On Face Vector Sets and on Alcoved Polytopes

Two Studies on Convex Polytopes

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CONTENTS

1

INTRODUCTION

This dissertation presents two studies on convex polytopes. The motivation behind these studies could be roughly summarized by the two questions “When is a vector of integers the f-vector of a polytope?” and “Do alcoved polytopes have unimodal h^*-vectors?”

In this chapter we introduce the topics of the dissertation. Section 1.1 explains the notation used throughout the dissertation. It gives a background on polytope theory. The concepts introduced here will be used in the whole dissertation, but they are particularly important for Part I.

The objects studied in Part I are *f-vector sets* and *flag vector sets* of polytopes. For a given convex polytope P of dimension d, the entries $f_i(P)$ of the f-vector $f(P) = (f_0(P), \dots, f_{d-1}(P))$ of P are the numbers of i-dimensional faces of P. The flag vector has as entries the numbers of chains of faces of P.

The complete set of all f-vectors of d-dimensional polytopes is only known up to dimension 3. For dimension 3, this set was described by Steinitz [82] in 1906. For higher dimensions complete classifications of the set of f-vectors are not known. Already in dimension 4 such a result seems unattainable.

Instead of considering the whole set of f-vectors of 4-dimensional polytopes, the projections to two of its four coordinates have been studied and have completely been classified in the 1960s and 1970s by Grünbaum [38, Sect. 10.4], Barnette–Reay [8] and Barnette [7]. In Chapter 2 we try to generalize these results in two different ways.

In Section 2.1 we look at the set of all extended f-vectors (or flag vectors) of 4-dimensional polytopes, and study coordinate projections of this set.

Our first main result of Chapter 2 is Theorem 2.1.5, which gives a complete description of the projection of the set of flag vectors of 4-polytopes to the entries f_0 and f_{03}.

In Section 2.2 we generalize the results by Grünbaum, Barnette and Reay in another way: We look at specific coordinate projections of the sets of all f-vectors of d-polytopes of a given dimension.

The second main result of this chapter is a description of the projection of the set of f-vectors of d-polytopes to the coordinates f_0 and f_{d-1}. For even dimensions d, the description is given in Theorem 2.2.2: With the exception of some small pairs (f_0, f_{d-1}), the set is completely described.

For odd dimensions d, Theorem 2.2.3 gives a description of the set with the exception of

some pairs (f_0, f_{d-1}) close to the boundary of the set.
Chapter 2 is based on a joint paper with Günter M. Ziegler [74].
In Chapter 3 we look at the f-vector sets described in the previous chapter under a different point of view. We want to describe how "complicated" these sets can be. In Section 3.1 we look at some classical questions regarding "complicated" sets. A famous example is given by Hilbert's tenth problem, which asks for an algorithm to find integer solutions of Diophantine equations. It was shown by Matiyasevich et al. that such an algorithm does not exist, not every Diophantine set is computable.
For our f-vector sets we develop in Section 3.3 a different notion of complexity, the *semi-algebraic sets of lattice points*. Section 3.4 gives many examples of f-vector sets that have a sufficiently "nice" description as the set of all lattice points inside some semi-algebraic set.
Our main results of Chapter 3 are Theorems 3.3.4 and 3.3.5, stating that the f-vector sets $\Pi_{12}(\mathcal{F}^4)$, the set of all edge and ridge numbers of 4-polytopes, and $\mathcal{F}^d$, the set of all f-vectors of d-dimensional polytopes, where d is greater or equal to 6, do not admit such a simple description, they are not semi-algebraic sets of lattice points.

Chapter 3 is based on another joint paper with Günter M. Ziegler [75].

Part II focuses on Ehrhart theory of lattice polytopes, in particular the h^*-vectors of alcoved polytopes. The work in this part is motivated by the conjectured unimodality of the h^*-vectors alcoved polytopes.
There is a variety of conjectures and theorems about the unimodality of h^*-vectors of certain lattice polytopes.

Unimodality in combinatorics often shows up as a consequence of some underlying algebraic properties. Most famously, the g-theorem (Theorem 1.1.3) for simplicial polytopes implies that the h-vectors of simplicial polytopes are unimodal.

Stanley conjectured that all lattice polytopes with integer decomposition property have unimodal h^*-vectors. (see [73, Question 1.1]). We study a certain class of lattice polytopes with integer decomposition property, alcoved polytopes. Alcoved polytopes have unimodular triangulations, which implies that their h^*-vectors are equal to the h-vectors of the triangulations. This fact allows us to use methods from Stanley–Reisner theory for simplicial complexes to study the h^*-vectors of alcoved polytopes.

Chapter 4 introduces background and notation used throughout this part. Section 4.1 gives the necessary notions from Ehrhart theory, and Section 4.2 introduces some important types of triangulations. Stanley–Reisner theory is briefly introduced in Section 4.3. Section 4.4 is concerned with various questions about unimodality in combinatorics.
In Chapter 5 we look at alcoved polytopes. The main theorems of this part are Theorem 5.3.2 and Theorem 5.3.3. Theorem 5.3.2 gives a condition under which alcoved polytopes have a unimodal h^*-vector.
Theorem 5.3.3 gives a bound for how far off alcoved polytopes can be from this restriction.
Chapter 5 is based on joint work with Christian Haase and Rainer Sinn.

Appendix A contains a list of polytopes with 7 and 8 vertices (given in terms of their facet lists) which were used for the constructive part of the proof of Theorem 2.1.5. Appendix B contains algorithms used to construct random alcoved polytopes and calculate their h^*-vectors.

1.1 NOTATION AND BACKGROUND ON POLYTOPE THEORY

In this section we give some important definitions and results on convex polytopes. For further references, we refer the reader to Ziegler [88] and Grünbaum [38].

A *convex polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$.
The *dimension* of a polytope is the dimension of its affine hull.
Let P be a d-dimensional polytope, or d-polytope for short. The unique point set V contained in all sets $S \subset \mathbb{R}^d$ such that $\operatorname{conv} S = P$, is called the *vertex set* of P and the points in V are the *vertices* of P. A description of P as the convex hull of finitely many points is called a $\mathcal{V}$*-description* of the polytope P.
P can also be described as an intersection of finitely many closed halfspaces in $\mathbb{R}^d$. Such a description of P is called an $\mathcal{H}$*-description* of P. In general the intersection of finitely many closed halfspaces in $\mathbb{R}^d$ is a *polyhedron*, a bounded polyhedron is a polytope.

A *face* of a d-polytope P is either the polytope P itself or the intersection of P with a hyperplane H in $\mathbb{R}^d$ such that P is entirely contained in one of the two closed halfspaces H^+, H^- defined by H. A hyperplane H defining a face $F = P \cap H$ of P will be called a *face-defining hyperplane* of F.
A face of a polytope is again a polytope.

The empty set is a face of every polytope. Its dimension is defined to be -1. The vertices of P are the 0-dimensional faces of P. Faces of dimension 1 are called *edges*, $(d-1)$-dimensional faces are called *facets* and $(d-2)$-dimensional faces are called *ridges*. All facets of P excluding P itself (but including the empty set) are called *proper facets*.

The poset $\mathcal{L}(P)$ of all faces of a d-polytope P, partially ordered by inclusion, is a graded lattice, called the *face lattice* of P [88, Thm. 2.2.7]. Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic. When it is clear from the context that we talk about combinatorial properties rather than geometric properties, we often write "the polytope" instead of "the combinatorial type of the polytope".

Let P be a full-dimensional polytope in $\mathbb{R}^d$ with 0 in the interior. The *polar* of P is

$$P^{\Delta} := \{y \in (\mathbb{R}^d)^* \mid yx \leq 1 \text{ for all } x \in P\}.$$

P^{Δ} is also a d-dimensional polytope with 0 in the interior and $P^{\Delta\Delta} = P$ [62, Thm. 2.13]. There is a bijection between the faces of P and P^{Δ} that reverses inclusion. In other words, the face lattice of P^{Δ} is the opposite of the face lattice of P:

$$\mathcal{L}(P^{\Delta}) = \mathcal{L}(P)^{\mathrm{op}}.$$

This gives rise to a combinatorial notion of polarity:
Two (combinatorial types of) polytopes P, Q are said to be *dual* to each other if $\mathcal{L}(Q) = \mathcal{L}(P)^{\mathrm{op}}$. The dual of a polytope P will be denoted by P^*.

For a d-dimensional polytope P, let $f_i = f_i(P)$ denote the number of i-dimensional faces of P. The *f-vector* of P is then $f(P) = (f_{-1}, f_0, f_1, \dots, f_{d-1}, f_d)$. We often omit the trivial faces, the empty set and P itself, and write the f-vector of P simply as $f(P) = (f_0, f_1, \dots, f_{d-1})$. The set of all f-vectors of d-polytopes is denoted by $\mathcal{F}^d \subset \mathbb{Z}^d$. The Euler–Poincaré formula

$$-f_{-1} + f_0 - f_1 + \ldots + (-1)^d f_d = 0$$

holds for the f-vectors of all non-empty d-dimensional polytopes P, so $\mathcal{F}^d \subset \mathbb{Z}^d$ lies on a hyperplane in $\mathbb{R}^d$. Even more, $\mathcal{F}^d$ spans this hyperplane: The Euler–Poincaré formula is up to scaling the only linear relation that is satisfied by all f-vectors of d-dimensional polytopes (Grünbaum [38, Thm. 8.1.1]).
For $S \subseteq \{0, \dots, d-1\}$, let $f_S = f_S(P)$ denote the number of chains $F_1 \subset \cdots \subset F_r$ of faces of P with $\{\dim F_1, \dots, \dim F_r\} = S$. The *flag vector* or *extended f-vector* of P is $(f_S)_{S\subseteq\{0,\dots,d-1\}}$. The set of all flag vectors of d-polytopes will be denoted by $\overline{\mathcal{F}}^d \subset \mathbb{Z}^{2^d}$. Its affine dimension is $c_d - 1$, where c_d is the d-th Fibonacci number, $c_1 = 1$, $c_2 = 2$, $c_d = c_{d-1} + c_{d-2}$ ([12], Thm. 2.6).

The coordinate projection of $\mathcal{F}^d$ to two of its coordinates, f_i and f_j, will be denoted by $\Pi_{i,j}(\mathcal{F}^d) \subset \mathbb{Z}^2$. Analogously, $\Pi_{S,T}(\overline{\mathcal{F}}^d) \subset \mathbb{Z}^2$ is the projection of $\overline{\mathcal{F}}^d$ to the two coordinates f_S and f_T.

A simplex is a d-dimensional polytope with $d+1$ vertices.
A polytope is called *simplicial* if all of its proper faces are simplices.
The dual notion is that of a *simple polytope*: A d-polytope is *simple* if all of its vertices lie in exactly d facets.

The *h-vector* of a simplicial d-polytope P is the vector $h(P) = (h_0(P), \dots, h_d(P))$, where

$$h_k(P) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(P).$$

The f-vector $f(P) = (f_{-1}(P), \dots, f_{d-1}(P))$ of P can also be expressed in terms of the h-vector of P:

$$f_{k-1}(P) = \sum_{i=0}^{k} \binom{d-i}{d-k} h_i(P).$$

The h-vectors of simplicial polytopes satisfy a set of linear relations, the so-called *Dehn–Sommerville equations.*

THEOREM 1.1.1 (Dehn–Sommerville equations [30], [76]). If P is a simplicial d-polytope, its h-vector satisfies:

$$h_k(P) = h_{d-k}(P).$$

A *simplicial complex* $\mathcal{C}$ is a finite collection of simplices such that

(i) $\emptyset \in \mathcal{C}$,

(ii) if $\Delta \in \mathcal{C}$, then all faces of Δ are also in $\mathcal{C}$,

(iii) the intersection $\Delta_1 \cap \Delta_2$ of any $\Delta_1, \Delta_2 \in \mathcal{C}$ is a face of both Δ_1 and Δ_2.

The elements of $\mathcal{C}$ are called *faces* of $\mathcal{C}$. The inclusion-maximal faces are called *facets*. The dimension of $\mathcal{C}$ is the largest dimension of its faces.
A simplicial complex is said to be *pure* if all of its facets have the same dimension.
The f-vector and h-vector of a simplicial complex are defined analogously to the f-vector and h-vector of a simplicial polytope.
As with polytopes we also distinguish between a combinatorial version and a geometric version of simplicial complexes. *Geometric simplicial complexes* are collections of "geometric" simplices, their vertices $v_1, \dots, v_n$ are realized in $\mathbb{R}^d$, and their faces are all of the form $\mathrm{conv}\{v_i\}_{i \in I}$ for some $I \subseteq \{1, \dots, n\}$.
Abstract simplicial complexes only contain the combinatorial information of the simplicial complex, its face lattice.
If $\mathcal{C}$ is an abstract simplicial complex on the vertices $v_1, \dots, v_n$, we denote the faces of $\mathcal{C}$ by their vertex sets, the face of $\mathcal{C}$ with vertex set $\{v_{i_1}, \dots, v_{i_s}\}$ is written as $\{v_{i_1}, \dots, v_{i_s}\}$.

Let $\mathcal{C}$ be a pure d-dimensional simplicial complex with m facets.
A *shelling* of $\mathcal{C}$ is an ordering $F_1, \dots, F_m$ of the facets of $\mathcal{C}$ such that either $\mathcal{C}$ is 0-dimensional or for every $j \in \{2, \dots, m\}$,

$$F_j \cap (\bigcup_{i=1}^{j-1} F_i)$$

is a pure $(d-1)$-dimensional simplicial complex.
A pure simplicial complex is *shellable* if it has a shelling.
Let $\mathcal{C}$ be a pure shellable simplicial complex with m facets and given shelling $F_1, \dots, F_m$.
Let F_j be a facet of $\mathcal{C}$ with vertex set $\mathrm{vert}(F_i) = \{v_1, \dots, v_{d+1}\}$.
We define the *restriction* of facet F_i as the set of all vertices v of F_i such that $\mathrm{vert}(F_i) \setminus \{v\}$ is contained in one of the facets $F_1, \dots, F_{i-1}$ that appear earlier in the shelling.
In particular the restriction of F_1 is the empty set.

THEOREM 1.1.2 (See [88, Thm. 8.19]). Let $\mathcal{C}$ be a pure d-dimensional simplicial complex. If $\mathcal{C}$ is shellable, then the entries h_i of its h-vector $h(\mathcal{C})$ count the facets in a shelling whose restriction contains i vertices.

The *g-vector* of a d-dimensional simplicial complex $\mathcal{C}$ is defined from its h-vector as:

$$g(\mathcal{C}) := (g_0(\mathcal{C}), \dots, g_{\lfloor \frac{d}{2} \rfloor}(\mathcal{C}))$$

with $g_0(\mathcal{C}) := h_0(\mathcal{C})$ and $g_i(\mathcal{C}) := h_i(\mathcal{C}) - h_{i-1}(\mathcal{C})$ for $i = 1, \dots, \lfloor \frac{d}{2} \rfloor$.

For all integers $m, k \geq 1$ there is a unique expression of m as

$$m = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \ldots + \binom{n_i}{i}$$

such that $n_k > n_{k-1} > \ldots > n_i \geq i \geq 1$. This expression is called the *k-canonical representation of m* (see Kruskal [54]). Now we define $\partial^k(m)$ to be

$$\partial^k(m) := \binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + \ldots + \binom{n_i - 1}{i-1}.$$

We also define $\partial^k(0) := 0$.
A sequence $m = (m_0, m_1, \ldots)$ of non-negative integers is called an *M-sequence* if $m_0 = 1$ and $m_{k-1} \geq \partial^k(m_k)$ for $k \geq 1$.

The next theorem, the g-theorem for simplicial polytopes, makes it possible to completely characterize the f-vectors of simplicial polytopes. We state here the numerical version of the theorem. The algebraic version and generalizations beyond polytopes are given in Section 4.3.

Theorem 1.1.3 (g-theorem for simplicial polytopes, Billera & Lee [17][16], Stanley [80]). Let $g = (g_0, g_1, \ldots, g_{\lfloor \frac{d}{2} \rfloor}) \in \mathbb{Z}^{\lfloor \frac{d}{2} \rfloor + 1}$. Then g is the g-vector of a simplicial d-polytope if and only if g is an M-sequence.

The Dehn–Sommerville equations together with the g-theorem describe the set of all h-vectors (and by that also of all f-vectors) of simplicial polytopes.

Part I

FACE AND FLAG VECTORS OF POLYTOPES

2

CHARACTERIZING FACE AND FLAG VECTOR PAIRS FOR POLYTOPES

While the f-vector set $\mathcal{F}^3$ of 3-polytopes was characterized (easily) by Steinitz [82] in 1906, a complete characterization of $\mathcal{F}^d$ is out of reach for any $d \geq 4$. For $d = 4$, the projections of the f-vector set $\mathcal{F}^4 \subset \mathbb{Z}^4$ onto two of the four coordinates have been determined in 1967–1974 by Grünbaum [38, Sect. 10.4], Barnette–Reay [8] and Barnette [7]. We will review these results in Section 2.1.1.

This chapter provides new results about coordinate projections of f-vector and flag vector sets: The first part is an extension to the flag vectors of 4-polytopes. In particular, in Theorem 2.1.5 we fully characterize the projection of the set of all flag vectors of 4-polytopes to the two coordinates f_0 and f_{03}. Our proof makes use of the classification of all combinatorial types of 4-polytopes with up to eight vertices by Altshuler and Steinberg [3, 4]. We have not used the classification of the 4-polytopes with nine vertices recently provided by Firsching [36].

In the second part we look at the set $\mathcal{F}^d$ of f-vectors of d-dimensional polytopes, for $d \geq 5$. Here even a complete characterization of the projection $\Pi_{0,d-1}(\mathcal{F}^d) \subset \mathbb{Z}^2$ to the coordinates f_0 and f_{d-1} seems impossible. We call (n, m) a *polytopal pair* if $(n, m) \in \Pi_{0,d-1}(\mathcal{F}^d)$, that is, if there is a d-polytope P with $f_0(P) = n$ and $f_{d-1}(P) = m$. These polytopal pairs must satisfy the *UBT inequality* $m \leq f_{d-1}(C_d(n))$ given by the Upper Bound Theorem [61] [88, Sect. 8.4], where $C_d(n)$ denotes a d-dimensional cyclic polytope with n vertices, and also $n \leq f_{d-1}(C_d(m))$, by duality.

Our second main result of this chapter, Theorem 2.2.2, states that for even $d \geq 4$, every (n, m) satisfying the UBT inequalities as well as $n + m \geq \binom{3d+1}{\lfloor d/2 \rfloor}$ is a polytopal pair. However, for even $d \geq 6$, there are pairs (n, m) with $n + m < \binom{3d+1}{\lfloor d/2 \rfloor}$ that satisfy the UBT inequalities, but for which there is no polytope: We call these *small exceptional pairs*. Theorem 2.2.3 states, in contrast, that for every odd $d \geq 5$ there are also *arbitrarily large exceptional pairs*.

2.1 FACE AND FLAG VECTOR PAIRS FOR 4-POLYTOPES

2.1.1 Face vector pairs for 4-polytopes

The 2-dimensional coordinate projections $\Pi_{i,j}(\mathcal{F}^4)$ of the set of f-vectors of 4-polytopes to the coordinate planes, as determined by Grünbaum, Barnette and Reay, are given by the following theorems. See also Figure 2.1.

THEOREM 2.1.1 (Grünbaum [38, Thm. 10.4.1]). The set of f-vector pairs (f_0, f_3) of 4-polytopes is equal to

$$\begin{aligned}\Pi_{0,3}(\mathcal{F}^4) = \{(f_0, f_3) \in \mathbb{Z}^2 : 5 \leq f_0 \leq \tfrac{1}{2} f_3(f_3 - 3),\\ 5 \leq f_3 \leq \tfrac{1}{2} f_0(f_0 - 3)\}.\end{aligned}$$

THEOREM 2.1.2 (Grünbaum [38, Thm. 10.4.2]). The set of f-vector pairs (f_0, f_1) of 4-polytopes is equal to

$$\begin{aligned}\Pi_{0,1}(\mathcal{F}^4) = \ & \{(f_0, f_1) \in \mathbb{Z}^2 : 10 \leq 2f_0 \leq f_1 \leq \tfrac{1}{2} f_0(f_0 - 1)\}\\ & \setminus\{(6,12), (7,14), (8,17), (10,20)\}.\end{aligned}$$

The existence parts of Theorems 2.1.1 and 2.1.2 are proved by taking neighborly polytopes, which yield the polytopal pairs on the upper bound, as well as dual neighborly polytopes for the polytopal pairs on the lower bound, and by finding some polytopes for examples of small polyhedral pairs. From these polytopes, polytopes with all other possible polytopal pairs are constructed by an inductive method of (generalized) stacking (see Sections 2.1.3 and 2.1.4).

THEOREM 2.1.3 (Barnette & Reay [8, Thm. 10]). The set of f-vector pairs (f_0, f_2) of 4-polytopes is equal to

$$\begin{aligned}\Pi_{0,2}(\mathcal{F}^4) = \{(f_0, f_2) \in \mathbb{Z}^2 :\ & 10 \leq \tfrac{1}{2}\left(2f_0 + 3 + \sqrt{8f_0 + 9}\right) \leq f_2 \leq f_0^2 - 3f_0,\\ & f_2 \neq f_0^2 - 3f_0 - 1\}\\ & \setminus\{(6,12), (6,14), (7,13), (7,15), (8,15),\\ & \quad (8,16), (9,16), (10,17), (11,20), (13,21)\}.\end{aligned}$$

The existence part of Theorem 2.1.3 is proved similarly to the proofs of Theorems 2.1.1 and 2.1.2, additionally considering all 4-dimensional pyramids, bipyramids and prisms ("cylinders").

THEOREM 2.1.4 (Barnette [7, Thm. 1], with corrections, cf. [49]). The set of f-vector pairs (f_1, f_2) of 4-polytopes is equal to

$$\begin{aligned}\Pi_{1,2}(\mathcal{F}^4) = \{(f_1, f_2) \in \mathbb{Z}^2 : \quad & 10 \le \tfrac{1}{2}f_1 + \left\lceil \sqrt{f_1 + \tfrac{9}{4}} + \tfrac{1}{2} \right\rceil + 1 \le f_2, \\ & 10 \le \tfrac{1}{2}f_2 + \left\lceil \sqrt{f_2 + \tfrac{9}{4}} + \tfrac{1}{2} \right\rceil + 1 \le f_1, \\ & f_2 \ne \tfrac{1}{2}f_1 + \sqrt{f_1 + \tfrac{13}{4}} + 2, \\ & f_1 \ne \tfrac{1}{2}f_2 + \sqrt{f_2 + \tfrac{13}{4}} + 2\} \\ \setminus\{&(12,12), (13,14), (14,13), (14,14), (15,15), \\ &(15,16), (16,15), (16,17), (16,18), (17,16), \\ &(17,20), (18,16), (18,18), (19,21), (20,17), (20,23), \\ &(20,24), (21,19), (21,26), (23,20), (24,20), (26,21)\}.\end{aligned}$$

For the proof of Theorem 2.1.4, a finite number of polytopes with few edges was found, and polytopes with all other possible polytopal pairs were constructed using an inductive method based on “facet splitting” (see Section 2.1.5).

The remaining f-vector projections are given by duality.

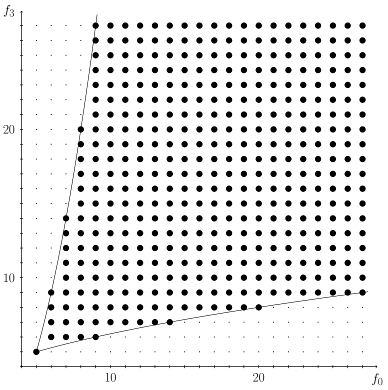
2.1.2 Flag vector pair (f_0, f_{03}) *for* 4-polytopes

In the following we will characterize the set

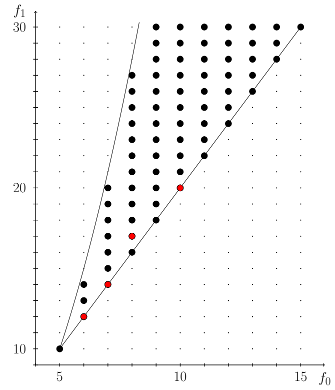
$$\Pi_{0,03}(\overline{\mathcal{F}}^4) = \{(f_0(P), f_{03}(P)) \in \mathbb{Z}^2 \mid P \text{ is a 4-polytope}\},$$

that is, we describe the possible number of vertex-facet incidences of a 4-polytope with a fixed number of vertices. Equivalently, this tells us the possible average number of facets of the vertex figures, $\frac{f_{03}}{f_0}$, for a given number f_0 of vertices.

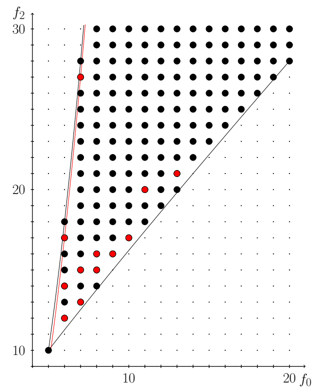
In 1984 Altshuler and Steinberg classified all combinatorial types of 4-polytopes with up to 8 vertices [3, 4]. This classification makes our proof much easier. We will use the classification to find examples of polytopes for certain small polytopal pairs and also to argue that some pairs cannot be polytopal pairs of any 4-polytope. The following is our first main theorem:



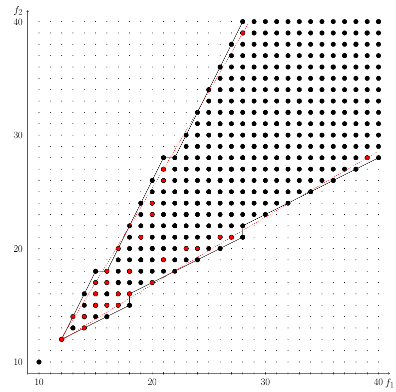
(a) $\Pi_{0,3}(\mathcal{F}^4)$



(b) $\Pi_{0,1}(\mathcal{F}^4)$



(c) $\Pi_{0,2}(\mathcal{F}^4)$



(d) $\Pi_{1,2}(\mathcal{F}^4)$

Figure 2.1: f-vector projections, red dots are exceptional pairs

THEOREM 2.1.5. There exists a 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ if and only if f_0 and f_{03} are integers satisfying

$$\begin{aligned} 20 \leq 4f_0 \leq f_{03} \leq 2f_0(f_0-3), \\ f_{03} \neq 2f_0(f_0-3) - k \text{ for } k \in \{1,2,3,5,6,9,13\} \end{aligned}$$

and (f_0, f_{03}) is not one of the 18 exceptional pairs

$$\begin{aligned} &(6,24),(6,25),(6,28),\\ &(7,28),(7,30),(7,31),\\ &(7,33),(7,34),(7,37),(7,40),\\ &(8,33),(8,34),(8,37),(8,40),\\ &(9,37),(9,40),(10,40),(10,43). \end{aligned}$$

See Figure 2.2 for a visualization of the projection in the plane $(f_0, f_{03} - 4f_0)$.

The proof of Theorem 2.1.5 follows the proofs of the projections of the f-vector ([7],

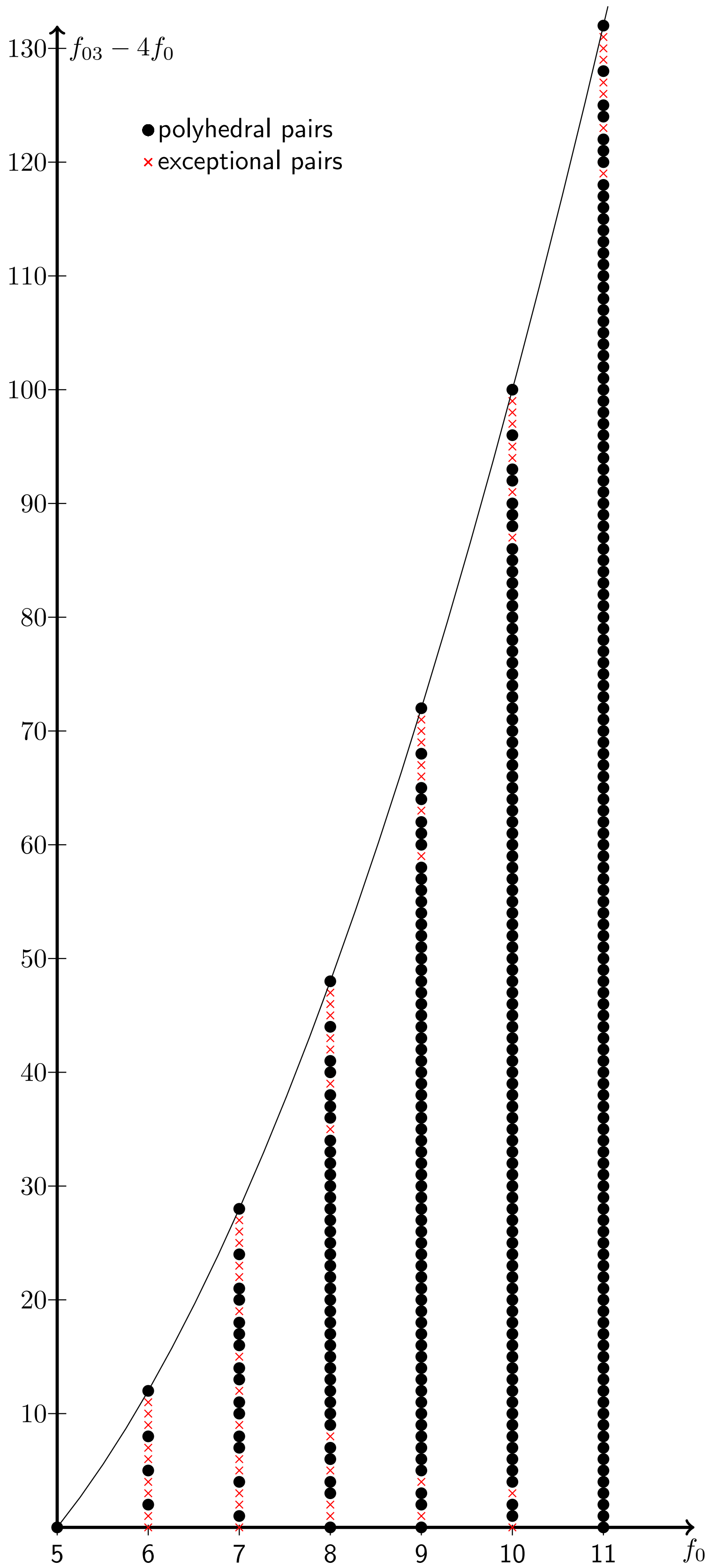


Figure 2.2: Projection $\Pi_{0,03}(\overline{\mathcal{F}}^4)$

[8], [38]), by taking small polytopal pairs as well as polytopal pairs on the boundaries and constructing new polytopal pairs from the given ones. The inductive methods used for this proof are the stacking and truncating constructions from Theorem 2.1.1, 2.1.2 and 2.1.3 and "facet splitting" methods generalized from the methods used in the proof of Theorem 2.1.4.

LEMMA 2.1.6. If P is a 4-dimensional polytope with f_0 vertices and f_{03} vertex-facet incidences, then

$$4f_0 \leq f_{03} \leq 2f_0(f_0 - 3).$$

Proof. Every vertex of a d-polytope lies in at least d facets, so clearly $4f_0 \leq f_{03}$ holds for all 4-dimensional polytopes, with equality if and only if P is simple.

The second inequality follows from a generalization of the upper bound theorem to flag vectors: For any d-dimensional polytope with n vertices and for any $S \subseteq \{0, \ldots, d-1\}$,

$$f_S \leq f_S(C_d(n)),$$

where $C_d(n)$ is the d-dimensional cyclic polytope with n vertices [15, Thm. 18.5.9]. In particular, 4-dimensional cyclic polytopes are simplicial, and for any 4-dimensional polytope P,

$$f_{03}(P) \leq f_{03}(C_4(n)) = 4f_3(C_4(n)) = 2n(n-3)$$

with equality if and only if P is neighborly. $\square$

LEMMA 2.1.7. There is no 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ if (f_0, f_{03}) is any of the following pairs:

$$\begin{aligned}
&(6,24), (6,25), (6,28),\\
&(7,28), (7,30), (7,31),\\
&(7,33), (7,34), (7,37), (7,40),\\
&(8,33), (8,34), (8,37), (8,40),\\
&(9,37), (9,40), (10,40), (10,43),\\
&(f_0, 2f_0(f_0-3)-k) \quad \text{for } k \in \{1,2,3,5,6,9,13\} \text{ and for any } f_0 \geq 6.
\end{aligned}$$

For the proof of this lemma we need some equations and inequalities which hold for the flag vector of any 4-polytope. The following are generalizations of the Dehn–Sommerville equations (Theorem 1.1.1) to all polytopes.

LEMMA 2.1.8 (Generalized Dehn–Sommerville equations, Bayer & Billera [12, Thm. 2.1]). Let P be a d-polytope and $S \subseteq \{0, 1, \ldots, d-1\}$. Let $\{i, k\} \subseteq S \cup \{-1, d\}$ such that $i < k-1$ and such that there is no $j \in S$ for which $i < j < k$. Then,

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}}(P) = f_S(P)(1 - (-1)^{k-i-1}).$$

For $d = 4$, $S = \{0\}$, $i = 0$, $k = 4$ and with the observation $f_{01} = 2f_1$ we obtain

$$f_{02} = -2f_0 + 2f_1 + f_{03}. \tag{2.1}$$

Lemma 2.1.9 (Bayer [11, Thm. 1.3, 1.4]). The flag vector of every 4-polytope satisfies the inequalities

$$f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0 \tag{2.2}$$
$$\text{and } -6f_0 + 6f_1 - f_{02} \geq 0. \tag{2.3}$$

Inequality (2.3) holds with equality if and only if the 4-polytope is center boolean, that is, if all its facets are simple.

Using Lemma 2.1.8 and the Euler–Poincaré formula for dimension 4 [38, Thm. 8.1.1] to rewrite Lemma 2.1.9, we obtain the inequalities

$$-3f_0 - 3f_3 + f_{03} + 10 \geq 0 \tag{2.4}$$
$$\text{and } 4f_0 - 4f_1 + f_{03} \leq 0. \tag{2.5}$$

Proof of Lemma 2.1.7. We first show that there is no polytope P with

$$(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - k) \quad \text{for } k \in \{1, 2, 3, 5, 6, 9, 13\}.$$

For $k = 1, 2, 3$ we prove the non-existence directly. For $k = 5, 6$ we show that if P is a polytope with $2f_0(f_0 - 3) - 7 \leq f_{03} \leq 2f_0(f_0 - 3) - 5$, then necessarily $f_{03} = 2f_0(f_0 - 3) - 7$. The proof for $k = 9$ and $k = 13$ follows similarly.

For $k > 0$ any 4-polytope with polytopal pair $(f_0, 2f_0(f_0 - 3) - k)$ cannot be neighborly, so

$$f_1 < \binom{f_0}{2}.$$

On the other hand, for $(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - k)$ Inequality (2.5) reads $\frac{1}{2}f_0(f_0 - 1) - \frac{k}{4} \leq f_1$. Both inequalities together give

$$\binom{f_0}{2} - \frac{k}{4} \leq f_1 < \binom{f_0}{2}. \tag{2.6}$$

There is no integer solution for $k = 1, 2, 3$.

For $k = 5, 6, 7$, the only possible integer value for f_1 is $\frac{1}{2}f_0(f_0 - 1) - 1$. Assume that P is a polytope with

$$f_1 = \binom{f_0}{2} - 1$$

and

$$2f_0(f_0 - 3) - 7 \leq f_{03} \leq 2f_0(f_0 - 3) - 5.$$

Since $f_1 = \binom{f_0}{2} - 1$, there is a unique pair v_1, v_2 of vertices of P not forming an edge.

We call such a pair of vertices a *non-edge*. Any facet of P which is not a simplex must contain this non-edge, since the only 3-polytope in which every two vertices form an edge is the simplex. Consider a facet F which is not a simplex, and therefore contains the unique non-edge. Such a facet F needs to exist, since if P were simplicial, $f_{03} \equiv 0 \mod 4$. Observe that if F would have more than five vertices, then we could find five vertices of F for which every two vertices form an edge. This subpolytope of F could not be d-dimensional, for $d \leq 3$. From this contradiction follows that F has five vertices. The only combinatorial types of 3-polytopes with five vertices are the square pyramid and the bipyramid over a triangle, only the latter has exactly one non-edge. So F is a bipyramid, and the non-edge is between the apices of F. If there were another non-tetrahedral facet of P, it would intersect F in a common face containing the non-edge. Such a face does not exist, and hence P is a polytope with one bipyramidal facet and t tetrahedral facets, for some integer t. This implies that

$$f_{03} = 4t + 5 \equiv 1 \mod 4.$$

From the assumption $2f_0(f_0 - 3) - 7 \leq f_{03} \leq 2f_0(f_0 - 3) - 5$ follows now

$$f_{03} = 2f_0(f_0 - 3) - 7.$$

Assume now that there is a polytope P with

$$(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - 9).$$

Inequality (2.6) implies that

$$f_1 = \binom{f_0}{2} - 2 \text{ or } f_1 = \binom{f_0}{2} - 1.$$

If $f_1 = \binom{f_0}{2} - 1$, then we have just proved that $f_{03} \equiv 1 \mod 4$. Since $f_{03}(P) \equiv 3 \mod 4$, it follows that P has two non-edges. The inequality $f_1 \leq 3f_0 - 6$ holds for 3-dimensional polytopes and any facet F has at most two non-edges:

$$\binom{f_0(F)}{2} - 2 \leq f_1(F) \leq 3f_0(F) - 6 \Rightarrow f_0(F) < 6.$$

Any non-tetrahedral facet is hence a polytope with five vertices, a bipyramid over a triangle or a square pyramid. Since $f_{03}(P) \equiv 3 \mod 4$, there have to be at least three non-tetrahedral facets. Bipyramids have one non-edge, not contained in any other facet. Square pyramids have two non-edges, which are both contained in exactly one other facet. This contradicts the fact that there are only two non-edges in P. In conclusion, there is no polytope with $(f_0, f_{03}) = (f_0, 2f_0(f_0 - 3) - 9)$.

Finally, assume that there exists a polytope P with polytopal pair

$$(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - 13).$$

From Inequality (2.6) it follows that P has $\binom{f_0}{2}-3$, $\binom{f_0}{2}-2$ or $\binom{f_0}{2}-1$ edges. Each facet F of P has at most three non-edges. For any facet F of P the inequality $f_1(F) \leq 3f_0(F)-6$ now yields $\binom{f_0(F)}{2} - 3 \leq 3f_0(F) - 6 \Rightarrow f_0(F) < 7$. If F has six vertices, it must have 12 edges and three non-edges. There are only two such combinatorially different 3-polytopes, which both are simplicial.

Assume that P has a facet F with six vertices. Then F contains three non-edges, all of them not in any 2-face of F and hence not in any other facet. So all other facets of P are tetrahedra. This is a contradiction to $f_{03}(P) \equiv 3 \mod 4$.

P is not simplicial, so there are non-tetrahedral facets, all of them with five vertices. Observe that since there are at most three non-edges, we cannot have more than three non-tetrahedral facets. Together with $f_{03}(P) \equiv 3 \mod 4$, this leaves us with two cases:

(i) The non-tetrahedral facets of P are three bipyramids over triangles.

(ii) The non-tetrahedral facets of P are two square pyramids and one bipyramid over a triangle.

In both cases, let t denote the number of tetrahedra in P. Then

$$\begin{aligned} f_{03}(P) &= 2f_0(f_0-3) - 13 = 4t + 3\cdot 5 \\ \Rightarrow\ t &= \tfrac{1}{2}f_0(f_0-3) - 7 \\ \Rightarrow\ f_3(P) &= t + 3 = \tfrac{1}{2}f_0(f_0-3) - 4. \end{aligned}$$

We can now calculate $f_2(P)$ in two ways. From the Euler–Poincaré formula,

$$\begin{aligned} f_2 &= f_1 + f_3 - f_0 \\ &= \binom{f_0}{2} - 3 + \tfrac{1}{2}f_0(f_0-3) - 4 - f_0 \\ &= f_0(f_0-3) - 7. \end{aligned}$$

Each 2-face lies in exactly two facets. The number of 2-faces of P can therefore also be calculated by counting the number of 2-faces in each facet. In case (i) this gives:

$$f_2 = \tfrac{1}{2}f_{23} = \tfrac{1}{2}(4t + 3\cdot 6) = f_0(f_0-3) - 5 \neq f_0(f_0-3) - 7.$$

In case (ii) we obtain:

$$f_2 = \tfrac{1}{2}f_{23} = \tfrac{1}{2}(4t + 2\cdot 5 + 6) = f_0(f_0-3) - 6 \neq f_0(f_0-3) - 7.$$

So there cannot be a polytope with polytopal pair $(f_0, 2f_0(f_0-3)-13)$.

It remains to show the non-existence of 18 pairs (f_0, f_{03}). All combinatorial types of 4-polytopes with up to 8 vertices have been classified by Altshuler and Steinberg [3, 4].

From this classification it follows that there are no polytopes with polytopal pairs

$$
\begin{aligned}
&(6,24),(6,25),(6,28),\\
&(7,28),(7,30),(7,31),\\
&(7,33),(7,34),(7,37),(7,40),\\
&(8,33),(8,34),(8,37) \text{ or } (8,40).
\end{aligned}
$$

To see that the four pairs $(9,37)$, $(9,40)$, $(10,40)$ and $(10,43)$ are exceptional pairs, we make use of the upper bound for the number of facets in terms of the number of vertices and vertex-facet incidences. If there were a polytope P with polytopal pair $(9,37)$, $(9,40)$, $(10,40)$ or $(10,43)$, due to Inequality (2.4) it would need to have less than 8 facets. By duality, this would give us a polytope P^* with $f_{03}(P^*) = 37, 40$ or 43 and $f_0(P^*) \leq 7$. From the upper bound $f_{03} \leq 2f_0(f_0-3)$ it follows that $f_0(P^*) = 7$. As seen above, polytopes with polytopal pair $(7,37)$ or $(7,40)$ do not appear in the classification. Pair $(7,43)$ is of the type $(f_0, 2f_0(f_0-3)-13)$, which is an exceptional pair. $\square$

We will use the classification of 4-dimensional polytopes with up to 8 vertices [3, 4] together with some classes of polytopes, such as cyclic polytopes, pyramids, and some additional polytopes, and from those polytopes and their polytopal pairs construct all other possible polytopal pairs. The methods needed for this construction are described in the following sections.

2.1.3 Stacking and truncating

The operations *stacking* and *truncating* (see [40, Sect. 16.2.1]) turn out to be essential in finding examples of polytopes for all possible polytopal pairs (f_0, f_{03}). Let P be a 4-polytope with at least one simplex facet F and v a point beyond F and beneath all other facets of P. Let $Q = \operatorname{conv}(\{v\} \cup P)$. Then

$$f_0(Q) = f_0(P) + 1 \text{ and } f_{03}(Q) = f_{03}(P) + 12.$$

Dually, let Q be a polytope obtained by truncating a simple vertex from a polytope P. Then

$$f_0(Q) = f_0(P) + 3 \text{ and } f_{03}(Q) = f_{03}(P) + 12.$$

The polytopes obtained through these two methods all have both a simple vertex and a simplex facet. This means that we can stack vertices on simplex facets and truncate simple vertices repeatedly. Truncating simple vertices and stacking vertices on simplex facets inductively, starting from a polytope with (f_0, f_{03}) with tetrahedral facet and simple vertex, we obtain new polytopes with

$$(f_0 + 2m + n, f_{03} + 12n) \text{ for } n \geq 0, \quad 0 \leq m \leq n.$$

Given a polytope P with a square pyramidal facet F, let v be a point beyond F and beneath all other facets of P. Let $Q = \operatorname{conv}(\{v\} \cup P)$. Then

$$f_0(Q) = f_0(P) + 1 \text{ and } f_{03}(Q) = f_{03}(P) + 16.$$

The results in this section are simple consequences from [38, Thm. 5.2.1], with corrections from [2].

2.1.4 *Generalized stacking on cyclic polytopes*

We need some more methods, especially to create polytopes with polytopal pair (f_0, f_{03}) close to the upper bound $f_{03} = 2f_0(f_0 - 3)$. For our next construction we need the observation that every cyclic 4-polytope with n vertices has edges that lie in exactly $n-2$ facets. Such edges are called *universal edges*. The following construction was used by Grünbaum [38, Sect. 10.4.1] for the characterization of the sets $\Pi_{0,3}(\mathcal{F}^4)$ and $\Pi_{0,1}(\mathcal{F}^4)$. Starting from a cyclic polytope with n vertices, we can obtain new polytopes by stacking a vertex onto it, such that the vertex lies beyond several facets. Let $R_i(n), i \in \{1, \ldots, n-3\}$, denote a polytope obtained from the cyclic polytope $C_4(n)$ with n vertices as the convex hull of $C_4(n)$ and a point v, where v is beyond i facets of $C_4(n)$ sharing a universal edge. Let $F_1, \ldots, F_i$ denote these i facets, such that F_j and F_{j+1} meet in a common 2-face, for all $j = 1, \ldots, i-1$. Then the new polytope $R_i(n)$ has one more vertex than $C_4(n)$ and the following facets:

(1) All $\frac{1}{2}n(n-3) - i$ facets of $C_4(n)$ which v lies beneath.

(2) Facets which are convex hulls of v and 2-faces of $C_4(n)$ that are contained in both a facet which v is beyond and a facet which v is beneath. There are two types of these facets:

(2a) Two such facets for each of the $i-2$ facets $F_2, \ldots, F_{i-1}$ which v lies beyond and which share two 2-faces with other facets which v lies beyond.

(2b) Three new facets for each of the two facets F_1 and F_i which v lies beyond and which share one 2-face with other facets which v lies beyond.

Note that all these facets are simplices. In conclusion, for $1 \leq i \leq n-3$,

$$f_0(R_i(n)) = n + 1 \text{ and } f_{03}(R_i(n)) = 2n(n-3) + 4i + 8.$$

Observe that

$$f_{03}(C_4(n+1)) = f_{03}(C_4(n)) + 4n - 4,$$

so if $i = n-3$, we obtain again a neighborly polytope, with $n+1$ vertices.

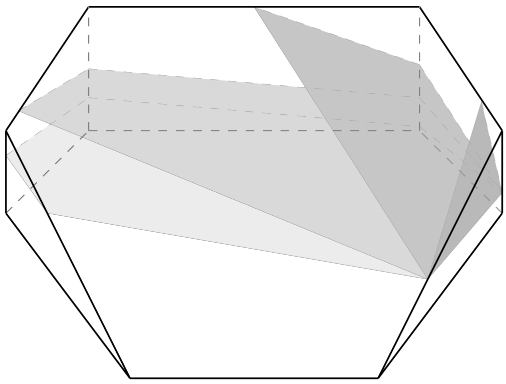
2.1.5 Facet splitting

We need to generalize the stacking method even more to obtain non-simplicial polytopes, compare to the *A-sewing construction* of Lee and Menzel [58]. For easier visualization, we choose to work in the dual setting. Instead of adding a new vertex to a polytope, we will create a new facet in the dual polytope. This method of *facet splitting* was used by Barnette [7] for the classification of $\Pi_{1,2}(\mathcal{F}^4)$: Consider a facet F of a 4-polytope P and a hyperplane H which intersects the relative interior of F in a polygon X. If on one side of H, the only vertices of P are simple vertices of F, then we can obtain a new polytope P' by separating facet F into two new facets by the polygon X. We say that P' is obtained from P by *facet splitting*.

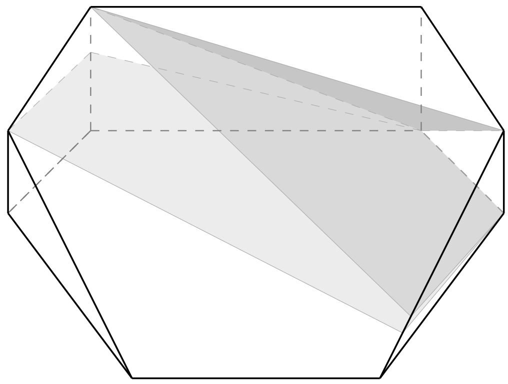
2.1.5.1 Dual of a cyclic polytope

We will split a facet of the dual of a cyclic polytope (see Barnette [7]). $C_4^*(n)$, the dual of the cyclic polytope with n vertices, is a simple polytope with n facets, each facet having $2(n-3)$ vertices. The facets are all *wedges* over $(n-2)$-gons, that is, polytopes with two triangular 2-faces, $n-5$ quadrilateral 2-faces and two $(n-2)$-gons meeting in an edge. Let G be a 2-dimensional plane in the affine hull of a facet F of $C_4^*(n)$. Let X be the intersection of F and G. All vertices of $C_4^*(n)$ are simple, so we can obtain a new polytope by facet splitting of $C_4^*(n)$ by choosing a hyperplane H which contains G such that on one side of H the only vertices of $C_4^*(n)$ are vertices of F. Such a hyperplane can be found by taking the facet-defining hyperplane of F and rotating it about G. The combinatorial properties f_1 and f_{03} of the polytope obtained through facet splitting depend on the choice of G: We can choose G not to intersect any vertices of F. Then, for any i such that $3 \leq i \leq n-2$, $X = G \cap F$ can be chosen to be an i-gon. Let $\delta_0(i,n)$ denote the polytope obtained through facet splitting for this choice of G (see Figure 2.3a). Now $\delta_0(i,n)$ has one more facet and i more vertices than $C_4^*(n)$. As $C_4^*(n)$ is a simple polytope, all of its edges lie in exactly three facets and each of the i new vertices of $\delta_0(i,n)$ lies in four facets. The new polytope has therefore $4i$ more vertex-facet incidences than $C_4^*(n)$. If we instead choose G to intersect exactly one vertex of F, X can again be any i-gon for $3 \leq i \leq n-2$. Call this polytope $\delta_1(i,n)$. It has one more facet and $i-1$ more vertices than $C_4^*(n)$. The $i-1$ new vertices are simple, and the one vertex of $C_4^*(n)$ which lies in X is contained in one additional facet. In total, f_{03} increases by $4i-3$. The polytopes $\delta_0(i,n)$ and $\delta_1(i,n)$ are used in the characterization of $\Pi_{1,2}(\mathcal{F}^4)$ [7].

Similarly, let $\delta_2(i,n)$ denote the polytope obtained when G intersects two vertices of F. As before, i can be chosen to be any integer between 3 and $n-2$. The new polytope has one more facet, $i-2$ more vertices and $4i-6$ more vertex-facet incidences. If we choose G to intersect F in three vertices, as the intersection of G and F we can obtain i-gons for $3 \leq i \leq n-3$ (see Figure 2.3b). The new polytope, denoted by $\delta_3(i,n)$, has one more facet, $i-3$ more vertices and $4i-9$ more vertex-facet incidences. Let us look at the duals of these polytopes. For $3 \leq i \leq n-2$ we obtain polytopes $\delta_0^*(i,n)$, $\delta_1^*(i,n)$



(a) $\delta_0(i,8)$



(b) $\delta_3(i,8)$

Figure 2.3: Facet of $C_4^*(8)$ split by an i-gon

and $\delta_2^*(i,n)$ with

$$\begin{aligned}(f_0(\delta_0^*(i,n)), f_{03}(\delta_0^*(i,n))) &= (n+1, 2n(n-3)+4i),\\ (f_0(\delta_1^*(i,n)), f_{03}(\delta_1^*(i,n))) &= (n+1, 2n(n-3)+4i-3),\\ (f_0(\delta_2^*(i,n)), f_{03}(\delta_2^*(i,n))) &= (n+1, 2n(n-3)+4i-6).\end{aligned}$$

For $3 \le i \le n-3$ we obtain polytopes $\delta_3^*(i,n)$ with

$$(f_0(\delta_3^*(i,n)), f_{03}(\delta_3^*(i,n))) = (n+1, 2n(n-3)+4i-9).$$

In particular, the polytopes $\delta_0^*(i,n)$, $\delta_1^*(i,n)$, $\delta_2^*(i,n)$ and $\delta_3^*(i,n)$ have simplex facets.

2.1.5.2 Polytopes with a bipyramidal facet

Given a polytope P with a facet B which is a bipyramid over a triangle, such that at least one apex v of B is a simple vertex, we can split the bipyramid into two tetrahedra by "moving" v outside the affine hull of B, along the unique edge which contains v and does not belong to B. The new polytope $\widetilde{P}$ has the same number of vertices and one more facet than P. The apices of the bipyramid still belong to the same number of facets as before, but the other three vertices now belong to one more facet. In total, the number of vertex-facet incidences increases by 3. Hence,

$$(f_0(\widetilde{P}), f_{03}(\widetilde{P})) = (f_0(P), f_{03}(P)+3).$$

Note that $\widetilde{P}$ has simplex facets and that any simple vertex of P is a simple vertex of $\widetilde{P}$.

2.1.6 Construction of polytopal pairs (f_0, f_{03})

We can now prove Theorem 2.1.5. First, we list some examples of polytopes with small polytopal pairs (f_0, f_{03}) for $f_{03} \leq 80$ with simplex facet and/or simple vertex, see Table 2.1. The second column in the table explains how the polytope is found. Polytopes P_i are polytopes with 7 or 8 vertices known from the classification of all polytopes with up to 8 vertices. Facet lists of all polytopes P_i can be found in Appendix A. *Dual of* (f_0, f_{03}) means that the polytope is the dual of the polytope with polytopal pair (f_0, f_{03}) in the table. A polytope P^* denotes the dual of a polytope P. The methods *stacking on a square pyramidal facet* and *splitting a bipyramidal facet* and the polytopes $R_i(n)$ are explained above.

Together with the inductive stacking and truncating methods from Section 2.1.3, this gives us all possible pairs for $f_{03} \leq 80$ and, in particular, polytopal pairs (f_0, f_{03}) with simple vertex and simplicial facet, for $f_0 \geq 9$, $53 \leq f_{03} \leq 64$ and $4f_0 \leq f_{03}$. See Figure 2.4. Stacking on simplex facets and truncating simple vertices of these 87 pairs of polytopes inductively will give all polytopal pairs (f_0, f_{03}) bounded by the lower bound $4f_0 \leq f_{03}$, $f_{03} \geq 53$, and a line with slope 12 going through $(9, 64)$. We have hence proved the following.

Lemma 2.1.10. There exists a 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ whenever

$$4f_0 \leq f_{03} \leq 12f_0 - 44 \text{ and } f_{03} \geq 53.$$

In the next step we construct polytopes with $12f_0 - 44 \leq f_{03} \leq 2f_0(f_0 - 3)$. In order to do so, we give examples of polytopes with simplex facet close to the upper bound. The cyclic polytopes have polytopal pairs

$$\begin{aligned}(f_0(C_4(n)), f_{03}(C_4(n))) &= (n, 2n(n-3)),\\ (f_0(C_4(n+1)), f_{03}(C_4(n+1))) &= (n+1, 2n(n-3) + 4n - 4).\end{aligned}$$

Our goal is to find polytopes with tetrahedral facets and polytopal pair

$$(n+1, 2n(n-3) + i), \text{ for } i = 0, \ldots, 4n-5.$$

If we find such polytopes, combined with the stacking and truncating operations from Section 2.1.3, this gives us all remaining polytopal pairs. In fact, by Lemma 2.1.7, there are no polytopes with polytopal pair $(n+1, 2n(n-3)+4n-k)$ for $k \in \{5, 6, 7, 9, 10, 13, 17\}$. In these cases, the "next" polytope in the stacking process, a polytope with polytopal pair $(n+2, 2n(n-3) + 4n - k + 12)$ for $k \in \{5, 6, 7, 9, 10, 13\}$, is given for $m = n + 1$ by the polytope with $(m+1, 2m(m-3) + 16 - k)$. For $k = 17$, the polytope with polytopal pair $(m+1, 2m(m-3) - 1)$ can be obtained through stacking a vertex onto two facets of $\delta_3^*(n-3, n)$ with polytopal pair $(n+1, 2n(n-3) + 4n - 21) = (m, 2m(m-3) - 17)$ (see Section 2.1.5.1). Stacking a vertex onto $\delta_3^*(n-3, n)$, such that the vertex is beyond two simplex facets which have a common 2-face, yields a new polytope with 16 more

(f_0, f_{03})	Description
Polytopes with Δ_3-facet and simple vertex	
$(5,20)$	4-simplex
$(6,26)$	2-fold pyramid over quadrangle
$(6,29)$	pyramid over triangular bipyramid
$(7,29)$	pyramid over triangular prism
$(7,32)$	2-fold pyramid over pentagon
$(7,35)$	P_1
$(7,36)$	P_2
$(7,39)$	P_3
$(7,45)$	P_5
$(8,35)$	P_1^*
$(8,36)$	P_2^*
$(8,38)$	2-fold pyramid over hexagon
$(8,39)$	P_8
$(8,42)$	P_9
$(8,45)$	P_{11}
$(8,46)$	P_{12}
$(8,49)$	P_{13}
$(8,52)$	P_{14}
$(8,55)$	P_{15}
$(8,59)$	P_{16}
$(8,62)$	P_{18}
$(9,39)$	P_3^*
$(9,42)$	P_9^*
$(9,45)$	split bipyramid in $(9,42)$
$(9,46)$	split bipyramid in $(9,43)$
$(9,49)$	split bipyramid in $(9,46)$
$(9,52)$	stack onto square pyramid in $(8,36)$
$(10,45)$	P_{11}^*
$(10,46)$	P_{12}^*
$(10,49)$	dual of $(9,49)$
$(10,52)$	split bipyramid in $(10,49)$
$(10,55)$	stack onto square pyramid in $(9,39)$
$(11,45)$	P_5^*
$(11,49)$	P_{13}^*
$(11,52)$	dual of $(9,52)$
$(11,55)$	dual of $(10,55)$
$(12,52)$	P_{14}^*
$(13,55)$	P_{15}^*
Polytopes with Δ_3-facet	
$(6,36)$	cyclic polytope $C_4(6)$
$(7,42)$	P_4
$(7,46)$	P_6
$(7,49)$	P_7
$(7,52)$	$R_2(6)$
$(7,56)$	cyclic polytope $C_4(7)$
$(8,43)$	P_{10}
$(8,60)$	P_{17}
$(8,63)$	P_{19}
$(8,65)$	P_{20}
$(8,66)$	P_{21}
$(8,68)$	P_{22}
$(8,69)$	P_{23}
$(8,70)$	P_{24}
$(8,72)$	P_{25}
$(8,73)$	P_{26}
$(8,76)$	P_{27}
$(8,80)$	cyclic polytope $C_4(8)$
$(9,79)$	stack onto square pyramid in $(8,63)$
Polytopes with simple vertex	
$(9,36)$	dual of cyclic polytope $C_4(6)$
$(9,43)$	P_{10}^*
$(10,42)$	P_4^*
$(11,46)$	P_6^*
$(12,49)$	P_7^*
$(13,52)$	$R_2(6)^*$

Table 2.1: Some polytopal pairs

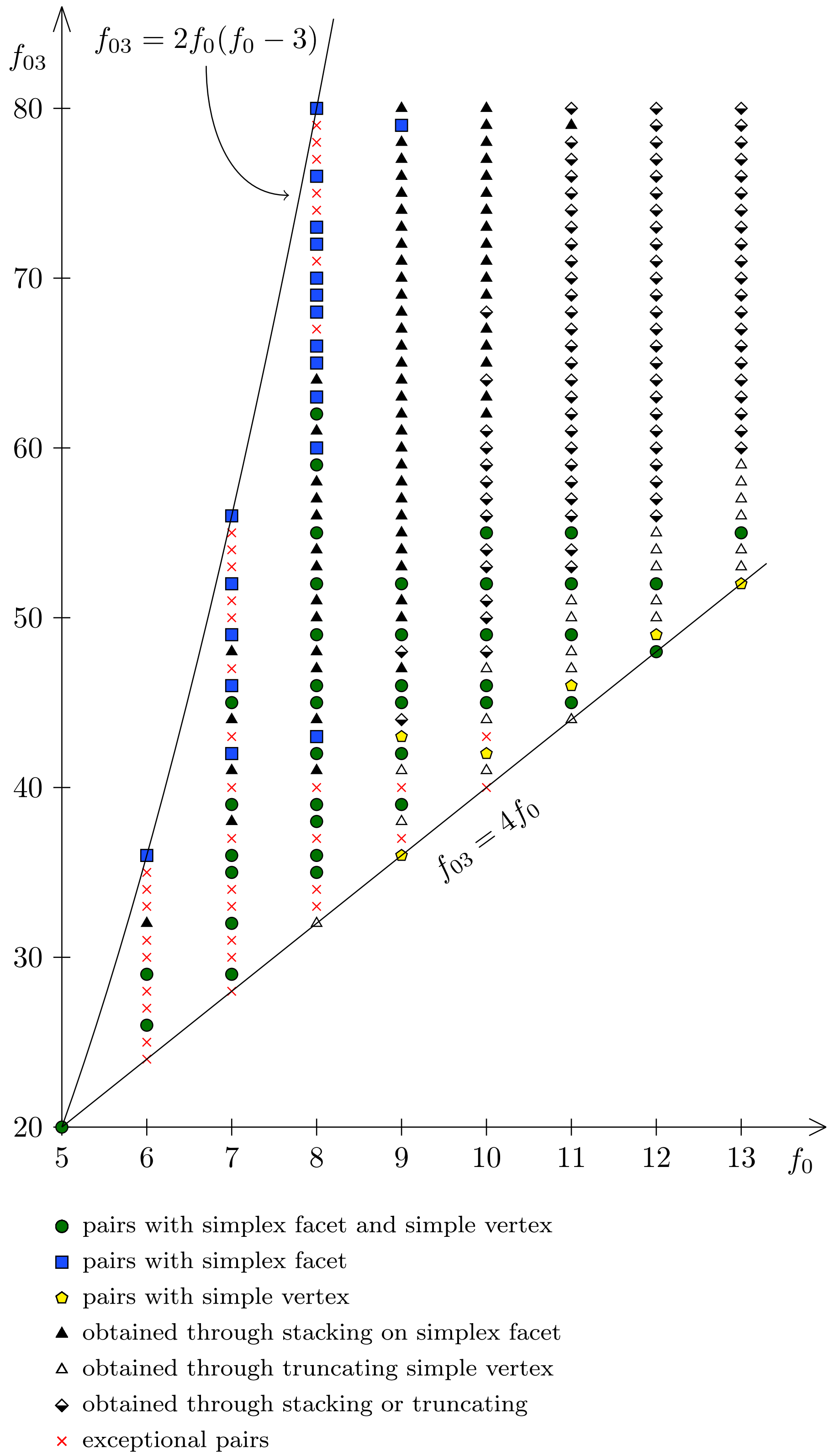


Figure 2.4: Polytopal pairs with $f_{03} \leq 80$

vertex-facet incidences and one additional vertex (cf. Section 2.1.4). So the new polytope has the required polytopal pair $(m+1, 2m(m-3)-1)$.

To find examples of polytopes with $(f_0, f_{03}) = (n+1, 2n(n-3)+i)$, for $f_0 = n+1 \geq 9$, $i = 0, \ldots, 4n-4$, $i \neq 4n-j$ for $j \in \{5, 6, 7, 9, 10, 13, 17\}$, we use the constructions from Sections 2.1.4 and 2.1.5. Table 2.2 shows how the polytopes are constructed.

For $f_0 \leq 8$ we use the fact that polytopes with up to 8 vertices have been classified (see Table 2.1). In particular, we can construct examples of polytopes with simplex facet, simple vertex and polytopal pair

$$(n+2, 2n(n-3)+i), \text{ for all } i = 0, \ldots, 4n-5, \; n \geq 7.$$

If we now inductively stack vertices on simplex facets and truncate simple vertices, we obtain polytopes with polytopal pairs (f_0, f_{03}) with $f_0 \geq 9$ bounded from above by $2f_0(f_0-3)$ and from below by a line of slope 4, going through $(9, 56)$. So we have found all polytopal pairs with

$$4f_0 + 20 \leq f_{03} \leq 2f_0(f_0-3) \tag{2.7}$$

for all $f_0 \geq 9$, with the only exceptions for each value of f_0 being the 7 pairs mentioned above. Lemma 2.1.10 and Inequality (2.7) together give all pairs (f_0, f_{03}) with $f_0 \geq 9$, $f_{03} \geq 53$ within the bounds, excluding the exceptional pairs. Since we classified all possible polytopal pairs with $f_{03} \leq 80$, and in particular all polytopal pairs with $f_0 \leq 8$, we have now proved Theorem 2.1.5.

2.1.7 Other flag vector pairs

The flag vector of a 4-polytope has 16 entries. Besides $f_\emptyset = 1$, the following nine entries depend on only one other entry:

$$\begin{aligned}
f_{01} &= 2f_1, & f_{12} &= f_{02}, & f_{13} &= f_{02},\\
f_{23} &= 2f_2, & f_{012} &= 2f_{02}, & f_{013} &= 2f_{02},\\
f_{023} &= 2f_{02}, & f_{123} &= 2f_{02}, & f_{0123} &= 4f_{02}.
\end{aligned}$$

These equations are some of the Generalized Dehn–Sommerville equations for 4-dimensional polytopes (Lemma 2.1.8). To obtain all 2-dimensional coordinate projections of the flag vectors of 4-polytopes, we therefore only have to consider the six entries f_0, f_1, f_2, f_3, f_{02} and f_{03}. We still need to determine the projections

$$\Pi_{0,02}(\overline{\mathcal{F}}^4), \Pi_{1,02}(\overline{\mathcal{F}}^4), \Pi_{1,03}(\overline{\mathcal{F}}^4) \text{ and } \Pi_{02,03}(\overline{\mathcal{F}}^4).$$

All other cases have already been done, or they follow directly, either by duality or by the linear dependence on a single entry.

For the projections $\Pi_{0,02}(\overline{\mathcal{F}}^4)$ and $\Pi_{1,02}(\overline{\mathcal{F}}^4)$, the pairs (f_0, f_{02}) in $\Pi_{0,02}(\overline{\mathcal{F}}^4)$ satisfy the fairly obvious bounds $6f_0 \leq f_{02} \leq 3f_0(f_0-3)$. Equality holds for simple and neighborly polytopes, respectively. Similarly the pairs (f_1, f_{02}) in $\Pi_{1,02}(\overline{\mathcal{F}}^4)$ satisfy

	f_{03}	Example of polytope
$f_{03} \equiv 0 \bmod 4$	$2n(n-3)$	stack onto Δ_3-facet of $R_{n-7}(n-1)$
	$2n(n-3)+4$	stack onto Δ_3-facet of $R_{n-6}(n-1)$
	$2n(n-3)+8$	stack onto Δ_3-facet of $R_{n-5}(n-1)$
	$2n(n-3)+12$	stack onto Δ_3-facet of $C_4(n)$
	$2n(n-3)+16$	$R_2(n)$
	$2n(n-3)+20$	$R_3(n)$
	$\dots$	$\dots$
	$2n(n-3)+4n-8$	$R_{n-4}(n)$
	$2n(n-3)+4n-4$	$C_4(n+1)$
$f_{03} \equiv 1 \bmod 4$	$2n(n-3)+1$	stack onto Δ_3-facet of $\delta_1^*(n-4,n-1)$
	$2n(n-3)+5$	stack onto Δ_3-facet of $\delta_1^*(n-3,n-1)$
	$2n(n-3)+9$	$\delta_1^*(3,n)$
	$\dots$	$\dots$
	$2n(n-3)+4n-11$	$\delta_1^*(n-2,n)$
	$2n(n-3)+4n-7$	does not exist
$f_{03} \equiv 2 \bmod 4$	$2n(n-3)+2$	stack onto Δ_3-facet of $\delta_2^*(n-3,n-1)$
	$2n(n-3)+6$	$\delta_2^*(3,n)$
	$\dots$	$\dots$
	$2n(n-3)+4n-14$	$\delta_2^*(n-2,n)$
	$2n(n-3)+4n-10$	does not exist
	$2n(n-3)+4n-6$	does not exist
$f_{03} \equiv 3 \bmod 4$	$2n(n-3)+3$	$\delta_3^*(3,n)$
	$\dots$	$\dots$
	$2n(n-3)+4n-21$	$\delta_3^*(n-3,n)$
	$2n(n-3)+4n-17$	does not exist
	$2n(n-3)+4n-13$	does not exist
	$2n(n-3)+4n-9$	does not exist
	$2n(n-3)+4n-5$	does not exist

Table 2.2: Polytopal pairs $(n+1, 2n(n-3)+i)$, $n \geq 8$

$3f_1 \le f_{02} \le 6f_1 - 3\sqrt{8f_1+1} - 3$, with equality for 2-simple polytopes (each edge is contained in exactly 3 facets) and neighborly polytopes, respectively.

The projection sets $\Pi_{1,03}(\overline{\mathcal{F}}^4)$ and $\Pi_{02,03}(\overline{\mathcal{F}}^4)$ are more difficult to describe even approximately. Upper bounds for f_{03} in terms of f_1 are achieved for neighborly polytopes, and in terms of f_{02} for center boolean polytopes. The problem of finding tight lower bounds for f_{03} in terms of f_1 and f_{02} is related to the open problem of finding an upper bound for the *fatness* $F = \frac{f_1+f_2-20}{f_0+f_3-10}$ of a polytope [35].

2.2 FACE VECTOR PAIR (f_0, f_{d-1}) FOR d-POLYTOPES

Now we work towards analogous results in higher dimensions. In one instance, recently the projection $\Pi_{0,1}(\mathcal{F}^5)$ of the f-vector of 5-polytopes to (f_0, f_1) was determined almost simultaneously by Kusunoki and Murai [56] and by Pineda-Villavicencio, Ugon and Yost [71].

We consider $\Pi_{0,d-1}(\mathcal{F}^d)$, the projection of the set of f-vectors of d-polytopes to (f_0, f_{d-1}). In the following, for given d, we will consider pairs of integers (n, m) and analyze under which conditions there are d-polytopes with n vertices and m facets.

Definition 2.2.1. For fixed dimension d, a pair $(n, m) \in \mathbb{N}^n$ is *d-large* if $n + m \ge \binom{3d+1}{\lfloor d/2 \rfloor}$; it is *$d$-small* otherwise. A pair (n, m) will be called an *exceptional pair* if $m \le f_{d-1}(C_d(n))$ and $n \le f_{d-1}(C_d(m))$, and if there is no d-polytope with n vertices and m facets.

The situation looks as follows:

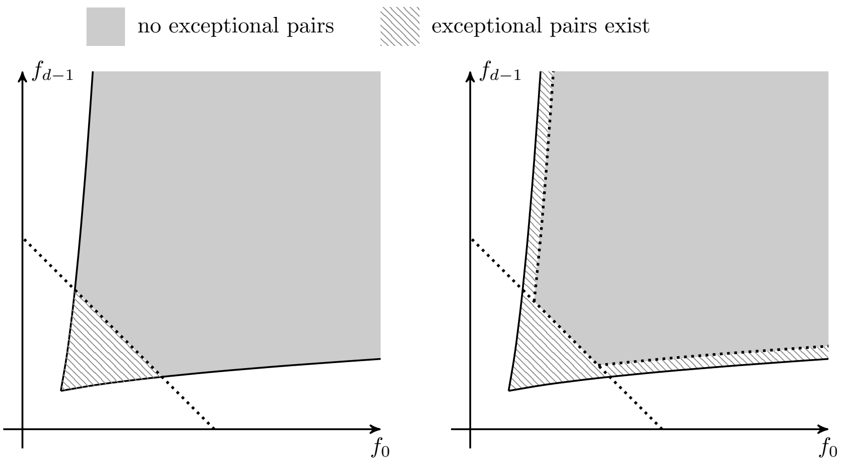
(1) If P is a d-polytope with n vertices and m facets, then

$$m \le f_{d-1}(C_d(n)),\ n \le f_{d-1}(C_d(m)).$$

(2) If (n, m) is a pair of integers, $n, m \ge d+1$ such that for a given dimension d, $m \le f_{d-1}(C_d(n)),\ n \le f_{d-1}(C_d(m))$, then there *usually* exists a d-polytope with n vertices and m facets:

- (2.1) For $d \le 4$ no exceptional pairs exist.
- (2.2) For even $d \ge 6$, only finitely many exceptional pairs exist, all of which are d-small (see Figure 2.5a).
- (2.3) For odd $d \ge 5$, there exist finitely many d-small exceptional pairs and additionally infinitely many d-large exceptional pairs for m odd and $f_{d-1}(C_d(n-1)) < m < f_{d-1}(C_d(n))$ and for n odd and $f_{d-1}(C_d(m-1)) < n < f_{d-1}(C_d(m))$ (see Figure 2.5b).

(1) are the UBT inequalities.
(2.1) holds trivially for $d \le 2$. For dimension 3, it is given by Steinitz' classification of all 3-dimensional polytopes [82]. For dimension 4, this is Theorem 2.1.1.
(2.2) is Theorem 2.2.2 and (2.3) is Theorem 2.2.3.



(a) even dimensions

(b) odd dimensions

Figure 2.5: Projections $\Pi_{0,d-1}(\mathcal{F}^d)$

THEOREM 2.2.2. Let $d \geq 2$ be even and (n, m) d-large. Then there exists a d-polytope P with n vertices and m facets if and only if

$$m \leq f_{d-1}(C_d(n)) \text{ and } n \leq f_{d-1}(C_d(m)).$$

The first inequality holds with equality if and only if P is neighborly, and the second inequality holds with equality if and only if P is dual-neighborly.

However, for $d \geq 6$ d-small exceptional pairs (n, m) exist.

Proof. The necessity of the conditions and the equality cases are direct consequences of the upper bound theorem (McMullen [61]). For the sufficiency, consider the g-vector of simplicial polytopes.

The $\frac{d}{2}$-th entry of the g-vector of a cyclic polytope $C_d(n)$ in even dimension $d = 2k$ with n vertices is

$$g_{d/2}(C_d(n)) = g_k(C_{2k}(n)) = \binom{n-k-2}{k}.$$

A consequence of the sufficiency part of the g-theorem (Billera & Lee [17, 16]) is that there exist simplicial $2k$-polytopes with n vertices, $g_i = g_i(C_{2k}(n))$ and $g_k = l$ for all $1 \leq i \leq k-1$ and for all $0 \leq l \leq \binom{n-k-2}{k}$.

For all simplicial $2k$-polytopes,

$$f_{2k-1} = (2k+1) + g_1(2k-1) + g_2(2k-3) + \cdots + g_{k-1} \cdot 3 + g_k.$$

Hence, there exist simplicial $2k$-polytopes with n vertices and $f_{2k-1}(C_{2k}(n)) - l$ facets, for $0 \leq l \leq \binom{n-k-2}{k}$. Observe that $\binom{n-k-2}{k} > f_{2k-1}(C_{2k}(n)) - f_{2k-1}(C_{2k}(n-1))$ for large n. In particular, this inequality holds for $n \geq 7k+2 = \frac{7}{2}d + 2$. This means that for

$n \geq \frac{7}{2}d + 2$ there are simplicial $2k$-polytopes with n vertices and m facets, for all integers m such that $f_{2k-1}(C_{2k}(n-1)) \leq m \leq f_{2k-1}(C_{2k}(n))$. Now we can stack a vertex on a facet of each of these polytopes and obtain polytopes with one more vertex and $d-1$ more facets. The new polytope has a simple vertex and simplex facets, so we can repeatedly stack vertices on simplex facets and truncate simple vertices. Truncating simple vertices gives a polytope with $d-1$ more (simple) vertices and one additional (simplex) facet. Consider the pair $(7k+2, m)$: We have just seen that this pair is not an exceptional pair as long as $m \geq f_{2k-1}(C_{2k}(7k+1))$. Stacking a vertex on a facet of a polytope with pair $(7k+2, f_{2k-1}(C_{2k}(7k+1)))$ gives a polytope with simplex facet, simple vertex and pair $(n_0, m_0) := (7k+3, f_{2k-1}(C_{2k}(7k+1)) + d - 1)$. Consider the line ℓ_1 of slope $\frac{1}{d-1}$ through (n_0, m_0). There are no exceptional pairs with $n \geq n_0$ above ℓ_1. The line ℓ_1 intersects the line $\ell_2 : m = n$ in a pair (n, n) such that

$$n = \frac{k+1}{12k+2}\binom{6k+1}{k} < \frac{1}{2}\binom{6k+1}{k}.$$

Together with the dual polytope, we have obtained all polytopes with pairs (n, m) within the bounds such that

$$\binom{3d+1}{\frac{d}{2}} \leq n + m.$$

Hence, there are no d-large exceptional pairs.

On the other hand, there are exceptional pairs for d-small (n, m). As an example consider d-polytopes with $d+2$ vertices. All d-polytopes with $d+2$ vertices are simplicial or (multiple) pyramids over some r-polytope with $r+2$ vertices [88, Sect. 6.5].

There are exactly $\lfloor \frac{d}{2} \rfloor = k$ different combinatorial types of simplicial d-polytopes with $d+2$ vertices ([38, Sect. 6.1]). One of these types is the stacked polytope with $2d$ facets. In particular, for $d \geq 6$, $2d \leq k^2+k+1$. Any non-simplicial d-polytope with $d+2$ vertices is a pyramid and has thus at most $f_{d-1}(\mathrm{Pyr}(C_{d-1}(d+1))) = f_{d-2}(C_{d-1}(d+1)) + 1 = k^2+k+1$ facets for $d = 2k$. This means that there are at most $k-1$ different combinatorial types of $(2k)$-polytopes with $2k+2$ vertices and more than k^2+k+1 facets.

The cyclic polytope with $d+2$ vertices has k^2+2k+1 facets for even dimensions $d = 2k$. So there are k pairs (n, m) for given n and $k^2+k+1 < m \leq f_{d-1}(C_d(n))$, but at most $k-1$ combinatorially non-equivalent polytopes. Therefore, for $n = d+2$ and even $d \geq 6$ there must be at least one exceptional pair. $\square$

An example is the pair $(n, m) = (8, 14)$ for dimension 6: There is no 6-polytope with 8 vertices and 14 facets [37], but there are 6-polytopes with 8 vertices and 13 or 15 facets.

THEOREM 2.2.3. Let $d \geq 3$ be odd. If (n, m) is d-large, then there exist d-polytopes with n vertices and m facets only if

$$m \leq f_{d-1}(C_d(n)) \text{ and } n \leq f_{d-1}(C_d(m))$$

with d-large exceptional pairs occurring only for $d \geq 5$, if m is odd and $f_{d-1}(C_d(n-1) \leq m$ and if n is odd and $f_{d-1}(C_d(m-1) \leq n$.

However, for $d \geq 5$ d-small exceptional pairs (n, m) exist.

Proof. The necessity follows again from the upper bound theorem (McMullen [61]). For the sufficiency, we follow the proof of Theorem 2.2.2.

A cyclic polytope $C_d(n)$ in odd dimension $d = 2k+1$ with n vertices has $g_{\lfloor \frac{d}{2} \rfloor}$ equal to

$$g_{\lfloor \frac{d}{2} \rfloor}(C_d(n)) = g_k(C_{2k+1}(n)) = \binom{n-k-3}{k}.$$

Again, by the g-theorem [17, 16, 80], there exist simplicial $(2k+1)$-polytopes with n vertices, $g_i = g_i(C_{2k}(n))$ and $g_k = l$ for all $1 \leq i \leq k-1$ and for all $0 \leq l \leq \binom{n-k-3}{k}$.

For all simplicial $(2k+1)$-polytopes,

$$f_{d-1} = (d+1) + g_1(d-1) + g_2(d-3) + \cdots + g_{k-1} \cdot 4 + g_k \cdot 2.$$

Hence, there exist simplicial $(2k+1)$-polytopes with n vertices and $f_{2k}(C_{2k+1}(n)) - 2l$ facets, for $0 \leq l \leq \binom{n-k-3}{k}$.

We have that $2\binom{n-k-3}{k} > f_{2k}(C_{2k+1}(n)) - f_{2k}(C_{2k+1}(n-1))$ holds for large n, in particular for $d = 5$ if $n \geq 9$ and for general d if $n \geq 5k+1 = \frac{5}{2}d - \frac{3}{2}$. With the same calculations as before, we obtain polytopes with n vertices and m facets for all pairs (n, m) if n and m are even and if

$$n + m \geq \frac{2}{2k-1}(4k\binom{4k-1}{k} + 4k^2 - 5k - 2).$$

For $d \geq 7$, this implies that

$$n + m \geq \binom{6k+4}{k} = \binom{3d+1}{\lfloor \frac{d}{2} \rfloor}.$$

For $d = 5$, we check that the constructions give us all polytopes with $n + m \geq 58$, where $\binom{3 \cdot 5+1}{2} > 58$. We can also construct polytopes with an odd number of facets, as long as $m \leq f_{d-1}(C_d(n-1))$. For this, we need a generalized stacking construction similar to the one described in Section 2.1.4. Starting with a simplicial polytope, we place a new vertex beyond one facet, inside the affine hull of a second facet and beneath all other facets. The new polytope has one new (simple) vertex and $d-2$ new facets. The polytope has one facet which is a bipyramid over a triangle. All other facets are simplices, so we can apply the inductive stacking and truncating method from before.

There are exceptional pairs (n, m) if m is odd and close to $f_{d-1}(C_d(n))$: Non-simplicial d-polytopes with n vertices have at most $f_{d-1}(C_d(n)) - \lfloor \frac{d}{2} \rfloor$ facets. This is a direct consequence of the upper bound theorem for *almost simplicial polytopes* by Nevo, Pineda-Villavicencio, Ugon & Yost [68]. These authors give upper bounds for the number of faces of the family $\mathcal{P}(d, n, s)$ of *almost simplicial polytopes*, d-polytopes on n vertices where one facet has $d + s \geq d + 1$ vertices and all other facets are simplices. Such polytopes have at most $f_{d-1}(C_d(n)) - \lfloor \frac{d}{2} \rfloor$ facets. (This follows from [68], Thm. 1.2 and Prop. 4.2.)

For any non-simplicial polytope P on n vertices there exists an almost-simplicial polytope on n vertices (i.e. a polytope with exactly one non-simplicial facet) that has at least as many i-faces as P: Let F be a non-simplicial facet of P. If we successively pull every vertex of $\operatorname{vert} P \setminus \operatorname{vert} F$ (in the sense of [33]) and then pull every vertex $v \in \operatorname{vert} F$ *within* the affine hull of F, then the resulting polytope is almost simplicial, with at least as many i-dimensional faces as P. So the i-faces of non-simplicial d-polytopes on n vertices are maximized among the almost simplicial d-polytopes on n vertices. In particular, for any non-simplicial d-polytope P, $f_{d-1}(P) \leq f_{d-1}(C_d(n)) - \lfloor \frac{d}{2} \rfloor$. Thus, for odd d, odd m, and

$$f_{d-1}(C_d(n)) - \left\lfloor \frac{d}{2} \right\rfloor < m < f_{d-1}(C_d(n)),$$

(n, m) is an exceptional pair.

The rest of the theorem for d-large (n, m) follows by duality. For d-small (n, m), the non-constructive proof for the existence of exceptional pairs in the even-dimensional case works as well in the odd-dimensional case. It can be slightly improved: All d-polytopes with $d + 2$ vertices are simplicial or (multiple) pyramids over some r-polytope with $r + 2$ vertices [88, Sect. 6.5]. In particular, for $d = 2k + 1$ and odd m, any polytope P with $d + 2$ vertices and m facets is a pyramid over some $(d - 1)$-polytope Q with $d + 1$ vertices and $m - 1$ facets. Hence,

$$m - 1 = f_{d-2}(Q) \leq f_{d-2}(C_{d-1}(d + 1)) = k^2 + 2k + 1.$$

Comparing this to

$$f_{d-1}(C_d(d + 2)) = k^2 + 3k + 2,$$

we see that there are $\left\lfloor \frac{k}{2} \right\rfloor$ exceptional pairs (n, m) for which there are no $(2k+1)$-polytopes, such that m is odd and

$$k^2 + 2k + 2 < m < k^2 + 3k + 2.$$

$\square$

Remark. This implies that for for odd d the projection sets $\Pi_{0,d-1}(\mathcal{F}^d)$ have infinitely many exceptional pairs, all of them near the boundary. For a complete characterization of d-large pairs in $\Pi_{0,d-1}(\mathcal{F}^d)$ one would need to analyze closely the possible facet numbers of non-simplicial polytopes with many facets.

For low dimensions, we can improve the bounds for the *d-large* pairs. We have seen that in dimension 5, a pair can be called d-large if $n + m \geq 58$. Similarly, for dimension

6, the bound for d-large pairs can be reduced to $n + m \geq 132$: It can be seen from the g-theorem that simplicial 6-polytopes with n vertices have $5n - 28$, $5n - 25$, $5n - 24$, or $5n - 22$ to $f_5(C_6(n))$ facets. For $n \geq 11$, it holds that $5n - 22 < f_5(C_6(n - 1))$. From this, the bound $n + m \geq 132$ for d-large pairs can be derived.

3

SEMI-ALGEBRAIC SETS OF F-VECTORS

The goal of this chapter is to analyze the “complexity” of f-vector sets of polytopes. We want to say that f-vector sets in general can be very complicated to describe. We will do this by defining “nice” ways to describe sets, and then we will show that some f-vector sets do not admit such a “nice” description.

First, we give a brief overview over some complexity measures of sets of integers, most importantly the notion of *Diophantine sets*. In Section 3.3 we modify this to the (more restrictive) notion of *semi-algebraic sets of lattice points*.

3.1 COMPUTABILITY AND HILBERT’S TENTH PROBLEM

In this section we talk about *algorithms* (or *programs*). Informally, by an *algorithm* or *program* we mean a finite sequence of instructions that can be executed by an abstract computer or machine. To formalize the notion of an *algorithm*, Turing [85] described in 1936 what he called “automatic machines” which became known as *Turing machines*. For a formal definition of a Turing machine, see for example [48, Sect. 8.2.2].

A set of natural numbers is called *computable* if there is an algorithm that decides within a finite amount of steps whether a given number belongs to the set or not. Computable sets are also known as *recursive* or *decidable* sets.

A set of natural numbers is called *recursively enumerable* (or *computably enumerable* or *Turing-recognizable*) if there is an algorithm which enumerates the numbers in the set. That is, given a number, if the number is from the set, the algorithm will determine after finitely many steps that the number is in the set. If the given number was not in the set, the algorithm might decide that the number is not in the set or it might run forever.

Computable sets are recursively enumerable, but not all recursively enumerable sets are computable. The classical example for a set which is recursively enumerable, but not computable comes from the *halting problem*:

The halting problem asks for a program (an algorithm) which takes as its input another program and an input for the program and decides in a finite amount of time whether the program halts with this input.

Turing showed in 1936 that such a program cannot exist [85]. The halting problem was one of the first examples of an *undecidable* problem, a decision problem (“yes” or “no” question) for which no algorithm exists that can give for any input a definite “yes” or

"no" answer. The halting set is the set of all tuples of programs and inputs such that the program halts with the given input.
The halting set is recursively enumerable but not computable.

A *Diophantine equation* is a polynomial equation $P(x_1, \ldots, x_n) = 0$ with integer coefficients.
Any set of solutions to a finite system of Diophantine equations can equivalently be described by a single Diophantine equation: Observe that $(x_1, \ldots, x_n)$ is a solution to the system of polynomial equations

$$P_1(x_1, \ldots, x_n) = 0, \ldots, P_n(x_1, \ldots, x_n) = 0$$

if and only if it is a solution to the polynomial equation

$$P_1(x_1, \ldots, x_n)^2 + \ldots + P_n(x_1, \ldots, x_n)^2 = 0.$$

A *Diophantine set* is a subset of $\mathbb{N}^m$ of the form

$$\left\{(a_1, \ldots, a_m) \in \mathbb{N}^m \,\middle|\, \exists\, b_1, \ldots, b_k \text{ so that } P(a_1, \ldots, a_m, b_1, \ldots, b_k) = 0\right\}$$

for some Diophantine equation $P(x_1, \ldots, x_m, y_1, \ldots, y_k) = 0$.
For a Diophantine set $S \subseteq \mathbb{N}^m$, the minimal number k such that there exists a Diophantine equation $P(x_1, \ldots, x_m, y_1, \ldots, y_k) = 0$ with

$$\left\{S = (a_1, \ldots, a_m) \in \mathbb{N}^m \,\middle|\, \exists\, b_1, \ldots, b_k \text{ so that } P(a_1, \ldots, a_m, b_1, \ldots, b_k) = 0\right\}$$

is called the *Diophantine rank* of S. Matiyasevich proved that every Diophantine set has *rank* at most 9 (see [51], and most recently Sun [83]).
The problem of finding solutions to a Diophantine equation in natural numbers is essentially equivalent to the problem of finding integer solutions to a Diophantine equation: By Lagrange's four-square theorem every natural number has a representation as a sum of four squares. This means that finding all solutions $(x_1, \ldots, x_n)$ in natural numbers to

$$P(x_1, \ldots, x_n) = 0$$

can be rewritten as the problem of finding all integer solutions
$(a_1, b_1, c_1, d_1, \ldots, a_n, b_n, c_n, d_n)$ to

$$P(a_1^2 + b_1^2 + c_1^2 + d_1^2, \ldots, a_n^2 + b_n^2 + c_n^2 + d_n^2) = 0.$$

On the other hand, the problem of finding all integer solutions $(x_1, \ldots, x_n)$ to

$$P(x_1, \ldots, x_n) = 0$$

can be rewritten as the problem of finding all solutions $(a_1, b_1, \ldots, a_n, b_n)$ in natural numbers to

$$P(a_1 - b_1, \ldots, a_n - b_n) = 0.$$

Diophantine equations can be surprisingly hard to solve. For example, the problem of the *sum of three cubes* asks for expressions of natural numbers as a sum of three cubes of integers, i.e. it is the problem to find solutions $(x, y, z) \in \mathbb{Z}^3$ to

$$x^3 + y^3 + z^3 = k$$

for given $k \in \mathbb{N}$. Numbers equal to 4 or 5 modulo 9 are known to have no expression as a sum of three cubes, all other natural numbers are conjectured to be a sum of three cubes. But these expressions can be hard to find. It was a long-standing open problem to find solutions for all $k = 1, \ldots, 100$, see for example Miller & Woollett [64] (1955). An integer solution to the last remaining equation

$$x^3 + y^3 + z^3 = 42$$

was only recently found by Booker and Sutherland in 2019, see [47]. The solution they found is

$$(x, y, z) = (-80538738812075974, 80435758145817515, 12602123297335631).$$

Many sets of integers turn out to be Diophantine, for example the prime numbers [52] or the Fibonacci numbers [50].

In 1900 at the conference of the International Congress of Mathematicians in Paris, David Hilbert presented 23 open problems which became famous as "Hilbert's problems". As of today, some of the problems have been solved, some partially answered, and some remain unsolved.
A problem that has been answered is *Hilbert's tenth problem*, concerning Diophantine equations. Here is Hilbert's tenth problem in its original formulation with an English translation by Winston [45]:

> *Eine Diophantische Gleichung mit irgend welchen Unbekannten und mit ganzen rationalen Zahlencoefficienten sei vorgelegt: man soll ein Verfahren angeben, nach welchem sich mittelst einer endlichen Anzahl von Operationen entscheiden läßt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.*
>
> Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

That is, Hilbert asks if there is an algorithm which can decide, given a Diophantine equation, whether its set of integer solutions is non-empty.

It turns out that Hilbert's tenth problem has a negative answer, there is no such algorithm. This follows from the next theorem:

THEOREM 3.1.1 (Matiyasevich's Theorem [59], see Davis [28] and Matiyasevich [60]). A set of integer points is a Diophantine set if and only if it is recursively enumerable.

Matiyasevich's Theorem is also known as the MRDP theorem after Matiyasevich, Robinson, Davis, and Putnam.

Because of the existence of sets like the halting set – sets which are recursively enumerable but not computable – Matiyasevich's Theorem shows that the algorithm Hilbert describes cannot exist.

Another possibility to measure the complexity of integer sets is given by measuring the computational complexity of an algorithm that decides whether an element is part of the set.
An example of a problem that is decidable, but not solvable in polynomial time is given by a bounded version of the halting problem: In this bounded version we ask for a program that again takes as its input another program and an input for this program and decides if the program halts with this input after at most n steps, where $n > 0$ is an integer encoded in binary form (see [31, Prop. 3.30]). The time it takes to solve this problem is in $\mathcal{O}(n)$, which is in exponential time in the number of bits, since n is encoded using $\mathcal{O}(\log_2 n)$ bits.

3.2 COMPLEXITY MEASURES OF f-VECTOR SETS

We have now seen some complexity measures for integer sets. In the subsequent sections we will modify these notions for a complexity measure for f-vector sets of polytopes.

As before, for any $d \geq 1$, let $\mathcal{F}^d \subset \mathbb{Z}^d$ let denote the set of all f-vectors $(f_0, f_1, \ldots, f_{d-1})$ of d-dimensional polytopes.

Grünbaum has noted in [38, Sect. 5.5] that f-vector sets of polytopes, like $\mathcal{F}^d$ and $\mathcal{F}^d_s$ for $d \geq 2$, are recursively enumerable. By Matiyasevich's Theorem (Theorem 3.1.1), this implies that they are Diophantine sets.

Nevo [67, Thm. 1.4] recently pointed out that deciding whether a vector $(f_0, \ldots, f_{d-1})$ belongs to the set $\mathcal{F}^d_s$ of f-vectors of simplicial d-dimensional polytopes can be done in polynomial time in the number of bits in the binary encoding of the vector, $\sum_{i=0}^{d-1} \lceil \log_2 f_i \rceil$.
For all the known complete descriptions of projections of f-vector sets $\Pi_{i,j}(\mathcal{F}^4)$, $\Pi_{0,1}(\mathcal{F}^5)$ and the flag vector projection $\Pi_{0,03}(\overline{\mathcal{F}}^4)$ described in Chapter 2 it follows directly from the descriptions that it is decidable in polynomial time in the number of bits in the binary encoding of the vector if a vector belongs to the set. For the sets $\mathcal{F}^d$, for $d \geq 4$, it is an open problem if it can be decided in polynomial time if a vector $(f_0, \ldots, f_{d-1})$ belongs to $\mathcal{F}^d$.

3.3 SEMI-ALGEBRAIC SETS OF LATTICE POINTS

In the following we define another complexity measure for f-vector sets. For this we need the notion of *semi-algebraic sets*.

A *basic semi-algebraic set* is a subset $S \subseteq \mathbb{R}^d$ that can be defined by a finite number of polynomial equations and inequalities. A *semi-algebraic set* is any finite union of basic semi-algebraic sets. The semi-algebraic set is *defined over* $\mathbb{Z}$ if the polynomials can be chosen with integral coefficients. In this case we will call this a $\mathbb{Z}$-*semi-algebraic set*.

An important property of semi-algebraic sets is the fact that they are closed under the projection operation:

THEOREM 3.3.1 (Tarski–Seidenberg theorem [10, Prop. 2.76]). The image of a semi-algebraic set $S \subset \mathbb{R}^{d+1}$ under the projection map

$$\begin{aligned} \Pi : \mathbb{R}^{d+1} &\longrightarrow \mathbb{R}^d \\ (x_1, \dots, x_d, x_{d+1}) &\longmapsto (x_1, \dots, x_d) \end{aligned}$$

is again a semi-algebraic set.

See Basu, Pollack & Roy [10] for more background on semi-algebraic sets.

We will now use these notions of semi-algebraic sets for a description of the complexity of the f-vector sets $\mathcal{F}^d \subset \mathbb{Z}^d$ and their projection sets. For dimensions smaller or equal to 3, we have explicit descriptions of $\mathcal{F}^d \subset \mathbb{Z}^d$. We have that $\mathcal{F}^1 = \{2\} \subset \mathbb{Z}$ and $\mathcal{F}^2 = \{(n,n) : n \geq 3\} \subset \mathbb{Z}^2$. In 1906, Steinitz [82] characterized the set $\mathcal{F}^3$ of f-vectors (f_0, f_1, f_2) of 3-dimensional polytopes P as

$$\mathcal{F}^3 = \big\{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2,\ f_2 \leq 2f_0 - 4,\ f_0 \leq 2f_2 - 4\big\}.$$

Thus for $d \leq 3$ the set $\mathcal{F}^d \subset \mathbb{Z}^d$ has a very simple structure: It is the set of all integer points in a $(d-1)$-dimensional rational cone.

Inspired by this, Grünbaum in 1967 [38, Sect. 10.4] and subsequently Barnette and Reay characterized the sets $\Pi_{ij}(\mathcal{F}^4)$ of all pairs (f_i, f_j) that occur for 4-dimensional polytopes. The results are Theorem 2.1.1, 2.1.2, 2.1.3 and 2.1.4. As we can see, they again got complete and reasonably simple answers: They found that in all cases this is the set of all integer points between some fairly obvious upper and lower bounds, with finitely many exceptional pairs and curves.

Here we start with a formal definition of what we mean by a “simple answer”:

Definition 3.3.2 (Semi-algebraic sets of integer points)**.** A set of $A \subset \mathbb{Z}^d$ is a *semi-algebraic set of integer points* if it is the set of all integer points in a semi-algebraic set, that is, if $A = S \cap \mathbb{Z}^d$ for some semi-algebraic set $S \subseteq \mathbb{R}^d$.

It turns out that Definition 3.3.2 is not quite general enough for f-vector theory, as we need to account for modularity constraints that may arise due to projections. For example, $A := \{(x, y) \in \mathbb{Z}^2 : x = 2y\}$ is a semi-algebraic set of integer points, but its projection to the first coordinate $\Pi_1(A) = 2\mathbb{Z}$ is not if we insist that the lattice is $\mathbb{Z}$. This is relevant for f-vector sets, as for example every simplicial 3-polytope satisfies $3f_2 = 2f_1$, so f_2 is even and f_1 is a multiple of 3. Consequently $\Pi_2(\mathcal{F}_s^3) = \{4, 6, 8, \dots\}$, the set of all possible facet numbers of simplicial 3-polytopes, is *not* a semi-algebraic set of integer points, but it is a semi-algebraic set of lattice points:

Definition 3.3.3 (Semi-algebraic sets of lattice points)**.** A subset $A \subset \mathbb{R}^d$ is a *semi-algebraic set of lattice points* if it is an intersection set of a semi-algebraic set with an affine lattice, that is, if $A = S \cap \Lambda$ for a suitable semi-algebraic set $S \subseteq \mathbb{R}^d$ and an affine lattice $\Lambda \subset \mathbb{R}^d$.

Here by an *affine lattice* we mean any translate of a linear lattice, that is, a discrete subset $\Lambda \subset \mathbb{R}^d$ that is closed under taking affine combinations $\lambda_1 a_1 + \cdots + \lambda_n a_n$ for $n \geq 1$ with $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ and $\lambda_1 + \cdots + \lambda_n = 1$. We will only consider *integer lattices*, that is, sublattices $\Lambda \subseteq \mathbb{Z}^d$. Moreover, without loss of generality we may always assume that the lattice is the affine lattice $\Lambda = \operatorname{aff}_{\mathbb{Z}} A$ spanned by A: The set of all affine combinations yields a lattice if $A \subset \mathbb{Z}^d$, and the lattice Λ has to contain $\operatorname{aff}_{\mathbb{Z}} A$.

With the generality of Definition 3.3.3, a great number of characterization results achieved in the f-vector theory of polytopes imply that full f-vector sets or coordinate projections (that is, single face numbers or face number pairs) are semi-algebraic sets of lattice points. We will summarize this in Section 3.4.

Semi-algebraic sets of lattice points $A \subset \mathbb{Z}^d$ are easy to identify and to characterize for $d = 1$; see the beginning of Section 3.5. However, already for sets in the plane $A \subset \mathbb{Z}^2$ this becomes non-trivial. For example, the answer depends on the field of definition: The set $\{(x, y) \in \mathbb{Z}^2 : y \geq \pi x\}$ is an $\mathbb{R}$-semi-algebraic set of integer points, but not a $\mathbb{Z}$-semi-algebraic set of lattice points.

Our two main results of this chapter are the following:

THEOREM 3.3.4. The set $\Pi_{12}(\mathcal{F}^4)$ of pairs (f_1, f_2) for 4-dimensional polytopes is not an $\mathbb{R}$-semi-algebraic set of lattice points.

THEOREM 3.3.5. For any $d \geq 6$, the set $\mathcal{F}^d$ of all f-vectors of d-dimensional polytopes is not an $\mathbb{R}$-semi-algebraic set of lattice points.

In Section 3.5 we develop proof techniques, including the "Strip lemma." Based on this, the proof of Theorem 3.3.4 is given in Section 3.6 and the proof of Theorem 3.3.5 in Section 3.7.

Coordinate projections of semi-algebraic sets of lattice points are not in general again semi-algebraic. Indeed, as we have seen before, the f-vector sets are Diophantine sets. Since every Diophantine set has rank at most 9, the f-vector sets of polytopes are the projections of the integer points of some semi-algebraic set defined over $\mathbb{Z}$ with at most 9 additional variables.

On the other hand, the semi-algebraic sets of lattice points that we consider in this paper are more restrictive than Diophantine sets: We are interested in the cases when a set cannot be described as the set of integer points of a semi-algebraic set (defined over $\mathbb{Z}$ or $\mathbb{R}$) without additional variables. With the proof of Theorem 3.3.4, we will see that, for example, the set $\Pi_{12}(\mathcal{F}^4)$ can be described using one additional variable (if we allow for inequalities, which is not usual in the context of Diophantine sets, but equivalent). The same is true for the f-vector set of simplicial 6-polytopes, $\mathcal{F}_s^6$.

In summary, we will show that many f-vector sets are semi-algebraic (Section 3.4), while some are not (Theorems 3.3.4 and 3.3.5). The crucial question that remains open concerns dimensions 4 and 5:

OPEN PROBLEM 1. Is the f-vector set of 4-polytopes $\mathcal{F}^4 \subset \mathbb{Z}^4$ semi-algebraic?

(The size/fatness projection of $\mathcal{F}^4$ displayed and discussed by Brinkmann & Ziegler [24] suggests that the answer is no.)

OPEN PROBLEM 2. Is the f-vector set of 5-polytopes $\mathcal{F}^5 \subset \mathbb{Z}^5$ semi-algebraic?

The lattices spanned by f-vector sets, as well as more general additive (semi-group) structures on them, are discussed by Ziegler in [87].

3.4 SEMI-ALGEBRAIC SETS OF f-VECTORS

The following theorem summarizes a great number of works in f-vector theory, started by Steinitz in 1906 [82] and re-started by Grünbaum in 1967 [38, Chap. 8-10].

THEOREM 3.4.1. The following sets of face numbers, face number pairs, and f-vectors, are $\mathbb{Z}$-semi-algebraic sets of lattice points:

(i) $\mathcal{F}^d$, the set of f-vectors of d-dimensional polytopes for $d \leq 3$,

(ii) $\mathcal{F}_s^d$ and $\mathcal{F}_{s*}^d$, the sets of f-vectors of simplicial and of simple d-dimensional polytopes for $d \leq 5$,

(iii) $\mathcal{F}_{cs}^3$, the set of f-vectors of 3-dimensional centrally-symmetric polytopes,

(iv) $\Pi_i(\mathcal{F}^d)$, the sets of numbers of i-faces of d-polytopes for all d and i,

(v) $\Pi_i(\mathcal{F}_s^d)$ and $\Pi_i(\mathcal{F}_{s*}^d)$, the sets of numbers of i-faces of simplicial and of simple d-polytopes,

(vi) $\Pi_{01}(\mathcal{F}^4)$, $\Pi_{02}(\mathcal{F}^4)$, and $\Pi_{03}(\mathcal{F}^4)$, sets of face number pairs of 4-polytopes,

(vii) $\Pi_{01}(\mathcal{F}^5)$, the set of pairs of "number of vertices and number of edges" for 5-polytopes,

(viii) $\Pi_0(\mathcal{F}_{cub}^d)$, the set of vertex numbers of cubical d-polytopes, for $d \leq 4$ and for all even dimensions d,

(ix) $\Pi_0(\mathcal{F}_{2s2s}^4)$, the set of vertex numbers of 2-simplicial 2-simple 4-polytopes, and

(x) $\Pi_{0,d-1}(\mathcal{F}^d)$, the set of pairs of "number of vertices and number of facets" of d-polytopes, for even dimensions d.

Proof. In each case, the set in question is described as all the integers or integer points that satisfy a number of polynomial equations, strict inequalities, non-strict inequalities, or inequalities:

(i) This is Steinitz's result [82], as quoted in the introduction. In this case, the equation and inequalities are linear. It also includes the information that the f-vector set of simplicial 3-polytopes is $\mathcal{F}_s^3 = \{(n, 3n-6, 2n-4) : n \geq 4\}$, which yields the case $d = 3$ of (ii).

(ii) $\mathcal{F}_s^4$ and $\mathcal{F}_s^5$ can be deduced from the g-Theorem (see Section 3.7):

$$\begin{aligned}\mathcal{F}_s^4 =& \{(f_0, f_1, -2f_0 + 2f_1, -f_0 + f_1) \in \mathbb{Z}^4 : f_0 \geq 5,\ 4f_0 - 10 \leq f_1 \leq \tfrac{1}{2} f_0(f_0 - 1)\},\\ \mathcal{F}_s^5 =& \{(f_0, f_1, -10f_0 + 4f_1 + 20, -15f_0 + 5f_1 + 30, -6f_0 + 2f_1 + 12) \in \mathbb{Z}^5 :\\ & \qquad f_0 \geq 6,\ 5f_0 - 15 \leq f_1 \leq \tfrac{1}{2} f_0(f_0 - 1)\}.\end{aligned}$$

(iii) The f-vector set $\mathcal{F}_{cs}^3$ of centrally-symmetric 3-polytopes spans the lattice $(2\mathbb{Z})^3$. Werner [86, Thm. 3.3.6] has described it as

$$\mathcal{F}_{cs}^3 = \{(f_0, f_1, f_2) \in (2\mathbb{Z})^3 : f_0 - f_1 + f_2 = 2, f_2 \leq 2f_0 - 4, f_0 \leq 2f_2 - 4, f_0 + f_2 \geq 14\}.$$

(iv),(v) Björner & Linusson [18] showed that for any integers $0 \leq i < d$ there are numbers $N(d, i)$ and $G(d, i)$ such that there is a simple d-polytope with $n > N(d, i)$ i-faces if and only if n is a multiple of $G(d,i)$. Additionally, $G(d,i) = 1$ for $i \geq \left\lfloor \frac{d+1}{2} \right\rfloor$. As a consequence the 1-dimensional coordinate projection of the set of f-vectors of all simple d-polytopes is a semi-algebraic set of integer points over $\mathbb{Z}$. The same holds for simplicial polytopes, by duality. Here, the number $G(d,i)$ is equal to 1 for $i \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 1$. This also implies that the 1-dimensional projection sets $\Pi_i(\mathcal{F}^d)$ are semi-algebraic sets of integer points for all choices of d and i with the possible exception of odd d and $i = \frac{d-1}{2}$. In order to show that $\Pi_i(\mathcal{F}^{2i+1})$ is a semi-algebraic set of integer points, we derive from [18]:

$$G(2i+1,i) = \begin{cases} p & \text{if } i+2 = p^s \text{ for some integer } s \geq 1 \text{ and some prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

Hence, $\Pi_i(\mathcal{F}^{2i+1})$ is a semi-algebraic set of integer points if $i+2 \neq p^s$ for all primes p and all integers s.

Let now $i+2 = p^s$ for some $s \geq 1$ and a prime p. Assume that we have a $(2i+1)$-polytope P with a simplex facet such that $\gcd(f_i(P), p) = 1$. Then using the construction of connected sums by Eckhoff [32] (see also [88, p. 274]) to successively add copies of P, its dual P^*, simple and simplicial polytopes, we obtain $(2i+1)$-polytopes with all possible numbers n of i-faces for all sufficiently large n, that is, for $n \geq M(d,i)$.

To complete our proof, we give a construction of the polytope P. We consider two different cases. In the first case, let $i+2 = 2^s$ for some $s \geq 2$.

Since $G(2i,i) = 1$ we can find a simple $2i$-polytope R with an odd number of i-faces. From [18] we get that

$$G(2i,i-1) = \begin{cases} 2 & \text{if } i+2 = 2^t \text{ for some integer } t, \\ 1 & \text{otherwise.} \end{cases}$$

Thus R has an even number of $(i-1)$-faces. Let Q be the connected sum $R \# R^*$ of R and its dual. Then $f_i(Q) = f_i(R) + f_i(R^*) = f_i(R) + f_{i-1}(R)$ is odd and Q has a simplex facet.

Let now P be the bipyramid over Q. Then $f_i(P) = 2f_{i-1}(Q) + f_i(Q)$ is odd and P has a simplex facet.

In the second case, $i+2 = p^s$ for some integer $s \geq 1$ and some odd prime p. Choose a simple $2i$-polytope R with $f_0(R) \geq i+1$ and $\gcd(f_i(R), p) = 1$. Such a polytope R exists since $G(2i,i) = 1$. Let P_1 be the prism over R and P_2 the pyramid over R^*. Then $f_i(P_1) = f_{i-1}(R) + 2f_i(R)$, $f_i(P_2) = f_{i-1}(R) + f_i(R)$, P_1 is a simple polytope and P_2 has $f_0(R) \geq i+1$ simplex facets. Let P be the connected sum of P_2 and $i+1$ copies of P_1:

$$P = (\cdot\cdot((P_2 \# \underbrace{P_1) \# P_1) \# \ldots P_1) \# P_1}_{i+1}.$$

The resulting $(2i+1)$-polytope P has a simplex facet and

$$\begin{aligned} f_i(P) &= f_i(P_2) + (i+1) f_i(P_1) \\ &= f_{i-1}(R) + f_i(R) + (i+1)(f_{i-1}(R) + 2f_i(R)) \\ &= (i+2)(f_{i-1}(R) + 2f_i(R)) - f_i(R) \\ &= p^s(f_{i-1}(R) + 2f_i(R)) - f_i(R), \end{aligned}$$

which is coprime to p, since $f_i(R)$ is coprime to p.

(vi) As we have seen in Section 2.1, the 2-dimensional coordinate projections $\Pi_{ij}(\mathcal{F}^4)$ have been characterized by Grünbaum [38, Thm. 10.4.1, 10.4.2], Barnette [7], and Barnette & Reay [8]: $\Pi_{03}(\mathcal{F}^4)$ consists of all the integer points between two parabolas, $\Pi_{01}(\mathcal{F}^4)$ is the set of all integer points between a line and a parabola, with four exceptions, and $\Pi_{02}(\mathcal{F}^4)$ is the set of all integer points between two parabolas, except for the integer points on an exceptional parabola, and ten more exceptional points.

(vii) The set $\Pi_{01}(\mathcal{F}^5)$ was recently determined independently by Kusunoki & Murai [56] and by Pineda-Villavicencio, Ugon & Yost [71]: It is the set of all integer points between a line and a parabola, except for the integer points on two lines and three more exceptional points.

(viii) The possible vertex numbers of cubical 3-polytopes are $\Pi_0(\mathcal{F}^3_{cub}) = \{8\} \cup \{n \in \mathbb{Z} : n \geq 10\}$. Blind & Blind [19] proved that the number of vertices f_0 as well as of edges f_1 are even for every cubical d-polytope if $d \geq 4$ is even. According to Blind & Blind [20, Cor. 1], there are "elementary" cubical d-polytopes C^d_k with $2^{d+1} - 2^{d-k}$ vertices, for $0 \leq k < d$. (In particular, C^d_{d-1} has $2^{d+1} - 2$ vertices.) As the facets of these polytopes are projectively equivalent to standard cubes, we can glue them in facets (as in Ziegler [87, Sect. 5.2]), and thus obtain all sufficiently large even vertex numbers. Thus $\Pi_0(\mathcal{F}^d_{cub})$ is a semi-algebraic subset of the lattice $2\mathbb{Z}$ for even $d \geq 4$.

(ix) Paffenholz & Werner [69] and Miyata [65] proved that the set of possible numbers of vertices for 2-simplicial 2-simple 4-polytopes is $\Pi_0(\mathcal{F}^4_{2s2s}) = \{5\} \cup \{n \in \mathbb{Z} : n \geq 9\}$.

(x) For even d and $n + m \geq \binom{3d+1}{\lfloor d/2 \rfloor}$ there exists a d-polytope P with n vertices and m facets if and only if $m \leq f_{d-1}(C_d(n))$ and $n \leq f_{d-1}(C_d(m))$, where $C_d(n)$ denotes the d-dimensional cyclic polytope with n vertices. This is Theorem 2.2.2. □

3.5 PROOF TECHNIQUES

It is easy to see that a subset $A \subseteq \mathbb{Z}$ is a semi-algebraic set of integer points if and only if it consists of a finite set of (possibly unbounded) intervals of integer points. Equivalently, a subset $A \subseteq \mathbb{Z}$ is *not* a semi-algebraic set of integer points if and only if there is a strictly monotone (increasing or decreasing) infinite sequence of integers, with $a_1 < a_2 < \cdots$ or $a_1 > a_2 > \cdots$, such that $a_{2i} \in S$ and $a_{2i+1} \in \mathbb{Z} \backslash S$.

The same characterization holds for semi-algebraic sets of lattice points $A \subset \mathbb{R}$, where $\operatorname{aff}_{\mathbb{Z}} A$ takes over the role of the integers $\mathbb{Z}$.

Examples of subsets of $\mathbb{Z}$ that are not $\mathbb{R}$-semi-algebraic sets of lattice points include the set of squares $\{n^2 : n \in \mathbb{Z}_{\geq 0}\}$, the set $\{n \in \mathbb{Z} : n \not\equiv 0 \mod 3\}$, and the set

$\{1, 2, 4, 6, 8, 10, \ldots\}$.

For subsets of $\mathbb{Z}^2$, or of $\mathbb{Z}^d$ for $d > 2$, we do not have – or expect – a complete characterization of semi-algebraic sets of integer points.

There are some obvious criteria: For example, every finite set of integer points is semi-algebraic, finite unions of semi-algebraic sets of lattice points with respect to the same lattice are semi-algebraic, products of semi-algebraic sets of lattice points are semi-algebraic, and so on.

However, these simple general criteria turn out to be of little use for studying the specific sets of integer points we are interested in. The "finite oscillation" criterion of the one-dimensional case suggests the following approach for subsets $A \subset \mathbb{Z}^d$:

LEMMA 3.5.1 (Curve lemma). If there is a semi-algebraic curve Γ that along the curve contains an infinite sequence of integer points $a_1, a_2, \ldots$ (in this order along the curve) with $a_{2i} \in \Gamma \cap A$ and $a_{2i+1} \in \Gamma \backslash A$, then A is *not* a semi-algebraic set of integer points. Similarly, if this holds with $a_1, a_2, \ldots \in \Lambda := \operatorname{aff}_{\mathbb{Z}} A$, then A is *not* a semi-algebraic set of lattice points.

However, for our examples the semi-algebraic curves Γ of Lemma 3.5.1 do not exist. Thus to show that a 2-dimensional set is not a semi-algebraic set of lattice points we develop a better criterion: Instead of the "curve lemma" we rely on a "strip lemma," which in place of single algebraic curves considers strips generated by disjoint translates of an algebraic curve.

In the following, we refer to Basu, Pollack and Roy [10] for notation and information about semi-algebraic sets.

Definition 3.5.2. Let $\gamma_0 = \{(x, f(x)) : x \geq 0\} \subset \mathbb{R}^2$ be a curve, where $f(x)$ is an algebraic function defined for all $x \geq 0$, and let c be a vector in $\mathbb{R}^2$. If the translates $\gamma_t = \gamma_0 + tc$ for $t \in [0, 1]$ are disjoint, then we refer to this family of curves $\mathcal{C} := \{\gamma_t\}_{t \in [0,1]}$ as a *strip of algebraic curves*. A *substrip* of $\mathcal{C}$ is a family $\mathcal{C}_J$ of all curves γ_t with $t \in J$, where J is any closed interval $J \subseteq [0, 1]$ of positive length.

LEMMA 3.5.3 (Strip lemma). Let $L \subset \mathbb{Z}^2$ be a set of integer points and $\Lambda = \operatorname{aff}_{\mathbb{Z}} L$ the affine lattice spanned by L. If there exists a strip of algebraic curves $\mathcal{C}$ such that every substrip $\mathcal{C}_J$ contains infinitely many points from $\Lambda \cap L$ and infinitely many points from $\Lambda \setminus L$, then L is not an $\mathbb{R}$-semi-algebraic set of lattice points.

See Figure 3.1 for a visualization.

Proof. Assume that L is $\mathbb{R}$-semi-algebraic, that is, there exists an $\mathbb{R}$-semi-algebraic set $S \subset \mathbb{R}^2$ such that $L = S \cap \Lambda$. The boundary of S is the intersection of the closure of S with the closure of $\mathbb{R}^2 \backslash S$, $\operatorname{bd}(S) = \overline{S} \cap \overline{\mathbb{R}^2 \backslash S}$. The Tarski–Seidenberg theorem (Thm. 3.3.1) yields that the closure of a semi-algebraic set in $\mathbb{R}^d$ is again a semi-algebraic set [10, Prop. 3.1]. The boundary $\operatorname{bd}(S)$ is the intersection of two semi-algebraic sets and hence itself a semi-algebraic set.

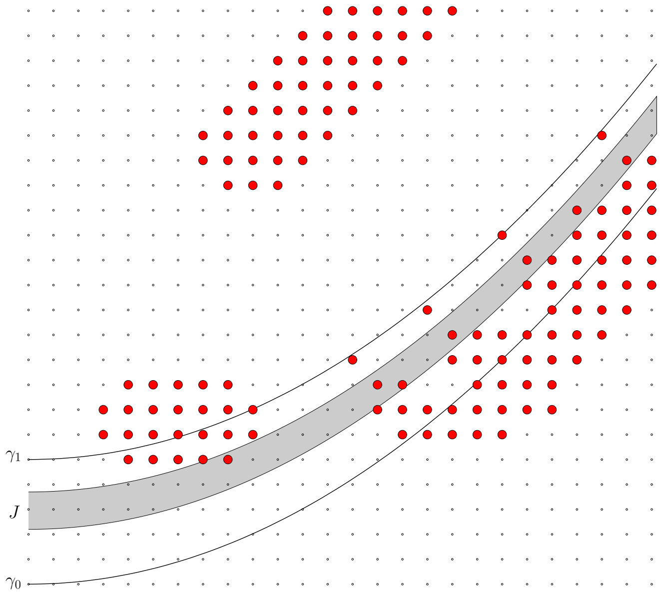


Figure 3.1: This sketch illustrates that for a semi-algebraic set L of lattice points there cannot be an infinite sequence of lattice points in L, as well as not in L, in every substrip between γ_0 and γ_1.

Any semi-algebraic set consists of finitely many connected components, all being semi-algebraic [10, Thm. 5.19].

From this we want to derive that for any strip of algebraic curves $\mathcal{C}$ there exists a substrip $\mathcal{C}_J$ of $\mathcal{C}$ such that for some $n \geq 0$, all lattice points $(a, b) \in \Lambda$ with $a \geq n$ in the substrip belong entirely to L, or all of them do not belong to L.

Denote by $\beta_1, \ldots, \beta_m$ all those connected components of $\mathrm{bd}(S)$ that contain points $(x, y) \in \mathbb{R}^2$ with arbitrarily large x in a strip $\mathcal{C}$. If such components do not exist, then either all points of $\mathcal{C} \cap \Lambda$ with sufficiently large x-coordinate (that is, all but finitely many of these points) lie in L, or all of them do not lie in L.

The intersection of a semi-algebraic component β_j and any semi-algebraic curve γ_t is again semi-algebraic, so it consists of finitely many connected components. Thus for any given β_j and γ_t, β_j has finitely many branches to infinity such that each branch eventually (for all sufficiently large x-coordinates) stays above γ_t, or below γ_t, or on γ_t. Thus by continued bisection we find that there exists some value $n' \geq 0$ such that the restriction of each β_j to $x \geq n'$ has finitely many components, each of which either lies on a curve γ_t, or it is a curvilinear asymptote to *some* curve γ_t. Let the components of $\{(x, y) \in \beta_j : x \geq n'\}$ be asymptotic to (or lie on) $\gamma_{t_1}, \ldots, \gamma_{t_k}$, with $0 \leq t_1 \leq \cdots \leq t_k \leq 1$, and let $[\delta_0, \delta_1] \subset [0, 1]$ be an interval of positive length (that is, with $\delta_0 < \delta_1$) that is disjoint from $\{0, t_1, \ldots, t_k, 1\}$. Then there exists an $n \geq 0$ such that the lattice points $(a, b) \in \Lambda$ with $a \geq n$ contained in the substrip obtained from $[\delta_0, \delta_1]$ either all belong to L or they all do not belong to L. $\square$

3.6 EDGE AND RIDGE NUMBERS OF 4-POLYTOPES

In this section we prove Theorem 3.3.4. The following theorem is a reformulation of Theorem 2.1.4.

THEOREM 3.6.1 (Barnette [7, Thm. 1], see also [49], with corrections). Let f_1 and f_2 be positive integers with $f_1 \geq f_2$. Then there is a 4-polytope P with $f_1(P) = f_1$ and $f_2(P) = f_2$ if and only if

$$\begin{aligned} f_2 &\geq \tfrac{1}{2}f_1 + \left\lceil \sqrt{f_1 + \tfrac{9}{4}} + \tfrac{1}{2} \right\rceil + 1, \\ f_2 &\neq \tfrac{1}{2}f_1 + \sqrt{f_1 + \tfrac{13}{4}} + 2, \end{aligned}$$

and (f_1, f_2) is not one of the 13 pairs

$$\begin{aligned} &(12,12),(14,13),(14,14),(15,15),(16,15),(17,16),(18,16),\\ &(18,18),(20,17),(21,19),(23,20),(24,20),(26,21). \end{aligned}$$

The case when $f_1(P) \leq f_2(P)$ is given by duality. See Figure 3.2.

Now we show that there is no semi-algebraic description of the set of pairs (f_1, f_2) by proving that the set

$$A \;:=\; \left\{(x,y) \in \mathbb{Z}^2 : x \geq 0, y \geq \tfrac{x}{2} + \left\lceil \sqrt{x + \tfrac{9}{4}} + \tfrac{1}{2} \right\rceil + 1\right\} \tag{3.1}$$

is not a semi-algebraic set of lattice points. See Figure 3.3.

The proof strategy is the following: In Lemma 3.6.2 we give an alternative description of the set. In Lemma 3.6.3 we observe that our set has the property described in Lemma 3.5.3, which implies that the set is not an $\mathbb{R}$-semi-algebraic set of lattice points.

LEMMA 3.6.2. Let x and y be nonnegative integers. Then

$$y \geq \tfrac{x}{2} + \left\lceil \sqrt{x + \tfrac{9}{4}} + \tfrac{1}{2} \right\rceil + 1 \tag{3.2}$$

if and only if

$$\left.\begin{aligned} & y \;\geq\; \tfrac{x}{2} + \sqrt{x + \tfrac{9}{4}} + 2 \\ \text{or}\quad & y \;=\; \tfrac{x}{2} + \sqrt{x + \tfrac{9}{4}} + \tfrac{3}{2} + r \\ & \quad\text{for some } r = i + \tfrac{1}{2} - \sqrt{(i + \tfrac{1}{2})^2 - 2j} \\ & \quad\text{with } i, j \in \mathbb{Z},\ i \geq 1,\ 0 \leq j \leq i. \end{aligned}\right\} \tag{3.3}$$

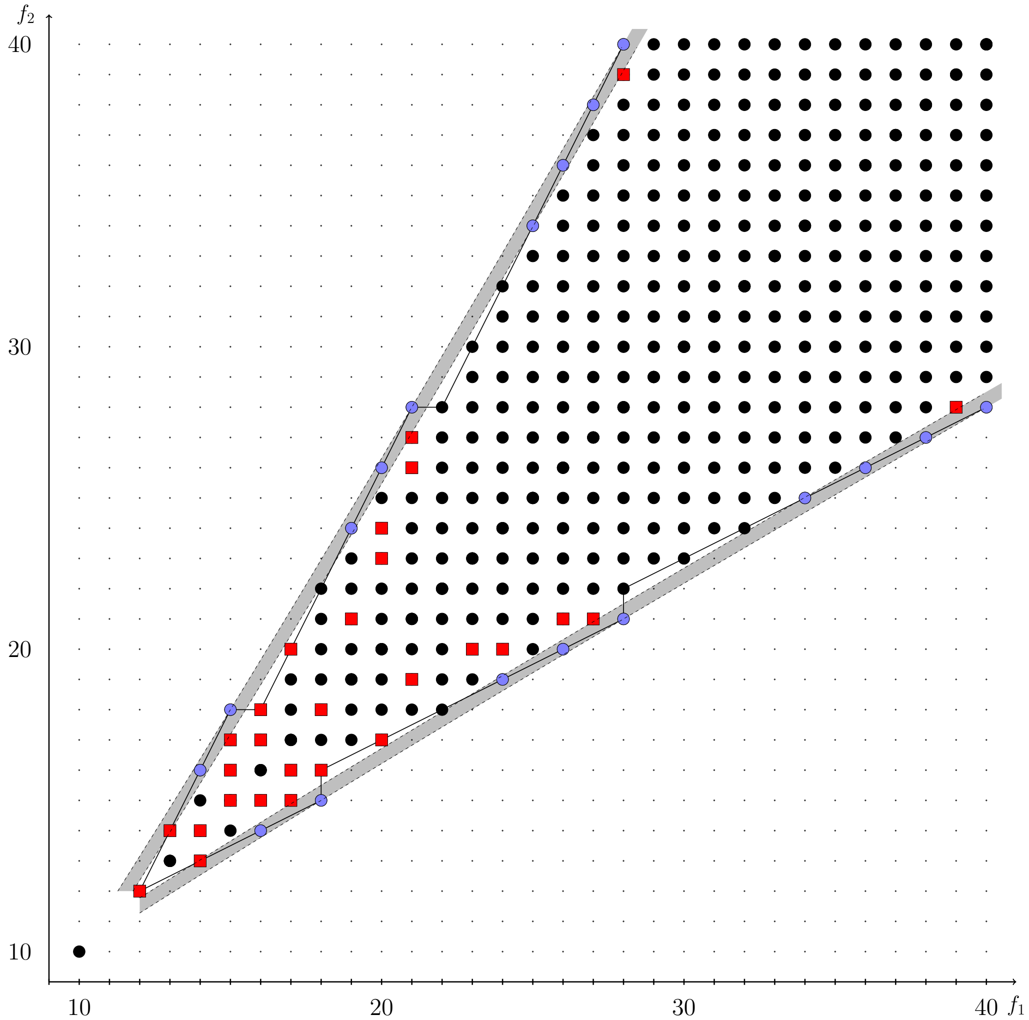


Figure 3.2: The set Π^4_{12} with the two strips that will play a crucial role in the proof that the set is not semi-algebraic, see Lemma 3.6.3 and its proof.

Proof. Let $x, y \geq 0$ be integers. We consider three separate cases:

Case a : $y \; > \; \frac{x}{2} + \sqrt{x + \frac{9}{4}} + \frac{5}{2},$

Case b : $y \; = \; \frac{x}{2} + \sqrt{x + \frac{9}{4}} + \frac{3}{2} + r$ for some $r \in [0, 1]$, *and*

Case c : $y \; < \; \frac{x}{2} + \sqrt{x + \frac{9}{4}} + \frac{3}{2}.$

In **Case a** the first part of condition (3.3) holds trivially. Since

$$y \; > \; \tfrac{x}{2} + \sqrt{x + \tfrac{9}{4}} + \tfrac{5}{2} \; > \; \tfrac{x}{2} + \left\lceil \sqrt{x + \tfrac{9}{4}} + \tfrac{1}{2} \right\rceil + 1,$$

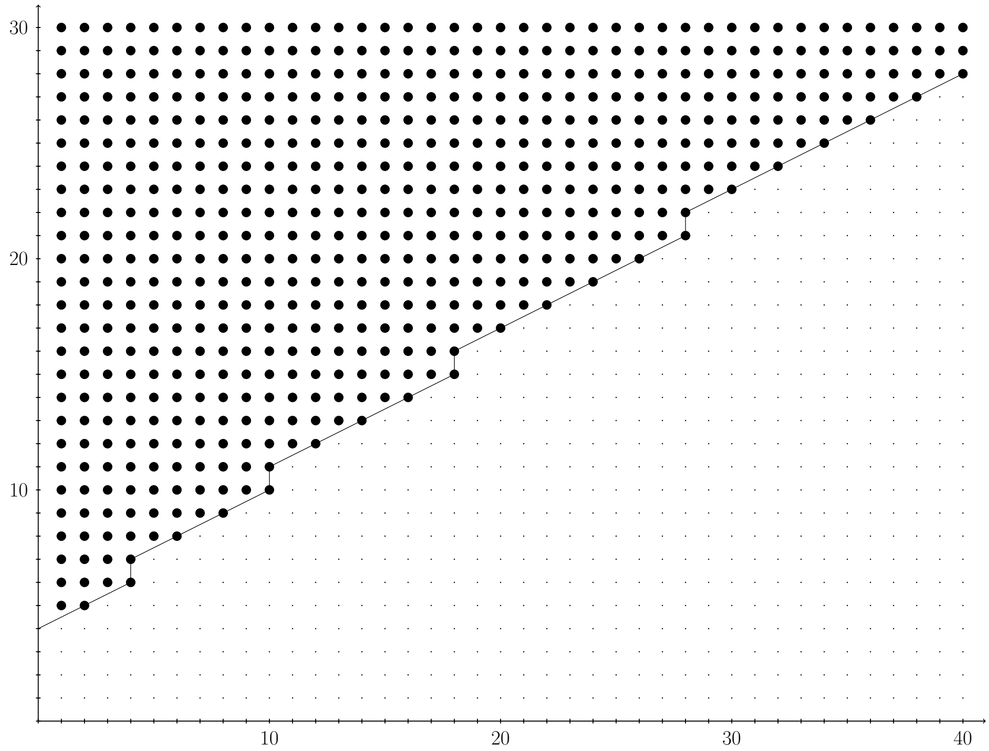


Figure 3.3: The set $A = \left\{(x, y) \in \mathbb{Z}^2 : x \geq 0, y \geq \frac{x}{2} + \left\lceil \sqrt{x + \frac{9}{4}} + \frac{1}{2} \right\rceil + 1\right\}$

condition (3.2) holds as well.

In **Case c**

$$y \ < \ \tfrac{x}{2} + \sqrt{x + \tfrac{9}{4}} + \tfrac{3}{2} \ \leq \ \tfrac{x}{2} + \left\lceil \sqrt{x + \tfrac{9}{4}} + \tfrac{1}{2} \right\rceil + 1,$$

hence condition (3.2) is not satisfied. On the other hand, observe that r lies in the range from 0 to 1 in the second part of condition (3.3). This shows us that condition (3.3) is not satisfied either.

In **Case b** we prove the equivalence of condition (3.2) and (3.3) first for odd x, then for even x.

Let x be odd, $y = \frac{x}{2} + \sqrt{x + \frac{9}{4}} + \frac{3}{2} + r$ and $r \in [0, 1]$. Assume $x = 2k + 1$ for some $k \geq 0$. We have

$$\sqrt{2k + \tfrac{13}{4}} \ = \ y - k - r - 2. \tag{3.4}$$

Now

$$\begin{aligned}
\tfrac{x}{2} + \left\lceil\sqrt{x+\tfrac{9}{4}} + \tfrac{1}{2}\right\rceil + 1 \quad &= \\
k + \left\lceil\sqrt{2k+\tfrac{13}{4}} + \tfrac{1}{2}\right\rceil + \tfrac{3}{2} \quad &\overset{(3.4)}{=} \\
k + \left\lceil y - k - r - \tfrac{3}{2}\right\rceil + \tfrac{3}{2} \quad &= \quad \begin{cases} y + \frac{1}{2} & \text{if } r \in [0, \frac{1}{2}[, \\ y - \frac{1}{2} & \text{if } r \in [\frac{1}{2}, 1]. \end{cases}
\end{aligned}$$

This shows that condition (3.2) holds if and only if $r \in [\frac{1}{2}, 1]$.

For $r \in [\frac{1}{2}, 1]$ condition (3.3) is trivially satisfied. It remains to show that condition (3.3) does not hold for $r \in [0, \frac{1}{2}[$. Assume by contradiction that condition (3.3) is satisfied for some x odd and $r \in [0, \frac{1}{2}[$. The first part of condition (3.3) does not hold. We will see that

$$r \;=\; i + \tfrac{1}{2} - \sqrt{(i+\tfrac{1}{2})^2 - 2j}$$

with $i \geq 1$ and $0 \leq j \leq i$ implies that x is even. Let

$$y \quad = \quad \tfrac{x}{2} + \sqrt{x+\tfrac{9}{4}} + \tfrac{3}{2} + i + \tfrac{1}{2} - \sqrt{(i+\tfrac{1}{2})^2 - 2j},$$

then

$$y - i - 2 \quad = \quad \tfrac{1}{2}(x + \sqrt{4x+9} - \sqrt{4(i^2+i-2j)+1}).$$

So $\sqrt{4x+9} - \sqrt{4(i^2+i-2j)+1}$ is an integer of the same parity as x.

Either $4x+9 = 4(i^2+i-2j)+1$ or both $\sqrt{4x+9}$ and $\sqrt{4(i^2+i-2j)+1}$ are integers. To see this, observe that if $a, b \in \mathbb{Z}, c \in \mathbb{Z}_{\neq 0}$, then $\sqrt{a} - \sqrt{b} = c \Rightarrow \frac{a-b-c^2}{2c} = \sqrt{b}$. This implies that $\sqrt{b}$ and hence $\sqrt{a}$ is a rational number, and since a and b are integers, $\sqrt{a}$ and $\sqrt{b}$ are integers as well.

In the first case, x is even. In the second case, if $\sqrt{4x+9}$ and $\sqrt{4(i^2+i-2j)+1}$ are integers, then they are odd integers. In both cases x is an even integer, which contradicts the assumption. Together, we obtain that conditions (3.2) and (3.3) are equivalent for odd x, for $r \in [0,1]$.

Let x now be even, $x = 2k$ for some $k \geq 0$, and $y = \frac{x}{2} + \sqrt{x+\frac{9}{4}} + \frac{3}{2} + r$ for some $r \in [0,1]$. We have

$$\begin{aligned}
y \quad &= \quad \tfrac{x}{2} + \sqrt{x+\tfrac{9}{4}} + \tfrac{3}{2} + r \\
&> \quad \tfrac{x}{2} + \left\lceil\sqrt{x+\tfrac{9}{4}} + \tfrac{1}{2}\right\rceil + r \\
&\geq \quad \tfrac{x}{2} + \left\lceil\sqrt{x+\tfrac{9}{4}} + \tfrac{1}{2}\right\rceil.
\end{aligned}$$

Since x is even, this also shows that

$$y \;\geq\; \tfrac{x}{2} + \left\lceil\sqrt{x+\tfrac{9}{4}} + \tfrac{1}{2}\right\rceil + 1,$$

so condition (3.2) holds. To see that condition (3.3) holds, we show if x and y are integers such that $x = 2k$ for some $k \geq 0$ and $y = \frac{x}{2} + \sqrt{x + \frac{9}{4}} + \frac{3}{2} + r$ for some $r \in [0,1]$, then r can be written as $r = i + \frac{1}{2} - \sqrt{(i+\frac{1}{2})^2 - 2j}$ with integers $i \geq 1$, $0 \leq j \leq i$. For this, note that

$$r \quad = \quad y - k - \tfrac{3}{2} - \sqrt{2k + \tfrac{9}{4}}. \tag{3.5}$$

Let $i := y - k - 2$ and $j := \frac{y(y-3)}{2} + \frac{k(k+1)}{2} - yk$. Then

$$\begin{aligned} r &= i + \tfrac{1}{2} - \sqrt{(i+\tfrac{1}{2})^2 - 2j}, \\ y &= \tfrac{i(i+3)}{2} - j + 1, &(3.6)\\ k &= \tfrac{i(i+1)}{2} - j - 1. &(3.7) \end{aligned}$$

Thus we have that $(y,k) \in \mathbb{Z}^2$ if and only if $(i,j) \in \mathbb{Z}^2$. Observe that if $y, k \geq 0$ and $r \in [0,1]$, then by (3.5), $y \geq k+3$, so $i \geq 1$. It follows that $r \in [0,1]$ if and only if $0 \leq j \leq i$. On the other hand, if $i \geq 1, j \geq 0$ and $r \in [0,1]$, then $j \leq i$, so from (3.6) it follows that $y \geq 0$ and from (3.7) it follows that $k \geq 0$ if $i \geq 2$. If $i = 1$ and $j = 0$, then $(y,k) = (3,0)$. We exclude the special case $(i,j) = (1,1), r = 1, (y,k) = (2,-1)$. This proves that y and k are non-negative integers with

$$r = y - k - \tfrac{3}{2} - \sqrt{2k + \tfrac{9}{4}} \in [0,1]$$

if and only if i and j are integers, $(i,j) \neq (1,1)$ with

$$i \geq 1,\ 0 \leq j \leq i,\ r = i + \tfrac{1}{2} - \sqrt{(i+\tfrac{1}{2})^2 - 2j},$$

so condition (3.3) is satisfied as well. □

Lemma 3.6.3. For $0 \leq r \leq 1$, let γ_r be the algebraic curve $y = \frac{x}{2} + \sqrt{x + \frac{9}{4}} + \frac{3}{2} + r$, restricted to $x \geq 0$. To each curve γ_{r_0} with $r_0 \in [0, \frac{1}{2}] \cap \mathbb{Q}$, there are two sequences of curves, $\gamma_{r_1(n)}$ and $\gamma_{r_2(n)}$, such that $|\gamma_{r_0} - \gamma_{r_1(n)}|$ and $|\gamma_{r_0} - \gamma_{r_2(n)}|$ converge to 0. Each $\gamma_{r_1(n)}$ contains an integer point $\big(x_1(n), y_1(n)\big)$ from the set A defined by (3.1) with $x_1(n) \to \infty$ as $n \to \infty$. Each $\gamma_{r_2(n)}$ contains a point $\big(x_2(n), y_2(n)\big)$ from $\mathbb{Z}^2_{\geq 0} \backslash A$ with $x_2(n) \to \infty$ as $n \to \infty$.

Proof. Let $r_0 \in [0, \frac{1}{2}[$ be a rational number, $r_0 = \frac{p}{q}, p, q \in \mathbb{Z}_{\geq 0}$. Let $i = nq, j = np$ for some $n \in \mathbb{Z}_{\geq 0}$, $r_1(n) = i + \frac{1}{2} - \sqrt{(i+\frac{1}{2})^2 - 2j}$. Then $i, j \in \mathbb{Z}_{\geq 0}, 0 \leq j < \frac{i}{2}, r_1(n) \in [0, \frac{1}{2}[, r_1(n) - r_0 \to 0$ as $n \to \infty$. If n is an integer such that $n \geq \frac{1}{q}$, then $i \geq 1$ and $0 \leq j \leq i$. Then, we have seen in Lemma 3.6.2 that

$$\big(x_1(n), y_1(n)\big) := \big(nq(nq+1) - 2np - 2, \tfrac{nq(nq+3)}{2} - np + 1\big)$$

is an integer point with $r_1(n) = y_1 - \frac{x_1}{2} - \frac{3}{2} - \sqrt{x_1 + \frac{9}{4}} \in [0, \frac{1}{2}[$ as $n \to \infty$. The point $(x_1(n), y_1(n))$ satisfies

$$y_1(n) \geq \tfrac{x_1(n)}{2} + \lceil \sqrt{x_1(n) + \tfrac{9}{4}} + \tfrac{1}{2} \rceil + 1,$$

which means it belongs to the set A defined in (3.1), and $x_1(n) \to \infty$ as $n \to \infty$.

Now let $i' = 2nq, j' = 2np$ for some $n \in \mathbb{Z}_{\geq 0}$, $r_2(n) = i' + 1 - \sqrt{(i')^2 + 2i' - 2j' + \frac{5}{4}}$. Then $i', j' \in \mathbb{Z}_{\geq 0}, 0 \leq j' < \frac{i'}{2}$ and $r_2(n) - r_0 \to 0$ as $n \to \infty$. The point

$$\big(x_2(n), y_2(n)\big) := (4n^2q^2 + 4nq - 4np - 1, 2n^2q^2 + 4nq - 2np + 2\big)$$

is an integer point with odd $x_2(n)$ and $r_2(n) = y_2 - \frac{x_2}{2} - \frac{3}{2} - \sqrt{x_2 + \frac{9}{4}} \in [0, \frac{1}{2}[$, where $x_2(n) \to \infty$ as $n \to \infty$. From Lemma 3.6.2 it follows that

$$y_2(n) < \tfrac{x_2(n)}{2} + \lceil \sqrt{x_2(n) + \tfrac{9}{4}} + \tfrac{1}{2} \rceil + 1,$$

hence the point $(x_2(n), y_2(n))$ does not belong to the set A. □

THEOREM 3.6.4. The set

$$A := \big\{(x, y) \in \mathbb{Z}^2_{\geq 0} : y \geq \tfrac{x}{2} + \lceil \sqrt{x + \tfrac{9}{4}} + \tfrac{1}{2} \rceil + 1\big\}$$

is not an $\mathbb{R}$-semi-algebraic set of lattice points.

Proof. It follows from the proof of Lemma 3.6.2 that A can be written as the disjoint union of the sets A_1 and A_2, where

$$A_1 := \Big\{(x, y) \in \mathbb{Z}^2_{\geq 0} : y \geq \tfrac{x}{2} + \sqrt{x + \tfrac{9}{4}} + 2\Big\}$$

and

$$\begin{aligned} A_2 := \Big\{(x, y) \in \mathbb{Z}^2_{\geq 0} : y = \tfrac{x}{2} + \sqrt{x + \tfrac{9}{4}} + \tfrac{3}{2} + r, \quad & r \in [0, \tfrac{1}{2}[, \\ r = i + \tfrac{1}{2} - \sqrt{(i + \tfrac{1}{2})^2 - 2j} & \\ \text{for some } i, j \in \mathbb{Z}_{\geq 0},\ i \geq 1,\ 0 \leq j \leq i\Big\}. & \end{aligned}$$

The affine lattice spanned by A is $\Lambda = \operatorname{aff}_{\mathbb{Z}} A = \operatorname{aff}_{\mathbb{Z}} A_1 = \operatorname{aff}_{\mathbb{Z}} A_2 = \mathbb{Z}^2$. The set A_1 is the intersection of $\mathbb{Z}^2$ with the semi-algebraic set

$$\begin{aligned} S_1 &:= \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, y \leq \tfrac{x}{2} + \sqrt{x + \tfrac{9}{4}} + 2\} \\ &= \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, \tfrac{x^2}{4} + y^2 - xy + x - 4y + \tfrac{7}{4} \leq 0\}. \end{aligned}$$

If there were a semi-algebraic set S such that $A = S \cap \mathbb{Z}^2$, then $S_2 := S \backslash S_1$ would be a semi-algebraic set and $A_2 = S_2 \cap \mathbb{Z}^2$. We will show that there is no such semi-algebraic set S_2 and hence no semi-algebraic set S.

Let γ_r denote the curve $y = \frac{x}{2} + \sqrt{x + \frac{9}{4}} + \frac{3}{2} + r$. For given r, $y \geq 0$ and $x \geq r - \frac{9}{4}$, γ_r is a semi-algebraic set:

$$\gamma_r = \{(x, y) \in \mathbb{R}^2 : \tfrac{x^2}{4} + y^2 - xy + (r + \tfrac{1}{2})x - (2r+3)y + r(r+3) = 0\}.$$

Set A_2 can now be written as

$$A_2 = \Big\{(x, y) \in \mathbb{Z}^2_{\geq 0} \cap \gamma_r : r \in [0, \tfrac{1}{2}[,\ r = i + \tfrac{1}{2} - \sqrt{(i + \tfrac{1}{2})^2 - 2j} \\ \text{for some } i, j \in \mathbb{Z}_{\geq 0}, i \geq 1, 1 \leq j \leq i\Big\}.$$

Consider an interval $[r, r+\varepsilon] \subset [0, \frac{1}{2}]$. Take a rational $r_0 \in]r, r+\varepsilon[$. By Lemma 3.6.3, there exist infinitely many integer points both from A_2 and from $\mathbb{R}^2/A_2$ with arbitrarily high x-coefficient in the interval $[r, r+\varepsilon]$. By Lemma 3.5.3, this implies that A_2 and hence A cannot be the intersection of $\mathbb{Z}^2$ with any semi-algebraic set. $\square$

Theorem 3.6.4 implies Theorem 3.3.4: Π^4_{12} is not a semi-algebraic set of lattice points.

3.7 THE SET $\mathcal{F}^d$ FOR DIMENSIONS 6 AND HIGHER

In this section we prove Theorem 3.3.5. For this we need the notion of g-vectors of simplicial polytopes as described in Section 1.1. We can also express the g-vector of a simplicial d-polytope P in terms of its f-vector:

$$g_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i+1}{d-k+1} f_{i-1} \quad \text{for } 0 \geq k \geq \lfloor \frac{d}{2} \rfloor.$$

Definition 3.7.1. Let G^d denote the set of g-vectors of simplicial d-polytopes and let G^d_{ij} denote the projection of the set of g-vectors of simplicial d-polytopes to the coordinates i and j,

$$G^d_{ij} \coloneqq \big\{(g_i(P), g_j(P)) : P \text{ is a simplicial } d\text{-polytope}\big\}.$$

LEMMA 3.7.2. The set G^d_{23} is not an $\mathbb{R}$-semi-algebraic set of lattice points for $d \geq 6$.

Proof. The g-theorem (Theorem 1.1.3) gives us

$$G^d_{23} = \big\{(g_2(P), g_3(P)) \in \mathbb{Z}^2 : g_2, g_3 \geq 0, \partial^3(g_3) \leq g_2\big\}.$$

Here,

$$\partial^3(g_3) \coloneqq \binom{n_3 - 1}{2} + \ldots + \binom{n_i - 1}{i}$$

where $n_i, \ldots, n_3$ are the unique integers such that $1 \leq i \leq n_i < \ldots < n_3$ and

$$g_3 = \binom{n_3}{3} + \ldots + \binom{n_i}{i}.$$

See Figure 3.4.

The affine lattice is $\Lambda = \mathbb{Z}^2$. We show that this set is not an $\mathbb{R}$-semi-algebraic set of lattice points. We will do this by considering the strip between the curve

$$\gamma_0 : g_3 = \tfrac{1}{2}g_2 + \tfrac{1}{3}g_2\sqrt{2g_2 + \tfrac{1}{4}}$$

through the points $(\binom{k}{2}, \binom{k+1}{3})$ for $k \in \mathbb{Z}_{\geq 0}$, and the same curve, shifted by the vector $(1,1) \in \mathbb{R}^2$,

$$\gamma_1 : g_3 = \tfrac{1}{2}(g_2 - 1) + \tfrac{1}{3}(g_2 - 1)\sqrt{2(g_2 - 1) + \tfrac{1}{4}} + 1.$$

We look at the points with $g_2 = \binom{k}{2}$ and $g_2 = \binom{k}{2} + 1$ for any integer $k \geq 2$. Observe that points with $g_2 = \binom{k}{2}$ in the strip satisfy $\partial^3(g_3) \leq g_2$ and points with $g_2 = \binom{k}{2} + 1$ in the strip satisfy $\partial^3(g_3) > g_2$. Additionally, if $k \to \infty$, the number of points with $g_2 = \binom{k}{2}$ and with $g_2 = \binom{k}{2} + 1$ in the strip goes to infinity. By Lemma 3.5.3 this implies that the strip, and hence the whole set G^d_{23}, is not a semi-algebraic set of lattice points. $\square$

Now we are ready to prove Theorem 3.3.5:

Proof of Theorem 3.3.5. The projection set G^d_{23} is not semi-algebraic by Lemma 3.7.2. This projection appears in the restriction of the set G^d to $g_1 := g_2$ and $g_i := 0$ for all $4 \leq i \leq \lfloor \frac{d}{2} \rfloor$. Therefore G^d is not an $\mathbb{R}$-semi-algebraic set of lattice points. The transformation from the g-vector to the f-vector is unimodular. Hence the set $\mathcal{F}^d_s$ of f-vectors of simplicial d-polytopes is not a semi-algebraic set of lattice points, for any $d \geq 6$. The set $\mathcal{F}^d$ of f-vectors of all d-polytopes is not a semi-algebraic set of lattice points, because its restriction to $2f_{d-2} = df_{d-1}$, the set of f-vectors of simplicial polytopes, is not a semi-algebraic set of lattice points. $\square$

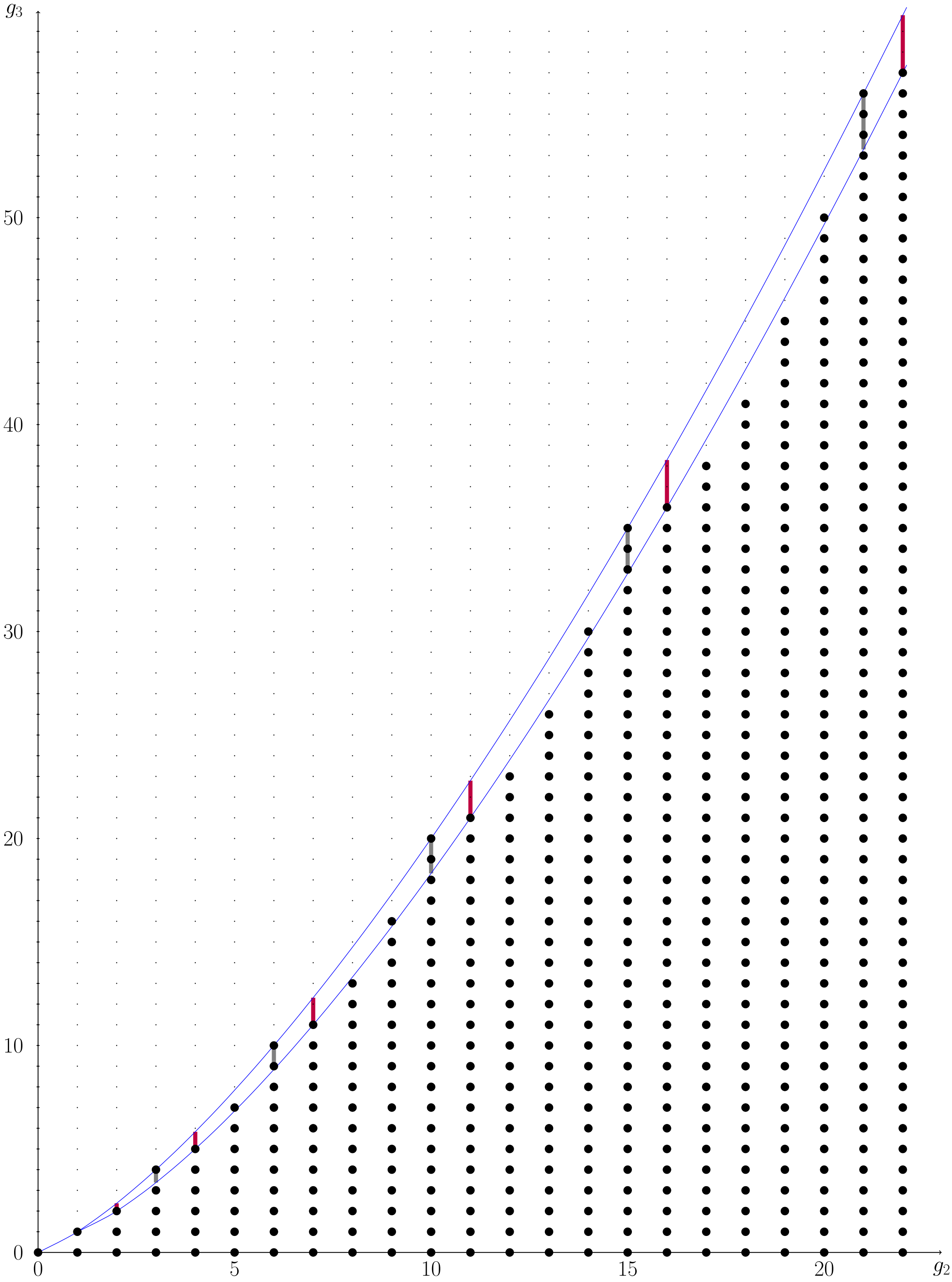


Figure 3.4: The set G^d_{23}, $d \geq 6$

Part II

ALCOVED POLYTOPES AND UNIMODALITY

4

UNIMODALITY

In this part we discuss whether a certain class of lattice polytopes, alcoved polytopes, have unimodal h^*-vectors. We start with some background on Ehrhart theory and Stanley–Reisner theory.

4.1 EHRHART THEORY

We briefly introduce concepts from Ehrhart theory used in this part. For more on Ehrhart theory, see Beck & Robins [13].
A polytope whose vertices all have integer coordinates is called a *lattice polytope.*

For a d-dimensional lattice polytope P, let $L_P(t)$ denote the number of lattice points in the t-th dilate of P:

$$L_P(t) \coloneqq |tP \cap \mathbb{Z}^d|.$$

The *Ehrhart series* of P is

$$\mathrm{Ehr}_P(z) \coloneqq 1 + \sum_{t \in \mathbb{N}_{>0}} L_P(t) z^t.$$

The following theorem is the main theorem of Ehrhart theory:

THEOREM 4.1.1 (Ehrhart [34, Thm. 2]). Let P be a d-dimensional lattice polytope. Then there exist complex numbers h_i^* such that $\mathrm{Ehr}_P(z)$ is a rational function

$$\mathrm{Ehr}_P(z) = \frac{h_0^*(P) + h_1^*(P)z + \ldots + h_d^*(P)z^d}{(1-z)^{d+1}}$$

where $h_0^*(P) + h_1^*(P) + \ldots + h_d^*(P) \neq 0$. A corollary of this theorem is that $L_P(t)$ can be expressed as a polynomial of degree d in the variable t, i.e. there exist numbers $q_0(P), q_1(P), \ldots, q_{d-1}(P), q_d(P)$ such that

$$L_P(t) = q_0(P) + q_1(P)t + \ldots + q_{d-1}(P)t^{d-1} + q_d(P)t^d$$

for all $t \in \mathbb{N}_{>0}$.
This polynomial is called the *Ehrhart polynomial* of P.
The vector $h^*(P) \coloneqq (h_0^*(P), h_1^*(P), \ldots, h_d^*(P))$ of coefficients of the numerator of the

Ehrhart series is called the h^*-*vector* (h-star-vector) of P.
In Ehrhart's original theorem, the coefficients h_i^* were only known to be complex numbers. By now, more is known about these coefficients:

THEOREM 4.1.2 (Stanley [78, Thm. 2.1]). The coefficients of the h^*-vector of a lattice polytope are non-negative integers.

Some entries of the h^*-vector have a combinatorial interpretation (see [41, Section 1]):

$$h_0^*(P) = 1,\ h_1^*(P) = |P \cap \mathbb{Z}^d| - (d+1),\ \text{and } h_d^*(P) = |\mathrm{int}(P) \cap \mathbb{Z}^d|.$$

The *lattice distance* between a hyperplane H and a lattice point p is 0 if $p \in H$ and otherwise $n+1$, where n is the number of hyperplanes parallel to H through lattice points which are lying strictly between p and H. In particular, a hyperplane H and a point $p \notin H$ have lattice distance 1 if there are no lattice points between H and the hyperplanes parallel to H containing p.
Analogously, the lattice distance between a facet F and a lattice point is the lattice distance between $\mathrm{aff}(F)$ and p.

A lattice polytope with 0 in its interior is called *reflexive* if its polar is also a lattice polytope.
Often, lattice polytopes are called reflexive if they are reflexive up to translations. Equivalently, a lattice polytope is reflexive (up to translations) if it has a unique interior lattice point and all facets have lattice distance 1 from the interior lattice point.
A lattice polytope $P \subset \mathbb{R}^n$ is *Gorenstein* of index k if kP, the k-th dilate of P, is a reflexive polytope for some $k \in \mathbb{N}_{>0}$.

THEOREM 4.1.3. [43, Hibi] A lattice d-polytope P in $\mathbb{R}^d$ is reflexive (up to unimodular equivalence) if and only if its h^*-vector is symmetric, i.e.:

$$h_i^* = h_{d-i}^* \text{ for } 0 \le i \le d.$$

A lattice polytope P is said to possess the *integer-decomposition property* (IDP) if every integer point in kP, for all $k \in \mathbb{N}_{>0}$, can be written as a sum of k integer points of P.
Polytopes which possess the IDP are called *IDP polytopes*, for short.

4.2 TRIANGULATIONS

Next we look at some triangulations of lattice polytopes.
Additional information on triangulations can be found in the book by De Loera, Rambau, and Santos [29].
A *triangulation* of a point configuration $\mathcal{A}$ is a simplicial complex with vertex set in $\mathcal{A}$ that covers $\mathrm{conv}(\mathcal{A})$.
With a triangulation of a lattice polytope P we always mean a triangulation of the point configuration $P \cap \mathbb{Z}^d$.

A full-dimensional lattice simplex S in $\mathbb{R}^d$ with vertices $v_0, \ldots, v_n$ is called a *unimodular simplex* if the vectors $v_n - v_0, v_{n-1} - v_0, \ldots, v_1 - v_0$ form a basis for $\mathbb{Z}^d$. All d-dimensional lattice simplices have the same volume, the volume $\frac{1}{d!}$. The volume of lattice polytopes is often "normalized" by the factor $d!$, so that a unimodular simplex is said to have *normalized volume* 1.
A triangulation of a lattice polytope is a *unimodular triangulation* if all its simplices are unimodular.
Lattice polytopes with unimodular triangulations also possess the IDP [39, Thm. 1.2.5].

A triangulation $\Delta(P)$ of a d-polytope P is called a *regular triangulation* if the following conditions hold:
P is the image $\pi(Q)$ of a polytope $Q \subset \mathbb{R}^{d+1}$ under the projection to the first d coordinates:

$$\pi : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$$
$$\begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \longmapsto x,$$

and $\Delta(P)$ is the image of all *lower faces* of P under the projection π. Here, a *lower face* F is a face whose outer normal vector has a negative last coordinate.

Lattice polytopes which have a unimodular triangulation satisfy particular nice conditions.
The following proposition gives an example of such a condition. It allows us to reduce all questions about h^*-vectors of lattice polytopes with unimodular triangulations to questions about the h-vectors of the triangulation:

PROPOSITION 4.2.1 (Betke & McMullen [14]). For any lattice polytope P which has a unimodular triangulation $\Delta(P)$, the h^*-vector of P is equal to the h-vector of $\Delta(P)$.

4.3 ALGEBRAIC BACKGROUND

Here we explain some notation from Stanley–Reisner theory that will be used in Chapter 5. More details can be found for example in the books by Stanley [77], Bruns & Herzog [25] and Miller & Sturmfels [63].
Let Δ be an abstract simplicial complex with vertices $x_1, \ldots, x_n$. Let $\mathbb{K}$ be a field and $\mathbb{K}[X_1, \ldots, X_n]$ be the polynomial ring over $\mathbb{K}$ where variable X_i corresponds to vertex x_i. The *Stanley-Reisner ideal* of Δ is the squarefree monomial ideal I_Δ of $\mathbb{K}[X_1, \ldots, X_n]$ generated by all the square-free monomials $X_{i_1} X_{i_2} \ldots X_{i_s}$ corresponding to the non-faces $\{x_{i_1}, x_{i_2} \ldots, x_{i_s}\}$ of Δ:

$$X_{i_1} X_{i_2} \ldots X_{i_s} \in I_\Delta \text{ if } \{x_{i_1}, x_{i_2} \ldots, x_{i_s}\} \notin \Delta.$$

The *face ring* (or *Stanley-Reisner ring*) $\mathbb{K}[\Delta]$ of Δ is the quotient of $\mathbb{K}[X_1, \dots, X_n]$ by the Stanley-Reisner ideal,

$$\mathbb{K}[\Delta] \coloneqq \mathbb{K}[X_1, \dots, X_n]/I_\Delta.$$

The Stanley–Reisner correspondence allows us to express many combinatorial problems of simplicial complexes in terms of homological algebra. We need the notion of a *Cohen–Macaulay ring*. In general, a commutative Noetherian local ring is called *Cohen–Macaulay* if its depth is smaller or equal to its Krull dimension.
Since we are only interested in Stanley-Reisner rings, we can simplify this definition to a characterization of Cohen–Macaulay rings for the case of certain quotient rings:

Proposition 4.3.1 (Hironaka's criterion, see [81], Prop. 4.1). Let $\mathbb{K}$ be an infinite field and let $R \coloneqq K[X_0, \dots, X_n]/I$ be the quotient of $K[X_0, \dots, X_n]$ by a homogeneous ideal I. Let d denote the Krull dimension of R. R is a *Cohen–Macaulay ring* if and only if there exist d homogeneous, linear elements $\theta_1, \dots, \theta_d$ from R and finitely many elements $\eta_1, \dots, \eta_n$ from R such that every $p \in R$ has a unique representation as

$$p = \sum_{i=1}^{n} \eta_i p_i(\theta_1, \dots, \theta_d),$$

where the $p_i(\theta_1, \dots, \theta_d)$ are elements in $K[\theta_1, \dots, \theta_d]$.

Equivalently, we can say that R is a free $K[\theta_1, \dots, \theta_d]$-module with basis $(\eta_1, \dots, \eta_n)$.

The system $\Theta \coloneqq (\theta_1, \dots, \theta_d)$ is called a *linear system of parameters (l.s.o.p.)* for R. If there exists a l.s.o.p. for a ring R, then any generic choice of $\theta_1, \dots, \theta_d$ will be a l.s.o.p. The term "generic" here refers to elements from a Zariski open subset of R_1^d [53].

A simplicial complex Δ with vertices $x_0, \dots, x_n$ is called *Cohen–Macaulay* over a field K if the Stanley-Reisner ring $K[X_0, \dots, X_n]/I_\Delta$ is a Cohen–Macaulay ring.

Reisner's criterion gives a characterization of Cohen–Macaulay complexes in terms of their homology groups:

Proposition 4.3.2 (Reisner's criterion [72]). A simplicial complex Δ is Cohen–Macaulay over a field K if and only if for any face F of Δ,

$$\dim_K(\widetilde{H}_i(\mathrm{link}_\Delta(F); k)) = 0 \text{ for } i < \dim(\mathrm{link}_\Delta(F)).$$

That is, Δ is Cohen–Macaulay over K if and only if the homology of each face's link vanishes below its top dimension.

In particular, this implies that pure shellable simplicial complexes are Cohen–Macaulay.

If we now have a pure shellable simplicial complex Δ of dimension $d-1$, then its face ring $K[\Delta]$ has Krull dimension d. According to Hironaka's criterion we can choose a l.s.o.p. $\Theta = (\theta_1, \dots, \theta_d)$ and consider the quotient ring $K[\Delta]/\Theta$. This ring can now be written as a direct sum

$$K[\Delta]/\Theta = (K[\Delta]/\Theta)_0 \oplus (K[\Delta]/\Theta)_1 \oplus \ldots \oplus (K[\Delta]/\Theta)_d,$$

where $\dim_K(K[\Delta]/\Theta)_i < \infty$ for $i = 0, \ldots, d$.

PROPOSITION 4.3.3 (see Stanley [77, Sect. 2.2]). Let Δ be defined as above with $K[\Delta]/\Theta = (K[\Delta]/\Theta)_0 \oplus (K[\Delta]/\Theta)_1 \oplus \ldots \oplus (K[\Delta]/\Theta)_d$. Let $h(\Delta) = (h_0, \ldots, h_d)$ be the h-vector of Δ. Then

$$\dim_K(K[\Delta]/\Theta)_i = h_i$$

for $i = 0, \ldots, d$.

Let Δ be a $(d-1)$-dimensional Cohen–Macaulay complex with face ring $K[X_1, \ldots, X_n]/I_\Delta = K[\Delta]$ and with l.s.o.p. Θ. An element $\omega \in K[X_1, \ldots, X_n]$ of degree 1 is called a *strong Lefschetz element* for $K[\Delta]/\Theta$ if the multiplication by ω^{d-2i},

$$\begin{aligned} \omega^{d-2i} : (K[\Delta]/\Theta)_i &\longrightarrow (K[\Delta]/\Theta)_{d-i} \\ m &\longmapsto \omega^{d-2i} m, \end{aligned}$$

is a bijection for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Following the notation from [55], we call ω an *almost strong Lefschetz element* for $K[\Delta]/\Theta$ if the multiplication by ω^{d-1-2i},

$$\begin{aligned} \omega^{d-1-2i} : (K[\Delta]/\Theta)_i &\longrightarrow (K[\Delta]/\Theta)_{d-1-i} \\ m &\longmapsto \omega^{d-1-2i} m, \end{aligned}$$

is an injection for $0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$.

A Cohen–Macaulay complex Δ is said to possess the *strong Lefschetz property* if there exists a strong Lefschetz element for $K[\Delta]/\Theta$.

The necessity of the g-theorem for simplicial polytopes (Thm. 1.1.3) follows from the following theorem, the algebraic g-theorem for simplicial polytopes:

THEOREM 4.3.4 (Stanley [80]). Boundary complexes of simplicial polytopes possess the strong Lefschetz property.

The generalization of the g-theorem to simplicial spheres is known as the *g-conjecture*. Adiprasito recently announced a proof of the g-conjecture for the more general class of simplicial rational homology spheres:

THEOREM 4.3.5 (Adiprasito [1]). Simplicial rational homology spheres have the strong Lefschetz property.

4.4 UNIMODALITY OF h-VECTORS AND h^*-VECTORS

A finite sequence $(s_1, s_2, \ldots, s_n)$ is called *unimodal* if there exists an index $k \in \{1, \ldots, n\}$ such that $s_1 \leq \ldots \leq s_k \geq \ldots \geq s_n$.

It follows directly from the definition of strong Lefschetz elements that the h-vectors of Cohen–Macaulay complexes with strong Lefschetz property are symmetric and unimodal.

In particular the h-vectors of simplicial polytopes and simplicial spheres are unimodal.

There is a variety of conjectures and theorems about the unimodality of h-vectors and h^*-vectors of different objects. We refer to Stanley [79], Brenti [23], Brändén [21] for surveys on unimodality in combinatorics. In the following we look at a few conjectures concerning unimodality in Ehrhart theory. More details can be found in the survey of Braun [22].

Let us start with a class of lattice polytopes that are known to have unimodal h^*-vectors.

THEOREM 4.4.1 (Hibi [42]). Reflexive lattice polytopes up to dimension 5 have unimodal h^*-vectors.

This theorem does not hold for higher dimensions. Mustaţă and Payne [66, Ex. 1.1], [70, Thm. 1.4] showed that there exist reflexive polytopes (even simplices) of all dimensions greater than 5 without unimodal h^*-vectors.
However, the next theorem shows that if a reflexive (or more generally a Gorenstein) polytope has a regular unimodular triangulation, then its h^*-vector is unimodal.

THEOREM 4.4.2 (Bruns & Römer [27, in the proof of Theorem 1]). Let P be a Gorenstein polytope with regular unimodular triangulation Δ. Then Δ has the strong Lefschetz property.
In particular, Gorenstein lattice polytopes with regular unimodular triangulation have unimodal h^*-vectors.

Using the g-theorem for simplicial spheres (Thm. 4.3.5), we can drop the condition of a regular triangulation: Any Gorenstein lattice polytope with unimodular triangulation has a unimodal h^*-vector (see [27, Sect. 1]).

It has been conjectured that the property of having a unimodular triangulation can be weakened even more to the more general condition of having the IDP.

CONJECTURE 4.4.3 (Hibi & Ohsugi [44]). A lattice polytope which is Gorenstein and IDP has unimodal h^*-vector.

Even more generally, there are no known examples of IDP polytopes without unimodal h^*-vectors. The next question or conjecture is part of a conjecture from Stanley that standard graded Cohen–Macaulay integral domains have unimodal h-vectors.

CONJECTURE 4.4.4 (Stanley, see [73, Question 1.1]). IDP polytopes have unimodal h^*-vectors.

Instead of considering all IDP polytopes, we can restrict ourselves to certain classes of IDP polytopes. A special class of polytopes that is conjectured to have unimodal

h^*-vectors are the *order polytopes*:
Let $P = P(\{p_1, \ldots, p_n\}, \preccurlyeq)$ be a finite poset. The *order polytope* $\mathcal{O}(P) \subset \mathbb{R}^n$ is defined by the inequalities:

$$
\begin{aligned}
&0 \leq x_i \leq 1 && \text{for all } i \in \{1, \ldots, n\}, \\
&x_i \leq x_j && \text{if } p_i \preccurlyeq p_j.
\end{aligned}
$$

Order polytopes have regular unimodular triangulations. We will see an example of such a triangulation for the more general class of alcoved polytopes in Definition 5.1.1.

In Chapter 5 we will consider lattice polytopes with regular unimodular triangulation (and which are not necessarily Gorenstein or reflexive).
The next theorem shows us what is known about their h^*-vectors.

THEOREM 4.4.5 (Hibi & Stanley, see Athanasiadis [5, Theorem 1.3]). Let P be a d-dimensional lattice polytope with a regular unimodular triangulation. Then:

$$
\begin{cases}
h_i^*(P) \geq h_{d+1-i}^*(P) & \text{for } 1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor, \\
h_{\lfloor \frac{d+1}{2} \rfloor}^*(P) \geq \ldots \geq h_{d-1}^*(P) \geq h_d^*(P), & \\
h_i^*(P) \leq \binom{h_1^*(P)+i-1}{i} & \text{for } 0 \leq i \leq d.
\end{cases}
$$

This theorem together with the known conditions $h^*(P) = (1, 0, \ldots, 0)$ if P is a simplex and $h_0^*(P) = 1 \leq |P \cap \mathbb{Z}^d| - (d+1) = h_1^*(P)$ otherwise, directly implies the following result.

COROLLARY 4.4.6. Lattice polytopes with regular unimodular triangulation of dimension smaller or equal 4 have unimodal h^*-vectors.

5

ALCOVED POLYTOPES

In this chapter we look at a particular class of polytopes, *alcoved polytopes.* Alcoved polytopes are an example of a class of lattice polytopes with regular unimodular triangulations or more generally, IDP polytopes. As we have seen, IDP polytopes are conjectured to have unimodular h^*-vectors (Conjecture 4.4.4). We will use the properties of alcoved polytopes to make some statements about their h^*-vectors.
Let us start with some definitions:

5.1 ALCOVED POLYTOPES

Definition 5.1.1. A hyperplane coming from an affine Coxeter arrangement of type $\mathcal{A}_d$ is a hyperplane of the form

$$H_d(i,j,k) = \{ \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \in \mathbb{R}^d \mid y_i - y_j = k\}$$

for some $k \in \mathbb{Z}$, $i,j \in \{0,\dots,d\}$ and $y_0 \coloneqq 0$. We will call such hyperplanes *alcove hyperplanes.*

A d-dimensional polyhedron is called an *alcoved polyhedron* of Lie type $\mathcal{A}$ if all of its facet-defining hyperplanes are alcove hyperplanes. If P is bounded it is called an *alcoved polytope* of Lie type $\mathcal{A}$.
In the following by an *alcoved polytope* we always mean an alcoved polytope of Lie type $\mathcal{A}$.
The $\mathcal{H}$-description of an alcoved d-polyhedron P with m facets can be given as

$$P = \{x \in \mathbb{R}^d \mid Mx \leq b\},$$

for $b \in \mathbb{Z}^m$ and M an $(m \times d)$-matrix with row vectors $a_k \in \{e_i, e_i - e_j\}$ for some $i,j \in 1,\dots,d$, for all $k \in \{1,\dots m\}$. M is a totally unimodular matrix, i.e. every minor of M is in $\{0,\pm 1\}$. In particular, this implies that alcoved polytopes are lattice polytopes (see [9, Sect. 7.1]).

Subdividing an alcoved d-polytope P by all alcove hyperplanes $H_d(i,j,k)$ gives a unimodular triangulation [57, Section 2.3], the *alcoved triangulation*. The simplices of the alcoved triangulation are called *alcoves*. The unimodular triangulation is regular, for example via the lifting function:

$$y_{d+1} = \sum_{i=1}^{d} y_i^2 + \sum_{\{i,j\}\in\{1,\ldots,d\}} (y_i - y_j)^2.$$

In Section 4.4 we have already seen an example of alcoved polytopes, the order polytopes. Another example is the class of *hypersimplices* $\Delta_{d-1,k}$. Hypersimplices can be defined as

$$\Delta_{d-1,k} \coloneqq \left\{x \in \mathbb{R}^{d-1} \,\middle|\, k-1 \leq x_1 + \ldots + x_{d-1} \leq k\right\} \cap [0,1]^{d-1},$$

the slice of the $(d-1)$-dimensional 0/1-cube between $\sum_{i=1}^{d-1} x_i = k-1$ and $\sum_{i=1}^{d-1} x_i = k$. Let $z_i \coloneqq x_1 + \ldots + x_i$ for all $i = 1, \ldots, d-1$. After this transformation of variables, the hypersimplex $\Delta_{d-1,k}$ can be expressed as the alcoved polytope given by the inequalities

$$\begin{aligned} 0 \leq \quad & z_1 && \leq 1, \\ k-1 \leq \quad & z_{d-1} && \leq k, \\ 0 \leq \ & z_i - z_{i-1} && \leq 1 \qquad \forall\, i = 2, \ldots, d-1. \end{aligned}$$

See Lam & Postnikov [57] for additional information and further examples of alcoved polytopes.

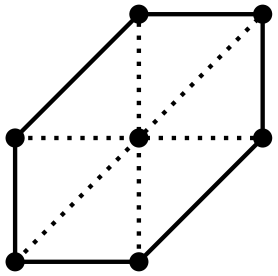
5.2 THE POLYTOPE Q_d

The polytope that we are about to define will be very useful for the proofs of our theorems. Since an alcoved polytope can only have a finite amount of admissible facet normals, there exists a unique alcoved polytope of minimal volume among all alcoved polytopes containing the origin in the interior. This polytope is obtained by taking the intersection of all facet-defining half-spaces that are defined by alcove hyperplanes and contain the origin in the interior:

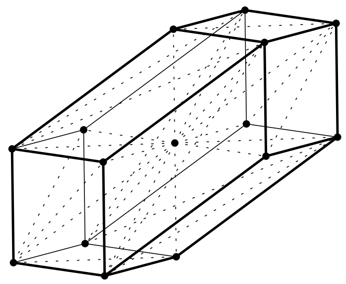
Definition 5.2.1. Let Q_d denote the alcoved polytope of minimal volume among all d-dimensional alcoved polytopes that have the origin in the interior:

$$Q_d \coloneqq \{\begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \in \mathbb{R}^d \mid y_i - y_j \leq 1 \text{ for } 0 \leq i,j \leq d, y_0 = 0\}.$$

The next proposition gives some idea about the combinatorial and geometric properties of the polytope Q_d.



(a) Alcoved triangulation of the hexagon Q_2



(b) Alcoved triangulation of Q_3

Figure 5.1: Examples of the polytope Q_d

PROPOSITION 5.2.2.

(i) Q_d is centrally symmetric.

(ii) Q_d has $2\binom{d+1}{2}$ facets.

(iii) Q_d is the convex hull of the union of the cubes $[-1,0]^d$ and $[0,1]^d$. This is a polytope that contains $2^{d+1}-1$ lattice points. It has one interior lattice point and all other $2^{d+1}-1$ lattice points are vertices.

(iv) The polytope Q_d is a projection of the $(d+1)$-dimensional unit cube $[0,1]^{d+1}$. Moreover, Q_d has the same h^*-vector as the unit cube $[0,1]^{d+1}$.

Proof of Proposition 5.2.2. (i) Central symmetry follows immediately from the hyperplane description.
(ii) To see that Q_d has $2\binom{d+1}{2}$ facets, observe that all $2\binom{d+1}{2}$ hyperplanes in the hyperplane description of Q_d are irredundant: The point with coordinates $x_i = 1$, $x_j = -1$ and $x_k = 0$ for all $k \in \{0, \dots d\} \setminus \{i,j\}$ is not contained in Q_d, but it is contained in the polyhedron obtained by removing $\{x \in R^d \mid x_i - x_j \leq 1\}$ from the hyperplane description of Q_d.
(iii) The cubes $[-1,0]^d$ and $[0,1]^d$ have 2^d lattice points each. The only lattice point in common is $\mathbf{0}$, so together they have $2^{d+1}-1$ lattice points.
It follows readily from the hyperplane description of Q_d that all vertices from the cubes $[-1,0]^d$ and $[0,1]^d$ are contained in Q_d. So the convex hull of $[-1,0]^d$ and $[0,1]^d$ is contained in Q_d. Since Q_d is an alcoved polytope, and hence a lattice polytope, in order to show that Q_d is equal to the convex hull it suffices to show that the vertices of the cubes $[-1,0]^d$ and $[0,1]^d$ are the only lattice points contained in Q_d. From the inequalities $x_i \leq 1$ and $-x_i \leq 1$ follows that all coordinates of the points in Q_d lie between -1 and 1. Because of the inequality $x_i - x_j \leq 1$, no point in Q_d can contain both $x_i = 1$ and $x_j = -1$ as coordinates. This shows that all lattice points in Q_d are vertices from $[-1,0]^d$ or from $[0,1]^d$, and hence that Q_d is the convex hull of $[-1,0]^d$ and $[0,1]^d$. The point $\mathbf{0}$ is the unique interior lattice point of Q_d, all other $2^{d+1}-2$ lattice points are vertices: No point with coordinates in $\{0,1\}$ or $\{-1,0\}$ besides the point $\mathbf{0}$ can be written as a

convex combination of the other points.
(iv) Q_d is the image of the unit cube $[0,1]^{d+1}$ under the projection $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}^d$, $e_i \mapsto e_i$ for $i = 1, \ldots, d$ and $e_{d+1} \mapsto -e_1 - \ldots - e_d$ with totally unimodular transformation matrix

$$\left(\begin{array}{cc} & -1 \\ I_d & \vdots \\ & -1 \end{array} \right).$$

The $(d+1)$-simplices in the alcoved triangulation of the $(d+1)$-cube are of the form $\operatorname{conv}\{0, e_{i_1}, e_{i_1} + e_{i_2}, \ldots, e_{i_1} + e_{i_2} + \ldots + e_{i_{d+1}}\}$.

They are mapped to $\operatorname{conv}\{0, e_{i_1}, e_{i_1} + e_{i_2}, \ldots, e_{i_1} + e_{i_2} + \ldots + e_{i_d}\}$, where $i_j \in \{0, \ldots, d\}$ with $e_0 := -e_1 - \ldots - e_d$. These are the d-simplices of the alcoved triangulation of Q_d. Intersections of $(d+1)$-simplices in the alcoved triangulation of the $(d+1)$-cube are mapped to the intersections of the corresponding d-simplices of Q_d. So if $\Delta_1, \ldots, \Delta_{(d+1)!}$ is a shelling order of the $(d+1)$-simplices of the alcoved triangulation of $[0,1]^{d+1}$, then $\varphi(\Delta_1), \ldots, \varphi(\Delta_{(d+1)!)}$ is a shelling order of the d-simplices of the alcoved triangulation of Q_d. The triangulations have therefore the same h-vectors. Since alcoved triangulations are unimodular triangulations, it follows that Q_d has the same h^*-vector as $[0,1]^{d+1}$. $\square$

5.3 ALCOVED POLYTOPES WITH INTERIOR LATTICE POINTS

In this section we focus on alcoved polytopes with interior lattice points. We do this because reflexive alcoved polytopes (which have 1 interior lattice point) are known to have unimodal h^*-vector. Given "nice" alcoved polytopes with interior lattice points, we can use reflexive alcoved polytopes as a base case for an inductive proof about the unimodality of the h^*-vectors. "Nice" here means polytopes that are unions of reflexive alcoved polytopes, or in other words alcoved polytopes that have interior lattice points and all facets have lattice distance 1 to the interior lattice points.

PROPOSITION 5.3.1. Let P be a d-dimensional alcoved polytope with interior lattice points such that all facets have lattice distance 1 to the interior lattice points. Denote by Δ the simplicial complex of the alcoved triangulation of P on the vertex set $P \cap \mathbb{Z}^d = \{x_1, \ldots, x_n\}$. Let $\mathbb{K}[\Delta] = \mathbb{K}[X_1, \ldots, X_n]/I_\Delta$ be the face ring of Δ and $\Theta = \{\theta_0, \ldots, \theta_d\}$ a linear system of parameters for $\mathbb{K}[\Delta]$. Then the multiplication by a generic element ω in $\mathbb{K}[X_1, \ldots, X_n]$ of degree 1,

$$\omega : (\mathbb{K}[\Delta]/\Theta)_j \longrightarrow (\mathbb{K}[\Delta]/\Theta)_{j+1},$$

is an injection for $0 \leq j \leq \lfloor \frac{d}{2} \rfloor - 1$.

We prove the proposition by induction on the number of interior lattice points. The proof is based on the proof of Theorem 1.1 of Kubitzke & Nevo [55], showing a similar condition for barycentric subdivisions of shellable simplicial complexes.
For background on homological algebra, especially the functor Tor, see [46, Sect. III.8].

Proof. Let P be a d-dimensional alcoved polytope with one interior lattice point and such that all of the facets of P have lattice distance 1 to the interior lattice point. P is (translation-equivalent to) a reflexive polytope. By Thm 4.4.2 reflexive (or more generally Gorenstein) polytopes with regular unimodular triangulation have the strong Lefschetz property. The induction assumption is hence satisfied.
Now assume P has at least two interior lattice points. Pick two interior lattice points p and q, and choose a facet F of P that is not parallel to the line through p and q. Let H_p be the hyperplane parallel to F through p and H_q the hyperplane parallel to F through q. H_p and H_q are disjoint by the choice of F. Let H_p^+ and H_p^- denote the two closed half-spaces defined by H_p. $P \cap H_p^+$ and $P \cap H_p^-$ are both d-dimensional alcoved polytopes. Let P_1 denote the one of these two polytopes containing point q. Similarly let P_2 denote the polytope containing p among the two polytopes $P \cap H_q^+$ and $P \cap H_q^-$. The facets of P_1 and P_2 all have lattice distance 1 to the interior lattice points. $P_1 \cap P_2$ is a d-dimensional alcoved polytope as well (possibly without interior lattice points). Let Δ, Δ_1 and Δ_2 denote the alcoved triangulations of P, P_1 and P_2, respectively. Let $\mathbb{K}[\Delta], \mathbb{K}[\Delta_1], \mathbb{K}[\Delta_2]$ and $\mathbb{K}[\Delta_1 \cap \Delta_2]$ be the corresponding face rings.
For some $\mathbb{K}[X_1, \ldots, X_n]$-module A, an ideal I of A and a quotient module A/I, let $(p)_{A/I}$ denote the image of an element $p \in A$ under the projection to A/I given by:

$$(X_i)_{A/I} = \begin{cases} X_i & \text{if } X_i \notin I \\ 0 & \text{otherwise.} \end{cases}$$

Consider the following Mayer–Vietoris sequence of $\mathbb{K}[X_1, \ldots, X_n]$-modules:

$$0 \to \mathbb{K}[\Delta] \xrightarrow{\varphi} \mathbb{K}[\Delta_1] \oplus \mathbb{K}[\Delta_2] \xrightarrow{\psi} \mathbb{K}[\Delta_1 \cap \Delta_2] \to 0, \tag{5.1}$$

where the homomorphisms are given in the following way:

$$\begin{aligned} \varphi : \mathbb{K}[\Delta] &\longrightarrow \mathbb{K}[\Delta_1] \oplus \mathbb{K}[\Delta_2] \\ p &\mapsto ((p)_{\mathbb{K}[\Delta_1]}, -(p)_{\mathbb{K}[\Delta_2]}) \end{aligned}$$

and

$$\begin{aligned} \psi : \mathbb{K}[\Delta_1] \oplus \mathbb{K}[\Delta_2] &\longrightarrow \mathbb{K}[\Delta_1 \cap \Delta_2] \\ (p, q) &\mapsto (p)_{\mathbb{K}[\Delta_1 \cap \Delta_2]} + (q)_{\mathbb{K}[\Delta_1 \cap \Delta_2]}. \end{aligned}$$

Explicitly this means

$$\varphi(X_i) = \begin{cases} (X_i, -X_i) & \text{for } X_i \notin I_{\Delta_1} \cup I_{\Delta_2} \\ (X_i, 0) & \text{for } X_i \in I_{\Delta_2} \setminus I_{\Delta_1} \\ (0, -X_i) & \text{for } X_i \in I_{\Delta_1} \setminus I_{\Delta_2} \end{cases}$$

and

$$\psi(X_i, X_j) = \begin{cases} X_i + X_j & \text{for } X_i, X_j \notin I_{\Delta_1 \cap \Delta_2} \\ X_i & \text{for } X_i \notin I_{\Delta_1 \cap \Delta_2}, X_j \in I_{\Delta_1 \cap \Delta_2} \\ X_j & \text{for } X_i \in I_{\Delta_1 \cap \Delta_2}, X_j \notin I_{\Delta_1 \cap \Delta_2} \\ 0 & \text{for } X_i, X_j \in I_{\Delta_1 \cap \Delta_2}. \end{cases}$$

Map φ is injective: $\varphi(p) = (0, 0)$ implies $p \in I_\Delta$.
Map ψ is surjective: $p \in \mathbb{K}[\Delta_1 \cap \Delta_2]$ is the image of $(p, 0)$ (and of $(0, p)$).
The image of φ is the kernel of ψ:
$\psi((p)_{\mathbb{K}[\Delta_1]}, -(p)_{\mathbb{K}[\Delta_2]}) = 0$ for all $p \in \mathbb{K}[\Delta]$, so $Im(\varphi) \subseteq \ker(\psi)$.
On the other hand, if $\psi(p, q) = 0$ for some $p \in \mathbb{K}[\Delta_1]$, $q \in \mathbb{K}[\Delta_2]$, then $(q)_{\mathbb{K}[\Delta_1 \cap \Delta_2]} = -(p)_{\mathbb{K}[\Delta_1 \cap \Delta_2]}$ and hence $\varphi(p \cup \{-q\}) = (p, q)$, and therefore $\ker(\psi) \subseteq Im(\varphi)$.
So the sequence is a short exact sequence.
Choose a linear system of parameters $\Theta = \{\theta_1, \ldots, \theta_d\}$ for $\mathbb{K}[\Delta]$ in such a way that $(\Theta)_{\Delta_1} := \{(\theta_1)_{\Delta_1}, \ldots, (\theta_d)_{\Delta_1}\}$, $(\Theta)_{\Delta_2}$ and $(\Theta)_{\Delta_1 \cap \Delta_2}$ are linear systems of parameters for $\mathbb{K}[\Delta_1], \mathbb{K}[\Delta_2]$ and $\mathbb{K}[\Delta_1 \cap \Delta_2]$, respectively. A generic choice of θ_i satisfies this condition: According to the definition, a l.s.o.p. for $\mathbb{K}[\Delta]$ is given by a generic choice of $\theta_1, \ldots, \theta_d$. Then the projections of $\theta_1, \ldots, \theta_d$ to $\mathbb{K}[\Delta_1], \mathbb{K}[\Delta_2]$ and $\mathbb{K}[\Delta_1 \cap \Delta_2]$ are still generic in the respective rings and therefore linear system of parameters for these rings.
We introduce the following notation for the sake of brevity:

$$\begin{aligned} S &:= \mathbb{K}[X_1, \ldots, X_n], \\ \mathbb{K}(\Delta) &:= \mathbb{K}[\Delta]/\Theta, \\ \mathbb{K}(\Delta_1) &:= \mathbb{K}[\Delta_1]/\Theta_{\Delta_1}, \\ \mathbb{K}(\Delta_2) &:= \mathbb{K}[\Delta_2]/\Theta_{\Delta_2}, \\ \mathbb{K}(\Delta_1 \cap \Delta_2) &:= \mathbb{K}[\Delta_1 \cap \Delta_2]/\Theta_{\Delta_1 \cap \Delta_2}. \end{aligned}$$

Tensoring the terms in (5.1) with S/Θ over S induces the following Tor-long exact sequence:

$$\begin{aligned} \ldots \quad & \to \mathrm{Tor}_1(\mathbb{K}[\Delta], S/\Theta) \to \mathrm{Tor}_1(\mathbb{K}[\Delta_1] \oplus \mathbb{K}[\Delta_2], S/\Theta) \\ \to \quad & \mathrm{Tor}_1(\mathbb{K}[\Delta_1 \cap \Delta_2], S/\Theta) \xrightarrow{\delta} \mathrm{Tor}_0(\mathbb{K}[\Delta], S/\Theta) \\ \to \quad & \mathrm{Tor}_0(\mathbb{K}[\Delta_1] \oplus \mathbb{K}[\Delta_2], S/\Theta) \to \mathrm{Tor}_0(\mathbb{K}[\Delta_1 \cap \Delta_2], S/\Theta) \to 0, \end{aligned}$$

where $\delta:\ \mathrm{Tor}_1(\mathbb{K}[\Delta_1\cap\Delta_2],S/\Theta)\to\mathrm{Tor}_0(\mathbb{K}[\Delta],S/\Theta)$ is the connecting homomorphism. Observe that

$$\mathrm{Tor}_0(\mathbb{K}[\Delta],S/\Theta)\cong\mathbb{K}(\Delta),$$
$$\mathrm{Tor}_0(\mathbb{K}[\Delta_1]\oplus\mathbb{K}[\Delta_2],S/\Theta)\cong\mathbb{K}(\Delta_1)\oplus\mathbb{K}(\Delta_2),$$
$$\mathrm{Tor}_0(\mathbb{K}[\Delta_1\cap\Delta_2],S/\Theta)\cong\mathbb{K}(\Delta_1\cap\Delta_2).$$

Also, $\mathrm{Tor}_1(\mathbb{K}[\Delta_1\cap\Delta_2],S/\Theta)=0$, since $\mathbb{K}[\Delta_1\cap\Delta_2]$ is free as an S-module.
We obtain the following exact sequence of S-modules:

$$0\to\mathbb{K}(\Delta)\to\mathbb{K}(\Delta_1)\oplus\mathbb{K}(\Delta_2)\to\mathbb{K}(\Delta_1\cap\Delta_2)\to 0,$$

and from this the following commutative diagram for $0\le j\le\lfloor\frac{d}{2}\rfloor-1$:

$$\begin{array}{ccccc}
0 & \to & \mathbb{K}(\Delta)_j & \to & \mathbb{K}(\Delta_1)_j\oplus\mathbb{K}(\Delta_2)_j \\
 & & \downarrow\omega & & \downarrow(\omega,\omega) \\
0 & \to & \mathbb{K}(\Delta)_{j+1} & \to & \mathbb{K}(\Delta_1)_{j+1}\oplus\mathbb{K}(\Delta_2)_{j+1}
\end{array}$$

where ω is a generic degree one element in S. By the induction hypothesis, the multiplications

$$\omega:(\mathbb{K}(\Delta_1))_j\longrightarrow(\mathbb{K}(\Delta_1))_{j+1}$$

and

$$\omega:(\mathbb{K}(\Delta_2))_j\longrightarrow(\mathbb{K}(\Delta_2))_{j+1}$$

are injective for $0\le j\le\lfloor\frac{d}{2}\rfloor-1$. From the commutative diagram it can be deduced that

$$\omega:(\mathbb{K}(\Delta))_j\longrightarrow(\mathbb{K}(\Delta))_{j+1}$$

is injective for $0\le j\le\lfloor\frac{d}{2}\rfloor-1$. □

In the above proposition we considered alcoved polytopes that are unions of reflexive alcoved polytopes. The proposition makes use of a certain property of the alcoved triangulation, namely the fact that the alcoved triangulation of any alcoved polytope with $m\ge 2$ interior lattice points can be described as a union of two simplicial complexes that are again alcoved triangulations of alcoved polytopes, each with less than m interior lattice points. This property does not hold for general regular unimodular triangulations. As an example consider the polygon with regular unimodular triangulation in Figure 5.2. The polygon has two interior lattice points p and q. There is no subcomplex of the triangulation whose underlying set is a convex polytope with unique interior lattice point p.

Now we are ready to prove our main theorem of this chapter, which tells us that alcoved polytopes that can be obtained as unions of reflexive alcoved polytopes have unimodal

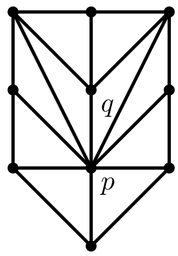


Figure 5.2: No proper subcomplex containing p in the interior has a convex underlying set.

h^*-vectors.

THEOREM 5.3.2. Let P be a d-dimensional alcoved polytope with interior lattice points such that all facets of P have lattice distance 1 to the interior lattice points. Then its h^*-vector is unimodal.

Proof. Proposition 5.3.1 implies that the entries of the h^*-vector increase until $h^*_{\lfloor \frac{d}{2} \rfloor}$:

$$h^*_0(P) \leq h^*_1(P) \leq \ldots \leq h^*_{\lfloor \frac{d}{2} \rfloor}(P).$$

The second part of the theorem follows from Theorem 4.4.5. For all d-dimensional lattice polytopes with regular unimodular triangulation, the entries of the h^*-vector decrease starting from $h^*_{\lfloor \frac{d+1}{2} \rfloor}(P)$:

$$h^*_{\lfloor \frac{d+1}{2} \rfloor}(P) \geq \ldots \geq h^*_{d-1}(P) \geq h^*_d(P).$$

□

For even dimensions d Theorem 5.3.2 tells us that the peak always occurs at the middle entry $h^*_{\frac{d}{2}}(P)$. For odd dimensions d the theorem only tells us that the peak occurs either at $h^*_{\frac{d-1}{2}}(P)$ or at $h^*_{\frac{d+1}{2}}(P)$. In Appendix B we describe the algorithms that we used to generate random alcoved polytopes and calculate their h^*-vectors. We tested around 20.000 alcoved polytopes of dimension up to 16. All h^*-vectors were unimodal. We tested around 1.000 alcoved polytopes with interior lattice points. The h^*-vectors of all of these polytopes had their peak at $h^*_{\lceil \frac{d-1}{2} \rceil}(P)$. See Appendix B.4 for some examples of h^*-vectors of some randomly generated alcoved polytopes.

Given an arbitrary alcoved polytope P with interior lattice points we can find an alcoved polytope P' with unimodal h^*-vector inside P by moving the facet-defining hyperplanes of P towards the interior lattice points until all facets have lattice distance 1 to the interior lattice points.

Since the h^*-vector of the alcoved polytopes are the h-vectors of the alcoved triangulations, it follows that $h^*_i(P) \geq h^*_i(P')$ for all $i \in \{0, \ldots, d\}$. A good approximation of P by P' gives a good approximation of $h^*(P)$ by $h^*(P')$.

It is therefore interesting to know how well P' approximates P. The next theorem tells us how far a facet can be from the interior lattice points.

THEOREM 5.3.3. Let P be a d-dimensional alcoved polytope with interior lattice points. Then the maximal lattice distance of a facet of P to the interior lattice points is $d-1$.

Proof. Let F be a facet of P. Let P' be the d-dimensional alcoved polyhedron obtained by removing the facet-defining hyperplane of F from the hyperplane description of P. We distinguish between two different cases: Either P' is an unbounded polyhedron or a polytope.
Case 1. If P' is an unbounded polyhedron, then F has lattice distance 1 to the interior lattice points. To see this, observe that the recession cone $\mathcal{C}$ of P' is an alcoved cone, i.e. an affine cone that is an alcoved polyhedron. The intersection of an alcoved polyhedron with alcove hyperplanes (hyperplanes parallel to facets of Q_d) is again an alcoved polyhedron, any possible vertices have to be lattice points by Definition 5.1.1. Let x be an interior lattice point of P (and hence of P'). Let $\mathcal{C}'$ denote the translate of $\mathcal{C}$ with apex x. $\mathcal{C}'$ is contained in the interior of P'. Let H be the hyperplane parallel to F that has distance 1 from x and separates x and F. The intersection of $\mathcal{C}'$ with H is a lattice polytope contained in the interior of P'. If F has distance larger than 1 from x, then $\mathcal{C}' \cap H$ is contained in the interior of P and its vertices are interior lattice points of P with smaller lattice distance to F than the distance between x and F. This shows that F has lattice distance 1 to the interior lattice points of P.
Case 2. See Figure 5.3 for an example in dimension 3. Let x be an interior lattice point

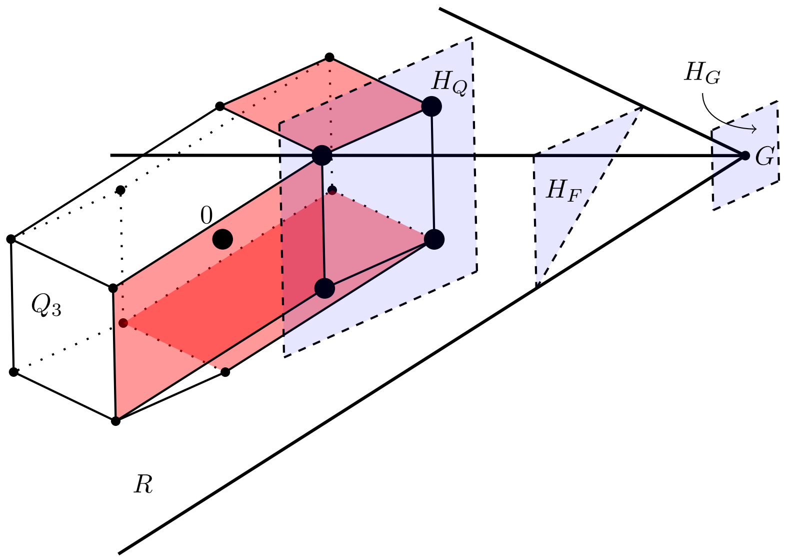


Figure 5.3: The hyperplanes containing the red facets of Q_3 intersect in the affine hull of face G of R.

of P closest to F. We may assume that $x = 0$. Let H_F be the hyperplane containing F. The polytope Q_d from Definition 5.2.1 is contained in all alcoved d-polytopes which contain the origin in the interior. Any facet of an alcoved d-polytope is parallel to two facets of Q_d. Among the two facet-defining hyperplanes of Q_d which are parallel to F, let H_Q denote the one separating 0 and F. The vertices of Q_d on H_Q are lattice points in P which are closer to F than 0. Since 0 is closest to F among all interior lattice points of P, these lattice points have to be in the boundary of P, each of the points has to be contained in at least one facet of P. Consider only the facets of P containing the vertices of Q_d in H_Q and (additionally) facet F. The facet-defining half-spaces of these facets define an alcoved polyhedron with 0 in the interior. If the polyhedron R obtained by removing the facet-defining half-space of F from the hyperplane description of the polyhedron is unbounded in the direction of the facet normal of F, then by case 1 H_F (and hence F) has lattice distance 1 from 0. Assume the polyhedron R is bounded in direction of the facet normal of F. Then there is a hyperplane H_G parallel to H_F which intersects R in a k-face G of R and such that H_F separates x and H_G. If $H_Q = H_G$, then $H_F = H_Q$, and H_F has distance 1 from 0. Assume H_G is not equal to H_Q. Then H_G is not equal to H_F either. We will show that H_G has lattice distance at most d from 0, and therefore H_F has at most lattice distance $d-1$ from 0. G is given as an intersection of $d-k$ facets of R. The hyperplane H_G containing G and parallel to F is of the form $\{x \in \mathbb{R}^d \mid x_i - x_j = l\}$ for some $i, j \in \{0, \ldots, d\}$ with $i \neq j$ and $x_0 \coloneqq 0$ and for some positive integer l.

We know that the difference $x_i - x_j = l$ is defined from some equations of the form $x_s - x_t = 1$, for $s, t \in \{0, \ldots, d\}$ and $s \neq t$. So the difference l between the two variables x_i and x_j can be obtained from setting the difference between some pairs of $d-1$ variables to 1. This shows that $l \in \{0, \ldots, d\}$.
We can also state this as a graph theoretical problem: Let G be a simple graph on $d+1$ vertices $v_0, \ldots, v_{d+1}$. There is an edge between vertex v_s and vertex v_t if and only if either $\{x \in \mathbb{R}^d \mid x_s - x_t = 1\}$ or $\{x \in \mathbb{R}^d \mid x_t - x_s = 1\}$ is a hyperplane of R intersecting in face G. If G would contain a cycle $(v_{s_1}, v_{s_2}), (v_{s_2}, v_{s_3}), \ldots, (v_{s_{r-1}}, v_{s_r}), (v_{s_r}, v_{s_1})$, then $x_{s_1} > x_{s_2} > \ldots > x_{s_r} > x_{s_1}$, a contradiction. So G does not contain any cycles, it is a forest. The condition that R is bounded in the direction of the facet-normal of F translates to the condition that vertex x_i and vertex x_j are path-connected. The longest possible path-length in a forest on $d+1$ vertices is d. The difference l is therefore at most d and facet F has lattice distance at most $d-1$ to 0. $\square$

We end this chapter with a proposition that shows that the bound from Theorem 5.3.3 is sharp for all dimensions d.

PROPOSITION 5.3.4. There is a d-dimensional alcoved polytope for any $d \in \mathbb{N}$ which has a facet with lattice distance $d-1$ to the interior lattice points.

Proof. We construct an example from the following polytope: Let P be the scaling of the order polytope of the chain of length d by $(d+1)$:

$$\begin{aligned} P = \{x \in \mathbb{R}^d : \; & x_1 \geq 0, \\ & x_2 \geq x_1, \\ & x_3 \geq x_2, \\ & \quad \vdots \\ & x_d \geq x_{d-1}, \\ & x_d \leq d+1\}. \end{aligned}$$

P is a reflexive alcoved simplex. The vertex description of P is given by

$$\begin{aligned} P = \operatorname{conv}\{ & (0,0,\ldots,0,0), (0,0,\ldots,0,d+1), \ldots, \\ & (0,d+1,\ldots,d+1,d+1), (d+1,d+1,\ldots,d+1,d+1)\}. \end{aligned}$$

Its unique interior lattice point is $p = (1,2,3,\ldots,d)$. The polytope $P \cap \{x \in \mathbb{R}^d \mid x_1 \leq d\}$ is still an alcoved polytope with unique interior lattice point p. The hyperplane $\{x \in \mathbb{R}^d \mid x_1 = d\}$ defining the new facet has distance $d-1$ from point p. □

For a visualization in dimension 3, see Figure 5.4.

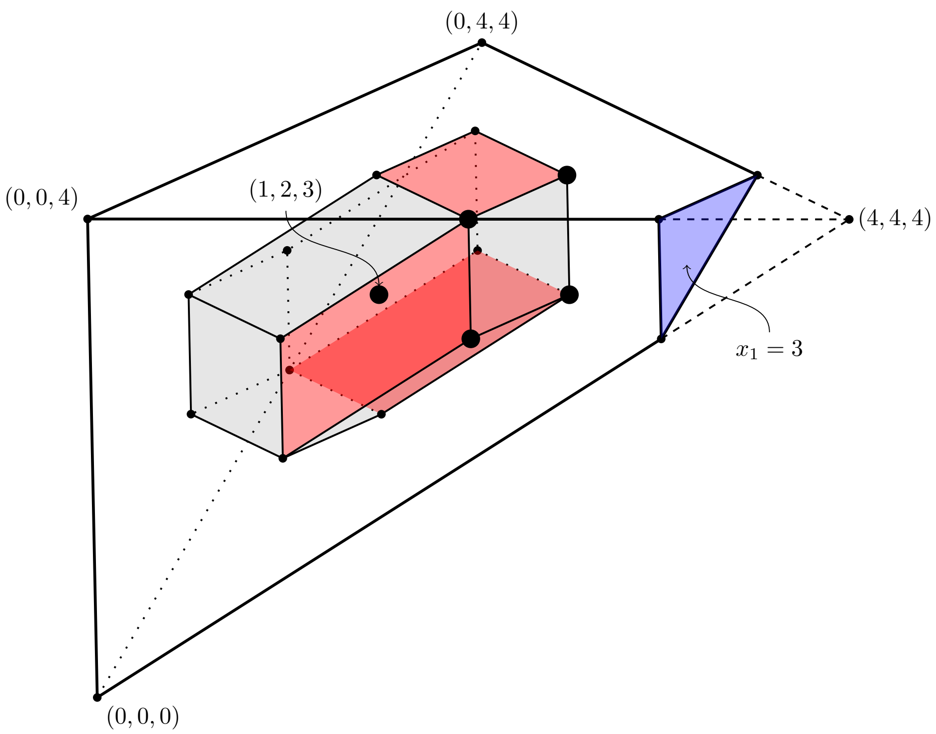


Figure 5.4: The blue facet has lattice distance 2 from the interior lattice point. The red facets of Q_3 are contained in facets of the bigger polytope.

Part III

APPENDIX

A

FACET LISTS OF POLYTOPES

Table A.1 lists all polytopes P_i with 7 and 8 vertices from Table 2.1 used in the construction of all possible pairs (f_0, f_{03}). The polytopes are given by their facet list. See Fukuda, Miyata & Moriyama [37] for a complete list of all 31 polytopes with 7 vertices and all 1294 polytopes with 8 vertices. Entry $7.x$ in the last column means that the polytope can be found as the x-th polytope listed in the classification of 4-polytopes with 7 vertices.

polytope	facet list	row
P_1	[654321][65430][6520][6420][5310][5210][4310] [4210]	7.3
P_2	[65432][65431][65210][64210][5320][5310][4320] [4310]	7.21
P_3	[65432][65431][65210][6421][5320][5310][4320] [4310][4210]	7.22
P_4	[65432][65410][6531][6431][5420][5321][5210] [4320][4310][3210]	7.11
P_5	[65432][6541][6531][6431][5421][5320][5310] [5210][4320][4310][4210]	7.16
P_6	[65432][65431][6521][6420][6410][6210][5320] [5310][5210][4320][4310]	7.24
P_7	[65432][6541][6531][6430][6410][6310][5421] [5320][5310][5210][4320][4210]	7.13
P_8	[765432][765410][76321][75310][64210][5430] [4320][3210]	8.186
P_9	[765432][76541][76310][75310][64210][6320] [5420][5410][5320]	8.285
P_{10}	[76543][76542][76321][75310][75210][64310] [64210][5430][5420]	8.1145
P_{11}	[765432][76541][76310][54310][7531][6421]	8.241

	[6320][6210][4320][4210]	
P_{12}	[765432][76541][76320][75310][54310][7610] [6421][6210][4320][4210]	8.353
P_{13}	[765432][76541][73210][63210][7631][7520] [7510][6420][6410][5420][5410]	8.201
P_{14}	[765432][76541][76310][7531][6430][6410] [5420][5410][5321][5210][4320][3210]	8.306
P_{15}	[765432][76510][7641][7541][6530][6421] [6321][6310][5420][5410][5320][4210][3210]	8.117
P_{16}	[76543][76521][76420][7542][6531][6431] [6410][6210][5432][5320][5310][5210][4320] [4310]	8.676
P_{17}	[76543][76542][73210][63210][7632][7531] [7520][7510][6431][6420][6410][5431][5420] [5410]	8.909
P_{18}	[76543][76521][7642][7542][6530][6510] [6432][6320][6210][5430][5421][5410][4321] [4310][3210]	8.778
P_{19}	[76543][76542][73210][7631][7621][7530] [7520][6431][6420][6410][6210][5431][5420] [5410][5310]	8.910
P_{20}	[76543][7652][7642][7531][7521][7431][7421] [6530][6521][6510][6430][6420][6210][5310] [4310][4210]	8.805
P_{21}	[76543][76542][7632][7531][7521][7320][7310] [7210][6431][6420][6410][6320][6310][5431] [5421][4210]	8.1227
P_{22}	[7654][7653][7643][7542][7532][7431][7421] [7321][6540][6530][6431][6410][6310][5420] [5320][4210][3210]	8.1262
P_{23}	[76543][7652][7642][7531][7521][7431][7421] [6530][6521][6510][6430][6420][6210][5310] [4321][4320][3210]	8.806
P_{24}	[76543][76542][7631][7621][7531][7520] [7510][7210][6430][6420][6321][6320][5431]	8.1041

	[5420][5410][4310][3210]	
P_{25}	[7654][7653][7643][7542][7532][7431][7421] [7321][6542][6530][6520][6430][6420][5321] [5310][5210][4310][4210]	8.1263
P_{26}	[76543][7652][7642][7541][7521][7420][7410] [7210][6530][6521][6510][6432][6320][6210] [5431][5310][4320][4310]	8.815
P_{27}	[7654][7653][7643][7542][7532][7431][7421] [7321][6542][6530][6520][6431][6420][6410] [6310][5321][5310][5210][4210]	8.1266

Table A.1: Polytopes P_i with 7 and 8 vertices

B

ALGORITHMS

Here we list algorithms used to calculated h^*-vectors of polytopes and test for unimodality. All algorithms are written as SageMath code [84]. We also give some examples of alcoved polytopes and their h^*-vectors.

B.1 CONVERT EHRHART POLYNOMIAL TO h^*-POLYNOMIAL

Given the Ehrhart polynomial of a lattice polytope P we can calculate the h^*-polynomial of P with help of the *Eulerian numbers* (see [13, Sect. 2.2]). The Eulerian number $A(d, k)$ for integers $d, k \geq 1$ is the number of permutations of the integers 1 to d with exactly $k - 1$ ascents.
Eulerian numbers can be defined recursively as:

$$A(d, 1) = 1,$$
$$A(d, k) = 0 \text{ for } d < k,$$
$$A(d, k) = (d - k + 1)A(d - 1, k - 1) + kA(d - 1, k) \text{ for } 2 \leq k \leq d.$$

Another possibility to define the Eulerian numbers is in the following way:

$$\sum_{j \geq 0} j^d z^j = \frac{\sum_{k=0}^{d} A(d, k) z^k}{(1 - z)^{d+1}}.$$

Let P be a d-dimensional lattice polytope with Ehrhart series $\mathrm{Ehr}_P(z)$ and Ehrhart polynomial $L_P(t) = q_0(P) + q_1(P)t + \ldots + q_{d-1}(P)t^{d-1} + q_d(P)t^d$.

The h^*-polynomial of P is

$$h_P^*(z) = (1 - z)^{d+1} \, \mathrm{Ehr}_P(z).$$

We can now calculate the h^*-polynomial from the Ehrhart polynomial using Eulerian numbers:

$$
\begin{aligned}
h_P^*(z) =&(1-z)^{d+1}\operatorname{Ehr}_P(z)\\
=&(1-z)^{d+1}\sum_{t\geq 0} L_P(t)z^t\\
=&(1-z)^{d+1}\sum_{t\geq 0}(\sum_{j=0}^{d} q_j t^j)z^t\\
=&(1-z)^{d+1}(q_0(1+z+z^2+\ldots)+q_1(z+2z^2+3z^3+\ldots)+\ldots\\
&+q_d(z+2^dz^2+3^dz^3+\ldots))\\
=&(1-z)^{d+1}\sum_{i=0}^{d} q_i(\sum_{j\geq 0} j^i z^j)\\
=&(1-z)^{d+1}\sum_{i=0}^{d} q_i\frac{\sum_{k=0}^{d}A(i,k)z^k}{(1-z)^{i+1}}\\
=&\sum_{i=0}^{d} q_i(\sum_{k=0}^{d}A(i,k)z^k)(1-z)^{d-i}.
\end{aligned}
$$

The following program was written together with Sophia Elia. This program converts the Ehrhart polynomial of a lattice polytope to the h^*-polynomial. The Ehrhart polynomials of lattice polytopes can be calculated using LattE Integrale [6]. There is also a built-in normaliz function [26] to calculate the Ehrhart series of a polytope, but it had too long run-time for our examples.

```
def eulerian_numbers(n):
    r'''
    Computes the Eulerian numbers up to A(n,n).

    OUTPUT:
    An n+1 by n+1 matrix of the Eulerian numbers up to A(n,n).
    '''
    A = zero_matrix(n+1,n+1)
    A[0,0] = 1
    for i in range(1,n+1):
        A[i,0] = 0
        A[i,1] = 1
    for j in range(2,n+1):
        for k in range(2,j+1):
            if j == k:
                A[j,k] = 1
            else:
                A[j,k] = (j-k+1)*A[j-1,k-1] +k*A[j-1,k]
    return(A)
```

```
def eulerian_polynomial(n):
    r'''
    Computes the nth Eulerian polynomial.
    '''
    R = PolynomialRing(ZZ, 't')
    t = R.gen()
    A = eulerian_numbers(n)
    return(R.sum( A[n,i]*t**i for i in range(n+1)))

def ehr_to_hstar(ehr_poly):
    r'''
    Convert the Ehrhart polynomial of a lattice polytope to the
    h*-polynomial.

    The Ehrhart series can be rewritten as follows:
    $$Ehr_P(t) = \sum_{m\geq 0}L_P(m)t^m
               = \sum_{m\geq 0}\sum_{j \geq 0}^{d}q_j m^j t^m$$,
    where L_P(m) = \sum_{j \geq 0}^{d}q_j m^j is the Ehrhart polynomial
    of P.

    The numerator of the rational expression for the series
    $\sum_{m\geq 0 }m^j t^m $ is an Eulerian polynomial. This function
    uses the Eulerian polynomials to transform from the Ehrhart
    polynomial to the h*-polynomial.

    INPUT:
    ``ehr_poly`` , a polynomial in 't' with rational coefficients, the
    output of the ``ehrhart_polynomial`` function.

    OUTPUT:
    The h*-polynomial as a polynomial in 't' with non-negative integral
    coefficients.

    EXAMPLE:

    The h*-polynomial of a unimodular simplex is always 1. Here we
    test the conversion for a 4-dimensional simplex:

        sage: p = polytopes.simplex(4)
        sage: e = p.ehrhart_polynomial()
        sage: ehr_to_hstar(e)
        1
    '''
    # change the polynomial into a vector
    Ring = PolynomialRing(QQbar, 't')
    t = Ring.gen()
    ehr_poly = ehr_poly.coefficients()
    # get the dimension of the polytope
    d = len(ehr_poly)-1
```

```
    # compute the h* polynomial
    factors = zero_vector(d+1)
    factors = factors.change_ring(Ring)
    for j in range(d+1):
        factors[j] = ehr_poly[j]*(1-t)**(d-j)*eulerian_polynomial(j)
    return sum(factors)
```

B.2 GENERIC ALCOVED POLYTOPES

We give an example of a function which allows us to create random alcoved polytopes. The polytopes are full-dimensional and inscribed in a $\dim(P)$-dimensional cube of side-length 5 (or smaller). This assures that the volume does not get too large for computation.

```
def alcoved_matrix(dim, vec):
    r'''
Computes a hyperplane representation matrix of an alcoved polytope.
INPUT: The dimension 'dim' of the polytope, and
        the coefficient vector 'vec' of dimension
        2*(dim choose 2)+2*dim for the hyperplanes.
OUTPUT: A (2*(dim choose 2)+2*dim)x(dim+1)-matrix
         for the H-description of an alcoved polytope.
    '''
    c = binomial(dim,2)
    M_help = Matrix (2*c,dim)
    M = Matrix (2*c+2*dim,dim+1)
    #
    # all hyperplanes of type x_i-x_j = constant:
    for i in range(c):
    # all sets of 2 indices out of all indices for each choice of
         (x_i,x_j)
        pairij = Combinations(range(dim),2).list()[i]
        M_help[2*i, pairij[0] ] = 1
        M_help[2*i,pairij[1] ] = -1
        M_help[2*i+1, pairij[0] ] = -1
        M_help[2*i+1, pairij[1] ] = 1
    for i in range(2*c):
        M[i, 0 ] = vec[i]
        for j in range(dim): M[i,j+1] = M_help[i,j]
    # all hyperplanes of type +x_i = constant
    # and -x_i = constant
    for i in range(2*c, 2*c+dim):
        M[i, 0] = vec[i]
        M[i+dim, 0] = vec[i+dim]
        M[i, i+1-2*c]=-1
        M[i+dim, i+1-2*c]=1
    return M
```

```
def random_vector(dim):
    r'''
    Computes a random (2*(dim choose 2)+2*dim)-dimensional coefficient
    vector for a generic alcoved polytope of dimension 'dim' contained
    in the cube [-2,3]^dim and containing the cube [0,1]^dim.
    '''
    rand=random_matrix(ZZ, 1, 2*binomial(dim,2),
    x=1,y=6).augment(random_matrix(ZZ, 1, dim, x=1,
    y=4).augment(random_matrix(ZZ, 1, dim, x=0, y=3)))[0]
    return rand

 def small_random_vector(dim):
    r'''
    Computes a random (2*(dim choose 2)+2*dim)-dimensional coefficient
    vector for a generic alcoved polytope of dimension 'dim' contained
    in the cube [0,3]^dim and containing the cube [0,1]^dim.
    This is better suited for higher dimensions, where the run-time
    would otherwise get too long.
    '''
    rand=random_matrix(ZZ, 1, 2*binomial(dim,2),
        x=1,y=4).augment(random_matrix(ZZ, 1, dim, x=1,
    y=3).augment(random_matrix(ZZ, 1, dim, x=0, y=2)))[0]
    return rand

def alcoved_polytope(dim,vec):
    r'''
    Computes an alcoved polytope of dimension 'dim' based on a vector
    'vec' that determines the position of the defining hyperplanes.
    INPUT: The dimension 'dim' of the polytope and a
          (2*(dim choose 2)+2*dim)-dimensional vector 'vec'.
    OUTPUT: An alcoved polytope of dimension 'dim'.
    '''
    P = Polyhedron(ieqs = alcoved_matrix(dim, vec), backend='normaliz')
    return P
```

B.3 UNIMODALITY

The programs in this section are a function that determines whether a list is unimodal and a function that tests the h^*-vectors of a given number of randomly generated alcoved polytopes of a given dimension for unimodality.
First we convert the h^*-polynomial to the h^*-vector:

```
# h^*-vector from h^*-polynomial
def Hvec(x): return
    (ehr_to_hstar(x.ehrhart_polynomial())).coefficients()
```

Then we determine if a list or tuple is unimodal:

```
def unimodal(ls):
    r'''
Determines if a list or tuple 'ls' is unimodal.
INPUT: List or tuple 'ls' of numbers.
OUTPUT: 'True' if the list is unimodal, 'False' otherwise.
    '''
    # determines index of the first decrease in the list
    decr= next((i for i in range(1,len(ls)) if ls[i]-ls[i-1]<0),False)
    if (decr == False) or (decr == len(ls)):
       return True
    else:
    # determines if there is an increase after the decrease
         if any(ls[i]-ls[i-1]>0 for i in range(decr+1,len(ls))):
           return False
         else: return True
```

Next we test several h^*-vectors of alcoved polytopes for unimodality:

```
def test_for_unimodality(num_of_tests, dim):
    r'''
This function tests the h^*-vectors of 'num_of_tests' many random
alcoved polytopes of dimension 'dim' for unimodality.
INPUT: The number of tests 'num_of_tests' and the dimension 'dim' of
        the polytopes that should be tested.
OUTPUT: Returns the vector that defines the first polytope with
         non-unimodal h^*-vector (with the alcoved_polytope-function)
         if there is any.
    '''
    for i in range(num_of_tests):
        rv = random_vector(dim)
        P = alcoved_polytope(dim,rv)
        hvector = Hvec(P)
        if unimodal(hvector) == False:
           print('There is an alcoved polytope with non-unimodal
               h-star-vector!')
           return rv
    print('All {} tested polytopes have unimodal
        h-star-vector.'.format(num_of_tests))
```

B.4 EXAMPLES

In Table B.1 there are some examples of (unimodal) h^*-vectors calculated with the function `Hvec` coming from alcoved polytopes randomly generated with the functions `random_vector` and `alcoved_polytope`. There is one example for each dimension between 3 and 16. For dimension 16 we used the `small_random_vector` to generate the polytopes.

Dim.	h^*-vector
3	1, 98, 188, 22
4	1, 104, 370, 146, 3
5	1, 356, 3216, 3965, 722, 6
6	1, 278, 3442, 7074, 2977, 189
7	1, 522, 9978, 33062, 26037, 4584, 102
8	1, 1962, 89574, 650410, 1210219, 643554, 86807, 1762
9	1, 5624, 409384, 4544444, 13413250, 12480018, 3617286, 265589, 2485
10	1, 11361, 1472932, 26848012, 130799314, 213186186, 123024451, 23477417, 1154189, 6839
11	1, 14350, 2256181, 51639116, 327390062, 729687382, 621049446, 198471087, 20964122, 521849, 1070
12	1, 19944, 5228717, 186169763, 1812852282, 6346402583, 8965318338, 5257171069, 1223272752, 98031585, 1936661, 3716
13	1, 91329, 44855680, 2470235309, 35303767765, 180399454984, 380503494555, 350270955412, 139818187839, 22527990348, 1233354464, 15460541, 13816
14	1, 92744, 61035357, 4461610156, 84498080329, 579384773749, 1683186967226, 2226507192578, 1364315287619, 375282480730, 42267367262, 1603500992, 13275503, 6792
15	1, 240485, 278806016, 30713524660, 831462306799, 8037261688378, 33174929171015, 63952385251690, 59735193988116, 26931894294332, 5599149137353, 484130852634, 14170655809, 90022039, 36218
16	1, 93896, 91469858, 10614760282, 323897816139, 3658410128555, 18169693219004, 43468545132385, 52316695445712, 31944633402938, 9656730519861, 1355225501529, 77874204105, 1436104725, 4957986, 586

Table B.1: Examples of h^*-vectors

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ZUSAMMENFASSUNG

Die vorliegende Arbeit behandelt zwei unterschiedliche Themenkomplexe aus dem Bereich der Polytoptheorie. Der erste Themenkomplex ist in Teil I enthalten und unterteilt in Kapitel 2 und 3. In diesem Teil beschäftigen wir uns mit den Mengen der f-Vektoren und Fahnenvektoren von Polytopen.

Ein neues Resultat aus Kapitel 2 ist die komplette Beschreibung der Projektion der Fahnenvektoren von 4-dimensionalen Polytopen auf die Einträge f_0, die Anzahl der Ecken, und f_{03}, die Anzahl der Ecken-Facetten-Inzidenzen. Dies ist Satz 2.1.5.

Weitere neue Resultate sind die Sätze 2.2.2 und 2.2.3. Darin wird die Menge aller Paare von Ecken- und Facettenanzahl von d-dimensionalen Polytopen beschrieben.

In Kapitel 3 wird das Konzept der *semi-algebraischen Mengen von Gitterpunkten* entwickelt. Dieses Konzept bietet uns eine Möglichkeit, die "Komplexität" der Mengen von f-Vektoren zu beschreiben. Wir betrachten eine Reihe von verschiedenen f-Vektor-Mengen und stellen fest, dass diese sich größtenteils als Menge aller Gitterpunkte in einer semi-algebraischen Menge beschreiben lassen. Unsere Hauptresultate in diesem Kapitel sind die Sätze 3.3.4 und 3.3.5, die besagen, dass zwei bestimmte f-Vektor-Mengen sich nicht als Menge aller Gitterpunkte in einer semi-algebraischen Menge beschreiben lassen. Diese Mengen sind erstens die Menge der Anzahl aller Kanten und 2-dimensionalen Seiten von 4-dimensionalen Polytopen, und zweitens die Menge aller f-Vektoren von d-dimensionalen Polytopen, mit d größer oder gleich 6.

In Teil II beschäftigen wir uns mit einer Fragestellung aus dem Bereich der Ehrharttheorie, die Frage, ob Alkovenpolytope unimodale h^*-Vektoren haben. Teil II ist aufgeteilt in Kapitel 4 und Kapitel 5.

In Kapitel 4 werden die notwendigen Konzepte aus Ehrharttheorie und Stanley–Reisner-Theorie erläutert, und einige Unimodalitätsvermutungen vorgestellt.

Kapitel 5 handelt von Alkovenpolytopen und deren h^*-Vektoren. Die beiden Hauptresultate in diesem Kapitel sind die Sätze 5.3.2 und 5.3.3. Satz 5.3.2 gibt eine Bedingung an, unter welcher Alkovenpolytope unimodale h^*-Vektoren haben: Alle Alkovenpolytope mit inneren Gitterpunkten, deren Facetten Gitterabstand 1 zu den inneren Gitterpunkten haben, haben unimodale h^*-Vektoren. Satz 5.3.3 begrenzt, wie sehr Alkovenpolytope diese Bedingung verfehlen können. Der Satz besagt, dass die Facetten von d-dimensionalen Alkovenpolytopen mit inneren Gitterpunkten höchstens Abstand $d-1$ zu den inneren Gitterpunkten haben.

SELBSTÄNDIGKEITSERKLÄRUNG

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin vom 8. Januar 2007 versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Zudem wurde die Arbeit auch noch nicht in einem früheren Promotionsverfahren eingereicht.

Berlin, 27. September 2020

Hannah Sjöberg