Mathematical Science Communication

a Study and a Case Study

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Introduction

The project Polytopia– *Adopt a Polyhedron* is the focal point of this thesis. It was developed in a science public relations context and is associated with the Collaborative Research Center *Discretization in Geometry and Dynamics*. The experimental character of the project is aimed at exploring the possibilities of a contemporary science communication project in mathematics. The outcome of this experiment is documented here. The central idea of the project is to put up polyhedra for adoption. One by one the geometrical objects would be chosen by the participants who then give them a name and build models of them. This idea originated from our goal to create a representation of mathematics in a unique way that is open to participation by incorporating aspects of citizen science.

The first chapter lays the groundwork for the classification of the project in the realm of science communication. First, we briefly overview the development within the science of science communication with a shift from focussing onto knowledge, through affection, to the discourse about trust in science, which we currently see. This progress resulted in an alteration of the practice of science communication as well. Whereas traditional formats focused on educating the public in a *downwards* manner, more modern ways of science communication are aiming at fostering a dialogue or transfer of knowledge on even ground.

In the second part of the first chapter, we focus on mathematical science communication and start this with looking into the sparse literature about it. In the book: "Raising the Public Awareness of Mathematics" by Behrends et al. [19], we determine that the deficit model was still prevailing in the discourse in 2012. This is documented in the authors' repeated attributions of a lack of knowledge about and affection towards mathematics in the public, as well as in the ascribed superiority of mathematics among the sciences. A further component of this second part is the definition of some notions to invigorate the discourse and prompt the development of more dialogical forms of mathematical science communication. It would be nice to say that these theoretical considerations lead the way in the development of our own project, but in fact it was the other way around. In classifying the objectives, motivations and methods of Polytopia, the lack of a framework became apparent. After a short digression into mathematical citizen science, the chapter closes with an agenda for future development of mathematical science communication.

The second chapter is dedicated to thoroughly describing the project. First, we classify its motivation, objectives and methods with regard to the terms developed in Chapter 1. Then, we introduce the preliminary considerations that lead to the development of the design and functionality of its website. Its development is documented by screenshots and descriptions. Furthermore, we give some detail on the technical implementation. Next up is the evaluation of the survey, which was filled out by the users of the website. The collected data measures demographics as well as user experience. The primary target group of students aged 7 to 13 years makes up about a quarter of the users. The chapter concludes with some reflections of the execution of the project and discusses future possibilities of including methods for evaluation into mathematical science communication projects.

In order to reach the primary target group of pupils through the math class, we developed school materials that can be implemented by the teachers. The third chapter starts by outlining the didactical principles that guided the conception of the school materials. *Project teaching*, *inquiry based learning*, and *dialogical learning* all have in common that they put the student in the position to guide the learning process and the teacher to assist it. We introduce the school materials and illustrate how their use put the didactical principles into practice. At the end of this chapter, we show how these principles were not just guiding the development of the school materials, but also how the underlaying ideas guided the overall design of the project.

The last chapter deals with the mathematical background of the project. The term *polyhedron* is not universally defined and holds different meanings in various mathematical contexts. First, we trace back the applications of the term *polyhedron*. The next part of the chapter is dedicated to rigorously defining the mathematical concepts that build the scientific backdrop for the project. Here, we also describe the data generation for the website. The last part of the chapter reproduces the progress made on an open problem that is closely related to our project and the polyhedral nets we used for building the models: Dürer's conjecture. It states the question whether every polyhedron possesses an overlap free edge-unfolding. While the painter and mathematician Dürer invented the edge-unfolding, the question itself can be attributed to Shephard [144], who posed it in 1975. A recent result by Ghomi [56] proved that at least for every combinatorial type of polyhedra Dürer's conjecture can be settled with a positive answer.

The conclusion reflects upon this thesis and its objective to closely knit together the four parts of the project that are covered in a chapter each: the theoretical framework and place within the discourse of (mathematical) science communication, the practice of implementing a mathematical science communication project, the relation to mathematics education and the contemporary guiding principles of its didactics, and finally, the mathematics itself that is displayed in the project *Polytopia – Adopt a Polyhedron*.

Chapter 1

Mathematical Science Communication

1.1 Science Communication

Claude Shannon introduced a mathematical model of communication in the 1940s [143]. His simplified model describes the process of transporting a message from one entity to another. On the left hand side is the information source, which puts the message into a transmitter that alters the message into a signal which can be sent through the channel towards the receiver. En route this signal might be disturbed by noise. This signal is then obtained by the receiver which transforms it back into a message delivered to the destination [143].

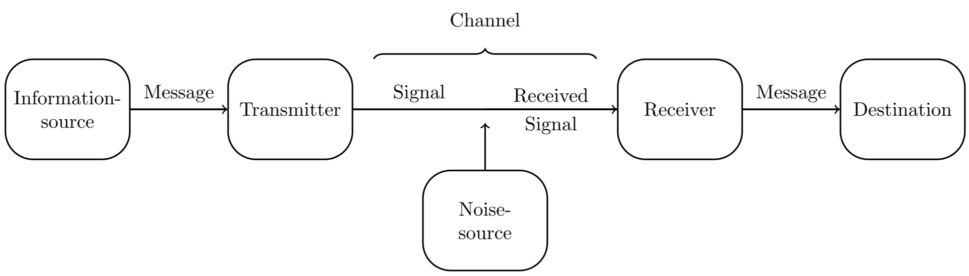


Figure 1.1: Communication model according to Shannon [143].

This communication model was developed for technical communication situations and is still applicable today for acts of communication like sending a text message, or broadcasting a radio show. Information theory investigates the impact of the noise source and tries to find ways to minimize it and ensure the message comes across mostly undisturbed.

As there are more subtleties, nuances and directions in human communication, we will later expand this model. But for now, this model of communication suffices to describe many instances of science communication like classical science journalism in a newspaper, public lectures given at the long night of science, and popular science books written by experts in their fields addressing an interested lay audience.

Example of Science Communication

We consider as an example, a scientist giving a public lecture on their latest work. The scientist themself is the *information source.* They may think of ways to transform the message in such a way that non-experts can understand it. The scientist uses non-technical language and finds meaningful metaphors to transport the essence without completely omitting the scientific rigor. These adjustments, together with the physical act of speaking, are the *transmitter.*

They are giving their lecture to an audience in a lecture hall. Of course the air in this hall, transporting the sound-waves, is part of the *channel.* But, more interestingly though, is the fact that every single visitor made the individual choice of being in it. Chatting seat neighbors may disturb the attendee. But *noise* can also take on other forms, as the visitor might be distracted by their itching foot or the question why the boss made a snide comment during today's meeting.

But even if we assume that the individual in the audience gets every word of the talk, they do not necessarily get the message in the same way as the lecturer intended. Their *receiver* is their preconceptions and prior knowledge about the subject in which they establish the new information (*destination*). A person more familiar with the subject might have picked up on other details of the talk than someone who is hearing about it for the first time.

The unidirectional transfer of knowledge has also been a characteristic of (science) education. Historically, education was at the heart of the objectives of science communication. The *Urania* in

Berlin is an illustrative example. It was founded in 1888 in the intellectual legacy of Alexander von Humboldt with the goal to educate the general public, especially the working class with scientific knowledge [155,168]. It still has not lost its significance and looking at today's program of the institution, one can see the successful combination of education and entertainment, which together form the portmanteau *edutainment*, is still widely offered [155].

1.1.1 The Role of Education in Science Communication

The initial mandate of science communication, education, is arguably still applicable to modern science communication. In education, as well as engagement and entertainment, science communication shares common goals with science education [14]. Even though the emphasis on education, entertainment, and engagement is allocated differently, their common goal is to foster *scientific literacy* in the public. While formal science education puts weight on the teaching of factual scientific knowledge, as science communication did in its beginnings, the latter now promotes the idea that "science not only has applications but implications" [14]. More on scientific literacy can be found in Section 1.1.3.2.

Learning happens through formal education in institutions of primary, secondary, and tertiary education, as well as informal education. Liu [93] and Baram-Tsabari et al. [14] argue that formal and informal education shall join forces better. Liu's argument is that as *free-choice learning opportunities*, as offered by television programs, science museums, or national parks, are widely used, informal learning makes up a major part of the lifelong learning journey anyways. "Given the ubiquity of free-choice science education, it is necessary to bridge formal and informal science education and consider them as a continuum [93]." Baram-Tsabari et al. put their focus on the scholarships in both fields. Given their similarities in goals and methods, they have surprisingly little common ground. In the editorial "Bridging science education and science communication research," they warn against reinventing the wheel, which they feel happened in the shift from the *deficit model* to the *dialogue model*, much like what occurred in formal education 20 years prior [14].

In formal education we often see a hierarchy between the teacher and the pupil, the professor and the student, or expert and layperson. This is due to different levels of knowledge and experience that lead to an imbalance in status and power. The correlation between knowledge and power is strong. Hence, the dicta "Knowledge is power". It is the ideal for contemporary science communication to establish forms of knowledge-transfer that operate on equal power, regardless of differences in knowledge.

1.1.2 Motivations of Science Communication

The differentiation between motivations and objectives for science communication is as follows: motivations are the reasons on the side of the *sender* to engage in science communication, while objectives are the effects that the sender intends to evoke on the receiving end of such practices. We now discuss some common motivations of science communication.

Science is Part of Culture Science plays a defining role in our culture and as such, it is intrinsically interesting to the members of this culture, similar to sports or politics [39].

Science is Publicly Funded As recipients of public funds, scientists and scientific institutions have a duty of transparency towards the citizens regarding the use of the tax money. Since the public is (ideally) also involved in debates of the further directions and developments of research that thorough information about the matters at hand should be shared, is a given.

Economy Rests upon Science and Innovation The economy in Germany and many other countries is based on technology and innovation and thus ultimately on science. The transfer of knowledge accumulated in scientific institutions is important for further progress in these areas.

Competition among Scientific Institutions Funding for research projects is increasingly based upon economical principles of competition among the applicants for certain grants. Thus, communicating and promoting their success is becoming a matter of course in the attention economy.

1.1.3 Objectives for Science Communication

As science communication developed and became a professional science and practice itself, different terms to describe the objectives were also developed. In the following, we want to introduce some of the guiding notions, as well as sum up the points that have been criticized by others, which led to the introduction of new terminology.

1.1.3.1 Public Understanding of Science

Public understanding of science still is an umbrella term used synonymously with science communication. The *public* is everybody who does not belong to the realm of science. Even scientists are members of the public, when facing different areas outside of their own field of study [29].

In a British study Durant et al. [39] evaluated the *public understanding of science* against self-claimed interest in science. The interest was measured by indication of whether a news article with a certain headline is likely to be read by the individual.

The term *understanding* is not easily defined. Durant defined the two dimensions of public understanding of science to be *knowledge* of elementary findings of science and the *process* of scientific inquiry. *Knowledge* was evaluated by asking factual science questions, like: "Antibiotics kill viruses as well as bacteria", "Diamonds are made of carbon", or "The centre of the earth is very hot" with which the participants could either agree, disagree or state that they don't know. The access to *process* of science was measured by questions about methodology such as basic statistical methods, the utilization of control groups in medical tests, and other procedures of establishing scientific knowledge [39].

The major finding was that (self-declared) interest in science is high in Britain but it does not correlate significantly with the two dimensions of understanding. High interest in scientific headlines was not an indicator for good results in the quiz. Therefore, a high interest in science does not necessarily mean high factual knowledge about science. However, the inverse correlation was "reasonably strong" [39]. Individuals who scored high in the public understanding of science also ranked high in interest.

Measures of *knowledge* and *understanding the process of science*, which constitute understanding of science in general, score "not much" [39] in British society. Durant et al. chose an illustration which depicts four people, three of them have a question mark on their head, which might represent their poor scores in the quiz. The planets drawn on one person's forehead may symbolize their understanding of science, see Fig. 1.2.

In the United States, Miller [109] conducted studies to measure the factual knowledge of certain scientific facts and certified a lack of general ignorance[1] similar to Durant et al..

[1] In [39] we find the quote: "If modern science is our greatest cultural achievement, then it is one of which most members of our culture are very largely ignorant." Surprisingly, the conclusion reads: "[...] although the public is largely uninformed, it is also largely interested in science".

Figure 1.2: Illustration from Durant et al. [39]

In short, the term *public understanding of science* is emphasizing the factual and procedural knowledge of science. It has close links to school education, where the roots of science communication lie.

1.1.3.2 Scientific Literacy

While *public understanding of science* and *scientific literacy* were historically used synonymously, their meaning differs today. In contrast to *public understanding of science*, which centers around science and is looking at the public as the "other", *scientific literacy* is a term used to describe the situation from the perspective of the individual. It is no longer an undefined particle of the public mass but a unique person with skills, needs and (social) conditions. Thus, using the term *scientific literacy* implies a change in the point of view. The question: "What does an individual need to know, to not appear ignorant in the eyes of the better educated?" is replaced by: "What does the individual need from science in their private, public and political life?"

Scientific literacy used to describe the ability to apply scientific knowledge in a way that serves the individual in their practical lives. Examples are: reading the supplement facts on the package of a granola bar, deciding which heating system for the home is the best in terms of costs, efficiency, and carbon emission, or deciding whether to take a certain medical examination and how to interpret the results[2]. Today, the concept of scientific literacy has broadened and this aspect is now called *practical scientific literacy* [29].

Zooming out from the private sphere to the social sphere, scientific literacy is an important skill to participate in political discourses. When we talk about this public dimension of scientific literacy, we cannot neglect to take a closer look at the political system in which the individual finds themselves. The literature on science communication is usually based on the assumption that an individual is a citizen within a democracy and therefore asks what skills and knowledge a citizen in a democracy needs to have in order to be able to actively participate in the democratic process. It is normative consensus that political decisions should be based on deliberation. The explicit demand for a more *deliberative democracy* can often be found, see the quote below. Thus, we briefly want to explain the concept. It was established by philosopher and sociologist Jürgen Habermas. The legitimation of a deliberative democracy is established by the discursive structure of opinion-forming and decision-making. Citizens can and should directly participate in the debates and discussions concerning political questions. The most critical variable for these public discourses to be fruitful is their quality [68].

In a declaration about the relationship between the public, the media, and the democratic state, made jointly by the three German academies of science[3], we read the following statement:

> *The problem of the alienation of a large part of the public from participation in political decisions has for some time contributed to a disenchantment with politics, as well as to demands for more participation and deliberative or even direct democracy, and has become a topic of public discussion. The problem thus described is equally relevant to science and science policy [36].*[4]

A high level of quality is necessary for political discourses, and this applies to debates about science as well. One of the guiding questions in the discourse of science communication is the question: "What skillset an individual must possess in order to participate in such a high quality discourse?"

[2]In the latter case, a whole area of research that investigates the individuals handling of statistics, exists which goes by the notion of *risk literacy*, see 1.2.2

[3]*Leopoldina*, *acatech*, and *Union der deutschen Akademien der Wissenschaften*

[4]original: *Das Problem der Entfremdung eines großen Teils der Öffentlichkeit von der Teilhabe an politischen Entscheidungen hat seit einiger Zeit zu einer Politikverdrossenheit sowie nach Forderungen nach mehr Partizipation und deliberativer oder auch direkter Demokratie beigetragen und ist zum Thema der öffentlichen Diskussion geworden. Das damit beschriebene Problem ist gleichermaßen in Bezug auf die Wissenschaft und die Wissenschaftspolitik relevant.* Translated by the author.

Miller [110] defines this skill set as *civic scientific literacy.* He expands the two-dimensional model for measuring factual and methodological knowledge by a third dimension that measures an "understanding of the impact of science and technology on individuals and society" [110]. Here we see that the burden to demonstrate a qualification to competently participate in the discourse still lies on the individual.

1.1.3.3 Deficit Model

The paper "Public Understanding of Science" by Wynne [166] is often cited when the "old way" of doing science communication is referred to as the *deficit model.* Wynne claims that the underlaying assumption is that the public is the main cause for social conflict about science. The public lack of understanding needs to be filled by education about scientific facts and methods in order to make the public agree with the experts' opinions of the matters at hand [114].

According to Wynne, social implications have been neglected. The public is problematized by their limited cognitive processes and capabilities, and the scientific knowledge, practices, and institutions are rendered unproblematic. Thus, "scientific knowledge is seen as encoding taken-for-granted norms, commitments, and assumptions that, when deployed in public, inevitably take on a social-prescripticve role [166]." The superior role of science leads to the paternalistic attitude of many science communication endeavors and neglects the circumstances in which individuals neglect scientific information which could be a shortage of social opportunity, power, or resources [166].

The second assumption of the *deficit model* is the amalgamation of the rational perception of and affection towards science. Wynne writes: "In many dominant formulations (e.g. Royal Society, 1985), PUS is equated with the public appreciation and support of science, [...] Thus, when publics resist or ignore a program advanced in the name of science , the cause is assumed to be their misunderstanding of the science [166]."

Two assumptions undergird the deficit model. The first is the belief that the public's lack of knowledge is responsible for resistance against science and thus, a lack of separation between understanding and appreciation and support for science. The second assumption is that experts' knowledge is always superior to other forms of knowledge.

This universal superiority was empirically proven wrong, for example in [165] and the implication that more understanding (i.e. factual and methodological knowledge) leads to more support also does not hold. Peters et al. found evidence that the relationship between knowledge about genetic engineering in food and support of such practices can be described by an U-shape. Individuals who are strongly opposed to food engineering and supporters measured higher in awareness and knowledge than persons who showed a more neutral attitude towards it [119]. This cannot be generalized, because generic food engineering is a highly emotionally charged debate but it is a counterexample.

Bauer shows that the assumption that more knowledge leads to more appreciation cannot completely be dismissed: socio-economic contexts also matters. In a study comparing knowledge and perception in India and the UK, it shows that the correlation between the two is much stronger in an India, while in the UK " 'familiarity might breed (some) contempt'(or at least a skeptical loyalty)." [17]

1.1.3.4 Public Awareness of Science

The deficit model attributed a lack or deficit to the public. This is a more rationalist view which assumes that more information and understanding must lead to support of science. When this view is omitted, we are left with the pure emotional response to science. Convincing the public to be more appreciative of science is then taking on the form of marketing and advertising. Bauer has pointed out that some form of a void is present in both cases and has shifted from a lack of knowledge towards a lack of attitude in the 1990s [17]. In Durant et al. [39] and Miller [109] we see examples for how the public is attributed with ignorance and uninformedness. *Knowledge* was replaced by *awareness* in the more recent discussions. *Awareness* denotes the public's affectional and emotional reactions towards science and its displays in form of support or appreciation.

Burns et al. [29] made an effort to collect contemporary definitions of science communication in 2003 and based the notion of *public awareness of science* on the work of Gilbert et al. [59] [5]:

> *Gilbert, Stocklmayer, and Garnett (1999) defined the* public awareness of science (PAS) as a set of positive attitudes towards science (and technology) that are evidenced by a series of skills and behavioral intentions. *[...] On occasion the term "public awareness of science" has been used as a synonym for "public understanding of science." Their aims are similar and their boundaries do overlap, bus PAS is pre-*

[5] Unfortunately we could not get a hold of the original paper, so we need to rely on secondary quotation here.

dominantly about attitudes *toward science. PAS may be regarded as a prerequisite – in fact, a fundamental component – of PUS and scientific literacy.*

1.1.3.5 Trust – the New Variable in the Discourse

Without going into any detail about the philosophy of science and its hermeneutics, we see that the communication of science is in predicament. Science in general and its disciplines in particular, have agreed upon certain procedures of truth determination. Science communication leaves that scientific realm and cannot claim the right to define the truth in the same way science can within. In order to eliminate a paternalistic tone it is necessary to abandon the unidirectional communication model and search for dialogical forms of science communication[6] [74]. These formats can, for example, take on the form of citizen forums that are organized around governmental plans to build a new research centre in the neighborhood wherein the local residents directly discuss with the scientists. Active participation of the individuals, for example as we find it in science centers, plays a major role [94]. In *citizen science*, the active participation of citizens in the scientific research process culminates. Citizen science is a scientific process that actively involves non-experts into the research process [24][7].

The attempt to provide all citizens with enough knowledge about science to answer every pressing question will fail. An illustrative example can be found in Baram-Tsabari et al. [14]: When the Large Hadron Collider at the CERN was put into operation, the question whether it could accidentally produce a black hole that would swallow the earth arose. A local french framer could not judge the truth of such claims without digging deep into atomic physics. In this scenario the crucial question for the farmer is: "Whom to trust?".

According to Fiske and Dupree [48] the identification of whom to trust follows systematic principles and is based around the perception of *intent* (warmth, being on ones side, friendliness) and *capabilities* (competence). They argue that a communicator needs both to be successful. Scientists have shown to be competent, but their credibility rests on a perception of impartiality and trustworthy intentions. Being able to identify the intent of a person and 'whose side they are on' is necessary to form stable interpersonal relationships. After deciding on the intent, the public estimated the capabilities to perform according to this intent is estimated. According

[6]As we will see in chapter 3 this is quite 'old news' in (science) education, where the shift from frontal teaching to more interactive forms has (theoretically) happened 20 years ago [14].

[7]For more on this topic and how it can be applied in the field of mathematics, see Section 1.2.2.1

to [48] competence and expertise is just the secondary dimension of trust. Friendliness, warmth and trustworthy intentions are the primary prerequisite. The authors then give advice to show “concern for humanity and the environment. Rather than persuading, we and our audience are better served by discussing, teaching and sharing information, to convey trustworthy intentions” [48].

Critical Science Literacy Priest [122] offers another perspective on trust in the scientific practice. In the paper “Critical Science Literacy: What Citizens and Journalists Need to Know to Make Sense of Science” she interprets science as a culture. Each culture contains a certain set of rules and knowledge about “how things work” that is crucial to orient oneself in this culture. The general aim of science communication is to make that culture a part of everyday culture. In order to do that, she asks: “What knowledge about science is of most central value to citizens in a contemporary democracy, in which many personal and policy decisions have some relationship to science or technology and most of the facts, observations, and conclusions of science (and pseudoscience) are available to us on computer screens.” [122]

Priest criticizes the measurement of scientific literacy by the knowledge of a collection of important scientific facts as shortsighted and claims that people need skills to evaluate scientific claims. This includes knowledge about handling uncertainty and the ability to recognize the social dimension of the scientific process and how trust is distributed in this process. Following up on the above example, if a professor of physics at a renown university claims that the fear of the earth being swallowed by a black whole lacks any substance, a critically scientific literate person would not go to the professor and ask to see their CV and certificates in original. They simply trust that the institution who employed the professor has done so, and has chosen a capable candidate, who knows enough about physics to be an authority. Here, trust comes from familiarity with the scientific system [122].

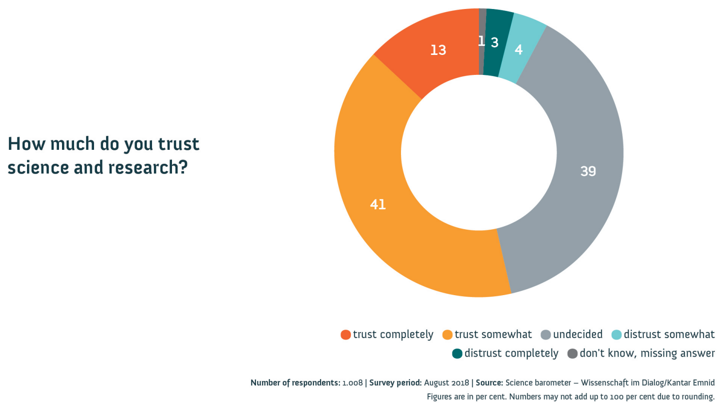
A key ingredient to this kind of trust is the underlying concept of scientific ethos. The ethics of science are well established in society. Though fraud may occur, the fact that it is heavily punished within the scientific community and the medial echo as a scientist is found guilty, manifests the prevailing ethics further [122].

1.1.3.6 What about the Trust in Science in Germany?

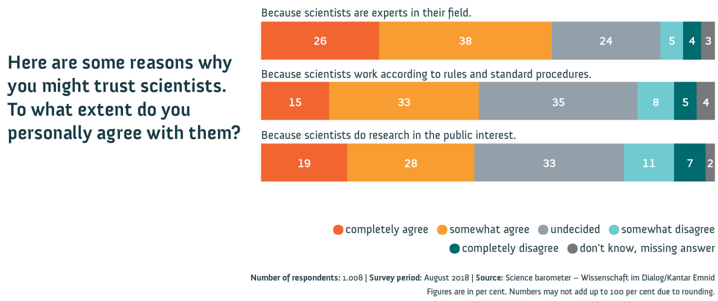
We can note that the general public trust in science is quite high. The organization *Wissenschaft im Dialog* surveyed the attitudes of the German population towards science and research in the study *Science Barometer* [162]. Figure 1.3a shows that more that 50 % of the respondents expressed confidence in science and research. The main reason for this trust are expertise, standard procedures, and commitment to the public interest, see Figure 1.3b. These findings resonate with the *Bungee Jump* model of Bauer. The model describes a scientist as being located on some high base, from which they sometimes jump down into the public discourse and loose some audiences over discussions about vaccination, evolution, or climate change [104]. The general high trust in science seems to dwindle when other interests come into play. These may be of an economic, or personal nature, see Figure 1.3c. But still, like a bungee jumper, the link to the high base is tight and besides some ruffled hair, not much damage is done.

Recent Trends on Trust in Science The recent pandemic crisis may have affected the general trust in science. Accurate assertions can be made once the immediate consequences have subsided. The initial effects are already apparent, for example, in the study by Allensbach commissioned by the *Frankfurter Allgemeine Zeitung* [120]. It showed a clear increase in trust in scientists and professors. The question "Who do you trust to tell the truth"[8] showed an increase from 30 to 43 percent among scientists and from 23 to 33 percent among professors. These increases were measured between 2015 and 2020. A even bigger effect was measured in the "Corona-Special" of the Science Barometer conducted again by *Wissenschaft im Dialog* [159]. The question "How much do you trust in science and research?" was asked in summer 2018, summer 2019, and in April, and May 2020. Initially, 54 % of the respondents answered with "trust completely" or "trust somewhat". This number dropped to 46% in 2019 but then significantly rose to 73% in April 2020 and slowly decreased to 71% in May. This change in the numbers can be explained by the drastic experience of the pandemic that caused major shifts in the every day life of nearly everyone. Scientists had a strong presence in the media coverage about the virus and their specific recommendations guided the political decisions addressing it. Time and the further development of the situation will show if this drastic experience will have long lasting effects.

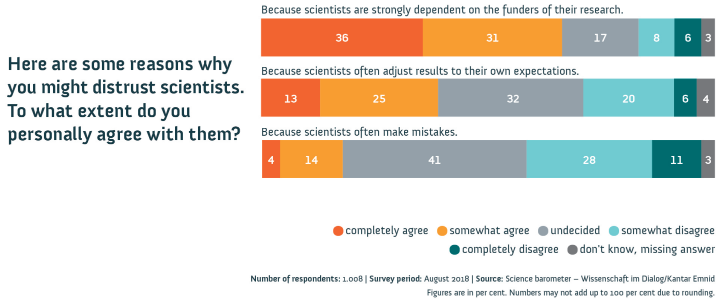
[8] Bei wem vertrauen Sie darauf, dass er die Wahrheit sagt?



(a) Trust in science and research [162]



(b) Reasons for trust in scientists [161]



(c) Reasons for distrust in scientists [160]

Figure 1.3: All Figures are taken from Science Barometer / Emnid [160–162].

1.1.3.7 Skepticism as an Opportunity.

As the discussion has shifted towards trust, this can be seen as the third incarnation of the deficit model. First a lack of knowledge, then of appreciation, and now it is the trust in science that the public lacks.

The difference is that trust – in contrast to knowledge and appreciation – requires mutuality. The public has not been trusted but is portrayed as deficient. Since trust is a *two way street*, the exchange is interrupted. Additionally, various crisis in the 1990s such as the *mad cow disease* scandal, and discussions over genetically modified food were diametral to trust building [17].

Bauer goes one step further and does not buy into the narrative that the public is lacking trust, but notes that the public does good in holding trust back in some cases:

> *I do not consider a skeptical public as a problem, rather as a resource that needs to be maintained and invested in. This is particularly important as science becomes a greater part of the private sector, operating with a commercial logic. [17]*

Knowledge societies base their economic wellbeing on developments in science and technology. The funding for these innovations stems from the private sector. Thus, science is increasingly financed by economically motivated companies. A growing number of actors near to the scientific institutions are communicating the results and innovative technologies to the public in order to attract sponsors or enhance the public status of the institution [18].

1.1.3.8 Untangling Science Journalism and Public Relations

At this point, it becomes necessary to talk about the different actors in science communication. On the one hand there is a direct communication of science from the scientific institutions and the people affiliated to them. An increase of competitive elements in the distribution of funding may lead to communication practices that promote the interest of the respective institutions. A statement by the German academies of science [36] reads:

> *On the part of science and its institutions there is [...] a trend to mix science journalism and science PR. For this reason, and from a normative perspective, a tendency to equate science PR and science journalism with regard to the public with*

information that is as independent as possible must be regarded as a serious quality deficit [36].[9]

Not only is the service of independent science journalism needed now more than ever, the media landscape has rapidly changed in the last two decades, which deprived the livelihood and power of the fourth estate [18, 36]. To compensate for the increasingly institutionalized science communication it is crucial to encourage and support science journalism.

In addition to the necessary promotion of independent science journalism, there are also efforts within the realm of science public relations that call for for *candid communication*[10] [36]. This claim was taken up by *Siggener Kreis*, a group of German science communicators, scientists and science journalists. They published guidelines for *Good Science PR* in 2016 [145]. These include respect for the point of view of all parties involved, strengthening the understanding of the scientific methods, but also takes the questions, needs, and attitudes of the public into account. Filtering out and promoting the information that is relevant for the public and not act on self-interest are guiding principles for *candid communication* [145].

Science PR should be devoted to science instead of scientific institutions [163]. When it is understood as part of science and not an intermediate, it is subject to the same standards as the research it reports on, which ensures impartiality [164].

[9] Seitens der Wissenschaft und ihren Institutionen ist [...] ein Trend zur Vermischung von Wissenschaftsjournalismus und Wissenschafts-PR zu beobachten. Aus diesem Grund sowie aus normativer Perspektive ist eine Tendenz zur Gleichsetzung von Wissenschafts-PR und Wissenschaftsjournalismus im Hinblick auf die Öffentlichkeit mit möglichst unabhängigen Informationen als gravierendes Qualitätsdefizit anzusehen.

[10] redliche Kommunikation

1.2 Mathematical Science Communication

This thesis is building bridges that are not just between mathematical science and the public, or mathematics, education, and science communication. During the process of writing, it became increasingly evident that the practices of communicating mathematics to the public and the science of science communication are missing crucial links. There is hardly any literature about mathematical science communication. In fact, we could only find one article [43] and one book [19]. This sparsity stands in stark contrast to the great number of projects and efforts being made to "bring mathematics to the public". There is a lack of terminology for mathematical science communication. For instance, there is no concise term for "bringing mathematics to the public". This section has two parts. First we examine the current state of the way we talk about mathematical science communication by analyzing the chapters of the more current book [19] and demonstrating that the deficit model is still predominant. A little detour takes us into a short discussion on the empirical evidence for the *image of mathematics*. In the second part of the section, the concepts from the previous section that address science communication in general are narrowed down to mathematics. The definitions are chosen in a way that steps away from the deficit model and creates space for new formats of more dialogical science communication.

1.2.1 Literature Review

The volume "Raising the Public Awareness of Mathematics" [19] is edited by Behrends, Crato and Rodrigues. It resulted from a workshop of the "Raising the Public Awareness" committee of the European Mathematical Society at a conference in Óbidos, Portugal in 2010, which had the goal to "[...]provide a forum for a general reflection with an international mix of experts on building the image of mathematics, ten years after the World Mathematical Year in 2000 (WMY2000)." Nowhere throughout the book do we find a reflection on what is actually meant by *awareness of mathematics*. A hint of its meaning can be found in the following quote from the introduction:

> *This book aims to encourage and inspire action to raise the public awareness of the importance of mathematical sciences for contemporary society through a cultural and historical perspective, and to provide mathematical societies, in Europe and the world, with ideas and details of concerted actions with other national or international*

> *organizations and societies with regards to raising the public awareness of science and technology and other important areas of society that have a strong mathematical component [19].*

The emphasis on the national mathematical societies can also be found in the structure of the book. The first part is dedicated to depicting the national experiences of mathematical science communication. We will take a closer look into it. Initiatives in the United Kingdom, France, Germany, the United States of America, Portugal, and Spain are described by authors from the respective countries who are directly involved into the activities.

The other three parts are *Exhibitions and Mathematical Museums, Popularization Activities* and *Popularization – Why and How.*

1.2.1.1 The Deficit Model – Presumed Dead yet Omnipresent

Recalling the structure of the deficit model, it is composed of two assumptions. One is that a lack of knowledge is responsible for a lack of appreciation. The other is the assumption that the experts' knowledge is superior to other forms of knowledge.

As we shall demonstrate below, we find both guiding motifs of the deficit model in the first part of the book: *National Experiences of Mathematical Science Communication.* Many authors describes a *lack of appreciation for mathematics* or the *enhancement of the image of mathematics* as the driving motivation. A *lack of knowledge* is also frequently mentioned. The public is repeatedly portrayed to be deficient in appreciation and knowledge about mathematics. Yet there is no evidence in current literature that a lack in abilities is responsible for a lack in appreciation for the science.

Another theme that presents itself throughout the articles is *superiority.* Mathematics is written about as being superior to the other sciences. Furthermore, it is expressed that mathematicians are the only ones capable of sharing the ideas of their work.

Deficit of Appreciation The public is attributed with a deficit. This deficit is twofold: more apparent in the *lack of appreciation* and less obvious in the *lack of knowledge.* The former is explicitly mentioned twice, once by Barrow and Wilson [15]:

> *It [the Millenium Mathematics Project,* author's note*] was set up in response to a perceived drop in the standards of teaching and learning of mathematics in the UK*

and a perceived lack of appreciation *of the role that mathematics plays in science, business and everyday life [15, p.7].*

Laubenbacher [90] mentions it in what they perceive to be the influence of politics in the United States of America:

> *In the United States, there used to be a broad political consensus that mathematics and science are key to the prosperity and security of the country. As this chapter is being written this consensus has eroded and funding for mathematics and science research and education is being threatened seriously in the effort to cut budgets and reduce the role of government. There is no doubt that in some part a* lack of appreciation *of the role of mathematics and science in our lives among the general public and, by extension, their elected representatives, is to blame. [90, p.53]*

The Spanish report by Torres [152] leads with the assumption of a negative image of mathematics in his opening paragraph:

> *Despite the social and cultural importance of mathematics, and also in education, their image and that of mathematicians is negative [152, p.67].*

Later they state the motivations of the outreach efforts in the World Mathematical Year 2000 (WMY 2000) as:

> *But in the WMY 2000, the mathematical community, in an extraordinary way, turned to work to* improve the social image of mathematics*, to disseminate mathematical culture to society so that society became aware that mathematics is a fundamental part of our society, our daily life, our culture, and that the economics, scientific, and technological development of a country would be impossible without them [152, p.67].*

As the social image of mathematics needs improvement, we can deduce that it is assumed to be lacking in the eye of the author. Torres explicates in the list of motivations for the commission of the Royal Spanish Mathematical Society:

> *(i) to change the negative attitudes of people towards mathematics; [152, p.68],*

and also in

(vi) to increase the appreciation of the mathematics in our surroundings by learning to watch reality with mathematical eyes; [152, p.68].

The idea that the "public's deficit" must be remedied can also be found in other articles in the book. An example occurs in the motivations to engage in more active outreach of the Portuguese Mathematical Society by Ramalho and Crato [124]:

> *We argue that some of the approaches adopted can result in* better public understanding of science role and in larger public support *for math education and research [124, p.57].*

The initiators of the German Year of Mathematics in 2008, decided to lead with a positive image. With their decision to omit the *negative image*, unfortunately they reminded the reader of its prevalence.

> *So we did not complain about the* negative image *that our subject might have in the eyes of the public and in some of the media, but proclaimed that mathematics is an exciting subject, [...] [171, p.38]*

Throughout the first part of the book, we find a multiplicity of occasions that mention *popularization of mathematics* as a term to describe the outreach practices. The choice of this notion fits well with the implicit meaning of *awareness of mathematics* that is used as an umbrella term for affectional and emotional responses towards mathematics, compare Section 1.1.3.4.

Deficit of Knowledge Next up is the assumption of a lack of mathematical knowledge in the public that can be found explicitly in the abstract of the first chapter by Barrow and Wilson [15]:

> *The United Kingdom has a track record of events to raise the public awareness of mathematics, although mathematics still remains a closed book to the vast majority of UK citizens, and prominent figures are quite happy to admit to their* lack of knowledge *of mathematics [...] [15, p.3]*

The school activities of the German year of Mathematics stood under the motto:

> *"Du kannst mehr Mathe, als du denkst!" ("You know more math than you think!") [171, p.39],*

While the message is formulated positively, it implies that the students may think that they do not know a lot of mathematics.

Superiority The leitmotif of *superiority* presents itself in a twofold manner. On the one hand, it is apparent in the assumed dominance of mathematics among the sciences and on the other hand, in the mathematicians' unparalleled ability to advocate for their subject.

The last part of the book takes on a more *fundamental* view. Brusse [27] opens up his opinion piece by establishing that mathematics is too hard for almost everybody, except for a select few, including himself as a professional mathematician.

> *[...] mathematics is at least one step too far away for almost everybody, and this step is a big jump because mathematics is hard for almost everybody. This is why mathematics has a problem [27, p.306].*

This fantasy of grandiosity is even further increased by Greuel [61]:

> *In an overall sense, mathematical thinking is, after speech, the most important human faculty. It was this skill especially that helped the human species in the struggle for survival and improved the competitive abilities of societies. I believe that mathematics has a special place in evolution [61, p.367].*

Hansen [70] harshly criticizes science writers for the quality of their work and only sees the prominent mathematician as capable of spreading the word.

> *Keeping the awareness of mathematics alive is a great challenge. The themes within mathematics of immediate appeal to the great public tend to be quickly written about by professional science writers without a proper mathematical background. Very often you feel there is a lack of authenticity in such writing, which seldom catches the essentials of the mathematics and in most cases fails to show that mathematics is one of the most creative areas of human thinking.*
> *Fortunately there are an increasing number of prominent mathematicians who care about disseminating the pearls of both classical and contemporary mathematics to a wider audience [70, p.395].*

format	number
exhibition(s)	321
workshop(s)	118
article(s)	96
lecture(s)	73
activity(s)	36
hands-on	18
participation	8
dialogue	4

Table 1.1: Total number of words describing the communication formats.

target	number
visitor(s)	96
audience	79
general public	61
participant(s)	25
viewer(s)	9
listener(s)	8
guest(s)	4

Table 1.2: Total number of words describing the targets of the communication formats.

Laubenbacher's article [90] leads to an interesting observation. In Laubenbacher's writing about the efforts made in the United States of America, they say the practice of mathematical science communication is an unpleasant necessity.

> *The burden of this outreach falls on all of us, [...]. While this takes us away from the things we love, whether it is teaching students or proving theorems, the continued health and prosperity of the profession demands that we all become engaged. And we need to leverage other stakeholders in a strong mathematics culture, such as industry and high-tech companies, as well as governments [90, p.54].*

1.2.1.2 Formats of Mathematical Science Communication

Science communication can take on many different formats [43]. We conducted a brief concordance analysis on the formats of science communication and words that were used to describe the recipients of the formats described in the first three parts of [19]. In Table 1.2 we see that by far the most used word is *exhibition*, which can be explained by the second part of the book being devoted to exhibitions and museums. The second most mentioned format is *workshop*. The word *article* takes rank three and is followed by *lecture*.

Three of the four top used words put the recipient in a passive role, as a visitor of an exhibition, reader of an article, or being in the audience of a lecture. In a workshop, a direct contact between the facilitator and the participant is more likely. A lot of the exhibitions have a participatory element in the form of hands-on exhibits or interactive displays. Words that link to more dialogical and engaging formats like *activity*, *hands-on*, or *dialogue* are lagging behind.

word	number
woman/women	3
gender	1
diversity	0
female	0
male	1

Table 1.3: Frequency of words, concerning gender and diversity issues.

The choice of words to describe the target groups of the communication formats shows a similar picture. The top two words here are *visitor(s)* and *audience*, which again assigns a passive role. When referred to as *audience*, the individual member of the target group is getting lost in the crowd. The target group in many cases is addressed as the *general public*, mentioned a total of 61 times.

1.2.1.3 Gender and Diversity

Given the quite manageable quantity of occasions in Behrends et al. [19] that mention issues concerning gender, see Table 1.3, we quote them all. Diversity is not addressed explicitly throughout the volume.

The first instance is aiming at enhancing the visibility of women in the history of mathematics in an exhibition, by Torres [152]. Note that the mathematicians are sorted by gender. Explicitly stating the female but not the male gender, effectively "others"[11] the former.

> *The featured mathematicians are: Pythagoras, Euclid, Archimedes, Apollonius, Al-Khwarizmi, Fibonacci, Cardano and Tartaglia, Fermat, Descartes, Newton, Lagrange, Cauchy, Galois, Abel, Leibniz, Euler, Gauss, Riemann, Hilbert, and Poincaré. But also some* women*: Hypatia, Madame du Châtelet, Sophie Germain, Sonia Kovaleskaia, and Emmy Noether [152, p.77].*

Brueckler [26] describes an activity that includes the participation of a volunteer and recommends choosing a male, reinforcing the stereotype that women are concerned about revealing their age.

[11] Different authors used and coined their definition of 'othering', however we work with Beauvoir's [34] concept of it .

Note that if performing it with an adult it is better to choose a male spectator since the person's age is disclosed to the public [26, p.209, footnote].

The following example shows an awareness of matters of gender and diversity in a school flyer for the Year of Mathematics in 2000:

[...] but also a variety of professions with different levels of maths qualifications and illustrating these, paying the necessary attention to different kinds of consideration such as gender, regional balance and also not limiting this to well developed countries *[30, p.225].*

In a book, distributed at schools in Denmark we find the issue of women and mathematics well hidden in a plethora of other topics:

*In the book you can find exciting mathematical unfoldings (with exercises) of similar phenomena under headlines such as: "Mathematics through the millennia"; "Mathematics and evolution"; "Fire!"; "How a vending machine actually works"; "Wavelets"; "Secret codes made public"; "Math in medicine"; "Tour de France mathematics"; "*Women *and mathematics"; "Error correcting codes"; "Beer and flat screens"; "Mathematics in the computer and vice versa"; "The science of the better"; "Artificial intelligence"; "Mathematical modeling of climate and energy"; "The Mars mission"; "The mathematics of shape [103, p.257]."*

And in a last one, Greuel [61] refers to the love of a beautiful women as an object to be *wanted*:

One way to put the dichotomy in a more philosophical or literary framework is to say that algebra is to the geometer what you might call the 'Faustian offer'. As you know, Faust in Goethe's story was offered whatever he wanted (in his case the love of a beautiful woman*), by the devil, in return for selling his soul [61, p.384].*

1.2.1.4 Digression: The Image of Mathematics

The pervasive accentuation of emotions is an interesting contrast to mathematics guiding principles of logic and rationality. It is also noticeable that the authors present no empirical evidence for the recurring assumption of public *lack of appreciation*.

In didactics and education efforts to understand the attitudes towards mathematics has been undertaken under the umbrella of *beliefs* and *epistemological worldviews* of students and math teachers, see [63, 151].

Empirical inquiry of the *image of mathematics* is sparse, see Sam [130] and Mendick [105]. Sam conducted a study in Britain in 1996. Its quantitative part surveyed 548 adults in an opportunity sample and conceptualized the term *image of mathematics* by eleven components like: stated attitudes, feelings, metaphors for mathematics, beliefs about the nature of mathematics, etc. Here we want to just focus on attitudes and feelings. About 44% of the respondents expressed an affectional response towards mathematics. The three mostly expressed emotional states were "difficult"[12], "boring", and "anxiety" with 70, 58, and 44 total mentions respectively. On rank four to six, we find more positive emotional descriptions; "enjoyable", "necessary", and "interesting", with 38, 40, and 35 mentions. An interesting point is that when the respondents were asked about mathematics in general, most of them related it to their experience of learning mathematics in school [130]. For an analysis of the impact of popular culture and its depiction of mathematics and mathematicians, see Mendick [105].

At this point we don't want to reproduce the results of the study any further, but go back one step further and, again, look at the ideas and concepts that lead the inquiries and the expected results. The design of the questionnaires as well as formulations in the articles about the studies provide hints. Sam et al. start their paper with an outcry in quotation marks: "Oh Gosh! Maths?". Another negative reaction by a respondent is described. Besides their study finding positively labelled emotions towards mathematics in a non neglectable number of people, no positive reaction is explicated in the beginning. Here, as well as in Ernest [43], we find the recurring motif of *mathematical myths*. This term describes *false* beliefs about mathematics. Without going into detail, this begs the question of who is sovereign in the interpretation of what is an *accurate* image of mathematics?

There is a need to close the empirical gap of the public perception of mathematics, but before approaching the practicalities of a survey it is necessary to reflect upon the underlaying assumptions. How does the image of mathematics compare to other subjects, like physics, or comparative literature? Even if the public image of mathematics is not inherently positive, how much public admiration would be *enough* to fill the void?

[12]The classification of "difficult", and "necessary" as emotional states was taken from the original.

1.2.2 Conceptualizing the Objectives of Mathematical Science Communication

The results of the analysis of the existing literature on mathematical science communication summed up in the first part of this chapter, paint rather gloomy picture: The deficit model is omnipresent, the discourse is vaguely fluttering around affectional responses and inclusion has been on hardly anyone's agenda.

In order to enhance the quality of the discourse, in the following we aim to coin terms that capture the motivations and objectives of mathematical science communication. By no means do we claim these definitions are complete. The goal is to invigorate the discourse by putting up a first version for debate and to inspire more contemporary formats of science communication that move beyond the paternalistic styles of the deficit model and create a dialogue or transfer of knowledge on eye-level. Additionally, further neglect of gender and inclusion is not acceptable in considerations of mathematics science communication.

For demarcation purposes we define the *old way* of doing mathematical science communication as:

Definition 1.2.1. The **Deficit Model of Mathematical Science Communication** is characterized by one or more of the following characteristics:

- attributions of a lack of knowledge, or
- attributions of a lack of appreciation in the public,
- assumptions of mathematic's superiority among the sciences, or
- the mathematicians' unique ability for outreach,
- and, as an extra, ignorance of diversity.

In contrary to the implicit notions of emotion and appreciation we have seen above, we define the awareness of mathematics as an emotionally neutral consciousness about mathematics. An example could be to know that there is a lot of mathematics in the hard- and software of a phone. And even though this object may be charged with positive emotions, they do not necessarily transfer to the subject of mathematics.

Definition 1.2.2. Public Awareness of Mathematics is to be conscious about the presence of mathematics in many aspects of society, technology and everyday life.

A positive emotional response towards mathematics, a better image and an active and even enjoyable active engagement merits being labeled for what it is.

Definition 1.2.3. Public Appreciation of Mathematics is a positive affectionate reaction towards mathematical activities and information about mathematics. The admiration for the beauty of mathematics falls into the same category.

Mathematical Literacy

Literacy in contrast to understanding puts the individual and its needs and desires in the center of attention. Thus it is crucial to look at the objectives of mathematical science communication through the eyes of an individual in society and figure what they may need *from mathematics* to thrive in their personal, social and political life. The practice of communicating mathematics can be seen as a service to the individual.

Before we go on and try to define *mathematical literacy*, we take a look at the definition used by the OECD (Organization for Economic Co-operation and Development) as a foundation for the tests of the PISA study in 2003, the year in which a special emphasis was put on mathematical literacy.

> *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. [50]*

In an accompanying explanation it is further elaborated that using and engaging with mathematics "implies a broader personal involvement though communicating, relating to, assessing and even *appreciating and enjoying* mathematics. Thus, the definition of mathematical literacy encompasses the functional use of mathematics in a narrow sense as well as preparedness for further study, and the aesthetic and recreational elements of mathematics [50]."

Here, enjoyment and appreciation is constituent to literacy. About half a page further down reads quite the opposite: "Mathematics related attitudes and emotions such as self-confidence, curiosity, feelings of interest and relevance, [...], are not components of the definition of mathematical literacy but nevertheless are important contributors to it." In the following, we aim to segregate rational abilities and emotional states in the definition.

Definition 1.2.4. Practical Mathematical Literacy is the ability to apply mathematics in daily life. Identifying a problem as mathematical, choosing an approach to solve it and knowing how to interpret the outcome are competencies that make up practical mathematical literacy.

Here, the linkage to formal education is strongest, and non-formal educational offerings cannot substitute years of training in school. Schools do their best to equip students with the necessary skills they will need in everyday life. The debate on which specific mathematical competencies and skills make one literate and therefore must be part of the curriculum is subject to debates over education and didactics that cannot be repeated here. In Section 3.4 we list the learning standards defined by the German conference of Ministers of Education and Cultural Affairs that build the framework of the curricula in each of the federal states. Note that practical mathematical literacy is also referred to as *numeracy* in the literature [49].

Statistical / Risk Literacy The field of risk literacy has great relevance for the individual in a private and also in a societal sense. In the private sphere it includes questions like: 'Shall I take an umbrella, when the weather forecast predicts a 60% probability of rain? Is it worth it to take that medical examination, when its false positive rate is 3% and it might cause an unnecessary health-scare?' Gigerenzer et al. showed that patients are not sufficiently supported by physicians in understanding the statistics, since physicians often lack the necessary abilities themselves [58, 76].

Definition 1.2.5. Risk Literacy

Risk Literacy is the individual's ability to assess the decision-making options and possible consequences in uncertain situations and ability to decide for one's personal benefit [125].

Algorithmic Literacy Algorithms make automated decisions from a vast set of data. With the rise of machine learning the functioning of these algorithms become more and more opaque not only to the users but to the developers as well. Many decisions that have far reaching consequences for an individual's life like credit worthiness, or prediction of a perpetrator's likelihood for future offenses are increasingly made by computer programs. Studies have shown that the outcome of these decisions can be heavily biased. A qualitative study by the Pew Research Center interviewed experts in the field of AI on the possible consequences of the development [123]. Many experts' voices predict a future in which decisions and therefore the negotiation of truth is more and more made by computers and call on transparency on the side of the developers and algorithmic literacy on the side of the users. We would like to add the question of how the opacity of the programs relates to trust in mathematics – the science responsible for the development of the algorithms that might have increasing power over people's lifes.

Defining algorithmic literacy can be done by zooming in on the competencies that an individual needs in order to behave and decide in a way that allows for the realization of the individual's personal goals. Given that the parameters and operation of the algorithms are inherently opaque, this kind of definition seems almost a bit cynical. Therefore, it is necessary that regulations concerning the transparency and biases are applied.

Trust and Critical Mathematical Literacy

It was established above that trust has become an important quality when it comes to communicating science. In accordance with Bauer [17] we do support the view that a skeptical public is an asset for science. In a knowledge society, science can grow by having to prove its trustworthiness to a skeptical public.

How does trust relate to mathematics – the science of proofs and *absolute* truths? To our knowledge the general trust in mathematics in contrast to science in general hasn't been studied, yet. Nevertheless, the fact that the empirical sciences base their entire scientific rigor on statistical methods and thus, mathematics, makes – at least within the scientific realm – arguing for the trust in mathematics obsolete.

A different aspect of the relationship between trust and mathematics becomes apparent when it comes to applications, like cryptography and data security in the digital domain. It is an open question whether typing in one's data to enter an online banking account, could be interpreted as an affirmation of trust in the encryption mechanisms behind it.

The trustworthiness of mathematics is most crucial when it concerns areas of the field touching ethical questions. These can arise in fields like mathematical finance [156,158], biased algorithms, or possible harm caused by self-driving cars. How does trust relate to artificial intelligence? Can citizens put trust into decisions made by algorithms? If computers become able to produce deep-fakes, what is the effect on trust in media and maybe even our own human senses, when we cannot trust anymore "what we have seen with our own eyes"? How does this disruption relate to the science that is responsible for it? The emerging of the field of Explainable AI (XAI) bears witness that the community is concerned with issues of trust.

Because the individual is eventually overburdened with understanding the many walks of mathematics behind these issues, it is now necessary to strengthen critical media coverage and thus, establish mathematical science journalism to serve its watch dog duty.

1.2.2.1 Mathematical Citizen Science[13]

Citizen science is a tool to engage the public in the scientific process. The method adds a new dimension of research and possible results due to the utilization of collective intelligence. Many scientific areas have a long tradition of collaborating with citizen researchers and rely on their participation. In other disciplines, like mathematics, the process of realizing its possibilities and thus implementation has just begun.

A Definition of Citizen Science There are many examples for the variety and success of implementing citizen science. The project *Foldit* invites participants to solve graphic puzzles in order to find optimal protein folding [154]. Another example is the German *Mückenatlas* [52]. Citizens all over the country can catch mosquitos and send them to a lab in which they are categorized to create map of the biodiversity of mosquitos within Germany.

Citizen science is not a clearly defined term and different interpretations and definitions exist. This one is based on the *Green Paper of Citizen Science Strategy in Germany* [24]:

Definition 1.2.6. Citizen science is a scientific practice that satisfies the following characteristics:

(a) involvement of individuals in the scientific process that are not institutionally bound to this scientific area,

(b) involvement can range from singular participation to long term investment of the individual's leisure time,

[13] Note that parts of the following section have been published in [72].

(c) the individual delves deeply into a scientific topic in collaboration with other scientists and volunteers,

(d) an academical education of the citizen scientist is not mandatory (but of course is no obstacle, either),

(e) accurate scientific practice and transparency of methodology and data collection is required, as well as

(f) public and open discussion of the results.

Mathematical Projects that have Adopted Aspects of Citizen Science. People have sought to open mathematics and the mathematical process to the public, and as a result involved citizens in ways which come close to our above established definition of citizen science. In the following we present projects that have implemented aspects of citizen science in a mathematical context. Note that each of the projects lack a different characteristic to be fully described as a citizen science project.

Historic Mass Computation[14] Utilizing non-professionals in mathematics is not a novel idea. In 1938, the Work Projects Administration, an American New Deal agency, established a computing organization that employed 450 office workers to carry out mass scientific computations. These computations were done by human computers who had little education other than the rudiments of arithmetic. They worked in an assembly line arrangement and handed the worksheets from table to table. Even though during the 1930s more advanced methods for calculation, like using logarithmic tables and punch cards were already known, the *Math Tables* project was formed under the ideals of the New Deal that aimed to counteract the effects of the depression by improving the situation for the workers and bettering their education [62].

This interesting project, whose success is mostly due to the engagement of its two leaders, adopts nearly all aspects of citizen science, except for the first and maybe most crucial; the not academically trained workers who carried out the scientific computations were institutionally bound, since they were paid for their work and one of the main goals of the project was the reduction of unemployment. But what still makes this project worth mentioning in the context

[14]The inclusion of this project in this paper is a good example for the power of the before mentioned forum, since it was given as an example in a discussion about citizen science and mathematics [80].

of citizen science is the involvement of a large number of non-academically trained workers. The project leaders had to find a computation method derived from only simple arithmetic, because the workers were not trained in mathematics or calculation. Over time these methods became more and more sophisticated and complex calculations could be tackled [62].

Distributed Computing As computers nowadays can do most of the computations for mathematicians and other scientists, collective human computer offices like in the *Math Tables* project are no longer necessary. However, there are still some calculations that exceed even the capabilities of the computers mathematicians usually have on hand. The runtime of certain problems is long and thus expensive. Distributed computing is a way to solve problems that share two characteristics: their runtime is long and they do not possess an ultimate goal.

The *GIMPS project (Great Internet Mersenne Prime Search)* was started in 1996 and aims to find Mersenne prime numbers. These are prime numbers of the form $p = 2^n - 1$. The Lucas-Lehmer algorithm can efficiently decide whether such a number is prime or not. George Woltman implemented the algorithm in an assembly language program that users can install on their computers. The program steadily works in the background and slowly tests a certain number on its primality. The project can be considered wildly successful. So far, more than 200,000 users have signed up and 17 Mersenne prime numbers have been found [106].

GIMPS is not the only distributed computing project. Wikipedia lists 62 currently running projects, of which 14 fall into the mathematics category [157]. These projects generally fulfill most of the requirements of citizen science. The only obstacle is their lack of involvement of the people and not just their computers. Each project is executed digitally through the internet and the users are not personally included in this exchange. All they need to do is install the program and donate their CPU time.

Collective Problem Solving The *Polymath Project* that was founded by the well-known mathematician and Fields medalist Timothy Gowers, is one of the most famous and successful projects that uses collective intelligence by removing logistical and institutionalized obstacles of participation. In 2009, Gowers asked in his blogpost: "Is massively collaborative mathematics possible?" [60]. He posted the first problem *Polymath1* which was the exploration of a particular combinatorial approach to the density Hales-Jewett theorem for $k = 3$ [8]. About a year after the proposal of the problem, a paper with the title *Density Hales-Jewett and Moser numbers* was published and the authors were referenced as: *D.H.J. Polymath* [15] [121]. In total 23 collaborators participated in the project and a "handful" contributed to the solution [112].

The *Polymath Project* clearly is a very successful example of how digital interconnection can help to collectively solve involved mathematical problems. It is still ongoing and so far project number 16 is being tackled. Despite there being no institutional thresholds in participating in the *Polymath Project*, the very nature of the problems excludes everyone who is unable to read the involved notation or does not know the underlying concepts and mathematical definitions.

Collectiveness in Mathematics In mathematics we can observe a strong movement towards a collectiveness of the scientific practice. The forum *mathoverflow* [6] is an example of how mathematics profits from an open forum and discussion in which everybody who is able to understand and apply the code of conduct can participate. As this code of conduct has a very high standard, the forum is not designed to be inclusive. In the *Polymath* Project, a high threshold of a very involved notation and prior expert knowledge makes participation exclusive.

The superpower of mathematics is its strong scientific practice, which Nielsen [113] identifies as an indicator and condition for the collectiveness and interconnectedness of scientific processes. Thus, the very characteristic that opens mathematics within its own community makes it harder for the public to participate.

[15] D.H.J. is an acronym for Density Hales-Jewett.

1.3 What's Next for Mathematical Science Communication?

In this chapter we reproduced the most important developmental steps of science communication and classified the corresponding terms. We then conducted a survey of the existing literature about mathematical science communication and identified a prevalence of the deficit model that was characterized by a presumed lack of appreciation and knowledge of math in the public. In the next step, we introduced definitions to describe the objectives of mathematical science communication. The goal was not to give an exhaustive list, but to invigorate the discourse. In these three steps, we laid the foundation to set up a four point agenda for the development of mathematical science communication.

- **Dialogue:** In order to overcoming the deficit model once and for all we propose a two step approach. The first step is to develop and implement formats that are based on a sincere eye-level dialogue. Goal of the bidirectional communication is to enhance the relationship and thus trust between science and the public. A good example for this eye-level dialogue is citizen science. Citizens are actively involved in the research process and in many cases their contribution is crucial for a research project because it produces fine-meshed data points or tackles big loads of work. Successfully marrying the method of citizen science with mathematics would create a room for the co-creation of mathematical knowledge.

- **Service:** The second step in overcoming the deficit model is the creation of opportunities that allow for transfer of knowledge without power issues. Obviously, dialogical formats may transfer knowledge as well. However, unidirectional transfer of knowledge from science to society is still one of the top priorities of science communication. A shift in the mindset to understand (mathematical) science communication as a service to the individual can make a difference in the power dynamic of the communication situation[16].

- **Mathematical Science Journalism:**

  Given the progress on digitalization and the algorithmic shift in the generation of knowledge, mathematical literacy is a crucial skill to make sense of the world and make informed

[16] We will see in chapter 3 that a similar shift has happened in education. In many modern didactical formats it is the teacher's role to assist the student to find their own path through subject matter.

decisions. Since the individual is often overburdened by keeping up with the development, we can view it as a need that has to be met on societal level. Thus, outsourcing it to the the forth estate of a democratic society, gives the individuals the possibility to keep up with the latest development without personally going into too much detail. The third point on the agenda is the strengthening of mathematical science journalism. First, this means allocating funds to secure professional journalists positions and second a high quality exchange between mathematicians and journalists.

- **The Science of Mathematical Science Communication**

  Giving mathematical science communication a seat at the table in the field of mathematics would lead to a professionalization of both, the discourse and the practice, as well as their mutual stimulation. A scholarly exchange about the motivations, goals and objectives is overdue.

  Professional evaluation of the many formats and projects of mathematical science communication is as mandatory today as it was in 1996, when it was already requested by Ernest [43]. Developing appropriate tools and methods is in its infancy.

  As we have seen above, and will expand further upon in Chapter 3, the development of science communication is following education in many ways. The didactics of mathematics, which form a separate field in the Department of Mathematics, could serve as a model for the establishment of the field of mathematical science communication as a scientific subject.

  The request for *sincere communication* that was formulated as a guiding principle for good science public relations would be met by imposing scientific principles on the communication as well.

  The science of communication belongs to the humanities, and so does the science of communicating mathematics. Analogous to didactics, science communication builds a bridge to a place outside of the scientific realm. For its credibility, it needs a strong foundation in mathematics as a science.

Chapter 2

POLYTOPIA – Adopt a Polyhedron

2.1 Motivation, Objectives and Methods

POLYTOPIA – ADOPT A POLYHEDRON is an experimental project to explore the connection between mathematics, mathematical science communication, and mathematics education. The bridges built in the project are not just between these fields, but also connect formal and informal learning of mathematics, combine elements of citizen science with art and show the process, methodology, and content of scientific mathematical research.

2.1.1 Motivation

The project is associated with the subproject *Communication and Presentation* in the Collaborative Research Centre *SFB 109 Transregio, Discretization in Geometry and Dynamics*. Thus, the motivation behind the project is to enhance the visibility of this collaborative research center in the media and to the public. Besides this motivation, which stems from the funding of the project, its goals are to explore the possibilities of more dialogical forms of mathematical science communication. Seeing a large number of non-experts actively engage in their scientific field, shows the public support of their research and thus is a motivation for the mathematical community, too.

2.1.2 Objectives

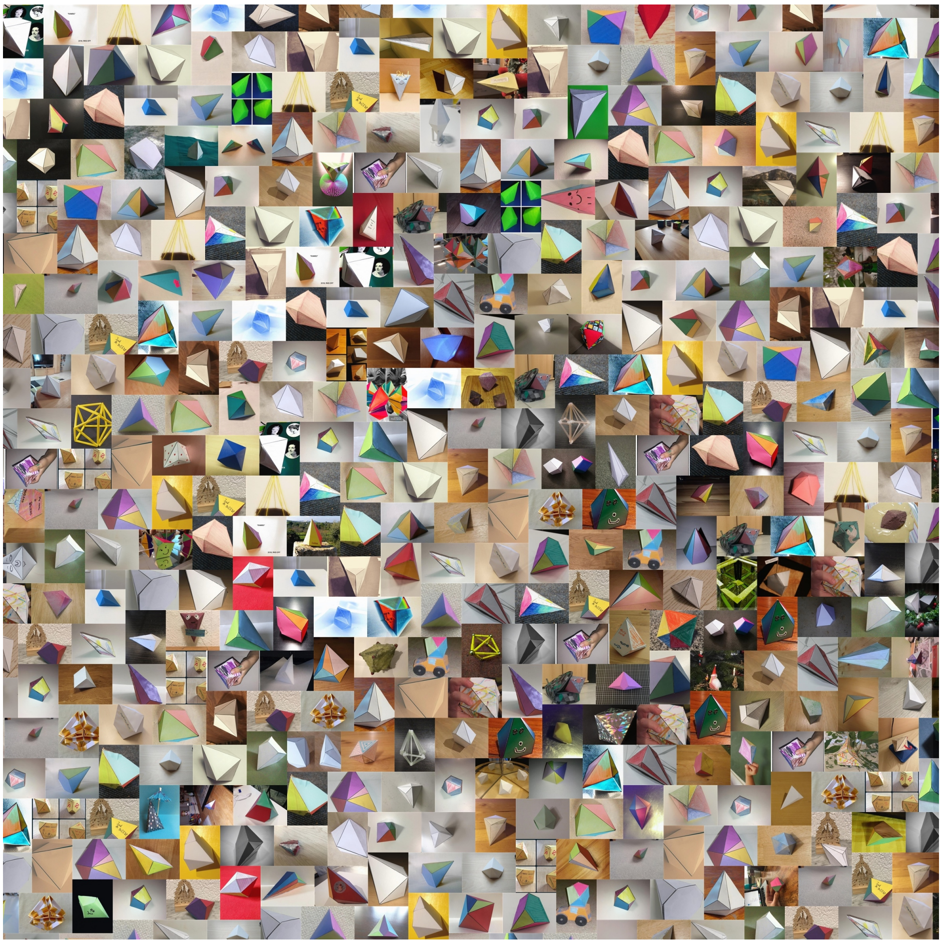


Figure 2.1: Mosaic of polyhedra models.

Dialogical and Participatory Elements The overall narrative of the project is to ask the public for help in naming the population of POLYTOPIA, the polyhedra. This approach sets the general tone of the project. A request is usually more successful if it is expressed in a friendly and humble manner. A dialogical format of science communication requires a two-way street of communication in which the participants can express themselves. This is accomplished by naming the polyhedra, building the models, and artistically crafting an *individualized* version of a polyhedron. Reporting back the *research questions* they found, see Section 3.3.2, adds an additional layer of feedback for the students. With uploading photographs of their models, the participants add to a decentralized art project. A first result was exhibited at the art show of the Bridges Conference in Linz in 2019 [31], see Figure 2.1 for a printed version of the picture that was on display.

Bridging Formal and Informal Learning The main target group of the project is pupils in the age-group 7 to 13. Besides media outlets, we address them through their teachers by providing school materials which are ready to be used in the math class. The main subject of the project – polyhedra – is not part of the official curriculum. Hence, we linked this topic closely to content that is included in the curriculum. For more on these linkage points see Section 3.4. In our general approach of the science communication project, we borrowed from the didactical principle of *inquiry based learning* and *dialogical learning*, see Sections 3.2.2 and 3.2.3, respectively. The *core idea* to adopt a geometrical object is presented to the participants as an invitation. No procedural or factual knowledge of mathematics is needed. The process of adopting and thus approaching mathematics on a relational level may lead to curiosity about the object which can be satisfied with the information provided on the website in the *glossary*, see Appendix.

2.1.3 Methods

The Narrative of the Project

All polyhedra were living happily in the realm of abstraction, when one day they notice that they are not all the same. Some are different. Very few of them possess a name and have been built as models in the *real world*. For example the *Cube*, the *Pyramid* or the *Associahedron*. The other polyhedra neither have a name, nor have they ever been realized in a physical model. But since there is an overwhelming amount of polyhedra – in fact infinitely many – the mathematical community needs the help of the public. Everyone is invited to adopt a polyhedron, give it a name, build a model, and thus help it come alive.

Adoption The focal idea of the project is the adoption of a polyhedron. One can adopt a star [3], a high or low pressure system [82], or even a revolution [2]. The identification of the individual participant with an individual object in a one-to-one setting makes the singular impact graspable and significant.

We applyed this concept as a vehicle of science communication for our project and thus, put up polyhedra for adoption. The invitation to actively contribute to science in a creative way by naming polyhedra and building models is offering an affectional and relational access to mathematics with a very low threshold for participation.

Citizen Science Citizen science is a valid scientific method that also has great potential to involve the public and raise public awareness, understanding, and literacy of science. It is a modern vehicle for science communication because it focuses on participation as a central element. Citizens, and this is everybody who is not institutionally bound to the endeavor at hand, are actively involved in the research process. As we have seen in Section 1.2.2.1, it is not easy to combine citizen science with mathematics. Hence, an artistic activity – building the models – has been added. We coined the term *citizen art* for this combination of citizen science and art [71, 73].

2.1.4 Classification

In accordance with the claims to transparency of science public relations made in Section 1.1.3.8, we denote that the project classifies as public relations for the Collaborative Research Center *SFB 109 Transregio Discretization in Geometry and Dynamics.* This is where the funding stems from and thus, its goal is to enhance the visibility of the research, researchers, content and organization.

Public Awareness of Mathematics The primary target group of the project is pupils. Obviously, it is not a direct goal to elicit any response towards the collaborative research center in the students. The publicity effect is more of a secondary nature: the project demonstrates that the obligation to provide information about the research content that is accessible has been fulfilled.

The students are invited to get a glimpse into mathematical research that is far beyond the scope of the school book, and yet is approachable. Polyhedra as objects are good candidates for such an undertaking because they are relatively familiar, intuitively graspable and yet of interest to the professional mathematician. A goal is to raise awareness in the students that there is such a thing as contemporary mathematical research and that the people behind it are making an effort to be approachable.

Mathematical Literacy The glossary of the website offers factual knowledge about polyhedra, their history, and research. This knowledge is not forced upon the user, but linked behind highlighted words. By clicking on a word, the user finds short explanatory articles. We choose this design to refrain from imposing a paternalistic style of communicating the scientific facts, and instead let the user decide how much information they want to seek.

In the process of building the model, learning something about that polyhedron and polyhedra in general is inevitable. The two forms of representation, a two-dimensional net and the emerging body in space are connected by the process of building the model. Additionally, when the participant decides to make an individualized polyhedron, they need to examine it thoroughly and analyze its structure.

Besides this factual knowledge, methodological knowledge is woven into the design. The users are invited to adopt a polyhedron and name it. By doing that, they are adding something, which is the crucial point. This possibility represents the openness and incompletion of the science of mathematics. Since mathematics is a very involved science, it is a long way to get to the boundary of existing knowledge in mathematics. The combination of mathematics and art creates a shortcut to this boundary. The representation of the polyhedra in a matrix and their order by number of vertices hints at classification, which is an important category in mathematical research.

Practical Mathematical Literacy The content of the school material and its links to the school curriculum is discussed in more depth in Chapter 3. In terms of practical mathematical literacy, one might ask whether active knowledge about polyhedra is of any value and applicable to everyday life. Thus, we would like to argue that learning to build precise models from paper might not be an activity considered very mathematical, but is a skill that is very handy in many life situations. The relationship of the different forms of representation of a polyhedron, the two-dimensional net that can be folded into the three-dimensional model, can be experienced first hand. The digital representation of the polyhedron in the viewer is on a flat screen and only through movement one gets the impression of an object in space. These different states of representations are described by the EIS model, see Section 3.3.

Algorithmic Literacy Even though the algorithms for the generation of the polyhedra are not explicit, the description of that process hints at the importance of computers in contemporary mathematical research. Thus, an awareness of the existence and importance of algorithms can be gained.

Public Appreciation of Mathematics As POLYTOPIA classifies as a public relations project, raising public appreciation has to be mentioned, since the affiliated CRC is prominently presented. The relational approach to mathematics by adopting and thus relating to a mathematical object is aiming for a positive emotional response which is emphasized by the choice of the mascots. *Ecki* and *Polly* are polyhedra stylized as stick figures with faces reminiscent of emojis.

The combination of mathematics with creativity generates a new access to the subject. It aims to also attract participants that are more drawn towards the art making process.

Trust As we have discusses in Section 1.1.3.5, forming trust is (at least) a two layered process. It involves warmth and approachability as well as competence and trustworthiness. We claim that mathematics does not need to prove its competence in any way. Its rigor is firmly anchored in its reputation, though it could use a little warmth and approachability. Both are displayed in the project. The design of the website, with its bright colors and friendly mascots, aims to create an inviting online environment. The invitation to bring a personal element, both in the adoption (naming) of a polyhedron and then building a model and uploading a picture, gives the participants the possibility to be an active part of the project. There is no requirement to "learn something". The activities are completely detached from the informative part, which is accessible in the glossary. The highlighted words in the articles hint at the existence of further information. The glossary explains the concepts and mathematical background in detail and in understandable language. As the user can freely choose to access this information, we hope to eliminate a paternalistic tones while providing expertise.

The link to the collaborative research centre and the project's nature of public relations is made explicit in the *Behind the Project* section which can be accessed through the footer of the homepage, or by clicking on the highlighted word *mathematicians* on the landing page.

2.2 Execution of the Project

In this section, we want to describe the practical issues of the execution of the project Polytopia – *Adopt a Polyhedron* with the website `www.polytopia.eu` as its mainstay. Polyhedra are gathered, presented, and the adoption process occurs. The preconsiderations for the web design, which was transcribed by a professional web designer according to our specifications, are linked with the final results. We will briefly sum up the implementation of the website and its user management system, which was required for the adoption process. The structure is represented in a sitemap. In the following, we describe the content of the website and then close the section by presenting the timeline of the implementation process.

Webdesign	light and clear multitude of polyhedra aims at kids invites adults as well interactive
Implementation	stable longevity not much administration data safety
Content	tells a story no explanations forced upon the user glossary can easily be found crafting sheets elements of gamification

Table 2.1: Preliminary considerations for the website.

2.2.1 Website

The website had to fulfill many requirements concerning the web design, implementation, and content. The design should align with our general approach of a friendly request for help. Therefore, the adjectives *light* and *clear* guided the development of the wireframes. The guiding principle for the implementation was its longevity and being low maintenance. Content wise we focused on the presentation of the narrative. These and other requirements are shown in Table 2.1. In the following we discuss and reflect the specific design decisions that were made in order to fulfill these goals.

2.2.1.1 Webdesign

Light and Clear The overall guideline for the web design was *light and clear*. Thus, we chose white as the background color, light and dark grey for the font, and blue for highlighting the links. There are no boxes or bars to bound the elements of the page except for the interactive visualization of the individual polyhedra. All the design elements are freely floating on the white background. The mascots add a little pop of color that can be found throughout all the elements of the project, such as in the header of the website, see Figure 2.2.

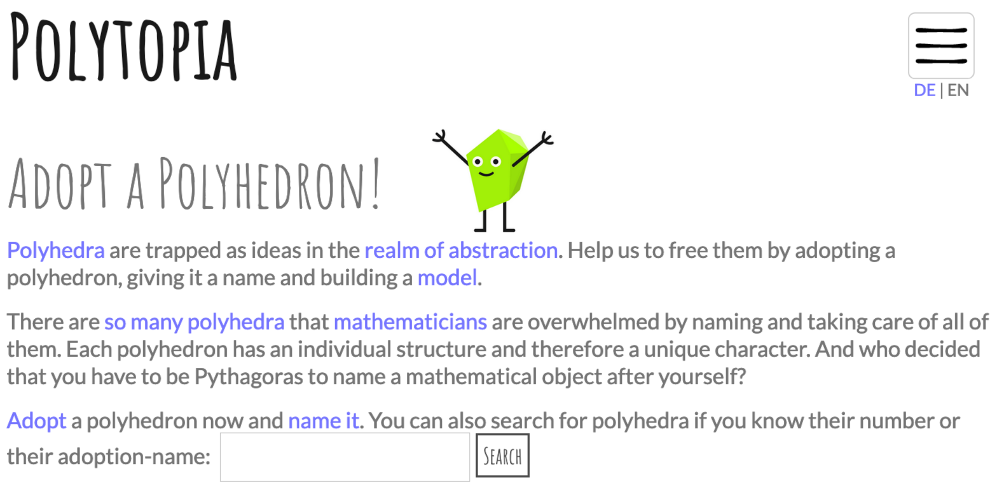


Figure 2.2: Screenshot of header of the website polytopia.eu/en.

Multitude of Polyhedra To present the multitude and variety of the many polyhedra in a non overwhelming way, we chose a representation of a pixelated matrix. The *pixels* are little squares where each one stands for a polyhedron. By clicking on the respective pixel, the user gets to the *detail view*. The color of the pixel indicates the status of the polyhedron, i.e. whether it is free (grey), adopted (green), realized (blue), or individualized (yellow).

Aims at Kids and Invites Adults The aim of the design is to be friendly and approachable by both kids and adults. Clarity is a guiding principle. To avoid a sterile impression, colorful accents and the mascots *Polly* and *Ecki* are added. The mascots are stylized polyhedra with smiling faces and stick limbs. *Polly* is green and raises their arms for a friendly hug, while *Ecki*

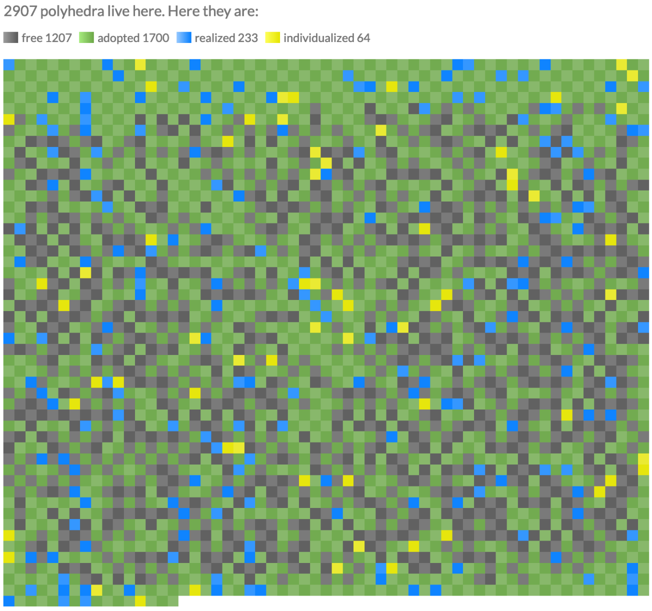


Figure 2.3: Screenshot of matrix of the polyhedra, taken on the February 10th 2020.

is pink, wears nerdy glasses and has accentuated front teeth. The mascots are placed on each page and are a splash of color in the mostly white and reduced design.

We chose the font *Amatic SC* for headlines and accentuation, see Figure 2.4. It is a handwritten font without embellishment or serifs that is a good fit to the clear and friendly design. The continuous text is written in *Lato Regular*, a simple, straight and sans serif font, see Figure 2.5.

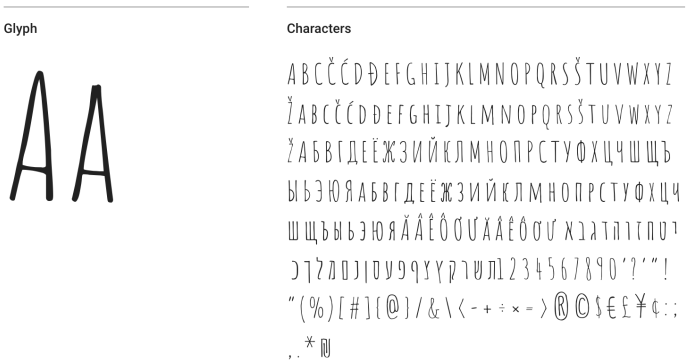


Figure 2.4: Screenshot of the font style example Amatic SC [4].

Figure 2.5: Screenshot of the font style example Lato Regular [5].

Interactive By clicking on the pixels in the polyhedra matrix, the detailed view for each polyhedron is opened. Figure 2.6 shows polyhedron *Flensi* as an example. The display of the name and a blue content looking mascot imply that this polyhedron has already been adopted.

Users can explore the polyhedron in the interactive visualization. They can turn the polyhedron around, vary its size, and click to show or hide the vertices, edges and faces. The faces can be colored by choosing from a predefined color set. The two arrows at the top of the frame allow the user to skip through various polyhedra without having to return to the matrix.

2.2.1.2 Implementation

The main objectives for the implementation of the website were its stableness, longevity and low-maintenance. Thus, we chose a static implementation, where every feature was coded from scratch in boilerplate `html` code. Using a content management system would have saved time in the implementation process, but would have required regular updates of the system and potential adaptions to new versions of the system. More than a year after completion of the website we can conclude that going for the static implementation was the right step, since it requires hardly any tending to at all.[1]

Some Technical Remarks Content related variables, like the number of adopted polyhedra, data base applications, and the registration emails are not seen by the users and run on the server using `PHP`. Visual elements like turning the polyhedron in the viewer or the gallery use `Java Script`, which is executed on the user's own device.

[1] Besides answering the occasional user who does not find the registration email in their spam folder.

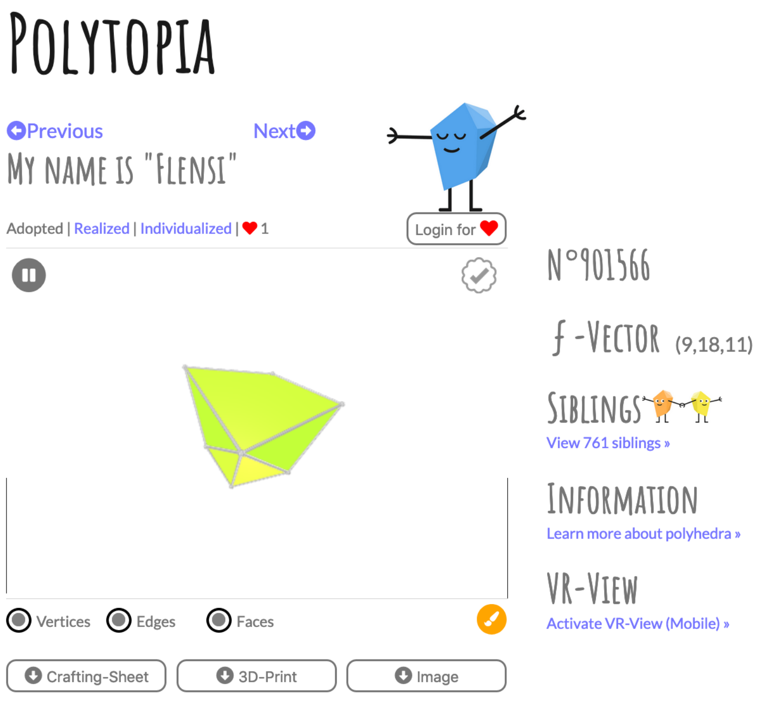


Figure 2.6: Screetshot of the detailed view of polyhedron *Flensi* [7]

The registration process is implemented in a state of the art double opt-in standard procedure. The user management system is coded from scratch in boilerplate `html`.[2] The user data is stored in the database and secured due to EU General Data Protection Regulation. See the Appendix for a detailed description of the privacy policy.

The viewer for polyhedra visualization is coded in `Javascript` using the `ThreeJS` environment. It bases the visualization on `.json` files of the polyhedra in which the coordinates of the vertices and their combinatorial structure is inscribed. `Javascript` and `ThreeJS` also build the framework for the Hamilton game.[3]

In combination with special cardboard glasses, a smartphone can be transformed into a virtual reality viewer for the polyhedra. The viewer's motions are mapped on the polyhedron and it turns accordingly. Its implementation is based on `AFrame` [1], which is a web framework for building virtual reality environments that bases on `ThreeJS`.

[2]Stefan Auerbach and Martin Skrodzki implemented the website and user management system.
[3]The viewer, the virtual reality viewer, and the Hamilton game were implemented by Marie-Charlotte Brandenburg.

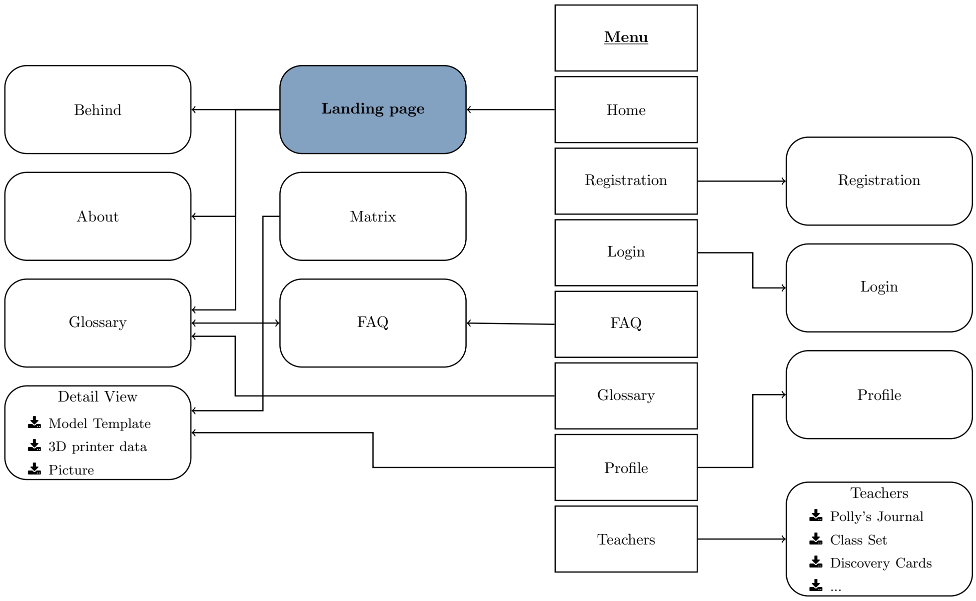


Figure 2.7: Sitemap of the website Polytopia

Sitemap as Directed Graph Figure 2.7 shows the structure of the website and the connections between the pages. The menu items are represented by rectangular boxes, while the pages have rounded corners. A link is symbolized by an arrow going from one page to another. Downloadable material is preceded with a small downwardly pointing arrow.

2.2.1.3 Content

The adoption process and visual presentation of the multitude of polyhedra are in the foreground of the website and presented on the homepage. No explanations about polyhedra, geometry and mathematics are on this site. If the users might have a need for explanation, highlighted words in the text link to the glossary and FAQ, which explain the underlaying mathematical terms in everyday language.

School material The primary target group of pupils between the ages of 7 to 13 are either addressed directly or via the math class. Therefore teachers find school material that is provided on the website in the category *teachers*. Chapter 3 is dedicated to giving detailed information about the didactical principles and the materials.

Hamilton Game The Hamilton game can be played on the graph of each polyhedron. In the upper right corner of the detail viewer is a small tick icon. In an extra frame, the skeleton of the polyhedron appears. The goal of the game is to find either a Hamiltonian path or cycle on the skeleton.

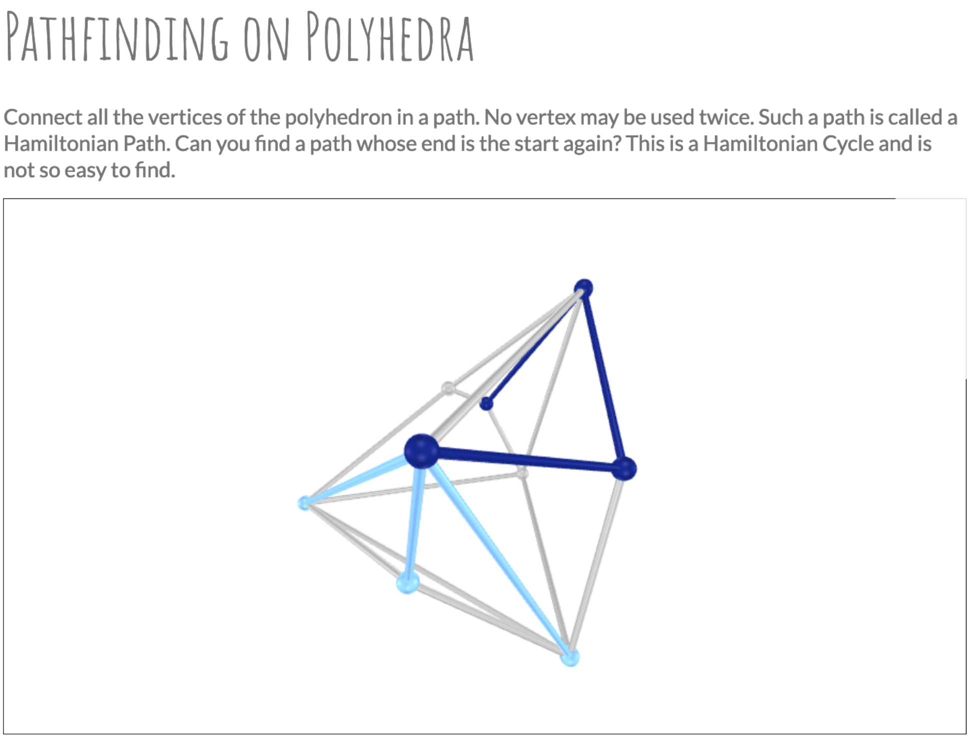
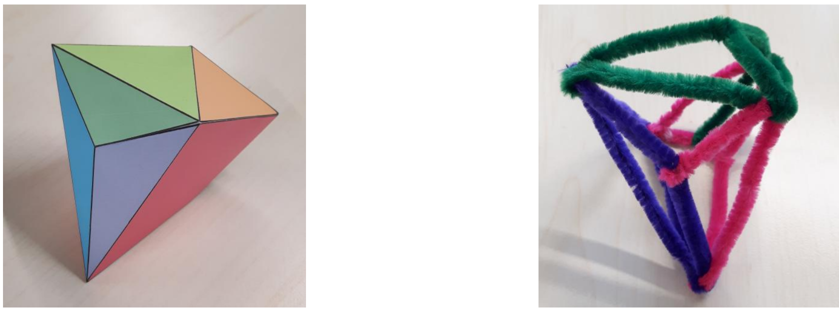


Figure 2.8: A screenshot of the Hamilton game

Pictures of the Polyhedra Models The users can upload one picture each for their *realized* model, i.e. a picture of the assembled paper model, and *individualized* model, i.e. a model built from a material of their choice. These pictures can either be found in the detail view when clicking on the respective tabs above the viewer or in the gallery where all polyhedra pictures are gathered.



(a) (b)

Figure 2.9: Realized and individualized models of the polyhedron *Flensi*.

Like Button Every successful homepage in today's internet needs a function to express adoration. Thus, we implemented a heart-shaped "like"–button which counts the number of times it has been clicked.

2.2.2 Timeline

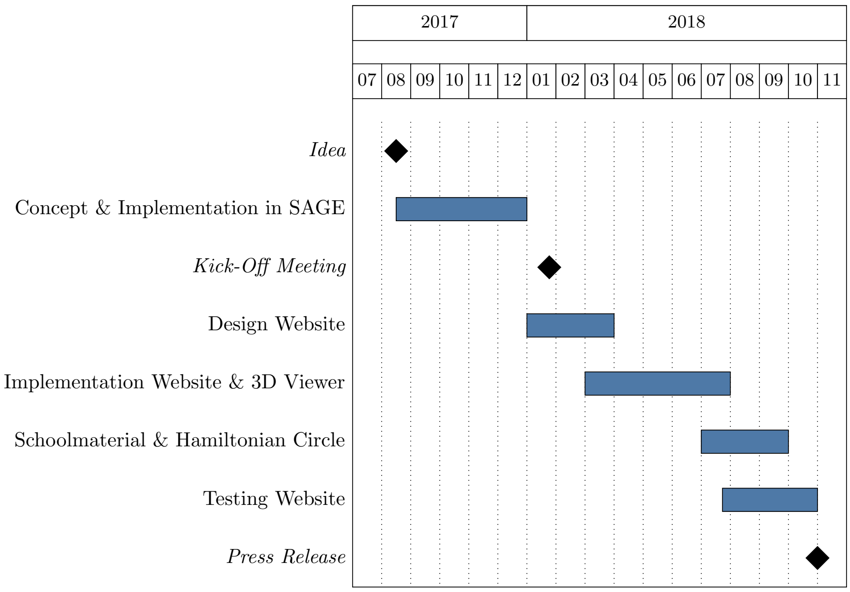


Figure 2.10: Timeline of the project's execution.

The project took about a year between being developed from idea to finishing the website and another three month for testing the website and launching it. The timeline, see Figure 2.10, displays the developmental phases and milestones of the execution process.

The Team

Realizing the many steps of implementing the project into its final form was a team effort. In the following, the members of the team and their main contributions are listed. Moritz Firsching provided the code for the Koebe-Andreev-Thurston realization of the polyhedra. His implementation follows Ziegler's exposition [170], which is based on the work of Bobenko and Springborn [23]. Max Pohlenz, a professional web designer is responsible for the appearance of the website. His brother Bendix created the mascots. Martin Skrodzki and Stefan Auerbach programmed the website in html boilerplates. Marie-Charlotte Brandenburg implemented the viewer, the Hamilton-game and the virtual-reality environment. Johanna Steinmeyer typeset the discovery cards in LaTeX and set many pictures of polyhedra in TikZ. Erin Henning translated the website and the school material from German to English. She also was in charge of social media and supported the project in workshops and events. Mara Kortenkamp was the latest addition to the team and typeset a presentation poster. She is currently working on an update of the code for the unfolding of the polyhedra. Pauline Linke helped conceptualizing *Polly's Journal* and took care of the typesetting. Brigitte Lutz-Westphal pointed out the importance of linking the school material to the curriculum and Tine Gärtner introduced me to the EIS-principle. The mathematics teachers Stefan Korntreff, Gudrun Tisch and Anikó Ramshorn-Bircsak and their students tested the school material.

2.3 Results of the Project

2.3.1 Defining Success

We defined the project to be successful when more that a thousand polyhedra were adopted in the first year after the launch of the website. Since the project was developed to run for a long time with little to no maintenance, neither technical nor in terms of actively drawing attention, a second goal was that the adoption process would go on steadily without advertising it in any form. Thus, if between 20 and 30 polyhedra would be adopted each month on average, we would consider the project to be effective.

2.3.2 User Numbers

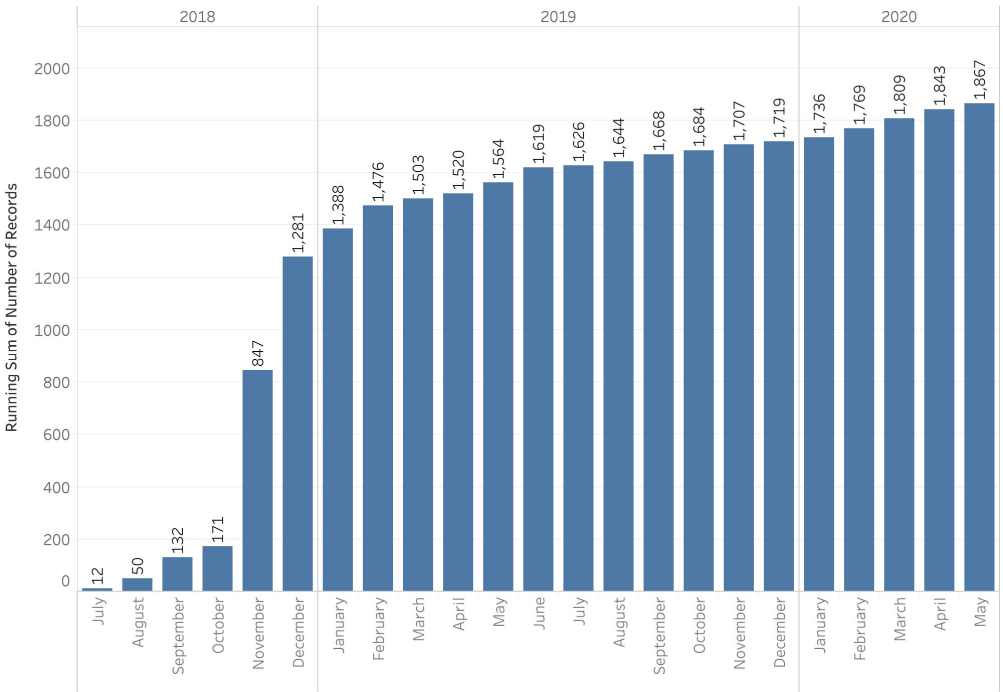
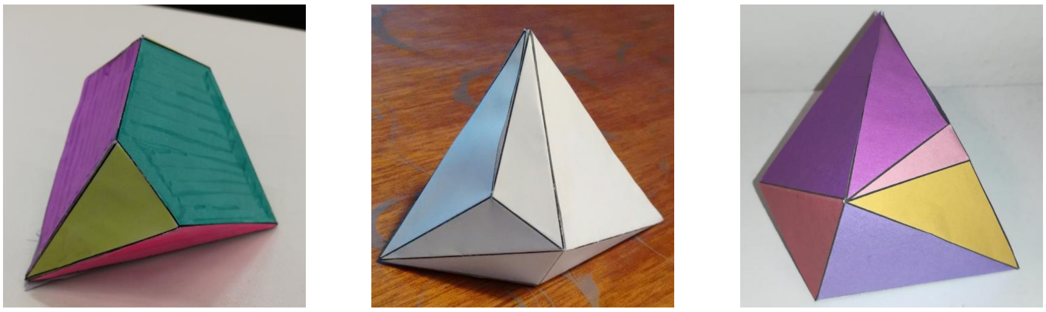


Figure 2.11: The total number of polyhedra that were adopted per month.

As we can see in Figure 2.11, both goals were reached. In the first year after finishing the website, 1626 requests to adopt a polyhedron were made. The jump that occurred in November and December 2018 can be traced back to the article that appeared at *Zeit Online* on the 9th of November 2018 and was featured on the landing page of the online newspaper [95]. Neglecting the immediate effect of this news report, we note in each month between March 2019 and May 2020, an average of 26 attempts (median 23.5 attempts) to adopt a polyhedron were made.



(a) (b) (c)

Figure 2.12: Examples of realized polyhedra: (a) "Guggeli Knubbeleder", (b) "Anne Elisabet", and (c) "Juweloeder".

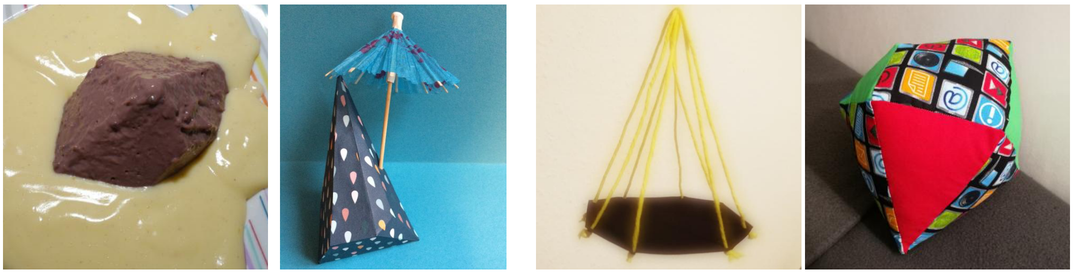
The very attentive reader might notice that the numbers in the graph and the total number of adopted polyhedra on the website are not identical. That is because the graph bases on the number of requests to adopt a polyhedron. For prevention measures, each to-be-given name has to be checked by the team and, unfortunately, some had to be declined.

The most impressive, but also rather meaningless number[4], is the total amount of clicks on the website, which is $2,315,318$. The total amount of adopted polyhedra is 1,812 of which 258 are realized, i.e. a photo of their paper model has been uploaded, see Figure 2.12 for examples. Figure 2.13 shows four of the 73 individualized polyhedra models. The counter on the website reports 2961 attempts for registration, of which 2097 (71 %) successfully finished the two step opt-in process and became active users. About 86 % of the active users adopted a polyhedron.

The heart shaped like-buttons were clicked 2469 times by 959 different users. The most liked polyhedron is the tetrahedron with 28 likes. This may be due to its prominent location as the first pixel in the upper left of the matrix. The Hamilton-game was played a total of 681 times by 283 different users, where three users had quite a pastime and played the game over 50 times each.

We implemented a counter for the German school materials in the database. Table 2.2 shows the number of downloads of the particular materials. Unfortunately, no counters for the English materials were set. Polly's Journal was the most popular download, followed by the worksheets

[4]This and all following numbers and statistics are constituted on May 31st 2020.



(a) (b) (c) (d)

Figure 2.13: Examples of individualized polyhedra and their materials (a) "Puhlyeder" (chocolate ice cream) (b) "Seems to be Piez" (cardboard, holding a paper umbrella), (c) "Octavius" (cardboard and string) and (d) "Flori" (soft toy, fabric and filling).

for the class set and the research questions. A total number of 343 downloads of the school materials can be noticed.

Type	number
Polly's Journal	93
Worksheet Class Set	70
Worksheet Research Questions	65
Discovery Cards	46
Teacher's manual Polly's Journal	44
Materials and Ideas	25
Total	343

Table 2.2: Number of downloads of the school materials.

2.3.3 User Survey

A quantative survey was implemented on the website and presented to the users right after they have completed the adoption process. If they decide to fill it out later it can be found in the menu on the right hand side. The complete survey[5] can be found in the Appendix. A pretest was run on the first 30 respondents. The data sets of the pretest are not part of the following evaluation.

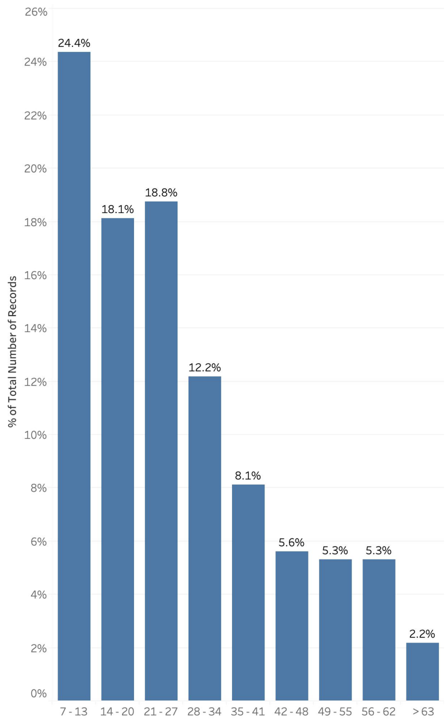


Figure 2.14: Bar diagram of the age groups

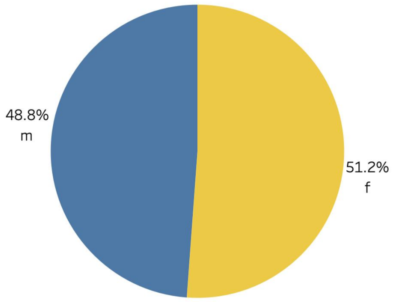
2.3.3.1 Demographical Data

In the time frame between October 15th 2018 and May 31st 2020, a total of 355 respondents took the time to answer the questions. That is 16.9% of the active users.

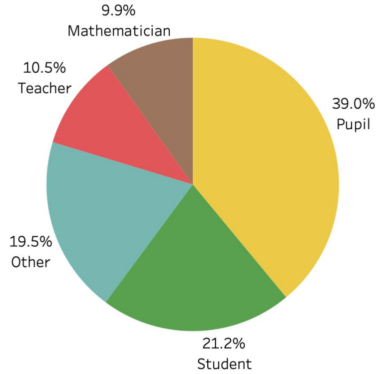
The primary target group of the project is pupils between 7 and 13 years old. 24.4 % of the users[6] fit into this group. A total of 61.3% of the users are younger that 28 years old. We note that the numbers of participants decline with higher age. This can be seen as evidence that the project is well suited for its younger target group. The project's online character is another explanation.

Looking at the occupations of the participants, we note that 39.0% of the users are pupils, 10.5 % teachers, and 21.2 % are university students. Professional mathematicians are quite highly represented with 9.9 %. Looking at the gender of the participants, we note that the number of females are a little higher (51.2 %) than male. In the design of the survey, we left out a third gender option, but did not make the answer obligatory.

[5] For the design of the survey we consulted the statistical consulting FU:STAT at Freie Univerität Berlin.
[6] In the following we will refer to users who filled out the survey, simply as users.



(a) Stated gender of the users in %



(b) Occupation of the users in %.

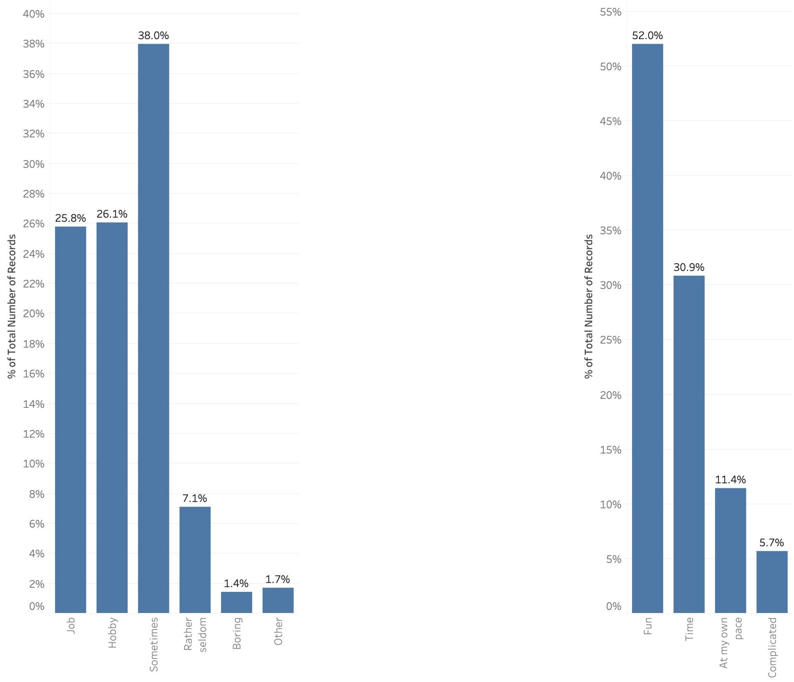
Figure 2.15

2.3.3.2 Self-Claimed Interest and Skill

In the survey we asked the respondents about their relationship towards mathematics. One question measures the self-claimed interest in mathematics and self-claimed practical skills in mathematics. The first question directly asks if the user is interested in mathematics. The first two items claim that the respondent is so interested in mathematics that they made it into a profession or a hobby. Here is the list of all possible items:

- Yes, very, it is actually part of my job.
- Yes, very, it is a hobby.
- Sometimes Mathematics is interesting.
- Rather seldom.
- No, I think Mathematics is boring.
- Other

We find that 25.8% are doing mathematics as a part of their job, and 26.1% enjoy it as a hobby, see Figure 2.16a. 38% find it sometimes interesting. Less than a total of 10 % find mathematics seldom interesting or choose the option “boring”. We do not have any comparative value in the general public, but can say that most of the users demonstrate a portion of interest in mathematics.



(a) Interest in mathematics.

(b) Self-claimed skill in mathematics.

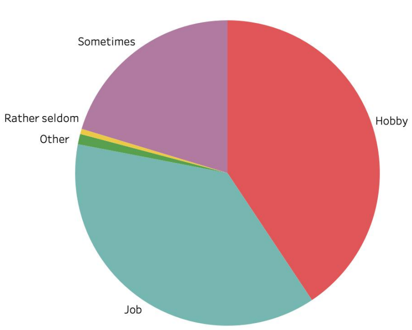
Figure 2.16

To measure self-claimed mathematical skills, we asked, which of the following statements they would agree with the most.

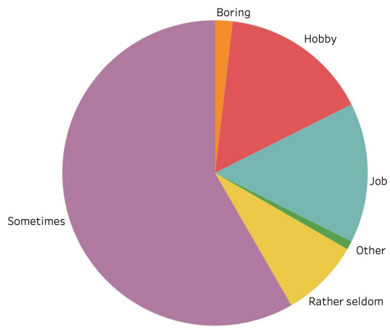
- I understand Mathematics fast and it is fun for me to discover connections myself.
- When I have some time to do Mathematics, I manage it quite well.
- As soon as someone explains things to me at my pace, I understand most of it.
- More often than not I find Mathematics to be rather complicated.

The results are depicted in Figure 2.16b. We find that more than half of the participants have confidence in their mathematical skills and decided for the first item. 30.9 % just need a little time to do well and 11.4 % might need some external help. About 5.7 % of the respondents find mathematics rather complicated.

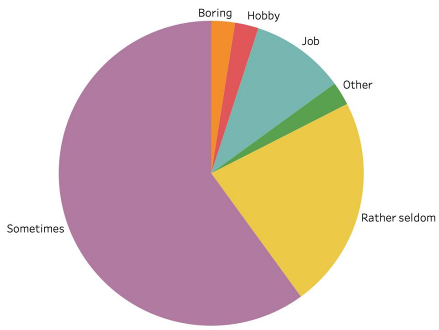
Contingency of Interest and Skill It may be reasonable to assume that self-claimed skill in mathematics correlates positively with self-claimed interest. Figure 2.17 depicts pie diagrams that show the distribution of self-claimed interest for each of the items of the question about skill. In Figure 2.17a we see that more than three quarters of those that chose the option "I understand Mathematics fast and its fun for me to discover connections myself." have also opted doing mathematics as a hobby or as a part of their job. Moving on to the second item, see Figure 2.17b, those who need a little time to understand mathematics have less often chosen mathematics as a hobby or part of their job, but chose "Sometimes Mathematics is interesting" by majority. Looking at the distributions of the options "Rather seldom" and "Mathematics is boring" we can find that the corresponding yellow and orange pieces of the pie are increasing with a decrease of self-claimed skill. Note that also in Figure 2.17d there is a proportion of people who do mathematics as a part of their job.



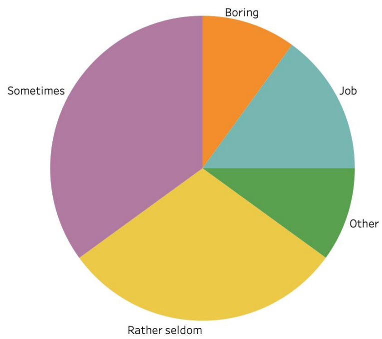
(a) I understand Math. fast and it is fun for me to discover connections myself.



(b) When I have some time to do Mathematics, I manage it quite well.



(c) As soon as someone explains things to me at my pace, I understand most of it.



(d) More often than not I find Mathematics to be rather complicated.

Figure 2.17: The contingency between self-claimed skill and interest in mathematics.

2.3.3.3 Did you Learn Something New about Math here?

The respondents self-assess whether they have learned something about mathematics on the website. We can see that by far the most chosen answer (42.1 %) was “A tiny bit”, followed by “An average amount” (21.8%). About 8.2 % of the users claimed to not have learned anything at all and 11.8 % of the users label themselves as “Poly-professionals”, see Figure 2.18.

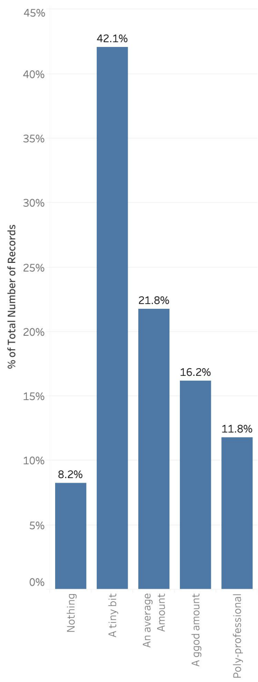
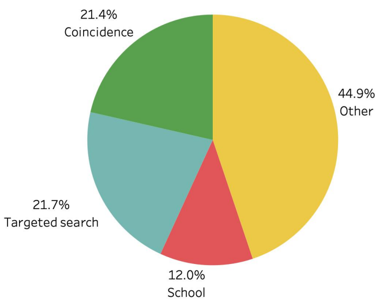


Figure 2.18: Did you learn something new about math here?

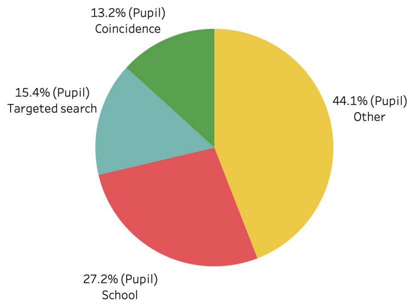
2.3.3.4 Pathways to the Project

Figure 2.19a illustrates how the users have come across our project in the first place. About a fifth landed on the page by coincidence and another fifth by targeted search. 12.0% of the total users have been introduced to the project in a school context. In Figure 2.19b we see that 27.2% of the pupils were introduced to the project at school.

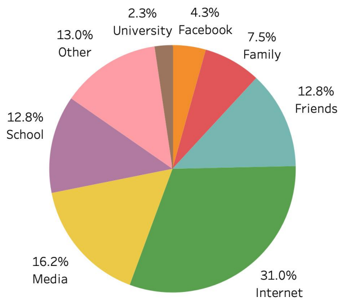
We added a question to specify how the users heard about the project, see Figure 2.19c. The majority, 31.0% found it via the internet, while social media, i.e. Facebook make up only 4.3 %. Media is a big item with another 16.2 %. Family and friends together make up 20.3 % of the pathways to the project.



(a) How did you find us?



(b) How did you find us? (by pupils)

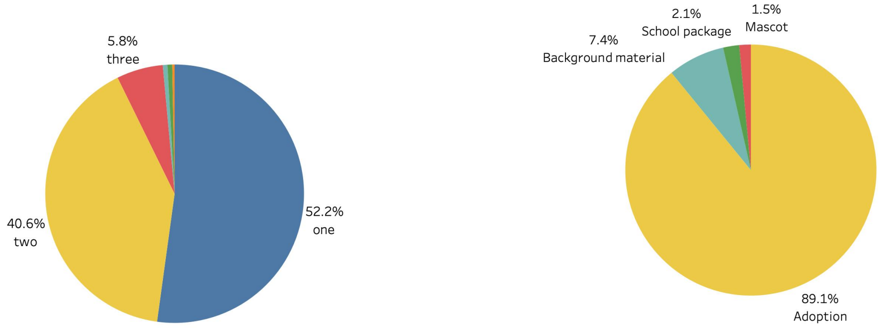


(c) How have you heard about the project?

Figure 2.19

2.3.3.5 How would you Rate this Page?

We asked the users to rate the page by giving marks between 'one' and 'six', where 'one' is *excellent* and 'six' is *not sufficient*. The general result is very positive. Over 50% of the respondents rated it with 'one' and about 40 % with 'two'. 1.4 % of the users found the project not good and graded it with a mark below 'three'.



(a) How would you rate this page? (b) What did you like best?

Figure 2.20

What do you like most about our page? The adoption of the polyhedra was what users by far liked most about the website. As Figure 2.20b shows, 89.1% choose the adoption process as their favorite part of the site. 7.4 % liked the background materials about mathematics best.

2.4 Conclusion

Looking at the numbers on the website, we can conclude that our two measurable goals were fulfilled. Finding a foster parent for over a thousand polyhedra within the first year was exceeded (1626 adoptions) and a steady rate of on average 26 adoptions (median 23.5) per month fell into our desired interval of 20 to 30 adoptions per month.

We defined our target group as pupils aged between 7 and 13 years old. About a fourth of the users fit into this age group. For future projects it would be advisable to specify more precise goals in numbers or percentages with respect to the target group.

Measuring the success of a science communication project beyond countable items like user numbers is a field in need of scholarly and practical attention. Defining the goals is a first step. The second step is the development of an inventory of items to quantify and evaluate the achievement of these objectives. The survey at hand was deliberately chosen to be brief and based on self-evaluation and user experience.

The transfer of knowledge was more a welcomed by-product than a central goal of the project. Mathematical literacy can be measured by the competencies that it comprises of. We wanted to refrain from quizzing the participants with questions about factual mathematical knowledge and thus asked them for their perceived increase in mathematical knowledge. This approach puts the individual and their experience in the center.

In terms of *public awareness of mathematics*, the positive rating of the project in the survey allows for the conclusion that the participants enjoyed the project. Especially the idea of "adopting a polyhedron" was met with approval, as about 9 of 10 users preferred it over other aspects. If these positive ratings can be interpreted as a positive affectional response that generalizes towards (scientific) mathematics, is open for further investigation.

Summing up, we can say that the project Polytopia – *Adopt a Polyhedron* built the backdrop for our study on mathematical science communication. Successfully putting an idea into practice will inevitably lead to what is called "experience". But it takes a second step, the reflection about it, to make this experience graspable for others. The practice will benefit from more clearly differentiated terminology to denote the objectives, motivations and methods of a project. Having a terminology at hand makes it easier to compare projects, learn from others' experience and is necessary for the development of evaluation methods.

Chapter 3

Mathematics Education

3.1 Bridging Formal and Informal Learning

Bridging formal and informal learning combines the efforts of science communication with education, takes mathematics out of the classroom and brings into other spheres of life [93]. The goal of the project Polytopia was to walk this bridge in both directions and bring science communication into the classroom as well.

Formal learning is structured by curricula in schools. Restrictions on how lesson time is used are placed on teachers and pupils. It can only sparsely be used for non-curricular subjects. Hence, in order to be applicable in the math class, the school material in Polytopia contains mostly contents from the curriculum and weaves in other mathematical aspects very carefully. In this chapter, we want to discuss the design of the school materials, demonstrate its links to the curriculum as well as the application of didactical principles in mathematical science education.

The Organization of the Curriculum in Germany and its Federal States The determination of the contents of the school curriculum is the responsibility of the individual federal states in Germany but have a common foundation in the decisions of the conference of Ministers of Education and Cultural Affairs[1]. These define the educational standards for the major subjects, including mathematics, for the three school degrees obtainable at German secondary

[1] Kultusministerkonferenz

schools [87–89]. In mathematics, the competence model [2] is structured by *general mathematical competencies*[3] which describe the process of doing mathematics [88, p.7] and *mathematical central themes*[4] which relate to factual knowledge [88, p.9], see Table 3.2.

On the Term *Mathematics Education* With regard to the design of the school material and the considerations it is based upon, it should be noted that the term fits strongly into the tradition of German *Mathematikdidaktik*. This term is not quite fittingly translated by the internationally used term *mathematics education* which denotes the actual teaching and learning process happening in schools. The direct translation would be *didactics of mathematics* which is not commonly used [146]. The way mathematics is taught and is spoken about is strongly linked to the respective country and the ideas of mathematics education prevailing there. Translations of of the technical terms of *Mathematik Didaktik* are given in the footnotes.

Preliminary Considerations for the Design of the School Material The main target group of POLYTOPIA is students between 7 and 13 years old. Incorporating the project into the math class at schools was an explicit goal. To ensure the adaptability of a science communication project into a formal learning environment, it had to meet a set of criteria. Teachers can, within the official framework of the curriculum, decide about the materials and methods used in their classrooms. Applying our materials in their classroom should not cost them more time in preparations but ideally simplify the process of lesson planning. Since the curriculum is densely packed with content related mathematical *central themes* and *general mathematical competencies*, not much room is left for extra subject matters. Thus, the materials must fit tightly into the curriculum and allow for a view into scientific mathematical process and contents without overextending the lesson time and the students' capabilities.

In accordance with our goals and objectives, we aimed for an eye-level dialogue with the students as well as the teachers which is reflected in the design. The school materials come in a package with copy templates for the students and a teachers' handbook that contains practical information about the background, organization, and conduct of the project, and classification in the competence model. Since not every teacher has come across the theory of polyhedra on

[2] Kompetenzmodell
[3] allgemeine mathematische Kompetenzen
[4] mathematische Leitidee

their own educational path, offering additional information about the subject was necessary. All school materials as well as the teachers' handbook can be found in the Appendix.

3.2 Didactical Concepts behind the School Materials

For the design and development of the school materials, we have employed three didactical principles: *project teaching* [5], *inquiry based learning* [6] and *dialogical learning* [7]. These together with considerations about the *EIS-model* that describes the interplay between different modes of representation build the theoretical framework of the educational material for the project. In the following a brief introduction to these principles and ideas is followed by a description of the materials and how they fulfill the didactical principles.

3.2.1 Project Teaching in the Math Class

Project teaching in the math class is a method to incorporate unique experiences in everyday school life that relate to a special topic or theme. The pupils often associate positive memories with it [97, p.62]. The framework curriculum should not be omitted during project teaching. We base our descriptions of project learning on Ludwig [97]. The conception of *project teaching* includes five characteristics.

- **Subject:** The basic requirements for a topic suitable for a project are self-containment, the possibility of *internal differentiation*[8], and a connection to the learners' environment.
- **Organizational frame:** An organizational framework for the project is required. This includes the composition of the learning group, a time frame, and whether it is a purely mathematical or cross-curricular project.
- **Student activity:** Reasonable activity is preferred to "hustle and bustle". Planning goal directed actions that lead to new knowledge and ideally new ways of thinking for the students is crucial.
- **Group work** Group work as a social form is a central element of *project teaching.* The students independently divide the work among each other and, where appropriate, the groups also exchange information.

[5] Projektunterricht
[6] Forschendes Lernen
[7] Dialogisches Lernen
[8] Binnendifferenzierung

- **Feedback** Feedback denotes the communication among learners and teacher. In one direction is consultation and evaluation by the teachers, and in the other direction the students give feedback about the project. The learners also give each other input on the project and its content.

These five characteristics are introduced in a dome-model, see Figure 3.1. For the stability of the dome and therefore the success of project teaching, all parts have to be there to stabilize each other. When one is missing, the dome collapses. Thus, it is crucial that each characteristic is fulfilled.

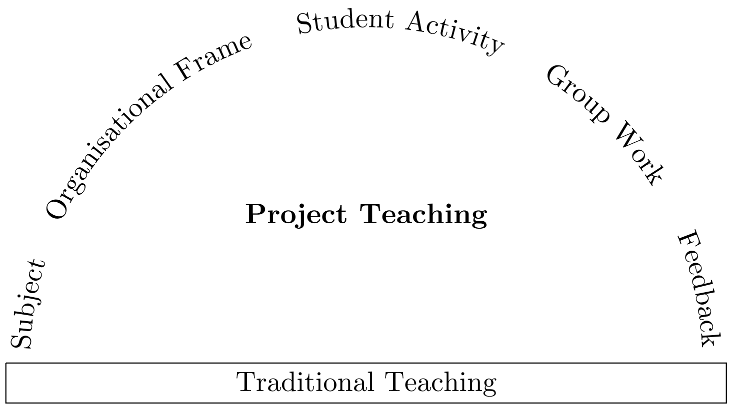


Figure 3.1: Dome model for project teaching, according to Ludwig [97, p.66].

3.2.2 Inquiry Based Learning

The main idea of *inquiry based learning* is that students are not first passively exposed to the subject matter and then made to answer questions posed by the teacher, but actively engage with a new topic and then find their own questions. With the help of the teacher they search for answers and acquire new content related and procedural knowledge in this self guided learning process [107]. Since the learners can freely explore the theme at hand and find answers to their own questions, the motivation to learn is of an intrinsic nature.

To define *inquiry based learning*, it is necessary to consider what constitutes to the scientific inquiry process in order to transfer these characteristics to the process of learning. In accordance with Roth et al. [127], we refer to John Dewey's characterization of inquiry:

> Inquiry is the controlled or directed transformation of an indeterminate situation into one that is so determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole [37].

Thus, the process of inquiry is trying to bring order and connections into a situation that lacked these properties in the first place. These connections result in a structure that is not constructed in a straight forward fashion but by a seeking motion. Accordingly *inquiry based learning* is a didactical principle that is based on the belief that the curiosity to get to the bottom of things a can serve as motivation for the learning process [127].

This attitude of a researcher, one of curiosity, the desire to know, approaching the world with questions and the goal to find answers is what justifies calling this approach *inquiry based learning*, when the most the learners can do is reacquire preexisting knowledge. Since it is new to the students, the structures in their minds are freshly constructed and their attitude is – ideally – similar to the one of a researcher [107].

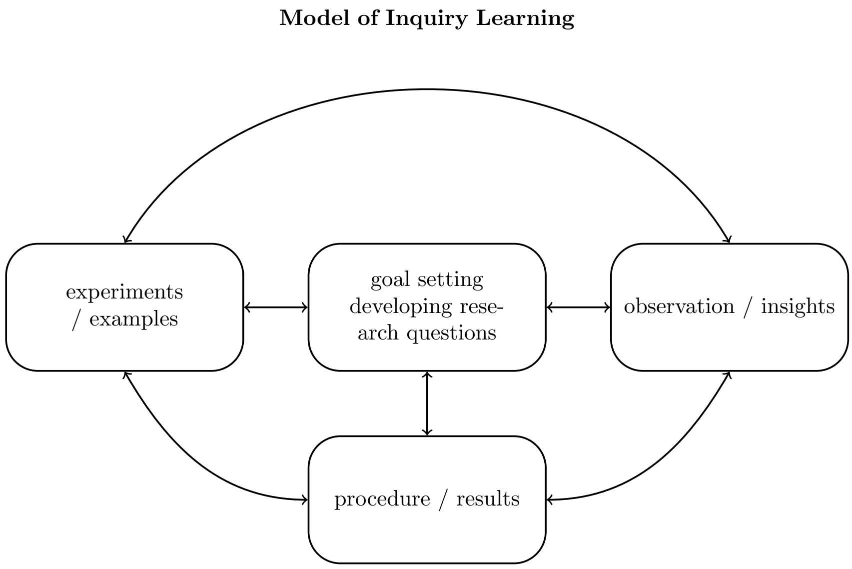


Figure 3.2: Model for inquiry based learning, according to Roth and Wiegand [127].

Structuring and planning are crucial aspect of research. It is not blindly browsing the unknown, but trying to establish a path that connects the new with preexisting knowledge. Before a plan can be set up, a goal in form of a *research question*[9] has to be set up. The development of these questions is central to the model of *inquiry based learning* by Roth and Wiegand [127], see Figure 3.2. It is related to each of the other three elements of the process of inquiry based learning: *experiments and examples*, *observation and insights*, and *procedure and results.*

The Process of Inquiry Based Learning The process of *inquiry based learning* begins with exposing the students with a new phenomenon or subject. The students get some time to pose questions, i.e. the *research questions*, about the subject and develop a goal for their research process. It is structured in a circular fashion around the goal. *Experiments* and systematic search for *examples* launch the first phase of the research process.

The second phase is dedicated to structuring and recording of *observations and insights.* Here, already known knowledge and new discoveries are linked together. The third phase is to reflect on the *results* and methods and find ways to represent them. The structuring of the newly acquired knowledge and the connection to previous knowledge form the main learning effect. In this phase, it may be noticed that the approach was not sufficient and a new question arises. The three phases are therefore not to be understood as subsequent, but they trigger each other and can always provide impulses for new research questions and interim goals [128].

Inquiry Based Learning in the Math Class In order to carry out *inquiry based learning* in the mathematics class, the special features of this discipline must be taken into account. The nature of the research process in mathematics differs from the cycle of examining the validity of a hypothesis by experiments. Instead of straightforwardly planned experimenting, *doing mathematics* can be described by a meandering movement of thought that examines the abstract objects by imagination [98]. Sketching and scribbling are the major tools to support this process. Ludwig et al. transfer *inquiry based learning* into the math class in consideration with the mathematical research process and introduce a cycle for inquiry based learning specifically for the math class [98]. Here, we just want to focus on the first step of this process; finding research questions.

According to Lutz-Westphal [99], formulating their own questions puts students in an unfamiliar position. Thus, finding interesting questions must be practiced. Encouragement and

[9] Forschungsfrage

acceptance towards all questions is an important first step. Next comes the collection and structuring of the questions. Introducing typical question words for mathematical questions can also prompt the process and lead to more questions of mathematical nature [99].

In mathematics, the generation of new knowledge is very much guided by and closely linked to already existing results. Thus inserting phases to present mathematical knowledge to the students is no contradiction to *inquiry based learning* [98].

3.2.3 Dialogical Learning

Every learner has an individual pace, style, and preexisting knowledge. *Dialogical learning* is a didactical principle which aims to enable each student to find and design their own suitable learning path. The subject matter is not divided and served in small pieces by the teacher, but the learner and their own perceptions and questions are in the center of the learning process. According to Gallin and Ruf, it is the eponymous dialogue between the learner and the learning content where comprehension occurs and learning happens. The moment when the learner reflects upon the distance travelled and thus puts sense into the context, is when the new connections are made [55, 129].

Similar to *inquiry based learning* and *project teaching*, *dialogical learning* starts with a moment of momentum, a *core idea*[10] that evokes real questions in the learners. These *core ideas* may have the form of a story, a riddle, or an impulse which fuels the learning process. A successful *core idea* creates productivity in the learning group because it inspires genuine curiosity. For such an impulse to have enough drive, its authenticity and a genuine connection to the personality of the teacher is crucial [55, 129].

Another important tool of dialogical learning is the *learning journal*[11]. It is the learner's travel log in which they record and reflect upon the learning process in writing. These diaries are freely created and documentations of the individual's learning path and results. The freedom of the design puts the responsibility of the learning process upon the learners. The students are prompted to write about the content and the process in their own everyday language which adapts it better to the individual perceptions [55, 129].

[10] Kernidee

[11] Reisetagebuch

Reflection on mathematical language gives insights about the process of doing and learning about mathematics, since language is giving formation to thought. An orientation towards technical language in the math class is reinforcing a perspective of mathematical learning that emphasizes rigidity and completeness, because technical terms are not connected to the learner's internal understanding [81]. Allowing for an atmosphere in which the learners can constructively rediscover mathematical phenomena and connect them to their individual prior knowledge in their own words is associated with a constructivist perspective on mathematics education. Ideally everyday experience, precision, redundancies and anticipation should be included in the writings about mathematics [81].

3.2.4 EIS-Principle

In the learning development of a child, according to Bruner which is cited in Zech [167], different levels of representation are developed . In a first state the child grasps the world mostly in a direct *enactive* form. The enactive representation registers an object through active handling with a concrete material. The next level is *iconic* representation that relies on pictures and graphics, independent from activities. Language development heralds the third *symbolic* level of representation, when the object itself is no longer needed in a direct or pictured form but replaced by a word or symbol. The new levels do not replace the preexisting ones but broaden the possibilities [167].

An integral approach to teaching mathematics is introduced by Heske, who encourages the use of all three levels of representation. Alternating the enactive, iconic and symbolic levels of representation prompts learning [75]. The acronym of the first letters prompts the name of the EIS-principle. The thought processes of transferring between the modes of representation are depicted and labelled in Figure 3.3. The understanding of the concept by the individual is deepened by this transfer.

Hole [77] extended the EIS model for the use of computers in the math class and developed the CEIS model. Using digital versions of the three representation levels adds a new dimension to the transfer capacity. The enactive layer of activity becomes animation when applied by a computer. Simulation is the digital extension of iconization, and the language of computers is even more symbolized than verbal expression. Thus, symbolization is the digital equivalent of the verbalization and formalization that describe the thought process of entering the third layer.

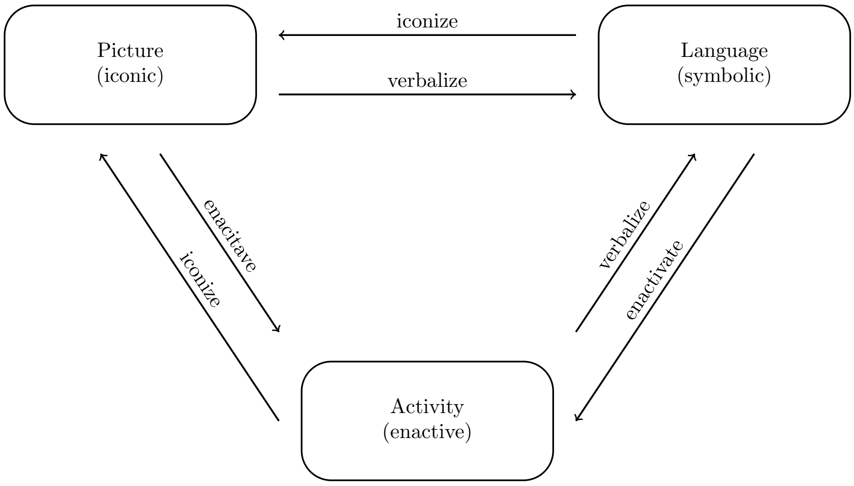


Figure 3.3: Model of the three representation levels of the EIS model and the transfer between them [77].

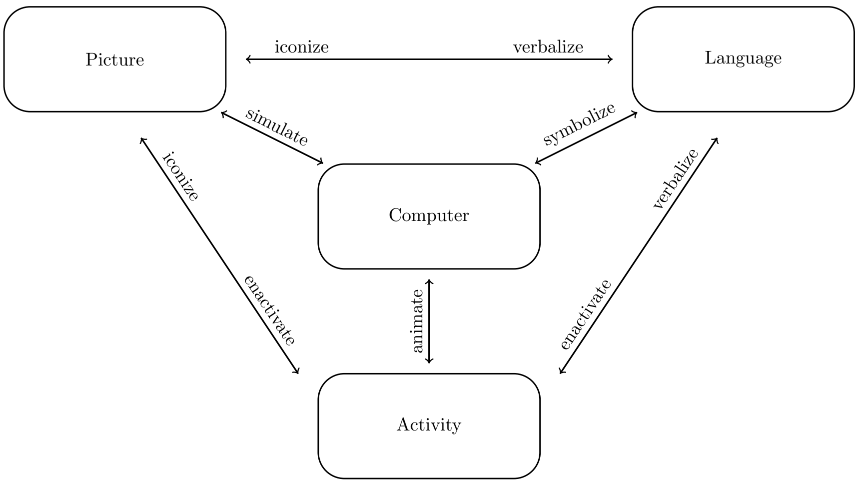


Figure 3.4: Model of the CEIS model. Application of computers adds a new dimension to the three levels of representation each [77].

Title	Age	Time	Learning Environment
Class Set	7 – 18	90 min	all
Research Questions	9 – 18	45 min	all, since the students adapt the questions their own level of prior knowledge
Polly's Journal	9 – 13	270 min	regular math class, suitable for independent work
Discovery Cards	12 – 18	90 min	math clubs, extra work for gifted students, independent work
Materials	7 – 18	180 min	interdisciplinar teaching in mathematics and art

Table 3.1: Overview of the school material and their appropriate age groups, time frame and learning environment.

3.3 School Material

Teachers can choose from different school materials to download on the website that suits different age groups and learning environments. Table 3.1 shows the items, their suitable age group, time frame, and learning environment. All materials can be found in the Appendix[12]. The school materials combine different aspects of the didactical concepts, mentioned above, according to the spot-light model [92]. This model utilizes the metaphor of a stage that is lighted by different spotlights which each symbolize a didactical method. To yield an optimal illumination of a subject matter multiple light sources, aka. learning methods, are required. In this section, we will first introduce the teaching materials, and then point out where the mix of didactical concepts is applied. Although more than one principle is applied in each of the packages, for each one its guiding didactical principle, or *main light source* to stay congruent with the metaphor, will be discussed.

3.3.1 Class Set

To simplify the supplies for students in a learning group, teachers can reserve a set of up to 36 polyhedra and print out all the nets in one go. Each net has an identification number to find the right polyhedron on the website. The class set, which is the smallest package of the school

[12]The Appendix contains the school materials in English. Since the website is bilingual, a German version is available, too.

material contains a front page with a tutorial for the assembly of the polyhedron, a worksheet that contains brief information about polyhedra and a reference to the project and its website. The teacher's handbook accompanying the class set gives a brief definition of polyhedra and states the learning objectives of the project. It contains a practical plan for conducting the project and an overview of the main components of the project as well as a description of its background. The class set is suitable if the teacher wants to introduce the class to the basic concept of the project POLYTOPIA.

EIS Principle guides the Process of Adoption The learning effect of partaking in the project with its different modes of representation of the polyhedra aligns with the (C)EIS-principle. In the process of manually assembling the net of the polyhedron into a three-dimensional model, the change of representation is done by the learners themselves. This enactive representation is the first presented to the students, who get a paper template of the net. With the number printed on each of these templates they find their personal polyhedra in an interactive, computer-aided, digital representation, i.e. in the iconic representation level. Students can change the color of their polyhedron or change the perspective by switching the corners, edges or faces on and off. Besides the students gaining some symbolic knowledge in learning the term *polyhedron*, the linguistic level has a certain twist. The students are invited to name their polyhedron themselves. The symbolic representation of a certain polyhedron is therefore chosen by the students and is not given, as is usual in mathematics and the natural sciences. This symbolizes the incompleteness of scientific mathematics.

3.3.2 Finding Research Questions

The class set can be accompanied by the worksheet *Finding Research Questions*. After adopting a polyhedron and crafting the paper model from the template, the students are invited to look at the subject of polyhedra with the mind set of a researcher and find research questions about the topic. The task is to find the most difficult question.

Inquiry Based Learning in Finding Research Questions Inquiry based learning is the guiding didactical principle in finding research questions. The direct link to the scientists of the collaborative research centre and the invitation to help them, gives insight to an important feature of mathematic as a scientific practice that according to a study by Lutz-Westphal and Schulte [100] is not apparent to high school students: mathematics as a stand alone scientific practice.

The small-sized study asked 60 students at a Berlin secondary school about their attitude towards mathematics as a scientific subject and found out that most respondents viewed it merely as a tool to support the natural sciences and emphasized the application orientation. Mathematics as a practice which is an end in itself was not recognized by the study participants who also expressed the belief that the science of mathematics is already "completed" [100]. According to Lutz-Westphal and Schulte these associations with mathematical research are cosistent with the beliefs about the subject of mathematics itself found by Grigutsch et al. [63].

Showcasing the science of mathematics as an end in itself and its incompleteness is an objective of our project Polytopia. The openness of scientific mathematics is showcased in the main narrative as the majority of polyhedra lack a name. The playful *call for help* ultimately opens the door to active participation in the science of mathematics.

According to Lutz-Westphal [99], we give prompts to invigorate mathematical thinking when it comes to finding suitable research questions. Of course teachers familiar with *inquiry based learning* in the math class are welcome to apply its full circle in their lessons, but we focus on finding the questions. The mascots on the worksheet invite the students to send back their questions in order to start a dialogue.

3.3.3 Polly's Journal

The learning journal "Polly's Journal" is designed as a learning journal according to the concept of *dialogical learning*. The successful implementation of the project hinges on the teacher and the learning group, but the learning journal allows for each of the five characteristics contained in Ludwig's model for *project teaching*.

"Polly's Journal" is divided into four parts and starts with an introduction of the mascot *Polly*. They tell the story behind the project and prompt the students to help finish their family photo album by adopting a polyhedron, giving it a name, building a model, and taking a picture. This story is the *core idea* of the project; it motivates the conduct of the project, sets the frame and defines a goal. Because the notion of polyhedra is not contained in the formal curriculum the following pages are dedicated to sizing up the learner's prior knowledge and building connections to the new definitions. In doing so, drawing oblique views and analyzing the nets of cubes and other bodies is repeated and skills are deepened.

The second part guides the learners through collaboratively building their *class polyhedron*. Before assembling, each face is to be enlarged by a factor. The group is separated into smaller groups that each take care of one face. A possibility for internal differentiation is presented by choosing faces with more than three vertices. Thus, reconstruction of the magnified faces is more or less involved, respectively. Teachers can preset the magnification factor, or let the students discover the quadratic relation between the size of the edges and the area of the face. Once the *class polyhedron* is ready, the group votes for a name and adopts the polyhedron on the website.

Subsequently the teacher hands out the paper templates for adopting the single polyhedra. Each learner gets a worksheet with a distinct polyhedron. The last step of the project invites the learners to take on the position of a researcher and find research questions. Additionally the teacher can either implement the discovery cards, see the Appendix, or add an inter-curricular element by expanding into the art and crafts class and letting the students build individualized models.

Dialogical Learning guides the Learning Journal As the learning journal "Polly's journal" is designed with the didactical concept of *dialogical learning* in mind, the students are guided through the project in a way that promotes self-regulating learning and prompts reflection upon their progress in writing. The core idea to adopt a polyhedron to help Polly get to know all her family members is not directly connected to the personality of the teacher, as encouraged by Ruf and Gallin [129], but to the makers of the project that can be personally contacted via email.

The approach of *dialogical learning* focuses on the use of language in the math class and emphasizes the importance of informally speaking and writing about mathematics. Using non-formal language makes it easier for the learners to adapt the newly acquired knowledge to their individual preexisting knowledge. The creative element of naming their own polyhedron playfully opens up the realm of technical terms in mathematics and allows the students to add a personal touch.

The EIS model is a guiding principle for the design of the learning journal as well. Besides the adoption process, the introduction of convex polyhedra is lead via different geometric representations. The definition of convex polyhedra is done as a perspective view that the learners are invited to draw out themselves.

3.3.4 Discovery Cards

The *discovery cards* are a learning opportunity for enthusiastic learners that can be used for internal differentiation or extracurricular activities. They serve the needs for gifted learners, since they exceed the regular curriculum and contain selected themes from the subject of Discrete Geometry that are centered around polyhedra. The cards are suitable for self-regulated learning. The students examine their polyhedron through the prompts on the learning cards that have topics ranging from counting the vertices, edges, and faces and assembling them in the f-vector, over investigating the graph of the polyhedron, to drawing Schlegel diagrams.

3.3.5 Project Teaching as the Backdrop for all Materials

All materials have in common that their application in the math class qualifies as *project teaching*. Recalling the dome model of this didactical principle, see Figure 3.1, we find the five characteristics of project learning that together constitute a successful project.

First is the *topic* that must be self-contained, includes possibilities for internal differentiation and is related to the surroundings of the learners. Adopting a polyhedron is a process that can be done in a 90 minute lesson from start to finish and has its standalone quality since it is not in the curriculum. Internal differentiation can be applied in a minimal form by choosing the complexity of the polyhedral nets. In "Polly's Journal" the students are invited to look for polyhedral shapes in the classroom and their surroundings.

The *organizational frame* is mostly given by the teacher, but the material is designed for an organization in a linear fashion that starts with prompting the *core idea* of adoption, a working phase and a specific endpoint.

The *student activity* is guided along the adoption process or the learning journal. Building the class polyhedron is a collective mathematical action that requires the learners to work tightly together to finish their geometrical figure. The new impulse of finding research questions turns the classical math class on its head, since finding the most unanswerable question is an explicit goal.

The social form of *group work* is most prominent in the building of the class polyhedron where each small group is responsible for one of the faces of the polyhedron that has to be assembled to a larger one. Only when all groups work sufficiently accurately is this possible. An exchange between the groups is prompted. Finding the *research questions* or working with the discovery cards allows for work in smaller groups as well.

An important possibility for *feedback* is the dialogue that can occur between the scientists and the students when they send their questions via email. On a smaller scale, collecting the learning journals and commenting on the thought processes that the students have expressed is a chance for teachers and learners alike.

Mathematical Competencies		Central Themes	
(K1)	Mathematical Argumentation	(L1)	Central Theme Numbers
(K2)	Mathematical Problem-Solving	(L2)	Central Theme Measurement
(K3)	Mathematical Modelling	(L3)	Central Theme of Space and Form
(K4)	Using Representations	(L4)	Central Theme Functional Relations
(K5)	Dealing with Symbolic, Formal and Technical Elements of Mathematics	(L5)	Central Theme of Data and Probability
(K6)	Communication		

Table 3.2: Guiding principles of the Competence Model in the Framework Curriculum 1-10 [138].

3.4 Linking Points to the Curriculum – Competencies and Central Themes

3.4.1 Primary Education

The main subject of the project, polyhedra, is not part of the curriculum in Germany as defined by the Conference of Ministers of Education and Cultural Affairs, which defines the framework for all federal states [89]. Nevertheless, the mathematical activities of the school materials have strong links to it. In the following, we specify these linking points to the framework curriculum[13] of the federal county Berlin. The skills and abilities are grouped into *general mathematical competencies*, which describe the process of doing mathematics, and the content related *central themes*, see Table 3.2. These themes range through all educational levels which are denoted with capital letters A through H. Level A contains the very basic skills of the central theme. The subject matters then build upon each other. Depending on the type of school, different levels are assigned to the grades. For internal differentiation teachers can tailor the levels for the individual skill level of the students.

Central theme *(L3) Space and Form* is paramount to the content of the school materials. We find aspects that belong to the levels C, D and E, which, depending on the type of school, accord to second to ninth grade [138, 140]. In central theme *(L2) Dimensions and Measurements*[14] level E is incorporated in the enlarging of geometrical objects. Table 3.3 shows the mathematical activities included in the school material and allocates it to the central themes and their levels.

[13] Rahmenlehrplan

[14] Note that in the federal county Berlin, the central themes are slightly differently denoted, but the strongly relate to the guidlined provided by the Conference of Ministers of Education and Cultural Affairs.

Level	Central Theme	Mathematical Activity
	Dimension and Measurements (L2)	
D	Size specifications of surface area, volume, angles in different units	Measuring angles and using the protractor in the reconstruction of the faces of the class polyhedron.
	Space and Form (L3)	
	Geometric objects	
C	Cube and cuboid nets	Polyhedra nets are introduced by nets of the cube. In the extension to general polyhedra level C is exceeded.
D	Oblique views of cubes and cuboids	Polyhedra are introduced as perspective oblique views.
E	Construction of triangles	In building the class polyhedron, each face of the polyhedron must be reconstructed.
	Geometric figures	
E	Enlarging and reducing the size of objects to scale	The faces of the class polyhedron are reconstructed in a different size.

Table 3.3: Classification of the mathematical activities in accordance with the levels of central themes (L2) Dimension and Measurements and (L3) Space and Form [138].

Three mathematical competencies get mainly encouraged: (K1) Mathematical argumentation, (K4) Using representations and (K6) Communication.

Recognition and determination of convex polyhedra on the basis of their three defining properties (plane side faces, straight edges and outwardly protruding corners) is practiced. The investigation of geometric bodies according to these criteria fall into the realm of (K1) Mathematical argumentation.

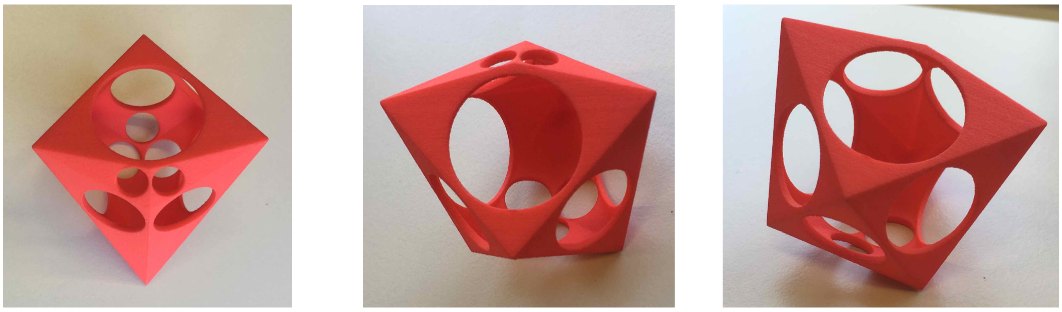
General mathematical competency *Using Representations (K4)* is a key element of the project and its school material. Each polyhedron is presented in seven different representations. These are: the individual pixel in the matrix (1), the digital visualization in the "viewer" (2), the paper template of the polyhedral net (3), its emerging physical 3D model (4), the 3D printing model (5), a VR model (6) and finally, its name chosen by the students (7).

Communication (K6) aligns with the didactical principle of *dialogical learning* which lead to the development of "Polly's Journal". In the learning journal the students write about their experiences. Thus, writing about mathematics in a reflecting way is practiced.

3.4.2 Secondary Education

Although the project aims at younger students, it is also suitable for secondary education. Its curriculum is structured by the same *central themes* and *mathematical competencies* as primary education. Both encompass more involved subjects and skills respectively [139].

Polyhedra serve well as visual aids and applications for the subject area of analytical geometry that is involved in the core idea [L3] *Space and Form*. Their faces are defined as planes spanned by the coordinates of the vertices. Two neighboring faces intersect in a straight line. These core concepts of analytical geometry can be derived from the 3D printer data which state the coordinates of the polyhedra and their incidences. For printing the polyhedra with a 3D printer each one has to be rotated so that one of the faces is parallel to the xy-plane. This operation can be executed by methods of analytical geometry, and holding the correctly printed polyhedron in their hands, the learners have a tangible proof for the accuracy of their calculation, see Figure 3.5.



(a) (b) (c)

Figure 3.5: Photographies of a 3D printed polyhedron.

3.5 Applying Didactical Principles into the Project Design

In Chapter 1.2 we described the shift that has occurred in science communication from knowledge over affection towards trust. This has lead to overcome the deficit model and the development of more dialogical formats of science communication that foster an eye-level encounter between science and the public. According to Baram-Tsabari and Osborne [14] this shift has happened in education two decades prior and the didactical principles described in this are fruits of this development. They have in common that the learner is in the center of the knowledge acquiring process. The teacher, traditionally in a position of power over the student, is put in a position to serve the student to find their own path by introducing subjects, offering new ideas and help organizing the process. Frontal teaching might still be the predominant practice of teaching, but at least scholarly it is overcome. With the observation in mind that science communication is in a way following the footsteps of education, we accelerated this process and applied the guiding ideas of the didactical principles in the overall design of the mathematical science communication project. The *core idea* of adoption is the key point of the entire project. A relationship between a mathematical object and its *forster partent* is established. This might lead to interest and thus *research questions* in the participants. We have already established above that the (C)EIS-principle is paramount to the modes of representation on the website that serves an informal educational purpose. And the *dialogical* elements in the format are not just coincidentally sharing the adjective with *dialogical learning*. With linking the science communication project Polytopia very closely to formal learning environments and borrowing from the didactical principles in the design of the project, we provide an example of mutual enrichment of the two practices.

Chapter 4

Mathematics

4.1 "Adopting a *What?*": On the Term *Polyhedron*

In the subtitle of the project POLYTOPIA – *Adopt a polyhedron* the participants stumble upon a mathematical notion that is not necessarily in their active vocabulary. The website gives a rather intuitive definition:

> Polyhedra consist of vertices, straight edges, and flat faces. In our project, we focus on convex polyhedra. This means that all inner angles between two edges or two faces are less or equal to 180°. No cavities, holes or indentations are allowed. The word polyhedron comes from the combination of Greek words poly- (many) and -hedron (face).

This definition is by no means commonly agreed upon. Mathematicians of all times and walks of mathematics picture rather different objects when something is called a polyhedron [66]. In this chapter we will take a closer look at the mathematical objects that are at the center of our project: polyhedra. We will take on three different perspectives. Firstly, we develop a genealogy of the terms that are historically used for our three-dimensional, flat faced, straight edged and convex bodies. Secondly, we rigorously define the term *polyhedra* as we used it in the project and introduce the notions and methods of modern mathematics used to visualize the models in the viewer and to generate the nets for the crafting sheets. However, the existence of such a net is not given. In fact the question whether every polyhedron possesses an unfolding whose faces do not overlap is an open problem, named *Dürer's conjecture* after the painter Albrecht Dürer, who invented the nets of polyhedra. The third part of this chapter gives a survey of this problem, sums up the development up to now and introduces related problems.

4.2 Genealogy of the Term *Polyhedron*

The term polyhedron changed its definition and meaning over the course of history. Even today, it is used to describe different mathematical concepts and objects and each author is inclined to define it accurately. This summary tells a short history of the term polyhedron. It is guided by two questions that lead us the way trough history[1]:

- Who used the term *polyhedron*, when and for what?
- What were the objects we use in the project and call *polyhedra* also called and by whom?

Ancient Greek Mathematicians

The first written description of polyhedra appears in Plato's dialog *Timaios*. He describes the construction of what today is called the *Platonic Solids* and assigns a regular polyhedron to each element of the universe: fire, water, earth, air and the universe itself. According to Plato every physical body is a polyhedron, as he writes:

> In the first place, that fire and earth and water and air are material bodies is evident to all. Every form of body has depth: and depth must be bounded by plane surfaces [11, 53 C].

In this sentences he uses the greek word *σώμα* (= soma) which means *body* in a general sense and denotes the biological body or corpse of a human or animal, anything with a corporal substance, as well as a mathematical figure of three dimensions [91, *σώμα*]. The expression *form of body* translates from the original *σώμας ϵῖδος* . In his further descriptions of the construction of the platonic solids, Plato mostly uses the term *ϵῖδος* [11, 54 C - 55 A]. Its English translation is: "that which is seen: form, shape", which resonates with a more mathematical application [91, *ϵῖδος*].

Plato believed that plane surfaces bound every form. This is very close to todays definition of an $\mathcal{H}$-polyhedron that is a region bounded by half-spaces, see Definition 4.3.5. If we would like

[1] For a more general approach on the history of polyhedra, see Malkevitch [101].

to guess whether Plato was aware that polyhedra make up a special class of three-dimensional geometrical objects, we would have to deny, because it seems that he assigned an angular nature to all objects in space.

Feeding the search engine of the *Thesaurus Linguae Graecae* [153], a comprehensive text collection of ancient greek authors, with the word $\pi o \lambda \upsilon \epsilon \delta \rho o \nu$ (poluedron) results in 45 hits. 34 of them stem from Euclid's *Elements*, five from Theon, three from Pappos and one each from Archimedes, Plutarch and the *Oracula Sybillina*.

Euclid [44, 149] did not use the word $\pi o \lambda \upsilon \epsilon \delta \rho o \nu$ (poluedron) in the claim and proof that the list of the Platonic solids is complete, where he refers to the solids as $\sigma \xi \eta \mu \alpha$ (sxhma) which can be translated to *shape, form, figure* [91, $\sigma \xi \eta \mu \alpha$]. The word $\pi o \lambda \upsilon \epsilon \delta \rho o \nu$ is used earlier, in the twelfth book. This book is mostly dedicated to propositions about objects in space, like pyramids, or prisms, but also cones and cylinders. Only proposition 17 is a statement about polyhedra in general. Thus, Euclid explicitly distinguished polyhedra from other three-dimensional geometrical objects.

Albrecht Dürer (1471–1528)

The German artist Albrecht Dürer was also very interested in mathematics. The polyhedron in his famous copper plate engraving *Melencholia I*, see Fig. 4.1, is up unto this day uncategorized and poses a riddle to art historians and mathematicians. The solid in the lower left of the picture emerges from an elongated combinatorial cube, whose vertices with the sharpest angles were chopped off and thus results in a shape with eight faces, six pentagons and two triangles, 18 edges, and twelve vertices. Fatumura et al. [54] pose a method for testing theories on the solid using its cross ratio and come up with their own. They propose the cross ratio of pentagon to be the golden ratio.

The invention of polyhedral nets is attributed to Dürer, who introduced them in his work *Underweysung der Messung mit dem Zirckel und Richtscheyt in Linien, Ebenen und Gantzen Corporen* [40]. As it can be derived from the title, Dürer denoted the *bodies in space* by the Early New High German term *Corpus*, which sounds similar to the German *Körper* (body). Strauss [41] translates it to *solid*. Amongst all the polyhedral nets, one quite surprisingly finds an "*unfolding*" of the sphere in the form of an orange peel. Despite Dürer not commenting on the fact that this orange peel unfolding will not result in a perfect sphere, he seems to be aware

Figure 4.1: Albrecht Dürer's *Melencholia I* [42].

that there is a difference between "solids with straight surfaces" and '[s]olids which are rounded on all sides" [41] as the former have corners and the latter do not. The unfoldings of polyhedra will be defined and further investigated later in this chapter, see Section 4.4.

Johannes Kepler (1571–1630)

In the second book of the work *Harmonice Mundi*, Johannes Kepler expands on Euclid's proof of the classification of the Platonic Solids. Even though he directly refers back to Euclid and uses his terminology to denote the Platonic Solids, he does not adopt the term $\pi o\lambda \upsilon \epsilon \delta \rho o \nu$ for general polyhedra, but uses the latin term *solidam figuram* (solid figure) instead. The adjective *solidum* is used here to distinguish the spacial from the plane figures (*Figurarum planarum*).

Leonhard Euler (1707–1783)

Euler wrote the articles *Elementa Doctrinae Solidorum* [46] and *Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita* [45] in an attempt to build a fundamental framework for geometry in space, just as Euclid had done for the geometry in the plane [131]. Euler restricts his elaborations to solids that are enclosed by planes, which he calls *solidorum*. Krömer translated this to *Körper* (solid/body) [85][2]. Euler developed a new way to denote polyhedra depending on the number of vertices and faces. For example, the three-sided prism is called a *pentahedrum hexagonum* (five-sided hexagon), or a triangular pyramid is denoted a *tetraedrum tetragonum* (four sided, four-gon) [46].

According to Malkevitch [101], Euler pioneered in interpreting polyhedra not just as geometrical, but also as combinatorial objects. Before Euler, Descartes came very close to finding Euler's formula, but Descartes' manuscript was lost for a long time and thus unknown to Euler [47].

Ludwig Schläfli (1814–1895)

According to Manning [102], Möbius [111] was the first mathematician who thought of the mathematical possibility of the fourth dimension in 1827. Following Manning's chronology of the discovery/invention[3] of the fourth dimension, it was only Grassmann in 1844 and Cayley in 1846 who preceded Schläfli in writing about four dimensional geometry. Schläfli, a Swiss

[2] A direct translations to English is not existent [86].
[3] Let's not go there...

mathematician wrote his treatise *Theorie der vielfachen Kontinuität* between 1850 and 1852, but could not find a publisher. When it was published in 1901, six years after his death, others had independently rediscovered large parts of his work [32] and thus, his groundbreaking work on higher dimensional geometry is missing in Manning's genealogy of the fourth dimension.

Schläfli [137] introduced multidimensional geometrical bodies and denoted them *Polyscheme*. The origins of this word can be traced back to the greek words $\pi o \lambda \upsilon$ (polu) and $\sigma \xi \eta \mu \alpha$ (schma) which we have already mentioned above. Due to the neglect of his work by the mathematicians of his time, this notion did not prevail.

Reinhard Hoppe (1816–1900)

The term for higher dimensional generalization of polyhedra that is still in common use is *polytope*. According to Coxeter [32] the birth of this term can be traced back to Hoppe [78] who wrote in 1882:

> Ebenso kann man die Grenze eines Polytops (Vielraum, so will ich die linear begrenzte Figur von 4 Dimensionen nennen) auf dem Raume abbilden durch ein Netz von Polyedern und ein sie alle umfassendes Polyeder als Schlussseite.[4]

Hoppe describes the composition of a four-dimensional polytope by its three-dimensional facets. This approach is similar to defining a polyhedron by its bounding faces.

Victor Schlegel (1843–1905)

The math teacher Schlegel chose his own terminology for polyhedra and polytopes. In 1883 he published *Theorie der zusammengesetzten Raumgebilde* (Theory of Composed Space-Constructions). Throughout this work he refers to 2-polytopes as polygons and to their three dimensional analogous as *Raumgebilde* (space-construction) [135].

Five years later, he explains the four-dimensional geometrical space in a popular scientific publication. Schlegel defines and enumerates the regular solids in three dimensional space and generalizes the definition of a regular solid for higher dimensions. He refers to the three dimensional objects as *Körper* (body) and their analogues in any dimension as *Gebilde* (Construction) [135].

[4] Likewise, the boundary of a polytope (multi-space, so I will call the linear bounded figure of four dimensions) can be mapped on space by a net of polyhedra and an enclosing polyhedron as a boundary facet.

Ernst Steinitz (1871–1928)

Steinitz' work *Vorlesung über die Theorie der Polyeder* [147] (Lecture on the Theory of Polyhedra) was posthumously published by Rademacher in 1934, who filled in the gaps left by the original author.

Steinitz refers to Möbius in his definition of polyhedra and follows the approach of seeing them as a collection of faces that meet in common edges or vertices. The characteristics of the polygons decide over the characteristic of the polyhedron. An *ordinary* polyhedron is composed of *ordinary* polygons. These are polygons that are simple and plane and whose edges have no common points other than the vertices. An *extraordinary* polygon can self-intersect.

This wider definition of polyhedra includes the star polyhedra and self-intersecting polyhedra that do not partition the space into an exterior and interior. Steinitz notes that the convex polyhedra form a subgroup of the *ordinary* polyhedra.

Aleksandr D. Aleksandrov (1912–1999)

The Russian geometer Alexandrov wrote a book about convex polyhedra in 1950 [118]. He points out explicitly that his definition does not specify whether a polyhedron is a solid that is bounded by finitely many polygons, or whether it is only a polygonal surface. However, the polygons that consist of finitely many straight lines, line segments or half-lines might not be bounded, i.e. are not lying in a circle of finite radius. An unbounded polygon has either at least two unbounded sides that are either half-lines or lines, or it is bounded by one line. Note that Alexandrov's definition of a polyhedron also allows for a collection of polygons that are meeting only on segments of their bounding edges.

Harold S. M. Coxeter (1907–2003)

Coxeter [32] includes remarks on the history of polyhedra in his work on regular polyhedra. His definition is very similar to Steinitz' and also describes the properties of the bounding polygons.

> A polyhedron may be defined as a finite, connected set of plane polygons, such that every side of each polygon belongs also to just one other polygon, with the proviso that the polygons surrounding each vertex form a single circuit (to exclude anomalies such as two pyramids with a common apex) [32].

Grünbaum (1929–2018)

In Grünbaum [67], we find a definition for a polyhedral set, or d-polyhedron:

> A set $K \in \mathbb{R}^d$ is called a *polyhedral set* provided K is the intersection of a finite family of closed halfspaces of $\mathbb{R}^d$.

Even though, this definition includes unbounded sets that even in $\mathbb{R}^3$ do not look like the polyhedra that Plato might have thought of, it is the property of being *bounded by plane surfaces*[5] that makes a set *polyhedral*.

We end our historical remarks with Grünbaum, as his definition of a polyhedral set is analogue to what Ziegler [169] will denote as an $\mathcal{H}$-polyhedron, see Definition 4.3.5. This does not mean that the question "What is a polyhedron?" is settled. For further reading, we refer to Grünbaum [66].

[5]Here the concept of a plane surface is generalized to any dimension by using halfspaces.

4.3 Defining the Term *Polyhedron*

What we call “a polyhedron” in the project is, in mathematical terms, the representative P of an equivalence class of combinatorially equivalent, convex 3-polytopes, where P is realized in a normalized Koebe-Andreev-Thurston representation.

In the following we will successively define these notions and analogously describe the generation of the data for the project. These definitions and notions are based on the standard work by Ziegler [169]. For the basic notations, we refer to the Linear Algebra standard by Axler [13].

4.3.1 Convex 3-Polytope

Let $\mathbb{R}^n$ be a vector space of dimension n, let $\mathbf{x}, \mathbf{y}$ and $\mathbf{z} \in \mathbb{R}^n$ be vectors/points and let a, b and λ be scalars. In this vector space we define two kinds of hulls: affine and convex.

Definition 4.3.1. Let $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}$ be (distinct) points in $\mathbb{R}^n$. The *affine hull* of these points is defined to be:

$$\text{affine}(\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \lambda_1 \mathbf{x_1} + ... + \lambda_k \mathbf{x_k} \text{ for } \lambda_i \in \mathbb{R}, \sum_{i=1}^{k} \lambda_i = 1\}.$$

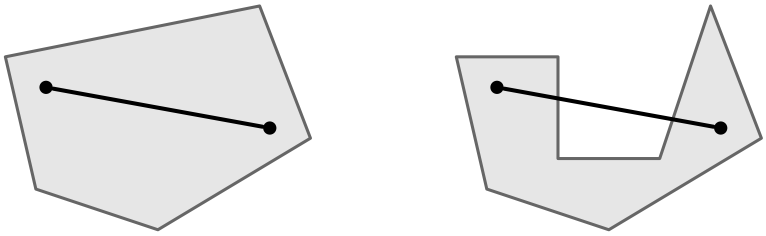
Definition 4.3.2. Let $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}$ be (distinct) points in $\mathbb{R}^n$. The *convex hull* of these points is defined to be:

$$\text{conv}(\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \lambda_1 \mathbf{x_1} + ... + \lambda_k \mathbf{x_k} \text{ for } \lambda_i \in \mathbb{R}_0^+, \sum_{i=1}^{k} \lambda_i = 1\}.$$

Note that in Definition 4.3.1 and Definition 4.3.2, the only minor but critical difference is the restriction that λ is non-negative. The result is that in the convex hull only the space *between* the points is included, whereas in the affine hull, the points *spread out* and support an entire affine subspace of $\mathbb{R}^n$.

Definition 4.3.3. A set $S \subset \mathbb{R}^n$ is *convex* if for any two points $\mathbf{x}$ and $\mathbf{y} \in S$ the convex hull conv(x, y), i.e. the connecting line between the points x and y, is also contained in S.

Now we are ready to give the first definition of a *polytope*.



(a) A convex set. (b) A non-convex set

Figure 4.2

Definition 4.3.4. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$. The dimension $d \leq n$ of the polytope is the dimension of the affine hull of these points. A d-dimensional polytope is called a d-polytope.

We say that two polyhedra are *geometrically equivalent* if they can be mapped into each other by a rigid motion or alteration of the size, i.e. if they are similar to each other.

Another possibility to define polyhedra is to *cut* them out of the surrounding space by dividing it by halfspaces and omitting one *half* of the space. The emerging object is called an "$\mathcal{H}$-polyhedron" by Ziegler [169], which is not to be confused with our application of the term polyhedron.

Definition 4.3.5. Let $\mathbf{a} \in \mathbb{R}^d$ and $a_0 \in \mathbb{R}$. A halfspace $H \subset \mathbb{R}^d$ is defined to be

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{a}, \mathbf{x}) \leq a_0\},$$

where $(\mathbf{a}, \mathbf{x})$ denotes the scalar product of $\mathbb{R}^d$. The intersection of a finite set of halfspaces is called an $\mathcal{H}$-polyhedron. A *polytope* is a bounded $\mathcal{H}$-polyhedron, i.e. an $\mathcal{H}$-polyhedron that does not contain a ray.

This alternative definition will become handy for defining the faces of the polytopes. A proof of the equivalence of both notions of $d-$polytopes can be found in [169]. Interestingly, this definition of a polytope reminds us of Plato, who when he wrote that every form must be bounded by plane surfaces, in fact came very close to this modern way of defining a polytope.

Now we have all the necessary notation to define the protagonists of our project:

Definition 4.3.6. A polyhedron is a convex 3-polytope.

Mentioning that the 3-polytope is convex is redundant, since by our definition a polytope is defined to be a convex hull and thus, must necessarily be convex. While we have concentrated on convex polyhedra in the project POLYTOPIA, we will use the non-convex polyhedra later as examples for the considerations on Dürer's conjecture, see Section 4.4.

4.3.2 Vertices, Edges and Faces of Polyhedra

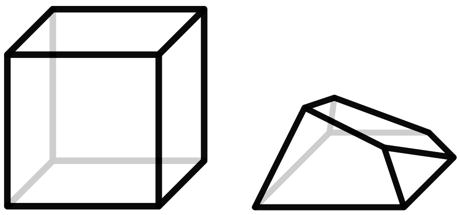
Here we want to define the building blocks of a polyhedron that we used in the intuitive definition: vertices, edges and faces.

Definition 4.3.7. Let P be a polyhedron in $\mathbb{R}^3$. A face of P is any set of the form

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^3 \mid (\mathbf{a}, \mathbf{x}) = a_0\}$$

where $\mathbf{a} \in \mathbb{R}^3$ and $a_0 \in \mathbb{R}$, and $(\mathbf{a}, \mathbf{x}) \leq a_0$ is an inequality that is satisfied for all points $\mathbf{x} \in P$. A k-face is a face of dimension k. The f-vector of P is the vector $f(P) = (f_{-1}, f_0, f_1, f_2, f_3)$, where f_k denotes the number of k-faces of P for $k \geq 0$ and $f_{-1} = 1$.

The f-vectors give us an option to put a classification upon the crowd of polyhedra. In the project POLYTOPIA we sorted them by the number of vertices and then grouped them by having the same f-vector. Two polyhedra having the same f-vector were called "siblings".



(a) The cube. (b) Not the cube.

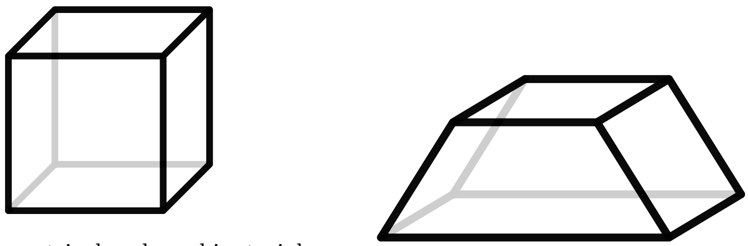
Figure 4.3: The cube and its only sibling. Both polyhedra have the f-vector $(1, 8, 12, 6)$.

4.3.3 Combinatorial Equivalence Classes

To identify the members of a combinatorial equivalence class of polyhedra, we need to distinguish between the geometric realization and the combinatorial type of a polyhedron. From now on, we focus on three-dimensional polytopes and omit the higher dimensional cases. Since the abstract system of edges and vertices of a polyhedron P form an undirected and simple graph $G(P)$, combinatorial equivalence becomes an easily graspable concept:

Definition 4.3.8. Two polyhedra P and Q are *combinatorially equivalent*, if and only if their graphs are isomorphic.

The two polyhedra in Fig. 4.4 belong to the same equivalence class of combinatorially equivalent polyhedra.



(a) A geometrical and combinatorial cube

(b) A combinatorial cube

Figure 4.4

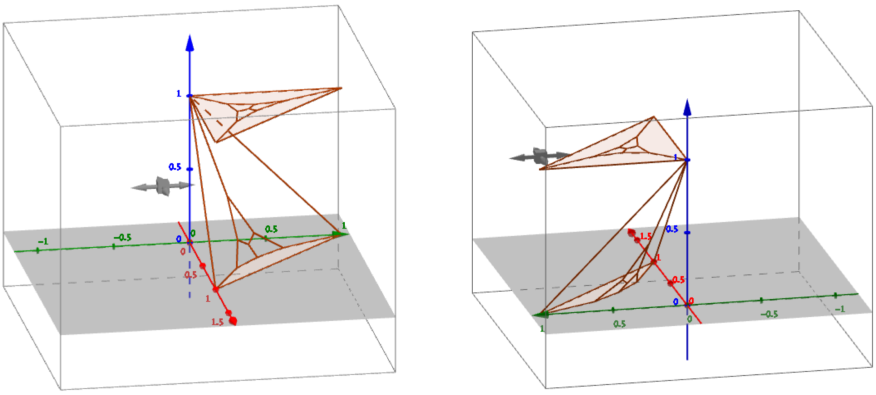
4.3.4 Steinitz' Theorem

There is even more to the connection between graphs and polyhedra. Steinitz' theorem tells us that from every simple, planar, and 3-connected graph, we can derive a polyhedron.

Theorem 4.3.9 (Steinitz' Theorem, [169, Theorem 4.1])**.** *A graph $G(P)$ is the graph of a polyhedron if and only if it is planar, 3-connected and simple.*

Since two polyhedra are combinatorially equivalent when their graphs are isomorphic, Steinitz' Theorem states that every 3-connected, simple, and planar graph corresponds to an equivalence class of combinatorially equivalent polyhedra.

While Steinitz' original proof that can also be found in Ziegler [169], is constructive, it is not very straight forward. Since our approach is of practical nature, as we want to generate a representative for each combinatorial equivalence class, we were looking for constructive proofs that are easier to implement. One version of the proof bases on Maxwell and Tutte and can be found in Richter-Gebert [126]: Let us assume that the graph G is planar, simple and 3-connected. It either contains a triangle, or a vertex with degree three. In the latter case, take the dual of the graph that must contain a triangle, proceed, and after constructing a polyhedron, again take the dual of the polyhedron. The graph is everted in a way that a triangular face is the exterior of the graph and all other faces are on the inside. The outer triangle is *fixed* as a frame of the graph. The interior edges can be imagined as being made from rubber bands that are linked together at the vertices and thus pull at them with even force. This state of the graph is called an *equilibrium state* as the forces level each other out at the vertices. In the next step the vertices are lifted from the plane such that a three-dimensional polyhedron emerges. Figure 4.5 shows an example of a polyhedron that was generated with GeoGebra. This simple geometry software is mostly used for educational purposes and its applicability accounts for the straightforwardness of the method.



(a) (b)

Figure 4.5: Two different perspectives on a polyhedron that has been lifted from a planar simple 3-connected graphs. Note that the lifting occurs downwards, such that the emerging polyhedron is below the graph, which hovers above the polyhedron.

The reason we did not choose this method to generate the polyhedra for the project is purely aesthetical. Since the polyhedron is lifted from a carrier triangle, most of the resulting polyhedra would look like some sort of triangular turtle shell and (combinatorial) symmetries would be hard to spot.

4.3.5 Koebe-Andreev-Thurston Generalization of Steinitz' Theorem

Luckily another constructive proof of Steinitz' theorem gives us a recipe to generate *prettier* polyhedra. The Koebe-Andreev-Thurston theorem is a refinement of Steinitz' theorem and states a specific realization of the polyhedron.

Theorem 4.3.10 (Koebe-Andreev-Thurston Theorem, see [170, Theorem 1.3]). *Each 3-connected planar graph can be realized as a 3-polytope, that has all edges tangent to the unit sphere. Moreover, this realization is unique up to Möbius transformations (projective transformations that fix the sphere).*

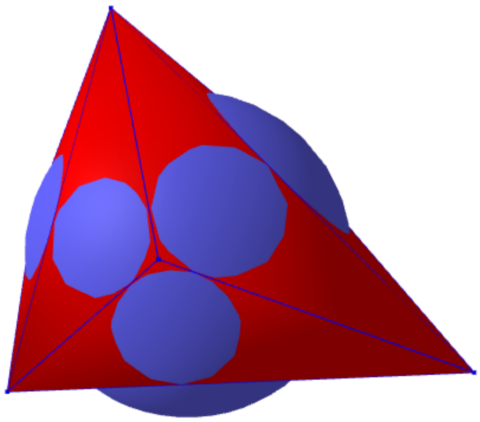


Figure 4.6: Polyhedron *Flensi* with edges tangent to a sphere.

Figure 4.6 shows a polyhedron whose edges are tangent to the *unit* sphere. We call such a polyhedron a *normalized* Koebe-Andreev-Thurston polyhedron. Note that the sphere permeates each face in an inscribed circle.

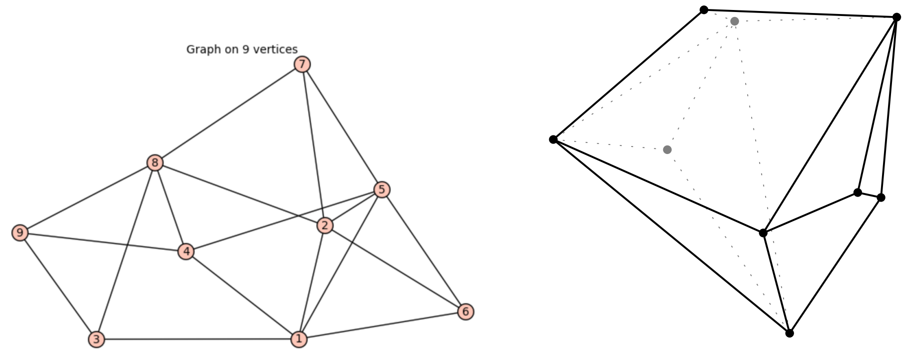
These definitions suffice to accurately define the protagonists of the project. Each "polyhedron" in the project POLYTOPIA is a representative of the class of combinatorially equivalent 3-polytopes. This representative is realized as a normalized Koebe-Andreev-Thurston polyhedron.

4.3.6 Putting it into Practice

In the digital representation of the website, each polyhedron is represented by a collection of data: an id-number, a name, a file for the visualization in the viewer, two crafting sheets to build the models and two files for the 3D printing. Here we want to touch upon the generation of the visualization files and the crafting sheets and exemplify the process of data generation along the polyhedron *Flensi*.

4.3.6.1 Visualization Files

We start our construction with the set of the all planar, simple, and 3-connected graphs with up to nine vertices, see Schaffer [132]. To generate all simple, planar, and 3-connected graphs with a specific number of vertices, we used the software `plantri` [25]. The graph that will later become polyhedron *Flensi* has the number 1566, see Figure 4.7a.



(a) Screenshot of *Flensi*'s simple, planar, and 3-connected graph in `SAGE`.

(b) The normalized Koebe-Andreev-Thurston realization of *Flensi* generated with `SAGE`.

Figure 4.7: The graph and Koebe-Andreev-Thurston realization of a polyhedron

From this graph a normalized Koebe-Andreev-Thurston realization is generated by an implementation[6] of the constructive proof of the Koebe-Andreev-Thurston theorem. The implementation follows Ziegler's exposition [170], which is based on the work of Bobenko and Springborn [23]. Figure 4.7b shows the representation of the normalized Koebe-Andreev-Thurston realization of *Flensi*.

From the representation of the polyhedron in `SAGE`, we derived a `.json` file that can be read by the viewer on the website. Each `.json` file contains the coordinates of the vertices, the edges, and facets that are denoted by the number of the corresponding vertices, see Figure 4.8 to see the file of our polyhedron *Flensi*.

4.3.6.2 Edge Unfoldings and Crafting Sheets

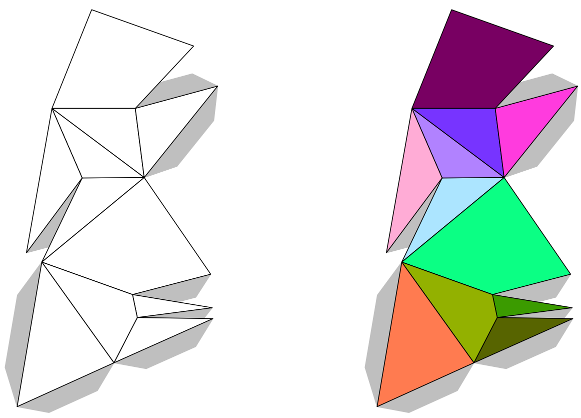
The crafting sheets used on the website are `.pdf` files that contain the net of the polyhedron with added tabs for glueing and the identification number of the polyhedron. Two crafting sheets are

[6]The implementation was done by Moritz Firsching in `SAGE` [150].

```
{
"vertices":
[[1.218763924, -0.03992110173, 0.2283440224],
[-0.02250956707, -1.6999595, -0.4284360003],
[-0.4266654456, 0.8860456128, 0.3778642779],
[-0.7564671372, -0.5217108954, 0.4413478661],
[-0.1438629377, -0.005876334581, 1.504060822],
[-1.31542261, 0.217685123, -0.131576948],
[0.5861189509, -0.1733527814, -0.8305600479],
[0.8330879378, 0.03377811496, -0.5747373192],
[0.3213208619, 1.220735125, -0.8779113015]],
"edges":
[[6, 8], [6, 7], [7, 8], [2, 8], [2, 5], [5, 8], [3, 5], [3, 4], [4, 5],
 [2, 4], [1, 4], [1, 3], [1, 5], [1, 6], [0, 4], [0, 1], [0, 8], [0, 7]],
"facets":
[[8, 6, 7], [8, 2, 5], [5, 3, 4], [5, 2, 4], [4, 1, 3], [5, 1, 3],
[8, 5, 1, 6], [4, 0, 1], [8, 0, 7], [2, 8, 0, 4], [7, 0, 1, 6]],
"colors":
 [null, null, null, null, null, null, null, null, null, null, null]
 }
```


Figure 4.8: The `.json` file for polyhedron *Flensi*.

provided for each polyhedron: a colorful one and a black and white option. Figure 4.9 shows the two crafting sheets for our example polyhedron. Figure 2.9 shows the emerging paper model and an individualized version.



(a) (b)

Figure 4.9: The colorful and black and white crafting sheets of polyhedron *Flensi*.

Descriptively spoken to derive a net of a (hollow) polyhedron, one has to cut it open along the edges in a way that allows for all the faces to fold flat upon a plane. To distinguish an unfolding that emerges from cutting along the edges rather than freely cutting through the faces, we will call the former an *edge unfolding*. If an unfolding is not overlapping, we will call it *simple*. Whether every polyhedron possesses such a simple net, is an open question that will be discussed in Section 4.4. Note that we will use the terms *net* and *unfolding* synonymously.

Definition 4.3.11 (Spanning Tree)**.** The spanning tree of the vertices of a polyhedron is a connected subgraph T of the graph G of the polyhedron (the one skeleton) that satisfies the following conditions:

- $V(T) = V(G)$, i.e. T and G have the same vertices,
- T has no cycles

Definition 4.3.12 (Cut Tree)**.** The spanning tree that is used for cutting a polyhedron in order to get an unfolding is called a *cut tree*. If this unfolding is simple, the cut tree is called a *simple cut tree*.

In the following we will describe the algorithm that generates the unfolding of a polyhedron.

Algorithm 4.3.13. Let P be a polyhedron and denote by T the spanning tree of its faces, i.e. the spanning tree of the vertices of the dual polyhedron. Note that the vertices of the faces must be sorted in such a way that they are cyclical and positively oriented with respect to the interior of the polyhedron.

1. Walking along the edges of the spanning tree in a depth-first search, we generate a list L of pairs $[F_i, F_j]$ of faces such that F_i is nearer to the root for every edge.
2. Then we rotate and transpose P such that the root face is located at the xy-plane and its first vertex is at the origin.
3. We take the first (next) pair from the list L.
4. The first face F_i of the pair is already located at the xy-plane. F_j is adjacent to F_i. We identify their two common vertices v and w at the connecting edge.

5. The face F_j is a 2-polytope in $\mathbb{R}^3$. By affinely and orthonormally mapping the face into its ambient space, we get a 2-dimensional representation $\bar{F}_j$ of F_j that is preserving the shape and size of F_j.

6. We now transpose $\bar{F}_j$ such that the location of vertices v and w remains identical.

Now repeat Steps 3 to 6 for all the pairs in the list.

4.4 Dürer's Conjecture

The crafting sheets for polyhedra that we create for the project Polytopia are visualizations of the unfoldings of the polyhedra on the plane. They can be cut out and glued together along the tabs in order to built a three-dimensional model. While doing this, we implicitly assume that every polyhedron has an unfolding whose faces do not overlap. The algorithm we used for generating the nets did not test for that[7].

The underlying question whether every geometrical type of polyhedron can in be developed onto the plane without any overlaps of the faces was discussed on many occasions [35, 117, 169]. This question remained open for a long time and mainly computational experiments lead the way to finding evidence for and against the conjecture. It is still open in its general form, but Ghomi has shown in 2014 that at least for each combinatorial type of polyhedron a realization with a simple unfolding can be found.

In the following, we will present the main developments that have been made in investigating Dürer's conjecture. First, we will introduce the problem and some related questions. Then, we will look into the computational experiments that have been conducted. Next, we will present some theoretical work together with Ghomi's recent result. The section closes with presenting some progress that has been made on related problems and the higher dimensional generalization of Dürer's conjecture.

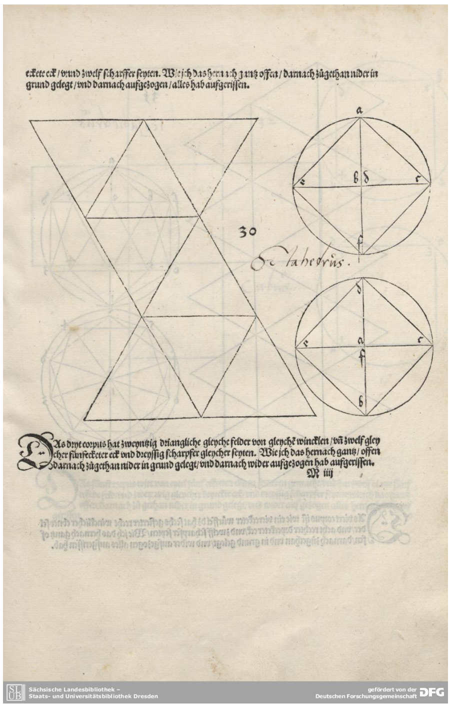
4.4.1 Dürer Invents the Edge Unfolding

With the discovery of drawing perspective in the end of the fifteenth century, new possibilities to represent space were assessable. The invention of polyhedral nets, which is attributed to the painter and mathematician Albrecht Dürer takes this development even further [51]. In his book *Underweysung der Messung, mit dem Zirckel und Richtscheyt, in Linien, Ebenen unnd gantzen corporen* [40] we find illustrations of polyhedra and their nets that date back to the year 1525. These nets are all simple, i.e. their faces do not overlap each other. Dürer's choice of words (e.g. "aufreissen", "aufthun", or "zusam leget") in the description of the nets and the polyhedra, suggests that he interprets nets as a process of unfolding, i.e. cutting a polyhedron open and

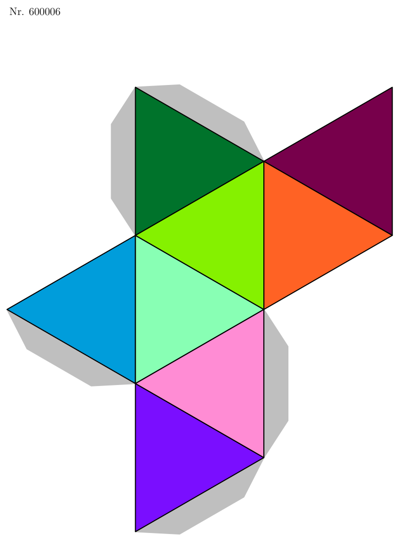
[7] To ensure all unfoldings on the crafting sheets are simple, we waited for the users to complain about overlap.

flattening it to the plane, as well as folding, i.e. glueing a net in order to yield a three dimensional body [51].

Figure 4.10a shows the historic print of an octahedron and its unfolding by Dürer, Figure 4.10b depicts the crafting sheet that we used for Polytopia.



(a) The octahedron and its unfolding by Dürer [40].



(b) The crafting sheet of the octahedron in Polytopia [142].

Figure 4.10

4.4.2 Shephard's Problem

The often cited question

> (A) Does every 3-polytope possess a simple unfolding?

was formulated only in 1975 by Shephard [144] who was more interested in investigating whether a polyhedron has a combinatorial equivalent that can be unfolded into a net which is a convex polygon itself, i.e. a *convex net*. His main result is to name classes of polyhedra, namely pyramids, bipyramids, prisms, antiprisms, cyclic polytopes and wedges that always have a combinatorial equivalent which is unfoldable into a convex net but he also gives a counterexample for a polyhedron with 17 vertices that does not satisfy this property.

Shephard also touches upon the question whether a net uniquely results in a specific polyhedron when assembled. This depends on two specifications. If the location of the interior edges are specified and also a rule for the assembly of the outer edges is given, then the resulting polyhedron is unique. As Shephard points out, and we will see in Section 4.4.6.4, for a convex polyhedron the specification of the interior edges is not even necessary. Just a rule for how to glue the boundary together suffices for uniqueness. Furthermore, Shephard shows that the specification of the inner folds is not sufficient and presents an example of a net that can be folded into either an octahedron or into a tetrahedron stacked on two of the triangular faces. Figure 4.11a shows an octahedron and Figure 4.11b a tetrahedron that is stacked on two of the triangular faces, where both polyhedra have the same net, see Figure 4.11c.

4.4.3 Grünbaum's Questions

Motivated by the absence of a counterexample, Grünbaum conjectured a positive answer for Dürer's Problem in the special case of convex polyhedra. [65]. His second, weaker conjecture:

> (B) Every convex polyhedron is combinatorially equivalent to a polyhedron that has a net [65].

was recently proven right by Ghomi [56]. We'll get back to this later in this section. Note that in Grünbaum's terminology a net is a simple unfolding.

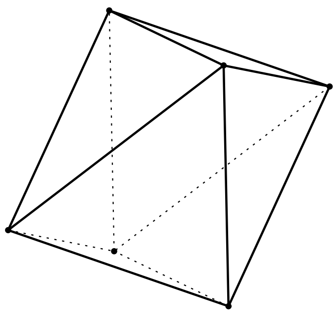
But Grünbaum also posed a stronger problem:

> (C) Does every spanning tree of an arbitrary polyhedron lead to its net [64]?

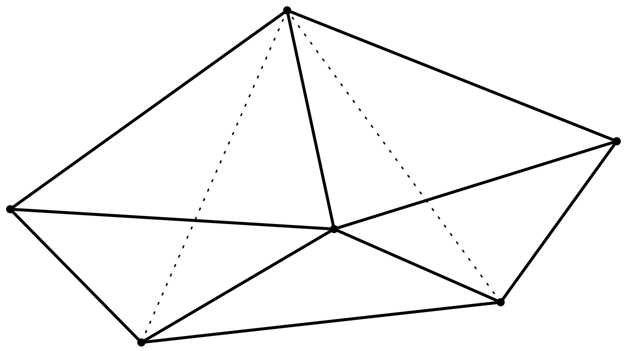
He directly falsified this claim by giving a counterexample.

In 2011, Horiyama and Shoji showed that the Platonic Solids develop in a simple unfolding for all possible spanning trees. The exhaustive computational proof calculated and tested the unfoldings for all $5,184,000$ spanning trees of the dodecahedron and icosahedron, as well as the 384 for the octahedron and the cube and the 16 possibilities for the tetrahedron [79].

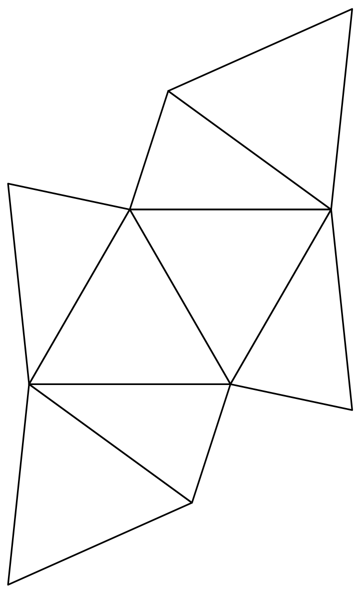
Grünbaum stated that every polyhedron combinatorially equivalent to a 3-sided prism or the tetrahedron has no overlapping unfoldings [64]. However, both statements do not hold. Namiki



(a) Octahedron that is a bipyramid on a rectangle with a golden ratio.



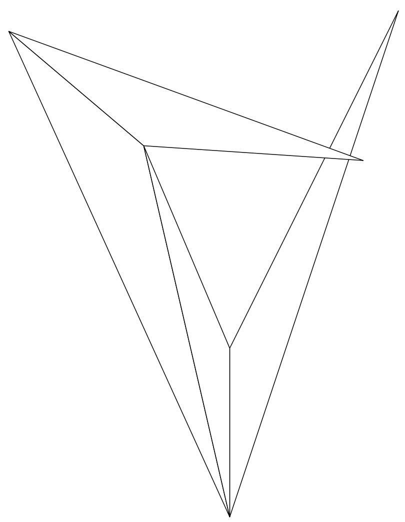
(b) Stacked tetrahedron.



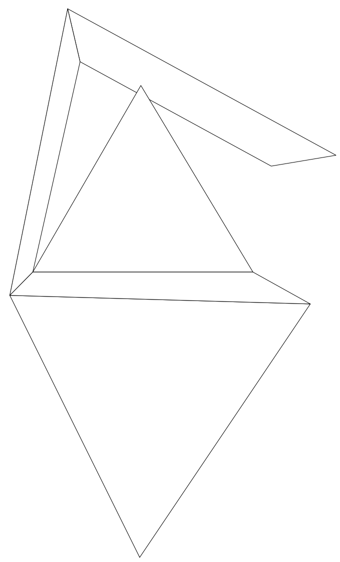
(c) The net of both polyhedra.

Figure 4.11: Both polyhedra were realized by Hafner [69].

namely constructed a tetrahedron as a minimal example for an overlapping unfolding, see Figure 4.12a. The overlapping triangular prism is taken from Ghomi [57], see Figure 4.12b.



(a) Self-intersecting net of a tetrahedron.



(b) Self-intersecting net of a prism.

Figure 4.12

4.4.4 Computational Experiments

Approaching the problem empirically by computer suggests itself since the problem has a very hands-on and illustrative nature. As the spanning tree determines the edge unfolding of a polyhedron, getting a grip on Dürer's conjecture became a quest for finding a suitable spanning tree that results in an overlap free unfolding.

4.4.4.1 Random Unfoldings – Schevon

In 1989, Schevon [134] conducted statistical experiments on randomly generated spherical and cubical polyhedra and unfolded them by also randomly generated spanning trees. It turned out that with a growing number of vertices the probability of overlap quickly approached 1. She concluded from her experiment that while each polyhedron may have *a* simple unfolding, most unfoldings of a polyhedron do overlap.

4.4.4.2 Attempts to Characterize the Simple Cut Tree – Fukuda

According to Schevon's experiments, applying randomly chosen spanning trees seemed to be an unfruitful approach. Thus, the next guess was that a certain type of spanning trees would result in a simple unfolding. Since the number of spanning trees grows rapidly when increasing the number of vertices, the desired spanning tree that would result in a simple unfolding does not carry enough weight in the random experiment.

There are various ways to compute a spanning tree for a graph. Here we present two such method that were suspected to always yield a simple unfolding.

Algorithm 4.4.1. To generate a minimal length tree for the graph of a polyhedron P repeat the following step as often as possible: Among the edges of G not yet chosen, choose the shortest edge, with respect to the length $l(e) :=\| e \|$ which does not form any loops with those edges already chosen [84].

Fukuda suggested that for all polyhedra, even those that do not contain a spanning-star, the minimal-length spanning tree would result in a simple unfolding. In 1997 he presented Dürer's conjecture in a workshop in Dagstuhl and suggested that each minimal length tree results in a simple unfolding. Rote however, found a counterexample to this conjecture [53, 83].

Lemma 4.4.2 (Spanning Star Lemma, [133])**.** *If the 1-skeleton [i.e. the graph] of a polyhedron contains a spanning star, i.e. one vertex is adjacent to all the others, then the unfolding using the spanning star as the cut tree is always simple.*

Fukuda proceeded to conjecture that creating an unfolding by cutting along a shortest-path spanning tree always results in a simple unfolding [53, 83]. This might be understood as a generalization of Schevon's spanning star lemma.

Algorithm 4.4.3. Let $v \in V(P)$ be the root vertex of a shortest path tree of a polyhedron P. For every other vertex $w \in V(P)\backslash v$ find a shortest possible path with respect to the length $l(e) :=\| e \|$ for every $e \in E(P)$, traveling only along the edges $E(P)$.

4.4.4.3 Systematic Approach – Schlickenrieder

Fukuda's second conjecture also did not stand for long. In the same year Schlickenrieder investigated the relation between a selection of spanning trees and their resulting unfoldings in a systematic computational experiment. He implemented 34 different rules to compute a cut tree and tested each of these rules on about 10.000 polyhedra of various types. Among the 34 spanning tree rules was also the shortest-path rule, which Fukuda conjectured to always result in a simple cut tree. Schlickenrieder implemented eight variations of this rule which chose different roots and weights to compute the shortest path tree. All eight versions failed to deliver simple unfoldings for all the tested polyhedra. The results of non-simple unfoldings varied between 2.1% and 19.2% for the versions of the shortest-path rule [136].

None of the 34 tested spanning tree rules was successful for *every* polyhedron in the test sample. Nevertheless, for *each* of the tested polyhedra, the experiment could find at least *one* spanning tree rule that resulted in a simple unfolding.

However, the most promising spanning tree rule was the *steepest-edge unfolding.* It is motivated by the intuition of peeling an orange and thus, cutting as straight as possible starting from a base. "Straightness" is translated to "steepness" with respect to a direction $c \in \mathbb{R}^3$. The direction c must lie in general position with respect to P, i.e. the height funtion $h(\cdot) = \langle \cdot, c \rangle$ has a unique minimizer v_{min} and maximizer v_{max} in $V(P)$. To construct a steepest edge tree, for each $v \in V(P)$ one has to find the steepest possible ascending path to the maximizer v_{max}.

Only a small percentage of the polyhedra from the sample, namely 1.2%, had no simple unfolding for a fixed direction c. This number remained relatively stable when choosing a different vector c. For each polyhedron of the sample, a vector c could be found, such that the steepest-edge rule with respect to the vector c yields a simple cut tree. Hence, Schlickenrieder conjectured that for every polyhedron a direction $c \in \mathbb{R}^3$ exists, such that the spanning tree that results from the steepest edge algorithm yields a simple unfolding.

4.4.4.4 Finding more Counterexamples – Lucier

A counterexample to Schlickenrieder's conjecture was presented by Lucier in 2007. Even though the construction is based on theoretical argument and not computation, we put it in this section, because it directly refers to the above claims and additionally gives another counterexample to Fukuda's conjecture for the non-overlapping unfoldings stemming from a shortest-path cut tree.

Lucier scrutinized possible overlaps that may occur in a non-simple unfolding in a local environment, which can be embedded into the faces of a polyhedron. Using locality, Lucier proves that a polyhedron can be constructed such that it results in an overlapping unfolding for possible steepest-edge unfoldings with respect to any possible direction $c \in \mathbb{R}^3$ [96].

Lucier claims that the class of normal order cut trees form a generalization of the steepest order cut trees and proves that even for this class of spanning trees, a counterexample can be constructed. The motivation for the normal order cut trees, reads: "As an informal intuition, the success of these [steepest-edge] unfoldings appears to derive from their tendency to 'expand outward' from a central point. It seems natural [...] that such unfoldings would have a high probability of being simple. [...] A motivation for the research presented in this paper was whether a simple normal order unfolding exists for every polyhedron. Unfortunately, despite our intuition, we shall now prove that this is not the case." [96]

4.4.5 Theoretical Work

Up to now we dealt with computational experiments in order to approach Dürer's conjecture. In this section we want to take a look at the theoretical progress that has been made. The first step was made by an undergraduate student who proved the existence of simple edge unfoldings for all simplicial polyhedra with up to six vertices by a studious case discrimination.

4.4.5.1 First Steps – DiBiase

A first theoretical approach at investigating Shephard's question was undertaken by DiBiase. The work proves that all simplicial polyhedra with up to six vertices have a simple edge unfolding. The case discrimination considers all possible planar maps of the combinatorial types of simplicial polyhedra with four, five, or six vertices and examines each case in detail. Since the method of case discrimination is becoming more and more complex as the number of vertices increase, it is not promising to try and generalize DiBiase's approach for polyhedra with more that six vertices [38].

4.4.5.2 Combinatorial Result – Ghomi

In their book "Unsolved Problems in Geometry", Croft, Falconer and Guy [33] have dedicated a section to Dürer's conjecture and related questions. One question that is directly derived from the question whether every polyhedron possess a simple unfolding, is whether each combinatorial type of polyhedron can be unfolded in a non-overlapping way. Ghomi [56] was able to verify in

2014 that every polyhedron can be affinely transformed and then unfolded in a simple net. The guiding principle is that the polyhedron is transformed to a slim and needle-shaped form and then cut lengthwise. The intuition is lead by the image of a banana that is peeled lengthwise and the strands of the peel spread out in a circular fashion.

In mathematical terms, firstly a stretching is performed on a polyhedron P in general position. Then for a sufficenty large λ, applying the affine transformation defined by:

$$A_\lambda(p) := \frac{1}{\lambda}(p + (\lambda - 1)\langle p, u\rangle u),$$

results in a needle shaped polyhedron P that peaks in the direction u. Now we can find a monotone spanning tree that has a root in the minimizing vertex and is strictly monotonly rising along its simple path towards the leafs. Cutting the polyhedron open along this trees results in a simple unfolding [56].

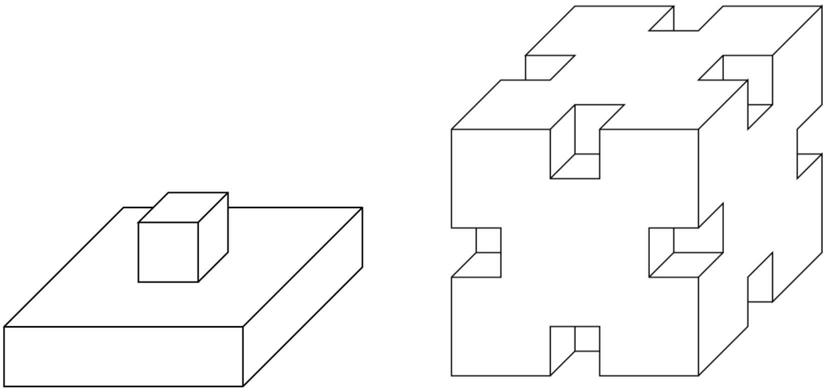
Sert and Zamora [141] picked up Ghomi's idea of streching the polyhedra and were able to simplify his proof and shorten it considerably. While Ghomi is relying on topological methods, they employ only elementary linear algebra in their straightforward proof.

4.4.6 Related Problems

The third part of this section about Dürer's conjecture is dedicated to related problems, such as the existence of simple unfoldings for non-convex polyhedra, cutting across faces and not just along the edges. We sum up some results on the inverted question, namely whether a given polygon can be folded into a polyhedron by joining its boundary and finish with a short glimpse into the higher dimensional generalization of Dürer's conjecture.

4.4.6.1 Non-convex Polyhedra

In Biedl et al. [21] a non-convex polyhedron that did not result in a simple unfolding was introduced. The polyhedra in Figure 4.14 are named *ortho-stacks*, since they are stacked from cubes and cubical elements. Looking at the polyhedron in Figure 4.13a, one can easily see that the small cube that is located on top of the bigger cuboid, cannot unfold within the annulus of the face it is attached to. A less trivial example that does only allow simply connected faces is a cube with little indentations at the center of the edges. All faces are simply connected. For two neighboring big faces two possibilities occur. Either they are still connected, which results in a lack of space for an unfolding of the indentation, or if they are separated and the indentation is fully unfolded the big faces overlap. Thus, no simple unfolding for this polyhedron exists [21].



(a) (b)

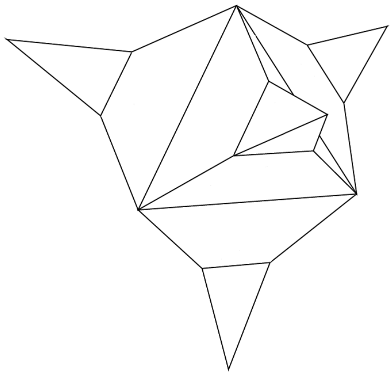
Figure 4.13: Two non-convex polyhedra with have no simple unfolding. Figure taken from [21].

When we look at the graph of the ortho-stack polyhedron in Figure 4.13b, we see that along the indentations it is not 3-connected. Thus, the graph does not satisfy Steinitz' theorem and the polyhedron is not topological convex, i.e. its graph is not isomorphic to a convex polyhedron [20].

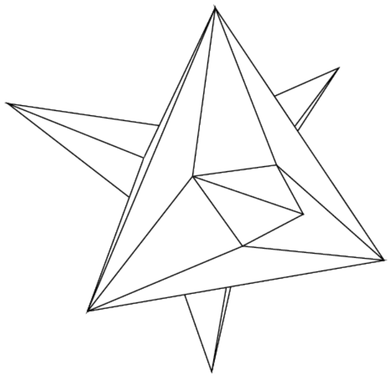
Bern et al. [20] settle the questions whether a non-convex, topologically convex polyhedron and a non-convex simplicial polyhedron can have no simple unfolding, by constructing two examples. The polyhedron in Figure 4.14a is constructed from four *basic hats* that cannot be simply unfolded. Twisting the spike in the middle of the hat by $60°$ results in a reverse orientation in comparison to the boundary and "breaks" the trapezoids into triangles, see Figure 4.14b. Thus, the emerging polyhedron is simplicial. Both of these polyhedra with 24 and 36 faces, respectively, do not possess a simple egde unfolding. Nevertheless, if cutting across faces is allowed, their boundary can be isometrically embedded into the plane. Thus, edge unfolding is a stronger restriction than general unfolding [20].

4.4.6.2 General Unfolding

In contrast to the edge unfolding, a general unfolding allows cutting across the faces, but still respects the vertices of the polyhedron as branches of the cut tree. Two methods that always



(a) A non-convex, topologically convex polyhedron that does not possess a simple unfolding.



(b) A non-convex, simplicial polyhedron that does not possess a simple unfolding.

Figure 4.14: Both figures taken from [20].

result in simple unfoldings are known: the *star unfolding* and the *source unfolding* [35]. The two are closely related and we will introduce the method of star-unfolding here. Let x be a point on a polyhedron P that has a unique shortest path to every vertex of P, where the shortest path is measured on the surface of P. Cutting along these shortest lines results in a simple unfolding for the polyhedron P. The star unfolding was invented by Alexandrov in 1950 [10] and its property to be simple was proved by Aronov and O'Rourke in 1992 [12]. See Figure 4.15 for an example of a net that emerges from a star unfolding of a triangular prism. Note that in this example, the point x is located in the middle of one of the quadrilateral faces of the prims.

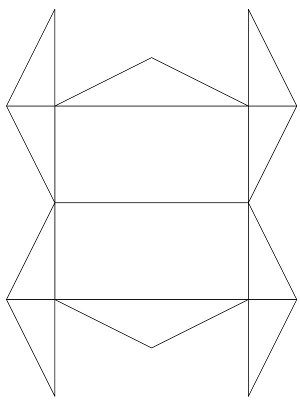


Figure 4.15: Star unfolding of a triangular prism.

4.4.6.3 Pseudo-Edge Unfolding by Ghomi and Barvinok

Ghomi and Barvinok [16] point out that the concept of an edge is not well understood "from the point of view of isometric embeddings". Thus, they investigate unfoldings with respect to *pseudo-edges*. A *pseudo-edge* connects two vertices of a polyhedron as a distance minimizing geodesic, as does an edge. The actual edges also *frame* the faces of the polyhedron, a characteristic not shared by the pseudo-edges. An unfolding that results from cutting along the pseudo-edges thus might cut across faces [16].

Definition 4.4.4. Let $P \in \mathbb{R}^3$ be a convex polyhedron. For two distinct points x, y on the boundary of P consider all shortest paths between them traveling along the surface. A *geodesic* is a path between x and y on the surface of P that is at least locally a shortest path.

Ghomi and Barvinok constructed so-called "almost flat convex caps" over an equilateral triangle and applied Tarasov's unfoldability criterion for pseudo-edge graphs [148]. The caps consist of 46 vertices each and are assembled to the faces of a tetrahedron. The resulting polyhedron proves to not be unfoldable in a non-overlapping way with respect to a pseudo-edge graph.

Since edges share many metrical characteristics with pseudo-edges, the counterexample they found for a pseudo-edge graph is an evidence but not a proof for the negation of Dürer's conjecture [16, 57]. The strength of this evidence is uncertain since the authors claim themselves that "edges do indeed exist intrinsically", as Alexandrov's theorem, see Theorem 4.4.5 below, and especially its constructive proof by Bobenko and Izmestiev [22] demonstrate. They just are not well understood at the current moment.

4.4.6.4 The Inverted Problem: Is a Polygon a Net of a Polyhedron?

If a simple edge-unfolding of a polyhedron exists, it is a simply connected non-overlapping polygon. Thus, the question whether a simple polygon is the net of polyhedron arises: Given a bounded polygon G embedded in the plane, can we find an allocation, i.e. glueing rule, for the boundary of the polygon that results in a polyhedron?

An existential answer to this question is given by Alexandrov's theorem. For polygon G, two conditions are required to establish an *Alexandrov glueing* [35], namely:

- "The positive curvature condition": for each vertex of the development [polygon], the sum of the angles glued together at this vertex must be at most 2π.

- "The Euler condition": if f, e, and v denote the number of faces, edges and vertices in a development [polygon], then the equality $f - e + v = 2$ must hold.

Then Alexandrov's theorem reads:

Theorem 4.4.5 (Alexandrov's Theorem, [Section 2, Theorem 1] [10])**.** *The stated conditions are not only necessary but also sufficient for a given development [polygon] to define a closed convex polyhedron by gluing. Moreover, there may be only two such polyhedra: one is the mirror image of the other, or, which is the same, one of them is the other "inside-out".*

Note that in this formulation a doubly covered polygon which for example emerges from folding a square at its diagonal, also counts as a polyhedron [10].

We can easily see that these conditions are always fulfilled for convex polygons P_{conv} when applying a semiperimeter-glueing: Choose a random point x at the boundary of P_{conv} and "pinch" the polyhedron together at x. From there glue all the opposing points at the boundary left and right of x together. Eventually you will reach the point y that is on the other side of the perimeter of x and "zip up" the boundary there. This gluing rule suffices both of Alexandrov's conditions. Since the polygon is convex, all the angles at the points of the boundary do not exceed π. Hence, the sum of two angles is not bigger that 2π [116], [35].

The original proof by Alexandrov only proves existence. Thus, it does not contain any information about the shape or type of the emerging convex polyhedron, nor does it give any conditions for the shape of the polygon that help us decide whether it folds into a convex polyhedron or not.

Croft, Falconer and Guy [33] have expressed the wish for "an algorithm to determine whether or not a given configuration of polygons is a net of some convex polyhedron" while hinting at Alexandrov's intrinsic metric. This approach was taken up by Bobenko and Izmestiev in 2008, who constructively proved Alexandrov's theorem and also provided a numerical algorithm to determine the convex polyhedron from a given polygon [22].

Alexandrov's theorem and the construction algorithm concern only convex polyhedra. Burago and Zalgaller answered this question in 1996 and showed that every simple polygon is an unfolding of a (mostly non-convex) non-selfintersecting polyhedron [28]. O'Rourke points out that this result was already published in 1960, but this paper is available only in Russian [115].

4.4.6.5 Higher Dimensions

The problem of polyhedral unfoldings can be generalized to higher dimensional polytopes as well. The cutting in dimension d occurs along $(d-1)$-dimensional facets of the polytope. An example can be seen in Dali's famous painting "Crucifixion (Corpus Hypercubus)". It depicts the body of Christ being crucified to a three dimensional unfolding of a four-dimensional hypercube. This unfolding occurs without intersection (i.e. overlap) of the cubes.

Little research has been done on higher dimensional unfoldings [35, 108]. Miller and Pak [108] investigated the interaction between the combinatorial structures of polytopes and their metric characteristic and called it the *metric combinatorics* of a polytope. Additionally they could prove that the higher dimensional generalization of the source unfolding always results in a non-overlapping unfolding.

Conclusion

This thesis studies mathematical science communication. The case study Polytopia – *Adopt a Polyhedron* is at the center of this study and builds the backdrop of all four areas of research that are touched upon in this interdisciplinary work.

A credible science communication project in mathematics needs a strong foundation in the science of mathematics. In our case this was the theory of polyhedra. After elaborating on the history of the term *polyhedron*, we describe our own use of this notion. For the website data for all combinatorial types of polyhedra with up to nine vertices was generated. The following digression into Dürer's conjecture shows the link between the playful and creative approach of our project and active mathematical research.

The project's link to mathematics education is twofold. Adapting the school materials to the curriculum in such a way that the students can practice tasks of curricular mathematics while glimpsing into scientific mathematics ensures the inclusion of learners at all skill levels. Borrowing from modern didactical principles for the overall design of the project shows the possibilities for informal learning to learn from formal didactics. We believe that knowledge transfer is a central part of the mandate of (mathematical) science communication. Thus, bringing together mathematics education and mathematical science communication can create vibrant formats of knowledge transfer.

The practical realization of the project involved many decisions on the design, the structure of the website, and its content. Besides reflecting on the process of execution, evaluation is an essential part of the documentation, as connections can be drawn, problems diagnosed and needs identified. Unfortunately, no methodology for measuring success of science communication projects is established yet. Therefore, the chosen evaluation approach must be understood as an experiment that builds the basis for further development.

The formalization of the objectives, methods and motivations of mathematical science communication is aligned with the development in the discourse of general science communication. Analysis of literature demonstrates that some steps in the development of mathematical science communication are still missing. It is important to emphasize that modern formats in this area do exist, but it is the discussion about them that is lacking precise formulations. If mathematical science communication wants to be consistent with its foundational science that lives by a very active scholarly exchange, it is essential to open up a lively dialogue about the projects and begin to negotiate on a terminology that identify its objectives, methods and motivations. Another important point in this dialogue is the clarification of the societal role of mathematical science communication. The economic argument of cultivating a mathematically and therefore technically skilled next generation workforce can be augmented with a wider view of mathematical literacy that for example includes the ability of orientation in an digitalized world that is increasingly based upon algorithmically assessed data.

The experimental approach turned out to be very fruitful because the process of bringing a project into a specific form results in many questions. Some of those are easy to answer, others require some research, whereas further questions reach far into the philosophy of science, or the political and social environment.

Still, there is an enormous potential for progress in filling the theoretical vacuum of the practice of mathematical science communication. The entire spectrum, with science on one end and the public on the other, should be subjected to the investigation for which the following prompts and questions can serve as impulses:

- Classification and comparison of projects of mathematical science communication. Collection of the methods, goals and motivations.
- Analysis of the constituency of mathematical literacy and its subsumable concepts like critical mathematical literacy or algorithmic literacy. What is their relevance for the individual beyond formal education?
- Development of methods for evaluation of science communication projects and identifying the peculiarities of mathematics.
- What constitutes to the *negative image* of mathematics? What does *negative* mean in this context? What public image do mathematicians wish to project?

- Advancing the development of mathematical citizen science, finding suitable research questions and implementing the collaborative process in a way that invites citizens and honors the scientific rigor.
- Does the practice of mathematical science communication and its particular projects further encourage existing structures of inequality?

This thesis is a first step towards studying mathematical science communication as an independent scientific field. The area of study is located between mathematics and its education, sociology, communication science, and journalism. For further understanding of the foundations of mathematical science communication it is necessary to thoroughly study more than one project and conduct a comparative analysis of many different projects in the field. Mathematical science communication is the practice that actively shapes the exchange between mathematics and the public. Therefore, we believe that the further study of mathematical science communication deserves attention.

Bibliography

[1] A-FRAME, A web framework for building virtual reality experiences. https://aframe.io/. Accessed: 2020-05-13.

[2] Adopt a Revolution. https://www.adoptrevolution.org/. Accessed: 2019-01-22.

[3] Adopt a Star, White Dwarf Research Corporation. https://adoptastar.org/. Accessed: 2020-06-08.

[4] Google Fonts: Amatic SC. https://fonts.google.com/specimen/Amatic+SC. Accessed: 2020-02-12.

[5] Google Fonts: Lato. https://fonts.google.com/specimen/Lato. Accessed: 2020-02-12.

[6] Mathoverflow. https://mathoverflow.net. Accessed: 2020-07-16.

[7] Polyhedron *Flensi*. https://www.polytopia.eu/en/detailansicht?id=901566. Accessed: 2020-05-13.

[8] Polymath1. http://michaelnielsen.org/polymath1/index.php?title=Polymath1. Accessed: 2019-04-29.

[9] The On-Line Encyclopedia of Integer Sequences, Sequence A000944. https://oeis.org/A000944. Accessed: 2018-02-28.

[10] Aleksandr D. Alexandrov. *Convex Polyhedra.* Springer Science & Business Media, 2005.

[11] Richard D. Archer-Hind. *The Timaeus of Plato.* C.J. Clay, M.A. and Sons, University Press, Cambridge, 1888. Reprinted in 1973 by Ayer Company Publshers, INC. in Salem, New Hampshire.

[12] Boris Aronov and Joseph O'Rourke. Nonoverlap of the star unfolding. *Discrete & Computational Geometry*, 8(3):219–250, Sep 1992.

[13] Sheldon Axler. *Linear Algebra Done Right.* Springer, 2015.

[14] Ayelet Baram-Tsabari and Jonathan Osborne. Bridging science education and science communication research. *Journal of Research in Science Teaching*, 52(2):135–144, 2015.

[15] John D. Barrow and Robin Wilson. Raising public awareness in the UK – Some snapshots. In *Raising Public Awareness of Mathematics*, pages 3–9. Springer, 2012.

[16] Nicholas Barvinok and Mohammad Ghomi. Pseudo-edge unfoldings of convex polyhedra. *Discrete & Computational Geometry*, pages 1–19, 2017.

[17] Martin W. Bauer. The evolution of public understanding of science – discourse and comparative evidence. *Science, Technology and Society*, 14(2):221–240, 2009.

[18] Martin W. Bauer. The knowledge society favours science communication, but puts science journalism into a clinch. *Science Communication Today, Paris, CNRS*, pages 145–166, 2013.

[19] Ehrhard Behrends, Nuno Crato, and José Francisco Rodrigues. *Raising Public Awareness of Mathematics*. Springer, 2012.

[20] Marshall Bern, Erik D. Demaine, David Eppstein, Eric Kuo, Andrea Mantler, and Jack Snoeyink. Ununfoldable polyhedra with convex faces. *Computational Geometry*, 24(2):51–62, 2003.

[21] Therese Biedl, Erik Demaine, Martin Demaine, Anna Lubiw, Mark Overmars, Joseph O'Rourke, Steven Robbins, and Sue Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proc. 10th Canad. Conf. Comput. Geom.*, 1 1998.

[22] Alexander I. Bobenko and Ivan Izmestiev. Alexandrov's theorem, weighted delaunay triangulations, and mixed volumes. In *Annales de l'Institut Fourier*, volume 58, pages 447–505, 2008.

[23] Alexander I. Bobenko and Boris Springborn. Variational principles for circle patterns and Koebe's theorem. *Transactions of the American Mathematical Society*, 356(2):659–689, 2004.

[24] Aletta Bonn, Anett Richter, Kathrin Vohland, Lisa Pettibone, Miriam Brandt, Reinart Feldmann, Claudia Goebel, Christiane Grefe, Susanne Hecker, Leonhard Hennen, et al. Green Paper Citizen Science Strategy 2020 for Germany. 2017. Accessed: 2020-07-16.

[25] Gunnar Brinkmann and Brendan McKay. *plantri (Version 5.0; January 30, 2018)*, 2018. https://users.cecs.anu.edu.au/~bdm/plantri/.

[26] Franka M. Brueckler. The maximin principle in the popularisation of mathematics: Maximum effect with minimum costs. In *Raising Public Awareness of Mathematics*, pages 197–214. Springer, 2012.

[27] Thomas F. Bruss. Image, influence and importance of mathematics as directives for public awareness. In *Raising Public Awareness of Mathematics*, pages 297–311. Springer, 2012.

[28] Yurii D. Burago and Victor A. Zalgaller. Isometric piecewise linear immersions of two-dimensional manifolds with polyhedral metrics into $\mathbb{R}^3$. *St Petersburg Mathematical Journal*, 7(3):369–386, 1996.

[29] Terry W. Burns, John D. O'Connor, and Susan M. Stocklmayer. Science communication: a contemporary definition. *Public understanding of science*, 12(2):183–202, 2003.

[30] Mireille Chaleyat-Maurel. WMY 2000: Ten Years on? In *Raising Public Awareness of Mathematics*, pages 215–228. Springer, 2012.

[31] Collective Artists. Citizen Art. http://gallery.bridgesmathart.org/exhibitions/2019-bridges-conference/anna-hartkopf. Accessed: 2020-05-29.

[32] Harold S.M. Coxeter. *Regular Polytopes.* Macmillan New York, 1963.

[33] Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy. *Unsolved problems in geometry*, volume 2. Springer Science & Business Media, 1991. 2nd printing, 1994.

[34] Simone de Beauvoir. *Le deuxième sexe.* Paris, Éditions Gallimard, 1949.

[35] Erik D. Demaine and Joseph O'Rourke. *Geometric folding algorithms: linkages, origami, polyhedra.* Cambridge University Press, 2007.

[36] Deutsche Akademie der Naturforscher Leopoldina e.V. – Nationale Akademie der Wissenschaften, acatech – Deutsche Akademie der Technikwissenschaften e.V. und Union der deutschen Akademien der Wissenschaften e.V. Zur Gestaltung der Kommunikation zwischen Wissenschaft, Öffentlichkeit und den Medien. *Schriftenreihe zur wissenschaftlichen Politikberatung*, 2014.

[37] John Dewey. *Logic, The Theory of Inquiry.* Henry Holt and Company, Inc., 1938.

[38] Julie DiBiase. *Polytope unfolding.* Bachelor thesis, Smith College, Northampton, Mass., 1990.

[39] John R. Durant, Geoffrey A. Evans, and Geoffrey P. Thomas. The public understanding of science. *Nature*, 340(6228):11, 1989.

[40] Albrecht Dürer. *Unterweysung der Messung mit dem Zyrkel und Richtscheydt.* Hieronymus Andreae, Nürnberg, 1525. SLUB Dresden / Digitale Sammlungen / S.B.616.

[41] Albrecht Dürer and Walter S. Strauss. *The painter's manual: A manual of measurement of lines, areas, and solids by means of compass and ruler assembled by Albrecht Dürer for the use of all lovers of art with appropriate illustrations arranged to be printed in the year MDXXV.* Abaris Books, 1977.

[42] Albrecht Dürer. Melencholia I. https://www.kunsthalle-karlsruhe.de/kunstwerke/Albrecht-D%C3%BCrer/Melencolia-I-Die-Melancholie/6ABE53364495E90C2D7B0E8D99C2B03B/, 1514. Accessed: 2020-06-08.

[43] Paul Ernest. Popularization: myths, massmedia and modernism. In *International handbook of mathematics education*, pages 785–817. Springer, 1996.

[44] Euclid. *Elements.* Perseus Digital Library. Accessed: 2020-07-11.

[45] Leonhard Euler. Demonstratio nonnullarum insignium proprieatatum, quibus solida hedris planis inclusa sunt praedita. *Novi commentarii academiae scientiarum Petropolitanae*, pages 140–160, 1758. http://eulerarchive.maa.org/, Accessed: 2020-07-11.

[46] Leonhard Euler. Elementa doctrinae solidorum. *Novi commentarii academiae scientiarum Petropolitanae*, pages 109–140, 1758. http://eulerarchive.maa.org/, Accessed: 2020-07-11.

[47] Pasquale J. Federico. *Descartes on Polyhedra*. Springer, 1982.

[48] Susan T. Fiske and Cydney Dupree. Gaining trust as well as respect in communicating to motivated audiences about science topics. *Proceedings of the National Academy of Sciences*, 111(Supplement 4):13593–13597, 2014.

[49] Gail E. Fitzsimons, Helga Jungwirth, Juergen Maaß, and Wolfgang Schloeglmann. Adults and mathematics (adult numeracy). In *International handbook of mathematics education*, pages 755–784. Springer, 1996.

[50] OECD Organisation for economic Co-operation and development. The PISA 2003 Assessment Framework – Mathematics, Reading, Science and Problem Solving Knowledge and Skills. http://archiv.ipn.uni-kiel.de/PISA/pisa2003/PISA2003Frameworks1.pdf. Accessed: 2020-05-06.

[51] Michael Friedman. From the sixteenth century onwards: Folding polyhedra – new epistemological horizons? In *A History of Folding in Mathematics*, pages 29–91. Springer, 2018.

[52] Friedrich-Loeffler-Institut (FLI) and Leibniz Centre for Agricultural Landscape Research (ZALF) e.V. Mückenatlas. https://mueckenatlas.com/about/. Accessed: 2019-02-19.

[53] Komei Fukuda. Strange Unfoldings of Convex Polytopes. https://inf.ethz.ch/personal/fukudak/unfold_home/unfold_open.html. Accessed: 2019-11-08.

[54] Fumiko Futamura, Marc Frantz, and Annalisa Crannell. The cross ratio as a shape parameter for Dürer's solid. *Journal of Mathematics and the Arts*, 8(3-4):111–119, 2014.

[55] Peter Gallin. Dialogisches Lernen. Von einem pädagogischen Konzept zum täglichen Unterricht. *Grundschulunterricht Mathematik*, 2:4–9, 2010.

[56] Mohammad Ghomi. Affine unfoldings of convex polyhedra. *Geometry & Topology*, 18(5):3055–3090, 2014.

[57] Mohammad Ghomi. Dürer's unfolding problem for convex polyhedra. *Notices of the AMS*, 65(1), 2018.

[58] Gerd Gigerenzer, Wolfgang Gaissmaier, Elke Kurz-Milcke, Lisa M. Schwartz, and Steven Woloshin. Helping doctors and patients make sense of health statistics. *Psychological Science in the Public Interest*, 8(2):53–96, 2007.

[59] John K. Gilbert, Susan M. Stocklmayer, and Robin Garnett. Mental modeling in science and technology centres: What are visitors really doing. In *International Conference on Learning Science in Informal Contexts, Canberra*, pages 16–32, 1999.

[60] Timothy Gowers. Gowers Weblog: Is massively collaborative mathematics possible? https://gowers.wordpress.com/2009/01/27/is-massively-collaborative-mathematics-possible/. Accessed: 2019-04-29.

[61] Gert-Martin Greuel. Mathematics between research, application, and communication. In *Raising Public Awareness of Mathematics*, pages 359–386. Springer, 2012.

[62] David A. Grier. The math tables project of the work projects administration: The reluctant start of the computing era. *IEEE Annals of the History of Computing*, 20(3):33–50, 1998.

[63] Stefan Grigutsch, Ulrich Raatz, and Günter Törner. Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematik-Didaktik*, 19(1):3–45, 1998.

[64] Branko Grünbaum. Nets of polyhedra. *Geombinatorics*, 1(2):5–9, 1991.

[65] Branko Grünbaum. Nets of polyhedra II. *Geombinatorics*, 1(3):5–10, 1991.

[66] Branko Grünbaum. Are your polyhedra the same as my polyhedra? In *Discrete & Computational Geometry*, pages 461–488. Springer, 2003.

[67] Branko Grünbaum, Victor Klee, Micha A. Perles, and Geoffrey C. Shephard. *Convex polytopes.* John Wiley & Sons, Ltd., 1967.

[68] Jürgen Habermas. *Faktizität und Geltung : Beiträge zur Diskurstheorie des Rechts und des demokratischen Rechtsstaats.* Suhrkamp, Frankfurt am Main, 3. edition, 1993.

[69] Izidor Hafner. Two convex polyhedra with a common net. http://vismath8.tripod.com/hafner5/Twopoly2Visual.html. Accessed: 2020-05-29.

[70] Vagn Lundsgaard Hansen. Keeping Mathematical Awareness Alive. In *Raising Public Awareness of Mathematics*, pages 387–396. Springer, 2012.

[71] Anna M. Hartkopf. Citizen Art–Collective Mathematical Art to Raise the Public Awareness of Mathematics. In *Bridges 2019 Conference Proceedings*, pages 355–358. Tessellations Publishing, 2019.

[72] Anna M. Hartkopf. Developments towards Mathematical Citizen Science. In *Forum Citizen Science 2019*, 2019.

[73] Anna M. Hartkopf and Günter M. Ziegler. Adopt a Polyhedron – A Citizen Art Project in Mathematics. In Carlo Séquin Eve Torrence, Bruce Torrence and Kristóf Fenyvesi, ed-

itors, *Proceedings of Bridges 2018: Mathematics, Art, Music, Architecture, Education, Culture*, pages 579–584, Phoenix, Arizona, 2018. Tessellations Publishing. Available online at http://archive.bridgesmathart.org/2018/bridges2018-579.pdf.

[74] Friederike Hendriks, Dorothe Kienhues, and Rainer Bromme. Trust in science and the science of trust. In *Trust and communication in a digitized world*, pages 143–159. Springer, 2016.

[75] Henning Heske. Ganzheitliches Lernen. *Mathematik-Didaktik. Berlin: Cornelsen Scriptor*, 2003.

[76] Ulrich Hoffrage, Samuel Lindsey, Ralph Hertwig, and Gerd Gigerenzer. Communicating statistical information. 2000.

[77] Volker Hole. *Erfolgreicher Mathematikunterricht mit dem Computer: methodische und didaktische Grundfragen in der Sekundarstufe I; mit praktischen Beispielen.* Auer, 1998.

[78] Reinhold Hoppe. Regelmässige linear begrenzte Figuren von vier Dimensionen. *Archiv der Mathematik und Physik*, 67:29–43, 1882.

[79] Takashi Horiyama and Wataru Shoji. Edge unfoldings of platonic solids never overlap. In *CCCG*, 2011.

[80] Alexandre Eremenko (https://mathoverflow.net/users/25510/alexandre eremenko). Can pure mathematics harness citizen science? MathOverflow. URL:https://mathoverflow.net/q/112762 (version: 2012-11-18).

[81] Stephan Hußmann. Umgangssprache–Fachsprache. *Mathematikdidaktik. Praxishandbuch für die Sekundarstufe I und II*, pages 60–75, 2003.

[82] Institut für Meteorologie, Freie Universität Berlin. Wetterpatenschaften. http://www.met.fu-berlin.de/wetterpate/. Accessed: 2020-10-21.

[83] Rolf Klein, Raimund Seidel, and Seth Teller. Dagstuhl Seminar Report 168 – Computational Geometry 9707. https://www.dagstuhl.de/Reports/97/9707.pdf. Accessed: 2019-11-08.

[84] Joseph B. Kruskal. On the shortest spanning subtree of a graph and the traveling salesman problem. *Proceedings of the American Mathematical society*, 7(1):48–50, 1956.

[85] Ralf Krömer. A German Translation of Euler's articles on the polyhedron formula now available on the worldwide web. *Historia Mathematica*, 35, 2008.

[86] Ralf Krömer. personal communication, 2019.

[87] Kultusministerkonferenz (Hrsg.). Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss (Jahrgangsstufe 10) (Beschluss der KMK vom 04.12.2003). https://www.kmk.org/fileadmin/veroeffentlichungen_beschluesse/2003/2003_12_04-Bildungsstandards-Mathe-Mittleren-SA.pdf, 2003. Accessed: 2020-02-24.

[88] Kultusministerkonferenz (Hrsg.). Bildungsstandards im Fach Mathematik für den Hauptschulabschluss (Jahrgangsstufe 9) (Beschluss der KMK vom 15.10.2004). https://www.kmk.org/fileadmin/veroeffentlichungen_beschluesse/2004/2004_10_15-Bildungsstandards-Mathe-Haupt.pdf, 2004. Accessed: 2020-02-24.

[89] Kultusministerkonferenz (Hrsg.). Bildungsstandards im Fach Mathematik für die Allgemeine Hochschulreife (Beschluss der Kultusministerkonferenz vom 18.10.2012). https://www.kmk.org/fileadmin/veroeffentlichungen_beschluesse/2012/2012_10_18-Bildungsstandards-Mathe-Abi.pdf, 2012. Accessed: 2020-02-28.

[90] Reinhard Laubenbacher. Mathematics in the Public Mind: The USA. In *Raising Public Awareness of Mathematics*, pages 47–55. Springer, 2012.

[91] Henry G. Liddell and Robert Scott. *A Greek-English Lexicon.* Oxford University Press, Oxford, 1843. Reprinted in 1989.

[92] Pauline Linke and Brigitte Lutz-Westphal. *Das "Spot-Modell" im Mathematikunterricht – Forschendes und Entdeckendes Lernen fundiert anwenden.* Universitätsbibliothek Dortmund, 2018.

[93] Xiufeng Liu. Beyond science literacy: Science and the public. *International Journal of Environmental and Science Education*, 4(3):301–311, 2009.

[94] Robert A Logan. Science mass communication: Its conceptual history. *Science Communication*, 23(2):135–163, 2001.

[95] Andreas Loos. Wer möchte Polytope taufen? https://blog.zeit.de/teilchen/2018/11/09/polytopia-polyeder-geometrie-forschungsgruppe-teilchen/. Accessed: 2020-05-18.

[96] Brendan Lucier. Local overlaps in special unfoldings of convex polyhedra. *Computational Geometry*, 42(5):495–504, 2009.

[97] Matthias Ludwig. *Projekte im Mathematikunterricht des Gymnasiums.* PhD thesis, Universität. Hildesheim, 1997.

[98] Matthias Ludwig, Brigitte Lutz-Westphal, and Volker Ulm. Forschendes Lernen im Mathematikunterricht, Mathematische Phänomene aktiv hinterfragen und erforschen. *Praxis der Mathematik in der Schule*, (73):2–9, 2017.

[99] Brigitte Lutz-Westphal. Das forschende Fragen lernen. Pflasterungen: scheinbar Bekanntes neu durchdringen. *Mathematik Lehren*, (184):16–19, 2014.

[100] Brigitte Lutz-Westphal and Alexander Schulte. *Mathematische Forschung-Was Forschendes Lernen im Mathematikunterricht aus der Praxis lernen kann.* Universitätsbibliothek Dortmund, 2016.

[101] Joseph Malkevitch. Milestones in the history of polyhedra. In *Shaping Space*, pages 53–63. Springer, 2013.

[102] Henry P. Manning. *Geometry in four dimensions*. University of Michigan Library, 1914.

[103] Steen Markvorsen. From PA (X) to RPAM (X). In *Raising Public Awareness of Mathematics*, pages 255–267. Springer, 2012.

[104] Martin W. Bauer. Science Barometer 2019: The Bigger Picture. https://www.wissenschaftskommunikation.de/science-barometer-2019-the-bigger-picture-32967/, 2019. Accessed: 2019-12-20.

[105] Heather Mendick, Debbie Epstein, and Marie-Pierre Moreau. Mathematical images and identities: entertainment, education, social justice. *Research in Mathematics Education*, 10(1):101–102, 2008.

[106] Mersenne Research, Inc. Great Internet Mersenne Prime Search GIMPS. https://www.mersenne.org/. Accessed: 2019-05-02.

[107] Rudolf Messner. *Schule forscht: Ansätze und Methoden zum forschenden Lernen*. Ed. Körber-Stiftung, 2009.

[108] Ezra Miller and Igor Pak. Metric combinatorics of convex polyhedra: cut loci and non-overlapping unfoldings, 2003. Preprint, .

[109] Jon D. Miller. Scientific literacy: A conceptual and empirical review. *Daedalus*, pages 29–48, 1983.

[110] Jon D. Miller. The measurement of civic scientific literacy. *Public understanding of science*, 7:203–223, 1998.

[111] August Ferdinand Möbius. *Der barycentrische Calcul, ein Hülfsmittel zur analytischen Behandlung der Geometrie (etc.)*. Barth, Leipzig, 1827.

[112] Michael Nielsen. The Polymath project: scope of participation. http://michaelnielsen.org/blog/the-polymath-project-scope-of-participation/. Accessed: 2019-04-29.

[113] Michael Nielsen. *Reinventing discovery: the new era of networked science*. Princeton University Press, 2011.

[114] Matthew C. Nisbet and Dietram A. Scheufele. What's next for science communication? Promising directions and lingering distractions. *American journal of botany*, 96(10):1767–1778, 2009.

[115] Joseph O'Rourke. On folding a polygon to a polyhedron, 2010.

[116] Joseph O'Rourke. *How to fold it: the mathematics of linkages, origami, and polyhedra*. Cambridge University Press, 2011.

[117] Joseph O'Rourke. Dürer's problem. In Marjorie Senechal, editor, *Shaping Space*, pages 77–86. Springer, 2013.

[118] А. Д. Александров . *Выпуклые многогранники, Изд-во Гостехиздат.* Москва , 1950.

[119] Hans-Peter Peters and Magdalena Sawicka. German reactions to genetic engineering in food production. *The public, the media and agricultural biotechnology. Wallingford (UK): CABI Publishing*, pages 57–96, 2007.

[120] Thomas Petersen. Die Stunde der Wissenschaft. *Frankfurter Allgemeine Zeitung, 2020-06-18*, page 6, 2020.

[121] D. H. J. Polymath. Density Hales-Jewett and Moser numbers, 2010. Preprint, arXiv:1002.0374.

[122] Susanna Priest. Critical science literacy: What citizens and journalists need to know to make sense of science. *Bulletin of Science, Technology & Society*, 33(5-6):138–145, 2013.

[123] Lee Rainie, Janna Anderson, and D. Page. Code-dependent: Pros and cons of the algorithm age. *Pew Research Center*, 8, 2017.

[124] Renata Ramalho and Nuno Crato. Balancing Math Popularization with Public Debate: A Mathematical Society's Continued Efforts to Raise the Public Awareness of Mathematics and for Youth Mathematical Education. In *Raising Public Awareness of Mathematics*, pages 57–66. Springer, 2012.

[125] Felix G. Rebitschek and Gerd Gigerenzer. Einschätzung der Qualität digitaler Gesundheitsangebote: Wie können informierte Entscheidungen gefördert werden? *Bundesgesundheitsblatt-Gesundheitsforschung-Gesundheitsschutz*, 2020.

[126] Jürgen Richter-Gebert. *Realization spaces of polytopes.* Springer, 1996.

[127] Jürgen Roth and Hans-Georg Weigand. Forschendes Lernen – Eine Annäherung an wissenschaftliches Arbeiten. *Mathematik Lehren*, 184:2–10, 2014.

[128] Jürgen Roth and Hans-Georg Weigand. Forschendes Lernen im Mathematikunterricht. *WTM-Verlag*, pages 999–1002, 2014.

[129] Urs Ruf and Peter Gallin. Dialogisches Lernen in Sprache und Mathematik (2 Bände). *Seelze-Felber: Kallmeyer*, 1999.

[130] Lim Chap Sam and Paul Ernest. A survey of public images of mathematics. *Research in Mathematics Education*, 2(1):193–206, 2000.

[131] Ed Sandifer. How Euler Did It, V, E and F, Part 1. http://eulerarchive.maa.org/hedi/HEDI-2004-06.pdf. Accessed: 2020-07-11.

[132] Gilles Schaeffer. Random sampling of large planar maps and convex polyhedra. In *Proceedings of the thirty-first annual ACM symposium on Theory of computing*, pages 760–769, 1999.

[133] Catherine A. Schevon. Unfolding polyhedra. https://www.ics.uci.edu/~eppstein/gina/unfold.html. Accessed: 2020-04-15.

[134] Catherine A. Schevon. *Algorithms for geodesics on convex polytopes.* PhD thesis, Johns Hopkins University, 1990.

[135] Victor Schlegel. *Theorie der homogen zusammengesetzten Raumgebilde.* Engelmann, Leipzig, Halle, 1883.

[136] Wolfram Schlickenrieder. *Nets of polyhedra.* Master's Thesis, Technische Universität Berlin, 1997.

[137] Ludwig Schläfli. *Theorie der vielfachen Kontinuität.* Zürcher & Furrer, 1901.

[138] Jugend und Familie Senatsverwaltung für Bildung. Framework Curriculum 1-10 compact. https://www.berlin.de/sen/bildung/unterricht/faecher-rahmenlehrplaene/rlp_kompakt_1_10_english.pdf. Accessed: 2020-02-26.

[139] Jugend und Familie Senatsverwaltung für Bildung. Rahmenlehrplan für die gymnasiale Oberstufe, Mathematik . https://www.berlin.de/sen/bildung/unterricht/faecher-rahmenlehrplaene/rahmenlehrplaene/mdb-sen-bildung-unterricht-lehrplaene-sek2_mathematik_neu2014.pdf. Accessed: 2020-02-26.

[140] Jugend und Familie Senatsverwaltung für Bildung. Rahmenlehrplan Teil C Mathematik Jahrgangsstufen 1 – 10. https://bildungsserver.berlin-brandenburg.de/fileadmin/bbb/unterricht/rahmenlehrplaene/Rahmenlehrplanprojekt/amtliche_Fassung/Teil_C_Mathematik_2015_11_10_WEB.pdf. Accessed: 2020-02-26.

[141] Gözde Sert and Sergio Zamora. On Unfoldings of Stretched Polyhedra. *arXiv preprint arXiv:1803.09828*, 2018.

[142] SFB Transregio 109, Discretization in Geometry and Dynamics. POLYTOPIA – Adopt a Polyhedron. https://www.polytopia.eu/en/. Accessed: 2019-05-02.

[143] Claude E. Shannon and Warren Weaver. *The mathematical theory of communication.* University of Illinois Press, 1949.

[144] Geoffrey C. Shephard. Convex polytopes with convex nets. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 78, pages 389–403. Cambridge University Press, 1975.

[145] Siggener Kreis / Initiative Qualität von Hochschulkommuni-kation (IQ_HKom). Leitlinien zur guten Wissenschafts-PR. https://www.wissenschaft-im-dialog.de/fileadmin/user_upload/

Ueber_uns/Gut_Siggen/Dokumente/LEITLINIEN_WISSPR_17_11_Druck_komprimiert.pdf. Accessed: 2020-01-20.

[146] Bharath Sriraman and Günter Törner. Political union / Mathematical Education Disunion: Building Bridges in European Didactic Traditions. *In L. English et al. (Eds.) The Handbook of International Research in Mathematics Education*, pages 660–694, 2008.

[147] Ernst Steinitz and Hans Rademacher. *Vorlesungen über die Theorie der Polyeder*. Springer-Verlag, Berlin, 1934. Reprint, Springer-Verlag 1976.

[148] Alexey S. Tarasov. Existence of a polyhedron which does not have a non-overlapping pseudo-edge unfolding. *arXiv preprint arXiv:0806.2360*, 2008.

[149] Clemens Thaer. *Euklid: Die Elemente*. Darmstadt, 1962.

[150] The SAGE Developers. *SAGEMath, the SAGE Mathematics Software System (Version 8.1.7)*, 2018. http://www.sagemath.org.

[151] Günter Törner. Epistemologische Beliefs–State-of-Art-Bemerkungen zu einem aktuellen mathematikdidaktischen Forschungsthema vor dem Hintergrund der Schraw-Olafson-Debatte. *Mathematikunterricht im Spannungsfeld von Evolution und Evaluation. Hildesheim, Franzbecker Verlag*, pages 324–333, 2005.

[152] Raúl Ibáñez Torres. The Butterfly Effect and the Popularization of Mathematics: Spain. In *Raising Public Awareness of Mathematics*, pages 67–84. Springer, 2012.

[153] University of California. Thesaurus Linguae Graecae – Text Search. http://stephanus.tlg.uci.edu/Iris/inst/tsearch.jsp#s=14, 2019. Word Index: $\pi o \lambda \upsilon \epsilon \delta \rho o \nu$, Accessed: 2019-08-23.

[154] University of Washington Center for Game Science, University of Washington Institute for Protein Design, Northeastern University, Vanderbilt University, University of California, Davis, and University of Massachusetts, Dartmouth. Foldit – solve puzzles for science. https://fold.it/portal/. Accessed: 2019-07-23.

[155] Urania Berlin. Geschichte der Urania: Die Urania – das erste Science Center der Welt , status 2018-02-05". https://www.urania.de/die-urania/geschichte, 2019. Accessed: 2019-11-27.

[156] Jason West. Money mathematics: Examining ethics education in quantitative finance. *Journal of Business Ethics Education*, 9(Special Issue):25–39, 2012.

[157] Wikipedia contributors. List of distributed computing projects — Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/List_of_distributed_computing_projects, 2019. Accessed: 2019-05-02.

[158] Paul Wilmott. The use, misuse and abuse of mathematics in finance. *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 358(1765):63–73, 2000.

[159] Wissenschaft im Dialog/Kantar. Wissenschaftsbarometer Corona Spezial . https://www.wissenschaft-im-dialog.de/fileadmin/user_upload/Projekte/Wissenschaftsbarometer/Dokumente_20/2020_WiD-Wissenschaftsbarometer_Corona_Spezial_Ergebnispraesentation.pdf. Accessed: 2020-07-07.

[160] Wissenschaft im Dialog/Kantar Emnid. Science Barometer 2018, Here are some reasons why you might distrust scientists. To what extend do you personally agree with them? https://www.wissenschaft-im-dialog.de/fileadmin/user_upload/Projekte/Wissenschaftsbarometer/Dokumente_18/Einzelgrafiken/EN/18_WB_SingleChart_08_ReasonDistrust.jpg. Accessed: 2019-12-20.

[161] Wissenschaft im Dialog/Kantar Emnid. Science Barometer 2018, Here are some reasons why you might trust scientists. To what extend do you personally agree with them? https://www.wissenschaft-im-dialog.de/fileadmin/user_upload/Projekte/Wissenschaftsbarometer/Dokumente_18/Einzelgrafiken/EN/18_WB_SingleChart_07_ReasonTrust.jpg. Accessed: 2019-12-20.

[162] Wissenschaft im Dialog/Kantar Emnid. Science Barometer 2018, How much do you trust science and research? https://www.wissenschaft-im-dialog.de/fileadmin/user_upload/Projekte/Wissenschaftsbarometer/Dokumente_18/Einzelgrafiken/EN/18_WB_SingleChart_06_TrustInScience.jpg. Accessed: 2019-12-20.

[163] Holger Wormer. Pro-faktisch statt post-faktisch. Wissenschaft und Wissenschafts-PR als Wissensmakler. *Wissenschaftsmanagement*, 5:32–36, 2016.

[164] Holger Wormer. Vom Public Understanding of Science zum Public Understanding of Journalism. In *Forschungsfeld Wissenschaftskommunikation*, pages 429–451. Springer, 2017.

[165] Brian Wynne. Sheepfarming after Chernobyl: A Case Study in Communicating Scientific Information. *Environment: Science and Policy for Sustainable Development*, 31(2):10–39, 1989.

[166] Brian Wynne. Public understanding of science. *Handbook of science and technology studies*, 1:361–388, 1995.

[167] Friedrich Zech. *Grundkurs Mathematikdidaktik*. Beltz, 1983.

[168] Indre Zetzsche and Peter Weingart. *Wissenschaftskommunikation: Streifzüge durch ein "neues" Feld*. Lemmens, 2004.

[169] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Revised edition, 1998; seventh updated printing 2007.

[170] Günter M. Ziegler. Convex polytopes: Extremal constructions and f-vector shapes. In E. Miller, V. Reiner, and B. Sturmfels, editors, *"Geometric Combinatorics", Proc. Park City Mathematical Institute (PCMI) 2004*, volume 13 of *IAS/Park City Math. Ser.*, pages 617–691, Providence, RI, 2007. Amer. Math. Society.

[171] Günter M. Ziegler and Thomas Vogt. Mathematics for the people. In *Raising Public Awareness of Mathematics*, pages 37–46. Springer, 2012.

Appendix

FAQ

How can I adopt a polyhedron?

You can choose a polyhedron from the matrix. Select your polyhedron by clicking on one of the black pixels. The green pixels are the polyhedra that have already been adopted, so you can look at them but not adopt them again. The black ones are free.

When you have clicked on a pixel, you will see a detailed view of the polyhedra. You can turn it around and look at it from all angles. If you and the polyhedron get along well and you decide to adopt it, you can click: Adopt me.

A new window pops up and you can sign up with your email address. We will send you a confirmation email. Go to your email-inbox and click on the link in the email.

Congratulations, you have adopted your very own mathematical object! Now you can give it a name. This can be your name or the name of your pet or a fantasy name. Try to come up with a name right away, your polyhedron will be very happy once it is named.

Realize your Polyhedron

In order to free the polyhedron from abstraction, it needs to take shape as a model. There are two options available. One possibility is to download the paper template below the polyhedron. You can cut it out and glue it together along the tabs. Here (Video is in production.) is a video to help you assemble the paper model.

Another possibility to make a model is via a 3D printer. The data can also be downloaded below the polyhedron. You may use an online 3D print service (We are looking for a sponsor for you.), and have it sent to your home.

Now your polyhedron has come to life! In order to prove that – you know that proofs are the bread and butter in mathematics – take a picture and upload it. Since the photo proof is visible for all users, we would like to ask you to make sure that the photo contains only the polyhedron and a rather neutral background. Quadratic pictures look best on our website.

Individualize your polyhedron

To add a little personality to your polyhedron you may build a creative model of it. There are no limits to your unique ideas, or the choice of material. Classical materials like wood, clay or cardboard are possible, but polyhedra are naturally keen to experiment and very happy if you build them out of more creative materials. Why not make a cheese polyhedron in contrast to the old cheese cube? If you have made an individual model, please also upload a picture.

What is a Polyhedron?

Polyhedra consist of vertices, straight edges, and flat faces. In our project, we focus on convex polyhedra. This means that all inner angles between two edges or two faces are less or equal to 180°. No cavities, holes or indentations are allowed. The word polyhedron comes from the combination of Greek words poly- (many) and -hedron (face).

Polyhedra are classical objects in geometry. The ancient philosopher Plato describes the class of regular polyhedra. Nowadays we call them Platonic solids. These bodies consist only of regular polyhedra, i.e. equilateral triangles, squares and equilateral pentagons. Euclid gave the first constructive proof of the completeness of that classification in his treatise Elements.

Up until today polyhedra and their higher-dimensional relatives, the polytopes are thoroughly investigated by mathematicians. You can find more information about polyhedra and their properties in the glossary.

Is every polyhedron really unique?

Yes, all polyhedra are different from each other. No two are identical. If you adopt a polyhedron, it really is unique. The polyhedra fundamentally differ in their combinatorial structure. This means that no two polyhedra have the same order of vertices, edges and polygons.

What does Realization of a polyhedron mean?

A polyhedron is labeled realized when a model has been built. This may be a paper model that can be crafted from the paper template, or a 3D printed object. In order to prove the realization, you must upload a picture that you have taken of the model.

What does Individualization of a polyhedron mean?

To get to know your polyhedron a little better and add some personality, you may also individualize it. This means that you make another model out of a creative material of your choice.

This could be wood, a sponge carved into the same form, a polyhedral marble-cake or a Plexiglas figure. It is easiest if you base the structure of your individualized model on the paper or 3D printed version in order to get the structure of the polyhedron right. Again, a picture proof is necessary.

Is the adoption of a polyhedron official?

Of course, mathematical objects do not belong to anyone. You can not register patents or sue someone who has stolen an idea for them. Similarly, the polyhedra in our project are free beings, like the stars in the universe, which you can also adopt but can not take home. So the adoption is solely authorized by everyone's acceptance. The project is backed by the Collaborative Research Center 109. This is an association of mathematicians of the Technical University Berlin and Technical University Munich, attached is also the Freie Universität Berlin.

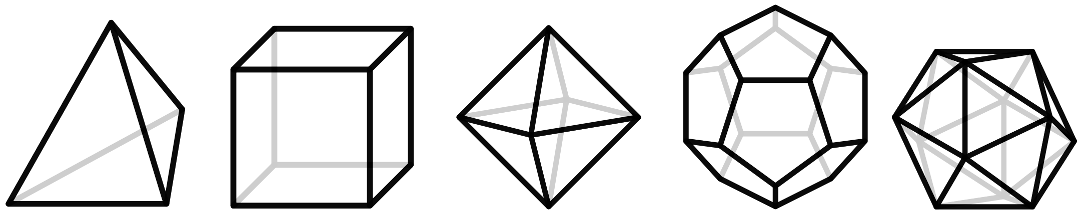
Glossary

Polyhedra

Polyhedra are geometrical bodies that consist of vertices, straight edges and flat faces. In our project we restrict ourselves to convex polyhedra. This means that all inner angles between two edges or two faces are less or equal to 180°. No cavities, holes or indentations are allowed.

The most prominent examples of polyhedra are the cube and the pyramid. You also may have encountered the prism or the octahedron. But there are so many more polyhedra.

The Platonic solids are a very symmetric and regular class of five polyhedra. They consist only of congruent (same shape and size), regular (all sides the same length) polygons. The tetrahedron, the octahedron and the icosahedron comprise of regular triangles. In every corner of the tetrahedron, three triangles meet. In the case of the octahedron four and with the icosahedron five triangles build a joint at each vertex. If you put six equilateral triangles at a vertex, which each have an interior angle of 60°, you get a full circle of 360° and they lie flat on the ground. Therefore, no new body emerges. Similarly, the cube is built from three squares and the dodecahedron from three equilateral pentagons at each vertex. Euclid used this approach in his proof to show that there can be no further polyhedra with these properties, thus classifying the five Platonic solids.



(a) (b) (c) (d) (e)

Figure 4.16: The platonic solids.

But symmetric polyhedra are not the only interest of mathematicians. For their research they are searching for polyhedra (or their higher-dimensional equivalent, the polytopes) that have specific properties. We have asked some geometers – this is what the mathematicians who study geometry are called – about their favorite polyhedra and this is what they said:

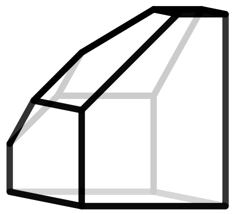


Figure 4.17: The associahedron.

"If polytopes could be viewed as rocks, then the associahedron is the diamond of polytopes. Diamonds are made of a very common element in nature – carbon – and likewise the associahedron can be realized via very common tools. Yet it enjoys such a unique and rare structure – and provides such a fascination– that no other polytope may ever be compared to the diamond."

Jean-Philippe Labbé

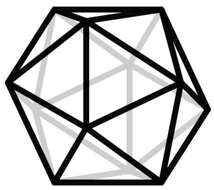


Figure 4.18: The icosahedron.

"My favorite 3-polytope is the icosahedron, for its complexity yet simplicity. If you start gluing equilateral triangles, five at a vertex, and no matter what you do you end up with this nice thing. When I got bored in high school (which, yes, happened), I drew icosahedra in the margins in my notebooks. Sometimes they were not totally regular; I amused myself making them look like faces. Last but not least, I am fascinated by the fact that you can decompose its 12 vertices into three golden rectangles intertwined as Borromean rings."

Francisco Santos

"My favourite polyhedron is "Miller's solid", also known as the "pseudo-rhombicuboctahedron" or as the "elongated square gyrobicupola", probably first found by D. M. Y. Sommerville in 1905 – an object that was often overlooked (already by Archimedes'), discovered and rediscovered (by J. C. P. Miller, among others). It is pretty, but if you look closely it has a certain twist, so it is not perfect. It looks classical, like an Archimedean solid, but it isn't really, at least according to the modern definition of an Archimedean solid. Thus it is a good reminder that we have to be careful with definitions in mathematics, and always look at possible exceptions and special cases."

Günter M. Ziegler

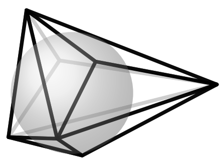


Figure 4.19: An example of a Koebe polyhedron.

"My favourite polyhedra are the Koebe polyhedra. All their edges touch a sphere. All faces of these polyhedra have inscribed discs. The discs of neighbouring faces touch. These exist an explicit dualization procedure that generates discrete minimal surfaces from Koebe polyhedra. The corresponding surface is then a discrete P-Schwarz surface and the Koebe polyhedron is its Gauss map."

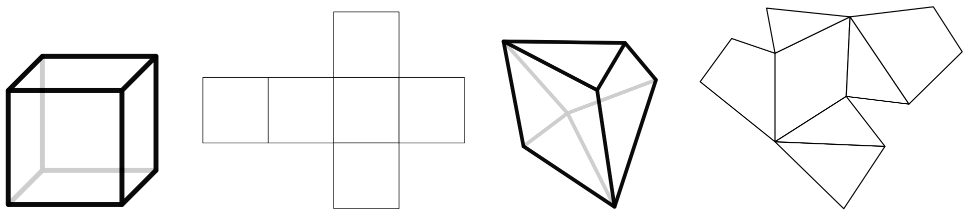
One can read more in: A.I. Bobenko, T. Hoffmann, B.A. Springborn, Minimal surfaces from circle patterns: Geometry from combinatorics, Ann. of Math. 164:1 (2006) 231-264 and can see in the animation movie "Koebe polyhedra and minimal surfaces" by Bobenko, Janek and Techter http://discretization.de/en/movies/koebe/

Alexander Bobenko

Polygon

A polygon is a two-dimensional polyhedron which consists of vertices and edges. The area bounded by the edges is the polygon itself. Three-dimensional polyhedra are made up of vertices, edges and polygons, i.e., the faces.

A special class of polygons are called the regular polygons. They are made up of equilateral edges (all the same length) and all their inner angles are the same. Some examples of regular polygons are the square, equilateral triangle, and equilateral pentagon. Regular polygons form the building blocks for the Platonic and Archimedean solids.



(a) (b) (c) (d)

Figure 4.20: A cube with its net and a polygon with seven corners and the associated net.

Polyhedral Nets

If you cut a hollow cube on enough edges, unfold it and lay it flat on the plane, you get what is called the net of the cube. If you draw this net on paper, you get an outline which you can cut out and glue together to form the cube. Of course, this method works for any polyhedra. In our project, we use exactly these templates to build our models. We simulated this process of cutting open and unfolding polyhedra with a computer and automatically created the nets of the polyhedra.

Archimedian Solids

Another very symmetric and hence 'beautiful' class of polyhedra are the Archimedean Solids. They also only consist of regular polygons, but here unlike the Platonic Solids a combination of them is allowed. The most common representative is the soccer ball. Mathematicians would rather speak of a truncated icosahedron, since it emerges when you chop off the tip of each vertex of an icosahedron.

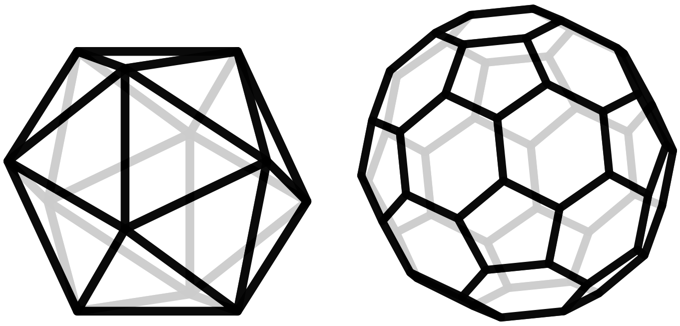


Figure 4.21: On the left an icosahedron and on the right its truncation - a soccer ball.

Vertices	Polyhedra
4	1
5	2
6	7
7	34
8	257
9	2.606
10	32.300
11	440.564
12	6.384.634
13	96.262.938
14	1.496.225.352
15	23.833.988.129
16	387.591.510.244
17	6.415.851.530.241
18	107.854.282.197.058
19	???

How many polyhedra are there?

For every fixed number of vertices, there are a certain number of polyhedra. In the table [9], the number of different types of polyhedra is given for the number of vertices. It is clear that the number of types increases rapidly. If you have four points in space, they are either all on the same level (not three-dimensional), or the shape will be a pyramid over a triangle. Therefore, there is only one polyhedron with four vertices, the tetrahedron.

For five vertices, there are two possibilities: the pyramid over the square, if four of the five vertices lie the same plane, or the double pyramid over a triangle. For six vertices, finding the seven different types starts to get more complicated.

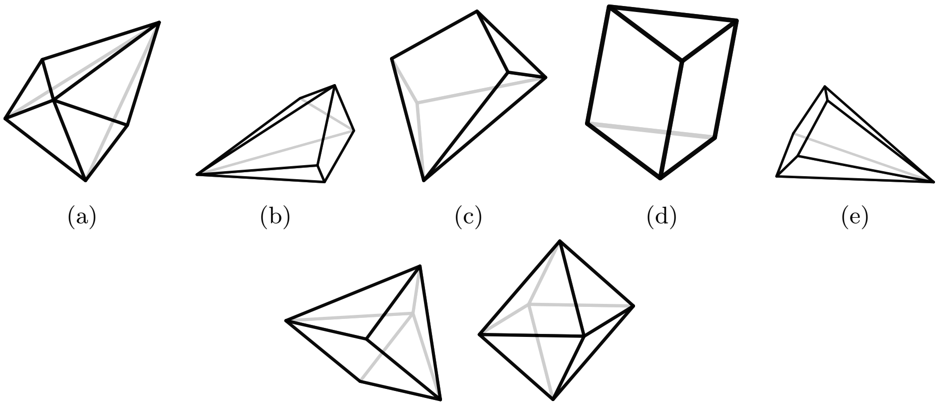


Figure 4.22: All seven types of polyhedra with six vertices.

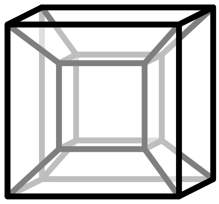
In order to find out how many polyhedra types actually exist for each number of vertices, we have to create and list them. But how do you know that this list is complete and no polyhedron is counted twice? In geometry, Steinitz's Theorem states that each polyhedron can be uniquely assigned to a graph with certain properties. (Here the notion of a graph is not referring to the ones living in coordinate systems but the ones that are subject to graph theory). These graphs are mathematically easier to grasp and therefore count. But even for this, you will need a computer because the numbers get very large very fast. The number of seven- and eight-vertice polyhedra, 34 and 257 respectively, were found back in 1899. For the discovery of the 2606 nine-vertice polyhedra in the year 1969 the invention of the computer was necessary.

Dimension

In mathematics, there are many ways to interpret dimensions. One way is to imagine dimensions as the number of variables. For example, the ingredients of an apple pie (flour, butter, sugar, eggs, baking soda and apples) can be understood as six variables and therefore the apple pie is a six-dimensional object.

By looking at photos and films, which are a representation of our three-dimensional world in a two-dimensional medium, we are used to seeing an extra dimension. This process of mapping a higher dimension into a lower one that is taking a two-dimensional photograph of the three-dimensional world, is called a projection in mathematics.

Unfortunately, it is not possible to truly represent four-dimensional space in the three-dimensional space surrounding us, but we can use projections to understand it. For example, if you look at a cube, its faces are squares. A square can be thought of as a two-dimensional cube, because all its sides are the same length, so the sides of the three-dimensional cube are two-dimensional cubes. This idea extends to higher dimensions. The side surfaces of a four-dimensional cube are three-dimensional cubes. The result is a so-called tesseract. Here is a link to a video where this relationship is graphically visualized.



Convex/Convexity

When we speak of polyhedra, we silently assume that they are convex polyhedra. Convex means that there are no indentations, cavities or holes. The mathematical definition of convexity states that for any two points that lie within a set, a straight line connecting them must lie completely within the set.

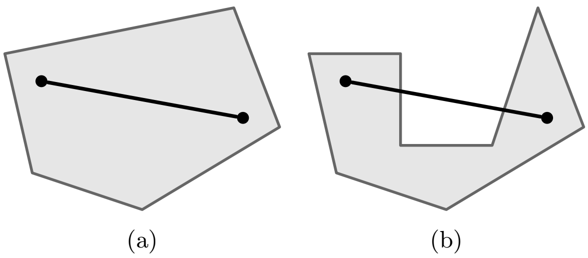


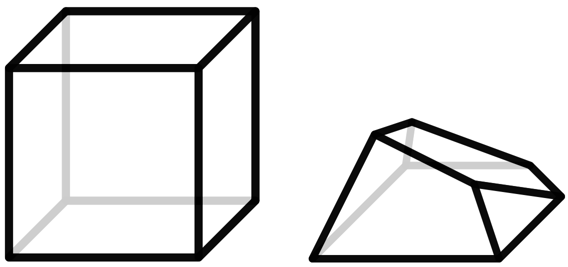
Figure 4.23: On the left a convex, on the right a non-convex object.

Combinatorial Type

Each polyhedron can be geometrically realized in different ways. It can be big or small, and its shape can also be changed, as long as the structure of the vertices, edges and surfaces remains the same. This structure, which is the number of edges meeting at the vertices, and the number of vertices belonging to each surface, is called the combinatorial type of a polyhedron. We call two polyhedra combinatorially equivalent if they possess the same combinatorial type, i.e., one can uniquely assign vertices to each other so that if two vertices in one polyhedron are connected by an edge, then the vertices in the other polyhedron are connected by an edge. Every polyhedron has an infinite number of different geometric interpretations. If you choose a polyhedron on Polytopia.eu, you will adopt the entire combinatorial type. So you have actually adopted infinitely many polyhedra. To make it less confusing and easier to make the model, we have chosen a clear realization of the polyhedron. These are the so-called Koebe-Adreev-Thurston realizations of polyhedra. In particular, these realizations have a sphere inscribed inside the polyhedra that touches each of the edges at exactly one point. In particular, each surface contains a circle that touches the edges just once.

f-vector

The f-vector of the polyhedron indicates how many vertices, edges, and faces it has. A vector in this case is not a geometric quantity but only the way of representing these numbers. The cube consists of 8 vertices, 12 edges, and 6 faces, and thus has the f-vector (8,12,6). However, the polyhedra are not uniquely determined by this vector. There may be other polyhedra with the same f-vector that have a completely different structure. We call these polyhedra siblings.



(a) (b)

Figure 4.24: Here we see a cube and its sister. She also possesses 6 faces, 12 edges, and 8 vertices but contains an entirely different structure that the cube.

Mathematical Models

Physical models and their construction have long played an important role in mathematics. For one thing, there was simply no other way to understand ideas in a three-dimensional environment. Of course, three-dimensional models can always be drawn, but then the drawing is only a projection of the model onto the plane, much like taking a picture of the model. When it comes to photos of familiar objects, recognizing the space does not cause us any problems, because we know, for example that a table is usually right-angled. If we see a perspectively distorted table in a photo, we intuitively know about the right angles. Of course, this intuition is not there when trying to understand the structure of an unfamiliar geometric object. In order to recognize certain properties, such as an axis of symmetry, it is very helpful to actually hold an object in your hand and turn it.

Models serve not only to gain knowledge but also to share knowledge. To make their research accessible to others, mathematicians needed a way to visualize it. Nowadays, this is done mainly with computers. There is a lot of software to generate mathematical and geometric graphics. Rotation of a model using this software also counteracts the problem of restriction to the flat screen.

Dürer's Conjecture

Although mathematicians have been dealing with polyhedra since ancient times, not everything is known about them. For one thing, every question that is answered only brings about new questions. For example, the number of three-dimensional polyhedra is known only up to 18 vertices. If somebody should find out how many polyhedra there are with 19 vertices, one can immediately ask about the number of polyhedra with 20 vertices. There are also questions that have been waiting a long time for an answer. A nice example, because it is easy to understand and yet still an unsolved problem, is the so-called Dürer conjecture. The painter Albrecht Dürer spent some years studying mathematics and the concept of the net of a polyhedron goes back to him. In his book, "The Painter's Manual" he drew nets of several polyhedra.

A net of a polyhedron is created by considering the polyhedron as an empty shell, which is cut along its edges in such a way that it remains connected but can be laid flat without distorting the faces. The question behind Dürer's conjecture is whether this is possible for every polyhedron such that the faces do not overlap when it is unfolded. In other words, does every polyhedron have a net?

To date, many mathematicians have considered this question and there are some intermediate results. For example, it is known that you can unfold any polyhedron without overlap if you pull and distort the faces and thus change its geometrical realization but not its structure (s. https://arxiv.org/pdf/1305.3231.pdf). The only polyhedron for which we know for sure that can always be unfolded without changing its geometrical structure is the tetrahedron.

Since we have automatically generated the unfoldings for the polyhedra in our project, it is possible that the net of your adopted polyhedron is overlapping. If so, write us an email!

Siblings

Polyhedra are siblings if they contain the same number of vertices, edges and faces, hence the same f-vector. Similar to human siblings, some polyhedral siblings do look alike each other while others have a completely different form. The cube consists of 8 vertices, 12 edges and 6 faces. These numbers do not uniquely define its structure. There are polyhedra who have the same f-vector but an entirely different structure.

Fields of Application of Polyhedra

From a purely mathematical perspective, polyhedra are, above all, beautiful and interesting, and their exploration requires no further justification. Nevertheless, one can obviously ask the question, which is almost as old as mathematics itself, what do you really need it for?

One important application of polyhedra is Linear Optimization. It is a method that is often used in business, among other areas, to make decisions that depend on many factors.

One example is making a timetable and network line for a public transportation system. There are many variables to be considered, such as arrival and departure times, operational costs, line capacities and so on. City planners want to meet public expectations for how often a train comes and also minimize the costs, run enough trains to carry enough passengers, but are also limited to the number of trains on the tracks for safety reasons. From these variables, a system of linear inequalities arise and their set of possible solutions form a polytope. The optimal solutions are located at the vertices of that polytope. So finding these vertices gives city planners optimal ways to build the most effective timetable possible.

How do the names of mathematical objects actually come about?

The Greek word for five is “penta”, so a pentagon is a five sided polygon. The hexagon, heptagon, and octagon get their names in the same way, but there is no trigon. Instead, a triangle is the 3-sided polygon and gets its name from its three angles. But then what is a square? Clearly, it is not enough that the name alone can give a definition. Although “square” does not describe the features, it is a commonly known shape. Therefore, it is necessary to actually use the name so that its meaning is well known.

Mathematical objects are also often named after mathematicians. More often than not, these objects and other concepts have been named after male mathematicians, but female mathematicians have also left a legacy behind. The Noetherian rings, named after Emmy Noether, and the Witch of Agnesi, after Maria Agnesi are some examples, but there is a need to close the gender gap.

Mostly, the objects that are named after mathematicians are given these names by scientists. The concept of a ring was already known, but to be able to distinguish the rings that Emmy Noether wrote about from the general ones, one talked about Noetherian rings. The convention of these rings came first and later a definition was established.

The Dürer conjecture was never proposed by the painter Albrecht Dürer himself but the underlying nets of polyhedra go back to him. The conjecture itself was posed by the mathematician G. C. Shephard in 1975. Why then it is known as Dürer’s and not Shephard’s conjecture, one can only speculate.

In summary, the rules and conventions for naming are rather ambiguous. It is similar to getting a nickname – if everyone knows who or what is meant, then the name sticks.

Data Protection

The "Polytopia" project aims to provide mathematics to students from a different angle, as well as to interested adults. On our website users can register and adopt an individual mathematical object, namely a polyhedron. Through this adoption, a relationship is created, which is intended to stimulate intensive study and to develop mathematical questions. Results of this activity can be presented as a photo in one of our galleries, to inspire other participants as well.

We take the protection of personal data very seriously and treat personal data confidentially in accordance with the statutory data protection regulations and this privacy policy.

We try to avoid the collection of personal data as much as possible. However, an inquiry of e.g. the email address is essential. Here, you will find the following information:

- Name and address of the responsible person
- General information about data processing
  - Extent of processing of personal data
  - Legal basis for the processing of personal data
  - Data erasure and storage duration
  - Creation of log files
  - Use of cookies
  - SSL encryption
- Forms
  - Registration
  - Contact form and e-mail contact
  - Newsletter
- Rights of the person concerned

Name and address of the responsible person

The person responsible within the meaning of the General Data Protection Regulation (GDPR) and other national data protection laws of the member states as well as other data protection regulations:

Prof. Dr. Alexander I. Bobenko Institut für Mathematik, MA 8-4, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany E-Mail: bobenko[at]math.tu-berlin.de

General information about data processing

Extent of processing of personal data

In principle, we process personal data of our users only insofar as is necessary to provide a functioning website and our content and services. The processing of personal data of our users takes place regularly only with the consent of the user. An exception applies to cases in which prior consent cannot be obtained for reasons of fact and the processing of the data is permitted by law.

Legal basis for the processing of personal data

Insofar as we obtain the consent of the data subject for processing of personal data, Art. 6 para. 1 lit. a EU General Data Protection Regulation (GDPR) serves as legal basis. In the processing of personal data necessary for the performance of a contract to which the data subject is a party, Art. 6 para. 1 lit. b GDPR serves as legal basis. This also applies to processing operations required to carry out pre-contractual actions. Insofar as the processing of personal data is required to fulfill a legal obligation which the persons responsible must fulfill, Art. 6 para. 1 lit. c GDPR serves as legal basis. In the event that vital interests of the data subject or another natural person require the processing of personal data, Art. 6 para. 1 lit. d GDPR serves as legal basis. If the processing is necessary to safeguard the legitimate interests of the persons responsible or of a third party, and if the interest, fundamental rights, and freedoms of the data subject do not prevail over the first interest, Art. 6 para. 1 lit. f GDPR serves as legal basis for processing.

Data erasure and storage duration

The personal data of the data subject will be deleted or blocked as soon as the purpose of the storage is deleted. In addition, such storage may be provided for by European or national legislators in EU regulations, laws, or other regulations to which those responsible are subject. Blocking or deletion of the data also takes place when a storage period prescribed by the standards mentioned expires, unless there is a need for further storage of the data for conclusion of a contract or fulfillment of the contract.

Creation of log files

Each time our website is accessed, our system automatically collects data and information from the computer system of the calling computer. The following data is collected here:

- The IP address of the user
- Time of access
- Websites from which the system of the user comes to our website
- Websites that are accessed by the user?s system on our website

A storage of this data takes place in a log file. This is regularly evaluated in order to identify and correct incorrect links and queries. The log file will then be deleted and the data will not be processed or passed on to third parties.

Use of cookies

Our website uses cookies. Cookies are text files that are stored in the Internet browser on the user?s computer system. When a user visits a website, a cookie may be stored in the user?s browser. This cookie contains a characteristic string that allows the browser to be uniquely identified when the website is reopened. We use cookies to make our website more user-friendly.

The following data is stored and transmitted in the cookies:

- Log-in information
- Consent to this privacy declaration

When visiting our website, users will be informed by an information banner about the use of cookies and are referred to this privacy policy.

SSL encryption

This site uses SSL encryption for security reasons and to protect the transmission of sensitive content, such as the requests you send to us as the site operator. You can recognize an encrypted connection by changing the address line of the browser from ?http://? to ?https://? by the lock symbol in your browser line. If SSL encryption is enabled, the data you submit to us cannot be read by third parties.

Forms

On our website we use different forms to make our offer available online. Below is a section for each form that explains this in more detail.

Registration

The following data is collected during the registration process.

- E-Mail address

At the time of registration, the following data will also be stored:

- Date and time of registration

As part of the registration process, the consent of the user to process this data is obtained. There is no disclosure to third parties in this context.

Registration allows the user to adopt a polyhedron and upload images. Account details can be deleted by e-mail request to info@polytopia.eu or via the profile of the user.

Contact form and e-mail contact

On our website, a contact form is available which can be used for electronic contact. If a user utilizes this option, the data entered in the input mask will be transmitted to us and saved.

This data is:

- Specified e-mail address
- Message body

Alternatively, contact via the provided e-mail address info@polytopia.eu is possible. In this case, the user?s personal data transmitted by e-mail will be stored.

For the processing of the data, the consent of the user is obtained in the context of the sending process and referred to this privacy policy. In this context, there is no disclosure of the data to third parties. The data is used exclusively for the conversation with the respective user and then deleted.

Newsletter

Visitors of the website can subscribe to a newsletter.

The following data is collected:

- E-mail address

Regularly, the responsible persons inform the registered website visitors, who have signed up for the newsletter about news on the project. The de-registration from the newsletter is made by the user by e-mail to info@polytopia.eu. The user is pointed to the opt-out option in every newsletter e-mail. There is no transfer of the collected data to third parties.

Rights of the person concerned

If your data is processed by our site, you have the following rights according to the GDPR:

- Revocation of consent (Art. 7 Abs. 3 GDPR)
- Information (Art. 15 GDPR)
- Corrigendum (Art. 16 GDPR)
- Deletion (Art. 17 Abs. 1 GDPR)
- Restriction of processing (Art 18 GDPR)
- Data transferability (Art. 20 GDPR)
- Opposition to processing (Art. 21 GDPR)
- Appeal to the supervisory authority (Art. 77 GDPR)

Feedback Questionnaire

Which of the following groups do you belong to?

- Pupils
- Teachers
- Mathematicians
- University Students
- Other

Are you interested in Mathematics?

- Yes, very, it is actually part of my job.
- Yes, very, it is a hobby.
- Sometimes Mathematics is interesting.
- Rather seldom.
- No, I think Mathematics is boring.
- Other

What statement do you agree the most with?

- I understand Mathematics fast and it is fun for me to discover connections myself.
- When I have some time to do Mathematics, I manage it quite well
- As soon as someone explains things to me at my pace, I understand most of it.
- More often than not I find Mathematics to be rather complicated.

What is your age?

Which gender do you most identify with?

How did you end up here?

- Randomly surfing the web
- Purposely searching
- We did this at school
- Other

How did you hear about us?

- Internet search
- Internet search
- School
- University
- Friends
- Family
- Media (Radio, Newspaper, Flyer, ...)
- Facebook
- Other

Did you learn something new about math here?

- Nope
- A tiny bit
- An average amount
- Quite a lot
- Yes! I'm an expert on polyhedra now

How would you rate this page on a scale from 1 to 6?

- 1 (excellent)
- 2 (very good)
- 3 (good)
- 4 (just okay)
- 5 (not so good)
- 6 (needs improvement)

What do you like most about our page?

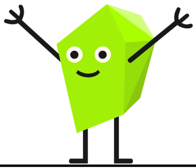
- Adopting the polyhedra
- The background material
- Our school packets
- The mascots Polly and Ecki

Do you have any other comments or tips for us?

School Material

Name: Class: Date:

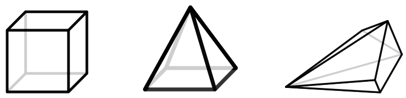
Adopt a Polyhedron



There are an infinite number of polyhedra. Unlike the cube or the pyramid, which are both polyhedra, most don’t have a name yet. You can help change that! Adopt a polyhedron and officially give it a name.

Convex polyhedra consist of:

- flat side surfaces
- straight edges
- outward pointing corners



How can I adopt a polyhedron?

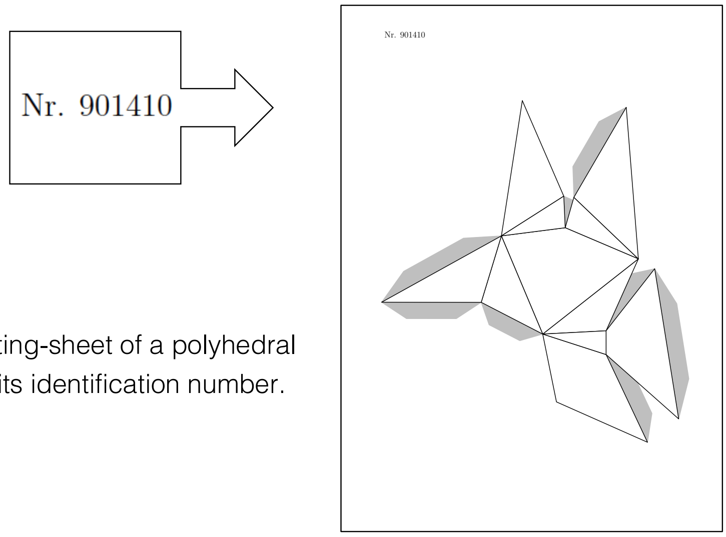
1. Build a model:
   Cut out the polyhedral net with the gray tabs attached. Now, fold it along the black edges and glue it together using the tabs.

2. Give your polyhedron a name:
   Register on the website www.polytopia.eu. There is a number on the craft sheet, which you can use to find your polyhedron by searching for it on the website. Give it a name and it’s yours!
   You can also upload a photo of your polyhedron model.

On www.polytopia.eu you can discover much more! There are games, a 3D polyhedron viewer and a ♥-feature.

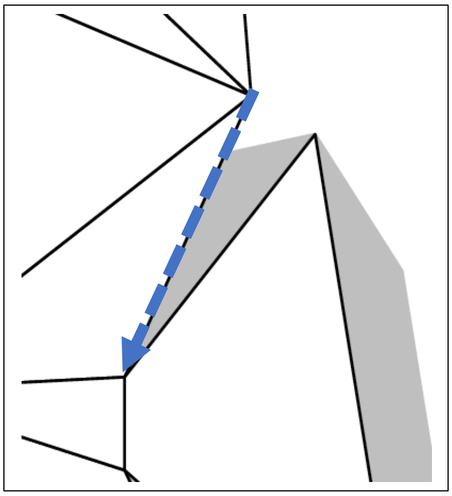
Class Set of Polyhedra

This class set contains up to 36 nets of polyhedra. It is freshly generated from our database so each time this class set is downloaded it will consist of different and individual polyhedra. Your students can officially adopt the polyhedra on our website www.polytopia.eu. There is a number printed on the crafting-sheet, which the students can use to find their polyhedra on the website.



The crafting-sheet of a polyhedral net with its identification number.

Tips for modeling:
- The gray areas are the tabs for gluing. These could also be cut off and the model can be assembled with tape.
- If a gluing tab and an adjacent surface overlap, as in the photo to the right, one has to cut all the way into the corner (the thick, dashed line to the tip of the arrow).
- All black edges should be folded before assembly. One tip to make them nice and straight is to fold them over a ruler or protractor.

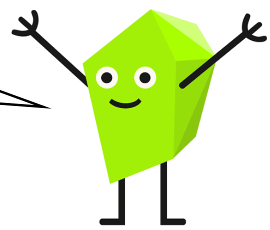


Name: Class: Date:

Finding Research Questions

Mathematical research always begins with searching for and coming up with research questions. Asking interesting questions is already halfway to interesting research. That's why it's so important to always be thinking about new questions.

For example, mathematical research questions usually begin with *why, when, how much, what happens if…*



Write down as many mathematical research questions as you can.

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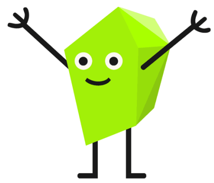
The special thing about mathematical research questions is that they may be very difficult to answer, even if they look simple. Therefore, it is not important if you can answer your research questions. Today, it's just about finding the questions.

Which of your questions do you think is the most difficult to answer? I would be happy if you sent me an email with this question!
My email is: ecki@polytopia.eu

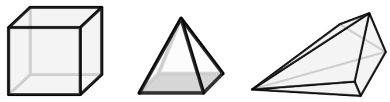
Teacher's Handbook for the packet „Class Set"



There are an infinite number of convex polyhedra. Unlike the cube or the pyramid, which are both polyhedra, most of them do not have a name. Together with your students, you can change that. Adopt the polyhedra and give them names.

Convex polyhedra consist of:

- flat side surfaces
- straight edges
- outward pointing corners



The cube, pyramid and an (still) unnamed polyhedron with six corners.

Learning Goals:

The main learning objectives of this project are the relation between space and form and understanding different mathematical representations. The handling of spatial geometric objects is practiced and the representation of the polyhedron as a polyhedral net invites students to consider forms in different dimensions. The two-dimensional net must be brought into its three-dimensional shape by the process of cutting, folding and gluing together.

On the website, students find their personal polyhedra in an interactive, computer-aided, digital representation. Their polyhedron can be changed by color or by switching the corners, edges and side surfaces viewer on and off. The students have the opportunity to give their personal polyhedron a name. The symbolic representation is therefore chosen by the students and is not given, as is usual in mathematics and the natural sciences. In this way, students learn that it is possible to participate in shaping science. This change in the perception of mathematics is one of the main goals of our project.

Polyhedra can be incorporated into many different lesson plans. For example, a series of lessons on surface area and volume calculations of geometric bodies or in the context of analytical geometry. For ages 10 to 14, we recommend accompanying the learning packet "Polly's Journal" to this class set, which can be found on our website.

What do you need for the implementation of the project?

- Roughly two hours

- A class set of polyhedral nets
- Adoption worksheet
- Finding research questions worksheet (optional)

- Scissors
- Tape or glue

- Internet enabled devices: cell phones, tablets, computers…
- Email addresses for registration

Components of the project

Adoption worksheet: The Adoption worksheet gives the definition of convex polyhedra, briefly summarizes the work assignments and contains the website address where the polyhedra can be adopted.

Polyhedral nets: Each student receives an individual polyhedral net with a number. The nets are cut out along the outer line, folded along the black edges and glued together using the gray tabs.

Model building: The models are created from the craft sheets. The class set of polyhedra is generated fresh from our database with each download. Therefore, these are various and randomly gathered polyhedra that are still available for adoption. The polyhedral nets have different complexities and therefore can be distributed to students based on their skill level.

Video tutorial: There is a video with instructions for building the models. You can watch it on our homepage and download it for use in the classroom.

Adoption: On the website www.polytopia.eu, your students can register with their email addresses. From experience, it is helpful to inquire beforehand if the students already have email addresses and if they have access to them. No further data is needed. Using the identification number on the crafting sheet, the respective polyhedron can be found on the website.

Finding Research Questions worksheet: This worksheet turns the usual mathematics lesson on its head. The goal is to find an interesting and not yet answered question about polyhedra. As the project “Adopt a Polyhedron” springs from scientific mathematics, we would like to invite the students to take on a research point of view of polyhedra.

Questions and Feedback: We welcome comments, questions and feedback on your project experience. Write us an email: schule@polytopia.eu

About the project, "Adopt a Polyhedron"

The project "Adopt a Polyhedron" is part of the public relations work of the Collaborative Research Center "Discretization in Dynamics and Geometry", which is funded by grants from the Deutschen Forschungsgemeinschaft (DFG) and is primarily involved with the structure and applications of discrete mathematics. Mathematicians from the Technische Univerität Berlin, the Technische Universität München, and the Freie Universität Berlin are investigating the discretization of differential geometry and dynamic systems. Discrete in a mathematical context means distinguishable. For example, the four corners of a square are clearly separated while a circle could be understood as a polygon with an infinite number of indefinable corners. Three-dimensional polyhedra, with their well-defined corners, edges and side surfaces belong to the classical research field of discrete geometry.

Goals

The goal of this project is to build models of "all" polyhedra in a collective endeavor. To accomplish this, we have initially released all polyhedra with up to nine vertices for adoption. It is not possible to realize all polyhedra, as there are an infinite number of them, but everyone can help bring as many as possible to life by adopting their own individual polyhedron, giving it a name and then building a model. In particular, we would like to invite students to actively participate in mathematics by focusing on the construction of geometric models. Modeling has long been a central discipline in (university) mathematics and has in recent decades been phased out by visualization with computers. However, the manual assembling of a model allows engagement and a deeper understanding of mathematics beyond abstract ideas.

Citizen Art

Lately, there has been an increasing effort in Citizen Science to actively engage everyday people in scientific research. As mathematicians, we also wish to offer interested people an opportunity to participate. Since the construction of models also has a creative and individual aspect and we want to emphasize the relationship between mathematics and art, we characterize our project under the term *Citizen Art*.

SFB TRR 109 Discretization in Geometry and Dynamics · Freie Universität Berlin · Technische Universität Berlin · Deutsche Mathematiker-Vereinigung · DFG Deutsche Forschungsgemeinschaft

Polly and her family

Name: ______________________

Name: Class: Date:

My name is Polly and I am a polyhedron. The great thing about my name is it describes what I am!

The word polyhedron comes from the ancient Greek words “poly”, meaning many and “hedron” meaning surface. I am made up of only flat surfaces that meet at straight edges and because polyhedra like to eat, we all have corners that stick outwards. As you can imagine, I have a very large family. My brother is a superstar. His name is Cube and I’m sure you already know him. You’ve probably seen my aunt, the pyramid, before too.

Sadly, I do not know all of my family members, so I would like to make a family photo album with everyone in it!

I asked my friends, the mathematicians at the Freie Universität Berlin, if they could help me, but they only know the structure of each of my family members and cannot build them all – there are just too many of us!

That’s why I need your help! Please adopt one of my family members and build a model. You can even give your own polyhedron an official name. The crafting sheets for building my relatives can be downloaded at www.polytopia.eu. After you build the model, please take a picture of my relative and upload it so I can get to know them.

But first, I want to show you what is so special about us polyhedra, how to build your model, and much more…

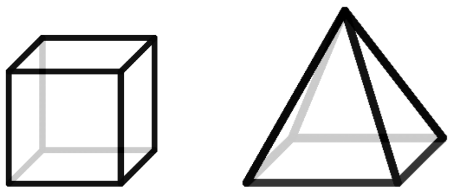
Name: Class: Date:

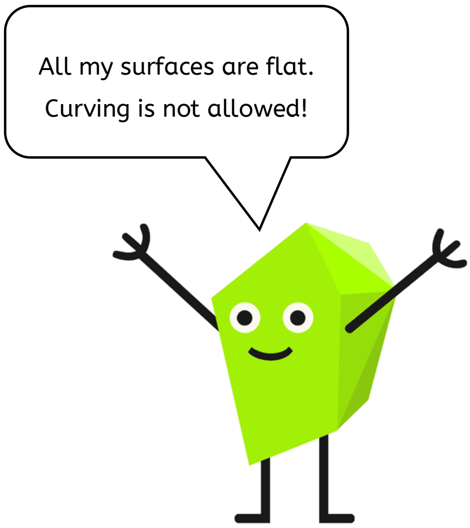
What does a polyhedron from Polly's family look like?

Polyhedra belonging to Polly's family are bodies that are made of

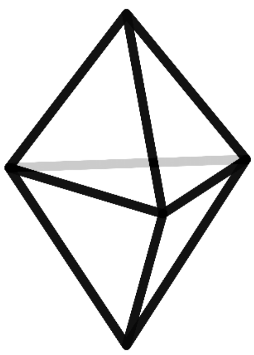
- flat surfaces
- straight edges
- outward corners

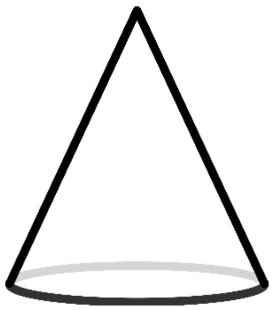
The cube and the pyramid are examples of Polly's relatives.

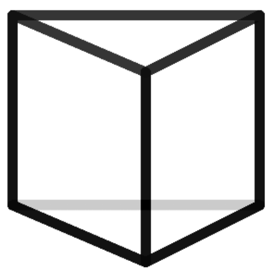




Mark which bodies are polyhedra:

☐ 

☐ 

☐ 

Name: Class: Date:

Do you know any more polyhedra from Polly’s family? Draw as many as you can.

Hint: Look around the classroom or think about your room. You can probably find a few polyhedra there.

Here you can draw bodies that are not part of Polly’s family or are not polyhedra.

Name: Class: Date:

WHICH STATEMENT BEST FITS YOU?

- □ I ALREADY KNEW ABOUT POLYHEDRA.
- □ I KNEW OF SUCH SHAPES, BUT THE TERM POLYHEDRON WAS NEW FOR ME.
- □ THIS IS ALL NEW FOR ME.
- □ I AM STILL NOT SURE WHAT POLYHEDRA ARE.
- □ ____.

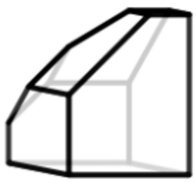
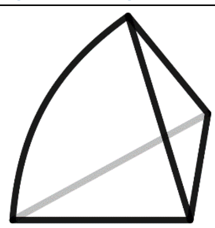
Want some more practice? Right this way! If you are already a polyhedron expert, continue to the next page.

Fill in the blanks with the correct words from the box.

straight, ball, round, inward, outward, prism, flat

Polyhedra from Polly's family are bodies made from _____________ surfaces, _____________ edges and ______________________ corners. The cube and the pyramid both belong to Polly's family. The _____________ is not a polyhedron.

Is it a polyhedron? Mark the box if it is and explain why or why not.

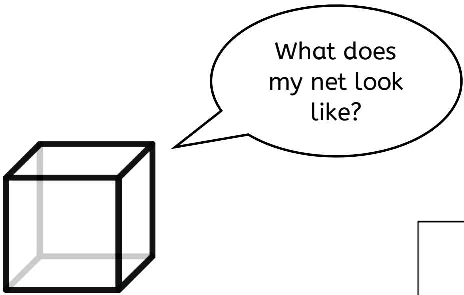
□ 	□ 	□ 

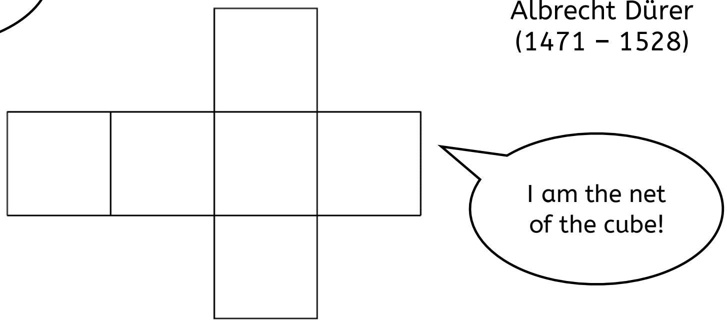
Name: Class: Date:

The net of a polyhedron

The painter and mathematician Albrecht Dürer first discussed the idea of polyhedral nets more than 500 years ago. If you cut open a polyhedron along its edges and unfold it, you get the net of the polyhedron.

Albrecht Dürer
(1471 – 1528)

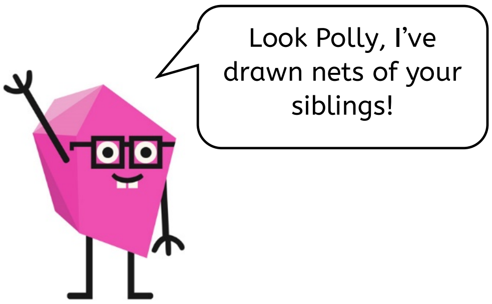


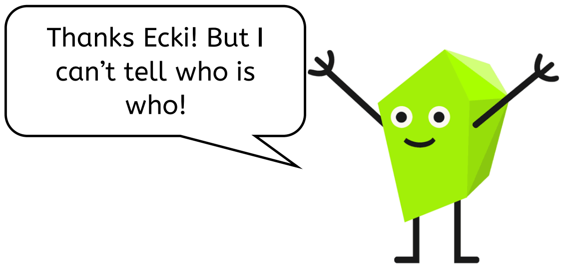


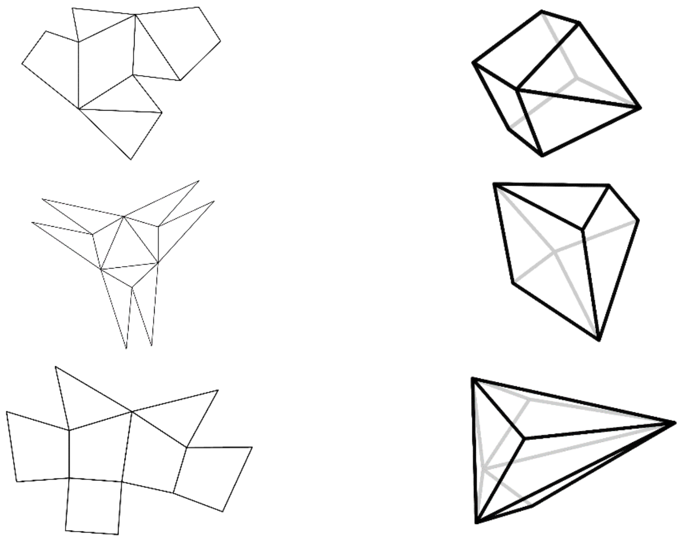
There are always several different ways to draw the net of any polyhedron. Draw as many cube nets as possible.

Name: Class: Date:

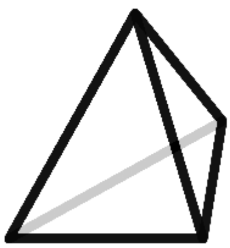
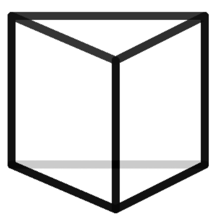
Polly’s friend Ecki has drawn nets of Polly’s siblings. However, Polly doesn’t know which net belongs to which sibling. Connect the polyhedra to the matching nets.







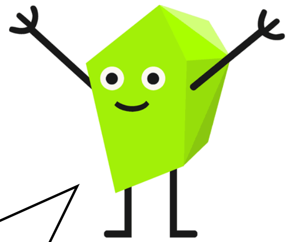
Now it’s your turn! Draw a net for at least one of the two polyhedra.

			
Tetrahedron		Prism	

For the quick drawers: Draw a polyhedron from Polly’s family and its net on a separate piece of paper.

Name: Class: Date:

Your whole class can help me bring one of my family members to life! To do this, you need to build a model of the polyhedron. Your teacher has already prepared the net of a polyhedron. The goal is to build a big cardboard model of this polyhedron together. The polyhedral net will be cut apart and each group will receive a piece. As a group, you are responsible for the enlargement of this piece. In the end, all the sides will be put together to make a large polyhedron.



First, describe your group's piece with a few words.

These questions can help you: How many edges does it have? What is the name of the shape of the surface? Are the angles acute, right or obtuse?

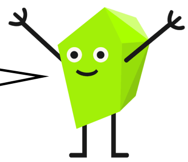
Name: Class: Date:

Draw a sketch of your piece in the box.

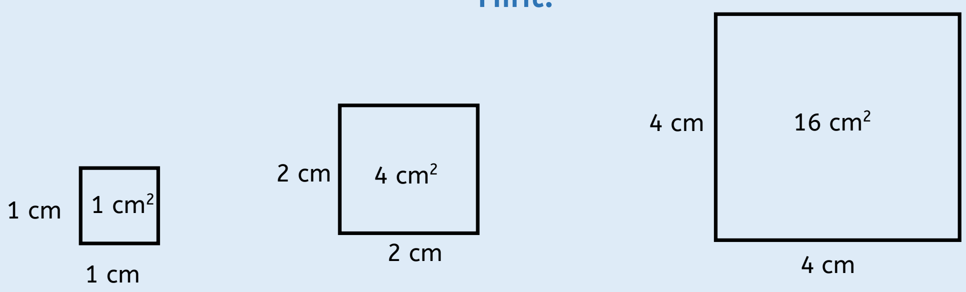
In a sketch, it is not important that the lengths and angles are exactly right. The surface just has to look something like the original.

Now measure the lengths of the edges and the degrees of the inner angles of your piece. Write down your results next to the edges and angles in your sketch. Also, calculate the surface area if you can.

Remember to use the correct scale when using the protractors!



Hint:



If the edge length doubles,
then the surface area increases by a factor of _____.

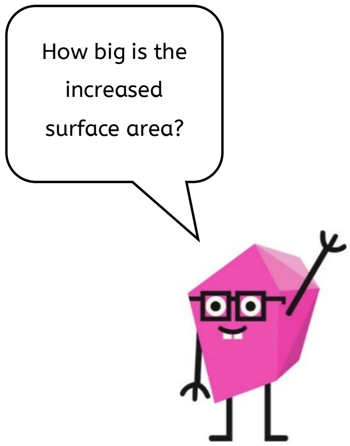
Name: Class: Date:

Write the measured lengths of your edges in the table. Discuss as a class by what factor you want to increase the edge lengths.

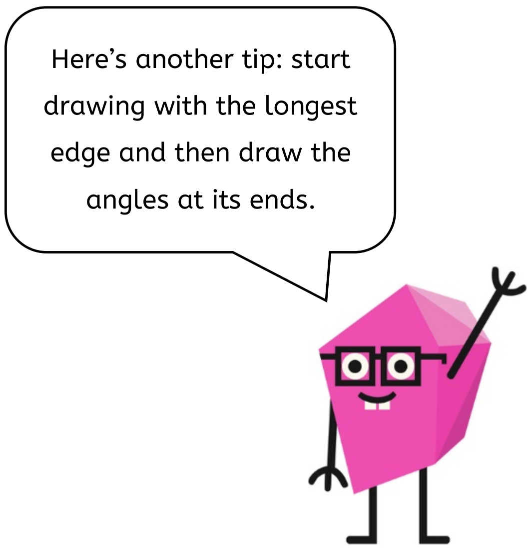
The edge lengths should be _______ times as long.

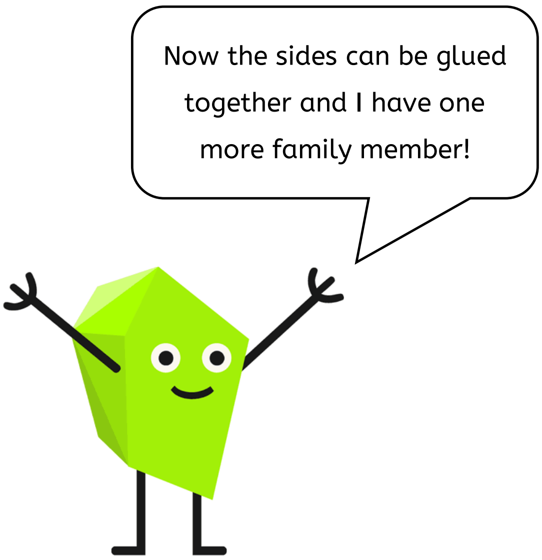
Write the calculated enlarged values in the table as well.

Original edge length	Enlarged edge length



Now we need the cardboard to draw and cut out the new enlarged side surface. Before you start cutting, make sure to compare your drawing to the template. Are all the angles and lengths correct?





Name: Class: Date:

Fact sheet for your class polyhedron

Name: ___________________________

Birthday: ___________________________

Number of side surfaces: ___________________________

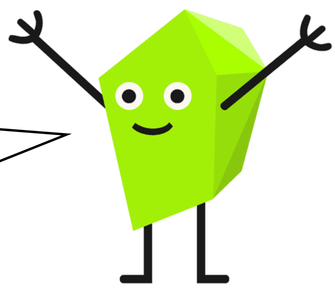
Number of edges: ___________________________

Number of corners: ___________________________

Paste a photo of your polyhedron here.

Please upload the photo of your polyhedron to www.polytopia.eu. Here you can also adopt your own polyhedron and build a model.

Thanks for all your help!



Name: Class: Date:

Fact sheet for your own polyhedron

Name: ___________________________

Birthday: ___________________________

Number of side surfaces: ___________________________

Number of edges: ___________________________

Number of corners: ___________________________

Paste a photo of your polyhedron here.

Write something about your polyhedron in the lines below:

____

____

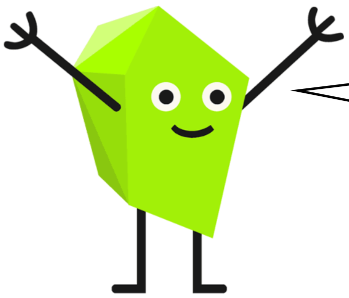
____

____

____

____

Name: Class: Date:



Did you know that mathematical research begins with looking for new questions? Let's research together and find interesting math questions!

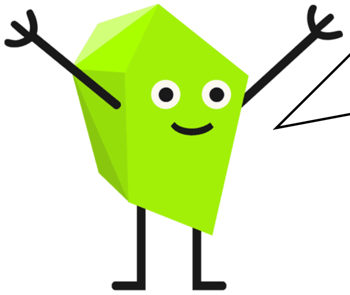
Write down as many mathematical questions about polyhedra as possible:

Mathematical research questions could begin with *why, when, how much, what happens if, …*

Name: Class: Date:

The special thing about mathematical research questions is that sometimes they cannot be solved quickly, even if they sound simple.

Therefore, it is not important if you can answer your research questions. Today, it is just about finding



Write down a mathematical research question that you think is difficult to answer:

Hint: If you think your own questions are too easy, discuss with your classmates. Everyone will have different and interesting questions!

By the way, we polyhedra love mathematical research questions. Feel free to send them to us:

polly@polytopia.eu

ecki@polytopia.eu

We look forward to your emails!

IMPRESSUM:

Published by the DFG- Collaborative Research Center "Discretization in Geometry and Dynamics", Transregio 109

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www.polytopia.eu, Anna Maria Hartkopf

Special educational support: Pauline Linke

Graphics: Johanna Steinmeyer und Max Pohlenz

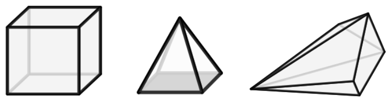
Translation: Erin Henning

Teacher's Handbook for "Polly's Journal"

There are an infinite number of convex polyhedra. Unlike the cube or the pyramid, which are both polyhedra, most of them do not have a name. Together with your students, you can change that. Adopt the polyhedra and give them names.

Convex polyhedra consist of:

- flat side surfaces
- straight edges
- outward pointing corners



The cube, the pyramid and an (still) unnamed polyhedron with six corners.

With the worksheets in "Polly's Journal", your students can get to know polyhedra while learning and practicing important content-related and process-related mathematical competences. In this handbook for teachers, you will find practical hints for the implementation of the project as well as useful links between the project and the curriculum as well as an overview of the learning prerequisites and objectives. In addition, a solution sheet is provided for the journal. At the end of this handbook, you will also find a feedback form for teachers. The email address for any feedback, as well as questions and comments is: schule@polytopia.eu.

Contents of this Handbook

Polly’s Journal as a guide for the math class

In order to connect the project “Adopt a Polyhedron” to a standard math class for ages 10 to 14, we have developed the learning packet “Polly’s Journal”. Learning journals are a major component of dialogical learning. The journal serves as a guide for the students where the learning process is simultaneously presented and individually controlled.

The journal contains activities of various kinds, such as reading, writing, drawing, matching, measuring and arithmetic. Through these exercises, the students enter into an active dialogue with the material. The booklet is designed so that the learners can work through it relatively quickly and thus an immediate experience of learning success is made possible. While working through the packet, students have the possibility for self-reflection and to do two differentiation tasks.

The journal is divided into four sections. First, the concept of polyhedra is introduced and practiced in tasks for recognizing and drawing polyhedra and non-polyhedral bodies. The next section deals with polyhedral nets. It starts with the already known cube net and is extended to general nets of polyhedra.

In the third section, the class builds a polyhedron model together. The students are divided into small groups and each group receives a different side surface of the polyhedron. The edge lengths of the side surfaces are to be increased by a fixed factor, which is either provided by the teacher or determined by the students themselves. Here is a more advanced challenge for the students is to discover the quadratic relationship between edge lengths and the surface area. While the individual enlarged side surfaces are being assembled into a polyhedron, the students collect suggested names for the polyhedron. Upon completion of the model, a vote is taken to determine a name. Additionally, there is the possibility that every student adopts a separate polyhedron. You can download an entire class set of polyhedra that is generated from our database, resulting in a new set with each download.

Now that the students have dealt extensively with polyhedra and even adopted one, they are invited in the last section to take a research-based look at the subject. The usual course of mathematics education is turned upside down, because now it's about developing interesting and possibly unsolvable questions. The students take on the role of mathematical researchers to improve their overall understanding of polyhedra.

Learning objectives

The main learning objectives of this packet are for students to understand the relationship between space and form, to understand different mathematical representations of a concept and to calculate area by measuring and arithmetic.

The convex polyhedra are explored in different geometric representations. The definition of convex polyhedra is presented by visual examples. Recognition and determination of convex polyhedra is first practiced on the basis of their three defining properties (flat side surfaces, straight corners, and outward pointing corners). Geometric bodies are analyzed according to these criteria and students must determine whether or not they are convex polyhedra.

Subsequently, polyhedra and polyhedral nets are associated with each other. In the process of assembling the two-dimensional polyhedral net into a three-dimensional model by hand, the representation is changed by the learners themselves. Through this action, the students also gain insight into how the same form can be perceived in different dimensions. The interactive visualization tool on the website offers another form of representation. For the students with access to VR glasses, another visualization experience is available to add even more perspective.

On the website, students find their personal polyhedra in an interactive, computer-aided, digital representation. Students can change the color of their polyhedron or change the perspective by switching the corner, edge or side surface viewers on and off. The students have the opportunity to give their personal polyhedron a name as well. The symbolic representation is therefore chosen by the students and is not given, as is usual in mathematics and the natural sciences. In this way, students learn that it is possible to participate in shaping science. This change in the perception of mathematics is one of the main goals of our project.

Finally, during the construction of the enlarged polyhedron, the lengths and angles of the side surfaces must be measured. Here, the handling of a protractor and its different scales is practiced. The measured quantities are entered into a table with reasonable accuracy. When enlarging the pieces, the lengths of the edges are multiplied but not the angles. Using a protractor will again be practiced when the students are drawing the enlarged side surface piece onto the cardboard.

Learning prerequisites:

The students do not need to be completely comfortable with the prerequisites listed below. Most of these are repeated in "Polly's Journal", giving students a chance to practice and recall. The central prerequisites are:

- Measuring polygon edges
- Measuring and drawing angles
- Handling units of measure, such as centimeters and degrees
- Drawing of triangles and other polygons
- Multiplication of decimal numbers with an integer factor
- The net of the cube

Practical information for preparation:

Print

- Learning Journal "Polly's Journal", one copy per student
- A class polyhedron, three times
- Set of polyhedral nets (if possible printed on stronger paper)
- Solution sheet for hanging in the classroom

Need

- Cardboard (if possible in different colors)
- Tape (if the class polyhedron is enlarged, the gluing tabs are cut off and needs to be assembled with tape)

Plan

- Remind students to bring scissors, glue and a protractor.
- Do the students have active and accessible email addresses?
- Consider whether a factor for the enlargement of the side surfaces is given or determined by the students themselves. (Pay attention to the dimensions of the cardboard and what is feasible).
- Number the side surfaces of the class polyhedron on all three sheets. Cut one of the nets apart, also cutting off the grey gluing tabs.

About the project, "Adopt a Polyhedron"

The project "Adopt a Polyhedron" is part of the public relations work of the Collaborative Research Center "Discretization in Dynamics and Geometry", which is funded by grants from the Deutschen Forschungsgemeinschaft (DFG) and is primarily involved with the structure and applications of discrete mathematics. Mathematicians from the Technische Universität Berlin, the Technische Universität München, and the Freie Universität Berlin are investigating the discretization of differential geometry and dynamic systems. Discrete in a mathematical context means distinguishable. For example, the four corners of a square are clearly separated while a circle could be understood as a polygon with an infinite number of indefinable corners. Three-dimensional polyhedra, with their well-defined corners, edges, and side surfaces belong to the classical research field of discrete geometry.

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The goal of this project is to build models of "all" polyhedra in a collective endeavor. To accomplish this, we have initially released all polyhedra with up to nine vertices for adoption. It is not possible to realize all polyhedra, as there are an infinite number of them, but everyone can help bring as many as possible to life by adopting their own individual polyhedron, giving it a name and then building a model,

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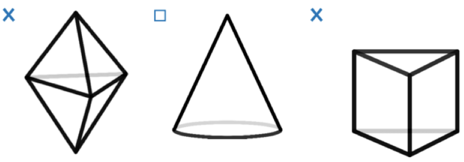
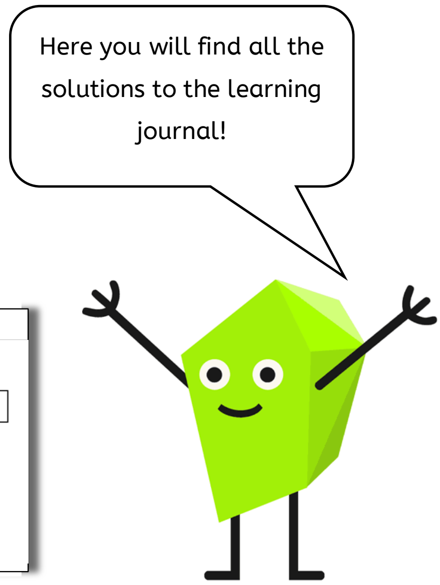
Lately, there has been an increasing effort in Citizen Science to actively engage the public in scientific research. As mathematicians, we also wish to offer interested people an opportunity to participate. Since the construction of models also has a creative and individual aspect and we want to emphasize the relationship between mathematics and art, we characterize our project under the term *Citizen Art*.

Solutions

Mark which bodies are polyhedra:

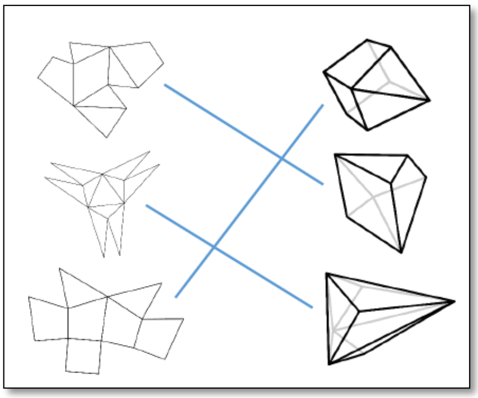




Fill in the blanks with the correct words from the box.

straight, ball, round, inward, outward, prism, flat

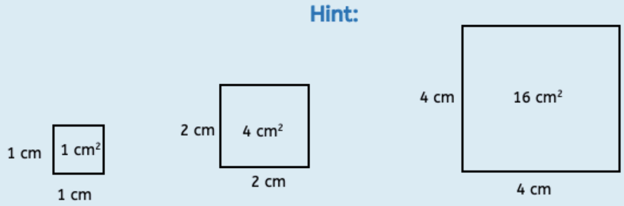
Polyhedra from Polly's family are bodies made from ____flat______ surfaces, ___straight____ edges and _____outward_______ corners. The cube and the pyramid both belong to Polly's family. The _____ball_____ is not a polyhedron.



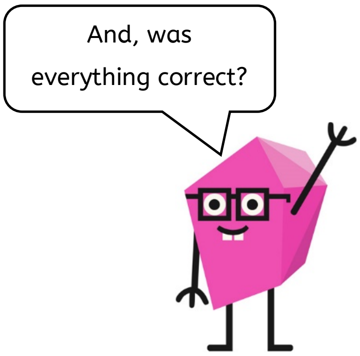
Is it a polyhedron? Mark the box if it is and explain why or why not.

X 	X 	☐ 
All faces are flat, all edges are straight, and all corners point outward.	All faces are flat, all edges are straight, and all corners point outward.	Two faces are not flat, and one edges is rounded.

Hint:



If the edge length doubles,
then the surface area increases by a factor of __4__.



Feedback:

We would like to know your thoughts on the project "Adopt a Polyhedron" and kindly ask you to complete this questionnaire. You can either copy these questions into an email and send them to schule@polytopia.eu, or by mail to: Anna Maria Hartkopf, Institut für Mathematik, Freie Universität Berlin, Arnimallee 2, 14195 Berlin.

1. In which grade / age level did you implement the project?

2. In which country / city is your school located?

3. How would you describe the performance of your students? Did the learning journal meet their learning level?

4. How much time did you plan for the project and how long did it actually take?

5. Did you complete the whole packet or only selected sections? If yes, which sections?

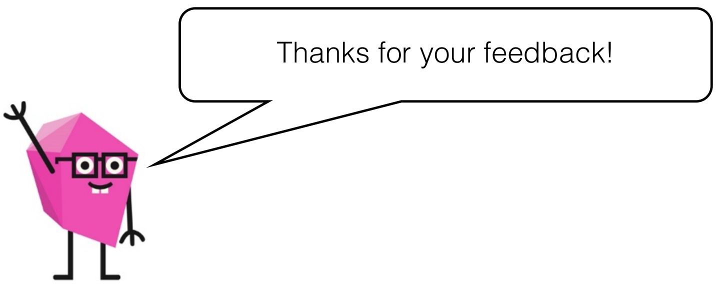
6. What worked well?

7. What did not work as well?

8. Which mathematical skills have your students newly learned or strengthened?

9. Were the students able to grasp a new view of mathematics through the project? If yes, how so?

10. Do you have any suggestions, tips or comments for improvement?



Materials and ideas for model construction

To build an individualized polyhedron, i.e. a more creative and artistic model, various materials are available. Below, we have put together some ideas for inspiration.

Clay, Play-Dough or Papier-Mâché (Solid Model)

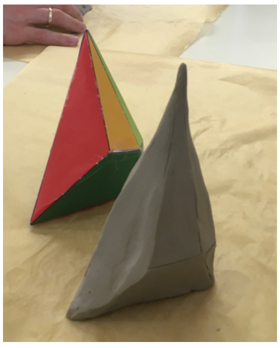
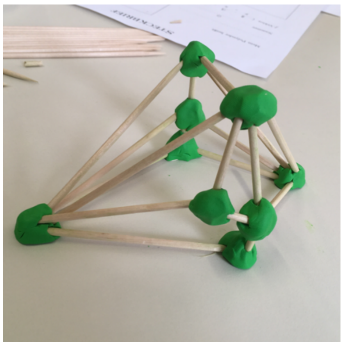
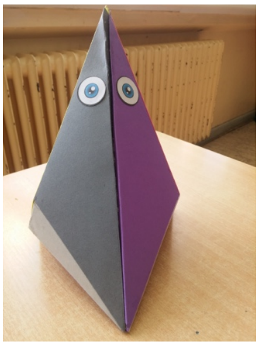
Using clay, play-dough or papier-mâché, the body of the polyhedron can be shaped. Papier-mâché can be slightly easier to handle. Solid clay models should not be fired in an oven because they are too thick, but they will harden completely when dry.

Chickpeas or Polymer Clay and Shish Kebab Sticks (Edge Model)

Soak the dry chickpeas overnight in water. The next day, they can be used as corner joints for the polyhedron edges. Let the model dry overnight so the chickpeas can solidify around the sticks. This will hold the model together nice and tightly. Polymer clay is also suitable as a connecting material for the polyhedron edges.

Cardboard or Fabric (Polyhedral Nets)

From cardboard, the net of the polyhedron can be redesigned while the size can also be changed. Anyone who can sew could also use the net of the polyhedron as a pattern for a soft toy polyhedron.

A clay polyhedron, an edge model and an enlarged polyhedral net model.

Profile

My polyhedron's name is

....

Number ....

f-vector (....... , ... , ...)
V E F

Glue your polyhedron model here.

The graph of your polyhedron

Mark the appropriate properties for your polyhedron.

- ☐ simple
- ☐ simplicial
- ☐ has an Euler path

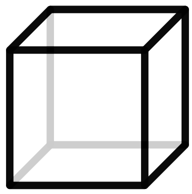
My family

same type: ....

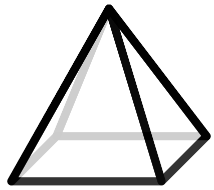
same f-vector: ....

& ....

Discover Card 0 The polyhedron



A polyhedron is made up of vertices (V), edges (E) and faces (F).
Well known examples are the cube and the pyramid.

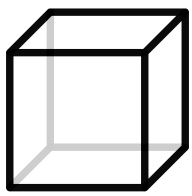


Exercises

1. Congratulations! You have just adopted your first polyhedron. Give your polyhedron a name.
2. To build a model, cut out the net of the polyhedron and glue it together using the tabs.
3. Now take a photo of your polyhedron model and upload it to the website.

✂ ....

Discover Card 0 The polyhedron



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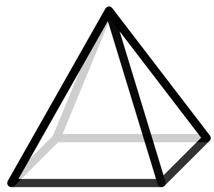

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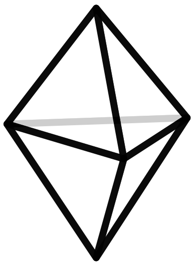
Discover Card 1 — The vertices, edges and faces of a polyhedron



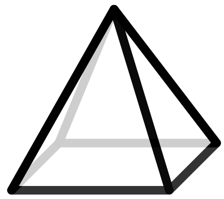
To start studying polyhedra, count the number of vertices, edges and faces. These numbers are written in the f-VECTOR.
The polyhedron on the left has 5 vertices, 9 edges and 6 faces.
Its f-vector is $(V, E, F) = (5, 9, 6)$.

Exercises

1. Count the number of vertices, edges and faces of the pyramid to the right.

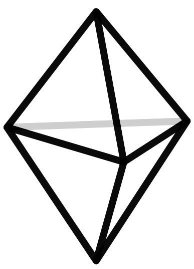
   Write them down in the f-vector below.

2. Find a polyhedron with the f-vector $(8, 12, 6)$. Draw it!
3. What is the f-vector of your polyhedron? Write it in the profile of your polyhedron.



$\Big(\underset{V}{\quad} , \underset{E}{\quad} , \underset{F}{\quad} \Big)$

Discover Card 1 — The vertices, edges and faces of a polyhedron



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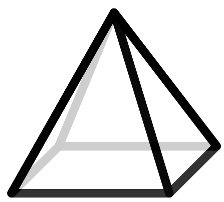

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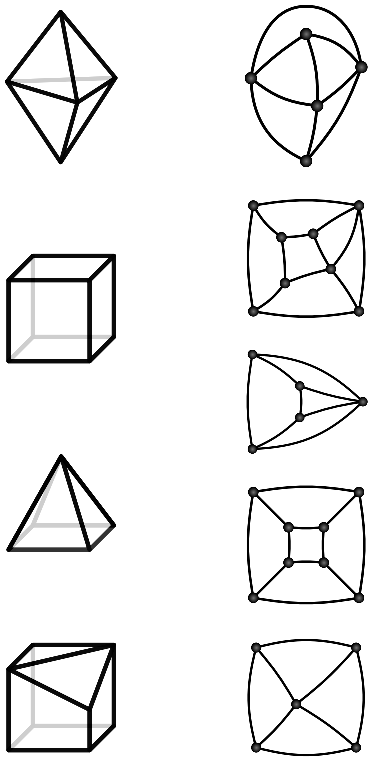
Discover Card 2 — The graph of a polyhedron

The GRAPH of a polyhedron is the structure of its vertices and edges. Imagine as if the polyhedron had rubber bands for edges so you could stretch them and stick all the vertices to the ground. The polyhedron now lies flat on the ground and this is what the graph of a polyhedron looks like.



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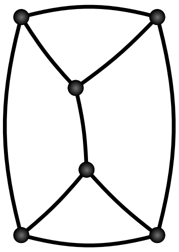

1. Match each of the polyhedra to its graph.

   One of the polyhedra has two graphs.

2. Which polyhedron has the graph to the left?

   Can you draw it?

3. Draw the graph of your polyhedron. It is not so easy! Draw several sketches.

   Once you are satisfied with a sketch, draw the graph on your polyhedron's profile.

✂ ....

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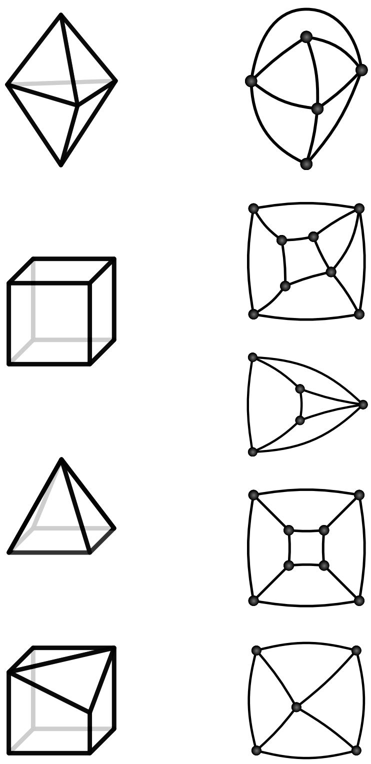

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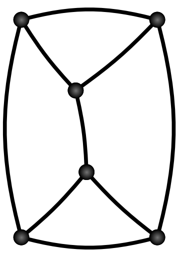

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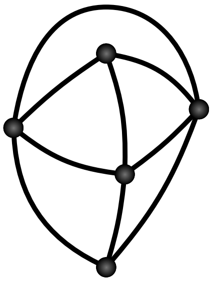
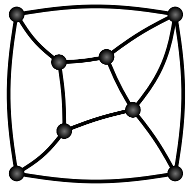
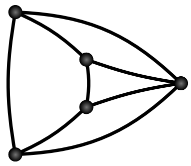
Discover Card 3 — Euler characteristic



The graph on the left has 5 vertices, 9 edges and divides the plane into 6 areas, or faces. Five of the faces are inside, bounded by the edges, and the sixth face is the whole outer area.
We add the number of vertices and faces and subtract the number of edges:
$V + F - E \ = \ 5 + 6 - 9 \ = ?$
This number is the Euler characteristic of the graph, named after the mathematician Leonhard Euler. For the graph of a polyhedron, it can be directly calculated from the f-vector, where the number of vertices, edges and faces are noted.

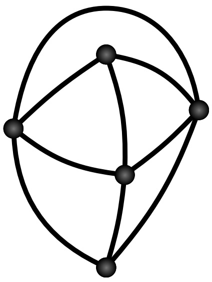
Exercises





1. Count the number of vertices, edges and faces of each of the graphs shown and use them to calculate their Euler characteristics.
2. What is the Euler characteristic of the graph of your polyhedron? Write it on the board.
3. What did the others write on the board? Try to guess what the Euler characteristic of the graph of a polyhedron must always be.

Discover Card 3 — Euler characteristic



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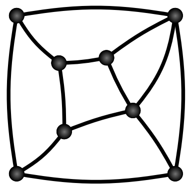

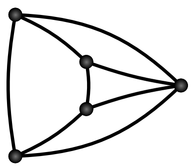

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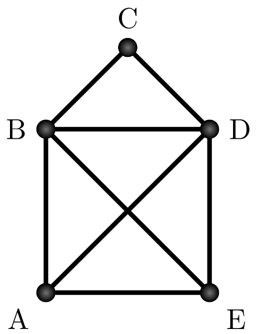
Discover Card 4   Euler path



To the left is the well-known *House of Santa Claus*. You can draw it without lifting your pencil off the paper and never draw the same edge twice.
See for yourself! Try the sequence of vertices, for example:
$A - B - C - D - E - B - D - A - E$.
Such a sequence of vertices containing each edge of the graph exactly once is called an Euler path.
Leonard Euler was a really important mathematician, both the Euler characteristic and the Euler path are named after him.

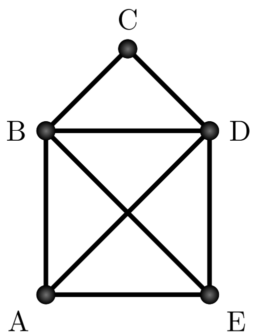
Exercises

1. Try to draw the House of Santa Claus by starting at different vertices. From which starting vertices can the house be drawn? Which vertex will you reach last?
2. What is special about these vertices?

   Hint: Count how many edges go out from each vertex.
3. Find a connected graph for which there is no Euler path. Why can the graph not have an Euler path?
4. Does the graph of your polyhedron have an Euler path? If no, why not? If yes, draw it on the graph of your polyhedron.

✂ ....

Discover Card 4   Euler path



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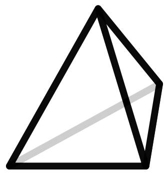
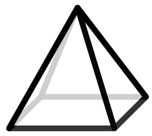
Discover Card 5 — Simple and Simplicial



Now we want to learn about two important types of polyhedra.

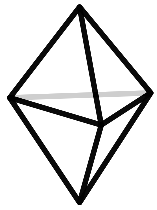
- A polyhedron is SIMPLE if exactly three edges meet at each vertex.
- A polyhedron is SIMPLICIAL if all its faces are triangles.

The tetrahedron shown on the left is the only polyhedron that is both simple and simplicial.



Exercises

1. Which of the polyhedra on the right are simple? Which are simplicial? Which are neither simple nor simplicial?

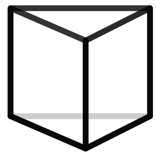


2. Is your polyhedron simple? Is it simplicial? Write the answers on the profile of your polyhedron.

Additional exercises

Imagine we have a simple polyhedron with 6 vertices. Even without knowing anything else about the polyhedron, we can find out how many edges it has. Each vertex of the simple polyhedron has exactly 3 edges coming out of it. Each edge in a polyhedron connects exactly two vertices. So the simple polyhedron has $(3 \cdot V) \div 2 \;=\; (3 \cdot 6) \div 2 \;=\; 9$ edges.

Question: *Can there be a simple polyhedron with 5 vertices?*



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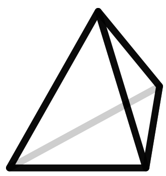
Discover Card 5 — Simple and Simplicial



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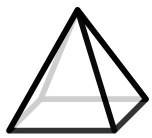

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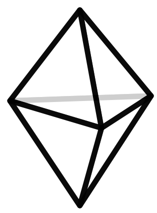

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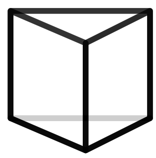
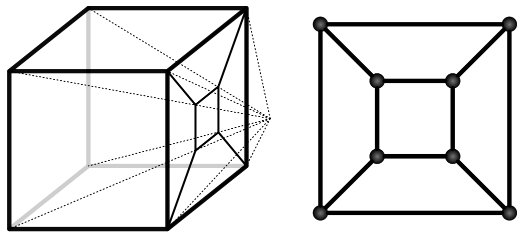


Additional Card

Schlegel diagram



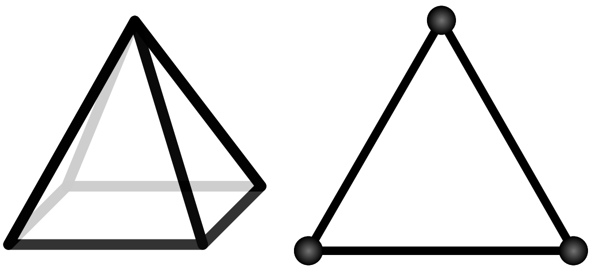
Imagine looking through one of the faces of the polyhedron shown on the left. The edges and vertices that you see form a graph without overlapping the edges.
The resulting graph is called the Schlegel diagram of the polyhedron. The Schlegel diagram is a special way to draw the graph of the polyhedron.

Exercises

1. Connect the Schlegel diagrams to the matching polyhedra.

   (Be careful, there are two different Schlegel diagrams for one of the polyhedra.)

2. Draw the Schlegel diagram for the square pyramid in the given triangle by looking through a triangular face.

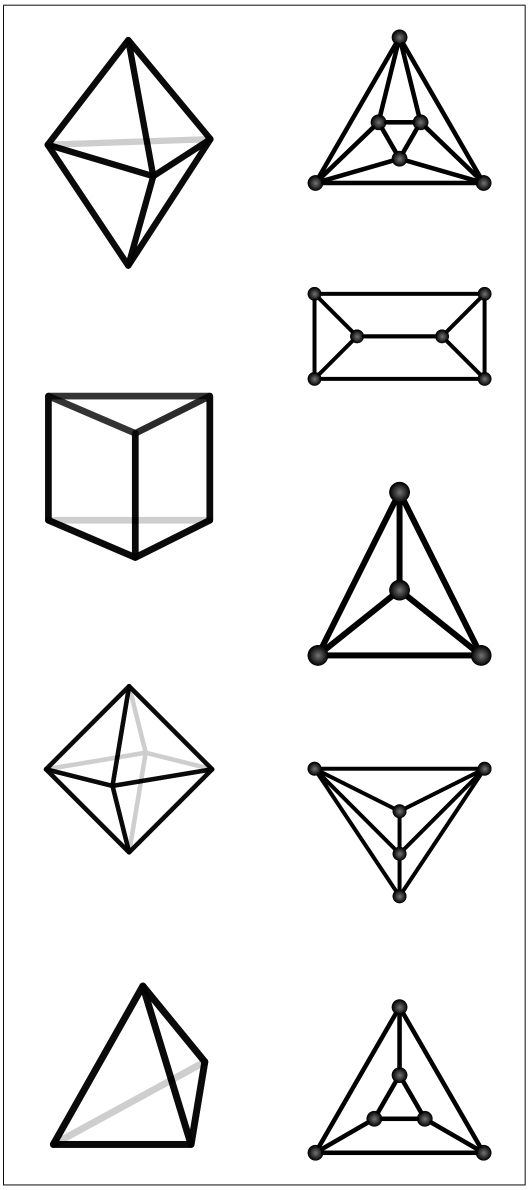


3. Now draw a Schlegel diagram by looking through the bottom square face.

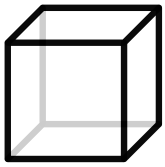
   How does the diagram change?

4. Pick a face of your polyhedron and draw the appropriate Schlegel diagram.

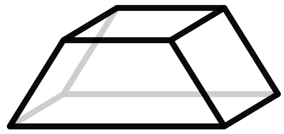
   Compare it with the graph of your polyhedron on the its profile page.



The type of a polyhedron



In mathematics, polyhedra are sorted by their f-vectors. There may be different polyhedra with the same f-vector. If two polyhedra have the same f-vector and the same graph then they are of the same TYPE.
The two 'dice' are of the same type. On the left is the typical geometric cube, on the right is a cube-type polyhedron.



Exercises

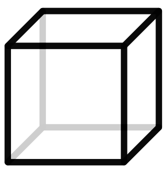
1. Search among your classmates for those whose polyhedra have the same f-vector and form a group.

2. Compare the graphs of your polyhedra. Which polyhedra are of the same type? Put the names of the polyhedra on your polyhedron's profile.

   Hint: Because the same graphs can look different, depending on how they are drawn, compare what shape the faces of the polyhedra are, how many of each shape, or how many edges meet at each vertex.

✂ ....

The type of a polyhedron



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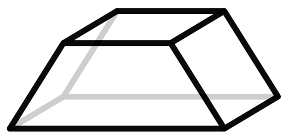

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   Hint: Because the same graphs can look different, depending on how they are drawn, compare what shape the faces of the polyhedra are, how many of each shape, or how many edges meet at each vertex.

Abstract

Diese Arbeit befasst sich mit der Theorie und Praxis von mathematischer Wissenschaftskommunikation. Zunächst geben wir eine Übersicht über die Entwicklungen der allgemeinen Wissenschaftskommunikationsforschung. Dies ist notwendig, um eine Grundlage zu schaffen anhand derer wir die sehr spärlich vorhandene Literatur zu mathematischer Wissenschaftskommunikation analysieren. Nachfolgend entwickeln wir eine Terminologie der Ziele, Methoden und Motivation von Wissenschaftskommunikation im Fach Mathematik.

Ein weiterer Fokus liegt auf der auf der Fallstudie POLYTOPIA – *Adoptiere ein Polyeder*. Dieses Projekt zum informellen Lernen von Mathematik schafft einen Zugang zur Mathematik auf der Beziehungsebene. Die vielen Entscheidungen über Design, Aufbau und Inhalt des Projektes werden reflektiert, die Ergebnisse der Nutzer*innenbefragung werden deskriptiv dargestellt. Weiter beleuchten wir die Bezüge zu den didaktischen Methoden, die einerseits dem Projektdesign insgesamt und andererseits dem dazugehörigen Schulmaterial zugrunde liegen.

Die Mathematik, die das Fundament für das Projekt bildet, wird von drei Seiten beleuchtet. Zunächst vollziehen wir die Entwicklung des Begriffs des *Polyeders* in einer historischen Perspektive nach. Dann definieren wir die im Projekt benötigten mathematischen Konzepte und beschreiben ihre Anwendung. Die dritte Perspektive ist eine Zusammenfassung über die Ergebnisse zu *Dürers Vermutung*. So wird die Frage bezeichnet, ob jedes Polyeder in ein überschneidungsfreies Netz zerlegt werden kann.

In dieser Arbeit werden Verbindungen zwischen diesen vier Bereichen geschaffen. Damit wird eine fundamentale Grundlage zur Etablierung der mathematischen Wissenschaftskommunikation als ein unabhängiges Wissenschaftsfeld gelegt und eine Agenda für ihre die weitere Entwicklung aufgesetzt.

Selbständigkeitserklärung

Hiermit vesichere ich, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Zudem wurde die Arbeit auch noch nicht in einem früheren Promotionsverfahren eingereicht.

Berlin, December 7, 2020

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