_1), \dots, \alpha^0_{\tilde{g}}(U_n))$$

is a chain of charge transporters for ϱ along the path (1.27), and hence

$$\begin{aligned}\alpha(\tilde{g})(\varrho, U^*A) &= (\varrho, Y_\varrho(\tilde{g})\alpha^0_g(U^*A)) \\ &\equiv \big(\varrho, Y_\varrho(\tilde{g}_1)^*\, Y_\varrho(\tilde{g}_2\tilde{g}_1) \cdots Y_\varrho(\tilde{g}_{N-1}\cdots\tilde{g}_1)^*\, Y_\varrho(\tilde{g})\; \alpha^0_{\tilde{g}}(U_n^*) \cdots \alpha^0_{\tilde{g}}(U_1^*)\; \alpha^0_{\tilde{g}}(A)\,\big)\end{aligned}$$

is in $\mathcal{F}(\tilde{g}\cdot\tilde{I})$. □

1.2. Statistics of Plektonic 'Fields'

We want to calculate the commutation relations of spacelike seperated fields. Let $\tilde{I}_1 \in \widetilde{\mathcal{K}}$ and $\tilde{I}_2 \in \widetilde{\mathcal{K}}$ be causally disjoint, i.e. ending at causally disjoint regions $I_1, I_2 \in \mathcal{K}$. Let further $\mathbf{B}_k = (\varrho_k, B_k) \in \mathcal{F}(\tilde{I}_k), k = 1, 2$. Then there are unitary intertwiners U_1 and U_2 such that for $k = 1, 2$ the endomorphism $\hat{\varrho}_k := {\rm Ad}U_k \circ \varrho$ is localized in I_k and $D_k := U_k B_k \in \mathcal{A}_{\rm u}(I_k)$. Writing[1] $\mathbf{D}_k = (\hat{\varrho}_k, D_k)$ and using identity (1.19), we calculate

$$\begin{aligned}\mathbf{B}_1\mathbf{B}_2 &= \mathbf{U}_1^*\mathbf{D}_1 \cdot \mathbf{U}_2^*\mathbf{D}_2 = (\mathbf{U}_2^* \times \mathbf{U}_1^*) \cdot \mathbf{D}_1\mathbf{D}_2 = (\mathbf{U}_2^* \times \mathbf{U}_1^*) \cdot \mathbf{D}_2\mathbf{D}_1 \\ &= (\mathbf{U}_2^* \times \mathbf{U}_1^*)(\mathbf{U}_1 \times \mathbf{U}_2) \cdot \mathbf{B}_2\mathbf{B}_1\,. \end{aligned} \tag{1.30}$$

In the third equation we have exploited the causal disjointness of the respective localization regions I_k of $\hat{\varrho}_k$ and D_k, to get $\hat{\varrho}_2\hat{\varrho}_1 = \hat{\varrho}_1\hat{\varrho}_2$, and further $\mathbf{D}_1\mathbf{D}_2 = \mathbf{D}_2\mathbf{D}_1$. Comparing the first and last expression in equation (1.30) we conclude that the unitary intertwiner

$$\varepsilon(\varrho_1, \varrho_2; \tilde{I}_1, \tilde{I}_2) := (\mathbf{U}_2^* \times \mathbf{U}_1^*) \circ (\mathbf{U}_1 \times \mathbf{U}_2) \quad \in \quad {\rm Int}(\varrho_2\varrho_1|\varrho_1\varrho_2) \tag{1.31}$$

from $\varrho_1\varrho_2$ to $\varrho_2\varrho_1$ is[2] , as indicated, independent of the choice of U_1 and U_2 and of the paths representing $\tilde{I}_1$ and $\tilde{I}_2$. To examine its dependence on $\tilde{I}_1$ and $\tilde{I}_2$, we attach to each of them one further region $J_1 \in \mathcal{K}$ and J_2, respectively, such that J_2 is causally disjoint from J_1 and such that for both $k = 1$ and 2 either $J_k \subset I_k$ or $J_k \supset I_k$. We denote the resulting homotopy classes of paths by $\tilde{J}_1$ and $\tilde{J}_2$, respectively. Then for $k = 1$ and 2, U_k has to be replaced by $\hat{U}_k U_k$, where $\hat{U}_k \in \mathcal{A}_{\rm u}(I_k \cup J_k)$ is a unitary intertwiner such that ${\rm Ad}\hat{U}_k \circ \hat{\varrho}$ is localized in J_k, and we get

$$\begin{aligned}\varepsilon(\varrho_1, \varrho_2; \tilde{J}_1, \tilde{J}_2) &= (\mathbf{U}_2^*\hat{\mathbf{U}}_2^* \times \mathbf{U}_1^*\hat{\mathbf{U}}_1^*) \circ (\hat{\mathbf{U}}_1\mathbf{U}_1 \times \hat{\mathbf{U}}_2\mathbf{U}_2) \\ &= (\mathbf{U}_2^* \times \mathbf{U}_1^*) \circ (\hat{\mathbf{U}}_2^* \times \hat{\mathbf{U}}_1^*) \circ (\hat{\mathbf{U}}_1 \times \hat{\mathbf{U}}_2) \circ (\mathbf{U}_1 \times \mathbf{U}_2) \\ &= \varepsilon(\varrho_1, \varrho_2; \tilde{I}_1, \tilde{I}_2)\,. \end{aligned} \tag{1.32}$$

In the last equation we have remembered the fact that $I_1 \cup J_1$ is causally disjoint from $I_2 \cup J_2$ to conclude that $\hat{\mathbf{U}}_1 \times \hat{\mathbf{U}}_2 = \hat{\mathbf{U}}_2 \times \hat{\mathbf{U}}_1$.Finite iterations of the operation $(\tilde{I}_1, \tilde{I}_2) \mapsto (\tilde{J}_1, \tilde{J}_2)$ define an equivalence relation $\sim$ on the set of pairs $\widetilde{\mathcal{K}}_2 := \{(\tilde{I}_1, \tilde{I}_2)\,,\, I_1 \subset I_2'\,\}$. By equation (1.32), the intertwiner ε depends on the pair $(\tilde{I}_1, \tilde{I}_2)$ only via its equivalence class, which we denote by $N(\tilde{I}_1, \tilde{I}_2)$:

$$\varepsilon(\varrho_1, \varrho_2; \tilde{I}_1, \tilde{I}_2) =: \varepsilon(\varrho_1, \varrho_2; N(\tilde{I}_1, \tilde{I}_2))\,. \tag{1.33}$$

As anticipated by the notation, the set of classes $\widetilde{\mathcal{K}}_2/_\sim$ is isomorphic to $\mathbb{Z}$, since the equivalence class of a pair $(\tilde{I}_1, \tilde{I}_2)$ is characterized by its relative winding number, which is defined as follows.

Definition 1.3. Let $\tilde{I}_k = (I_k, [\gamma_k]_{I_k})$, $k = 1, 2$.

i) We write $\tilde{I}_1 < \tilde{I}_2$ if $I_1 \subset I_2'$ and $\int_{\gamma_1} d\theta < \int_{\gamma_2} d\theta$, where θ is the angle function in a fixed Lorentz frame. Clearly, the last relation is independent of the frame and of the representants $\gamma_k \in [\gamma_k]_{I_k}$.

[1] with the announced abuse of notation – note that $\mathbf{D}_k$ is *not* in $\mathcal{F}(\tilde{I}_k)$.

[2] Note that the set of equivalence classes of localized morphisms, which has the structure of a semigroup by equation (0.11), is an Abelian semigroup due to the existence of such intertwiners.

ii) Let $I_1 \subset I_2'$. The *relative winding number* of $\tilde{I}_1$ w.r.t. $\tilde{I}_2$, in symbols $N(\tilde{I}_1, \tilde{I}_2)$, is the unique integer $N \in \mathbb{Z}$ satisfying

$$(0, 2\pi N) \cdot \tilde{I}_2 < \tilde{I}_1 < (0, 2\pi(N+1)) \cdot \tilde{I}_2 . \tag{1.34}$$

Now we define the *statistics operator* $\varepsilon(\varrho_1, \varrho_2)$ by equation (1.31) for some pair $(\tilde{I}_1, \tilde{I}_2) \in \widetilde{\mathcal{K}}_2$ with winding number $N(\tilde{I}_1, \tilde{I}_2) = 0$, i.e.

$$\varepsilon(\varrho_1, \varrho_2) := \varepsilon(\varrho_1, \varrho_2; 0) . \tag{1.35}$$

The case of arbitrary winding number N can be reduced to the case $N = 0$ by the use of global self-intertwiners. A *global self-intertwiner* of $\varrho \in \Delta(S_0)$ is an intertwiner V_ϱ from ϱ to itself satisfying

$$\pi_0(V_\varrho) = \mathbb{1} \quad \text{ and } (\varrho, V_\varrho B) \in \mathcal{F}((0, 2\pi) \cdot \tilde{I}) \text{ if } (\varrho, B) \in \mathcal{F}(\tilde{I}). \tag{1.36}$$

V_ϱ is uniquely characterized by these conditions if ϱ is irreducible. It can be constructed as follows. Let $V \in \mathcal{B}(\mathcal{H}_0)$ be a unitary such that the representation $\mathrm{Ad}V \circ \pi_0 \circ \varrho$ is localized in some cone $S_1 \subset S_0'$, and let $J_+, J_- \in \mathcal{K}$ be two regions containing $S_0 \cup S_1$ such that $(S_0, J_-, S_1) < (S_0, J_+, S_1)$ in the sense of Definition 1.3. Then $V_\varrho := i^{J_+}(V^*)\, i^{J_-}(V)$ satisfies the requirements (1.36).

LEMMA 1.4. *Let* $\mathbf{B}_k = (\varrho_k, B_k) \in \mathcal{F}(\tilde{I}_k)$, $k = 1, 2$, *where* $\tilde{I}_k \in \widetilde{\mathcal{K}}$ *are causally disjoint, and let* $N := N(\tilde{I}_1, \tilde{I}_2)$. *Then*

$$\mathbf{B}_1 \mathbf{B}_2 = \varepsilon(\varrho_1, \varrho_2; N)\, \mathbf{B}_2 \mathbf{B}_1 \quad \textit{with} \tag{1.37}$$

$$\varepsilon(\varrho_1, \varrho_2; N) = (\mathbf{V}_{\varrho_2}^{-N} \times \mathbf{1})\, \varepsilon(\varrho_1, \varrho_2)\, (\mathbf{1} \times \mathbf{V}_{\varrho_2}^{N}) . \tag{1.38}$$

PROOF. Let $\tilde{J}_2 := (0, 2\pi N) \cdot \tilde{I}_2$. Then $\mathbf{V}_{\varrho_2}^N \mathbf{B}_2 =: \mathbf{D}_2$ is in $\in \mathcal{F}(\tilde{J}_2)$, and $N(\tilde{I}_1, \tilde{J}_2) = 0$. Hence $\mathbf{B}_1 \mathbf{D}_2 = \varepsilon(\varrho_1, \varrho_2)\, \mathbf{D}_2 \mathbf{B}_1$, and equation (1.37) follows from identity (1.19). □

This formula allows to calculate commutation relations of more than two generalized field operators: It implies that, given n localized endomorphisms $\boldsymbol{\varrho} = (\varrho_1, \ldots, \varrho_n)$, n mutually causally disjoint paths $\tilde{\mathbf{I}} = (\tilde{I}_1, \ldots, \tilde{I}_n)$ in $\mathcal{K}$, and permutations $\pi, \pi' \in S_n$, there is a unitary intertwiner $\hat{\varepsilon}_{\boldsymbol{\varrho},\tilde{\mathbf{I}}}(\pi', \pi)$ from $\varrho_{\pi^{-1}(1)} \cdots \varrho_{\pi^{-1}(n)}$ to $\varrho_{(\pi'\pi)^{-1}(1)} \cdots \varrho_{(\pi'\pi)^{-1}(n)}$ changing the order of factors in a product of n generalized field operators $\mathbf{B}_k \in \mathcal{F}(\tilde{I}_k)$ according to

$$\mathbf{B}_{(\pi'\pi)^{-1}(n)} \cdots \mathbf{B}_{(\pi'\pi)^{-1}(1)} = \hat{\varepsilon}_{\boldsymbol{\varrho},\tilde{\mathbf{I}}}(\pi', \pi)\, \mathbf{B}_{\pi^{-1}(n)} \cdots \mathbf{B}_{\pi^{-1}(1)} . \tag{1.39}$$

Associativity of the group product in S_n then implies that the family of operators $\hat{\varepsilon} = \hat{\varepsilon}_{\boldsymbol{\varrho},\tilde{\mathbf{I}}}$ must obey $\hat{\varepsilon}(\pi''\pi', \pi) = \hat{\varepsilon}(\pi'', \pi'\pi) \circ \hat{\varepsilon}(\pi', \pi)$. Also, $\hat{\varepsilon}(\pi'', \hat{\pi})$ and $\hat{\varepsilon}(\pi', \pi)$ can be composed only if $\hat{\pi} = \pi'\pi$. In other words, the map $(\pi', \pi) \mapsto \hat{\varepsilon}(\pi', \pi)$ establishes a homomorphism from the permutation groupoid $\widetilde{S}_n$, defined below, into the groupoid of unitary intertwiners. The homomorphism $\hat{\varepsilon}_{\boldsymbol{\varrho},\tilde{\mathbf{I}}}$ can best be described by decomposing it into two parts: The localization regions $\tilde{\mathbf{I}} = (\tilde{I}_1, \ldots, \tilde{I}_n)$ determine a homomorphism $\varphi_{\tilde{\mathbf{I}}}$ from $\widetilde{S}_n$ into the groupoid of coloured cylinder braids $\widetilde{B}_n(C)$ (see Definition 1.5), and the endomorphisms $\boldsymbol{\varrho} = (\varrho_1, \ldots, \varrho_n)$ determine a homomorphism $\varepsilon_{\boldsymbol{\varrho}}$ from $\widetilde{B}_n(C)$ into $\widetilde{\mathrm{Int}}(\varrho_1, \ldots, \varrho_n)$, and

$$\hat{\varepsilon}_{\boldsymbol{\varrho},\tilde{\mathbf{I}}} = \varepsilon_{\boldsymbol{\varrho}} \circ \varphi_{\tilde{\mathbf{I}}} . \tag{1.40}$$

The groupoids $\widetilde{B}_n(C)$ and $\widetilde{S}_n$ are defined as follows.

DEFINITION 1.5. i) The *groupoid of coloured cylinder braids* $\widetilde{B}_n(C)$ consists of cylinder braids whose strands carry colours $\{\alpha_1, \ldots, \alpha_n\}$. Two coloured braids can be composed only if the colours of the strands to be connected coincide. More precisely, $\widetilde{B}_n(C)$ is $B_n(C) \times S_n$ as a set. A product of two elements (b', π') and (b, π) is declared only if $\pi' = \nu(b)\pi$, where ν is the natural homomorphism $B_n(C) \to S_n$, in which case it is defined by

$$(b', \nu(b)\pi)(b, \pi) := (b'b, \pi) . \tag{1.41}$$

The permutation π of a coloured braid (b, π) will be viewed as a map associating to each colour α_k the position $\pi(k) \in \{1, \ldots, n\}$ of a strand carrying the colour α_k. I.e., the colour of the k^{th} strand is $\alpha_{\pi^{-1}(k)}$. Graphically, we represent (b, π) like the braid b (as done on page 8 for $b = t_2$), with the k^{th} strand labelled by $\pi^{-1}(k)$.

ii) The *permutation groupoid* $\widetilde{S}_n$ is defined analogously, with $B_n(C)$ replaced by S_n : $(\pi', \pi\sigma)(\pi, \sigma) := (\pi'\pi, \sigma)$.

LEMMA AND DEFINITION 1.6. *Let $\varrho_1, \dots, \varrho_n \in \Delta(S_0)$. There is a groupoid homomorphism $\varepsilon_{\boldsymbol{\varrho}}$ from $\widetilde{B}_n(C)$ into the groupoid of unitary intertwiners such that*

$$\varepsilon_{\boldsymbol{\varrho}}(b,\pi) \in \mathrm{Int}(\, \varrho_{(\nu(b)\pi)^{-1}(1)} \cdots \varrho_{(\nu(b)\pi)^{-1}(n)} \mid \varrho_{\pi^{-1}(1)} \cdots \varrho_{\pi^{-1}(n)}\,), \tag{1.42}$$

which is uniquely characterized by

1. $\varepsilon_{\boldsymbol{\varrho}}(1,\pi) = \mathbf{1}_{\,\varrho_{\pi^{-1}(1)}\cdots\varrho_{\pi^{-1}(n)}}$
2. $\varepsilon_{\boldsymbol{\varrho}}(t_k,\pi) = \mathbf{1}_{\,\varrho_{\pi^{-1}(1)}\cdots\varrho_{\pi^{-1}(k-1)}} \times \varepsilon(\varrho_{\pi^{-1}(k)}, \varrho_{\pi^{-1}(k+1)}) \times \mathbf{1}_{\,\varrho_{\pi^{-1}(k+2)}\cdots\varrho_{\pi^{-1}(n)}}$
3. $\varepsilon_{\boldsymbol{\varrho}}(c_1,\pi) = \mathbf{V}_{\varrho_{\pi^{-1}(1)}} \times \mathbf{1}_{\,\varrho_{\pi^{-1}(2)}\cdots\varrho_{\pi^{-1}(n)}}$.

It also satisfies

$$\varepsilon_{\boldsymbol{\varrho}}(c_k,\pi) = \mathbf{1}_{\,\varrho_{\pi^{-1}(1)}\cdots\varrho_{\pi^{-1}(k-1)}} \times \mathbf{V}_{\varrho_{\pi^{-1}(k)}} \times \mathbf{1}_{\,\varrho_{\pi^{-1}(k+1)}\cdots\varrho_{\pi^{-1}(n)}} \,. \tag{1.43}$$

Here $c_k := t_{k-1}\cdots t_1 c_1 t_1 \cdots t_{k-1}$. Note that from the definition follows immediately the identity

$$\varepsilon_{\boldsymbol{\varrho}}(b,\pi) = \varepsilon_{\boldsymbol{\varrho}\cdot\pi^{-1}}(b,1)\,. \tag{1.44}$$

PROOF. $\varepsilon = \varepsilon_{\boldsymbol{\varrho}}$ can be extended from the generators to arbitrary braids by the requirement to be a homomorphism iff the relations of the generators are respected:

$$\begin{aligned}
\varepsilon(t_k,\tau_i\pi)\varepsilon(t_i,\pi) &= \varepsilon(t_i,\tau_k\pi)\varepsilon(t_k,\pi) \quad \text{if } |i-k|\ge 2\,, && (1.45)\\
\varepsilon(t_k,\tau_{k+1}\tau_k\pi)\varepsilon(t_{k+1},\tau_k\pi)\varepsilon(t_k,\pi) &= \varepsilon(t_{k+1},\tau_k\tau_{k+1}\pi)\varepsilon(t_k,\tau_{k+1}\pi)\varepsilon(t_{k+1},\pi) && (1.46)\\
\varepsilon(c_1,\pi)\varepsilon(t_1,\tau_1\pi)\varepsilon(c_1,\tau_1\pi)\varepsilon(t_1,\pi) &= \varepsilon(t_1,\tau_1\pi)\varepsilon(c_1,\tau_1\pi)\varepsilon(t_1,\pi)\varepsilon(c_1,\pi)\,. && (1.47)
\end{aligned}$$

Equation (1.45) is just a special case of the intertwiner identity $(\mathbf{T}_1\times\mathbf{1})\circ(\mathbf{1}\times\mathbf{T}_2) = \mathbf{T}_1\times\mathbf{T}_2 = (\mathbf{1}\times\mathbf{T}_2)\circ(\mathbf{T}_1\times\mathbf{1})$. Equation (1.46) follows from associativity of the product $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3$, if we choose $\mathbf{B}_j$ localized in $\tilde{I}_j \in \widetilde{\mathcal{K}}$ with $\tilde{I}_3 < \tilde{I}_2 < \tilde{I}_1 < (0,2\pi)\cdot\tilde{I}_3$, and apply Lemma 1.4 repeatedly. To verify equation (1.47), let $\tilde{J}_1, \tilde{J}_2 \in \widetilde{\mathcal{K}}$ with $N(\tilde{J}_1,\tilde{J}_2) = -1$, and let $\mathbf{D}_i = (\hat{\varrho}_i, D_i) \in \mathcal{F}(\tilde{J}_i)$, $i = 1,2$. Then $N(\tilde{J}_2,\tilde{J}_1) = 0$ and Lemma 1.4 implies

$$\begin{aligned}
\mathbf{D}_2\mathbf{D}_1 &= \varepsilon(\hat{\varrho}_2,\hat{\varrho}_1)\,\mathbf{D}_1\mathbf{D}_2\,, && \text{since } N(\tilde{J}_2,\tilde{J}_1)=0\,,\\
&= \varepsilon(\hat{\varrho}_2,\hat{\varrho}_1)\,(\mathbf{V}_{\hat{\varrho}_2}\times\mathbf{1}_{\,\hat{\varrho}_1})\,\varepsilon(\hat{\varrho}_1,\hat{\varrho}_2)\,(\mathbf{1}_{\,\hat{\varrho}_1}\times\mathbf{V}_{\hat{\varrho}_2}^{-1})\,\mathbf{D}_2\mathbf{D}_1\,, && \text{since } N(\tilde{J}_1,\tilde{J}_2)=-1\,.
\end{aligned}$$

Hence

$$\varepsilon(\hat{\varrho}_2,\hat{\varrho}_1)\,(\mathbf{V}_{\hat{\varrho}_2}\times\mathbf{1}_{\,\hat{\varrho}_1})\,\varepsilon(\hat{\varrho}_1,\hat{\varrho}_2) = (\mathbf{1}_{\,\hat{\varrho}_1}\times\mathbf{V}_{\hat{\varrho}_2}), \tag{1.48}$$

which we exploit in the first and last equations of the sequence

$$\begin{aligned}
&(\mathbf{V}_{\hat{\varrho}_1}\times\mathbf{1}_{\,\hat{\varrho}_2})\,\varepsilon(\hat{\varrho}_2,\hat{\varrho}_1)\,(\mathbf{V}_{\hat{\varrho}_2}\times\mathbf{1}_{\,\hat{\varrho}_1})\,\varepsilon(\hat{\varrho}_1,\hat{\varrho}_2)\\
&= (\mathbf{V}_{\hat{\varrho}_1}\times\mathbf{1}_{\,\hat{\varrho}_2})\,(\mathbf{1}_{\,\hat{\varrho}_1}\times\mathbf{V}_{\hat{\varrho}_2}) = (\mathbf{1}_{\,\hat{\varrho}_1}\times\mathbf{V}_{\hat{\varrho}_2})\,(\mathbf{V}_{\hat{\varrho}_1}\times\mathbf{1}_{\,\hat{\varrho}_2})\\
&= \varepsilon(\hat{\varrho}_2,\hat{\varrho}_1)\,(\mathbf{V}_{\hat{\varrho}_2}\times\mathbf{1}_{\,\hat{\varrho}_1})\,\varepsilon(\hat{\varrho}_1,\hat{\varrho}_2)\,(\mathbf{V}_{\hat{\varrho}_1}\times\mathbf{1}_{\,\hat{\varrho}_2})\,.
\end{aligned}$$

Putting $\hat{\varrho}_i = \varrho_{\pi^{-1}(k-2+i)}$, we see that this implies equation (1.47). Successive application of Equation (1.48) leads to Equation (1.43). □

LEMMA AND DEFINITION 1.7. *Let $\tilde{\mathbf{I}} = (\tilde{I}_1, \dots, \tilde{I}_n)$ where $\tilde{I}_k \in \widetilde{\mathcal{K}}$ are mutually causally disjoint. There is a groupoid homomorphism $\varphi_{\tilde{\mathbf{I}}} : \widetilde{S}_n \to \widetilde{B}_n(C)$ uniquely characterized by*

$$\varphi_{\tilde{\mathbf{I}}}(\tau_k,\pi) = (c_k^{-N} t_k c_{k+1}^{N}, \pi)\,, \quad \textit{where } N = N(\tilde{I}_{\pi^{-1}(k)}, \tilde{I}_{\pi^{-1}(k+1)})\,. \tag{1.49}$$

Here we used the notation $c_k := t_{k-1}\cdots t_1 c_1 t_1 \cdots t_{k-1}$. In the geometric representation[3] *of $\widetilde{B}_n(C)$, namely as the fundamental group of*

$$\left((\mathbb{R}^2 \setminus \{0\})^{\times n} \setminus \Delta_n\right) / S_n$$

with base point $\big((1,0),\dots,(n,0)\big)\cdot S_n$, the image of (p,π) under $\varphi_{\tilde{\mathbf{I}}}$ is given as follows. Let $\tilde{r}_k \in \widetilde{H}_i$ correspond to $\tilde{I}_k$ for $k = 1,\dots,n$. Fixing a timelike unit vector e, we project the

[3] see Appendix A.1 for more details.

spacelike directions $\mathbb{R}^+ r_k$ onto the spacelike hyperplane $e^\perp$ and identify them with points on the unit circle $S^1 \subset \mathbb{R}^2$. Let γ_k be paths in S^1 from r_0 to r_k representing $\tilde{r}_k$ after this identification. Then $\varphi_{\tilde{\mathbf{I}}}(p,\pi) = (b,\pi)$, where b is the homotopy class of the S_n-orbit of the path $(\beta_1,\dots,\beta_n)$ in $(\mathbb{R}^2 \setminus \{0\})^{\times n} \setminus \Delta_n$ which is defined by

$$\beta_i = (p(i), \gamma^{-1}_{\pi^{-1}(i)}) \, * \, (i \mapsto p(i), r_{\pi^{-1}(i)}) \, * \, (i, \gamma_{\pi^{-1}(i)}) \, . \tag{1.50}$$

Here we have identified $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{R}^+ \times S^1$ such that $(x,0)$ is identified with $(|x|, r_0)$.

Note, that $\varphi_{\tilde{\mathbf{I}}}(p,\pi)$ can be written in the form

$$\varphi_{\tilde{\mathbf{I}}}(p,\pi) =: \left(\varphi_{\tilde{\mathbf{I}}_{\pi^{-1}}}(p), \pi\right) \, , \tag{1.51}$$

where $\varphi_{\tilde{\mathbf{I}}_{\pi^{-1}}}(p)$ is the braid b which is geometrically described in the Definition. Further, this braid is mapped to p under the natural homomorphism ν from B_n onto S_n.

PROOF. The geometric description of $\varphi_{\tilde{\mathbf{I}}}(\pi,\sigma)$ by equation (1.50) together with equation (1.51) clearly yields a groupoid homomorphism. So we only have to check that it coincides with equation (1.49) on the generators (τ_k,π), $k = 1,\dots,n$. After relabelling, it is sufficient to take $\pi = 1$. For $N(\tilde{I}_k, \tilde{I}_{k+1}) = 0$, both equations (1.50) and (1.49) lead to $\varphi_{\tilde{\mathbf{I}}}(\tau_k, 1) = (t_k, 1)$. If $N(\tilde{I}_k, \tilde{I}_{k+1}) = N \neq 0$, let $\tilde{I}'_{k+1} := (0, 2\pi N) \cdot \tilde{I}_{k+1}$. Then $\gamma'_{k+1} = \gamma_{k+1} \cdot c^{-N}$, where c denotes the generator of $\pi_1(S^1, r_0)$, which encircles S^1 once in clockwise direction. Let further $(\beta_1,\dots,\beta_n)$ be the strands defining $\varphi_{\tilde{\mathbf{I}}}(\tau_k, 1)$ according to equation (1.50), and $(\beta_1,\dots,\beta'_{k+1},\dots,\beta_n)$ those defining $\varphi_{(\tilde{I}_1,\dots,\tilde{I}'_{k+1},\dots,\tilde{I}_n)}(\tau_k, 1)$. They are related by

$$\beta_{k+1} = (k, c^{-N}) \, * \, \beta'_{k+1} \, * \, (k+1, c^N) \, ,$$

which implies

$$\varphi_{\tilde{\mathbf{I}}}(\tau_k, 1) = (c_k^{-N}, \tau_k) \, \varphi_{(\tilde{I}_1,\dots,\tilde{I}'_{k+1},\dots,\tilde{I}_n)}(\tau_k, 1) \, (c_{k+1}^N, 1) \, . \tag{1.52}$$

But $\varphi_{(\tilde{I}_1,\dots,\tilde{I}'_{k+1},\dots,\tilde{I}_n)}(\tau_k, 1) = (t_k, 1)$, since $N(\tilde{I}_k, \tilde{I}'_{k+1}) = 0$. Hence, equation (1.52) implies the claimed equation (1.49). □

Now we can describe the commutation relations of a multiple product of generalized fields in terms of the intertwiner $\varepsilon_{\boldsymbol{\varrho}} \circ \varphi_{\tilde{\mathbf{I}}}$:

THEOREM 1.8. *Let for $k = 1,\dots,n$, $\mathbf{B}_k = (\varrho_k, B_k) \in \mathcal{F}(\tilde{I}_k)$, where $\tilde{I}_k \in \widetilde{\mathcal{K}}$ are mutually causally disjoint. Let $\boldsymbol{\varrho} := (\varrho_1,\dots,\varrho_n)$ and $\tilde{\mathbf{I}} := (\tilde{I}_1,\dots,\tilde{I}_n)$. Then for all $\pi', \pi \in S_n$ the following commutation relations hold:*

$$\mathbf{B}_{(\pi'\pi)^{-1}(n)} \cdots \mathbf{B}_{(\pi'\pi)^{-1}(1)} = \left(\varepsilon_{\boldsymbol{\varrho}} \circ \varphi_{\tilde{\mathbf{I}}}\right)(\pi', \pi) \, \mathbf{B}_{\pi^{-1}(n)} \cdots \mathbf{B}_{\pi^{-1}(1)} \, . \tag{1.53}$$

PROOF. As emphasized after equation (1.39), a family $\varepsilon(\pi', \pi)$ of intertwiners in terms of which the commutation relations (1.53) are satisfied, must be a groupoid homomorphism. Hence it suffices to check equation (1.53) on the generators $(\pi', \pi) = (\tau_k, \pi)$ for $k = 1,\dots,n-1$. For these, the claim follows from Lemma 1.4, which now reads

$$\mathbf{B}_1 \, \mathbf{B}_2 = \varepsilon_{\boldsymbol{\varrho}}(c_1^{-N} t_1 c_1^N \, , \, 1) \, \mathbf{B}_2 \, \mathbf{B}_1 \quad , N = N(\tilde{I}_1, \tilde{I}_2) \, .$$

□

Finally. we introduce the a concept which is important in the classification of the representations of the braid group arising in local quantum physics.

DEFINITION 1.9. Let $\varrho \in \Delta(S_0)$. A *left inverse* of ϱ is positive linear endomorphism ϕ_ϱ of the universal algebra, which leaves $\mathcal{A}_{\mathrm{u}}(S_0)$ invariant and satisfies

$$\begin{aligned} \phi_\varrho(\mathbf{1}) &= \mathbf{1} && \text{and} \\ \phi_\varrho\big(\varrho(A) B \varrho(C)\big) &= A \, \phi_\varrho(B) \, C && \text{for all } A, B, C \in \mathcal{A}_{\mathrm{u}} \, . \end{aligned} \tag{1.54}$$

Note that in particular $\phi_\varrho \circ \varrho = \mathrm{id}$, justifying the name. If ϱ is an irreducible massive single particle representation, then it has a unique left inverse, see e.g. [Haa92]. We shall have occasion to exploit the following observation. If T is a local self-intertwiner of ϱ, then equation (1.54) and the intertwiner propery imply that $\phi_\varrho(T)$ commutes with all elements in $\mathcal{A}_\mathrm{u}$, and is hence a multiple of unity.

One gets a numerical invariant of an irreducible sector $[\varrho]$ describing its statistics, the statistics parameter $\lambda_{[\varrho]}$, by applying ϕ_ϱ to the statistics operator $\varepsilon_\varrho := \varepsilon(\varrho, \varrho)$:

$$\lambda_{[\varrho]} \mathbb{1} := \phi_\varrho(\varepsilon_\varrho) \,.$$

Its phase $\omega_{[\varrho]}$ in the polar decomposition is called the statistics phase. If ϱ is in particular an automorphism, the statistics operator ε_ϱ must be a multiple of unity because it commutes with $\varrho^2(\mathcal{A}_\mathrm{u})$, and the statistics phase is then the corresponding factor. We will only consider sectors with finite statistics, i.e. with $\lambda_{[\varrho]} \neq 0$.

1.3. Single Particle Space of Plektons

We are interested in representations ϱ of $\mathcal{A}_\mathrm{u}$ which describe massive one particle states, and in addition allow for the construction of multiparticle states via Haag-Ruelle scattering theory. Mathematically, this may be accomplished by requiring ϱ to be a *massive single particle representation*, i.e. it is irreducible and covariant, with the corresponding energy momentum spectrum containing an isolated mass shell H_m as its lower boundary, see equation (0.9). The subspace of $\mathcal{H}_0$ corresponding to the H_m part of the spectrum will be denoted by $\mathcal{H}_\varrho^{(1)}$ and considered as the space of single particle state vectors with mass m in the sector ϱ :

$$\mathcal{H}_\varrho^{(1)} := \{\psi \in \mathcal{H}_0 \,|\, \mathrm{sp}_{P_\varrho}\, \psi \subset H_m \,\} = \{\psi \in \mathcal{H}_0 \,|\, P_\varrho^2 \psi = m^2\, \psi \,\} \,. \tag{1.55}$$

Following Wigner's analysis of irreducible ray representations of the Poincaré group, elements of $\mathcal{H}_\varrho^{(1)}$ can be identified with functions on H_m with values in a "little Hilbert space" V, square integrable with respect to the Lorentz invariant measure $\mathrm{d}\mu$ on H_m. In view of the representation $U_\varrho|\mathcal{H}_\varrho^{(1)}$, the role of H_m is that it is an orbit of $\tilde{L}_+^\uparrow$ in momentum space $\mathbb{R}^3$, and the role of V is that it carries a representation D of the "little group", i.e. the subgroup of $\tilde{L}_+^\uparrow$ which leaves a fixed point $p_0 \in H_m$ invariant. Choosing the point $p_0 = (m, 0, 0)$, the little group is easily seen to coincide with the universal covering group of the rotation subgroup, i.e. with $\mathbb{R}$. Wigner's analysis asserts that there is a unitary

$$W_\varrho : \mathcal{H}_\varrho^{(1)} \to L^2(H_m, \mathrm{d}\mu; V) \tag{1.56}$$

which intertwines $U_\varrho|\mathcal{H}_\varrho^{(1)}$ with the following representation $\tilde{U}$ of $\tilde{P}_+^\uparrow$ in $L^2(H_m, \mu; V)$:

$$(\tilde{U}_\varrho(x, \tilde{g})\, \psi)(p) = e^{ix\cdot p}\, D(\Omega(\tilde{g}, p))\, \psi(g^{-1}\cdot p) \quad \text{ for all } \tilde{g} \in \tilde{P}_+^\uparrow \,. \tag{1.57}$$

Here, $\Omega(\tilde{g}, p)$ is the element of the little group called Wigner rotation, see equation (A.18) in the Appendix. We assume D to be a *finite* sum of irreducible, hence one dimensional, representations of the little group $\mathbb{R}$, each characterized by a real parameter s_l:

$$V = \mathbb{C}^g \,, \quad D(\omega) = \bigoplus_{l=1}^{g} e^{is_l\omega}\, 1 \,. \tag{1.58}$$

$U_\varrho|\mathcal{H}_\varrho^{(1)}$ then decomposes into g irreducible representations of $\tilde{P}_+^\uparrow$, each corresponding to one particle type carrying the charge quantum numbers of ϱ. Physically, m is interpreted as the mass of the particles, and $s_1, \ldots, s_g$ as their spins. Note that the spins may differ only by integers, since $U_\varrho(2\pi)$ must be a multiple $\exp 2\pi i s[\varrho]$ of unity due to the irreducibility of ϱ, see equation (0.8):

$$s_l = s[\varrho] \bmod 1 \,, \quad l = 1, \ldots, g \,.$$

1.4. Charge Conjugation and P_1CT -Theorem

A representation $\pi_0\bar{\varrho}$ of the observable algebra is said to be *conjugate* to $\pi_0\varrho$, if the product $\pi_0\bar{\varrho}\varrho$ contains the vacuum representation. It is unique up to equivalence. If ϱ describes one particle states, then $\bar{\varrho}$ is meant to describe the corresponding antiparticles.[4] Physically, one can think of antiparticle states arising by creating a chargeless particle-antiparticle state from the vacuum, and then shifting the particle "behind the moon". Mathematically, this idea can be made rigorous as a constructive definition for $\bar{\varrho}$, using chains of charge transporters for ϱ. Here, we will use instead a different representative $\bar{\varrho}$ of the conjugate sector which is defined via modular theory.

"Modular" Charge Conjugation. Let W_1 be the wedge in $\mathcal{M}^3$ defined by

$$W_1 := \{x\,,\,|x^0| < x^1\}\,, \tag{1.59}$$

and j be the reflexion at the vertex line of W_1, i.e. the proper Poincaré transformation given by

$$j\,(x^0, x^1, x^2) = (-x^0, -x^1, x^2)\,. \tag{1.60}$$

Let further $S_0(W_1)$ be the Tomita operator of the von Neumann algebra $\mathcal{A}(W_1)$, i.e. the closed antilinear operator characterized by

$$S_0(W_1)\,A\Omega = A^*\Omega \quad \text{for all } A \in \mathcal{A}(W_1)\,. \tag{1.61}$$

It has a unique polar decomposition $S_0(W_1) = J_0\,\Delta_0^{\frac{1}{2}}$, where J_0 is an antilinear isometry satisfying $J_0^2 = \mathbb{1}$ and Δ_0 is a positive operator (see, e.g. [BR87]). J_0 is called the *modular conjugation* and Δ_0 the *modular operator* of $\mathcal{A}(W_1)$, and the pair is called the *modular objects* of $\mathcal{A}(W_1)$. As an input for the further analysis we assume that the modular conjugation has a direct geometric significance, which has been shown by Bisognano and Wichmann to hold for Wightmann fields [BW75]:

ASSUMPTION 1 (Modular Covariance of the Observables). The vacuum representation U_0 of $P_+^\uparrow$ extends to a representation of P_+ by the definition

$$U_0(j) := J_0\,, \tag{1.62}$$

such that $\mathrm{Ad}\,U_0(j) : \mathcal{A}(I) \to \mathcal{A}(j \cdot I)$.

Then also the representation α^0 of $P_+^\uparrow$ by automorphisms of $\mathcal{A}_\mathrm{u}$ extends to a representation of P_+ satisfying in particular

$$\mathrm{Ad}U_0(j) \circ \pi_0 = \pi_0 \circ \alpha_j^0 \quad \text{and} \tag{1.63}$$

$$\alpha_j^0 : \mathcal{A}_\mathrm{u}(I) \to \mathcal{A}_\mathrm{u}(j \cdot I)\,. \tag{1.64}$$

From now on we assume that S_0 is invariant under j, which is compatible with our earlier requirement that S_0 be invariant under time reflection. In fact, we will later choose S_0 to be centered along the positive x_2-axis.

DEFINITION AND LEMMA 1.10. Let $\varrho \in \Delta(S_0)$ with finite statistics, i.e. $\lambda_{[\varrho]} \neq 0$. Then the endomorphism

$$\bar{\varrho} := \alpha_j^0 \circ \varrho \circ \alpha_j^0 \tag{1.65}$$

is also localized in S_0, and it is a conjugate of ϱ, i.e. $\pi_0\bar{\varrho}\varrho \supset \pi_0$. Further, it is irreducible if ϱ is. $\bar{\varrho}$ will be called the *modular conjugate* of ϱ. A left inverse of $\bar{\varrho}$ is given by

$$\phi_{\bar{\varrho}} = \alpha_j^0 \circ \phi_\varrho \circ \alpha_j^0\,,$$

where ϕ_ϱ is a left inverse of ϱ.

PROOF. This is shown in the article [GL92] of D. Guido and R. Longo. □

This choice of a conjugate representation has the advantage that one can easily derive the well-known results concerning the particle-antiparticle symmetry, and enjoys the property that $\bar{\bar{\varrho}} = \varrho$ and $\overline{\varrho\sigma} = \bar{\varrho}\,\bar{\sigma}$.

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[4]That it actually does, is the content of Proposition 1.11.

Particle-Antiparticle Symmetry. We recall that $\tilde{P}_+$ can be obtained from $\tilde{P}_+^\uparrow$ by adjoining an element $\tilde{j}$ satisfying the relations

$$\tilde{j}^2 = 1 \quad \text{and} \quad \tilde{j}\ (x,(\gamma,\omega))\ \tilde{j} = (jx,(\bar{\gamma},-\omega))\ . \tag{1.66}$$

Under the homomorphism $\Lambda : \tilde{P}_+ \to P_+$ it projects to the proper Poincaré transformation $\Lambda(\tilde{j}) =: j$ of $\mathcal{M}^3$ from equation (1.60). The representation of $\tilde{P}_+^\uparrow$ defined by

$$U_{\bar{\varrho}}(\tilde{g}) := U_0(j)\, U_\varrho(\tilde{j}\tilde{g}\tilde{j})\, U_0(j) \tag{1.67}$$

implements $\alpha^0(\tilde{P}_+^\uparrow)$ in the representation $\pi_0\bar{\varrho}$, i.e. satisfies

$$\mathrm{Ad}U_{\bar{\varrho}}(\tilde{g}) \circ \pi_0\bar{\varrho} = \pi_0\bar{\varrho} \circ \alpha^0_{\tilde{g}}\ . \tag{1.68}$$

Further, the representation $U_{\bar{\varrho}}$ is fixed by this property if ϱ is irreducible. To see this, let $\bar{U}$ be another representation satisfying (1.68). Then $\tilde{g} \mapsto \bar{U}(\tilde{g})^{-1}U_{\bar{\varrho}}(\tilde{g})$ is a representation of $\tilde{P}_+^\uparrow$ in $\pi_0\bar{\varrho}(\mathcal{A}_\mathrm{u})' = \mathbb{C}1\!\!1$, which must be trivial. Also, relation (1.67) carries over to products $\varrho_1\varrho_2$ of endomorphisms, if $U_{\varrho_1\varrho_2}$ is related to U_{ϱ_1} and U_{ϱ_2} by formula (1.7). This is shown by first deriving the formula $Y_{\bar{\varrho}}(\tilde{g}) = \alpha^0_j(Y_\varrho(\tilde{j}\tilde{g}\tilde{j}))$ from equation (1.67).

PROPOSITION 1.11 (Particle-Antiparticle Symmetry). *i) If $\pi_0\varrho$ is an irreducible representation of $\mathcal{A}_\mathrm{u}$, then the spin phases and the statistics parameters of of $[\varrho]$ and $[\bar{\varrho}]$ coincide:*

$$s[\bar{\varrho}] = s[\varrho] \bmod 1 \quad \textit{and} \quad \lambda_{[\bar{\varrho}]} = \lambda_{[\varrho]}\ .$$

In addition, the statistics operators are related by $\varepsilon_{\bar{\varrho}} = \alpha^0_j(\varepsilon_\varrho)^{-1}$.

ii) Let ϱ be a massive single particle representation of $\mathcal{A}_\mathrm{u}$ with finite degeneracies, see equation (1.58)*. Then the conjugate representation $\bar{\varrho}$ contains particles with the same mass, spins and degeneracies:*

$$U_{\bar{\varrho}}\,|_{\mathcal{H}^{(1)}_{\bar{\varrho}}} \cong U_\varrho\,|_{\mathcal{H}^{(1)}_\varrho}\ . \tag{1.69}$$

REMARK. The last statement (1.69) still holds if P_ϱ^2 has several eigenvalues and $\mathcal{H}^{(1)}_\varrho$ is defined as the span of the eigenvectors of P_ϱ^2.

PROOF. i) The first statement follows immediately from equation (1.67) and antilinearity of $U_0(\tilde{j})$. To show the statement about $\varepsilon_{\bar{\varrho}}$, let V be a charge transporter for ϱ from S_0 to some cone $S \subset S_0'$ along a path $\tilde{I}$ with $N(S_0,\tilde{I}) = 0$. Then $\alpha^0_j(V)$ transports $\bar{\varrho}$ from S_0 to jS along the path $\tilde{j}\tilde{I}$ which has $N(S_0,\tilde{j}\tilde{I}) = -1$. Hence, according to equation (1.31), we have

$$\begin{aligned} \varepsilon_\varrho(0) &= \qquad V^*\,\varrho(V) \quad \text{and} \\ \varepsilon_{\bar{\varrho}}(-1) &= \alpha^0_j(V)^*\,\bar{\varrho}(\alpha^0_j(V)) \equiv \alpha^0_j(V^*\,\varrho(V))\ , \end{aligned}$$

and the claim follows from $\varepsilon_{\bar{\varrho}}(-1) = \varepsilon_{\bar{\varrho}}^{-1}$.

ii) Equation (1.67) implies, remembering that $U_0(j)$ is antilinear, that for all $f \in \mathcal{S}(\mathcal{M}^3)$

$$U_0(j)\, f(P_{\bar{\varrho}}) = f^j(P_\varrho)\, U_0(j)\ , \qquad \text{where} \qquad f^j(p) := \overline{f(-j\cdot p)}\ . \tag{1.70}$$

This in turn implies that

$$\mathrm{sp}_{P_\varrho}\, \psi = -j\, \mathrm{sp}_{P_{\bar{\varrho}}}\, U_0(j)\psi \qquad \text{for all } \psi \in \mathcal{H}_0\ . \tag{1.71}$$

Recalling that the energy momentum spectrum is Lorentz invariant, this shows that $\mathrm{spec}\, P_{\bar{\varrho}} = \mathrm{spec}\, P_\varrho$ and in particular that $\bar{\varrho}$ is a massive single particle representation with the same mass m as ϱ, and that $U_0(j)$ is an antiunitary map from $\mathcal{H}^{(1)}_\varrho$ onto $\mathcal{H}^{(1)}_{\bar{\varrho}}$ and vice versa. We define an antilinear operator $\hat{J}$ on $\mathcal{H}^{(1)}_\varrho = L^2(H_m, \mathrm{d}\mu; \mathbb{C}^g)$ by

$$(\hat{J}\psi)_l(p) := \overline{\psi_l(-jp)}\ . \tag{1.72}$$

$\hat{J}$ is a conjugation which extends $\tilde{U}_\varrho$ to a representation of $\tilde{P}_+$:

$$\begin{aligned} \left(\hat{J}\,\tilde{U}_\varrho(x,\tilde{g})\,\hat{J}\,\psi\right)_l(p) &= e^{-ix\cdot(-j)p}\, e^{-is_l\,\Omega(\tilde{g},-jp)}\, \psi_l((-j)g^{-1}(-j)\,p) \\ &= e^{i(jx)\cdot p}\, e^{is_l\,\Omega(\tilde{j}\tilde{g}\tilde{j},p)}\, \psi_l((jgj)^{-1}\,p) \\ &= \left(\tilde{U}_\varrho(\tilde{j}(x,\tilde{g})\tilde{j})\,\psi\right)_l(p)\ . \end{aligned} \tag{1.73}$$

Here we have used formula (A.20) from the appendix: $\Omega(\tilde{g}, -jp) = -\Omega(\tilde{j}\tilde{g}\tilde{j}, p)$. From equations (1.67) and (1.73) follows that the unitary operator $U_0(j)\, W_{\bar{\varrho}}^{-1} \hat{J} W_\varrho$ from $\mathcal{H}_\varrho^{(1)}$ onto $\mathcal{H}_{\bar{\varrho}}^{(1)}$ intertwines $U_{\bar{g}}|_{\mathcal{H}_{\bar{\varrho}}^{(1)}}$ with $U_\varrho|_{\mathcal{H}_\varrho^{(1)}}$, as claimed. □

From part ii) of the Proposition we know that we can choose the intertwiners W_ϱ and $W_{\bar{\varrho}}$ from equation (1.56) such that they intertwine the restrictions of U_ϱ and $U_{\bar{\varrho}}$ to the respective single particle spaces with one and the same representation $\tilde{U}_\varrho \equiv \tilde{U}_{\bar{\varrho}}$ on $L^2(H_m, \mathrm{d}\mu; \mathbb{C}^g)$.

LEMMA 1.12. *Let W_ϱ and $W_{\bar{\varrho}}$ be chosen as above. Then*

$$W_{\bar{\varrho}}\, U_0(\tilde{j})\,|_{\mathcal{H}_{\bar{\varrho}}^{(1)}} = \tilde{U}_\varrho(\tilde{j})\, W_\varrho\,, \qquad \textit{where} \tag{1.74}$$

$$(\tilde{U}_\varrho(\tilde{j})\,\psi)(p) := c_\varrho\, \overline{\psi(-j\cdot p)} \qquad \textit{for all } \psi \in L^2(H_m, \mathrm{d}\mu; \mathbb{C}^g)\,. \tag{1.75}$$

Here c_ϱ is a matrix in $U(g)$. The matrices c_ϱ and $c_{\bar{\varrho}}$ are transposed to each other, $c_{\bar{\varrho}} = c_\varrho^t$.

PROOF. We have chosen W_ϱ and $W_{\bar{\varrho}}$ such that the operator $W_{\bar{\varrho}}^{-1} W_\varrho$ from $\mathcal{H}_\varrho^{(1)}$ to onto $\mathcal{H}_{\bar{\varrho}}^{(1)}$ intertwines $U_{\bar{\varrho}}|\mathcal{H}_{\bar{\varrho}}^{(1)}$ with $U_\varrho|\mathcal{H}_\varrho^{(1)}$, just as the operator $U_0(j)\, W_{\bar{\varrho}}^{-1} \hat{J} W_\varrho$, as we have seen in the last proof. Recall that $\hat{J}$ has been defined in equation (1.72). This implies that $W_{\bar{\varrho}} U_0(\tilde{j}) W_\varrho^{-1} \hat{J}$ is in $\tilde{U}_\varrho(\tilde{P}_+^\uparrow)'$, which coincides with $\mathbf{1} \otimes \mathrm{End}(\mathbb{C}^g)$. Being unitary, it must be an element c_ϱ in $U(g)$. This shows equation (1.75). Let $c_{\bar{\varrho}}$ be defined analogously, with ϱ and $\bar{\varrho}$ interchanged in the above argumentation. Then we conclude that

$$\mathbf{1} = W_\varrho\, U_0(\tilde{j})^2\, W_\varrho^{-1} = c_\varrho \hat{J}\, c_{\bar{\varrho}} \hat{J} = c_\varrho \overline{c_{\bar{\varrho}}}\,,$$

which shows that $c_{\bar{\varrho}} = c_\varrho^t$. □

P₁CT -Operator. $U_0(j)$ induces an operator $U(\tilde{j})$ in $\mathcal{H}$ which we may interpret as a PCT operator. Let

$$U(\tilde{j})\,(\varrho, \psi) := (\bar{\varrho}, U_0(j)\psi) \qquad \text{for all } (\varrho, \psi) \in \mathcal{H} \text{ and} \tag{1.76}$$

$$\alpha_{\tilde{j}}\,(\varrho, B) := (\bar{\varrho}, \alpha_j^0(B)) \qquad \text{for all } (\varrho, B) \in \mathcal{F}\,. \tag{1.77}$$

Let further $\tilde{j}$ act in $\widetilde{\mathcal{K}}$ as follows. If $\tilde{I}$ be the homotopy class of a path $(I_0 = S_0, \dots, I_m)$ in $\mathcal{K}$, then we define $\tilde{j} \cdot \tilde{I}$ as the homotopy class of the path $(j \cdot I_0, \dots, j \cdot I_m)$. By this definition, the action of $\tilde{P}_+^\uparrow$ on $\widetilde{\mathcal{K}}$ declared in equation (1.28) is extended to an action of $\tilde{P}_+$.

PROPOSITION 1.13 (P₁CT -Theorem). *$U(\tilde{j})$ and $\alpha_{\tilde{j}}$ extend the actions U and α of $\tilde{P}_+^\uparrow$ on $\mathcal{H}$ and the field bundle $\mathcal{F}$, respectively*[5] *to actions of $\tilde{P}_+$ in the sense that they satisfy*

$$U(\tilde{j})U(\tilde{g})U(\tilde{j})(\varrho, \psi) = U(\tilde{j}\tilde{g}\tilde{j})(\varrho, \psi) \qquad \textit{and} \tag{1.78}$$

$$\alpha_{\tilde{j}}\alpha_{\tilde{g}}\alpha_{\tilde{j}}(\varrho, B) = \alpha_{\tilde{j}\tilde{g}\tilde{j}}(\varrho, B)\,. \tag{1.79}$$

$U(\tilde{j})$ implements $\alpha_{\tilde{j}}$ in the sense that

$$U(\tilde{j})\,\mathbf{B} \cdot U(\tilde{j})\,\psi = \alpha_{\tilde{j}}(\mathbf{B}) \cdot \psi\,. \tag{1.80}$$

In addition, $\alpha_{\tilde{j}}$ acts geometrically on the field bundle $\mathcal{F}$:

$$\alpha_{\tilde{j}} : \mathcal{F}(\tilde{I}) \to \mathcal{F}(\tilde{j} \cdot \tilde{I})\,. \tag{1.81}$$

PROOF. Equations (1.78) and (1.79) follow immediately from equation (1.67). To prove equation (1.81), let $\tilde{I}$ be as above and $(\varrho, B) \in \mathcal{F}(\tilde{I})$. We have to exhibit a sequence of intertwiners $\bar{U}_k \in \mathcal{A}_\mathrm{u}(j \cdot I_k \cup j \cdot I_{k+1}), k = 0, \dots, m-1$, such that $\mathrm{Ad}(\bar{U}_k \cdots \bar{U}_1) \circ \bar{\varrho} \in \Delta(j \cdot I_k)$ and $\bar{U}_m \cdots \bar{U}_1\, \alpha_j^0(B) \in \mathcal{A}_\mathrm{u}(j \cdot I_m)$. Let U_k be a choice of intertwiners by virtue of which (ϱ, B) is localized in the path $(I_0, \dots, I_m)$, i.e. in particular $\varrho_k := \mathrm{Ad}(U_k \cdots U_1) \circ \varrho \in \Delta(I_k)$. Now let $\bar{U}_k := \alpha_j^0(U_k)$. This intertwiner does the job due to equation (1.64), and to the fact that $\alpha_j^0 \circ \varrho_k \circ \alpha_j^0 \in \Delta(j \cdot I_k)$ since $\varrho_k \in \Delta(I_k)$. □

To summarize, $U(\tilde{j})$ can be interpreted as a P₁CT operator: It is an antilinear representor of the geometrical transformation $\tilde{j}$,[6] mapping $\mathcal{H}_\varrho$ onto the conjugate 'sector' $\mathcal{H}_{\bar{\varrho}}$, effecting the

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[5]see equation (1.23) and (1.24)

[6]The subscript '1' indicates, following Bernd Kuckert [Kuc95], that the spatial reflection is performed only at the 1-axis.

particle-antiparticle symmetry stated in Proposition 1.11, and implementing the geometrical action $\alpha_{\tilde{j}} : \mathcal{F}(\tilde{I}) \to \mathcal{F}(\tilde{j} \cdot \tilde{I})$. In chapter 3 we will also show that it has the familiar commutation relation with the S-matrix: $U(\tilde{j})\, S = S^{-1}\, U(\tilde{j})$. Finally in the case of anyons, where one has a genuine field algebra replacing the field bundle, $U(\tilde{j})$ is essentially the modular conjugation of the field algebra associated to W_1, see Proposition 4.16.