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Bounding the resources for thermalizing many-body localized systems

Carlo Sparaciari[1,2], Marcel Goihl[3], Paul Boes[3], Jens Eisert[3]✉ & Nelly Huei Ying Ng[3,4]

Understanding under which conditions physical systems thermalize is a long-standing question in many-body physics. While generic quantum systems thermalize, there are known instances where thermalization is hindered, for example in many-body localized (MBL) systems. Here we introduce a class of stochastic collision models coupling a many-body system out of thermal equilibrium to an external heat bath. We derive upper and lower bounds on the size of the bath required to thermalize the system via such models, under certain assumptions on the Hamiltonian. We use these bounds, expressed in terms of the max-relative entropy, to characterize the robustness of MBL systems against externally-induced thermalization. Our bounds are derived within the framework of resource theories using the convex split lemma, a recent tool developed in quantum information. We apply our results to the disordered Heisenberg chain, and numerically study the robustness of its MBL phase in terms of the required bath size.

[1] Department of Computing, Imperial College London, London SW7 2AZ, UK. [2] Department of Physics and Astronomy, University College London, London WC1E 6BT, UK. [3] Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany. [4] School of Physical and Mathematical Sciences, Nanyang Technological University, 637371 Nanyang, Singapore. ✉email: jense@zedat.fu-berlin.de

When pushed out of equilibrium, closed interacting quantum many-body systems generically relax to an equilibrium state, where local subsystems can be described using thermal ensembles that only depend on the energy of the initial state. While this behavior is plausible from the perspective of quantum statistical mechanics[1–5], it is far from clear which local properties are responsible for the emergence of thermalization. The discovery of many-body localization (MBL) offers a fresh perspective into this question, as this effect occurs in interacting many-body systems preventing them from actually thermalizing[6,7]. Examples of systems that are non-thermalizing, like integrable systems, have been known before, but these have been fine-tuned such that small perturbations restore thermalization. Many-body localization is strikingly different in this respect, as its non-thermalizing behavior appears to be robust to changes in the Hamiltonian[8]. A related open question, recently considered in several papers, is whether MBL is stable with respect to its own dynamics when small ergodic regions are present[9–13], or when the system is in contact with an actual external environment[14–17]. This is both a critical question for the experimental realization of systems exhibiting MBL properties, and for its fundamental implications on the process of thermalization in quantum systems.

The main focus of this work is to investigate the robustness of the MBL phase under instances of external dissipative processes. To do so, we introduce a physically realistic class of interaction models, describing the interaction between a many-body system and a finite-sized thermal environment. Within these models, the interactions are described in terms of energy-preserving stochastic collisions occurring between the system (or regions thereof) and sub-regions of the bath. During the interactions, the system and bath can be either weakly or strongly coupled. For this class of processes, we are able to derive analytical bounds on the minimum size of the bath required to thermalize the many-body system. We apply these bounds to the setting where the system is in the MBL phase, so as to characterize the robustness of this phase with respect to the coupling with an external environment. It is worth noting, however, that the bounds obtained hold for general many-body systems out of thermal equilibrium.

It is key to the approach taken here—and one of the merits of this work—that in order to arrive at our results, we make use of tools from quantum information theory, tools that might at first seem somewhat alien to the problem at hand, but which turn out to provide a powerful machinery. We demonstrate this by using a technical result known as the convex split lemma[18,19], to derive the quantitative bounds on the bath size required for a region of the spin-lattice to thermalize. Given that MBL phases are challenging to study theoretically, and most known results are numerical in nature[20–26], our work provides a fresh approach in understanding such phases from an analytical perspective. The convex split lemma has originally been derived in the context of quantum Shannon theory, which is the study of compression and transmission rates of quantum information. Our main contribution is to connect this mathematical result to a class of thermodynamic models that can be used to describe thermalization processes in quantum systems. This gives rise to surprisingly stringent and strong results. Note, however, in the approach taken for thermalizing processes, it is assumed that systems thermalize close to exactly, a requirement that will be softened in future work.

As part of our results, we find that the max-relative entropy[27,28] and its smoothed version emerge as operationally significant measures, that quantify the robustness of the MBL phase in a spin-lattice. The max-relative entropy is an element within a family of entropic measures that generalize the Rényi divergences[29] to the quantum setting. In order to illustrate the practical relevance of our results, we consider a specific system exhibiting the MBL phase, namely the disordered Heisenberg chain. Employing exact diagonalization, we numerically compute the value of the max-relative entropy as a function of the disorder and of the size of the lattice region that we are interested in thermalizing. Our findings suggest that the MBL phase is robust to thermalization despite being coupled to a finite external bath, under our collision models, indicating that such models allow for a conceptual understanding of the MBL phase stability. This extends the narrative of refs. [14–16], which find that MBL is thermalized when coupled to an infinite sized bath. Moreover, our numerical simulations show that the max-relative entropy signals the transition from the ergodic to the MBL phase.

Results

Thermalization setting. We first set up some basic notation. Given some Hamiltonian H_S of a system S with Hilbert space $\mathcal{H}_S$, we define the thermal state with respect to inverse temperature $\beta = 1/(k_B T)$ as the quantum state

$$\tau_\beta(H_S) := \frac{e^{-\beta H_S}}{\mathrm{Tr}(e^{-\beta H_S})}. \tag{1}$$

In what follows, we model the process of thermalization of S with an external heat bath B. In particular, let H_S and H_B denote the Hamiltonian of S and B, respectively. If S is initially in a state ρ, then, for fixed β and any $\epsilon > 0$, we say that a global process $\mathcal{E} : \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ ϵ-thermalizes the system S if

$$\left\| \mathcal{E}\left(\rho \otimes \tau_\beta(H_B)\right) - \tau_\beta(H_S) \otimes \tau_\beta(H_B) \right\|_1 \leq \epsilon, \tag{2}$$

where $\|\cdot\|_1$ is the trace norm. Intuitively, this corresponds to the situation where the process $\mathcal{E}$ acts on the compound of the initial state of the system ρ and the bath state $\tau_\beta(H_B)$, and brings the system state close to its thermal state $\tau_\beta(H_S)$ while leaving the bath mostly invariant. We write

$$\rho \xrightarrow[H_B,\mathcal{E}]{\epsilon} \tau_\beta(H_S) \tag{3}$$

if Eq. (2) holds. It is worth noting that ϵ-thermalization requires the global system SB to be close to thermal after the channel $\mathcal{E}$ is applied. Monitoring the bath as well is necessary in order to avoid the possibility of trivial thermalization processes in which the non-thermal state ρ is simply swapped into the bath, which would merely move the excitation out of the considered region, rather than describing a physically realistic dissipation process. Thus, our notion of thermalization is different from previously considered ones, where for instance the sole system's evolution is considered. At the same time, and as mentioned before, it is a rather stringent measure, in that close to full global thermalization is required.

Having introduced the basic notation and terminology, we now turn to the model used in this work. We consider a spin-lattice V, where each site is described by a finite-dimensional Hilbert space $\mathcal{H}$. The Hamiltonian of the system is composed of local operators, i.e.,

$$H_V = \sum_x H_x, \tag{4}$$

where x is labeling a specific subset of adjacent sites in the lattice and H_x is the corresponding Hamiltonian operator whose support is limited to these sites. Within the lattice, we consider a local region $R \subseteq V$ with Hilbert space $\mathcal{H}_R$. We are interested in the stability of the MBL phase with respect to stochastic collisions between the lattice region R and an external thermal bath B. In order to precisely re-cast this problem in terms of ϵ-thermalization, we first need to detail our choices for the initial state of the

region R, the Hamiltonians H_B and H_R, and the class of maps $\mathcal{E}$ that model the thermalization process.

Given an initial state vector of the lattice $|\psi(0)\rangle$, if we consider closed evolution generated by the Hamiltonian dynamics, then the global system is always in a pure state. However, if the local subsystems eventually equilibrate, then the equilibrium state is given by partially tracing over the global infinite-time average[30],

$$\omega := \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t \; |\psi(t)\rangle\langle\psi(t)|, \tag{5}$$

so that the state that describes region R at the time of its first interaction with the bath is $\omega_R = \mathrm{Tr}_{R^c}[\omega]$, where we trace out the remaining of the lattice $R^c = V\backslash R$. To define a valid Hamiltonian H_R, a natural approach is often to disregard interactions between R and the rest of the lattice R^c. As such, we consider the Hamiltonian $H_R = \sum_{x:x\subseteq R} H_x$, which includes only terms H_x whose support is contained in R, and denote the corresponding thermal state $\tau_\beta(H_R)$. It is worthwhile to point out that that there is an alternative natural approach to defining thermal states of subsystems, namely $\hat{\tau}_\beta(H_R) = \mathrm{Tr}_{R^c}\left[\tau_\beta(H_V)\right]$ the reduced state over the complement R of the thermal state of the full lattice[31]. These two thermal states are close to each other whenever the interaction terms between R and R^c in H_V are small. During our later simulations, we check the values of max-relative entropy using both versions of thermal states, and find that they produced similar values, implying that these different alternatives are actually not too dissimilar from each other for the disordered Heisenberg chain, which is expected, given that interactions are 2-local.

There has been a large body of work in thermalization, where a large class of many-body systems can effectively act as their own "environment"[4,30], thus local observables tend to equilibrate towards the corresponding thermal values. However, there are exceptions to this, in particular whenever there are non-negligible interactions between subsystems, of which MBL systems are an example. Nevertheless, most such systems still do equilibrate, and this is the assumption we make throughout the paper, that ω as defined in Eq. (5) exists.

We model the external thermal bath as a collection of $n-1$ copies of the region R in thermal equilibrium. More formally, B is a system with Hilbert space $\mathcal{H}_B = \mathcal{H}_R^{\otimes n-1}$ and Hamiltonian

$$H_B = \sum_{i=1}^{n-1} H_R^{(i)}, \tag{6}$$

where the operator $H_R^{(i)} = \mathbb{I}_1 \otimes \ldots \otimes \mathbb{I}_{i-1} \otimes H_R \otimes \mathbb{I}_{i+1} \otimes \ldots \otimes \mathbb{I}_{n-1}$ only acts non-trivially on the i-th subsystem of the bath. With this choice of Hamiltonian, the initial state on B is $\tau_\beta(H_B) = \tau_\beta(H_R)^{\otimes n-1}$. Such a choice for the bath is crucial to make our problem analytically tractable but is also physically relevant for experimental setups, where it is possible to engineer one-dimensional systems which are then coupled to a bath per site[32,33] or a mixture of a system and bath species that interact via contact interactions[34]. Moreover, we note that for the model of system-bath interactions that we introduce below, any state transition that can be realized on R with any heat bath Hamiltonian H_B can also be realized, for some n, with a Hamiltonian of the form (6)[35]. We also refer the interested reader to Supplementary Note 3, for a further discussion on bath choices.

We now turn to our model of the system-bath interactions, described via the following master equation,

$$\frac{\partial\, \rho_{RB}(t)}{\partial\, t} = \sum\nolimits_k \frac{1}{r_k} \left[U_{RB}^{(k)}\, \rho_{RB}(t)\, U_{RB}^{(k)\dagger} - \rho_{RB}(t) \right], \tag{7}$$

where ρ_{RB} is the global state on R and B. This equation models a series of collisions, each described by a unitary operator $U_{RB}^{(k)} \in \mathcal{B}(\mathcal{H}_R \otimes \mathcal{H}_B)$ acting non-trivially on the region R and a subset of bath components, occurring at a given rate $r_k^{-1} > 0$ in time according to a Poissonian distribution (see Supplementary Note 1 for details). We consider elastic collisions, that is, we require that each unitary operator $U_{RB}^{(k)}$ conserves the global energy

$$[U_{RB}^{(k)}, H_R + H_B] = 0. \tag{8}$$

It is worth noting that the above condition does not imply that the total energy of the lattice is conserved. In fact, this is in general not the case, due to those operators H_x in Eq. (4), with support on both R and R^c. However, since by assumption these operators have support on at most k adjacent lattice sites, if the region R is sufficiently large one expects these boundary terms to contribute less and less to the total energy of the region. For high temperatures, such notions that boundary terms are negligible have rigorously been established[31]. Note that Eq. (7) is already in standard Lindblad form, and therefore describes a Markovian dynamical semi-group on RB. It is also important to note, however, that the process happening on the region R is in principle non-Markovian, since B is modeled here explicitly, and can be a very small heat bath, which retains the memory of the system's initial state. See Fig. 1 for a graphical illustration of the setup.

In summary, the interaction model is general, in the sense that the unitary operators can act on a single subsystem (thus generating local Hamiltonian dynamics, for example), on two subsystems (these are the standard two-body interactions), or many more subsystems. Long-range interaction terms are also allowed, with the only restriction of Eq. (8)—in the hope that one may work toward relaxation in the future. We are nonetheless neglecting the interaction terms between the region R and R^c. While this assumption is critical for allowing us to apply our framework to this problem, we note that there are situations where it is physically relevant—for example, if the interaction between R and B occurs on a much shorter timescale than

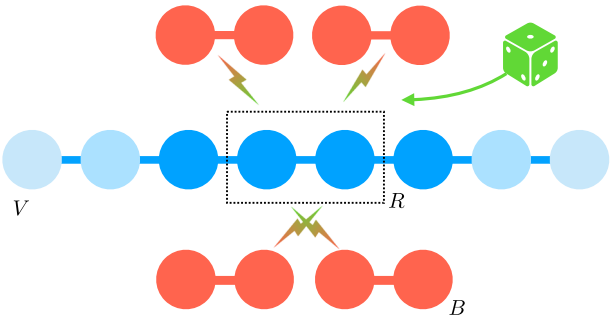

Fig. 1 Thermalization setting. Thermalization of a region R of an equilibrated lattice V via stochastic collisions with an external bath B (red). The collisions are modeled as randomly distributed, energy-preserving unitary interactions between R and B, where these interactions could be either between single subsystems of B (top) or multiple ones (bottom). Our framework is inspired by the resource-theoretic framework of thermal operations[54,55], which have been used to investigate a wide variety of questions, such as the notion of work for microscopic systems[35,56], the quantum fluctuation theorems[57], the third law[58,59], and several other topics[60–62].

between R and R^c, which is allowed by our class of interactions, as the rates $\{r_k\}_k$ can be chosen to be high enough. Moreover, our results on bath size are independent on the system-bath coupling strength.

We are finally in a position to define our central measure of robustness to thermalization. Let $\mathfrak{E}_n$ denote the set of quantum channels (i.e., completely-positive trace-preserving maps acting over the quantum states of a system[36]) on RB that can be generated via the above collision process for a bath of size $n-1$. Each channel in this set is realized through a different choice of unitary operations $\{U_{RB}^{(k)}\}$, collision rates $\{r_k^{-1}\}$, and final time t. Given the region initial state ω_R, its Hamiltonian H_R, and the bath Hamiltonian H_B defined as in Eq. (6), we define n_ϵ as the minimum integer such that there exists an element in $\mathfrak{E}_{n_\epsilon+1}$ that ϵ-thermalizes R,

$$n_\epsilon := \min\left\{ n \in \mathbb{N} \mid \exists\, \mathcal{E} \in \mathfrak{E}_{n+1} \;:\; \omega_R \underset{\epsilon, H_B}{\overset{\mathcal{E}}{\to}} \tau_\beta(H_R) \right\} - 1. \quad (9)$$

The integer n_ϵ then quantifies the smallest size of a thermal environment required to thermalize region R under stochastic collisions, and hence provides a natural measure to quantify the robustness of an MBL system against thermalization.

Upper and lower bounds for thermalization. Our main results are upper and lower bounds on n_ϵ that are essentially tight for a wide range of Hamiltonians H_R. These bounds are stated using the (smooth) max-relative entropy, an entropic quantity that has received considerable attention in recent years in quantum information and communication theoretical research[35,37,38]. Once again, it is worth noting that our results are applicable in general for finite-dimensional quantum systems where equilibration occurs, of which MBL is a particularly interesting example. The max-relative entropy[27] between two quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ such that $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ is defined as

$$D_{\max}(\rho||\sigma) = \inf\{\lambda \in \mathbb{R} \;:\; \rho \le 2^\lambda \sigma\}, \quad (10)$$

while the smooth max-relative entropy between the same two quantum states, for some $\epsilon > 0$, is defined as

$$D_{\max}^\epsilon(\rho||\sigma) = \inf_{\tilde{\rho} \in B_\epsilon(\rho)} D_{\max}(\tilde{\rho}||\sigma), \quad (11)$$

where $B_\epsilon(\rho)$ is the ball of radius ϵ around the state ρ with respect to the distance induced by the trace norm.

Upper bound. We first present and discuss the upper bound, which can be easily stated in terms of the quantities just introduced.

Theorem 1 (Upper bound on the size of the bath). *For a given Hamiltonian H_R, inverse temperature β, and a constant $\epsilon > 0$, we have that*

$$n_\epsilon \le \frac{1}{\epsilon^2}\, 2^{D_{\max}\left(\omega_R||\tau_\beta(H_R)\right)}. \quad (12)$$

The above theorem provides a quantitative bound on the size of the thermal bath needed to ϵ-thermalize a lattice region R, when the coupling is mediated by stochastic collisions. For this specific dynamics, the region can be ϵ-thermalized if the size of the bath (the number of components) is proportional to the exponential of the max-relative entropy between the state of the region ω_R and its thermal state $\tau_\beta(H_R)$.

Theorem 1 is proven in Supplementary Note 2. Here, we present a sketch of the proof in two steps. In the first step, we show that $\mathfrak{E}_n$ can be connected to so-called random unitary channels[39]. In the second step, we use this connection to find a particular channel in $\mathfrak{E}_n$ that achieves the upper bound of the above theorem. A central ingredient to the second step is a result known as convex split lemma[18,19].

Turning to the first step, recall that a random unitary channel is a map of the form

$$\mathcal{E}(\cdot) = \sum_k p_k\, U_k \;\cdot\; U_k^\dagger, \quad (13)$$

where $\{p_k\}_k$ is a probability distribution, and $\{U_k\}_k$ is a set of unitary operators. For a given number $n-1$ of bath subsystems, we define the class of energy-preserving random unitary channels $\mathfrak{R}_n$ as those random unitary channels on RB for which each unitary operator $U_k \in \mathcal{B}(\mathcal{H}_R \otimes \mathcal{H}_B)$ commutes with the Hamiltonian of the global system, i. e., $[U_k, H_R + H_B] = 0$. In Supplementary Note 1, we show that for any $n \ge 1$, $\mathfrak{E}_n \subseteq \mathfrak{R}_n$, therefore allowing us to analyze any element of $\mathfrak{E}_n$ as a random unitary channel.

Turning to the second step, we use a stochastic collision model with a simple representation in terms of random unitary channels. Let us first recall that the thermal bath B is described by $n-1$ copies of $\tau_\beta(H_R)$, the Gibbs state of the Hamiltonian H_R at inverse temperature β. The collisions occur either between the region R and one subsystem of the bath, or between two bath subsystems. The rate of collisions is uniform, and given by $r^{-1} > 0$. During a collision involving the ith and jth subsystems of RB, the states of the two colliding components are swapped, so that the interaction is described by the unitary operator $U_{\mathrm{swap}}^{(i,j)}$. The action of this operator over two quantum systems, described by the state vectors $|\psi\rangle_1$ and $|\phi\rangle_2$, respectively, is given by $U_{\mathrm{swap}}^{(1,2)}|\psi\rangle_1 \otimes |\phi\rangle_2 = |\phi\rangle_1 \otimes |\psi\rangle_2$. For an initial global state $\rho_{RB}^{(\mathrm{in})} = \omega_R \otimes \tau_\beta(H_R)^{\otimes n-1}$, the steady state obtained through this process is

$$\rho_{RB}^{(\mathrm{ss})} = \sum_{m=1}^{n} \frac{1}{n}\, \tau_\beta(H_R)^{\otimes m-1} \otimes \omega_R \otimes \tau_\beta(H_R)^{\otimes n-m}, \quad (14)$$

where the a-thermality of the region has been uniformly hidden into the different components of the bath. It is worth noting that, under the stochastic collision model described above, the global system reaches its steady-state exponentially quickly in the collision rate r^{-1}, see Supplementary Note 2 for more details.

The mapping from the initial state of region and bath to the steady-state is achieved by the following random unitary channel

$$\bar{\mathcal{E}}_n(\cdot) = \sum_{i=1}^{n} \frac{1}{n}\, U_{\mathrm{swap}}^{(1,i)} \;\cdot\; U_{\mathrm{swap}}^{(1,i)\,\dagger}, \quad (15)$$

which uniformly swaps each of the bath subsystems with R. Such channels have been studied before in the context of entropy production[40,41], see also ref. [42] for a similar example. Since all subsystems share the same Hamiltonian H_R, it is easy to see that each one of the $U_{\mathrm{swap}}^{(1,i)}$ commutes with the joint Hamiltonian, so that $\bar{\mathcal{E}}_n \in \mathfrak{R}_n$. Finally, we can invoke the convex split lemma, see Supplementary Note 2, which allows us to show that, for any $\epsilon > 0$, the channel $\bar{\mathcal{E}}_n$ can ϵ-thermalize the region R when the number of subsystems is $n = \epsilon^{-2} 2^{D_{\max}\left(\omega_R||\tau_\beta(H_R)\right)}$.

The collision model presented here already encompasses a wide range of possible and physically realistic thermalization processes. However, before turning to a lower bound, we note that, due to the particularly simple nature of (15), one can use the above construction to upper bound the required size of a bath for other thermalization models as well. For example, by noting that the channel (15) is permutation-symmetric (in the sense that it has permutation-invariant states as its fixed points), it follows that thermalization models with permutation-symmetric dynamics

(i.e., those that allow for any permutation-symmetric channel) are also subject to the above upper bound.

Lower bound and optimality results. We now turn to derive a lower bound on n_ϵ. This bound is obtained through a further assumption on the Hamiltonian H_R, which we call the energy subspace condition (ESC).

Definition 2 (Energy subspace condition). *Given a Hamiltonian H_R, we say that it fulfills the ESC iff for any $n \in \mathbb{N}$, given the set of energy levels $\{E_k\}_{k=1}^d$ of the Hamiltonian H_R, we have that for any vectors $m, m' \in \mathbb{N}^d$ with the same normalization factor, namely,*

$$\sum_k m_k E_k \neq \sum_k m'_k E_k. \tag{16}$$

Let us here briefly discuss the physical significance of the ESC condition. The ESC entails (but is not equivalent to) that energy levels cannot be exact integer multiples of one another, which also implies full non-degeneracy. Furthermore, note that the ESC is not approximate, in other words, it still holds even if energy levels are very close to each other. Having exact integer multiple energy levels is a very fine-tuned situation that breaks as soon as randomness is introduced in the Hamiltonian[43]. Let us for example consider how likely it is for MBL systems to have degenerate energy levels. In the ergodic phase, the level statistics are Wigner–Dyson, and therefore non-degeneracy is enforced by level repulsion. On the other hand, in the strong MBL phase, level statistics are Poissonian, which means that the probability density function is maximum for zero-energy gaps. Despite this, the probability of exact degeneracy would correspond to a zero-volume integral of the Poissonian distribution (which is bounded from above), and therefore still amounts to zero probability of having degenerate gaps.

It is clear, however, that the ESC is much more stringent than requiring non-degeneracy; If we require Eq. (16) to be satisfied for all $n \in \mathbb{N}$, this implies that the energy levels of the Hamiltonian H_R need to be irrational. Nevertheless, one can relax this condition by asking it to hold for all $n \leq N$, for some sufficiently large $N \in \mathbb{N}$, for example with the upper bound on n_ϵ in Theorem 1. We refer to this as the ESC being satisfied up to N. In the next section, we discuss how a paradigmatic MBL system relates to this condition.

We can now state the following theorem, proved in Supplementary Note 4, on the optimality of the channel associated with the convex split lemma.

Theorem 3 (Optimal thermalization processes). *If H_R satisfies the ESC and the state ω_R is diagonal in the energy eigenbasis, then the channel $\bar{\mathcal{E}}_n$ in Eq. (15) provides the optimal thermalization process, that is, for any $n \in \mathbb{N}$,*

$$\bar{\mathcal{E}}_n \in \arg\min_{\mathcal{E} \in \mathfrak{E}_n} \left\| \mathcal{E}\left(\omega_R \otimes \tau_\beta(H_R)^{\otimes n-1}\right) - \tau_\beta(H_R)^{\otimes n} \right\|_1. \tag{17}$$

Theorem 3 shows that, for Hamiltonians satisfying the ESC, the channel $\bar{\mathcal{E}}_n$ provides the optimal thermalization of R, that is, no other random energy-preserving channel acting on the same global system can achieve a smaller value of ϵ in Eq. (2). The above result applies to initial states that are diagonal in the energy eigenbasis; this is in general not the case for the reduced state ω_R of the infinite-time average of Eq. (5), since it might have coherence in the eigenbasis of the reduced Hamiltonian H_R. For states with coherence, the channel of Eq. (15) is not necessarily optimal anymore, but we can still bound the difference in thermalization achieved by this channel and an optimal one, see Supplementary Note 4 for the proof.

Theorem 4 (Thermalization bound for coherent states). *Fix $n \in \mathbb{N}$, and assume that H_R satisfies the ESC. Consider the channel $\mathcal{E}_{\text{opt}} \in \mathfrak{E}_n$ achieving optimal thermalization*

$$\epsilon_{\text{opt}} = \left\| \mathcal{E}_{\text{opt}}\left(\omega_R \otimes \tau_\beta(H_R)^{\otimes n-1}\right) - \tau_\beta(H_R)^{\otimes n} \right\|_1, \tag{18}$$

and the decohering channel $\Delta(\cdot) = \sum_E \Pi_E \cdot \Pi_E$, where Π_E is the eigenprojector onto the energy subspace associated with E. We define the parameter $\delta = \|\omega_R - \Delta(\omega_R)\|_1$, quantifying the amount of coherence contained in the state of the region. Then, the thermalization achieved by the channel $\bar{\mathcal{E}}_n$ is bounded as

$$\left\| \bar{\mathcal{E}}_n\left(\omega_R \otimes \tau_\beta(H_R)^{\otimes n-1}\right) - \tau_\beta(H_R)^{\otimes n} \right\|_1 \leq \epsilon_{\text{opt}} + \delta. \tag{19}$$

The above theorem provides a quantitative bound on the thermalization achieved by the channel $\bar{\mathcal{E}}_n$ when the input system has coherence in the energy eigenbasis. In the case of MBL systems, the eigenstates of the Hamiltonian are close to product states, see for instance ref. [44], and therefore the reduced state of the infinite-time average ω_R is expected to have small and strongly-decaying coherence. Thus, Theorem 4 shows that the stochastic collision model introduced in the previous section is able to effectively thermalize the region of MBL systems. From the two theorems stated above, we can derive the following corollary, providing a lower bound on the size of the thermal bath needed to thermalize a given quantum system.

Corollary 5 (Lower bound to size of the bath). *For a given β and $\epsilon > 0$, and some Hamiltonian H_R satisfying the ESC, we have*

$$n_\epsilon \geq 2^{D_{\max}^{2\sqrt{\epsilon}+\delta}\left(\omega_R \| \tau_\beta(H_R)\right)}, \tag{20}$$

where $\delta = \|\omega_R - \Delta(\omega_R)\|_1$ and $\Delta(\omega_R)$ is the decohered version of the state ω_R.

Note that this lower bound is arising from the stringent model of thermalization of Eq. (2), and that less stringent models will potentially lead to smaller lower bounds. When H_R does not satisfy the ESC, it is easy to find counter-examples to the optimality of the channel $\bar{\mathcal{E}}$, as we show in Supplementary Note 4. The idea is that this channel is optimal only when it is able to produce a uniform distribution within each energy subspace of the global system, and this is possible if each subspace is fully characterized by a different frequency of single-system eigenvectors, which is exactly given by the ESC. Indeed, when the ESC is maximally violated, i.e., when the system Hamiltonian is completely degenerate, then no bath is required at all. See "Discussion" section for a discussion of this and its relation to known bounds from randomness extraction.

The disordered Heisenberg chain. Our results from the previous section show that, for systems that satisfy the ESC, the max-relative entropy between the local state of a lattice region and its thermal state provides a natural measure for the robustness of that region to thermalization for a broad family of interactions. This includes many-body systems close to the transition between the ergodic and MBL phase, where both level repulsion and randomness effects favor a lack of exact degeneracies so that it seems reasonable to expect for such systems to satisfy the ESC to sufficient order n. In this section, we use these results to study the robustness of the MBL phase to the thermal noise for a concrete system. Specifically, we consider the disordered Heisenberg chain, a one-dimensional spin-$\frac{1}{2}$ lattice system composed of L sites, governed by the Hamiltonian

$$H_V = \sum_i^L \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z\right) + \Delta \sum_i^L h_i \sigma_i^z, \tag{21}$$

where $\sigma^x, \sigma^y, \sigma^z \in \mathcal{B}(\mathbb{C}^2)$ are the Pauli operators, Δ is the

(dimensionless) disorder strength, and each parameter $h_i \in [-1, 1]$ is drawn uniformly at random. We employ periodic boundary conditions.

It has been demonstrated both theoretically[45] and experimentally[7] that this system undergoes a localization transition above the critical disorder strength $\Delta_c \approx 7$. The transition manifests itself in a breakdown of conductance[7,45], and a slowdown of entanglement growth after a quench[23,24]. Moreover, a phenomenological model in terms of quasi-local constants of motions exists which provides an explanation for the non-thermal behavior of the system[46,47].

To relate the model to our theoretical results, we note that when the region R is a single qubit ($|R| = 1$) with a non-zero-energy gap, then the ESC is always satisfied for all $n \in \mathbb{N}$. For $|R| = 2$, we have verified the ESC condition across a large range of disorder strengths $\Delta \in [0.01, 20]$, up to $N = 25$ (out of ~2000 realizations, all of them satisfy the ESC); we have additionally considered a small number of realizations for $N = 35$, for all of which the ESC holds. As $|R|$ increases, higher values of N become significantly harder to check numerically. However, we have verified that for example $|R| = 4$ always satisfies the ESC up to $N = 5$ (for 2000 generated realizations), while the condition is satisfied with high probability when $N = 6$ (90% of the realizations).

In our simulation, we choose as initial state vector $|\Psi(0)\rangle$ a variation of the Néel state with support on the total-magnetization sectors $M = \pm 1, 0$. Our choice is motivated by the fact that this state, due to its increased overlap over different symmetric subspaces of the Hamiltonian, thermalizes more easily during the ergodic phase. For each random realization, we numerically compute the infinite-time average of $|\Psi\rangle$ as defined in Eq. (5), using exact diagonalization. We then trace out part of the lattice so as to obtain the state ω_R, describing the infinite-time averaged state reduced to the region R. Notice that in the ergodic phase, when the disorder strength $\Delta < \Delta_c$, this state is expected to be close to thermal, with a temperature which depends on the energy of the initial state of the lattice. However, when the disorder strength Δ passes its critical value, the state ω_R is not thermal anymore[7].

To numerically compute the max-relative entropy for this system, we use the Gibbs state of the reduced Hamiltonian H_R, obtained from the Hamiltonian in Eq. (21) by only considering terms with full support on the region R. The inverse temperature β is obtained by constructing the global Gibbs state of the lattice and requiring its energy to equal to the one of the initial state vector $|\Psi(0)\rangle$. We compute $D_{\max}(\omega_R||\tau_\beta(H_R))$ for different disorder strengths Δ, and different sizes of the region R (Fig. 2).

We find that in the ergodic phase the state is approximately thermal, and the max-relative entropy remains almost constant as $|R|$ increases. For big enough sizes of the region, the max-relative entropy starts increasing even in the ergodic case. However, this effect is due to the finite size of the lattice in our simulation, and it can be mitigated by increasing the number of lattice sites (at the expense of a higher computational cost). As Δ approaches the critical value, we find that the max-relative entropy scales linearly in the region size $|R|$, with a linear coefficient which increases with the disorder strength (Fig. 2a). As a result, the size of the external thermal bath n_ϵ scales exponentially in the region size, due to the bounds we have obtained in the previous section. This exponential scaling in the size of the bath suggests robustness of the MBL phase with respect to the dynamics given by Eq. (7), since the relative size of the bath $n_\epsilon/|R|$ needs to diverge as $|R|$ tends to infinity. In other words, for the MBL phase to be destroyed one needs, under the interaction models we consider, an exponentially vast amount of thermal noise. It is worth noting

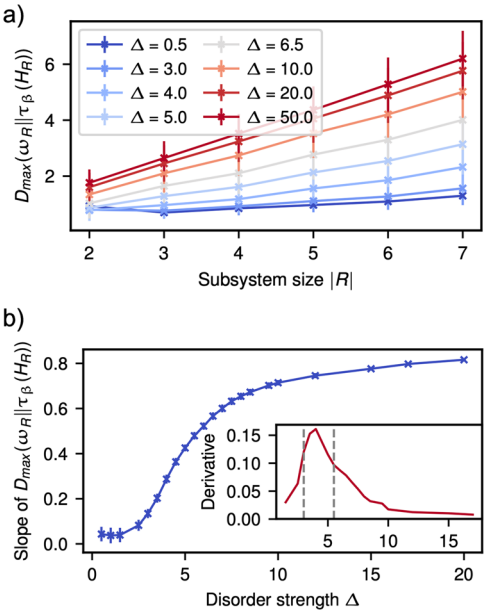

Fig. 2 Max-relative entropy for the disordered Heisenberg chain. a Max-relative entropy $D_{\max}\left(\omega_R||\tau_\beta(H_R)\right)$ as a function of subsystem size $|R|$ for a lattice of $L = 15$ sites. The plots show an average of over 100 disorder realizations. The states were calculated employing exact diagonalization. For low values of disorder Δ the max-relative entropy is almost constant as $|R|$ increases, while for higher values of Δ it scales linearly in $|R|$, hinting toward robustness of the MBL phase with respect to the class of interaction models we are considering. **b** The slope of the max-relative entropy as a function of the disorder Δ provides information on the phase transition. Indeed, we can see that this quantity abruptly increases in the proximity of the expected phase transition from the ergodic to the MBL phase. The slope is obtained by a linear fit with error bars indicating least-squares errors. The inset shows the derivative of the slope, with the gray lines, indicate a possible transition region.

that our characterization of the robustness of the MBL phase to thermal noise is distinctly different from others found in the literature[14–17]. Indeed, we couple the system with a finite-sized thermal bath, and we quantify the robustness in terms of its size. Furthermore, our notion of thermalization accounts for the evolution of both the system and bath, rather than focusing on the system only. Other works instead consider infinite thermal reservoirs and quantified the robustness as a function, for instance, of the coupling between system and environment. A promising experimental realization is recent optical lattice experiments[32–34]. However, to connect to our findings one would need full state tomography on both system and bath which so far is out of reach for these platforms.

We additionally study the first derivative of the max-relative entropy with respect to the region size $|R|$, as a function of the disorder strength, shown in Fig. 2b. We find that, during the ergodic phase, the derivative remains constant and small. As Δ approaches the critical value, the derivative increases, and for $\Delta \gg \Delta_c$ the derivative becomes constant again. Thus, we find that the derivative of the max-relative entropy with respect to the region size is an order parameter for the MBL phase transition. We then use this order parameter to estimate the critical value $\Delta_c^{(L)}$ for the finite-length spin chain we are considering, obtaining a value of ~4.5 for $L = 15$ sites. While the critical value for infinite-length spin chains is considered to be $\Delta_c \approx 7$, we find that our value, which we stress is obtained for a finite number of sites,

seems to be in good accord with known results found in the literature using other measures[48,49].

Discussion

We show that mathematical results originally developed to study quantum information processing may find their applications in many-body physics as well, in particular for the study of MBL in this paper. We demonstrate this by applying the recently developed convex split lemma technique, to derive upper and lower bounds for the size of the external thermal bath required to thermalize an MBL system. The class of interaction models between the lattice and the thermal bath is described by the master equation (7) and consists of stochastic energy-preserving collisions between the system and bath components. The bounds we obtain depend on the max-relative entropy between the state we aim to thermalize and its thermal state.

We make use of these analytic results to study a specific and at the same time much ubiquitous system exhibiting MBL features, known as the disordered Heisenberg chain. We show that the MBL phase in this system is in fact robust with respect to the thermalization processes considered here and that the derivative of the max-relative entropy with respect to region size serves as an approximate order parameter of the ergodic to MBL transition. We emphasize that this is not in contradiction with previous results, where a breakdown of localization was reported[14–16], as the size of the baths considered in these works was unbounded. Resource-theoretic frameworks offer another potentially useful approach for studying thermalization with infinite-dimensional baths; the framework of elementary thermal operations[50] which involves a single bosonic bath that is coupled only with two levels of the system of interest. One may then study the resources (the number of bosonic baths with different frequencies) required to achieve thermalization. Also, and more technically, it would be interesting to study the extent to which both the ESC condition and the requirement of exact commutation in our framework can be relaxed to only approximately hold true and how this, in turn, affects the lower bound of Corollary 5 (Eq. 20). These questions we leave to be studied in future work.

The success of our application implies that, potentially, other information-theoretic tools could be employed to study the thermalization of MBL systems—and non-equilibrium dynamics of many-body systems in more generality, for that matter. For instance, results in randomness extraction[51] might be useful to provide new bounds. In randomness extraction, a weakly random source is converted into an approximately uniform distribution, with the use of seed (a small, uniformly distributed auxiliary system). In analogy, thermalization requires a non-thermal state to be mapped into an almost thermal state, with the help of an external bath (the seed). Thus, it seems possible that results from randomness extraction might be modified to study this setting and to obtain bounds on the thermal seed.

It has been shown that excited states of one-dimensional MBL systems are well-approximated by matrix product states (MPS) with a low bond dimension[44,52] if the system features an information mobility gap. These states have several interesting properties, and in particular, they feature an area law for the entanglement entropy which is logarithmic in the bond dimension[53]. Since our result is based on a particular entropic quantity, it might be possible to use the properties of MPS to derive a fully analytical bound on the robustness of these systems with respect to thermal noise. It is the hope that our work stimulates further cross-fertilization between the fields of quantum thermodynamics and the study of quantum many-body systems out of equilibrium.

Data availability

The data sets generated during the numerical simulation are available from the corresponding author on reasonable request.

Code availability

The code used for the numerical simulations is written in Python, and it is available from the corresponding author on reasonable request.

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Author contributions

C.S., P.B., and N.H.Y.N. have formulated the proofs, M.G. has written the code and produced the numerics, and all authors C.S, M.G., P.B., J.E., and N.H.Y.N. have contributed to research design and the writing of the paper.

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Correspondence and requests for materials should be addressed to J.E.

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