Introduction

The present thesis is concerned with the theory of identical relativistic particles in three-dimensional space-time which do not obey Bose-Einstein or Fermi-Dirac statistics. Despite considerable efforts by many theoretical physicists, the quantum theory of such particles is still poorly understood, and the aim of this thesis is to clarify some of its aspects. In particular we will address the issue of free relativistic fields associated to such particles. We recall that in quantum field theories, like e.g. quantum electrodynamics, free fields are usually the starting point for a discussion of the more realistic dynamics of interacting fields, at least perturbatively.

It was Leinaas and Myrheim who first realized that in two space dimensions there may be particles other than bosons or fermions [LM77]. Statistics means in the context of quantum mechanics the effect of exchanging identical particles. In group-theoretic terms it is usually described by a one-dimensional representations of the permutation group corresponding to symmetric or antisymmetric wavefunctions respectively, depending on the nature of the particles which it describes, namely bosons or fermions. Leinaas and Myrheim realized that the arguments implying this alternative were not cogent in two space dimensions, and proposed a new statistics where the relevant group is the so-called braid group.

Let us briefly sketch their arguments leading to this conclusion. In quantum mechanics one assumes that a state of n identical particles can be described by labelling the particles with indices $1, \ldots, n$ and associating a wavefunction ψ to it. Indistinguishability of the n particles is then taken into account by requiring that a wavefunction ψ_π which arises from a different labelling $\pi(1), \ldots, \pi(n)$ must yield the same expectation values for observables. Then one can indeed conclude that ψ must be either symmetric, $\psi_\pi = \psi$, or antisymmetric, $\psi_\pi = \operatorname{sgn}\pi \cdot \psi$. Leinaas and Myrheim criticized that the introduction of particle indices at the beginning of this analysis cannot be operationally justified. They took the indistinguishability of the particles into account already in the classical configuration space: A configuration of n indistinguishable particles should not be described by an n-tupel of single particle configurations, rather, all tupels which differ only by a permutation should be identified. This had already been proposed by Laidlaw and DeWitt [LD71]. From here the analysis of Leinaas and Myrheim consistently leads to a formulation, where the behaviour of the wavefunctions under the exchange of particles depends on paths along which the particles are exchanged. In this formulation, a wavefunction describing e.g. two particles in the plane picks up a phase factor ω if these are exchanged in counter-clockwise direction, and ω^{-1} if they are exchanged in clockwise direction. Since these paths are topologically distinct, ω may be different from ω^{-1}. In the usual quantum mechanical discussion there is no dependence on the paths, which implies that here ω can only be 1 or -1: Thus bosons and fermions appear as a special case in the formalism proposed by Leinaas and Myrheim, and an arbitrary phase $\omega \neq \pm 1$ indicates that we describe a 'braiding' and not only a permutation. Indeed, in the case of n particles the phases involved establish a one-dimensional representation of the braid group[1], and the wavefunctions transform under exchange of particles (along paths) according to this representation.

The braid group for n particles, being a non-Abelian group of infinite order, is much more complicated than the permutation group and its irreducible representations are still not fully classified. There are however, as in the above example, one-dimensional representations, characterized by a phase ω. The associated particles are called *anyons* [Wil82], since 'any' phase is

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[1] The braid group is in this context the fundamental group of the n-particle configuration space, supposed that one excludes the set of points where two or more particles occupy the same configuration. But this can be justified *a posteriori* [Wil90].

possible. If the representation of the braid group is irreducible and not one-dimensional, one speaks of *plektons* [FRS92]. It is noteworthy that in three-dimensional space the two paths considered above are topologically equivalent, which implies that here ω can only be 1 or -1, corresponding to the bosonic or fermionic case. Thus the anyonic braid group statistics is a generalization which interpolates between Bose and Fermi statistics and may occur in two space dimensions. Their analysis and the occurrence of braid group statistics also extends to one space dimension, but this situation will not be considered here. Quantum mechanical models of anyons have first been discussed by Wilczek [Wil82]. In the sequel a strong interest of physicists in braid group statistics arose because of its possible relevance for the explanation of up to then poorly understood 'two-dimensional' condensed matter phenomena. In particular, there were several proposals to explain the fractional quantum Hall effect with such models [Lau83, Hal84, ASW84], as well as with nonrelativistic quantum field theoretic models exhibiting anyonic quasiparticles [ZHK89, FK91]. Further, arguments have been proposed that fractional statistics might play a role in high-T_c superconductivity [Lau88, Wil90]. Inspired by the increasing interest in such models, it has been shown by Fredenhagen and by Fröhlich and Marchetti [Fre89, FM89] that also in the framework of algebraic quantum field theory the possibility of braid group statistics arises in $d = 2 + 1$, if the relevant charge is not localizable in bounded spacetime regions, but only in regions which extend to infinity in some spacelike direction[2].

As already mentioned, despite considerable efforts the quantum theory of particles with braid group statistics is still poorly understood, and the aim of this thesis is to clarify some of its aspects. Apart from giving guidelines for the construction of models, such structural analysis may also help to recognize in which physical phenomena plektons could play a role, and to predict new phenomena which may be described in terms of plektons or anyons. The analysis will be performed in the framework of algebraic quantum field theory and employ its well-developped technical arsenal. In this theory one has clearly stated and physically motivated assumptions resting strongly on the principle of locality, which are expected to be met by reasonable models. It has turned out successful in analyzing the general strucure of quantum field theories, e.g. it has given reasons for the particle-antiparticle symmetry and for the connection of spin and statistics. Within this set-up, which offers a conceptionally clear-cut notion of exchange statistics, many investigations on plektons have been performed, achieving a number of model-independent results. Thus, a 'weak' spin-statistics connection has been established: In two space dimensions the spin is not quantized, and models whose spin is not integer or half integer necessarily have to obey braid group statistics [FM89, Fre89]. Also, the associated (Haag-Ruelle) scattering theory is understood [FM91, FGR96]. Unfortunately, none of the hitherto proposed explicit models of relativistic anyons in (continuous) three dimensional spacetime [Sem88, Swa90, JW90, BCS93, CP96] is physically completely satisfying, in particular from the viewpoint of algebraic quantum field theory. But a relativistic model satisfying these axioms is highly desirable, not at least because non-relativistic quantum theory should be obtainable as the limiting case of an underlying relativistic theory. It seems particularly desirable to have, as a starting point for further models, a 'free field' of relativistic anyons, having suitable localization properties. In fact, it has been shown that such fields cannot be pointlike localized, rather they must be localized around 'semi-infinite strings', which extend from space-time points to spacelike infinity. The difficulties associated with the construction of local free fields become apparent on the level of the plektonic Hilbert space [FGR96, FM91, MS95], which is also introduced in this thesis. Recall that in the case of spin $\frac{1}{2}$ particles, one has to go over from a spin (or 'Wigner') to a spinor (or 'covariant') basis in the momentum space formulation of the one particle theory, in order to obtain anticommuting local spinor fields. But in the case of unquantized spin, one has no satisfactory corresponding procedure. The construction of free anyonic fields was in fact the original aim of this thesis.

However, it turned out that no such model, in a quite general sense, can exist. This is one of our main results (Theorem 5.3). In order to provide firm ground on which further constructive attempts shall be performed, the main part of the thesis is devoted to establishing

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[2]which is in fact the typical situation for massive particles [BF82].

a number of model-independent results for plektons and, most of all, anyons in the framework of algebraic quantum field theory. Most notably among the results are a version of the PCT theorem adapted to the three dimensional situation with braid group statistics (Proposition 1.13 and Theorem 4.14) and a spin-statistics theorem for anyons (Theorem 4.15). We recall that the famous PCT theorem in local quantum physics states that (in four dimensions) the combined transformation of the total space-time inversion (PT) and charge conjugation (C) is an anti-unitarily implemented symmetry in every model of quantum field theory [Jos57]. A large part of the thesis is concerned with the Hilbert space of scattering states for plektons and anyons. Referring to this space, explicit formulae are given for the ray representation of the Poincaré group (Propositions 3.4 and 4.20), for the PCT operator (Propositions 3.7 and 4.20) and for the Tomita operators of the anyonic field algebra associated to wedge regions, modulo the S-matrix (Lemma 4.21 and Proposition 4.22). The family of Tomita-operators of the local field algebras encodes some amount of information about the latter, in paricular it carries over the localization concept to the Hilbert space. Thus our construction of the 'incoming' Tomita-operators for wedge regions should be helpful for further model-building. Before a more detailed overview of the results can be given, a brief introduction into some of the concepts of algebraic quantum field theory is in order, which will also indicate how plektons arise in this framework.

Algebraic Quantum Field Theory and Plektons

The development of algebraic quantum field theory [Haa59, HK64] was motivated by the wish to have a mathematically well-defined and consistent frame capable of describing the phenomena of elementary particle physics, which is based entirely on observable quantities, and which incorporates the principle of locality into its foundation.

The notion of 'observable' used here is connected with restrictions to the quantum mechanical superposition principle, so-called superselection rules, whose existence in quantum field theory has been pointed out by Wick, Wightman and Wigner in [WWW52]. For example, in quantum electrodynamics operators corresponding to observable quantities must have vanishing matrix elements between state vectors carrying different electrical charges. Otherwise, the relative phase between states with different charges would be measurable – which is not the case: no one has ever prepared a coherent superposition of such states. A second example is the so-called univalency superselection rule. A 360° rotation must act as the identity on observables. Hence they must have vanishing matrix elements between a vector with integer and one with half integer spin. Conventionally, quantum field theory is formulated in terms of quantum fields which are in general unobservable, like e.g. the Dirac field which is not observable according to both of the mentioned superselection rules. The use of unobservable objects is naturally connected with some amount of arbitrariness, which has to be fixed by physically poorly motivated conventions: e.g., normal commutation relations can be altered by a Klein transformation with no consequences on the observable level. From an operational point of view it is desirable and should be possible to eliminate the unobservable fields from the description and base the theory on observable quantities only.

The principle of 'locality' in this context asserts that measurements are performed in finite regions of spacetime, and that measurements performed in space-like separated regions should be compatible, i.e. they should not influence each other.

Assumptions. The formalism of algebraic quantum field theory (AQFT in the following) takes these requirements into account as follows. Associated to every bounded spacetime region $\mathcal{O}$ there is a von Neumann algebra $\mathcal{A}(\mathcal{O})$, encoding the observables measurable within $\mathcal{O}$. A von Neumann algebra is, up to isomorphism, an algebra of bounded operators acting on some Hilbert space, which is stable under taking adjoints and is closed in the weak (or equivalently, in the strong) topology. By the use of bounded operators instead of the unbounded quantum fields of conventional quantum field theory and also of the Wightman framework, the domain problems of the unbounded operators are avoided, whose physical significance is questionable. This step is justified because one can reconstruct an unbounded operator by the algebra of bounded functions of it, all of which mathematically represent one and the same measurement device, only with different scales. This point of view has been advocated already by I. E. Segal [Seg47].

The algebras $\mathcal{A}(\mathcal{O})$ are in the origial framework of AQFT assumed to be abstract von Neumann algebras, but in our context it is *a fortiori* justified to consider them as given in the vacuum representation, i.e. as concrete operator algebras acting in a Hilbert space $\mathcal{H}_0$. The basic idea of AQFT is that the assignment

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

encodes all relevant information of the theory, and the axioms of AQFT refer to this assignment. It is assumed to be inclusion preserving

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2 \quad (\textit{Isotony}) , \tag{0.1}$$

and to satisfy microcausality:

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2' \quad (\textit{Locality}) . \tag{0.2}$$

Here $\mathcal{A}(\mathcal{O})'$ denotes the commutant of $\mathcal{A}(\mathcal{O})$ in $\mathcal{B}(\mathcal{H}_0)$, and $\mathcal{O}'$ is the causal complement of $\mathcal{O}$ in Minkowski space, i.e. the set of all point which are spacelike to all points in $\mathcal{O}$. The family of von Neumann algebras is assumed to be *Poincaré covariant*, i.e. $\mathcal{H}_0$ carries a strongly continuous unitary representation U_0 of the Poincaré group $P_+^\uparrow$ such that

$$\begin{aligned} \text{Ad}U_0(g) : \mathcal{A}(\mathcal{O}) &\to \mathcal{A}(g \cdot \mathcal{O}) \qquad \text{for all } g \in P_+^\uparrow \\ A &\mapsto U_0(g) \, A \, U_0(g)^{-1} . \end{aligned} \tag{0.3}$$

These three assumptions incorporate the principle of locality. Positivity of the energy (or stability) is implemented by the assumption that the energy-momentum operators, defined as the 3 generators of the representation $x \mapsto U_0(x)$ of space-time translations, satisfy the *spectrum condition*, i.e. their joint spectrum is contained in the closed forward light cone. Further, the existence of a unique, up to a phase, normalized vector is assumed which is invariant under translations – the vacuum vector Ω. Correspondingly, we will call $\mathcal{H}_0$ the vacuum Hilbert space. In our context only theories without massless particles will be considered, and hence we require the spectrum condition in the more restrictive form

$$\text{spec}P_0 \subset \{0\} \cup \{p \in \mathbb{R}^3 \, / \, p^2 > 0, \, p_0 > 0\} . \tag{0.4}$$

Here $\text{spec}P_0$ denotes the joint spectrum of the energy-momentum operators. The (defining) vacuum representation of the net of observables is assumed to be *cyclic*:

$$\sum_{\mathcal{O}} \mathcal{A}(\mathcal{O}) \, \Omega \quad \text{is dense in } \mathcal{H}_0 ,$$

and *irreducible*:[3]

$$\left(\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O}) \right)' = \mathbb{C}\mathbb{1} .$$

The last property is a completeness requirement, since by virtue of it every bounded operator in $\mathcal{H}_0$ can be approximated by local observables. For an unbounded region G of Minkowski space we denote by $\mathcal{A}(G)$ the von Neumann algebra generated by all local algebras $\mathcal{A}(\mathcal{O})$ with $\mathcal{O} \subset G$, i.e. $\mathcal{A}(G) = (\cup_{\mathcal{O} \subset G} \mathcal{A}(\mathcal{O}))''$. We will be particularly interested in spacelike cones. A *spacelike cone* is an open cone in Minkowski space $\mathcal{M}$ which extends to infinity in some spacelike direction. More precisely, it is a convex region of the form

$$S = a + \cup_{\lambda > 0} \lambda \mathcal{O} ,$$

where $a \in \mathcal{M}$ is the apex of the region and $\mathcal{O}$ is the intersection of a forward and a backward lightcone:

$$\mathcal{O} = x + V_+ \cap y + V_-$$

for some $x, y \in \mathcal{M}$ spacelike to the origin. We denote by $\mathcal{K}$ the class of spacelike cones S and their causal complements S'. In our context, the discussion will proceed in terms of the family

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[3]in fact, irreducibility is a consequence of cyclicity, the spectrum condition and uniqueness of the vacuum.

$\mathcal{A}(I)_{I \in \mathcal{K}}$ of observable algebras assigned to these regions. This family is assumed to satisfy a considerable strengthening of the microcausality (0.2) called *Haag duality*:

$$\mathcal{A}(I') = \mathcal{A}(I)' \quad \text{for all } I \in \mathcal{K} . \tag{0.5}$$

As pointed out at the beginning of this section, in the conventional formulation of quantum field theory, superselection rules appear in the form of coherent subspaces (in our examples characterized by the electrical charge and univalency, respectively), between which observables cannot effect transitions. In the context of AQFT, superselection rules arise as inequivalent *representations* of the observable algebra. By such a a representation we mean a family $\{\pi\} = \{\pi^I\}_{I \in \mathcal{K}}$ of normal representations $\pi^I : \mathcal{A}(I) \to \mathcal{B}(\mathcal{H}_\pi)$ in a Hilbert space $\mathcal{H}_\pi$ which is consistent in the sense that one has isotony of these representations in the form

$$\pi^J \mid_{\mathcal{A}(I)} = \pi^I \quad \text{if } I \subset J . \tag{0.6}$$

Two representations $\{\pi\}$ and $\{\hat{\pi}\}$ are said to be equivalent, if there is a unitary $V : \mathcal{H}_{\hat{\pi}} \to \mathcal{H}_\pi$ such that $\pi^I = \text{Ad} V \circ \hat{\pi}^I$ for all $I \in \mathcal{K}$. In this context, a (superselection) *sector* means an equivalence class $[\pi]$ of irreducible representations, and charges are characteristic properties distinguishing the sectors. Mathematically, there will in general be a host of inequivalent representations, and one needs criteria to single out the physically interesting ones. A minimal requirement is that the representations allow to implement the space-time translations by a strongly continuous group of unitaries, and that the corresponding generators satisfy the spectrum condition, so that the dynamics can be implemented and is stable. Here, we will only be concerned with *covariant* representations $\{\pi\}$ which by definition allow to implement the universal covering group $\tilde{P}_+^\uparrow$ of the Poincaré group[4]. More precisely, there is a strongly continuous unitary representation U_π of $\tilde{P}_+^\uparrow$ acting on $\mathcal{H}_0$ such that

$$\text{Ad} U_\pi(\tilde{g}) \circ \pi^I = \pi^{gI} \circ \text{Ad} U_0(g) \text{ for all } \tilde{g} \in \tilde{P}_+^\uparrow , I \in \mathcal{K} . \tag{0.7}$$

Here $\tilde{g} \mapsto g$ denotes the canonical covering homomorphism as explained in the Appendix. Note that if $\{\pi\}$ is irreducible, a 2π-rotation must be represented by a multiple of unity, which only depends on the sector $[\pi]$:

$$U_\pi(2\pi) =: e^{2\pi i s[\pi]} . \tag{0.8}$$

The real number $s[\pi]$, determined modulo one, is called the *spin* of the sector $[\pi]$. Of particular interest in our context are *massive single particle representations*, which by definition are irreducible and covariant, with the corresponding energy momentum spectrum containing an isolated mass shell $H_m := \{p \in \mathbb{R}^3 \mid p^2 = m^2, p_0 > 0\}$ as its lower boundary:

$$H_m \subseteq \text{spec} P_\pi \subseteq H_m \cup \{p \in \mathbb{R}^3 \mid p^2 > M^2, p_0 > 0\} \text{ for some } M > m > 0 . \tag{0.9}$$

Buchholz and Fredenhagen have shown in [BF82], that every massive single particle representation is localizable in spacelike cones relative to some vacuum representation π_0 (which they construct for a given massive single particle representation). In our context it is appropriate to identify π_0 with the defining representation on $\mathcal{H}_0$, and to require this property from all considered representations $\{\pi\}$. In precise terms, this means that for every spacelike cone S there is a unitary $V_S : \mathcal{H}_0 \to \mathcal{H}_\pi$ such that

$$\pi^{S'}(A) = V_S A V_S^{-1} \quad \text{for all } A \in \mathcal{A}(S') . \tag{0.10}$$

Some Known Results. Having set up the frame and introduced criteria to select representations corresponding to physically realistic situations, let us briefly review the relevant properties of the corresponding charges, which can be derived in this frame. The salient results concern the following three items, into which we will enter subsequently in some detail.

(1) Charge composition and -conjugation,

(2) Exchange statistics of identical charges (and particles, in view of (3))

(3) Multiparticle states.

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[4]Relevent are in fact the unitary ray representations of the Poincaré group, but these are in one-to-one correspondence with the unitary representations of $\tilde{P}_+^\uparrow$. This has been shown by Wigner and Bargmann for four dimensions, and also holds in three dimensions.

Ad (1): Charge composition. The family of local von Neumann algebras $\mathcal{A}(I)$ can be embedded into an abstract C*-algebra $\mathcal{A}_\mathrm{u}$ in such a way that all local relations are preserved, and every representation $\{\pi\}$ of the family of local observable algebras uniquely lifts to a representation of $\mathcal{A}_\mathrm{u}$. In particular, the defining representation on $\mathcal{H}_0$ gives rise to a representation of $\mathcal{A}_\mathrm{u}$ which we now call the vacuum representation and denote by π_0. As an immediate consequence of Haag duality (0.5) and the Buchholz-Fredenhagen criterion (0.10), any representation π of $\mathcal{A}_\mathrm{u}$ arising as above is unitarily equivalent to a representation on $\mathcal{H}_0$ of the form

$$\pi \cong \pi_0 \circ \varrho \, ,$$

where ϱ is a *localized endomorphism* of $\mathcal{A}_\mathrm{u}$. By definition, this means that ϱ is an algebra-homomorphism of $\mathcal{A}_\mathrm{u}$ into itself which preserves adjoints and (as a consequence) the norm, and which is

- *localized* in some region $I_0 \in \mathcal{K}$, i.e.

  $$\varrho(A) = A \quad \text{for all } A \in \mathcal{A}_\mathrm{u}(I_0') \, ,$$

  and
- *transportable* to other regions along paths in $\mathcal{K}$. By a *path* in $\mathcal{K}$ we mean a finite sequence $(I_0,\dots,I_n)$ of regions in $\mathcal{K}$ such that either $I_{k-1} \subset I_k$ or $I_{k-1} \supset I_k$ for $k = 1,\dots,n$. Projected onto some spacelike (two dimensional) hypersurface, the situation looks geometrically as follows:
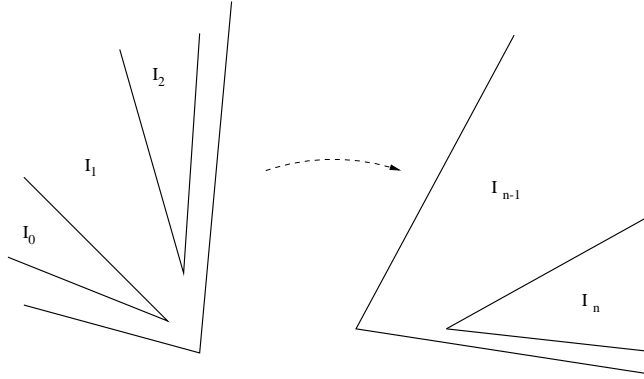
  Then for each k there is a unitary $U_k \in \mathcal{A}_\mathrm{u}(I_{k-1} \cup I_k)$ such that $\varrho_k := \mathrm{Ad}(U_k \cdots U_1) \circ \varrho$ is localized in I_k. We say that ϱ_n arises by 'transporting ϱ from I_0 to I_n along the path' $\tilde{I} = (I_0,\dots,I_n)$, and that $(U_1,\dots,U_n)$ is a 'chain of charge transporters' for ϱ along that path.

Being an injective endomorphism of an infinite dimensional Banach space, such a ϱ needs not be surjective and hence an automorphism. In case it actually is an automorphism, we say that the sector of ϱ is an *Abelian sector*. As we shall see soon, this is the case exactly for anyons.

A *state* of the system corresponds to a continuous positive normalized functional on the universal algebra. Having arranged all relevant representations $\pi_0\varrho$ to act on the same Hilbert space $\mathcal{H}_0$, a vector $\psi \in \mathcal{H}_0$ determines a (pure) vector state in the sense of expectation values only after specifying an (irreducible) representation in which the observables are meant to act on $\mathcal{H}_0$[5]:

$$A \mapsto \langle \psi, \pi_0\varrho(A)\psi \rangle, \ A \in \mathcal{A}_\mathrm{u} \, .$$

The set of such vector states falls, according to the equivalence classes of representations, into disjoint connected components[6] which are also called sectors, or *folia*.

Going over from representations π to endomorphisms ϱ is the crucial step towards the definition of a charge composition. Namely, the composition $\varrho_1\varrho_2 := \varrho_1 \circ \varrho_2$ of two endomorphisms localized in I is again an endomorphism localized in I, and the sector of the representation

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[5]$\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathcal{H}_0$.
[6]in the associated norm topology

$\pi_0 \varrho_1 \varrho_2$ only depends on the sectors of $\pi_0 \varrho_1$ and $\pi_0 \varrho_2$. This allows one to define a *composition of sectors* via

$$[\pi_0 \varrho_1] \cdot [\pi_0 \varrho_2] := [\pi_0 \varrho_1 \varrho_2] \,. \tag{0.11}$$

A result which can be obtained in this framework is that every sector has a *conjugate* sector, which means that the composition of the sector with its conjugate contains the (equivalence class of the) vacuum representation as a subrepresentation. There is a symmetry between charges and anticharges and between particles and antiparticles: The spins of a sector and of its conjugate sector coincide, and if π is a massive single particle representation, then the conjugate sector contains the same particles.

Ad (2): Statistics. There is an intrinsic notion of statistics in this frame, which describes the effect of exchanging identical localized charges. To give an idea, we consider an endomorphism ϱ of $\mathcal{A}_u$ which is localized in a spacelike cone I. Let I_1 and I_2 be two causally disjoint spacelike cones, and let ϱ_1 and ϱ_2 be endomorphisms which arise by transporting ϱ from I to I_1 and I_2, respectively. The paths along which they are transported are required to be such that ϱ_1 is transported in a counter-clockwise direction with respect to ϱ_2, and such that they stay causally disjoint after having left I :
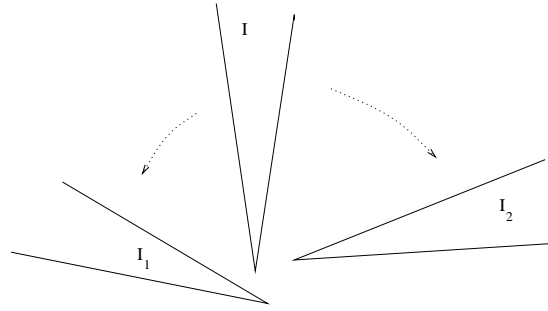
Being localized in causally disjoint regions, ϱ_1 and ϱ_2 commute. Hence their composition $\varrho_1 \varrho_2$, which has two disjoint localization regions, may be transported back, giving rise to ϱ^2, in two ways – according to wether ϱ_1 is considered the first or the second factor. One calculates

$$\begin{aligned}
\varrho_1 \varrho_2 &= \mathrm{Ad}\big(V_1 \varrho(V_2)\big) \circ \varrho^2 \\
\varrho_2 \varrho_1 &= \mathrm{Ad}\big(V_2 \varrho(V_1)\big) \circ \varrho^2 \,,
\end{aligned} \tag{0.12}$$

where V_1 and V_2 are products of the charge transporters which have effected the transport of ϱ from I to I_1 and I_2 along the respective paths. Hence, the effect of transporting the two identical charges into disjoint regions, exchanging and transporting back reads

$$\varrho^2 = \mathrm{Ad}\varepsilon_\varrho \circ \varrho^2 \,, \tag{0.13}$$

where ε_ϱ is the unitary element of $\mathcal{A}_u(I)$ given by $\big(V_2 \varrho(V_1)\big)^* V_1 \varrho(V_2)$. In fact, ε_ϱ turns out to be independent of the charge transporters, of the regions I_1 and I_2 and of the paths as long as the geometric conditions on the paths is satisfied. In addition, ε_ϱ only depends on the sector $[\pi_0 \varrho]$. It is called the *statistics operator* of that sector. If on the other hand ϱ_1 is transported in a clockwise direction with respect to ϱ_2, the same procedure leads to ε_ϱ^{-1} instead of ε_ϱ in the above equation. A priori, ε_ϱ^{-1} may differ from ε_ϱ. In four dimensional space time, in contrast, there is no toplogical difference between the two prescriptions 'clockwise' or 'counter-clockwise', and hence here $\varepsilon_\varrho^2 = \mathbf{1}$. We shall now see that the failure of $\varepsilon_\varrho^2 = \mathbf{1}$, proper to low dimensions, is the reason for the occurrence of braid group statistics. The *braid group* B_n of n strands is by definition generated by 'elementary braids' $t_1, \ldots, t_{n-1}$ which satisfy the relations

$$\begin{aligned}
t_k \, t_{k+1} \, t_k &= t_{k+1} \, t_k \, t_{k+1} \quad &&\text{for } k = 1, \ldots, n-2 \,, \\
t_k \, t_l &= t_l \, t_k \quad &&\text{if } |k - l| \ge 2 \,.
\end{aligned} \tag{0.14}$$

As suggested by the name, an element of this group may be pictured as a braiding of n strands, the group operation being defined in the obvious way by appending one braid to the other.

The elementary braids correspond to twisting two neighbouring strands once in a clockwise[7] direction:
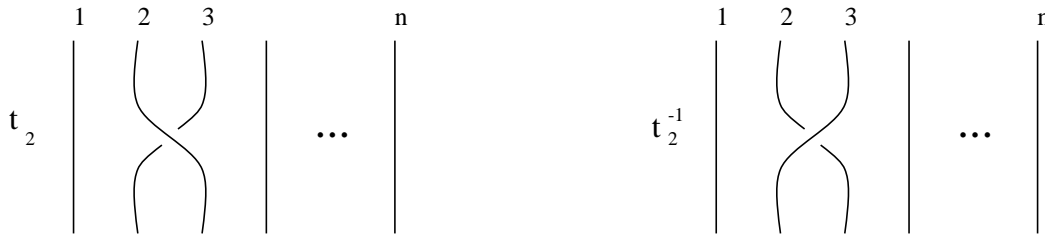
Since for $n < m$ the group B_n is naturally embedded in B_m, the infinite braid group B_∞ can be defined as the braid group of an unspecified number of strands.

The statistics operator ε_ϱ satisfies the relations

$$\varepsilon_\varrho \, \varrho(\varepsilon_\varrho) \, \varepsilon_\varrho = \varrho(\varepsilon_\varrho) \, \varepsilon_\varrho \, \varrho(\varepsilon_\varrho)$$

$$\varrho^2(\varepsilon_\varrho) \, \varepsilon_\varrho = \varepsilon_\varrho \, \varrho^2(\varepsilon_\varrho) \,,$$

and therefore the map $t_k \mapsto \varrho^{k-1}(\varepsilon_\varrho)$ is a representation of the braid group B_∞ by unitary elements in $\mathcal{A}_{\mathrm{u}}(I)$. If $\varepsilon_\varrho^2 = \mathbf{1}$, this is effectively a representation of the permutation group, since the latter is generated by the transpositions $\tau_k := (k\,k+1)$ which satisfy the relations (0.14) and the additional relation $\tau_k^2 = 1$. In particular the permutation group is the homomorphic image of the Braid group under the map $t_k \mapsto \tau_k$. Thus canonically associated to each sector is in $d = 2+1$ a representation of the braid group, and in $d = 3+1$ a representation of the permutation group.

This representation is indeed analogous to the action of the permutation group on wave-functions in quantum mechanics. To indicate the analogy, we return to the situation of equation (0.13). The two states $\omega_i := \langle \Omega \,, \pi_0 \varrho_i(\,\cdot\,)\Omega \rangle$ describe a charge $[\pi_0\varrho]$ localized in I_i, $i = 1,2$, i.e. they look like the vacuum state $\langle \Omega \,, \pi_0(\,\cdot\,)\Omega \rangle$ in I_i'. We consider the state

$$\omega_{12} := \langle \Omega \,, \pi_0 \varrho_1 \varrho_2(\,\cdot\,)\Omega \rangle$$

in the doubly charged sector $[\pi_0\varrho^2]$. It describes two identical charges, one localized in I_1 and the other one in I_2, i.e. it looks like ω_1 in I_2' and like ω_2 in I_1'. Since ϱ_1 and ϱ_2 commute, ω_{12} coincides with the (analogously defined) state ω_{21}. That is to say, one cannot tell whether the first or the second factor of the product ϱ^2 has been moved to I_1 : identical charges cannot be labelled on this (observable) level of description. The effect of exchanging the charges in I_1 and I_2 becomes apparent if we refer to the vectors corresponding to the state ω_{12}. By virtue of equations (0.12), the state $\omega_{12} = \omega_{21}$ is a vector state in the reference representation $\pi_0\varrho^2$, to which two vectors Ω_{12} and Ω_{21} are naturally assigned, corresponding to the two possible labellings:

$$\langle \Omega_{12} \,, \pi_0\varrho^2(\,\cdot\,)\Omega_{12} \rangle = \omega_{12} \equiv \omega_{21} = \langle \Omega_{21} \,, \pi_0\varrho^2(\,\cdot\,)\Omega_{21} \rangle \,,$$

where $\Omega_{12} := \pi_0\big(V_1\varrho(V_2)\big)^* \Omega$, and Ω_{21} is defined analogously. Ω_{12} describes the doubly charged state in which the first charge has been transported into I_1, counterclockwise relative to the second one. Though yielding one and the same (observable) state ω_{12}, these vectors may be distinct: they are related by

$$\Omega_{21} = \pi_0(\varepsilon_\varrho) \, \Omega_{12} \,. \tag{0.15}$$

Thus ε_ϱ describes the effect of exchanging identical localized charges.

From ε_ϱ one can obtain a numerical invariant of the sector $[\pi_0\varrho]$, the so-called statistics parameter, whose phase (in the polar decomposition) is called statistics phase and denoted by $\omega_{[\varrho]}$. Clearly, in the case of permutation group statistics ($\varepsilon_\varrho^2 = \mathbf{1}$), this phase is just a sign – corresponding to (para-) bosons of fermions. If ϱ is in particular an automorphism (Abelian sector), the statistics operator ε_ϱ must be a multiple of unity because it commutes with $\varrho^2(\mathcal{A}_{\mathrm{u}})$ by equation (0.13), and the statistics phase is then the factor occurring in $\varepsilon_\varrho = \omega_{[\varrho]} \, \mathbf{1}$. In this case the representation defined by ε_ϱ is Abelian, i.e. ϱ describes anyons.

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[7]the relevant convention is given in the appendix.

There is a 'weak' connection of spin and statistics, according to which the statistics phase coincides with the spin phase $\exp 2\pi i s[\pi_0\varrho]$ up to a sign [Fre89, FM89]. The sign cannot be determined without further input (to be assumed below), but note that this connection already implies that a sector $[\pi_0\varrho]$ has non-trivial braid group statistics if and only if its spin is not semi-integer, i.e. not in $\frac{1}{2}\mathbb{Z}$.

Ad (3): Particle States. Following Wigner [Wig39], an elementary particle should be quantum mechanically described by an irreducible representation of the universal covering group of the Poincaré group. In three as well as in four dimensional space-time, the physically relevant representations – and hence the occurring particle types – are classified by two numerical invariants. One is the mass $m \geq 0$, and the other one is, in the massive case $m > 0$, the spin s which labels an irreducible representation of the universal covering group of rotation subgroup. Hence in three dimensional space-time s may be *any* real number, in contrast to four dimensions where s is an integer or half-integer.

If $\pi_0\varrho$ is a massive single particle representation (recall equation (0.9)), then the subspace corresponding to the H_m-part of the spectrum describes elementary particles of mass m in the sector $[\pi_0\varrho]$. There may be different particle types, distinguished by different spins, but all spins occuring must coincide with the spin of the sector $[\pi_0\varrho]$ up to an integer. Thus, in view of the spin statistics connection, if there are fields creating particles whose spin is not semi-integer, they must obey braid group statistis: Spin-anyons are connected with statistical anyons, and both phenomena are peculiar to $2+1$ dimensions.

Starting from massive single particle representations, one can construct multparticle states by a procedure known as Haag-Ruelle scattering theory. It has been originally formulated [Haa58, Rue62] in the framework of conventional quantum field theory, where one has unobservable fields creating the charged single particle states, which transform under some global gauge group and have well-defined spacelike (namely, normal) commutation relations. For the case of non-trivial (and non-Abelian) braid group statistics, however, the existence of such a field algebra has not yet been etablished (for results in this direction, see [Reh90, MS90]). But for some purposes it can be substituted by a simple construct in the algebraic framework which has been proposed in [DHR71], the so-called *field bundle*. Using this construction, K. Fredenhagen, M. Gaberdiel and S. Rüger adapted the Haag-Ruelle scattering theory to the case of particles with braid group statistics [FGR96].

The structure of the field bundle is rich enough to carry a localization concept for the objects which substitute charge carrying fields in this formalism, such that one has definite space-like commutation relations. In the plektonic case, the localization concept is based on paths of spacelike cones, and the commutation relations reflect the braid group statistics.

The resulting n-particle scattering vectors depend on the n single particle vectors and, peculiar to the plektonic case, on n paths of spacelike cones originating from the 'field operators' which have created the particles. The statistics of charges determines the statistics of these particle states, which essentially reads as equation (0.15), now Ω_{12} representing a two-*particle* state. Since the n-particle state vectors depend on paths, in contrast to the single particle vectors, they cannot be identified with (appropriately symmetrized) tensor products of the single particle vectors in a canonical way. More seriously, the representation of $\tilde{P}_+^\uparrow$ on the space of n-particle states is not isomorphic to an n-fold tensor product of single particle representations [Fre90b]. A multiparticle Hilbert space carrying an appropriate representation of $\tilde{P}_+^\uparrow$ has been proposed by R. Schrader and the author [MS95], employing the theory of fibre bundles and builing on earlier ideas of R. Schrader [Sch89]. In fact, the structure of the Haag-Ruelle scattering states has been revealed by K. Fredenhagen et.al. [FGR96] and for the anyonic case by J. Fröhlich and P. Marchetti [FM91], and essentially coincides with our construction. We briefly sketch the structure. The sets (i.e. tupels modulo permutations) of n noncoinciding relativistic velocities form the points of a manifold which we denote by $^nH_1$. This manifold has the braid group B_n as its fundamental group, and hence its universal covering manifold can be viewed as a principal fibre bundle with the braid group as structure group. Further, the braid group acts on a finite dimensional Hilbert space F via a representation ε, which is fixed by the statistics operators of the considered massive single particle representations and the 'fusion rules' governing the decomposition of products of the latter into irreducible representations.

Given this representation ε of the structure group, there is an associated vector bundle with fibre F over the base space ${}^nH_1$. Now the Hilbert space of n-particle scattering states is isomorphic to the space of square integrable sections in this bundle. The representation of the covering group of the Poincaré group, whose existence we have required by only considering covariant representations of $\mathcal{A}_u$, is fixed on the space of scattering states by its restriction to the single particle vectors and by the representation ε of the braid group [FGR96].

Overview of the Results and Structure of the Thesis

The central concern of this thesis is to present a number of model independent properties of plektons, particularly anyons, in the framework of algebraic quantum field theory.

Our first result concerns the existence of an analogon to the PCT operator in the plektonic field bundle. An additional input to our analysis is that we assume the observable algebra to have the *Bisognano-Wichmann* property [BW75]. This means that the Tomita operator[8] $S(W_1)$: $A\,\Omega \mapsto A^*\,\Omega$, $A \in \mathcal{A}(W_1)$, of the observable algeba associated to the wedge region

$$W_1 := \{x \in \mathcal{M}\,,\, |x^0| < x^1\} \tag{0.16}$$

has a direct geometrical meaning: Let $S(W_1) = J\,\Delta^{\frac{1}{2}}$ be the polar decomposition of this operator. Then the modular operator Δ can be expressed in terms of the representation U_0 of the Poincaré group, and the modular conjugation J represents antilinearly the reflexion j at the vertex line of W_1 (which is the x^2-axis) in such a way that U_0 is extended to a representation of the proper Poincaré group P_+, and such that $\mathrm{Ad}J$ acts geometrically on the observable algebra:

$$\mathrm{Ad}J \,:\, \mathcal{A}(\mathcal{O}) \to \mathcal{A}(j\mathcal{O})\,.$$

The operator J may be used to associate a particular representant $\pi_0\bar{\varrho}$ of the conjugate sector to a given representation $\pi_0\varrho$ of the observable algebra [GL92]. An immediate application is that J also extends the representation $U_{\pi_0\varrho} \oplus U_{\pi_0\bar{\varrho}}$ of $\tilde{P}^\uparrow_+$ to a representation of $\tilde{P}_+$, such that $\mathrm{Ad}J$ acts geometrically in the field bundle (Proposition 1.13), which to our knowledge had not been established yet for plektons in $2+1$ dimensions. Further, this operator effects directly the charge-anticharge and particle-antiparticle symmetry (Proposition 1.11), namely that a sector and its conjugate sector have the same spin and statistics parameter, and the same particle content (masses, spins and degeneracies of single particle spaces). Thus, J is a rudimentary analogon of the PCT-operator.

A major concern of the thesis is to further clarify the structure of the Hilbert space of scattering states. The isometric Møller opertators W^+ and W^- in the present context are presented from the 'physical' Hilbert spaces of outgoing and incoming scattering states, respectively, onto the abovementioned 'reference' Hilbert space of sections in a vector bundle. They are defined by using concepts which are naturally adapted to the structure of the physical space of scattering vectors, so that several points become more transparent now. One of them concerns the S-matrix, which is the map from the space of the incoming to the space of the outgoing scattering states usually defined by $S := (W^+)^*\,W^-$. But this definition presupposes that both Møller operators map onto the same reference Hilbert space, or at least that one has a canonical comparison map between the images of W^+ and of W^-. This is achieved by the Møller operators presented here (Theorem 2.9). As a further advantage of our Møller operators, it now becomes transparent that they indeed translate the 'physical' representation of $\tilde{P}^\uparrow_+$ on the space of scattering states into the canonical representation on the reference Hilbert space (Proposition 3.4). A new result is that they also relate the 'PCT-operator' J, which turns out to map outgoing onto incoming scattering states and vice versa (Lemma 3.5), with a canonical conjugation operator which we define on the reference Hilbert space (Proposition 3.7). Consistently, our S-matrix commutes with Poincaré transformations and is intertwined with its inverse by the operator J (Propositions 3.4 and 3.7).

In the sequel, we restrict attention to anyons. Selecting a suitable set of Abelian sectors one can, in contrast to the non-Abelian case, construct a *field algebra* [Reh90]. Thus the conventional picture of the algebraic structure of quantum field theory can be set up, with appropriate modifications: One has charged fields localized in paths of spacelike cones, which transform

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[8]see Section 1.4 for a more precise definition.

covariantly under a compact (here: Abelian) gauge group and have spacelike commutation relations governed by the statistics phases. The observable algebra can be regained as the algebra of gauge invariant operators. In our Proposition 4.3 it is made precise in what sense this construct is the unique anyonic field algebra associated to a given observable algebra and set of (suitably chosen) sectors, and shown that it satisfies a 'twisted' version of Haag duality (0.5). A refined version of the operator J, which inherits all its profitable properties mentioned above, acts geometrically on the field algebra and can hence be interpreted as a PCT-operator (Theorem 4.14 and Proposition 4.16). In addition it coincides, up to a twist operator, with the modular conjugation of the field algebra associated to the wedge region[9] W_1 (Theorem 4.14). Similarly, the modular operator of this field algebra is given in terms of the representation of $\tilde{P}_+^\uparrow$. Thus the Bisognano-Wichmann property of the observable algebra lifts to the field algebra. This has been shown in the case of permutation group statistics by D. Guido and R. Longo [GL95] and by B. Kuckert [Kuc95], and is transferred to the anyonic case here for the first time. The same holds for an important immediate consequence of the Bisognano-Wichmann property, namely the strong version of the spin-statistics connection: the statistics phase of a sector and its spin phase coincide – not only up to a sign (Theorem 4.15).

For anyons we have achieved quite explicit formulae for all the abovementioned objects concerning the space of scattering states: Here, the fusion rules are trivial and hence the representation ε of the braid group is determined by the set of statistics phases of the massive single particle representations under consideration. Hence, the Hilbert space of scattering states and the representation of $\tilde{P}_+^\uparrow$ on it are explicitely given, and so is, consequently, the *modular operator* of the field algebra associated to the wedge W_1.

A new and important result is that also the 'incoming PCT-operator', i.e. the product of the PCT operator with the S-matrix, can be written down (Proposition 4.20). Consequently, the same holds for the *modular conjugation* of the field algebra associated to the wedge W_1. In effect, we have thereby constructed the model-independent 'incoming Tomita-operator' of the anyonic field algebra associated to W_1, i.e. the Tomita operator times the S-matrix. This opens up new possibilities for the construction of models of anyons – in particular for free anyons, where one expects a trivial S-matrix.

In 3+1 dimensions one has for every particle type a canonical model of relativistic quantum field theory which describes a system of arbitrarily many noninteracting identical particles, and also offers a way to discuss interaction: the *free field*. For anyons in 2+1 dimensions, in contrast, one is not in this profitable situation[10]. There are two obvious difficulties: Firstly, the free fields in $d = 3+1$ are operator valued quantum fields acting in the symmetrized or antisymmetrized Fock space over the one particle space, whereas the n-particle Hilbert space for anyons does not have a canonical tensor product structure. Secondly, for nonzero spin the construction of the free fields in $d = 3+1$ exploits the existence of so-called u- and v-intertwiners, which serve to implement the covariant transformation properties of the free fields under a finite dimensional (non-unitary) representation of the universal covering group $SL(2,\mathbb{C})$ of the Lorentz group. In the 2+1-dimensional situation, the universal covering group of the Lorentz group is the universal covering group of $SL(2,\mathbb{R})$, and no analogon of the u- and v-intertwiners into some finite-dimensional representation of this group is known. Both difficulties might be circumvented, the first one by a non-canonical trivialization of the vector bundle in terms of which the anyonic Hilbert space is described [Mun92, MS95], and the second one by using an infinite dimensional representation of the universal covering group of $SL(2,\mathbb{R})$[11]. Nonetheless, constructive attempts at a direct substitute of the free Fock space field have failed. We have finally found reasons for this failure and will present them in this thesis.

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[9]more precisely, to a specific path ending at W_1.

[10]D.R.Grigore has constructed free fields in $d = 2+1$ for any spin [Gri94] in a bosonic Hilbert space, but in contradiction to the weak spin statistics holding in algebraic quantum field theory [Fre89, FM89], all of these fields have bosonic statistics. Presumably, this is due to the fields having infinitely many components.

[11]This route has been proposed by R. Schrader and the author in [MS95], where the ribbon braid group is considered instead of the braid group. Grigore has realized a similar solution of the 'intertwiner problem' in [Gri94].

In view of the mentioned difficulties we have to specify what we mean by a 'free field' for anyons. The requirement of a trivial S-matrix seems a poor starting point for model building, since e.g. in $3 + 1$ dimensions the class of fields which lead to a trivial S-matrix is quite large, larger in fact than the Borchers class of the free Fock space field. As the characterizing property of a free field we consider the fact that the field algebra is completely determined by the single particle spaces — altogether with its localization structure, which in the anyonic context is based on paths of spacelike cones. This implies that already on the single particle level there is a notion of localization. In fact, by the the Tomita operators the localization structure of the field algebra is carried over to the Hilbert space on which it acts [Sch97]. We require that for free fields, conversely, the family of 'local single particle subspaces' determines the family of local field algebras. To be specific, we assume that the local algebras of a free theory are generated by basic 'fields'[12] which create only single particle states out of the vacuum (Definition 5.1). But under slightly stronger assumptions, we can establish no-go results for free anyons: In Section 5.2 we assume that the basic fields are *determined* by the single particle vectors which they create from the vacuum. Using our results on the explicit form of the Tomita operator for wedge regions, we show that in the case of non-zero spin this assumption, together with a certain intersection property of the field algebra, which is usually satisfied if one has local quantum fields (i.e. distributions), is in conflict with the Bisognano-Wichmann property. This conflict is due to the representation of the Wigner rotation and is peculiar to $2 + 1$ dimensions. In Section 5.3 we show that the basic fields cannot satisfy the condition that for two spacelike seperated fields φ_1 and φ_2, the norm of the vector $\varphi_1\, U(x)\, \varphi_2\, \Omega$ be polynomially bounded for large x. If they satisfy this mild regularity condition, we can establish, via a theorem à la Jost-Schroer, commutation relations of the fields which are not compatible with anyonic statistics [Mun98]. It is noteworthy that anyonic commutation relations of this kind have appeared in the literature, but the fact that they are inconsistent has not.

The thesis is organized as follows.

- In Chapter 1 some of the concepts of algebraic quantum field theory mentioned are introduced in more detail, in particular the correspondence of representations of the observable algebra to endomorphisms of the universal algebra, and the field bundle construction. It is pointed out how the commutation relations between spacelike localized elements of the field bundle are governed by a representation of the braid group. After a short exposition of the structure of the single particle spaces of plektons, consequences of the Bisognano-Wichmann property of the observable algebra are elaborated: Namely, a rudimentary version of the PCT-operator for the field bundle can be defined, which implements the particle-antiparticle symmetry and acts geometrically on the field bundle.

- Chapters 2 and 3 are concerned with the Hilbert space of scattering states of plektons. The construction of Haag-Ruelle scattering states in the present framework as developped in [FGR96] is briefly sketched. We define the Hilbert spaces of outgoing and incoming scattering vectors containing all particle numbers $n \geq 0$ and all sectors which have n-particle states, and Møller operators from these into a reference Hilbert space. It is shown that they translate the representation of the covering group of the Poincaré group on the respective spaces of scattering states into the canonical representation on the reference Hilbert space, and that they translate the product of the PCT-operator with the S-matrix (i.e. the incoming PCT-operator) to a canonical conjugation operator on the reference Hilbert space.

- In Chapter 4 the anyonic case is treated in more detail. An anyonic field algebra is constructed from the observable algebra and an appropriately chosen set of Abelian sectors. An algebraic PCT-theorem and the spin-statistics theorem are derived from the Bisognano-Wichmann property of the observable algebra. This is accomplished by lifting the latter property, in a 'twisted' version, to the field algebra. Finally, explicit formulae are given concerning the spaces of scattering states: Namely, for the representation of the braid group on which the construction of the reference Hilbert space depends, and

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[12]We speak of 'fields' although these operators are not assumed to be pointlike localized like Wightman distributions (not even stringlike as in [Ste82]).

consequently for the representation of the Poincaré group and for the incoming PCT-operator. By the Bisognano-Wichmann property, we thus have in particular constructed the incoming Tomita-operator for anyonic models.

- In Chapter 5 we show that for anyons one cannot expect to have 'free fields' with the best possible localization properties, namely in paths of spacelike cones. Free fields are here characterized as operators which create only single particle states from the vacuum and generate the local (i.e. spacelike cone localized) field algebras. A localization concept on the Hilbert space using the Tomita modular theory is introduced and elaborated, and, using the results of the last chapter, it is shown that the idea of a free field with all desired properties, in particular the Bisognano-Wichmann property, can hardly be held up. Finally we show that free fields for anyons, if they can be localized in smaller regions than wedge regions, would have to violate the mentioned mild regularity condition which holds e.g. in Wightman theories.

This work is partly based on a previous joint publication with R. Schrader [MS95]. The no-go result for free relativistic anyons has been published in [Mun98].