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Research Article

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On singularities of real algebraic sets and applications to kinematics

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Abstract: We address the question of identifying non-smooth points in $\mathbf{V}_{\mathbb{R}}(I)$ the real part of an affine algebraic variety. Two simple algebraic criteria will be formulated and proven. As an application, we investigate the configuration spaces of the planar four-bar linkage and the delta robot and prove that all singularities are CS-singularities.

Keywords: real algebraic geometry, singularities, planar linkages, kinematics

MSC 2020: 14B05, 14P05, 14P15, 13P25, 70B15

1 Introduction

For any zero set $X = \mathbf{V}_{\mathbb{R}}(I)$ of an ideal $I = (g_1,\ldots, g_k) \leq \mathbb{R}[\bar{x}]$, $\bar{x} = (x_1,\ldots, x_n)$, there is the question of identifying points where X is not locally a submanifold of $\mathbb{R}^n$.

The standard approach to this problem is to look for points $p \in X$, where the rank of the Jacobian of $(g_1,\ldots, g_k)$ drops below the height of I, which is the codimension of $\mathbf{V}_{\mathbb{C}}(I)$. Unfortunately, this is in general not enough to imply that X is **not** locally a submanifold. Obviously problems arise, if I is not radical or equidimensional (cf. Examples 1.1(ii) and (iii)) and techniques to handle this are well known (although not computationally feasible in some cases), but there are more intricate difficulties for real algebraic sets, where the localization of the reduced coordinate ring is not regular and $X = \mathbf{V}_{\mathbb{R}}(I)$ is still a smooth submanifold of $\mathbb{R}^n$ at this point (cf. Examples 1.1(vi)).

The following examples illustrate different kinds of behaviour of real algebraic sets at points where the Jacobian drops rank.

Examples 1.1
In all examples, we use the notation $\mathfrak{m} := \langle \bar{x} \rangle \leq \mathbb{R}[\bar{x}]$, $A = \mathbb{R}[\bar{x}]/I$.

(i) The simple node $I = \langle y^2 - x^2 - x^3 \rangle \leq \mathbb{R}[x, y]$ shows the expected behaviour. $A_{\mathfrak{m}}$ is not regular and $X = \mathbf{V}_{\mathbb{R}}(I)$ is not locally a manifold at the origin.

(ii) Let $I = \langle x^2, xy \rangle \leq \mathbb{R}[x, y]$. Then $X = \mathbf{V}_{\mathbb{R}}(I)$ is just the y-axis, which is locally a manifold at the origin, although $A_{\mathfrak{m}}$ is not regular. The problem here is clearly that I is not a radical ideal, i.e. $A_{\text{red}} = A/\sqrt{(0)} = \mathbb{R}[x, y]/\sqrt{I}$ localized at $\mathfrak{m}$ is regular. In theory, $\sqrt{I}$ can be calculated algorithmically with Gröbner base methods (e.g. `radical(I)` calculates the radical in the computer algebra system (CAS) Singular). Unfortunately, the computation is unfeasible in many cases. But we will see a useful criterion to decide if A is already reduced, which is the case for many polynomial systems, which originate from engineering problems, see Proposition 2.3.

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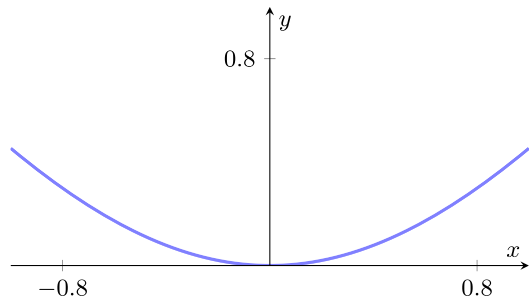

Figure 1: $\mathbf{V}_{\mathbb{R}}(y^3 + 2x^2y - x^4)$.

(iii) Let $I = \langle (z-1)xy, z(z-1)\rangle \trianglelefteq \mathbb{R}[x, y, z]$. Then $\mathbf{V}_{\mathbb{R}}(I)$ is the union of the x-axis, the y-axis and the plane given by $z = 1$. At the origin X is certainly not locally a submanifold, but the rank of the Jacobian at the origin equals $\operatorname{ht} I = 1$. Note that I is radical, but not equidimensional and $A_{\mathfrak{m}}$ is not regular. In this case, we need to calculate an equidimensional decomposition before applying the Jacobian criterion. Again this is possible with Gröbner base methods (`primdecGTZ(I)` calculates a primary decomposition in Singular) but as hard as the computation of the radical. However, for many problems in kinematics A is (locally) a complete intersection ring, which implies that I is equidimensional, see Proposition 2.3.

(iv) Let $I = \langle x^2 + y^2\rangle \trianglelefteq \mathbb{R}[x, y, z]$. Then $X = \mathbf{V}_{\mathbb{R}}(I)$ is the z-axis, which is a submanifold of $\mathbb{R}^3$, although the rank of the Jacobian drops at any point of X below $\operatorname{ht} I$ and I is radical and equidimensional. This difficulty only appears in real geometry, since $X_{\mathbb{C}} = \mathbf{V}(I)$ is **not** locally a complex manifold at any point of the z-axis. The problem is that $I \neq \mathbf{I}_{\mathbb{R}}(X) = \{f \in \mathbb{R}[x, y] \mid f|_X \equiv 0\} = \langle x, y\rangle$. Clearly, $(\mathbb{R}[x, y, z]/\langle x, y\rangle)_{\mathfrak{m}}$ is a regular local ring.

There are algorithms to compute the real radical $\sqrt[\mathbb{R}]{I} = \mathbf{I}_{\mathbb{Q}}(X)$ from $I \trianglelefteq \mathbb{Q}[\bar{x}]$ (e.g. `realrad(I)` computes the real radical over $\mathbb{Q}$ in Singular) but this computation is harder than that of the usual radical. Also, we have in general $\sqrt[\mathbb{R}]{I} \cdot \mathbb{R}[\bar{x}] \neq \sqrt[\mathbb{R}]{I \cdot \mathbb{R}[\bar{x}]} = \mathbf{I}_{\mathbb{R}}(X)$ (see Example (v)), in contrast to the usual radical. If this is the case not much can be gained by computing $\mathbf{I}_{\mathbb{Q}}(X)$. We present a very useful criterion by T. Y. Lam [1] to check for an ideal I whether $\mathbf{I}_{\mathbb{R}}(X) = I$, see Proposition 2.6.

(v) Let $I = \langle x^3 - 5y^3\rangle \trianglelefteq \mathbb{Q}[x, y]$ and $X = \mathbf{V}_{\mathbb{R}}(I)$, which is just the line given by $x = \sqrt[3]{5}y$. The Jacobian drops rank at the origin but X is an analytic submanifold of $\mathbb{R}^2$. Note that $I_{\mathbb{Q}}(X) = I$, but $I_{\mathbb{R}}(X) = \langle x - \sqrt[3]{5}y\rangle$.

(vi) This example motivated this paper. Let $I = \langle y^3 + 2x^2y - x^4\rangle \trianglelefteq \mathbb{R}[x, y]$ and $X = \mathbf{V}_{\mathbb{R}}(I)$. We will see that $A_{\mathfrak{m}}$ is not regular and even $\mathbf{I}_{\mathbb{R}}(X) = I$, but $X = \mathbf{V}_{\mathbb{R}}(I)$ is the analytic submanifold of $\mathbb{R}^2$ shown in Figure 1. Thus, we have established that $I = \mathbf{I}_{\mathbb{R}}(X)$ and $A_{\mathfrak{m}}$ not regular does not imply that X is nonsmooth at the origin.

The reason here is that some analytic branches are not visible in the real picture. We can decompose $y^3 + 2x^2y - x^4$ in the ring of convergent power series $\mathbb{R}\{x, y\}$:

$$y^3 + 2x^2y - x^4 = (x^2 - y(1 + \sqrt{1+y})) \cdot (x^2 - y(1 - \sqrt{1+y})). \tag{1}$$

Now $y(1 - \sqrt{1+y})$ is negative for y close to zero, hence the real zero set of $g(x, y) = x^2 - y(1 + \sqrt{1+y}) \in \mathbb{R}\{x, y\}$ coincides with the real zero set of $y^3 + 2x^2y - x^4$ on the domain of g. Since $(\partial_x g, \partial_y g) \neq (0, 0)$ at the origin, X_2 is clearly a submanifold there.

(vii) Let $I = \langle y^3 - x^{10}\rangle \trianglelefteq \mathbb{R}[\bar{x}]$ and $X = \mathbf{V}_{\mathbb{R}}(I)$. Here $A_{\mathfrak{m}}$ is not regular and $\mathbf{I}(X) = I$ again. But in this case X is not locally an analytic submanifold at the origin although the real picture looks very "smooth," which is because X is a C^3-submanifold (but not C^4).

It is well known that any real algebraic set which is (locally) a smooth (C^∞)-manifold is also a real analytic manifold (see Proposition 3.3), so any "nonanalytic" point is at the most "finitely differentiable." This example emphasizes the need for an algebraic criterion to algebraically discern between the singularities seen in the last two examples, because the real picture can be very deceiving. Criteria to identify points that are not locally topological submanifolds are beyond the scope of this article, although we will see that we can rule out this case in a lot of situations.

In this paper we want to show strategies how to deal effectively with all the problems seen in the examples, arising in the study of singular points of real algebraic sets. This is of great importance in the theory of linkages [2,3], when studying local kinematic properties, since the configuration space of a linkage will usually be given as a real algebraic set. Thus far, there is no broad consensus in the kinematics community how to handle points with rank drop in the Jacobian of the constraint equations. However, it is largely accepted that a configuration space (CS)-singularity should be defined as nonmanifold point of the configuration space. But then it is difficult to conclusively identify those points as the previous examples show.

This problem gained momentum when it was observed [4] that a closed 6R-chain exists with rank drop in the Jacobian of the constraint equations but smooth configuration space nevertheless. In this example, the singular configuration turned out to be an embedded point of the configuration space and would vanish upon taking the radical of the ideal of algebraic constraint equations. The same phenomenon can be seen in Example (ii). Proposition 2.3 will show that this can only happen for overconstrained mechanism.

For further discussion we refer the reader to Sections 7 and 8, where we investigate the configuration spaces of two types of linkages with the developed techniques. This will demonstrate how to identify CS-singularities for a large class of linkages. We will be able to address all questions raised in [5].

As mentioned earlier, the problem underlying Example (vi) requires some novel theoretical machinery. One of our main results will be:

Theorem 1.2. *Let Y be an irreducible $\mathbb{R}$-variety embedded in $\mathbb{C}^n$. Assume Y is normal at $p \in Y$ and p is in the euclidean closure of the real nonsingular points of Y. Then p is a manifold point of $Y_{\mathbb{R}} = Y \cap \mathbb{R}^n$ if and only if Y is nonsingular at p.*

Here, **manifold point** means any point $p \in Y_{\mathbb{R}}$ such that $Y_{\mathbb{R}}$ is locally a smooth submanifold of $\mathbb{R}^n$ at p, see Definition 3.2. We will see that this property and the fact that p is a limit point of real nonsingular points are intrinsic to Y, so we could also formulate Theorem 1.2 in a coordinate invariant way (without specifying an embedding).

Note that any variety which is locally a complete intersection and for which the codimension of the singular locus is greater than 1 is normal according to Serre's criterion [6, Theorem 39]. Thus, Theorem 1.2 provides a useful tool for the analysis of many algebraic sets $Y_{\mathbb{R}}$ with $\dim Y > 1$. For example, an immediate corollary for dimensions greater than 1 is that any isolated hypersurface singularity is either isolated in $Y_{\mathbb{R}}$ or a nonmanifold point.

Note also that if p is not a limit point of the real nonsingular points of Y one can find a euclidean neighbourhood U of p such that U does not contain any real nonsingular points of Y. We can now replace Y by the Zariski closure Y' of $U \cap Y_{\mathbb{R}}$. If we iterate this procedure we find an algebraic set $\tilde{Y}$ and a euclidean neighbourhood $\tilde{U}$ of p such that $\tilde{Y}_{\mathbb{R}} \cap \tilde{U} = Y_{\mathbb{R}} \cap \tilde{U}$ and p is in the euclidean closure of the nonsingular real points of $\tilde{Y}$. Since p will be a manifold point of $Y_{\mathbb{R}}$ if and only if it is a manifold point of $Y_{\mathbb{R}}$ we can now analyse $\tilde{Y}$ in place of Y.

If Y is not normal, one has the option to calculate a normalization $\varphi\colon Z \to Y$. We will see in Example 5.3 how to utilize φ and Theorem 1.2 to analyse many algebraic sets with $\dim Y > 1$. We will also see in which cases this approach does not work.

For algebraic curves normalization yields the following result:

Theorem 1.3. *Let Y be an $\mathbb{R}$-variety with $\dim Y = 1$ and $\varphi\colon Z \to Y$ a normalization of Y. Any real point $p \in Y_{\mathbb{R}} = Y \cap \mathbb{R}^n$ is a manifold point of $Y_{\mathbb{R}}$ if and only if there is exactly one real point in the fibre $\varphi^{-1}(p)$ and this is a simple root in $\varphi^{-1}(p)$.*

In the example of the curve $f(x, y) = y^3 + 2yx^2 - x^4$ with coordinate ring $A = \mathbb{R}[x, y]/\langle f \rangle$ the normalization $B = A[u]$ of A is generated by the integral element $u = \frac{y}{x} \in Q(A)$ and we have the normalization

$$\psi : \mathbb{R}[x, y]/\langle f \rangle \to B \cong \mathbb{R}[x, y, u]/\langle u^3 + 2u - x, ux - y \rangle.$$

Thus, lying over the origin is

$$\langle x, y\rangle_B = \langle u, x, y\rangle_B \cap \langle u^2 + 2, x, y\rangle_B. \quad (2)$$

Hence, there is exactly one real point $(0, 0, 0)$ lying over the origin and this is a simple root. According to Theorem 1.3, the origin must be a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$.

Further curve examples and also a discussion on how the conditions of Theorem 1.3 can be checked symbolically by CAS like Singular can be found in Example 6.5.

Before the examples from kinematics in Sections 7 and 8, the paper is structured as follows: in Sections 2 and 3 we review some well-known facts from commutative algebra, real algebra and differential geometry, which will enable us to make precise the notion of manifold point and deal with Examples (i)–(v). We will also focus on base extensions of affine algebras, which is very handy if one needs to extend results gained by algorithmic calculations in polynomial rings over $\mathbb{Q}$ to polynomial rings over $\mathbb{R}$.

In Section 4, we build the theoretical foundation for local analysis of real algebraic sets. Central to the exposition is Theorem 4.3, which gives an algebraic condition for manifold points and shows together with Risler's analytic Nullstellensatz that this is an intrinsic property. Finally, we can derive Theorem 1.2 from the previous theory and a criterion by G. Efroymson about local reality.

Section 5 deals with the problem that the extensions of a prime ideal of $\mathbb{R}[\bar{x}]$ to the ring of formal power series $\mathbb{R}[[\bar{x}]]$ will not be prime in general and symbolic calculations in $\mathbb{R}[[\bar{x}]]$ are not possible. Instead, we investigate the integral closure of the local ring to divide the associated primes of the extended ideal. The main result Proposition 5.2 goes back to Zariski and Samuel [7] and was extended by Ruiz [8] to a complete description of the normalization of $\mathcal{F}$, where $\mathcal{F}$ is the local ring $\mathbb{R}[[\bar{x}]]/(I \cdot \mathbb{R}[[\bar{x}]])$.

Ultimately, in Section 6 we formulate and prove Theorem 6.4, which is a ring-theoretic formulation of Theorem 1.3 and decides the case completely for real algebraic curves. This extends results of [9].

This paper is mostly build on the work of Risler [10], Efroymson [11], and Ruiz [8]. For further reading regarding local properties of real algebraic sets the author also recommends O'Shea and Charles [12] for their work on geometric Nash fibres and real tangent cones.

2 Algebraic preliminaries

In this section, let $\mathbb{K}$ be a field such that $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$, $\{f_1, \ldots, f_n\}$ a set of polynomials in $\mathbb{K}[\bar{x}]$, where $\bar{x} = (x_1, \ldots, x_n)$, and $I = \langle f_1, \ldots, f_n\rangle \leq \mathbb{K}[\bar{x}]$ the ideal generated by all f_i. We set $A = \mathbb{K}[\bar{x}]/I$ and consider two sets associated with A:

$$Y := \{x \in \mathbb{C}^n \mid f(x) = 0, \text{ for all } f \in \mathrm{I}\} = \mathbf{V}(I),$$
$$X := Y_{\mathbb{R}} = \{x \in \mathbb{R}^n \mid f(x) = 0, \text{ for all } f \in \mathrm{I}\} = \mathbf{V}_{\mathbb{R}}(I).$$

Sometimes we will call X the **real part** of Y. Since we can only perform symbolic computations over the rational numbers we need to investigate base changes of A. For any extension field $\mathbb{K} \subset \mathbb{L}$ and any ideal $J \leq A$ we set

$$I_{\mathbb{L}} := \mathbb{L} \otimes_{\mathbb{K}} I = I \cdot \mathbb{L}[\bar{x}], \quad A_{\mathbb{L}} := \mathbb{L} \otimes_{\mathbb{K}} A = \mathbb{L}[\bar{x}]/I_{\mathbb{L}}$$
$$J_{\mathbb{L}} := \mathbb{L} \otimes_{\mathbb{K}} J = (\hat{J} \cdot \mathbb{L}[\bar{x}])/I_{\mathbb{L}}, \quad \text{where } \hat{J} \leq \mathbb{K}[\bar{x}] \text{ with } \hat{J}/I = J.$$

If $\mathbb{L} = \mathbb{C}$, we call $A_{\mathbb{C}}$, $I_{\mathbb{C}}$ or $J_{\mathbb{C}}$ the complexification of A, I or J, respectively. Finally, for any $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$ we define the maximal ideal

$$\mathfrak{m}_p = \langle x_1 - p_1, \ldots, x_n - p_n\rangle \subset \mathbb{C}[\bar{x}].$$

Definition 2.1. The **singular locus** of A is the set of all prime ideals $\mathfrak{p} \in \operatorname{Spec} A$ such that $A_{\mathfrak{p}}$ is not regular. A point $p \in Y$ is called a **singularity** of Y if $((A_{\mathbb{C}})_{\mathrm{red}})_{\mathfrak{m}_p}$ is not regular, i.e. $\mathfrak{m}_p$ is in the singular locus of $(A_{\mathbb{C}})_{\mathrm{red}}$.

Recall that a local ring $(B, \mathfrak{m})$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim B$ with $k = A/\mathfrak{m}$, see [13, p. 123]. For localizations of affine rings this is usually checked with the Jacobian criterion [14, Theorem 5.7.1].

Remark. $(A_{\mathbb{C}})_{\mathrm{red}}$ denotes the reduction $A_{\mathbb{C}}/\sqrt{(0)}$ without nilpotents. The stacking of subscripts in $((A_{\mathbb{C}})_{\mathrm{red}})_{\mathfrak{m}_p}$ is admittedly horrible but we will see in Proposition 2.2 that there is some freedom in the choice of the coefficient field. So we can get rid of the complexification and/or the reduction in Definition 2.1 if I is radical and/or $\mathfrak{m}_p \trianglelefteq A$.

2.1 Base change

We review some facts from commutative algebra regarding extensions of the coefficient field.

Recall that for any prime ideal $\mathfrak{p} \trianglelefteq A$ of a commutative ring the **height** of $\mathfrak{p}$ is defined as the supremum of lengths of chains of prime ideals ending at $\mathfrak{p}$. Thus, the height of $\mathfrak{p}$ coincides with $\dim A_{\mathfrak{p}}$. For an arbitrary ideal $I \trianglelefteq A$, the height of I means the infimum of the heights of all prime ideals containing I. We will write $\operatorname{ht} I$ to denote the height of I. It is well known that A noetherian implies $\operatorname{ht} I < \infty$, for all ideals $I \trianglelefteq A$.

If A is a polynomial ring $k[\bar{x}]$ over an arbitrary field k, then it is an important fact that $\dim A/I = \dim A - \operatorname{ht} I$ [15, Corollary 13.4].

Proposition 2.2. *Let $\mathbb{L}$ be any field extension of $\mathbb{K}$. Then*

(*i*) $I_{\mathbb{L}} \cap \mathbb{K}[\bar{x}] = I$.

(*ii*) $\operatorname{ht} I_{\mathbb{L}} = \operatorname{ht} I$, $\dim A_{\mathbb{L}} = \dim A$.

(*iii*) $\sqrt{I_{\mathbb{L}}} = \sqrt{I}\,\mathbb{L}[\bar{x}]$.

(*iv*) *Let $\mathfrak{p} \trianglelefteq A$ prime. Then $A_{\mathfrak{p}}$ is regular iff $(A_{\mathbb{L}})_{\mathfrak{P}}$ is regular for one and then all associated primes $\mathfrak{P}$ of $\mathfrak{p}_{\mathbb{L}}$.*

Remark. Since we require $\mathbb{Q} \subset \mathbb{K}$, $\mathbb{K}$ is a perfect field in particular and therefore $\mathbb{L}$ separable over $\mathbb{K}$. This means (note that $\mathbb{K} \subset \mathbb{L}$ does not need to be algebraic) that every finitely generated subextension is separably generated over $\mathbb{K}$, see [15, A1.2]. (i) and (ii) would work for any field extension of any field $\mathbb{K}$, whereas (iii) and (iv) are in general wrong if $\mathbb{K} \subset \mathbb{L}$ is not separable.

Proof. (i) and (ii) follow because $\mathbb{K}[\bar{x}] \subset \mathbb{L}[\bar{x}]$ is a faithfully flat ring extension. (iii) is a consequence of the fact that any reduced $\mathbb{K}$-algebra is geometrically reduced [16, Lemmas 10.42.6 and 10.44.6]. We will show (iv) with the general Jacobian criterion [14, Theorem 5.7.1], since there appears to be no reference in the usual literature on commutative algebra.

First choose any associated prime $\mathfrak{P}$ of $\mathfrak{p}_{\mathbb{L}}$ and let $\hat{\mathfrak{p}}, \hat{\mathfrak{P}}$ denote the preimages of $\mathfrak{p}$ and $\mathfrak{P}$ in $\mathbb{K}[\bar{x}]$ and $\mathbb{L}[\bar{x}]$, respectively. Now write K for the quotient field of $\mathbb{K}[\bar{x}]/\hat{\mathfrak{p}}$ and K' for the quotient field of $\mathbb{L}[\bar{x}]/\hat{\mathfrak{P}}$. Since $\hat{\mathfrak{P}} \cap \mathbb{K}[\bar{x}] = \hat{\mathfrak{p}}$ [7, VII Theorem 36], K is clearly subfield of K'. For any K-vector space V we then have $\dim_K V = \dim_{K'} K' \otimes_K V$, since the tensor product commutes with direct sums. Consequently,

$$\operatorname{rank}\left[\frac{\partial f_i}{\partial x_j} \bmod \hat{\mathfrak{p}}\right]_{i,j} = \operatorname{rank}\left[\frac{\partial f_i}{\partial x_j} \bmod \hat{\mathfrak{P}}\right]_{i,j} =: h,$$

where $\langle f_1, \dots, f_n \rangle = I$, as stated in the beginning of Section 2.

Now assume $A_{\mathfrak{p}}$ is a regular local ring and choose an associated prime $\mathfrak{q}$ of I with $\mathfrak{q} \subset \hat{\mathfrak{p}}$ (note that there should be only one prime with this property, otherwise $A_{\mathfrak{p}}$ would not be regular). Then we conclude that $\operatorname{ht} \mathfrak{q} = h$ from the Jacobian criterion. Now any associated prime of $\mathfrak{q}_{\mathbb{L}}$ has height h as well [7, VII Theorem 36] and one of them is contained in $\hat{\mathfrak{P}}$. But then $(A_{\mathbb{L}})_{\mathfrak{P}}$ is regular according to the general Jacobian criterion.

On the contrary, assume that $(A_{\mathbb{L}})_{\mathfrak{P}}$ is regular. Then there exists an associated prime $\mathfrak{Q}$ of $I_{\mathbb{L}}$ with $\mathfrak{Q} \subset \hat{\mathfrak{P}}$ and $\operatorname{ht} \mathfrak{Q} = h$. Now since $\mathfrak{Q}$ is associated with $I_{\mathbb{L}}$, it is associated with $\mathfrak{r}_{\mathbb{L}}$ for a primary ideal $\mathfrak{r} \in \mathbb{K}[\bar{x}]$, which is part of a primary decomposition of I (use $(J_1 \cap J_2)\mathbb{L}[\bar{x}] = J_1\mathbb{L}[\bar{x}] \cap J_2\mathbb{L}[\bar{x}]$ for ideals $J_1, J_2 \leq \mathbb{K}[\bar{x}]$ and $\mathfrak{Q} = (I_{\mathbb{L}} : \langle b \rangle)$ for some $b \in \mathbb{L}[\bar{x}]$). So $\mathfrak{q} := \sqrt{\mathfrak{r}}$ is a prime ideal associated with I. Now

$$\mathfrak{r} = \mathfrak{r}_{\mathbb{L}} \cap \mathbb{K}[\bar{x}] \subset \mathfrak{Q} \cap \mathbb{K}[\bar{x}] \subset \hat{\mathfrak{P}} \cap \mathbb{K}[\bar{x}] = \hat{\mathfrak{p}}.$$

But then $\mathfrak{q} = \sqrt{\mathfrak{r}} \subset \hat{\mathfrak{p}}$. Also, $h = \operatorname{ht}\mathfrak{Q} \geq \operatorname{ht}\mathfrak{q}$. Consequently, $A_{\mathfrak{p}}$ is regular according to the general Jacobian criterion. □

2.2 Locally complete intersection rings

For polynomial systems $\langle f_1,\ldots,f_k\rangle \leq \mathbb{K}[\bar{x}]$ arising from engineering problems, it is often the case that $\dim \mathbb{K}[\bar{x}]/I = n - k$, i.e. the codimension of the variety defined by $f_1,\ldots,f_k$ is just k. Then $\mathbb{K}[\bar{x}]/I$ is locally a complete intersection [15, p. 462] and in particular Cohen-Macaulay. This has very useful implications:

Proposition 2.3. *Let $I = \langle f_1,\ldots,f_k\rangle \leq \mathbb{K}[\bar{x}]$ be generated by $k \leq n$ elements. Assume also that* $\dim I = n - k$ *or equivalently* $\operatorname{ht} I = k$. *Then*
(*i*) *I is equidimensional, i.e. any associated prime of I has the same dimension $n - k$.*
(*ii*) *Let J be the ideal of the k-minors of the Jacobian of $(f_1,\ldots,f_k)$. If* $\operatorname{ht}(I + J) > k$, *then I is radical.*

Proof. The first statement is just the unmixedness theorem [15, Corollary 18.14]. Note also that any associated prime of I is minimal over I.

The second statement follows from [15, Theorem 18.15] because $\mathbb{K}[\bar{x}]/I$ is Cohen-Macaulay according to [15, Proposition 18.13]. □

2.3 Real algebra

We review some facts from real algebra. Most of them can be found in [1] or [17].

Definition 2.4. Let B be any commutative ring and $I \leq B$ an ideal. B is called (formally) **real**, if and only if any equation

$$b_1^2 + \ldots + b_k^2 = 0, \quad k \geq 1,$$

implies $b_1 = \ldots = b_k = 0$. I is called **real**, if B/I is real. Also, we define the **real radical**

$$\sqrt[\mathbb{R}]{I} = \{x \in B \mid x^{2r} + b_1^2 + \ldots + b_k^2 \in I, \text{ for } r, k \geq 0, b_i \in B\},$$

which is either the smallest real ideal containing I or B if there are no real ideals between I and B, cf. [17]. Therefore, I is real if and only if $\sqrt[\mathbb{R}]{I} = I$.

The analogue to Hilbert's Nullstellensatz in real algebraic geometry is as follows:

Proposition 2.5. (Risler's Real Nullstellensatz [1]) *Let $I \leq \mathbb{K}[\bar{x}]$ be any ideal. Then*

$$\mathbf{I}_{\mathbb{K}}(\mathbf{V}_{\mathbb{R}}(I)) = \sqrt[\mathbb{R}]{I}.$$

The next result is the primary tool to check ideals $I \leq \mathbb{R}[\bar{x}]$ for realness.

Proposition 2.6. (Simple Point Criterion [1]) *Let $\mathbb{K} = \mathbb{R}$ and $I \leq \mathbb{R}[\bar{x}]$. Then I is real if and only if I is radical and for every associated prime $\mathfrak{p}$ of I there exists $x \in \mathbf{V}_{\mathbb{R}}(\mathfrak{p})$, with $A_{\mathfrak{m}_x}$ regular.*

Remarks.

(i) We have the following generalization for ideals $I \le \mathbb{Q}[\bar{x}]$: I is real if and only if I is radical and for every associated prime $\mathfrak{p}$ of I, there exists $x \in \mathbf{V}_{\mathbb{R}}(\mathfrak{p})$ with $(A_{\mathbb{R}})_{\mathfrak{m}_x}$ regular. See [18, Theorem 1.3].

(ii) There are algorithms to compute the real radical of an ideal $J \le \mathbb{Q}[\bar{x}]$ (e.g. `realrad` in Singular), but to the author's knowledge, all implemented algorithms so far only compute over $\mathbb{Q}$ since there is ambiguity in the ordering of field extensions of $\mathbb{Q}$ (in Singular we get $\texttt{realrad}(x^3 - 5y^3) = x^3 - 5y^3$).

Examples 2.7

(i) $\mathbb{C}$ is clearly not real, since $1^2 + i^2 = 0$, but $\mathbb{Q}$ and $\mathbb{R}$ are. Also, any domain B is real if and only if its field of fractions $Q(B)$ is real. $Q(B)$ can be ordered in this case.

(ii) Consider the ideal $I = \langle x^2 + y^2 \rangle \le \mathbb{R}[x, y, z]$ from Example 1.1(iv). I is not real, since $x, y \notin I$. We see easily from the definition that $x, y \in \sqrt[r]{I}$ and from the real Nullstellensatz follows that $1 \notin \sqrt[r]{I}$. Hence, $\sqrt[r]{I} = \langle x, y \rangle$.

(iii) Let $I = \langle x^3 - 5y^3 \rangle \le \mathbb{Q}[x, y]$ from Example 1.1(v). Then I is prime in $\mathbb{Q}[x, y]$. Since there exist points $p \in \mathbf{V}_{\mathbb{R}}(I)$ with $(\mathbb{R}[x, y]/I_{\mathbb{R}})_{\mathfrak{m}_p}$ regular, I must be real in $\mathbb{Q}[x, y]$, see remark (i) after Proposition 2.6. $I_{\mathbb{R}}$ is not real, however, since $\sqrt[r]{I_{\mathbb{R}}} = \mathbf{I}_{\mathbb{R}}(\mathbf{V}_{\mathbb{R}}(I_{\mathbb{R}})) = \langle x - \sqrt[3]{5}y \rangle$. This is different for the standard radical, see Proposition 2.2.

(iv) $f(x, y) = y^3 + 2x^2y - x^4$ from Example 1.1(vi) is an irreducible polynomial in $\mathbb{R}[\bar{x}]$ and for any $x_0 \neq 0$; there exists a real solution $y_0 \in \mathbb{R}$ of $f(x_0, y) = 0$, since this is a polynomial of degree 3. Also, the local ring at (x_0, y_0) is regular with the Jacobian criterion, hence according to Proposition 2.6, $I = \langle f \rangle$ is a real ideal of $\mathbb{R}[x, y]$.

(v) Any zero-dimensional ideal $J \le \mathbb{R}[\bar{x}]$ is real if and only if J is radical and $\mathbf{V}(J) \subset \mathbb{R}^n$.

(vi) With the real Nullstellensatz it is clear that I is real if and only if $\mathbf{I}_{\mathbb{K}}(\mathbf{V}_{\mathbb{R}}(I)) = I$. This means $I \le \mathbb{R}[\bar{x}]$ is real if and only if $\mathbf{V}_{\mathbb{R}}(I)$ is Zariski-dense in $\mathbf{V}(I)$.

3 Analytic preliminaries

In this section, we set $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Any open subset $U \subset \mathbb{K}^n$ is meant to be euclidean open. f is called **analytic** at $p \in U$ (or holomorphic for $\mathbb{K} = \mathbb{C}$), if

$$f(z) = \sum c_{i_1}, \dots c_{i_n}(z_1 - p_1)^{i_1} \dots (z_n - p_n)^{i_n}$$

in a neighbourhood of p.

A d-dimensional smooth (analytic, complex) **submanifold** of $\mathbb{K}^n$ is a set $X \subset \mathbb{K}^n$ such that for every p in X there exists an open neighbourhood $U \subset \mathbb{K}^n$ of p and a C^∞-diffeomorphism (C^ω, biholomorphism) $\phi : U \to V$ to an open set $V \subset \mathbb{K}^n$, with

$$X \cap U = \{x \in U \mid \phi_{d+1}(x) = \dots = \phi_n(x) = 0\}.$$

A set $X \subset \mathbb{K}^n$ with point $p \in X$ is locally at p **the graph** of an analytic (smooth, holomorphic) mapping (in the first d coordinates), if there exists an open neighbourhood U of p and an analytic (smooth, holomorphic) mapping $\psi : \rho(U) \to \mathbb{K}^{n-d}$ such that

$$X \cap U = \{(y, \psi(y)) \mid y \in \rho(U)\},$$

where $\rho : \mathbb{K}^n \to \mathbb{K}^d$ is the projection to the first d coordinates. Note that it needs to be checked that this gives a local definition. We leave this task to the reader.

Proposition 3.1. *Let $X \subset \mathbb{K}^n$ be any set and $p \in X$. The following conditions are equivalent:*

(*a*) *There is an open neighbourhood U of p such that $X \cap U$ is an analytic (smooth, holomorphic) submanifold of $\mathbb{K}^n$.*

(*b*) *There exists a permutation* $\pi\colon \mathbb{K}^n \to \mathbb{K}^n$ *of coordinates such that*

$$\pi(X) = \{\pi(x) \mid x \in X\}$$

is locally the graph of an analytic (smooth, holomorphic) mapping at $\pi(p)$.

(*c*) *For a generic choice of* $A \in \mathrm{GL}(n, \mathbb{K})$, $A(X)$ *is locally the graph of an analytic (smooth, holomorphic) mapping at* Ap.

Remark. Generic choice means as usual the complement of a proper algebraic subset of $\mathrm{GL}(n, \mathbb{K})$. Such a subset is dense in $\mathrm{GL}(n, \mathbb{K})$ in the euclidean topology.

Definition 3.2. A point p of a set $X \subset \mathbb{K}^n$ is an analytic (smooth, holomorphic) **manifold point** of X, if any of the equivalent conditions of Proposition 3.1 is fulfilled.

Any smooth mapping parameterizing a real algebraic set will be a smooth semi-algebraic mapping whose component functions are known to be Nash functions [17, Definition 2.9.3, Proposition 8.1.8] and in particular analytic. We get the following proposition:

Proposition 3.3. *Let* $\mathbb{K} = \mathbb{R}$ *and* $X \subset \mathbb{R}^n$ *be a real algebraic set, with* $p \in X$. p *is an analytic manifold point of* X *if and only if* p *is a smooth manifold point of* X.

In light of Proposition 3.3, it is enough to work with analytic manifold points if one considers algebraic subsets of $\mathbb{R}^n$. From now on manifold point means analytic/holomorphic manifold point.

4 Local real algebraic geometry

We now assume $\mathbb{K} = \mathbb{R}$ and that the origin is a singular point of a real algebraic set X. So we have an ideal $I \trianglelefteq \mathbb{R}[\bar{x}]$ with $I \subset \langle \bar{x} \rangle =: \mathfrak{m}$ and $A_{\mathfrak{m}}$ is not regular, where $A = \mathbb{R}[\bar{x}]/I$. As we have seen in Example 1.1(vi) we need to investigate the extension of I in the ring of convergent/formal power series. The following notations will be used in the rest of the paper:

Definition 4.1.
(i) $I_{\mathfrak{m}} = I \cdot \mathbb{R}[\bar{x}]_{\mathfrak{m}}, \quad \mathcal{R} = \mathbb{R}[\bar{x}]_{\mathfrak{m}}/I_{\mathfrak{m}} = A_{\mathfrak{m}}, \quad \mathfrak{r} = \mathfrak{m}\mathcal{R}.$
(ii) $I' = I \cdot \mathbb{R}\{\bar{x}\}, \quad \mathcal{O} = \mathbb{R}\{\bar{x}\}/I', \quad \mathfrak{o} = \mathfrak{m}\mathcal{O}.$
(iii) $I'' = I \cdot \mathbb{R}[[\bar{x}]], \quad \mathcal{F} = \mathbb{R}[[\bar{x}]]/I'', \quad \mathfrak{f} = \mathfrak{m}\mathcal{F}.$

Since the ring extensions $\mathbb{R}[\bar{x}]_{\mathfrak{m}} \to \mathbb{R}\{\bar{x}\} \to \mathbb{R}[[\bar{x}]]$ are faithfully flat as flat local homomorphism [6, Corollary 4.A], we have the following chain of local rings:

$$\mathcal{R} \subset \mathcal{O} \subset \mathcal{F}.$$

We will also need the fact that $\mathcal{F}$ is the $\mathfrak{r}$-adic completion of $\mathcal{R}$:

$$\mathcal{F} = \varprojlim_k \mathcal{R}/\mathfrak{r}^k.$$

Now we define the following ideal of $\mathbb{R}\{\bar{x}\}$, which is usually called the vanishing ideal of the set germ $(X,0)$ [10]. We can do a similar construction in $\mathbb{R}[[\bar{x}]]$, but since $f(p)$ is not defined in general for elements $p \in \mathbb{R}^n$, $f \in \mathbb{R}[[\bar{x}]]$, we need to replace points in $\mathbb{R}^n$ with tuples of formal Puiseux series without constant term. See [8, Definition IV.4] for this approach.

Definition 4.2.

$$\hat{I} = \left\{ f \in \mathbb{R}\{\bar{x}\} \;\middle|\; \begin{array}{l} \exists U \ni 0 \text{ euclidean neighbourhood with } f \\ \text{converging on } U \text{ and } f \equiv 0 \text{ on } X_{\mathbb{R}} \cap U \end{array} \right\}, \quad \hat{O} = \mathbb{R}\{\bar{x}\}/\hat{I}.$$

Theorem 4.3. *The origin is a manifold point of X if and only if $\hat{O}$ is regular.*

Proof. First, let the origin be a manifold point of X (of dimension d). According to Proposition 3.1, we find w.l.o.g an analytic parameterization of X:

$$\Psi : U \mapsto \mathbb{R}^n,$$

$$(x_1, \dots, x_d) \to (x_1, \dots, x_d, \psi_1(x_1, \dots, x_d), \dots, \psi_{n-d}(x_1, \dots, x_d)),$$

where U is an euclidean neighbourhood of the origin in $\mathbb{R}^d$ and $\Psi(0) = 0$. We set

$$L := \langle x_{d+1} - \psi_1(x_1, \dots, x_d), \dots, x_n - \psi_{n-d}(x_1, \dots x_d) \rangle \leq \mathbb{R}\{\bar{x}\}$$

and claim that $L = \hat{I}$.

Clearly we have $L \subset \hat{I}$, so let $a \in \hat{I}$. Since $\Psi(0) = 0$, we can compose a and Ψ and get a converging power series

$$a(x_1, \dots x_d, \psi_1(x_1, \dots, x_d), \dots \psi_{n-d}(x_1, \dots x_d)) = 0, \tag{3}$$

which follows because $a \circ \Psi$ is identically zero close to the origin. We now set $\varphi_i := x_{d+i} - \psi_i(x_1, \dots x_d) \in \mathbb{R}\{\bar{x}\}$, for $i = 1, \dots, n-d$ and have that φ_i is of x_{d+i}-order 1. According to the Weierstrass division theorem [8, Proposition 3.2] we have a representation

$$a = q_1 \cdot \varphi_1 + r,$$

with $q_1 \in \mathbb{R}\{\bar{x}\}$ and $r \in \mathbb{R}\{x_1, \dots, x_d, x_{d+2}, \dots, x_{n-1}\}$. If we iterate this process with r instead of a, we have a decomposition

$$a = q_1 \cdot \varphi_1 + \dots + q_{n-d} \cdot \varphi_{n-d} + r,$$

with $r \in \mathbb{R}\{x_1, \dots, x_d\}$. Because of (3) and

$$\varphi_i(x_1, \dots x_d, \psi_1(x_1, \dots, x_d), \dots, \psi_{n-d}(x_1, \dots, x_d)) = \psi_i(x_1, \dots, x_d) - \psi_i(x_1, \dots, x_d) = 0,$$

we have

$$r(x_1, \dots, x_d) = 0,$$

so $r = 0$ and therefore $a \in L$.

It now remains to check that $\mathbb{R}\{\bar{x}\}/L$ is a regular local ring. We will use Nagata's Jacobian criterion [8, Proposition 4.3]. With $\mathfrak{m}'$ denoting the maximal ideal of $\mathbb{R}\{\bar{x}\}$, it is enough to show that $\mathfrak{m}' \not\supset J_{n-d}(L)$ and $\operatorname{ht}(L) \leq n-d$, where $J_{n-d}(L)$ is the Jacobian ideal of L of order $n-d$ [8, Section 4.1]. Since L is generated by $n-d$ elements $\operatorname{ht}(L) \leq n-d$ follows easily from Krull's height theorem [13, Corollary 11.16]. Now the Jacobian

$$\frac{D(\varphi_1, \dots, \varphi_{n-d})}{D(x_{d+1}, \dots, x_n)} = \det \left[\frac{\partial \varphi_i}{\partial x_j} \right]_{\substack{i=1,\dots,n-d \\ j=d+1,\dots,n}} = 1,$$

hence $J_{n-d}(L) = \mathbb{R}\{\bar{x}\} \not\subset \mathfrak{m}'$. This means that $\mathbb{R}\{\bar{x}\}/L$ is a regular local ring and $\dim \mathbb{R}\{\bar{x}\}/L = d$.

Now on the contrary let $\hat{O}$ be regular with $\dim \hat{O} = d$. According to Nagata's Jacobian criterion, it is $\mathfrak{m}' \not\supset J_{n-d}(\hat{I})$. For that reason there are $g_1, \dots, g_{n-d} \in \hat{I}$ such that w.l.o.g

$$\frac{D(g_1, \dots, g_{n-d})}{D(x_1, \dots, x_{n-d})} \notin \mathfrak{m}'.$$

This means that the submatrix comprising the first $(n-d)$ columns of the Jacobian matrix of $(g_1,\dots,g_{n-d})$ evaluated at the origin has full rank. Now let U be a euclidean environment of the origin in $\mathbb{R}^n$ such that U is contained in the region of convergence of all g_i. We set

$$X' := \{x \in U \mid g_i(x) = 0, \quad \text{for all } i\}.$$

According to the analytic implicit function theorem [19, Theorem 1.8.3] the origin is a manifold point of X' and we only have to show that X' agrees with X on a neighbourhood of the origin. This follows easily if we can prove $L := \langle g_1,\dots,g_{n-d}\rangle = \hat{I}$.

Since $g_1,\dots,g_{n-d} \in \hat{I}$ we clearly have $L \subset \hat{I}$. On the other hand, because $\mathfrak{m}' \not\supset J_{n-d}(L)$ and $\mathrm{ht}(L) \le n-d$ we can apply Nagata's Jacobian criterion again to see that $\mathbb{R}\{\bar{x}\}/L$ is a regular local ring of dimension d. Then $\mathbb{R}\{\bar{x}\}/L$ is also an integral domain [13, Lemma 11.23], so L is a prime ideal. Because $\mathbb{R}\{\bar{x}\}$ is local and Cohen-Macaulay (or use [8, Proposition 2.4]) we have $\mathrm{ht}(L) = \dim \mathbb{R}\{\bar{x}\} - \dim \mathbb{R}\{\bar{x}\}/L = n-d$. But $\hat{I}$ is prime as well with $\mathrm{ht}(\hat{I}) = n-d$. Since $L \subset \hat{I}$, we have $L = \hat{I}$. This completes the proof. □

The following theorem due to J.-J. Risler allows the calculation of $\hat{I}$.

Theorem 4.4. (Real Analytic Nullstellensatz [10, Theorem 4.1]).

$$\hat{I} = \sqrt[r]{I'} = \left\{ f \in \mathbb{R}\{\bar{x}\} \,\middle|\, f^{2n} + b_1^2 + \ldots + b_k^2 \in I',\ r,\ k \ge 0, b_i \in \mathbb{R}\{\bar{x}\} \right\}, \quad \hat{O} = O/\sqrt[r]{(0)}\,.$$

The next proposition collects some well-known facts on the relationship of the rings $\mathcal{R}$, O and $\mathcal{F}$.

Proposition 4.5. *Let $\mathcal{R}$, O and $\mathcal{F}$ be as in Definition 4.1. Then*
(*a*) *$\mathcal{R}$ reduced $\Leftrightarrow$ O reduced $\Leftrightarrow$ $\mathcal{F}$ reduced.*
(*b*) *$\mathcal{R}$ normal domain $\Leftrightarrow$ O normal domain $\Leftrightarrow$ $\mathcal{F}$ normal domain.*
(*c*) *$\mathcal{R}$ regular $\Leftrightarrow$ O regular $\Leftrightarrow$ $\mathcal{F}$ regular.*
(*d*) *I' real $\Leftrightarrow$ I'' real, i.e. O real $\Leftrightarrow$ $\mathcal{F}$ real.*
(*e*) *$O/\sqrt[r]{(0)}$ is regular if and only if $\mathcal{F}/\sqrt[r]{(0)}$ is regular.*
(*f*) *If $\mathcal{F}, O$ regular, then $\mathcal{F}, O$ real.*

Proof of Proposition 4.5. The proofs for (a), (b), (c) and (d) can be found in [8, Chapter V,VI]. (e) is also an easy consequence of results in [8]. We will carry out a proof for completeness sake.

We only have to show that $\sqrt[r]{I'} \cdot \mathbb{R}[[\bar{x}]] = \sqrt[r]{I''}$. Then the statement follows from [8, Proposition V.4.5]. $\sqrt[r]{I'} \cdot \mathbb{R}[[\bar{x}]] \subset \sqrt[r]{I''}$ is clear, so we proceed to show $\sqrt[r]{I'} \cdot \mathbb{R}[[\bar{x}]] \supset \sqrt[r]{I''}$ by using the argument in the proof of [8, Theorem V.4.2]. Let $f \in \sqrt[r]{I''}$, which means

$$f^{2s} + p_1^2 + \ldots + p_k^2 \in I'',$$

for elements $p_1,\dots,p_k \in \mathbb{R}[[\bar{x}]]$. Consequently, we have

$$\bar{f}^{2s} + \bar{p}_1^2 + \ldots + \bar{p}_k^2 = 0,$$

in $\mathcal{F}$. According to M. Artin's approximation theorem in the form of [8, Proposition V.4.1] we find elements $\hat{f}, \hat{p}_1,\dots,\hat{p}_k \in O$, for every $\alpha \ge 1$, such that

$$\hat{f}^{2s} + \hat{p}_1^2 + \ldots \hat{p}_k^2 = 0,$$

and $\bar{f} = \hat{f} \bmod \mathfrak{f}^\alpha$ (recall that $\mathfrak{f}$ is the maximal ideal of $\mathcal{F}$). Then for every $\alpha \ge 0$:

$$f \in \sqrt[r]{I'} \cdot \mathbb{R}[[\bar{x}]] + (\mathfrak{m}'')^\alpha,$$

where $\mathfrak{m}'' = \mathfrak{m} \cdot \mathbb{R}[[\bar{x}]]$ is the maximal ideal of $\mathbb{R}[[\bar{x}]]$. It follows

$$f \in \bigcap_\alpha \left(\sqrt[r]{I'} \cdot \mathbb{R}[[\bar{x}]] + (\mathfrak{m}'')^\alpha \right) = \sqrt[r]{I'} \cdot \mathbb{R}[[\bar{x}]],$$

since any ideal of $\mathbb{R}[[\bar{x}]]$ is closed in the $\mathfrak{m}''$-adic topology.

Now we go on to show (f). Suppose $\mathcal{F}$ is a regular local ring. Because $\mathcal{F}/\mathfrak{f} \cong \mathcal{R}/\mathfrak{r} \cong \mathbb{R}$ is real, $\mathcal{F}$ must be real according to [1, Proposition 2.7]. ☐

Remark. With some minor modifications all the theory so far in Section 4 (except the statements about realness) would also work if we exchange $\mathbb{R}$ with $\mathbb{C}$ and Theorem 4.4 with Rückert's analytic Nullstellensatz [20, Theorem 2.20, Theorem 3.7], which states that $\hat{I} = \sqrt{I'}$ in the complex setting. Then we see easily from Proposition 4.5 why there is no need in complex algebraic geometry to consider the completion of $\mathcal{R}$ to decide if $\hat{\mathcal{O}}$ is regular. Because for $\mathbb{K} = \mathbb{C}$ we have $\hat{\mathcal{O}} = \mathcal{O}/\sqrt{(0)} = \mathcal{O}$ if $\mathcal{R}$ is reduced, and $\mathcal{O}$ is regular iff $\mathcal{R}$ is regular.

In the real case, it is not enough for I to be real to imply the realness of I', see Example 1.1(vi), hence $\hat{I}$ is in general bigger than I' and the nonregularity of $\mathcal{R}$ does not imply the nonregularity of $\hat{\mathcal{O}}$. On the other hand, if $\mathcal{R}$ is regular, then $\mathcal{O}$ is regular and real, hence also $\hat{\mathcal{O}} = \mathcal{O}$.

Corollary 4.6. *Let I'' or I' be real. The origin is a manifold point of $X = \mathbf{V}_{\mathbb{R}}(I)$, if and only if the origin is nonsingular.*

In Section 6, we analyse thoroughly how to check that I'' is real for $\dim I = 1$. But there is also a useful criterion for many higher-dimensional algebraic sets.

Theorem 4.7. (G. Efroymson [11]) *Let $\mathbb{K} = \mathbb{R}$, $I \leq \mathbb{R}[\bar{x}]$ a real prime with $I \subset \langle \bar{x} \rangle$ and $\mathcal{R}$ integrally closed. $\mathcal{F}$ is real if and only if the origin is contained in the euclidean closure of the nonsingular points of $\mathbf{V}_{\mathbb{R}}(I)$.*

This leads directly to the first of our main results listed in the introduction. We repeat it here for completeness sake:

Theorem 1.2. *Let Y be an irreducible $\mathbb{R}$-variety embedded in $\mathbb{C}^n$. Assume Y is normal at $p \in Y$ and p is in the euclidean closure of the real nonsingular points of Y. Then p is a manifold point of $Y_{\mathbb{R}} = Y \cap \mathbb{R}^n$ if and only if Y is nonsingular at p.*

Proof. According to Corollary 4.6 and Theorem 4.7, the only thing we need to show is that $\mathbf{I}(Y)$ is real. But this follows from the simple point criterion, Proposition 2.6. ☐

Corollary 4.8. *Let X be a normal, irreducible real algebraic variety embedded in euclidean space. Any isolated singularity of X is either a nonmanifold point of X or isolated in X.*

See Example 5.3 for a demonstration of Theorem 1.2 and Corollary 4.8. If X is not normal, we can calculate a normalization of X and utilize Efroymson's criterion again. This will be the subject of Section 5.

5 Normalization and analytic branches

In order to decompose the extended ideal I'', we examine the normalization of $\mathcal{F}$, which can be compared to the normalization of $\mathcal{R}$. We assume again $\mathbb{K} = \mathbb{R}$, but also require $I \leq \mathbb{R}[\bar{x}]$ to be a radical ideal, with minimal decomposition

$$I = \mathfrak{p}_1{}' \cap \ldots \cap \mathfrak{p}_k{}', \quad \mathfrak{p}_i{}' \leq \mathbb{R}[\bar{x}] \quad \text{prime}, \quad i = 1, \ldots, k.$$

Now let w.l.o.g $\mathfrak{p}_1{}', \ldots, \mathfrak{p}_s{}' \subset \mathfrak{m} = \langle \bar{x} \rangle$ and $\mathfrak{p}'_{s+1}, \ldots, \mathfrak{p}'_n \not\subset \mathfrak{m}$. In $\mathcal{R} = (\mathbb{R}[\bar{x}]/I)_{\mathfrak{m}}$, we then have the following minimal primary decomposition of the zero ideal:

$$(0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s, \tag{4}$$

where $\mathfrak{p}_i$ is the prime ideal generated by $\mathfrak{p}_i{}'$ in $\mathcal{R}$ for $i = 1,\ldots, s$. From now on we will use the notation $\mathcal{R}_i = \mathcal{R}/\mathfrak{p}_i$ and for any reduced ring A we will write $\overline{A}$ for the integral closure of A in its total ring of fractions. The following lemma collects some well-known facts about the integral closure of reduced local rings. Recall that $\mathfrak{r}$ is the maximal ideal of $\mathcal{R}$.

Lemma 5.1. *It is*

$$\overline{\mathcal{R}} = \overline{\mathcal{R}_1} \times \ldots \times \overline{\mathcal{R}_s}$$

a product of semi-local normal domains. Additionally, we have

$$\sqrt{\mathfrak{r} \cdot \overline{\mathcal{R}}} = (\mathfrak{n}_{11} \cap \ldots \cap \mathfrak{n}_{1k_1}) \cap \ldots \cap (\mathfrak{n}_{s1} \cap \ldots \cap \mathfrak{n}_{sk_s}),$$

where $\mathfrak{n}_{ij}$ *are the maximal ideals of* $\overline{\mathcal{R}}$ *in the form:*

$$\mathfrak{n}_{ij} = \overline{\mathcal{R}_1} \times \ldots \times \overline{\mathcal{R}_{i-1}} \times \mathfrak{n}'_{ij} \times \overline{\mathcal{R}_{i+1}} \times \ldots \times \overline{\mathcal{R}_s},$$

and $\mathfrak{n}_{ij}{}'$ *is one of the* k_i *maximal ideals of* $\overline{\mathcal{R}_i}$. *We also have the following minimal primary decomposition* $\sqrt{\mathfrak{r} \cdot \overline{\mathcal{R}_i}} = \mathfrak{n}'_{i1} \cap \ldots \cap \mathfrak{n}'_{ik_i}$ *and*

$$\overline{\mathcal{R}_{\mathfrak{n}_{ij}}} \cong (\overline{\mathcal{R}_i})_{\mathfrak{n}'_{ij}}.$$

We now want to compare the normalization of $\mathcal{F}$ and the completion of $\overline{\mathcal{R}}$, so we need to investigate what form $\overline{\mathcal{R}}_\mathfrak{n}$ can take for $\mathfrak{n} \leq \overline{\mathcal{R}}$ maximal. Since $\overline{\mathcal{R}}_\mathfrak{n} = (\overline{\mathcal{R}}_i)_{\mathfrak{n}'}$ for some i and $\mathfrak{n}' \leq \overline{\mathcal{R}}_i$ maximal we assume that $\mathcal{R}$ is a domain. The following exposition is taken from [8, Section VI.4], which can be checked for details. Because

$$\mathbb{R} = \mathcal{R}/\mathfrak{r} \subset \overline{\mathcal{R}}/\mathfrak{n}$$

is an algebraic field extension it must be $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{C}, \mathbb{R}$. We distinguish between the following three cases:

(a) $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{R}$. Since $\overline{\mathcal{R}}$ is finitely generated over $\mathcal{R}$ [8, Proposition III.2.3], we can extend a surjection $\mathbb{R}[\bar{x}]_\mathfrak{m} \to \mathcal{R}$ to a surjection $\mathbb{R}[\bar{x}, \bar{y}]_{\langle \bar{x},\bar{y}\rangle} \to \overline{\mathcal{R}}_\mathfrak{n}$, for new variables $\bar{y} = (y_1,\ldots, y_m)$. Hence, $\overline{\mathcal{R}}_\mathfrak{n} \cong \mathbb{R}[\bar{x}, \bar{y}]_{\langle \bar{x},\bar{y}\rangle}/J$ and its formal completion $(\overline{\mathcal{R}}_\mathfrak{n})^* = \mathbb{R}[[\bar{x}, \bar{y}]]/J\mathbb{R}[[\bar{x}, \bar{y}]]$ is the $\mathfrak{n}$-adic completion of $\overline{\mathcal{R}}_\mathfrak{n}$.

(b) $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{C}$ and $\sqrt{-1} \in Q(\mathcal{R})$. Since $\overline{\mathcal{R}}$ is integrally closed, $\mathbb{C} \subset \overline{\mathcal{R}}$. Then we get a surjection $\mathbb{C}[\bar{x}, \bar{y}]_{\langle \bar{x},\bar{y}\rangle} \to \overline{\mathcal{R}}_\mathfrak{n}$. Hence, $\overline{\mathcal{R}}_\mathfrak{n} \cong \mathbb{C}[\bar{x}, \bar{y}]_{\langle \bar{x},\bar{y}\rangle}/J$ and the formal completion $(\overline{\mathcal{R}}_\mathfrak{n})^* = \mathbb{C}[[\bar{x}, \bar{y}]]/J\mathbb{C}[[\bar{x}, \bar{y}]]$ is the $\mathfrak{n}$-adic completion of $\overline{\mathcal{R}}_\mathfrak{n}$.

(c) $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{C}$ and $\sqrt{-1} \notin Q(\mathcal{R})$. Now we need to adjoin $\sqrt{-1}$ to $\overline{\mathcal{R}}$ and we get a unique maximal ideal $\mathfrak{n}'$ in $\overline{\mathcal{R}}[\sqrt{-1}\,]$ over $\mathfrak{n}$ and the formal completion $(\overline{\mathcal{R}}_\mathfrak{n})^*$ is considered as the formal completion of $(\overline{\mathcal{R}}[\sqrt{-1}\,])_{\mathfrak{n}'}$ as in (b). It can be shown that this is isomorphic to the $\mathfrak{n}$-adic completion of $\overline{\mathcal{R}}_\mathfrak{n}$.

Now for $i = 1,\ldots, s$ we set $\mathcal{F}_i \coloneqq \mathcal{F}/(\mathfrak{p}_i \cdot \mathcal{F}) = \mathbb{R}[[\bar{x}]]/(\mathfrak{p}_i{}' \cdot \mathbb{R}[[\bar{x}]])$.

Proposition 5.2. (Ruiz, Zariski [8, Proposition VI.4.4]) *With the notations in this section and* Lemma 5.1, *we have for any* $i = 1,\ldots, s$:

$$\overline{\mathcal{F}_i} = [(\overline{\mathcal{R}_i})_{\mathfrak{n}_{i1}}]^* \times \ldots \times [(\overline{\mathcal{R}_i})_{\mathfrak{n}_{ik_i}}]^*$$

and $[(\overline{\mathcal{R}_i})_{\mathfrak{n}_{ij}}]^* \cong \overline{\mathcal{F}_i/\mathfrak{q}_{ij}}$, *where* $\mathfrak{q}_{i1},\ldots, \mathfrak{q}_{ik_i}$ *are the associated primes of* (0) *in* $\mathcal{F}_i$. *Additionally,*

$$\overline{\mathcal{F}} = \overline{\mathcal{F}_1} \times \ldots \times \overline{\mathcal{F}_s}.$$

Remark. The importance of Proposition 5.2 for us lies in the fact that $\mathcal{F}$ is real if and only if $\overline{\mathcal{F}}$ is real, so we can check realness on completions of local rings of normal irreducible varieties. This is very useful if we want to apply Efroymson's criterion, Theorem 4.7. See Example 5.3.

Proof. The only thing missing from the proof in [8] is to take into account nondomains $\mathcal{R}$, so we need to check

$$\overline{\mathcal{F}} = \overline{\mathcal{F}_1} \times \ldots \times \overline{\mathcal{F}_s}.$$

According to Chevalley's theorem [8, Proposition VI.2.1], we have a minimal primary decomposition

$$\mathfrak{p}_i \cdot \mathcal{F} = \mathfrak{q}'_{i1} \cap \ldots \cap \mathfrak{q}'_{ik_i},$$

with $\mathfrak{q}'_{ij}$ prime of height $\operatorname{ht} \mathfrak{p}_i =: d_i$ and $\mathfrak{q}_{ij} = \mathfrak{q}'_{ij} \cdot \mathcal{F}_i$. It remains to show that

$$(0) = (\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s) \cdot \mathcal{F} = \mathfrak{q}'_{11} \cap \ldots \cap \mathfrak{q}'_{1k_1} \cap \ldots \cap \mathfrak{q}'_{s1} \cap \ldots \cap \mathfrak{q}'_{sk_s}$$

is a minimal primary decomposition of (0) in $\mathcal{F}$, because then

$$\overline{\mathcal{F}} = \bigtimes_{i,j} \overline{\mathcal{F}/\mathfrak{q}'_{ij}} = \overline{\mathcal{F}_1} \times \ldots \times \overline{\mathcal{F}_s}.$$

Now suppose w.l.o.g $\mathfrak{q}'_{11} \supset \bigcap_{i,j \neq 1,1} \mathfrak{q}'_{ij}$. Then because $\mathfrak{q}'_{11}$ is prime, there exists $\mathfrak{q}'_{ij} \subset \mathfrak{q}'_{11}$, where clearly $i \neq 1$. If we can show that $\mathfrak{q}'_{ij} \cap \mathcal{R} = \mathfrak{p}_i$, we are done, since (4) is a minimal decomposition.

Assume $\mathfrak{p}_i \subsetneq \mathfrak{a} := \mathfrak{q}'_{ij} \cap \mathcal{R}$. Since $\mathfrak{a}$ is prime it follows $\operatorname{ht} \mathfrak{a} > d_i = \operatorname{ht} \mathfrak{p}_i$. Consequently, according to Chevalley's theorem every associated prime of $\mathfrak{a} \cdot \mathcal{F}$ is of height greater than d_i. Since $\mathfrak{a}\mathcal{F} \subset \mathfrak{q}'_{ij}$ and $\operatorname{ht} \mathfrak{q}'_{ij} = d_i$, this is a contradiction. □

Example 5.3.

(i) Let $I = \langle x^2 + y^2 - z^3 \rangle \trianglelefteq \mathbb{R}[x, y, z]$ and $A = \mathbb{R}[x, y, z]/I$. The real part of $Y = \mathbf{V}(I)$ is plotted in Figure 2. Since the singular locus of Y is just the origin and A is a complete intersection ring, Y must be normal according to Serre's criterion [6, Theorem 39] or use [15, Theorem 18.15].

Thus, because the origin is not isolated in $\mathbf{V}_{\mathbb{R}}(I)$, it must be a nonmanifold point of $\mathbf{V}_{\mathbb{R}}(I)$ according to Corollary 4.8.

(ii) Let $I = \langle x^2 - y^2 z \rangle \trianglelefteq \mathbb{R}[x, y, z]$. The real part of $Y = \mathbf{V}(I)$ is called the Whitney umbrella and plotted in Figure 3. It demonstrates nicely the scope and limits of Theorem 1.2 and Corollary 4.6.

Let $A = \mathbb{R}[x, y, z]/\langle x^2 - y^2 z \rangle$ be the coordinate ring of Y. Since Y is not normal we cannot apply Theorem 1.2 directly. But we can easily calculate a normalization of Y. The element $u = \frac{x}{y} \in Q(A)$ is integral over A, since $u^2 - z = 0$ in $Q(A)$. Then, we have the monomorphism

$$\psi : A \hookrightarrow B = A[u] \cong \mathbb{R}[x, y, z, u]/\langle u^2 - z, yu - x \rangle.$$

Since B is the coordinate ring of a nonsingular variety Z it must be integrally closed and ψ is the normalization of A.

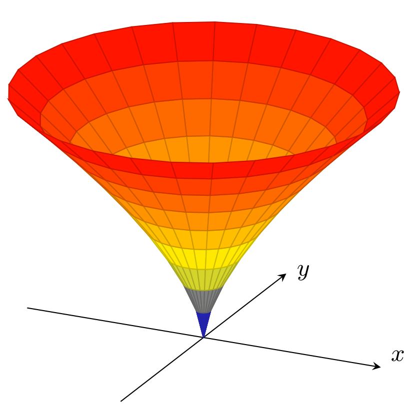

Figure 2: $\mathbf{V}_{\mathbb{R}}(x^2 + y^2 - z^3)$.

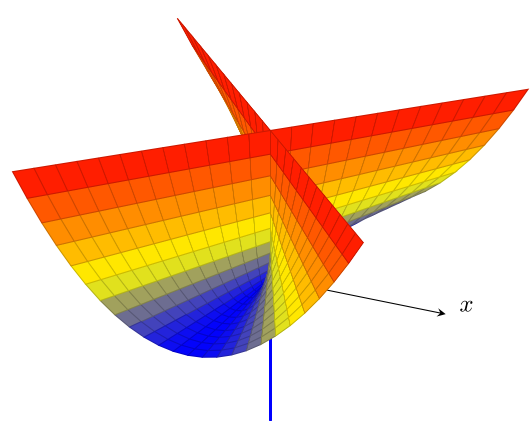

Figure 3: $\mathbf{V}_{\mathbb{R}}(x^2 - y^2 z)$.

The singular locus of Y is clearly given by the z-axis $x, y = 0$. Let $p = (0, 0, z_0)$ be any point on the real part of this axis. We want to use Corollary 4.6 to decide if p is a manifold point of $Y_{\mathbb{R}} = \mathbf{V}_{\mathbb{R}}(I)$, so we have to check, if the completion $\mathcal{F}$ of the local ring of Y at p is real.
Because $\mathcal{F}$ is real if and only if $\overline{\mathcal{F}}$ is real and normalization commutes with completion in the sense of Proposition 5.2 it is enough to show that all points $q_1, \dots, q_k$ lying over p in the normalization are real and the completion of the local ring of Z at q_i is real for all i.[1] Now

$$\langle x, y, z - z_0 \rangle_B = \langle u^2 - z_0, z - z_0, x, y \rangle_B.$$

First, we assume that p lies on the cross axis, i.e. $z_0 > 0$. Then, the real points $q_\pm = (0, 0, z_0, \pm\sqrt{z_0})$ are all points lying over p. But Z is a nonsingular variety, hence the completions of the local rings of Z at $q_\pm$ are both real according to Proposition 4.5(f). Therefore, $\mathcal{F}$ must be real and p a nonmanifold point of $Y_{\mathbb{R}}$ according to Corollary 4.6.

Now assume p is the origin, i.e. $z_0 = 0$. Then, there is just one real point $q = (0, 0, 0, 0)$ lying over p and $\mathcal{F}$ must be real, because Z is nonsingular at q. Thus, p is a nonmanifold point of $Y_{\mathbb{R}}$.
Finally, we assume p is on the handle, i.e. $z_0 < 0$. In this case, the nonreal points

$$q_\pm = (0, 0, z_0, \pm i\sqrt{-z_0})$$

are all points lying over p (or put ring-theoretically: the nonreal maximal ideal $\langle u^2 - z_0, x, y, z - z_0 \rangle_B$ lies over $\langle x, y, z - z_0 \rangle_A$). It follows that $\mathcal{F}$ is not real so we cannot use Corollary 4.6.

However, since there are no real points lying over p, this means that p is not a limit point of the real nonsingular points of Y [11]. Thus, we can find an euclidean neighbourhood U of p not containing any real nonsingular points of Y. The Zariski-closure of $\mathbf{V}_{\mathbb{R}}(I) \cap U$ is then just the z-axis $\tilde{Y} = \mathbf{V}(x, y)$ and the real part of $\tilde{Y}$ is a submanifold of $\mathbb{R}^2$.

6 Real algebraic curves

Now we apply the theory of Section 5 to singularities of real algebraic curves. Let $\dim I = 1$ with $I \subset \mathfrak{m}$. Then, the structure of $\hat{\mathcal{F}} = \mathcal{F}/\sqrt[r]{(0)}$ will be especially nice, since the real radical of an associated prime $\mathfrak{q}$ of I'' will be either $\mathfrak{q}$ itself or the maximal ideal $\mathfrak{m}'' = \mathfrak{m} \cdot \mathbb{R}[[\bar{x}]]$ of $\mathbb{R}[[\bar{x}]]$:

Lemma 6.1. *Let* $\mathfrak{q} \leq \mathbb{R}[[\bar{x}]]$ *be a prime with* $\operatorname{ht} \mathfrak{q} = n - 1$. *Then*

$$\sqrt[r]{\mathfrak{q}} = \begin{cases} \mathfrak{q} & \mathfrak{q} \text{ real}, \\ \mathfrak{m}'' & \mathfrak{q} \text{ not real}. \end{cases}$$

Proof. Any ideal is real if and only if it is radical and all associated primes are real [11]. So $\sqrt[r]{\mathfrak{q}}$ is the intersection of all real primes containing $\mathfrak{q}$. We only have to show that $\mathfrak{m}''$ is real, but this is clear, since $\mathbb{R}[[\bar{x}]]/\mathfrak{m}'' \cong \mathbb{R}$ is real. □

Lemma 6.2. *Let* $(A, \mathfrak{m}_A) \subset (B, \mathfrak{m}_B)$ *be a finite extension of local rings. Assume* $\mathfrak{m}_A B = \mathfrak{m}_B$ *and*

$$B/\mathfrak{m}_B = A/\mathfrak{m}_A$$

Then $A = B$.

1 Since we only want to show that $\mathcal{F}/\sqrt[r]{(0)}$ is not regular, a sufficient condition would also be that there are two real points q_1, q_2 lying over p and the completion of the local ring of Z at q_i is real, i.e. q_1 and q_2 are in the euclidean closure of the real nonsingular locus of Z (Theorem 4.7). Then, $\mathcal{F}/\sqrt[r]{(0)}$ would not be a domain because there exist at least two real minimal primes of $\mathcal{F}$.

Proof. For any element $b \in B$, there exists $a \in A$ with $b - a \in \mathfrak{m}_B = \mathfrak{m}_A B$ since $A/\mathfrak{m}_A = B/\mathfrak{m}_B$. It follows

$$B = A + \mathfrak{m}_A B.$$

Since m_A is the Jacobson radical of A and B is a finite A-module, the statement of the lemma follows from Nakayama's lemma. □

Lemma 6.3. *Let A be a noetherian ring with ideal $J \leq A$. If $\mathfrak{p}$ is an associated prime of J, we have $J \cdot A_\mathfrak{p} = \mathfrak{p} \cdot A_\mathfrak{p}$ if and only if $\mathfrak{p}$ appears in one and then every minimal primary decomposition of J.*

Proof. First note that any prime appearing in a minimal primary decomposition is isolated, so it is independent of the decomposition. Now let $J \cdot A_\mathfrak{p} = \mathfrak{p} \cdot A_\mathfrak{p}$ and choose a minimal primary decomposition $J = \bigcap_{i=1}^{s} \mathfrak{s}_i$. Since $\mathfrak{p}$ is an associated prime of J we have w.lo.g $\sqrt{\mathfrak{s}_1} = \mathfrak{p}$. According to [13, Proposition 4.9], we have

$$\mathfrak{p} \cdot A_\mathfrak{p} = J \cdot A_\mathfrak{p} = \bigcap_{\sqrt{\mathfrak{s}_i} \subset \mathfrak{p}} \mathfrak{s}_i \cdot A_\mathfrak{p} \subset \mathfrak{s}_1 A_\mathfrak{p}.$$

Consequently, $\mathfrak{p} \cdot A_\mathfrak{p} = \mathfrak{s}_1 \cdot A_\mathfrak{p}$. But $\mathfrak{s}_1$ is a $\mathfrak{p}$-primary ideal, so $\mathfrak{p} = \mathfrak{s}_1$ according to [13, Proposition 4.9].

On the other hand, if $\mathfrak{p}$ appears in a minimal primary decomposition of J it is an isolated prime and $\mathfrak{p} \cdot A_\mathfrak{p} = J \cdot A_\mathfrak{p}$ follows with [13, Proposition 4.9] again. □

If $J \cdot A_\mathfrak{p} = \mathfrak{p} \cdot A_\mathfrak{p}$ like in the previous lemma, we say J is $\mathfrak{p}$-**reduced**. We can now formulate the main result of this section. The following is a ring-theoretic formulation of Theorem 1.3 from the introduction. Note that localization commutes with normalization [13, Proposition 5.12].

Theorem 6.4. *Let $\dim I = 1$ and I radical. The origin is a manifold point of $\mathbf{V}_\mathbb{R}(I)$ if and only if one of the following two conditions is true*

(*a*) *There is exactly one real maximal ideal $\mathfrak{n} \leq \overline{\mathcal{R}}$ lying over $\mathfrak{r} = \mathfrak{m} \cdot \mathcal{R}$ and $\mathfrak{r} \cdot \overline{\mathcal{R}}$ is $\mathfrak{n}$-reduced.*

(*b*) *All the maximal ideals $\mathfrak{n} \leq \overline{\mathcal{R}}$ are not real. In this case, the origin is an isolated point of $\mathbf{V}_\mathbb{R}(I)$.*

Proof. First, let $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be a minimal primary decomposition in $\mathcal{F}$. According to Chevalley's theorem $\mathcal{F}$ is reduced [8, Proposition 2.1], hence all the $\mathfrak{q}_i$ are prime. Using Theorem 5.2 and Lemma 5.1, we find ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_q$ maximal in $\overline{\mathcal{R}}$ with □

$$\overline{\mathcal{F}/\mathfrak{q}_i} \cong \left(\overline{\mathcal{R}}_{\mathfrak{n}_i}\right)^*.$$

Now we will show the following assertion:

$$\mathcal{F}/\mathfrak{q}_i \text{ real} \Leftrightarrow \mathfrak{n}_i \text{ real}. \tag{5}$$

Clearly $\overline{\mathcal{F}/\mathfrak{q}_i}$ is real if and only if $\mathcal{F}/\mathfrak{q}_i$ is real, since they are contained in the quotient field of $\mathcal{F}/\mathfrak{q}_i$, so we need to show that $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ is real if and only if $\mathfrak{n}_i$ is real.

If $\mathfrak{n}_i$ is not real, then $\overline{\mathcal{R}}/\mathfrak{n}_i \cong \mathbb{C}$ and one can see from the construction before Theorem 5.2 that $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ will not be real (since $\mathbb{C} \subset (\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$).

On the other hand, let $\mathfrak{n}_i$ be real, then $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ will be the $\mathfrak{n}_i$-adic completion of the local ring $\overline{\mathcal{R}}_{\mathfrak{n}_i}$. Since $\overline{\mathcal{R}}$ is normal of dimension 1, we also know that $\overline{\mathcal{R}}_{\mathfrak{n}_i}$ is regular according to Serre's regularity criterion R_1 [6, Theorem 39]. Then $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ is regular too [13, Proposition 11.24], with residue field $\overline{\mathcal{R}}/\mathfrak{n}_i = \mathbb{R}$. Hence, $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ must be real because of [1, Proposition 2.7]. This proves assertion (5).

Next, we consider

$$\sqrt[r]{I''} = \sqrt[r]{\mathfrak{q}_1'} \cap \ldots \cap \sqrt[r]{\mathfrak{q}_r'}, \tag{6}$$

where $\mathfrak{q}_i'$ is the preimage of $\mathfrak{q}_i$ in $\mathbb{R}[[\bar{x}]]$. As one checks easily $\mathfrak{q}_i$ is real if and only if $\mathfrak{q}_i'$ is real.

If none of the $\mathfrak{n}_i$ is real, then none of the $\mathfrak{q}_i'$ is real and according to Lemma 6.1, we would get $\sqrt[r]{I''} = \mathfrak{m}''$ from (6). Since $\hat{I} = \sqrt[r]{I'} = \sqrt[r]{I''} \cap \mathbb{R}\{\bar{x}\}$ (see the proof of Proposition 4.5(e)), we would also have $\hat{I} = \mathfrak{m}' = \mathfrak{m} \cdot \mathbb{R}\{\bar{x}\}$ and one checks easily with Definition 4.2 that the origin must be an isolated point of $X = \mathbf{V}_{\mathbb{R}}(I)$.

If two of the $\mathfrak{n}_i$ are real, then $\hat{\mathcal{F}} = \mathbb{R}[[\bar{x}]]/\sqrt[r]{I''}$ would not be a domain and therefore not regular. Then the origin cannot be a manifold point of X according to Theorem 4.3 and Proposition 4.5.

Now we examine the case that exactly one $\mathfrak{n}_i$ is real, w.l.o.g. we choose $\mathfrak{n}_1$ real. Then $\sqrt[r]{I''} = \mathfrak{q}_1'$. We have the following commutative diagram:

$$\begin{array}{ccccc} \overline{\mathcal{R}}_{\mathfrak{n}_1} & & \xrightarrow{\quad\psi\quad} & & (\overline{\mathcal{R}}_{\mathfrak{n}_1})^* \\ l\uparrow & & & & \eta\updownarrow\cong \\ \overline{\mathcal{R}} & \longrightarrow & \overline{\mathcal{F}} & \longrightarrow & (\overline{\mathcal{F}/\mathfrak{q}_1}) \\ \uparrow & & \uparrow & & \iota\uparrow \\ \mathcal{R} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/\mathfrak{q}_1 \end{array} \tag{7}$$

First, we assume that $\mathfrak{r} \cdot \overline{\mathcal{R}}_{\mathfrak{n}_1} = \mathfrak{n}_1 \cdot \overline{\mathcal{R}}_{\mathfrak{n}_1}$. We proceed in two steps.

(1) $\iota(\mathfrak{o})$ generates the maximal ideal of $\overline{\mathcal{F}/\mathfrak{p}_1}$, where $\mathfrak{o}$ is the maximal ideal of $\mathcal{F}/\mathfrak{q}_1$: since ψ is the $\mathfrak{n}_1\overline{\mathcal{R}}_{\mathfrak{n}_1}$-adic completion of $\overline{\mathcal{R}}_{n_1}$ we know that $\psi(\mathfrak{r} \cdot \overline{\mathcal{R}}_{\mathfrak{n}_1}) = \psi(\mathfrak{n}_1 \cdot \overline{\mathcal{R}}_{\mathfrak{n}_1})$ generates the maximal ideal of $(\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$. But $\langle\psi(\mathfrak{r} \cdot \overline{\mathcal{R}}_{\mathfrak{n}_1})\rangle = \langle\eta(\iota(\mathfrak{o}))\rangle$ and because η is an isomorphism we conclude that $\iota(\mathfrak{o})$ generates the maximal ideal of $\overline{\mathcal{F}/\mathfrak{p}_1}$.

(2) $\mathcal{F}/\mathfrak{p}_1$ is regular: we have already seen that the residue field of $(\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$ is $\mathbb{R}$, hence the same is true for the residue field of $\overline{\mathcal{F}/\mathfrak{p}_1}$. Also, we know that $\overline{\mathcal{F}/\mathfrak{p}_1}$ is finite over $\mathcal{F}/\mathfrak{p}_1$ [8, Proposition III.2.3] and from (1) we know that the maximal ideal of $\mathcal{F}/\mathfrak{p}_1$ generates the maximal ideal of $\overline{\mathcal{F}/\mathfrak{p}_1}$.

Now we are exactly in the situation of Lemma 6.2 with $A = \mathcal{F}/\mathfrak{p}_1$ and $B = \overline{\mathcal{F}/\mathfrak{p}_1}$. It follows $\overline{\mathcal{F}/\mathfrak{p}_1} = \mathcal{F}/\mathfrak{p}_1$. Then $\mathcal{F}/\mathfrak{p}_1$ is a normal local ring of dimension at most 1. With Serre's regularity criterion R_1, we see that $\hat{\mathcal{F}} = \mathcal{F}/\mathfrak{p}_1$ is regular and according to Theorem 4.3 and Proposition 4.5 the origin must be a manifold point of X.

Now suppose on the contrary that $\mathcal{F}/\mathfrak{p}_1$ is regular. Then it is a Cohen-Macaulay domain. It fulfils S_2 and R_1 and is normal by Serre's normality criterion [6, Theorem 39]. Therefore, $\mathcal{F}/\mathfrak{p}_1 = \overline{\mathcal{F}/\mathfrak{p}_1} \cong (\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$.

We set $\mathfrak{b} = \mathfrak{r} \cdot \overline{\mathcal{R}}_{\mathfrak{n}_1}$. Because diagram (7) commutes and ι is an isomorphism we have that $\psi(\mathfrak{b})$ generates the maximal ideal $\mathfrak{a}$ of $(\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$. But ψ is faithfully flat as a flat local homomorphism [6, Corollary 4.A]. Hence,

$$\mathfrak{b} = \langle\psi(\mathfrak{b})\rangle \cap \overline{\mathcal{R}}_{\mathfrak{n}_1} = \mathfrak{a} \cap \overline{\mathcal{R}}_{\mathfrak{n}_1} = \mathfrak{n}_1\overline{\mathcal{R}}_{\mathfrak{n}_1}.$$

So $\mathfrak{r}\overline{\mathcal{R}}$ is $\mathfrak{n}_1$-reduced. This completes the proof.

Example 6.5. Theorem 6.4 is useful because its conditions can be tested with all CAS which have a normalizing algorithm for polynomial rings and an algorithm for primary decomposition implemented. For this demonstration, we use Singular with the libraries `normal.lib` and `primdec.lib` [21–25] for normalization and primary decomposition. See [21] for more details and an implementation of an algorithm checking the conditions of Theorem 6.4.

(i) $f(x, y) = y^3 + 2yx^2 - x^4$. $\mathbf{V}_{\mathbb{R}}(f)$ is the curve from Example (vi) of the introduction. Consider the following execution in Singular:

```
LIB "normal.lib";
ideal I = y^3 + 2*y*x^2 - x^4;
def nor = normal(I); def S = nor[1][1];
setring S;
ideal W = norid + ideal(x,y);
primdecGTZ(W);
```

Listing 1: Manifold test for $y^3 + 2yx^2 - x^4$ at the origin.

This gives:

```
[1]:                              [2]:
    [1]:                              [1]:
        _[1]=T(2)                         _[1]=T(2)^2+2
        _[2]=y                            _[2]=y
        _[3]=x                            _[3]=x
        _[4]=-T(2)^2+T(1)-2               _[4]=-T(2)^2+T(1)-2
    [2]:                              [2]:
        -- same                           -- same
```

Let $A = \mathbb{Q}[x, y]/\langle f\rangle$ and $\psi\colon A \hookrightarrow B$ be the normalization of A. Singular calculates B as a quotient $\mathbb{Q}[x, y, T_1, T_2]/N$, where N is referenced in Singular by the handle norid. The previous output is a minimal primary decomposition over $\mathbb{Q}$ of the ideal J generated by $\psi(\langle x, y\rangle)$ in B. See also equation (2), where we calculated just that for the same example.

The second entry in both lists is the radical of the respective primary component of J. Since they are the same, the primary ideals in this decomposition must be prime and J is radical, i.e. any zero of J is simple. We see that there is exactly one real zero lying over the origin (from ideal [1] in the primary decomposition) and this is a simple zero. Thus, the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$.

We should note that $\mathbb{R} \otimes_{\mathbb{Q}} B$ is the normalization of $\mathbb{R} \otimes_{\mathbb{Q}} A$, because any normal $\mathbb{Q}$-algebra is geometrically normal [16, Section 10.160.4]. This means there is no problem with calculating the integral closure of A in Singular over the rational numbers.

(ii) $g(x, y) = x^2 - y^5$. $\mathbf{V}_{\mathbb{R}}(g)$ is the cusp from Figure 4. Let us run Listing 1 with the second line replaced by ideal I = x^2 – y^5. Then, we get the output:

```
[1]:
    [1]:
        _[1]=T(1)^2
        _[2]=y
        _[3]=x
    [2]:
        _[1]=T(1)
        _[2]=y
        _[3]=x
```

Again we set $A = \mathbb{Q}[x, y]/\langle f\rangle$ and write $\psi\colon A \hookrightarrow B$ for the normalization of A. Also, we denote with J the ideal generated by $\psi(\langle x, y\rangle)$ in B.

We see from the output above that there is exactly one real zero q lying over the origin. But this is not a simple zero, because the maximal ideal $\mathfrak{n}_q$ is not a primary component of J, i.e. J is not $\mathfrak{n}_q$-reduced. According to Theorem 6.4 the origin is not a manifold point of $\mathbf{V}_{\mathbb{R}}(g)$.

(iii) $h(x, y) = y^3 - y^4 + yx^4 - x^6$. The real curve $\mathbf{V}_{\mathbb{R}}(g)$ is plotted in Figure 5. Listing 1 with the second line replaced by ideal I = y^3 – y^4 + y*x^4 – x^6 gives the following output:

```
[1]:
    [1]:
        _[1]=T(2)^3-T(2)^2-2*T(2)-1
        _[2]=y
        _[3]=x
        _[4]=-T(2)^2+T(1)+T(2)+1
    [2]:
        -- same
```

We set $A = \mathbb{Q}[x, y]/\langle h\rangle$ and write $\psi\colon A \hookrightarrow B$ for the normalization of A. Also, we write J for the ideal generated by $\psi(\langle x, y\rangle)$ in B.

From the output we see that J is reduced and a prime ideal over $\mathbb{Q}$. This means any root of J in $\mathbb{C}$ is simple. It remains to check how many real roots J has. For this we could analyse the cubic, which is the first

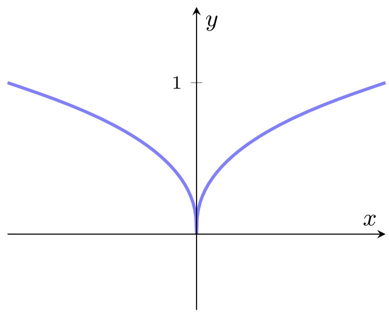

Figure 4: $\mathbf{V}_{\mathbb{R}}(x^2 - y^5)$.

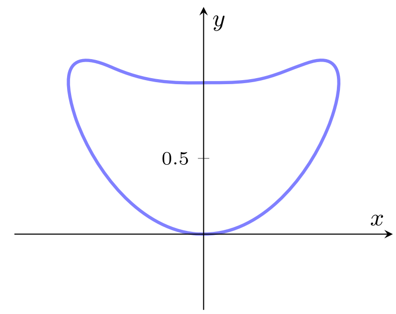

Figure 5: $\mathbf{V}_{\mathbb{R}}(y^3 - y^4 + yx^4 - x^6)$.

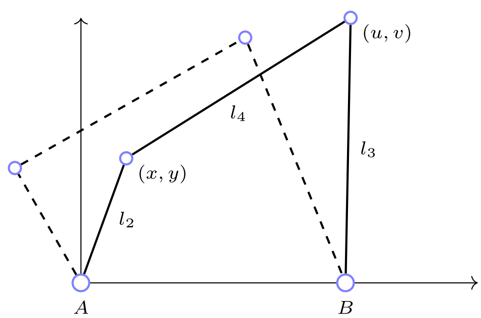

Figure 6: The four-bar linkage.

generator of J in the output above. But we can also use the real root counter of the Singular library `rootsmr.lib` [27]:

```
LIB "rootsmr.lib";
nrRootsDeterm(M); //--> gives 1
```

Hence, there is only one real root lying over the origin and this is a simple root. Thus, the origin is a manifold-point of $\mathbf{V}_{\mathbb{R}}(h)$.

7 CS-singularities of the four-bar linkage

In a basic description a **linkage** is a collection of rigid bodies connected by joints. Its configuration space is the set of all assembly configurations and can in general be represented by a real algebraic set, see Figure 6 and equation (8) for an example. A nonmanifold point of the configuration space is called a **CS-singularity**. In such a configuration, the linkage will exhibit degenerate kinematic behaviour. Recently, efforts have been made in the kinematics community to define and categorize kinematic singularities of linkages in a rigorous way [2,3,28].

As pointed out in the introduction and in [5, p. 227] one needs to check for every singularity of the configuration space whether it is a CS-singularity. In this section, we apply the theory developed so far to identify all CS-singularities of the four-bar linkage (Figure 6). This linkage is one of the oldest and most widely used planar mechanisms in Kinematics and Mechanical Engineering. In its common form it consists of four bars of length l_1, l_2, l_3, l_4 connected in a circular arrangement by revolute joints (hinges) with one bar fixed to the ground.

Conditions on the design parameters l_i such that there exist points with a rank drop in the constraint equations are well known, see e.g. [5,29] (Grashof condition (9)). Lesser known are methods to show that these points are CS-singularities, i.e. nonmanifold points. We will do this in a purely computational algebraic way with our theory on algebraic curves. See also [30, Theorem 1.6] or [31] for a different method based on differential topology and Morse theory.

The configuration space of the four-bar can be represented by the real algebraic set $X = \mathbf{V}_{\mathbb{R}}(I)$, where $I = \langle p_1, p_2, p_3 \rangle \leq \mathbb{R}[x, y, u, v]$ is generated by the polynomials:

$$\begin{aligned} p_1 &= x^2 + y^2 - l_2^2, \\ p_2 &= (u-2)^2 + v^2 - l_3^2, \\ p_3 &= (u-x)^2 + (v-y)^2 - l_4^2, \end{aligned} \tag{8}$$

which are just the length constraints in euclidean coordinates. We fixed the length $l_1 = |AB| = 2$ of the ground bar, since any other length can be treated by scaling the system. The choice $l_1 = 2$ is completely arbitrary.

7.1 Dimension of I

We will assume $l_2 \neq 2$, $l_4 \neq 2$, since the complementary cases can be analysed in much the same way. Now we calculate a pseudo Gröbner basis [14] of I with respect to the polynomial ordering $(dp(2), dp(2))$ and the enumeration v, y, u, x. We can do all the calculations in $B = \mathbb{Q}(l_2, l_3, l_4)[x, y, u, v]$ but we have to be careful to avoid dividing by elements of $\mathbb{Q}(l_2, l_3, l_4) \setminus \mathbb{Q}$ in all Gröbner base calculations, since these could be zero for valid parameters l_2, l_3, l_4. In Singular, we can achieve this by setting `option(intStrategy)` and `option(contentSB)`.

We get six polynomials $g_1, \ldots, g_6$ with the leading terms:

$$\begin{aligned} \mathrm{LT}(g_1) &= -16u^2x, & \mathrm{LT}(g_4) &= y^2, \\ \mathrm{LT}(g_2) &= (-2l_2^2 + 8)vu, & \mathrm{LT}(g_5) &= 2vy, \\ \mathrm{LT}(g_2) &= -2vx^2, & \mathrm{LT}(g_6) &= v^2. \end{aligned}$$

According to Exercise 2.3.8 of [14] $\{g_1, \ldots, g_6\}$ is a Gröbner basis of I as long as $l_2 \neq \pm 2$ which we assumed in the beginning but then we can calculate the dimension of I with

$$\dim I = \dim \langle u^2x, vu, vx^2, y^2, vy, v^2 \rangle.$$

With a simple combinatorial argument [32, Proposition 9.1.3] we see that the dimension of the right ideal is 1 and consequently $\dim I = 1$. Since $I = \langle p_1, p_2, p_3 \rangle$ can be generated by three elements, $A := \mathbb{R}[x, y, u, v]/I$ must be a locally complete intersection ring and consequently equidimensional Cohen-Macaulay [15, Proposition 18.13].

7.2 Singular locus

According to [5] there only exist singular points in X, if and only if

$$\pm l_2 \pm l_3 \pm l_4 = 2. \tag{9}$$

We restrict our investigation to the case $l_2 - l_3 + l_4 = 2$, i.e. $l_3 = l_2 + l_4 - 2 > 0$, since other cases can be treated in a similar way. Since $\dim I = 1$ equidimensional we need to analyse the ideal J generated by I and all the 3-minors of the Jacobian of (p_1, p_2, p_3). With a Singular Gröbner base computation we get $J = \langle s_1, s_2, s_3, s_4 \rangle$, with

$$\begin{aligned} s_1 &= q_1(l_2, l_4)x + c_1(l_2, l_4), \\ s_2 &= q_2(l_2, l_4)u + r_2(l_2, l_4)x + c_2(l_2, l_4), \\ s_3 &= q_3(l_2, l_4)y, \\ s_4 &= q_4(l_2, l_4)v + f(l_2, l_4, x, y, u), \end{aligned}$$

where all the coefficients are polynomials in l_2, l_4 or l_2, l_4, x, y, u, respectively. We need to examine the coefficients of the leading monomials of the s_i to make sure that $\{s_1, s_2, s_3, s_4\}$ is a Gröbner basis of J.

A quick calculation in SINGULAR shows:

$$\begin{aligned}
q_1(l_2, l_4) &= l_4^2 \cdot (l_4 - 2) \cdot (l_2 + l_4) \cdot (l_2 + l_4 - 2)^2 \cdot (l_2 + 2)^2 \cdot l_2 \cdot (3l_2 - 8),\\
q_2(l_2, l_4) &= l_4^2 \cdot (l_2 + 2l_4 - 2) \cdot (l_2 + l_4 - 2)^2 \cdot (l_2 + 2)^2 \cdot (l_2 - 2),\\
q_3(l_2, l_4) &= l_4^2 \cdot (l_2 + l_4 - 2)^2 \cdot (l_2 + 2),\\
q_4(l_2, l_4) &= l_2^2 \cdot (l_2 + 2) \cdot (l_2 - 2),
\end{aligned}$$

Taking into account our assumptions that $l_4, l_2 > 0$, $l_2 + l_4 - 2 = l_3 > 0$, $l_2 \neq 2$, $l_3 \neq 2$ and in addition $l_2 \neq \frac{8}{3}$ (which we will also need to check separately), we see that none of the q_i will vanish and s_1, s_2, s_3, s_4 form a Gröbner basis of J for all assumed values of l_2, l_4. Clearly, then $\dim J = 0$ and since A is Cohen-Macaulay, we can infer from [15, Theorem 18.15] that I must be a radical ideal. But then the singular locus of A is given by all the prime ideals containing J.

We now set $p = (l_2, 0, l_2 + l_4, 0) \in R^4$. As we can check quickly by substitution in (s_1, s_2, s_3, s_4), we have $J \leq \mathfrak{m}_p$, so p is the only singularity of $X_{\mathbb{C}} = \mathbf{V}(I)$.

7.3 Manifold points

To check whether p is a nonmanifold point with Theorem 6.4 we need to calculate the integral closure C of $A_{\mathfrak{m}_p}$. We could do this by applying the normalization algorithm described in [14] and implemented in SINGULAR but it has proven difficult to check the validity of the Gröbner base calculations in each step for the investigated values of l_2, l_4. We could still analyse the situation for generic values of l_2, l_4 but we want a statement for all valid values.

Instead we will determine the blow up $\pi: \tilde{X} \to X$ at p, since $\tilde{X}$ will be nonsingular after one blow up and is then the normalization of X. First, we move p to the origin and consider $I_{\mathrm{bl}} = \langle p_1', p_2', p_3', b_1, b_2, b_3, b_4, b_5, b_6 \rangle \leq \mathbb{R}[x, y, u, v, \hat{x}, \hat{y}, \hat{u}, \hat{v}]$ given by

$$\begin{aligned}
p_1' &= p_1(x + l_2, y, u + l_2 + l_4, v) = x^2 + y^2 + 2l_2 x,\\
p_2' &= p_2(x + l_2, y, u + l_2 + l_4, v) = u^2 + v^2 + (2l_2 + 2l_4 - 4)u,\\
p_3' &= p_3(x + l_2, y, u + l_2 + l_4, v) = x^2 + y^2 - 2xu + u^2 - 2yv + v^2 - 2l_4 x + 2l_4 u,
\end{aligned}$$

and the homogeneous polynomials

$$\begin{aligned}
b_1 &= x\hat{y} - y\hat{x}, & b_4 &= y\hat{u} - u\hat{y},\\
b_2 &= x\hat{u} - u\hat{x}, & b_5 &= y\hat{v} - v\hat{y},\\
b_3 &= x\hat{v} - v\hat{x}, & b_6 &= u\hat{v} - v\hat{u}.
\end{aligned}$$

Then we go to the chart $\hat{y} = 1$ and get the isomorphic system in $\mathbb{R}[\hat{x}, y, \hat{u}, \hat{v}]$:

$$\begin{aligned}
p_1'' &= y \cdot (y\hat{x}^2 + y + (2l_2)\hat{x}),\\
p_2'' &= y \cdot (\hat{u}^2 + y\hat{v}^2 + (2l_2 + 2l_4 - 4))\hat{u},\\
p_3'' &= y \cdot (y\hat{x}^2 - 2y\hat{x}\hat{u} + y\hat{u}^2 + y\hat{v}^2 - 2y\hat{v} + y + (-2l_4)\hat{x} + (2l_4)\hat{u}).
\end{aligned}$$

We set $I_y := \langle p_1''/y, p_2''/y, p_2''/y \rangle \leq \mathbb{R}[\hat{x}, y, \hat{u}, \hat{v}]$. To get the equations of the strict transform on this chart, we need to remove the exceptional divisor, so we have to calculate the saturation

$$J := (I_y : \langle y \rangle^\infty).$$

This can easily be achieved with the command `sat` in SINGULAR. But again we have to be careful to check whether the Gröbner basis calculations stay valid for all assumed values for l_2, l_4, so we will calculate the saturation manually. First we calculate $I_y \cap \langle y \rangle$, which we get by eliminating t of

$$I_y t + \langle (1 - t)y \rangle.$$

Now we divide any generator of $I_y \cap \langle y \rangle$ by y and after checking that all coefficients of the leading monomials will not be zero after substitution of values for l_2, l_4 we normalize the generators and get the following Gröbner basis of $J = (I_y : \langle y \rangle)$:

$$\begin{aligned}
f_1 &= \hat{u}^2\hat{x}^2 + \frac{-2l_2 - 2l_4}{l_2 + 2}\hat{u}\hat{x}^3 + \frac{l_2^2 + 2l_2l_4 + l_4^2}{l_2^2 + 4l_2 + 4}\hat{x}^4 + \frac{l_2^2 - 4l_2 + 4}{l_2^2 + 4l_2 + 4}\hat{u}^2 + \frac{-2l_2^2 - 2l_2l_4 + 4l_2 - 4l_4}{l_2^2 + 4l_2 + 4}\hat{u}\hat{x} + \frac{l_2^2 + 2l_2l_4 + l_4^2}{l_2^2 + 4l_2 + 4}\hat{x}^2,\\
f_2 &= y\hat{x}^2 + y + (2l_2)\hat{x},\\
f_3 &= y\hat{u}^2 - y\hat{u}\hat{x} + \frac{l_2^2 + 4l_2 + 4}{4}\hat{u}^2\hat{x} + \frac{-l_2^2 - l_2l_4 - 2l_2 - 2l_4}{2}\hat{u}\hat{x}^2 + \frac{l_2^2 + 2l_2l_4 + l_4^2}{4}\hat{x}^3,\\
f_4 &= \hat{v}\hat{x} + \frac{l_2 + 2}{2l_2}\hat{u}\hat{x}^2 + \frac{-l_2 - l_4}{2l_2}\hat{x}^3 + \frac{-l_2 + 2}{2l_2}\hat{u} + \frac{-l_2 - l_4}{2l_2}\hat{x},\\
f_5 &= \hat{v}\hat{u} + \frac{-3l_2^2 - 3l_2l_4 + 6l_2 - 2l_4}{l_2^2 - 4l_2 + 4}\hat{v}\hat{x} + \frac{l_2^2 + 4l_2 + 4}{2l_2^2 - 4l_2}\hat{u}^2\hat{x}^3 + \frac{-l_2^2 - l_2l_4 - 2l_2 - 2l_4}{l_2^2 - 2l_2}\hat{u}\hat{x}^4 + \frac{l_2^2 + 2l_2l_4 + l_4^2}{2l_2^2 - 4l_2}\hat{x}^5\\
&\quad + \frac{3l_2^2 + 4}{2l_2^2 - 4l_2}\hat{u}^2\hat{x} + \frac{-4l_2^3 - 4l_2^2l_4 + 6l_2^2 - 2l_2l_4 + 4l_2 + 4l_4}{l_2^3 - 4l_2^2 + 4l_2}\hat{u}\hat{x}^2\\
&\quad + \frac{5l_2^3 + 10l_2^2l_4 - 10l_2^2 + 5l_2l_4^2 - 12l_2l_4 - 2l_4^2}{2l_2^3 - 8l_2^2 + 8l_2}\hat{x}^3 + \frac{2l_2^2 + 4l_2l_4 - 4l_2 + 2l_4^2 - 4l_4}{l_2^2 - 4l_2 + 4}\hat{x},\\
f_6 &= \hat{v}y + y\hat{u}\hat{x} + \frac{-2l_2 - l_4 + 2}{2l_2}y\hat{x}^2 + \frac{-2l_2 - l_4 + 2}{2l_2}y + (l_2 - 2)\hat{u} + (-l_2 + 2)\hat{x},\\
f_7 &= \hat{v}^2 + \frac{-2l_2 - 2l_4 + 4}{l_2 - 2}\hat{v} + \hat{u}^2 + \frac{-2l_2 - 2l_4 + 4}{l_2 - 2}\hat{u}\hat{x} + \frac{l_2^2 + 2l_2l_4 - 2l_2 + l_4^2 - 2l_4}{l_2^2 - 2l_2}\hat{x}^2 + \frac{l_2^2 + 2l_2l_4 - 2l_2 + l_4^2 - 2l_4}{l_2^2 - 2l_2}.
\end{aligned}$$

If we try to repeat the process we get the same ideal, hence $J = (I : \langle y \rangle^\infty)$ is the ideal of the strict transform of X on the chart $\hat{y} = 1$. Now we check that the ideal of the 3-minors of the Jacobian of $(f_1, \dots, f_7)$ is the whole ring and $\tilde{X}$ nonsingular on $\hat{y} = 1$. We can do this as before in the calculation of the singular locus of X.

Now we need to identify all points q in the fibre over the origin, so we calculate a pseudo Gröbner basis of $J + \langle y \rangle$ and get

$$\begin{aligned}
g_1 &= (2l_2)\hat{x},\\
g_2 &= (l_2 - 2)\hat{u} + (-l_2 + 2)\hat{x},\\
g_3 &= y,\\
g_4 &= \hat{v}^2 + \frac{-2l_2 - 2l_4 + 4}{l_2 - 2}\hat{v} + \hat{u}^2 + \frac{-2l_2 - 2l_4 + 4}{l_2 - 2}\hat{u}\hat{x}\\
&\quad + \frac{l_2^2 + 2l_2l_4 - 2l_2 + l_4^2 - 2l_4}{l_2^2 - 2l_2}\hat{x}^2 + \frac{l_2^2 + 2l_2l_4 - 2l_2 + l_4^2 - 2l_4}{l_2^2 - 2l_2}.
\end{aligned}$$

We see that this is a Gröbner base for all considered values of l_2, l_4. Now we substitute $\hat{x} = 0$, $\hat{u} = 0$ from g_1, g_2 into g_4 and multiply with $(l_2^2 - 2l_2)$. Then we get

$$g'(\hat{v}) = (l_2^2 - 2l_2)\hat{v}^2 - l_2(2l_2 - 2l_4 + 4)\hat{v} + (l_2^2 + 2l_2l_4 - 2l_2 + l_4^2 - 2l_4).$$

g' is a quadratic equation in $\hat{v}$ with discriminant

$$8l_2l_4(l_2 + l_4 - 2) = 8l_2l_3l_4 > 0.$$

Consequently, both points lying over the origin in the chart $\hat{y} = 1$ are real. According to Theorem 6.4, $p = (l_2, 0, l_2 + l_4, 0)$ is then not a manifold point of X. After checking that for all other charts the same points (if any) are lying over the origin we have even proved that the extension $I_p\mathbb{R}[[x, y, u, v]]$ is real, where I_p is the translated ideal.

8 CS-singularities of the delta robot

In this last section, we want to apply some of our results to a linkage with a configuration space of dimension greater than 1. We will do this for the delta robot of Figure 7, but the same argument can be used for many other linkages, e.g. the five-bar linkage or the 3RRR-planar parallel manipulator of [5].

The delta robot is a parallel linkage that consists of three identical limbs which carry a platform serving as a Cartesian positioning device (Figure 7). It was developed in 1985 by a research team under the supervision of Reymond Clavel who described it first in his PhD thesis [33].

Applications of the delta robot include pick and place tasks like packaging and pre-assembly work. Recently, delta robots can also be found in many 3D-printers to carry the extruder. Unlike depicted in Figure 7 the actual platform is mounted upside down in almost all applications.

In the original design and most currently manufactured delta robots (Fanuc M1, ABB IRB 360, Yaskawa MPP3H), each limb comprises a solid upper arm A of length a and a parallelogram-linkage B with sides of length b (Figure 8). B is attached to the upper arm with a revolute joint G. This allows the tip H of the parallelogram to travel on a spherical surface around G.

Finally, the complete limb is connected to the base and the moving platform with revolute joints F and H. Both the joint-connections to the base and to the moving platform of each limb are placed at the vertices of equilateral triangles with apothems r_2 and r_1, respectively.

In applications the base joints F will be actuated (motor-driven). The composition with the parallelogram linkages in the limbs forces the platform to always maintain the same orientation. This means that the platform will perform translational movements under actuation of the base joints. $a, b > 0$ and $d := r_1 - r_2$ are parameters of the delta robot. We will assume $d \neq 0$ like in most applications to keep the analysis short. In this way, we can scale the system and set $d = 1$.

To give equation for the configuration space we follow the discussion in [34, Section 3.6], which can be checked for details. Depending on the parameters a and b we set:

$$Y_{a,b} = \mathbf{V}(\{s_i, c_j, l_k \,|\, i, j = 1, 2, 3 \text{ and } k = 1, \ldots, 6\}) \subset \mathbb{C}^{15},$$
$$X_{a,b} = \mathbf{V}_{\mathbb{R}}(\{s_i, c_j, l_k \,|\, i, j = 1, 2, 3 \text{ and } k = 1, \ldots, 6\}) = Y_{a,b} \cap \mathbb{R}^{15},$$

where the polynomials s_i, c_j, l_k are defined as follows:

$$\begin{aligned} s_1 &:= x_1^2 + y_1^2 + z_1^2 - b^2, & c_1 &:= ca_1^2 + sa_1^2 - a^2, \\ s_2 &:= x_2^2 + y_2^2 + z_2^2 - b^2, & c_2 &:= ca_2^2 + sa_2^2 - a^2, \\ s_3 &:= x_3^2 + y_3^2 + z_3^2 - b^2, & c_3 &:= ca_3^2 + sa_3^2 - a^2, \end{aligned} \tag{10}$$

and

$$(l_1, l_2, l_3)^T := v_1 - Av_2, \qquad (l_4, l_5, l_6)^T := v_1 - A^{-1}v_3, \tag{11}$$

with

$$v_i := \begin{pmatrix} 1 + ca_i + x_i \\ y_i \\ z_i + sa_i \end{pmatrix}, \qquad A := \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3, \mathbb{R}).$$

The polynomials s_i,c_j,l_k depend on the 15 variables

$$\mathbf{w} = (y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, ca_1, sa_1, ca_2, sa_2, ca_3, sa_3).$$

Finally, we set

$$I_{a,b} := \langle \{s_i, c_j, l_k \,|\, i, j = 1, 2, 3 \text{ and } k = 1, \ldots, 6\} \rangle \leq (\mathbb{Q}[\sqrt{3}])[\mathbf{w}].$$

Values for (x_i, y_i, z_i) and (ca_i, sa_i) in $X_{a,b}$ are just coordinates of the direction vectors $\overrightarrow{FG}$ and $\overrightarrow{GH}$ corresponding to the ith limb in the coordinate system A^i, $i = 0, 1, 2$.

With Theorem 1.2 from Section 4 we can prove the following:

Theorem 8.1. (Delta) *For generic $a, b > 0$, $Y_{a,b}$ has 24 singularities counted with multiplicity. Any real singularity of $Y_{a,b}$ is either isolated in X or a nonmanifold point of X, i.e. a CS-singularity of the delta robot.*

Remarks.
(i) By inspection of the singular locus of Y (see Listing 2) and utilization of elimination orderings in Singular one can find formula depending on a, b for the coordinates of the singularities of $Y_{a,b}$. It can be shown that all singularities of $Y_{a,b}$ are distinct and real for $a, b > 0$.
(ii) There is a symmetry in $Y_{a,b}$ induced by 120°-rotation of the delta pose around the vertical axis through the centre of the base triangle and reflection of the delta pose at the ground plane. This means the dihedral group D_3 acts on $Y_{a,b}$ and also on its singularities S. For each of the four orbits in S one singular configuration of the delta robot is sketched in Figure 9.

Proof. When we execute Listing 2 in Singular, we see that $\dim I_{a,b} = 3$ and $\dim K = 0$ for the ideal K generated by $I_{a,b}$ and the 12-minors of the Jacobian of the generators of $I_{a,b}$. Refer to [26] for a more detailed description of the listing. We calculate over the field $\mathbb{Q}(a, b)$, therefore, the results hold only for generic a, b. The listing also shows $\dim_{\mathbb{R}} \mathbb{R}[\mathbf{w}]/K_{\mathbb{R}} = 24$, hence $Y_{a,b}$ has 24 singularities counted with multiplicity. Note that we need to divide the number from the Singular-output by 2 to take into account the algebraic field extension $\mathbb{Q}[\sqrt{3}]$, which is of degree 2 over $\mathbb{Q}$, see [35, Theorem 1].

We know with [15, Proposition 18.13] that the coordinate ring $A = \mathbb{R}[\mathbf{w}]/(I_{a,b} \cdot \mathbb{R}[\mathbf{w}])$ is Cohen-Macaulay, since $I_{a,b}$ is generated by 12 elements. Now, by Proposition 2.3 and Theorem 18.15 of [15] we conclude:
(a) $I_{a,b}$ is equidimensional and radical.
(b) The singular locus of $Y_{a,b}$ is zero-dimensional.

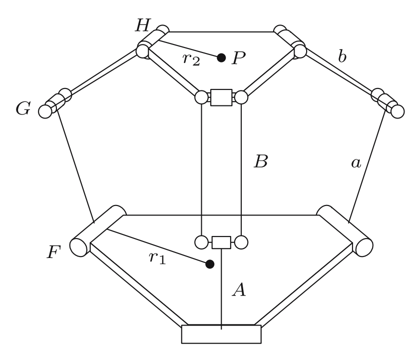

Figure 7: Illustration of the delta robot.

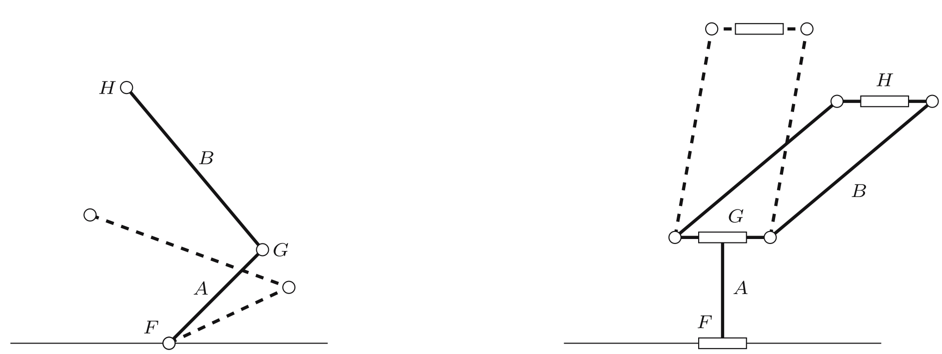

Figure 8: One arm of the delta robot. Front and side views in different configurations.

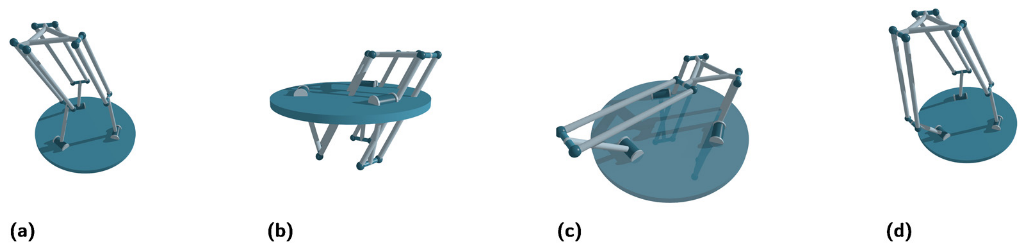

Figure 9: Singular poses of the delta robot, $\frac{a}{d} = 6$, $\frac{b}{d} = 10$. (a) First orbit, (b) second orbit, (c) third orbit, (d) fourth orbit.

(c) A is a product of normal domains. In particular, the irreducible components of $Y_{a,b}$ (if there are more than one) have empty intersection.

With (a)–(c) above we can apply Corollary 4.8 and have the result that any real singularity of $Y_{a,b}$ is either a nonmanifold point of $X_{a,b}$ or isolated in $X_{a,b}$. □

```
LIB "primdec.lib";

ring s = (0,s3),(z1,z2,z3,sa1,sa2,sa3,y1,y2,y3,x1,x2,x3,ca1,ca2,ca3),dp;
minpoly = s3^2 - 3;
ring r = (0,a,b),(x3,y3,z3,sa3,ca3,x2,y2,z2,sa2,ca2,x1,y1,z1,sa1,ca1,s3),(dp(15),dp(1));

poly k1 = x1^2 + y1^2 + z1^2 - b^2;
poly k2 = x2^2 + y2^2 + z2^2 - b^2;
poly k3 = x3^2 + y3^2 + z3^2 - b^2;
poly k2d1 = ca1^2 + sa1^2 - a^2;
poly k2d2 = ca2^2 + sa2^2 - a^2;
poly k2d3 = ca3^2 + sa3^2 - a^2;
poly root3 = s3^2 - 3;

matrix A[3][3] = -1/2,-s3/2,0,s3/2,-1/2,0,0,0,1;
matrix Ai[3][3] = -1/2,s3/2,0,-s3/2,-1/2,0,0,0,1;
matrix kl1[3][1] = 1 + ca1 + x1,y1,z1+sa1;
matrix kl2[3][1] = 1 + ca2 + x2,y2,z2+sa2;
matrix kl3[3][1] = 1 + ca3 + x3,y3,z3+sa3;

matrix gl1 = kl1 - A*kl2;
matrix gl2 = kl1 - Ai*kl3;
int t;
ideal I = root3,ideal(gl1),ideal(gl2),k1,k2,k3,k2d1,k2d2,k2d3;
matrix Jacp = jacob(I);
matrix Jac[12][15] = Jacp[2..13,1..15];
ideal J = minor(Jac,12);

setring s;
ideal J = imap(r,J);
list L = minAssGTZ(J);
int my_time, my_time_m, my_time_s;

setring r;
ideal sing; ideal sing_gr;
list L = imap(s,L);
int num_sol = 0;

for (int i=1; i <= size(L); i++) {
    sing = L[i],I;
    "";"Component " + string(i);
    "---------------------------";
    t = timer;
    sing_gr = std(sing); // i=3 => calculation takes about 5 min on intel core i5. takes much longer (~2h) if older
          ↪ version of Singular (< 4.0.3) is used.
    my_time_s = timer-t;
    my_time_m = my_time_s div 60;
    my_time_s = my_time_s - 60*my_time_m;
    "The computation took " + string(my_time_m) + "m " + string(my_time_s) + "s.";
    "The dimensions is " + string(dim(sing_gr)) + " and the number of zeros with multiplicities is " +
          ↪ string(vdim(sing_gr));
    num_sol = num_sol + vdim(sing_gr);
}
"";"The number of all zeros with multiplicities in the singular locus is " + string(num_sol);
//needs to be divided by 2 because of root3 in the ideal
```

Listing 2: Analysis of the singular locus of the delta configuration space.

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