p-primary torsion part of Bloch's cycle complex

from Grothendieck's coherent duality point of view

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SELBSTSTÄNDIGKEITSERKLÄRUNG

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Abstract

Let X be a separated scheme of finite type over k with k being a perfect field of positive characteristic p. In this thesis we define a complex $K_{n,X,log}$ via Grothendieck's duality theory of coherent sheaves following [Kat87] and build up a quasi-isomorphism from the Kato-Moser complex of logarithmic de Rham-Witt sheaves $\widetilde{\nu}_{n,X}$ to $K_{n,X,log}$ for the étale topology, and also for the Zariski topology under the extra assumption $k = \overline{k}$. Combined with Zhong's quasi-isomorphism from Bloch's cycle complex $\mathbb{Z}^c_X$ to $\widetilde{\nu}_{n,X}$ [Zho14, 2.16], we deduce certain vanishing, étale descent properties as well as invariance under rational resolutions for higher Chow groups of 0-cycles with $\mathbb{Z}/p^n$-coefficients.

Zusammenfassung

Sei k ein vollkommener Körper der Charakteristik $p > 0$. Sei X ein separiertes k-Schema vom endlichen Typ. In dieser Doktorarbeit definieren wir ein Komplex $K_{n,X,log}$ über Grothendiecks Dualitätstheorie kohärenter Garben nach [Kat87] und ein Quasiisomorphismus von dem Kato-Moser-Komplex der logarithmischen de Rham-Witt Garben $\widetilde{\nu}_{n,X}$ nach $K_{n,X,log}$ für die étale Topologie und auch für die Zariski Topologie unter der zusätzlichen Annahme $k = \overline{k}$. In Kombination mit Zhongs Quasiisomorphismus vom Blochs Zykelkomplex $\mathbb{Z}^c_X$ nach $\widetilde{\nu}_{n,X}$ [Zho14, 2.16], leiten wir bestimmte Eigenschaften für Höhere Chowgruppen von 0-Zyklen mit $\mathbb{Z}/p^n$-Koeffizienten ab.

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Contents

Introduction

In this work, we show that Bloch's cycle complex of zero cycles mod p^n is quasi-isomorphic to the Cartier operator fixed part of a certain dualizing complex. From this we obtain new vanishing results for the higher Chow groups of zero cycles with mod p^n coefficients for singular varieties.

Let X be a separated scheme of finite type over k of dimension d with k being a perfect field of positive characteristic p. Bloch introduced his cycle complex $\mathbb{Z}^c_X(m)$ in [Blo86] as the first candidate for a motivic complex under the framework of Beilinson-Lichtenbaum. Let m, i be integers, and $\Delta^i = \operatorname{Spec} k[T_0, \dots, T_i]/(\sum T_j - 1)$. Here $\mathbb{Z}^c_X(m) := z_m(-, -\bullet - 2m)$ is a complex of sheaves in the Zariski or the 'etale topology. The global sections of its degree $(-i-2m)$-term $z_m(X, i)$ is the free abelian group generated by dimension $(m+i)$-cycles in $X \times \Delta^i$ intersecting all faces properly and the differentials are the alternating sums of the cycle-theoretic intersection of the cycle with each face (cf. Section 2). In this article we define a complex $K_{n,X,log}$ via Grothendieck's duality theory of coherent sheaves following the idea in [Kat87] and build up a quasi-isomorphism from the Kato-Moser complex of logarithmic de Rham-Witt sheaves $\widetilde{\nu}_{n,X}$ (namely the Gersten complex of logarithmic de Rham-Witt sheaves, which is introduced and studied in [Kat86a, §1][Mos99, (1.3)-(1.5)]) to $K_{n,X,log}$ for the étale topology and also for the Zariski topology under the extra assumption $k = \overline{k}$. Combined with Zhong's quasi-isomorphism from Bloch's cycle complex $\mathbb{Z}^c_X := \mathbb{Z}^c_X(0)$ to $\widetilde{\nu}_{n,X}$ [Zho14, 2.16], we deduce certain vanishing and finiteness properties as well as invariance under rational resolutions for higher Chow groups of 0-cycles with $\mathbb{Z}/p^n$-coefficients. The proofs in this article are self-contained in respect to Kato's work [Kat87].

Let us briefly recall Kato's work in [Kat87] and introduce our main object of studies: $K_{n,X,log}$. Let $\pi : X \to \operatorname{Spec} k$ be the structure morphism of X. Let $W_nX := (|X|, W_n\mathcal{O}_X)$, where $|X|$ is the underlying topological space of X, and $W_n\mathcal{O}_X$ is the sheaf of length n truncated Witt vectors. Let $W_n\pi : W_nX \to \operatorname{Spec} W_nk$ be the morphism induced from π via functoriality. According to Grothendieck's duality theory, there exists an explicit Zariski complex $K_{n,X}$ of quasi-coherent sheaves representing $(W_n\pi)^! W_nk$ (such a complex $K_{n,X}$ is called a residual complex, cf. [Har66, VI 3.1]. We will collect some related facts in Section 1.1.). There is a natural Cartier operator $C' : K_{n,X} \to K_{n,X}$, which is compatible with the classical Cartier operator $C : W_n\Omega^d_X \to W_n\Omega^d_X$ in the smooth case via Ekedahl's quasi-isomorphism (see Theorem 1.17). Here $W_n\Omega^d_X$ denotes the degree $d := \dim X$ part of the de Rham-Witt complex. We define the complex $K_{n,X,log}$ to be the mapping cone of $C' - 1$. What Kato did in [Kat87] is the FRP counterpart, where FRP is the "flat and relatively perfect" topology (this is a topology with étale coverings and with the underlying category lying in between the small and the big étale site). Kato then showed that $K_{n,X,log}$ in the topology FRP acts as a dualizing complex in a rather big triangulated subcategory of the derived category of $\mathbb{Z}/p^n$-sheaves, containing all coherent sheaves and sheaves like logarithmic de Rham-Witt sheaves [Kat87, 0.1]. Kato also showed that in the smooth setting, $K_{n,X,log}$ is concentrated in one degree and this only nonzero cohomology sheaf is the top degree logarithmic de Rham-Witt sheaf [Kat87, 3.4]. For the latter, an analogy on the small étale site naturally holds. Rülling later observed that with a trick from p^{-1}-linear algebra, [Kat87, 3.4] can be done on the Zariski site as well, as long as one assumes $k = \overline{k}$ (cf. Proposition 1.24). Comparing this with the Kato-Moser complex $\widetilde{\nu}_{n,X}$, which is precisely the Gersten resolution of the logarithmic de Rham-Witt sheaf in the smooth setting, one gets an identification in the smooth setting $\widetilde{\nu}_{n,X} \simeq K_{n,X,log}$ on the Zariski topology. Similar as in [Kat87, 4.2] (cf. Proposition 1.32), Rülling also built up the localization sequence for $K_{n,X,log}$ on the Zariski site in his unpublished notes (cf. Proposition 1.33). Compared with the localization sequence for $\mathbb{Z}^c_X$ [Blo94, 1.1] and for $\widetilde{\nu}_{n,X}$ (which trivially holds in the Zariski topology), it is reasonable to expect a chain map relating these objects in general.

The aim of this article is to build a quasi-isomorphism $\overline{\zeta}_{log} : \widetilde{\nu}_{n,X} \xrightarrow{\simeq} K_{n,X,log}$ in the singular setting, such that when pre-composed with Zhong's quasi-isomorphism $\overline{\psi} : \mathbb{Z}^c_X \to \widetilde{\nu}_{n,X}$ [Zho14, 2.16], it gives another perspective of Bloch's cycle complex with $\mathbb{Z}/p^n$-coefficients in terms of Grothendieck's coherent duality theory. More precisely, we prove the following result.

Theorem 0.1 (Theorem 5.10, Theorem 6.4)**.** *Let X be a separated scheme of finite type over k with k being a perfect field of positive characteristic p. Then there exists a chain map*

$$\overline{\zeta}_{log,\text{ét}} : \widetilde{\nu}_{n,X,\text{ét}} \xrightarrow{\simeq} K_{n,X,log,\text{ét}},$$

and when $k = \overline{k}$, a chain map

$$\overline{\zeta}_{log,\text{Zar}} : \widetilde{\nu}_{n,X,\text{Zar}} \xrightarrow{\simeq} K_{n,X,log,\text{Zar}}$$

which are quasi-isomorphisms.

Composed with Zhong's quasi-isomorphism $\overline{\psi}$*, we have the following composition of chain maps*

$$\overline{\zeta}_{log,\text{ét}} \circ \overline{\psi}_{\text{ét}} : \mathbb{Z}^c_{X,\text{ét}}/p^n \xrightarrow{\simeq} K_{n,X,log,\text{ét}}$$

and when $k = \overline{k}$*, the composition of chain maps*

$$\overline{\zeta}_{log,\mathrm{Zar}} \circ \overline{\psi}_{\mathrm{Zar}} : \mathbb{Z}^c_{X,\mathrm{Zar}}/p^n \xrightarrow{\simeq} K_{n,X,log,\mathrm{Zar}}$$

which are quasi-isomorphisms.

We explain more on the motivation behind the definition of $K_{n,X,log}$. In the smooth setting, the logarithmic de Rham-Witt sheaves can be defined in two ways: either as the subsheaves of $W_n\Omega^d_X$ generated by log forms, or as the invariant part under the Cartier operator C. In the singular case, these two perspectives give two different (complexes of) sheaves. The first definition can also be done in the singular case, and this was studied by Morrow [Mor15]. For the second definition one has to replace $W_n\Omega^d_X$ by a dualizing complex on W_nX: for this Grothendieck's duality theory yields a canonical and explicit choice, and this is what we have denoted by $K_{n,X}$. And then this method leads naturally to Kato's and also our construction of $K_{n,X,log}$. Now with our main theorem one knows that $\mathbb{Z}^c_X/p^n$ sits in a distinguished triangle

$$\mathbb{Z}^c_X/p^n \to K_{n,X} \xrightarrow{C'-1} K_{n,X} \xrightarrow{+1}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$, in either the étale topology, or the Zariski topology with $k = \overline{k}$ assumption. In particular, when X is Cohen-Macaulay of pure dimension d, then the triangle above becomes

$$\mathbb{Z}^c_X/p^n \to W_n\omega_X[d] \xrightarrow{C'-1} W_n\omega_X[d] \xrightarrow{+1} .$$

where $W_n\omega_X$ is the only non-vanishing cohomology sheaf of $K_{n,X}$ (when $n = 1$, $W_1\omega_X = \omega_X$ is the usual dualizing sheaf on X), and $\mathbb{Z}^c_X/p^n$ is concentrated at degree $-d$ (cf. Proposition 8.1). This is a generalization of the top degree case of [GL00, 8.3], which in particular implies the above triangle in the smooth case.

As corollaries, we arrive at some properties of the higher Chow groups of 0-cycles with p-primary torsion coefficients. (The versions stated here are not necessarily the most general ones. See the main text for more general statements.)

Corollary 0.2 (Proposition 8.2, Corollary 8.3, Corollary 8.7, Corollary 8.11)**.** *Let* X *be a separated scheme of finite type over* k *with* $k = \overline{k}$*.*

(1) (Cartier invariance)

$$\mathrm{CH}_0(X, q; \mathbb{Z}/p^n) = H^{-q}(W_nX, K_{n,X,\mathrm{Zar}})^{C'_{\mathrm{Zar}}-1}.$$

(2) (Affine vanishing) Suppose X *is affine and Cohen-Macaulay of pure dimension* d*. Then*

$$\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = 0$$

for $q \neq d$*.*

(3) (Étale descent) Suppose X *is affine and Cohen-Macaulay of pure dimension* d*. Then*

$$R^i\epsilon_*(\mathbb{Z}^c_{X,\text{ét}}/p^n) = R^i\epsilon_*\widetilde{\nu}_{n,X,\text{ét}} = 0, \quad i \neq -d.$$

(4) (Invariance under rational resolution) For a rational resolution of singularities $f : \widetilde{X} \to X$ *(cf. Definition 8.8) of an integral* k*-scheme* X *of pure dimension, the trace map induces an isomorphism*

$$\mathrm{CH}_0(\widetilde{X}, q; \mathbb{Z}/p^n) \xrightarrow{\simeq} \mathrm{CH}_0(X, q; \mathbb{Z}/p^n)$$

for each q*.*

Now we give a more detailed description of the structure of this article.

In Part 1, we review the basic properties of the chain complexes to appear. Section 1 is devoted to the properties of the complex $K_{n,X,log}$, the most important object of our studies. Section 1.1 is a preliminary subsection on residual complexes and Grothendieck's duality theory. After this, we study the Zariski version in Section 1.2-Section 1.5. Following the idea in [Kat87], we define the Cartier operator C' for the residual complex $K_{n,X}$, and then define the complex $K_{n,X,log}$ to be the mapping cone of $C' - 1$ in Section 1.2. We compare our C' with the classical definition of the Cartier operator C for top degree de Rham-Witt sheaves in Section 1.3. To avoid interruption of a smooth reading we collect the calculation in the next two subsections (Section 1.3.2-Section 1.3.3). The localization sequence appears in Section 1.4. In these subsections, the most important ingredients are a surjectivity result of $C' - 1$ (cf. Proposition 1.24. See also Section A for a short discussion on σ-linear algebra), the trace map of a nilpotent thickening (cf. Proposition 1.32), and the localization sequence (cf. Proposition 1.33). They are observed already by Rülling and are only re-presented here by the author. After a short discussion on functoriality in Section 1.5, we move to the étale case in Section 1.6. Most of the properties hold true in a similar manner, except that the surjectivity of $C_{\text{ét}} - 1 : W_n\Omega^d_{X,\text{ét}} \to W_n\Omega^d_{X,\text{ét}}$ over a smooth k-scheme X holds true without any extra assumption of the base field (except perfectness, which is already needed in defining the Cartier operator). This enables us to build the quasi-isomorphism $\zeta_{log,\text{ét}}$ without assuming k being algebraically closed in the next part. The rest of the sections in Part 1 are introductory treatments of Bloch's cycle complex $\mathbb{Z}^c_X(m)$, Kato's complex of Milnor K-theory $C^M_{X,t}(m)$ and the Kato-Moser complex of logarithmic de Rham Witt sheaves $\widetilde{\nu}_{n,X,t}(m)$, respectively.

In Part 2 we construct the quasi-isomorphism $\overline{\zeta}_{log} : \widetilde{\nu}_{n,X} \xrightarrow{\simeq} K_{n,X,log}$ and study its properties in Section 5. We first build a chain map $\zeta : C^M_X \to K_{n,X}$ and then we show that it induces a chain map $\zeta_{log} : C^M_X \to K_{n,X,log}$. This map actually factors through a chain map $\overline{\zeta}_{log} : \widetilde{\nu}_{n,X} \to K_{n,X,log}$ via the Bloch-Gabber-Kato isomorphism [BK86, 2.8]. We prove that $\overline{\zeta}_{log}$ is a quasi-isomorphism for $t = \text{ét}$, and also for $t = \text{Zar}$ with an extra $k = \overline{k}$ assumption. In Section 6, we review the main results of [Zho14, §2] and compose Zhong's quasi-isomorphism $\overline{\psi} : \mathbb{Z}^c_X/p^n \to \widetilde{\nu}_{n,X}$ with our $\overline{\zeta}_{log}$. This composite map enables us to use tools from the coherent duality theory in calculation of certain higher Chow groups of 0-cycles.

In Part 3 we discuss the applications. Section 7 mainly serves as a preparation section for Section 8. In Section 8 we arrive at several results for higher Chow groups of 0-cycles with p-primary torsion coefficients: affine vanishing, finiteness (reproof of a theorem of Geisser), étale descent, and invariance under rational resolutions.

Notations and conventions

(1) Basic settings. k will always be a perfect field of characteristic $p > 0$. k-schemes will be assumed to be separated schemes of finite type over k, unless otherwise stated. (In particular, in subsection Section 1.1 we shall allow more general schemes.) Let X be a k-scheme. Let $\pi : X \to k$ be the structure morphism of X. Let $W_nX := (|X|, W_n\mathcal{O}_X)$, where $|X|$ is the underlying topological space of X, and $W_n\mathcal{O}_X$ is the sheaf of truncated Witt vectors, and let $W_n\pi : W_nX \to W_nk$ be the morphism induced from π via functoriality. F_X denotes the absolute Frobenius map of X, W_nF_X is the map induced from F_X via functoriality. When $X = \operatorname{Spec} A$ is affine, we also write F_A (resp. W_nF_A) for $F_{\operatorname{Spec} A}$ (resp. $W_nF_{\operatorname{Spec} A}$).

(2) Topologies. $X_{\text{Zar}}, X_{\text{ét}}$ denote the small Zariski site and the small étale site, respectively (we will use a subscript t when the topology t is unspecified). Their structure sheaves are denoted by $\mathcal{O}_X$ and $\mathcal{O}_{X,\text{ét}}$. Let ϵ_* be the restriction functor from the category of étale abelian sheaves to the category of Zariski abelian sheaves. Denote by $R\epsilon_*$ the right derived functor of ϵ_*. The functor ϵ_* can be restricted to a functor from the category of $\mathcal{O}_{X,\text{Zar}}$-modules to the category of $\mathcal{O}_{X,\text{ét}}$-modules, and let ϵ^* be the left adjoint of this restricted functor. Then one has $\epsilon_* \circ \epsilon^* = id$. The functor ϵ^* (resp. ϵ_*) can be restricted to the category of quasi-coherent sheaves on X_{Zar} (resp. $X_{\text{ét}}$), and the pair (ϵ_*, ϵ^*) induces a categorical equivalence between quasi-coherent étale sheaves and quasi-coherent Zariski sheaves by étale descent. We follow [Stacks, Tag 01BE] for the notion of quasi-coherence on the small étale site (see also [Stacks, Tag 03DX]).

We clarify a possible ambiguity here. Fix $n \in \mathbb{N}_{>0}$. Let $\mathcal{G}$ be an étale $\mathbb{Z}/n$-sheaf. Let

$$f_{\text{Zar}} : (\text{Zariski } \mathbb{Z}/n\text{-sheaves}) \to (\text{Zariski abelian sheaves})$$

$$f_{\text{ét}} : (\text{étale } \mathbb{Z}/n\text{-sheaves}) \to (\text{étale abelian sheaves})$$

be the forgetful functors, which are clearly seen to be fully faithful. Let

$$\epsilon'_* : (\text{étale } \mathbb{Z}/n\text{-sheaves}) \to (\text{Zariski } \mathbb{Z}/n\text{-sheaves})$$

be the restriction functor. The functor f_{Zar} and $f_{\text{ét}}$ are clearly exact, and $f_{\text{ét}}$ sends injective étale $\mathbb{Z}/n$-sheaves to ϵ_*-acyclic objects (this is because $R^i\epsilon_*\mathcal{G}$ is the sheaf associated to presheaf $U \mapsto H^i(U_{\text{ét}}, \mathcal{G})$; and injective étale $\mathbb{Z}/n$-sheaves are flasque by [SGA4-2, Exposé V 4.10(2)] and thus Čech acyclic for any Čech cover [SGA4-2, Exposé V 4.5] which is equivalent to $\Gamma(U_{\text{ét}}, -)$-acyclic for all $U \in X_{\text{ét}}$ [Mil80, III 2.12]). Then the Leray spectral sequence implies that

$$R\epsilon_* \circ f_{\text{ét}}(\mathcal{G}) = R(\epsilon_* \circ f_{\text{ét}})(\mathcal{G}) = R(f_{\text{Zar}} \circ \epsilon'_*)(\mathcal{G}) = f_{\text{Zar}} \circ R\epsilon'_*(\mathcal{G})$$

This means the i-th cohomology sheaves of ϵ_* and ϵ'_* are the same for étale $\mathbb{Z}/n$-modules.

Denote by $D^b(X_t, \mathbb{Z}/n)$ the derived category of $\mathbb{Z}/n$-modules in the topology t with bounded cohomologies. The forgetful functor f_t induces a triangulated functor $f_t : D^b(X_t, \mathbb{Z}/n) \hookrightarrow D^b(X_t, \mathbb{Z})$ for both $t = \text{Zar}$ and $t = \text{ét}$, which is exact and faithful (the faithfulness can be seen from the description via homotopy category [Har66, I.4.7]: whenever two chain maps between complexes of $\mathbb{Z}/n$-modules are $\mathbb{Z}$-linearly chain homotopic, they are naturally $\mathbb{Z}/n$-linearly chain homotopic). As a functor between derived categories, f_t is compatible with the derived restriction functors $R\epsilon_*, R\epsilon'_*$ as explained above.

The subscript t will be omitted when $t = \mathrm{Zar}$. We will also omit this subscript t occasionally for maps between étale sheaves, when it is clear from the context. (These rules do not apply to the introduction, where we have deliberately cut down symbols to avoid heavy notations).

(3) De Rham-Witt theory. Let X be a k-scheme. Denote by $W_n\Omega^\bullet_X$ or $W_n\Omega^\bullet_{X/k}$ the de Rham-Witt complex on X as defined in [Ill79]. Since k is a perfect field of positive characteristic p, Illusie's de Rham-Witt complex agrees with the relative version of Langer-Zink in [LZ04] (we will only use this relative version in Lemma 1.21), thus our notations shall cause no confusion. When $n = 1$, $W_n\Omega^\bullet_X$ is also denoted by $\Omega^\bullet_X$ or $\Omega^\bullet_{X/k}$, which is the same as the complex of Kähler differentials. In particular, we have the following four maps according to [Ill79, I] and an observation of Hesselholt-Madsen that the V-filtration equals the R-filtration in general [HM03, 3.2.4] (see also [Mor15, §2.3]. By V-filtration of $W_n\Omega^m_X$ we mean the decreasing filtration $\{V^iW_{n-i}\Omega^m_X + dV^iW_{n-i}\Omega^{m-1}_X\}_i$ of $W_n\Omega^m_X$ indexed by i): the restriction map

$$R : W_n\Omega^m_X \to R_*W_{n-1}\Omega^m_X,$$

the lift-and-multiplication-by-p map

$$\underline{p} : R_*W_{n-1}\Omega^m_X \to W_n\Omega^m_X,$$

the Verschiebung map

$$V : R_*(W_{n-1}F_X)_*W_{n-1}\Omega^m_X \to W_n\Omega^m_X,$$

and the Frobenius map

$$F : W_n\Omega^m_X \to R_*(W_{n-1}F_X)_*W_{n-1}\Omega^m_X.$$

Here by abuse of notation we denote by $R : W_{n-1}X \hookrightarrow W_nX$ the closed immersion induced by the restriction map $R : W_n\mathcal{O}_X \to W_{n-1}\mathcal{O}_X$ on structure sheaves. All the four maps stated above are $W_n\mathcal{O}_X$-linear. We will denote by $W_n\Omega'^i_X$ the abelian sheaf $F(W_{n+1}\Omega^i_X)$ regarded as a $W_n\mathcal{O}_X$-submodule of $(W_nF_X)_*W_n\Omega^i_X$. We sometimes erase the subscript X when there's no confusion.

When we write an element in $W_n\Omega^m_X$ in terms of a product with respect to an totally ordered index set, we make the following assumptions: when an index set is empty, the respective factor of the product does not occur; when an index set is non-empty, the factors of the product are ordered such that the indices are increasing. With these assumptions we avoid any confusion concerning signs.

(4) Coherent duality theory. We follow [Har66][Con00] for the Grothendieck duality theory, and in particular we adopt the sign conventions from [Con00]. We will be working with residual complexes as defined in [Con00, §3.2]. When X is a k-scheme, X is equipped with a canonical residual complex $K_{n,X}$ for every $n \geq 1$ (see Section 1 below). For $f : X \to Y$ being a morphism of finite type between k-schemes, we use $f^\triangle$ instead of $f^!$ to denote the extraordinary inverse image functor for residual complexes as in [Har66, VI 3.1]. An introduction to the functor $f^\triangle$ and some related facts of the Grothendieck duality theory are collected in Section 1.1. When X is Cohen-Macaulay of pure dimension d, $K_{n,X}$ is concentrated in degree $-d$ [Con00, 3.5.1]. This only non-vanishing cohomology sheaf is denoted by $W_n\omega_X$. When $n = 1$, this is denoted by ω_X (ω_X is the usual dualizing sheaf for coherent sheaves on X).

(5) Local cohomology. Let $Y = \operatorname{Spec} B$ be an affine scheme and $Z \subset Y$ be a closed subscheme of pure codimension c. Suppose Z is defined by a sequence $t = \{t_1, \dots, t_c\} \subset B$. Define the Koszul complex associated to sequence t

$$\bigwedge^c B^c \xrightarrow{d_c} \bigwedge^{c-1} B^c \to \dots \xrightarrow{d_3} \bigwedge^2 B^c \xrightarrow{d_2} B^c \xrightarrow{d_1} B$$

with $K^{-q}(t) = K_q(t) = \bigwedge^q B^c$ for $q = 0, \dots, c$. Denote by $\{e_1, \dots, e_c\}$ the standard basis of B^c, and $e_{i_1,\dots,i_q} := e_{i_1} \wedge \dots \wedge e_{i_q} \in K_q(t)$. Then the differential is given by

$$d^{-q}_{K^\bullet}(e_{i_1,\dots,i_q}) = d^{K^\bullet}_q(e_{i_1,\dots,i_q}) = \sum_{j=1}^q (-1)^{j+1} t_{i_j} e_{i_1,\dots,\widehat{i_j},\dots,i_q}$$

(This is consistent with the conventions in [Con00, p.17].) When t is a regular sequence, $K_\bullet(t)$ is a free resolution of $B/(t)$ as B-modules, where (t) denotes the ideal $(t_1,\dots,t_c)\subset B$.

Let M be a B-module. Define

$$K^\bullet(t,M) := \mathrm{Hom}_B(K^{-\bullet}(t),M).$$

Its differential is therefore given by

$$d^q_{K^\bullet(t,M)}(g)(e_{i_1,\dots,i_{q+1}}) = \sum_{j=1}^{q+1}(-1)^{j+1}t_{i_j}g(e_{i_1,\dots,\widehat{i_j},\dots,i_{q+1}})$$

with $g\in \mathrm{Hom}_B(K^{-q}(t),M)$. The map $g\mapsto g(e_{1,\dots,c})$ thus induces an isomorphism

$$H^c(K^\bullet(t,M))\simeq M/(t)M.$$

When t is a regular sequence, this is the only non-vanishing cohomology of the complex $K^\bullet(t,M)$ by [EGAIII-1, Ch. III (1.1.4)].

Denote by t^N the sequence $t_1^N,\dots,t_c^N$. Let $\widetilde{M}$ be the associated quasi-coherent sheaf of M on Y. Then by [SGA2, Exposé II Prop. 5], there is an natural isomorphism

$$\mathrm{colim}_N\, H^c(K^\bullet(t^N,M))\simeq H^c_Z(Y,\widetilde{M}).$$

We denote by $\begin{bmatrix} m \\ t \end{bmatrix}$ the image of $m\in M$ under the composition

$$M\to \mathrm{Hom}_B(\bigwedge^c B^c, M)\to H^c(K^\bullet(t,M))\to H^c_Z(Y,\widetilde{M}),$$

where the first map is associating $m\in M$ the B-linear homomorphism $[e_{1,\dots,c}\mapsto m]$. Notice that this composition restricted to $(t)M$ is the zero map.

Our convention for $\begin{bmatrix} m \\ t \end{bmatrix}$ is consistent with the definitions in [CR11, §A][CR12], but differs from the definition in [BER12, (4.1.2)] by a sign when t is a regular sequence.

Part 1. The complexes

1. Kato's complex $K_{n,X,log,t}$

Let X be a separated scheme of finite type over k of dimension d. In this section, we aim to define and analyse a complex $K_{n,X,log,t}$ for $t=$ Zar and $t=$ ét over a separated scheme X of finite type over k. The original idea of this complex comes from [Kat87, §3], except that Kato is working in a different topology. Our treatment here is self-contained, but the influence of Kato's work [Kat87] on this work is definitely inevitable. We will be working in $t=$ Zar exclusively in Section 1.2-Section 1.5, and will omit the subscript $t=$ Zar in these subsections. Then we will be working in the étale topology in Section 1.6. But before all these, we review Grothendieck's duality theory for coherent sheaves with an emphasis on residual complexes and the functor $f^\triangle$ in Section 1.1.

1.1. **Preliminaries: Residual complexes and Grothendieck's duality theory.** Grothendieck's duality theory aims to generalize the Serre duality for coherent sheaves from the smooth case to the singular case. More substantially, one needs a well-formulated functor $f^!$ in the derived category and a trace map $\mathrm{Tr}_f: Rf_*\circ f^!\to id$ for a proper morphism f. To overcome the difficulty of gluing objects in the derived category, Grothendieck defined the notion of residual complexes, which are certain objects in the category of complexes of quasi-coherent sheaves with coherent cohomology sheaves, to serve as a "concrete" substitution for dualizing complexes. $f^!$ (denoted by $f^\triangle$ in this case) and Tr_f could now be defined locally for residual complexes, and then one has the respective global maps by gluing these local maps. In this subsection we collect some basic facts for Grothendieck's duality with an emphasis on residual complexes and the functor $f^\triangle$. The general references for this topic are [Har66][Con00]. The topology will be the Zariski topology throughout this subsection.

Notation 1.1. All schemes in this subsection Section 1.1 will be assumed to be noetherian with finite Krull dimension (the finite Krull dimension condition is a necessary condition for a scheme to admit a dualizing complex, cf Remark 1.3(2)). $D(X)$ denotes the derived category of $\mathcal{O}_X$-modules. $D_{\mathrm{qc}}(X)$ (resp. $D_c(X)$) denotes the full subcategory of $D(X)$ consisting of complexes whose cohomology sheaves are quasi-coherent (resp. coherent) $\mathcal{O}_X$-modules. $D^+(X)$ (resp. $D^-(X)$, $D^b(X)$) denotes the full subcategory of $D(X)$ consisting of complexes that are cohomologically bounded below (resp. bounded above, bounded). Combinations of these notations might also appear, like $D^b_{\mathrm{qc}}(X)$, $D^b_c(X)$, etc.

According to [Har66, I 4.8], the natural inclusion functor from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X$-modules induces a categorical equivalence from the full subcategory of the derived category of quasi-coherent $\mathcal{O}_X$-modules consisting of complexes cohomologically bounded below to $D^+_{\mathrm{qc}}(X)$.

1.1.1. *Dualizing complexes and residual complexes.* The general references for this part are [Har66, V, VI], [Con00, §3.1, §3.2].

Definition 1.2 (Dualizing complex).

(1) ([Con00, p.118]) A *dualizing complex* on a scheme X is a complex $R \in D^b_c(X)$ such that
- R has finite injective dimension (i.e. R is isomorphic in $D(X)$ to a bounded complex of injective $\mathcal{O}_X$-modules), and
- the natural map
$$\eta_R : id \to R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(-, R), R)$$
is an isomorphism in the derived category $D_c(X)$.

(2) ([Con00, p.123]) For any dualizing complex R on a scheme X and any $x \in X$, there exists an unique integer $d = d_R(x)$ such that $\mathcal{H}^{-d}(R\mathcal{H}om_{\mathrm{Spec}\,\mathcal{O}_{X,x}}(k(x), R_x)) \neq 0$ (cf. [Har66, V.3.4 and V.7.1]). We call d_R the *codimension function on* X *associated to* R, and define the *associated filtration* $Z^\bullet(R)$ of X by
$$Z^p(R) = \{x \in X \mid d_R(x) \geq p\}.$$

We remark that a complex in $D^+_{\mathrm{qc}}(X)$ is of finite injective dimension if and only if it is quasi-isomorphic to a bounded complex of quasi-coherent injective $\mathcal{O}_X$-modules (cf. [Har66, I.7.6(i) and p.83 Def.]). For this one only needs to show the "only if" part. In fact, such a complex is isomorphic in $D(X)$ to a cohomologically bounded complex of quasi-coherent $\mathcal{O}_X$-modules by the categorical equivalence [Har66, I.4.8], and by applying the canonical truncation functor one can assume this complex is bounded. This bounded complex of quasi-coherent $\mathcal{O}_X$-modules is again isomorphic in $D(X)$ to a bounded below complex of quasi-coherent injective $\mathcal{O}_X$-modules by [Har66, II 7.18] and the Cartan-Eilenberg resolution. Applying the canonical truncation functor again we get a bounded complex of quasi-coherent injective $\mathcal{O}_X$-modules. In the end one notices that every morphism in $D^+(X)$ with an injective target can actually be represented by a chain map [Wei94, 10.4.7].

Remark 1.3.

(1) Connection with pointwise analogs. In [Con00, p.120], Conrad defined the notion of weak (resp. strong) pointwise dualizing complexes) on a locally noetherian scheme. In general one has dualizing implies strongly pointwise dualizing implies weakly pointwise dualizing. Note that under our assumption Notation 1.1, these three notions coincide (cf. [Con00, p.120], [Har66, V 8.2]).

(2) Existence and examples [Har66, V §10][Con00, p.133].
   (a) A scheme X is said to be *Gorenstein* if every local ring of X is a Gorenstein local ring [Har66, p.296]. One of the equivalent definitions of a *Gorenstein local ring* is a noetherian local ring admitting a finite injective resolution [Har66, V 9.1]. Examples of Gorenstein rings include $\mathbb{Z}$, fields, regular rings, $W_n k$. When a scheme X is Gorenstein, $\mathcal{O}_X[0]$ is a dualizing complex [Har66, V §10 p.299 1.]. In particular, any regular scheme has a dualizing complex $\mathcal{O}_X[0]$.
   (b) If $f : X \to Y$ is a morphism of finite type and Y admits a dualizing complex R, then $f^! R$ (as defined in [Con00, (3.3.6)]) is a dualizing complex on X. In particular,
      (i) any scheme of finite type over a Gorenstein ring (of finite Krull dimension) admits a dualizing complex;
      (ii) when X is smooth of pure dimension d over a Gorenstein ring A (with finite Krull dimension), then $\omega_{X/A}[d] := \Omega^d_{X/A}[d]$ is a dualizing complex on X.

   To mention a necessary condition: If a scheme admits a dualizing complex then it must be catenary and have finite Krull dimension. ([Har66, p.300 1.2.])

(3) Uniqueness, [Har66, V 3.1][Con00, p.123-p.124]. Let X be a scheme and with dualizing complexes R, R'. Then there exists a unique locally constant $\mathbb{Z}$-valued function $n = n(R, R')$ on X (n is just an integer when X is connected), and a unique line bundle $\mathcal{L} = \mathcal{L}(R, R')$, such that there is an isomorphism
$$\beta_{R,R'} : R' \simeq \mathcal{L}[n] \otimes^L R.$$
From the construction in [Har66, V 3.1], $\mathcal{L}(R, R')$ is defined to be $\mathcal{H}^{-n}(R\mathcal{H}om_{\mathcal{O}_X}(R, R'))$.

(4) Connection with the ring-theoretic version. For a noetherian ring A (to fit in our general assumption we also assume A has finite Krull dimension), one can likewise define the notion of

a dualizing complex. Then the notions of a dualizing complex on A and on $\operatorname{Spec} A$ coincide. ([Con00, 3.1.4])

Definition 1.4 (Residual complex)**.** (1) ([Con00, p.125]) A *residual complex* on a scheme X is a complex K such that

- K is bounded as a complex,
- all the terms of K are quasi-coherent and injective $\mathcal{O}_X$-modules,
- the cohomology sheaves are coherent, and
- there is an isomorphism of $\mathcal{O}_X$-modules

$$\bigoplus_{q\in\mathbb{Z}} K^q \simeq \bigoplus_{x\in X} i_{x*}J(x),$$

where $i_x : \operatorname{Spec}\mathcal{O}_{X,x} \to X$ is the canonical map and $J(x)$ is the quasi-coherent sheaf on $\operatorname{Spec}\mathcal{O}_{X,x}$ associated to an injective hull of $k(x)$ over $\mathcal{O}_{X,x}$ (i.e. the unique injective $\mathcal{O}_{X,x}$-module up to non-unique isomorphisms which contains $k(x)$ as a submodule and such that, for any $0 \neq a \in J(x)$, there exists an element $b \in \mathcal{O}_{X,x}$ with $0 \neq ba \in k(x)$. For a discussion on injective hulls, see [Lam99, §3D and §3J]). $J(x)$ as a sheaf on $\operatorname{Spec}\mathcal{O}_{X,x}$ is supported on the closed point, therefore $i_{x*}J(x)$ as a sheaf on X is supported on $\overline{\{x\}}$.

Unlike dualizing complexes, residual complexes are regarded as objects in the category of complexes of $\mathcal{O}_X$-modules instead of the derived category.

(2) ([Con00, p.125]) Given a residual complex K on X and a point $x \in X$, there is a unique integer $d_K(x)$, such that $i_{x*}J(x)$ is a direct summand of K^q, i.e.,

$$K^q \simeq \bigoplus_{d_K(x)=q} i_{x*}J(x).$$

The assignment $x \mapsto d_K(x)$ is called the *codimension function on X associated to K* (cf. [Har66, IV, 1.1(a)]). We define the *associated filtration*

$$Z^\bullet(K) = \{x \in X \mid d_K(x) \geq p\}.$$

As a first property, residual complexes can be regarded as dualizing complexes via the natural functor

$$Q : (\text{complexes of } \mathcal{O}_X\text{-modules}) \to D(X),$$

according to [Har66, chapter VI, 1.1 a)] and Remark 1.3(1).

Next we want to define a functor $E_{Z^\bullet}$ relating dualizing and residual complexes. First of all we need make some proper assumptions on the filtration $Z^\bullet$ involved.

Definition 1.5 ([Har66, p.240][Con00, p.105])**.** Let X be a scheme and let $Z^\bullet = \{Z^p\}$ be a decreasing filtration of X by subsets Z^p such that

- it is compatible with specialization, i.e., each Z^p is stable under specialization, and each $x \in Z^p - Z^{p+1}$ is not a specialization of any other point of Z^p, and
- it is stationary on above and separated, i.e., $X = Z^p$ for some sufficiently negative p and $\bigcap Z^p = \emptyset$, so X is disjoint union of $Z^p - Z^{p+1}$ over $p \in \mathbb{Z}$.

If $Z^\bullet$ is such a filtration, we denote by $Z^\bullet[n]$ the filtration with $Z^\bullet[n]^p = Z^{p+n}$.

According to the first item, the intuition for such a filtration should be that Z^p consists precisely of those points in the closure of the points in $Z^p - Z^{p+1}$, i.e.,

$$Z^p = \bigcup_{x\in Z^p - Z^{p+1}} \overline{\{x\}}$$

where $\overline{\{x\}}$ denotes the closure of the point x in X, and this relation is strict in the sense that

$$Z^p \supsetneq \bigcup_{x\in Z^p - Z^{p+1} - \{x_0\}} \overline{\{x\}}$$

for any $x_0 \in Z^p \setminus Z^{p+1}$.

Example 1.6. Recall that the (co)dimension of a point $x \in X$ is defined to be the (co)dimension of its closure $\overline{\{x\}}$ as a topological space.

(1) The *dimension filtration*

$$Z^p = \{x \in X \mid \dim \overline{\{x\}} \leq -p\}$$

of X and its shifts

$$Z^\bullet[n]^p = \{x \in X \mid \dim \overline{\{x\}} \leq n - p\}$$

are examples of filtrations satisfying Definition 1.5.

(2) A more standard example of Definition 1.5 is the *codimension filtration*

$$Z^p = \{x \in X \mid \dim \mathcal{O}_{X,x} \geq p\}$$

of X and its shifts

$$Z^\bullet[n]^p = \{x \in X \mid \dim \mathcal{O}_{X,x} \geq p + n\}.$$

On each irreducible component of X, the dimension filtration is clearly a shift of the codimension filtration, which is a descending filtration in a more natural way. The terminology of the dimension filtration is actually non-standard, we include it here just for convenience reasons.

Now review some terminology from local cohomology ([SGA2, Exposé I], see also [Har66, IV][Har67]). Let Z be a locally closed subset of X. Then there exists an open subset V of X containing Z as a closed subset (e.g. $V = X \setminus (\overline{Z} \setminus Z)$ where $\overline{Z}$ denotes the closure of Z). Let $\mathcal{F}$ be an abelian sheaf on X. Define the following subgroup of $\mathcal{F}(V)$

$$\Gamma_Z(\mathcal{F}) := \Gamma_Z(\mathcal{F}|_V) := \{s \in \mathcal{F}(V) \mid \text{any } x \text{ with germ } s_x \neq 0 \text{ lies in } Z\}.$$

One can check that this definition of $\Gamma_Z(\mathcal{F})$ is independent of the choice of V: indeed, one has

$$\Gamma_Z(\mathcal{F}|_V) = \mathrm{Ker}(\Gamma(V, \mathcal{F}) \to \Gamma(V - Z, \mathcal{F}))$$

and for any open subset $V' \supset V$ such that V' contains Z as a closed subset, the restriction map induces an isomorphism $\Gamma_Z(\mathcal{F}|_{V'}) \simeq \Gamma_Z(\mathcal{F}|_V)$. The functor $\Gamma_Z(-)$ is easily seen to be left exact, and its q-th derived functor will be denoted by $H^q_Z(-)$.

Define $\underline{\Gamma}_Z$ to be the functor which assigns to any abelian sheaf $\mathcal{F}$ on X the sheaf

$$U \mapsto \Gamma_{U \cap Z}(\mathcal{F}|_U)$$

on X. $\underline{\Gamma}_Z(\mathcal{F})$ is not necessarily a subsheaf of $\mathcal{F}$. Clearly,

$$\mathrm{Supp}\, \underline{\Gamma}_Z(\mathcal{F}) := \{x \in X \mid \underline{\Gamma}_Z(\mathcal{F})_x \neq 0\} \subset Z.$$

The functor $\underline{\Gamma}_Z(-)$ is left exact, and its q-th derived functor will be denoted by $\mathcal{H}^q_Z(-)$.

For any open subset W of Z, there exists an open subset V of X such that $W = \overline{Z} \cap V$ where $\overline{Z}$ is the closure of Z in X (e.g. $V = W' \cap (X \setminus (\overline{Z} \setminus Z))$ where W' is any open subset of X such that $W = W' \cap Z$). This means that W is closed in V, and thus locally closed in X. Let

$$i : Z \hookrightarrow X$$

be a canonical immersion of the locally closed subset Z of X with any possible structure sheaf on Z (i.e., only the underlying topological space of Z matters). Define $i^!\mathcal{F}$ to be the sheaf

$$W \mapsto \Gamma_W(\mathcal{F})$$

on Z. This is a subsheaf of $i^{-1}\mathcal{F}$ on Z: for each open $W \subset Z$, let V be an open subset of X such that $W = \overline{Z} \cap V$ (and thus $W = Z \cap V$). Then the composition

$$\Gamma_W(\mathcal{F}) := \Gamma_W(\mathcal{F}|_V) \hookrightarrow \Gamma(V, \mathcal{F}) \to \Gamma(V \cap Z, i^{-1}\mathcal{F}) = \Gamma(W, i^{-1}\mathcal{F})$$

is injective: for any $s \in \Gamma_W(\mathcal{F}|_V)$ having zero image in $\Gamma(W, i^{-1}\mathcal{F})$, by definition of the inverse image functor we have $s = 0$ in $\Gamma(V', \mathcal{F})$ for an open subset V' of X with $V' \supset W$. But we also have $s = 0$ in $\Gamma(V - W, \mathcal{F})$, and thus $s = 0$ in $\Gamma(V, \mathcal{F})$. The functor $i^!$ is left exact, and its q-th derived functor is denoted by $R^q i^!$. Note that for an immersion $i : Z \hookrightarrow X$ between separated k-schemes of finite type, the symbol $i^!$ is not the same as the extraordinary inverse image functor between derived categories from duality theory (cf. [Con00, (3.3.6)]). In particular, $i^!$ applied to an $\mathcal{O}_X$-module does not give out an $\mathcal{O}_X$-module in general. Under either interpretation, the symbol $(-)^!$ in this paper serves only for heuristic purposes, and thus shall not cause any confusion for understanding the main part of this paper.

For a given sheaf $\mathcal{F}$ on X, the sheaf $\underline{\Gamma}_Z(\mathcal{F})$ on X and the sheaf $i^!\mathcal{F}$ on Z are related by

$$\underline{\Gamma}_Z(\mathcal{F}) = i_* i^! \mathcal{F}.$$

Thus one has identification of stalks for any point $z \in Z$:

$$(\underline{\Gamma}_Z(\mathcal{F}))_z = (i^!\mathcal{F})_z.$$

The right hand side is due to the facts that an immersion can always be decomposed into a closed immersion and an open immersion, and that the pushforward via either such a closed immersion or an open immersion preserves stalks over points in Z. This identification of stalks implies an identification of sheaves on Z

$$i^!\mathcal{F} = i^{-1}\underline{\Gamma}_Z(\mathcal{F}).$$

Since the functor i^{-1} is exact, one has an identification of the derived functors

$$R^q i^!\mathcal{F} = i^{-1}\mathcal{H}^q_Z(\mathcal{F}).$$

In particular:

- When Z is closed, denote by $j : U = X \setminus Z \hookrightarrow X$ its open complement. Then we have
$$\underline{\Gamma}_Z(\mathcal{F}) = \mathrm{Ker}(\mathcal{F} \to j_*j^{-1}\mathcal{F}).$$
In this case we have a canonical injective map
$$\underline{\Gamma}_Z(\mathcal{F}) \hookrightarrow \mathcal{F}.$$
- When Z is open, one easily sees
$$i^!\mathcal{F} = i^{-1}\mathcal{F} \quad \text{and} \quad \Gamma_Z(\mathcal{F}) = \Gamma(Z, \mathcal{F}).$$
Thus we have an isomorphism of sheaves
$$\underline{\Gamma}_Z(\mathcal{F}) = i_*i^{-1}\mathcal{F}$$
and a canonical homomorphism of sheaves
$$\mathcal{F} \to \underline{\Gamma}_Z(\mathcal{F})$$
given by the adjunction. One can also canonically interpret the local cohomology groups in this case as
$$H^q_Z(\mathcal{F}) = H^q(Z, i^{-1}\mathcal{F}) \quad \text{and} \quad \mathcal{H}^q_Z(\mathcal{F}) = R^q i_*(i^{-1}\mathcal{F}).$$

Consider a set Z which is stable under specialization. One notices that sets Z^p from Definition 1.5, and in particular, the sets from the codimension filtration, i.e., $Z^p = \{x \in X \mid \dim \mathcal{O}_{X,x} \geq p\}$ for some p, are typical examples of such a Z. These sets are far away from being locally closed in general. Following [Har66, IV §1, Var. 1 Motif D and p223 5.], define the abelian sheaf

$$\underline{\Gamma}_Z(\mathcal{F}) := \mathrm{colim}_A\, \underline{\Gamma}_A(\mathcal{F})$$

where A runs over all closed subsets of X of codimension $\geq p$. Note that the set of all such A is a well-defined family of supports as defined in [Har66, IV §1 Var. 1], and all these abelian sheaves $\underline{\Gamma}_A(\mathcal{F})$ together with the connecting homomorphisms

$$\underline{\Gamma}_A(\mathcal{F}) \hookrightarrow \underline{\Gamma}_{A'}(\mathcal{F})$$

for all $A \subset A'$ (induced by the natural injective map $\Gamma_A(\mathcal{F}) \hookrightarrow \Gamma_{A'}(\mathcal{F})$) form an inductive system.

Suppose from now on that $Z^\bullet$ is a descending filtration as in Definition 1.5. Denote

$$\underline{\Gamma}_{Z^p/Z^{p+1}}(\mathcal{F}) := \underline{\Gamma}_{Z^p}(\mathcal{F})/\underline{\Gamma}_{Z^{p+1}}(\mathcal{F}).$$

The functor $\underline{\Gamma}_{Z^p/Z^{p+1}}$ is in general not left exact. But since the category of abelian sheaves has enough injectives, one can still define the derived functor $R\underline{\Gamma}_{Z^p/Z^{p+1}} : D^+(X) \to D(X)$ using injective resolutions (note that the construction of the derived functor does not rely on any one-sided exactness, cf. [Har66, I 5.1]), and therefore can define

$$\mathcal{H}^i_{Z^p/Z^{p+1}}(\mathcal{F}^\bullet) := H^i(R\underline{\Gamma}_{Z^p/Z^{p+1}}(\mathcal{F}^\bullet))$$

for a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules. Note that with this definition of $\mathcal{H}^i_{Z^p/Z^{p+1}}$, one only has an injection $\underline{\Gamma}_{Z^p/Z^{p+1}}(\mathcal{F}) \hookrightarrow \mathcal{H}^0_{Z^p/Z^{p+1}}(\mathcal{F})$ for a general sheaf $\mathcal{F}$. When $\mathcal{F}$ is flasque, this injection is an isomorphism.

The functors $\underline{\Gamma}_{Z^p/Z^{p+1}}$ and $\mathcal{H}^i_{Z^p/Z^{p+1}}$ are closely related to another functor Γ_x with x being a point in X. For $x \in X$, $\mathcal{F}$ a $\mathcal{O}_X$-module, define the abelian group (cf. [Har66, §1 Var. 8])

$$\Gamma_x(\mathcal{F}) := \mathrm{colim}_{U \ni x}\, \Gamma_{\overline{\{x\}} \cap U}(\mathcal{F}|_U),$$

where U runs through all open neighborhoods of x. Naturally $\Gamma_x(\mathcal{F}) \subset \mathcal{F}_x$. Since a filtered colimit preserves exactness in the category of abelian sheaves, Γ_x is left-exact and the derived functor $R\Gamma_x$ and cohomology groups $H^i_x(\mathcal{F})$ are thereby defined. Due to the compatibility of the colimit with taking cohomologies, one has an identification of $\mathcal{O}_{X,x}$-modules

$$H^i_x(\mathcal{F}) = (\mathcal{H}^i_{\overline{\{x\}}}(\mathcal{F}))_x.$$

Moreover, one has a canonical functorial isomorphism [Har66, p.226][Con00, (3.1.4)]

$$\mathcal{H}^i_{Z^p/Z^{p+1}}(\mathcal{F}^\bullet) \xrightarrow{\simeq} \bigoplus_{x \in Z^p - Z^{p+1}} i_{x*}(H^i_x(\mathcal{F}^\bullet)), \tag{1.1.1}$$

where $i_x : \operatorname{Spec} \mathcal{O}_{X,x} \to X$ is the canonical map and by slight abuse of notations we use $H^i_x(\mathcal{F}^\bullet)$ to denote the quasi-coherent sheaf on $\operatorname{Spec} \mathcal{O}_{X,x}$ associated to the $\mathcal{O}_{X,x}$-module $H^i_x(\mathcal{F}^\bullet)$. $H^i_x(\mathcal{F}^\bullet)$ as a sheaf on $\operatorname{Spec} \mathcal{O}_{X,x}$ is supported on the closed point if it is nonzero.

Definition 1.7 (Cousin functor $E_{Z^\bullet}$)**.** Let $Z^\bullet$ be as in Definition 1.5. For any bounded below complex $\mathcal{F}^\bullet$, choose a bounded below injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}^\bullet$. Then one has a natural decreasing exhaustive filtration by subcomplexes of $\mathcal{I}^\bullet$:

$$\cdots \supset \underline{\Gamma}_{Z^p}(\mathcal{I}^\bullet) \supset \underline{\Gamma}_{Z^{p+1}}(\mathcal{I}^\bullet) \supset \dots.$$

This filtration is stalkwise bounded below. Now consider the E_1-spectral sequence associated to this filtration

$$E_1^{p,q} \Rightarrow H^{p+q}(\mathcal{F}^\bullet).$$

The *Cousin complex* ([Con00, p.105]) $E_{Z^\bullet}(\mathcal{F}^\bullet)$ associated to $\mathcal{F}^\bullet$ is defined to be the 0-th line of the E_1-page, namely

$$E_{Z^\bullet}(\mathcal{F}^\bullet) := (E_1^{p,0} = \mathcal{H}^p_{Z^p/Z^{p+1}}(\mathcal{F}), d_1^{p,0}).$$

We will also use the shortened notation E for $E_{Z^\bullet}$ when the filtration $Z^\bullet$ is clear from the context.

In the end, we recall below the categorical equivalence between dualizing complexes and residual complexes.

Proposition 1.8 ([Con00, 3.2.1])**.** *Let X be a scheme and $Z^\bullet$ be a filtration on X which is a shift of the codimension filtration on each irreducible component of X. Suppose X admits a residual complex. Then $E_{Z^\bullet}$ and Q induce quasi-inverses*

$$\begin{pmatrix} \textit{dualizing complexes whose} \\ \textit{associated filtration is } Z^\bullet \end{pmatrix} \underset{E_{Z^\bullet}}{\overset{Q}{\rightleftarrows}} \begin{pmatrix} \textit{residual complexes whose} \\ \textit{associated filtration is } Z^\bullet \end{pmatrix}.$$

1.1.2. *The functor $f^\triangle$.* Let $f : X \to Y$ be a finite type morphism between noetherian schemes of finite Krull dimension and let K be a residual complex on Y with associated filtration $Z^\bullet := Z^\bullet(K)$ and codimension function d_K. Define the function $d_{f^\triangle K}$ on X to be ([Con00, (3.2.4)])

$$d_{f^\triangle K}(x) := d_K(f(x)) - \operatorname{trdeg}(k(x)/k(f(x)))$$

(so far the subscript $f^\triangle K$ is simply regarded as a formal symbol), and define $f^\triangle Z^\bullet$ accordingly

$$f^\triangle Z^\bullet = \{x \in X \mid d_{f^\triangle K}(x) \geq p\}.$$

Notice that when f has constant fiber dimension r, $f^\triangle Z^\bullet$ is simply $f^{-1}Z^\bullet[r]$.

Following [Har66, VI, 3.1], [Con00, 3.2.2], we list some properties of the functor $f^\triangle$ below.

Proposition 1.9. *There exists a functor*

$$f^\triangle : \begin{pmatrix} \textit{residual complexes on } Y \\ \textit{with filtration } Z^\bullet \end{pmatrix} \to \begin{pmatrix} \textit{residual complexes on } X \\ \textit{with filtration } f^\triangle Z^\bullet \end{pmatrix}$$

having the following properties (we assume all schemes are noetherian schemes of finite Krull dimension, and all morphisms are of finite type).

(1) If f is finite, there is an isomorphism of complexes ([Har66, VI 3.1])

$$\psi_f : f^\triangle K \xrightarrow{\simeq} E_{f^{-1}Z^\bullet}(\overline{f}^* R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)) \simeq \overline{f}^* \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K),$$

where $\overline{f}^* := f^{-1}(-) \otimes_{f^{-1}f_*\mathcal{O}_X} \mathcal{O}_X$ is the pullback functor associated to the map of ringed spaces $\overline{f} : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. Since $\overline{f}$ is flat, the pullback functor $\overline{f}^*$ is exact. The last isomorphism is due to the fact that $\overline{f}^* \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$ is a residual complex with respect to filtration $f^{-1}Z^\bullet$ (see [Har66, VI, 4.1], [Con00, (3.4.5)]).

(2) *If f is smooth and separated of relative dimension r, there is an isomorphism of complexes ([Har66, VI 3.1])*

$$\varphi_f : f^{\triangle}K \xrightarrow{\simeq} E_{f^{-1}Z^{\bullet}[r]}(\Omega^r_{X/Y}[r] \otimes^{L}_{\mathcal{O}_X} Lf^*K) = E_{f^{-1}Z^{\bullet}[r]}(\Omega^r_{X/Y}[r] \otimes_{\mathcal{O}_X} f^*K).$$

The last equality is due to the flatness of f and local freeness of $\Omega^r_{X/Y}$.

When f is étale (or more generally residually stable, see (5) below), this becomes

$$\varphi_f : f^{\triangle}K \xrightarrow{\simeq} E_{f^{-1}Z^{\bullet}}(f^*K) \simeq f^*K.$$

The last isomorphism is due to [Har66, VI 5.3]. In particular, when $f = j : X \hookrightarrow Y$ is an open immersion, $j^{\triangle}K = j^*K$ is a residual complex with respect to filtration $X \cap Z^{\bullet}$ ([Con00, p.128]).

(3) *When f is finite étale, the chain maps ψ_f, φ_f are compatible. Namely, for a given residual complex K on Y, there exists an isomorphism of complexes $\overline{f}^*\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K) \xrightarrow{\simeq} f^*K$ as defined in [Con00, (2.7.9)], such that the following diagram of complexes commutes*

$$\begin{array}{ccc} & \overset{\psi_f}{\underset{\simeq}{\nearrow}} & \overline{f}^*\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K) \\ f^{\triangle}(K) & & \downarrow \simeq \\ & \underset{\varphi_f}{\overset{\simeq}{\searrow}} & f^*K. \end{array}$$

(4) *(Composition) If $f : X \to Y$ and $g : Y \to Z$ are two such morphisms, there is an natural isomorphism of functors ([Con00, (3.2.3)])*

$$c_{f,g} : (gf)^{\triangle} \xrightarrow{\simeq} f^{\triangle}g^{\triangle}.$$

(5) *(Residually stable base change) Following [Con00, p.132], we say a (not necessarily locally finite type) morphism $f : X \to Y$ between locally noetherian schemes is* residually stable *if*
- *f is flat,*
- *the fibers of f are discrete and for all $x \in X$, the extension $k(x)/k(f(x))$ is algebraic, and*
- *the fibers of f are Gorenstein schemes.*

As an example, an étale morphism is residually stable. For more properties of residually stable morphisms, see [Har66, VI, §5]. Let f be a morphism of finite type, and u be a residually stable morphism. Let

(1.1.2)
$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

be a cartesian diagram. Then there is an natural transformation between functors ([Har66, VI 5.5])

$$d_{u,f} : f'^{\triangle}u^* \xrightarrow{\simeq} u'^*f^{\triangle}.$$

(6) *$f^{\triangle}$ is compatible with translation and tensoring with an invertible sheaf. More precisely, for an invertible sheaf $\mathcal{L}$ on Y and a locally constant $\mathbb{Z}$-valued function n on Y, one has canonical isomorphisms of complexes [Con00, (3.3.9)]*

$$f^{\triangle}(\mathcal{L}[n] \otimes K) \simeq (f^*K)[n] \otimes f^{\triangle}K \simeq (f^*\mathcal{L} \otimes f^{\triangle}K)[n].$$

More properties and compatibility diagrams can be found in [Con00, §3.3] and [Har66, VI, §3, §5].

1.1.3. *Trace map for residual complexes.*

Proposition 1.10. *Let $f : X \to Y$ be a* proper *morphism between noetherian schemes of finite Krull dimensions and let K be a residual complex on Y. Then there exists a map of complexes*

$$\mathrm{Tr}_f : f_*f^{\triangle}K \to K,$$

such that the following properties hold ([Con00, §3.4]).

(1) *When f is finite, Tr_f at a given residual complex K agrees with the following composite as a map of complexes ([Con00, (3.4.8)]):*

(1.1.3)
$$\mathrm{Tr}_f : f_*f^{\triangle}K \underset{\simeq}{\xrightarrow{\psi_f}} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K) \xrightarrow{\text{ev. at 1}} K.$$

(2) *When $f : \mathbf{P}^d_Y \to Y$ is the natural projection, then the trace map Tr_f at K, as a map in the derived category $D^b_c(Y)$, agrees with the following composite ([Con00, p.151])*

$$f_*f^\triangle K \xrightarrow[\simeq]{\varphi_f} Rf_*(\Omega^n_{\mathbf{P}^n_X/X}[n]) \otimes_{\mathcal{O}_Y} K \to K.$$

The first map is induced from φ_f followed by the projection formula ([Con00, (2.1.10)]), and the second map is induced by base change from the following isomorphism of groups([Con00, (2.3.3)])

$$\mathbb{Z} \xrightarrow{\simeq} H^d(\mathbf{P}^d_{\mathbb{Z}}, \Omega^d_{\mathbf{P}^d_{\mathbb{Z}}/\mathbb{Z}}) = \check{H}^d(\mathfrak{U}, \Omega^d_{\mathbf{P}^d_{\mathbb{Z}}/\mathbb{Z}}), \quad 1 \mapsto (-1)^{\frac{d(d+1)}{2}} \frac{dt_1 \wedge \cdots \wedge dt_d}{t_1 \ldots t_d},$$

where $\mathfrak{U} = \{U_0, ..., U_d\}$ is the standard covering of $\mathbf{P}^d_{\mathbb{Z}}$ and the t_i's are the coordinate functions on U_0.

(3) *(Functoriality, [Con00, 3.4.1(1)])* Tr_f *is functorial with respect to residual complexes with the same associated filtration.*

(4) *(Composition, [Con00, 3.4.1(2)]) If $g : Y \to Z$ is another proper morphism, then*

$$\operatorname{Tr}_{gf} = \operatorname{Tr}_g \circ g_*(\operatorname{Tr}_f) \circ (gf)_* c_{f,g}.$$

(5) *(Residually stable base change, [Har66, VI 5.6]) Notations are the same as in diagram (1.1.2), and we assume f proper and u residually stable. Then the diagram*

$$\begin{array}{ccc} u^*Rf_*f^\triangle & \xrightarrow{u^*\operatorname{Tr}_f} & u^* \\ \downarrow \simeq & & \uparrow \operatorname{Tr}_{f'} \\ Rf'_*u'^*f^\triangle & \xrightarrow[\simeq]{Rf'_*(d_{u,f})} & Rf'_*f'^\triangle u^* \end{array}$$

commutes.

(6) Tr_f *is compatible with translation and tensoring with an invertible sheaf ([Con00, p.148]).*

(7) *(Grothendieck-Serre duality, [Con00, 3.4.4]) If $f : X \to Y$ is proper, then for any $\mathcal{F} \in D^-_{\mathrm{qc}}(X)$, the composition*

$$Rf_*R\mathcal{H}om_X(\mathcal{F}, f^\triangle K) \to R\mathcal{H}om_Y(Rf_*\mathcal{F}, Rf_*f^\triangle K) \xrightarrow{\operatorname{Tr}_f} R\mathcal{H}om_Y(Rf_*\mathcal{F}, K)$$

is an isomorphism in $D^+_{\mathrm{qc}}(Y)$.

More properties and compatibility diagrams can be found in [Con00, §3.4] and [Har66, VI, §4-5, VII, §2].

1.2. **Definition of** $K_{n,X,log}$**.** Let W_nk be the ring of Witt vectors of length n of k. Notice that W_nk is an injective W_nk-module, which means $\operatorname{Spec} W_nk$ is a Gorenstein scheme by [Har66, V. 9.1(ii)], and its structure sheaf placed at degree 0 is a residual complex (with codimension function being the zero function and the associated filtration being $Z^\bullet(W_nk) = \{Z^0(W_nk)\}$, where $Z^0(W_nk)$ is the set of the unique point in $\operatorname{Spec} W_nk$) by [Har66, p299 1.] and the categorical equivalence Proposition 1.8 (note that in this case the Cousin functor $E_{Z^\bullet(W_nk)}$ applied to W_nk is still W_nk). This justifies the symbol $(W_nF_k)^\triangle$ to appear. To avoid possible confusion we will distinguish the source and target of the absolute Frobenius by using the symbols $k_1 = k_2 = k$. Absolute Frobenius is then written as $F_k : (\operatorname{Spec} k_1, k_1) \to (\operatorname{Spec} k_2, k_2)$, and the n-th Witt lift is written as $W_nF_k : (\operatorname{Spec} W_nk_1, W_nk_1) \to (\operatorname{Spec} W_nk_2, W_nk_2)$. There is a natural isomorphism of W_nk_1-modules (the last isomorphism is given by Proposition 1.9(1))

(1.2.1)
$$\begin{aligned} W_nk_1 &\xrightarrow{\simeq} \overline{W_nF_k}^* \operatorname{Hom}_{W_nk_2}((W_nF_k)_*(W_nk_1), W_nk_2) \simeq (W_nF_k)^\triangle(W_nk_2), \\ a &\mapsto a \otimes [(W_nF_k)_*1 \mapsto 1] \quad (= [(W_nF_k)_*a \mapsto 1]), \end{aligned}$$

where $\overline{W_nF_k} : (\operatorname{Spec} W_nk_1, W_nk_1) \to (\operatorname{Spec} W_nk_2, (W_nF_k)_*(W_nk_1))$ is the natural map of ringed spaces, and the Hom set is given the $(W_nF_k)_*(W_nk_1)$-module structure via the first place. In fact, it is clearly a bijection: identify the target with W_nk_2 via the evaluate-at-1 map, then one can see that the map (1.2.1) is identified with $a \mapsto (W_nF_k)^{-1}(a)$.

Let X be a separated scheme of finite type over k with structure map $\pi : X \to k$. Recall that $W_nX := (|X|, W_n\mathcal{O}_X)$, where $|X|$ is the underlying topological space of X, and $W_n\mathcal{O}_X$ is the sheaf of length n truncated Witt vectors. $W_n\pi : W_nX \to W_nk$ is the morphism induced from π via functoriality. Since W_nk is a Gorenstein scheme as we recalled in the last paragraph,

$$K_{n,X} := (W_n\pi)^\triangle W_nk$$

is a residual complex on W_nX, associated to the codimension function $d_{K_{n,X}}$ with

$$d_{K_{n,X}}(x) = -\dim \overline{\{x\}},$$

and the filtration $Z^\bullet(K_{n,X}) = \{Z^p(K_{n,X})\}$ with

$$Z^p(K_{n,X}) = \{x \in X \mid \dim \overline{\{x\}} \leq -p\}$$

(cf. Proposition 1.9). That is, the filtration $Z^\bullet(K_{n,X})$ is precisely the dimension filtration in the sense of Example 1.6(1), which is a shift of the codimension filtration on each irreducible component. In particular, $K_{n,X}$ is a bounded complex of injective quasi-coherent $W_n\mathcal{O}_X$-modules with coherent cohomologies sitting in degrees

$$[-d, 0].$$

When $n = 1$, we set $K_X := K_{1,X}$. Now we turn to the definition of C'. We denote the level n Witt lift of the absolute Frobenius F_X by $W_nF_X : (W_nX_1, W_n\mathcal{O}_{X_1}) \to (W_nX_2, W_n\mathcal{O}_{X_2})$. The structure maps of W_nX_1, W_nX_2 are $W_n\pi_1, W_n\pi_2$ respectively. These schemes fit into a commutative diagram

$$\begin{array}{ccc} W_nX_1 & \xrightarrow{W_nF_X} & W_nX_2 \\ {\scriptstyle W_n\pi_1}\downarrow & & \downarrow{\scriptstyle W_n\pi_2} \\ \operatorname{Spec} W_nk_1 & \xrightarrow{W_nF_k} & \operatorname{Spec} W_nk_2. \end{array}$$

Denote

$$K_{n,X_i} := (W_n\pi_i)^\triangle(W_nk_i), \quad i = 1, 2.$$

Via functoriality, one has a $W_n\mathcal{O}_{X_1}$-linear map

$$\begin{aligned} K_{n,X_1} = (W_n\pi_1)^\triangle(W_nk_1) &\xrightarrow[\simeq]{(W_n\pi_1)^\triangle(1.2.1)} (W_n\pi_1)^\triangle(W_nF_k)^\triangle(W_nk_2) \\ &\simeq (W_nF_X)^\triangle(W_n\pi_2)^\triangle(W_nk_2) \simeq (W_nF_X)^\triangle K_{n,X_2}. \end{aligned} \tag{1.2.2}$$

Here the isomorphism at the beginning of the second line is given by Proposition 1.9(4). Then via the adjunction with respect to the morphism W_nF_X, one has a $W_n\mathcal{O}_{X_2}$-linear map

$$C' := C'_n : (W_nF_X)_*K_{n,X_1} \xrightarrow[\simeq]{(W_nF_X)_*(1.2.2)} (W_nF_X)_*(W_nF_X)^\triangle K_{n,X_2} \xrightarrow{\operatorname{Tr}_{W_nF_X}} K_{n,X_2}, \tag{1.2.3}$$

where the last trace map is Proposition 1.10. We call it the (level n) *Cartier operator* for residual complexes. We sometimes omit the $(W_nF_X)_*$-module structure of the source and write simply as $C' : K_{n,X} \to K_{n,X}$.

Now we come to the construction of $K_{n,X,log}$ (cf. [Kat87, §3]). Define

$$K_{n,X,log} := \operatorname{Cone}(K_{n,X} \xrightarrow{C'-1} K_{n,X})[-1]. \tag{1.2.4}$$

This is a complex of abelian sheaves sitting in degrees

$$[-d, 1].$$

When $n = 1$, we set $K_{X,log} := K_{1,X,log}$. Writing more explicitly, $K_{n,X,log}$ is the following complex

$$(K_{n,X}^{-d} \oplus 0) \to (K_{n,X}^{-d+1} \oplus K_{n,X}^{-d}) \to \ldots \to (K_{n,X}^{0} \oplus K_{n,X}^{-1}) \to (0 \oplus K_{n,X}^{0}).$$

The differential of $K_{n,X,log}$ at degree i is given by

$$\begin{aligned} d_{log} = d_{n,log} : K^i_{n,X,log} &\to K^{i+1}_{n,X,log} \\ (K^i_{n,X} \oplus K^{i-1}_{n,X}) &\to (K^{i+1}_{n,X} \oplus K^{i}_{n,X}) \\ (a, b) &\mapsto (d(a), -(C'-1)(a) - d(b)), \end{aligned}$$

where d is the differential in $K_{n,X}$. The sign conventions we adopt here for shifted complexes and the cone construction are the same as in [Con00, p6, p8]. And naturally, one has a distinguished triangle

$$K_{n,X,log} \to K_{n,X} \xrightarrow{C'-1} K_{n,X} \xrightarrow{+1} K_{n,X,log}[1]. \tag{1.2.5}$$

Explicitly, the first map is in degree i given by

$$\begin{aligned} K^i_{n,X,log} = K^i_{n,X} \oplus K^{i-1}_{n,X} &\to K^i_{n,X}, \\ (a, b) &\mapsto a. \end{aligned}$$

The "+1" map is given by

$$K^i_{n,X} \to (K_{n,X,log}[1])^i = K^{i+1}_{n,X,log} = (K^{i+1}_{n,X} \oplus K^i_{n,X}),$$

$$b \mapsto (0, b).$$

Both maps are indeed maps of chain complexes.

1.3. **Comparison of $W_n\Omega^d_{X,log}$ with $K_{n,X,log}$.** Recall the following result from classical Grothendieck duality theory [Har66, IV, 3.4][Con00, 3.1.3] and Ekedahl [Eke84, §1] (see also [CR12, proof of 1.10.3 and Rmk. 1.10.4]).

Proposition 1.11 (Ekedahl). *When X is smooth and of pure dimension d over k, then there is a canonical quasi-isomorphism*

$$W_n\Omega^d_X[d] \xrightarrow{\simeq} K_{n,X}.$$

Remark 1.12. Suppose X is a separated scheme of finite type over k of dimension d. Denote by U the smooth locus of X, and suppose that the complement Z of U is of dimension e. Suppose moreover that U is non-empty and equidimensional (it is satisfied for example, when X is integral). Then Ekedahl's quasi-isomorphism Proposition 1.11 gives a quasi-isomorphism of dualizing complexes

$$W_n\Omega^d_U[d] \xrightarrow{\simeq} K_{n,U}. \tag{1.3.1}$$

Note that by the very definition, the associated filtrations of quasi-isomorphic dualizing complexes are the same (cf. [Har66, 3.4]). As explained above, the associated filtration of $K_{n,U}$ is its dimension filtration. Let $Z^\bullet$ be the codimension filtration of U (cf. Example 1.6). Since U is of pure dimension d, we know that its dimension filtration is just a shift of the codimension filtration, i.e., $Z^\bullet[d]$. Apply the Cousin functor associated to the shifted codimension filtration $Z^\bullet[d]$ (cf. Definition 1.7) to the quasi-isomorphism (1.3.1) between dualizing complexes, we have an isomorphism of residual complexes

$$E_{Z^\bullet[d]}(W_n\Omega^d_U[d]) \xrightarrow{\simeq} K_{n,U}$$

with the same filtration $Z^\bullet[d]$ (cf. Proposition 1.8). Since W_nj is an open immersion, we can canonically identify the residual complexes $(W_nj)^*K_{n,X} \simeq K_{n,U}$ by Proposition 1.9(2). Since $K_{n,X}$ is a residual complex and in particular is a Cousin complex (cf. [Con00, p. 105]), the adjunction map $K_{n,X} \to (W_nj)_*(W_nj)^*K_{n,X} \simeq (W_nj)_*K_{n,U}$ is an isomorphism at degrees $[-d, -e-1]$. Thus the induced chain map

$$K_{n,X} \to (W_nj)_*E_{Z^\bullet[d]}(W_n\Omega^d_U[d])$$

is an isomorphism at degrees $[-d, -e-1]$.

1.3.1. *Compatibility of C' with the classical Cartier operator C.* We review the absolute Cartier operator in classical literature (see e.g. [BK05, Chapter 1 §3], [Ill79, §0.2], [Katz70, 7.2], [IR83, III §1]). Let X be a k-scheme. The *(absolute) inverse Cartier operator* γ_X of degree i on a scheme X is affine locally, say, on $\operatorname{Spec} A \subset X$, given additively by the following expression ($\mathcal{H}^i(-)$ denotes the cohomology sheaf of the complex)

$$\begin{aligned} \gamma_A : \quad \Omega^i_{A/k} \quad &\to \quad \mathcal{H}^i(F_{A,*}\Omega^\bullet_{A/k}) \\ a\,da_1 \dots da_i &\mapsto a^p a_1^{p-1} da_1 \dots a_i^{p-1} da_i, \end{aligned} \tag{1.3.2}$$

where $a, a_1, \dots a_i \in A$. Here $\mathcal{H}^i(F_{A,*}\Omega^\bullet_{A/k})$ denotes the A-module structure on $\mathcal{H}^i(\Omega^\bullet_{A/k})$ via the absolute Frobenius $F_A : A \to A, a \mapsto a^p$ (note that $F_{A,*}\Omega^\bullet_{A/k}$ is a complex of A-modules in positive characteristic). For each degree i, γ_A thus defined is an A-linear map. These local maps patch together and give rise to a map of sheaves

$$\gamma_X : \Omega^i_X \to \mathcal{H}^i(F_{X,*}\Omega^\bullet_X) \tag{1.3.3}$$

which is $\mathcal{O}_X$-linear. When X is smooth of dimension d, γ_X is a isomorphism of $\mathcal{O}_X$-modules, which is called the *(absolute) Cartier isomorphism*. See [BK05, 1.3.4] for a proof (note that although the authors there assumed the base field to be algebraically closed, the proof of this theorem works for any perfect field k of positive characteristic).

This can be generalized to the de Rham-Witt case.

Lemma 1.13 (cf. [Kat86b, 4.1.3]). *Denote by $W_n\Omega'^i_X$ the abelian sheaf $F(W_{n+1}\Omega^i_X)$ regarded as a $W_n\mathcal{O}_X$-submodule of $(W_nF_X)_*W_n\Omega^i_X$. When X is smooth of dimension d, the map*

$$\overline{F} : W_n\Omega^i_X \to W_n\Omega'^i_X / dV^{n-1}\Omega^{i-1}_X$$

induced by Frobenius $F : W_{n+1}\Omega^i_X \to R_*(W_nF_X)_*W_n\Omega^i_X$ is an isomorphism of $W_n\mathcal{O}_X$-modules.

In particular, when $i = d$,

$$\overline{F} : W_n\Omega^d_X \to (W_nF_X)_*W_n\Omega^d_X / dV^{n-1}\Omega^{d-1}_X$$

is an isomorphism of $W_n\mathcal{O}_X$-modules.

Proof. Since

$$\mathrm{Ker}(R : W_{n+1}\Omega^i \to W_n\Omega^i) = V^n\Omega^i + dV^n\Omega^{i-1},$$

$FV^n\Omega^i = 0$ and $FdV^n\Omega^{i-1} = dV^{n-1}\Omega^{i-1}$, $F : W_{n+1}\Omega^i \to W_n\Omega^i$ reduces to

$$\overline{F} : W_n\Omega^i \to W_n\Omega^i/dV^{n-1}.$$

Surjectivity is clear. We show injectivity. Suppose $x \in W_{n+1}\Omega^i$, $y \in \Omega^{i-1}$, such that $F(x) = dV^{n-1}y$. Then $F(x - dV^n y) = 0$, which implies by [Ill79, I (3.21.1.2)] that $x - dV^n y \in V^n\Omega^i$.

The second claim follows from the fact that $F : W_{n+1}\Omega^d \to R_*(W_nF_X)_*W_n\Omega^d$ is surjective on top degree d [Ill79, I (3.21.1.1)], and therefore $W_n\Omega'^d = (W_nF_X)_*W_n\Omega^d$ as $W_n\mathcal{O}_X$-modules. □

Definition 1.14 ((absolute) Cartier operator)**.** Let X be a smooth scheme of dimension d over k.

(1) The composition

$$C := C_X : Z^i(F_{X,*}\Omega^\bullet_X) \to \mathcal{H}^i(F_{X,*}\Omega^\bullet_X) \xrightarrow{(\gamma_X)^{-1}} \Omega^i_X \tag{1.3.4}$$

$$\big(\text{with } Z^i(F_{X,*}\Omega^\bullet_X) := \mathrm{Ker}(F_{X,*}\Omega^i_X \xrightarrow{d} F_{X,*}\Omega^{i+1}_X)\big)$$

is called the *(absolute) Cartier operator* of degree i, denoted by C or C_X.

(2) (cf. [Kat86b, 4.1.2, 4.1.4]) More generally, for $n \geq 1$, define the *(absolute) Cartier operator* $C_n := C_{n,X}$ *of level* n to be the composite

$$C_n : W_n\Omega'^i_X \twoheadrightarrow W_n\Omega'^i_X/dV^{n-1}\Omega^{i-1}_X \xrightarrow[\simeq]{\overline{F}^{-1}} W_n\Omega^i_X, \tag{1.3.5}$$

where $\overline{F} : W_n\Omega^i_X \xrightarrow{\simeq} W_n\Omega'^i_X/dV^{n-1}\Omega^{i-1}_X$ is the map in Lemma 1.13. When $i = d$ is the top degree we obtain the $W_n\mathcal{O}_X$-linear map

$$C_n : (W_nF_X)_*W_n\Omega^d_X \twoheadrightarrow (W_nF_X)_*W_n\Omega^d_X/dV^{n-1}\Omega^{d-1}_X \xrightarrow{\overline{F}^{-1}} W_n\Omega^d_X. \tag{1.3.6}$$

Remark 1.15. (1) According to the explicit formula for F, we have $C = C_1$ [Ill79, I 3.3].

(2) C_n (for all n) are compatible with étale pullbacks. Actually any de Rham-Witt system (e.g. $(W_n\Omega^\bullet_X, F, V, R, \underline{p}, d)$) is compatible with étale base change [CR12, 1.3.2].

(3) The n-th power of Frobenius F induces a map

$$\overline{F}^n : W_n\Omega^i_X \xrightarrow{\simeq} \mathcal{H}^i((W_nF_X)^n_*W_n\Omega^\bullet_X),$$

which is the same as [IR83, III (1.4.1)].

(4) Notice that on $\mathrm{Spec}\, W_nk$, $C_n : W_nk \to W_nk$ is simply the map $(W_nF_k)^{-1}$, because $F : W_{n+1}k \to W_nk$ equals $R \circ W_{n+1}F_k$ in characteristic p.

Some notational remarks following classical literature:

a) We sometimes omit "$(W_nF_X)_*$" in the source. But one should always keep that in mind and be careful with the module structure.

b) We will simply write C for C_n sometimes. This shall not cause any confusion according to (1).

Before we move on, we state a remark on étale schemes over W_nX.

Remark 1.16. (1) Notice that every étale W_nX-scheme is of the form $W_ng : W_nU \to W_nX$, where $g : U \to X$ is an étale X-scheme. In fact, there are two functors

$$F : \{\text{étale } W_nX\text{-schemes}\} \leftrightarrows \{\text{étale } X\text{-schemes}\} : G$$
$$V \mapsto V \times_{W_nX} X$$
$$W_nU \leftarrow U$$

The functor F is a categorical equivalence according to [EGAIV-4, Ch. IV, 18.1.2]. The functor G is well-defined (i.e. produces étale W_nX-schemes) and is a right inverse of F by [Hes15, Thm. 1.25]. We want to show that there is a natural isomorphism $GF \simeq id$, and this is the consequence of the following purely categorical statement.

Categorically, if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are two functors satisfying both F being a categorical equivalence and $FG \simeq id$, then G is a quasi-inverse of F, i.e., there exists a canonical natural isomorphism $GF \simeq id$.

To show this, one first notices that G is fully faithful and essentially surjective. Indeed,

- Fully faithfulness: for any $U_1, U_2 \in \mathcal{B}$,

$$\mathrm{Hom}_{\mathcal{B}}(U_1,U_2) \xrightarrow{G} \mathrm{Hom}_{\mathcal{A}}(G(U_1),G(U_2)) \xrightarrow[\simeq]{F} \mathcal{H}om_{\mathcal{B}}(FG(U_1),FG(U_2)) \simeq \mathrm{Hom}_{\mathcal{B}}(U_1,U_2)$$

  The last $\simeq$ is induced by the natural isomorphism $FG \simeq id$. Thus the composition is the identity map, and therefore the first map $G : \mathrm{Hom}_{\mathcal{B}}(U_1,U_2) \to \mathrm{Hom}_{\mathcal{A}}(G(U_1),G(U_2))$ is an isomorphism of sets.
- Essential surjectivity: for any $V \in \mathcal{A}$, we want to show that there is a functorial isomorphism $GF(V) \simeq V$. Since $FG \simeq id$, we know that there is a functorial isomorphism $FGF(V) \simeq F(V)$. The fully faithfulness of F then gives a canonical choice of a map $GF(V) \to V$, which must be an isomorphism again by the fully faithfulness of F.

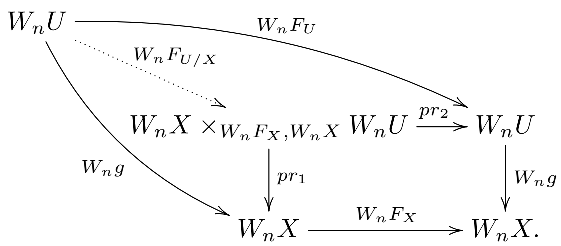
As a result, G admits a quasi-inverse functor $H : \mathcal{A} \to \mathcal{B}$. Now

$$GF \simeq GF \circ GH \simeq G \circ (FG) \circ H \simeq GH \simeq id.$$

(2) The square

$$\begin{CD} W_nU @>{W_nF_U}>> W_nU \\ @V{W_ng}VV @VV{W_ng}V \\ W_nX @>{W_nF_X}>> W_nX. \end{CD}$$

is a cartesian square. For this, consider the following cartesian diagram



$W_nF_{U/X}$ is an isomorphism, since $F_{U/X}$ is [Fu15, 10.3.1].

We shall now state the main result in this subsection, which seems to be an old folklore (cf. proof of [Kat87, 3.4]). To eliminate possible sign inconsistency of the Cartier operator with the Grothendieck trace map calculated via residue symbols [Con00, Appendix A], we reproduce the proof by explicit calculations (see Section 1.3.2-Section 1.3.3). And at the same time, this result justifies our notation for C': The classical Cartier operator C is simply the $(-d)$-th cohomology of our C'.

Theorem 1.17 (Compatibility of C' with C)**.** *Suppose that X is a smooth scheme of dimension d over a perfect field k of characteristic $p > 0$. Then the top degree classical Cartier operator*

$$C : (W_nF_X)_*W_n\Omega^d_{X/k} \to W_n\Omega^d_{X/k}$$

as defined in Definition 1.14, agrees with the $(-d)$-th cohomology of the Cartier operator for residual complexes

$$C' : (W_nF_X)_*W_n\Omega^d_{X/k} \to W_n\Omega^d_{X/k}$$

as defined in (1.2.3) via Ekedahl's quasi-isomorphism Proposition 1.11.

Proof. The Cartier operator is stable under étale base change, i.e., for any étale morphism $W_ng : W_nX \to W_nY$ (which must be of this form according to Remark 1.16(1)), we have

$$C_X \simeq (W_ng)^*C_Y : (W_nF_X)_*W_n\Omega^d_X \to W_n\Omega^d_X.$$

We claim that the map C' defined in (1.2.3) is also compatible with étale base change. That is, whenever we have an étale morphism $W_ng : W_nX \to W_nY$, there is a canonical isomorphism

$$C'_X \simeq (W_ng)^*C'_Y : (W_nF_X)_*K_{n,X} \to K_{n,X}.$$

First of all, the Grothendieck trace map $\mathrm{Tr}_{W_nF_X}$ for residual complexes is compatible with étale base change by Proposition 1.10(5), i.e.,

$$\mathrm{Tr}_{W_nF_X} \simeq g^*\,\mathrm{Tr}_{W_nF_Y} : (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X} \to K_{n,X}.$$

Secondly, because of the cartesian square in Remark 1.16(2) and the flat base change theorem

$$(W_ng)^*(W_nF_X)_* \simeq (W_nF_X)^*(W_ng)_*,$$

we are reduced to show that (1.2.2) is compatible with étale base change. And this is true, because we have

$$(W_n g)^* \simeq (W_n g)^{\triangle}$$

by Proposition 1.9(2), and the compatibility of $(-)^{\triangle}$ with composition by Proposition 1.9(4). This finishes the claim.

Note that the question is local on $W_n X$. Thus to prove the statement for smooth k-schemes X, using the compatibility of C and C' with respect to étale base change, it suffices to prove for $X = \mathbf{A}^d_k$. That is, we need to check that the expression given in Lemma 1.23 for C' agrees with the expression for C given in Lemma 1.19. This is apparent. ☐

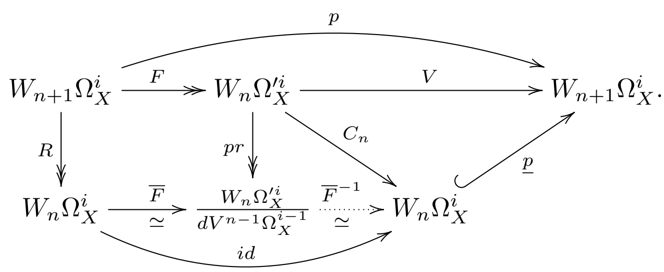
1.3.2. *Proof of Theorem 1.17: C for the top Witt differentials on the affine space.* Let k be a perfect field of positive characteristic p. The aim of this subsection is to provide the formula for the Cartier operator on the top degree de Rham-Witt sheaf over the affine space (Lemma 1.19). But before this, we first show a lemma which will be used in the calculation of Lemma 1.19.

Lemma 1.18 (cf. [Kat86b, 4.1.2]). *Let X be a smooth k-scheme. Then*

$$V = \underline{p} \circ C_n : R_* W_n \Omega'^i_X \to W_{n+1}\Omega^i_X,$$

where $W_n\Omega'^q_X$ denotes the abelian sheaf $F(W_{n+1}\Omega^i_X)$ regarded as a $W_n\mathcal{O}_X$-submodule of $(W_nF_X)_* W_n\Omega^i_X$.

Proof. Consider the following diagram



Notice that $\overline{F} : W_n\Omega^i \to W_n\Omega'^i / dV^{n-1}\Omega^{i-1}$ is an isomorphism by Lemma 1.13, and therefore we can take the inverse. All the small parts commute by definition (among these one notices that the top part commutes because X is of characteristic p), except the triangle on the right. Moreover one has the outer diagram commutes, due to the definition of $\underline{p}$ [Ill79, I 3.4]. Since $F : W_{n+1}\Omega^d_X \to W_n\Omega'^d_X$ is surjective, commutativity of the right triangle follows from the known commutativities. ☐

Lemma 1.19 (C_n on $\mathbf{A}^d$). *Let $X = \mathbf{A}^d_k$. Then the Cartier operator (cf. Definition 1.14)*

$$C := C_n : W_n\Omega^d_X \to W_n\Omega^d_X$$

is given by the following formula:

$$\begin{aligned}
&C\Bigg(\alpha\Big(\prod_{i\in I, v_p(j_i)\geq 1} [X_i^{j_i-1}]_n d[X_i]_n\Big) \cdot \Big(\prod_{i\in I, v_p(j_i)=0} [X_i^{j_i-1}]_n d[X_i]_n\Big) \\
&\quad \cdot \Big(\prod_{i\notin I, s_i\neq n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big) \cdot \Big(\prod_{i\notin I, s_i=n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big)\Bigg) \\
&= (W_nF_k)^{-1}(\alpha)\Big(\prod_{i\in I, v_p(j_i)\geq 1} [X_i^{j_i/p-1}]_n d[X_i]_n\Big) \cdot \Big(\prod_{i\in I, v_p(j_i)=0} \frac{1}{j_i} dV([X_i^{j_i}]_{n-1})\Big) \\
&\quad \cdot \Big(\prod_{i\notin I, s_i\neq n-1} dV^{s_i+1}([X_i^{j_i}]_{n-s_i-1})\Big) \cdot \Big(\prod_{i\notin I, s_i=n-1} 0\Big),
\end{aligned}$$

where $\alpha \in W_n k$.

We remind the reader of our assumptions. When we write an element in some de Rham-Witt sheaf in terms of a product with respect to an totally ordered index set, we make the following assumptions: when an index set is empty, the respective factor of the product does not occur; when an index set is non-empty, the factors of the product are ordered such that the indices are increasing. With these assumptions we avoid any confusion concerning signs.

Proof. Write $\boldsymbol{X} = \{X_1, \dots, X_{d-1}\}$ (empty when $d = 1$) and $S = X_d$. According to [HM04, (4.2.1)], any element in $W_n\Omega^d_{k[\boldsymbol{X},S]}$ is uniquely written as

$$\sum_{j\geq 1} b^{(n)}_{0,j}[S^{j-1}]_n d[S]_n + \sum_{s=1}^{n-1}\sum_{p\nmid j} dV^s(b^{(n-s)}_{s,j}[S^j]_{n-s}), \tag{1.3.7}$$

where $b^{(n)}_{0,j} \in W_n\Omega^{d-1}_{k[\boldsymbol{X}]}$, and for $s \in [1, n-1]$, $b^{(n-s)}_{s,j} \in W_{n-s}\Omega^{d-1}_{k[\boldsymbol{X}]}$. (Here we have used $W_n\Omega^d_{k[\boldsymbol{X}]} = 0$ and $W_{n-s}\Omega^d_{k[\boldsymbol{X}]} = 0$.)

Now compute

$$F \quad : \quad W_{n+1}\Omega^d_{k[\boldsymbol{X},S]} \to W_n\Omega^d_{k[\boldsymbol{X},S]},$$

$$b^{(n+1)}_{0,j}[S^{j-1}]_{n+1}d[S]_{n+1} \mapsto F(b^{(n+1)}_{0,j})[S^{pj-1}]_n d[S]_n; \tag{1.3.8}$$

$$dV^s(b^{(n+1-s)}_{s,j}[S^j]_{n+1-s}) \mapsto dV^{s-1}(b^{(n+1-s)}_{s,j}[S^j]_{n+1-s}); \tag{1.3.9}$$

and

$$V \quad : \quad W_n\Omega^d_{k[\boldsymbol{X},S]} \to W_{n+1}\Omega^d_{k[\boldsymbol{X},S]},$$

$$b^{(n)}_{0,j}[S^{j-1}]_n d[S]_n \mapsto (-1)^{d-1}\frac{p}{j}dV(b^{(n)}_{0,j}[S^j]_n) \quad \text{when } v_p(j) = 0. \tag{1.3.10}$$

In the last equation we used $db^{(n)}_{0,j} \in W_n\Omega^d_{k[\boldsymbol{X}]} = 0$.

Therefore, according to

(1) $p = \underline{p} \circ R$ where $\underline{p}$ is injective, by [Ill79, I 3.4],
(2) $V = \underline{p} \circ C_n$, by Lemma 1.18, and
(3) $C_n \circ F = R$ (because of (1)(2)),
(4) $F : W_{n+1}\Omega^d_{k[\boldsymbol{X},S]} \to W_n\Omega^d_{k[\boldsymbol{X},S]}$ is surjective,

one gets

$$C_n \quad : \quad W_n\Omega^d_{k[\boldsymbol{X},S]} \to W_n\Omega^d_{k[\boldsymbol{X},S]} \tag{1.3.11}$$

$$\begin{array}{c} b^{(n)}_{0,j}[S^{j-1}]_n d[S]_n \\ (j\geq 1) \end{array} \mapsto \begin{cases} C_n(b^{(n)}_{0,j})[S^{j/p-1}]_n d[S]_n, & v_p(j)\neq 0; \quad (\text{by } (1.3.8)) \\ \frac{(-1)^{d-1}}{j} dV(R(b^{(n)}_{0,j})[S^j]_{n-1}), & v_p(j) = 0. \quad (\text{by } (1.3.10)) \end{cases}$$

$$\begin{array}{c} dV^s(b^{(n-s)}_{s,j}[S^j]_{n-s}) \\ (s\in[1,n-1], p\nmid j) \end{array} \mapsto \begin{cases} dV^{s+1}(R(b^{(n-s)}_{s,j})[S^j]_{n-s-1}), & 1\leq s\leq n-2; \quad (\text{by } (1.3.9)) \\ 0, & s = n-1. \quad (\text{by } C_n = \overline{F}^{-1}\circ pr) \end{cases}$$

Note that $C_n(b^{(n)}_{0,j})$ is computed via the induction on d: when $d = 1$,

$$C_n(b^{(n)}_{0,j}) = (W_nF_k)^{-1}(b^{(n)}_{0,j}) \in W_nk$$

because $F = R \circ W_nF_k : W_nk \to W_{n-1}k$ (note that $\mathrm{char} k = p$).

Since $b^{(n)}_{0,j} \in W_n\Omega^{d-1}_{k[\boldsymbol{X}]}$ could also be written in expression (1.3.7), we could further write (1.3.11) out. That is to say, every element in $W_n\Omega^d_{\mathbf{A}^d_k}$ is uniquely written as a sum of expressions of the form

$$\alpha\Big(\prod_{i\in I}[X_i^{j_i-1}]_n d[X_i]_n\Big)\Big(\prod_{i\in[1,d]\setminus I} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big), \tag{1.3.12}$$

where $\alpha \in W_nk$, $I \subset [1, d]$ an index subset (I is the set of indices taken the form $[X_i^{j_i-1}]d[X_i]$ and the rest indices takes the form $dV^{s_i}([X_i^{j_i}]_{n-s_i})$), and

$$\{j_i\}_{i\in[1,d]}, \quad \{s_i\}_{i\in[1,d]\setminus I}$$

some integers, satisfying

- $j_i \geq 1$, when $i \in I$, and
- $v_p(j_i) = 0$ and $s_i \in [1, n-1]$ when $i \in [1, d] \setminus I$.

C_n maps each of the factors of (1.3.12) in the following way:

$$\alpha \mapsto W_n(F_k)^{-1}(\alpha), \quad \alpha \in W_nk,$$

$$[X_i^{j_i-1}]_n d[X_i]_n \mapsto \begin{cases} [X_i^{j_i/p-1}]_n d[X_i]_n, & v_p(j_i) \geq 1; \\ \frac{1}{j_i} dV([X_i^{j_i}]_{n-1}), & v_p(j_i) = 0. \end{cases}$$

$$dV^{s_i}([X_i^{j_i}]_{n-s_i}) \mapsto \begin{cases} dV^{s_i+1}([X_i^{j_i}]_{n-s_i-1}), & s_i \neq n-1; \\ 0, & s_i = n-1. \end{cases}$$

□

1.3.3. *Proof of Theorem 1.17: C' for the top Witt differentials on the affine space.* The aim of Section 1.3.3 is to calculate C' for top de Rham-Witt sheaves on the affine space (Lemma 1.23). To do this, one needs to first calculate the trace map of the canonical lift of the absolute Frobenius.

1.3.3.1. *Trace map of the canonical lift $\widetilde{F}_{\widetilde{X}}$ of absolute Frobenius F_X.* Let k be a perfect field of positive characteristic p. Let $X = \mathbf{A}_k^d$, and let $\widetilde{X} := \operatorname{Spec} W_n(k)[X_1, \dots, X_d]$ be the canonical smooth lift of X over $W_n(k)$. To make explicit the module structures, we distinguish the source and the target of the absolute Frobenius of $\operatorname{Spec} k$ and write it as

$$F_k : \operatorname{Spec} k_1 \to \operatorname{Spec} k_2$$

Similarly, write the absolute Frobenius on X as

$$F_X : X = \operatorname{Spec} k_1[X_1, \dots, X_d] \to Y = \operatorname{Spec} k_2[X_1, \dots, X_d].$$

There is a canonical lift $\widetilde{F}_{\widetilde{X}}$ of F_X over $\widetilde{X}$, and we write it as

$$\widetilde{F}_{\widetilde{X}} : \widetilde{X} = \operatorname{Spec} W_n(k_1)[X_1, \dots, X_d] \to \widetilde{Y} := \operatorname{Spec} W_n(k_2)[Y_1, \dots, Y_d].$$

$\widetilde{F}_{\widetilde{X}}$ is given by

$$\begin{aligned} \widetilde{F}_{\widetilde{X}}^* : \quad \Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) = W_n(k_2)[Y_1, \dots, Y_d] &\to W_n(k_1)[X_1, \dots, X_d] = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}), \\ W_n k_2 \ni \alpha &\mapsto W_n(F_k)(\alpha), \\ Y_i &\mapsto X_i^p. \end{aligned}$$

on the level of global sections. Clearly $\widetilde{F}_{\widetilde{X}}$ restricts to F_X on X. Let

$$\pi_X : X \to \operatorname{Spec} k_1, \quad \pi_Y : Y \to \operatorname{Spec} k_2, \quad \pi_{\widetilde{X}} : \widetilde{X} \to W_n k_1, \quad \pi_{\widetilde{Y}} : \widetilde{Y} \to W_n k_2$$

be the structure maps. The composition $\widetilde{F}_{\widetilde{X}} \circ \pi_{\widetilde{Y}} : \widetilde{X} \to \operatorname{Spec} W_n k_2$ gives $\widetilde{X}$ a $W_n k_2$-scheme structure, and the map $\widetilde{F}_{\widetilde{X}}$ is then a map of $W_n k_2$-schemes. Therefore the trace map

$$\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}} : \widetilde{F}_{\widetilde{X},*} \widetilde{F}_{\widetilde{X}}^{\triangle} K_{\widetilde{Y}} \to K_{\widetilde{Y}}$$

makes sense. Consider the following map of complexes

$$\begin{aligned} \widetilde{F}_{\widetilde{X},*} K_{\widetilde{X}} \simeq \widetilde{F}_{\widetilde{X},*} \pi_{\widetilde{X}}^{\triangle} W_n k_1 \xrightarrow[\sim]{\widetilde{F}_{\widetilde{X},*}\pi_{\widetilde{X}}^{\triangle}(1.2.1)} \widetilde{F}_{\widetilde{X},*} \pi_{\widetilde{X}}^{\triangle} W_n F_k^{\triangle} W_n k_2 \simeq \\ \widetilde{F}_{\widetilde{X},*} \widetilde{F}_{\widetilde{X}}^{\triangle} \pi_{\widetilde{Y}}^{\triangle} W_n k_2 \simeq \widetilde{F}_{\widetilde{X},*} \widetilde{F}_{\widetilde{X}}^{\triangle} K_{\widetilde{Y}} \xrightarrow{\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}}} K_{\widetilde{Y}}. \end{aligned}$$

Taking the $(-d)$-th cohomology, it induces a map

$$\widetilde{F}_{X,*} \Omega^d_{\widetilde{X}/W_n k_1} \to \Omega^d_{\widetilde{Y}/W_n k_2} \tag{1.3.13}$$

In the following lemma we will compute this map.

Lemma 1.20. *The notations are the same as above. The map (1.3.13) has the following expression:*

$$\begin{aligned} \Omega^d_{\widetilde{X}/W_n k_1} &\xrightarrow{(1.3.13)} \quad \Omega^d_{\widetilde{Y}/W_n k_2} \\ \alpha \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X} &\mapsto \begin{cases} (W_n F_k)^{-1}(\alpha) \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{Y}, & \text{when } \lambda_i = p-1 \text{ for all } i; \\ 0, & \text{when } \lambda_i \neq p-1 \text{ for some } i. \end{cases} \end{aligned} \tag{1.3.14}$$

Here $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_d\}$, $\boldsymbol{\mu} = \{\mu_1, \dots, \mu_d\}$ are multi-indices, and $\boldsymbol{X}^{\boldsymbol{\lambda}} := X_1^{\lambda_1} \dots X_d^{\lambda_d}$ (similar for $\boldsymbol{Y}^{\boldsymbol{\mu}}$, $\boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}}$, etc.), $d\boldsymbol{X} := dX_1 \dots dX_d$ (similar for $d\boldsymbol{Y}$, etc.).

Proof. Construct a regular immersion of $\widetilde{X}$ into $\widetilde{P} = \mathbf{A}^d_{\widetilde{Y}}$ associated to the following homomorphism of rings:

$$\begin{aligned} \Gamma(\widetilde{P}, \mathcal{O}_{\widetilde{P}}) = W_n(k_2)[Y_1, \dots, Y_d, T_1, \dots, T_d] &\to W_n(k_1)[X_1, \dots, X_d] = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}), \\ \alpha &\mapsto (W_n F_k)(\alpha), \quad \alpha \in W_n(k_2), \\ Y_i &\mapsto X_i^p, \quad i = 1, \dots, d, \\ T_i &\mapsto X_i, \quad i = 1, \dots, d. \end{aligned}$$

Its kernel is

$$I = (T_1^p - Y_1, \dots, T_d^p - Y_d).$$

Denote

$$t_i = T_i^p - Y_i, i = 1, \dots, d.$$

Obviously the t_i's form a regular sequence in $\Gamma(\widetilde{P}, \mathcal{O}_{\widetilde{P}})$. Denote by i the associated closed immersion. Then one has a factorization of $\widetilde{F}_{\widetilde{X}}$:

(1.3.15)
$$\begin{array}{ccc} \widetilde{X} = \operatorname{Spec} W_n(k_1)[X_1, \dots, X_d] & \overset{i}{\hookrightarrow} & \widetilde{P} = \operatorname{Spec} W_n(k_2)[Y_1, \dots, Y_d, T_1, \dots, T_d] \\ & \searrow^{\widetilde{F}_{\widetilde{X}}} & \downarrow \pi \\ & & \widetilde{Y} = \operatorname{Spec} W_n(k_2)[Y_1, \dots, Y_d]. \end{array}$$

Regarding $\widetilde{X}$ as a $W_n k_2$-scheme via the composite map $\widetilde{F}_{\widetilde{X}} \circ \pi_{\widetilde{Y}}$, the diagram (1.3.15) is then a diagram in the category of $W_n k_2$-schemes.

A general element in $\Gamma(\widetilde{X}, \Omega^d_{\widetilde{X}/W_n k_1})$ is a sum of expressions of the form

(1.3.16)
$$\alpha \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X}, \quad \alpha \in W_n k_1, \boldsymbol{\lambda} \in [0, p-1]^d, \boldsymbol{\mu} \in \mathbb{N}^d.$$

The element (1.3.16) in $\Gamma(\widetilde{X}, \Omega^d_{\widetilde{X}/W_n k_1})$ corresponds to

(1.3.17)
$$(W_n F_k)^{-1}(\alpha) \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X}, \quad \alpha \in W_n k_2, \boldsymbol{\lambda} \in [0, p-1]^d, \boldsymbol{\mu} \in \mathbb{N}^d,$$

in $\Gamma(\widetilde{X}, \Omega^d_{\widetilde{X}/W_n k_2})$ under $(-d)$-th cohomology of the map $\widetilde{F}_{\widetilde{X},*}\pi^{\triangle}_{\widetilde{X}}$(1.2.1), and

$$(W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{T}, \quad \alpha \in W_n k_2, \boldsymbol{\lambda} \in [0, p-1]^d, \boldsymbol{\mu} \in \mathbb{N}^d$$

is a lift of (1.3.17) to $\Gamma(\widetilde{P}, \Omega^d_{\widetilde{P}/W_n k_2})$. Write

$$\begin{aligned} \beta := & \, dt_d \dots dt_1 \cdot (W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{T} \\ = & \, (-1)^d dY_d \dots dY_1 \cdot (W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{T} \end{aligned}$$

in $\Gamma(\widetilde{P}, \omega_{\widetilde{P}/W_n k_2})$ ($\omega_{\widetilde{P}/W_n k_2}$ denotes the dualizing sheaf with respect to the smooth morphism $\widetilde{P} \to W_n k_2$). One can write out the image of β under map [Con00, p.30 (a)], i.e.,

$$\begin{aligned} \omega_{\widetilde{P}/W_n k_2} & \simeq \omega_{\widetilde{P}/\widetilde{Y}} \otimes_{\mathcal{O}_{\widetilde{P}}} \pi^* \omega_{\widetilde{Y}/W_n k_2}, \\ \beta & \mapsto (-1)^{\frac{d(3d+1)}{2}} (W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} \otimes \pi^* \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{Y}. \end{aligned}$$

where $\omega_{\widetilde{P}/\widetilde{Y}}$ and $\omega_{\widetilde{Y}/W_n k_2}$ denote the dualizing sheaf with respect to the smooth morphisms $\pi : \widetilde{P} \to \widetilde{Y}$ and $\widetilde{Y} \to W_n k_2$. It's easily seen that $\widetilde{F}_{\widetilde{X}}$ is a finite flat morphism between smooth $W_n k_2$-schemes. Applying [CR11, Lemma A.3.3], one has

$$\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}}((W_n F_k)^{-1}(\alpha) \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X}) = (W_n F_k)^{-1}(\alpha) \operatorname{Res}_{\widetilde{P}/\widetilde{Y}} \begin{bmatrix} \boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} \\ t_1, \dots, t_d \end{bmatrix} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{Y},$$

where $\operatorname{Res}_{\widetilde{P}/\widetilde{Y}} \begin{bmatrix} \boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} \\ t_1, \dots, t_d \end{bmatrix} \in \Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$ is the residue symbol defined in [Con00, (A.1.4)], and $\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}}$ is the trace map on top differentials [Con00, (2.7.36)].

We consider the following cases (in the following (RN) with $N \in [1, 10]$ being a positive integer means the corresponding property from [Con00, §A]):

- When $(\lambda_1, \dots, \lambda_n) \neq (p-1, \dots, p-1)$, $\boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} = d\eta$ for some $\eta \in \Omega^{d-1}_{\widetilde{P}/\widetilde{Y}}$. Suppose without loss of generality that $\lambda_1 \neq p-1$. Then we can take

  $$\eta = \frac{1}{\lambda_1 + 1} T_1^{\lambda_1+1} T_2^{\lambda_2} \dots T_d^{\lambda_d} dT_2 \dots dT_d.$$

  Noticing that

  $$dt_i = d(T_i^p - Y_i) = pT_i^{p-1} dT_i$$

  in $\Omega_{\widetilde{P}/\widetilde{Y}}$, and that $\lambda_1 + mp + 1$ ($m \in \mathbb{Z}_{>0}$) is not divisible by p when $\lambda_1 + 1$ is so. Now we calculate

$$\operatorname{Res}_{\widetilde{P}/\widetilde{Y}} \begin{bmatrix} \boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} \\ t_1, \dots, t_n \end{bmatrix} = \frac{1}{\lambda_1 + 1} \operatorname{Res}_{\widetilde{P}/\widetilde{Y}} \begin{bmatrix} d(T_1^{\lambda_1+1} T_2^{\lambda_2} \dots T_d^{\lambda_d} dT_2 \dots dT_d) \\ t_1, t_2, \dots, t_n \end{bmatrix}$$

$$
\begin{aligned}
&= \frac{p}{\lambda_1+1}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix} T_1^{\lambda_1+p}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d \\ t_1^2, t_2, \dots, t_n \end{bmatrix} \quad \text{(by (R9))}\\
&= \frac{p}{(\lambda_1+1)(\lambda_1+p+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix} d(T_1^{\lambda_1+p+1}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_2\dots dT_d) \\ t_1^2, t_2, \dots, t_n \end{bmatrix}\\
&= \frac{2p^2}{(\lambda_1+1)(\lambda_1+p+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix} T_1^{\lambda_1+2p}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d \\ t_1^3, t_2, \dots, t_n \end{bmatrix} \quad \text{(by (R9))}\\
&= \frac{2p^2}{\prod_{i=0}^{2}(\lambda_1+ip+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix} d(T_1^{\lambda_1+2p+1}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_2\dots dT_d) \\ t_1^3, t_2, \dots, t_n \end{bmatrix}\\
&= \frac{6p^3}{\prod_{i=0}^{2}(\lambda_1+ip+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix} T_1^{\lambda_1+3p}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d \\ t_1^4, t_2, \dots, t_n \end{bmatrix} \quad \text{(by (R9))}\\
&= \dots\\
&= \frac{(\prod_{i=1}^{n} i)\cdot p^n}{\prod_{i=0}^{n-1}(\lambda_1+ip+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix} T_1^{\lambda_1+np}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d \\ t_1^{n+1}, t_2, \dots, t_n \end{bmatrix} \quad \text{(by (R9))}\\
&= 0.
\end{aligned}
$$

The last step is because $p^n = 0$ in $\Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$.

- When $(\lambda_1, \dots, \lambda_n) = (p-1, \dots, p-1)$, consider

(1.3.18)
$$
\begin{array}{ccc}
X' := \operatorname{Spec}\frac{\mathbb{Z}[Y'_1,\dots,Y'_d,T'_1,\dots,T'_d]}{(T_1'^p-Y'_1,\dots,T_d'^p-Y'_d)} & \hookrightarrow & \operatorname{Spec}\mathbb{Z}[Y'_1,\dots,Y'_d,T'_1,\dots,T'_d] =: P' \\
& \searrow^{f} & \downarrow \\
& & \operatorname{Spec}\mathbb{Z}[Y'_1,\dots,Y'_d] =: Y'.
\end{array}
$$

f is given by $f^*(Y'_i) = Y'_i = T_i'^p$ in $\Gamma(X', \mathcal{O}_{X'})$. This is a finite locally free morphism of rank p^d. Consider the map

$$
\begin{aligned}
h : \Gamma(Y', \mathcal{O}_{Y'}) = \mathbb{Z}[Y'_1, \dots, Y'_d] &\to W_n(k_2)[Y_1, \dots, Y_d] = \Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}),\\
Y'_i &\mapsto Y_i \text{ for all } i,
\end{aligned}
$$

that relates the two diagrams (1.3.18) and (1.3.15). In $\Gamma(Y', \mathcal{O}_{Y'})$, we have

$$
\begin{aligned}
p^d\cdot\operatorname{Res}_{P'/Y'}\begin{bmatrix} T_1'^{p-1}\dots T_d'^{p-1}dT'_1\dots dT'_d \\ T_1'^p-Y'_1,\dots,T_d'^p-Y'_d \end{bmatrix} &= \operatorname{Res}_{P'/Y'}\begin{bmatrix} d(T_1'^p-Y'_1)\dots d(T_d'^p-Y'_d) \\ T_1'^p-Y'_1,\dots,T_d'^p-Y'_d \end{bmatrix}\\
&\overset{(R6)}{=} \operatorname{Tr}_{X'/Y'}(1)\\
&= p^d.
\end{aligned}
$$

The symbol $\operatorname{Tr}_{X'/Y'}$ denotes the classical trace map associated to the finite locally free ring extension $\Gamma(Y', \mathcal{O}_{Y'}) \to \Gamma(X', \mathcal{O}_{X'})$. (We are following [Con00] for this notation. The meaning of this symbol is hidden in [Con00, (R6)] and its proof.) As for the last equality, $\operatorname{Tr}_{X'/Y'}(1) = p^d$ because f is a finite locally free map of rank p^d. Since p^d is a non-zerodivisor in $\Gamma(Y', \mathcal{O}_{Y'})$, one deduces

$$
\operatorname{Res}_{P'/Y'}\begin{bmatrix} T_1'^{p-1}\dots T_d'^{p-1}dT'_1\dots dT'_d \\ T_1'^p-Y'_1,\dots,T_d'^p-Y'_d \end{bmatrix} = 1.
$$

Set

$$
\boldsymbol{T^{p-1}} = T_1^{p-1}\dots T_d^{p-1},
$$

which is the canonical lift of $\boldsymbol{X^{\lambda}}$ via the map $i : \widetilde{X} \hookrightarrow \widetilde{P}$ in our current case. Pulling back to $\Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$ via h, one has

(1.3.19)
$$
\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix} \boldsymbol{T^{p-1}}d\boldsymbol{T} \\ t_1,\dots,t_d \end{bmatrix} \overset{(R5)}{=} h^*\operatorname{Res}_{P'/Y'}\begin{bmatrix} T_1'^{p-1}\dots T_d'^{p-1}dT'_1\dots dT'_d \\ T_1'^p-Y'_1,\dots,T_d'^p-Y'_d \end{bmatrix} = 1.
$$

Altogether, we know that the map (1.3.13) takes the following expression

$$
\begin{aligned}
\Omega^d_{\widetilde{X}/W_nk_1} &\to \Omega^d_{\widetilde{Y}/W_nk_2}\\
\alpha\boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}}d\boldsymbol{X} &\mapsto \begin{cases} (W_nF_k)^{-1}(\alpha)\boldsymbol{Y}^{\boldsymbol{\mu}}d\boldsymbol{Y}, & \text{when } \lambda_i = p-1 \text{ for all } i;\\ 0, & \text{when } \lambda_i \neq p-1 \text{ for some } i. \end{cases}
\end{aligned}
$$

□

1.3.3.2. *C' for top Witt differentials.* Now we turn to the W_n-version. The aim of this subsection is to calculate C' for top Witt differentials on $\mathbf{A}^d_k$ (Lemma 1.23).

Let k be a perfect field of positive characteristic p. Let $\operatorname{Spec} k$ be the base scheme and $f : X \to Y$ a finite morphism between smooth, separated and equidimensional k-schemes, which both have dimension d. Same as in the main text, we denote by $\pi_X : X \to k$ and $\pi_Y : Y \to k$ the respective structure maps. $K_{n,X} := (W_n\pi_X)^{\triangle} W_n k$, $K_{n,Y} = (W_n\pi_Y)^{\triangle} W_n k$ are residual complexes on X and Y. Then we define the trace map

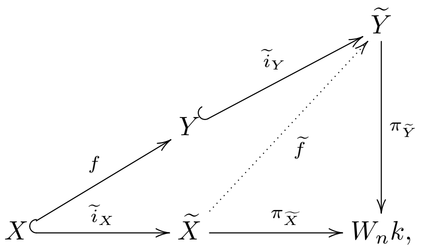
$$\operatorname{Tr}_{W_n f} : (W_n f)_*(W_n\Omega^d_X) \to W_n\Omega^d_Y \tag{1.3.20}$$

to be the $(-d)$-th cohomology map of the composition

$$\operatorname{Tr}_{W_n f} : (W_n f)_* K_{n,X} \simeq \mathcal{H}om_{W_n\mathcal{O}_Y}((W_n f)_* W_n\mathcal{O}_X, K_{n,Y}) \xrightarrow{\text{ev. at } 1} K_{n,Y} \tag{1.3.21}$$

via Ekedahl's isomorphism $W_n\Omega^d_X \simeq \mathcal{H}^{-d}(K_{n,X})$ [Eke84, §I].

Computation of the trace map is a local problem on Y. Thus by possibly shrinking Y we could assume that Y and therefore also X is affine. In this case, there exist smooth affine $W_n k$-schemes $\widetilde{X}$ and $\widetilde{Y}$ which lift X and Y. Denote the structure morphisms of $\widetilde{X}, \widetilde{Y}$ by $\pi_{\widetilde{X}}$ and $\pi_{\widetilde{Y}}$, respectively. We claim that there exists a finite $W_n k$-morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ lifting $f : X \to Y$. In fact, by the formal smoothness property of $\widetilde{Y}$,



we know there exists a morphism (not unique in general) $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ lifting $f : X \to Y$. Any such lift $\widetilde{f}$ is proper and quasi-finite, because its reduced morphism f is. Thus $\widetilde{f}$ is finite. This proves the claim.

Consider the map of abelian sheaves [Eke84, I (2.3)]

$$\begin{aligned} \varrho^*_Y : \quad & W_n\mathcal{O}_Y \xrightarrow[\simeq]{\vartheta_Y} \mathcal{H}^0(\Omega^\bullet_{\widetilde{Y}/W_n k}) \hookrightarrow \mathcal{O}_{\widetilde{Y}}, \\ & \sum_{i=0}^{n-1} V^i([a_i]) \mapsto \widetilde{a}_0^{p^n} + p\widetilde{a}_1^{p^{n-1}} + \cdots + p^{n-1}\widetilde{a}_{n-1}^p, \end{aligned} \tag{1.3.22}$$

where $a_i \in \mathcal{O}_Y$, and $\widetilde{a}_i \in \mathcal{O}_{\widetilde{Y}}$ being arbitrary liftings of a_i. The map ϑ_Y appearing above is the $i = 0$ case of the canonical isomorphism defined in [IR83, III. 1.5]

$$\vartheta_Y : W_n\Omega^i_Y \xrightarrow{\simeq} \mathcal{H}^i(\Omega^\bullet_{\widetilde{Y}/W_n k}). \tag{1.3.23}$$

In the following lemma we will use $W_n\Omega^\bullet_{\widetilde{Y}/W_n k}$ to denote the relative de Rham-Witt complex defined by [LZ04].

Lemma 1.21. *Notations are the same as above.*

1. *(Ekedahl) $\varrho^*_Y : W_n\mathcal{O}_Y \to \mathcal{O}_{\widetilde{Y}}$ is a morphism of sheaves of rings. And it induces a finite morphism $\varrho_Y : W_n Y \to \widetilde{Y}$.*
2. *([BER12, 8.4(ii)]) There is a commutative diagram*

$$\begin{array}{ccc} W_{n+1}\Omega^q_{\widetilde{Y}/W_n k} & \xrightarrow{F^n} & Z^q(\Omega^\bullet_{\widetilde{Y}/W_n k}) \\ \big\downarrow & & \big\downarrow \\ W_n\Omega^q_Y & \xrightarrow{\vartheta_Y} & \mathcal{H}^q(\Omega^\bullet_{\widetilde{Y}/W_n k}). \end{array}$$

In particular,

$$\vartheta_Y(\sum_{i=0}^{n-1} dV^i([a_i])) = \sum_{i=0}^{n-1} F^n dV^i([\widetilde{a}_i]) = \widetilde{a}_0^{p^n-1} d\widetilde{a}_0 + \widetilde{a}_1^{p^{n-1}-1} d\widetilde{a}_1 + \cdots + \widetilde{a}_{n-1}^{p-1} d\widetilde{a}_{n-1}. \tag{1.3.24}$$

Proof. (1) Let $\widetilde{a}_i \in \mathcal{O}_{\widetilde{Y}}$ be any lift of $a_i \in \mathcal{O}_Y$ for $0 \le i \le n-1$. $\widetilde{a}_n$ is an arbitrary element in $\mathcal{O}_{\widetilde{Y}}$. Since $p^n \widetilde{a}_n = 0$ in $\mathcal{O}_{\widetilde{Y}}$, one has the equality

$$\varrho_Y^*(\sum_{i=0}^{n-1} V^i([a_i])) = \mathrm{gh}_n(\widetilde{a}_0, \dots, \widetilde{a}_n),$$

where gh_n is the n-th component of the ghost map [Ill79, 0 (1.1.3)]

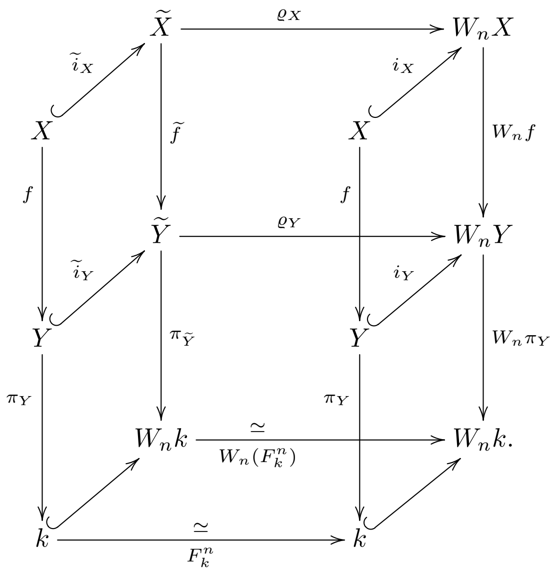
$$\mathrm{gh} : W_{n+1}\mathcal{O}_{\widetilde{Y}} \to (\mathcal{O}_{\widetilde{Y}}^{\oplus(n+1)}, +, *),$$
$$(\widetilde{a}_0, \dots, \widetilde{a}_n) \mapsto (\widetilde{a}_0, \widetilde{a}_0^p + p\widetilde{a}_1, \dots, \sum_{i=0}^{n} p^i \widetilde{a}_i^{p^{n-i}}).$$

The ghost map is a ring map, where $(\mathcal{O}_{\widetilde{Y}}^{\oplus(n+1)}, +, *)$ denotes $\mathcal{O}_{\widetilde{Y}}^{\oplus(n+1)}$ equipped with the ring structure of the termwise addition and multiplication. Therefore ϱ_Y^* is a ring map. The second statement on finiteness is proven in [Eke84, I, paragraph after (2.4)].

(2) One just need to check for $q = 0$: for higher q, both ϑ_Y and F^n are generated by the $q = 0$ case as morphisms of differential graded algebras. Take $\sum_{i=0}^{n-1} V^i([a_i]_{n-i}) \in W_n\mathcal{O}_Y$. Then $\sum_{i=0}^{n} V^i([\widetilde{a}_i]_{n+1-i}) \in W_{n+1}\mathcal{O}_{\widetilde{Y}}$ is a lift. $F^n(\sum_{i=0}^{n} V^i([\widetilde{a}_i]_{n+1-i})) = \sum_{i=0}^{n} p^i F^{n-i}([\widetilde{a}_i]_{n+1-i}) = \sum_{i=0}^{n} p^i \widetilde{a}_i^{p^{n-i}}$. We are done with the relation $p^n \widetilde{a}_n = 0$ in $\mathcal{O}_{\widetilde{Y}}$.

□

With the help of Lemma 1.21(1), we have the following commutative diagram of schemes (cf. [Eke84, I. (2.4)])



Lemma 1.22. *Notations are the same as in Lemma 1.21. Set* $K_{\widetilde{X}} = \pi_{\widetilde{X}}^{\triangle} W_n k$, *and* $K_{\widetilde{Y}} = \pi_{\widetilde{Y}}^{\triangle} W_n k$. *The* $(-d)$*-th cohomology of the map* $\mathrm{Tr}_{\widetilde{f}} : \widetilde{f}_* K_{\widetilde{X}} \to K_{\widetilde{Y}}$ *gives a map* $\widetilde{f}_*\Omega^d_{\widetilde{X}} \to \Omega^d_{\widetilde{Y}}$, *which we again denote by* $\mathrm{Tr}_{\widetilde{f}}$. *Then by passing to quotients, this map* $\mathrm{Tr}_{\widetilde{f}}$ *induces a well-defined map*

$$\tau_{\widetilde{f}} : \mathcal{H}^d(\widetilde{f}_*\Omega^\bullet_{\widetilde{X}}) \to \mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}}).$$

Moreover, the map $\tau_{\widetilde{f}}$ *is compatible with* $\mathrm{Tr}_{W_n f}$ *defined in (1.3.21) :*

$$\begin{array}{ccc} (W_n f)_* W_n\Omega^d_X & \xrightarrow{\mathrm{Tr}_{W_n f}} & W_n\Omega^d_Y \\ {\scriptstyle (W_n f)_*\vartheta_X}\downarrow \simeq & & {\scriptstyle \vartheta_Y}\downarrow \simeq \\ (\varrho_Y \widetilde{f})_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{X}}) & \xrightarrow{(\varrho_Y)_*\tau_{\widetilde{f}}} & (\varrho_Y)_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}}) \end{array}$$

Proof. We do it the other way around, namely we define the map $\tau_{\widetilde{f}} : \mathcal{H}^d(\widetilde{f}_*\Omega^\bullet_{\widetilde{X}}) \to \mathcal{H}^d(\widetilde{\Omega}^\bullet_{\widetilde{Y}})$ via $\mathrm{Tr}_{W_n f} : (W_n f)_* W_n\Omega^d_X \to W_n\Omega^d_Y$, and then show that this is the reduction of $\mathrm{Tr}_{\widetilde{f}} : \widetilde{f}_*\Omega^d_{\widetilde{X}} \to \Omega^d_{\widetilde{Y}}$.

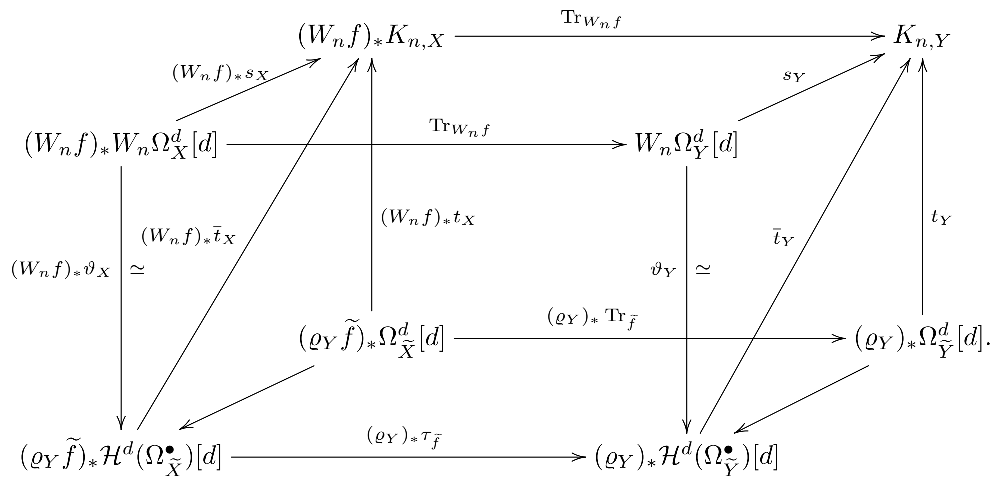
First of all, via isomorphisms ϑ_X, ϑ_Y, the map $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$ defined in (1.3.21 induces a well-defined map $\tau_{\widetilde{f}} : \mathcal{H}^d(\widetilde{f}_*\Omega^\bullet_{\widetilde{X}}) \to \mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}})$. To show compatibility with $\mathrm{Tr}_{\widetilde{f}}$, one needs an observation of Ekedahl: Ekedahl observed that the composite

$$t_Y : (\varrho_Y)_*\Omega^d_{\widetilde{Y}/W_nk}[d] \xrightarrow{\simeq} (\varrho_Y)_*K_{\widetilde{Y}} \simeq (\varrho_Y)_*\pi^\triangle_{\widetilde{Y}}W_nk \xrightarrow[\simeq]{(\varrho_Y)_*\pi^\triangle_{\widetilde{Y}}(1.2.1)^n} (\varrho_Y)_*\pi^\triangle_{\widetilde{Y}}(W_nF^n_k)^\triangle W_nk \simeq (\varrho_Y)_*(\varrho_Y)^\triangle(W_n\pi_Y)^\triangle W_nk \xrightarrow{\mathrm{Tr}_{\varrho_Y}} K_{n,Y}$$

factors through $\bar{t}_Y : (\varrho_Y)_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk})[d] \to K_{n,Y}$ (cf. [Eke84, §1 (2.6)]). Then he defined the map $W_n\Omega^d_Y[d] \to K_{n,Y}$ to be the composite

$$s_Y : W_n\Omega^d_Y[d] \xrightarrow[\simeq]{\vartheta_Y} \mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk})[d] \xrightarrow{\bar{t}_Y} K_{n,Y}. \tag{1.3.25}$$

Now consider the following diagram of complexes of sheaves



The unlabeled arrows are given by the natural quotient maps. The front commutes by the definition of $\tau_{\widetilde{f}}$. The top commutes by the definition of $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$. The triangles in the right (resp. the left) side commute due to the definition of $\bar{t}_Y$ and s_Y (resp. $\bar{t}_X$ and s_X). The back square commutes, because the trace map $\mathrm{Tr}_{\widetilde{f}}$ is functorial with respect to maps residual complexes with the same associated filtration by Proposition 1.10(3). We want to show that the bottom square commutes. To this end, it suffices to show $(\varrho_Y)_*\,\mathrm{Tr}_{\widetilde{f}} : (\varrho_Y\widetilde{f})_*\Omega^d_{\widetilde{X}} \to (\varrho_Y)_*\Omega^d_{\widetilde{Y}}$ is compatible with $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$ via ϑ_X and ϑ_Y. Because the map $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$ is determined by the degree $-d$ part of the map $\mathrm{Tr}_{W_nf} : (W_nf)_*K_{n,X} \to K_{n,Y}$, we are reduced to show compatibility of $(\varrho_Y)_*\,\mathrm{Tr}_{\widetilde{f}} : (\varrho_Y\widetilde{f})_*\Omega^d_{\widetilde{X}} \to (\varrho_Y)_*\Omega^d_{\widetilde{Y}}$ with $\mathrm{Tr}_{W_nf} : (W_nf)_*K_{n,X} \to K_{n,Y}$ via $(W_nf)_*(s_X \circ \vartheta_X^{-1})$ and $s_Y \circ \vartheta_Y^{-1}$. By commutativity of the left and right squares, this is reduced to the commutativity of the square on the back, which is known. Therefore the bottom square commutes as a result. ☐

$\tau_{\widetilde{f}}$ is just a temporary notation for the lemma above. Later we will denote $\tau_{\widetilde{f}}$ by $\mathrm{Tr}_{\widetilde{f}}$.

Lemma 1.23. *Let $X = \mathbf{A}^d_k$. Let*

$$C' = C'_n : W_n\Omega^d_X \to W_n\Omega^d_X.$$

be the map given by the $-d$-th cohomology of the level n Cartier operator for residual complexes (cf. (1.2.3)). Then C' is given by the following formula:

$$C'\Bigg(\alpha\Big(\prod_{i\in I, v_p(j_i)\geq 1}[X_i^{j_i-1}]_n d[X_i]_n\Big)\Big(\prod_{i\in I, v_p(j_i)=0}[X_i^{j_i-1}]_n d[X_i]_n\Big)$$
$$\Big(\prod_{i\notin I, s_i\neq n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big)\Big(\prod_{i\notin I, s_i=n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big)\Bigg)$$
$$= (W_nF_k)^{-1}(\alpha)\Big(\prod_{i\in I, v_p(j_i)\geq 1}[X_i^{j_i/p-1}]_n d[X_i]_n\Big)\Big(\prod_{i\in I, v_p(j_i)=0}\frac{1}{j_i}dV([X_i^{j_i}]_{n-1})\Big)$$

$$\Big(\prod_{i\notin I, s_i\neq n-1} dV^{s_i+1}([X_i^{j_i}]_{n-s_i-1})\Big)\Big(\prod_{i\notin I, s_i=n-1} 0\Big),$$

where $\alpha\in W_nk$.

Proof. Consider the map $W_nF_X : W_nX \to W_nX$ with $X := \mathbf{A}^d_k$. It is not a map of W_nk-schemes a priori, but after labeling the source by $W_nX := W_n\mathbf{A}^d_{k_1}$ and the target by $W_nY := W_n\mathbf{A}^d_{k_2}$, one can realize W_nF_X as a map of W_nk_2-schemes (the W_nk_2-scheme structure of W_nX is given by $W_nF_X\circ W_n\pi_Y$, where $\pi_Y : Y \to k_2$ denotes the structure morphism of the scheme Y). Write

$$\widetilde{X} = \mathbf{A}^d_{W_nk_1} = \operatorname{Spec} W_nk_1[X_1,\dots,X_d] \quad (\text{resp. } \widetilde{Y} = \mathbf{A}^d_{W_nk_2} = \operatorname{Spec} W_nk_2[X_1,\dots,X_d]),$$

and take the canonical lift $\widetilde{F}_{\widetilde{X}}$ of F_X as in example Lemma 1.20. Consider
(1.3.26)

$$\begin{array}{ccccc}
(W_nF_X)_*W_n\Omega^d_{X/k_1} & \xrightarrow[\simeq]{(W_nF_X)_*(1.2.2)} & (W_nF_X)_*W_n\Omega^d_{X/k_2} & \xrightarrow{\mathrm{Tr}_{W_nF_X}} & W_n\Omega^d_{Y/k_2} \\
\downarrow{\scriptstyle (W_nF_X)_*\vartheta_X} & & \downarrow{\scriptstyle (W_nF_X)_*\vartheta_X} & & \downarrow{\scriptstyle \vartheta_Y} \\
(\varrho_Y\widetilde{F}_{\widetilde{X}})_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{X}/W_nk_1}) & \xrightarrow[\simeq]{(\varrho_Y\widetilde{F}_{\widetilde{X}})_*\pi^{\triangle}_{\widetilde{X}}(1.2.1)} & (\varrho_Y\widetilde{F}_{\widetilde{X}})_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{X}/W_nk_2}) & \xrightarrow{(\varrho_Y)_*(\mathrm{Tr}_{\widetilde{F}_{\widetilde{X}}})} & (\varrho_Y)_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk_2}).
\end{array}$$

The composite map of the top row is C' (cf. (1.2.3) and Ekedahl's quasi-isomorphism Proposition 1.11). The composite of the bottom row is induced from $\varrho_{Y,*}$(1.3.13). The right side commutes due to Lemma 1.22. The left side commutes by naturality.

Given an index set $I\subset[1,d]$ and integers $\{j_i\}_{i\in[1,d]}$, $\{s_i\}_{i\in[1,d]\setminus I}$ satisfying

- $j_i\geq 1$, when $i\in I$, and
- $v_p(j_i)=0$ and $s_i\in[1,n-1]$ when $i\in[1,d]\setminus I$,

recall that a general element in $W_n\Omega^d_{X/k_1}$ is the sum of expressions of the following form (cf. (1.3.12))

$$\alpha\Big(\prod_{i\in I}[X_i^{j_i-1}]_n d[X_i]_n\Big)\Big(\prod_{i\notin I} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big), \tag{1.3.27}$$

where $\alpha\in W_nk_1$. One can also write this expression (1.3.27) in terms of finer index sets:

$$\alpha\Big(\prod_{i\in I, v_p(j_i)\geq 1}[X_i^{j_i-1}]_n d[X_i]_n\Big)\Big(\prod_{i\in I, v_p(j_i)=0}[X_i^{j_i-1}]_n d[X_i]_n\Big)$$
$$\Big(\prod_{i\notin I, s_i\neq n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big)\Big(\prod_{i\notin I, s_i=n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big).$$

where $\alpha\in W_nk_1$ (which might differ from the α in (1.3.27) by a sign because we might have changed the order of the factors in the (non-commutative) product). As we explained after diagram (1.3.26), we can decompose C' in the following way:

$$C' = \vartheta_Y^{-1}\circ(1.3.13)\circ\vartheta_X : W_n\Omega^d_{X/k_1}\to W_n\Omega^d_{Y/k_2}.$$

According to the explicit formula (1.3.14) for the map (1.3.13), and the explicit formula (1.3.24) for the maps ϑ_X,ϑ_Y, one could perform the following calculations:

$$C'\Bigg(\alpha\Big(\prod_{i\in I, v_p(j_i)\geq 1}[X_i^{j_i-1}]_n d[X_i]_n\Big)\Big(\prod_{i\in I, v_p(j_i)=0}[X_i^{j_i-1}]_n d[X_i]_n\Big)$$
$$\Big(\prod_{i\notin I, s_i\neq n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big)\Big(\prod_{i\notin I, s_i=n-1} dV^{s_i}([X_i^{j_i}]_{n-s_i})\Big)\Bigg)$$
$$= (\vartheta_Y^{-1}\circ(1.3.13))\Bigg(W_n(F_k)^n(\alpha)\Big(\prod_{i\in I, v_p(j_i)\geq 1} X_i^{p^nj_i-1}dX_i\Big)\Big(\prod_{i\in I, v_p(j_i)=0} X_i^{p^nj_i-1}dX_i\Big)$$
$$\Big(\prod_{i\notin I, s_i\neq n-1} j_iX_i^{p^{n-s_i}j_i-1}dX_i\Big)\Big(\prod_{i\notin I, s_i=n-1} j_iX_i^{p^{n-s_i}j_i-1}dX_i\Big)\Bigg)$$
$$= \vartheta_Y^{-1}\Bigg(W_n(F_k)^{n-1}(\alpha)\Big(\prod_{i\in I, v_p(j_i)\geq 1} X_i^{p^{n-1}j_i-1}dX_i\Big)\Big(\prod_{i\in I, v_p(j_i)=0} X_i^{p^{n-1}j_i-1}dX_i\Big)$$

$$\Big(\prod_{i\notin I, s_i\neq n-1} j_iX_i^{p^{n-s_i-1}j_i-1}dX_i\Big)\Big(\prod_{i\notin I, s_i=n-1} j_iX_i^{p^{n-s_i-1}j_i-1}dX_i\Big)\Big)$$
$$= (W_nF_k)^{-1}(\alpha)\Big(\prod_{i\in I, v_p(j_i)\geq 1}[X_i^{j_i/p-1}]_n d[X_i]_n\Big)\Big(\prod_{i\in I, v_p(j_i)=0}\frac{1}{j_i}dV([X_i^{j_i}]_{n-1})\Big)$$
$$\Big(\prod_{i\notin I, s_i\neq n-1} dV^{s_i+1}([X_i^{j_i}]_{n-s_i-1})\Big)\Big(\prod_{i\notin I, s_i=n-1} 0\Big).$$

□

1.3.4. *Criterion for surjectivity of $C'-1$.* The following proposition is proven in the smooth case by Illusie-Raynaud-Suwa [Suw95, 2.1]. The proof presented here is due to Rülling.

Proposition 1.24 (Raynaud-Illusie-Suwa). *Let $k=\bar{k}$ be an algebraically closed field of characteristic $p>0$. X is a separated scheme of finite type over k. Then for every i, $C'-1$ induces a surjective map on global cohomology groups*

$$H^i(W_nX, K_{n,X}) := H^i(R\Gamma(W_nX, K_{n,X})) \xrightarrow{C'-1} H^i(W_nX, K_{n,X}).$$

Proof. Take a Nagata compactification of X, i.e., an open immersion

$$j : X \hookrightarrow \overline{X}$$

such that $\overline{X}$ is proper over k. The boundary $\overline{X}\setminus X$ is a closed subscheme in $\overline{X}$. By blowing up in $\overline{X}$ one could assume $\overline{X}\setminus X$ is the closed subscheme associated to an effective Cartier divisor D on $\overline{X}$. We could thus assume j is an affine morphism. Therefore

$$W_nj : W_nX \hookrightarrow W_n\overline{X}$$

is also an affine morphism.

For any quasi-coherent sheaf $\mathcal{M}$ on $W_n\overline{X}$, the difference between $\mathcal{M}$ and $(W_nj)_*(W_nj)^*\mathcal{M}$ are precisely those sections that have poles (of any order) at $\mathrm{Supp}D = W_n\overline{X}\setminus W_nX$. Suppose that the effective Cartier divisor D is represented by $(U_i, f_i)_i$, where $\{U_i\}_i$ is an affine cover of $\overline{X}$, and $f_i\in\Gamma(U_i,\mathcal{O}_X)$. Recall that $\mathcal{O}_{\overline{X}}(mD)$ denotes the line bundle on $\overline{X}$ which is the inverse (as line bundles) of the m-th power of the ideal sheaf of $\overline{X}\setminus X\hookrightarrow\overline{X}$. Locally, one has an isomorphism

$$\mathcal{O}_{\overline{X}}(mD)\mid_{U_i}\simeq \mathcal{O}_{U_i}\cdot\frac{1}{f_i^m}$$

for each i. Denote by $W_n\mathcal{O}_{\overline{X}}(mD)$ the line bundle on $W_n\overline{X}$ such that

$$W_n\mathcal{O}_{\overline{X}}(mD)\mid_{U_i}\simeq W_n\mathcal{O}_{U_i}\cdot\frac{1}{[f_i]^m},$$

where $[-]=[-]_n$ denotes the Teichmüller lift. Denote

$$\mathcal{M}(mD) := \mathcal{M}\otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD).$$

The natural map

$$\mathcal{M}(*D) := \mathrm{colim}_m\,\mathcal{M}(mD) \xrightarrow{\simeq} (W_nj)_*(W_nj)^*(\mathcal{M}(mD)) = (W_nj)_*(W_nj)^*\mathcal{M} \tag{1.3.28}$$

is an isomorphism of sheaves. Here the inductive system on the left hand side is given by the natural map

$$\mathcal{M}(mD) := \mathcal{M}\otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD) \to \mathcal{M}\otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}((m+1)D)$$

induced from the inclusion $W_n\mathcal{O}_{\overline{X}}(mD)\hookrightarrow W_n\mathcal{O}_{\overline{X}}((m+1)D)$, i.e., locally on U_i, this inclusion is the map

$$W_n\mathcal{O}_{\overline{X}}(mD)\mid_{U_i}\hookrightarrow W_n\mathcal{O}_{\overline{X}}((m+1)D)\mid_{U_i}$$
$$\frac{a}{[f_i]^m}\mapsto\frac{a[f_i]}{[f_i]^{m+1}}.$$

where $a\in W_n\mathcal{O}_{U_i}$. As a result,

$$\begin{aligned}H^i(W_nX,(W_nj)^*\mathcal{M}) &= H^i(R\Gamma(W_n\overline{X}, R(W_nj)_*(W_nj)^*\mathcal{M}))\\ &= H^i(R\Gamma(W_n\overline{X},(W_nj)_*(W_nj)^*\mathcal{M})) && (W_nj \text{ is affine})\\ &= H^i(R\Gamma(W_n\overline{X},\mathrm{colim}_m\,\mathcal{M}(mD))) && (1.3.28)\\ &= \mathrm{colim}_m\,H^i(W_n\overline{X},\mathcal{M}(mD)).\end{aligned}$$

Apply this to the bounded complex $K_{n,\overline{X}}$ of injective quasi-coherent $W_n\mathcal{O}_{\overline{X}}$-modules. (1.3.28) immediately gives an isomorphism of complexes

$$K_{n,\overline{X}}(*D) := \mathrm{colim}_m K_{n,\overline{X}}(mD) \xrightarrow{\simeq} (W_n j)_* K_{n,X}, \tag{1.3.29}$$

and an isomorphism of W_nk-modules

$$\mathrm{colim}_m H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)) = H^i(W_nX, K_{n,X}).$$

Via the projection formula and tensoring

$$C' : (W_nF_X)_* K_{n,\overline{X}} \to K_{n,\overline{X}}$$

with $W_n\mathcal{O}_{\overline{X}}(mD)$, one gets a map

$$\begin{aligned}(W_nF_X)_*(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(pmD)) &\simeq (W_nF_X)_*(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} F_X^* W_n\mathcal{O}_{\overline{X}}(mD)) \\ &\simeq (W_nF_X)_*(K_{n,\overline{X}}) \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD) \xrightarrow{C'\otimes id_{W_n\mathcal{O}_{\overline{X}}(mD)}} K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD).\end{aligned}$$

Pre-composing with the natural map

$$(W_nF_X)_*(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD)) \to (W_nF_X)_*(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(pmD)),$$

and taking global section cohomologies, one gets

$$C' : H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)) \to H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)).$$

To show surjectivity of

$$C' - 1 : H^i(W_nX, K_{n,X}) \to H^i(W_nX, K_{n,X}),$$

it suffices to show surjectivity for

$$C' - 1 : H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)) \to H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)).$$

Because $\mathcal{H}^q(K_{n,\overline{X}})$ are coherent sheaves for all q, $\mathcal{H}^q(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD)) = \mathcal{H}^q(K_{n,\overline{X}}) \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD)$ are also coherent, therefore the local-to-global spectral sequence implies that

$$M := H^i(W_n\overline{X}, K_{n,\overline{X}}(mD))$$

is a finite W_nk-module. Now M is equipped with an endomorphism C' which acts p^{-1}-linearly (cf. Definition A.14). The proposition is then a direct consequence of the following Lemma 1.25. □

Lemma 1.25. *Let k be a separably closed field of characteristic p and M be a finite W_nk-module. Let T be a $p^{\pm 1}$-linear map on M. Then*

$$T - 1 : M \to M$$

is surjective.

This lemma is adapted from [SGA7-II, Exposé XXII], where it is stated in p-linearity version and for a k-vector space. We remark that for a perfect field to be separably closed, it is equivalent for it to be algebraically closed. For its proof, see Appendix Section A, Proposition A.15 and Remark A.13.

The following proposition is a corollary of [Suw95, Lemma 2.1]. We restate it here as a convenient reference.

Proposition 1.26 (Raynaud-Illusie-Suwa)**.** *Assume $k = \overline{k}$. When X is separated smooth over k of pure dimension d,*

$$C - 1 : W_n\Omega^d_X \to W_n\Omega^d_X$$

is surjective.

Proof. Apply affine locally the H^{-d}-case of Proposition 1.24. Then Ekedahl's quasi-isomorphism $W_n\Omega^d_X[d] \simeq K_{n,X}$ from Proposition 1.11 together with compatibility of C' and C from Theorem 1.17 gives the claim. □

Remark 1.27. When X is Cohen-Macaulay of pure dimension d, W_nX is also Cohen-Macaulay of pure dimension d, and thus $K_{n,X,t}$ is concentrated at degree $-d$ for all n [Con00, 3.5.1]. Denote by $W_n\omega_X$ the only nonzero cohomology sheaf of $K_{n,X}$ in this case. Then the same reasoning as in Proposition 1.26 shows that when $k = \overline{k}$ and X is Cohen-Macaulay over k of pure dimension, the map

$$C' - 1 : W_n\omega_X \to W_n\omega_X$$

is surjective.

1.3.5. *Comparison between $W_n\Omega^d_{X,log}$ and $K_{n,X,log}$.* Let X be a k-scheme. Denote by $d\log$ the following map of abelian étale sheaves

$$\begin{aligned} d\log : (\mathcal{O}^*_{X,\text{ét}})^{\otimes q} &\to W_n\Omega^q_{X,\text{ét}},\\ a_1\otimes\cdots\otimes a_q &\mapsto d\log[a_1]_n\ldots d\log[a_q]_n, \end{aligned}$$

where $a_1,\ldots,a_q\in\mathcal{O}^*_{X,\text{ét}}$, $[-]_n:\mathcal{O}_{X,\text{ét}}\to W_n\mathcal{O}_{X,\text{ét}}$ denotes the Teichmüller lift, and $d\log[a_i]_n:=\frac{d[a_i]_n}{[a_i]_n}$. We will denote its sheaf theoretic image by $W_n\Omega^q_{X,log,\text{ét}}$ and call it the étale sheaf of log forms. We denote by $W_n\Omega^q_{X,log}:=W_n\Omega^q_{X,log,\text{Zar}}:=\epsilon_*W_n\Omega^q_{X,log,\text{ét}}$, and call it the Zariski sheaf of log forms.

Lemma 1.28 ([CSS83, lemme 2], [GS88a, 1.6(ii)]). *Let X be a smooth k-scheme. Then we have the following left exact sequences*

$$0\to W_n\Omega^q_{X,log}\to W_n\Omega^q_X\xrightarrow{1-\overline{F}} W_n\Omega^q_X/dV^{n-1}, \tag{1.3.30}$$

$$0\to W_n\Omega^q_{X,log}\to W_n\Omega'^q_X\xrightarrow{C-1} W_n\Omega^q_X, \tag{1.3.31}$$

where $W_n\Omega'^q_X:=F(W_{n+1}\Omega^q_X)$. The last maps are also surjective when $t=$ ét.

Still we assume X to be a smooth k scheme of pure dimension d. Use shortened notation

$$D_n:=\text{Cone}(W_n\Omega^d_X[d]\xrightarrow{C_n-1} W_n\Omega^d_X[d])[-1].$$

We have a map of distinguished triangles in general

$$\begin{array}{ccccccc} K_{n,X,log} & \longrightarrow & K_{n,X} & \xrightarrow{C'_n-1} & K_{n,X} & \xrightarrow{+1} & \\ \uparrow & & \uparrow & & \uparrow & & \\ D_n & \longrightarrow & W_n\Omega^d_X[d] & \xrightarrow{C_n-1} & W_n\Omega^d_X[d] & \xrightarrow{+1} & . \end{array} \tag{1.3.32}$$

Its commutativity is guaranteed by Theorem 1.17.

The following proposition collects what we have done so far.

Proposition 1.29 (cf. [Kat87, Prop. 4.2]). *X is smooth of pure dimension d over a perfect field k. Then*

(1) *The natural map $D_n\to K_{n,X,log}$ is a quasi-isomorphism. Moreover,*

$$\begin{aligned} \mathcal{H}^{-d}(K_{n,X,log}) &= W_n\Omega^d_{X,log},\\ \mathcal{H}^{-d+1}(K_{n,X,log}) &= \text{Coker}(W_n\Omega^d_X\xrightarrow{C_n-1} W_n\Omega^d_X),\\ \mathcal{H}^{i}(K_{n,X,log}) &= 0,\quad \textit{for all } i\neq -d,-d+1. \end{aligned}$$

(2) *When $k=\overline{k}$, the natural map*

$$W_n\Omega^d_{X,log}[d]\to K_{n,X,log}$$

is a quasi-isomorphism of complexes of abelian sheaves (equivalently, one has $\mathcal{H}^{-d+1}(K_{n,X,log})=0$).

Proof. (1) The map $D_n\to K_{n,X,log}$ is a quasi-isomorphism by the five lemma and $W_n\Omega^d_X[d]\xrightarrow{\simeq} K_{n,X}$ by Proposition 1.11. We have $\mathcal{H}^{-d}(D_n)=W_n\Omega^d_{X,log}$ by the exact sequence (1.3.31).

(2) Proposition 1.26+(1) above.

∎

1.4. Localization triangle associated to $K_{n,X,log}$.

1.4.1. *Definition of* $\text{Tr}_{W_nf,log}$.

Proposition 1.30 (Proper pushforward, cf. [Kat87, (3.2.3)]). *Let $f:X\to Y$ be a proper map between separated schemes of finite type over k. Then so is $W_nf:W_nX\to W_nY$, and we have a map*

$$\text{Tr}_{W_nf,log}:(W_nf)_*K_{n,X,log}\to K_{n,Y,log}$$

of complexes that fits into the following commutative diagram of complexes, where the two rows are distinguished triangles in $D^b(W_nX, \mathbb{Z}/p^n)$

$$\begin{array}{ccccccc}
(W_nf)_*K_{n,X,log} & \longrightarrow & (W_nf)_*K_{n,X} & \xrightarrow{C'-1} & (W_nf)_*K_{n,X} & \longrightarrow & \\
\downarrow{\scriptstyle \mathrm{Tr}_{W_nf,log}} & & \downarrow{\scriptstyle \mathrm{Tr}_{W_nf}} & & \downarrow{\scriptstyle \mathrm{Tr}_{W_nf}} & & \\
K_{n,Y,log} & \longrightarrow & K_{n,Y} & \xrightarrow{C'-1} & K_{n,Y} & \longrightarrow & .
\end{array}$$

Moreover $\mathrm{Tr}_{W_nf,log}$ *is compatible with composition and open restriction.*

This is the covariant functoriality of $K_{n,X,log}$ with respect to proper morphisms. Thus we also denote $\mathrm{Tr}_{W_nf,log}$ by f_*.

Proof. It suffices to show the following diagrams commute.

$$\begin{array}{ccc}
(W_nF_Y)_*(W_nf)_*K_{n,X} & \xrightarrow[\simeq]{(W_nF_Y)_*(W_nf)_*(1.2.2)} & (W_nF_Y)_*(W_nf)_*(W_nF_X)^{\triangle}K_{n,X} \\
\downarrow{\scriptstyle (W_nF_Y)_*\mathrm{Tr}_{W_nf}} & & \downarrow{\scriptstyle (W_nF_Y)_*\mathrm{Tr}_{W_nf}} \\
(W_nF_Y)_*K_{n,Y} & \xrightarrow[\simeq]{(W_nF_Y)_*(1.2.2)} & (W_nF_Y)_*(W_nF_Y)^{\triangle}K_{n,Y},
\end{array}$$

$$\begin{array}{ccccc}
(W_nF_Y)_*(W_nf)_*(W_nF_X)^{\triangle}K_{n,X} & \xrightarrow{\simeq} & (W_nf)_*(W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X} & \xrightarrow{(W_nf)_*\mathrm{Tr}_{W_nF_X}} & (W_nf)_*K_{n,X} \\
\downarrow{\scriptstyle (W_nF_Y)_*\mathrm{Tr}_{W_nf}} & & & & \downarrow{\scriptstyle \mathrm{Tr}_{W_nf}} \\
(W_nF_Y)_*(W_nF_Y)^{\triangle}K_{n,Y} & & \xrightarrow{\mathrm{Tr}_{W_nF_Y}} & & K_{n,Y},
\end{array}$$

where Tr_{W_nf} on the right of the first diagram and the left of the second diagram denotes the trace map of residual complex $(W_nF_Y)^{\triangle}K_{n,Y}$:

$$\mathrm{Tr}_{W_nf} : (W_nf)_*(W_nF_X)^{\triangle}K_{n,X} \simeq (W_nf)_*(W_nf)^{\triangle}(W_nF_Y)^{\triangle}K_{n,Y} \to (W_nF_Y)^{\triangle}K_{n,Y}.$$

Commutativity of the first diagram is due to functoriality of the trace map with respect to residual complexes with the same associated filtration Proposition 1.10(3). Commutativity of the second is because of compatibility of the trace map with composition of morphisms Proposition 1.10(4). ☐

1.4.2. $\mathrm{Tr}_{W_nf,log}$ *in the case of a nilpotent immersion.* Before stating the main result of this section, we state a lemma on compatibilities.

Lemma 1.31. *(1) The following diagram is commutative for any finite morphism* $f : X \to Y$ *of* k*-schemes*

$$\begin{array}{ccc}
(W_nf)_*K_{n,X} & \xrightarrow{\simeq} & \mathcal{H}om_{W_n\mathcal{O}_Y}((W_nf)_*(W_n\mathcal{O}_X), K_{n,Y}) \\
{\scriptstyle \mathrm{Tr}_{W_nf}}\downarrow & \swarrow{\scriptstyle \mathrm{ev}_1} & \downarrow{\scriptstyle ((W_nf)^*)^{\vee}} \\
K_{n,Y} & \xrightarrow[\simeq]{} & \mathcal{H}om_{W_n\mathcal{O}_Y}(W_n\mathcal{O}_Y, K_{n,Y}).
\end{array}$$

Here ev_1 *denotes the evaluation-at-1 map, and the map on top is Proposition 1.9(1) associated to the finite map* W_nf*.*

Similarly, for any finite morphism $f : X \to Y$ *of* W_nk*-schemes, write* $K_X := \pi_X^{\triangle}W_nk$ *and* $K_Y := \pi_Y^{\triangle}W_nk$*, where* π_X, π_Y *are the structure maps of* X *and* Y*. Then the following diagram commutes*

$$\begin{array}{ccc}
f_*K_X & \xrightarrow{\simeq} & \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K_Y) \\
{\scriptstyle \mathrm{Tr}_f}\downarrow & \swarrow{\scriptstyle \mathrm{ev}_1} & \downarrow{\scriptstyle (f^*)^{\vee}} \\
K_Y & \xrightarrow[\simeq]{} & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, K_Y),
\end{array}$$

where the map on top is Proposition 1.9(1) associated to the finite map f*.*

(2) *The Cartier operator $C' : (W_nF_X)_*K_{n,X_1} \to K_{n,X_2}$. for residual complexes can be decomposed in the following way: i.e. the following diagram commutes*

$$
\begin{array}{ccc}
(W_nF_X)_*K_{n,X} & \xrightarrow{\simeq} & (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_X}(W_n\mathcal{O}_X, K_{n,X}) \\
 & & \downarrow \\
 & & \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*W_n\mathcal{O}_X, (W_nF_X)_*K_{n,X}) \\
\simeq \big\downarrow (1.2.2) & & \simeq \big\downarrow (W_nF_X)_*(1.2.2)\circ \\
 & & \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*W_n\mathcal{O}_X, (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X}) \\
 & & \big\downarrow \mathrm{Tr}_{W_nF_X}\circ \\
(W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X} & \xrightarrow{\simeq} & \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*(W_n\mathcal{O}_X), K_{n,X}) \\
\mathrm{Tr}_{W_nF_X} \big\downarrow & & \big\downarrow ((W_nF_X)^*)^{\vee} \\
K_{n,X} & \xrightarrow[\simeq]{} & \mathcal{H}om_{W_n\mathcal{O}_X}(W_n\mathcal{O}_X, K_{n,X}).
\end{array}
$$

Here the composite of the two left vertical arrows is C', and the horizontal arrow in the middle is Proposition 1.9(1) associated to the finite map W_nF_X.

Proof. (1) One only need to note that the trace map for finite morphisms between residual complexes is given by evaluate-at-1 map by Proposition 1.10(1). Both of the claims are then direct.

(2) For the second part, we only need to prove the commutativity of the top square. One notices that we can alter the order of the maps in the second column. Then the commutativity of the following diagram gives the claim.

$$
\begin{array}{ccc}
(W_nF_X)_*K_{n,X} & \xrightarrow{\simeq} & (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_X}(W_n\mathcal{O}_X, K_{n,X}) \\
 & & \big\downarrow (1.2.2)\circ \\
 & & (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_X}(W_n\mathcal{O}_X, (W_nF_X)^{\triangle}K_{n,X}) \\
\simeq \big\downarrow (1.2.2) & \swarrow \mathrm{ev}_1 & \big\downarrow \\
 & & \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*W_n\mathcal{O}_X, (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X}) \\
 & & \big\downarrow \mathrm{Tr}_{W_nF_X}\circ \\
(W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X} & \xrightarrow{\simeq} & \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*(W_n\mathcal{O}_X), K_{n,X}).
\end{array}
$$

□

The proof of the proposition presented below is due to Rülling.

Proposition 1.32 (cf. [Kat87, 4.2]). *Let $i : X_0 \hookrightarrow X$ be a nilpotent immersion (thus so is $W_ni : W_n(X_0) \to W_nX$). Then the natural map*

$$\mathrm{Tr}_{W_ni,log} : (W_ni)_*K_{n,X_0,log} \to K_{n,X,log}$$

is a quasi-isomorphism.

Proof. Put $I_n := \mathrm{Ker}(W_n\mathcal{O}_X \to (W_ni)_*W_n\mathcal{O}_{X_0})$. Apply $\mathcal{H}om_{W_n\mathcal{O}_X}(-, K_{n,X})$ to the sequence of $W_n\mathcal{O}_X$-modules

$$0 \to I_n \to W_n\mathcal{O}_X \to (W_ni)_*W_n\mathcal{O}_{X_0} \to 0, \tag{1.4.1}$$

we get again a short exact sequence of complexes of $W_n\mathcal{O}_X$-modules

$$0 \to (W_ni)_*K_{n,X_0} \xrightarrow{\mathrm{Tr}_{W_ni}} K_{n,X} \to Q_n := \mathcal{H}om_{W_n\mathcal{O}_X}(I_n, K_{n,X}) \to 0.$$

We know the first map is Tr_{W_ni}, because of Lemma 1.31(1). The restriction of the map $(W_nF_X)^* : W_n\mathcal{O}_X \to (W_nF_X)_*W_n\mathcal{O}_X$ to I_n gives a map

$$(W_nF_X)^* \mid_{I_n}: \quad I_n \to (W_nF_X)_*I_n,$$

$$\sum_{i=0}^{n-1} V([a_i]) \mapsto \sum_{i=0}^{n-1} V([a_i^p]).$$

Define

$$(1.4.2)\qquad \begin{aligned} C'_{I_n} : (W_nF_X)_*Q_n &= (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_X}(I_n, K_{n,X}) \\ &\to \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*I_n, (W_nF_X)_*K_{n,X}) \\ &\xrightarrow[\simeq]{(W_nF_X)_*(1.2.2)\circ} \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*I_n, (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X}) \\ &\xrightarrow{\mathrm{Tr}_{W_nF_X}\circ} \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*I_n, K_{n,X}) \\ &\xrightarrow{((W_nF_X^*)|_{I_n})^{\vee}} \mathcal{H}om_{W_n\mathcal{O}_X}(I_n, K_{n,X}) = Q_n. \end{aligned}$$

According to Lemma 1.31(2), C' is compatible with C'_{I_n}, and thus one has the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & (W_nF_X)_*(W_ni)_*K_{n,X_0} & \xrightarrow{(W_nF_X)_*\mathrm{Tr}_{W_ni}} & (W_nF_X)_*K_{n,X} & \to & (W_nF_X)_*Q_n & \to & 0 \\ & & \downarrow{\scriptstyle C'} & & \downarrow{\scriptstyle C'} & & \downarrow{\scriptstyle C'_{I_n}} & & \\ 0 & \to & (W_ni)_*K_{n,X_0} & \xrightarrow{\mathrm{Tr}_{W_ni}} & K_{n,X} & \to & Q_n & \to & 0. \end{array}$$

Replace C' by $C'-1$, and C'_{I_n} by $C'_{I_n}-1$, we arrive at the two lower rows of the following diagram. Denote

$$Q_{n,log} := \mathrm{Cone}(Q_n \xrightarrow{C'_{I_n}-1} Q_n)[-1].$$

Taking into account the shifted cones of $C'-1$ and $C'_{I_n}-1$, we get the first row of the following diagram which is naturally a short exact sequence. Now we have the whole commutative diagram of complexes, where all the three rows are exact, and all the three columns are distinguished triangles in the derived category:

$$\begin{array}{ccccccccc} 0 & \to & (W_ni)_*K_{n,X_0,log} & \xrightarrow{\mathrm{Tr}_{W_ni,log}} & K_{n,X,log} & \to & Q_{n,log} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (W_ni)_*K_{n,X_0} & \xrightarrow{\mathrm{Tr}_{W_ni}} & K_{n,X} & \to & Q_n & \to & 0 \\ & & \downarrow{\scriptstyle C'-1} & & \downarrow{\scriptstyle C'-1} & & \downarrow{\scriptstyle C'_{I_n}-1} & & \\ 0 & \to & (W_ni)_*K_{n,X_0} & \xrightarrow{\mathrm{Tr}_{W_ni}} & K_{n,X} & \to & Q_n & \to & 0. \\ & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & & \end{array}$$

We want to show that $\mathrm{Tr}_{W_ni,log}$ is a quasi-isomorphism. By the exactness of the first row, it suffices to show $Q_{n,log}$ is an acyclic complex. Because the right column is a distinguished triangle, it suffices to show $C'_{I_n}-1 : Q_n \to Q_n$ is a quasi-isomorphism. Actually it's even an isomorphism of complexes: since $(W_nF_X)^* \mid_{I_n}: I_n \to (W_nF_X)_*I_n$ is nilpotent (because $I_1 = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_{X_0})$ is a finitely generated nilpotent ideal of $\mathcal{O}_X$), $C'_{I_n} : Q_n \to Q_n$ is therefore nilpotent (because one can alter the order of the three labeled maps in (1.4.2) in the obvious sense), and $C'_{I_n}-1$ is therefore an isomorphism of complexes. □

1.4.3. *Localization triangles associated to $K_{n,X,log}$.* Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ its open complement. Recall (cf. Section 1.1.1)

$$(1.4.3)\qquad \underline{\Gamma}_Z(\mathcal{F}) := \mathrm{Ker}(\mathcal{F} \to j_*j^{-1}\mathcal{F})$$

for any abelian sheaf $\mathcal{F}$. Denote its i-th derived functor by $\mathcal{H}^i_Z(\mathcal{F})$. Notice that

- $\Gamma_{Z'}(\mathcal{F}) = \Gamma_Z(\mathcal{F})$ for any nilpotent thickening Z' of Z (e.g. $Z' = W_nZ$),
- $\mathcal{F} \to j_*j^{-1}\mathcal{F}$ is surjective whenever $\mathcal{F}$ is flasque, and
- flasque sheaves are $\underline{\Gamma}_Z$-acyclic ([Har67, 1.10]) and f_*-acyclic for any morphism f.

Therefore, for any complex of flasque sheaves $\mathcal{F}^\bullet$ of $\mathbb{Z}/p^n$-modules on W_nX,

$$0 \to \underline{\Gamma}_Z(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to (W_nj)_*(\mathcal{F}^\bullet|_{W_nU}) \to 0$$

is a short exact sequence of complexes. Thus the induced triangle

(1.4.4) $$\underline{\Gamma}_Z(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to (W_n j)_*(\mathcal{F}^\bullet|_{W_nU}) \xrightarrow{+1}$$

is a distinguished triangle in $D^b(W_nX, \mathbb{Z}/p^n)$, whenever $\mathcal{F}^\bullet$ is a flasque complex with bounded cohomologies. In particular, since $K_{n,X,log}$ is a bounded complex of flasque sheaves, this is true for $\mathcal{F}^\bullet = K_{n,X,log}$.

The following proposition is proven in the smooth case by Gros-Milne-Suwa [Suw95, 2.6]. The proof presented here is due to Rülling.

Proposition 1.33 (Rülling). *Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ its open complement. Then*

(1) The map

$$(W_n i)_* K_{n,Z,log} = \underline{\Gamma}_Z((W_n i)_* K_{n,Z,log}) \xrightarrow{\mathrm{Tr}_{W_n i,log}} \underline{\Gamma}_Z(K_{n,X,log})$$

is a quasi-isomorphism of complexes of sheaves.

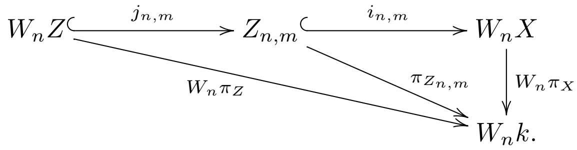
(2) (Localization triangle) The following

(1.4.5) $$(W_n i)_* K_{n,Z,log} \xrightarrow{\mathrm{Tr}_{W_n i,log}} K_{n,X,log} \to (W_n j)_* K_{n,U,log} \xrightarrow{+1}$$

is a distinguished triangle in $D^b(W_nX, \mathbb{Z}/p^n)$.

Note that we are working on the Zariski site and abelian sheaves on W_nX can be identified with abelian sheaves on X canonically. Thus we can replace $(W_n i)_* K_{n,Z,log}$ by $i_* K_{n,Z,log}$, and $(W_n j)_* K_{n,U,log}$ by $j_* K_{n,U,log}$ freely.

Proof. (1) Let I_n be the ideal sheaf associated to the closed immersion $W_n i : W_nZ \hookrightarrow W_nX$, and let $Z_{n,m}$ be the closed subscheme of W_nX determined by m-th power ideal I_n^m. In particular, $Z_{n,1} = W_nZ$. Denote by $i_{n,m} : Z_{n,m} \hookrightarrow W_nX$ and by $j_{n,m} : W_nZ \hookrightarrow Z_{n,m}$ the associated closed immersions. In this way one has a series of decomposition of $W_n i$ as maps of W_nk-schemes indexed by m:



Denote $K_{Z_{n,m}} := (\pi_{Z_{n,m}})^\triangle(W_nk)$, where $\pi_{Z_{n,m}} : Z_{n,m} \to W_nk$ is the structure morphism. We have a canonical isomorphism

(1.4.6) $$i_{n,m,*}\mathcal{H}^i(K_{Z_{n,m}}) \simeq \mathcal{E}xt^i_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X})$$

by Proposition 1.9(4) and Proposition 1.9(1) associated to the closed immersion $i_{n,m}$. The trace maps associated to the closed immersions

$$Z_{n,m} \hookrightarrow Z_{n,m+1}$$

for different m make the left hand side of (1.4.6) an inductive system. The right hand side also lies in an inductive system when m varies: the canonical surjections

$$i_{n,m+1,*}\mathcal{O}_{Z_{n,m+1}} \to i_{n,m,*}\mathcal{O}_{Z_{n,m}}$$

induce the maps

(1.4.7) $$\mathcal{H}om_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \to \mathcal{H}om_{W_n\mathcal{O}_X}(i_{n,m+1,*}\mathcal{O}_{Z_{n,m+1}}, K_{n,X})$$

whose i-th cohomologies are the connecting homomorphisms of the inductive system. According to the second part of Lemma 1.31(1), the map (1.4.7) is the trace map associated to the closed immersion $Z_{n,m} \hookrightarrow Z_{n,m+1}$, and thus is compatible with the inductive system on the left hand side of (1.4.6).

Consider the trace map associated to the closed immersion $i_{n,m} : Z_{n,m} \hookrightarrow W_nX$, i.e., the evaluation-at-1 map

$$\mathcal{H}om_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \to K_{n,X}.$$

Its image naturally lies in $\Gamma_{W_nZ}(K_{n,X})$. After taking colimit on m, it is an isomorphism

$$\mathrm{colim}_m\, \mathcal{E}xt^i_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \xrightarrow[\simeq]{\mathrm{ev}_1} \mathcal{H}^i_Z(K_{n,X})$$

by [Har66, V 4.3].

Now we consider

$$\operatorname{colim}_m i_{n,m,*}\mathcal{H}^i(K_{Z_{n,m}}) \simeq \operatorname{colim}_m \mathcal{E}xt^i_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \xrightarrow[\simeq]{\mathrm{ev}_1} \mathcal{H}^i_Z(K_{n,X}). \tag{1.4.8}$$

The composite map of (1.4.8) is $\operatorname{colim}_m \mathrm{Tr}_{i_{n,m}}$. On the other hand, consider the log trace associated to the closed immersion $i_{n,m}$ (cf. Proposition 1.30)

$$\mathrm{Tr}_{i_{n,m},log} : \mathcal{H}^i(i_{n,m,*}K_{Z_{n,m},log}) = \mathcal{H}^i(\underline{\Gamma}_Z(i_{n,m,*}K_{Z_{n,m},log})) \to \mathcal{H}^i(\underline{\Gamma}_Z(K_{n,X,log})) = \mathcal{H}^i_Z(K_{n,X,log}). \tag{1.4.9}$$

The maps (1.4.8), (1.4.9) give the vertical maps in the following diagram (due to formatting reason we omit $i_{n,m,*}$ from every term of the first row) which are automatically compatible by Proposition 1.30:

$$\begin{array}{ccccccccc}
\mathcal{H}^{i-1}(K_{Z_{n,m}}) & \xrightarrow{C'-1} & \mathcal{H}^{i-1}(K_{Z_{n,m}}) & \longrightarrow & \mathcal{H}^i(K_{Z_{n,m},log}) & \longrightarrow & \mathcal{H}^i(K_{Z_{n,m}}) & \xrightarrow{C'-1} & \mathcal{H}^i(K_{Z_{n,m}}) \\
\downarrow{\scriptstyle \mathrm{Tr}_{i_{n,m}}} & & \downarrow{\scriptstyle \mathrm{Tr}_{i_{n,m}}} & & \downarrow{\scriptstyle \mathrm{Tr}_{i_{n,m},log}} & & \downarrow{\scriptstyle \mathrm{Tr}_{i_{n,m}}} & & \downarrow{\scriptstyle \mathrm{Tr}_{i_{n,m}}} \\
\mathcal{H}^{i-1}_Z(K_{n,X}) & \xrightarrow{C'-1} & \mathcal{H}^{i-1}_Z(K_{n,X}) & \longrightarrow & \mathcal{H}^i_Z(K_{n,X,log}) & \longrightarrow & \mathcal{H}^i_Z(K_{n,X}) & \xrightarrow{C'-1} & \mathcal{H}^i_Z(K_{n,X}).
\end{array}$$

Taking the colimit with respect to m, the five lemma immediately gives that $\operatorname{colim}_m \mathrm{Tr}_{i_{n,m},log}$ is an isomorphism. Then $\mathrm{Tr}_{W_n i,log}$, which is the composition of

$$(W_n i)_*\mathcal{H}^i(K_{n,Z,log}) \xrightarrow[\text{Proposition 1.32},\simeq]{\operatorname{colim}_m \mathrm{Tr}_{j_{n,m},log}} \operatorname{colim}_m i_{n,m,*}\mathcal{H}^i(K_{Z_{n,m},log}) \xrightarrow[\simeq]{\operatorname{colim}_m \mathrm{Tr}_{i_{n,m},log}} \mathcal{H}^i_Z(K_{n,X,log}),$$

is an isomorphism. This proves the statement.

(2) Since $\underline{\Gamma}_Z(K_{n,X,log}) \to K_{n,X,log} \to (W_n j)_*K_{n,U,log} \xrightarrow{+1}$ is a distinguished triangle, the second part follows from the first part.

□

1.5. **Functoriality.** The push-forward functoriality of $K_{n,X,log}$ has been done in Proposition 1.30 for proper f. Now we define the pullback map for an étale morphism f. Since $W_n f$ is then also étale, we have an isomorphism of functors $(W_n f)^* \simeq (W_n f)^\triangle$ by Proposition 1.9(2). Define a chain map of complexes of $W_n\mathcal{O}_Y$-modules

$$f^* : K_{n,Y} \xrightarrow{\mathrm{adj}} (W_n f)_*(W_n f)^*K_{n,Y} \simeq (W_n f)_*(W_n f)^\triangle K_{n,Y} \simeq (W_n f)_*K_{n,X}. \tag{1.5.1}$$

Here adj stands for the adjunction map of the identity map of $(W_n f)^*K_{n,Y}$.

Proposition 1.34 (Étale pullback)**.** *Suppose $f : X \to Y$ is an étale morphism. Then*

$$f^* : K_{n,Y,log} \to (W_n f)_*K_{n,X,log},$$

defined by termwise applying (1.5.1), is a chain map between complexes of abelian sheaves.

Proof. It suffices to prove that C' is compatible with f^* defined above. Consider the following diagram in the category of complexes of $W_n\mathcal{O}_Y$-modules

$$\begin{array}{ccccc}
(W_nF_Y)_*K_{n,Y} & \xrightarrow[\simeq]{(1.2.2)} & (W_nF_Y)_*(W_nF_Y)^\triangle K_{n,Y} & \xrightarrow{\mathrm{Tr}_{W_nF_Y}} & K_{n,Y} \\
\downarrow{\scriptstyle \mathrm{adj}} & a) & \downarrow{\scriptstyle \mathrm{adj}} & d) & \downarrow{\scriptstyle \mathrm{adj}} \\
(W_nf)_*(W_nf)^*(W_nF_Y)_*K_{n,Y} & \xrightarrow[\simeq]{(1.2.2)} & (W_nf)_*(W_nf)^*(W_nF_Y)_*(W_nF_Y)^\triangle K_{n,Y} & \xrightarrow{\mathrm{Tr}_{W_nF_Y}} & (W_nf)_*(W_nf)^*K_{n,Y} \\
{\scriptstyle \alpha}\downarrow{\scriptstyle \simeq} & b) & {\scriptstyle \beta}\downarrow{\scriptstyle \simeq} & & \\
(W_nf)_*(W_nF_X)_*(W_nf)^*K_{n,Y} & \xrightarrow[\simeq]{(1.2.2)} & (W_nf)_*(W_nF_X)_*(W_nf)^*(W_nF_Y)^\triangle K_{n,Y} & e) & \downarrow{\scriptstyle \simeq} \\
\downarrow{\scriptstyle \simeq} & c) & \downarrow{\scriptstyle \simeq} & & \\
(W_nf)_*(W_nF_X)_*K_{n,X} & \xrightarrow[\simeq]{(1.2.2)} & (W_nf)_*(W_nF_X)_*(W_nF_X)^\triangle K_{n,X} & \xrightarrow{\mathrm{Tr}_{W_nF_X}} & (W_nf)_*K_{n,X}.
\end{array}$$

In this diagram we use shortened notations for the maps due to formatting reasons, e.g. we write (1.2.2) instead of $(W_nf)_*(W_nF_X)_*$(1.2.2), etc.. The maps labelled α and β are base change maps, and they are isomorphisms because W_nf is flat (actually W_nf is étale because f is étale) [Har66, II 5.12]. The composite of the maps on the very left and very right are $(W_nF_Y)_*(f^*)$ and f^* (where f^* is as defined in

(1.5.1)). The composite of the maps on the very top and very bottom are C'_Y and $(W_nf)_*C'_X$. Diagrams $a), b), c), d)$ commute due to naturality. Diagram $e)$ commutes, because we have a cartesian square

$$\begin{array}{ccc} W_nX & \xrightarrow{W_nF_X} & W_nX \\ {\scriptstyle W_nf}\downarrow & & \downarrow{\scriptstyle W_nf} \\ W_nY & \xrightarrow{W_nF_Y} & W_nY \end{array}$$

by Remark 1.16(2), and then the base change formula of the Grothendieck trace map as given in Proposition 1.10(5) gives the result. ☐

Lemma 1.35. *Consider the following cartesian diagram*

$$\begin{array}{ccc} W & \xrightarrow{f'} & Z \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with g being proper, and f being étale. Then we have a commutative diagram of residual complexes

$$\begin{array}{ccccc} (W_ng)_*K_{n,Z} & \xrightarrow{f'^*} & (W_ng)_*(W_nf)_*K_{n,W} & \xrightarrow{\simeq} & (W_nf)_*(W_ng)_*K_{n,W} \\ \downarrow{\scriptstyle \mathrm{Tr}_{W_ng}} & & & & \downarrow{\scriptstyle \mathrm{Tr}_{W_ng'}} \\ K_{n,Y} & & \xrightarrow{f^*} & & (W_nf)_*K_{n,X}. \end{array}$$

Proof. We decompose the diagram into the following two diagrams and show their commutativity one by one.

$$\begin{array}{ccc} (W_ng)_*K_{n,Z} & \xrightarrow{\mathrm{adj}} & (W_ng)_*(W_nf')_*(W_nf')^*K_{n,Z} \\ \Big| & & \downarrow{\scriptstyle \simeq} \\ \Big| & & (W_nf)_*(W_ng')_*(W_nf')^*(W_ng)^\triangle K_{n,Y} \\ \Big|{\scriptstyle \mathrm{Tr}_{W_ng}} & & {\scriptstyle \simeq}\uparrow{\scriptstyle \alpha} \\ \Big| & & (W_nf)_*(W_nf)^*(W_ng)_*(W_ng)^\triangle K_{n,Y} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{W_ng}} \\ K_{n,Y} & \xrightarrow{\mathrm{adj}} & (W_nf)_*(W_nf)^*K_{n,Y}. \end{array}$$

Here α denotes the base change map, it is an isomorphism because W_nf is flat [Har66, II 5.12]. This diagram commutes by naturality. Next consider

$$\begin{array}{ccc} (W_ng)_*(W_nf')_*(W_nf')^*K_{n,Z} & \xrightarrow{\simeq} & (W_ng)_*(W_nf')_*K_{n,W} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ (W_nf)_*(W_ng')_*(W_nf')^*(W_ng)^\triangle K_{n,Y} & \xrightarrow{\simeq} & (W_nf)_*(W_ng')_*K_{n,W} \\ {\scriptstyle \simeq}\uparrow{\scriptstyle \alpha} & & \Big| \\ (W_nf)_*(W_nf)^*(W_ng)_*(W_ng)^\triangle K_{n,Y} & & \Big|{\scriptstyle \mathrm{Tr}_{W_ng'}} \\ \downarrow{\scriptstyle \mathrm{Tr}_{W_ng}} & & \downarrow \\ (W_nf)_*(W_nf)^*K_{n,Y} & \xrightarrow{\simeq} & (W_nf)_*K_{n,X}. \end{array}$$

The top part commutes by naturality. The bottom part commutes by the base change formula of the Grothendieck trace maps with respect to étale morphisms Proposition 1.10(5). ☐

Since both f^* for log complexes in Proposition 1.34 and $g_* := \mathrm{Tr}_{W_ng,log}$ are defined termwise, we arrive immediately the following compatibility as a consequence of Lemma 1.35.

Proposition 1.36. *Notations are the same as Lemma 1.35. One has a commutative diagram of complexes*

$$\begin{array}{ccccc} (W_ng)_*K_{n,Z,log} & \xrightarrow{f'^*} & (W_ng)_*(W_nf)_*K_{n,W,log} & \xrightarrow{\simeq} & (W_nf)_*(W_ng)_*K_{n,W,log} \\ \downarrow g_* & & & & \downarrow g'_* \\ K_{n,Y,log} & & \xrightarrow{f^*} & & (W_nf)_*K_{n,X,log}. \end{array}$$

1.6. **Étale counterpart** $K_{n,X,log,\text{ét}}$**.** Let X be a separated scheme of finite type over k of dimension d. In this subsection we will use $t = \text{Zar}, \text{ét}$ to distinguish objects, morphisms on different sites. When t is omitted, it means $t = \text{Zar}$ unless otherwise stated.

Denote the structure sheaf on the small étale site $(W_nX)_{\text{ét}}$ by $W_n\mathcal{O}_{X,\text{ét}}$. Denote

$$(\epsilon_*, \epsilon^*) : ((W_nX)_{\text{ét}}, W_n\mathcal{O}_{X,\text{ét}}) \to ((W_nX)_{\text{Zar}}, W_n\mathcal{O}_X)$$

the module-theoretic functors. Recall that every étale W_nX-scheme is of the form $W_ng : W_nU \to W_nX$, where $g : U \to X$ is an étale X-scheme by Remark 1.16(1). Now let $\mathcal{F}$ be a $W_n\mathcal{O}_{X,\text{ét}}$-module on $(W_nX)_{\text{ét}}$. Consider the following map (cf. [Kat87, p. 264])

$$\tau : (W_nF_X)_*\mathcal{F} \to \mathcal{F}, \tag{1.6.1}$$

which is defined to be

$$\begin{aligned}((W_nF_X)_*\mathcal{F})(W_nU \xrightarrow{W_ng} W_nX) &= \mathcal{F}(W_nX \times_{W_nF_X, W_nX} W_nU \xrightarrow{pr_1} W_nX) \\ &\xrightarrow[\simeq]{W_nF^*_{U/X}} \mathcal{F}(W_nU \xrightarrow{W_ng} W_nX)\end{aligned}$$

for any étale map $W_ng : W_nU \to W_nX$ (here we use pr_1 to denote the first projection map of the fiber product). This is an automorphism of $\mathcal{F}$ as an abelian étale sheaf, but changes the $W_n\mathcal{O}_{X,\text{ét}}$-module structure of $\mathcal{F}$.

Lemma 1.37. *(1) The map τ is a map of $W_n\mathcal{O}_{X,\text{ét}}$-modules. That is, suppose α, β are the maps defining $W_n\mathcal{O}_{X,\text{ét}}$-module structure on $\mathcal{F}$ and $(W_nF_X)_*\mathcal{F}$ respectively, the following diagram commutes*

$$\begin{array}{ccc} W_n\mathcal{O}_{X,\text{ét}} \times (W_nF_X)_*\mathcal{F} & \xrightarrow{W_nF_X^* \times \tau} & W_n\mathcal{O}_{X,\text{ét}} \times \mathcal{F} \\ \downarrow \beta & & \downarrow \alpha \\ (W_nF_X)_*\mathcal{F} & \xrightarrow{\tau} & \mathcal{F}. \end{array}$$

(2) Given an étale sheaf $\mathcal{F}$ of $W_n\mathcal{O}_{X,\text{ét}}$-modules, the restriction of $(W_nF_X)_*\mathcal{F} \xrightarrow[\simeq]{\tau} \mathcal{F}$ to the Zariski open subsets W_nX is simply the identity map on the underlying complex of abelian sheaves.

(3) τ is functorial with respect to $\mathcal{F}$ in the category of $W_n\mathcal{O}_{X,\text{ét}}$-modules. I.e. for any homomorphism $f : \mathcal{F} \to \mathcal{G}$ of $W_n\mathcal{O}_{X,\text{ét}}$-modules, the following diagram of abelian étale sheaves on $(W_nX)_{\text{ét}}$

$$\begin{array}{ccc} (W_nF_X)_*\mathcal{F} & \xrightarrow{\tau} & \mathcal{F} \\ \downarrow (W_nF_X)_*f & & \downarrow f \\ (W_nF_X)_*\mathcal{G} & \xrightarrow{\tau} & \mathcal{G}. \end{array}$$

is commutative.

Proof. (1) In fact, suppose $\mathcal{F}$ is equipped with the $W_n\mathcal{O}_{X,\text{ét}}$-module structure

$$\alpha : W_n\mathcal{O}_{X,\text{ét}} \times \mathcal{F} \to \mathcal{F}.$$

Namely, on an étale section $W_ng : W_nU \to W_nX$, we have a map

$$\alpha_{W_ng} : \Gamma(W_n\mathcal{O}_U) \times \mathcal{F}(W_nU \xrightarrow{W_ng} W_nX) \to \mathcal{F}(W_nU \xrightarrow{W_ng} W_nX).$$

Then $(W_nF_X)_*\mathcal{F}$ is equipped with the following $W_n\mathcal{O}_{X,\text{ét}}$-module structure

$$\beta : W_n\mathcal{O}_{X,\text{ét}} \times (W_nF_X)_*\mathcal{F} \xrightarrow{W_nF_X^* \times id} (W_nF_X)_*W_n\mathcal{O}_{X,\text{ét}} \times (W_nF_X)_*\mathcal{F} \xrightarrow{(W_nF_X)_*\alpha} (W_nF_X)_*\mathcal{F}.$$

Namely, on an étale section $W_ng : W_nU \to W_nX$,

$$\beta_{W_ng} : \quad \Gamma(W_n\mathcal{O}_U) \times \mathcal{F}(W_nX \times_{W_nF_X, W_nX} W_nU \xrightarrow{pr_1} W_nX) \xrightarrow{(1 \otimes id_{\Gamma(W_n\mathcal{O}_X)}) \times id}$$

$$\Gamma(W_n\mathcal{O}_X)\otimes_{W_nF_X,\Gamma(W_n\mathcal{O}_X)}\Gamma(W_n\mathcal{O}_U)\times\mathcal{F}(W_nX\times_{W_nF_X,W_nX}W_nU\xrightarrow{pr_1}W_nX)$$
$$\xrightarrow{(W_nF_X)_*(\alpha_{W_ng})}\mathcal{F}(W_nX\times_{W_nF_X,W_nX}W_nU\xrightarrow{pr_1}W_nX).$$

From the explicit expressions of α and β, one deduces that the following diagram commutes

$$\begin{array}{ccc}
\Gamma(W_n\mathcal{O}_U)\times\mathcal{F}(W_nX\times_{W_nF_X,W_nX}W_nU\xrightarrow{pr_1}W_nX) & \xrightarrow{W_nF_X^*\times W_nF_{U/X}^*} & \Gamma(W_n\mathcal{O}_U)\times\mathcal{F}(W_nU\xrightarrow{W_ng}W_nX) \\
\downarrow{\scriptstyle\beta_{W_ng}} & & \downarrow{\scriptstyle\alpha_{W_ng}} \\
\mathcal{F}(W_nX\times_{W_nF_X,W_nX}W_nU\xrightarrow{pr_1}W_nX) & \xrightarrow[\simeq]{W_nF_{U/X}^*} & \mathcal{F}(W_nU\xrightarrow{W_ng}W_nX).
\end{array}$$

(2) This is because of the definition of the small Zariski site X_{Zar}: the morphism set of two open subsets of X_{Zar} can either be the empty set or a one element set consisting of an open immersion. In particular, $W_nX\times_{W_nF_X,W_nX}W_nU$ is an Zariski open subset of W_nX via identification with W_nU through $W_nF_{U/X}$. This explains our claim.

(3) This is direct, because the restriction maps of étale sheaves $\mathcal{F},\mathcal{G}$ induced by $W_nF_{U/X}$ are compatible:

$$\begin{array}{ccc}
\mathcal{F}(W_nX\times_{W_nF_X,W_nX}W_nU\xrightarrow{pr_1}W_nX) & \xrightarrow[\simeq]{W_nF_{U/X}^*} & \mathcal{F}(W_nU\xrightarrow{W_ng}W_nX) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
\mathcal{G}(W_nX\times_{W_nF_X,W_nX}W_nU\xrightarrow{pr_1}W_nX) & \xrightarrow[\simeq]{W_nF_{U/X}^*} & \mathcal{G}(W_nU\xrightarrow{W_ng}W_nX).
\end{array}$$

□

Define
$$K_{n,X,\text{ét}}:=\epsilon^*K_{n,X}$$
to be the complex of étale $W_n\mathcal{O}_{X,\text{ét}}$-modules associated to the Zariski complex $K_{n,X}$ of $W_n\mathcal{O}_X$-modules. This is still a complex of quasi-coherent sheaves with coherent cohomologies. For a proper map $f:X\to Y$ of k-schemes, define
$$\mathrm{Tr}_{W_nf,\text{ét}}:(W_nf)_*K_{n,X,\text{ét}}=\epsilon^*((W_nf)_*K_{n,X})\xrightarrow{\epsilon^*\mathrm{Tr}_{W_nf}}K_{n,Y,\text{ét}}$$
to be the étale map of $W_n\mathcal{O}_{Y,\text{ét}}$-modules associated to the Zariski map $\mathrm{Tr}_{W_nf}:K_{n,X}\to K_{n,X}$ of $W_n\mathcal{O}_X$-modules. Define the Cartier operator $C'_{\text{ét}}$ for étale complexes to be the composite
$$C'_{\text{ét}}:K_{n,X,\text{ét}}\xrightarrow[\simeq]{\tau^{-1}}(W_nF_X)_*K_{n,X,\text{ét}}=\epsilon^*((W_nF_X)_*K_{n,X})\xrightarrow{\epsilon^*(1.2.3)}K_{n,X,\text{ét}}.$$
Define
$$K_{n,X,log,\text{ét}}:=\mathrm{Cone}(K_{n,X,\text{ét}}\xrightarrow{C'_{\text{ét}}-1}K_{n,X,\text{ét}})[-1].$$

We also have the sheaf-level Cartier operator. Let X be a smooth k-scheme. Recall that by definition, $C_{\text{ét}}$ is the composition of the inverse of transfer-of-module-structure (1.6.1) with the module-theoretic etalization of the $W_n\mathcal{O}_X$-linear map (1.3.6):
$$C_{\text{ét}}:W_n\Omega^d_{X,\text{ét}}\xrightarrow[\simeq]{\tau^{-1}}(W_nF_X)_*W_n\Omega^d_{X,\text{ét}}=\epsilon^*((W_nF_X)_*W_n\Omega^d_X)\xrightarrow{\epsilon^*(1.3.6)}W_n\Omega^d_{X,\text{ét}}.$$
This is precisely the same as the classical definition that appeared in Lemma 1.28 before, because τ is the identity map when restricted to $(\text{Ét}/X)_{\mathrm{Zar}}$ by Lemma 1.37(2). (Here $(\text{Ét}/X)_{\mathrm{Zar}}$ denotes the site with the underlying category being the category of all étale X-schemes and coverings being Zariski coverings.)

Proposition 1.38 (cf. Theorem 1.17)**.** *$C'_{\text{ét}}$ is the natural extension of C' to the small étale site, i.e.,*
$$\epsilon_*C'_{\text{ét}}=C':K_{n,X}\to K_{n,X}.$$
When X is smooth, $C_{\text{ét}}$ is the natural extension of C to the small étale site
$$\epsilon_*C_{\text{ét}}=C:W_n\Omega^d_X\to W_n\Omega^d_X.$$
And one has compatibility
$$C_{\text{ét}}=\mathcal{H}^{-d}(C'_{\text{ét}}).$$

Proof. The first two claims are direct from Lemma 1.37(2). The last one comes from Lemma 1.37(2) and the compatibility of C and C' in the Zariski case Theorem 1.17. □

Proposition 1.39 (cf. Proposition 1.24)**.** *Let X be a separated scheme of finite type over k with $k = \overline{k}$. Then*

$$H^i(W_nX, K_{n,X,\text{ét}}) := H^i(R\Gamma((W_nX)_t, K_{n,X,\text{ét}})) \xrightarrow{C'_{\text{ét}}-1} H^i(W_nX, K_{n,X,\text{ét}})$$

is surjective for every i.

Proof. One notices the following identifications

$$\begin{aligned} H^i(W_nX, K_{n,X,\text{ét}}) : &= H^i(R\Gamma((W_nX)_{\text{ét}}, K_{n,X,\text{ét}})) = H^i(R\Gamma((W_nX)_{\text{ét}}, \epsilon^* K_{n,X,\text{Zar}})) \\ &= H^i(R\Gamma((W_nX)_{\text{Zar}}, K_{n,X,\text{Zar}})) =: H^i(W_nX, K_{n,X,\text{Zar}}). \end{aligned}$$

The first equality of the second row is due to the exactness of ϵ^* and the fact that ϵ^* maps injective quasi-coherent Zariski sheaves to injective quasi-coherent étale sheaves. The surjectivity then follows from the compatibility of C' and $C'_{\text{ét}}$ Proposition 1.38 and the Zariski case Proposition 1.24. □

In the étale topology and for any perfect field k, the surjectivity of

$$C_{\text{ét}} - 1 : W_n\Omega^d_{X,\text{ét}} \to W_n\Omega^d_{X,\text{ét}}$$

is known without the need of Proposition 1.39 (cf. Lemma 1.28). For the same reasoning as in Proposition 1.29, we have

Proposition 1.40 (cf. Proposition 1.29)**.** *Assume X is smooth of pure dimension d over a perfect field k. Then the natural map*

$$W_n\Omega^d_{X,log,\text{ét}}[d] \to K_{n,X,log,\text{ét}}$$

is a quasi-isomorphism of complexes of abelian sheaves.

We go back to the general non-smooth case. The proper pushforward property in the étale setting is very similar to the Zariski case.

Proposition 1.41 (Proper pushforward, cf. Proposition 1.30)**.** *For $f : X \to Y$ proper, we have a well-defined map of complexes of étale sheaves*

$$\text{Tr}_{W_nf,log,\text{ét}} : (W_nf)_*K_{n,X,log,\text{ét}} \to K_{n,X,log,\text{ét}} \tag{1.6.2}$$

given by applying $\text{Tr}_{W_nf,\text{ét}}$ termwise.

Proof. In fact, because τ^{-1} is functorial Lemma 1.37(3), we have the following commutative diagram of complexes of abelian étale sheaves

$$\begin{array}{ccc} (W_nf)_*K_{n,X,\text{ét}} & \xrightarrow[\simeq]{\tau^{-1}} & (W_nf)_*(W_nF_X)_*K_{n,X,\text{ét}} \\ \downarrow{\scriptstyle \text{Tr}_{W_nf,\text{ét}}} & & \downarrow{\scriptstyle (W_nF_Y)_*\text{Tr}_{W_nf,\text{ét}}} \\ K_{n,Y,\text{ét}} & \xrightarrow[\simeq]{\tau^{-1}} & (W_nF_Y)_*K_{n,Y,\text{ét}}. \end{array}$$

The rest of the proof goes exactly as in Proposition 1.30. □

Proposition 1.42 (cf. Proposition 1.32)**.** *Let $i : X_0 \hookrightarrow X$ be a nilpotent immersion. Then the natural map*

$$\text{Tr}_{W_ni,log,\text{ét}} : (W_ni)_*K_{n,X_0,log,\text{ét}} \to K_{n,X,log,\text{ét}}$$

is a quasi-isomorphism.

Proof. We adopt the same notations as in the proof of Proposition 1.32. Almost all steps of the proof go through directly, except that the map

$$\begin{aligned} C'_{I_n,\text{ét}} : Q_{n,\text{ét}} &= \mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}(I_{n,\text{ét}}, K_{n,X,\text{ét}}) \\ &\xrightarrow[\simeq]{\tau^{-1}} (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}(I_{n,\text{ét}}, K_{n,X,\text{ét}}) \\ &\to \mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}((W_nF_X)_*I_{n,\text{ét}}, (W_nF_X)_*K_{n,X,\text{ét}}) \\ &\xrightarrow[\simeq]{(W_nF_X)_*(1.2.2)\circ} \mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}((W_nF_X)_*I_{n,\text{ét}}, (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X,\text{ét}}) \\ &\xrightarrow{\text{Tr}_{W_nF_X}\circ} \mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}((W_nF_X)_*I_{n,\text{ét}}, K_{n,X,\text{ét}}) \\ &\xrightarrow{((W_nF_X^*)|_{I_{n,\text{ét}}})^\vee} \mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}(I_{n,\text{ét}}, K_{n,X,\text{ét}}) = Q_{n,\text{ét}} \end{aligned}$$

is nilpotent. To this end, it suffices to show τ^{-1} commutes with all the maps appearing in the above composition, e.g.,

$$\begin{array}{ccc} \mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}((W_nF_X)_*I_{n,\text{ét}}, K_{n,X,\text{ét}}) & \xrightarrow{((W_nF_X^*)|_{I_{n,\text{ét}}})^\vee} & \mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}(I_{n,\text{ét}}, K_{n,X,\text{ét}}) \\ \downarrow{\tau^{-1}} & & \downarrow{\tau^{-1}} \\ (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}((W_nF_X)_*I_{n,\text{ét}}, K_{n,X,\text{ét}}) & \xrightarrow{(W_nF_X)_*((W_nF_X^*)|_{I_{n,\text{ét}}})^\vee} & (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_{X,\text{ét}}}(I_{n,\text{ét}}, K_{n,X,\text{ét}}). \end{array}$$

And these diagrams are commutative due to Lemma 1.37(3). □

Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ being the open complement as before. Define

$$\underline{\Gamma}_Z(\mathcal{F}) := \mathrm{Ker}(\mathcal{F} \to j_*j^{-1}\mathcal{F})$$

for any étale abelian sheaf $\mathcal{F}$ on X, just as in the Zariski case (cf. (1.4.3)). Replacing Z (resp. X) by a nilpotent thickening will define the same functor as $\underline{\Gamma}_Z(-)$, because the étale site of any scheme is the same as the étale site of its reduced scheme [EGAIV-4, Ch. IV, 18.1.2]. Recall that when $\mathcal{F} = \mathcal{I}$ is an injective $\mathbb{Z}/p^n$-sheaf,

$$0 \to \underline{\Gamma}_Z(\mathcal{I}) \to \mathcal{I} \to j_*j^{-1}\mathcal{I} \to 0$$

is exact. In fact, because $j_!\mathbb{Z}/p^n$ is a subsheaf of the constant sheaf $\mathbb{Z}/p^n$ on X, we have that the map $\mathrm{Hom}_X(\mathbb{Z}/p^n, \mathcal{I}) \to \mathrm{Hom}_X(j_!\mathbb{Z}/p^n, \mathcal{I})$ is surjective. Thus $\mathrm{Hom}_X(\mathbb{Z}/p^n, \mathcal{I}) \to \mathrm{Hom}_U(\mathbb{Z}/p^n, j^{-1}\mathcal{I}) = \mathrm{Hom}_X(\mathbb{Z}/p^n, j_*j^{-1}\mathcal{I})$ is surjective. This implies that for any complex $\mathcal{F}^\bullet$ of étale $\mathbb{Z}/p^n$-sheaves with bounded cohomologies,

$$R\underline{\Gamma}_Z(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to j_*j^{-1}\mathcal{F}^\bullet \xrightarrow{+1} \tag{1.6.3}$$

is a distinguished triangle in $D^b(X, \mathbb{Z}/p^n)$ (cf. (1.4.4)).

Proposition 1.43 (Localization triangle, cf. Proposition 1.33)**.** *Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ being the open complement as before. Then*

(1) We can identify canonically the functors

$$(W_ni)_* = R\underline{\Gamma}_Z \circ (W_ni)_* : D^b((W_nZ)_{\text{ét}}, \mathbb{Z}/p^n) \to D^b((W_nX)_{\text{ét}}, \mathbb{Z}/p^n).$$

The composition of this canonical identification with the trace map

$$(W_ni)_*K_{n,Z,log,\text{ét}} = R\underline{\Gamma}_Z((W_ni)_*K_{n,Z,log,\text{ét}}) \xrightarrow{\mathrm{Tr}_{W_ni,log,\text{ét}}} R\underline{\Gamma}_Z(K_{n,X,log,\text{ét}})$$

is a quasi-isomorphism of complexes of étale $\mathbb{Z}/p^n$-sheaves.

(2)

$$(W_ni)_*K_{n,Z,log,\text{ét}} \xrightarrow{\mathrm{Tr}_{W_ni,log,\text{ét}}} K_{n,X,log,\text{ét}} \to (W_nj)_*K_{n,U,log,\text{ét}} \xrightarrow{+1}$$

is a distinguished triangle in $D^b((W_nX)_{\text{ét}}, \mathbb{Z}/p^n)$.

Proof. (1) One only needs to show that $(W_ni)_* = R\underline{\Gamma}_Z \circ (W_ni)_*$, and then the rest of the proof is the same as in Proposition 1.33(1). Let $\mathcal{I}$ be an injective étale $\mathbb{Z}/p^n$-sheaf on W_nZ. Since $\mathrm{Hom}_{W_nX}(-, (W_ni)_*\mathcal{I}) = \mathrm{Hom}_{W_nZ}((W_ni)^{-1}(-), \mathcal{I})$ and $(W_ni)^{-1}$ is exact, we know $(W_ni)_*\mathcal{I}$ is an injective abelian sheaf on $(W_nX)_{\text{ét}}$. This implies that $R(\underline{\Gamma}_Z \circ (W_ni)_*) = R\underline{\Gamma}_Z \circ (W_ni)_*$ by the Leray spectral sequence, and thus $(W_ni)_* = R(W_ni)_* = R(\underline{\Gamma}_Z \circ (W_ni)_*) = R\underline{\Gamma}_Z \circ (W_ni)_*$.

(2) One only need to note that $(W_nj)_*K_{n,U,log,\text{ét}} = R(W_nj)_*K_{n,U,log,\text{ét}}$. In fact, the terms of $K_{n,U,log,\text{ét}}$ are quasi-coherent $W_n\mathcal{O}_{X,\text{ét}}$-modules which are $(W_nj)_*$-acyclic in the étale topology (because $R^if_*(\epsilon^*\mathcal{F}) = \epsilon^*(R^if_*\mathcal{F})$ for any quasi-coherent Zariski sheaf $\mathcal{F}$ and any quasi-compact quasi-separated morphism f [Stacks, Tag 071N].). Now the first part and the distinguished triangle (1.6.3) imply the claim.

□

2. Bloch's cycle complex $\mathbb{Z}^c_{X,t}(m)$

Let X be a separated scheme of finite type over k of dimension d. Let

$$\Delta^i = \mathrm{Spec}\, k[T_0, \ldots, T_i]/(\sum T_j - 1).$$

Define $z_m(X, i)$ to be the free abelian group generated by closed integral subschemes $Z \subset X \times \Delta^i$ that intersect all faces properly and

$$\dim Z = m + i.$$

We say two closed subschemes Z_1, Z_2 of a scheme Y *intersect properly* if for every irreducible component W of the schematic intersection $Z_1 \cap Z_2 := Z_1 \times_Y Z_2$, one has

$$\dim W \leq \dim Z_1 + \dim Z_2 - \dim Y \tag{2.0.1}$$

(cf. [Gei05, A.1]). A subvariety of $X \times \Delta^i$ is called a *face* if it is determined by some $T_{j_1} = T_{j_2} = \cdots = T_{j_s} = 0$ $(0 \leq j_1 < \cdots < j_s \leq i)$. Note that a face is Zariski locally determined by a regular sequence of $X \times \Delta^i$. Therefore the given inequality condition in dimension (2.0.1) in the definition of $z_m(X, i)$ is equivalent to the equality condition [Gei05, (53)].

The above definition defines a sheaf $z_m(-, i)$ in both the Zariski and the étale topology on X ([Gei04, Lemma 3.1]). Notice that $z_m(-, i)$ is not a flasque sheaf even on the Zariski site, because cycles meeting faces properly on $U \times \Delta^i$ can have closures in $X \times \Delta^i$ that do not. Define the complex of sheaves

$$\to z_m(-, i) \xrightarrow{d} z_m(-, i-1) \to \ldots z_m(-, 0) \to 0$$

with differential map

$$d(Z) = \sum_j (-1)^j [Z \cap V(T_j)].$$

Here we mean by $V(T_j)$ the closed integral subscheme determined by T_j and by $[Z \cap V(T_j)]$ the linear combination of the reduced irreducible components of the scheme theoretic intersection $Z \cap V(T_j)$ with coefficients being intersection multiplicities. $z_m(X, \bullet)$ is then a homological complex concentrated in degree $[\max\{0, -m\}, \infty)$. By a shift of degree and labeling cohomologically, following the notation in [Gei10] we set

$$\mathbb{Z}^c_X(m)^i = z_m(-, -i - 2m).$$

This complex is nonzero in degrees

$$(-\infty, \min\{-2m, -m\}].$$

We write $\mathbb{Z}^c_X := \mathbb{Z}^c_X(0)$.

Define the higher Chow group

$$\mathrm{CH}_m(X, i) := H_i(z_m(X, \bullet)) = H^{-i-2m}(\mathbb{Z}^c_X(m)(X))$$

for any i and any m.

Proposition 2.1 (Bloch, Zariski descent)**.** *Suppose X has equidimension d over k. Then*

$$\mathrm{CH}_m(X, i) = R^{-i-2m}\Gamma(X_{\mathrm{Zar}}, \mathbb{Z}^c_X(m)).$$

Proof. Let $Z \hookrightarrow X$ be a closed immersion. The restriction map

$$z_m(X, \bullet)/z_m(Z, \bullet) \to z_m(X - Z, \bullet)$$

is a quasi-isomorphism of complexes of abelian groups by the moving lemma (it was claimed in [Blo86, 3.3] and later proved in [Blo94, 0.1]), and it induces a quasi-isomorphism of complexes of abelian Zariski sheaves

$$\mathcal{F}^\bullet \xrightarrow{\simeq} \mathbb{Z}^c_X(m),$$

where $\mathcal{F}^i(U) := z_m(X, -i-2m)/z_m(X-U, -i-2m)$ with U being a Zariski open in X. The Zariski sheaf $\mathcal{F}^i$ defined in this way is flasque for each i [Blo86, 3.4]. Therefore, we have $R^{-i-2m}\Gamma(X_{\mathrm{Zar}}, \mathbb{Z}^c_X(m)) = H^{-i-2m}(\mathcal{F}^\bullet(X)) = H_i(z_m(X, \bullet))$. □

Higher Chow groups are indeed a generalization of the Chow group, as

$$\mathrm{CH}_m(X, 0) = z_m(X, 0)/dz_m(X, 1) = \mathrm{CH}_m(X)$$

agrees with the classical definition of a Chow group (cf. [Ful98, §1.3]). The higher Chow groups with coefficients in an abelian group A will be denoted

$$\mathrm{CH}_m(X, i; A) := H^{-i-2m}(\mathbb{Z}^c_X(m)(X) \otimes_{\mathbb{Z}} A).$$

Proposition 2.2 (Functoriality, [Blo86, Prop. 1.3]. See also [Lev98, Part I, Ch. 2, Rmk. 2.1.7(i)])**.** *The complex $\mathbb{Z}^c_{X,t}(m)$, either $t = \mathrm{Zar}$ or $t = \text{ét}$, is covariant for proper morphisms, and contravariant for flat morphisms. More precisely, for proper $f : X \to Y$, we have a well-defined chain map of complexes of abelian sheaves*

$$f_* : f_*\mathbb{Z}^c_{X,t}(m) \to \mathbb{Z}^c_{Y,t}(m)$$

by pushforward of cycles, and for flat $f : X \to Y$ of equidimension c (i.e. fiber at each point of Y is either empty or of dimension c), we have a well-defined chain map of complexes of abelian sheaves

$$f^* : \mathbb{Z}^c_{Y,t}(m - c)[2c] \to f_*\mathbb{Z}^c_{X,t}(m)$$

by pullback of cycles.

3. Kato's complex of Milnor K-theory $C^M_{X,t}(m)$

Recall that given a field L, the q-th Milnor K-group $K^M_q(L)$ of L is defined to be the q-th graded piece of the non-commutative graded ring

$$\bigoplus_{q\geq 0} K^M_q(L) = \frac{\bigoplus_{q\geq 0}(L^*)^{\otimes q}}{(a\otimes(1-a)\mid a, 1-a\in L^*)},$$

where $(a\otimes(1-a)\mid a,1-a\in L^*)$ denotes the two-sided ideal of the non-commutative graded ring $\bigoplus_{q\geq 0}(L^*)^{\otimes q}$ generated by elements of the form $a\otimes(1-a)$ with $a, 1-a\in L^*$. The image of an element $a_1\otimes\cdots\otimes a_q\in(L^*)^{\otimes q}$ in $K^M_q(L)$ is denoted by $\{a_1,\ldots,a_q\}$.

Let X be a separated scheme of finite type over k of dimension d. We denote by

$$K^M_q(x) := K^M_q(k(x))$$

the q-th Milnor K-group of the field $k(x)$, and we have $K^M_q(x)=0$ when $q<0$. Kato defined in [Kat86c] a Gersten complex of Milnor K-groups

$$\bigoplus_{x\in X_{(d)}} K^M_{d-m}(x) \xrightarrow{d^M} \ldots \xrightarrow{d^M} \bigoplus_{x\in X_{(1)}} K^M_{1-m}(x) \xrightarrow{d^M} \bigoplus_{x\in X_{(0)}} K^M_{-m}(x).$$

Here the superscript M stands for Milnor, and the notation $X_{(q)}$ denotes the set of dimension q points of X. We briefly review its sheafified constructions in this section. Our sign conventions are the same as in [Ros96].

We firstly make clear the definition of a Milnor K-sheaf on a point $X=\operatorname{Spec} L$, where L is a field. Then $\mathcal{K}^M_{\operatorname{Spec} L,q,\mathrm{Zar}}$ is the constant sheaf associated to the abelian group $K^M_q(L)$ (without the assumption that L is an infinite field, cf. [Ker10, Prop. 10(4)]), and $\mathcal{K}^M_{\operatorname{Spec} L,q,\text{ét}}$ is the étale sheaf associated to the presheaf

$$L'\mapsto K^M_q(L');\quad L'/L \text{ finite separable.}$$

Choose a separable closure L^{sep} of L. Then the geometric stalk at the geometric point $\operatorname{Spec} L^{\mathrm{sep}}$ over $\operatorname{Spec} L$ is $\operatorname{colim}_{L\subset L'\subset L^{\mathrm{sep}}} K^M_q(L')$, which is equal to $K^M_q(L^{\mathrm{sep}})$ because the filtered colimit commutes with the tensor product and the quotient. Now by Galois descent of the étale sheaf condition, the sheaf $\mathcal{K}^M_{\operatorname{Spec} L,q,\text{ét}}$ is precisely

$$L'\mapsto K^M_q(L^{\mathrm{sep}})^{\mathrm{Gal}(L^{\mathrm{sep}}/L')};\quad L'/L \text{ finite separable.}$$

Here the Galois action is given on each factor, according to the very definition of the étale presheaf $\mathcal{K}^{M,\mathrm{pre}}_{X,q,\text{ét}}$.

Now with the topology $t=\mathrm{Zar}$ or $t=\text{ét}$, we have the corresponding Gersten complex of Milnor K-theory, denote by $C^M_{X,t}(m)$ (the differentials d^M will be introduced below):

(3.0.1)
$$\bigoplus_{x\in X_{(d)}} \iota_{x*}\mathcal{K}^M_{x,d-m,t} \xrightarrow{d^M} \ldots \xrightarrow{d^M} \bigoplus_{x\in X_{(1)}} \iota_{x*}\mathcal{K}^M_{x,1-m,t} \xrightarrow{d^M} \bigoplus_{x\in X_{(0)}} \iota_{x*}\mathcal{K}^M_{x,-m,t},$$

where $\iota_x:\operatorname{Spec} k(x)\hookrightarrow X$ the natural inclusion map. As part of the convention,

$$C^M_{X,t}(m)^i = \bigoplus_{x\in X_{(-i-m)}} \iota_{x,*}\mathcal{K}^M_{x,-i-2m,t}.$$

In other words, (3.0.1) sits in degrees

$$[-d-m, \min\{-m,-2m\}].$$

We set $C^M_{X,t}=C^M_{X,t}(0)$. It remains to introduce the differential maps.

When $t=\mathrm{Zar}$, the differential map d^M in (3.0.1) is defined in the following way. Let $x\in X_{(q)}$ be a dimension q point, and $\rho: X'\to\overline{\{x\}}$ be the normalization of $\overline{\{x\}}$ with generic point x'. Define

$$(d^M)^x_y: K^M_{q-m}(x) = K^M_{q-m}(x') \xrightarrow{\sum\partial^{x'}_{y'}} \bigoplus_{y'|y} K^M_{q-m-1}(y') \xrightarrow{\sum \mathrm{Nm}_{y'/y}} K^M_{q-m-1}(y).$$

Here the notation $y'|y$ means that $y'\in X'^{(1)}$ is in the fiber of y,

(3.0.2)
$$\partial^{x'}_{y'}: K^M_{q-m}(x')\to K^M_{q-m-1}(y')$$

is the Milnor tame symbol defined by y', and

(3.0.3)
$$\mathrm{Nm}_{y'/y}: K^M_{q-m-1}(y')\to K^M_{q-m-1}(y)$$

is the Milnor norm map of the finite field extension $k(y) \hookrightarrow k(y')$. The differential d^M of this complex is given by

$$d^M := \sum_{x\in X_{(q)}} \sum_{y\in X_{(q-1)}\cap\overline{\{x\}}} (d^M)^x_y : \bigoplus_{x\in X_{(q)}} K^M_{q-m}(x) \to \bigoplus_{y\in X_{(q-1)}} K^M_{q-m-1}(y).$$

There are different sign conventions concerning the tame symbol in the literature. Here we clarify the sign convention we adopt. Following [Ros96, p.328], we define the tame symbol $\partial^{\operatorname{Spec} L}_{\operatorname{Spec} k(v)} := \partial_v : K^M_n(L) \to K^M_{n-1}(k(v))$ for a field L, a normalized discrete valuation v on L and $k(v)$ the residue field with respect to v, via

$$\partial_v(\{\pi_v, u_1, \dots, u_{n-1}\}) = \{\overline{u}_1, \dots, \overline{u}_{n-1}\}. \tag{3.0.4}$$

Here π_v is a local parameter with respect to v, $u_1, \dots u_{n-1}$ are units in the valuation ring of v, and $\overline{u}_1, \dots, \overline{u}_{n-1}$ are the images of $u_1, \dots u_{n-1}$ in the residue field $k(v)$.

When $t =$ ét, set $x \in X_{(q)}$, $y \in X_{(q-1)} \cap \overline{\{x\}}$. Denote by $\rho : X' \to \overline{\{x\}}$ the normalization map and denote by x' the generic point of X'. One can canonically identify the étale abelian sheaves $\mathcal{K}^M_{x,q-m,\text{ét}}$ and $\rho_*\mathcal{K}^M_{x',q-m,\text{ét}}$ on $\overline{\{x\}}$, and thus identify $\iota_{x,*}\mathcal{K}^M_{x,q-m,\text{ét}}$ and $\iota_{x,*}\rho_*\mathcal{K}^M_{x',q-m,\text{ét}}$ on X. Let $y' \in X'^{(1)}$ such that $\rho(y') = y$. Then the componentwise differential map

$$(d^M)^x_y : \iota_{x,*}\mathcal{K}^M_{x,q-m,\text{ét}} \to \iota_{y,*}\mathcal{K}^M_{y,q-m-1,\text{ét}}$$

is defined to be the composition

$$(d^M)^x_y = \iota_{y,*}(\mathrm{Nm}) \circ \rho_*(\partial).$$

Here $\partial := \sum_{y'\in X'^{(1)}\cap\rho^{-1}(y)} \partial^{x'}_{y'}$, where

$$\partial^{x'}_{y'} : \iota_{x',*}\mathcal{K}^M_{x',q-m,\text{ét}} \to \iota_{y',*}\mathcal{K}^M_{y',q-m-1,\text{ét}} \tag{3.0.5}$$

on X' is defined to be the sheafification of the tame symbol on the presheaf level. Indeed, the tame symbol is a map of étale presheaves by [Ros96, R3a]. And $\mathrm{Nm} := \sum_{y'\in X'^{(1)}\cap\rho^{-1}(y)} \mathrm{Nm}_{y'/y}$, where

$$\mathrm{Nm}_{y'/y} : \rho_*\mathcal{K}^M_{y',q-m-1,\text{ét}} \to \mathcal{K}^M_{y,q-m-1,\text{ét}} \tag{3.0.6}$$

on y is defined to be the sheafification of the norm map on the presheaf level. The norm map is a map of étale presheaves by [Ros96, R1c].

Proposition 3.1 (Functoriality, [Ros96, (4.6)(1)(2)])**.** *The complex $C^M_{X,t}(m)$, either $t = \mathrm{Zar}$ or $t =$ ét, is covariant for proper morphisms and contravariant for flat equidimensional morphisms. More precisely, for proper $f : X \to Y$, we have a well-defined chain map of complexes of abelian sheaves*

$$f_* : f_*C^M_{X,t}(m) \to C^M_{Y,t}(m)$$

induced by the norm map of Milnor K-theory. When $f : X \to Y$ is flat and of equidimension c (i.e. the fiber at each point of Y is either empty or of dimension c), we have a well-defined chain map of complexes of abelian sheaves

$$f^* : C^M_{Y,t}(m-c)[2c] \to f_*C^M_{X,t}(m),$$

by the natural pullback maps of Milnor K-sheaves on fields.

Proof. The case $t = \mathrm{Zar}$ is given in [Ros96, (4.6)(1)(2)]. As for $t =$ ét, f_* is a well-defined map of étale presheaves at each term [Ros96, R1c], thus induces a chain map of étale presheaves, and then induces a chain map of étale sheaves. And f^* is a map of étale sheaves on each term, therefore induces a chain map of étale sheaves. □

4. Kato-Moser's complex of logarithmic de Rham-Witt sheaves $\widetilde{\nu}_{n,X,t}(m)$

Let X be a separated scheme of finite type over k of dimension d. Kato first defined the Gersten complex of the logarithmic de Rham-Witt sheaves in [Kat86a, §1]. Moser in [Mos99, (1.3)-(1.5)] sheafified Kato's construction on the étale site and studied its dualizing properties. We will adopt here the sign conventions in [Ros96].

Let Y be a k-scheme. Let $q \in \mathbb{N}$ be an integer. Recall that in Section 1.3.5, we have defined $W_n\Omega^q_{Y,log,t}$, with either $t = \mathrm{Zar}$ or $t =$ ét, to be the abelian subsheaf of $W_n\Omega^q_{Y,t}$ étale locally generated by log forms.

Lemma 4.1 (Bloch-Gabber-Kato isomorphism, [BK86, 2.8]))**.** *Let L be a field of characteristic p. The $d\log$ map induces an isomorphism of sheaves for both $t = \mathrm{Zar}$ and $t =$ ét over $\operatorname{Spec} L$:*

$$d\log : \mathcal{K}^M_{\operatorname{Spec} L,q,t}/p^n \xrightarrow{\simeq} W_n\Omega^q_{\operatorname{Spec} L,log,t}.$$

Proof. The original Bloch-Gabber-Kato isomorphism states that

$$d\log : K_q^M(L)/p^n \xrightarrow{\simeq} \Gamma(\operatorname{Spec} L, W_n\Omega^q_{\operatorname{Spec} L, log, \text{ét}})$$

is an isomorphism of abelian groups. This is precisely the statement for the sheaf theoretic version for $t = \text{Zar}$, because

$$\Gamma(\operatorname{Spec} L, \mathcal{K}^M_{\operatorname{Spec} L, q, \text{Zar}}) = K_q^M(L)$$

and

$$\Gamma(\operatorname{Spec} L, W_n\Omega^q_{\operatorname{Spec} L, log, \text{Zar}}) = \Gamma(\operatorname{Spec} L, \epsilon_* W_n\Omega^q_{\operatorname{Spec} L, log, \text{ét}}) = \Gamma(\operatorname{Spec} L, W_n\Omega^q_{\operatorname{Spec} L, log, \text{ét}}),$$

where ϵ_* is the restriction functor from the category of étale abelian sheaves to the category of Zariski abelian sheaves. As for $t = \text{ét}$, the lemma follows directly from the definition of the étale sheaf $W_n\Omega^q_{\operatorname{Spec} L, log, \text{ét}}$, which says it is the sheaf theoretic image of the $d\log$ map (cf. Section 1.3.5). □

We will freely use $W_n\Omega^q_{L, log, t}$ for $W_n\Omega^q_{\operatorname{Spec} L, log, t}$ below. Since the étale sheaf $W_n\Omega^q_{\operatorname{Spec} L, log, \text{ét}}$ is precisely the association $H \mapsto W_n\Omega^q_{\operatorname{Spec} H, log, \text{ét}}(H)$ for any finite separable extension H over L, our notation $W_n\Omega^q_{L, log, t}$ shall make no confusion.

Now let X be a separated scheme of finite type over k of dimension d. Define the Gersten complex $\widetilde{\nu}_{n,X,t}(m)$, in the topology $t = \text{Zar}$ or ét, to be the complex of t-sheaves isomorphic to $C^M_{X,t}(m)/p^n$ via the Bloch-Gabber-Kato isomorphism:

$$0 \to \bigoplus_{x \in X_{(d)}} \iota_{x,*} W_n\Omega^{d-m}_{k(x), log, t} \to \ldots \to \bigoplus_{x \in X_{(1)}} \iota_{x*} W_n\Omega^{1-m}_{k(x), log} \to \bigoplus_{x \in X_{(0)}} \iota_{x,*} W_n\Omega^{-m}_{k(x), log, t} \to 0. \tag{4.0.1}$$

Here $\iota_x : \operatorname{Spec} k(x) \to X$ is the natural map. We will still denote by ∂ the reduction of the tame symbol ∂ mod p^n (cf. (3.0.2)(3.0.5)), but denote by tr the reduction of Milnor's norm Nm mod p^n (cf. (3.0.3)(3.0.6)). The reason for the later notation will be clear from Lemma 5.3. As part of the convention,

$$\widetilde{\nu}_{n,X,t}(m)^i = \bigoplus_{x \in X_{(-i-m)}} \iota_{x*} W_n\Omega^{-i-2m}_{k(x), log, t},$$

i.e. $\widetilde{\nu}_{n,X}(m)$ is concentrated in degrees

$$[-d-m, -2m].$$

Notice that $\widetilde{\nu}_{n,X,t}(m)$ is the zero complex when $m < 0$. Set $\widetilde{\nu}_{n,X,t} := \widetilde{\nu}_{n,X,t}(0)$.

Proposition 4.2. *Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ its open complement. We have the following short exact sequence for $t = \text{Zar}$:*

$$0 \longrightarrow i_*\widetilde{\nu}_{n,Z,\text{Zar}}(m) \longrightarrow \widetilde{\nu}_{n,X,\text{Zar}}(m) \longrightarrow j_*\widetilde{\nu}_{n,U,\text{Zar}}(m) \longrightarrow 0.$$

For $t = \text{ét}$, when $m = 0$, one has the localization triangle

$$i_*\widetilde{\nu}_{n,Z,\text{ét}} \to \widetilde{\nu}_{n,X,\text{ét}} \to Rj_*\widetilde{\nu}_{n,U,\text{ét}} \xrightarrow{+1}.$$

Proof. $\widetilde{\nu}_{n,X,\text{Zar}}(m)$ is a complex of flasque sheaves (therefore $Rj_*(\widetilde{\nu}_{n,X,\text{Zar}}(m)) = j_*\widetilde{\nu}_{n,X,\text{Zar}}(m)$), and one has the sequence being short exact in this case. When $t = \text{ét}$, the purity theorem holds for $m = 0$ [Mos99, Corollary on p.130], i.e., $i_*\widetilde{\nu}_{n,Z,\text{ét}} = \underline{\Gamma}_Z(\widetilde{\nu}_{n,X,\text{ét}}) \xrightarrow{\simeq} R\underline{\Gamma}_Z(\widetilde{\nu}_{n,X,\text{ét}})$. We are done with the help of the distinguished triangle (1.6.3) in the étale topology. □

Remark 4.3. As pointed out by [Gro85, p.45 Remarque] and [Mil86, Rmk. 2.4], the purity theorem does not hold for general m (i.e., $i_*\widetilde{\nu}_{n,Z,\text{ét}}(m) = \Gamma_Z(\widetilde{\nu}_{n,X,\text{ét}}(m)) \to R\underline{\Gamma}_Z(\widetilde{\nu}_{n,X,\text{ét}}(m))$ is not a quasi-isomorphism for general m) even in the smooth case.

Functoriality of $\widetilde{\nu}_{n,X,t}(m)$ is the same as that of $C^M_{X,t}(m)$ via $d\log$. We omit the statement.

Part 2. The maps

5. Construction of the chain map $\zeta_{n,X,log,t} : C^M_{X,t} \to K_{n,X,log,t}$

5.1. Construction of the chain map $\zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,t}$.

Let $x \in X_{(q)}$ be a dimension q point. $\iota_x : \operatorname{Spec} k(x) \to X$ is the canonical map and $i_x : \overline{\{x\}} \hookrightarrow X$ the closed immersion. At degree $i = -q$, and over a point x, we define the degree i map to be $\zeta^i_{n,X,t} := \sum_{x \in X_{(q)}} \zeta^i_{n,x,t}$, with

$$\begin{aligned} \zeta^i_{n,x,t} : (W_n\iota_x)_*\mathcal{K}^M_{x,q,t} &\xrightarrow{d\log} (W_n\iota_x)_* W_n\Omega^q_{k(x), log, t} \subset (W_n\iota_x)_* W_n\Omega^q_{k(x), t} \\ &= (W_n i_x)_* K^i_{n, \overline{\{x\}}, t} \xrightarrow{(-1)^i \operatorname{Tr}_{W_n i_x}} K^i_{n,X,t}. \end{aligned} \tag{5.1.1}$$

We will use freely the notation $\zeta^i_{n,X,t}$ with some of its subscript or superscript dropped.

It's worth noticing that all the maps of étale sheaves involved here are given by the sheafification of the respective Zariski maps on the étale presheaf level. So to check commutativity of a composition of such maps between étale sheaves, it suffices to check on the $t = \mathrm{Zar}$ level. Keeping the convention as before, we usually omit the subscript Zar when we are working over the Zariski topology.

Proposition 5.1. *Let X be a separated scheme of finite type over k with k being a perfect field. For $t = \mathrm{Zar}$ and $t = \text{ét}$, the map*

$$\zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,t},$$

as defined termwise in (5.1.1), is a chain map of complexes of sheaves on the site $(W_nX)_t$.

Note that we have a canonical identification $(W_nX)_t = X_t$ for both $t = \mathrm{Zar}$ and $t = \text{ét}$. We use $(W_nX)_t$ just for the convenience of describing the $W_n\mathcal{O}_X$-structure of residual complexes appearing later.

Proof. To check $\zeta_{n,X,t}$ is a map of complexes, it suffices to check that the diagram

$$\begin{array}{ccc} (C^M_{X,t})^i & \xrightarrow{d^M_X} & (C^M_{X,t})^{i+1} \\ \big\downarrow{\scriptstyle \zeta^i_{n,X,t}} & & \big\downarrow{\scriptstyle \zeta^{i+1}_{n,X,t}} \\ (K_{n,X,t})^i & \xrightarrow{d_X} & (K^{i+1}_{n,X,t})^{i+1} \end{array}$$

commutes for $t = \mathrm{Zar}$. To this end, it suffices to show: for each $x \in X_{(q)}$, and $y \in X_{(q-1)}$ which is a specialization of x, the diagram

(5.1.2)
$$\begin{array}{ccc} (W_n\iota_x)_*\mathcal{K}^M_{x,q} & \xrightarrow{(d^M_X)^x_y} & (W_n\iota_y)_*\mathcal{K}^M_{y,q-1} \\ & & \big\downarrow{\scriptstyle \zeta_{n,y}} \\ \big\downarrow{\scriptstyle \zeta_{n,x}} & & (W_ni_{y,x})_*K^{-q+1}_{n,\overline{\{y\}}} \\ & & \big\downarrow{\scriptstyle -\mathrm{Tr}_{W_ni_{y,x}}} \\ K^{-q}_{n,\overline{\{x\}}} & \xrightarrow{d_{\overline{\{x\}}}} & K^{-q+1}_{n,\overline{\{x\}}} \end{array}$$

commutes ($i_{y,x} : \overline{\{y\}} \hookrightarrow \overline{\{x\}}$ denotes the canonical closed immersion).

Since the definition of the differential maps in C^M_X involves normalization, consider the normalization $\rho : X' \to \overline{\{x\}}$ of $\overline{\{x\}}$, and form the cartesian square

$$\begin{array}{ccc} \overline{\{y\}} \times_{\overline{\{x\}}} X' & \hookrightarrow & X' = \overline{\{x'\}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ \overline{\{y\}} & \xrightarrow{i_{y,x}} & \overline{\{x\}}. \end{array}$$

Denote the generic point of X' by x'. Suppose y' is one of the generic points of the irreducible components of $\overline{\{y\}} \times_{\overline{\{x\}}} X'$, and denote by Y' the irreducible component corresponding to y'. In particular, y' is a codimension 1 point in the normal scheme X', thus is regular. Because the base field k is perfect, y' is also a smooth point in X'. According to Remark 1.12, the degree $[-q, -q+1]$ terms of $K_{n,X'}$ are of the form

$$(W_n\iota_{x'})_*H^0_{x'}(W_n\Omega^q_{X'}) \xrightarrow{\delta} \bigoplus_{y' \in X'_{(q-1)}} (W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) \to \dots,$$

where δ denotes the differential map of the Cousin complex $K_{n,X}$. Notice that δ can be calculated via the boundary map of the localization sequence of local cohomologies, because of smoothness around y' (cf. [CR11, A.1.2]). After localizing at a single $y' \in X'^{(1)}$ in the Zariski sense, one gets

$$(W_n\iota_{x'})_*H^0_{x'}(W_n\Omega^q_{X'}) \xrightarrow{\delta_{y'}} (W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) \to \dots.$$

Consider the following diagrams. Write $\iota_{x'} : \operatorname{Spec} k(x') \hookrightarrow X'$, $\iota_{y'} : \operatorname{Spec} k(y') \hookrightarrow X'$ the inclusions, $i_{y',x'} : Y' = \overline{\{y'\}} \hookrightarrow X'$ the closed immersion,

(5.1.3)
$$\begin{array}{ccc}
(W_n\iota_{x'})_*\mathcal{K}^M_{x',q} & \xrightarrow{\partial^{x'}_{y'}} & (W_n\iota_{y'})_*\mathcal{K}^M_{y',q-1} \\
\downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\
(W_n\iota_{x'})_*W_n\Omega^q_{k(x')} & & (W_n\iota_{y'})_*W_n\Omega^{q-1}_{k(y')} \\
& \searrow{\scriptstyle \delta_{y'}} & \downarrow{\scriptstyle -\operatorname{Tr}_{W_n(i_{y',x'})}} \\
& & (W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}).
\end{array}$$

For any $y' \in \rho^{-1}(y) \subset X'^{(1)}$,

(5.1.4)
$$\begin{array}{ccc}
(W_n\rho)_*\mathcal{K}^M_{y',q-1} & \xrightarrow{\operatorname{Nm}_{y'/y}} & \mathcal{K}^M_{y,q-1} \\
\downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\
(W_n\rho)_*W_n\Omega^{q-1}_{k(y')} & \xrightarrow{\operatorname{Tr}_{W_n\rho}} & W_n\Omega^{q-1}_{k(y)}.
\end{array}$$

And write $i_{y',x'} : Y' = \overline{\{y'\}} \hookrightarrow X'$, $i_{y,x} : \overline{\{y\}} \hookrightarrow \overline{\{x\}}$,

(5.1.5)
$$\begin{array}{ccc}
(W_n\rho)_*(W_n\iota_{y'})_*W_n\Omega^{q-1}_{k(y')} & \xrightarrow{\operatorname{Tr}_{W_n\rho}} & (W_n\iota_y)_*W_n\Omega^{q-1}_{k(y)} \\
\downarrow{\scriptstyle \operatorname{Tr}_{W_n(i_{y',x'})}} & & \downarrow{\scriptstyle \operatorname{Tr}_{W_n(i_{y,x})}} \\
(W_n\rho)_*(W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) & \xrightarrow{\operatorname{Tr}_{W_n\rho}} & K^{-(q-1)}_{n,\overline{\{x\}}},
\end{array}$$

(5.1.6)
$$\begin{array}{ccc}
(W_n\iota_{x'})_*W_n\Omega^q_{k(x')} & \xrightarrow{d_{X'}=\sum \delta_{y'}} & \bigoplus_{y'\in\rho^{-1}(y)}(W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^{q-1}_{X'}) \\
\downarrow{\scriptstyle \operatorname{Tr}_{W_n\rho}\ \simeq} & & \downarrow{\scriptstyle \operatorname{Tr}_{W_n\rho}} \\
K^{-q}_{n,\overline{\{x\}}} & \xrightarrow{d_{\overline{\{x\}}}} & K^{-(q-1)}_{n,\overline{\{x\}}}.
\end{array}$$

All the trace maps above are trace maps of residual complexes at a certain degree. (5.1.5) is the degree $q-1$ part of the diagram

$$\begin{array}{ccc}
(W_n\rho)_*(W_ni_{y',x'})_*K_{n,Y'} & \xrightarrow{\operatorname{Tr}_{W_n\rho}} & (W_ni_{y,x})_*K_{n,\overline{\{y\}}} \\
\downarrow{\scriptstyle \operatorname{Tr}_{W_n(i_{y',x'})}} & & \downarrow{\scriptstyle \operatorname{Tr}_{W_n(i_{y,x})}} \\
(W_n\rho)_*K_{n,X'} & \xrightarrow{\operatorname{Tr}_{W_n\rho}} & K_{n,\overline{\{x\}}}
\end{array}$$

(the trace map on top is the trace map of the restriction of $W_n\rho$ to W_nY'), and thus is commutative by the functoriality of the Grothendieck trace map with respect to composition of morphisms (Proposition 1.10(4)). (5.1.6) is simply the degree $-q$ to $-q+1$ part of the trace map $\operatorname{Tr}_{W_n\rho} : (W_n\rho)_*K_{n,X'} \to K_{n,\overline{\{x\}}}$, thus is also commutative. It remains to check the commutativity of (5.1.3) and (5.1.4). And they follow from Lemma 5.2 and Lemma 5.3.

One notices that diagram (5.1.2) decomposes into the four diagrams (5.1.3)-(5.1.6):

$$
\begin{array}{ccccc}
(W_n\rho)_*(W_n\iota_{x'})_*\mathcal{K}^M_{x',q} & \xrightarrow{\bigoplus_{y'|y}\partial^{x'}_{y'}} & \bigoplus_{y'|y}(W_n\rho)_*(W_n\iota_{y'})_*\mathcal{K}^M_{y',q-1} & \xrightarrow{\sum_{y'/y}\mathrm{Nm}_{y'/y}} & (W_n\iota_y)_*\mathcal{K}^M_{y,q-1} \\
\downarrow{\scriptstyle d\log} & (5.1.3) & \downarrow{\scriptstyle d\log} & (5.1.4) & \downarrow{\scriptstyle d\log} \\
(W_n\rho)_*(W_n\iota_{x'})_*W_n\Omega^q_{k(x')} & & \bigoplus_{y'|y}(W_n\rho)_*(W_n\iota_{y'})_*W_n\Omega^{q-1}_{k(y')} & \xrightarrow{\mathrm{Tr}_{W_n\rho}} & (W_n\iota_y)_*W_n\Omega^{q-1}_{k(y)} \\
 & \searrow{\scriptstyle \bigoplus_{y'|y}\delta_{y'}} & \downarrow{\scriptstyle \bigoplus_{y'|y}-\mathrm{Tr}_{W_n(i_{y',x'})}} & (5.1.5) & \downarrow{\scriptstyle -\mathrm{Tr}_{W_n(i_{y,x})}} \\
\downarrow{\scriptstyle \mathrm{Tr}_{W_n\rho}} & & \bigoplus_{y'|y}(W_n\rho)_*(W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) & \xrightarrow{\mathrm{Tr}_{W_n\rho}} & K^{-(q-1)}_{n,\{x\}} \\
 & & (5.1.6) & & \| \\
K^{-q}_{n,\{x\}} & & \xrightarrow{\quad d_{\overline{\{x\}}}\quad} & & K^{-(q-1)}_{n,\{x\}}
\end{array}
$$

Here by symbol $y'|y$ we mean that $y' \in \rho^{-1}(y)$. Notice that we have added a minus sign to both vertical arrows of (5.1.5) in the corresponding square above, but this does not affect its commutativity. Since one can canonically identify

$$(W_n\rho)_*(W_n\iota_{x'})_*\mathcal{K}^M_{x',q} \quad \text{with} \quad (W_n\iota_x)_*\mathcal{K}^M_{x,q},$$

to show the commutativity of the diagram (5.1.2), it only remains to show Lemma 5.2 and Lemma 5.3. $\square$

Lemma 5.2. *For an integral normal scheme X', with $x' \in X'$ being the generic point and $y' \in X'^{(1)}$ being a codimension 1 point, the diagram (5.1.3) is commutative.*

Proof. Given a $y' \in X'^{(1)}$ lying over y, $K^M_q(x')$ is generated by

$$\{\pi', u_1, \ldots, u_{q-1}\} \text{ and } \{v_1, \ldots, v_{q-1}, v_q\}$$

as an abelian group, where $u_1, \ldots, u_{q-1}, v_1, \ldots, v_{q-1}, v_q \in \mathcal{O}^*_{X',y'}$, and π' is a chosen uniformizer of the dvr $\mathcal{O}_{X',y'}$. Thus it suffices to check the commutativity for these generators. We will use our convention (5) at the beginning of this paper for the computation of local cohomologies.

In the first case, the left-bottom composition gives

$$
\begin{aligned}
(\delta_{y'} \circ d\log)(\{\pi', u_1, \ldots, u_{q-1}\}) &= \delta_{y'}(d\log[\pi']_n d\log[u_1]_n \ldots d\log[u_{q-1}]_n) \\
&= \begin{bmatrix} d[\pi']_n d\log[u_1]_n \ldots d\log[u_{q-1}]_n \\ [\pi']_n \end{bmatrix}.
\end{aligned}
$$

The last equality above is given by the boundary map of the localization sequence of local cohomologies [CR11, A.1.2]. Here we have used the fact that $[\pi']$ is a regular element in W_nX', since π' is regular in X'. The top-right composition gives

$$
\begin{aligned}
&(-\mathrm{Tr}_{W_n(i_{y',x'})} \circ d\log \circ \partial^{x'}_{y'})(\{\pi', u_1, \ldots, u_{q-1}\}) \\
&= (-\mathrm{Tr}_{W_n(i_{y',x'})} \circ d\log)\{\overline{u}_1, \ldots, \overline{u}_{q-1}\} \\
&= -\mathrm{Tr}_{W_n(i_{y',x'})}(d\log[\overline{u}_1]_n \ldots d\log[\overline{u}_{q-1}]_n) \\
&= \begin{bmatrix} d[\pi']_n d\log[\overline{u}_1]_n \ldots d\log[\overline{u}_{q-1}]_n \\ [\pi']_n \end{bmatrix}.
\end{aligned}
$$

The last equality is given by [CR12, 2.4.1]. So the diagram (5.1.3) is commutative in this case.

In the second case, since $\partial^{x'}_{y'}(\{v_1, \ldots, v_q\}) = 0$, we need to check the left-bottom composite also gives zero. In fact,

$$
\begin{aligned}
(\delta_{y'} \circ d\log)(\{v_1, \ldots, v_q\}) &= \delta_{y'}(d\log[v_1]_n \ldots d\log[v_q]_n) \\
&= \begin{bmatrix} [\pi']_n \cdot d\log[v_1]_n \ldots d\log[v_q]_n \\ [\pi']_n \end{bmatrix} \\
&= 0.
\end{aligned}
$$

The second equality is due to [CR11, A.1.2]. The last equality is because $[\pi']_n \cdot d\log[v_1]_n \ldots d\log[v_q]_n$ lies in the submodule $([\pi']_n)W_n\Omega^q_{k(x')} \subset W_n\Omega^q_{k(x')}$. $\square$

Lemma 5.3 (Compatibility of Milnor norm and Grothendieck trace)**.** *Let F/E be a finite field extension with both fields E and F being of transcendence degree $q-1$ over k. Suppose there exists a finite morphism g between integral finite type k-schemes, such that F is the function field of the source of g and E is the function field of the target of g, and the field extension F/E is induced via the map g. Then the following diagram commutes*

$$\begin{array}{ccc} K^M_{q-1}(F) & \xrightarrow{\mathrm{Nm}_{F/E}} & K^M_{q-1}(E) \\ \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\ W_n\Omega^{q-1}_F & \xrightarrow{\mathrm{Tr}_{W_n g}} & W_n\Omega^{q-1}_E. \end{array}$$

Here the norm map $\mathrm{Nm}_{F/E}$ denotes the norm map from Milnor K-theory, and $\mathrm{Tr}_{W_n g}$ denotes the Grothendieck trace map associated to the finite morphism g.

Remark 5.4. (1) Apparently Lemma 5.3 implies the commutativity of the diagram (5.1.4) (i.e., take $F = k(y')$ and $E = k(y)$, and g to be the restriction of the normalization map ρ to a neighborhood of y'. In fact, whenever E is essentially of finite type over k, then F as a finite extension of E is also essentially of finite type over k, and there exists such a finite morphism g satisfying the assumptions in the lemma.)

(2) The localized trace map $\mathrm{Tr}_{W_n g}$ at the generic point of the source of g does not depend on the choice of g inducing the same field extension $E \subset F$. In fact, since the function field extension $E \subset F$ is given, the birational class of g is fixed.

(3) The compatibility of the trace map with the norm and the pushforward of cycles in various settings has been a folklore, and many definitions/properties of the trace map in the literature reflect this viewpoint. To list a few,

- Kato defined a trace map between Kähler differentials via the Milnor norm map (cf. [GO08, §2.2.3, ii.]).
- Rülling defined a trace map for generalized de Rham-Witt complex for finite field extensions in the odd characteristic in [Rül07, 2.6], and showed its compatibility with Milnor norm map as a consequence of [NS89, 4.7] and [Rül07, 3.18(iii)] (using the same notations as in [Rül07, §3], one notices that there is a natural map $\mathrm{CH}^n(k,n) \to \mathrm{CH}^{n+1}(A_k(m),n)$ induced by the inclusion of the point 1: $\operatorname{Spec} k \simeq \operatorname{Spec} k[T]/(T-1) \hookrightarrow \operatorname{Spec} k[T] = \mathbf{A}^1_k$). This restriction on the characteristic is removed later in the appendix of [KPR20].
- Along the line of the second item, Krishna and Park extended the trace in [Rül07, 2.6] to the case of finite extensions of regular semi-local k-algebras essentially of finite type for an arbitrary field k, and also to the case of finite extensions of regular k-algebras essentially of finite type for a perfect field k [KPR20, 7.8, 7.9].

But since we have not found a proof of the compatibility of the Milnor norm with the trace map defined via the Grothendieck duality theory, we include a proof here.

Proof. We start the proof by some reductions. Since both $\mathrm{Nm}_{F/E}$ and $\mathrm{Tr}_{F/E}$ are independent of the choice of towers of simple field extensions, without loss of generality, one could suppose F is a finite simple field extension over E. Now $F = E(a) = \frac{E[T]}{f(T)}$ for some monic irreducible polynomial $f(T) \in E[T]$ with $a \in F$ being one of its roots. This realizes $\operatorname{Spec} F$ as an F-valued point P of $\mathbf{P}^1_E$, namely,

$$\begin{array}{ccc} \operatorname{Spec} F = P & \xhookrightarrow{i_P} & \mathbf{P}^1_E \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \pi} \\ & & \operatorname{Spec} E. \end{array}$$

All the three morphisms on above are morphisms of finite type (although not between schemes of finite type over k), so it makes sense to talk about the associated trace maps for residual complexes (cf. Proposition 1.9). But for the particular residual complexes we are interested in, we need to enlarge the schemes involved to schemes of finite type over k, while preserving the morphism classes (e.g., closed immersion, smooth morphism, etc) of the morphisms between them.

To this end, take Y to be any separated smooth connected scheme of finite type over k with E being the function field. Since $\mathbf{P}^1_E$ is the generic fiber of $Y \times_k \mathbf{P}^1_k$, by possibly shrinking Y to an affine neighborhood $\operatorname{Spec} B$ of $pr_1(P)$ (here $pr_1 : Y \times_k \mathbf{P}^1_k \to Y$ is the first projection map) one can extend the above diagram to the following:

$$\begin{array}{ccc} \operatorname{Spec} F \in W & \xhookrightarrow{\ i_W\ } & \mathbf{P}^1_Y \\ & \searrow^{g} & \downarrow{\scriptstyle \pi} \\ & & Y = \operatorname{Spec} B \ni \operatorname{Spec} E. \end{array}$$

Here $W := \overline{\{P\}}^{\mathbf{P}^1_Y}$ is the closure of the point P in $\mathbf{P}^1_Y$. This is a commutative diagram of finite type k-schemes. In particular, it makes sense to talk about the residual complexes $K_{n,Y}, K_{n,W}$ and $K_{n,\mathbf{P}^1_Y}$.

Now it remains to show the commutativity of the following diagram

$$\begin{array}{ccc} K^M_{q-1}(E(a)) & \xrightarrow{\operatorname{Nm}_{E(a)/E}} & K^M_{q-1}(E) \\ \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\ W_n\Omega^{q-1}_{E(a)} & \xrightarrow{\operatorname{Tr}_{W_n g}} & W_n\Omega^{q-1}_E, \end{array} \tag{5.1.7}$$

where $\operatorname{Tr}_{W_n g}$ denotes the trace map for residual complexes $\operatorname{Tr}_{W_n g} : (W_n g)_* K_{n,W} \to K_{n,Y}$ at degree $-(q-1)$.

We do induction on $[E(a) : E]$. When $[E(a) : E] = 1$, then both the Grothendieck trace $\operatorname{Tr}_{W_n g} : W_n\Omega^{q-1}_{E(a)/k} \to W_n\Omega^{q-1}_{E/k}$ and the norm map $\operatorname{Nm}_{E(a)/E} : K^M_{q-1}(E(a)) \to K^M_{q-1}(E)$ are the identity, therefore the claim holds. Now the induction step. Suppose the diagram (5.1.7) commutes for $[E(a) : E] \le r-1$. We will need to prove the commutativity for $[E(a) : E] = r$.

First note that $\operatorname{Tr}_{W_n g} : (W_n g)_* K_{n,W} \to K_{n,Y}$ naturally decomposes into

$$(W_n g)_* K_{n,W} \xrightarrow{(W_n\pi)_* \operatorname{Tr}_{W_n i_P W}} (W_n\pi)_* K_{n,\mathbf{P}^1_Y} \xrightarrow{\operatorname{Tr}_{W_n\pi}} K_{n,Y}. \tag{5.1.8}$$

by Proposition 1.10(4). $H^1_P(W_n\Omega^q_{\mathbf{P}^1_Y})$ is a direct summand of the degree $-(q-1)$ part of $K_{n,\mathbf{P}^1_Y}$. We claim that one can canonically identify

$$H^1_P(W_n\Omega^q_{\mathbf{P}^1_Y}) = H^1_P(W_n\Omega^q_{\mathbf{P}^1_E}), \tag{5.1.9}$$

via pulling back along the natural map $\mathbf{P}^1_E \hookrightarrow \mathbf{P}^1_Y$. In fact, it suffices to show $H^1_P(W_n\Omega^q_{\mathbf{A}^1_Y}) = H^1_P(W_n\Omega^q_{\mathbf{A}^1_E})$, with a choice of $\mathbf{A}^1_E$ containing P. Notice that $\mathbf{A}^1_E = \operatorname{Spec} E \times_Y \mathbf{A}^1_Y = \operatorname{Spec} S^{-1}(B[T])$ where S is a multiplicatively closed subset consisting of nonzero elements in B (notice that B is an integral domain by the assumption). Let $\mathfrak{p} \subset E[T]$ be the prime ideal corresponding to the point P. Notice that $S = B \setminus 0 \subset B[T] \setminus \mathfrak{p}$. Indeed, since $\mathfrak{p}$ is a principal ideal, it is generated by a non-constant polynomial with coefficients in field E. Thus the inclusion holds. Now $H^1_P(W_n\Omega^q_{\mathbf{A}^1_Y}) = S^{-1}H^1_P(W_n\Omega^q_{\mathbf{A}^1_Y}) = H^1_P(S^{-1}W_n\Omega^q_{\mathbf{A}^1_Y}) = H^1_P(W_n\Omega^q_{\mathbf{A}^1_E})$, where the last equality is due to the compatibility of localization and the de Rham-Witt sheaves. So the claim holds. Thus on degree $-(q-1)$ and at the point P, the map (5.1.8) is canonically identified with

$$W_n\Omega^{q-1}_{E(a)} \xrightarrow{\operatorname{Tr}_{W_n i_P W}} H^1_P(W_n\Omega^q_{\mathbf{P}^1_E}) \xrightarrow{\operatorname{Tr}_{W_n\pi}} W_n\Omega^{q-1}_E.$$

Consider

$$\begin{array}{ccccc} K^M_q(E(T)) & \xrightarrow{\partial_P} & K^M_{q-1}(E(a)) & \xrightarrow{\operatorname{Nm}_{E(a)/E}} & K^M_{q-1}(E) \\ & & \downarrow{\scriptstyle d\log} & & \\ \downarrow{\scriptstyle d\log} & (-1) & W_n\Omega^{q-1}_{E(a)} & \searrow^{\operatorname{Tr}_{W_n g}} & \downarrow{\scriptstyle d\log} \\ & & \downarrow{\scriptstyle \operatorname{Tr}_{W_n i_P W}} & & \\ W_n\Omega^q_{E(T)} & \xrightarrow{\delta_P} & H^1_P(W_n\Omega^q_{\mathbf{P}^1_E}) & \xrightarrow{\operatorname{Tr}_{W_n\pi}} & W_n\Omega^{q-1}_E. \end{array}$$

We have used the identification (5.1.9) in this diagram. We have seen that the left square is commutative up to sign -1, as a special case of Lemma 5.2 (i.e. take normal scheme $X' = \mathbf{P}^1_E$ and $y' := P = \operatorname{Spec} F$). Since ∂_P is surjective, to show the commutativity of the trapezoid on the right, it suffices to show that the composite square is commutative up to -1. For any element

$$s := \{s_1, \ldots, s_{q-1}\} \in K^M_{q-1}(E(a)),$$

one can always find a lift

$$\widetilde{s} := \{f, \widetilde{s}_1, \dots, \widetilde{s}_{q-1}\} \in K_q^M(E(T)),$$

such that each of the $s_i = s_i(T)$ is a polynomial of degree $\leq r-1$ (e.g. decompose $E(a)$ as a r-dimensional E-vector space $E(a) = \bigoplus_{j=0}^{r-1} Ea^j$ and suppose $s_i = \sum_{j=0}^{r-1} b_{i,j}a^j$ with $b_{i,j} \in E$, then $\widetilde{s}_i = \widetilde{s}_i(T) = \sum_{j=0}^{r-1} b_{i,j}T^j$ satisfies the condition), and $\partial_P(\widetilde{s}) = s$. Denote by

$$y_{i,1}, \dots, y_{i,a_i} \quad (1 \leq i \leq q-1)$$

the closed points of $\mathbf{P}_E^1$ corresponding to the irreducible factors of the polynomials $\widetilde{s}_1, \dots, \widetilde{s}_{q-1}$. Note that the local section $\widetilde{s}_{i,l}$ cutting out $y_{i,l}$ is by definition an irreducible factor of $\widetilde{s}_i$, and therefore $\deg \widetilde{s}_{i,l} < r$ for all i and all l.

We claim that

$$\sum_{y \in (\mathbf{P}_E^1)_{(0)}} (\mathrm{Tr}_{W_n\pi})_y \circ \delta_y = 0 : W_n\Omega^q_{E(T)/k} \to W_n\Omega^{q-1}_{E/k}. \tag{5.1.10}$$

In fact,

$$0 \to W_n\Omega^q_{\mathbf{P}_E^1} \to W_n\Omega^q_{E(T)} \to \bigoplus_{y \in (\mathbf{P}_E^1)_{(0)}} (W_n\iota_y)_* H_y^1(W_n\Omega^q_{\mathbf{P}_E^1}) \to 0 \tag{5.1.11}$$

is an exact sequence [CR12, 1.5.9], where $\iota_y : y \hookrightarrow \mathbf{P}_E^1$ is the natural inclusion of the point y. Taking the long exact sequence with respect to the global section functor, one arrives at the following diagram with the row being a complex

$$\begin{array}{ccccc}
W_n\Omega^q_{E(T)} & \xrightarrow{\delta} & \bigoplus_{y \in (\mathbf{P}_E^1)_{(0)}} H_y^1(W_n\Omega^q_{\mathbf{P}_E^1}) & \longrightarrow & H^1(\mathbf{P}_E^1, W_n\Omega^q_{\mathbf{P}_E^1}) \\
 & & & \searrow^{\sum_y (\mathrm{Tr}_{W_n\pi})_y} & \downarrow {\scriptstyle \mathrm{Tr}_{W_n\pi}} \\
 & & & & W_n\Omega^{q-1}_{E/k}.
\end{array}$$

The trace maps on the skewed arrow of the above are induced from the degree 0 part of $\mathrm{Tr}_{W_n\pi} : (W_n\pi)_* K_{n,\mathbf{P}_Y^1} \to K_{n,Y}$. The trace map on the vertical arrow of the above is induced also by $\mathrm{Tr}_{W_n\pi} : (W_n\pi)_* K_{n,\mathbf{P}_Y^1} \to K_{n,Y}$, while the global cohomology group is calculated via (5.1.11), i.e., one uses the last two terms of (5.1.11) as an injective resolution of the sheaf $W_n\Omega^q_{\mathbf{P}_E^1}$, and then $\mathrm{Tr}_{W_n\pi} : (W_n\pi)_* K_{n,\mathbf{P}_Y^1} \to K_{n,Y}$ induces the map of complexes on global sections (placing at degrees $[-1, 0]$), and then the map of cohomologies on degree 0 gives our trace map $H^1(\mathbf{P}_E^1, W_n\Omega_E^q) \to W_n\Omega_E^{q-1}$ on the right. From the construction of these trace maps, the diagram above is by definition commutative. Therefore (5.1.10) holds.

One notices that $\delta_y \circ d\log(\widetilde{s}) = 0$ unless $y \in \{p, y_{1,1}, \dots, y_{q-1,a_{q-1}}, \infty\}$. Now calculate

$$\begin{aligned}
&(\mathrm{Tr}_{W_n g} \circ d\log)(s) \\
&\quad = (\mathrm{Tr}_{W_n g} \circ d\log \circ \partial_P)(\widetilde{s}) \\
&\quad = -((\mathrm{Tr}_{W_n\pi})_P \circ \delta_P \circ d\log)(\widetilde{s}) \quad \text{(Lemma 5.2)} \\
&\quad = \sum_{y \in \{y_{1,1}, \dots, y_{q-1,a_{q-1}}, \infty\}} ((\mathrm{Tr}_{W_n\pi})_y \circ \delta_y \circ d\log)(\widetilde{s}) \quad (5.1.10) \\
&\quad = -\sum_{y \in \{y_{1,1}, \dots, y_{q-1,a_{q-1}}, \infty\}} (d\log \circ \mathrm{Nm}_{E(k(y))/E} \circ \partial_y)(\widetilde{s}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(induction hypothesis)} \\
&\quad = (d\log \circ \mathrm{Nm}_{E(a)/E} \circ \partial_P)(\widetilde{s}) \quad ([\text{Ros96}, 2.2 \text{ (RC)}]) \\
&\quad = (d\log \circ \mathrm{Nm}_{E(a)/E})(s).
\end{aligned}$$

This finishes the induction. □

5.2. **Functoriality of $\zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,t}$.** Let k denote a perfect field of positive characteristic p.

Proposition 5.5 (Proper pushforward)**.** *ζ is compatible with proper pushforward. I.e., for $f : X \to Y$ a proper map, the following diagram is commutative*

$$\begin{array}{ccc} (W_nf)_*C^M_{X,t} & \xrightarrow{\zeta_{n,X,t}} & (W_nf)_*K_{n,X,t} \\ \downarrow f_* & & \downarrow f_* \\ C^M_{Y,t} & \xrightarrow{\zeta_{n,Y,t}} & K_{n,Y,t}. \end{array}$$

Here f_* on the left denotes the pushforward map for Kato's complex of Milnor K-theory (cf. Proposition 3.1), and f_* on the right denotes the Grothendieck trace map $\mathrm{Tr}_{W_nf,t}$ for residual complexes.

Proof. We only need to prove the proposition for $t = \mathrm{Zar}$ and for degree $i \in [-d, 0]$. Then by the very definition of the ζ map and the compatibility of the trace map with morphism compositions (Proposition 1.10(4)), it suffices to check the commutativity at points $x \in X_{(q)}$, $y \in Y_{(q)}$, where $q = -i$:

$$\begin{array}{ccc} K^M_q(x) & \xrightarrow{d\log} & W_n\Omega^q_{k(x)} \\ f_* \downarrow & & \downarrow f_* \\ K^M_q(y) & \xrightarrow{d\log} & W_n\Omega^q_{k(y)}. \end{array}$$

(1) When $y \neq f(x)$, both pushforward maps are zero maps, therefore we have the desired commutativity.
(2) When $y = f(x)$, by definition of ζ and the pushforward maps, we need to show commutativity of the following diagram for finite field extension $k(y) \subset k(x)$

$$\begin{array}{ccc} K^M_q(x) & \xrightarrow{d\log} & W_n\Omega^q_{k(x)} \\ \mathrm{Nm}_{k(x)/k(y)} \downarrow & & \downarrow \mathrm{Tr}_{W_nf} \\ K^M_q(y) & \xrightarrow{d\log} & W_n\Omega^q_{k(y)}. \end{array}$$

This is precisely Lemma 5.3.

□

Proposition 5.6 (Étale pullback)**.** *ζ is compatible with étale pullbacks. I.e., for $f : X \to Y$ an étale morphism, the following diagram is commutative*

$$\begin{array}{ccc} C^M_{Y,t} & \xrightarrow{\zeta_{n,Y,t}} & K_{n,Y,t} \\ \downarrow f^* & & \downarrow f^* \\ (W_nf)_*C^M_{X,t} & \xrightarrow{\zeta_{n,X,t}} & (W_nf)_*K_{n,X,t}. \end{array}$$

Here f^* on the left denotes the pullback map for Kato's complex of Milnor K-theory (cf. Proposition 3.1), and f^* on the right denotes the pullback map for residual complexes (1.5.1).

Proof. It suffices to prove the proposition for $t = \mathrm{Zar}$. Take $y \in Y_{(q)}$. Consider the cartesian diagram

$$\begin{array}{ccc} X \times_Y \overline{\{y\}} =: W & \xrightarrow{f|_W} & \overline{\{y\}} \\ \downarrow i_W & & \downarrow i_y \\ X & \xrightarrow{f} & Y. \end{array}$$

Then the desired diagram at point y decomposes in the following way at degree $-q$:

$$\begin{array}{ccccc} K^M_q(y) & \xrightarrow{d\log} & W_n\Omega^q_{k(y)} = K^{-q}_{n,\overline{\{y\}}} & \xrightarrow{\mathrm{Tr}_{W_ni_y}} & K^{-q}_{n,Y} \\ f^* \downarrow & & \downarrow (f|_W)^* & & \downarrow f^* \\ \bigoplus_{x \in W_{(q)}} K^M_q(x) & \xrightarrow{d\log} & \bigoplus_{x \in W_{(q)}} W_n\Omega^q_{k(x)} = K^{-q}_{n,W} & \xrightarrow{\mathrm{Tr}_{W_ni_W}} & K^{-q}_{n,X}. \end{array}$$

The right square commutes due to Lemma 1.35. As for the left, it follows from the fact that both f^* and $(f\mid_W)^*$ are induced by the natural map $f^*:\mathcal{O}_Y \to f_*\mathcal{O}_X$. $\square$

5.3. **Extend to $K_{n,X,log,t}$.** Recall the complex $K_{n,X,log,t} := \mathrm{Cone}(K_{n,X,t} \xrightarrow{C'_t - 1} K_{n,X,t})[-1]$, i.e.

$$K^i_{n,X,log,t} = K^i_{n,X,t} \oplus K^{i-1}_{n,X,t}.$$

The differential $K^i_{n,X,log,t} \to K^{i+1}_{n,X,log,t}$ is given by

$$\begin{aligned} d_{log}: K^i_{n,X,t} \oplus K^{i-1}_{n,X,t} &\to K^{i+1}_{n,X,t} \oplus K^i_{n,X,t} \\ (a,b) &\mapsto (d(a), -(C'-1)(a) - d(b)). \end{aligned}$$

Notice that

$$K_{n,X,t} \to K_{n,X,log,t}, \quad a \mapsto (a,0) \tag{5.3.1}$$

is not a chain map. Nevertheless,

Proposition 5.7. *We keep the same assumptions as in Proposition 5.1. The chain map $\zeta_{n,X,t}: C^M_{X,t} \to K_{n,X,t}$ composed with (5.3.1) gives a chain map*

$$\zeta_{n,X,log,t} := (5.3.1) \circ \zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,log,t}$$

of complexes of abelian sheaves on $(W_nX)_t$.

We will also use the shortened notation $\zeta_{log,t}$ for $\zeta_{n,X,log,t}$. When $t = \mathrm{Zar}$, the subscript Zar will also be omitted.

Proof. Given $x \in X_{(q)}$, we prove commutativity of the following diagram

$$\begin{array}{ccccccc} \iota_{x*}\mathcal{K}^M_{x,q,t} & \xrightarrow{d\log} & \iota_{x*}W_n\Omega^q_{k(x),log,t} & \xrightarrow{\mathrm{Tr}_{W_n i_x,t}} & K^{-q}_{n,X,t} & \xrightarrow{C'_{X,t}-1} & K^{-q}_{n,X,t} \\ \| & & \downarrow & & & & \| \\ \iota_{x*}\mathcal{K}^M_{x,q,t} & \xrightarrow{d\log} & \iota_{x*}W_n\Omega^q_{k(x),t} & \xrightarrow{C'_{\overline{\{x\}},t}-1} & (i_{x,*}K_{n,\overline{\{x\}},t})^{-q} & \xrightarrow{\mathrm{Tr}_{W_n i_x,t}} & K^{-q}_{n,X,t}. \end{array}$$

The left square naturally commutes. The right square also commutes, because C' is compatible with the Grothendieck trace map $\mathrm{Tr}_{W_n i_x}$. (The proofs of Proposition 1.30 and Proposition 1.41 give the case for $t = \mathrm{Zar}$ and $t = \text{ét}$, respectively). Now because $C'_{\overline{\{x\}},t} - 1 : W_n\Omega^q_{k(x),t} \to W_n\Omega^q_{k(x),t}$, which is identified with $C_{\overline{\{x\}},t} - 1$ as a result of Theorem 1.17 and Proposition 1.38, annihilates $W_n\Omega^q_{k(x),log,t}$, the composite of the second row is zero. Thus the composite of the first row is zero. This yields a unique chain map

$$\zeta_{n,X,log,t} : C^M_{X,t} \to K_{n,X,log,t},$$

which on degree $i = -q$ writes

$$\begin{aligned} \zeta^i_{n,X,log,t} : (C^M_{X,t})^i = \mathcal{K}^M_{x,q,t} &\to K^i_{n,X,log,t} = K^i_{n,X,t} \oplus K^{i-1}_{n,X,t}, \\ s = \{s_1 \ldots, s_q\} &\mapsto (\zeta^i_{n,X,t}(s), 0). \end{aligned}$$

$\square$

As a direct corollary of Proposition 5.5 and Proposition 5.6, one has the following proposition.

Proposition 5.8 (Functoriality). *(1) $\zeta_{log,t}$ is compatible with proper pushforward. I.e., for $f: X \to Y$ a proper map, the following diagram of complexes is commutative*

$$\begin{array}{ccc} (W_nf)_*C^M_{X,t} & \xrightarrow{\zeta_{n,X,log,t}} & (W_nf)_*K_{n,X,log,t} \\ \downarrow f_* & & \downarrow f_* \\ C^M_{Y,t} & \xrightarrow{\zeta_{n,Y,log,t}} & K_{n,Y,log,t}. \end{array}$$

Here f_* on the left denotes the pushforward map for Kato's complex of Milnor K-theory (cf. Proposition 3.1), and f_* on the right denotes $\mathrm{Tr}_{W_nf,log,t}$ as defined in Proposition 1.30 and Proposition 1.41.

(2) *$\zeta_{log,t}$ is compatible with étale pullbacks. I.e., for $f : X \to Y$ an étale morphism, the following diagram of complexes is commutative*

$$\begin{array}{ccc} C^M_{Y,t} & \xrightarrow{\zeta_{n,Y,log,t}} & K_{n,Y,log,t} \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ (W_n f)_* C^M_{X,t} & \xrightarrow{\zeta_{n,X,log,t}} & (W_n f)_* K_{n,X,log,t}. \end{array}$$

Here f^* on the left denotes the pullback map for Kato's complex of Milnor K-theory (cf. Proposition 3.1), and f^* on the right denotes the pullback map defined in Proposition 1.34.

5.4. $\overline{\zeta}_{n,X,log,t} : C^M_{X,t}/p^n \simeq \widetilde{\nu}_{n,X,t} \to K_{n,X,log,t}$ **is a quasi-isomorphism.** Let X be a separated scheme of finite type over a perfect field k of positive characteristic p. Since $\zeta_{n,X,t}$ is termwise defined via the $d\log$ map, it annihilates $p^n C^M_{X,t}$. Therefore $\zeta_{n,X,log,t}$ annihilates $p^n C^M_{X,t}$ as well, and induces a chain map

$$\overline{\zeta}_{n,X,log,t} : C^M_{X,t}/p^n \to K_{n,X,log,t}.$$

Since the $d\log$ map induces an isomorphism of complexes $C^M_{X,t}/p^n \simeq \widetilde{\nu}_{n,X,t}$, to show $\overline{\zeta}_{n,X,log,t}$ is a quasi-isomorphism, it is equivalent to show

$$\overline{\zeta}_{n,X,log,t} : \widetilde{\nu}_{n,X,t} \to K_{n,X,log,t}$$

is a quasi-isomorphism.

Lemma 5.9. *Suppose X is separated smooth over the perfect field k. Then for any level n the following chain maps*

$$\overline{\zeta}_{n,X,log,\text{ét}} : \widetilde{\nu}_{n,X,\text{ét}} \to K_{n,X,log,\text{ét}};$$
$$\overline{\zeta}_{n,X,log,\text{Zar}} : \widetilde{\nu}_{n,X,\text{Zar}} \to K_{n,X,log,\text{Zar}}, \quad \text{when } k = \overline{k};$$

are quasi-isomorphisms.

Proof. This is a local problem, thus it suffices to prove the statement for each connected component of X. Therefore we assume X is of pure dimension d over k. Then for any level n, we have a quasi-isomorphism ([GS88b, Cor 1.6])

$$W_n\Omega^d_{X,log,t}[d] \xrightarrow{\simeq} \widetilde{\nu}_{n,X,t}.$$

We also have

$$W_n\Omega^d_{X,log,\text{ét}}[d] \xrightarrow{\simeq} K_{n,X,log,\text{ét}} \quad \text{(by Proposition 1.40)},$$
$$W_n\Omega^d_{X,log,\text{Zar}}[d] \xrightarrow{\simeq} K_{n,X,log,\text{Zar}} \quad \text{when } k = \overline{k} \text{ (by Proposition 1.29)}.$$

On degree $-d$, we have a diagram

$$\begin{array}{ccc} \widetilde{\nu}^{-d}_{n,X,t} = \bigoplus_{x \in X^{(0)}} (W_n\iota_x)_* W_n\Omega^d_{k(x),log,t} & \xrightarrow{\overline{\zeta}^{-d}_{n,x,log,t}} & K^{-d}_{n,X,log,t} = \bigoplus_{x \in X^{(0)}} (W_n\iota_x)_* H^0_x(W_n\Omega^d_{X,t}) \\ \uparrow & & \uparrow \\ W_n\Omega^d_{X,log,t} & \xrightarrow{(-1)^d} & W_n\Omega^d_{X,log,t} \end{array}$$

which is naturally commutative, due to the definition of $\overline{\zeta}_{n,X,log,t}$. It induces quasi-isomorphisms as stated in the lemma. □

Theorem 5.10. *Let X be a separated scheme of finite type over k with k being a perfect field. Then the chain maps*

$$\overline{\zeta}_{n,X,log,\text{ét}} : \widetilde{\nu}_{n,X,\text{ét}} \to K_{n,X,log,\text{ét}};$$
$$\overline{\zeta}_{n,X,log,\text{Zar}} : \widetilde{\nu}_{n,X,\text{Zar}} \to K_{n,X,log,\text{Zar}}, \quad \text{when } k = \overline{k};$$

are quasi-isomorphisms.

Proof. One can assume that X is reduced. In fact, the complex $\widetilde{\nu}_{n,X,t}$ is defined to be the same complex as $\widetilde{\nu}_{n,X_{\text{red}},t}$ (see (4.0.1)), and we have a quasi-isomorphism $K_{n,X_{\text{red}},log,t} \xrightarrow{\simeq} K_{n,X,log,t}$ given by the trace map, according to Proposition 1.32 and Proposition 1.42. One notices that $\overline{\zeta}_{n,X_{\text{red}},log,t}$ is compatible with

$\overline{\zeta}_{n,X,log,t}$ because of the functoriality of the map $\zeta_{log,t}$ with respect to proper maps Proposition 5.8(1). As long as we have a quasi-isomorphism

$$\overline{\zeta}_{n,X_{\mathrm{red}},log,t} : \widetilde{\nu}_{n,X_{\mathrm{red}},t} \to K_{n,X_{\mathrm{red}},log},$$

we get automatically that

$$\overline{\zeta}_{n,X,log,t} : \widetilde{\nu}_{n,X_{\mathrm{red}},t} = \widetilde{\nu}_{n,X,t} \xrightarrow{\overline{\zeta}_{n,X_{\mathrm{red}},log,t}} K_{n,X_{\mathrm{red}},log} \xrightarrow{\simeq} K_{n,X,log,t}$$

is a quasi-isomorphism.

Now we do induction on the dimension of the reduced scheme X. Suppose X is of dimension d, and suppose $\overline{\zeta}_{n,Y,log,t}$ is a quasi-isomorphism for schemes of dimension $\leq d-1$. Now decompose X into the singular part Z and the smooth part U

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z.$$

Then Z has dimension $\leq d-1$. Consider the following diagram in the derived category of complexes of $\mathbb{Z}/p^n$-modules

(5.4.1)
$$\begin{array}{ccccccc}
i_*\widetilde{\nu}_{n,Z,t} & \longrightarrow & \widetilde{\nu}_{n,X,t} & \longrightarrow & Rj_*\widetilde{\nu}_{n,U,t} & \xrightarrow{+1} & i_*\widetilde{\nu}_{n,Z,t}[1] \\
\downarrow{\scriptstyle i_*\overline{\zeta}_{n,Z,log,t}} & & \downarrow{\scriptstyle \overline{\zeta}_{n,X,log,t}} & & \downarrow{\scriptstyle Rj_*\overline{\zeta}_{n,U,log,t}} & & \downarrow{\scriptstyle i_*\overline{\zeta}_{n,Z,log,t}[1]} \\
i_*K_{n,Z,log,t} & \xrightarrow{\mathrm{Tr}_{W_n i,log}} & K_{n,X,log,t} & \longrightarrow & Rj_*K_{n,U,log,t} & \xrightarrow{+1} & i_*K_{n,Z,log,t}[1],
\end{array}$$

where the two rows are distinguished triangles coming from Proposition 1.33, Proposition 1.43 and Proposition 4.2. We show that the three squares in (5.4.1) are commutative in the derived category. The left square is commutative because of Proposition 5.8(1). The middle square of (5.4.1) is induced from the diagram

(5.4.2)
$$\begin{array}{ccc}
\widetilde{\nu}_{n,X,t} & \longrightarrow & j_*\widetilde{\nu}_{n,U,t} \\
\downarrow{\scriptstyle \overline{\zeta}_{n,X,log,t}} & & \downarrow{\scriptstyle j_*\overline{\zeta}_{n,U,log,t}} \\
K_{n,X,log,t} & \longrightarrow & j_*K_{n,U,log,t}
\end{array}$$

of chain complexes. Let $x \in X_{(q)}$. When $x \in X_{(q)} \cap U$, both $\widetilde{\nu}_{n,X,t} \to j_*\widetilde{\nu}_{n,U,t}$ and $K_{n,X,log,t} \to j_*K_{n,U,log,t}$ give identity maps at x, therefore the square (5.4.2) commutes in this case. When $x \in X_{(q)} \cap Z$, both of these give the zero map at x, therefore the square (5.4.2) is also commutative. The right square of (5.4.1) can be decomposed in the following way (cf. (1.4.4) and (1.6.3)):

$$\begin{array}{ccccc}
Rj_*\widetilde{\nu}_{n,U,t} & \xrightarrow{+1} & R\underline{\Gamma}_Z(\widetilde{\nu}_{n,X,t})[1] & \xleftarrow[\simeq]{i_*} & i_*\widetilde{\nu}_{n,Z,t}[1] \\
\downarrow{\scriptstyle Rj_*\overline{\zeta}_{n,U,log,t}} & & \downarrow{\scriptstyle R\underline{\Gamma}_Z(\overline{\zeta}_{n,X,log,t})[1]} & & \downarrow{\scriptstyle i_*\overline{\zeta}_{n,Z,log,t}[1]} \\
Rj_*K_{n,U,log,t} & \xrightarrow{+1} & R\underline{\Gamma}_Z(K_{n,X,log,t})[1] & \xleftarrow[\simeq]{i_*} & i_*K_{n,Z,log,t}[1].
\end{array}$$

The map i_* on the first row is induced by the norm map of Milnor K-theory Proposition 3.1. It is clearly an isomorphism of complexes when $t = \mathrm{Zar}$. It is a quasi-isomorphism when $t = \text{ét}$ due to the purity theorem [Mos99, p.130 Cor.]. The map i_* on the second row is induced from $\mathrm{Tr}_{W_n i,log,t}$ as defined in Proposition 1.30 and Proposition 1.41, and it is an isomorphism due to Proposition 1.33(1) when $t = \mathrm{Zar}$, and Proposition 1.43 when $t = \text{ét}$. The first square commutes by naturality of the $+1$ map. The second commutes because of the compatibility of $\zeta_{log,t}$ with the proper pushforward Proposition 5.8(1). We thus deduce that the right square of (5.4.1) commutes.

Now consider over any perfect field k for either of the two cases:

(1) $t = \text{ét}$ and k a perfect field, or
(2) $t = \mathrm{Zar}$ and $k = \overline{k}$.

The left vertical arrow of (5.4.1) is a quasi-isomorphism because of the induction hypothesis. The third one counting from the left is also a quasi-isomorphism because of Lemma 5.9. Thus so is the second one. □

6. Combine $\psi_{X,t}(m) : \mathbb{Z}^c_{X,t}(m) \to C^M_{X,t}(m)$ with $\zeta_{n,X,log,t} : C^M_{X,t} \to K_{n,X,log,t}$

6.1. The map $\psi_{X,t}(m) : \mathbb{Z}^c_{X,t}(m) \to C^M_{X,t}(m)$. In [Zho14, 2.14], the author constructed a map of abelian groups $\psi_{X,t}(m) : \mathbb{Z}^c_X(m)(X) \to C^M_{X,\mathrm{Zar}}(m)(X)$ based on the Nesterenko-Suslin-Totaro isomorphism [NS89, Thm. 4.9][Tot92]. First we briefly recall his construction and make sure his results pass to the sheaf-theoretic case.

Let k be a perfect field of characteristic p, and X be a separated scheme of finite type over k of dimension d. Since the $t = \mathrm{Zar}$ case is written in [Zho14, 2.14], we write only $t = \text{ét}$ below. Define a map of complexes of étale sheaves

$$\psi := \psi_{X,\text{ét}}(m) : \mathbb{Z}^c_{X,\text{ét}}(m) \to C^M_{X,\text{ét}}(m)$$

in the following way. Let i be a given degree, and $U \in X_{\text{ét}}$ be a section. We denote by $U_x = x \times_X U$ the fiber above x. Define

$$z_m(U, -i-2m) \to \bigoplus_{x \in X_{(-i-m)}} K^M_{x,-i-2m,t}(U_x) = \bigoplus_{x \in X_{(-i-m)}} \bigoplus_{u \in U_x} K^M_{-i-2m}(k(x)^{\mathrm{sep}})^{\mathrm{Gal}(k(u))}.$$

Let $Z \in z_m(U, -i-2m)$ be a prime cycle.

- When $i \in [-d-m, \min\{-2m, -m\}]$ and $\dim p_U(Z) = -i-m$, Z, as a cycle of dimension $-i-m$ in $U \times \Delta^{-i-2m}$, is dominant over some $u(Z) \in U_{(-i-m)}$ under projection $p_U : U \times \Delta^{-i-2m} \to U$. Since $U \to X$ is of relative dimension 0, we have $u(Z) \in U_x$ for some $x \in X_{(-i-m)}$. By slight abuse of notation, we denote by $T_0, \dots, T_{-i-2m} \in k(Z)$ the pullbacks of the corresponding coordinates via $Z \hookrightarrow U \times \Delta^{-i-2m}$. Since Z intersects all faces properly, $T_0, \dots, T_{-i-2m} \in k(Z)^*$. Thus $\{\frac{-T_0}{T_{-i-2m}}, \dots, \frac{-T_{-i-2m-1}}{T_{-i-2m}}\} \in K^M_{-i-2m}(k(Z))$ is well-defined. Then one applies the norm map $\mathrm{Nm}_{k(Z)/k(u(Z))} : K^M_{-i-2m}(k(Z)) \to K^M_{-i-2m}(k(u(Z)))$ and the natural map $K^M_{-i-2m}(k(u(Z))) \to K^M_{-i-2m}(k(x)^{\mathrm{sep}})^{\mathrm{Gal}(k(u(Z)))}$. Denote this composite map again by $\mathrm{Nm}_{k(Z)/k(u(Z))}$. Define
$$\psi(Z) := \mathrm{Nm}_{k(Z)/k(u(Z))}\{\frac{-T_0}{T_{-i-2m}}, \dots, \frac{-T_{-i-2m-1}}{T_{-i-2m}}\} \in K^M_{-i-2m}(k(x)^{\mathrm{sep}})^{\mathrm{Gal}(k(u(Z)))}.$$
- When $i \notin [-d-m, \min\{-2m, -m\}]$ or $\dim p_U(Z) \neq -i-m$, define $\psi(Z) := 0$.

Remark 6.1. One can define a similar map as ψ in terms of the cubical description of the cycle complex [Lev09, §1.1-1.2], cf. [RS18, §3.1]. But we shall not need this.

Proposition 6.2. *$\psi_{X,t}(m)$ is a well-defined map of complexes of sheaves for $t = \mathrm{Zar}$ and $t = \text{ét}$.*

Proof. $t = \mathrm{Zar}$ case is clear from [Zho14, 2.15], thus it suffices to show the claim for $t = \text{ét}$.

We first claim that $\psi_{X,\text{ét}}(m)$ is a well-defined map of étale sheaves on each term. To this end, take $g : V \to U$ an étale map over X. Fix a point $x \in X_{(-i-m)}$, and take $Z \in z_m(U, -i-2m)$ a prime cycle with generic point z. We need to check commutativity of the following diagram

$$\begin{array}{ccc} z_m(V,-i-2m) & \xrightarrow{\psi_V} & \bigoplus_{x\in X_{(-i-m)}} \bigoplus_{v\in V_x} K^M_{-i-2m}(k(x)^{\mathrm{sep}})^{\mathrm{Gal}(k(v))} \\ \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle g^*} \\ z_m(U,-i-2m) & \xrightarrow{\psi_U} & \bigoplus_{x\in X_{(-i-m)}} \bigoplus_{u\in U_x} K^M_{-i-2m}(k(x)^{\mathrm{sep}})^{\mathrm{Gal}(k(u))}, \end{array}$$

where g^* on the two vertical maps denote the restrictions in the respective étale sheaves. Denote by $g \times id : V \times \Delta^{-i-2m} \to U \times \Delta^{-i-2m}$ the product morphism, and by $p_U : U \times \Delta^{-i-2m} \to U$, $p_V : V \times \Delta^{-i-2m} \to V$ the natural projections. Firstly, note that $\dim p_V((g \times id)^{-1}Z) \le \dim p_U(Z)$, because the image of $p_V((g \times id)^{-1}Z)$ under the map g lies in $p_U(Z)$. So it remains to check the following three cases:

1. $\dim p_U(Z) \neq -i-m$ and $\dim p_V((g \times id)^{-1}Z) \neq -i-m$. In this case we have both composite maps map Z to zero.
2. $\dim p_U(Z) = -i-m$ and $\dim p_V((g \times id)^{-1}Z) \neq -i-m$. This can only happen when $(g \times id)^{-1}Z = \emptyset$: otherwise there will be a generic point, say z', of $(g \times id)^{-1}Z$ mapping to the generic point z of Z via $g \times id$. But $p_U(z)$ would be the generic point of $p_U(Z)$ and is a dimension $-i-m$ point, and $p_V(z')$ would be a generic point of $p_V((g \times id)^{-1}Z)$ and is a point of dimension strictly smaller than $-i-m$. But $g(p_V(z')) = p_U(z)$. This contradicts the fact that g is of relative dimension 0.

   This tells us $\psi_V(g^*(Z)) = 0$. We need to show $g^*(\psi_U(Z)) = 0$ as well. Fix $u \in U_x$.

(a) If $u \neq p_U(z)$, then $\psi_U(Z) = 0$ at u.

(b) If $u = p_U(z)$ then $g^{-1}(u) = g^{-1}(p_U(z)) = \emptyset$. Because otherwise $p_V^{-1}(g^{-1}(u))$ would be non-empty due to the surjectivity of p_V, and $(g \times id)^{-1}(z)$ lies in the intersection $p_V^{-1}(g^{-1}(u)) \cap (g \times id)^{-1} Z$. This contradicts $(g \times id)^{-1} Z = \emptyset$. Now we've proved $g^*(\psi_U(Z)) = 0$ at u.

(3) $\dim p_U(Z) = \dim p_V((g \times id)^{-1} Z) = -i - m$. Fix $u \in U_x, v \in V_x$, with $g(v) = u$.

(a) When $u = p_U(z)$, each point in $p_V^{-1}(v)$ will be the generic point of some irreducible component Z_i of $(g \times id)^{-1} Z$. Note that g is étale. The classical commutativity from Milnor K-theory [Ros96, R1c],

$$\begin{array}{ccc} \bigoplus_{Z_i} K^M_{-i-2m}(k(Z_i)) & \xrightarrow{\sum_{v(Z_i)=v} \mathrm{Nm}_{k(Z_i)/k(v)}} & K^M_{-i-2m}(k(v)) \\ \uparrow & & \uparrow \\ K^M_{-i-2m}(k(Z)) & \xrightarrow{\mathrm{Nm}_{k(Z)/k(u(Z))}} & K^M_{-i-2m}(k(u)), \end{array}$$

implies the commutativity of the required diagram, with the down-right composition equals $g^*(\psi_U(Z))$, and the left-top composition equals $\psi_V(g^*(Z))$.

(b) When $u \neq p_U(z)$ then we must have $v \neq p_V(z')$ for any generic point z' of $(g \times id)^{-1} Z$. In this case both composite maps Z to zero, and we still have the desired commutativity.

$\psi_{X,t}(m)$ is also a well-defined map of complexes. For $t =$ ét, it suffices to check this on the presheaf level, which is equivalent to check for $t =$ Zar. This is done already in [Zho14, 2.14]. ☐

6.2. **Functoriality.** Zhong in [Zho14, 2.15] proved that $\psi_{X,t}(m)$ is covariant with respect to proper morphisms, and contravariant with respect to quasi-finite flat morphisms. We improve his contravariant statement from quasi-finite flat morphisms to flat morphisms.

Proposition 6.3. *Let X, Y be separated schemes of finite type over k. $d := \dim X$.*

(1) [Zho14, 2.15] For proper $f : X \to Y$, the following diagram is commutative:

$$\begin{array}{ccc} f_*\mathbb{Z}^c_{X,t}(m) & \xrightarrow{\psi_{X,t}(m)} & f_* C^M_{X,t}(m) \\ \downarrow f_* & & \downarrow f_* \\ \mathbb{Z}^c_{Y,t}(m) & \xrightarrow{\psi_{Y,t}(m)} & C^M_{Y,t}(m). \end{array}$$

(2) For flat $f : X \to Y$ of equidimension c, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{Z}^c_{Y,t}(m-c)[2c] & \xrightarrow{\psi_{Y,t}(m-c)[2c]} & C^M_{Y,t}(m-c)[2c] \\ \downarrow f^* & & \downarrow f^* \\ f_*\mathbb{Z}^c_{X,t}(m) & \xrightarrow{\psi_{X,t}(m)} & f_* C^M_{X,t}(m). \end{array}$$

Proof. It suffices to prove the $t =$ Zar case. Covariant functoriality is proved by Zhong. It remains to check contravariant functoriality for flat $f : X \to Y$ of equidimension c (therefore $0 \le c \le d$). Notice firstly that

- $\mathbb{Z}^c_{X,t}(m)$ is concentrated in degrees $(-\infty, \min\{-2m, -m\}]$,
- $\mathbb{Z}^c_{Y,t}(m-c)[2c]$ is concentrated in degrees $(-\infty, \min\{-2m, -m-c\}]$,
- $C^M_{X,t}(m)$ is concentrated in degrees $[-d-m, \min\{-2m, -m\}]$, and
- $C^M_{Y,t}(m-c)[2c]$ is concentrated in degrees $[-d-m, \min\{-2m, -m-c\}]$.

We discuss commutativity of the following diagram

$$\begin{array}{ccc} z_{m-c}(Y, -i-2m) & \xrightarrow{\psi_Y} & \bigoplus_{Y_{(-i-m-c)}} K^M_{-i-2m}(y) \\ \downarrow f^* & & \downarrow f^* \\ z_m(X, -i-2m) & \xrightarrow{\psi_X} & \bigoplus_{X_{(-i-m)}} K^M_{-i-2m}(x). \end{array}$$

Let $Z \in z_{m-c}(Y, -i-2m)$ be a $(-i-m-c)$-dimensional cycle in $Y \times \Delta^{-i-2m}$. f^* sends Z to the cycle-theoretic pullback

$$[(f \times 1)^{-1}(Z)] = \sum \mathrm{mult}_{Z'} \cdot Z' \in z_m(X, -i-2m),$$

where Z' are the irreducible components of $(f\times 1)^{-1}(Z)$ with reduced schematic structure, $\mathrm{mult}_{Z'} := \mathrm{lgth}_{\mathcal{O}_{(f\times 1)^{-1}(Z),Z'}}(\mathcal{O}_{(f\times 1)^{-1}(Z),Z'})$ is the multiplicity of Z' (lgth denotes the length). Each Z' is of pure dimension $-i-m$.

- When $i \in [-d-m, \min\{-2m,-m\}]$, ψ_X sends the cycle $[(f\times 1)^{-1}(Z)]$ to

$$\sum_{Z'} \mathrm{mult}_{Z'}\cdot \mathrm{Nm}_{k(Z')/k(x(Z'))}\{\frac{-T_0}{T_{-i-2m}},\dots,\frac{-T_{-i-2m-1}}{T_{-i-2m}}\} \in \bigoplus_{X_{(-i-m)}} K^M_{-i-2m}(x(Z')). \tag{6.2.1}$$

  Here Z' runs over all irreducible components of $(f\times 1)^{-1}(Z)$ such that $\dim p_X(Z') = -i-m$, but is equipped with the reduced schematic structure; and for each norm symbol, $x(Z') \in X_{(-i-m)}$ denotes the point that is dominated by Z' via p_X. (6.2.1) is simply zero when there's no such Z' satisfying the dimension condition, according to convention.
- When $i \notin [-d-m, \min\{-2m,-m\}]$, ψ_X sends the cycle $[(f\times 1)^{-1}(Z)]$ to zero.
- When $i \in [-d-m, \min\{-2m,-m-c\}]$ and $\dim p_Y(Z) = -i-m-c$, denote by $y(Z) \in Y_{(-i-m-c)}$ the point dominated by Z via projection p_Y, ψ_Y sends Z to

$$\mathrm{Nm}_{k(Z)/k(y(Z))}\{\frac{-T_0}{T_{-i-2m}},\dots,\frac{-T_{-i-2m-1}}{T_{-i-2m}}\} \in K^M_{-i-2m}(y(Z)).$$

- When $i \notin [-d-m, \min\{-2m,-m-c\}]$ or $\dim p_Y(Z) \neq -i-m-c$, $\psi_Y(Z) = 0$.

So altogether we need to check commutativity in the following cases:

a) when $i \in [-d-m, \min\{-2m,-m-c\}]$, consider the following diagram

$$\begin{array}{ccc} \bigoplus_{Z'} K^M_{-i-2m}(k(Z')) & \xrightarrow{\sum \mathrm{mult}_{Z'}\cdot \mathrm{Nm}_{k(Z')/k(x(Z'))}} & \bigoplus_{x(Z')} K^M_{-i-2m}(k(x(Z'))) \\ {\scriptstyle (f\times 1)^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ K^M_{-i-2m}(k(Z)) & \xrightarrow{\mathrm{Nm}_{k(Z)/k(y(Z))}} & K^M_{-i-2m}(k(y(Z))). \end{array}$$

Here Z' runs over all irreducible components of $(f\times 1)^{-1}Z$ with reduced schematic structure. Z' is automatically of dimension $-i-m$. And $x(Z')$ runs over all the dimension $-i-m$ points of X which is dominated by some Z'. This is commutative because of [Ros96, R1c]. In this case, $f^*(\psi_Y(Z))$ equals the image of

$$\{\frac{-T_0}{T_{-i-2m}},\dots,\frac{-T_{-i-2m-1}}{T_{-i-2m}}\} \tag{6.2.2}$$

under the bottom-right composite, and $\psi_X([(f\times 1)^{-1}(Z)])$ equals the image of same element (6.2.2) under the left-top composite, and thus they are equal.

b) When $i \notin [-d-m, \min\{-2m,-m-c\}]$, both maps send Z to zero. The commutativity trivially holds.

□

6.3. $\overline{\zeta}_{n,X,log,t}\circ\overline{\psi}_{X,t} : \mathbb{Z}^c_{X,t}/p^n \xrightarrow{\simeq} K_{n,X,log,t}$ **is a quasi-isomorphism.** In [Zho14, 2.16] Zhong proved: The map $\psi_{X,\text{ét}}(m)$ defined above is a map of complexes, and combined with the Bloch-Gabber-Kato isomorphism, it induces a quasi-isomorphism of complexes by modding out p^n in the étale topology for all m (note that when $m > d$, both complexes are zero complexes):

$$\overline{\psi}_{X,\text{ét}}(m) : \mathbb{Z}^c_{X,\text{ét}}/p^n(m) \xrightarrow{\simeq} \widetilde{\nu}_{n,X,\text{ét}}(m).$$

In the proof, Zhong actually showed that these two complexes of sheaves on each section of the big Zariski site over X are quasi-isomorphic. Therefore by restriction to the Zariski site, we have

$$\overline{\psi}_{X,\text{Zar}}(m) : \mathbb{Z}^c_{X,\text{Zar}}/p^n(m) \xrightarrow{\simeq} \widetilde{\nu}_{n,X,\text{Zar}}(m).$$

Set $m = 0$ and combine with the result in last section Theorem 5.10:

Theorem 6.4. *Let X be a separated scheme of finite type over k with k being a perfect field of positive characteristic p. Then the following composition of chain maps*

$$\overline{\zeta}_{n,X,log,\text{ét}}\circ\overline{\psi}_{X,\text{ét}} : \mathbb{Z}^c_{X,\text{ét}}/p^n \xrightarrow{\simeq} K_{n,X,log,\text{ét}},$$

and when $k = \overline{k}$, the following composition of chain maps

$$\overline{\zeta}_{n,X,log,\text{Zar}}\circ\overline{\psi}_{X,\text{Zar}} : \mathbb{Z}^c_{X,\text{Zar}}/p^n \xrightarrow{\simeq} K_{n,X,log,\text{Zar}},$$

are quasi-isomorphisms.

Remark 6.5. From the construction of the maps $\overline{\zeta}_{n,X,log,t}$ and $\overline{\psi}_{X,t}$, we can describe explicitly their composite map. We write here only the Zariski case, and the étale case is just given by the Zariski version on the small étale site and then doing the étale sheafification.

Let U be a Zariski open subset of X. Let $Z \in (\mathbb{Z}^c_{X,\mathrm{Zar}})^i(U) = z_0(U, -i)$ be a prime cycle.

- When $i \in [-d, 0]$ and $\dim p_U(Z) = -i$, set $q = -i$. Then Z as a cycle of dimension q in $U \times \Delta^q$, is dominant over some $u = u(Z) \in U_{(q)}$ under projection $p_U : U \times \Delta^q \to U$. By slight abuse of notation, we denote by $T_0, \dots, T_q \in k(Z)$ the pullbacks of the corresponding coordinates via $Z \hookrightarrow U \times \Delta^q$. Since Z intersects all faces properly, $T_0, \dots, T_q \in k(Z)^*$. Thus $\{\frac{-T_0}{T_q}, \dots, \frac{-T_{q-1}}{T_q}\} \in K^M_q(k(Z))$ is well-defined. Take the Zariski closure of $\operatorname{Spec} k(Z)$ in $U \times \Delta^q$, and denote it by Z'. Then p_U maps Z' to $\overline{\{u\}}^U = \overline{\{u\}}^X \cap U$. Denote by $i_u : \overline{\{u\}}^X \hookrightarrow X$ the closed immersion, and denote the composition
$$Z' \xrightarrow{p_U} \overline{\{u\}}^U \hookrightarrow \overline{\{u\}}^X \xhookrightarrow{i_u} X$$
by h. h is clearly generically finite, then there exists an open neighborhood V of u in X such that the restriction $h : h^{-1}(V) \to V$ is finite. Then $W_n h : W_n(h^{-1}(V)) \to W_n V$ is also finite. Therefore it makes sense to consider the trace map $\mathrm{Tr}_{W_n h}$ near the generic point of Z'. Similarly, it makes sense to consider the trace map $\mathrm{Tr}_{W_n p_U}$ near the generic point of Z'. Then we calculate
$$\begin{aligned}\zeta_{log}(\psi(Z)) &= (-1)^i \mathrm{Tr}_{W_n i_u}(d\log(\mathrm{Nm}_{k(Z)/k(u(Z))}\{\frac{-T_0}{T_q}, \dots, \frac{-T_{q-1}}{T_q}\})) \\ &= (-1)^i \mathrm{Tr}_{W_n i_u}(\mathrm{Tr}_{W_n p_U} d\log\{\frac{-T_0}{T_q}, \dots, \frac{-T_{q-1}}{T_q}\}) \quad \text{(Lemma 5.3)} \\ &= (-1)^i \mathrm{Tr}_{W_n h}(\frac{T_q dT_0 - T_0 dT_q}{T_0 T_q} \cdots \frac{T_q dT_{q-1} - T_{q-1} dT_q}{T_{q-1} T_q})\end{aligned}$$
Here in the last step we have used the functoriality of the trace map with respect to composition of morphisms Proposition 1.10(4).
- When $i \notin [-d, 0]$ or $\dim p_U(Z) \neq -i$, we have $\zeta_{log}(\psi(Z)) = 0$.

Combining Proposition 6.3 and Proposition 5.8, one arrives at the following proposition.

Proposition 6.6 (Functoriality). *The composition $\overline{\zeta}_{n,X,log,t} \circ \overline{\psi}_{X,t} : \mathbb{Z}^c_{X,t}/p^n \xrightarrow{\simeq} K_{n,X,log,t}$ is covariant with respect to proper morphisms, and contravariant with respect to étale morphisms for both $t = \mathrm{Zar}$ and $t = \text{ét}$.*

Part 3. Applications

7. De Rham-Witt analysis of $\widetilde{\nu}_{n,X,t}$ and $K_{n,X,log,t}$

Let X be a separated scheme of finite type over k of dimension d. In this section we will use terminologies as defined in [CR12, §1], such as Witt residual complexes, etc.

Recall that Ekedahl defined a map of complexes of $W_n\mathcal{O}_X$-modules (cf. [CR12, Def. 1.8.3])
$$\underline{p} := \underline{p}_{\{K_{n,X}\}_n} : R_* K_{n-1,X,t} \to K_{n,X,t}.$$
Recall that by abuse of notation, we denote by $R : W_{n-1}X \hookrightarrow W_n X$ the closed immersion induced by the restriction map on the structure sheaves $R : W_n\mathcal{O}_X \to W_{n-1}\mathcal{O}_X$.

Lemma 7.1. *The map $\underline{p} : R_* K_{n-1,X,t} \to K_{n,X,t}$ induces a map of complexes of abelian sheaves*
$$\underline{p} : K_{n-1,X,log,t} \to K_{n,X,log,t} \tag{7.0.1}$$
by applying $\underline{p}$ on each summand.

Proof. It suffices to show that $C'_t : K_{n,X,t} \to K_{n,X,t}$ commutes with $\underline{p}$ for both $t = \text{ét}$ and $t = \mathrm{Zar}$. For $t = \text{ét}$, $C'_{\text{ét}}$ is the composition of $\tau^{-1} : K_{n,X,\text{ét}} \to (W_n F_X)_* K_{n,X,\text{ét}}$ and $\epsilon^*(C'_{\mathrm{Zar}}) : (W_n F_X)_* K_{n,X,\text{ét}} \to K_{n,X,\text{ét}}$. With the help of Lemma 1.37(3), we know that
$$\begin{array}{ccc} R_* K_{n-1,X,\text{ét}} & \xrightarrow{\tau^{-1}} & (W_n F_X)_* R_* K_{n-1,X,\text{ét}} \\ \downarrow{\underline{p}} & & \downarrow{\underline{p}} \\ K_{n,X,\text{ét}} & \xrightarrow{\tau^{-1}} & (W_n F_X)_* K_{n,X,\text{ét}} \end{array}$$

is commutative, thus it suffices to prove the proposition for $t = \mathrm{Zar}$. That is, it suffices to show the diagrams (7.0.2) and (7.0.3) commute:

(7.0.2)
$$\begin{array}{ccc} R_*K_{n-1,X} & \xrightarrow[\simeq]{R_*(1.2.2)} & R_*(W_{n-1}F_X)^\triangle K_{n-1,X} \\ \downarrow{\scriptstyle \underline{p}} & & \downarrow{\scriptstyle \underline{p}_{\{(W_nF_X)^\triangle K_{n,X}\}_n}} \\ K_{n,X} & \xrightarrow[\simeq]{(1.2.2)} & (W_nF_X)^\triangle K_{n,X}, \end{array}$$

(7.0.3)
$$\begin{array}{ccccc} (W_nF_X)_*R_*(W_{n-1}F_X)^\triangle K_{n-1,X} & \xrightarrow{\simeq} & R_*(W_{n-1}F_X)_*(W_{n-1}F_X)^\triangle K_{n-1,X} & \xrightarrow{R_*\,\mathrm{Tr}_{W_{n-1}F_X}} & R_*K_{n-1,X} \\ \downarrow{\scriptstyle (W_nF_X)_*\underline{p}_{\{(W_nF_X)^\triangle K_{n,X}\}_n}} & & & & \downarrow{\scriptstyle \underline{p}} \\ (W_nF_X)_*(W_nF_X)^\triangle K_{n,X} & & \xrightarrow{\mathrm{Tr}_{W_nF_X}} & & K_{n,X}. \end{array}$$

Here $\underline{p} := \underline{p}_{\{K_{n,X}\}_n}$ is the lift-and-multiplication-by-p map associated to the Witt residual complex $\{K_{n,X}\}_n$, while $\underline{p}_{\{(W_nF_X)^\triangle K_{n,X}\}_n}$ denotes the one associated to Witt residual system $\{(W_nF_X)^\triangle K_{n,X}\}_n$ (cf. [CR12, 1.8.7]). By definition,

$$\underline{p}_{\{(W_nF_X)^\triangle K_{n,X}\}_n} : R_*(W_{n-1}F_X)^\triangle K_{n-1,X} \to (W_nF_X)^\triangle K_{n,X}$$

is given by the adjunction map of

$$(W_{n-1}F_X)^\triangle K_{n-1,X} \xrightarrow[\simeq]{(W_{n-1}F_X)^\triangle({}^a\underline{p})} (W_{n-1}F_X)^\triangle R^\triangle K_{n,X} \simeq R^\triangle(W_nF_X)^\triangle K_{n,X},$$

where ${}^a\underline{p}$ is the adjunction of $\underline{p}$ for residual complexes (cf. [CR12, Def. 1.8.3]). The second diagram (7.0.3) commutes because the trace map $\mathrm{Tr}_{W_nF_X}$ induces a well-defined map between the Witt residual complexes [CR12, Lemma 1.8.9].

It remains to show the commutativity of the diagram (7.0.2). According to the definition of $\underline{p}_{(W_nF_X)^\triangle K_{n,X}}$ in [CR12, 1.8.7], we are reduced to show the adjunction square commutes:

$$\begin{array}{ccccc} R^\triangle K_{n,X} & \xrightarrow{R^\triangle(1.2.2)} & R^\triangle(W_nF_X)^\triangle K_{n,X} & \xrightarrow{\simeq} & (W_{n-1}F_X)^\triangle R^\triangle K_{n,X} \\ \uparrow{\scriptstyle {}^a\underline{p}} & & & & \uparrow{\scriptstyle (W_{n-1}F_X)^\triangle({}^a\underline{p})} \\ K_{n-1,X} & & \xrightarrow{(1.2.2)} & & (W_{n-1}F_X)^\triangle K_{n-1,X}. \end{array}$$

And this is $(W_{n-1}\pi)^\triangle$ applied to the following diagram

$$\begin{array}{ccccc} R^\triangle W_nk & \xrightarrow{R^\triangle(1.2.1)} & R^\triangle(W_nF_k)^\triangle W_nk & \xrightarrow{\simeq} & (W_{n-1}F_k)^\triangle R^\triangle W_nk \\ \uparrow{\scriptstyle {}^a\underline{p}} & & & & \uparrow{\scriptstyle (W_{n-1}F_k)^\triangle({}^a\underline{p})} \\ W_{n-1}k & & \xrightarrow{(1.2.1)} & & (W_{n-1}F_k)^\triangle W_{n-1}k. \end{array}$$

We are reduced to show its commutativity. Notice that this diagram is over $\operatorname{Spec} W_{n-1}k$, where the only possible filtration is the one-element set consisting of the unique point of $\operatorname{Spec} W_{n-1}k$. This means that the Cousin functor associated to this filtration sends any dualizing complex to itself, and the map ${}^a\underline{p}$ in the sense of a map either between residual complexes [CR12, Def. 1.8.3] or between dualizing complexes [CR12, Def. 1.6.3] actually agree.

Now we start the computation. Formulas for (1.2.1) and for ${}^a\underline{p}$ (in the sense of a map between dualizing complexes) are explicitly given in Section 1.2 and [CR12, 1.6.4(1)], respectively. To make things clear, we label the source and target of W_nF_k by $\operatorname{Spec} W_nk_1$ and $\operatorname{Spec} k_2$ respectively, as we did in the beginning of Section 1.2. Take $a \in W_{n-1}k_1$. Denote $\overline{W_nF_k} : (\operatorname{Spec} W_nk_1, W_nk_1) \to (\operatorname{Spec} W_nk_2, (W_nF_k)_*(W_nk_1))$, and $\overline{R} : (\operatorname{Spec} W_{n-1}k_i, W_{n-1}k_i) \to (\operatorname{Spec} W_nk_i, R_*W_{n-1}k_i)$ $(i = 1, 2)$ the natural maps of ringed spaces. Now the down-right composition $((W_{n-1}F_k)^\triangle({}^a\underline{p})) \circ (1.2.1)$ equals to the Cousin functor $E_{(W_{n-1}F_k)^\triangle R^\triangle Z^\bullet(W_nk)}$ applied to the following composition

$$W_{n-1}k_1 \xrightarrow[\simeq]{(1.2.1)} \overline{W_{n-1}F_k}^*\,\mathrm{Hom}_{W_{n-1}k_2}((W_{n-1}F_k)_*(W_{n-1}k_1), W_{n-1}k_2)$$

$$\xrightarrow[\simeq]{^a\underline{p}} \overline{W_{n-1}F_k}^* \operatorname{Hom}_{W_{n-1}k_2}((W_{n-1}F_k)_*(W_{n-1}k_1), \overline{R}^* \operatorname{Hom}_{W_nk_2}(R_*W_{n-1}k_2, W_nk_2)),$$
$$a \mapsto [(W_{n-1}F_k)_*1 \mapsto (W_{n-1}F_k)^{-1}(a)]$$
$$\mapsto [(W_{n-1}F_k)_*1 \mapsto [R_*1 \mapsto \underline{p}(W_{n-1}F_k)^{-1}(a)]].$$

And $R^\triangle$(1.2.1) $\circ$ $(^a\underline{p})$ equals to the Cousin functor $E_{(W_{n-1}F_k)^\triangle R^\triangle Z^\bullet(W_nk)}$ applied the following composition

$$W_{n-1}k_1 \xrightarrow[\simeq]{^a\underline{p}} \overline{R}^* \operatorname{Hom}_{W_nk_1}(R_*W_{n-1}k_1, W_nk_1)$$
$$\xrightarrow{(1.2.1)} \overline{R}^* \operatorname{Hom}_{W_nk_1}(R_*W_{n-1}k_1, \overline{W_nF_k}^* \operatorname{Hom}_{W_nk_2}((W_nF_k)_*(W_nk_1), W_nk_2)),$$
$$a \mapsto [R_*1 \mapsto \underline{p}(a)]$$
$$\mapsto [R_*1 \mapsto [(W_nF_k)_*1 \mapsto (W_nF_k)^{-1}\underline{p}(a)]].$$

It remains to identify $\underline{p}((W_{n-1}F_k)^{-1}a)$ and $(W_nF_k)^{-1}\underline{p}(a)$. And this is straightforward: write $a = \sum_{i=0}^{n-2} V^i[a_i] \in W_{n-1}k_1$,

$$(W_nF_k)^{-1}\underline{p}(a) = \sum_{i=0}^{n-2}(W_nF_k)^{-1}\underline{p}(V^i[a_i]) = \sum_{i=0}^{n-2}(W_nF_k)^{-1}(V^{i+1}[a_i^p])$$
$$= \sum_{i=0}^{n-2}(V^{i+1}[a_i]) = \underline{p}\sum_{i=0}^{n-2}(V^i[a_i^{1/p}]) = \underline{p}((W_{n-1}F_k)^{-1}a). \tag{7.0.4}$$

Hence we finish the proof. □

However we don't naturally have a restriction map R between residual complexes. Nevertheless, we could use the quasi-isomorphism $\overline{\zeta}_{n,X,log,t} : \widetilde{\nu}_{n,X,t} \xrightarrow{\simeq} K_{n,X,log,t}$ to build up a map

$$R : K_{n,X,log,t} \to K_{n-1,X,log,t} \tag{7.0.5}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$. For this we will need to show that $\underline{p}$ and R induce chain maps for $\widetilde{\nu}_{n,X,t}$. This should be well-known to experts, we add here again due to a lack of reference.

Lemma 7.2.

$$\underline{p} : \widetilde{\nu}_{n,X,t} \to \widetilde{\nu}_{n+1,X,t}, \quad R : \widetilde{\nu}_{n+1,X,t} \to \widetilde{\nu}_{n,X,t}$$

given by $\underline{p}$ and R termwise, are well defined maps of complexes for both $t = \mathrm{Zar}$ and $t = \text{ét}$.

Proof. It suffices to prove for $t = \mathrm{Zar}$. Let $x \in X_{(q)}$ be a point of dimension q. Let $\rho : X' \to \overline{\{x\}}$ be the normalization of $\overline{\{x\}}$. Let x' be the generic point of X' and $y' \in X'^{(1)}$ be a codimension 1 point. Denote $y := \rho(y')$. It suffices to check the commutativity of the following diagrams in (1) and (2).

(1) Firstly,

$$\begin{array}{ccc} W_n\Omega^q_{x',log} & \xrightarrow{\partial} & W_n\Omega^{q-1}_{y',log} \\ \downarrow \underline{p} & & \downarrow \underline{p} \\ W_{n+1}\Omega^q_{x',log} & \xrightarrow{\partial} & W_{n+1}\Omega^{q-1}_{y',log}, \end{array} \qquad \begin{array}{ccc} W_{n+1}\Omega^q_{x',log} & \xrightarrow{\partial} & W_{n+1}\Omega^{q-1}_{y',log} \\ \downarrow R & & \downarrow R \\ W_n\Omega^q_{x',log} & \xrightarrow{\partial} & W_n\Omega^{q-1}_{y',log}. \end{array}$$

Notice that $p = \underline{p} \circ R$. Suppose π' is a uniformizer of the dvr $\mathcal{O}_{X',y'}$ and $u_1, \ldots, u_q$ are invertible elements in $\mathcal{O}_{X',y'}$. Calculate

$$\begin{aligned} &\underline{p}(\partial(d\log[\pi']_n d\log[u_2]_n \ldots d\log[u_q]_n)) \\ &= \underline{p}(d\log[u_2]_n \ldots d\log[u_q]_n) \\ &= p(d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1}) \\ &= p(\partial(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &= \partial(p(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &= \partial(\underline{p}(d\log[\pi']_n d\log[u_2]_n \ldots d\log[u_q]_n)), \end{aligned}$$

and

$$\begin{aligned} &\underline{p}(\partial(d\log[u_1]_n d\log[u_2]_n \ldots d\log[u_q]_n)) \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
&= p(\partial(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \dots d\log[u_q]_{n+1}))\\
&= \partial(p(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \dots d\log[u_q]_{n+1}))\\
&= \partial(\underline{p}(d\log[u_1]_{n} d\log[u_2]_{n} \dots d\log[u_q]_{n})).
\end{aligned}
$$

This proves the first diagram. Now we prove the commutativity of the second.

$$
\begin{aligned}
&R(\partial(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \dots d\log[u_q]_{n+1}))\\
&\quad = R(d\log[u_2]_{n+1} \dots d\log[u_q]_{n+1})\\
&\quad = d\log[u_2]_{n} \dots d\log[u_q]_{n}\\
&\quad = \partial(d\log[\pi']_{n} d\log V[u_2]_{n} \dots d\log V[u_q]_{n})\\
&\quad = \partial(R(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \dots d\log[u_q]_{n+1})),
\end{aligned}
$$

and

$$
\begin{aligned}
&R(\partial(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \dots d\log[u_q]_{n+1}))\\
&\quad = 0\\
&\quad = \partial(d\log[u_1]_{n} d\log[u_2]_{n} \dots d\log[u_q]_{n})\\
&\quad = \partial(R(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \dots d\log[u_q]_{n+1})).
\end{aligned}
$$

(2) Secondly,

$$
\begin{array}{ccc}
W_n\Omega^{q-1}_{y',log} & \xrightarrow{\mathrm{tr}} & W_n\Omega^{q-1}_{y,log}\\
\downarrow{\scriptstyle \underline{p}} & & \downarrow{\scriptstyle \underline{p}}\\
W_{n+1}\Omega^{q-1}_{y',log} & \xrightarrow{\mathrm{tr}} & W_{n+1}\Omega^{q-1}_{y,log},
\end{array}
\qquad
\begin{array}{ccc}
W_{n+1}\Omega^{q-1}_{y',log} & \xrightarrow{\mathrm{tr}} & W_{n+1}\Omega^{q-1}_{y,log}\\
\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R}\\
W_{n}\Omega^{q-1}_{y',log} & \xrightarrow{\mathrm{tr}} & W_{n}\Omega^{q-1}_{y,log}.
\end{array}
$$

Notice that $\rho : X' \to \overline{\{x\}}^X$ can be restricted to a map from $\overline{\{y'\}}^{X'}$ to $\overline{\{y\}}^X$ ($\overline{\{x\}}^X$ denotes the closure of x in X, and similarly for $\overline{\{y'\}}^{X'}$, $\overline{\{y\}}^X$). Furthermore, y' (resp. y) belongs to the smooth locus of $\overline{\{y'\}}^{X'}$ (resp. $\overline{\{y\}}^X$), and there $\underline{p}$ and R come from the restriction of the $\underline{p}$ and R on the respective smooth locus. The map tr, induced by Milnor's norm map, agrees with the Grothendieck trace map $\mathrm{Tr}_{W_n\rho}$ due to Lemma 5.3. And according to compatibility of the Grothendieck trace map with the Witt system structure (i.e. de Rham-Witt structure with zero differential) on canonical sheaves [CR12, 4.1.4(6)], we arrive at the desired commutativity.

□

Lemma 7.3. *Assume either*

- $t = \mathrm{Zar}$ *and* $k = \overline{k}$, *or*
- $t = \text{ét}$.

Then we have the following short exact sequence

(7.0.6) $$0 \to \widetilde{\nu}_{i,X,t} \xrightarrow{\underline{p}^j} \widetilde{\nu}_{i+j,X,t} \xrightarrow{R^i} \widetilde{\nu}_{j,X,t} \to 0$$

and distinguished triangles

(7.0.7) $$K_{i,X,log,t} \xrightarrow{\underline{p}^j} K_{i+j,X,log,t} \xrightarrow{R^i} K_{j,X,log,t} \xrightarrow{+1}$$

in the derived category $D^b(X_t, \mathbb{Z}/p^n)$.

Proof. (1) Because of Lemma 7.2, it suffices to show

$$0 \to W_i\Omega^q_{x,log,t} \xrightarrow{\underline{p}^j} W_{i+j}\Omega^q_{x,log,t} \xrightarrow{R^i} W_j\Omega^q_{x,log,t} \to 0$$

is short exact for any given point $x \in X_{(q)}$. And this is true for $t = \text{ét}$ because of [CSS83, Lemme 3]. And for $t = \mathrm{Zar}$, one further needs $R^1\epsilon_* W_n\Omega^q_{x,log,\text{ét}} = 0$ for any $x \in X_{(q)}$ when $k = \overline{k}$, which is proved in [Suw95, Cor. 2.3].

(2) Now it suffices to show that $\underline{p}$ and R for the system $\{K_{n,X,log,t}\}_n$ are compatible with $\underline{p}$ and R of the system $\{\widetilde{\nu}_{n,X,t}\}_n$, via the quasi-isomorphism $\overline{\zeta}_{n,X,log,t}$. The compatibility for R is clear by definition. It remains to check the compatibility for $\underline{p}$. Because $\overline{\zeta}_{n,X,log,t} = (5.3.1) \circ \overline{\zeta}_{n,X,t}$, it

suffices to check compatibility of $\underline{p}: \widetilde{\nu}_{n-1,X,t} \to \widetilde{\nu}_{n,X,t}$ with $\underline{p}: K_{n-1,X,t} \to K_{n,X,t}$ via $\overline{\zeta}_{n,X,t}$. At a given degree $-q$ and given point $x \in X_{(q)}$, the map $\overline{\zeta}_{n,X,t}: \widetilde{\nu}_{n,X,t} \to K_{n,X,t}$ factors as

$$(W_n\iota_x)_* W_n\Omega^q_{x,log,t} \to (W_n i_x)_* W_n\Omega^q_{x,t} = (W_n i_x)_* K^q_{n,\overline{\{x\}},t} \xrightarrow{(-1)^q \operatorname{Tr}_{W_n i_x,t}} K^q_{n,X,t}.$$

The first arrow is the inclusion map. And compatibility of the $\underline{p}$ via the trace map is given in [CR12, Lemma 1.8.9].

□

8. Higher Chow groups of zero cycles

Let X be a separated scheme of finite type over k of dimension d.

8.1. Vanishing and finiteness results.

Proposition 8.1. *There is a distinguished triangle*

$$\mathbb{Z}^c_{X,\text{ét}}/p^n \to K_{n,X,\text{ét}} \xrightarrow{C'_{\text{ét}}-1} K_{n,X,\text{ét}} \xrightarrow{+1}$$

in the derived category $D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$. When $k = \overline{k}$, one also has the Zariski counterpart. Namely, we have a distinguished triangle

(8.1.1) $$\mathbb{Z}^c_X/p^n \to K_{n,X} \xrightarrow{C'-1} K_{n,X} \xrightarrow{+1}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$.

In particular, when $k = \overline{k}$ and X is Cohen-Macaulay of pure dimension d, then $\mathbb{Z}^c_X/p^n$ is concentrated at degree $-d$, and the triangle (8.1.1) becomes

$$\mathbb{Z}^c_X/p^n \to W_n\omega_X[d] \xrightarrow{C'-1} W_n\omega_X[d] \xrightarrow{+1}$$

in this case. Here $W_n\omega_X$ is the only non-vanishing cohomology sheaf of $K_{n,X}$ (when $n = 1$, $W_1\omega_X = \omega_X$ is the usual dualizing sheaf on X).

Proof. This is direct from the main result Theorem 6.4 and Remark 1.27. □

Proposition 8.2. *Assume $k = \overline{k}$. Then higher Chow groups of zero cycles equals the C'-invariant part of the cohomology groups of Grothendieck's coherent dualizing complex, i.e.,*

$$\mathrm{CH}_0(X, q; \mathbb{Z}/p^n) = H^{-q}(W_nX, K_{n,X})^{C'-1},$$

$$R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = H^{-q}(W_nX, K_{n,X,\text{ét}})^{C'_{\text{ét}}-1}.$$

Proof. This follows directly from the Proposition 1.24 and Proposition 1.39 and the main result Theorem 6.4. □

Corollary 8.3 (Vanishing)**.** *Suppose X is affine and Cohen-Macaulay of pure dimension d. Then*

(1) When $t = \mathrm{Zar}$ and $k = \overline{k}$,

$$\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = 0$$

for $q \neq d$.

(2) When $t = \text{ét}$,

$$R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = 0$$

for $q \neq d, d-1$. If we assume furthermore $k = \overline{k}$ or smoothness, then we also have

$$R^{-d+1}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = 0.$$

Proof. When X is Cohen-Macaulay of pure dimension d, W_nX is also Cohen-Macaulay of pure dimension d, and $K_{n,X,t}$ is concentrated at degree $-d$ for all n [Con00, 3.5.1]. Now Serre's affine vanishing theorem implies $H^{-q}(W_nX, K_{n,X,t}) = 0$ for $q \neq d$. This implies that $R^{-q}\Gamma(W_nX, K_{n,X,log,t}) = 0$ unless $q = d, d-1$. With the given assumptions, Theorem 6.4 implies that $\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = 0$ unless $q = d, d-1$. If one also assumes $k = \overline{k}$, Proposition 8.2 gives the vanishing result for $q = d-1$.

When X is smooth, $C_{\text{ét}} - 1 : W_n\Omega^d_{X,\text{ét}} \to W_n\Omega^d_{X,\text{ét}}$ is surjective by [GS88a, 1.6(ii)] (see (1.3.31)). By compatibility of $C_{\text{ét}}$ and $C'_{\text{ét}}$ Proposition 1.38, one deduces that $C' - 1 : \mathcal{H}^{-d}(K_{n,X,\text{ét}}) \to \mathcal{H}^{-d}(K_{n,X,\text{ét}})$ is surjective. □

Generalizing Bass's finiteness conjecture for K-groups (cf. [Wei13, IV 6.8]), the finiteness of higher Chow groups in various arithmetic settings has been a "folklore conjecture" in literature (expression taken from [KS12, §9]). The following result was first proved by Geisser [Gei10, §5, eq. (12)] using the finiteness result from étale cohomology theory, and here we deduce it as a corollary of our main theorem, which essentially relies on the finiteness of coherent cohomologies on a proper scheme. We remark that Geisser's result is more general than ours in that he allows arbitrary torsion coefficients.

Corollary 8.4 (Finiteness, Geisser). *Assume $k = \overline{k}$. Let X be proper over k. Then for any q,*

$$\mathrm{CH}_0(X, q; \mathbb{Z}/p^n) \quad \text{and} \quad R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_{X,\text{ét}}/p^n)$$

are finite $\mathbb{Z}/p^n$-modules.

Proof. According to Theorem 6.4, $R^{-q}\Gamma(X_t, \mathbb{Z}^c_{X,t}/p^n) = R^{-q}\Gamma(X_t, K_{n,X,log,t})$ for $t = \text{Zar}$ and $t = \text{ét}$. Thus it suffices to show that for every i, $R^i\Gamma(X_t, K_{n,X,log,t})$ is a finite $\mathbb{Z}/p^n$-module. First of all, since $R^i\Gamma(X_t, K_{n,X,log,t})$ is the C'_t-invariant part of $R^i\Gamma(X_t, K_{n,X,t})$ by Proposition 1.24 and Proposition 1.39, $R^i\Gamma(X_t, K_{n,X,log,t})$ is a module over the invariant ring $(W_nk)^{1-W_nF_X^{-1}} = \mathbb{Z}/p^n$. Because X is proper, $R^i\Gamma(X_t, K_{n,X,t})$ is a finite W_nk-module by the local-to-global spectral sequence. Then Proposition A.16 gives us the result.

Alternatively, we can also do induction on n. In the $n = 1$ case, because $R^i\Gamma(X_t, K_{X,log,t})$ is the C'_t-invariant part of the finite dimensional k-vector space $H^i(X, K_{X,t})$ again by Proposition 1.24 and Proposition 1.39, it is a finite $\mathbf{F}_p$-module by p^{-1}-linear algebra Proposition A.12. The desired result then follows from the long exact sequence associated to (7.0.7) by induction on n. □

8.2. **Étale descent.** The results Proposition 8.5, Proposition 8.6 in this subsection are well-known to experts.

Proposition 8.5 (Gros-Suwa). *Assume $k = \overline{k}$. Then one has a canonical isomorphism*

$$\widetilde{\nu}_{n,X,\text{Zar}} = \epsilon_*\widetilde{\nu}_{n,X,\text{ét}} \xrightarrow{\simeq} R\epsilon_*\widetilde{\nu}_{n,X,\text{ét}}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$.

Proof. When $k = \overline{k}$, terms of the étale complex $\widetilde{\nu}_{n,X,\text{ét}}$ are ϵ_*-acyclic according to [GS88a, 3.16]. □

The étale descent of Bloch's cycle complex with $\mathbb{Z}$-coefficients is shown in [Gei10, Thm 3.1], assuming the Beilinson-Lichtenbaum conjecture. Looking into the proof one sees that the mod p^n version holds conjecture-free, and is a corollary of [GL00, 8.4] (we thank Geisser for pointing this out) via an argument of Thomason [Tho85, 2.8]. But one could also deduce this as a corollary of Proposition 8.5 via Zhong's quasi-isomorphism in Section 6.3 (which is again dependent on the main result of Geisser-Levine [GL00, 1.1]).

Proposition 8.6 (Geisser-Levine). *Assume $k = \overline{k}$. Then one has a canonical isomorphism*

$$\mathbb{Z}^c_{X,\text{Zar}}/p^n = \epsilon_*\mathbb{Z}^c_{X,\text{ét}}/p^n \xrightarrow{\simeq} R\epsilon_*\mathbb{Z}^c_{X,\text{ét}}/p^n.$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$.

Proof. Clearly, we have the compatibility

$$\begin{array}{ccc} \mathbb{Z}^c_{X,\text{Zar}}/p^n & \xrightarrow{\simeq} & R\epsilon_*\mathbb{Z}^c_{X,\text{ét}}/p^n \\ \downarrow{\scriptstyle \psi_{X,\text{Zar}}} & & \downarrow{\scriptstyle R\epsilon_*\psi_{X,\text{ét}}} \\ \widetilde{\nu}_{n,X,\text{Zar}} & \xrightarrow{\simeq} & R\epsilon_*\widetilde{\nu}_{n,X,\text{ét}}. \end{array}$$

Thus $\mathbb{Z}^c_{X,\text{Zar}}/p^n \xrightarrow[\simeq]{\overline{\psi}_{X,\text{Zar}}} \widetilde{\nu}_{n,X,\text{Zar}} = \epsilon_*\widetilde{\nu}_{n,X,\text{ét}} \xrightarrow[\simeq]{\text{Proposition 8.5}} R\epsilon_*\widetilde{\nu}_{n,X,\text{ét}} \xrightarrow[\simeq]{R\epsilon_*\overline{\psi}_{X,\text{ét}}} R\epsilon_*\mathbb{Z}^c_{X,\text{ét}}/p^n$. □

Corollary 8.7. *Assume $k = \overline{k}$. Suppose X is affine and Cohen-Macaulay of pure dimension d. Then*

$$R^i\epsilon_*(\mathbb{Z}^c_{X,\text{ét}}/p^n) = R^i\epsilon_*\widetilde{\nu}_{n,X,\text{ét}} = 0, \quad i \neq -d.$$

Proof. This is a direct consequence of Proposition 8.6, Proposition 8.5 and Corollary 8.3. □

8.3. **Birational geometry and rational singularities.** Recall the following definition of *resolution-rational singularities*, which are more often called *rational singularities* before in the literature, but here we follow the terminology from [Kov17] (see also Remark 8.10(1)).

Definition 8.8 (cf. [Kov17, p. 9.1])**.** An integral k-scheme X is said to have *resolution-rational singularities*, if

(1) there exists a birational proper morphism $f : \widetilde{X} \to X$ with $\widetilde{X}$ smooth (such a f is called a *resolution of singularities* or simply a *resolution* of X), and
(2) $R^i f_* \mathcal{O}_{\widetilde{X}} = R^i f_* \omega_{\widetilde{X}} = 0$ for $i \geq 1$. And $f_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$.

Such a map $f : \widetilde{X} \to X$ is called a *rational resolution* of X.

Note that the cohomological condition (2) is equivalent to the following condition

(2') $\mathcal{O}_X \simeq Rf_* \mathcal{O}_{\widetilde{X}}$, $f_* \omega_{\widetilde{X}} \simeq Rf_* \omega_{\widetilde{X}}$ in the derived category of abelian Zariski sheaves.

A necessary condition for an integral scheme to have such singularities is being Cohen-Macaulay. This is well-known, but we write it again here for the convenience of the reader.

Lemma 8.9 (cf. [KM98, 5.10, 5.12])**.** *Let X be an integral k-scheme of pure dimension d admitting a Macaulayfication $f : \widetilde{X} \to X$ (i.e., a proper birational morphism with its source being Cohen-Macaulay, [Kov17, 4.2]). Suppose cohomological condition (2) holds for f. Then X is Cohen-Macaulay, and*

$$f_* \omega_{\widetilde{X}} \simeq \omega_X.$$

If we further assume that $\widetilde{X}$ is normal, then X is also normal.

In particular, an integral equidimensional k-scheme with rational singularities is Cohen-Macaulay.

Proof. Consider the following diagram

$$\begin{array}{ccccccc}
Rf_* K_{\widetilde{X}} & \underset{\simeq}{\xleftarrow{\mathrm{ev}_1}} & Rf_* R\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}, K_{\widetilde{X}}) & \xrightarrow{\simeq} & Rf_* R\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}, f^{\triangle} K_X) & \longrightarrow & R\mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_{\widetilde{X}}, Rf_* f^{\triangle} K_X) \\
\downarrow \mathrm{Tr}_f & & & & & \searrow^{\simeq} & \downarrow \mathrm{Tr}_f \\
 & & & & & & R\mathcal{H}om(f_* \mathcal{O}_{\widetilde{X}}, K_X) \\
 & & & & & & \simeq \downarrow (f^*)^\vee \\
K_X & & & \underset{\simeq}{\xleftarrow{\mathrm{ev}_1}} & & & R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, K_X).
\end{array}$$

For the triangle on the top right corner, the skewed arrow is defined to be the composite of the horizontal and the vertical arrows, and it is by definition the duality morphism and is an isomorphism in the derived category (cf. Proposition 1.10(7)). The map $(f^*)^\vee : R\mathcal{H}om(f_* \mathcal{O}_{\widetilde{X}}, K_X) \to R\mathcal{H}om(\mathcal{O}_X, K_X)$ is given by applying the dualizing functor $R\mathcal{H}om_{\mathcal{O}_X}(-, K_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(-, K_X)$ to the given isomorphism $\mathcal{O}_X \underset{\simeq}{\xrightarrow{f^*}} f_* \mathcal{O}_{\widetilde{X}}$. Note that we have used the cohomological condition that $f_* \mathcal{O}_{\widetilde{X}} \simeq Rf_* \mathcal{O}_{\widetilde{X}}$ in this diagram. The whole diagram is commutative, because if we start from $\alpha \in \mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}, K_{\widetilde{X}}) \simeq R\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}, K_{\widetilde{X}})$, we arrive at $\mathrm{Tr}_f(\alpha(1)) \in K_X$ under both composite maps along the clockwise and the counterclockwise directions.

This being done, we know that the top-right-down composition

$$Rf_* K_{\widetilde{X}} \simeq Rf_* R\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}, K_{\widetilde{X}}) \simeq Rf_* R\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}, f^{\triangle} K_X) \xrightarrow{\simeq} R\mathcal{H}om_{\mathcal{O}_X}(Rf_* \mathcal{O}_{\widetilde{X}}, K_X)$$
$$\simeq \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_{\widetilde{X}}, K_X) \underset{\simeq}{\xrightarrow{(f^*)^\vee}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, K_X) \simeq K_X.$$

is the same as the map on the left, i.e., the trace map Tr_f, in the derived category. This implies that Tr_f is an isomorphism in the derived category. Since $\widetilde{X}$ is Cohen-Macaulay of pure dimension d, $\omega_{\widetilde{X}}[d] \simeq K_{\widetilde{X}}$, thus $Rf_* \omega_{\widetilde{X}}[d] \simeq Rf_* K_{\widetilde{X}} \underset{\simeq}{\xrightarrow{\mathrm{Tr}_f}} K_X$. Together with the given condition $f_* \omega_{\widetilde{X}} \simeq Rf_* \omega_{\widetilde{X}}$, we have $f_* \omega_{\widetilde{X}}[d] \simeq K_X$ via Tr_f. The Cohen-Macaulay part of the lemma then follows from [Con00, 3.5.1].

The normality part of the lemma follows simply from the factorization of f via the normalization morphism of X. ☐

Remark 8.10. (1) According to this lemma, we know that on integral k-schemes of pure dimension, our definitions for resolution-rational singularities and for rational resolutions are the same as the ones in [Kov17, 9.1].

(2) A Macaulayfication of a scheme separated and of finite type over k always exists, cf, [Kov17, 4.3].

(3) According to [Kov17, 9.6], resolution-rational singularities are pseudo-rational. By definition [Kov17, 1.2], a k-scheme X is said to have *pseudo-rational singularities*, if it is normal Cohen-Macaulay, and for every normal scheme X', every projective birational morphism $f : X' \to X$, the composition $f_*\omega_{X'} \to Rf_*\omega_{X'} \xrightarrow{\mathrm{Tr}_f} \omega_X$ is an isomorphism.

Corollary 8.11. *Let X and Y be integral k-schemes of pure dimensions which have pseudo-rational singularities and are properly birational, i.e., there are proper birational k-morphisms $Z \to X$ and $Z \to Y$ with Z being some integral scheme. Then we have*

$$R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_{\widetilde{X},\text{ét}}/p^n) = R^{-q}\Gamma(Y_{\text{ét}}, \mathbb{Z}^c_{Y,\text{ét}}/p^n)$$

for all q and all $n \geq 1$. If we assume furthermore $k = \overline{k}$, we also have

$$\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = \mathrm{CH}_0(Y, q, \mathbb{Z}/p^n)$$

for all q and all $n \geq 1$.

In particular, since for any rational resolution of singularities $f : \widetilde{X} \to X$, $\widetilde{X}$ and X are properly birational as k-schemes (i.e., take Z to be $\widetilde{X}$), one can compute the higher Chow groups of zero cycles of X via those of $\widetilde{X}$.

Proof. Using Chow's Lemma [Kov17, 4.1], we know that there exist projective birational morphisms $f' : Z_1 \to Z$ and $g' : Z_2 \to Z$ such that the compositions $Z_1 \xrightarrow{f'} Z \xrightarrow{f} X$ and $Z_2 \xrightarrow{g'} Z \xrightarrow{g} Y$ are also birational and projective. Let $U \subset Z$ be an open dense subset such that f' and g' restricted to the preimage of U are isomorphisms. Take Z' be the Zariski closure of the image of the diagonal of U in $Z_1 \times_Z Z_2$ with the reduced scheme structure. Then the two projections $Z' \to Z_1$ and $Z' \to Z_2$ are also projective and birational. This means that by replacing Z' with Z, f with $Z' \to X$ and g with $Z' \to Y$, we can assume our $f : Z \to X$, $g : Z \to Y$ to be projective birational and our Z to be integral. Using Macaulayfication [Kov17, 4.3, 4.4] we can additionally assume that Z is Cohen-Macaulay. This implies that f and g are pseudo-rational modifications by [Kov17, 9.7].

Suppose that X is of pure dimension d. Then so is Z. Now [Kov17, 8.6] implies that the trace map of f induces an isomorphism

$$\mathrm{Tr}_f : Rf_*\omega_{Z,t}[d] \xrightarrow{\simeq} \omega_{X,t}[d]$$

in $D^b(X_t, \mathbb{Z}/p)$. Thus

$$\mathrm{Tr}_{f,log} : Rf_*K_{Z,log,t} \xrightarrow{\simeq} K_{X,log,t}$$

is also an isomorphism in $D^b(X_t, \mathbb{Z}/p)$. Consider the diagram

(8.3.1)
$$\begin{array}{ccccccc}
f_*K_{Z,log,t} & \xrightarrow{\underline{p}^{n-1}} & f_*K_{n,Z,log,t} & \xrightarrow{R} & f_*K_{n-1,Z,log,t} & \xrightarrow{+1} & f_*K_{Z,log,t}[1] \\
\downarrow{\scriptstyle \mathrm{Tr}_{f,log}} & & \downarrow{\scriptstyle \mathrm{Tr}_{W_nf,log}} & & \downarrow{\scriptstyle \mathrm{Tr}_{W_{n-1}f,log}} & & \downarrow{\scriptstyle \mathrm{Tr}_{f,log}[1]} \\
K_{X,log,t} & \xrightarrow{\underline{p}^{n-1}} & K_{n,X,log,t} & \xrightarrow{R} & K_{n-1,X,log,t} & \xrightarrow{+1} & K_{X,log,t}[1]
\end{array}$$

in $D^b(X_t, \mathbb{Z}/p)$. The first row is Rf_* applied to the triangle (7.0.7) on Z. The second row is the triangle (7.0.7) on X. The left square commutes on the level of complexes by compatibility of the trace map with $\underline{p}$ [CR12, 1.8.9]. To prove commutativity of the middle square in the derived category, it suffices to show the square

$$\begin{array}{ccc}
f_*\widetilde{\nu}_{n,Z,t} & \xrightarrow{R} & f_*\widetilde{\nu}_{n-1,Z,t} \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\widetilde{\nu}_{n,X,t} & \xrightarrow{R} & \widetilde{\nu}_{n-1,X,t}
\end{array}$$

commutes on the level of complexes. Since the vertical maps f_* for Kato-Moser complexes are tr (cf. §4), which are by definition the reduction of the norm maps for Milnor K-theory, they agree with the Grothendieck trace maps Tr_{W_nf}, $\mathrm{Tr}_{W_{n-1}f}$ by Lemma 5.3. And according to the compatibility of R with the Grothendieck trace maps [CR12, 4.1.4(6)], we arrive at the desired commutativity. The right square in (8.3.1) commutes by naturality of the "+1" map. With all these commutativities we conclude that the vertical maps in (8.3.1) define a map of triangles. By induction on n we deduce that

$$\mathrm{Tr}_{W_nf,log} : Rf_*K_{n,Z,log,t} \xrightarrow{\simeq} K_{n,X,log,t}$$

is an isomorphism in $D^b(X_t, \mathbb{Z}/p^n)$ for every n. The main result Theorem 6.4 thus implies

$$R^{-q}\Gamma(Z_{\text{ét}}, \mathbb{Z}^c_{\widetilde{X},\text{ét}}/p^n) = R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_{X,\text{ét}}/p^n)$$

for all q and n. When $k = \overline{k}$, the same theorem also implies that

$$\mathrm{CH}_0(Z, q, \mathbb{Z}/p^n) = \mathrm{CH}_0(X, q, \mathbb{Z}/p^n)$$

for all q and n.

Now replacing f with g everywhere in the above argument and we get the result.

□

Appendix

A. σ-linear algebra

The author thanks Yun Hao for his notes and careful discussion on this topic.

Definition A.1. Let k be a field, and V a finite dimensional k-vector space. Let $\sigma \in \mathrm{End}(k)$ be a field endomorphism of k (therefore σ being surjective is equivalent to being an automorphism). A *σ-linear operator* or a *σ-linear map* on V is a map $T : V \to V$, such that

$$T(v + w) = T(v) + T(w), \quad T(cv) = \sigma(c)T(v), \quad v, w \in V, c \in k.$$

Notice that this is equivalent to a k-linear map $V \to \sigma_* V$.

Notation A.2. Through out this appendix, we will keep these notations k, σ, T, V without further notice. We don't consider the 0-vector space therefore assume $\dim V > 0$.

In particular, when k is of characteristic p and σ is the p-th power Frobenius F_k, T is called *p-linear*, and when furthermore k is *perfect* and σ is the map F_k^{-1} of taking p-th roots, then T is called *p^{-1}-linear*. Similarly, one could define p^n-linear for $n \in \mathbb{Z}$ (when $n = 0$, T is simply k-linear by assumption).

Remark A.3. Let $\sigma \in \mathrm{End}(k)$. For any σ-linear map T,

(1) the kernel
$$\mathrm{Ker}(T) := \{v \in V \mid Tv = 0\}$$
of T is always a k-vector subspace. But
$$\mathrm{Im}(T) := \{v \in V \mid Tw = v \text{ for some } w \in V\}$$
may not be a k-vector subspace. We denote by $\langle \mathrm{Im}(T)\rangle$ the k-vector subspace generated by $\mathrm{Im}(T)$. When σ is surjective (i.e. $\sigma \in \mathrm{Aut}(k)$ is a field automorphism), then $\mathrm{Im}(T)$ is a k-vector subspace.

(2) Denote by
$$\kappa := k^{1-\sigma} = \{c \in k \mid \sigma(c) = c\}$$
the set of fixed points of k by field endomorphism σ. Then κ is a nonzero subfield of k. The fixed points of T
$$V^{1-T} := \{v \in V \mid T(v) = v\}$$
is naturally a κ-vector space. An element in V^{1-T} are also called a *T-invariant vector.*
$$\mathrm{Im}(1 - T), \quad \mathrm{Coker}(1 - T)$$
are also naturally κ-vector spaces.

Definition A.4. A σ-linear map $T : V \to V$ is

(1) *semi-simple*, if $\langle \mathrm{Im}T\rangle = V$. When σ is surjective, this is equivalent to T being surjective.
(2) *nilpotent* if $V = \mathrm{Ker}(T^N)$ for some $N \in \mathbb{N}$.

Consider chains of k-vector subspaces

$$\mathrm{Ker}\, T \subset \mathrm{Ker}\, T^2 \subset \cdots \subset \mathrm{Ker}\, T^n \subset \dots,$$
$$\langle \mathrm{Im}T\rangle \supset \langle \mathrm{Im}T^2\rangle \supset \cdots \supset \langle \mathrm{Im}T^n\rangle \supset \dots.$$

Since V is finite dimensional, both of them become stationary for some large $N \in \mathbb{N}$. Define

$$V_{\text{nil}} := \bigcup_{n\geq 1} \mathrm{Ker}(T^n) = \mathrm{Ker}(T^N) = \mathrm{Ker}(T^{N+1}) = \dots,$$

$$V_{\text{ss}} := \bigcap_{n\geq 1} \langle \mathrm{Im}(T^n)\rangle = \langle \mathrm{Im}(T^N)\rangle = \langle \mathrm{Im}(T^{N+1})\rangle = \dots.$$

Obviously,

(1) V_{nil} and V_{ss} are k-vector subspaces of V that is stable under T.
(2) T is nilpotent on V_{nil}. T is injective if and only if $V_{\mathrm{nil}} = 0$. $1 - T$ is invertible on V_{nil}, with inverse $1 + T + \cdots + T^{N-1}$ where N is the smallest number with $T^N = 0$ on V_{nil}.
(3) T is semi-stable on V_{ss}: because $\langle T(V_{\mathrm{ss}})\rangle = \langle T(\langle \mathrm{Im}(T^N)\rangle)\rangle = \langle T(\mathrm{Im}(T^N))\rangle = \langle \mathrm{Im}(T^{N+1})\rangle = V_{\mathrm{ss}}$.
(4) $V^{1-T} \subset V_{\mathrm{ss}}$.

Lemma A.5. $V_{\mathrm{nil}} \cap V_{\mathrm{ss}} = 0$.

In particular, T is always injective on V_{ss}.

Proof. Since $\mathrm{Im}T^{2N}$ generates V_{ss} as a k-vector space, one can find a k-basis of V_{ss} consisting of elements in $\mathrm{Im}T^{2N}$. Suppose

$$T^{2N}(v_1), \dots, T^{2N}(v_r)$$

is such a basis ($r = \dim V_{\mathrm{ss}}$), with $v_1, \dots, v_r \in V$. Obviously,

$$T^N(v_1), \dots, T^N(v_r)$$

are k-linearly independent: otherwise applying T^N to their linear relation will give a linear relation for $T^{2N}(v_1), \dots, T^{2N}(v_r)$ (notice that σ as a field endomorphism is always injective). Now take $v = \sum a_i T^N(v_i) \in V_{\mathrm{ss}}$, with $a_i \in k$. If $v \in V_{\mathrm{nil}}$, then $0 = T^N(v) = \sum \sigma^N(a_i) T^{2N}(v_i)$ implies $\sigma^N(a_i) = 0$ for all i, which implies $a_i = 0$ for all i (again because σ is injective). That is, $v = 0$. □

Proposition A.6 (Fitting decomposition)**.** *Suppose $\sigma \in \mathrm{Aut}(k)$. Then V admits a decomposition of k-vector spaces*

$$V = V_{\mathrm{nil}} \oplus V_{\mathrm{ss}},$$

such that

(1) the k-vector subspaces V_{nil} and V_{ss} are stable under T.
(2) T is nilpotent on V_{nil}, and semi-simple and bijective on V_{ss}.

Proof. Because σ is surjective, one has $\mathrm{Im}T^n = \langle \mathrm{Im}T^n\rangle$. As in the discussion above, the k-vector subspaces V_{nil} and V_{ss} satisfy both conditions. (In this case $V_{\mathrm{ss}} = \mathrm{Im}(T^N) = \mathrm{Im}(T^{N+1}) = \dots$ and therefore T is surjective on V_{ss}.) Together with Lemma A.5, it remains to show $V = V_{\mathrm{nil}} + V_{\mathrm{ss}}$.

Take $v \in V$. Then $T^N(v) \in \mathrm{Im}T^N = \mathrm{Im}T^{2N} = V_{\mathrm{ss}}$. So there is some w such that $T^N(v) = T^{2N}(w)$, i.e. $T^N(v - T^N(w)) = 0$. So $v - T^N(w) \in \mathrm{Ker}\, T^N = V_{\mathrm{nil}}$. In other words, $v = (v - T^N(w)) + T^N(w) \in V_{\mathrm{nil}} + V_{\mathrm{ss}}$. This implies $V = V_{\mathrm{nil}} + V_{\mathrm{ss}}$. □

Proposition A.7 (Change of basis)**.** *Let $(e_1, \dots, e_d)$ be a k-basis for V. Let $(e'_1, \dots, e'_d)$ be another basis, such that*

$$(e'_1, \dots, e'_d) = (e_1, \dots, e_d) \cdot \mathbf{P}$$

with $\mathbf{P} \in \mathrm{GL}_d(k)$ is an invertible matrix. If T has matrix representation $\mathbf{T}$ with respect to (e_i), i.e.

$$T(e_1, \dots, e_d) = (e_1, \dots, e_d) \cdot \mathbf{T},$$

then the matrix representation $\mathbf{T}'$ of T with respect to (e'_i) is

$$\mathbf{T}' = \mathbf{P}^{-1}\mathbf{T}\mathbf{P}^{\sigma},$$

where $\mathbf{P}^{\sigma}$ is the matrix obtained by applying σ to each entry of $\mathbf{P}$.

Proof. This is direct:

$$T(e'_1, \dots, e'_d) = T((e_1, \dots, e_d) \cdot \mathbf{P}) = (e_1, \dots, e_d) \cdot \mathbf{T}\mathbf{P}^{\sigma} = (e'_1, \dots, e'_d) \cdot \mathbf{P}^{-1}\mathbf{T}\mathbf{P}^{\sigma}.$$

□

Lemma A.8. *Notations k, σ, T, V as above (in particular $\sigma \in \mathrm{End}(k)$). Suppose*

(1) T is not nilpotent on the whole of V (i.e. $V \setminus V_{\mathrm{nil}} \neq \emptyset$), and
(2) for any $n \in \mathbb{N}$, any sequence $b_0, \dots b_n \in k$ with at least one $b_j \neq 0$, there exists a nonzero $x \in k$ such that

$$x = \sum_{i=0}^{n} \sigma^{i+1}(x)\sigma^i(b_{n-i}). \tag{A.0.1}$$

Then for any $e \in V \setminus V_{\mathrm{nil}}$, the k-vector space

$$V_e := \langle e, Te, T^2e, \dots \rangle$$

generated by the sequence $e, Te, T^2e, \dots$, contains a nonzero *T-invariant vector v.*

Remark A.9. When T is $p^{\pm 1}$-linear, any $x \in k$ satisfying (A.0.1) is separable over k. Indeed, when T is p-linear then (A.0.1) is a polynomial. By taking derivation with respect to x one sees that (A.0.1) is an separable polynomial with indeterminant x. When T is p^{-1}-linear, solutions of (A.0.1) are the same as that of the polynomial obtained by taking iterated p-th power. And the smallest such power of (A.0.1) is clearly a separable polynomial in x.

Therefore if moreover T is semisimple and k is separably closed, we know there is a nonzero T-invariant vector in V.

Proof. Take arbitrary $e \in V \setminus V_{\mathrm{nil}}$. Consider a sequence of vectors

$$e, Te, T^2e, \dots.$$

Let n be the biggest integer such that $e, Te, T^2e, ..., T^ne$ are k-linearly independent. Then $0 \le n \le d := \dim_k V$, and $V_e = \langle e, Te, \dots, T^ne\rangle$. Therefore $T^{n+1}e \in V_e$ has expression

$$T^{n+1}e = \sum_{i=0}^{n} b_i T^i e$$

for some $b_i \in k$. Since $T^{n+1}e \neq 0$ (because $e \in V \setminus V_{\mathrm{nil}}$), at least one b_j is nonzero, for $0 \le j \le n$.

Consider a vector $v = \sum_{i=0}^{n} a_i T^i e \in V_e$. v is nonzero if and only if some $a_i, (0 \le i \le n)$ is nonzero, and it is T-invariant if and only if the a_i's satisfy

$$\begin{aligned} 0 &= \sum_{i=0}^{n} \sigma(a_i) T^{i+1} e - \sum_{i=0}^{n} a_i T^i e \\ &= (\sigma(a_n) b_0 - a_0) e + \sum_{i=1}^{n} (\sigma(a_{i-1}) + \sigma(a_n) b_i - a_i) T^i e. \end{aligned}$$

And these both happen if and only if the following system of equations with indeterminants $a_0, \dots, a_n$ has a *nonzero* solution in k:

$$\begin{aligned} a_0 &= \sigma(a_n) b_0, \\ a_1 &= \sigma(a_0) + \sigma(a_n) b_1 = \sigma^2(a_n)\sigma(b_0) + \sigma(a_n) b_1, \\ a_2 &= \sigma(a_1) + \sigma(a_n) b_2 = \sigma^3(a_n)\sigma^2(b_0) + \sigma^2(a_n)\sigma(b_1) + \sigma(a_n) b_2, \\ &\dots \\ a_{n-1} &= \sigma(a_{n-2}) + \sigma(a_n) b_{n-1} = \sum_{i=0}^{n-1} \sigma^{i+1}(a_n)\sigma^i(b_{n-1-i}), \\ a_n &= \sigma(a_{n-1}) + \sigma(a_n) b_n = \sum_{i=0}^{n} \sigma^{i+1}(a_n)\sigma^i(b_{n-i}). \end{aligned}$$

The last equation involves only one unknown a_n, and the rest of the a_i's $(0 \le i \le n-1)$ are expressed in terms of a_n. And apparently, when $a_n = 0$ all the other a_i's are zero. Therefore this system of equations have a *nonzero* solution in k if and only if the last equation in a_n has a *nonzero* solution in k. And this is guaranteed by assumption because at least one of the b_j's is nonzero. □

Proposition A.10 (Existence of T-invariant basis)**.** *Notations k, σ, T, V as before. Suppose*

(1') T is semisimple on V,

(2) Same as (2) in Lemma A.8, i.e., for any $n \in \mathbb{N}$, any sequence $b_0, \dots b_n \in k$ with at least one $b_j \neq 0$, there exists a nonzero $x \in k$ such that

$$x = \sum_{i=0}^{n} \sigma^{i+1}(x)\sigma^i(b_{n-i});$$

(3) for any $c \in k$, there exists a $y \in k$ such that

$$\sigma(y) - y + c = 0. \tag{A.0.2}$$

Then

(a) there exists a k-basis of V consisting of T-invariant elements. In other words,

$$V \simeq V^{1-T} \otimes_{\kappa} k.$$

(b) $1 - T$ is surjective on $V = V_{\mathrm{ss}}$.

Remark A.11. When T is $p^{\pm 1}$-linear, similar as we explained in the last remark, any element satisfying (A.0.2) is separable over k. Therefore if moreover T is semisimple and k is separably closed, we have results (a)(b).

Proof. (a) Do induction on the $\dim V$.

When $\dim V = 1$, then according to Lemma A.8 we have a nonzero T-invariant vector v_1, and any nonzero vector is a k-basis of V. We are done in this case.

Now suppose the proposition is true for $\dim V \leq d-1$. We prove for $\dim V = d$. According to Lemma A.8, there exists a nonzero T-invariant vector $v_1 \in V$. Passing T to the quotient

$$\overline{T} : \overline{V} := V/\langle v_1\rangle \to V/\langle v_1\rangle, \quad w + \langle v_1\rangle \mapsto T(w) + \langle v_1\rangle.$$

This is clearly a well-defined σ-linear map and semisimple. By induction hypothesis, we can find a T-invariant basis of $V/\langle v_1\rangle$: $(\overline{v}_2, \dots, \overline{v}_d)$. Take $v_i' \in V$ to be any lift of $\overline{v}_i \in V/\langle v_1\rangle$. Then for each $2 \leq i \leq d$, we have

$$Tv_i' - v_i' = c_i v_1, \quad \text{for some } c_i \in k.$$

According to assumption (3), we can find a_i such that $\sigma(a_i) - a_i + c_i = 0$, $2 \leq i \leq d$. Then

$$T(v_i' + a_i v_1) = v_i' + a_i v_1, \quad 2 \leq i \leq d.$$

Define $v_i := v_i' + a_i v_1$, $2 \leq i \leq d$. $(v_1, v_2, \dots, v_d)$ are k-linearly independent because $(\overline{v}_2, \dots, \overline{v}_d)$ are k-linearly independent in $V/\langle v_1\rangle$. Therefore they form a T-invariant basis of V, or in other words,

$$V \simeq \langle v_1, v_2, \dots v_d\rangle_\kappa \otimes_\kappa k = V^{1-T} \otimes_\kappa k.$$

(b) By (a) we have T-invariant basis $(v_1, \dots, v_d)$ for V. Now for any $v := \sum b_i v_i \in V$ with $b_i \in k$, there exists $a_i, 1 \leq i \leq d$ satisfies

$$(1-T)(\sum a_i v_i) = v$$

if and only if a_i satisfy $a_i - \sigma(a_i) = b_i$ for each $i \in [1, d]$. Assumption (3) guarantees that such a_i's exist. Therefore $1 - T$ is surjective on V.

□

Proposition A.12. *Suppose $\sigma \in \mathrm{Aut}(k)$, and the pair (k, σ) satisfies assumption (2)(3) in Proposition A.10. T, V as before (in particular, V is a finite dimensional k-vector space). Then*

$$1 - T : V \to V$$

is surjective. And

$$V_{\mathrm{ss}} \simeq V^{1-T} \otimes_\kappa k,$$

which in particular means V^{1-T} is a finite dimensional κ-vector space with $\dim_\kappa V^{1-T} = \dim_k V_{\mathrm{ss}}$.

Remark A.13. When T is $p^{\pm 1}$-linear and k is separably closed, (2)(3) are satisfied by the remarks above, therefore we have $1 - T$ being surjective.

Proof. The second claim is direct from $V^{1-T} \subset V_{\mathrm{ss}}$ and Proposition A.10. Now we prove the first.

Notice the $1-T$ is invertible on V_{nil} (2), assumption (2)(3) and Proposition A.10 applied to V_{ss} implies that $1 - T$ is surjective on V_{ss}. Use Fitting decomposition Proposition A.6 and consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{\mathrm{ss}} & \longrightarrow & V & \longrightarrow & V_{\mathrm{nil}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1-T} & & \downarrow{\scriptstyle 1-T} & & \simeq\downarrow{\scriptstyle 1-T} & & \\ 0 & \longrightarrow & V_{\mathrm{ss}} & \longrightarrow & V & \longrightarrow & V_{\mathrm{nil}} & \longrightarrow & 0. \end{array}$$

The snake lemma (for abelian groups, note that $1 - T$ is not linear) immediately gives the desired result. □

We generalize the definition of a σ-linear map.

Definition A.14. Let R be a ring, and M be a finitely generated R-module. Let $\sigma \in \mathrm{End}(R)$ be a ring endomorphism of R. A *σ-linear operator* or a *σ-linear map* on M is a map $T : M \to M$, such that

$$T(v + w) = T(v) + T(w), \quad T(cv) = \sigma(c)T(v), \quad v, w \in M, c \in R.$$

Notice that this is equivalent to a R-linear map $M \to \sigma_* M$.

In particular, when $R = W_n k$ with k being a characteristic p field and $\sigma = W_n F_k$ (F_k is the p-th power Frobenius), T is called *p-linear*. When k is furthermore *perfect* and $\sigma = W_n F_k^{-1}$, T is called *p^{-1}-linear*. Similarly, one could define p^n-linear for $n \in \mathbb{Z}$ (when $n = 0$, T is simply R-linear by assumption).

Proposition A.15. *Let $(R,(p))$ be a local ring of p^n-torsion with some $n \in \mathbb{N}_{>0}$, and M be a finite R-module. Let $\sigma \in \mathrm{End}(R)$ be a ring endomorphism of R and T be a σ-linear map. Then $\sigma : R \to R$ has a well-defined reduction $\overline{\sigma} : k := R/p \to k$ by modding out ideal (p). Suppose that $\overline{\sigma} \in \mathrm{Aut}(k)$, and that the pair $(k, \overline{\sigma})$ satisfies assumption (2)(3) in Proposition A.10 (cf. Remark A.13). Then*

$$1 - T : M \to M$$

is surjective.

Proof. Take $v \in M$. Because M is finite as a R-module, M/pM is a finite dimensional k-vector space. Then Proposition A.12 implies that there exists a $w \in M$, such that $(1-T)(w) - v \in pM$. That is, there exists a $v_1 \in M$ such that

$$(1-T)(w) = v + pv_1.$$

Do the same process with v_1 instead of v, one gets a $w_1 \in M$ and a $v_2 \in M$ such that

$$(1-T)(w_1) = v_1 + pv_2.$$

Thus

$$(1-T)(w - pw_1) = v - p^2 v_2.$$

Repeat this process. After finitely many times, because $p^n = 0$ in R,

$$(1-T)(w - pw_1 + \cdots + (-1)^{n-1} p^{n-1} w_{n-1}) = v.$$

□

Proposition A.16. *Let $(R,(p))$, k, σ, M, T satisfy the same assumptions as in Proposition A.15 (so in particular we have $1 - T : M \to M$ being surjective). Suppose furthermore that the natural map*

$$R^{1-\sigma} \to (R/p)^{1-\overline{\sigma}}$$

is surjective. Then M^{1-T} is a finite $R^{1-\sigma}$-module.

Proof. Since R is of p^n-torsion for some $n > 0$, we know that $p^m M = 0$ for some $m \leq n$. Do induction on the smallest number m such that $p^m M = 0$. When $m = 1$, $M = M/pM$ is a finite R/p-module, thus by Proposition A.12 we know that M^{1-T} is a finite dimensional $(R/p)^{1-\overline{\sigma}}$-vector space. Since $R^{1-\sigma} \to (R/p)^{1-\overline{\sigma}}$ is surjective, M^{1-T} is a finite $R^{1-\sigma}$-module.

Now we assume $m > 1$. Note that T induces a σ-linear map on pM and pM is a finite R-module, so by Proposition A.15 the map $1-T$: $pM \to pM$ is surjective. Now we have the two rows on the bottom of the following diagram being exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M^{1-T}/(pM)^{1-T} & \to & M/p & \xrightarrow{1-T} & M/p & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M^{1-T} & \to & M & \xrightarrow{1-T} & M & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & (pM)^{1-T} & \to & pM & \xrightarrow{1-T} & pM & \to & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The vertical maps between the last two rows are natural inclusions, and the first row is the cokernels of these inclusion maps. The snake lemma implies that the first row is exact, which means that $M^{1-T}/(pM)^{1-T} = (M/p)^{1-T}$. This is a finite $R^{1-\sigma}$-module by the case $m = 1$, because M/p is a finite R-module with $p \cdot M/p = 0$. On the other hand, since $p^{m-1} \cdot pM = 0$, the induction hypothesis applied to the R-module pM gives $(pM)^{1-T}$ is a finite $R^{1-\sigma}$-module. Now the vertical exact sequence on the left gives that M^{1-T} is a finite $R^{1-\sigma}$-module. □

Remark A.17. When $R = W_n k$ for a perfect field k of positive characteristic p and $\sigma = (W_n F_X)^{\pm 1}$, it satisfies the assumption for ring R in Proposition A.15 and Proposition A.16. In fact, one has $(W_n k)^{1-\sigma} = \mathbb{Z}/p^n$ in this case.

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