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Hilbert Functions

a connection between algebra, geometry and combinatorics

Habilitationsschrift

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Contents

Introduction

The first part of this introduction is intended for a broad audience. I will briefly describe some basic notions which are common to all chapters and state two conjectures which shaped the research area of this dissertation. As motivation, I will show how the two conjectures imply some classical results.

Imagine the topics in this dissertation as spanned by two problems. The first comes from algebraic geometry and aims at understanding the degrees of freedom allowed for polynomial equations. The second is a discrete problem and asks for a numerical understanding of triangulations of spheres. Instances of the two problems can be traced far back in history, long before modern mathematical language was developed. Today, commutative algebra is a crucial tool in both situations. Homology, parameter systems, regular sequences, Lefschetz elements, the Cohen-Macaulay and the Gorenstein property play important roles, but most important for our purpose here is the central role played by Hilbert functions of finitely generated graded commutative algebras over a field $\mathbb{K}$. The following definitions, while not the most general ones, will suffice for highlighting the connections with a classical context.

What are Hilbert functions? Let $R = \bigoplus_{i \geq 0} R_i$ be graded commutative $\mathbb{K}$-algebra, generated by finitely many elements of degree one. The *Hilbert function* $\mathrm{HF}_R : \mathbb{N} \longrightarrow \mathbb{N}$ of R is given by the $\mathbb{K}$-vector space dimensions of the graded components:

$$\mathrm{HF}_R(i) = \dim_{\mathbb{K}} R_i.$$

The associated *Hilbert Series* $\mathrm{HS}_R(t) = \sum_{i \geq 0} \mathrm{HF}_R(d) t^i$ was proven by Hilbert [Hil90] in the late 19Tm century to be rational of the form

$$\mathrm{HS}_R(t) = \frac{h(t)}{(1-t)^d},$$

where $h(t)$ is a polynomial with integer coefficients, called the *h-polynomial*, and d is the Krull dimension of R. This implies that there exists a unique polynomial $\mathrm{HP}_R(t)$, called the *Hilbert polynomial*, which has degree d and rational coefficients, and which satisfies $\mathrm{HP}_R(i) = \mathrm{HF}_R(i)$ for i large enough.

Connection to classical geometry. The first problem can be traced back to a literally ancient theorem [Cox03; EGH96] about points and lines in the Euclid plane.

Theorem (Pappus of Alexandria, 4th Century A.D.). *Let ℓ_1 and ℓ_2 be two lines meeting in one point. Let P_1, P_2, P_3 be three distinct points on ℓ_1 and Q_1, Q_2, Q_3 be three distinct points on ℓ_2. Assume all six points are different from $\ell_1 \cap \ell_2$. Then, the three intersection points*

$$A_{i,j} = P_i Q_j \cap P_j Q_i, \qquad 1 \leq i < j \leq 3$$

are collinear.

There is a fascinating trail of generalisations of this statement, passing through the Theorem of Pascal, the Theorem of Chasles, and the Cayley-Bacharach Theorem. The survey of Eisenbud, Green and

Harris [EGH96] offers a wonderful historical overview. We will state here one conjectured generalisation due to Eisenbud, Green, and Harris[EGH93] which has been a major driving force for research on the topic [Val98; Fra04; CM08; MPS08; MM11; CS13; MM11; FR07; CK13; Abe15; HWW17; GH18; CS18].

Conjecture 1 (Eisenbud, Green, Harris, 1992)**.** *Let $I \subseteq S = \mathbb{K}[x_0,\dots,x_n]$ be a homogeneous ideal containing a regular sequence $f_1,\dots,f_r$ with $\deg(f_i) = d_i$. Then, there exists a lex-segment ideal L such that I and $L+(x_0^{d_1},\dots,x_{r-1}^{d_r})$ have the same Hilbert function.*

A *lex-segment ideal*[1] is a monomial ideal $L \subseteq S$ with the property that, if two monomials m, m' with $\deg m = \deg m'$ satisfy $m \in L$ and $m' \geq_{Lex} m$, then $m' \in L$. Macaulay proved that for any homogeneous ideal there exists a lex-segment ideal with the same Hilbert function [Mac27]. This leads to a full numerical characterisation of all sequences of integers which may appear as a Hilbert function of a homogeneous ideal of the standard grade polynomial ring. Schützenberger [Sch59], Kruskal [Kru63], and Katona [Kat68] independently gave a similar characterisation for all graded ideals containing $x_0^2,\dots,x_n^2$. Clements and Lindström [CL69] extended this to Hilbert functions of graded ideals containing arbitrary powers of the variables. The Eisenbud, Green, Harris Conjecture is clearly a common generalisation of these statements. Peeva and Stillman [PS09] and Migliore [Mig07] offer overviews of related open problems.

Here is why Conjecture 1 is a generalisation of Pappus's Theorem. To avoid case distinctions for parallel lines, it is convenient to consider the Theorem of Pappus as a statement in the projective plane $\mathbb{P}^2$. A curve $C \subset \mathbb{P}^2$ of degree d is the zero locus of a homogeneous polynomial $f \in S = \mathbb{K}[x_0,x_1,x_2]$ of degree d. We write $C = V(f)$. For C to contain a given finite set of points $X \subset \mathbb{P}^2$, its coefficients must satisfy a system of linear equations. The Hilbert function of X measures the codimension of the corresponding solution set for each degree. By the Hilbert function of X we we understand $\mathrm{HF}_X := \mathrm{HF}_{S/I(X)}$, where $I(X)$ is the defining homogeneous ideal of X. In particular, from a certain degree on we have $\mathrm{HF}_X(d) = \mathrm{HP}_X(d) = |X|$, the number of points in X. In Pappus's Theorem two degenerate cubics are given:

$$\begin{aligned} C_1 = V(f_1) &= P_1Q_2 \cup P_2Q_3 \cup P_3Q_1 \\ C_2 = V(f_2) &= P_1Q_3 \cup P_2Q_1 \cup P_3Q_2. \end{aligned}$$

These intersect in nine distinct points: $\{P_1,P_2,P_3,Q_1,Q_2,Q_3,A_{1,2},A_{1,3},A_{2,3}\} =: X \subset \mathbb{P}^2$. Thus $I(X) = (f_1,f_2)$, and f_1,f_2 form a regular sequence with $\deg f_1 = \deg f_2 = 3$. Consider $X' = X \setminus \{A_{2,3}\}$, whose defining ideal $I = I(X')$ contains f_1,f_2. By Conjecture 1, which is a theorem in this case, there exists a lex-segment ideal L such that S/J, where $J = L+(x_0^3,x_1^3)$, has the same Hilbert function as X'. As X' consists of 8 points, for large d we must have $\mathrm{HF}_{S/J}(d) = 8$. It is an easy combinatorial check now to see that $\mathrm{HF}_{S/J}(3) = \mathrm{HF}_{X'}(3)$ is forced to be 8 as well. So the vector space of cubic polynomials vanishing on X' is two-dimensional, and contains the linearly independent set $\{f_1,f_2\}$. Consider a third cubic:

$$C_3 = V(f_3) = \ell_1 \cup \ell_2 \cup A_{1,2}A_{1,3}.$$

Clearly f_3 vanishes at X' by definition, thus it must be a linear combination of f_1 and f_2. As both vanish at $A_{2,3}$ as well, we have $A_{2,3} \in C_3$. By construction, $A_{2,3} \notin \ell_1 \cup \ell_2$, so $A_{2,3} \in A_{1,2}A_{1,3}$.

Connection to discrete geometry. Even though "solids" (or polyhedra, or convex 3-dimensional polytopes) have been studied since Pythagoras time, our story begins much later in this case. The reason is that the so called Euler formula

$$V - E + F = 2$$

[1]"lex" refers to the lexicographic order on monomials $\geq_{Lex}$.

connecting the number of vertices (V), the number of edges (E), and the number of faces (F) of a polyhedron was for a long time unnoticed. Euler was certainly the first to publish this result. It seems that a very similar relation was known to Descartes, However, his formula, which was discovered much later in a copied manuscript, was in terms of planar angles. There is again a long and interesting trail of generalisations and reinterpretations of this result: Poincaré's proof in terms of homology [Poi93], the theorems of Dehn [Deh05] and of Sommerville [Som27], lower and upper bound theorems [MW71; Bar71; Bar73; Sta75; Kal87; Cla93; Nov98; MN+13], the g-theorem [BL80; Sta80]. All are statements about the relations between the number of vertices, edges, faces, and their higher dimensional counterparts. The book of Stanley [Sta07] and the book of Ziegler[Zie95] offer excellent overviews of this vast area. Kalai's blog [Kal+] is a great source for both background and current updates.

To prove Euler's formula, it is enough to look at boundaries of 3-dimensional polytopes whose faces are triangles. Such polytopes are called simplicial. The reason for this is that subdividing faces into triangles by drawing diagonals leaves the relation unchanged: $V-(E+1)+(F+1)=V-E+F$. Let us consider more generally simplicial spheres, that is simplicial complexes Δ which are homeomorphic to spheres. Denote by f_i the number of i-dimensional faces of Δ. The f-vector of Δ is then $f(\Delta)=(f_{-1},f_0,\dots,f_{d-1})$. It was noticed that a linear transformation of the f-vector, called the h-vector and denoted by $h(\Delta)=(h_0,h_1,\dots,h_d)$, makes many statements more elegant. We will see that the name is no coincidence: it is actually the h-vector of graded $\mathbb{K}$-algebra. One problem which shaped research in the past decades is due to McMullen [McM71] and is known as the g-conjecture[2] for simplicial spheres.

Conjecture 2 (McMullen's g-conjecture[3], 1971)**.** *Let $(h_0,\dots,h_d)\in\mathbb{Z}^{d+1}$. The following two conditions are equivalent.*

1. *There exists a triangulation of a sphere with h-vector $(h_0,\dots,h_d)$.*
2. *$h_i=h_{d-i}$ for all i and $(h_0,h_1-h_0,\dots,h_{\lfloor\frac{d}{2}\rfloor}-h_{\lfloor\frac{d}{2}\rfloor-1})$ is the Hilbert function of some standard graded $\mathbb{K}$-algebra.*

This statement is a theorem for boundaries of simplicial polytopes. One direction was proved in 1980 by Billera and Lee [BL80; BL81] who constructed simplicial polytopes for each h-vector satisfying Condition 2. In the same year, Stanley proved the other implication for simplicial polytopes [Sta80]. As all simplicial 2-spheres are polytopal, the g-theorem returns the classical full description of their face numbers. In higher dimension things get more interesting [Kal88]: most simplicial spheres are not boundaries of polytopes. Given the g-theorem, the g-conjecture states that h-vectors cannot be used as a criterion for distinguishing polytopal from nonpolytopal spheres.

Stanley's proof is algebraic and uses a correspondence with commutative rings which he had already used earlier in the proof of the upper bound conjecture. His insight is an important mile stone for relation between commutative algebra and combinatorics. Hochster and Reisner brought significant contributions to the foundations of this theory [Hoc77; Rei76]. Let us have a glimpse at this correspondence. An abstract simplicial complex on the vertex set $[n]=\{1,\dots,n\}$ is a collection of subsets $\Delta\subseteq 2^{[n]}$ which is closed under taking subsets. To every simplicial complex Δ one can attach a standard graded $\mathbb{K}$-algebra, denoted by $\mathbb{K}[\Delta]$, and known now as the *Stanley-Reisner ring* (or the *face ring*) of Δ. This $\mathbb{K}$-algebra is defined as a quotient of the standard graded polynomial ring by the monomials not supported at faces of Δ:

$$\mathbb{K}[\Delta]=S/I_\Delta,\quad \text{where } I_\Delta=(x_{i_1}\dots x_{i_r}\ :\ \{i_1,\dots,i_r\}\notin\Delta)\subset S=\mathbb{K}[x_1,\dots,x_n].$$

The Stanley-Reisner ring is itself graded: $\mathbb{K}[\Delta]=\bigoplus_{i\geq 0}\mathbb{K}[\Delta]_i$. Furthermore, in each degree, the monomials supported on faces of Δ form a $\mathbb{K}$-basis of $\mathbb{K}[\Delta]_i$. This is why the f-vector of Δ is a linear transformation of the h-vector of $\mathbb{K}[\Delta]$. Let us look only at the 2-dimensional case. By definition we have

[2] In the last days of 2018 Adiprasito has presented a proof for this conjecture [Adi18].

[3] g stands for the g-vector which is defined as $(h_0,h_1-h_0,\dots,h_{\lfloor\frac{d}{2}\rfloor}-h_{\lfloor\frac{d}{2}\rfloor-1})$

that

$$\mathrm{HF}_{\mathbb{K}[\Delta]}(0) = 1 \quad \text{and} \quad \mathrm{HF}_{\mathbb{K}[\Delta]}(i) = f_0 + i \cdot f_1 + \binom{i}{2} f_2 \quad \text{for } i > 0.$$

This means, that the Hilbert series of $\mathbb{K}[\Delta]$ is

$$\begin{aligned}
\mathrm{HS}_{\mathbb{K}[\Delta]}(t) &= 1 + f_0 \cdot t \sum_{i \geq 0} t^i + f_1 \cdot t^2 \sum_{i \geq 0} i \cdot t^i + f_2 \cdot t^3 \sum_{i \geq 0} \binom{i}{2} \cdot t^i \\
&= 1 + f_0 \frac{t}{1-t} + f_1 \frac{t^2}{(1-t)^2} + f_2 \frac{t^3}{(1-t)^3} \\
&= \frac{1 + (f_0 - 3)t + (f_1 - 2f_0 + 3)t^2 + (-1 + f_0 - f_1 + f_2)t^3}{(1-t)^3}.
\end{aligned}$$

So, from $h_0 = h_3$ in the g-theorem for simplicial polytopes, we recover Euler's formula:

$$1 = -1 + f_0 - f_1 + f_2.$$

From $h_1 = h_2$ we obtain $f_1 = 3f_0 - 6$. The two of them combined give $f_2 = 2f_0 - 4$, so, for a triangulation of a 2-sphere, the number of vertices uniquely determines the number of edges and triangles. Finally, the g-condition is that $(1, f_0 - 4)$ is some Hilbert function, which means in this case that $f_0 \geq 4$. We thus recover the classical characterization of face numbers of simplicial 2-spheres.

We conclude this part with a few words on Stanley's proof of the g-theorem [Sta80]. His strategy was to associate to the simplicial polytope a projective complex toric variety, which has the same complex dimension as the polytope. Thus the real dimension is twice the dimension of the polytope. As the polytope is simplicial, the singular cohomology ring over $\mathbb{R}$ behaves well enough: Poincaré duality and the Hard Lefschetz Theorem hold. The fact that the polytope is simplicial plays a crucial role here. By a fundamental result of toric geometry, one can quotient the Stanley-Reisner ring of Δ by a linear system of parameters to obtain an Artinian graded Algebra isomorphic to the singular cohomology ring over $\mathbb{R}$ of the toric variety. The Hilbert function of this Artinian quotient is equal to the h-vector of Δ. Poincaré duality implies the symmetry of the h-vector. The Hard Lefschetz theorem gives the existence of a linear form ω such that linear map between consecutive graded components given by multiplication with ω always has maximal rank. This implies the unimodality of the h-vector, and that quotienting by ω one obtains an algebra with the g-vector as Hilbert function.

A third conjecture Matroids were introduced as an abstraction of the general concept of independence. There are many ways to define these combinatorial structures [Oxl11]. For the purpose of this work, it is most convenient to think of a matroid as a particular type of simplicial complex. Roughly speaking, a *matroid* is an abstract simplicial complex whose maximal faces satisfy an exchange property analogous to the exchange property for bases of a finite dimensional vector space. By looking at all possible restrictions to subsets of the vertex set, one obtains another definition: A simplicial complex is a matroid if every restriction is Cohen-Macaulay, equivalently if every restriction is shellable, equivalently if every restriction is pure (all maximal faces under inclusion have the same dimension). In general, the implications "shellable $\Rightarrow$ Cohen-Macaulay $\Rightarrow$ pure" are strict. However, when this is asked for every restriction, all three are equivalent to the simplicial complex being a matroid. Cohen-Macaulayness implies that the h-vectors of matroids are Hilbert functions (also known in the finite case as *O-sequences*, or *M-sequences*). A possible description of the Hilbert functions of matroids was conjectured by Stanley [Sta77].

Conjecture 3 (Stanley 1977)**.** *The h-vector of a matroid is a pure O-sequence.*

Stanley proved that the face ring of a matroid is more than just Cohen-Macaulay, it is *level*. This means, that it is Cohen-Macaulay and the socle is concentrated in one degree alone. For Cohen-Macaulay rings, the dimension of the socle is known as the *Cohen-Macaulay type*. When the ring is level, the Cohen-Macaulay type equals the last entry of the h-vector. In particular, Cohen-Macaulay type one means that the ring is Gorenstein. So level algebras are in between Cohen-Macaulay and Gorenstein algebras. Stanley's result implies that h-vectors of matroids are Hilbert functions of Artinian level algebras. Stanley's conjecture states that there exists even a *monomial* Artinian algebra with the same Hilbert function. Hilbert functions of such algebras are called *pure O-sequences*. There is also a combinatorial way to define them, namely as f-vectors of pure multicomplexes.

Just as the Eisenbud, Green, Harris Conjecture and as McMullen's g-conjecture for simplicial spheres, this problem has motivated many researchers in the field [Hib92; NPS02; Hau05; BJR09; Spe09; Var11; HK12; Mer+12; DKK12; HSZ13; Huh15; AHK17; AHK18]. While some special cases are known, it is still considered widely open. Conjecture 3 is part of the motivation in the final three chapters of this thesis. An overview of the state of the art is given there.

Summing up

Hilbert functions developed from classical mathematical concepts. In algebraic geometry, the coefficients of the Hilbert polynomial are among the most important invariants. This information is often presented in the equivalent form of Chern classes of a coherent sheaf. For a projective variety, the degree of the Hilbert polynomial is the dimension of the variety. So, when dealing with projective curves there are only two coefficients, and they determine the degree and the genus of the curve, giving thus a topological classification of the embedded curve. Moreover, the celebrated Riemann-Roch formula arises from the computation the Hilbert polynomials for a suitable class of modules (see [Eis95, pag. 44]). Hilbert functions made their way to combinatorics in the 1970s. Stanley's proof of the upper bound conjecture using commutative rings was one of the highlights of that period, establishing the connection between commutative algebra and discrete geometry as a permanent player in the field. Stanley's Conjecture 3 is one connection between Hilbert functions a other mathematical areas: combinatorial design theory, real algebraic geometry, tropical geometry, optimization and approximation theory. I believe there is one more reason why the interest in Hilbert functions has increased in the past few decades: Hilbert functions are constant on the fibres of a flat family. In particular, they are constant under Gröbner degeneration. The development of Gröbner bases theory has brought a new and powerful tool to the game: computer algebra. The development of computational power in the case of polynomial equations has made complicated examples - impossible to do "by hand"- easy to work with. This allowed researchers to look for patters and to test conjectures in an extremely efficient way.

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Summary

The main body of this dissertation consists of six research articles which have Hilbert functions as a central topic [CV13; CCV14; CV11; CV15; CKV14; CM15]. Each of them forms a chapter where the published version is reproduced exactly, with some updates to bibliography. Each chapter contains its own detailed introduction. We will focus here on the highlights.

Chapter 1: On the h-vectors of Cohen-Macaulay Flag Complexes

Alexandru Constantinescu and Matteo Varbaro,
Mathematica Scandinavica, vol. 112, issue 1, pp. 87-111, (2013)
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One can see Macaulay's characterisation of Hilbert functions as a compete characterisation of h-vectors of Cohen-Macaulay standard graded $\mathbb{K}$-algebras. Schützenberger, Kruskal and Katona characterised f-vectors of arbitrary simplicial complexes completely. This chapter deals with what happens when extra properties (of the $\mathbb{K}$-algebra or of the simplicial complex) are assumed. For instance, the essence of the Eisenbud, Green, Harris Conjecture 1 boils down to the following inclusion of sets.

$$\left\{ \begin{array}{c} h\text{-vectors of} \\ \text{quadratic Artinian } \mathbb{K}\text{-algebras} \end{array} \right\} \subseteq \left\{ \begin{array}{c} f\text{-vectors of} \\ \text{simplicial complexes} \end{array} \right\}.$$

Our starting point was a particular case of the above conjecture which was formulated by Kalai [Fro08].

Conjecture 4 (Kalai)**.**

$$\left\{ \begin{array}{c} \textit{h-vectors of Cohen-Macaulay,} \\ \textit{flag simplicial complexes} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \textit{f-vectors of} \\ \textit{balanced simplicial complexes} \end{array} \right\}.$$

A simplicial complex is flag if and only if its Stanely-Reisner ideal is quadratic. A d-dimensional simplicial complex is balanced if its underlying graph is $(d+1)$-colorable. In particular, face rings of balanced complexes have a special system of parameters, given by monochromatic linear forms. The main results of this chapter are grouped as follows.

1. We prove a particular case of Conjecture 4 by replacing "Cohen-Macaulay" with "vertex decomposable". Vertex decomposability is a recursive combinatorial property, which implies shellability and, thus also Cohen-Macaulayness.

2. In the proof of the above inclusion we could not conclude that the f-vector we obtained was also of a flag complex. However, we felt that this might be the case when one starts with a balanced complex, so made the following conjecture.

   Conjecture 5.

$$\left\{ \begin{array}{c} \textit{h-vectors of vertex decomposable,} \\ \textit{balanced, flag simplicial complexes} \end{array} \right\} = \left\{ \begin{array}{c} \textit{f-vectors of} \\ \textit{flag simplicial complexes} \end{array} \right\}.$$

We proved "$\supseteq$" in the above equality, and "$\subseteq$" with the weaker assumption of *quasi-flag* instead of flag on the right hand side. We introduced this notion recursively. Essentially, quasi-flag means that the f-vector of the complex can be written via deletion-contraction as a sum of f-vectors of smaller quasi-flag complexes. Not all quasi-flag complexes are flag, but by our conjecture, this feature should not be distinguishable by looking at the f-vector alone.

3. A further generalisation of our Conjecture 5 is

Conjecture 6.

$$\left\{ \begin{array}{c} \textit{h-vectors of Cohen-Macaulay,} \\ \textit{flag simplicial complexes} \end{array} \right\} = \left\{ \begin{array}{c} \textit{f-vectors of} \\ \textit{flag simplicial complexes} \end{array} \right\}.$$

This conjecture is much stronger, and we were only able to prove it for h-vectors of the form $(1,n,m)$. However, we believe that even a counterexample to this statement should shed some light of the relationships between all the above sets of vectors.

Chapter 2: Note on a Conjecture by Kalai

Giulio Caviglia, Alexandru Constantinescu and Matteo Varbaro;
Israel Journal of Mathematics, 0021-2172, pp. 1–7, (2014)
https://doi.org/10.1007/s11856-014-1115-y

This short chapter is a continuation of the previous one. The main results are the following.

1. The Eisenbud, Green, Harris conjecture holds for quadratic monomials ideals. Our proof relies on finding a regular sequence $f_1,\dots,f_g$ such that each f_i can be written as a product of linear forms and then conclude by the result of Abedelfatah [Abe15].

2. As a corollary we obtain Kalai's Conjecture 4.

Chapter 3: Koszulness, Krull Dimension and Other Properties of Graph Algebras

Alexandru Constantinescu and Matteo Varbaro;
Journal of Algebraic Combinatorics vol. 34, issue 3, pp. 375-400, (2011)
https://doi.org/10.1007/s10801-011-0276-6

This project was motivated by a series of questions raised by Jürgen Herzog. The main topic of this chapter are vertex cover algebras, which are a particular type of semigroup rings. The algebra generators correspond to sets of vertices which intersect every edge of a fixed (hyper)graph. On the one hand, these algebras can be interpreted as symbolic fiber cones of monomial ideals of pure codimension (greater than) two, they are thus a particular type of blowup algebra. Vertex covers on the other had are closely related to practical problems which are modeled by graphs. Here are the main results.

1. We found a combinatorial description of the Krull dimension of vertex cover algebras. This is done by introducing a new invariant for graphs, called the *ordered matching number*, and proving that this invariant gives the degree of the Hilbert polynomial (of some Veronese embedding). Ordered matchings turned up surprisingly in my research in a seemingly unrelated setting: liaison theory[CG18]. As a consequence of this we find an upper bound for the arithmetic rank of monomial ideals of codimension two.

2. We proved that vertex cover algebras are Koszul when the graph is bipartite. We actually proved more, namely that in this setting the vertex cover algebra has a structure of algebra with straightening laws. These were introduced in 1982 by De Concini, Eisenbud and Procesi in an attempt to gather under one definition several classical coordinate rings. Determinantal rings (thus coordinate rings of Grassmannians also) and Pfaffian rings are among the main examples of such structures.

3. We give a combinatorial criterion for Cohen-Macaulayness for edge ideals of graphs, when the associated vertex cover algebra is a domain. We show that this is equivalent to the vertex cover algebra being a Hibi ring, and thus obtain a characterization of those bipartite graphs whose vertex cover algebras are Gorenstein domains.

Chapter 4: h-vectors of Matroid Complexes

Alexandru Constantinescu and Matteo Varbaro;
Proceedings of CoMeTA 2013, Springer INdAM series, (2015)
https://doi.org/10.1007/978-3-319-20155-9_29

This is the first chapter with Stanley's Conjecture 3 as a starting point. Duality is an important feature of matroids. In general, the dual of a simplicial complex is obtained by taking the complex generated by the complements of the maximal faces. When the complex is a matroid, then so is its dual. Algebraically, the Stanley-Reisner ideal of the dual is the vertex cover ideal of the original complex. So Stanley's conjecture can be rephrased in dual terms, and this is the point of view we adopt in this chapter. We prove the following.

1. We proved that the 1-skeleton of a matroid is a complete p-partite graph. We grouped matroids according to dimension and the partition $\mathbf{p}$ their 1-skeleton, and for each group we found a minimal and a maximal h-vector.

2. The matroids realising this minimum and maximum are a simultaneous generalisation of uniform and partition matroids. We denoted them by $\Delta_{\max}(\mathbf{p},d)$ and $\Delta_{\min}(\mathbf{p},d)$. They come actually in a larger group, one for each $t = 0,\dots,d-2$, with the maximal one obtained for $t = 0$ and the minimal for $t = d-2$. We then proved that for all these $\Delta_t(\mathbf{p},d)$ Stanley's conjecture holds.

3. We were able to compute the Cohen-Macaulay type of all $\Delta_t(\mathbf{p},d)$, and obtained as a consequence, that Stanley's conjecture holds for Cohen-Macaulay type 2.

4. We found a counterexample to the Interval Conjecture of Pure O-sequence, formulated by Boij, Migliore, Mirò-Roig, Nagel, Zanello in [Boi+12]. This conjecture stated that if $(h_0,\dots,h_s)$ and $(h'_0,\dots,h'_s)$ are two pure O-sequences with $h_i = h'_i$ for all but one index i_0, in which case we have $h_{i_0} < h'_{i_0}$, then
$$(h_0,\dots,h_{i_0-1},\alpha,h_{i_0+1},\dots,h_s)$$
is also a pure O-sequence for every $h_{i_0} \leq \alpha \leq h'_{i_0}$. Our counterexample is $(1,4,10,13,12,9,3)$ and $(1,4,10,13,14,9,3)$, and was found by computer experiments. It was later proven that this is the smallest possible counterexample [HSZ13].

Chapter 5: Generic and Special Constructions of Pure O-sequences

Alexandru Constantinescu, Thomas Kahle and Matteo Varbaro;
Bulletin of the London Mathematical Society, vol 46, (5). pp. 924–942, (2014)
https://doi.org/10.1112/blms/bdu047

This chapter is partially a continuation of the previous one, especially the results for small Cohen-Macaulay type. The novelty in this part is a completely new approach which involves a generic change of coordinates. The idea is that to prove Stanley's conjecture it is enough to pass to a *monomial* Artinian reduction of the Stanley-Reisner ring. This rarely exists, but when it does, it would be level by Stanley's result. If one performs first a generic change of coordinates and then a degeneration to the reverse-lexicographic initial ideal, the h-vector stays unchanged. Furthermore, in characteristic zero, the new monomial ideal has a good combinatorial property: it is strongly stable. This implies the existence of a monomial Artinian reduction. Unfortunately, the level property is lost this way. We had two ideas: find classes of ideals for which the level property is preserved, find a more subtle change of coordinates which is not generic, preserves the level property and allows for a monomial Artinian reduction. For the latter, a binomial regular sequence would suffice. Here is an overview of the results in this chapter.

1. Using the strategy described above, we proved that any truncation of a matroid satisfies Stanley's conjecture. Truncating a d-dimensional matroid means passing to some k-skeleton for $k < d$. Not all matroids are truncations. For example, if removing parallel elements reduces the matroid to a simplex, then the original matroid was not a truncation of another matroid. However, some nice classes of matroids are truncations. For instance, Schubert matroids (also known as shifted matroids, or PI-matroids, or generalised Catalan matroids) are truncations and thus our result implies Stanley's conjecture for them.

2. We showed by direct combinatorial techniques, that duals of rank d matroids with at most $d+2$ parallel classes also satisfy Stanley's conjecture.

3. Building on techniques developed in the previous chapter, we proved Stanley's conjecture for matroids of Cohen-Macaulay type ≤ 5. This means that, if $(h_0,\dots,h_s)$ is the h-vector of a matroid and $h_s \leq 5$, then it is a pure O-sequence.

4. Combining the previous two cases with computational experiments, we eliminated "brute force" (i.e. checking all possible pure O-sequences) as a method for finding counterexamples.

Chapter 6: Determinantal Schemes and Pure O-sequences

Alexandru Constantinescu and Matey Mateev;
Journal of Pure and Applied Algebra, 219, pp. 3873-3888, (2015)
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Part of this chapter was driven by disbelief in Stanley's Conjecture 3. One reason for this is the elusive structure of pure O-sequences. Completely characterising these is believed to "solve all basic problems in combinatorial design theory" [Zie00]. Furthermore, criteria for not being a pure O-sequence are very rare. As a hint to the difficulty of the problem, consider the following statement: A fine projective plane of order q exists if and only if the vector $(1,h_1,\dots,h_{q+1})$, with

$$h_i = (q^2+q+1)\binom{q+1}{i}, \qquad \forall\, i = 1,\dots,q+1$$

is a pure O-sequence [MNZ13]. The case $q = 12$ is still open, and direct computation cannot solve it. Our approach to finding a criterion for non pure O-sequences was inspired by a positive result about the h-vectors of standard determinantal varieties. These are codimension c projective schemes defined by maximal minors of homogeneous $t \times (t+c-1)$-matrices. Their Hilbert function to depends only on the degrees of the entries of the matrix, which we collect in a *degree matrix*. We proved the following.

1. If the degrees are constant in the columns of the matrix, then the h-vector of the determinantal scheme is a log-concave pure O-sequence. To prove that these h-vectors are pure O-sequences, we showed that they the h-vectors of some $\Delta_t(p,d)$ matroid from Chapter 4. For log-concavity, namely the property that $h_i^2 \geq h_{i-1}h_{i+1}$, we used Huh's result [Huh15] which later was generalised in a joint work with Katz [HK12].

2. We conjectured that if the degree matrix does not have the above property, then the h-vector is indeed not a pure O-sequence. This would provide a method to find a large non-obvious family of non-pure O-sequences. We proved this conjecture in codimension two, and in any codimension with the assumption that there is a unique component-wise maximal row in the degree matrix.

3. We conclude this chapter by listing the experiments which prove our conjecture for all accessible small cases.

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Chapter 1

On the h-vectors of Cohen-Macaulay Flag Complexes

Alexandru Constantinescu, Matteo Varbaro
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Division of Labor

The collaboration on this paper was very intense and a clear separation of the contributions is not possible. It is however possible to say that both authors contributed in an equal maner to the conception, execution and writing of this work.

1.1 Introduction

The f-vectors of simplicial complexes and the h-vectors of standard graded K-algebras are fascinating subjects in combinatorics and commutative algebra. These topics have been the object of study for many researchers in the past decades (for instance see [BFS87; FFK88; EGH93; Fro08]). The f-vectors of simplicial complexes have been completely characterized by Kruskal and Katona, and the h-vectors of Cohen-Macaulay standard graded K-algebras have been characterized by Macaulay. However, many questions regarding both f- and h-vectors remain open, when extra properties are assumed for the simplicial complex, respectively for the standard graded algebra.

An unpublished conjecture of Kalai stated that for any flag simplicial complex there exists a balanced simplicial complex with the same f-vector. This fact has been recently proven by Frohmader in [Fro08]. This conjecture of Kalai has also a second part which is still open, namely:

Conjecture 1.1 (Kalai)**.** *The following inclusion holds true:*

$$\left\{\begin{array}{c} f\text{-vectors of Cohen-Macaulay,} \\ \text{flag simplicial complexes} \end{array}\right\} \subseteq \left\{\begin{array}{c} f\text{-vectors of Cohen-Macaulay,} \\ \text{balanced simplicial complexes} \end{array}\right\}.$$

As the h-vector of a simplicial complex is uniquely determined by its f-vector, the above conjecture can be also stated replacing f-vectors with h-vectors. By a theorem of Björner, Frankl and Stanley [BFS87] the h-vectors of Cohen-Macaulay, balanced simplicial complexes are f-vectors of simplicial complexes. Therefore the following would be a consequence of Kalai's Conjecture 1.1:

Conjecture 1.2. *The following inclusion holds true:*

$$\left\{\begin{array}{c} h\text{-vectors of Cohen-Macaulay,} \\ \text{flag simplicial complexes} \end{array}\right\} \subseteq \left\{\begin{array}{c} f\text{-vectors of} \\ \text{simplicial complexes} \end{array}\right\}.$$

Actually, the above inclusion is a particular case of a more general conjecture by Eisenbud, Green and Harris (see [EGH93] or the lecture notes by Valla [Val98]), which can be stated as:

Conjecture 1.3 (Eisenbud, Green and Harris)**.** *The following inclusion holds true:*

$$\left\{\begin{array}{c}\textit{h-vectors of}\\ \textit{quadratic Artinian } K\textit{-algebras}\end{array}\right\} \subseteq \left\{\begin{array}{c}f\textit{-vectors of}\\ \textit{simplicial complexes}\end{array}\right\}.$$

After introducing most of the terminology that we will need, in Section 2 we present a few results and remarks that we will use throughout this paper. In particular, in Theorem 1.8, we will extend results of Crupi, Rinaldo and Terai from [CRT11] and of the two authors from [CV11].

In the third section we will prove Conjecture 1.2 for vertex decomposable, flag simplicial complexes (Theorem 1.12). This section also includes an example of a h-vector of a quadratic Artinian algebra, which is the f-vector of a balanced complex, but not the h-vector of a Cohen-Macaulay, flag simplicial complex (Example 1.13). The section ends with a few comments on some technical aspects appearing in the proof of Theorem 1.12.

In Section 4 we will first notice that the f-vector of a flag simplicial complex is always the h-vector of a vertex decomposable, balanced, flag simplicial complex (Proposition 1.15). This result led us to the statement:

Conjecture 1.4. *The following equality holds true:*

$$\left\{\begin{array}{c}\textit{h-vectors of vertex decomposable,}\\ \textit{balanced, flag simplicial complexes}\end{array}\right\} = \left\{\begin{array}{c}f\textit{-vectors of}\\ \textit{flag simplicial complexes}\end{array}\right\}. \tag{1.1}$$

We were not able to find a proof for the above equality. However, relaxing the requests on the right hand side or strengthening the ones on the left we will be able to prove the hard inclusion of Conjecture 1.4. First, in Definition 1.16 we introduce a new class of simplicial complexes – the quasi-flag simplicial complexes. It turns out that flag complexes are quasi-flag and in general the converse is not true. However, we are not aware of any quasi-flag simplicial complex whose f-vector is not the one of a flag simplicial complex. We will then prove the following inclusion (Theorem 1.17):

$$\left\{\begin{array}{c}h\text{-vectors of vertex decomposable,}\\ \text{balanced, flag simplicial complexes}\end{array}\right\} \subseteq \left\{\begin{array}{c}f\text{-vectors of}\\ \text{quasi-flag simplicial complexes}\end{array}\right\}.$$

In the fifth section we are going to discuss a natural extension of Conjecture 1.4:

Conjecture 1.5. *The following equality holds true:*

$$\left\{\begin{array}{c}\textit{h-vectors of Cohen-Macaulay,}\\ \textit{flag simplicial complexes}\end{array}\right\} = \left\{\begin{array}{c}f\textit{-vectors of}\\ \textit{flag simplicial complexes}\end{array}\right\}.$$

In Proposition 1.21 we will see that the above conjecture is true when the h-vector is of the form $(1,n,m)$. We will then prove the following result (Theorem 1.22):

$$\left\{\begin{array}{c}h\text{-vectors of } (d-1)\text{-dimensional}\\ \text{Cohen-Macaulay, flag simplicial complexes}\\ \text{on } [2d]\text{, without cone points}\end{array}\right\} = \left\{\begin{array}{c}f\text{-vectors of flag}\\ \text{simplicial complexes on } [d]\end{array}\right\}.$$

In a certain sense the above result is a first step towards proving Conjecture 1.5. This is because when Δ is a Cohen-Macaulay, $(d-1)$-dimensional, flag simplicial complex on $[n]$, without cone points, we have $n \geq 2d$.

In the last section we will come back to Conjecture 1.4. We introduce two properties of simplicial complexes and show that for each of them, if added on the left hand side of (1.1), the conjecture holds. We also include examples of simplicial complexes with and without these properties.

Many results in this paper have been suggested and double-checked by extensive computer algebra experiments performed with CoCoA [CoC].

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1.2 Preliminaries

Let us start by introducing some terminology and notation that we will use throughout the paper. For general aspects on the topics presented below we refer the reader to the books of Stanley [Sta96] , of Bruns and Herzog [BH93] and of Lovász and Plummer [PL86].

For a positive integer n denote by $[n]$ the set $\{1,\dots,n\}$. A simplicial complex Δ on $[n]$ is a collection of subsets of $[n]$ such that $F \in \Delta$ and $F' \subset F$ imply $F' \in \Delta$. We will also require that for every $i \in [n]$ we have $\{i\} \in \Delta$. Each element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* and we will denote by $\mathscr{F}(\Delta)$ the set of facets of Δ. We call a vertex v a *cone point* of Δ if $v \in F$ for any $F \in \mathscr{F}(\Delta)$. A simplicial complex is called *pure* if all facets have the same cardinality. The dimension of a face F is $|F|-1$ and the *dimension of* Δ is $\max\{\dim F \ : \ F \in \Delta\}$.

Let $f_i = f_i(\Delta)$ denote the number of faces of Δ of dimension i, in particular $f_{-1} = 1$ and $f_0 = n$. The sequence $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$, where $d-1$ is the dimension of Δ, is called the *f-vector* of Δ.

Denote by $S = K[x_1,\dots,x_n]$ the polynomial ring in n variables over a field K and let Δ be a simplicial complex on $[n]$. For each subset $F \subset [n]$ we set

$$\mathrm{x}_F = \prod_{i\in F} x_i.$$

The *Stanley-Reisner ideal* of Δ is the ideal I_Δ of S generated by the square-free monomials x_F, with $F \notin \Delta$. That is

$$I_\Delta = (\mathrm{x}_F \ : \ F \text{ is a minimal nonface of } \Delta).$$

We will denote by $K[\Delta] = S/I_\Delta$ the *Stanley-Reisner ring* of Δ. It is a well known fact that $\dim K[\Delta] = \dim\Delta + 1$. We will denote by $h(\Delta) = (h_0, h_1, \dots, h_s) = h(K[\Delta])$, the h-vector of the graded algebra $K[\Delta]$. In other words, if $H_{K[\Delta]}(t)$ is the Hilbert series of $K[\Delta]$, we have

$$H_{K[\Delta]}(t) = \frac{h_0 + h_1 t + \ldots + h_s t^s}{(1-t)^d},$$

where d is the Krull dimension of $K[\Delta]$ and $h_s \neq 0$. The sequence $h(\Delta)$ is called the *h-vector* of Δ. The h-vector of Δ can be determined directly from the f-vector of Δ using the relation:

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

Comparing the coefficients we obtain the formula:

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}. \tag{1.2}$$

It is well known that $s \leq d$. So, as opposed to the f-vector, the h-vector does not contain precise information about the dimension of the simplicial complex. In other words, the f-vector can be determined from the h-vector only if the dimension of Δ is also known.

Let Δ be a simplicial complex and F a face of Δ. The *link* of F in Δ is the following simplicial complex:

$$\mathrm{link}_\Delta F = \{F' \in \Delta \ : \ F' \cup F \in \Delta \text{ and } F' \cap F = \emptyset\}.$$

For a set of vertices $W \subset [n]$, the *restriction* of Δ to W is the following subcomplex of Δ:

$$\Delta_W = \{F \in \Delta \ : \ F \subset W\}.$$

The subcomplex Δ_W is also called the subcomplex of Δ *induced by* the vertex set W. If $[n] \setminus W = F$ is a face of Δ, the subcomplex Δ_W is called the *face deletion* of F in Δ. We will abuse notation and write $\Delta \setminus F = \{F' \in \Delta \ : \ F \not\subset F'\}$ for the face deletion of $F \in \Delta$. Whenever F is a 0-dimensional face $\{v\}$ we will just write $\Delta \setminus v$ for the face deletion of $\{v\}$ and $\mathrm{link}_\Delta v$ for the link of $\{v\}$.

Consider $\Delta' \subseteq \Delta$ a subcomplex and let Γ be a simplicial complex with vertex set disjoint from the vertex set of Δ. We define the *star* of Δ with Γ along Δ' to be the simplicial complex:

$$\Delta *_{\Delta'} \Gamma = \Delta \bigcup \{F' \cup F \ : \ F' \in \Delta' \text{ and } F \in \Gamma\}.$$

It is easy to see that, for any $F \in \Delta$ the three definitions above are connected in the following way:

$$\Delta = (\Delta \setminus F) *_{\mathrm{link}_\Delta F} \langle F \rangle.$$

A simplicial complex Δ on $[n]$ is said to be k-colorable, for some $k \in \mathbb{N}$, if there exists a function $\mathrm{col} : [n] \longrightarrow [k]$ such that if $\mathrm{col}(i) = \mathrm{col}(j)$ for $i \neq j$, then no face of Δ contains both i and j. Obviously, if the dimension of Δ is $d-1$, then $k \geq d$. A $(d-1)$-dimensional simplicial complex is called *balanced* if it is d-colorable. For a balanced simplicial complex and for every $i \in [d]$ we denote by $V_i = \{v \in [n] \ : \ \mathrm{col}(v) = i\}$ the set of vertices colored i. Fixing a coloring, the Stanley-Reisner ring of a balanced simplicial complex has a canonical linear system of parameters (see [Sta96, Proposition 4.3]), given by

$$\theta_i = \sum_{j \in V_i} x_j.$$

A simplicial complex Δ is called Cohen-Macaulay (CM for short) over a field K if and only if the ring $K[\Delta]$ is Cohen-Macaulay. If Δ is CM over any field K then we simply say that Δ is CM. There are several combinatorial properties of simplicial complexes that imply Cohen-Macaulayness. In this paper we will focus on the following one. A pure simplicial complex Δ is recursively defined to be *vertex decomposable* if it is either a simplex or else has some vertex v such that:

1. both $\Delta \setminus v$ and $\mathrm{link}_\Delta v$ are vertex decomposable,
2. no face of $\mathrm{link}_\Delta v$ is a facet of $\Delta \setminus v$.

A vertex satisfying condition 2. above is called a *shedding vertex*. As we mentioned above, a vertex decomposable simplicial complex is always CM. The other implication is known to be false in general.

Remark 1.6. (a) The notion of vertex decomposable complex exists also in the nonpure case. In the pure case, vertex decomposability can be defined in a more compact way, without condition (2) above and starting from the 0-dimensional simplex (see Björner's manuscript [Bjo95]). However, we prefer to use the above definition, which is the restriction of the more general one to the pure case.

(b) If Δ is vertex decomposable and balanced, the sets V_i that we defined above are uniquely determined.

A simplicial complex is called *flag* if all its minimal nonfaces have cardinality two. In other words, if its Stanley-Reisner ideal is generated by square-free monomials of degree two. Flag simplicial complexes are closely related to simple graphs, i.e. finite graphs with neither loops nor multiple edges. Let G be a (simple) graph on the vertex set $V(G) = [n]$ and denote by $E(G)$ the set of its edges. We define the edge ideal of G as the ideal:

$$I(G) = (x_i x_j \ : \ \{i, j\} \in E(G)) \subset S.$$

For a flag simplicial complex Δ we will denote by G_Δ, or just G if no confusion arrises, the graph of minimal nonfaces of Δ. In particular $I_\Delta = I(G_\Delta)$.

Given the correspondence between Stanley-Reisner ideals of flag simplicial complexes and edge ideals of simple graphs we also need to introduce some terminology related to graphs. For a vertex $v \in V(G)$ we denote by $N(v) = \{w \in V(G) \, : \, \{v,w\} \in E(G)\}$ the *open neighborhood* of v in G. By $N[v]$ we denote the *closed neighborhood* of v, i.e. $N(V) \cup \{v\}$. For a subset of vertices $W \in V(G)$ we define:

$$N(W) = (\bigcup_{v\in W} N(v)) \setminus W.$$

A *perfect matching* of G is a collection of disjoint edges $\{e_1,\dots,e_r\}$ of G such that every vertex belongs to one of the edges, i.e. $V = \cup e_i$. An *independent* set in G is a collection of vertices $\{v_1,\dots,v_r\}$ such that $\{v_i,v_j\} \notin E(G)$ for any $i,j \in \{1,\dots,r\}$. An independent set is called maximal if it is not strictly included in any other independent set of G. Notice that the independent sets of G form a simplicial complex, which we will denote by $\Delta(G)$. It is easy to see that $G_{\Delta(G)} = G$ and $\Delta(G_\Delta) = \Delta$. A *vertex cover* of G is a collection of vertices $C = \{v_1,\dots,v_t\}$ such that $e \cap C \neq \emptyset$ for any $e \in E(G)$. A vertex cover is called *minimal* if no proper subset of C is again a vertex cover. The smallest cardinality of minimal vertex covers of G is called the *covering number* of G and we will denote it by $\tau(G)$.

Lemma 1.7. *Let G be a graph without isolated vertices on $[2d]$ such that $\tau(G) = d$. Suppose that any vertex of G belongs to a maximal independent set of cardinality d. Then G admits a perfect matching.*

Proof. Let $C = \{v_1,\dots,v_d\} \subseteq V(G)$ be a minimal vertex cover of cardinality d. Notice that for any $i = 1,\dots,d$ there exists a maximal independent set H, of cardinality d, such that $v_i \in H$. So there exist $k \leq d$ maximal independent sets $H_1,\dots,H_k$ of cardinality d, such that

$$C \subseteq \bigcup_{j=1}^{k} H_j.$$

Set $F = V(G) \setminus C$. By definition F is a maximal independent set of G of cardinality d. For any $j = 1,\dots,k$, set $C_j = C \cap H_j$. Notice that $|F \cap N(C_j)| = |C_j|$ for any $j = 1,\dots,k$. In fact, since H_j is a maximal independent set, it is easy to show that $F \cap N(C_j) = F \setminus H_j$, so

$$|F \cap N(C_j)| = |F \setminus H_j| = |F| - |F \cap H_j| = d - (d - |C_j|) = |C_j|.$$

For any $j = 1,\dots,k$, set $A_j = C_j \setminus (\bigcup_{p=1}^{j-1} C_p)$ and $B_j = (F \cap N(C_j)) \setminus (\bigcup_{p=1}^{j-1} (F \cap N(C_p)))$.

Claim 1. For any $j = 1,\dots,k$ we have $|A_j| = |B_j|$.

Set $\widetilde{C}_j = C_j \cap (\bigcup_{p=1}^{j-1} C_p)$. If we had $|\widetilde{C}_j| < |F \cap N(\widetilde{C}_j)|$, then $(C \setminus \widetilde{C}_j) \cup (F \cap N(\widetilde{C}_j))$ would be a vertex cover of cardinality less than d. Thus

$$|\widetilde{C}_j| \geq |F \cap N(\widetilde{C}_j)|.$$

Putting everything together we obtain

$$|B_j| = |F \cap N(C_j)| - |F \cap N(\widetilde{C}_j)| \leq |C_j| - |\widetilde{C}_j| = |A_j|.$$

But then $d = \sum_{j=1}^{k} |B_j| \leq \sum_{j=1}^{k} |A_j| = d$, from which we get the claim.

For any $j = 1,\dots,k$ let G^j denote the subgraph of G induced by $\bigcup_{p=1}^{j} (A_p \cup B_p)$.

Claim 2. For any $j = 1,\dots,k$ the graph G^j has a perfect matching.

We will prove Claim 2 by induction. Notice that G^1 is a bipartite graph with bipartition

$$C_1 \cup (F \cap N(C_1)).$$

The covering number of G^1 is $|C_1| = |F \cap N(C_1)|$. In fact, if C' were a vertex cover of G^1 of cardinality less than $|C_1|$, then $C' \cup (C \setminus C_1)$ would be a vertex cover of G of cardinality less than d, a contradiction. Therefore G^1 has a perfect matching by König's theorem ([PL86, Theorem 1.1.1]).

Assume that G^{j-1} has a perfect matching. Consider the bipartite subgraph of G induced on the vertices of $C_j \cup (F \cap N(C_j))$. As above, by König's theorem, it has a perfect matching. Moreover, such a perfect matching restricts to a perfect matching of the subgraph of G induced by $A_j \cup B_j$, since

$$F \cap N(\widetilde{C}_j) \subseteq B_j.$$

So we can extend the perfect matching of G^{j-1} to a perfect matching of G^j. $\square$

Before we state the next theorem we recall a graph theoretical notion from [CV11]. An edge e of a graph G is called *right edge* if $|C \cap e| = 1$ for any minimal vertex cover C of G. By the paper of the second author with Benedetti [BV11], $e = \{i, j\}$ is right if and only if $\forall \{i,i'\},\{j,j'\} \in E(G) \Rightarrow \{i',j'\} \in E(G)$. Finally, recall that G satisfies the *weak square condition* if every vertex of G belongs to a right edge.

Theorem 1.8. *Let $\Delta = \Delta(G)$ be a $(d-1)$-dimensional flag simplicial complex on $[2d]$ without cone points. The following are equivalent:*

1. *G has a perfect matching of right edges, $\{\{u_1, v_1\}, \dots, \{u_d, v_d\}\}$, such that $\{u_1, \dots, u_d\}$ is an independent set and if $\{u_i, v_j\}$ is an edge of G then $i \le j$.*
2. *Δ is strongly connected.*
3. *Δ is Cohen-Macaulay over any field.*
4. *G has a unique perfect matching and it is unmixed.*
5. *Δ is vertex decomposable.*

Proof. The equivalence of the first four points is known from [CV11, Theorem 4.7] for graphs that satisfy the weak square condition. So we only need to check that every vertex of G belongs to a right edge. Each of the first four properties implies that Δ is pure. In particular any vertex of G belongs to an independent set of cardinality d. So by Lemma 1.7 G has a perfect matching, say $\{e_1, \dots, e_d\} \subseteq E(G)$. Since Δ is pure of dimension $d-1$, for any minimal vertex cover $C \subseteq V(G)$ we have $|C \cap e_i| = 1$ for any $i = 1, \dots, d$. This means that G satisfies the weak square condition, so [CV11, Theorem 4.7] implies that the properties (1), (2), (3) and (4) are equivalent.

Since a vertex decomposable simplicial complex is always CM, (5) $\Rightarrow$ (3) follows. We will argue by induction on d to prove that (1) $\Rightarrow$ (5). If $d = 1$ it is trivial, since any 0-dimensional simplicial complex is vertex decomposable.

Therefore consider $d \ge 2$. Clearly v_d is a shedding vertex of Δ, and $\Delta \setminus v_d$ and $\mathrm{link}_\Delta v_d$ are flag simplicial complexes. Precisely they are $\Delta \setminus v_d = \Delta(G_1)$ and $\mathrm{link}_\Delta v_d = \Delta(G_2)$ where G_1 is the subgraph of G induced on the set of vertices $V(G) \setminus \{v_d\}$ and G_2 is the subgraph of G induced on the set of vertices $V(G) \setminus N[v_d]$. Notice that $\Delta \setminus v_d$ is a $(d-1)$-dimensional simplicial complex as well as Δ. Clearly the graph G_1^{red} obtained from G_1 after removing its (unique) isolated vertex, is a graph on $2(d-1)$ vertices such that (1) is easily seen holding true. So $\Delta(G_1^{red})$ is vertex decomposable by induction, and since $\Delta \setminus v_d$ is obtained from $\Delta(G_1^{red})$ adding some cone points, it is vertex decomposable too. We want to show that (1) holds true also for G_2^{red}. To see this, assume that u_i is not a vertex of G_2 for some $i < d$. Then, using the fact that $\{u_i, v_i\}$ is right, it is easy to see that v_i is an isolated vertex in G_2. Analogously, if v_i is not a vertex of G_2 then u_i is an isolated vertex of G_2. Hence the perfect matching of G induces a perfect matching on G_2^{red}. At this point it is easy to see that (1) holds true for G_2^{red}, so using the above argument $\mathrm{link}_\Delta v_d$ is vertex decomposable by induction. Therefore Δ is vertex decomposable. $\square$

We conclude this section with a useful remark. Let $A = S/J$ an Artinian K-algebra. We will say that A is a *quadratic Artinian K-algebra* if J is generated by quadrics, and that A is a *monomial Artinian K-algebra* if J is generated by monomials.

Remark 1.9. Let Δ be a simplicial complex on $[n]$. Construct the ideal

$$J_\Delta = I_\Delta + (x_1^2, \ldots, x_n^2) \subseteq S.$$

It is straightforward to verify that S/J_Δ is a *monomial Artinian K-algebra* such that

$$h(S/J_\Delta) = f(\Delta).$$

On the other hand, if $A = S/J$ is a monomial Artinian K-algebra such that $x_i^2 \in J$ for any $i = 1, \ldots, n$, then $J = I_\Delta + (x_1^2, \ldots, x_n^2)$ for some simplicial complex Δ on $[n]$. Once again we have

$$h(A) = f(\Delta).$$

Therefore the set of h-vectors of monomial Artinian K-algebras whose defining ideal contains the square of each variable is equal to the set of f-vectors of simplicial complexes. By the same argument, characterizing the f-vectors of flag simplicial complexes is equivalent to classifying the h-vectors of quadratic monomial Artinian K-algebras.

1.3 h-vectors of Vertex Decomposable Flag Simplicial Complexes

In this section we are going to prove Conjecture 1.2 when Δ is vertex decomposable. First of all, we want to remark that the inclusion in Conjecture 1.3 is strict. To this aim let us take a look at the next example.

Example 1.10. Consider the f-vector of the empty triangle, $(1,3,3)$. If a quadratic Artinian K-algebra with h-vector $(1,3,3)$ existed, then it would be of the kind:

$$A = K[x,y,z]/(f_1, f_2, f_3),$$

where the f_i's are degree 2 homogeneous polynomials of $K[x,y,z]$. Clearly we have the inequality $\dim_K((f_1,f_2,f_3)_3) \leq 9$, while $\dim_K(K[x,y,z]_3) = 10$. This implies $\dim_K(A_3) \geq 1$, a contradiction. Another way to obtain a contradiction is to notice that the ideal (f_1, f_2, f_3) is a complete intersection, thus the h-vector of A has to be symmetric.

Before stating the main result of this section we will prove the following algebraic lemma.

Lemma 1.11. *Let A be a standard graded, Noetherian, d-dimensional, Cohen-Macaulay K-algebra and $J \subseteq A$ a height 1 ideal generated by elements of degree 1 such that A/J is Cohen-Macaulay. If K is infinite, then for any $i \in \mathbb{N}$ we have*

$$h_i(A/J) \leq h_i(A).$$

Proof. By [BH93, Proposition 1.5.12] we can choose a degree 1 homogeneous element $x \in J$ which is A-regular. Thus for any i we have that $h_i(A/(x)) = h_i(A)$. Moreover $A/(x)$ and A/J have the same dimension. Let us extend x to a regular sequence for A of degree 1 elements, say $x, x_2, \ldots, x_d$ where $d = \dim(A)$. It turns out that $x_2, \ldots, x_d$ is a system of parameters for A/J. Because A/J is Cohen-Macaulay, $x_2, \ldots, x_d$ is a regular sequence for A/J. So there is a graded surjection

$$A/(x, x_2, \ldots, x_d) \longrightarrow A/(J + (x_2, \ldots, x_d)),$$

from which we get the desired inequality:

$$h_i(A/J) \leq h_i(A).$$

$\square$

We are ready to prove the main theorem of this section.

Theorem 1.12. *Let Δ be a vertex decomposable, flag simplicial complex. Then there exists a simplicial complex Γ such that $f(\Gamma) = h(\Delta)$.*

Proof. Suppose that Δ is d-dimensional on $[n]$. If Δ is the d-simplex, then it is enough to choose $\Gamma = \{\emptyset\}$. So we can assume that Δ is not a simplex and use induction on d and n.

Let v be a shedding vertex of Δ such that $\Delta_1 = \Delta \setminus \{v\}$ and $\Delta_2 = \mathrm{link}_\Delta v$ are vertex decomposable simplicial complexes. We may assume $v = n$, so it turns out that Δ_1 is of dimension d on $[n-1]$, whereas Δ_2 is of dimension $d-1$. For any $i = 0, \ldots, d$ we have

$$f_i(\Delta) = |\{i\text{-faces of } \Delta \text{ not containing } v\}| + |\{i\text{-faces of } \Delta \text{ containing } v\}| = f_i(\Delta_1) + f_{i-1}(\Delta_2).$$

Using (1.2) it is not difficult to show that the same formula holds at the level of h-vectors:

$$h_i(\Delta) = h_i(\Delta_1) + h_{i-1}(\Delta_2) \quad \text{for every} \quad i = 1, \ldots, (d+1).$$

Before proceeding with the induction we will prove the following:

Claim. For any i we have $h_i(\Delta_2) \leq h_i(\Delta_1)$.

By definition we have that

$$I_{\Delta_1} = (x_{i_1}x_{i_2} \; : \; \{i_1, i_2\} \notin \Delta \text{ and } v \notin \{i_1, i_2\}),$$

$$I_{\Delta_2} = (x_{i_1}x_{i_2} \; : \; \{i_1, i_2\} \notin \Delta, \; v \notin \{i_1, i_2\} \text{ and both } \{i_1, v\}, \{i_2, v\} \in \Delta).$$

Moreover $K[\Delta_1] = K[x_i : i \neq v]/I_{\Delta_1}$ and $K[\Delta_2] = K[x_i : i \neq v \text{ and } \{i, v\} \in \Delta]/I_{\Delta_2}$. Therefore

$$K[\Delta_2] = K[\Delta_1]/(x_i \; : \; \{i, v\} \notin \Delta).$$

Since Δ_1 and Δ_2 are vertex decomposable, $K[\Delta_1]$ and $K[\Delta_2]$ are Cohen-Macaulay. So we are in the situation of Lemma 1.11. Hence

$$h_i(\Delta_2) = h_i(K[\Delta_2]) \leq h_i(K[\Delta_1]) = h_i(\Delta_1),$$

and the claim follows.

By induction there exist two simplicial complexes, Γ_1 and Γ_2, such that $f(\Gamma_1) = h(\Delta_1)$ and $f(\Gamma_2) = h(\Delta_2)$. We want to construct the desired simplicial complex Γ starting from them. By the Kruskal-Katona theorem (for instance see [Sta96, Theorem 2.1]) we can assume that both Γ_1 and Γ_2 are rev-lex complexes. Therefore, since by the claim $f_i(\Gamma_2) \leq f_i(\Gamma_1)$, actually Γ_2 is a subcomplex of Γ_1. So it makes sense to construct the simplicial complex

$$\Gamma = \Gamma_1 *_{\Gamma_2} \{u\},$$

where u is a new vertex. It is straightforward to check that

$$f_i(\Gamma) = f_i(\Gamma_1) + f_{i-1}(\Gamma_2) = h_i(\Delta_1) + h_{i-1}(\Delta_2) = h_i(\Delta).$$

□

The reader might think at this point that h-vectors of quadratic Artinian K-algebras are h-vectors of Cohen-Macaulay flag simplicial complexes. The following example will show that this is not the case.

Example 1.13. Let $h = (1,4,5,1)$ be a sequence of integers (notice that h is the f-vector of a balanced simplicial complex). In the paper of Roos [Roo95] we found the quadratic Artinian K- algebra $A = K[x_1,x_2,x_3,x_4]/I$, where I is the ideal

$$I = (x_1x_2 + x_3^2,\ x_1x_4,\ x_1^2 + x_3^2 + x_4^2,\ x_2^2,\ x_2x_3 + x_3x_4),$$

with $h(A) = (1,4,5,1)$.

If there existed a Cohen-Macaulay flag simplicial complex Δ with $h(\Delta) = h$, then there would exist an Artinian Koszul K-algebra B with $h(B) = h$. In fact, if $\theta = \theta_1,\dots,\theta_d$ is a system of parameters for $K[\Delta]$, it is enough to take $B = K[\Delta]/(\theta)$. This follows from the theorem of Fröberg [Frö75] and the result of Backelin and Fröberg [FB85, Theorem 4]. This implies that

$$\frac{1}{1-4z+5z^2-z^3} = \sum_{i\geq 0} \dim_K(\mathrm{Tor}_i^B(K,K))z^i,$$

(for instance see [FB85, p. 87]). Computing the coefficients on the left hand side we obtain $\dim_K(\mathrm{Tor}_9^B(K,K)) = -174$, obviously a contradiction.

In light of Examples 1.10 and 1.13, we conclude this section discussing whether the simplicial complex Γ of Theorem 1.12 could be chosen with some extra properties. First of all we have a remark.

Remark 1.14. It is easy to see that the following holds true: A simplicial complex Δ on $[n]$ is flag if and only if $\Delta = \{\emptyset\}$ or there exists a vertex v of Δ such that $\Delta \setminus v$ is flag and $\mathrm{link}_\Delta v = \Delta_W$ for some $W \subseteq [n]$ (in particular $\mathrm{link}_\Delta v$ is flag).

Let $\Gamma_2 \subseteq \Gamma_1$ be two simplicial complexes, with $\Gamma_2 = (\Gamma_1)_W$ induced by a subset of vertices $W \subseteq [n]$. Then

$$\frac{K[x_i\ :\ i \in W]}{J_{\Gamma_2}} \cong \frac{S}{J_{\Gamma_1} + (x_j\ :\ j \notin W)},$$

where J_{Γ_1} and J_{Γ_2} are the ideals defined in Remark 1.9. Therefore

$$f(\Gamma_2) = h\left(\frac{S}{J_{\Gamma_1} + (x_j\ :\ j \notin W)}\right).$$

Thus we are in the situation in which there exists a monomial Artinian K-algebra A and an ideal $I \subseteq A$ generated by variables such that

$$f(\Gamma_1) = h(A) \ \text{ and } \ f(\Gamma_2) = h(A/I).$$

Moreover, if A is quadratic, then by Remark 1.14 the complex $\Gamma_1 *_{\Gamma_2} \{v\}$ is flag. In the proof of Theorem 1.12 we have that $K[\Delta_2] = K[\Delta_1]/I$, where I is an ideal generated by variables. Since $K[\Delta_1]$ and $K[\Delta_2]$ are both Cohen-Macaulay, going modulo a generic regular sequence, we could restrict to the Artinian case. The problem is that the quadratic Artinian reduction A of $K[\Delta_1]$ is not necessarily monomial. This is why, even assuming that Γ_1 and Γ_2 are flag, we could not conclude that $\Gamma_1 *_{\Gamma_2} \{v\}$ is also flag. In other words, if in the proof of Theorem 1.12 we assume by induction that Γ_1 and Γ_2 are flag, we do not see how to construct a flag simplicial complex Γ, because Γ_2 might not be a subcomplex of Γ_1 induced by some set of vertices. However, the behavior of the f-vector of Γ_2 is similar to that of the f-vector of an induced subcomplex of Γ_1. For instance, if $f_0(\Gamma_2) = f_0(\Gamma_1)$, it follows by the proof of Theorem 1.12 that $f_i(\Gamma_2) = f_i(\Gamma_1)$ for any i. In the next section we present more precise results in this direction under the assumption that Δ is also balanced (see Definition 1.16 and Theorem 1.17).

1.4 Balanced, Vertex Decomposable, Flag Complexes

The reason for which we study balanced, vertex decomposable, flag simplicial complexes is given by Proposition 1.15. We conjecture that the converse of this proposition is true. In Theorem 1.17 we will prove a weaker version of the equality in Conjecture 1.4. Finally we will prove that the conjecture holds for $(d-1)$-dimensional, balanced, flag, vertex decomposable simplicial complexes on $[2d]$, without cone points.

Proposition 1.15. *Let Γ be a flag simplicial complex. Then there exists a balanced, flag, vertex decomposable simplicial complex Δ such that $h(\Delta) = f(\Gamma)$.*

Proof. Set $n = \dim\Gamma + 1$ and as in Remark 1.9 consider the ideal

$$J_\Gamma = I_\Gamma + (x_1^2, \dots, x_n^2) \subseteq S.$$

Consider the polarization of J_Γ:

$$J_\Gamma^{pol} = I_\Gamma + (x_1y_1, \dots, x_ny_n) \subseteq P = K[x_1, \dots, x_n, y_1, \dots, y_n].$$

Since polarization is a particular distraction, it preserves the height and the graded Betti numbers (see the paper of Bigatti, Conca and Robbiano [BCR05]). Particularly

$$h(P/J_\Gamma^{pol}) = h(S/J_\Gamma) = f(\Gamma),$$

where the last equality follows from Remark 1.9. The simplicial complex Δ associated to J_Γ^{pol} is flag. More precisely $\Delta = \Delta(G)$, where G is the graph on $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ whose edges are $\{u_i, v_i\}$ for $i = 1, \dots, n$ and $\{v_i, v_j\}$ such that $\{i, j\}$ is not a face of Γ. Then, by Theorem 1.8, Δ is vertex decomposable. Moreover Δ is easily seen to be balanced setting $\operatorname{col}(u_i) = \operatorname{col}(v_i) = i$ for any $i = 1, \dots, n$. □

We conjecture that the converse of Proposition 1.15 is also true:

Conjecture 1.4. The following equality holds:

$$\left\{ \begin{array}{c} h\text{-vectors of vertex decomposable} \\ \text{balanced and flag simplicial complexes} \end{array} \right\} = \left\{ \begin{array}{c} f\text{-vectors of} \\ \text{flag simplicial complexes} \end{array} \right\}.$$

Next, we are going to prove a result in support of the above conjecture. This next theorem will be a version of Conjecture 1.4, in which we will prove that the hard inclusion $(\subseteq)$ holds with weakened conditions on the right hand side of the equality. In Theorem 1.19 and in the two lemmas of Section 6 we will prove that equality holds when adding some stronger conditions on the left hand side. First we need to define a new class of simplicial complexes, suggested by Remark 1.14.

Definition 1.16. Let Δ be a simplicial complex on $[n]$. Then Δ is *quasi-flag* if and only if $n = 0$ or there exists a vertex v of Δ such that

1. $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex,
2. $\operatorname{link}_\Delta v = \Delta_W$ for some $W \subseteq [n]$ and the f-vector of $\operatorname{link}_\Delta v$ is that of a quasi-flag simplicial complex.

Theorem 1.17. *Let Δ be a balanced, vertex decomposable, flag simplicial complex on $[n]$. Then there exists a quasi-flag simplicial complex Γ such that $f(\Gamma) = h(\Delta)$.*

Proof. If Δ is a simplex then we can choose $\Gamma = \{\emptyset\}$. If Δ is not a simplex we can choose a shedding vertex v such that $\Delta_1 = \Delta \setminus \{v\}$ and $\Delta_2 = \operatorname{link}_\Delta v$ are vertex decomposable, flag simplicial complexes. As in the proof of Theorem 1.12, we have

$$K[\Delta_2] = K[\Delta_1]/(x_i : i \in W),$$

where $W = \{i \ : \ \{i,v\} \notin \Delta\}$. Let $\mathrm{col} : [n] \to [d]$ be a d-coloring of Δ, where $\dim\Delta = d-1$. For any $j = 1,\dots,d$ we set $V_j = \{i \in [n] : \mathrm{col}(i) = j\}$. We can assume that $v = n \in V_d$. Notice that the coloring on Δ induces a d-coloring on Δ_1 and a $(d-1)$-coloring on Δ_2, so that Δ_1 and Δ_2 are both balanced. So we have the following system of parameters for $K[\Delta_1]$:

$$\theta_i = \sum_{\substack{j\in V_i\\ j\neq n}} x_j, \quad i = 1,\dots,d.$$

It turns out that θ_i, where $i = 1,\dots,(d-1)$, provides also a system of parameters for $K[\Delta_2]$. Note that θ_d is zero in $K[\Delta_2]$. We may assume that $i \in V_i$ for any $i = 1,\dots,d$ and that $i \notin W$ for any $i = 1,\dots,(d-1)$. Consider the ideal of $K[x_{d+1},\dots,x_{n-1}]$:

$$I = (x_i x_j,\ x_i(\sum_{\substack{k\in V_h\\ k\neq h}} x_k) \ : \ d+1 \le i,j \le n-1,\ h = 1,\dots,d, \text{ and } \{i,j\},\{i,h\} \notin \Delta_1).$$

Going modulo the θ_i's, it is easy to see that

$$\frac{K[\Delta_1]}{(\theta_1,\dots,\theta_d)} \cong \frac{K[x_{d+1},\dots,x_{n-1}]}{I} = A.$$

Moreover

$$\frac{K[\Delta_2]}{(\theta_1,\dots,\theta_d)} = \frac{K[\Delta_2]}{(\theta_1,\dots,\theta_{d-1})} \cong \frac{A}{(x_i : i \in W)} = B.$$

Since Δ_1 and Δ_2 are both Cohen-Macaulay,

$$h(\Delta_1) = h(A) \text{ and } h(\Delta_2) = h(B).$$

Notice that $x_i^2 \in I$ for any $i = d+1,\dots,n-1$. So for any term-order $\prec$ in $K[x_{d+1},\dots,x_{n-1}]$ there exists a simplicial complex Γ_1 such that

$$\mathrm{LT}_\prec(I) = J_{\Gamma_1}.$$

If we consider as $\prec$ a deg-rev-lex term-order such that the smallest variables are the x_i's with $i \in W$, we have

$$\mathrm{LT}_\prec(I + (x_i : i \in W)) = J_{\Gamma_1} + (x_i : i \in W),$$

see for instance the book of Eisenbud [Eis95, Proposition 15.12]. By the above discussion we have $f(\Gamma_1) = h(\Delta_1)$ and $f((\Gamma_1)_W) = h(\Delta_2)$. By induction Γ_1 and $\Gamma_2 = (\Gamma_1)_W$ have both the f-vector of quasi-flag simplicial complexes. So

$$\Gamma = \Gamma_1 *_{\Gamma_2} \{u\},$$

where u is a new vertex, is a quasi-flag simplicial complex. As in the proof of Theorem 1.12 we have $f(\Gamma) = h(\Delta)$, thus we conclude (notice that since Γ_2 is already contained in Γ_1 this time we need not use the Kruskal-Katona theorem). □

Remark 1.18. By Remark 1.14 the flag simplicial complexes are quasi-flag. However notice that not all f-vectors of simplicial complexes are f-vectors of quai-flag simplicial complexes. For instance take $f = (1,n,\binom{n}{2})$. The up to isomorphism unique complex with such an f-vector is the complete graph K_n. However the link of any vertex of K_n is not a subcomplex of K_n induced by a set of vertices. Thus K_n is not quasi-flag.

Another example is also provided by the f-vector $(1,4,5,1)$. The up to isomorphism unique complex Δ which has such an f-vector is the one whose set of facets is

$$\mathscr{F}(\Delta) = \{\{1,2\},\{1,3\},\{2,3,4\}\}.$$

The unique vertex v such that $\text{link}_\Delta v$ is an induced subcomplex of Δ is 4. However the f-vector of $\Delta\setminus 4$ is $(1,3,3)$, which is not the f-vector of a quasi-flag simplicial complex by the above considerations. Therefore Δ is not quasi-flag.

We are not aware of any example of quasi-flag simplicial complex whose f-vector is not flag.

Some evidence in favor of Conjecture 1.4 is also provided by the following theorem.

Theorem 1.19. *The following equality holds true:*

$$\left\{\begin{array}{c} h(\Delta)\ :\ \Delta \textit{ is a } (d-1)\textit{-dimensional} \\ \textit{balanced, flag, vertex decomposable} \\ \textit{simplicial complex on } [2d]\textit{, without cone points} \end{array}\right\} = \left\{\begin{array}{c} f(\Gamma)\ :\ \Gamma \textit{ is a flag simplicial} \\ \textit{complex on } [d] \end{array}\right\}.$$

Proof. It is easy to see that the proof of Proposition 1.15 yields that the set on the right hand side is a subset of the one on the left. For the other inclusion let $\{\{u_1,v_1\},\dots,\{u_d,v_d\}\}$ be the perfect matching of $G=G_\Delta$ described in (1) of Theorem 1.8. Also denote by

$$P=K[x_1,\dots,x_d,y_1,\dots,y_d]$$

the polynomial ring containing I_Δ, where x_i is the variable associated to u_i and y_i the one associated to v_i. Notice that Δ is balanced, so by [Sta96, Proposition 4.3] the set

$$\{\theta_i=x_i+y_i\ :\ i=1,\dots,d\}$$

is a system of parameters for $K[\Delta]$. Thus we have the graded isomorphism

$$\frac{K[\Delta]}{(\theta_1,\dots,\theta_d)} \longrightarrow \frac{K[z_1,\dots,z_d]}{(z_i^2,\ z_hz_k\ :\ i=1,\dots,d,\ \{v_h,v_k\} \text{ or } \{u_h,v_k\} \text{ is an edge})}$$

which maps y_i to z_i and x_i to $-z_i$. Since Δ is Cohen-Macaulay over K we have

$$h\left(\frac{K[\Delta]}{(\theta_1,\dots,\theta_d)}\right)=h(\Delta).$$

So, by the above graded isomorphism, we have

$$h(\Delta)=h\left(\frac{K[z_1,\dots,z_d]}{(z_i^2,z_hz_k\ :\ i=1,\dots,d,\ \{v_h,v_k\} \text{ or } \{u_h,v_k\} \text{ is an edge})}\right).$$

Using Remark 1.9 we obtain the desired conclusion. □

1.5 h-vectors of Cohen-Macaulay Flag Complexes

In this section we are going to discuss a natural generalization of Conjecture 1.4, namely:

Conjecture 1.5 The following equality holds true:

$$\left\{\begin{array}{c} h\text{-vectors of Cohen-Macaulay,} \\ \text{flag simplicial complexes} \end{array}\right\} = \left\{\begin{array}{c} f\text{-vectors of} \\ \text{flag simplicial complexes} \end{array}\right\}.$$

One reason for the above conjecture is given by the following remark.

Remark 1.20. Conjecture 1.5 implies Kalai's Conjecture 1.1.

Proof. If Δ is a d-dimensional, CM, flag simplicial complex then, if Conjecture 1.5 were true, there would exist a s-dimensional, flag simplicial complex Γ' with $f(\Gamma') = h(\Delta)$, where $s \leq d$. By [Fro08] there exists also a s-dimensional, balanced simplicial complex Γ'', with $f(\Gamma'') = f(\Gamma')$. By [BFS87, Theorem 1], there exists a s-dimensional CM, balanced simplicial complex Γ''' with $h(\Gamma''') = f(\Gamma'')$. Thus $h(\Gamma''') = h(\Delta)$. Adding $d-s$ cone points to Γ''' we get a d-dimensional simplicial complex Γ that is still CM and balanced. Furthermore $h(\Gamma) = h(\Gamma''') = h(\Delta)$. Since $\dim\Gamma = \dim\Delta$, we get $f(\Gamma) = f(\Delta)$. □

The set on the right hand side of the equality in Conjecture 1.5 is contained in the one on the left by Proposition 1.15. So the hard part of the conjecture is to prove that for any Cohen-Macaulay, flag simplicial complex Δ there exists a flag simplicial complex Γ with $f(\Gamma) = h(\Delta)$.

First of all, notice that as an easy consequence of a more general theorem of Conca, Trung and Valla ([CTV01]), we obtain the validity of Conjecture 1.5 when the h-vector of Δ is "short enough". Here is the precise statement:

Proposition 1.21. *Let Δ be a Cohen-Macaulay, flag simplicial complex with h-vector $(1,n,m)$. Then there exists a flag simplicial complex Γ with $f(\Gamma) = h(\Delta)$.*

Proof. The K-algebra $K[\Delta]$ is Koszul by [Frö75]. Taking a regular sequence of linear forms $\theta_1,\ldots,\theta_d$, where $d-1 = \dim\Delta$, we get that $A = K[\Delta]/(\theta_1,\ldots,\theta_d)$ is a Koszul Artinian K-algebra by [FB85, Theorem 4]. Since $h(A) = h(\Delta) = (1,n,m)$, we have $m \leq n^2/4$ by [CTV01, Theorem 3.1]. Under this condition it is easy to construct a bipartite graph with n vertices and m edges. Such a bipartite graph can also be seen as a 1-dimensional, flag simplicial complex with f-vector $(1,n,m)$. □

In particular the above proposition implies that Conjecture 1.5 is true when the dimension of Δ is 1. The following theorem brings more evidence in favor of Conjecture 1.5.

Theorem 1.22. *The following equality holds true:*

$$\left\{ \begin{array}{c} h(\Delta)\ :\ \Delta \textit{ is a } (d-1)\textit{-dimensional, CM, flag} \\ \textit{simplicial complex on } [2d], \textit{ without cone points} \end{array} \right\} = \left\{ \begin{array}{c} f(\Gamma)\ :\ \Gamma \textit{ is a flag} \\ \textit{simplicial complex on } [d] \end{array} \right\}.$$

Proof. If Δ is a $(d-1)$-dimensional, CM, flag simplicial complex on $[2d]$ without cone points then Δ is vertex decomposable and balanced by Theorem 1.8. Thus Theorem 1.19 yields the conclusion. □

Suppose Δ is a CM, flag simplicial complex, without cone points and G_Δ is bipartite with partition of the vertex set $A \cup B$. As both A and B are minimal vertex covers, by the purity of Δ we have $|A| = |B|$. This implies the following corollary of the above theorem.

Corollary 1.23. *The following inclusion holds true:*

$$\left\{ \begin{array}{c} h(\Delta)\ :\ \Delta \textit{ CM, flag simplicial complex,} \\ \textit{with } G_\Delta \textit{ bipartite} \end{array} \right\} \subseteq \left\{ \begin{array}{c} f(\Gamma)\ :\ \Gamma \textit{ is a flag} \\ \textit{simplicial complex} \end{array} \right\}.$$

We conclude this section with the following remark.

Remark 1.24. If $\Delta = \Delta(G)$ is a flag, CM simplicial complex, then $\Delta(G)$ is pure. In particular any vertex of G belongs to an independent set of cardinality $\dim\Delta + 1$. Therefore, if Δ is a $(d-1)$-dimensional, flag, CM simplicial complex on $[n]$ without cone points, then $n \geq 2d$ by the result of Gitler and Valencia [GV05, Theorem 2.1].

In the spirit of the previous remark, Theorem 1.22 can be seen as the first step towards proving Conjecture 1.5.

1.6 Further Results and Examples

In this last section we will present two rather technical properties of flag simplicial complexes. We will show that the first property (which we call *balanced cone-face property* - (1.3)) implies Cohen-Macaulayness (Proposition 1.26) and that the h-vector of such a simplicial complex is the f-vector of a flag simplicial complex (Lemma 1.25). For simplicial complexes with the second property (1.4) we will construct a new complex, with the same h-vector, which will satisfy the hypothesis of Theorem 1.19. We will also present examples of simplicial complexes with and without these properties.

Lemma 1.25. *Suppose Δ is a balanced, flag simplicial complex of dimension $d-1$ and let $F_0 = \{a_1,\dots,a_d\}$ be a facet of Δ with the property that:*

$$\forall\, v \in V(\Delta), \exists\ 1 \le i \le d \text{ such that } (F_0 \setminus \{a_i\}) \cup \{v\} \text{ is a facet of } \Delta. \tag{1.3}$$

Then we have $h(\Delta) = f(\Delta \setminus F_0)$.

Proof. For simplicity, we will denote by $(h_0,\dots,h_r)$ and $(f_{-1},\dots,f_{d-1})$ the h-vector, respectively the f-vector, of Δ. The f-vector of $\Delta \setminus F_0$ will be denoted by $(f'_{-1},\dots,f'_s)$. We will prove the lemma by induction. First of all, it is clear that $h_0 = f'_{-1} = 1$ and $h_1 = f'_0 = n-d$. Suppose that we already have $h_j = f'_{j-1}$ for all $j \le i$.

The following observation is the key of the proof. As Δ is flag, for any $d \ge i > j$, if $\{v_1,\dots,v_{i-j}\}$ and $\{w_1,\dots,w_j\}$ are two faces of Δ such that $\{v_k,w_l\} \in \Delta$ for any k and l, then $\{v_1,\dots,v_{i-j},w_1,\dots,w_j\} \in \Delta$.

Every i-dimensional face $F \in \Delta$ is a disjoint union: $(F \setminus F_0) \cup (F \cap F_0)$. We will count the i-faces of Δ with $|F \setminus F_0| = j$. As Δ is balanced, the number of vertices of F_0 that are colored different from all the vertices of $F \setminus F_0$ is exactly $d-j$. Choose an $(i-j)$-face $G \subset F_0$ supported on these vertices. It is easy to notice that, by our hypothesis and the above observation, $G \cup (F \setminus F_0) \in \Delta$. As there are $\binom{d-j}{i+1-j}$ different ways to choose G, we get that the number of i-faces of Δ with $|F \setminus F_0| = j$ is

$$f'_j \cdot \binom{d-j}{i+1-j}.$$

Decomposing the set of i-faces of Δ according to the cardinality of $F \setminus F_0$, we obtain

$$f_i = \sum_{j=0}^{i+1} \binom{d-j}{i+1-j} f'_{j-1}.$$

As the the f-vector of Δ can be computed from the h-vector of Δ by the formula:

$$f_i = \sum_{j=0}^{i+1} \binom{d-j}{i+1-j} h_j,$$

we obtain by the inductive hypothesis that $h_{i+1} = f'_i$. □

Notice we did not request in Lemma 1.25 that Δ is Cohen-Macaulay. This is because, under the hypothesis of the above lemma, Δ is always CM.

Proposition 1.26. *If Δ is a simplicial complex with the same properties as in the statement of Lemma 1.25 then Δ is Cohen-Macaulay.*

Proof. As we have seen in the preliminaries section, a balanced simplicial complex has a canonical linear system of parameters, namely $\{\theta_i = \sum_{\mathrm{col}(j)=i} x_j \ :\ i = 1,\dots,d\}$. It is easy to see that the property (1.3) is equivalent to

$$x_{a_i}x_v \in \mathrm{Gens}(I_\Delta) \Rightarrow \mathrm{col}(a_i) = \mathrm{col}(v), \quad \forall\, i = 1,\dots,d.$$

Notice also that if V_i is the set of vertices of color i, then $x_v x_w \in \mathrm{Gens}(I_\Delta)$ for any $v, w \in V_i$ and $\forall\ i = 1,\dots,d$. If we denote by $W = [n] \setminus F_0$, considering the above observation, it is not difficult to see that

$$\frac{K[\Delta]}{(\theta_1,\dots,\theta_d)} \simeq \frac{K[x_i\ :\ i \in W]}{(x_i^2, x_i x_j\ :\ \{i,j\}\ \text{ minimal nonface of } \Delta_W)}.$$

The isomorphism is obtained by sending $x_i \mapsto x_i$ if $i \notin F_0$ and

$$x_i \mapsto -\sum_{\mathrm{col}(j)=\mathrm{col}(i)} x_j \quad \text{if } i \in F_0.$$

By Remark 1.9 we obtain that

$$h\left(\frac{K[\Delta]}{(\theta_1,\dots,\theta_d)}\right) = f(\Delta_W).$$

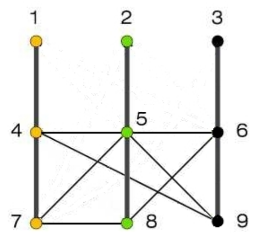
As $\Delta_W = \Delta \setminus F_0$, by Lemma 1.25 we also have that

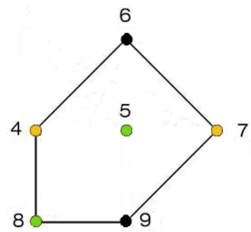
$$h\left(\frac{K[\Delta]}{(\theta_1,\dots,\theta_d)}\right) = h(K[\Delta]),$$

which by [Sta96, Lemma 2.6] implies that Δ is Cohen-Macaulay. □

Let us present now an example of a simplicial complex satisfying (1.3). First let us establish a graphical convention. Throughout this section, the thicker vertical lines in the pictures of graphs represent the fact that the subgraphs induced by the vertices in one column are complete (e.g. in the next figure, the subgraphs induced by each of the vertex sets $\{1,4,7\}$, $\{2,5,8\}$ and $\{3,6,9\}$ are complete).

Example 1.27. The independence complex Δ of the graph on the left is an example of simplicial complex satisfying the hypothesis of Lemma 1.25. It is easy to see that $F_0 = \{1,2,3\}$ satisfies property (1.3). One can check that Δ is pure, of dimension 2 and that $h(\Delta) = (1,6,5)$.





On the right hand side you can see a picture of the 1-dimensional simplicial complex $\Delta \setminus F_0$. One can notice that $\Delta \setminus F_0$ is no longer pure, nor balanced. The only property inherited from Δ, apart from flagness, is the 3-colorability.

In the remaining part of this section we will show that under certain conditions, a flag, balanced, CM simplicial complex may be "modified" such that it satisfies the hypothesis of Theorem 1.19. Let Δ be a CM, flag balanced $(d-1)$-dimensional simplicial complex on $[n]$. As we have seen, if $n = 2d$ and Δ has no cone points, we know that there exists a flag simplicial complex Γ such that $h(\Delta) = f(\Gamma)$. Suppose now that $n > 2d$. Adding $n - 2d$ cone points to Δ we still obtain a CM, flag and balanced simplicial complex, and the dimension of this new complex is one less than half the number of vertices.

In order to simplify notation, suppose that Δ is already a $(d-1)$-dimensional, CM, flag, balanced simplicial complex on $[2d]$, with r cone points $z_1,\dots,z_r$. Let $[2d] = \cup_{i=1}^d V_i$ be the partition of the vertices corresponding to the coloring. Without loss of generality we may also assume that $V_{d+1-j} = \{z_j\}$ for $j = 1,\dots,r$. We will denote by $G = G_\Delta$ the graph of minimal nonfaces of Δ. Suppose that Δ has the property that in G for every $i \in 1,\dots,d$ with $|V_i| > 2$ we have

$$\exists\ y_{i,1}, y_{i,2} \in V_i \text{ such that } \forall\ x \in V_i \text{ we have } N[y_{i,1}] \subseteq N[x] \text{ or } N[y_{i,2}] \subseteq N[x]. \tag{1.4}$$

Denote by $\overline{V_i} = V_i \setminus \{y_{i,1}, y_{i,2}\}$ and by $\overline{V} = \cup \overline{V_i}$ the union over all $i = 1, \ldots, (d-r)$ with $|V_i| > 2$. Notice that the cardinality of $\overline{V}$ satisfies $|\overline{V}| = r$, where r is the number of cone points. For any $x \in \overline{V}$ denote by y_x the element of property (1.4). If for both $y_{i,1}$ and $y_{i,2}$ the inclusion of the closed neighborhoods is satisfied, then randomly choose one of them as y_x. We will denote by $\mathrm{Gens}(I_\Delta)$ the set of minimal generators of the Stanley-Reisner ideal of Δ. If no confusion may arise, we will denote the variables with the same letters as the vertices of Δ. With the above notation we have:

Lemma 1.28. *The flag simplicial complex $\widetilde{\Delta}$ corresponding to the square-free monomial ideal generated by*

$$\left(\mathrm{Gens}(I_\Delta) \setminus \left(\bigcup_{x \in \overline{V}} \{xy_x\}\right)\right) \cup \left(\bigcup_{x \in \overline{V}} \{xz_j\}\right) \tag{1.5}$$

is balanced, Cohen-Macaulay and has the same f-vector as Δ.

Proof. It is easy to see that it will be enough to prove the lemma for $r = 1$. We call a "step" the deletion of xy_x from $\mathrm{Gens}(I_\Delta)$ together with the adding of xz_j to $\mathrm{Gens}(I_\Delta)$ for some $x \in \overline{V}$. Notice that after "taking a step" property (1.4) still holds in the new complex. To prove the lemma we have to show that at each step the f-vector does not change and that properties 1. and 2. below hold. It is clear that each step reduces r, the number of cone points, by one. We will not need to prove Cohen-Macaulayness at each step, as it follows from properties 1. and 2. when there are no more cone points.

Suppose $r = 1$ and that i is the color for which $|V_i| > 2$. Let $x, y \in V_i$ be two vertices with $N[y] \subseteq N[x]$ and let z be a cone point.

We will first prove that $f(\widetilde{\Delta}) = f(\Delta)$. As z is a cone point for Δ, it will also be a cone point for the simplicial complex $\mathrm{link}_\Delta x$. We will denote by $L_{xz} = \mathrm{link}_\Delta\{x, z\}$. By definition $V(L_{xz}) \cap N[x] = \emptyset$, so property (1.4) implies that $V(L_{xz}) \cap N[y] = \emptyset$ as well. This ensures that deleting the generator xy we obtain the new faces $\{F \cup \{x, y\} \ : \ F \in L_{xz}\} = \widetilde{\Delta} \setminus \Delta$. On the other hand, adding xz as a generator we delete exactly the faces $\{F \cup \{x, z\} \ : \ F \in L_{xz}\}$. This means we have for every $i \in \{-1, \ldots, d-1\}$:

$$f_i(\widetilde{\Delta}) = f_i(\Delta) - f_{i-2}(L_{xz}) + f_{i-2}(L_{xz}),$$

where $f_j = 0$ for $j < -1$.

Notice that $\widetilde{\Delta}$ is still balanced. The only vertex that changes color is x, which will be colored with the same color as z. We will write $\cup_{i=1}^{d} \widetilde{V}_i$ for the partition of the vertices induced by the coloring. In order to prove that $\widetilde{\Delta}$ remains CM we will prove that

1. $\widetilde{\Delta}$ is pure.
2. $\widetilde{\Delta}_S$ is a connected, 1-dimensional complex for any subset of vertices $S = \widetilde{V}_i \cup \widetilde{V}_j$ with $1 \le i < j \le d$.

Notice that (also for $r > 1$) $\widetilde{\Delta}$ is a $(d-1)$-dimensional simplicial complex on $[2d]$, without cone points. It is easy to check that conditions 1. and 2. above imply the first point of Theorem 1.8 and thus imply Cohen-Macaulayness.

To prove 1. we only have to check that the facets of the form $\{x, y\} \cup F$ with $F \in L_{xz}$ are of dimension $d-1$. But the maximal faces under inclusion in L_{xz} are all of cardinality $d-2$ by the purity of Δ, so $\widetilde{\Delta}$ is also pure.

To prove 2. we have to check three cases. Fix $S = \widetilde{V}_i \cup \widetilde{V}_j$ with $1 \le i < j \le d$.

Case 1. $S \cap \{x, y, z\} = \emptyset$. In this case $\widetilde{\Delta}_S = \Delta_S$, so by [Sta96, Theorem 4.5] it is CM, thus connected.

Case 2. $S \cap \{x, y, z\} = \{y\}$. The inclusion $N[y] \subseteq N[x]$ is equivalent to

$$\{v, x\} \in \Delta \Rightarrow \{v, y\} \in \Delta.$$

Let v, w be two vertices in $\widetilde{\Delta}_S$. Again by [Sta96, Theorem 4.5] in $\Delta_{S \cup \{x\}}$ there exists a path connecting them: $v = v_1, v_2, \ldots, v_t = w$. Suppose $v_k = x$ for some k. By the above observation $\{v_{k-1}, y\}, \{y, v_{k+1}\} \in \widetilde{\Delta}_S$, so we can modify the path to $v_1, \ldots, v_{k-1}, y, v_{k+1}, \ldots, v_t$. Hence $\widetilde{\Delta}_S$ is also connected.

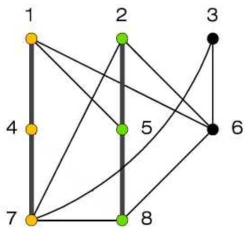
Case 3. $S \cap \{x,y,z\} \supseteq \{x,z\}$. Suppose $z \in \widetilde{V}_j$. As z is a cone point in Δ, it is connected to all vertices of $\widetilde{V}_i$. If $y \in \widetilde{V}_i$ then it is enough to notice that $\{x,y\} \in \widetilde{\Delta}_S$. Otherwise, as $\widetilde{\Delta}$ is pure and balanced, there exists at least one vertex $v \in \widetilde{V}_i$ such that $\{x,v\}$ is an edge. □

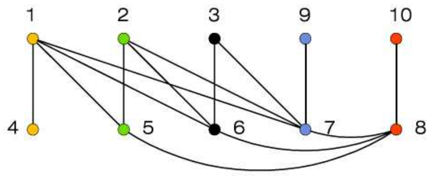
Using the above lemma together with Theorem 1.19 we obtain the following corollary.

Corollary 1.29. *If Δ is a Cohen-Macaulay, flag, balanced simplicial complex satisfying property* (1.4)*, there exists a flag simplicial complex Γ such that $h(\Delta) = f(\Gamma)$.*

In the next example we will see how Lemma 1.5 works.

Example 1.30. Let Δ' be the flag, balanced simplicial complex corresponding to the graph on $\{1,\dots,8\}$ represented on the left hand side. Consider Δ to be the independence complex of the whole graph G on $\{1,\dots,10\}$. Notice that Δ is obtained from Δ' by adding the cone points 9 and 10. It is not difficult to check that Δ is CM, (actually vertex decomposable).

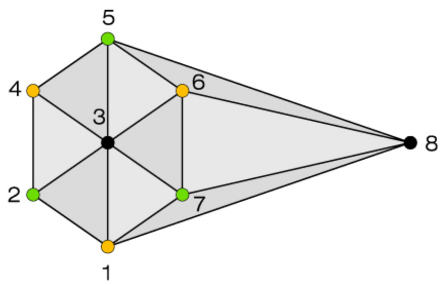


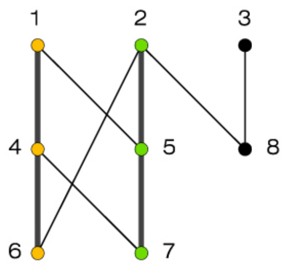


Now we construct the simplicial complex $\widetilde{\Delta}$ as the independence complex of the graph $\widetilde{G}$ depicted on the right hand side. If we set $V_1 = \{1,4,7\}$ and $V_2 = \{2,5,8\}$, using the notation of Lemma 1.5 we have $\overline{V} = \overline{V_1} \cup \overline{V_2} = \{7\} \cup \{8\}$. As $V_1 = N[4] \subseteq N[7] = V_1 \cup \{2,3,8\}$ and $V_2 \cup \{6,7\} = N[2] \subseteq N[8] = V_2 \cup \{6,7\}$ we may choose $y_7 = 4$ and $y_8 = 2$. Deleting the edges $\{4,7\}$ and $\{2,8\}$ and adding the edges $\{7,9\}$ and $\{8,10\}$ we find ourselves in the hypothesis of Lemma 1.5, so $\widetilde{\Delta}$ is flag, balanced, CM and $f(\widetilde{\Delta}) = f(\Delta)$.

Unfortunately, property (1.4) is not satisfied in general. The following simplicial complex turned up in several contexts as a counter-example to the strategy we were trying to use in order to prove Conjecture 1.4.

Example 1.31. Let Δ be the 2-dimensional simplicial complex on $\{1,\dots,8\}$ represented below on the left hand side. The picture on the right hand side represents the graph $G = G_\Delta$ of minimal nonfaces.





Notice that Δ is balanced and it is also easy to check that it is vertex decomposable. One vertex decomposition is obtained by removing in order the vertices $8,\ 7,\ 6,\ 5,\ 4$. Let $V_1 = \{1,4,6\}$, $V_2 = \{2,5,7\}$ and $V_3 = \{3,8\}$ be the disjoint sets of vertices of the same color. Notice that these sets are uniquely determined, i.e. there is a unique 3-coloring modulo a permutation of the colors. From G we can easily read that $N[1] = V_1 \cup \{5\}$, $N[4] = V_1 \cup \{7\}$ and $N[6] = V_1 \cup \{2\}$, so Δ does not satisfy property (1.4). It is also easy to check that Δ does not satisfy the conditions of Lemma 1.25. However, $h(\Delta) = (1,5,3)$ is clearly the f-vector of a flag simplicial complex.

We would also like to notice that $\mathrm{link}_\Delta 8$ is vertex decomposable, but its vertex decomposition cannot be induced by the vertex decomposition of Δ, because 7 is not a shedding vertex for $\mathrm{link}_\Delta 8$. Notice that for $\Delta \setminus 8$ both the lexicographic and the reversed lexicographic order on $\mathscr{F}(\Delta \setminus 8)$ are shelling orders. However, this is no longer true for $(\Delta \setminus 8)_{\{1,7,6,5\}}$.

The above observations also underline the fact that even if vertex decomposability strongly encourages proofs by induction, in the case of Conjecture 1.4 this strategy works only in the presence of extra assumptions or leads to weaker conclusions.

The flag, balanced, pure simplicial complexes with having property (1.3) are exactly the independence complexes of the clique-whiskered graphs introduced by Cook II and Nagel in [CN12]. Both Lemma 1.25 and Proposition 1.26 have a correspondent in the above mentioned paper.

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Chapter 2

On a conjecture by Kalai

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Division of Labor

This project started while the three authors were visiting the MSRI in 2012. Most of the conceptual work was done there. The dense exchange of ideas in that short period of time make a clear separation of the contributions is not possible. It is however possible to say that the three authors contributed in an equal manner to the conception, execution and writing of this work.

2.1 Introduction

An unpublished conjecture of Gil Kalai, which was also independently phrased by Jürgen Eckhoff in [Eck88] and recently verified by Frohmader [Fro08], states that for any flag simplicial complex Δ there exists a balanced simplicial complex Γ with the same f-vector. Here a $(d-1)$-dimensional simplicial complex is balanced if you can use d colors to label its vertices so that no face contains two vertices of the same colour. Kalai's conjecture has also a second part which is still open: If Δ happens to be Cohen-Macaulay (CM), then Γ is required to be CM as well.

In this note we show that for any CM flag simplicial complex Δ there exists a CM balanced simplicial complex Γ with the same h-vector; notice that the equality of the h-vectors implies the equality of the f-vectors only if the simplicial complexes involved have the same dimension. Equivalently, by [BFS87, Theorem 1], we prove that the h-vector of a CM flag simplicial complex satisfies the Kruskal-Katona's conditions (that is the inequalities satisfied by the f-vector of a simplicial complex, see [Sta96, Chapter II, Theorem 2.1]).

Such a result has been proved in [CV13, Theorem 3.3] under the additional assumption that Δ is vertex decomposable. Other recent developments concerning Kalai's conjecture and related topics can be found in [CN12; BV12]. To this purpose we will show a stronger statement, Theorem 2.1, namely that the *Eisenbud-Green-Harris conjecture* (**EGH**) holds for quadratic monomial ideals.

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring over a field K. The **EGH** conjecture, in the general form, states that for every homogeneous ideal I of S containing a regular sequence $f_1,\dots,f_r$ of degrees $d_1 \leq \dots \leq d_r$ there exists a homogeneous ideal $J \subseteq S$, with the same Hilbert function as I (i.e. $\mathrm{HF}_I = \mathrm{HF}_J$) and containing $x_1^{d_1},\dots,x_r^{d_r}$. Furthermore, by the main theorem in the paper [CL69] of Clements and Lindstöm and specifically by Corollary 2 of that paper, the ideal J, when it exists, can be chosen to be the sum of the ideal $(x_1^{d_1},\dots,x_r^{d_r})$ and a lex-segment ideal of S. The result of Clements and Lindström recovers Kruskal-Katona theorem when $2 = d_1 = \ldots = d_r$. Recently, these results have been substantially improved by Murai and Mermin in [MM11, Theorem 8.1], who dealt with a related question of Evans known as Lex-Plus-Powers Conjecture. We refer to [EGH93] and [EGH96] for the original formulation of the **EGH** conjecture. The only large classes for which the **EGH** conjecture is known are: when $f_1,\dots,f_r$ are Gröbner basis (by a deformation argument), when $d_i > \sum_{j<i}(d_j-1)$ for all $i=2,\dots,r$ ([CM08]) and when each f_i factors as product of linear forms ([Abe15, Corollary 4.3] for the case $r=n$, and [Abe15] together with the argument in the proof of [CM08, Proposition 10] for the general case). [1]

2.2 The result

Below ht$I$ stands for the height of an ideal I and HF_M for the Hilbert function of a graded module M.

Theorem 2.1. *Let $I \subseteq S = K[x_1,\dots,x_n]$ be a monomial ideal generated in degree 2, of height $\mathrm{ht}I = g$. Then, there exists a monomial ideal $J \in S$, such that $(x_1^2,\dots,x_g^2) \subseteq J$ and*

$$\mathrm{HF}_I = \mathrm{HF}_J.$$

Furthermore J can be chosen with the same projective dimension as I.

Proof. Since the Hilbert function and the projective dimension are invariant with respect to field extension, we can assume without loss of generality that K is infinite. We will prove that I contains a regular sequence of the form $x_1\ell_1,\dots,x_g\ell_g$, where ℓ_i is a linear form for every $i \in [g] = \{1,\dots,g\}$. Then we will infer the theorem by a result in [Abe15].

As the minimal primes of a monomial ideal are also monomial, after possibly relabeling the indeterminates, we may assume without loss of generality that $(x_1,\dots,x_g)$ is a minimal prime of I. Thus, we may decompose the degree 2 component of I as

$$I_2 = x_1V_1 \oplus \dots \oplus x_gV_g,$$

where each V_i is a linear space generated by indeterminates. Our goal is to find g linear forms $\ell_i \in V_i$, such that:

$(*)$ for all $A \subseteq [g]$, the K-vector space $\langle x_i : i \in A\rangle + \langle \ell_i : i \in [g]\setminus A\rangle$ has dimension g.

Clearly, finding a subset A not satisfying $(*)$ produces a prime ideal containing I of height $< g$, contradicting the fact that $x_1\ell_1,\dots,x_g\ell_g$ is a regular sequence ([Mat80, Theorem 17.4]). So, to see that $(*)$ is equivalent to $x_1\ell_1,\dots,x_g\ell_g$ being a S-regular sequence (from now on we will just write regular sequence for S-regular sequence), consider the following short exact sequence (where $C = (x_1\ell_1,\dots,x_g\ell_g)$):

$$0 \to (S/(C:\ell_g))(-1) \to S/C \to S/(C+(\ell_g)) \to 0,$$

To conclude the proof of the claim, recall that g homogeneous polynomials form a regular sequence if and only if they generate a height g ideal [Mat80, Theorem 17.4].

Notice that $S/(C:\ell_g) = S/(B+(x_g))$, where B is an ideal containing $(x_1\ell_1,\dots,x_{g-1}\ell_{g-1})$. Denoting by ℓ_i' the image of ℓ_i by going modulo x_g and by $R = K[x_1,\dots,x_{g-1},x_{g+1},\dots,x_n]$, we have that

$$S/(x_1\ell_1,\dots,x_{g-1}\ell_{g-1},x_g) \cong R/(x_1\ell_1',\dots,x_{g-1}\ell_{g-1}').$$

We can therefore apply the induction on g and infer that $x_1\ell_1',\dots,x_{g-1}\ell_{g-1}'$ is an R-regular sequence. So, the Krull dimension of $S/(C:\ell_g)$ is at most $n-g$.

Similarly, we can use the induction to infer that $S/(C+(\ell_g))$ has Krull dimension $n-g$ (notice that, because the assumption that $\langle x_1,\dots,x_{g-1},\ell_g\rangle$ has dimension g, the image of x_i modulo ℓ_g can be still thought as x_i if $i<g$).

So, both the extremes of the above exact sequence are graded modules of Krull dimension not exceeding $n-g$, or equivalently, the degrees of the corresponding Hilbert polynomials are at most $n-g-1$. Due to the additivity of the Hilbert function over graded exact sequences, the Hilbert polynomial of S/C has to have degree $< n-g$ (thus Krull dimension $\leq n-g$). Therefore $\mathrm{ht}(C) \geq g$. However we know by Krull's Hauptidealsatz that C may have height at most g, so $\mathrm{ht}(C) = g$ and $x_1\ell_1, \ldots, x_g\ell_g$ is a regular sequence.

So we have to seek $\ell_i \in V_i$ satisfying $(*)$. Let $A = \{i_1, \ldots, i_a\}$ be a subset of $[g]$. We define $U_A \subseteq \prod_{i \in A} V_i$ to be the following set:

$$U_A = \{(v_{i_1}, \ldots, v_{i_a}) \in \prod_{i \in A} V_i \ : \ \langle v_{i_1}, \ldots, v_{i_a} \rangle + \langle x_j : j \in [g] \setminus A \rangle \text{ has dimension } g\}.$$

As the condition of linear dependence is obtained by imposing certain determinantal relations to be zero, U_A is a Zariski open set of $\prod_{i \in A} V_i$. Thus the $\widetilde{U}_A$ below is a Zariski open set of $\prod_{i=1}^{g} V_i$

$$\widetilde{U}_A = U_A \times \prod_{i \in [g] \setminus A} V_i \subseteq \prod_{i=1}^{g} V_i.$$

This construction can be done for every $A \subseteq [g]$, and thus we can define the open set

$$U = \bigcap_{A \subseteq [g]} \widetilde{U}_A \subset \prod_{i=1}^{g} V_i.$$

Any element $(\ell_1, \ldots, \ell_g) \in U$ will automatically satisfy $(*)$, so our goal is to show that $U \neq \emptyset$. As $\prod_{i=1}^{g} V_i$ is irreducible, it is enough to show that all the open sets $\widetilde{U}_A$'s are nonempty. For any $A \subseteq [g]$ we have

$$\dim_K \left(\sum_{i \in A} V_i + \sum_{j \in [g] \setminus A} \langle x_j \rangle \right) \geq g, \tag{2.1}$$

otherwise $(\sum_{i \in A} V_i + \sum_{j \in [g] \setminus A} \langle x_j \rangle)$ would be a prime ideal containing I of height $< g$.

Given $A \subseteq [g]$, we define a bipartite graph G_A. The vertex set of G_A has the following partition: $V(G_A) = \{x_1, \ldots, x_n\} \cup \{1, \ldots, g\}$, and the edge set of G_A is given by:

$$\{x_i, j\} \in E(G_A) \iff \begin{cases} x_i \in V_j & , \quad \text{if } j \in A \\ i = j & , \quad \text{if } j \notin A \end{cases}$$

We fix A and prove that G_A satisfies the hypothesis of the Marriage Theorem. For a subset $B \subseteq V(G_A)$, we denote by $N(B)$ the set of vertices adjacent to some vertex in B. Choose now $B \subseteq \{1, \ldots, g\}$. By applying (2.1) to the set $A \cap B \subseteq [g]$, we can deduce that

$$\dim_K \left(\sum_{i \in A \cap B} V_i + \sum_{j \in ([g] \setminus A) \cap B} \langle x_j \rangle \right) \geq \dim_K \left(\sum_{i \in A \cap B} V_i + \sum_{j \in [g] \setminus (A \cap B)} \langle x_j \rangle \right) - \dim_K \left(\sum_{j \in [g] \setminus B} \langle x_j \rangle \right) \geq |B|.$$

Furthermore, notice that the dimension of the leftmost vector space above is $|N(B)|$, thus we can apply the Marriage Theorem and infer the existence of a matching in G_A of the form $\{x_{i_j}, j\}_{j \in [g]}$. Therefore U_A is nonempty for A nonempty as it contains $(x_{i_j} : j \in A)$, and thus $\widetilde{U}_A$ is nonempty for every A. So we found a regular sequence of quadrics $f_1, \ldots, f_g$ in I consisting of products of linear forms.

Let $\mathrm{pd}(I)$ be the projective dimension of I and assume that $\mathrm{pd}(I) = p-1$. By applying a linear change of coordinates, we may assume that $x_{p+1}, \ldots, x_n$ is a S/I-regular sequence. Going modulo $(x_{p+1}, \ldots, x_n)$, the image $I' \subseteq K[x_1, \ldots, x_p]$ of I may not be monomial, but still contains a regular sequence of quadrics which are products of linear forms, namely the images of $f_1, \ldots, f_g$. So we find $J' \subseteq K[x_1, \ldots, x_p]$ containing $(x_1^2, \ldots, x_g^2)$ with the same Hilbert function of I' by [Abe15, Corollary 4.3] and, when g is less then p, by the same argument used to prove [CM08, Proposition 10].

Clearly $\mathrm{pd}(J') = \mathrm{pd}(K[x_1, \ldots, x_p]/J') - 1 \leq p-1$. Furthermore we have $\mathrm{pd}(J') \geq \mathrm{pd}(I) = p-1$ by [CS13, Theorem 4.4]. Defining $J = J'S$, so we have $(x_1^2, \ldots, x_g^2) \subseteq J$, $\mathrm{pd}(J) = \mathrm{pd}(I)$ and $\mathrm{HF}_I = \mathrm{HF}_J$. $\square$

The following example shows that the above proof cannot be extended to prove EGH for all monomial ideals.

Example 2.2. The ideal $I = (x_1^2x_2, x_2^2x_3, x_1x_3^2) \subseteq K[x_1,x_2,x_3]$ does not contain a regular sequence of the form $\ell_1\ell_2\ell_3, q_1q_2q_3$, where all ℓ_i and q_j are linear forms. Elementary direct computations allow one to see that the generators are the only products of three linear forms, which are contained in I. Clearly any choice of two of them does not produce a regular sequence.

The following corollary is the main motivation for this note.

Corollary 2.3. *For any CM flag simplicial complex Δ there exists a CM balanced simplicial complex Γ with the same h-vector.*

Proof. Let g be the height of the Stanley-Reisner ideal I_Δ. By Theorem 2.1, there exists an ideal $J \subseteq S$, containing $(x_1^2,\dots,x_g^2)$ and with the same Hilbert function and projective dimension as I_Δ. By Auslander-Buchsbaum [Mat80, Theorem 19.1], S/J is CM as well, thus all the associated primes of J have the same height by [Mat80, Theorem 17.2]. As $(x_1^2,\dots,x_g^2) \subseteq J$, every associated prime must also contain $(x_1,\dots,x_g)$, thus the generators of J are monomials in the first g variables. So J is the extension to S of a monomial ideal $J' \subseteq K[x_1,\dots,x_g]$, whose Hilbert function $\mathrm{HF}_{K[x_1,\dots,x_g]/J'}$ equals the h-vector of Δ. The CM balanced Γ is the simplicial complex associated to the polarization of J'. $\square$

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Chapter 3

Koszulness, Krull Dimension and Other Properties of Graph-Related Algebras

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This project started during the PRAGMATIC summer school in 2008. Part of the conceptual work was done there, and part was done while both authors were employed by the University of Genova. All creative work was done together at the blackboard or with pen and paper, so a clear separation of the important contributions is not possible. It is however possible to say that both authors contributed in an equal manner to the conception, execution and writing of this work.

Introduction

Due to their relation with the resolution of singularities of schemes, blowup algebras are an important subject in both Commutative Algebra and Algebraic Geometry. On the other hand, the vertex covers of a graph are important objects in Graph Theory, having many practical applications. In this paper we are going to study blowup algebras related to graphs, merging the above topics. In fact, these algebras have an interpretation in terms of the vertex covers, more precisely the k-covers, of the graph. Many ring-theoretic properties are thus described in terms of the combinatorics of the graph.

Given a graph G on n vertices its cover ideal is the ideal $J(G) = \bigcap(x_i, x_j) \subseteq K[x_1, \dots, x_n]$, where the intersection runs over the edges of G. The symbolic Rees algebra of this ideal is also known as the vertex cover algebra of G. In their paper [HHT07] Herzog, Hibi and Trung have studied this algebra in the more general context of hypergraphs. In the present paper we study the symbolic fiber cone of $J(G)$, denoted by $\bar{A}(G)$. There are three main results:

(a) *A combinatorial characterization of the Krull dimension of $\bar{A}(G)$ (Theorem 3.8)*. This problem was raised by Herzog in 2008. As a nice consequence we give an upper bound for the number of equations defining up to radical a monomial ideal of codimension 2, refining a result of Lyubeznik obtained in [Lyu88].

(b) *The Koszul property of $\bar{A}(G)$ for a bipartite graph G (Theorem 3.18)*. This problem was suggested by Herzog too, during an informal conversation at Oberwolfach in 2009. Actually we prove more: If G is bipartite, then $\bar{A}(G)$ has a natural structure of homogeneous algebra with straightening laws.

From the arising poset we give many examples of bipartite graphs for which $\bar{A}(G)$ has or has not certain ring-theoretic properties.

(c) *A combinatorial criterion for the Cohen-Macaulyness of edge ideals of graphs satisfying the weak square condition (Theorem3.33).* To characterize the graphs for which the edge ideal is Cohen-Macaulay is a wide open question and a very studied problem. Our result generalizes a theorem by Herzog and Hibi obtained in [HH05], where they characterize the bipartite graphs for which the edge ideal is Cohen-Macaulay.

Let us describe how the paper is organized.

In the first section we recall the definition and basic properties of the symbolic fiber cone of the cover ideal of a graph, namely $\bar{A}(G)$. In Section 2 we compute in terms of the combinatorics of the graph the dimension of $\bar{A}(G)$. The combinatorial invariant that we introduce is called the *ordered matching number*. It turns out it has a lower bound given by the paired-domination number and an upper bound given by the matching number of the graph. When the base field is infinite the dimension of $\bar{A}(G)$ is an upper bound for the arithmetical rank of $J(G)$ localized at the maximal irrelevant ideal, so we get interesting upper bounds for the arithmetical rank of a monomial ideal of pure codimension 2 after having localized at the maximal irrelevant ideal, thus improving a result of [Lyu88].

In the third section of the paper we prove that for a bipartite graph G, the algebra $\bar{A}(G)$ is Koszul. The Koszul property follows from the homogeneous ASL structure which we can give to $\bar{A}(G)$. In a joint paper with Benedetti [BCV08], we gave for a bipartite graph a combinatorial condition equivalent to $\bar{A}(G)$ being a domain. This combinatorial property is called weak square condition (WSC). The ASL structure provides in the bipartite case another equivalent condition: $\bar{A}(G)$ is a domain if and only if $\bar{A}(G)$ is a Hibi ring. Using this structure and a result of Hibi from [Hib87] we are able to characterize for bipartite graphs the Gorenstein domains. The non-integral case turns out to be more complicated. However, from the description of the poset on which $\bar{A}(G)$ is an ASL we can deduce some nice consequences. For instance, we can produce many examples of bipartite graphs such that $\bar{A}(G)$ is not Cohen-Macaulay, using results of Kalkbrener and Sturmfels [KS95] and of the second author [Var09]. With some additional assumption on the combinatorics of the graph we can prove that $\bar{A}(G)$ is Cohen-Macaulay if and only if it is equidimensional.

In the fourth and last section we focus our attention on the edge ideal of the graph, namely $I(G) = (x_ix_j : \{i,j\} \text{ is an edge of } G) \subseteq S = K[x_1,\dots,x_n]$. Two problems that have recently caught the attention of many authors (see for instance [Frö90; HV08; HH05; Kat06; Kum09; Zhe04]) are the characterization in terms of the combinatorics of G of the Cohen-Macaulay property and the Castelnuovo-Mumford regularity of $S/I(G)$. Our approach is to restrict the problem to a subgraph $\pi(G)$ of G which maintains some useful properties of the edge ideal. This graph is constructed passing through another graph, namely G^{0-1}, introduced by Benedetti and the second author in [BV11]. Using this tool we are able to extend a result of [HH05] regarding the Cohen-Macaulay property and a result of Kummini from [Kum09] regarding the Castelnuovo-Mumford regularity.

The authors wish to thank Jürgen Herzog for suggesting this topic and for many useful discussions which led to new stimulating questions and interesting observations. We also wish to thank Aldo Conca and Bruno Benedetti for their useful comments.

3.1 Terminology and Preliminaries

For the convenience of the reader we include in this short section the standard terminology and the basic facts about the algebra of basic covers of a graph.

For a natural number $n \geq 1$ we denote by $[n]$ the set $\{1,\dots,n\}$. By a graph G on $[n]$ we understand a graph with n vertices without loops or multiple edges. If we do not specify otherwise, we also assume that a graph has no isolated points. We denote by $V(G)$ (respectively $E(G)$) the vertex set (respectively the edge set) of G. From now on G will always denote a graph on $[n]$ and we will write, when it does not raise confusion, just V for $V(G)$ and E for $E(G)$. A subset $V' \subseteq V$ is called a *vertex cover* of G, if for any $e \in E$ we have $e \cap V' \neq \emptyset$. A vertex cover V' is called minimal if no proper subset of V' is again a vertex cover. More generally, a non-zero function $\alpha : V(G) \to \mathbb{N}$, is a *k-cover* of G ($k \in \mathbb{N}$) if $\alpha(i) + \alpha(j) \geq k$ whenever $\{i,j\} \in E(G)$. A k-cover α is *decomposable* if $\alpha = \beta + \gamma$ where β is an h-cover and γ is a $(k-h)$-cover; α is *indecomposable* if it is not decomposable. A k-cover α is called *basic* if it is not decomposable as a 0-cover plus a k-cover (equivalently if no function $\beta < \alpha$ is a k-cover). Notice the correspondence between basic 1-covers and minimal vertex covers.

Throughout the paper K will be a field, $S = K[x_1,\dots,x_n]$ will denote the polynomial ring with n variables over K and $\mathfrak{m} = (x_1,\dots,x_n)$ will be the irrelevant maximal ideal of S. The *edge ideal* of G, denoted by $I(G)$, is the square-free monomial ideal of S

$$I(G) = (x_i x_j : \{i,j\} \in E(G)) \subseteq S.$$

A graph G is called *Cohen-Macaulay over K* if $S/I(G)$ is a Cohen-Macaulay ring. A graph is called just *Cohen-Macaulay* if it is Cohen-Macaulay over any field (equivalently over $\mathbb{Z}$). The *cover ideal* of G is the Alexander dual of the edge ideal, and we denote it by $J(G)$. So

$$J(G) = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j).$$

As said in the introduction, in this paper we study the symbolic fiber cone of $J(G)$. To introduce it, we recall the definition of the symbolic Rees algebra of an ideal $I \subseteq S$:

$$R(I)_s = \bigoplus_{k \geq 0} I^{(k)} t^k \subseteq S[t],$$

where $I^{(k)}$ denotes the kth symbolic power of I, i.e. $I^{(k)} = (I^k S_W) \cap S$, where W is the complement in S of the union of the associated primes of I and S_W denotes the localization of S at the multiplicative system W. If I is a square-free monomial ideal then $I^{(k)}$ is just the intersection of the (ordinary) k-powers of the minimal prime ideals of I. Therefore

$$(J(G))^{(k)} = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j)^k.$$

The symbolic fiber cone of I is $R(I)_s / \mathfrak{m} R(I)_s$. We will denote by $\bar{A}(G)$ the symbolic fiber cone of $J(G)$.

There is a more combinatorial way to construct $\bar{A}(G)$, given by the relation between basic covers and $J(G)$:

$$J(G)^{(k)} = (x_1^{\alpha(1)} \cdots x_n^{\alpha(n)} : \alpha \text{ is a basic } k\text{-cover}).$$

Thus $R(J(G))_s = K[x_1^{\alpha(1)} \cdots x_n^{\alpha(n)} t^k : \alpha \text{ is a } k\text{-cover}] \subseteq S[t]$. For more details on this interpretation of these algebras see [HHT07], in which this symbolic Rees algebra is denoted by $A(G)$. The authors of that paper proved many properties of $A(G)$. First of all they noticed that $A(G)$ is a finitely generated K-algebra, since it is generated in degree less than or equal to 2. Moreover $A(G)$ is a standard graded S-algebra if and only if G is bipartite. They also proved that $A(G)$ is always a Gorenstein normal domain.

Since $\bar{A}(G) = A(G)/\mathfrak{m}A(G)$, we have that

$$\bar{A}(G) = K \oplus \Big(\bigoplus_{k \geq 1} < x_1^{\alpha(1)} \cdots x_n^{\alpha(n)} t^k : \alpha \text{ is a basic } k\text{-cover} > \Big),$$

where the multiplication table is given by

$$x_1^{\alpha(1)}\cdots x_n^{\alpha(n)}t^k\cdot x_1^{\beta(1)}\cdots x_n^{\beta(n)}t^h=\begin{cases} x_1^{\gamma(1)}\cdots x_n^{\gamma(n)}t^{h+k} & \text{if } \gamma=\alpha+\beta \text{ is a basic } (h+k)\text{-cover,}\\ 0 & \text{otherwise.}\end{cases}$$

With the above presentation it is clear that the Hilbert function of $\bar{A}(G)$ counts the basic k-covers of G, i.e.

$$\mathrm{HF}_{\bar{A}(G)}(k)=\dim_K(\bar{A}(G)_k)=|\{\text{basic } k\text{-covers of } G\}|.$$

It turns out that the number of basic $2h$-covers of a graph grows as a polynomial in h of degree $\dim\bar{A}(G)-1$, namely the Hilbert polynomial $\mathrm{HP}_{\bar{A}(G)^{(2)}}$ of the second Veronese subring of $\bar{A}(G)$, which is standard graded (see Remark 3.6). This simple fact will be crucial in the characterization of the Krull dimension of $\bar{A}(G)$ in terms of G.

From the above discussion it follows that $\bar{A}(G)$ is a standard graded K-algebra (equivalently it is the ordinary fiber cone of $J(G)$) if and only if G is bipartite. The graphs for which $\bar{A}(G)$ is a domain have been characterized in [BCV08] in the bipartite case and in [BV11] in general. Moreover, if $\bar{A}(G)$ is a domain then it is Cohen-Macaulay, but it may be non-Gorenstein. When G is bipartite, even if $\bar{A}(G)$ is not a domain, the projective scheme defined by $\bar{A}(G)$ is connected, but not necessarily equidimensional, and therefore it may be non-Cohen-Macaulay (for more details see [BCV08]).

3.2 The Krull Dimension of $\bar{A}(G)$

In this section we will introduce the notion of ordered matching number. This notion extends the one of graphical dimension of a bipartite graph introduced in [BCV08]. In [BCV08] it was conjectured that for a bipartite graph, the Krull dimension of $\bar{A}(G)$ is equal to the graphical dimension of G, which as we will see in a moment is equal to one plus the ordered matching number of G. We will prove that this is true not only in the case of bipartite graphs, but for any graph G. As consequences of this result we are able to give interesting upper bounds for the arithmetical rank of monomial ideals of pure codimension 2 in the localization $S_{\mathfrak{m}}$, refining in this case an upper bound given in [Lyu88].

Given a graph G, we recall that a set $M\subseteq E(G)=E$ of edges is a *matching* of G if any two distinct edges of M have empty intersection. A matching is called *maximal* if it has maximal cardinality among all matchings of G. The *matching number* of G, denoted by $\nu(G)$, is the cardinality of a maximal matching of G. A matching M is called *perfect* if every vertex in V belongs to an edge in M. A set of vertices $V'\subseteq V(G)=V$ is called *independent* if $\{v,w\}\notin E$ for any $v,w\in V'$. Let $M=\{\{a_i,b_i\}: i=1,\dots,r\}$ be a nonempty matching of G. We will say that M is an *ordered matching* if:

- $\{a_1,\dots,a_r\}=A\subseteq V$ is a set of independent vertices,
- $\{a_i,b_j\}\in E$ implies $i\leq j$.

In this case we will call A a *free parameter set* and $B=\{b_1,\dots,b_r\}\subseteq V$ a *partner set* of A.

Definition 3.1. Let G be a graph. We define the *ordered matching number* of G as:

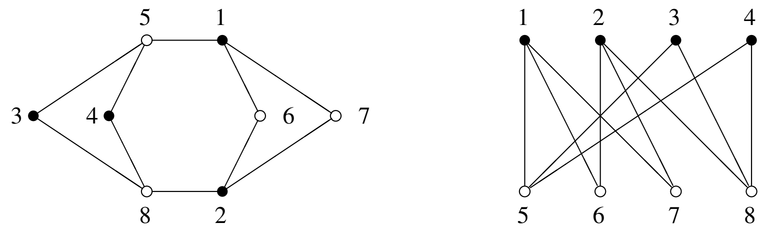
$$\nu_o(G):=\max\{|M|\ :\ M\subseteq E\ \text{ is an ordered matching}\}.$$

Remark 3.2.
1. Being an ordered matching depends on the labeling of the vertices in both A and B.
2. In the case of bipartite graphs it is not difficult to verify that the notion of ordered matching number is equivalent to that of graphical dimension given in [BCV08]. In fact, using the notation of [BCV08], we have $\nu_o(G)=\mathrm{gdim}(G)-1$ for each bipartite graph G.
3. In general, B is not necessarily a set of independent vertices.

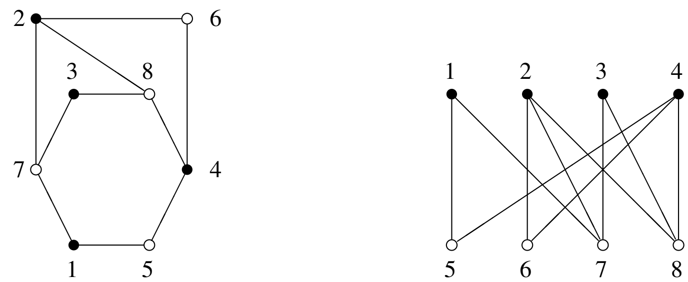
The ordered matching number of a graph is not always easy to compute and we were not able to express it in terms of classical invariants of graphs in general. In the following example we will see that it does not depend on the local degree of the vertices. By local degree of a vertex we understand the number of edges incident in that vertex.

Example 3.3. Let G and G' be the bipartite graphs represented below. If $V(G) = A \cup B$ and $V(G') = A' \cup B'$ it turns out that all four sets have two vertices of local degree 2 and two vertices of local degree 3. However, we have $\nu_o(G) = 2$ and $\nu_o(G') = 3$.

G :



G' :



For G an ordered matching of maximal cardinality is $\{\{1,5\},\{2,6\}\}$. For G' we have that $\{\{1,5\},\{2,6\},\{3,7\}\}$ is an ordered matching of maximal cardinality. In general these ordered matchings are not unique. For instance, another ordered matching of cardinality 2 for G is $\{\{2,6\},\{3,8\}\}$.

A subset $V' \subseteq V$ is a *point cover* of G if for each $v \in V \setminus V'$, there exists a vertex $w \in V'$ such that $\{v,w\} \in E$. Notice that a vertex cover is a point cover, but the converse is false. An easy example is given by the triangle $G = K_3$: any vertex of G is a point cover, but not a vertex cover.

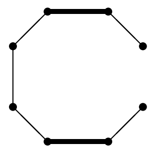
Remark 3.4. We recall that a set $S \subseteq V$ is called a *paired-dominating set* of G if S is a point cover of G and if the subgraph induced by S has at least one perfect matching. The minimum cardinality of a paired-dominating set is called the *paired-domination number* of G and is denoted by $\gamma_{\mathrm{P}}(G)$. The following inequalities hold true:

$$\frac{\gamma_{\mathrm{P}}(G)}{2} \leq \nu_o(G) \leq \nu(G).$$

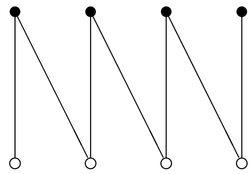
The second inequality is straightforward from the definition. To see the first one, suppose that $A = \{a_1,\ldots,a_r\}$ is a free parameter set with partner set $B = \{b_1,\ldots,b_r\}$. If $\gamma_{\mathrm{P}}(G) > 2r$, then there is a vertex v in $V \setminus (A \cup B)$ adjacent to none of the vertices of $A \cup B$. Choose a vertex w adjacent to v, and set $a_{r+1} = v$, $b_{r+1} = w$. It turns out that $\{a_1,\ldots,a_r,a_{r+1}\}$ is a free parameter set with partner set $\{b_1,\ldots,b_r,b_{r+1}\}$.

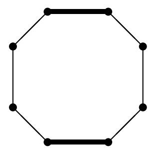
Example 3.5. In this example we will see that the ordered matching number may reach both the upper and lower bound given in the previous remark. The thick lines in the pictures on the left represent the edges of a perfect matching of a minimal paired dominating set.

G :

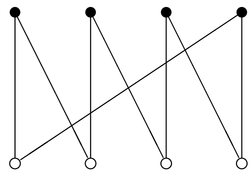


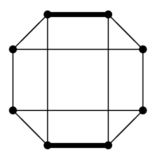
$\nu(G) = 4$
$\nu_o(G) = 4$
$\gamma_P(G)/2 = 2$



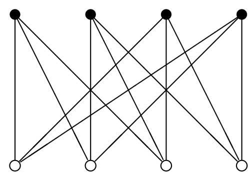


$\nu(G) = 4$
$\nu_o(G) = 3$
$\gamma_P(G)/2 = 2$





$\nu(G) = 4$
$\nu_o(G) = 2$
$\gamma_P(G)/2 = 2$



In spite of the examples above, the ordered matching number is easy to compute at least for trees. In this case $\nu_o(G) = \nu(G)$ (Proposition 3.10), and there are many algorithms that compute the matching number of a bipartite graph.

To prove the main result of this section, the following remark and lemma are crucial.

Remark 3.6. There exists a polynomial $P \in \mathbb{Q}[t]$ of degree $\dim(\bar{A}(G)) - 1$ such that, for $h \gg 0$,

$$P(h) = |\{\text{basic } 2h\text{-covers of } G\}|.$$

To see this, consider the second Veronese subring of $\bar{A}(G)$, namely $\bar{A}(G)^{(2)} = \bigoplus_{h\geq 0} \bar{A}(G)_{2h}$. By [HHT07, Theorem 5.1.a] we have that $\bar{A}(G)^{(2)}$ is a standard graded K-algebra. So it has a Hilbert polynomial, denoted by $\mathrm{HP}_{\bar{A}(G)^{(2)}}$, such that $\mathrm{HP}_{\bar{A}(G)^{(2)}}(h) = \dim_K(\bar{A}(G)_{2h})$ for $h \gg 0$. Notice that $\bar{A}(G)$ is a finite $\bar{A}(G)^{(2)}$-module, so $\dim(\bar{A}(G)) = \dim(\bar{A}(G)^{(2)})$, which is the degree of $\mathrm{HP}_{\bar{A}(G)^{(2)}}$ minus 1. So it is enough to take $P = \mathrm{HP}_{\bar{A}(G)^{(2)}}$.

Lemma 3.7. *Let G be a graph, $k > 0$ a natural number and α a basic k-cover of G. Denote by $A_{k/2} := \{v \in V \ : \ \alpha(v) \leq k/2\}$.*

(a) *The set $A_{k/2}$ is a point cover of G and α is uniquely determined by the values it takes on the vertices in $A_{k/2}$.*
(b) *Suppose that $\dim(\bar{A}(G)) > s$. Then there exist $k > 0$ and a basic k-cover α such that $|\{\alpha(v) \ : \ v \in A_{k/2}\}| \geq s$.*

Proof. (a) Denote by $W = V \setminus A_{k/2} = \{w \in V \ : \ \alpha(w) > k/2\}$. As α is basic, for each vertex $w \in W$ there exists a vertex v such that $\{w,v\} \in E$ and $\alpha(w) + \alpha(v) = k$. As $\alpha(w) > k/2$ we must have that $\alpha(v) < k/2$. So the set $A_{k/2}$ is a point cover of G. It is easy to see that the only possible choice to extend α on the set W is:

$$\alpha(w) = \max\{k - \alpha(v) \ : \ \{v,w\} \in E, \text{ and } v \in A_{k/2}\}.$$

As $A_{k/2}$ is a point cover, the set we are considering is not empty for any $w \in W$. In order to obtain a k-cover, we need to assign to $\alpha(w)$ at least the maximum considered above. But in order to obtain a basic k-cover we need to assign exactly this value.

(b) Suppose there are no k and α as we claim. Then, for every $k \geq 0$, there is a function

$$\{\text{basic } k\text{-covers of } G\} \longrightarrow \{(a_1,\dots,a_n) : \ 0 \leq a_i \leq k/2 \text{ and } |\{a_1,\dots,a_n\}| < s\},$$

given by associating to each basic k-cover α, a vector which has the same values as α on $A_{k/2}$ and is 0 in all the other positions. Point (a) guarantees that this is actually an injection. It is not difficult to see that the cardinality of the set on the right-hand side is equal to $C \cdot k^{s-1}$, where C is a constant depending on n and s. Therefore Remark 3.6 implies $\dim(\bar{A}(G)) \leq s$, a contradiction. □

Now we can prove the main result of this section.

Theorem 3.8. *Let $\bar{A}(G)$ be the symbolic fiber cone of the cover ideal of a graph G. Then*

$$\dim(\bar{A}(G)) = \nu_o(G) + 1.$$

Proof. We will first prove that $\dim(\bar{A}(G)) \geq \nu_o(G)+1$. By Remark 3.6 we have to show that $|\{\text{basic } 2h\text{-covers of } G\}|$ grows as a polynomial in h of degree at least $\nu_o(G)$.

Let $M = \{\{a_i, b_i\} : i = 1,\dots,r\}$ an ordered matching of maximal cardinality for G. Denote by $A = \{a_1,\dots,a_r\}$ the free parameter set and by $B = \{b_1,\dots,b_r\}$ the partner set of A. Furthermore set $X = A \cup B$. So $\nu_o(G) = r$. Let $k > 2r$ be an even natural number. We will construct a basic k-cover of G for every decreasing sequence of numbers:

$$\frac{k}{2} \geq i_1 > i_2 > \ldots > i_r \geq 0.$$

As the number of decreasing sequences as above is $\binom{k/2+1}{r}$, this will imply that the degree of $\mathrm{HP}_{\bar{A}(G)^{(2)}}$ is at least r, so also that $\dim(\bar{A}(G)) \geq \nu_o(G)+1$. For a decreasing sequence as above and for all $j \in \{1,\dots,r\}$ we define:

$$\begin{aligned} \alpha(a_j) &= i_j, \\ \alpha(b_j) &= k - i_j. \end{aligned}$$

As G is connected, if $V \setminus X \neq \emptyset$, there exists a vertex $v \in V \setminus X$ such that there exists at least one edge between v and X. We define:

$$\alpha(v) = \max\{k - \alpha(w) \ : \ w \in X \text{ and } \{v,w\} \in E\},$$

append v to X and continue in the same way until α is defined for all vertices of G. It is easy to see that by construction, for each edge $\{v,w\}$ with $v \notin X$ or $w \notin X$ (or both), we have $\alpha(v)+\alpha(w) \geq k$ and that for each vertex $v \notin X$ there exists another vertex v' such that $\alpha(v)+\alpha(v') = k$. So to check that we defined a basic k-cover we need to focus on the vertices in X. Let $\{v,w\}$ be an edge with $v,w \in X$. As A is a set of independent vertices, we can assume that $w = b_j \in B$ and check the following two cases: If $v = a_h \in A$ then by definition $h \leq j$, and by construction:

$$\alpha(a_h) + \alpha(b_j) = i_h + k - i_j \geq k.$$

If $v = b_l \in B$ then:

$$\alpha(b_l) + \alpha(b_j) = k - i_l + k - i_j \geq k.$$

So α is a k-cover. The fact that $\{a_j, b_j\} \in E$ for each $1 \leq j \leq r$ guarantees that α is a basic k-cover. So we may conclude that $\dim(\bar{A}(G)) \geq \nu_o(G)+1$.

Assume now that $\dim(\bar{A}(G)) = s+1$. To prove that $\dim(\bar{A}(G)) \leq \nu_o(G)+1$, consider $k > 0$ and a basic k-cover α as in Lemma 3.7, (b). Denote by

$$\{i_1,\ldots,i_r\} = \{\alpha(v) \ : \ v \in V \text{ and } \alpha(v) \leq k/2\}.$$

By Lemma 3.7, (b), we have $r \geq s$. We can also assume that $i_1 > i_2 > \ldots > i_r$. For each $1 \leq j \leq r$ choose a vertex $a_j \in V$ such that $\alpha(a_j) = i_j$ and denote by

$$A = \{a_1,\ldots,a_r\}.$$

As α is a basic k-cover, for each $1 \leq j \leq r$ there exists a vertex $b_j \in V$ such that $\alpha(a_j)+\alpha(b_j) = k$. Choose one such b_j for each j and denote by

$$B = \{b_1,\ldots,b_r\}.$$

It is not difficult to see that A is a free parameter set with the partner set B, so

$$\nu_o(G) \geq r \geq s = \dim(\bar{A}(G)) - 1.$$

□

We recall that the analytic spread of a homogeneous ideal $I \subseteq S$, denoted by $\ell(I)$, is the dimension of its ordinary fiber cone. When K is an infinite field, Northcott and Rees proved in [NR54] that $\ell(I)$ is the cardinality of a set of minimal generators of a minimal reduction of $IS_{\mathfrak{m}}$, i.e. an ideal $\mathfrak{a} \subseteq S_{\mathfrak{m}}$ minimal by inclusion and such that there exists k for which $\mathfrak{a}(IS_{\mathfrak{m}})^k = (IS_{\mathfrak{m}})^{k+1}$.

Corollary 3.9. *Let G be a bipartite graph. Then*

$$\ell(J(G)) = \nu_o(G)+1.$$

Proof. As said in the preliminaries, in [HHT07, Theorem 5.1.b] the authors showed that G is bipartite if and only if $A(G)$ is a standard graded S-algebra. This is equivalent to $A(G)$ being the ordinary Rees algebra of $J(G)$. Therefore, when G is bipartite, $\bar{A}(G)$ is the ordinary fiber cone of $J(G)$, so the corollary follows by Theorem 3.8. □

Before we state the next proposition, let us establish some notation that we will use in its proof. Let G be a bipartite graph with bipartition of the vertex set $V_1 \cup V_2$. In order to compute the ordered matching number we only need to look at free parameter sets $A_0 \subseteq V_1$ with partner sets $B_0 \subseteq V_2$. Notice that the graph induced by the set of vertices $A_0 \cup B_0$ may not be connected. Denote this graph by G_0 and denote its connected components by C_1, C_2, and so on. Notice that if G is a tree, then for any vertex $v \notin A_0 \cup B_0$, if there exists an edge $\{v, w_0\}$, with w_0 in some C_i, then $\{v,w\}$ is not an edge for any $w \in C_i$, $w \neq w_0$. In other words, a vertex outside G_0 is “tied" to a connected component of G_0 by at most one edge.

Proposition 3.10. *If G is a tree, then* $\dim\bar{A}(G) = \nu_o(G)+1 = \nu(G)+1$, *where $\nu(G)$ is the matching number of G.*

Proof. By Remark 3.4 and Theorem 3.8 we only have to prove that $\nu_o(G) \geq \nu(G)$ whenever G is a tree. Choose $A_0 = \{a_1,\ldots,a_r\}$ a maximal free parameter set with partner set $B_0 = \{b_1,\ldots,b_r\}$ and suppose that the matching $M = \{\{a_i,b_i\}\}_{i=1,\ldots,r}$ is not maximal. By a classical result of Berge (for instance see the book of Lovász and Plummer [PL86, Theorem 1.2.1]) we get that there must exist an augmenting path in G relative to M. As G is bipartite it is easy to see that this path must be of the form $P = a', b_{i_1}, a_{i_1}, \ldots, b_{i_k}, a_{i_k}, b'$, and as A_0 is a free parameter set the indices must be ordered in the following way $1 \leq i_1 < \ldots < i_k \leq r$. We will construct a new ordered matching with $r+1$ elements. Notice that a' and b' are not vertices of G_0. Denote by C the connected component of G_0 to which the vertices in

$P \cap (A_0 \cup B_0)$ belong. We reorder the connected components such that the C_i's to which b' is connected come first, C comes next and the connected components to which a' is connected come last. Inside C we relabel the vertices such that $a_{i_k}, a_{i_{k-1}}, \ldots, a_{i_1}, a'$ are the first $k+1$ with partners $b', b_{i_k}, \ldots, b_{i_2}, b_{i_1}$. It is easy to see now that, as there are no cycles in G, we obtain a new ordered matching of cardinality $r+1$, a contradiction. □

Given an ideal I of some ring R we recall that the arithmetical rank of I is the integer

$$\operatorname{ara}(I) = \min\{r : \exists f_1, \ldots, f_r \in R \text{ for which } \sqrt{I} = (f_1, \ldots, f_r)\}.$$

If R is a factorial domain, geometrically $\operatorname{ara}(I)$ is the minimal number of hypersurfaces that define set-theoretically the scheme $\mathscr{V}(I)$ in $\operatorname{Spec}(R)$. As we said in the beginning of this section we can obtain interesting upper bounds for this number in the case of monomial ideals of pure codimension 2 in $S_{\mathfrak{m}}$.

Corollary 3.11. *Let K be an infinite field, and G a graph. Then*

$$\operatorname{ara}(J(G)S_{\mathfrak{m}}) \leq \nu_o(G) + 1.$$

In particular, $\operatorname{ara}(J(G)S_{\mathfrak{m}}) \leq \nu(G) + 1$.

Proof. Let us consider the second Veronese subring of $\bar{A}(G)$, i.e.

$$\bar{A}(G)^{(2)} = \bigoplus_{i \geq 0} \bar{A}(G)_{2i}.$$

By [HHT07, Theorem 5.1.a] we have $J(G)^{(2i)} = (J(G)^{(2)})^i$, so that $\bar{A}(G)^{(2)}$ is the ordinary fiber cone of $J(G)^{(2)}$. Since $\bar{A}(G)$ is finite as a $\bar{A}(G)^{(2)}$-module, the Krull dimensions of $\bar{A}(G)$ and the one of $\bar{A}(G)^{(2)}$ are the same. Therefore, using Theorem 3.8, we get

$$\nu_o(G) + 1 = \dim \bar{A}(G)^{(2)} = \ell(J(G)^{(2)}) = \ell((J(G)S_{\mathfrak{m}})^{(2)}).$$

By a result in [NR54, p.151], since K is infinite, the analytic spread of $(J(G)S_{\mathfrak{m}})^{(2)}$ is the cardinality of a set of minimal generators of a minimal reduction of it. The radical of such a reduction is clearly the radical of $(J(G)S_{\mathfrak{m}})^{(2)}$, i.e. $J(G)S_{\mathfrak{m}}$. So we get the desired inequality. □

Remark 3.12. The author of [Lyu88] proved that the arithmetical rank of a monomial ideal of pure codimension 2, once localized at $\mathfrak{m}$, is at most $\lfloor n/2 \rfloor + 1$, where n is the numbers of variables. But every squarefree monomial ideal of codimension 2 is obviously of the form $J(G)$ for some graph on $[n]$. So, since $\nu(G)$ is at most $\lfloor n/2 \rfloor$, Corollary 3.11 refines the result of [Lyu84].

For the next result, let us recall that a set $E' \subseteq E(G) = E$ of edges of a graph G is said to be *pairwise disconnected* if it is a matching and for any two different edges of E' there is no edge in E connecting them.

Corollary 3.13. *Let G be a graph for which $\nu_o(G)$ is equal to the maximum size of a set of pairwise disconnected edges. If K is an infinite field, then*

$$\operatorname{ara}(J(G)S_{\mathfrak{m}}) = \nu_o(G) + 1.$$

Proof. By a result of Katzman ([Kat06, Proposition 2.5]) the maximum size of a set of pairwise disconnected edges of G provides a lower bound for the Castelnuovo-Mumford regularity of $S/I(G)$. Therefore, $\operatorname{reg}(S/I(G)) \geq \nu_o(G)$. But $J(G)$ is the Alexander dual of $I(G)$, so a result of Terai ([Ter99]) implies that $\operatorname{pd}(S/J(G)) \geq \nu_o(G) + 1$. Now, Lyubeznik showed in [Lyu84] that $\operatorname{pd}(S/I) = \operatorname{cd}(S, I) = \operatorname{cd}(S_{\mathfrak{m}}, IS_{\mathfrak{m}})$ (cohomological dimension) for any square-free monomial ideal I. Since the cohomological dimension provides a lower bound for the arithmetical rank, we get $\operatorname{ara}(J(G)S_{\mathfrak{m}}) \geq \nu_o(G) + 1$. Now we get the conclusion by Corollary 3.11. □

Corollary 3.14. *Let $I \subseteq S = K[x_1,\dots,x_n]$ be a square-free monomial ideal of pure codimension 2, and let d be the minimum degree of a non-zero monomial in I. Assume that the field K is infinite. Then*

$$\mathrm{ara}(IS_{\mathfrak{m}}) \le \min\{d+1, n-d+1\}$$

Proof. The inequality $\mathrm{ara}(IS_{\mathfrak{m}}) \le n-d+1$ is well known. One way to see this is by defining the following partial order on the set of the square-free monomials of S:

$$m \le n \iff n|m \quad \text{for any square-free monomials } m,n \text{ of } S.$$

It is easy to see that S is a (non-homogeneous) algebra with straightening laws (see Bruns and Vetter [BV88] for the definition) on this poset over K. Notice that I comes from a poset ideal. This means that $I = \Omega S$, where Ω is a subset of the square-free monomials such that: $n \in \Omega$, $m \le n \implies m \in \Omega$. Then by [BV88, Proposition 5.20] we get $\mathrm{ara}(I) \le n-d+1$. This obviously implies that $\mathrm{ara}(IS_{\mathfrak{m}}) \le n-d+1$.

To prove the inequality $\mathrm{ara}(IS_{\mathfrak{m}}) \le d+1$, notice that $I = J(G)$ for a graph G on $[n]$ ($\{i,j\}$ is an edge of G if and only if (x_i,x_j) is a minimal prime of I). Then Corollary 3.11 implies that $\mathrm{ara}(IS_{\mathfrak{m}}) \le \nu(G)+1$. It is well known and easy to show, that the matching number is at most the least cardinality of a vertex cover of G. It turns out that this number is equal to d. □

3.3 Koszul Property and ASL structure of $\bar{A}(G)$

During an informal conversation at Oberwolfach in 2009, Herzog asked whether $\bar{A}(G)$ is Koszul provided that G is bipartite. In this section we answer this question positively, showing even more: if G is bipartite, then $\bar{A}(G)$ has a structure of homogeneous ASL.

Algebras with straightening laws (ASL's for short) were introduced by De Concini, Eisenbud and Procesi in [DEP82]. These algebras provide an unified treatment of both algebraic and geometric objects that have a combinatorial nature. For example, the coordinate rings of some classical algebraic varieties (such as determinantal rings and Pfaffian rings) have an ASL structure. For more details on this topic the reader can consult [BV88]. First, we will recall the definition of homogeneous ASL on posets.

Let $(P,<)$ be a finite poset and denote by $K[P] = K[X_p : p \in P]$ the polynomial ring over K whose variables correspond to the elements of P. Denote by I_P the following monomial ideal of $K[P]$:

$$I_P = (X_pX_q \ : \ p \text{ and } q \text{ are incomparable elements of } P).$$

Definition 3.15. Let $A = K[P]/I$, where I is a homogeneous ideal with respect to the standard grading. The graded algebra A is called a homogeneous ASL on P if

(ASL1) The residue classes of the monomials not in I_P are linearly independent in A.

(ASL2) For every $p,q \in P$ such that p and q are incomparable the ideal I contains a polynomial of the form

$$X_pX_q - \sum \lambda X_sX_t$$

with $\lambda \in K$, $s,t \in P$, $s \le t$, $s < p$ and $s < q$. The above sum is allowed to run on the empty-set.

The polynomials in (ASL2) give a way of rewriting in A the product of two incomparable elements. These relations are called the *straightening relations* or straightening laws.

A total order $<'$ on P is called a linear extension of the poset $(P,<)$ if $x < y$ implies $x <' y$. It is known that if τ is a revlex term order with respect to a linear extension of $<$, then the polynomials in (ASL2) form a Gröbner basis of I and $\mathrm{in}_\tau(I) = I_P$.

We will prove now that when G is a bipartite graph, $\bar{A}(G)$ has an ASL structure. Let us first fix some notation. Let G be a bipartite graph with the partition of the vertex set $[n] = A \cup B$ and suppose that $|A| \leq |B|$. We denote by $\mathscr{C}(G)$ the set of 1-covers of G which take values in $\{0,1\}$ (not necessarily basic). Equivalently $\mathscr{C}(G)$ is the set of vertex covers of G. We define on $\mathscr{C}(G)$ the following partial order: Given $\alpha, \beta \in \mathscr{C}(G)$, we say that

$$\alpha \leq \beta \quad \Longleftrightarrow \quad \alpha(a) \leq \beta(a) \ \forall \ a \in A \ \text{ and } \ \alpha(b) \geq \beta(b) \ \forall \ b \in B.$$

Actually with this partial order $\mathscr{C}(G)$ becomes a distributive lattice, as we are going to explain. We recall that a poset $\mathscr{L}$ is a *lattice* if every two elements $l, l' \in L$ have a supremum, denoted by $l \vee l'$, and a infimum, denoted by $l \wedge l'$. Furthermore we say that $\mathscr{L}$ is *distributive* if $l \vee (l' \wedge l'') = (l \vee l') \wedge (l \vee l'')$.

Remark 3.16. The poset structure we gave to $\mathscr{C}(G)$ actually confers a distributive lattice structure to $\mathscr{C}(G)$. Given $\alpha, \beta \in \mathscr{C}(G)$, set

$$(\alpha \vee \beta)(v) = \begin{cases} \max\{\alpha(v), \beta(v)\} & \text{if } v \in A, \\ \min\{\alpha(v), \beta(v)\} & \text{if } v \in B. \end{cases}$$

$$(\alpha \wedge \beta)(v) = \begin{cases} \min\{\alpha(v), \beta(v)\} & \text{if } v \in A. \\ \max\{\alpha(v), \beta(v)\} & \text{if } v \in B; \end{cases}$$

Clearly $\alpha \vee \beta$ and $\alpha \wedge \beta$ belong to $\mathscr{C}(G)$, and are respectively the supremum and the infimum of α and β. Moreover it is straightforward to verify the distributivity of these operations.

Let $\mathscr{P}(G)$ be the set of basic 1-covers of G. One has $\mathscr{P}(G) \subseteq \mathscr{C}(G)$, so the partial order on $\mathscr{C}(G)$ induces a poset structure also on $\mathscr{P}(G)$. Unfortunately, even if α and β are basic, it may happen that $\alpha \vee \beta$ or $\alpha \wedge \beta$ are not basic. So in general $\mathscr{P}(G)$ does not inherit the lattice structure from $\mathscr{C}(G)$.

Remark 3.17. Notice that the poset structure on $\mathscr{P}(G)$ can be read off only from A, or B. In fact, if α and β are basic 1-covers, we have $\alpha(a) \leq \beta(a) \ \forall \ a \in A \iff \alpha(b) \geq \beta(b) \ \forall \ b \in B$. Therefore, for all $\alpha, \beta \in \mathscr{P}(G)$, we have

$$\alpha \leq \beta \quad \Longleftrightarrow \quad \alpha(a) \leq \beta(a) \ \forall \ a \in A \quad \Longleftrightarrow \quad \alpha(b) \geq \beta(b) \ \forall \ b \in B.$$

For any $\alpha, \beta \in \mathscr{C}(G)$, it is easy to check the following equality:

$$\alpha + \beta = \alpha \wedge \beta + \alpha \vee \beta,$$

where the sum is componentwise. The above equality translates to a relation among the generators of $\bar{A}(G)$ in the following way. Denote by $R = K[\mathscr{P}(G)] = K[X_\alpha : \alpha \in \mathscr{P}(G)]$. We have the following natural presentation of $\bar{A}(G)$:

$$\begin{array}{rccc} \Phi: & R & \longrightarrow & \bar{A}(G) \\ & X_\alpha & \longmapsto & x_1^{\alpha(1)} \cdots x_n^{\alpha(n)} t \end{array}$$

For simplicity we set $X_{\alpha \vee \beta}$ (respectively $X_{\alpha \wedge \beta}$) to be 0 (as elements of R) whenever they are not basic 1-covers. Using this convention it is obvious that the polynomial

$$X_\alpha X_\beta - X_{\alpha \wedge \beta} X_{\alpha \vee \beta}$$

belongs to the kernel of Φ for any pair of basic 1-covers α and β. The main result of this section is the following theorem.

Theorem 3.18. *Let G be a bipartite graph. The algebra $\bar{A}(G)$ has a homogeneous ASL structure on $\mathscr{P}(G)$ over K. With the above notation, the straightening relations are*

$$\Phi(X_\alpha)\Phi(X_\beta) = \begin{cases} \Phi(X_{\alpha \wedge \beta})\Phi(X_{\alpha \vee \beta}) & \textit{if both } \alpha \vee \beta \textit{ and } \alpha \wedge \beta \textit{ are basic 1-covers,} \\ 0 & \textit{otherwise;} \end{cases}$$

for any α and β incomparable basic 1-covers. In particular we have

$$\ker \Phi = (X_\alpha X_\beta - X_{\alpha \wedge \beta} X_{\alpha \vee \beta} \ : \ \alpha \textit{ and } \beta \textit{ are incomparable basic 1-covers}).$$

Before proving Theorem 3.18, we will prove the following lemma.

Lemma 3.19. *Let G be a bipartite graph. Set*

$$\mathscr{M} = \{U \in R \,:\, U = X_{\alpha_1} \cdots X_{\alpha_d},\ d \in \mathbb{N},\ \alpha_1 \leq \ldots \leq \alpha_d\}.$$

The subset $\Phi(\mathscr{M}) \subseteq \bar{A}(G)$ consists of linearly independent elements of $\bar{A}(G)$.

Proof. First of all we will show that $\Phi(U) \neq 0$ for any $U \in \mathscr{M}$. By contradiction, suppose that there are basic 1-covers $\alpha_1 \leq \ldots \leq \alpha_d$ such that $U = X_{\alpha_1} \cdots X_{\alpha_d}$ is in the kernel of Φ. In other words the d-cover γ that associates to a vertex v the value $\gamma(v) = \alpha_1(v) + \ldots + \alpha_d(v)$ is non-basic. So there exists a vertex v_0 of G such that $\gamma(v_0) + \gamma(w) > d$ for any w adjacent to v_0. Let us assume that $v_0 \in A$ (otherwise the issue is similar). Set $q = \min\{i = 1, \ldots, d : \alpha_i(v_0) = 1\}$; if $\alpha_i(v_0) = 0$ for any i we set $q = d+1$. Since α_q is a basic 1-cover, there exists a vertex w_0 adjacent to v_0 such that $\alpha_q(v_0) + \alpha_q(w_0) = 1$. As $\alpha_1 \leq \ldots \leq \alpha_d$, we have $\alpha_i(v_0) = 0$ for all $i < q$, and $\alpha_j(w_0) = 0$ for all $j \geq q$ (because $w_0 \in B$ and $\alpha_q(w_0) = 0$). This implies that

$$\gamma(v_0) + \gamma(w_0) = \sum_{i=q}^{d} \alpha_i(v_0) + \sum_{j=1}^{q-1} \alpha_j(w_0) = (d-q+1) + (q-1) = d,$$

a contradiction.

Since $\{x_1^{\gamma(1)} \cdots x_n^{\gamma(n)} \;:\; \gamma \text{ is a basic } d\text{-cover}, d \in \mathbb{N}\}$ is a K-basis of $\bar{A}(G)$, it is enough to show that $\Phi(U) \neq \Phi(V)$ whenever U and V are different elements of $\mathscr{M}$. Suppose that $U = X_{\alpha_1} \cdots X_{\alpha_d}$ and $V = X_{\beta_1} \cdots X_{\beta_e}$. If $d \neq e$, using the facts proved above, we have $\Phi(U) \neq \Phi(V)$. Thus consider the case $d = e$. Since $U \neq V$, there exists an index $j = 1, \ldots, d$ such that $\alpha_j \neq \beta_j$. So there exists a vertex v_0 of G such that $\alpha_j(v_0) \neq \beta_j(v_0)$. Let us assume that $v_0 \in A$, $\alpha_j(v_0) = 0$ and $\beta_j(v_0) = 1$. The other cases are analog. Furthermore, up to a relabeling we can assume $v_0 = 1$. Since $\alpha_1 \leq \ldots \leq \alpha_d$ and $\beta_1 \leq \ldots \leq \beta_d$, we get $\alpha_i(1) = 0$ for all $i \leq j$ and $\beta_h(1) = 1$ for all $h \geq j$. So we have that $\Phi(U)$ has degree less than or equal to $d - j$ with respect to x_1, whereas $\Phi(V)$ has degree at least $d - j + 1$ with respect to it. Therefore they cannot be equal. □

Proof of Theorem 3.18. We have seen that $\bar{A}(G) = R/\ker\Phi$. Because G is bipartite, the graded K-algebra $\bar{A}(G)$ is generated by the elements $x^\alpha = x_1^{\alpha(1)} \cdots x_n^{\alpha(n)}$, with α a basic 1-cover. Moreover the degree of x^α is 1 if α is a basic 1-cover. So $\ker\Phi$ is homogeneous with respect to the standard grading of R . We need to see now that (ASL1) and (ASL2) are satisfied.

The first condition follows by Lemma 3.19. From the discussion preceding Theorem 3.18 we get that the polynomials $X_\alpha X_\beta - X_{\alpha\wedge\beta} X_{\alpha\vee\beta}$ belong to $\ker\Phi$. By construction $\alpha\wedge\beta < \alpha\vee\beta$, $\alpha\wedge\beta < \alpha$ and $\alpha\wedge\beta < \beta$ hold (whenever $\alpha\wedge\beta$ and $\alpha\vee\beta$ are basic 1-covers). So (ASL2) holds as well. The last part of the statement follows immediately from [BV88, Proposition 4.2]. □

As we said in the beginning of this section, the homogeneous ASL structure of $\bar{A}(G)$ implies that the straightening relations form a quadratic Gröbner basis. This implies the following corollary.

Corollary 3.20. *If G is a bipartite graph, then $\bar{A}(G)$ is a Koszul algebra.*

Remark 3.21. Independently and by different methods Rinaldo showed in [Rin11, Corollary 3.9] a particular case of Corollary 3.20. Namely he proved that $\bar{A}(G)$ is Koszul provided that G is a bipartite graph satisfying the weak square condition (see the definition below). Actually we will show in Corollary 3.22 that for such a graph $\bar{A}(G)$ is even a Hibi ring.

A special class of algebras with straightening laws are the so called Hibi rings. They were constructed in [Hib87] as an example of integral ASLs. The poset that supports their structure is a distributive lattice $\mathscr{L}$ and the straightening relations are given for any two incomparable elements $p, q \in \mathscr{L}$ by

$$X_p X_q - X_{p\wedge q} X_{p\vee q}.$$

In [BCV08] the following property for bipartite graphs was introduced, which was then extended in [BV11] for any graph. A graph G is said to have the weak square condition (WSC for short) if for every vertex $v \in V$, there exists an edge $\{v,w\} \in E$ containing it such that

$$\left.\begin{array}{c}\{v,v'\} \in E\\ \{w,w'\} \in E\end{array}\right\} \Rightarrow \{v',w'\} \in E.$$

We have the following corollary.

Corollary 3.22. *Let G be a bipartite graph. The following are equivalent:*

(i) *G satisfies the WSC;*

(ii) *$\bar{A}(G)$ is a domain;*

(iii) *$\bar{A}(G)$ is a Hibi ring on $\mathscr{P}(G)$ over K.*

Proof. The equivalence between (i) and (ii) was already proved in [BCV08, Theorem 1.9] and we present it here only for completeness. The fact that (iii) implies (ii) was proved by Hibi in the same paper where he introduced these algebras (see [Hib87, p. 100]). So we only need to prove that (ii) implies (iii).

For every $\alpha,\beta \in \mathscr{P}(G)$ that are incomparable, we must have $X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta} \in \ker\Phi$ by Theorem 3.18. Since $\bar{A}(G)$ is a domain, then both $\alpha\wedge\beta$ and $\alpha\vee\beta$ have to be basic 1-covers. In other words, in this case the poset $\mathscr{P}(G)$ inherits the lattice-structure from $\mathscr{C}(G)$. So by [Hib87, p.100] and by Theorem 3.18 we conclude. □

A classical structure theorem of Birkhoff [Bir67, p.59] states that for each distributive lattice $\mathscr{L}$ there exists a unique poset P such that $\mathscr{L} = J(P)$, where $J(P)$ is the set of poset ideals of P, ordered by inclusion. By Corollary 3.22 we have that if a bipartite graph G satisfies the WSC, then the poset of basic 1-covers $\mathscr{P}(G)$ is a distributive lattice. So by Birkhoff's result there exists a unique poset P_G such that $\mathscr{P}(G) = J(P_G)$. We use now another result of Hibi which describes completely the Gorenstein Hibi rings (see [Hib87, p.105]) to obtain the following corollary.

Corollary 3.23. *Let G be a bipartite graph satisfying the WSC. The following conditions are equivalent:*

(i) *$\bar{A}(G)$ is Gorenstein;*

(ii) *the poset P_G defined above is pure.*

We want to close this section showing some tools to deduce properties of $\bar{A}(G)$ from the combinatorics of $\mathscr{P}(G)$. In particular we will focus on the Cohen-Macaulayness of $\bar{A}(G)$, but one can read off by $\mathscr{P}(G)$ also the dimension, the multiplicity, and the Hilbert series of $\bar{A}(G)$.

The main technique is to consider the "canonical" initial ideal of the ideal defining $\bar{A}(G)$. Let I be the ideal, which we described above in terms of its generators, such that $\bar{A}(G) = R/I$ (recall that $R = K[\mathscr{P}(G)]$). Denote by $\mathrm{in}(I)$ the initial ideal of it with respect to a degrevlex term order associated to a linear extension of the partial order on $\mathscr{P}(G)$. From the results of this section it follows that $\mathrm{in}(I)$ is a square-free monomial ideal, so we can associate to it a simplicial complex $\Delta = \Delta(\mathrm{in}(I))$. Moreover it is easy to show that Δ is the order complex of $\mathscr{P}(G)$, i.e. its faces are the chains of $\mathscr{P}(G)$.

Example 3.24. *$\bar{A}(G)$ non Cohen-Macaulay*. Let G be a path of length $n-1 \geq 5$. So G is a graph on n vertices with edges:

$$\{1,2\},\ \{2,3\},\ \ldots,\ \{n-1,n\}.$$

For any $i = 1,\ldots,\lfloor n/2 \rfloor$ define the basic 1-cover

$$\alpha_i(j) = \begin{cases} 1 & \text{if } j = 2k \text{ and } k \leq i,\\ 1 & \text{if } j = 2k-1 \text{ and } k > i,\\ 0 & \text{otherwise.}\end{cases}$$

Then define also the basic 1-cover

$$\beta(j) = \begin{cases} 1 & \text{if } j = 1,3 \text{ or } j = 2k, \text{ with } k \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots \leq \alpha_{\lfloor n/2 \rfloor}$ and $\beta \leq \alpha_3 \leq \ldots \leq \alpha_{\lfloor n/2 \rfloor}$ are maximal chains of $\mathscr{P}(G)$. So $\mathscr{P}(G)$ is not pure. Therefore the order complex of $\mathscr{P}(G)$ is not pure. So $\bar{A}(G)$ is not an equidimensional ring by [KS95, Corollary 1]. In particular, if G is a path of length at least 5, $\bar{A}(G)$ is not Cohen-Macaulay.

Before stating the following result we recall some notion regarding posets. A poset P is *bounded* if it has a least and a greatest element. An element $x \in P$ *covers* $y \in P$ if $y \leq x$ and there not exists $z \in P$ with $y < z < x$. The poset P is said to be *locally upper semimodular* if whenever v_1 and v_2 cover u and $v_1, v_2 < v$ for some v in P, then there exists $t \in P$, $t \leq v$, which covers v_1 and v_2.

Theorem 3.25. *Let G be a bipartite graph and $A \cup B$ a bipartition of the vertex set with $|A| \leq |B|$. Moreover, let Δ be the order complex of $\mathscr{P}(G)$. If* $\operatorname{rank}(\mathscr{P}(G)) = |A|$, *then the following are equivalent:*

(i) $\bar{A}(G)$ *is equidimensional;*

(ii) $\mathscr{P}(G)$ *is a pure poset;*

(iii) Δ *is shellable;*

(iv) $\bar{A}(G)$ *is Cohen-Macaulay.*

Proof. (iv) $\Rightarrow$(i) is well known. As the Cohen-Macaulayness of $R/\operatorname{in}(I)$ implies the Cohen-Macaulayness of $R/I \cong \bar{A}(G)$, (iii) $\Rightarrow$ (iv) is also true. (i) $\Rightarrow$ (ii) follows by [KS95, Corollary 1].

(ii) $\Rightarrow$ (iii) To prove that Δ is shellable we will use a result of Björner (see [Bjö80, Theorem 6.1]), stating that it is enough to show that $\mathscr{P}(G)$ is a bounded locally upper semimodular poset. The poset $\mathscr{P}(G)$ is obviously bounded, so let α and β be two elements of $\mathscr{P}(G)$ which cover γ. The fact that $\operatorname{rank}(\mathscr{P}(G)) = |A|$ together with the pureness of $\mathscr{P}(G)$, imply that for a basic 1-cover ξ we have $\operatorname{rank}(\xi) = \sum_{v \in A} \xi(v)$. If α and β cover γ, since all the unrefinable chains between two comparable elements of a bounded pure complex have the same length, it follows that $s = \operatorname{rank}(\alpha) = \operatorname{rank}(\beta) = \operatorname{rank}(\gamma) + 1$. But $\gamma(v) \leq \min\{\alpha(v), \beta(v)\}$, for each $v \in A$, so if we look at the rank of the elements involved we obtain $\gamma(v) = \min\{\alpha(v), \beta(v)\}$ for all $v \in A$. Consider the (non necessarily basic) 1-cover, defined at the beginning of this section: $\alpha \vee \beta$. It is easy to see that, to make it basic, we can reduce its value at some vertex in B, and not in A. Let δ be the basic 1-cover obtained from $\alpha \vee \beta$. Then

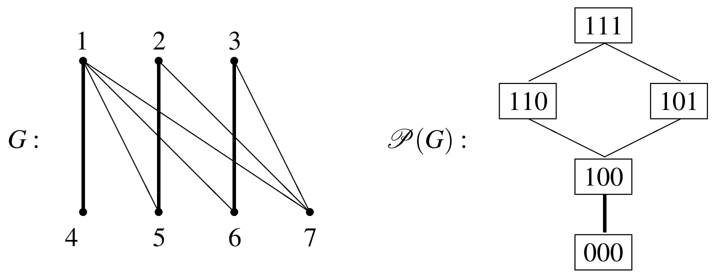
$$\operatorname{rank}(\delta) = \sum_{v \in A} \delta(v) = \sum_{v \in A} (\alpha \vee \beta)(v) = s + 1,$$

which implies that δ covers α and β. $\square$

By Theorems 3.8 and 3.18, we have that $\operatorname{rank}(\mathscr{P}(G)) = \nu_o(G)$, so the hypothesis of the theorem concerns just the combinatorics of the graph.

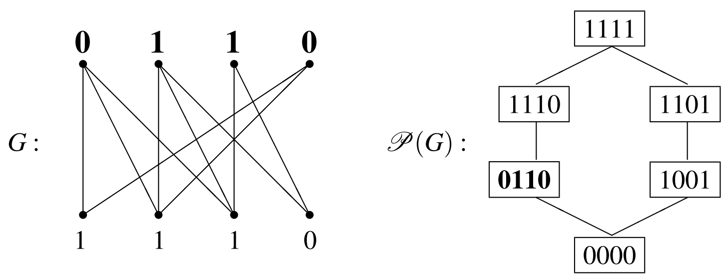
We showed in [BCV08] that $\bar{A}(G)$ domain implies $\bar{A}(G)$ Cohen-Macaulay. Given the above example and theorem it is natural to ask the following questions: “Can $\bar{A}(G)$ be Cohen-Macaulay and not a domain?". “Are there examples of graphs for which $\mathscr{P}(G)$ is pure but $\bar{A}(G)$ is not Cohen-Macaulay?". Both answers are positive and they are provided by the following examples.

Example 3.26. 1. $\bar{A}(G)$ *Cohen-Macaulay but not domain.* Consider the graph G on seven vertices below:



It is easy to see that G does not satisfy the WSC. We order the basic 1-covers component-wise with respect to the values they take on the vertex set $\{1,2,3\}$. It is clear from the Hasse diagram above that $\mathscr{P}(G)$ is pure. Moreover $\text{rank}(\mathscr{P}(G)) = \nu_o(G) = 3 = |A|$, so Theorem 3.25 implies that $\bar{A}(G)$ is Cohen-Macaulay.

2. $\mathscr{P}(G)$ *pure but* $\bar{A}(G)$ *not Cohen-Macaulay.* Consider the graph G in the picture below. It is not difficult to see that it has only six basic 1-covers. On the right you can see the Hasse diagram of the poset $\mathscr{P}(G)$. The values written next to the vertices represent the basic 1-cover written in bold on the right. Notice that the partial order is defined component-wise with respect to the values taken on the "upper" vertices of G.



The poset $\mathscr{P}(G)$ is pure, but the ordered complex of it is not strongly connected. Then I has an initial ideal not connected in codimension 1, so [Var09, Corollary 2.13] implies that $\bar{A}(G)$ is not Cohen-Macaulay.

3.4 Cohen-Macaulayness of Edge Ideals of Graphs with the Weak Square Condition

An interesting open problem, far to be solved, is to characterize in a combinatorial fashion all the Cohen-Macaulay graphs. The authors of [HH05] gave a complete answer when G is bipartite. On the other hand if G is Cohen-Macaulay then it is unmixed, and for bipartite unmixed graphs $\bar{A}(G)$ is the ordinary fiber cone of an ideal generated in one degree, so it is a domain. This means that a bipartite Cohen-Macaulay graph satisfies the WSC. Since many of these graphs are not bipartite (see [BV11] for details), a natural extension of the theorem of Herzog and Hibi would be to characterize all the graphs satisfying the WSC which are Cohen-Macaulay. We are able to do this defining for each graph G a "nicer" graph $\pi(G)$. This association behaves like a projection.

We start with a definition that makes sense by [BV11, Lemma 2.1].

Definition 3.27. We say that an edge $\{i,j\}$ of G is a *transversal edge* if one of the following equivalent conditions is satisfied:

(i) for any basic 1-cover α of G we have $\alpha(i)+\alpha(j)=1$;
(ii) for any basic k-cover α of G we have $\alpha(i)+\alpha(j)=k$;
(iii) if $\{i,i'\}$ and $\{j,j'\}$ are edges of G, then $\{i',j'\}$ is an edge of G as well (in particular $i'\neq j'$).

Notice that a graph satisfies the WSC if and only if every vertex belongs to a transversal edge. We recall that these graphs are of interest because they are exactly those graphs for which $\bar{A}(G)$ is a domain. In [BV11] the authors constructed from G a graph G^{0-1}, possibly with isolated vertices, in order to characterize the graphs for which all the symbolic powers of $J(G)$ are generated in one degree. We recall the definition:

1. $V(G^{0-1})=V(G)$;
2. $E(G^{0-1})=\{\{i,j\}\in E(G):\ \{i,j\}$ is a transversal edge of $G\}$.

It was proved in [BV11] that for any G the graph G^{0-1} is the disjoint union of some complete bipartite graphs $K_{r,s}$ (with $s\geq r\geq 1$) and some isolated points. Moreover G^{0-1} has no isolated vertices if and only if G satisfies the WSC.

We construct a new graph, that we will denote by $\pi(G)$, as follows: assume that

$$G^{0-1}=\Big(\bigcup_{i=1}^{m}K_{r_i,s_i}\Big)\bigcup\Big(\bigcup_{j=1}^{t}\{v_j\}\Big),$$

where the unions are disjoint unions of graphs, $r_i\geq s_i\geq 1$ and $v_j\in V(G)$. Denote by (A_i,B_i) the bipartition of K_{r_i,s_i} and choose for each i one vertex $a_i\in A_i$ and one vertex $b_i\in B_i$. We define the vertex set of $\pi(G)$ as

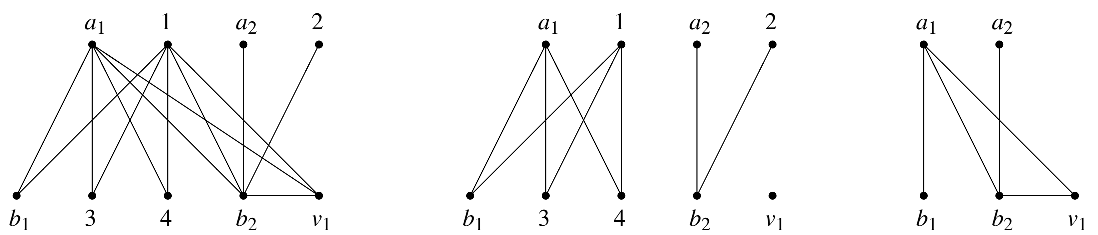
$$V(\pi(G))=\{a_i,b_i,v_j:\ i=1,\dots,m \text{ and } j=1,\dots,t\}.$$

The graph $\pi(G)$ will be the restriction of G to $V(\pi(G))\subseteq V(G)$. In particular

$$E(\pi(G))=\{\{i,j\}\in E(G): i,j\in V(\pi(G))\}.$$

By [BV11, Lemma 2.6] the definition of $\pi(G)$ does not depend from the choice of the vertices a_i and b_i. This is because, for any U and $W\in\{A_i,B_i,\{v_j\}: i=1,\dots,m \text{ and } j=1,\dots,t\}$, the existence of an edge from U to W is equivalent to the fact that the induced subgraph of G on the vertices of $U\cup W$ is bipartite complete. The notation π comes from the fact that the operator π is a projection, in the sense that $\pi(\pi(G))=\pi(G)$.

In the following picture we present an example of how this construction works:



G $\qquad$ G^{0-1} $\qquad$ $\pi(G)$

The following result is one of the reasons for introducing $\pi(G)$.

Proposition 3.28. *For every graph G, there is a well defined 1-1 correspondence*

$$\pi : \{\textit{basic covers of } G\} \longrightarrow \{\textit{basic covers of } \pi(G)\}$$

that associates to a basic k-cover α of G the basic k-cover $\pi(\alpha)$ of $\pi(G)$, with $\pi(\alpha)(v) = \alpha(v)$ for all $v \in V(\pi(G))$. Moreover this correspondence induces a graded isomorphism

$$\bar{A}(G) \cong \bar{A}(\pi(G)).$$

Proof. Using the fact that the edges between each A_i and B_i are transversal, it is straightforward to check that α has the same value on all vertices in A_i (resp. in B_i) for every $i = 1,\dots,m$. This implies that the definition of π does not depend on the choice of a_i and b_i for any i. It is easy to see that π is a bijection between the basic k-covers of G and those of $\pi(G)$; moreover this operation is compatible with the multiplicative structure on $\bar{A}(G)$ and of $\bar{A}(\pi(G))$. Therefore we also have a graded isomorphism between the algebras $\bar{A}(G)$ and $\bar{A}(\pi(G))$. □

Remark 3.29.
1. The previous Proposition provides another proof of the fact that $\bar{A}(G)$ is a Hibi ring when G is a bipartite graph satisfying the WSC. In fact in this case $\pi(G)$ is unmixed bipartite, so it is known that $\bar{A}(\pi(G))$ is a Hibi ring (for instance see [BCV08, Theorem 3.3]).
2. Proposition 3.28 shows also that $\mathscr{P}(G) = \mathscr{P}(\pi(G))$. So in order to study $\mathscr{P}(G)$ it can be convenient to pass to the projection and work on a graph with less vertices.

In some cases $\pi(G) = G$, for instance if G is a cycle on $n \neq 4$ vertices. The usefulness of $\pi(G)$ arises especially when G satisfies the WSC. As we already said in the above remark, in this case $\pi(G)$ is unmixed. Less trivially, we can strengthen this fact, but first we need a technical lemma.

Lemma 3.30. *Let G be a graph satisfying the WSC. Then there exists a unique perfect matching $M = \{\{u_i, v_i\} : i = 1,\dots,r\}$ of $\pi(G)$, where $r = |\pi(G)|/2$. Moreover it is possible to label the vertices of $\pi(G)$ in such a way that $\{v_1,\dots,v_r\}$ is an independent set of vertices of $\pi(G)$ and that the relation $v_i \prec v_j$ if and only $\{u_i, v_j\}$ is an edge defines a partial order on $V = \{v_1,\dots,v_r\}$.*

Proof. Since G satisfies the WSC, G^{0-1} has no isolated points, so we obtain a perfect matching $M = \{\{u_i, v_i\} : i = 1,\dots,r\}$ directly by construction. Moreover, since the edges of M are transversal, it immediately follows that for each 1-cover α of $\pi(G)$ we have $\sum_{v\in\pi(G)} \alpha(v) = r$. This implies that if N is another perfect matching of $\pi(G)$ then the r edges of N must be transversal. But the only transversal edges of $\pi(G)$ are those of M, therefore $M = N$.

We prove now that we can assume that $\{v_1,\dots,v_r\}$ is an independent set of vertices. In fact, suppose that there exist $i < j$ such that $\{v_i, v_j\}$ is an edge, and take the least j with this property. First notice that there exists no edge $\{u_j, v_k\}$ of $\pi(G)$ with $k < j$. The existence of such an edge would imply that also $\{v_k, v_i\}$ is an edge (as $\{u_j, v_j\}$ is transversal) and this would contradict the minimality of j. Now switch v_j and u_j. As we have seen that there are no edges $\{u_j, v_k\}$ with $k < j$, we can proceed with the same argument and assume that $\{v_1,\dots,v_r\}$ is an independent set of vertices.

To conclude we have to show that the relation

$$v_i \prec v_j \iff \{u_i, v_j\} \text{ is an edge of } \pi(G)$$

defines a partial order on V.

1. *Reflexivity* is obvious.
2. *Transitivity* is straightforward because $\{u_i, v_i\}$ is a transversal edge of $\pi(G)$, $\forall\, i = 1,\dots,r$.
3. *Anti-symmetry*: suppose there exist $i \neq j$ such that $v_i \prec v_j$ and $v_j \prec v_i$. Then $\{u_i, v_j\}$ and $\{u_j, v_i\}$ are both edges of $\pi(G)$. This contradicts [BV11, Lemma 2.6, point (3)].

□

We recall that if $I \subseteq S$ is a square-free monomial ideal we can associate to it the simplicial complex $\Delta(I)$ on the set $[n]$ such that $\{i_1, \dots, i_s\}$ belongs to $\Delta(I)$ if and only if $x_{i_1} \cdots x_{i_s}$ does not belong to I.

To prove the next result we need a theorem from [MRV08], that we are going to state in the case of graphs. We recall that a graph G has a perfect matching of *König type* if it has a perfect matching of cardinality $\mathrm{ht}(I(G))$.

Theorem 3.31. *(Morey, Reyes and Villareal [MRV08, Theorem 2.8]). Let G be an unmixed graph which admits a matching of König type. Assume that for any vertex v the induced subgraph on all the vertices of G but v has a leaf. Then $\Delta(I(G))$ is shellable.*

Thus we are ready to show the following.

Theorem 3.32. *Let G be a graph satisfying the WSC, and let $\Delta = \Delta(I(\pi(G))$. Then Δ is shellable. In particular $\pi(G)$ is a Cohen-Macaulay graph.*

Proof. We want to use Theorem 3.31. It is clear that $\pi(G)$ is unmixed because it has a perfect matching of transversal edges. Furthermore such a matching is obviously of König type. It remains to show that for any $v \in V(\pi(G))$, the induced subgraph of $\pi(G)$ on $V(\pi(G)) \setminus \{v\}$ has a leaf. Label the vertices of $\pi(G)$ as in Lemma 3.30 and in such a way that $v_i \prec v_j$ implies $i \leq j$. Since v_1 is a leaf, the only problem could arise when we remove from $\pi(G)$ either u_1 or v_1. If we remove u_1, then v_2 becomes a leaf, so we must show that the graph induced by $\pi(G)$ on $V(\pi(G)) \setminus \{v_1\}$ has a leaf.

Suppose there are no leaves. Then, denoting by $r = |V(\pi(G))|/2$, we can choose the minimum i such that $\{u_i, u_r\}$ is an edge (because u_r is not a leaf and by Lemma 3.30 these are the only possible edges, different from $\{u_r, v_r\}$, containing u_r). We claim that $i = 1$. If not, since v_i is not a leaf, there exists $j < i$ such that $\{u_j, v_i\}$ is an edge. But, since $\{u_i, v_i\}$ is a transversal edge, it follows that $\{u_j, u_r\}$ is an edge, contradicting the minimality of i. Now, since v_r is not a leaf, there exists a minimal $k < r$ such that $\{u_k, v_r\}$ is an edge. Arguing as above we have that $k = 1$. Then $\{u_1, u_r\}$ and $\{u_1, v_r\}$ are both edges, and this contradicts the fact that $\{u_r, v_r\}$ is transversal.

Therefore Δ is shellable by Theorem 3.31, and it is well known that this implies that $\pi(G)$ is a Cohen-Macaulay graph (for instance see the book of Bruns and Herzog [BH93, Theorem 5.1.13]). $\square$

For the following result we recall that an ideal $I \subseteq S$ is connected in codimension 1 if any two minimal primes $\wp, \wp'$ of I are 1-connected: i.e. there exists a path $\wp = \wp_1, \dots, \wp_m = \wp'$ of minimal primes of I such that $\mathrm{ht}(\wp_i + \wp_{i+1}) = \mathrm{ht}(I) + 1$. If $I = I_\Delta$ is a square-free monomial ideal, then I is connected in codimension 1 if and only if Δ is strongly connected, i.e. if and only if it is possible to walk from a facet to another passing through faces of codimension 1 in Δ.

Theorem 3.33. *Let G be a graph satisfying the WSC and set $\Delta = \Delta(I(G))$ the simplicial complex associated to the edge ideal. The following conditions are equivalent:*

(i) *G has a unique perfect matching;*
(ii) *G has a unique perfect matching of transversal edges;*
(iii) *$\pi(G) = G$;*
(iv) *Δ is shellable;*
(v) *G is Cohen-Macaulay;*
(vi) *$I(G)$ is connected in codimension 1.*

Proof. (iii) $\Rightarrow$ (iv) is Theorem 3.32. (iv) $\Rightarrow$ (v) follows from [BH93, Theorem 5.1.13]. (v) $\Rightarrow$ (vi) is a general fact proved by Hartshorne in [Har62]. (iii) $\Rightarrow$ (ii) follows immediately from Lemma 3.30.

We want to show that (vi) $\Rightarrow$(iii). Suppose $\pi(G) \neq G$. This means that there is a bipartite complete subgraph of G, say H, with more than two vertices and such that any edge of H is a transversal edge of G. Let $V(H) = A \cup B$ be the bipartition of the vertex set of H, and assume that $|A| \geq 2$. It is easy to construct a basic 1-cover α that associates 1 to the vertices in A and 0 to the vertices in B, and a basic 1-cover β that associates 0 to the vertices in A and 1 to the ones in B. Consider the two ideals of S

$$\begin{aligned} \wp_\alpha &= (x_i : \alpha(i) = 1), \\ \wp_\beta &= (x_i : \beta(i) = 1). \end{aligned}$$

The ideals $\wp_\alpha$ and $\wp_\beta$ are minimal prime ideals of $I(G)$. We claim that they are not 1-connected. If they were, then there would be a minimal prime ideal $\wp$ of $I(G)$ such that there exist $i, j \in A$ with $x_i \in \wp$ and $x_j \notin \wp$. Therefore the basic 1-cover γ associated to $\wp$ satisfies $\gamma(i) = 1$ and $\gamma(j) = 0$. As H is a complete bipartite graph each vertex of A is connected to all vertices of B. So, because γ is a 1-cover, it must also associate 1 to every vertex of B, and this contradicts the fact that H consists of transversal edges.

Now we are going to show that (ii) $\Rightarrow$ (iii). If G has a perfect matching of transversal edges it is straightforward to check that it is unmixed. By [BV11, Theorem 2.8], the connected components of G^{0-1} are all of the type $K_{r,r}$ for some $r \geq 1$. If G were different from $\pi(G)$, then at least one of the r's would be greater than 1. So we could find another perfect matching of transversal edges of G by changing the matching of $K_{r,r}$ induced by the initial matching on G.

For the implication (i) $\Rightarrow$ (ii), let $M = \{\{a_1,b_1\},\ldots,\{a_m,b_m\}\}$ be the unique perfect matching of G. Suppose that an edge in M, say $\{a_1,b_1\}$, is not transversal. Since G satisfies the WSC there is an $i > 1$ such that $\{a_1,b_i\}$ (resp. $\{a_1,a_i\}$) is a transversal edge. But then $\{b_1,a_i\}$ (resp. $\{b_1,b_i\}$) is an edge by the weak square condition. So $M' = \{\{a_1,b_i\}, \{a_2,b_2\},\ldots, \{a_i,b_1\}, \ldots, \{a_m,b_m\}\}$ (resp. $M' = \{\{a_1,a_i\}, \{a_2,b_2\},\ldots,\{b_i,b_1\}, \ldots,\{a_m,b_m\}\}$) is another matching, a contradiction.

It remains to show that (ii) $\Rightarrow$ (i). But we already proved that if (ii) holds then G is Cohen-Macaulay. In particular G is unmixed, so any other perfect matching of G is forced to consist of transversal edges. □

Whereas graphs whose edge ideal has a linear resolution have been completely characterized by Fröberg in [Frö90], it is still an open problem (even in the bipartite case) to characterize in a combinatorial fashion the Castelnuovo-Mumford regularity of the edge ideal. A general result in [Kat06] asserts that a lower bound for $\operatorname{reg}(S/I(G))$ is the maximum size of a pairwise disconnected set of edges of G. Moreover by the present paper it easily follows that the ordered matching number of G provides an upper bound for $\operatorname{reg}(S/I(G))$ (see the remark below). In [Zhe04] Zheng showed that if G is a tree, then $\operatorname{reg}(S/I(G))$ is actually equal to the maximum number of disconnected edges of G. Later, in [HV08], Hà and Van Tuyl showed that the same conclusion holds true for chordal graphs, and recently, the author of [Kum09] showed this equality in the bipartite unmixed case, too. As another application of the operator π, we show in Theorem 3.36 that this equality holds also for any bipartite graph satisfying the WSC, extending the result of Kummini. First notice that to prove his theorem Kummini defined a new graph, called the acyclic reduction, starting from a bipartite unmixed graph ([Kum09, Discussion 2.8]). It is possible to show that this new graph coincides with $\pi(G)$. So in some sense $\pi(G)$ can be seen as an extension to the class of all graphs of the acyclic reduction defined in [Kum09].

Remark 3.34. We showed in Corollary 3.11 that, for any graph G, we have $\operatorname{ara}(J(G)S_\mathfrak{m}) \leq \nu_o(G)+1$, provided the field K is infinite. Recall that the cohomological dimension of $J(G)S_\mathfrak{m}$ and the one of $J(G)$ agree, i.e. $\operatorname{cd}(S_\mathfrak{m}, J(G)S_\mathfrak{m}) = \operatorname{cd}(S, J(G))$. But by a result in [Lyu84] the cohomological dimension of $J(G)$ is equal to the projective dimension of $S/J(G)$. Since the cohomological dimension is a lower bound for the arithmetical rank, we have that $\operatorname{pd}(S/J(G)) \leq \nu_o(G)+1$. As $I(G)$ is the Alexander dual of $J(G)$, it follows by [Ter99] that

$$\operatorname{reg}(S/I(G)) \leq \nu_o(G).$$

The above inequality holds true also if K is finite, since the extension of scalars from $S/I(G)$ to $S/I(G) \otimes_K \bar{K}$, where $\bar{K}$ is the algebraic closure of K, does not change the regularity. Since $\nu_o(G)$ is less than or equal to the matching number of G by definition, the above inequality strengthens [HV08, Theorem 1.5].

Lemma 3.35. *Let G be any graph. Then*

$$\operatorname{reg}(S/I(G)) = \operatorname{reg}(S'/I(\pi(G))),$$

where $S' = K[y_1, \ldots, y_p]$ is the polynomial ring in $p = |V(\pi(G))|$ variables over K.

Proof. For any $i = 1, \ldots, p$ call V_i the set of vertices of G that collapses to the vertex i of $\pi(G)$. Then consider the homomorphism

$$\begin{array}{rcl} \phi : S' & \longrightarrow & S \\ y_i & \mapsto & \prod_{j \in V_i} x_j =: m_i \end{array}$$

By the correspondence of basic 1-covers of G and $\pi(G)$ described in Proposition 3.28, one easily sees that $\phi(J(\pi(G)))S = J(G)$. Moreover it is obvious that $m_1, \ldots, m_p$ form a regular sequence of S, so by a theorem of Hartshorne ([Har66, Proposition 1]) S is a flat S'-module via ϕ. Then if $F_\bullet$ is a minimal free resolution of $S'/J(\pi(G))$ over S' it follows that $F_\bullet \otimes_{S'} S$ is a minimal free resolution of $S/J(G)$ over S. Therefore the total Betti numbers of $S'/J(\pi(G))$ and of $S/J(G)$ are the same, and in particular $\operatorname{pd}(S/J(G)) = \operatorname{pd}(S'/J(\pi(G)))$. Thus [Ter99] yields the conclusion. □

Theorem 3.36. *Let G be a bipartite graph satisfying the WSC. Then the Castelnuovo-Mumford regularity of $S/I(G)$ is equal to the maximum size of a pairwise disconnected set of edges of G.*

Proof. By Lemma 3.35, using the same notation, $\operatorname{reg}(S/I(G)) = \operatorname{reg}(S'/I(\pi(G)))$. Moreover, the maximum size of a pairwise disconnected set of edges in G is equal to the same number for $\pi(G)$. Since $\pi(G)$ is Cohen-Macaulay by Theorem 3.32, one can deduce the conclusion using [HH05, Corollary 2.2.b]. □

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Chapter 4

h-vectors of matroid complexes

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Division of Labor

This project started during Matteo Varbaro's visit at the University of Basel where I was employed. All creative work was done together with pen and paper, so a clear separation of the important contributions is not possible. It is however possible to say that both authors contributed in an equal manner to the conception, execution and writing of this work.

Introduction

In 1977 Stanley conjectured that the h-vectors of matroids are pure O-sequences [Sta77, p.59], that is they are h-vectors of Artinian monomial level algebras or, equivalently, f-vectors of pure order ideals. Ever since, the h-vectors of matroids have been in the focus of many researchers (see [Hib89; Hib92; Cha97; Mas+97; Swa05; Spe09]). Pure O-sequences themselves have attracted a lot of attention as well, quite a few conjectures being made regarding their shape ([Boi+12] gives an overview of the topic). Although several researchers have approached Stanley's conjecture, to our knowledge only very specific cases have been proven. The case of cographic matroids was proven in [Cha97; Mer01], that of lattice path matroids in [Sch10] and more generally the one of cotransversal matroids in [Oh10]. Low rank and degree situations were recently investigated in [Sto08; Sto09; HSZ13].

In the present paper we prove Stanley's conjecture in several cases, which appear in every rank and codimension. As a particular case, we obtain the conjecture for all matroid complexes of Cohen-Macaulay type 2. For any positive integers n and d, we divide the $(d-1)$-dimensional matroids on n vertices in different classes, which are indexed by the partitions of n with length at least d. For each class we build the set of all possible h-vectors of the duals of the matroids in the respective class. We then identify two special matroids whose duals have minimal, respectively maximal h-vectors in that set. For all these extremal matroids we prove in a constructive way that Stanley's conjecture holds.

Our approach passes via an equivalent phrasing of Stanley's conjecture. The h-vector of a matroid Δ is defined as the h-vector of the corresponding Stanley-Reisner ring and we will denote it by h_Δ. To a simplicial complex in general, apart from the Stanley-Reisner ideal I_Δ, one can associate its vertex cover ideal $J(\Delta)$. We will denote the h-vector of the quotient ring of $J(\Delta)$ by h^Δ. If we denote by Δ^c the dual of Δ (that is the simplicial complex generated by the complements of the facets in the vertex set), we have

that
$$J(\Delta^c) = I_\Delta \qquad \text{and} \qquad h^{\Delta^c} = h_\Delta.$$
A classical theorem of matroid theory says that Δ is a matroid if and only if Δ^c is a matroid. This implies the following equivalent formulation of Stanley's conjecture:

Conjecture (Stanley)**.** *For any matroid Δ, the vector h^Δ is a pure O-sequence.*

Let us summarize the contents of the paper. Section 4.1 is mainly devoted to preliminary results and establishing the notation. Nevertheless, we show in Corollary 4.4 an equality involving the cover ideals of certain matroid complexes. This equality supplies the exact sequence (4.1), which will be a crucial tool throughout the paper. The existence of this exact sequence depends heavily on the properties of matroids. In Remark 4.10 we also present a counterexample to the Interval Conjecture for Pure O-sequences formulated by Boij et al. in [Boi+12].

In Section 4.2, we first provide some structural results for matroid complexes. We show that the 1-skeleton of a matroid is a complete p-partite graph. The division of the matroids into classes will be done in correspondence with these partitions of the vertex set. In each class we then define $d-1$ matroids: $\Delta_t(d,p,\mathbf{a})$, for $t = 0,\dots,d-2$, where $\mathbf{a}$ is the partition of n. All these matroids are representable over fields with "enough" elements, and in most cases they are neither graphic nor transversal. We will call $\Delta_0(d,p,\mathbf{a})$ complete p-partite matroids. These are a simultaneous generalization of both uniform and partition matroids.

Later on in this section, we attach to each matroid Δ another matroid ${}^{\mathrm{si}}\Delta$, named simplified matroid, of the same dimension but on less vertices. The simplified matroid reflects many properties of the original matroid. For example, the total Betti numbers of $J(\Delta)$ and $J({}^{\mathrm{si}}\Delta)$ are the same (Proposition 4.18). In Proposition 4.20 we provide a formula which computes h^Δ for 1-dimensional matroids. It turns out that the set of h-vectors of matroid complexes of the type $(1,2,h_2,\dots,h_s)$ coincides with the set of pure O-sequences of the form $(1,2,h_2,\dots,h_s)$.

In Section 4.3 we prove the conjecture of Stanley in various instances. In Theorem 4.27 we show that h^Δ is a pure O-sequence whenever Δ is a $(d-1)$-dimensional complete p-partite matroid for some $p \geq d$. Using Theorem 4.27, we prove the more general statement that $\mathrm{h}^{\Delta_t(d,p,\mathbf{a})}$ is a pure O-sequence for all $t = 0,\dots,d-2$ (Theorem 4.29).

In Section 4.4, for any partition $\mathbf{a}$ of n with $p \geq d$ parts, we denote by $\mathscr{M}(d,p,\mathbf{a})$ the set of $(d-1)$-matroids on n vertices, whose 1-skeleton is p-partite and the cardinalities of the partition sets correspond to $\mathbf{a}$. By the results of Section 4.2, every matroid belongs to exactly one of these sets. In Theorems 4.34 and 4.37, we show that
$$\mathrm{h}^{\Delta_{d-2}(d,p,\mathbf{a})} \leq \mathrm{h}^\Delta \leq \mathrm{h}^{\Delta_0(d,p,\mathbf{a})}, \ \ \forall\, \Delta \in \mathscr{M}(d,p,\mathbf{a}).$$
We are able to compute the Cohen-Macaulay type of each $\Delta_t(d,p,\mathbf{a})$. From this and the above inequalities we can settle Stanley's conjecture whenever the Cohen-Macaulay type of S/I_Δ is less than or equal to two. In other words, we establish Stanley conjecture for all the h-vectors of type $(h_0,h_1,h_2,\dots,h_{s-1},2)$.

4.1 Preliminaries

In this section we will recall most of the algebraic and combinatorial notions that we will use throughout the paper. For general aspects on the topics presented below we refer the reader to the books of Stanley [Sta96], of Bruns and Herzog [BH93] and of Oxley [Oxl11].

For a positive integer n denote by $[n]$ the set $\{1,\dots,n\}$. A simplicial complex Δ on $[n]$ is a collection of subsets of $[n]$ such that $F \in \Delta$ and $F' \subset F$ imply $F' \in \Delta$. Notice that we are not requiring that $\bigcup_{F\in\Delta} F = [n]$, therefore Δ can be viewed as a simplicial complex on any overset of $\bigcup_{F\in\Delta} F$. Each element $F \in \Delta$ is called a *face* of Δ. The dimension of a face F is $|F|-1$ and the *dimension of* Δ is $\max\{\dim F \ : \ F \in \Delta\}$. A maximal face of Δ with respect to inclusion is called a *facet* and we will denote by $\mathscr{F}(\Delta)$ the set of facets of Δ. A simplicial complex is called *pure* if all facets have the same cardinality. We call a vertex v a *cone point* of Δ if $v \in F$ for any $F \in \mathscr{F}(\Delta)$. If $F_1, \dots, F_m$ are subsets of $[n]$, then we denote by $\langle F_1, \dots, F_m \rangle$ the smallest simplicial complex on $[n]$ containing them. Explicitly:

$$\langle F_1, \dots, F_m \rangle = \{F \subset [n] \ : \ \exists\, i \in \{1, \dots, m\} : F \subset F_i\}.$$

We say that $F_1, \dots, F_m$ generate the simplicial complex $\langle F_1, \dots, F_m \rangle$. Clearly every simplicial complex is generated by its set of facets. For any face F the *link* of F in Δ is the following simplicial complex:

$$\mathrm{link}_\Delta F = \{F' \in \Delta \ : \ F' \cup F \in \Delta \text{ and } F' \cap F = \emptyset\}.$$

For a set of vertices $W \subset [n]$, the *restriction* of Δ to W is the following subcomplex of Δ:

$$\Delta|_W = \{F \in \Delta \ : \ F \subset W\}.$$

The subcomplex $\Delta|_W$ is also called the subcomplex of Δ *induced by* the vertex set W. If F is a face of Δ, then the *face deletion* of F in Δ is $\Delta \setminus F = \{F' \in \Delta \ : \ F \not\subseteq F'\}$. Whenever F is a 0-dimensional face $\{v\}$ we will just write $\Delta \setminus v$ for the face deletion of $\{v\}$ and $\mathrm{link}_\Delta v$ for the link of $\{v\}$. Notice that $\Delta \setminus v = \Delta|_{[n]\setminus\{v\}}$ for all $v \in [n]$. The *dual complex* of Δ is the simplicial complex Δ^c on $[n]$ with facets:

$$\mathscr{F}(\Delta^c) = \{[n] \setminus F \ : \ F \in \mathscr{F}(\Delta)\}.$$

For any integer $0 \le k \le \dim\Delta$, the *k-skeleton* of Δ is defined as the simplicial complex with facet set $\{F \in \Delta \ : \ \dim F = k\}$.

We will now associate to a simplicial complex two square-free monomial ideals. We will then see how these ideals are related via the dual complex. Denote by $S = \Bbbk[x_1, \dots, x_n]$ the polynomial ring in n variables over a field $\Bbbk$. For each subset $F \subset [n]$ define the monomial $\mathtt{x}_F$ and the prime ideal $\mathscr{P}_F$ as follows:

$$\begin{aligned} \mathtt{x}_F &= \prod_{i \in F} x_i, \\ \mathscr{P}_F &= (x_i \ : \ i \in F). \end{aligned}$$

The *Stanley-Reisner ideal* of Δ is the ideal I_Δ of S generated by the square-free monomials $\mathtt{x}_F$, with $F \notin \Delta$. In particular we have

$$I_\Delta = (\mathtt{x}_F \ : \ F \text{ is a minimal nonface of } \Delta).$$

The second square-free monomial ideal we can associate to Δ is the *cover ideal* of Δ:

$$J(\Delta) = \bigcap_{F \in \mathscr{F}(\Delta)} \mathscr{P}_F.$$

The name "cover ideal" comes from the following fact. A collection of vertices $A \subset [n]$ is called a *vertex cover* of Δ if $A \cap F \neq \emptyset$ for any $F \in \mathscr{F}(\Delta)$. A vertex cover A is called *basic* if no proper subset of A is again a vertex cover. It is easy to check that we have

$$J(\Delta) = (\mathtt{x}_A \ : \ A \text{ is a basic vertex cover of } \Delta).$$

It is a well known fact that the prime decomposition of the Stanley-Reisner ideal is

$$I_\Delta = \bigcap_{F \in \mathscr{F}(\Delta)} \mathscr{P}_{[n]\setminus F}.$$

The following equality, which follows directly from the definition, will be very important for the approach of this paper:

$$J(\Delta) = I_{\Delta^c}.$$

We denote by $\Bbbk[\Delta] = S/I_\Delta$ the *Stanley-Reisner ring* of Δ. Let $h_{\Bbbk[\Delta]} = (h_0, h_1, \ldots, h_s)$ be its h-vector. If $\mathrm{HS}_{\Bbbk[\Delta]}(t)$ is the Hilbert series of $\Bbbk[\Delta]$, then we have

$$\mathrm{HS}_{\Bbbk[\Delta]}(t) = \frac{h_0 + h_1 t + \ldots + h_s t^s}{(1-t)^d},$$

where $h_s \neq 0$ and $d = \dim \Bbbk[\Delta] = \dim \Delta + 1$.

In the classical terminology, the h-vector of a simplicial complex is the h-vector of its Stanley-Reisner ring. As we will mainly deal with cover ideals, in order to avoid the over-use of the word dual, we will fix the following notation and terminology.

Notation 4.1. Let Δ be any simplicial complex.

1. We denote the h-vector of $\Bbbk[\Delta]$ by h_Δ.
2. We denote the h-vector of $S/J(\Delta)$ by h^Δ.
3. We will refer throughout this paper to h^Δ as *the h-vector of* Δ.

Notice that we have the correspondence: $h^\Delta = h_{\Delta^c}$.

For a $(d-1)$-dimensional simplicial complex Δ on $[n]$ such that $S/J(\Delta)$ is Cohen-Macaulay, we denote by $\mathrm{type}(\Delta)$ the last total Betti number in the minimal free resolution of $S/J(\Delta)$, namely

$$\mathrm{type}(\Delta) = \beta_d(S/J(\Delta)) = \dim_\Bbbk \mathrm{Tor}_d^S(S/J(\Delta), \Bbbk).$$

Matroid theory was born out of the need to study the concept of dependence in an abstract way. In this paper we will view matroids as simplicial complexes whose faces correspond to the independent sets. A characteristic of matroids is that they admit many different but equivalent definitions (see [Oxl11] and [Sta96, Chapter III.3]). We present here three of them.

Definition 4.2. A simplicial complex Δ is called a *matroid complex* (or just matroid) if one of the following equivalent properties hold:

1. *The augmentation axiom:* For any two faces $F, G \in \Delta$ with $|F| < |G|$ there exists $i \in G$ such that $F \cup \{i\} \in \Delta$.
2. *The exchange property:* For any two facets $F, G \in \mathscr{F}(\Delta)$ and for any $i \in F$ there exists a $j \in G$ such that $(F \setminus \{i\}) \cup \{j\} \in \Delta$.
3. For any subset $W \subset [n]$ the restriction $\Delta|_W$ is pure.

A basic result in matroid theory that we will exploit a lot is the following:

Theorem 4.3 ([Oxl11]Theorem 2.1.1)**.** *A simplicial complex Δ on $[n]$ is a matroid if and only if Δ^c is a matroid.*

An algebraic characterization of matroid complexes has been given in [MT11] and [Var11], namely: a simplicial complex Δ is a matroid iff all the symbolic powers of I_Δ are Cohen-Macaulay. Another algebraic property that will be important for us (even if it does not characterize matroids) is the following: the Stanley-Reisner ring of a matroid is level ([Sta96, Chapter III, Theorem 3.4]). This means that $\Bbbk[\Delta]$ is Cohen-Macaulay and the socle of its Artinian reduction lies in exactly one degree. A prototype of level algebras are the Gorenstein algebras, which correspond to socle dimension 1. An important consequence of $S/J(\Delta)$ being level is that the type can be expressed only in terms of the last entry of the h-vector, namely

$$\mathrm{type}(\Delta) = \mathrm{h}^{\Delta}(s) \quad \text{where } \ s = \max\{i : \mathrm{h}^{\Delta}(i) \neq 0\}.$$

The following lemma is important because it provides a recursive formula for the h-vectors. This formula will be a main ingredient in many of our proofs. The lemma itself can also be interpreted from a liaison-theoretical point of view, namely it is easy to check that the ideal relation provides a basic double link.

Lemma 4.4. *If Δ is a matroid on $[n]$ and $v \in \Delta$ is a vertex that is not a cone point, then*

$$J(\Delta) = x_v J(\Delta \setminus v) + J(\mathrm{link}_{\Delta} v).$$

Proof. We will first make a general observation. Let Γ be a simplicial complex on $[n]$ and consider $\Gamma \setminus n$ and $\mathrm{link}_{\Gamma} n$ as simplicial complexes on $[n-1]$. It is straightforward to show that, if $\Gamma \setminus n$ is pure and of the same dimension as Γ, then $(\Gamma \setminus n)^c = \mathrm{link}_{\Gamma^c} n$.

We may obviously assume that $v = n$. Since Δ is a matroid, $\Delta \setminus n$ is pure. Since n is not a cone point, it has the same dimension as Δ. Therefore, by the above observation we have

$$(\Delta \setminus n)^c = \mathrm{link}_{\Delta^c} n.$$

As we assumed that $n \in \Delta$, we have that n is not a cone point of Δ^c. By Theorem 4.3 and the general observation we obtain also that

$$\Delta^c \setminus n = ((\Delta^c \setminus n)^c)^c = (\mathrm{link}_{(\Delta^c)^c} n)^c = (\mathrm{link}_{\Delta} n)^c.$$

For all simplicial complexes Γ on $[n]$ we have the following equality for Stanley-Reisner ideals:

$$I_{\Gamma} = x_n I_{\mathrm{link}_{\Gamma} n} + I_{\Gamma \setminus n}.$$

Exploiting it for $\Gamma = \Delta^c$, together with $I_{\Delta^c} = J(\Delta)$ and all the above observations, we conclude. □

Remark 4.5. Whereas the equality $I_{\Delta} = x_n I_{\mathrm{link}_{\Delta} n} + I_{\Delta \setminus n}$ holds true for any simplicial complex, the equality $J(\Delta) = x_v J(\Delta \setminus v) + J(\mathrm{link}_{\Delta} v)$ depends strongly on the fact that Δ is a matroid. For instance, Lemma 4.4 already fails for any vertex of a path of length three.

Remark 4.6. From the point of view of Gorenstein liaison, Lemma 4.4 implies that the ideals $J(\Delta)$ can be linked to a complete intersection. A more general statement in this direction has been proven by Nagel and Römer in [NR08].

Remark 4.7. Lemma 4.4 gives rise to the exact sequence:

$$0 \longrightarrow J(\mathrm{link}_{\Delta} v)(-1) \longrightarrow J(\Delta \setminus v)(-1) \oplus J(\mathrm{link}_{\Delta} v) \longrightarrow J(\Delta) \longrightarrow 0.$$

The above exact sequence yields the following relation for the Hilbert functions:

$$\mathrm{HF}_{J(\Delta)}(k) = \mathrm{HF}_{J(\Delta \setminus v)}(k-1) + \mathrm{HF}_{J(\mathrm{link}_{\Delta} v)}(k) - \mathrm{HF}_{J(\mathrm{link}_{\Delta} v)}(k-1) \quad \forall\, k \in \mathbb{Z},$$

which in turn yields:

$$\mathrm{HF}_{S/J(\Delta)}(k) = \mathrm{HF}_{S/J(\Delta \setminus v)}(k-1) + \mathrm{HF}_{S/J(\mathrm{link}_{\Delta} v)}(k) - \mathrm{HF}_{S/J(\mathrm{link}_{\Delta} v)}(k-1) \quad \forall\, k \in \mathbb{Z}.$$

Eventually, taking differences, for every matroid Δ and every $v \in \Delta$ that is not a cone point:

$$\mathrm{h}^{\Delta}(k) = \mathrm{h}^{\Delta \setminus v}(k-1) + \mathrm{h}^{\mathrm{link}_{\Delta} v}(k) \quad \forall\, k \in \mathbb{Z}. \tag{4.1}$$

Formula (4.1) will be crucial throughout the paper.

An *order ideal* is a finite collection Γ of monomials of some standard graded polynomial ring, such that $M \in \Gamma$ and N divides M imply $N \in \Gamma$. The partial order given by the divisibility of monomials gives Γ a poset structure. An order ideal is called *pure* if all maximal monomials have the same degree. To every order ideal Γ we associate its f-vector: $f(\Gamma) = (f_0(\Gamma), \dots, f_s(\Gamma))$, where for every $i = 0, \dots, d$ we have

$$f_i(\Gamma) = |\{M \in \Gamma \,:\, \deg(M) = i\}|.$$

A *pure O-sequence* is a vector $h = (h_0, \dots, h_s)$ that can be obtained as the f-vector of some pure order ideal.

Remark 4.8. Pure O-sequences can also be presented as the h-vectors of Artinian monomial level algebras, i.e. Artinian level algebras A which are isomorphic to R/I for some polynomial ring R and some monomial ideal $I \subset R$. It is very easy to see that, in this situation, if A is Gorenstein then I is forced to be a complete intersection. So the pure O-sequences of type $(h_0, h_1, \dots, h_{s-1}, 1)$ are well understood: they are h-vectors of complete intersections. In particular, it emerges that pure O-sequences are much more special than h-vectors of level algebras in general.

A characterization of pure O-sequences of the type $(h_0, h_1, \dots, h_{s-1}, 2)$, i.e. when the Artinian monomial level algebra A has Cohen-Macaulay type 2, is not known (see [Boi+12]).

In [Sta77] Stanley phrased his conjecture in terms of the h-vector of the Stanley-Reisner ring. By Theorem 4.3 an equivalent statement is the following:

Conjecture 4.9 (Stanley)**.** *If Δ is a matroid, then the h-vector of $S/J(\Delta)$ is a pure O-sequence.*

Conjecture 4.9 is known for some families; we list here the most general of them.

1. When $S/J(\Delta)$ is Gorenstein, see [Sto08, Theorem 4.4.10].

2. When $\mathrm{h}^{\Delta} = (1, h_1, h_2, h_3)$, see [Sto08] and [HSZ13].

3. When Δ is a graphic matroid, see [Mer01].

4. When Δ is a transversal matroid, see [Oh10].

5. When $\mathrm{h}^{\Delta} = (1, 2, h_2, \dots, h_s)$. Indeed, one can see by the Hilbert-Burch theorem that, in the codimension 2 case, pure O-sequences coincide with h-vectors of level algebras (see [Boi+12, Proposition 4.5] for the precise proof), so one can deduce the validity of the conjecture in this case by [Sta96, Chapter III, Theorem 3.4].

Computational experiments using the computer algebra system CoCoa [CoC] were an important part in the preparation of this work. In our investigation, we found a counterexample to the Interval Conjecture for Pure O-sequences (see [Boi+12]).

Remark 4.10. One can check that the vectors $(1, 4, 10, 13, 12, 9, 3)$ and $(1, 4, 10, 13, 14, 9, 3)$ are pure O-sequences. Indeed, the order ideals are generated by $\{x^3y^2z,\ x^3yt^s2,\ x^3z^2t\}$, respectively by $\{x^4y^2,\ x^3yzt,\ x^2z^2t^2\}$. Looking at all possible choices of three monomials of degree 6 in 4 variables, it is possible to compute all the pure O-sequences of the form $(1, 4, h_2, \dots,\ h_5, 3)$. Checking the obtained list, one can realize that $(1, 4, 10, 13, 13, 9, 3)$ does not appear among the pure O-sequences, a contradiction to the above-mentioned conjecture.

4.2 The structure of matroids

In this paper we will stratify the set of matroids of fixed dimension and on a fixed vertex set in terms of partitions of the vertex set. To this aim, in this section we will prove some technical facts. Most of these are well known facts for matroid theory specialists, however we consider it convenient to provide proofs as well. We will then present the simplified matroid associated to any given matroid. This matroid has only trivial parallel classes, but important information, such as the total Betti numbers $\beta_i(S/J(\Delta))$, is preserved. We conclude the section presenting a formula that computes the h-vector of a codimension two Stanley-Reisner ring of a matroid.

From now on, unless otherwise stated, we will consider simplicial complexes Δ on $[n]$ with the property that $v \in \Delta$ for all $v \in [n]$. Notice that the number of vertices not belonging to Δ does not influence h^{Δ}, so this is no restriction in terms of our goals. This assumption can be also expressed as $[n] = \bigcup_{F \in \Delta} F$ and if Δ is a $(d-1)$-dimensional matroid on $[n]$, a remark in [Sta96, p. 94] implies that:

$$n - d = \max\{i : \mathrm{h}^{\Delta}(i) \neq 0\}.$$

The following easy remark is the starting point for many of the following technical results.

Remark 4.11. If Δ is a 1-dimensional simplicial complex on $[n]$, then Δ is a matroid if and only if for any $v, w \in [n]$ with $\{v, w\} \notin \Delta$ we have that $\mathrm{link}_{\Delta}(v) = \mathrm{link}_{\Delta}(w)$.

One-dimensional simplicial complexes can be viewed as graphs on the same vertex set; the edges are the faces of dimension one. For this reason we will switch between graph and simplicial complex whenever we find ourselves in this case. Let us recall that a graph is called a *complete p-partite* graph if and only if its vertex set can be partitioned into p disjoint nonempty sets $A_1, \dots, A_p$ such that $\{v, w\}$ is an edge if and only if v and w lie in different sets of the partition. The following proposition shows that one-dimensional matroids and complete p-partite graphs are actually the same thing.

Proposition 4.12. *If Δ is a 1-dimensional matroid, then Δ is a complete p-partite graph, for some integer $p \geq 2$.*

Proof. We will proceed by induction on n, the number of vertices. Assume that $n \geq 2$, choose v a vertex of Δ and consider the set $A_v = \{w \in \Delta \ : \ \{v, w\} \notin \Delta\}$. As Δ is a matroid, we have by Remark 4.11 that $\mathrm{link}_{\Delta} v = \mathrm{link}_{\Delta} w$ for any $w \in A_v$. This implies that A_v is an independent set of vertices. Clearly $\mathrm{link}_{\Delta} v$ is a 0-dimensional simplicial complex whose faces correspond to the elements in $[n] \setminus A_v$. Moreover, as one can check by definition, the restriction $\Delta|_{[n]\setminus A_v}$ is also a matroid.

If $\dim \Delta|_{[n]\setminus A_v} = 1$, we have by induction that $\Delta|_{[n]\setminus A_v}$ is a complete p-partite graph, with p-partition of the vertex set $A_1 \cup \ldots \cup A_p$. In this case it follows that Δ is a complete $(p+1)$-partite graph with partition $[n] = A_v \cup A_1 \cup \ldots \cup A_p$.

If $\dim \Delta|_{[n]\setminus A_v} = 0$ then $[n] \setminus A_v$ is an independent set of vertices, so Δ is a complete bipartite graph with bipartition $[n] = A_v \cup ([n] \setminus A_v)$. □

The next corollary gives a stratification of the set of all $(d-1)$-dimensional matroids on $[n]$ that will be crucial throughout this work. Clearly, the k-skeleton of a matroid is again a matroid, so we have the following.

Corollary 4.13. *Let Δ be a simplicial complex. If Δ is a matroid, then there exists a positive integer $p \geq 2$ such that the 1-skeleton of Δ is a complete p-partite graph.*

Before showing the next technical lemmas let us fix more notation. From now on, exploiting Corollary 4.13, Δ will be a $(d-1)$-dimensional matroid on $[n]$, with p-partition of its 1-skeleton $A_1, \dots, A_p$. We will call the sets of independent vertices given by the p-partition *parallel classes*. Whenever necessary we will denote the vertices of a given parallel class as follows

$$A_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,a_i}\}.$$

For any integer $r \in \{1,\ldots,p\}$ and any indices $1 \le i_1 < \ldots < i_r \le p$ we denote by $\Delta_{i_1,\ldots,i_r}$ the restriction of Δ to the vertex set $A_{i_1} \cup \ldots \cup A_{i_r}$. We call $\Delta_{i_1,\ldots,i_r}$ the restriction of Δ to the parallel classes $A_{i_1},\ldots,A_{i_r}$.

Lemma 4.14. *If for $r \le d$ parallel classes $A_{i_1},\ldots,A_{i_r}$, with $1 \le i_1 < \ldots < i_r \le p$, there exist r vertices $v_{i_j} \in A_{i_j}$ such that $\{v_{i_1},\ldots,v_{i_r}\} \in \Delta$, then for any r vertices $u_{i_j} \in A_{i_j}$ we have that $\{u_{i_1},\ldots,u_{i_r}\} \in \Delta$.*

Proof. We may assume without loss of generality that $i_j = j$, for $j = 1,\ldots,r$. Choose now r vertices $u_j \in A_j$ and assume that $\{u_1,\ldots,u_r\} \notin \Delta$. Let $s < r$ be the maximum size of a subset of $\{u_1,\ldots,u_r\}$ that belongs to Δ. Again we may assume that actually $\{u_1,\ldots,u_s\} \in \Delta$. The simplicial complex $\Delta_{1,\ldots,r}$ is a matroid. Since $\{v_1,\ldots,v_r\} \in \Delta_{1,\ldots,r}$ and the 1-skeleton of $\Delta_{1,\ldots,r}$ is complete r-partite, we have $\dim\Delta_{1,\ldots,r} = r-1$. As a matroid is pure, we have that u_{s+1} belongs to some $(r-1)$-dimensional facet F of $\Delta_{1,\ldots,r}$. Notice that, by the r-partition of $\Delta_{1,\ldots,r}$'s 1-skeleton, the facet F has to contain exactly one vertex from each parallel class. By the augmentation axiom, we know that there exist $r-s$ vertices $w_1,\ldots,w_{r-s} \in F$ such that $\{u_1,\ldots,u_s,w_1,\ldots,w_{r-s}\} \in \Delta_{1,\ldots,r}$. As a face cannot contain two vertices from the same parallel class, we obtain that $w_i = u_{s+1}$ for some $1 \le i \le r-s$. In particular $\{u_1,\ldots,u_s,u_{s+1}\} \in \Delta$, a contradiction to the maximality of s. □

Lemma 4.15. *Let A_i be one of the parallel classes of Δ and let $v,w \in A_i$. Then*

$$\mathrm{link}_\Delta v = \mathrm{link}_\Delta w.$$

Proof. Choose $\{a_1,\ldots,a_{d-1}\} \in \mathrm{link}_\Delta v$. The restriction $\Delta|_{\{v,w,a_1,\ldots,a_{d-1}\}}$ is a $(d-1)$-dimensional pure complex. As $\{v,w\} \notin \Delta$ we obtain that $\{a_1,\ldots,a_{d-1},w\} \in \Delta|_{\{v,w,a_1,\ldots,a_{d-1}\}}$ and thus $\{a_1,\ldots,a_{d-1}\} \in \mathrm{link}_\Delta w$. □

Exploiting the results of Lemma 4.14 and Lemma 4.15 we will simplify notation in the following way. We will write $A_{i_1}\ldots A_{i_r} \in \Delta$ if there exist vertices $v_{i_j} \in A_{i_j}$ for all $j = 1,\ldots,r$ such that $\{v_{i_1},\ldots,v_{i_r}\} \in \Delta$. By Lemma 4.14 this holds for any choice of r vertices, one in each parallel class. As by Lemma 4.15 the link of all the vertices in one parallel class is the same, we will denote by $\mathrm{link}_\Delta A_i$ the link of some vertex $v \in A_i$. These two lemmas lead us to the following definition (see [Oxl11, p. 49] for the classical matroid-theoretical definitioin).

Definition 4.16. Let Δ be a simplicial complex with complete p-partite 1-skeleton, satisfying Lemma 4.14. Let $A_1 \cup \ldots \cup A_p$ be the p-partition and choose for each $i = 1,\ldots,p$ a vertex $v_{i,1} \in A_i$. We define the associated *simplified* complex as:

$${}^{\mathrm{si}}\Delta = \Delta|_{\{v_{1,1},\ldots,v_{p,1}\}}.$$

We will call a parallel class of Δ a *cone class* if the corresponding vertex in ${}^{\mathrm{si}}\Delta$ is a cone point of ${}^{\mathrm{si}}\Delta$. This is clearly equivalent to every facet of Δ containing a vertex of that parallel class.

Remark 4.17. Let Δ be a simplicial complex with complete p-partite 1-skeleton. Then, using Lemma 4.14, we have

$$\Delta \text{ is a matroid } \iff {}^{\mathrm{si}}\Delta \text{ is a matroid.}$$

The next proposition shows the close relation between a matroid Δ and ${}^{\mathrm{si}}\Delta$.

Proposition 4.18. *Given a matroid Δ on $[n]$, we have $\beta_i(S/J(\Delta)) = \beta_i(S/J({}^{\mathrm{si}}\Delta))$ for all i. In particular,* $\mathrm{type}(\Delta) = \mathrm{type}({}^{\mathrm{si}}\Delta)$.

Proof. Set $R = \Bbbk[y_1,\ldots,y_p]$, and consider the $\Bbbk$-algebra homomorphism:

$$\begin{array}{rcl}\phi : R & \longrightarrow & S \\ y_i & \mapsto & \prod_{j\in A_i} x_j = m_i\end{array}$$

One can check that $\phi(J({}^{\mathrm{si}}\Delta))S = J(\Delta)$. Moreover it is obvious that $m_1,\ldots,m_p$ form a regular sequence of S, so by a theorem of Hartshorne ([Har66, Proposition 1]) S is a flat R-module via ϕ. So, if $F_\bullet$ is a minimal free resolution of $R/J({}^{\mathrm{si}}\Delta)$ over R, then it follows that $F_\bullet \otimes_R S$ is a minimal free resolution of $S/J(\Delta)$ over S. Therefore we may conclude. $\square$

Remark 4.19. With the notation of Proposition 4.18, notice that ϕ allows also to recover the *graded* Betti numbers of $J(\Delta)$ from those of $J({}^{\mathrm{si}}\Delta)$. Provided that the partition of the 1-skeleton of Δ is known, it is enough to consider the natural $\mathbb{Z}^p$-grading both on R and on S. The $\mathbb{Z}^p$-grading on S is given by the p-partition.

We will conclude this section with a first application of Equation (4.1). We will find a formula the h-vectors h^Δ where Δ is a 1-dimensional matroid. By Theorem 4.3, this is equivalent to describing the h-vectors of $\Bbbk[\Delta]$, where Δ is a matroid such that its Stanley-Reisner ideal has height 2.

By Proposition 4.12, a 1-dimensional matroid Δ is actually a complete p-partite graph on n vertices. For all $k = 1,\ldots,n-1$, let us set:

$$c_k(\Delta) = |\{i \in \{1,\ldots,p\} : |A_i| \geq k\}| - 1.$$

Proposition 4.20. *Let Δ be a 1-dimensional matroid on $[n]$. For all $k = 0,\ldots,n-2$, we have*

$$\mathrm{h}^\Delta(k) = \sum_{i=1}^{n-k-1} c_i(\Delta).$$

Proof. Let us choose a vertex $v \in A_p$. Clearly, the cover ideal of the link of v is the principal ideal:

$$J(\mathrm{link}_\Delta v) = \Bigg(\prod_{i\in[n]\setminus A_p} x_i\Bigg).$$

In particular, we have

$$\mathrm{h}^{\mathrm{link}_\Delta v}(i) = \begin{cases} 1 & \text{if } 0 \leq i < n - |A_p|, \\ 0 & \text{otherwise.}\end{cases}$$

The partition sets of the matroid $\Delta \setminus v$ are $A_1, A_2, \ldots, A_{p-1}, A_p \setminus \{v\}$, so we have

$$c_k(\Delta\setminus v) = \begin{cases} c_k(\Delta) & \text{if } k \neq |A_p|, \\ c_k(\Delta) - 1 & \text{if } k = |A_p|.\end{cases}$$

By induction we have

$$\mathrm{h}^{\Delta\setminus v}(k) = \sum_{i=1}^{n-k-2} c_i(\Delta\setminus v),$$

for all $k = 0,\ldots,n-3$. On the other side, by (4.1) we have

$$\mathrm{h}^\Delta(k) = \mathrm{h}^{\Delta\setminus v}(k-1) + \mathrm{h}^{\mathrm{link}_\Delta v}(k), \quad \forall\, k = 0,\ldots,n-1.$$

Therefore,

$$\mathrm{h}^\Delta(k) = \begin{cases} \sum_{i=1}^{n-k-1} c_i(\Delta) - 1 + \mathrm{h}^{\mathrm{link}_\Delta v}(k) & = & \sum_{i=1}^{n-k-1} c_i(\Delta) & \text{if } k \leq n - |A_p| - 1, \\ \\ \sum_{i=1}^{n-k-1} c_i(\Delta) + \mathrm{h}^{\mathrm{link}_\Delta v}(k) & = & \sum_{i=1}^{n-k-1} c_i(\Delta) & \text{otherwise.}\end{cases}$$

$\square$

Corollary 4.21. *For a sequence $h = (1,2,h_3,\dots,h_s)$, the following are equivalent:*

(i) *There is a matroid Δ such that h is the h-vector of $\Bbbk[\Delta]$.*

(ii) *There is a matroid Δ such that h is the h-vector of $S/J(\Delta)$.*

(iii) *h is a pure O-sequence.*

(iv) *h is the h-vector of a level algebra.*

(v) *$h_{i+1} \leq 2h_i + h_{i-1}$ for all $i = 1,\dots,s$.*

Proof. The equivalence between (i) and (ii) follows by Theorem 4.3, whereas (iii) is equivalent to (iv) by the Hilbert-Burch theorem. The equivalence between (iv) and (v) was shown by Iarrobino in [Iar84]. As $S/J(\Delta)$ is level, and thus (ii) implies (iv), we just need to prove that (v) implies (ii) and this follows easily from Proposition 4.20. □

Corollary 4.22. *If Δ is a 1-dimensional matroid, then* $\operatorname{type}(\Delta) = p-1$, *where Δ is p-partite.*

4.3 Stanley's Conjecture

The main result of this section is Theorem 4.29, in which we prove that Stanley's conjecture holds for certain matroids which we identify in a natural way. The first discussion of this section and Theorem 4.27 are particular cases of the main result. They are the starting point of the inductive procedure in the proof of Theorem 4.29. For a better understanding of the construction which we present here, we will start with a closer look at an already known case of Stanley's Conjecture 4.9, namely the codimension two case. In this first part we will concentrate on examples which hopefully provide the necessary intuition for the more technical proofs.

Consider a 1-dimensional matroid Δ on $[n]$, thus by Proposition 4.12 it is a complete p-partite graph. Recall that we denote the partition sets of the graph by A_i and for $i = 1,\dots,p$ we have $a_i = |A_i|$. For simplicity, we assume for the moment that $a_1 \leq \ldots \leq a_p$. We will now present an inductive method to compute the h-vector of Δ.

When we restrict to the first layer A_1, we obtain a 0-dimensional matroid on a_1 vertices. It is clear that in this case $J(\Delta_1) = (x_1 \cdots x_{a_1})$ so the h-vector of Δ_1 is the vector of length a_1: $(1,1,\dots,1)$. Let $v_{2,1}$ be the first vertex of the parallel class A_2. This vertex will be a cone-point of the 1-dimensional matroid $\Delta|_{A_1\cup\{v_{2,1}\}}$, so the h-vector will be the same as the one of Δ_1. We will now use the recursive formula (4.1) to compute the h-vector of $\Delta_{1,2}$. By Lemma 4.15 we have

$$\operatorname{link}_{\Delta_{1,2}} v_{2,i} = A_1, \quad \forall\, v_{2,i} \in A_2.$$

So the h-vector of $\Delta|_{A_1\cup\{v_{2,1},v_{2,2}\}}$ is computed as follows:

$$\begin{array}{cccccc} & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \\ \hline 1 & 2 & 2 & \dots & 2 & 1 \end{array}$$

where the first row represents the h-vector of $\Delta|_{A_1\cup\{v_{2,1}\}}$, the second row represents the h-vector of the link, i.e. of Δ_1. The last row is the h-vector of $\Delta|_{A_1\cup\{v_{2,1},v_{2,2}\}}$. To compute the h-vector of $\Delta|_{A_1\cup\{v_{2,1},v_{2,2},v_{2,3}\}}$

we proceed in the same way. All together we have to apply this procedure $a_2 - 1$ times. This can be done also directly in the following way:

$$\begin{array}{cccccccccc}
 & & & & & 1 & \dots & 1 & 1 & 1 \\
 & & & & 1 & 1 & \dots & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & \dots & 1 & 0 & 0 \\
 & & & & & & & & & \\
 & & & & & & \dots & & & \\
 & & & & & & & & & \\
1 & 1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & 0 \\
\hline
1 & 2 & 3 & \dots & a_1 & a_1 & \dots & 3 & 2 & 1
\end{array}$$

The h-vector of $\Delta_{1,2,3}$ of is computed in a similar way. The only difference is that $v_{3,1}$ will no longer be a cone point. Thus the first row will be $h^{\Delta_{1,2}}$ and the number of shifted rows will be a_3. Repeating this procedure, we can imagine that h^{Δ} is computed summing the columns of the staircase in Figure 4.1. Notice that the last nonzero entry of h^{Δ} is $p-1$.

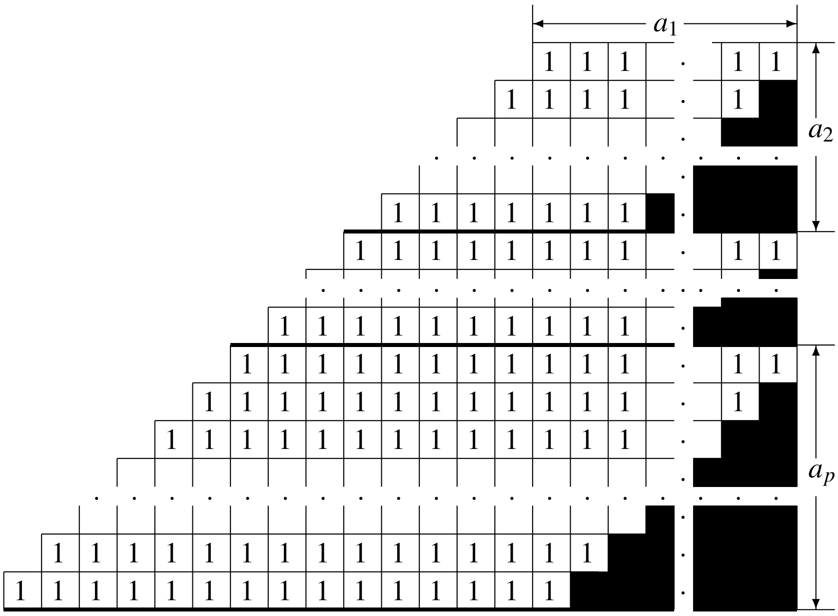

Figure 4.1: Computing h^{Δ} for $d = 2$

In Figure 4.2 we can see one example of how the corresponding order ideal is constructed in the case when Δ is the 1-dimensional matroid on 15 vertices, with 4-partition (3,3,4,5). Notice that the columns contain monomials of the same degree and that the exponent of x is constant on the rows. Depending on the order of the parallel classes we can build a total of 12 different staircases, each one producing an order ideal. Eliminating the symmetry given by exchanging x and y, we are left with 6 different order ideals with the right f-vector. For example ordering the partition as $(4,3,3,5)$ we obtain the order ideal generated by $\{x^4y^9, x^7y^6, x^{10}y^3\}$.

In higher dimensions the picture becomes more complicated. One can either imagine d-dimensional staircases, where each cube has value 1, or 2-dimensional staircases, where each row is the h-vector of the link of a parallel class. As we already saw, the order of the a_i's plays no role in the computation of h^{Δ}, providing us with several ways to construct an order ideal with the same f-vector. A complicated

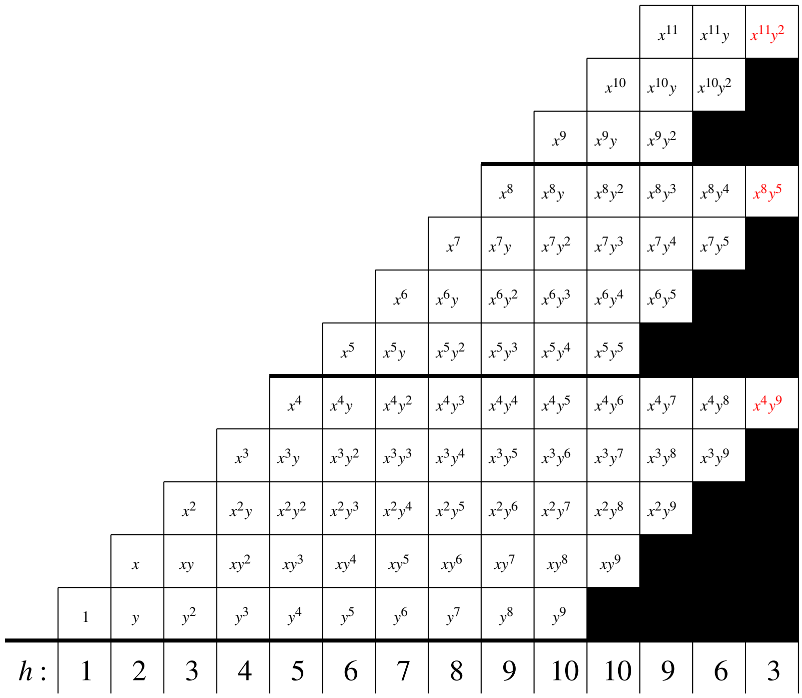

Figure 4.2: One order ideal which produces h^Δ the 4-partition $(3,3,4,5)$

example in dimension 2, with 6-partite 1-skeleton shows that unfortunately with this method there is no "canonical" choice. By canonical we understand a construction that should be independent of the values of the a_i's.

There is one case in which the choice of the order ideal is unique, namely the case when $d = p$. As we will see in Remark 4.35, this is equivalent to $J(\Delta)$ being Gorenstein.

Lemma 4.23. *If Δ is a $(d-1)$-dimensional, d-partite matroid with partition $(a_1,\dots,a_d)$, then*

$$h^\Delta = f(\langle y_1^{a_1-1}\cdots y_d^{a_d-1}\rangle).$$

Proof. The minimal generators of $J(\Delta)$ are the monomials corresponding to the basic covers of Δ. In this situation, $A_1,\dots,A_d$ are the unique basic covers of Δ, so $J(\Delta)$ is a complete intersection with d generators of degrees $a_1,\dots,a_d$. The conclusion follows because the h-vector of a complete intersection depends only on the degree of its minimal generators. □

We will now define a class of matroids and prove that the Stanley conjecture holds for this class. When one fixes the dimension and the p-partition of the vertex set, these matroids will have all the admissible faces, thus they are in a sense a generalization of the Gorenstein matroids.

Definition 4.24. Let Δ be a $d-1$-dimensional matroid on $[n]$ with p-partite 1-skeleton. We say that Δ is a *complete p-partite matroid* if

$$A_{i_1}\dots A_{i_d} \in \Delta, \quad \text{for any subset } \{i_1,\dots,i_d\} \subset \{1,\dots,p\}.$$

Whenever p is clear from the context, we will just call Δ *complete*. Notice that a complete matroid is uniquely determined by the cardinalities of the parallel classes $a_1,\dots,a_p$ and by d. It is also clear that a matroid is complete iff its simplification ${}^{\mathrm{si}}\Delta$ is the *uniform matroid* $U_{d,p}$ (see [Oxl11, p. 17]). Complete matroids also generalize *partition matroids* (see [Oxl11, p. 18]), which correspond to the case $p = d$. In Proposition 4.12 we proved that for $d = 2$ all matroids are complete. For $d > 2$ this is no longer true, as the following easy example shows.

Example 4.25. Let $n = 4$ and $\Delta = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$. It is clear that Δ is a matroid. The 1-skeleton of Δ is

$$\Delta^1 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\} = K_4,$$

so it is a complete 4-partite graph. This means that $a_1 = a_2 = a_3 = a_4 = 1$. Clearly this matroid is not complete, as the face $\{2,3,4\}$ is missing. The complete 2-dimensional matroid corresponding to the above a_i's is $\Delta' = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.

Remark 4.26. Let Δ be a complete p-partite matroid. We have

(i) For any subset of vertices $M \subset [n]$ the restriction of Δ to M is also a complete matroid.

(ii) For any parallel class A_i, the link in Δ of any of its vertices $\mathrm{link}_\Delta A_i$ is also a complete matroid.

Theorem 4.27. *Let Δ be a complete, $(d-1)$-dimensional matroid with p-partition of the 1-skeleton $A_1,\ldots,A_p$. For $i = 1,\ldots,p$ we denote by $a_i = |A_i|$. Let Γ be the pure multi-complex on $\{y_1,\ldots,y_d\}$ with facets*

$$\mathscr{F}(\Gamma) = \{y_1^{(\sum_{i=l_0}^{l_1-1} a_i)-1} y_2^{(\sum_{i=l_1}^{l_2-1} a_i)-1} \cdots y_d^{(\sum_{i=l_{d-1}}^{p} a_i)-1} \ : \ \forall\ 1 = l_0 < l_1 < l_2 < \ldots < l_{d-1} \le p\}.$$

Then we have that

$$\mathrm{h}^\Delta = f(\Gamma),$$

where h^Δ is the h-vector of the algebra $S/J(\Delta)$.

Before we start the proof, let us make a few easy remarks and introduce some notation. For each $i \in \{d,\ldots,p\}$, we denote the link of the i-parallel class in the restriction of Δ to the first i parallel classes by

$$L_i = \mathrm{link}_{\Delta_{1,\ldots,i}} A_i.$$

Notice that L_i is the $(d-2)$-skeleton of $\Delta_{1,\ldots,i-1}$. We will write $r(i)$ for the length of the h-vector of L_i. As the number of vertices of L_i is $a_1 + \ldots + a_{i-1}$ and its dimension is $d-2$, we have that

$$r(i) = 2 - d + \sum_{j=1}^{i-1} a_j.$$

Proof. We will prove this theorem by simultaneous induction on d and $p-d$. The case $d = 1$ is trivially true and by Lemma 4.23 we know that the theorem is true for $p = d$.

For each i, denote by Γ^{L_i} the pure multi-complex corresponding to L_i which is given by the inductive hypothesis. We assume now that $p > d > 1$ and that $\mathrm{h}^{\Delta_{1,\ldots,p-1}} = f(\Gamma_{p-1})$, where

$$\Gamma_{p-1} = \langle y_1^{(\sum_{i=1}^{l_1-1} p_i)-1} y_2^{(\sum_{i=l_1}^{l_2-1} p_i)-1} \cdots y_d^{(\sum_{i=l_{d-2}}^{p} p_i)-1} \ : \ \forall\ 1 < l_1 < l_2 < \ldots < l_{d-1} \le p-1\rangle.$$

We will use $\mathrm{h}^{\Delta_{1,\ldots,p-1}}$ and h^{L_p} to compute h^Δ via the formula given in (4.1). Clearly this formula has to be applied a_p times, once for every vertex in A_p. So we obtain

$$\mathrm{h}^\Delta(j) = \mathrm{h}^{\Delta_{1,\ldots,p-1}}(j - a_p) + \sum_{k=0}^{a_p-1} \mathrm{h}^{L_p}(j-k), \tag{4.2}$$

for all $0 \le j \le 1 - d + \sum_{k=1}^{p} a_k$. To conclude we just need to check that the f-vectors of Γ, Γ_{p-1} and Γ^{L_p} satisfy the same formula. To this purpose, for any $j \in \mathbb{Z}$, let us denote $F_j = \{M \in \Gamma : \deg M = j\}$, $G_j = \{M \in \Gamma^{L_p} : \deg M = j\}$ and $H_j = \{M \in \Gamma_{p-1} : \deg M = j\}$. Let us furthermore partition F_j as

$$F_j = F_{j,\ge a_p} \bigcup \left(\bigcup_{k=0}^{a_p-1} F_{j,k} \right),$$

where $F_{j,\geq a_p} = \{M \in F_j : y_d^{a_p} \mid M\}$ and $F_{j,k} = \{M \in F_j : y_d^k \mid M \text{ and } y_d^{k+1} \nmid M\}$. It is easy to check the bijections of sets

$$\begin{array}{rcl} G_{j-a_p} & \stackrel{\cong}{\to} & F_{j,\geq a_p} \\ M & \mapsto & M \cdot y_d^{a_p} \end{array}$$

and, for all $k = 0, \ldots, a_p - 1$,

$$\begin{array}{rcl} H_{j-k} & \stackrel{\cong}{\to} & F_{j,k} \\ M & \mapsto & M \cdot y_d^k \end{array}$$

Therefore we get the formula:

$$f_j(\Gamma) = f_{j-a_p}(\Gamma_{p-1}) + \sum_{k=0}^{a_p-1} f_{j-k}(\Gamma^{L_p}) \quad \forall\, j \in \mathbb{Z},$$

which, together with (4.2), yields the conclusion by induction. □

Fixing two positive integers d and n and a vector $\mathbf{a} = (a_1, \ldots, a_p) \in (\mathbb{Z}_+)^p$ such that $p \geq d$, $a_1 + \ldots + a_p = n$, we introduce the class

$$\mathscr{M}(d, p, \mathbf{a}),$$

consisting of all $(d-1)$-dimensional matroids with p-partite 1-skeleton, where the partition sets A_i have cardinality a_i for all $i = 1, \ldots, p$. Note that the classes $\mathscr{M}(d, p, \mathbf{a})$ depend only on the set $\{a_1, \ldots, a_p\}$. That is, $\mathscr{M}(d, p, \mathbf{a})$ coincides with $\mathscr{M}(d, p, \mathbf{a}^\sigma)$ for any permutation σ of p elements ($\mathbf{a}^\sigma$ means $(a_{\sigma(1)}, \ldots, a_{\sigma(p)})$). Furthermore notice that, if $d = 2$ or $p = d$, $\mathscr{M}(d, p, \mathbf{a})$ consists of a single matroid, but this happens only in these cases. To see this, it is enough to consider for $t = 0, \ldots, d-2$, the following simplicial complexes

$$\Delta_t(d, p, \mathbf{a}) = \langle \{v_1, v_2, \ldots, v_t, v_{i_1}, \ldots, v_{i_{d-t}}\} \ : \ t < i_1 < \ldots < i_{d-t} \leq p \text{ where } v_i \in A_i \rangle. \tag{4.3}$$

It is easy to see that $\Delta_t(d, p, \mathbf{a})$ are elements of $\mathscr{M}(d, p, \mathbf{a})$. Moreover, one can show that, if $p > d$, they are not isomorphic by twos - the easiest way to show this is to notice that they have a different number of facets. The matroid $\Delta_0(d, p, \mathbf{a})$ is just the complete p-partite matroid whose partition sets $A_1, \ldots, A_p$ satisfy $|A_i| = a_i$ for all $i = 1, \ldots, p$. Notice that, a part from the case $t = 0$, the matroid $\Delta_t(d, p, \mathbf{a})$ depends on the vector $\mathbf{a}$, not just on the set of its entries.

Remark 4.28. For every t, d, n and $\mathbf{a}$ as above the matroids $\Delta_t(d, p, \mathbf{a})$ are representable. To see this it is enough to notice that their simplification satisfies

$$^{\mathrm{si}}\Delta_t(d, p, \mathbf{a}) = \{v_1, \ldots, v_t\} * U_{d-t, p-t} = \langle \{v_1, \ldots, v_t\} \cup F \ : \ F \in \mathscr{F}(U_{d-t, p-t}) \rangle,$$

where $U_{d-t,p-t}$ is the uniform matroid of rank $d-t$ on $p-t$ vertices. Thus, a representation of $\Delta_t(d, p, \mathbf{a})$ is obtained by taking a_i copies of the ith column ($i = 1, \ldots, p$) in a representation of $\{v_1, \ldots, v_t\} * U_{d-t,p-t}$. Furthermore, it is easy to check that in order to obtain a representation over a field $\mathbb{F}$, its cardinality has to be "large enough".

As a first thing, we want to show that Stanley's conjecture holds true for all $\Delta_t(d, p, \mathbf{a})$.

Theorem 4.29. *Let $d, p \in \mathbb{N}$ be such that $p \geq d \geq 1$ and $\mathbf{a} = (a_1, \ldots, a_p) \in (\mathbb{Z}_+)^p$. Then $h^{\Delta_t(d,p,\mathbf{a})}$ is a pure O-sequence for all $t = 0, \ldots, d-2$.*

Proof. The case $t=0$ has already been treated in Theorem 4.27. So, we will use induction on t, assuming that $t \geq 1$. Let us write Δ_t for $\Delta_t(d,p,\mathbf{a})$. The restricted simplicial complex $\Delta_t' = (\Delta_t)_{2,3,\ldots,p}$ is just $\Delta_{t-1}(d-1,p-1,\tilde{\mathbf{a}})$, where $\tilde{\mathbf{a}} = (a_2,\ldots,a_p)$. Therefore, we know by induction that $h^{\Delta_t'}$ is a pure O-sequence. Set $A_1 = \{v_{1,1},\ldots,v_{1,a_1}\}$ and $\Delta_t^i \subset \Delta_t$ the sub-complex induced by the vertices $A_2 \cup \ldots \cup A_p \cup \{v_{1,1},\ldots,v_{1,i}\}$ for all $i=1,\ldots,a_1$. We have $h^{\Delta_t^1} = h^{\Delta_t' * a_{1,1}} = h^{\Delta_t'}$. Moreover, for all $i \geq 2$ and $k \in \mathbb{Z}$, we have:

$$h^{\Delta_t^i}(k) = h^{\Delta_t^{i-1}}(k-1) + h^{\Delta_t'}(k).$$

Particularly, since $\Delta_t = \Delta_t^{a_1}$, we get:

$$h^{\Delta_t}(k) = \sum_{j=0}^{a_1-1} h^{\Delta_t'}(k-j) \quad \forall\, k \in \mathbb{Z}. \tag{4.4}$$

We know that $h^{\Delta_t'}$ is a pure O-sequence, so let Γ' be the order ideal such that $f_{\Gamma'} = h^{\Delta_t'}$. Let us suppose that the set of maximal degree monomials of Γ' is:

$$\mathscr{F}_{\Gamma'} = \{u_1,\ldots,u_s : u_i \in \Bbbk[y_1,\ldots,y_{d-1}] \text{ and } \deg(u_i) = a_2+\ldots+a_p-d+1\}.$$

Let Γ be the pure order ideal with the following set of maximal monomials:

$$\mathscr{F}(\Gamma) = \{u_1 y_d^{a_1-1},\ldots,u_s y_d^{a_1-1}\}.$$

One can easily see that

$$f_\Gamma(k) = \sum_{j=0}^{a_1-1} f_{\Gamma'}(k-j), \quad \forall\, k \in \mathbb{Z},$$

so (4.4) yields the conclusion. □

Putting together Theorem 4.27 and the proof of Theorem 4.29 we obtain an explicit construction for an order ideal with the f-vector we are looking for. Namely, we obtain the following corollary.

Corollary 4.30. *If we denote by $\Gamma_t(d,p,\mathbf{a})$ the following order ideal:*

$$\langle y_1^{a_1-1}\cdots y_t^{a_t-1} y_{t+1}^{(\sum_{i=t+1}^{l_1-1} a_i)-1} \cdots y_d^{(\sum_{i=l_{d-1}}^{p} a_i)-1} \;:\; \forall\, t+1 < l_1 < l_2 < \ldots < l_{d-1} \leq p\rangle,$$

we have that

$$h^{\Delta_t(d,p,\mathbf{a})} = f(\Gamma_t(d,p,\mathbf{a})).$$

In particular,

$$\operatorname{type}(S/J(\Delta_t(d,p,\mathbf{a}))) = \binom{p-t-1}{d-t-1} \tag{4.5}$$

A consequence of Theorem 4.29 is the following interesting fact:

Corollary 4.31. *Let $d \geq 1$. For all $\mathbf{a} \in \mathbb{N}^{d+1}$ and $\Delta \in \mathscr{M}(d,d+1,\mathbf{a})$, h^Δ is a pure O-sequence.*

Proof. We want to show that Δ actually is $\Delta_t(d,p,\mathbf{a})$ for some $t=0,\ldots,d-2$, so that Theorem 4.29 would give the thesis. Passing to ${}^{\mathrm{si}}\Delta$, a proof in the case $\mathbf{a} = \mathbf{1} = (1,1,\ldots,1) \in \mathbb{N}^p$ is enough. Notice that any $(d-1)$-dimensional pure simplicial complex on the vertex set $\{1,\ldots,d+1\}$ is a matroid. In order to have the complete graph on $d+1$ vertices as 1-skeleton, ${}^{\mathrm{si}}\Delta$ must have $m \geq 3$ facets. Moreover, if Δ is a $(d-1)$-simplicial complex on $d+1$ vertices with $m \geq 3$ facets, then it is easy to prove that Δ is isomorphic to the matroid $\Delta_{d-m+1}(d,d+1,\mathbf{1})$. □

4.4 Minimal and maximal h-vectors

Among the matroids described in (4.3), two play a fundamental role:

$$\begin{array}{rcl} \Delta_{\max}(d,p,\mathbf{a}) & = & \Delta_0(d,p,\mathbf{a}), \\ \Delta_{\min}(d,p,\mathbf{a}) & = & \Delta_{d-2}(d,p,\mathbf{a}^{\sigma}), \end{array} \tag{4.6}$$

where σ is a permutation of p elements such that $a_{\sigma(1)} \leq \ldots \leq a_{\sigma(p)}$. In this section we will see that, for any $\Delta \in \mathscr{M}(d,p,\mathbf{a})$, we have

$$h^{\Delta_{\min}(d,p,\mathbf{a})} \leq h^{\Delta} \leq h^{\Delta_{\max}(d,p,\mathbf{a})}$$

component-wise.

Given a matroid Δ with parallel classes $A_1,\ldots,A_p$, we need to consider in the following lemma the matroid $\Delta_{r\leftrightarrow s}$, where the parallel classes A_r and A_s are switched. Let us give a more rigorous definition: The matroid $\Delta_{r\leftrightarrow s}$ has as facets the subsets $F = \{v_{i_1},\ldots,v_{i_d}\}$ of $[n]$ such that
(i) $|F \cap (A_r \cup A_s)| \in \{0,2\}$ and $F \in \mathscr{F}(\Delta)$,
(ii) $v_{i_j} \in A_r$, $F \cap A_s = \emptyset$ and there exists $v \in A_s$ such that $(F \setminus \{v_{i_j}\}) \cup \{v\} \in \mathscr{F}(\Delta)$,
(iii) $v_{i_k} \in A_s$, $F \cap A_r = \emptyset$ and there exists $u \in A_r$ such that $(F \setminus \{v_{i_k}\}) \cup \{u\} \in \mathscr{F}(\Delta)$.

Lemma 4.32. *Let $p > d$ and $\mathbf{a} = (a_1,\ldots,a_p) \in (\mathbb{Z}_+)^p$ be a vector such that $a_1 \leq \ldots \leq a_p$. Let $\Delta \in \mathscr{M}(d,p,\mathbf{a})$ be a matroid such that A_p is a cone class for Δ. Pick $\ell \in \{1,\ldots,p-1\}$ such that A_ℓ is not a cone class for Δ (it exists because $p > d$). Then*

$$h^{\Delta_{\ell\leftrightarrow p}} \leq h^{\Delta}.$$

Proof. Set $L_p = \mathrm{link}_{\Delta} A_p$ and $\overline{L}_\ell = \mathrm{link}_{\Delta_{\ell\leftrightarrow p}} A_\ell$. Furthermore, let $L'_p = L_p \setminus A_\ell$ and $\overline{L'}_\ell = \overline{L}_\ell \setminus A_p$. Notice that $L'_p \cong \overline{L'}_\ell$ and that $T = \mathrm{link}_{L_p} A_\ell \cong \mathrm{link}_{\overline{L'}_\ell} A_p = U$. For all $k \in \mathbb{Z}$, we have

$$h^{\Delta}(k) = \sum_{i=0}^{a_p-1} h^{L'_p}(k-a_\ell-i) + \sum_{i=0}^{a_p-1}\sum_{j=1}^{a_\ell} h^{T}(k-a_\ell-i+j) \tag{4.7}$$

and

$$h^{\Delta_{\ell\leftrightarrow p}}(k) = \sum_{i=0}^{a_\ell-1} h^{\overline{L'}_\ell}(k-a_p-i) + \sum_{i=0}^{a_\ell-1}\sum_{j=1}^{a_p} h^{U}({}_{k-a_p-i+j}). \tag{4.8}$$

From the above discussion we have $h^{L'_p}(r) = h^{\overline{L'}_\ell}(r) =: h'_r$ and $h^T(r) = h^U(r) =: h''_r$ for all $r \in \mathbb{Z}$. Let us set

$$M_1 = \sum_{i=0}^{a_p-1} h'_{k-a_\ell-i}, \quad M_2 = \sum_{i=0}^{a_p-1}\sum_{j=1}^{a_\ell} h''_{k-a_\ell-i+j}$$

and

$$N_1 = \sum_{i=0}^{a_\ell-1} h'_{k-a_p-i}, \quad N_2 = \sum_{i=0}^{a_\ell-1}\sum_{j=1}^{a_p} h''_{k-a_p-i+j}.$$

Because $a_\ell \leq a_p$, obviously $N_1 \leq M_1$. Moreover we claim that $N_2 = M_2$. To see this, it is enough to notice that

$$h''_{k-a_p-i+j} = h''_{k-a_\ell-(a_p-j)+(a_\ell-i)}.$$

So, we get that (4.8) is less than or equal to (4.7). $\square$

We need one more technical lemma.

Lemma 4.33. *Let $A_1,\ldots,A_p$ and $B_1,\ldots,B_q$ be partitions of $\{1,\ldots,n\}$ of cardinality $|A_i| = a_i$ and $|B_j| = b_j$, where $p \geq d$ and $q \geq d$, such that*

(i) $a_1 \leq \ldots \leq a_p$,

(ii) $b_1 \leq \ldots \leq b_q$,

(iii) $B_j = \bigcup_{k=1}^{r_j} A_{i_{j,k}}$,

(iv) $\bigcup_{i=1}^{d} A_i \subset \bigcup_{i=1}^{d} B_i$.

Set $\tilde{\mathbf{a}} = (a_1, a_2, \ldots, a_{d-1}, a_d + \ldots + a_p)$ *and* $\mathbf{b} = (b_1, \ldots, b_q)$. *If* Γ *is the only* $(d-1)$*-dimensional matroid in* $\mathscr{M}(d,d,\tilde{\mathbf{a}})$*, then:*

$$h^{\Gamma} \leq h^{\Delta} \quad \forall\, \Delta \in \mathscr{M}(d,q,\mathbf{b}).$$

Proof. If $q = d$, then the assertion can be deduced by inspection on the h-vectors of Δ and Γ, described in Theorem 4.27. In fact, using this theorem, one can show a more general statement: Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}_+)^d$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in (\mathbb{Z}_+)^d$ be vectors such that $\alpha_1 \leq \ldots \leq \alpha_d$, $\beta_1 \leq \ldots \leq \beta_d$, $\sum_{i=1}^{d} \alpha_i = \sum_{i=1}^{d} \beta_i$ and $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, d-1$. Then, the h-vector of the only matroid in $\mathscr{M}(d,d,\boldsymbol{\alpha})$ is less than or equal to the h-vector of the only matroid in $\mathscr{M}(d,d,\boldsymbol{\beta})$. We leave to the reader the easy proof of this fact.

We will use induction on p. Notice that, as we always have $d \leq q \leq p$, the case $p = d$, implies $q = d$, so we are done by the above discussion.

If $p > d$ and $q > d$, then $i_{q,k} > d$ for all $k = 1, \ldots, r_q$. Consider the sub-complex $\Gamma' \subset \Gamma$ induced by the vertices not in B_q and set $L = \mathrm{link}_{\Gamma} B_q$. As B_q is a subset of a parallel class in Γ, L is well defined and for all $k \in \mathbb{Z}$ we have

$$h^{\Gamma}(k) = h^{\Gamma'}(k - b_q) + \sum_{i=1}^{b_q} h^{L}(k - b_q + i).$$

In the same vein, we can consider the sub-complex $\Delta' \subset \Delta$ induced by all the vertices of Δ not in B_q and we set $K = \mathrm{link}_{\Delta} B_q$. Once again we have, for all $k \in \mathbb{Z}$,

$$h^{\Delta}(k) = h^{\Delta'}(k - b_q) + \sum_{i=1}^{b_q} h^{K}(k - b_q + i).$$

By Lemma 4.32 we can assume that B_q is not a cone class of Δ, so that Δ' has dimension $d-1$. Therefore by the induction on p we immediately get $h^{\Gamma'} \leq h^{\Delta'}$.

On the other hand, L is the unique $(d-1)$-partite $(d-2)$-dimensional matroid on the partition $(a_1, \ldots, a_{d-1})$, whereas K is a $(d-2)$-dimensional matroid on a certain partition $C_1, \ldots, C_r$. For sure $r \geq d-1$ and, provided that $|C_1| \leq \ldots \leq |C_r|$, we get also that $a_i \leq b_i \leq |C_i|$ for all $i = 1, \ldots d-1$. Take a facet $\{v_{i_1}, \ldots, v_{i_{d-1}}\}$ of K and suppose that each $v_{i_k} \in C_{i_k}$. Then the sub-complex $K' \subset K$ induced by the vertices of $C_{i_1} \cup \ldots \cup C_{i_{d-1}}$ is a complete $(d-1)$-partite $(d-2)$-dimensional matroid. We can assume $i_1 < \ldots < i_d$, so that $a_k \leq |C_{i_k}|$ for all $k = 1, \ldots, d-1$. So we can choose a_k vertices in each one of the C_{i_k}s. It turns out that L is isomorphic to the sub-complex of K' induced by these vertices. Therefore L is isomorphic to an induced sub-complex of K, which implies $h^L \leq h^K$. So we can conclude. □

Theorem 4.34. *If* $d \geq 1$ *and* $\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{N}^p$ *with* $p \geq d$*, then:*

$$h^{\Delta_{\min}(d,p,\mathbf{a})} \leq h^{\Delta} \quad \forall\, \Delta \in \mathscr{M}(d,p,\mathbf{a}).$$

Proof. Since neither the matroid $\Delta_{\min}(d,p,\mathbf{a})$ nor the set $\mathscr{M}(d,p,\mathbf{a})$ depend on the order of $a_1, \ldots, a_p$, we are allowed to assume that $a_1 \leq \ldots \leq a_p$. We induct on p. If $p = d$, then the theorem is trivial, since $\mathscr{M}(d,d,\mathbf{a})$ consists in only one matroid, namely $\Delta_{\min}(d,d,\mathbf{a})$. If $p > d$, let us set $\Delta_{\min}(d,p,\mathbf{a})' \subset \Delta_{\min}(d,p,\mathbf{a})$ and $\Delta' \subset \Delta$ the sub-complexes induced by the vertices of $A_1 \cup \ldots \cup A_{p-1}$. Furthermore set

$\mathbf{a}' = (a_1,\dots,a_{p-1})$. We have that $\Delta_{\min}(d,p,\mathbf{a})' = \Delta_{\min}(d,p-1,\mathbf{a}')$. Exploiting Lemma 4.32, we can assume that $\dim\Delta' = d-1$, so that $\Delta' \in \mathscr{M}(d,p-1,\mathbf{a}')$. So, by induction, we get

$$h^{\Delta_{\min}(d,p,\mathbf{a})'} \leq h^{\Delta'}.$$

Now set $L = \mathrm{link}_{\Delta_{\min}(d,p,\mathbf{a})} A_p$ and $K = \mathrm{link}_\Delta A_p$. It turns out that L is the unique $(d-1)$-partite $(d-2)$-dimensional matroid on the partition $(a_1,\dots,a_{d-2},a_{d-1}+\dots+a_{p-1})$. Instead K will be a $(d-2)$-dimensional matroid on a certain partition $(b_1,\dots,b_q)$. Such partitions satisfy the hypotheses of Lemma 4.33, so we get

$$h^L \leq h^K.$$

This yields the conclusion, since for all $k \in \mathbb{Z}$

$$\begin{aligned} h^{\Delta_{\min}(d,p,\mathbf{a})}(k) &= h^{\Delta_{\min}(d,p,\mathbf{a})'}(k-a_p) + \sum_{i=1}^{a_p} h^L(k-a_p+i) \quad \text{and} \\ h^\Delta(k) &= h^{\Delta'}(k-a_p) + \sum_{i=1}^{a_p} h^K(k-a_p+i). \end{aligned}$$

□

Remark 4.35. By Theorem 4.34 and (4.5) one has that, for all $\Delta \in \mathscr{M}(d,p,\mathbf{a})$,

$$\mathrm{type}(S/J(\Delta)) \geq p-d+1. \tag{4.9}$$

This implies that, for any matroid Δ, S/I_Δ is Gorenstein if and only if I_Δ is a complete intersection if and only if $p = d$. So we recover [Sto08, Theorem 4.4.10].

Equation (4.9) and Corollary 4.31 imply the following:

Theorem 4.36. *If Δ is a matroid on $\{1,\dots,n\}$ such that* $\mathrm{type}(S/I_\Delta) \leq 2$*, then $h(\Delta) = h(\Bbbk[\Delta])$ is a pure O-sequence.*

To show that $\Delta_{\max}(d,p,\mathbf{a})$ has maximal h-vector among the matroids $\Delta \in \mathscr{M}(d,p,\mathbf{a})$ is much easier.

Theorem 4.37. *If $d \geq 1$ and $\mathbf{a} = (a_1,\dots,a_p) \in \mathbb{N}^p$ with $p \geq d$, then:*

$$h^\Delta \leq h^{\Delta_{\max}(d,p,\mathbf{a})} \quad \forall\, \Delta \in \mathscr{M}(d,p,\mathbf{a}).$$

Proof. It is harmless to assume that $\Bbbk$ is infinite; otherwise we can tensor with its algebraic closure. Looking at the respective vertex covers, it is clear that $J(\Delta_{\max}(d,p,\mathbf{a})) \subset J(\Delta)$ for all $\Delta \in \mathscr{M}(d,p,\mathbf{a})$. Since both $S/J(\Delta_{\max}(d,p,\mathbf{a}))$ and $S/J(\Delta)$ are $(n-d)$-dimensional Cohen-Macaulay rings, we can choose $n-d$ linear forms which are both $S/J(\Delta_{\max}(d,p,\mathbf{a}))$- and $S/J(\Delta)$- regular (the generic ones have this property). Passing to the Artinian reduction, the inclusion is preserved, so we infer the desired inequality.

□

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Chapter 5

Generic and special constructions of pure O-sequences

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Division of Labor

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[1]

Introduction

The f-vector and h-vector are fundamental invariants of a simplicial complex, encoding the number of faces that the complex has in each dimension. What can be said in general about these vectors? Starting from Euler's polyhedron formula in the middle of the 18th century, different conditions and eventually characterizations have been found. It seems natural to ask for a description of the set of f- or equivalently h-vectors of all simplicial complexes or all pure simplicial complexes in a given dimension. The situations for these two classes are quite different. There is a precise characterization of the set of f-vectors of all simplicial complexes due to Schützenberger, Kruskal, and Katona [Sta96, Theorem II.2.1]. The opposite is the case for pure simplicial complexes—a characterization is believed to be intractable. As Ziegler points out, it would *solve all basic problems in design theory* [Zie00, Exercise 8.16]. The celebrated g-theorem characterizes h-vectors of simplicial polytopes ([BL80; BL81; Sta80]) and it is conjectured that this characterization also applies to simplicial spheres (of which there are many more than boundaries of simplicial polytopes [Kal88]). This indicates that subclasses of pure complexes—like Gorenstein, Cohen-Macaulay, or matroid complexes—may be feasible. It is known for a long time, essentially due to Macaulay, that the sets of vectors that arise as h-vectors of Cohen-Macaulay complexes consist exactly of *O-sequences*—Hilbert functions of Artinian algebras [Mac27]. Although necessary conditions are known, characterizations for matroid or Gorenstein complexes are open and may be out of reach.

In this paper we focus on matroids. They were originally introduced by Whitney as a way to study the concept of independence [Whi35]. Subsequently they appeared in a wide range of mathematical areas from linear algebra, (real) algebraic geometry, and combinatorial geometry to graph theory, optimization, and approximation theory. The new edition of Oxley's book [Oxl11] provides an excellent guide to the theory. Interest in algebraic properties of matroids is still growing as witnessed by recent work of DeConcini-Procesi [DP11], Holtz-Ron [HR11], Lenz [Len11], Moci [Moc12], and Huh [Huh12a; Huh12b].

What properties should the h-vector of a matroid have? Since matroids are Cohen-Macaulay, their h-vectors must be O-sequences. In [Sta77] Stanley shows that they are also Hilbert functions of Artinian algebras whose socle is concentrated in one degree. He conjectured that for any matroid one can even find a *monomial* algebra with this property. In this case, its Hilbert function is called a *pure O-sequence*.

Conjecture ([Sta77, p.59])**.** *The h-vector of a matroid complex is a pure O-sequence.*

For an abstract simplicial complex Δ on $[n] := \{1,\dots,n\}$, let $f_i(\Delta)$ be the number of faces of size i. Let $d = \max\{i : f_i \neq 0\}$ be the *rank* of Δ. The vector $f = (f_0,\dots,f_d)$ is the *f-vector* of Δ. It encodes the same information as the *h-vector* $h(\Delta) = (h_0,\dots,h_s)$ whose component h_i is the coefficient of x^{d-i} in the polynomial $\sum_{i=0}^{d} f_i(x-1)^{d-i}$. A central tool for the study of the h-vector is the *Stanley-Reisner ring* $\mathbb{K}[\Delta] := \mathbb{K}[x_1,\dots,x_n]/I_\Delta$, where $I_\Delta = (\prod_{i\in G} x_i : G \notin \Delta)$ is the *Stanley-Reisner ideal*. In this setting, the h-vector appears as the coefficient vector of the numerator polynomial of the Hilbert series of $\mathbb{K}[\Delta]$ (see [Sta96]). The field $\mathbb{K}$ in this definition is arbitrary, and homological properties of $\mathbb{K}[\Delta]$ may depend on the characteristic. However, Stanley's conjecture is field independent.

The problem raised by Stanley is extremely difficult and the authors are not strong believers in the validity of the conjecture. The complications are in part due to the strange properties of pure O-sequences. For instance, they need not be unimodal, and it is likely that they can not be characterized well [Boi+12]. On the positive side, it is known that both pure O-sequences and h-vectors of matroid complexes satisfy a common set of inequalities [Cha97; Hib89]:

$$h_0 \le h_1 \le \cdots \le h_{\lfloor \frac{s}{2} \rfloor}, \qquad h_i \le h_{s-i} \text{ for } 0 \le i \le \lfloor \frac{s}{2} \rfloor.$$

In contrast, the Brown-Colbourn inequalities

$$\text{for any } b \ge 1 \quad (-1)^j \sum_{i=0}^{j} (-b)^i h_i \ge 0, \quad 0 \le j \le s.$$

hold for h-vectors of matroids, but not pure O-sequences [BC92]. Other than this our understanding is poor. Positive answers to Stanley's conjecture are known for short h-vectors [DKK12; HSZ10], and for special classes of matroids [Mer01; Mer+12; Oh10]. In the present paper we prove that Stanley's conjecture holds for matroids that are truncations of other matroids and for matroids whose h-vector $(1,h_1,\dots,h_s)$ satisfies $h_s \le 5$ (with no restriction on s). We employ two completely different methods of proof, both of which have potential for generalizations. As a consequence of our results, the search for counterexamples is pushed closer to today's computational limits.

Generic pure O-sequences

The Stanley-Reisner ring $\mathbb{K}[\Delta]$ of a matroid Δ is level. To produce a pure O-sequence which equals the h-vector of Δ it would suffice to pass to a *monomial* Artinian reduction. Unfortunately, a monomial ideal rarely has one. In this context, the generic initial ideal may come to mind. It has the same h-vector as the original ideal and (in characteristic zero) is strongly stable. Therefore it possesses a regular sequence of variables and a monomial Artinian reduction. However, this does not prove Stanley's conjecture as typically the quotient modulo the generic initial ideal is not level. We envision an approach to Stanley's conjecture in which one interpolates between these two objectives with a less drastic version of the generic initial ideal (Remark 5.5). In Section 5.1 we study this genericity of matroids and show that a generalization of Stanley's conjecture holds for all simplicial complexes that are truncations (skeletons) of matroids (Theorem 5.10).

Special pure O-sequences

In matroid theory duality is central. If Δ is a matroid, then the complex Δ^c whose facets are the complements of facets of Δ is the *dual matroid*. Directly from the definitions, its Stanley-Reisner ideal I_{Δ^c} equals the *cover ideal* $J(\Delta)$ of Δ. In this paper h_Δ is the h-vector of (the quotient by) I_Δ and h^Δ that of (the quotient by) $J(\Delta)$. By matroid duality it suffices to prove Stanley's conjecture for either of the classes. Several known results on matroid complexes are stated in terms of the dual matroid [DKK12; Mer01; Oh10], which may be taken as an indication that that the cover ideal is a natural object. This perspective permeates the work of the first and third author and also our Section 5.2, where we aim at a generalization of the construction of pure O-sequences in [CV15]. This construction is recursive and relies on finding pure O-sequences for links and deletions in the matroid. When trying to generalize the construction we require a *compatibility condition* (Lemma 5.19 and Definition 5.24) the checking of which remains an obstacle. Carefully keeping track of the contributions in the recursion allows us to prove Stanley's conjecture for duals of matroids with at most rank $+\,2$ parallel classes (Theorem 5.36). Exploiting the constraints on the h-vectors of matroids whose dual has a fixed number of parallel classes, proved in [CV15], we can show Stanley's conjecture when the type is at most five (Theorem 5.39).

The search for a counterexample

Matroids on nine or fewer elements have been enumerated by Mayhew and Royle [MR08] and Stanley's conjecture has been confirmed for all of them in [DKK12]. Beyond nine vertices, mostly due to the lack of a good list of candidates, only sporadic experiments have been carried out. Our results have implications for the search for a counterexample. By Theorem 5.39, any candidate counterexample must be of Cohen-Macaulay type at least six. To confirm such a counterexample in silico would include enumeration of all $\binom{N}{6}$ socles where N is a binomial coefficient (see Example 5.42). The methods of Section 5.2, in particular Lemma 5.19, imply faster searches for pure O-sequences realizing the h-vector of the cover ideal of a given matroid. In Section 5.4 we discuss our computational efforts. As part of this project we developed a small `C++`-library which can be used to enumerate pure O-sequences The source code is available at `https://github.com/tom111/GraphBinomials` and is licensed under the GPL. We also made intensive use of Cocoa [CoC], Macaulay2 [GS] and Sage [Ste+].

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5.1 Linear resolutions and the generic initial ideal

Let $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$. For any ideal $I \subseteq S$ we denote $\operatorname{gin}(I)$ the *generic initial ideal* with respect to the graded reverse lexicographic term order. Any graded S-module M has a minimal graded free resolution:

$$0 \to F_p \xrightarrow{\delta_p} F_{p-1} \to \dots \to F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \to 0,$$

in which $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$. Let $Z_i(M) = \ker \delta_i$ be the ith syzygy module of M. The module M has a *k-linear resolution* if $\beta_{i,j}(M) = 0$ whenever $j \neq i+k$. It is *componentwise linear* if $M_{\langle k\rangle}$ has k-linear resolution for all $k \in \mathbb{Z}$, where $M_{\langle k\rangle}$ is the submodule of M generated by all homogeneous elements of degree k. It is not difficult to show that, if M has a linear resolution, then it is componentwise linear, for example, using [CH03, Corollary 2.5]. Linearity of the free resolution is a genericity condition. This intuition is justified by

Theorem 5.1 ([AJT00, Theorem 1.1])**.** *Let* $char(\mathbb{K}) = 0$. *An ideal* $I \subset S$ *is componentwise linear if and only if* $\beta_{i,j}(S/I) = \beta_{i,j}(S/\operatorname{gin}(I))$ *for all* i, j.

Since $I = Z_0(S/I)$, one may ask which conclusions are implied if $Z_i(S/I)$ is componentwise linear. The following result gives one direction.

Proposition 5.2 ([CV13, Theorem 5.7])**.** *Let* $I \subset S$ *be a graded ideal such that* $\beta_{i,j}(S/I) = \beta_{i,j}(S/\operatorname{gin}(I))$ *for all* $i > s+1$ *and* $j \in \mathbb{Z}$. *Then* $Z_s(S/I)$ *is componentwise linear.*

In general, the other implication in Proposition 5.2 does not hold (Example 5.4). In fact, it would imply Stanley's conjecture for cover ideals of simple matroids. To see this, let $I \subset S$ be an ideal such that $S/\operatorname{gin}(I)$ is level. In characteristic zero, the generic initial ideal is strongly stable and thus $x_n, x_{n-1}, \ldots, x_{d+1}$ is a regular sequence in $S/\operatorname{gin}(I)$. The Artinian reduction $S/(\operatorname{gin}(I) + (x_n, x_{n-1}, \ldots, x_{d+1}))$ is an Artinian level monomial algebra with the same h-vector as S/I. In fact, having a binomial regular sequence would suffice to ensure monomiality of the quotient (see Remark 5.5). Consequently, the h-vector of S/I is a pure O-sequence. If the converse of Proposition 5.2 were true, then the h-vector of any level algebra whose second to last syzygy module is componentwise linear would be a pure O-sequence. This is the case for cover ideals of *simple* matroids, that is matroids without parallel elements:

Proposition 5.3. *Let* Δ *be a rank* d *simple matroid on* n *vertices. Then* $\beta_{d-1,j}(S/J(\Delta)) \neq 0$ *only for* $j = n-1$. *In particular,* $Z_{d-2}(S/J(\Delta))$ *is componentwise linear.*

Proof. Let $\Gamma = \Delta^c$. Hochster's formula implies:

$$\beta_{d-1,j}(S/J(\Delta)) = \beta_{d-1,j}(S/I_\Gamma) = \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_{\Bbbk} \tilde{H}_{j-d}(\Gamma_W, \Bbbk),$$

where Γ_W denotes the restriction of Γ to the vertex subset W. If $j > n-1$, then the only summand that could occur is $\dim_{\Bbbk} \tilde{H}_{n-d}(\Gamma, \Bbbk) = 0$ in the case $j = n$. If $j < n-1$, then we can find two distinct vertices outside of W. Since Δ is simple, they must be contained in a facet F of Δ. Therefore, $G := [n] \setminus F$ is a facet of Γ, and $|G \cap ([n] \setminus W)| \leq n - j - 2$. Thus $\dim(\Gamma_W) \geq j - d + 1$. By Reisner's criterion $\widetilde{H}_{j-d}(\Gamma_W, \Bbbk) = 0$ since Γ_W is a matroid and can thus have only top-dimensional homology. $\square$

Example 5.4. Let Δ be the rank three simple matroid on $\{1, \ldots, 7\}$ with the following facets

$$123, 124, 125, 127, 135, 136, 137, 145, 146, 147, 156, 167, 234, 235,$$
$$236, 246, 247, 256, 257, 267, 345, 346, 347, 357, 367, 456, 457, 567,$$

commonly known as the Fano matroid. A quick computation with Macaulay2 shows that the cover ideal is level of Cohen-Macaulay type 8 while its generic initial ideal is not level ($\beta_3(S/\operatorname{gin}(J(\Delta)) = 10$). Since Δ is simple, Proposition 5.3 shows that $Z_1(S/J(\Delta))$ is componentwise linear.

Remark 5.5. Propositions 5.2 and 5.3 inspired the search for a less generic initial ideal in which the coordinate transform has block structure. The hope was to find a construction that balances between preserving the last Betti number—yielding a *level* quotient—and maintaining the existence of a binomial regular sequence—needed to have a *monomial* quotient. However, we did not find a definition that realizes just the right balance.

If the generic initial ideal of I_Δ is level, then h_Δ is a pure O-sequence since it equals the Hilbert function of the Artinian reduction of $\operatorname{gin}(I_\Delta)$ by variables. To implement this strategy we employ the following two general lemmas. Following [HH99], let $I_{<k}$ denote the subideal of a homogeneous ideal I generated by the homogeneous elements of I of degree less than k.

Lemma 5.6. *Let $I \subset S$ be a homogeneous ideal of projective dimension p and regularity k. If* $\mathrm{pd}(I_{<k}) < p$ *and char*$(\Bbbk) = 0$*, then* $\beta_p(I) = \beta_{p,p+k}(I) = \beta_{p,p+k}(\mathrm{gin}(I)) = \beta_p(\mathrm{gin}(I))$.

Proof. Let $J_1 = \mathrm{gin}(I)_{<k}$ and $J_2 = \mathrm{gin}(I_{<k})_{<k}$. It is easy to see that $J_1 = J_2$. In characteristic zero, the generic initial ideal is strongly stable and [BS87, Theorem 2.4(a)] shows $\mathrm{pd}(J_2) < p$. Using the Eliahou-Kervaire resolution [EK90, Theorem 2.1], we get that no monomial $x_{p+1}u$ with $u \in \Bbbk[x_1,\dots,x_{p+1}]$ is a minimal generator of $J_2 = J_1$. Therefore any minimal generator of $\mathrm{gin}(I)$ of the form $x_{p+1}u$ with $u \in \Bbbk[x_1,\dots,x_{p+1}]$ must be of degree at least k. Since by [BS87, Theorem 2.4(b)] we have $\mathrm{reg}(I) = \mathrm{reg}(\mathrm{gin}(I))$ it must be of degree exactly k.

The Eliahou-Kervaire formula [HH11, Corollary 7.2.3] gives one of the equations: $\beta_p(\mathrm{gin}(I)) = \beta_{p,p+k}(\mathrm{gin}(I))$. Since $\beta_{p,p+k}(I)$ is an extremal Betti number, we have $\beta_{p,p+k}(\mathrm{gin}(I)) = \beta_{p,p+k}(I)$ by [BCP99, Corollary 1.3]. Finally, it is a general fact (see for example [MS05, Theorem 8.29]) that $\beta_{p,p+j}(I) \leq \beta_{p,p+j}(\mathrm{gin}(I))$ for any j, so actually $\beta_p(I) = \beta_{p,p+k}(I) = \beta_{p,p+k}(\mathrm{gin}(I)) = \beta_p(\mathrm{gin}(I))$. □

Lemma 5.7. *Let Δ be a Cohen-Macaulay complex of dimension d, and F a minimal non-face of cardinality $d+1$. Then $\Delta \cup F$ is Cohen-Macaulay.*

Proof. Let $\langle F \rangle$ denote the complex on $[n]$ with one facet F. By construction $\langle F \rangle \cap \Delta$ is the boundary of a d-simplex. In particular $\Bbbk[\langle F \rangle \cap \Delta]$ is a d-dimensional Cohen-Macaulay ring. So the statement follows at once from the exact sequence

$$0 \to \Bbbk[\Delta \cup F] \to \Bbbk[\Delta] \oplus \Bbbk[\langle F \rangle] \to \Bbbk[\langle F \rangle \cap \Delta] \to 0,$$

and the depth inequalities. □

The following theorem is the main result of this section. We state it for Stanley-Reisner ideals.

Theorem 5.8. *Let Δ be the $(d-1)$-skeleton of a d-dimensional Cohen-Macaulay complex. Then h_Δ is a pure O-sequence. Furthermore, if char*$(\mathbb{K}) = 0$*, then $\mathbb{K}[\Delta]$ is level.*

Proof. By Hochster's formula $\mathrm{reg}(\Bbbk[\Delta]) \leq d$ and since I_Δ has a generator of degree $d+1$, $\mathrm{reg}(\Bbbk[\Delta]) = d$. Write $J = (I_\Delta)_{<d+1}$ and let Γ be the corresponding simplicial complex. The result follows from Lemma 5.6 once we show $\mathrm{depth}(\Bbbk[\Gamma]) > d$, which, in turn, is equivalent to the d-skeleton of Γ being Cohen-Macaulay. The d-skeleton of Γ is the complex Γ_d that arises from Δ by turning all non-faces of size $d+1$ into facets. Now, Δ is the $(d-1)$-skeleton of a d-dimensional Cohen-Macaulay complex Ω. There are two kinds of facets of Γ_d: those that are facets of Ω and those that are not. Those that are not, are minimal non-faces in Ω. By Lemma 5.7, Γ_d is Cohen-Macaulay. The statement about the h-vector is characteristic-free because the h-vector of a simplicial complex does not depend on the coefficient field. □

It is equivalent to say that a vector is the Hilbert function of an Artinian monomial algebra and that it is the f-vector of an *order ideal of monomials*, also known as a *multicomplex*. In this language pure O-sequences are f-vectors of pure multicomplexes. Similar to simplicial complexes, there are theories of shellability of multicomplexes (such as M-shellability) and the work of Chari suggests that a characterization of f-vectors of shellable multicomplexes may be possible [Cha97]. He also conjectures that the h-vector of any coloop-free matroid is a *shellable O-sequence* [Cha97, Conjecture 3] which would imply Stanley's conjecture.

Remark 5.9. Let $I \subset S = \mathbb{K}[x_1,\dots,x_r]$ be a strongly stable ideal such that S/I is an Artinian level ring. In this case the h-vector of S/I is the f-vector of an M-shellable multicomplex.

Proof. By the Eliahou-Kervaire resolution, the variable x_r appears only in the minimal generators of I of maximal degree. Let k be this maximal degree, and let $u_1, \dots, u_t$ be the degree k minimal generators of I divisible by x_r. Write $u_i = v_i x_r$ for all $i = 1, \dots, t$. One easily checks that $v_1, \dots, v_t$ generate the order ideal of S/I. Let $\prec$ be the graded revlex order induced by $x_r > \dots > x_1$. We can assume $v_1 \prec \dots \prec v_t$. Now write $v_i = v_i' x_r^{e_i}$, where e_i is the maximum power of x_r dividing v_i, and let $V_i = \{v_i' x_r^j : j = 0, \dots e_i\}$. We claim that $V_t, \dots, V_1$ is a shelling of the multicomplex S/I. It remains to show that, if u is a monomial of degree e dividing v_i, then there exists $j \geq i$ such that $v_j = u x_r^{k-e-1}$. Let m be the monomial of degree $k-e-1$ such that $v_i = um$. If no such j existed, then $u x_r^{k-e-1}$ would be in I, so there would exist a minimal generator u' of I, say of degree a, such that $u = u'u''$ for some u''. Then $u' x_r^{e-a}$ would be in I as well. Since I is strongly stable, $u = u' x_r^{e-a} / x_r^{e-a} \cdot u'' \in I$. This is a contradiction to u_i being a minimal generator. □

In matroid theory, passing from a matroid of rank d to its k-skeleton for $k < d-1$ is called a *truncation*. The rank function of the truncation is $A \mapsto \min\{\mathrm{rk}(A), k+1\}$. The shift of one arises because the k-skeleton is of dimension k which means rank $k+1$. All together we have

Theorem 5.10. *Any truncation of a matroid satisfies Chari's conjecture and consequently also Stanley's conjecture.*

Proof. If $I \subset S$ is a strongly stable ideal such that S/I is level, then the h-vector of S/I is the f-vector of an M-shellable multicomplex by Remark 5.9. By Theorem 5.8, the h-vectors of truncated matroids satisfy Chari's and consequently also Stanley's conjecture. □

Evidently the next question is: Which matroids are truncations? Certainly not all of them.

Example 5.11. Any complete bipartite graph is a rank two matroid that is not the truncation of a matroid. More generally, any matroid that becomes a simplex after identifying parallel elements is not a truncation.

Remark 5.12. If a rank d matroid Γ is a truncation, then it is a truncation of a rank $d+1$ matroid Δ. In this case, any facet of Δ is a *spanning circuit* of Γ, that is, a minimal non-face of size $d+1$. In particular, the facets of Δ are contained in the spanning circuits of Γ. Moreover, if Γ has no spanning circuit, then it is not the truncation of a matroid.

Example 5.13. The dual of the Fano matroid from Example 5.4 has no spanning circuit.

Remark 5.14. Let Δ be a matroid which has a spanning circuit. In [Bry86] Brylawski gives an algorithm that decides if there exists a matroid Γ such that Δ is the truncation of Γ, and constructs the *freest* such matroid whenever possible.

In the remainder of the section we discuss Schubert matroids (also known as shifted matroids, PI-matroids, and generalized Catalan matroids [Fin10]). They play an important role in the study of Hopf algebras of (poly)matroids [DF10].

Definition 5.15. Let $1 \leq s_1 < s_2 < \dots < s_d \leq n$ be a sequence of strictly ascending integers. The *Schubert matroid $SM_n(s_1, \dots, s_d)$* is the rank d matroid on $[n]$ with facets:

$$\left\{\{i_1, \dots, i_d\} \;:\; i_j \leq s_j\right\}. \tag{5.1}$$

Remark 5.16. For any simplicial complex Δ, the ideal $(I_\Delta)_{\langle k \rangle}$, generated by the degree k part of I_Δ is generated by all monomials corresponding to non-faces of size k.

Lemma 5.17. *If $\Delta = SM_n(s_1, \dots, s_d)$ is a Schubert matroid of rank d and $s_1 \geq 2$, then for any $k < d+1$, $(I_\Delta)_{\langle k \rangle}$ is the ideal generated by the degree k part of the Stanley-Reisner ideal of $SM_n(s_1 - 1, s_1, \dots, s_d)$.*

Proof. If $\{j_1,\dots,j_{d+1}\}$ is a facet of $SM_n(s_1-1,s_1,\dots,s_d)$ then it is a minimal non-face of $SM_n(s_1,\dots,s_d)$ since any $\{j_1,\dots,\widehat{j_l},\dots,j_{d+1}\}$ satisfies (5.1). On the other hand, if $\{j_1,\dots,j_{d+1}\}$ is a non-face of $SM_n(s_1-1,s_1,\dots,s_d)$, then $\{j_2,\dots,j_{d+1}\}$ is a non-face of $SM_n(s_1,\dots,s_d)$, assuming without loss of generality that $j_1<j_2<\dots<j_{d+1}$. By Remark 5.16 the statement holds for any $k<d+1$. □

Theorem 5.18. *Schubert matroids have componentwise linear Stanley-Reisner ideals and in particular satisfy Chari's (and thus Stanley's) conjecture.*

Proof. If $s_1=1$, then $SM_n(s_1,\dots,s_d)\cong SM_{n-1}(s_1-1,\dots,s_d-1)*\{v\}$. The Stanley-Reisner ideal of $SM_{n-1}(s_1-1,\dots,s_d-1)*\{v\}$ does not use the variable of v and is componentwise linear if and only if the Stanley-Reisner ideal of $SM_n(s_1,\dots,s_d)$ is componentwise linear. If $s_d<n$, then $SM_n(s_1,\dots,s_d)\cong SM_{s_d}(s_1,\dots,s_d)$. The Stanley-Reisner ideal of $SM_{s_d}(s_1,\dots,s_d)$ equals that of $SM_n(s_1,\dots,s_d)$ plus variables. One is componentwise linear if and only the other is. Consequently, assume $1<s_1<s_2<\dots<s_d=n$. We proceed by induction on the corank $n-d$. The base case is $n-d=1$ in which I_Δ is principal. To check that I is componentwise linear, it suffices to check $I_{\langle k\rangle}$ for any k in which I has minimal generators [HH99], and I_Δ has minimal generators in degrees $\le d+1$. Since $\operatorname{reg}(I_\Delta)=d+1$, the ideal $(I_\Delta)_{\langle d+1\rangle}$ has a linear resolution [EG84, Proposition 1.1]. By Lemma 5.17 and the induction hypothesis we conclude. □

5.2 Matroids with $d+2$ parallel classes

In the remainder of the paper we focus on duals of matroids, or equivalently, h-vectors of cover ideals. If Δ is a matroid, then $h^\Delta=h_{\Delta^c}$ is the h-vector of $S/J(\Delta)$, the quotient by the cover ideal of Δ. The one-dimensional skeleton of a matroid is a complete p-partite graph whose groups of vertices correspond to the partition of the vertex set of the matroid set into *parallel classes* [CV15, Corollary 2.3]. The main result of this section (Theorem 5.36) says that Stanley's conjecture holds for cover ideals of matroids whose number of parallel classes is at most two more than the rank. Due to the technical nature of the proof, we divide it into several smaller results, give various examples along the way, and state the general theorem at the very end.

Our notation follows closely that of [CV15]. Let Δ be a matroid of rank d, with parallel classes $A_1,\dots,A_p$, of cardinalities $a_1,\dots,a_p$. Such matroids are *p-partite*. The *simplification* ${}^{\mathrm{si}}\Delta$ of Δ is the matroid that arises from Δ by replacing each parallel class by a single vertex. We begin with a technical condition to be used in many inductive constructions.

Lemma 5.19. *Let $\Gamma'=\langle N_1,\dots,N_u\rangle$ be a pure order ideal in variables $y_1,\dots,y_d$, and let $\Gamma''=\langle M_1,\dots,M_v\rangle$ be a pure order ideal in the variables $y_1,\dots,\widehat{y_r},\dots,y_d$, that is, not using y_r. Assume that $h^{\Delta\setminus A_p}=f(\Gamma')$ and that $h^{\operatorname{link}_\Delta A_p}=f(\Gamma'')$. Suppose that $\forall\, i\in[u]$, $\exists\, j\in[v]$ such that*

$$\frac{N_i}{y_r^{n_i}}\mid M_j,\quad \textit{where } n_i=\max\{m: y_r^m\mid N_i\}. \tag{5.2}$$

Then h^Δ equals the f-vector of the pure order ideal

$$\Gamma=\langle y_r^{a_p}N_1,\dots,y_r^{a_p}N_u,y_r^{a_p-1}M_1,\dots,y_r^{a_p-1}M_v\rangle.$$

Proof. By [CV15] we have for any $i\ge 0$ that

$$h_i^\Delta=h_{i-a_p}^{\Delta\setminus A_p}+\sum_{j=0}^{a_p-1}h_{i-j}^{\operatorname{link}_\Delta A_p}.$$

It suffices to show the corresponding formula for Γ:

$$f_i(\Gamma) = f_{i-a_p}(\Gamma') + \sum_{j=0}^{a_p-1} f_{i-j}(\Gamma'').$$

Fix an index i and write $\Gamma_i = \{M \in \Gamma : \deg M = i\}$. We write Γ_i as the disjoint union $G_{\geq a_p} \sqcup G_{a_p-1} \sqcup \ldots \sqcup G_0$, where $G_j = \{M \in \Gamma_i : y_r^j \mid M \text{ but } y_r^{j+1} \nmid M\}$, and $G_{\geq a_p} = \{M \in \Gamma_i : y_r^{a_p} \mid M\}$. If a generator of Γ is divisible by $y_r^{a_p}$, then it cannot come from generators of Γ''. Hence $f_{i-a_p}(\Gamma') = |G_{\geq a_p}|$, and it suffices to check that $f_{i-j}(\Gamma'') = |G_{a_p-j-1}|$. The inequality $f_{i-j}(\Gamma'') \leq |G_{a_p-j-1}|$ follows from the definition of Γ. To obtain equality we confirm that each monomial in G_{a_p-j-1} divides some generator $y_r^{a_p-1}M_l$. Assume there exists a monomial $M = y_r^{a_p-j-1}M' \in \Gamma$ (with $y_r \nmid M'$), such that $M \mid y_r^{a_p}N_k$, for some k. By (5.2), there exists l such that

$$\frac{N_k}{y_r^{n_k}} \mid M_l.$$

This implies that $M' \mid M_l$, and as $a_p - j - 1 \leq a_p - 1$ we conclude. □

In our inductive proofs, the matroids Γ' are special simplicial complexes for which Stanley's conjecture is known by [CV15]. They are defined as follows.

Definition 5.20. Let $\mathbf{a} = (a_1, \ldots, a_p)$ be a vector of positive integers. Fix integers $2 \leq d \leq p$ and $0 \leq t \leq d-2$. Let $A_1, \ldots, A_p$ be disjoint sets of vertices with $|A_i| = a_i$ for any i. The matroid $\Delta_t(d, p, \mathbf{a})$ is the rank d matroid on $\sum_i a_i$ vertices with facets

$$A_{i_1} \ldots A_{i_{d-t}} A_{p-t+1} \ldots A_p \quad \text{where } 1 \leq i_1 < \cdots < i_{d-t} \leq p-t.$$

Here $A_{j_1} \ldots A_{j_k}$ stands for all sets $\{v_{j_1}, \ldots, v_{j_k}\}$ such that $v_{j_i} \in A_{j_i}$. The matroid $\Delta_0(d, p, \mathbf{a})$ is the *complete matroid* of rank d with p parallel classes of sizes $a_1, \ldots, a_p$.

The simplification of $\Delta_t(d, p, \mathbf{a})$ is isomorphic to $\Delta_t(d, p, \mathbf{1})$, which in turn equals the simplicial join of the uniform matroid $U_{d-t,p-t}$ of rank $d-t$ on $p-t$ vertices, with a simplex on t vertices. The matroids Δ_t appear in [CV15] with a different numbering of the parallel classes, but here we find this convention more natural. The h-vector of the cover ideal of $\Delta_t(d, p, \mathbf{a})$ is a pure O-sequence by [CV15, Theorem 3.7] and we give its order ideal in Example 5.22, after setting up a useful notation.

Notation 5.21. Fix positive integers $(a_1, \ldots, a_p)$. For any set partition $\mathscr{P} = P_1 \sqcup \cdots \sqcup P_d$ of $[p]$, denote by $[\mathscr{P}] = [P_1|P_2|\ldots|P_d]$ the monomial in d variables:

$$y_1^{-1+\sum_{j\in P_1} a_j} \cdot \ldots \cdot y_d^{-1+\sum_{j\in P_d} a_j}.$$

When no confusion may arise, we will use this notation for the corresponding partition as well.

Example 5.22. Fix integers t, d, p such that $0 \leq t \leq d-2 \leq p-2$, and an integer vector $\mathbf{a} = (a_1, \ldots, a_p)$. For any ascending sequence $1 = l_0 < l_1 < \cdots < l_d = p+1$ of integers, let $\mathscr{P}(l_0, \ldots, l_d)$ be the d-partition into sets $P_i = \{l_{i-1}, \ldots, l_i - 1\}$. We define the following pure order ideal:

$$\Gamma_t(d, p, \mathbf{a}) := \langle [\mathscr{P}(l_0, \ldots, l_{d-t})|p-t+1|\ldots|p] : \text{ for all } 1 = l_0 < l_1 < \cdots < l_{d-t} = p-t+1 \rangle.$$

In particular, when $t = 0$ we have

$$\Gamma_0(d, p, \mathbf{a}) := \langle [\mathscr{P}(l_0, \ldots, l_d)] : \text{ for all } 1 = l_0 < l_1 < \cdots < l_d = p+1 \rangle.$$

By [CV15, Theorem 3.7] the vector $h^{\Delta_t(d,p,\mathbf{a})}$ equals the f-vector of $\Gamma_t(d,p,\mathbf{a})$. This equality is not easy to check in general. One may prove it by induction for complete matroids, then notice that

$$\Delta_t(d,p,(a_1,\dots,a_p)) = \Delta_0(d-t,p-t,(a_1,\dots,a_{d-t})) * \Delta_0(t,t,(a_{p-t+1},\dots,a_p)),$$

check that a similar equality holds for the pure order ideals (viewed as multicomplexes), and finally use the behavior of h-vectors and f-vectors over star products. In this section we are mainly interested in the case $p = d+1$, where $\Gamma_t(d,d+1,\mathbf{a})$ is generated by

$$\begin{array}{cccccccccccccccccc}
[& 1 & | & 2 & | & \cdots & | & d-t+1 & | & d-t,d-t+1 & | & d-t+2 & | & \cdots & | & d+1 &] \\
[& 1 & | & 2 & | & \cdots & | & d-t+1,d-t & | & d-t+1 & | & d-t+2 & | & \cdots & | & d+1 &] \\
 & & & & & & & & \vdots & & & & & & & & \\
[& 1 & | & 2,3 & | & \cdots & | & d-t & | & d-t+1 & | & d-t+2 & | & \cdots & | & d+1 &] \\
[& 1,2 & | & 3 & | & \cdots & | & d-t & | & d-t+1 & | & d-t+2 & | & \cdots & | & d+1 &].
\end{array}$$

In particular, for $t=1, d=3, p=4$ and some $\mathbf{a}$ we obtain:

$$\begin{aligned}
\Gamma_1(3,4,\mathbf{a}) &= \langle [\mathscr{P}(l_0,l_1,l_2) \mid 4] : \text{ for all } 1 = l_0 < l_1 < l_2 = 4 \rangle \\
&= \langle [\mathscr{P}(1,2,4) \mid 4], [\mathscr{P}(1,3,4) \mid 4] \rangle \\
&= \langle [1 \mid 2,3 \mid 4], [1,2 \mid 3 \mid 4] \rangle \\
&= \langle y_1^{a_1-1} y_2^{a_2+a_3-1} y_3^{a_4-1},\ y_1^{a_1+a_2-1} y_2^{a_3-1} y_3^{a_4-1} \rangle.
\end{aligned}$$

Plugging in various values for $\mathbf{a}$ one can directly check $h^{\Delta_1(3,4,\mathbf{a})} = f_{\Gamma_1(3,4,\mathbf{a})}$.

Definition 5.23. Let $[P_1|\cdots|P_d]$, $[Q_1|\cdots|Q_d]$ be d-partitions of subsets of $[p]$. For every vector of positive integers $\mathbf{a} = (a_1,\dots,a_p)$, let $\leq_{\mathbf{a}}$ be the partial order defined by

$$[P_1|\cdots|P_d] \leq_{\mathbf{a}} [Q_1|\cdots|Q_d] \quad\Longleftrightarrow\quad \sum_{j\in P_i} a_j \leq \sum_{j\in Q_i} a_j, \text{ for all } i = 1,\dots,d.$$

For any $(d-1)$-partition $[Q'_1|\dots|Q'_{d-1}]$ of $[p]$ and integer $r \in [d]$, a partial order $\leq^r_{\mathbf{a}}$ is defined by

$$[P_1|\cdots|P_d] \leq^r_{\mathbf{a}} [Q'_1|\cdots|Q'_{d-1}] \quad\Longleftrightarrow\quad [P_1|\dots|\widehat{P_r}|\dots|P_d] \leq_{\mathbf{a}} [Q'_1|\cdots|Q'_{d-1}].$$

The compatibility condition (5.2) in Lemma 5.19 can be rewritten using the new notation.

Definition 5.24. Let $\mathrm{P} = \{\mathscr{P}_1,\dots\mathscr{P}_s\}$ be a set of d-partitions of $[p]$, $\mathrm{Q} = \{\mathscr{Q}_1,\dots,\mathscr{Q}_r\}$ a set of $(d-1)$-partitions of $[p]$. For every $r \in [d]$ we say that the sets P,Q satisfy the *r-compatibility condition* if for each $\mathscr{P} \in \mathrm{P}$ there exists a $\mathscr{Q} \in \mathrm{Q}$ such that $\mathscr{P} \leq^r_{\mathbf{a}} \mathscr{Q}$.

Example 5.25. The sets of partitions $\mathrm{P} = \{[1|2|3,4],[1|2,3|4],[1,2|3|4]\}$ and $\mathrm{Q} = \{[1,2|3,4]\}$ are 3-compatible if and only if $a_2 \leq a_4$, while the collections $\mathrm{P}' = \{[1|2,3|4,5],[1,2|3|4,5]\}$ and $\mathrm{Q}' = \{[1|2,3,4,5],[1,2|3,4,$ are i-compatible for any $\mathbf{a}$ and any $i = 1,2,3$.

In the new notation, the *gluing* in Lemma 5.19 takes two sets Γ' and Γ'' of partitions of $[p-1]$ and produces a set of d-partitions of $[p]$. The procedure consists of

- adding the element p to each rth set of a partition in Γ',
- inserting the set $\{p\}$ into each partition of Γ'' as the rth set, shifting the index of the last $d-r$ sets by one.

Here is an example of how Lemma 5.19 can be applied. It is one of the base cases in the proof of Proposition 5.35.

Example 5.26. Let Δ be the rank 3 matroid with 5 parallel classes and facets:

$$A_1A_2A_3, A_1A_2A_4, A_1A_3A_4, A_2A_3A_4, A_1A_3A_5, A_1A_4A_5, A_2A_3A_5, A_2A_4A_5.$$

As $\Delta \setminus A_5 = \Delta_0(3,4,(a_1,a_2,a_3,a_4))$, it holds that $h^{\Delta\setminus A_5} = f(\Gamma_0(3,4,(a_1,a_2,a_3,a_4))$, corresponding to

$$\mathrm{P} = \{[1|2|3,4],[1|2,3|,4],[1,2|3|4]\}.$$

The rank 2 matroid $\mathrm{link}_\Delta A_5$ is the complete bipartite graph $\Delta_0(2,2,(a_1+a_2,a_3+a_4))$, and thus its h-vector is obtained from the order ideal generated by $\mathrm{Q} = \{[1,2|3,4]\}$. Example 5.25 shows that P and Q are 3-compatible if and only if $a_2 \leq a_4$. Switching the pairs (A_1,A_2) and (A_3,A_4) in Δ gives an isomorphic matroid, therefore we may assume without loss of generality that $a_2 \leq a_4$, and obtain by Lemma 5.19 that $h^\Delta = f(\Gamma)$ for

$$\Gamma = \langle [1|2|3,4,5],[1|2,3|4,5],[1,2|3|4,5],[1,2|3,4|5]\rangle.$$

A crucial property of $(d+2)$-partite matroids is that they possess a dual graph, which together with the vector $(a_1,\ldots,a_p)$ completely encodes their structure.

Definition 5.27. Let Δ be a matroid of rank d with $d+2$ parallel classes and let ${}^{\mathrm{si}}\Delta$ be its simplification. The graph G_Δ is the rank two matroid $({}^{\mathrm{si}}\Delta)^{\mathrm{c}}$.

By construction G_Δ is a complete q-partite graph on $[d+2]$, for some $q \in \{2,\ldots,d+2\}$. If G_Δ is a complete graph on $d+2$ vertices (i.e. if $q = d+2$), then its dual is the complete $(d+2)$-partite matroid, for which Stanley's conjecture holds by [CV15, Theorem 3.5]. However, not all complete q-partite graphs have simple matroids as their duals.

Remark 5.28. For every $d \geq 2$, the bipartite graph with partition $\{1,2\}\cup\{3,\ldots,d+2\}$ and the tripartite graph with partition $\{1\}\cup\{2\}\cup\{3,\ldots,d+2\}$ have duals in which 1 and 2 are parallel and these are the only n-partite graphs with this property.

Proof. The set $\{1,2\}$ is a minimal non-face in the dual of a complete n-partite graph G if and only if every edge of G has at least one of 1 and 2 as a vertex. □

Remark 5.29. The $[(d-1)+2]$-partite matroid $\mathrm{link}_\Delta A_i$ of rank $d-1$ corresponds to the deletion of i in G_Δ, that is $G_{\mathrm{link}_\Delta A_i} = G_\Delta \setminus i$. The $(d+1)$-partite matroid $\Delta\setminus A_i$ of rank d corresponds to $\mathrm{link}_{G_\Delta} i$ viewed as a matroid on $[d+2]\setminus\{i\}$. That is, if j is parallel to i in G_Δ, it is a loop in the rank one matroid $\mathrm{link}_{G_\Delta} i$. If the parallel class in G_Δ of $d+2$ (the vertex corresponding to the parallel class A_{d+2} in Δ) has cardinality s, then

$$\Delta\setminus A_{d+2} \cong \Delta_{s-1}(d,d+1,(a_1,\ldots,a_{d+1})).$$

Similar isomorphisms hold for the deletions of the other parallel class A_i and each one is determined by which vertices of G_Δ are parallel to i.

Our proof of Theorem 5.36 is an induction on the number of vertices of G_Δ. Remark 5.28 implies that there are three different bases of induction to consider, dividing the proof into three cases:

1. G_Δ has at most one parallel class of cardinality ≥ 2.

2. G_Δ is bipartite.

3. G_Δ is r-partite for $r \geq 3$, and at least two parallel classes of cardinality ≥ 2.

Proposition 5.30. *If G_Δ is complete n-partite on $\{1,\ldots,r\}\cup\{r+1\}\cup\ldots\cup\{d+2\}$, for some $r \geq 1$, then h^Δ is a pure O-sequence.*

Proof. The proof is by induction on the number $d+2-r$ of singleton classes. By Remark 5.28, the base case is $d+2-r=3$, since for larger r the graph G_Δ is not the dual of a simple matroid. Decompose Δ into deletion and link at A_{d+2}. By Remark 5.29, it holds that $\Delta \setminus A_{d+2} = \Delta_0(d,d+1,(a_1,\dots,a_{d+1}))$, thus its h-vector is realized by $\Gamma' = \Gamma_0(d,d+1,(a_1,\dots,a_{d+1}))$, which is generated by

$$\mathrm{P} = \{[1|2|\dots|d-1|d,d+1],[1|2|\dots|d-1,d|d+1],\dots,[1,2|3|\dots|d|d+1]\}.$$

By Remark 5.28, A_d and A_{d+1} are parallel in $\mathrm{link}_\Delta A_{d+2}$, so by Remark 5.29 we have that $\mathrm{link}_\Delta A_{d+2}$ is the matroid $\Delta_0(d-1,d,(a_1,\dots,a_{d-1},a_d+a_{d+1}))$. Thus $h^{\mathrm{link}_\Delta A_{d+2}} = f(\Gamma'')$, with Γ'' generated by

$$\mathrm{Q} = \{[1|2|\dots|d-1,d,d+1],[1|2|\dots|d,d+1],\dots,[1,2|3|\dots|d,d+1]\}.$$

It is easy to check that P and Q are d-compatible.

In the induction step Γ' is as above and Γ'' is given by the inductive hypothesis. That is to say, we may assume that we applied Lemma 5.19 $(d-r-1)$ times already, and thus, from the last application we have that

$$\Gamma'' \supseteq \langle [1|2|\dots|d-1,d,d+1],[1|2|\dots|d,d+1],\dots,[1,2|3|\dots|d,d+1] \rangle.$$

Compatibility is again straightforward and we conclude. □

The second case, when G_Δ is bipartite, follows from a general fact about the join of simplicial complexes (or multicomplexes). Let Δ and Δ' be two simplicial (multi)complexes on disjoint vertex sets. Their *join* is the (multi)complex $\Delta * \Delta' = \{\sigma \cup \sigma' \ : \ \sigma \in \Delta \text{ and } \sigma' \in \Delta'\}$. The join operation commutes with duals: $(\Delta * \Delta')^{\mathrm{c}} = \Delta^{\mathrm{c}} * \Delta'^{\mathrm{c}}$. The tensor product of the Stanley-Reisner rings is the Stanley-Reisner ring of their join, and by duality, the same statement holds for tensor product of the quotients by their cover ideals. In the following remark, the simplicial join of two order ideals is computed by viewing them as multicomplexes.

Remark 5.31. Let Δ and Δ' be two matroids, and let Γ and Γ' be two order ideals. If $h^\Delta = f(\Gamma)$ and $h^{\Delta'} = f(\Gamma')$, then $h^{\Delta * \Delta'} = f(\Gamma * \Gamma')$.

In the next proposition we allow also bipartite graphs with partitions of cardinality two (i.e. Δ is $(d+1)$ partite). This turns out useful in the third case.

Proposition 5.32. *If G_Δ is bipartite with partition $\{1,\dots,s\} \cup \{s+1,\dots,d+2\}$, then the h-vector of the cover ideal of Δ is a pure O-sequence.*

Proof. From the bipartition of G_Δ we obtain

$$\Delta = \Delta_0(s-1,s,\mathbf{a}') * \Delta_0(d+1-s,d+2-s,\mathbf{a}''),$$

where $\mathbf{a}' = (a_1,\dots,a_s)$ and $\mathbf{a}'' = (a_{s+1},\dots,a_{d+2})$. Thus [CV15, Theorem 3.5] and Remark 5.31 show that Δ satisfies Stanley's conjecture. □

Example 5.33. If $h^\Delta = f(\Gamma_0(s-1,s,\mathbf{a}') * \Gamma_0(d+1-s,d+2-s,\mathbf{a}''))$, then an explicit description of the order ideal generators follows from Example 5.22:

$$\begin{array}{cccccccccccccccccc}
[& 1 & | & \dots & | & s-2 & | & s-1,s & | & s+1 & | & \dots & | & d & | & d+1,d+2 &] \\
[& 1 & | & \dots & | & s-2 & | & s-1,s & | & s+1 & | & \dots & | & d,d+1 & | & d+2 &] \\
 & & & & & & & & \vdots & & & & & & & & \\
[& 1 & | & \vdots & | & s-2 & | & s-1,s & | & s+1,s+2 & | & \dots & | & d+1 & | & d+2 &] \\
[& 1 & | & \dots & | & s-2,s-1 & | & s & | & s+1 & | & \dots & | & d & | & d+1,d+2 &] \\
 & & & & & & & & \vdots & & & & & & & & \\
 & & & & & & & & \vdots & & & & & & & & \\
[& 1,2 & | & \dots & | & s-1 & | & s & | & s+1,s+2 & | & \dots & | & d+1 & | & d+2 &].
\end{array}$$

Lemma 5.34. *If G_Δ is tripartite, with partition $\{1,\dots,s\}\cup\{s+1,\dots,d+1\}\cup\{d+2\}$, where $s\geq 2$ and $d\geq 4$, then h^Δ is a pure O-sequence. It equals $f(\Gamma)$, where Γ is the pure order ideal obtained by applying Lemma 5.19 to*

$$\Gamma' = \Gamma_0(d,d+1,(a_1,\dots,a_{d+1})), \quad \textit{and}$$
$$\Gamma'' = \Gamma_0(s-1,s,(a_1,\dots,a_s)))*\Gamma_0(d+1-s,d+2-s,(a_{s+1},\dots,a_{d+2})).$$

Proof. Without loss of generality, assume $a_s\leq a_{s+1}\leq \dots, a_{d+1}$. The matroid $\Delta\setminus A_{d+2}$ equals $\Delta_0(d,d+1,(a_1,\dots,a_{d+1}))$, so $\Gamma'=\Gamma_0(d,d+1,(a_1,\dots,a_{d+1}))$. The matroid $\mathrm{link}_{\Delta_0}A_{d+2}$ corresponds to the bipartite graph from Proposition 5.32, thus Γ'' can be chosen as in the statement and Example 5.33. To apply Lemma 5.19, we check d-compatibility of the generators of Γ', and Γ''. Let $P=[1|\dots|i,i+1|\dots|d|d+1]$ be a generator of Γ'.

- If $i\leq s-1$, then choose $Q=[1|\dots|i,i+1|\dots|s|s+1|\dots|d,d+1]$ and $P\leq^d_{\mathbf{a}} Q$ for any $\mathbf{a}$.
- If $s\leq i\leq d$, then choose $Q=[1|\dots|s-1,s|s+1|\dots|i+1,i+2|\dots|d+1]$. For $j<s$ the inequality of the jth entries is clear. For $j\geq s$, and $j\neq i$ the a_j are again ordered, because we assume that $a_j\leq a_{j+1}$ whenever $j\geq s$. Their ith entries correspond to $\{i,i+1\}$ and $\{i+1,i+2\}$, thus as also $a_i\leq a_{i+2}$ we conclude.
- If $i=d+1$, then $[1|2|\dots|d-1|\widehat{d,d+1}]\leq_{\mathbf{a}}[1|2|\dots|d-1,d|\widehat{d+1}]$ for any $\mathbf{a}$ and we conclude by the previous case. ☐

Example 5.26 reproduced the above construction in the case $d=s=2$. We are now ready to prove the third and most complicated case.

Proposition 5.35. *If G_Δ is q-partite with $q\geq 3$ and has at least two parallel classes of cardinality ≥ 2, then the h-vector h^Δ is a pure O-sequence.*

Proof. The proof is a repeated application of Lemma 5.19 with the tripartite graph of Lemma 5.34 as the base case. This is possible because of the two parallel classes of cardinality ≥ 2. Order the vertices of G_Δ such that each parallel class contains consecutive vertices. With this convention, there are only two cases to consider:

Case 1: $d+2$ is parallel to $d+1$ in G_Δ.

Case 2: $d+2$ is not parallel to any vertex is G_Δ.

We use the notation of Lemma 5.19 for Γ' and Γ''.

Case 1 Let $\{r,\dots,d+1,d+2\}$ be the parallel class of $d+1$ in G_Δ. By Remark 5.29, $\Delta\setminus A_{d+2}=\Delta_{d+2-r}(d,d+1,(a_1,\dots,a_{d+1}))$, we can choose $\Gamma'=\Gamma_{d+2-r}(d,d+1,(a_1,\dots,a_{d+1}))$. The matroid $\mathrm{link}_\Delta A_{d+2}$ corresponds to $G_\Delta\setminus\{d+2\}$, thus by the inductive hypothesis there exists an order ideal Γ'' such that $h^{\mathrm{link}_\Delta A_{d+2}}=f(\Gamma'')$. We may also assume that Γ'' was obtained by a repeated application of Lemma 5.19, and thus among its generators has:

$$[1|2|\dots|r-2,r-1|r|\dots|d,d+1],\dots,[1,2|3|\dots|r-1|r|\dots|d,d+1].$$

These generators appear from generators of the Γ' at the previous step because $\mathrm{link}_{G_\Delta}(d+1)$ is isomorphic to $\mathrm{link}_{G_\Delta}(d+2)$. Compatibility is easy to confirm.

Case 2 Let $\{r,\dots,d+1\}$ be the parallel class of $d+1$ in G_Δ. Define a permutation σ of the vertices of $G_\Delta\setminus\{d+2\}$. In order to not complicate notation more than necessary, do this inductively on the parallel classes. The first two parallel classes remain unchanged. For every other parallel, reverse the order of its vertices. More precisely, assume for every $i<r$ that σ is already defined. For every $j\in\{r,\dots,d+1\}$, set $\sigma(j)=r+d+1-j$. As $d+2$ is not parallel to any vertex in G_Δ, Remark 5.29 implies that the deletion $\Delta\setminus A_{d+2}$ is $\Delta_0(d,d+1,(a_1,\dots,a_{d+1}))$. Now use [CV15, Theorem 3.5] with the vertices permuted by σ. That is we have $h^{\Delta\setminus A_{d+2}}=f(\Gamma')$, with Γ' generated by:

$$\begin{array}{lc|c|c|c|c|c|c|cr}
[& 1 & 2 & \dots & m & d+1 & d & \dots & r+1,r &] \\
 & & & & & \vdots & & & & \\
[& 1 & 2 & \dots & m & d+1,d & d-1 & \dots & r &] \\
[& 1 & 2 & \dots & m,d+1 & d & d-1 & \dots & r &] \\
 & & & & & \vdots & & & & \\
[& 1,2 & 3 & \dots & d+1 & d & d-1 & \dots & r &],
\end{array}$$

for some m which plays no role in the proof. Inductively construct Γ'' such that $h^{\mathrm{link}_\Delta A_{d+2}}=f(\Gamma'')$. Assume that Γ'' was constructed using the same strategy of permuting and applying Lemma 5.19 just with $(r-1)$-compatibility. For each $j=r+1,\dots,d+1$, there are $r-1$ generators of Γ'' which have been added at the jth step. This is due to the fact that the simplification of $\Delta|_{A_1,\dots,A_{j-1}}$ is dual to the discrete matroid on $j-1$ vertices with $j-r$ loops, thus its h-vector is obtained from $\Gamma_{j-r}(j-1,j,(a_{\sigma(1)},\dots,a_{\sigma(j)}))$. After applying the gluing from Lemma 5.19, the generators are:

$$\begin{array}{lc|c|c|c|c|c|c|c|cr}
[& 1 & 2 & \dots & m',m & d+1 & \dots & j,j-1 & \dots & r &] \\
 & & & & & \vdots & & & & & \\
[& 1,2 & 3 & \dots & m & d+1 & \dots & j,j-1 & \dots & r &],
\end{array}$$

where m and m' depend on the cardinality of the parallel class of $r-1$ in G_Δ. Their precise description is not needed, as they take the same values for both Γ'' and Γ'.

To check $(r-1)$-compatibility, let $P=[1|2|\dots|\sigma(i),\sigma(i+1)|\dots|r]$ be a generator of Γ'. If $i<r-1$, then choose Q among the generators added at the $(d+1)$th step, namely

$$Q=[1|2|\dots|\sigma(i),\sigma(i+1)|\dots|m|d+1,d|\dots|r].$$

If $i>r-1$, then choose Q among the generators added at the $\sigma(i)$th step, namely

$$Q=[1|2|\dots|m',m|d+1|\dots|\sigma(i),\sigma(i+1)|\dots|r].$$

It is easy to see that in both cases $P\leq_{\mathbf{a}}^{r-1}Q$ for any vector $\mathbf{a}$. Finally, the proof of Case 1 works identically also if σ is applied to the inductive hypothesis. □

Propositions 5.30, 5.32, and 5.35, together with the $(d+1)$-partite case [CV15, Corollary 3.9] imply the main theorem of this section.

Theorem 5.36. *If Δ is a rank d matroid with at most $d+2$ parallel classes, then the h-vector of the quotient by its cover ideal is a pure O-sequence.*

5.3 Small type

If $h^\Delta=h_{\Delta^c}$ is the h-vector of the cover ideal of a matroid Δ, then its last entry is the Cohen-Macaulay type of $\mathbb{K}[\Delta^c]$. If it is small, then the parallel classes of the matroid must be few thanks to [CV15, Remark

4.4]: Precisely, if a matroid is of rank d and has p parallel classes, then its type is at least $p-d+1$. Theorem 5.39 exploits this fact to prove that h^Δ is a pure O-sequence whenever the type is at most five. We start with a proposition that shows that among the simple matroids there is only one of rank d with p parallel classes and whose type is $p-d+1$.

Proposition 5.37. *Let Δ be a p-partite matroid of rank d. Then* $\operatorname{type}(S/J(\Delta)) = p-d+1$ *if and only if* ${}^{\mathrm{si}}\Delta = \Delta_{d-2}(d,p,\mathbf{1})$.

Proof. By [CV15, Proposition 2.8] we can assume that Δ is simple. [CV15, Remark 4.4] shows that $\operatorname{type}(S/J(\Delta)) \geq p-d+1$, and equality holds if $\Delta = \Delta_{d-2}(d,p,\mathbf{1})$. Assume that Δ satisfies $\operatorname{type}(S/J(\Delta)) = p-d+1$. The proof is by induction on $p-d$. The base case is when $d=p$ in which case ${}^{\mathrm{si}}\Delta$ is a simplex. Now assume that $p-d$ is positive. Without loss of generality assume that the vertex p is not a cone point (otherwise relabel the vertices). By [CV15, Remark 1.7] we have

$$h_k^\Delta = h_{k-1}^{\Delta\setminus p} + h_k^{\operatorname{link}_\Delta p} \quad \forall\, k \in \mathbb{Z}.$$

Again by [CV15, Remark 4.4] and since $\operatorname{type}(S/J(\Delta)) = p-d+1$, we get $\operatorname{type}(S/J(\Delta\setminus p)) = p-d$ and $\operatorname{type}(S/J(\operatorname{link}_\Delta p)) = 1$. The matroid $\operatorname{link}_\Delta p$ is $(d-1)$-partite and, by the induction hypothesis, $\Delta\setminus p = \Delta_{d-2}(d,p-1,\mathbf{1})$. After potentially relabeling the vertices, $\{1,2,\dots,d-2,i,j\}$ is a face of Δ for all $i,j\in\{d-1,\dots,p-1\}$. If $\{1,2,\dots,d-2,p\}$ was not a face of Δ, then there is some $k\in\{1,\dots,d-2\}$ such that $\{1,\dots,\hat{k},\dots,d-2,i,j,p\}$ is a face of Δ for all i and j in $\{d-1,\dots,p-1\}$. This would imply that $\{i,j\}\in\operatorname{link}_\Delta p$ for all $i,j\in\{1,\dots,p-1\}\setminus\{k\}$ and $\operatorname{link}_\Delta p$ would be $(p-2)$-partite—a contradiction. Therefore $\{1,2,\dots,d-2,p\}$ is a face of Δ. We now show that, for fixed $i\in\{d-1,\dots,p-1\}$, the set $\{i,k\}$ is a face of $\operatorname{link}_\Delta p$ for all $k\in\{1,\dots,d-2\}$. If not, then $\{1,\dots,d-2,j,p\}$ is a facet of Δ for all $j\in\{d-1,\dots,p-1\}\setminus\{i\}$. Pick $r,s\in\{d-1,\dots,p-1\}\setminus\{i\}$. Certainly $B=\{1,\dots,d-2,r,s\}$ is a facet of Δ. Since i is parallel to some $k\in\{1,\dots,d-2\}$ (in $\operatorname{link}_\Delta p$), also $B' = (\{1,\dots,d-2,r,p\}\setminus\{k\})\cup\{i\}$ is a facet of Δ. Removing k from B, the only way to satisfy basis exchange among B and B' is that $\{r,s,p\}$ is a face of Δ. In this case, however, $\operatorname{link}_\Delta p$ would be d-partite, since the restriction of its 1-skeleton to the vertices $\{1,\dots,d-2,r,s\}$ would be a complete graph. □

Remark 5.38. Theorem 4.3 in [CV15] says that $h^{\Delta_{d-2}(d,p,\mathbf{1})}$ is a componentwise lower bound for all simple matroids of rank d on p vertices.

Theorem 5.39. *Let Δ be a matroid and $h^\Delta = (1,h_1,\dots,h_s)$ its h-vector. If $h_s \leq 5$, then h^Δ is a pure O-sequence.*

Remark 5.40. By duality, Theorem 5.39 also holds for Stanley-Reisner ideals.

Proof of Theorem 5.39. By [CV15, Remark 4.4] $\operatorname{type}(S/J(\Delta)) \geq p-d+1$ which in our case implies $p\leq d+4$. The cases $p=d$ and $p=d+1$ are trivial, and $p=d+2$ is the content of Theorem 5.36. By Proposition 5.37, if $p=d+4$, then ${}^{\mathrm{si}}\Delta = \Delta_{d-2}(d,p,\mathbf{1})$ and the result follows from [CV15, Theorem 3.7]. It remains to check the case $p=d+3$, however, there are no simple matroids with cover ideal of type five such that $p=d+3$. To see this, assume Δ is such a matroid and consider its dual Δ^{c}. The simplification ${}^{\mathrm{si}}\Delta$ has the same type, so we can assume that Δ is simple and consequently Δ^{c} is of rank three. Let G be the complete q-partite graph which is the 1-skeleton of Δ^{c}. Since Δ^{c} is of rank three, $q\geq 3$. Let $b_1\geq\cdots\geq b_q$ be the sizes of the parallel classes in G which we can assume ordered nonincreasingly. Let $h^{\Delta^{\mathrm{c}}} = (1,h_1,h_2,5)$ be the h-vector. By the Brown-Colbourn inequalities [BC92, Theorem 3.1], $1-h_1+h_2\leq 5$. If $n\leq d+3$ is the number of vertices of G and e the number of edges, then $h_1=n-3$ and $h_2 = 3-2n+e$. It follows that $e\leq 3n-2$. Now, if $q=3$, then $b_i\geq 3$ for $i=1,\dots,q$ and $e>3n-2$. If $q=4$, then $b_i\geq 2$ for $i=1,\dots,q$, except for one graph in which $b_4=1$ and $b_2=b_3=b_4=2$. If $q=5$, there are five possible graphs. If $q=6$, then K_6 the complete graph is the only possible graph. When the graph is fixed, the h-vector of Δ^{c} is fixed. Table 5.1 summarizes the possible graphs and their

q	$(b_1,\dots,b_q)$	h^Δ
4	$(2,2,2,1)$	$(1,4,7,5)$
5	$(1,1,1,1,1)$	$(1,2,3,5)$
5	$(2,1,1,1,1)$	$(1,3,5,5)$
5	$(2,2,1,1,1)$	$(1,4,8,5)$
5	$(3,1,1,1,1)$	$(1,4,7,5)$
5	$(4,1,1,1,1)$	$(1,5,9,5)$
6	$(1,1,1,1,1,1)$	$(1,3,6,5)$

Table 5.1: Possible q-partite graphs in the proof of Theorem 5.39

h-vectors. Using the database of Mayhew and Royle [MR08], a simple for-loop in Sage enumerates all matroids of rank three, filters those with the given h-vectors, computes their duals, and confirms that none is simple. □

Remark 5.41. The matroid $\Delta_{d-2}(d,p,\mathbf{1})$ is the only matroid of type t which satisfies $p=d+t-1$ and in the proof of Theorem 5.39 we showed that, if $t=5$, then there is no matroid of type five such that $p=d+3$. It would be interesting to understand for which t there is a such a gap in the allowable number of parallelism classes.

5.4 The search for counterexamples

As soon as the number of variables d, the socle degree s, and the type t are fixed, one can enumerate all pure O-sequences with these characteristics. A pure order ideal with these data is generated by t monomials of degree s. Let $N_{d,s}=\binom{s+(d-1)}{d-1}$ be the number of monomials of degree s in d variables. A priori, there are $\binom{N_{d,s}}{t}$ generating sets of order ideals to consider and our program loops over these, computing their f-vectors. Naturally, many of those socles will be equivalent after relabeling the variables, or have the same f-vector even if they are not equivalent. One may hope to reduce the number of combinations by exploiting this symmetry. However, it is not clear how to do so. Checking if two socles are equivalent after permuting the variables is computationally more expensive than just computing the f-vectors of the order ideals they generate. One shortcut that is easy to implement is to require the lexicographically first monomial in each socle to have weakly increasing exponent vector. This can be achieved by a permutation of the variables and is quick to check. Further improvements are possible if one is not interested in all pure O-sequences, but just wants to check a particular example. The computation of the face numbers of an order ideal descends degree by degree. In each step, the program searches for monomials that divide the given monomials in the previous degree. If a candidate h-vector is given, then one can stop the degree descent as soon as there is disagreement between the candidate vector and the number of monomials in the current degree. Our software implements all of these shortcuts.

Example 5.42. By Theorem 5.39 and [DKK12] any candidate counterexample for Stanley's conjecture must be on at least ten vertices and of Cohen-Macaulay type six. Assume that Δ is of rank four. For h-vectors of cover ideals, checking an example with this data amounts to enumerating order ideals generated by six monomials of degree six in four variables. Our implementation handles approximately 30000 order ideals per second on a standard laptop. Checking all $\binom{84}{6}=406481544$ potential socles would take approximately four hours. However, this number grows quickly. If a counterexample exists and was of rank five on twelve vertices and type seven, then a back-of-the-envelope calculation estimates the computational time as around 173 CPU years.

Lemma 5.19 inspires a method to search for pure order ideals.

Method 5.43. Let Δ be a p-partite matroid of rank d with parallel classes $A_1,\ldots,A_p$ which we may choose ordered such that $A_1\ldots A_d\in\Delta$, that is $\{v_1,\ldots,v_d\}$ is a facet whenever $v_i\in A_i$ for all $i=1,\ldots,d$. To find a pure order ideal whose f-vector equals h^Δ, instead of enumeration, one may proceed as follows.

1. For each $i\in\{d,\ldots,p\}$ let G_i be the set of generators of $\Gamma_0(d-1,i-1,(a_1,\ldots,a_{i-1}))$.
2. Compute c_i, the last entry of the h-vector of $\mathrm{link}_{\Delta|_{A_1\cup\cdots\cup A_{i-1}}}A_i$.
3. For every $i\in\{d,\ldots,p\}$ choose a c_i-subset H_i of G_i.
4. Define $\Gamma=\langle\overline{H}_d\cup\ldots\cup\overline{H}_p\rangle$, where the collection of partitions $\overline{H}_j$ is obtained by adding the set $\{j,\ldots,p\}$ to every $(d-1)$-partition of $[j-1]$ contained in H_j.
5. Check if $h^\Delta=f(\Gamma)$.

The gist of this method is, instead of searching all socles, to only search order ideal generators among the monomials that could potentially arise from a repeated application of Lemma 5.19. The method starts at the complete matroid $\Delta|_{A_1\cup\cdots\cup A_d}$ and reconstructs Δ by gluing the remaining parallel classes. In this process it mimics the construction of Lemma 5.19 in many different ways. The compatibility condition is never checked. It is faster to just confront the f-vector of the final result with h^Δ.

The choice of ordering of the A_i fixes the order in which Lemma 5.19 would be applied (and one may try different orderings). Step (1) creates lists of candidates for the generators of Γ'' (in the notation of the lemma). Steps (2) and (3) enumerate the sets of order ideal generators that may result from the choices. Finally, Step (4) implements the gluing in Lemma 5.19. Evidently, if the procedure does not find an order ideal whose f-vector is h^Δ we have not found a counterexample.

Example 5.44. In specific examples, the number of orderings of the parallel classes can be reduced using symmetries of the matroid. For instance in Example 5.26 the pairs (A_1,A_2) and (A_3,A_4), and also the classes in each pair, could be exchanged. Given that $A_1A_2A_5$ and $A_3A_4A_5$ are not in Δ, the only orderings to check in this case are A_1,A_2,A_3,A_4,A_5 and A_1,A_3,A_5,A_2,A_4.

Example 5.45. Let Δ be the simple rank four matroid on eight vertices with the following facets:

$$\begin{gathered}1235,1236,1237,1238,1245,1246,1247,1248,1256,1257,1268,1278,1345,1346,1347,\\1348,1357,1358,1367,1368,1456,1458,1467,1478,1567,1568,1578,1678,2356,2357,\\2358,2456,2457,2458,2568,2578,3456,3457,3458,3567,3568,4567,4578,5678.\end{gathered}$$

Precisely, Δ is a series-extension (15 is a cocircuit) of the Fano matroid. The largest example that we tried our method on is the rank four matroid $\Delta_{\mathbf{a}}$ on 20 vertices whose simplification is Δ and whose parallel classes have sizes $(1,2,3,4,1,3,4,2)$. We have

$$h^{\Delta_{\mathbf{a}}}=(1,4,9,16,25,36,49,64,81,100,112,116,111,96,70,40,14)$$

which means that enumeration of order ideals is entirely pointless. However, using Method 5.43 we found that this vector is a pure O-sequence. It equals the f-vector of the order ideal

$$\begin{aligned}\Gamma=\langle &bc^2d^{13},bc^6d^9,b^4c^3d^9,bc^{10}d^5,b^8c^3d^5,bc^{12}d^3,b^4c^9d^3,\\&a^9b^3c^4,a^5b^9c^2,bc^{15},a^5b^3c^8,b^{14}c^2,a^2b^{12}c^2,a^2b^{10}c^4\rangle.\end{aligned}$$

The Artinian monomial level algebra with $\mathbb{K}$-basis Γ is $\mathbb{K}[a,b,c,d]/I$ where

$$\begin{aligned}I=\big(&a^{10},a^6b^4,a^3b^{10},ab^{13},b^{15},a^3b^4c^3,b^{11}c^3,a^6c^5,ab^4c^5,b^5c^5,ac^9,b^2c^{10},\\&c^{16},ad,b^9d,b^5c^4d,c^{13}d,b^2c^4d^4,c^{11}d^4,b^5d^6,c^7d^6,b^2d^{10},c^3d^{10},d^{14}\big).\end{aligned}$$

Remark 5.46. The number of different h-vectors of coloop free matroids is equal to the number of different f-vectors of coloop free matroids. Since matroids are very particular pure multicomplexes, the number of their f-vectors is smaller than the number of pure O-sequences (which are f-vectors of pure multi-complexes). Therefore, it seems plausible that the probability of finding a pure O-sequence equal to the h-vector of a matroid tends to zero as the parameters grow. This limits the usefulness of random search for order ideals in larger examples.

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Chapter 6

Determinantal Schemes and Pure O-sequences

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Division of Labor

Throughout this project the two authors shared an office at the University of Basel. The daily exchange of ideas in that period make a clear separation of the contributions is not possible. It is however possible to say that both authors contributed in an equal manner to the conception, execution and writing of this work.

[1]

Introduction

Classical determinantal rings have made their way from algebraic geometry to commutative algebra more than fifty years ago, and have been an active research topic ever since. Over the years, the study has been extended to Pfaffian ideals of generic skew-symmetric matrices and to determinantal ideals of ladders, of symmetric matrices and of homogeneous polynomial matrices. Defining ideals of Segre varieties, Veronese varieties, rational normal scrolls and rational normal curves are all examples of such objects. We refer to the books of W. Bruns and U. Vetter [BV88], of R.M. Miró-Roig [Mir08], and of C. Băeţică [Băe06] for overviews of this vast subject.

We study the Hilbert functions of standard determinantal rings. Ideals defined by the maximal minors of a homogeneous, polynomial, $t \times (t+c-1)$ matrix M are called standard determinantal if they define a scheme of the "expected codimension", i.e. if their height is c. These ideals are Cohen-Macaulay, and a graded minimal free resolution for them is given by the Eagon-Northcott complex [EN62]. Their Hilbert function and their graded Betti numbers are determined by the degrees of the polynomials in M. Hilbert functions of determinantal ideals have been studied, among many others, by S. Abhyankar [Abh88], W. Bruns, A. Conca and J. Herzog [CH94; BC03], S. Ghorpade [Gho96; Gho02], N. Budur, M. Casanellas and E. Gorla [BCG04].

Our main result (Theorem 6.8) states that if in each column of M all polynomials have the same degree, then the h-vector of the corresponding standard determinantal ring is a log-concave pure O-sequence. The idea of the proof is to obtain the h-vectors of such matrices as h-vectors of some representable matroids, and then use the results of J. Huh [Huh15] for log-concavity, and those of the first author with M. Varbaro [CV15] to prove that they are pure O-sequences. We conjecture that the converse of this theorem also holds, namely if the h-vector of a standard determinantal ideal is a pure O-sequence, then all the degrees in each column of its defining matrix must be equal (Conjecture 6.10).

A pure O-sequence is the Hilbert function of some monomial, artinian, level algebra. Equivalently, a pure O-sequence can be described as the f-vector of a pure multicomplex, or of a pure order ideal. In [Hib89], T. Hibi proved that if $(h_0,\ldots,h_s)$ is a pure O-sequence, then $h_i \leq h_{s-i}$ for all $i=0,\ldots,\lfloor s/2 \rfloor$. Other than the Hibi inequalities and some *ad hoc* methods, we are not aware of any criteria which imply non-purity for an O-sequence. In most specific examples, an exhaustive computer listing of all pure O-sequences with some fixed parameters is needed to check non-purity. Moreover, while a complete characterization of pure O-sequences is considered to "solve all basic problems of design theory" (G. Ziegler [Zie95, Exercise 8.16]), such a goal is expected to be "nearly impossible" by several experts (see M. Boij, J. Migliore, R.M. Miró-Roig, U.Nagel, F. Zanello [Boi+12]). The validity of Conjecture 6.10, together with the computational formulae we find, would provide a fast way to construct (for fixed codimension, socle degree and type) large families of O-sequences which are not pure.

The key to most of our proofs is provided by Lemma 6.1. Using a basic double link from Gorenstein liaison theory, we describe a recursive formula for the h-vector of the standard determinantal ring corresponding to M in terms of h-vectors corresponding to submatrices of M. Using this lemma we find simple formulae for the length and the last entries of the h-vectors (Lemma 6.4 and Proposition 6.6), as well as an explicit formula for the h-polynomial for every standard determinantal ring (Proposition 6.3).

Using the Eagon-Northcott resolution, we show that a standard determinantal ideal is level (i.e. its socle is concentrated in one degree) if and only if in each column of M all polynomials have the same degree. In the third part of the paper we prove several cases of Conjecture 6.10. In particular, we prove that the statement is true for matrices with all entries of positive degree and for matrices in which the degrees in the second row are strictly smaller than the degrees in the first row.

Many of the results in this paper have been suggested and double-checked using intensive computer experiments. The last section of this paper is dedicated to the computational aspects. A data base of pure and non-pure O-sequences, as well as implementations in CoCoA [CoC] of the formula from Proposition 6.3 and for checking particular cases of Conjecture 6.10 can be found, licensed under the GPL, at `https://github.com/alexconstantinescu/PureOSequences`.

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6.1 Preliminaries

We first recall most of the algebraic and geometric notions that we use, and then prove the key lemma of this paper.

Let $\Bbbk$ be an infinite field and $S=\Bbbk[x_0,\ldots,x_n]$ be the polynomial ring over $\Bbbk$. For any two integers $t,c \geq 1$, a matrix M of size $t \times (t+c-1)$, with polynomial entries, is called *homogeneous* if it represents a homogeneous map of degree zero between graded free S-modules

$$\bigoplus_{i=1}^{t} S(b_i) \xrightarrow{M} \bigoplus_{j=1}^{t+c-1} S(a_j).$$

Let $f_{i,j} \in S$ be the entries of M. The homogeneity condition implies that $\deg f_{i,j} = a_j - b_i$ for all i,j. Whenever $b_i > a_j$ we have $f_{i,j}=0$. Without loss of generality, we may assume that M does not contain invertible elements (i.e. $f_{i,j}=0$ when $a_j=b_i$). Alternatively, a matrix with polynomial entries is homogeneous if and only if all its minors are homogeneous polynomials (if and only if all its 2×2 minors are homogeneous). We will denote by $I_{\max}(M)$ the ideal generated by the maximal minors of the matrix M.

An ideal $I \subseteq S$ of height c is a *standard determinantal ideal* if it is generated by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix. As these ideals are saturated, they define a projective scheme $X \subset \mathbb{P}^n$. We call all such schemes *standard determinantal schemes*. The matrix $A = (a_{i,j}) \in \mathbb{Z}^{t \times (t+c-1)}$, with $a_{i,j} = b_j - a_i$, is called the *degree matrix* of the ideal I. We will assume that $a_1 \leq \cdots \leq a_t$ and $b_1 \leq \cdots \leq b_{t+c-1}$, so the entries of A increase from left to right and from the bottom to the top. Since $a_{i,i} \leq 0$ implies that all the minors containing the first i columns are zero, we can assume without loss of generality that $a_{i,i} > 0$ for all i.

For the Hilbert series of a standard graded $\mathbb{k}$-algebra S/I we will use the notation $\mathrm{HS}_{S/I}$. We will write the Hilbert series in rational form as

$$\mathrm{HS}_{S/I}(z) = \frac{\mathrm{HP}(z)}{(1-z)^d},$$

where d is the Krull dimension of S/I. The numerator $\mathrm{HP}(z) = 1 + h_1 z + h_2 z^2 + \cdots + h_s z^s$, with $h_s \neq 0$, is called the *h-polynomial* of S/I, and its coefficients form the *h-vector* of S/I, $h_{S/I} = (h_0, h_1, \ldots, h_s)$. The degree matrix A of I determines the minimal free resolution of S/I (given by the Eagon-Northcott complex) and therefore also $h_{S/I}$. As in this paper we study the h-vectors of standard determinantal ideals, we will write h^A and $\mathrm{HP}^A(z)$ instead of $h_{S/I_{max}(M)}$, respectively $\mathrm{HP}_{S/I_{max}(M)}(z)$. We denote by $\tau(h^A)$ the degree of the h-polynomial.

Our key lemma is based on liaison theory. We recall here briefly the notion of basic double link. If $\mathfrak{b} \subseteq \mathfrak{a} \subseteq S$ are two homogeneous ideals such that $\mathfrak{b}$ is Cohen-Macaulay, $ht(\mathfrak{a}) = ht(\mathfrak{b}) + 1$, and $f \in NZD_S(S/\mathfrak{b})$ is a homogeneous non-zero-divisor, then the ideal $I = f \cdot \mathfrak{a} + \mathfrak{b}$ is called a *basic double link* of $\mathfrak{a}$. The terminology is motivated by Gorenstein liaison theory: In the above notation, I can be Gorenstein linked to $\mathfrak{a}$ in two steps if $\mathfrak{a}$ is unmixed, and $S/\mathfrak{b}$ is Cohen-Macaulay and generically Gorenstein (see [Kle+01, Proposition 5.6] and [Har07, Theorem 3.5]). In [Gor07], Gorla constructed basic double links in which all the ideals involved are standard determinantal (see also [Kle+01] for more general results in this direction). We will use this construction to prove the following recursive formula for the h-vector of a standard determinantal ideal.

For any matrix A and positive integers k and l we use the following notation: $A^{(k,l)}$ is the matrix obtained from A by deleting the k-th row and l-th column. By convention, $A^{(k,0)}$ (respectively $A^{(0,l)}$) means that only the k-th row (respectively the l-th column) has been deleted.

Lemma 6.1. *Let $A = (a_{i,j}) \in \mathbb{Z}^{t \times (t+c-1)}$ be a degree matrix. For any $k = 1, \ldots, t$ and $l = 1, \ldots, t+c-1$, such that $a_{k,l} \geq 0$, we have*

$$\mathrm{HP}^A(z) = z^{a_{k,l}} \mathrm{HP}^{A^{(k,l)}}(z) + (1 + \cdots + z^{a_{k,l}-1}) \mathrm{HP}^{A^{(0,l)}}(z).$$

Proof. We distinguish two cases:
Case 1: $a_{k,l} > 0$. Without loss of generality we can assume that $k, l = 1$. Consider the homogeneous matrix

$$M = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,t+c-1} \\ 0 & f_{2,2} & \cdots & f_{2,t+c-1} \\ \vdots & \vdots & & \vdots \\ 0 & f_{t,2} & \cdots & f_{t,t+c-1} \end{pmatrix},$$

where the $f_{i,j}$'s are generically chosen forms in $S = \mathbb{k}[x_0, \ldots, x_n]$, with $n \geq c-1$ and $\deg(f_{i,j}) = a_{i,j}$. Such forms exist because the field $\mathbb{k}$ is infinite. Let $\mathfrak{a} = I_{\max}(M^{(1,1)})$ and $\mathfrak{b} = I_{\max}(M^{(0,1)})$ be two ideals which by the generic choice of the forms $f_{i,j}$ are standard determinantal. Thus, by construction, we have

$ht(\mathfrak{b}) = ht(\mathfrak{a}) - 1$ and $f_{1,1}$ is a non-zero-divisor in $S/\mathfrak{b}$. If $I = I_{\max}(M)$, then by direct computation on the generators we obtain that

$$I = f_{1,1}\mathfrak{a} + \mathfrak{b},$$

so I is a basic double link of $\mathfrak{a}$. By [Gor07, Theorem 3.1], the ideal I is also standard determinantal. Notice that the corresponding degree matrices of I, $\mathfrak{a}$ and $\mathfrak{b}$ are A, $A^{(1,1)}$, respectively $A^{(0,1)}$. From the short exact sequence where the first map is given by $g \mapsto (g, f_{1,1} \cdot g)$ and the second by $(g,h) \mapsto gf_{1,1} - h$, it follows that, if $d = n - c + 1$, then

$$\begin{aligned}\mathrm{HS}_{S/I}(z) = \frac{\mathrm{HP}^A}{(1-z)^d} &= z^{a_{1,1}}\mathrm{HS}_{S/\mathfrak{a}}(z) + (1 - z^{a_{1,1}})\mathrm{HS}_{S/\mathfrak{b}}(z) \\ &= \frac{z^{a_{1,1}}\mathrm{HP}^{A^{(1,1)}}(z)}{(1-z)^d} + \frac{(1-z^{a_{1,1}})\mathrm{HP}^{A^{(0,1)}}(z)}{(1-z)^{d+1}} \\ &= \frac{z^{a_{1,1}}\mathrm{HP}^{A^{(1,1)}}(z) + (1 + \cdots + z^{a_{1,1}-1})\mathrm{HP}^{A^{(0,1)}}(z)}{(1-z)^d}\end{aligned}$$

and we conclude.

Case 2: $a_{k,l} = 0$. By induction on t and c we will show that

$\mathrm{HP}^A(z) = \mathrm{HP}^{A^{(k,l)}}(z)$. By the ordering of the entries in A, and because $a_{i,i} > 0$ for all i, if $a_{k,l} = 0$, then $k > l$ (i.e. $a_{k,l}$ lies below the diagonal).

When $c = 1$, the h-vector corresponding to A is just a sequence of 1's of length $\mathrm{tr}(A) = \sum_{i=1}^t a_{i,i}$. Notice that

$$\mathrm{tr}(A^{(k,l)}) = \sum_{i=1}^{l-1} a_{i,i} + \sum_{i=l}^{k-1} a_{i,i+1} + \sum_{i=k+1}^{t} a_{i,i}.$$

By the homogeneity of A, we have $\mathrm{tr}(A^{(k,l)}) = \mathrm{tr}(A^{(k,l)}) + a_{k,l} = \mathrm{tr}(A)$.

The first row has only positive entries, so $t \geq 2$. For $t = 2$, since $a_{2,1} = 0$, from *Case 1* applied to the indices $(2, c+1)$, it follows that

$$\mathrm{HP}^A(z) = z^{a_{2,c+1}}\mathrm{HP}^{(a_{1,1},\ldots,a_{1,c})}(z) + (1 + \cdots + z^{a_{2,c+1}-1})\mathrm{HP}^{A^{(0,c+1)}}(z). \tag{6.1}$$

The h-polynomial of a 1-row degree matrix is the h-polynomial of the corresponding complete intersection, namely:

$$\mathrm{HP}^{(a_{1,1},\ldots,a_{1,c})}(z) = \prod_{i=1}^{c}\left(1 + \cdots + z^{a_{1,i}-1}\right).$$

By induction on c we have $\mathrm{HP}^{A^{(0,c+1)}}(z) = \mathrm{HP}^{(a_{1,2},\ldots,a_{1,c})}(z)$, so (6.1) becomes

$$\begin{aligned}\mathrm{HP}^A(z) &= z^{a_{2,c+1}}\prod_{i=1}^{c}\left(1 + \cdots + z^{a_{1,i}-1}\right) + \left(1 + \cdots + z^{a_{2,c+1}-1}\right)\prod_{i=2}^{c}\left(1 + \cdots + z^{a_{1,i}-1}\right) \\ &= \left(1 + \cdots + z^{a_{1,1}+a_{2,c+1}-1}\right)\prod_{i=2}^{c}\left(1 + \cdots + z^{a_{1,i}-1}\right).\end{aligned}$$

As A corresponds to the degrees in a homogeneous matrix, we have $a_{1,1} + a_{2,c+1} = a_{2,1} + a_{1,c+1}$ and we conclude.

When $t > 2$, there exists some positive entry $a_{i,i}$, with $i \neq k, l$. The matrices $A^{(i,i)}$ and $A^{(0,i)}$ contain $a_{k,l} = 0$. Applying *Case 1* for $a_{i,i}$, and using the induction on t and c we obtain

$$\begin{aligned}\mathrm{HP}^A(z) &= z^{a_{i,i}}\mathrm{HP}^{A^{(i,i)}}(z) + \left(1 + \cdots + z^{a_{i,i}-1}\right)\mathrm{HP}^{A^{(0,i)}}(z) \\ &= z^{a_{i,i}}\mathrm{HP}^{(A^{(i,i)})^{(k,l)}}(z) + \left(1 + \cdots + z^{a_{i,i}-1}\right)\mathrm{HP}^{(A^{(0,i)})^{(k,l)}}(z) \\ &= \mathrm{HP}^{A^{(k,l)}}(z)\end{aligned}$$

□

Remark 6.2. Lemma 6.1 implies the following recursive formula for the h-vector of A:

$$h_i^A = h_{i-a_{k,l}}^{A^{(k,l)}} + \sum_{k=0}^{a_{k,l}-1} h_{i-k}^{A^{(0,l)}}.$$

In particular, if some entry $a_{k,l} = 0$, then $h^A = h^{A^{(k,l)}}$. As we are interested in studying the h-vectors of standard determinantal ideals, we may assume from now on that none of the degree matrices contain zeros.

6.2 Formulae

In this section we find a general formula for the h-polynomial in terms of the entries of the degree matrix (Proposition 6.3). We then compute the length and the last entry of the h-vector (Lemma 6.4). Finally, we give a more explicit description of the last entries of the h-vector when all the rows in the degree matrix are equal (Proposition 6.6).

Recall that the h-polynomial of a complete intersection generated in degrees $(d_1, \ldots, d_c)$ is

$$\mathrm{HP}^{(d_1,\ldots,d_c)}(z) = \prod_{i=1}^{c} \left(1 + z + \cdots + z^{d_i - 1}\right). \tag{6.2}$$

We fix the following notation. Let $a, b \geq 0$ be two integers. For any increasing sequence of integers $0 < i_1 < \cdots < i_b < a+b$ and any matrix $A = (a_{i,j}) \in \mathbb{Z}^{a \times (a+b)}$, we define two ordered sets of integers:

$$\begin{aligned} \{j_1, \ldots, j_{i_b - b}\} &= \{1, \ldots, i_b\} \setminus \{i_1, \ldots, i_b\} \\ \mathrm{g}_A(i_1, \ldots, i_b) &= \{a_{i_1,i_1},\, a_{i_2-1,i_2},\, \ldots,\, a_{i_b-(b-1),i_b},\, \sum_{i=i_b-(b-1)}^{a} a_{i,i+b}\} \end{aligned}$$

To the first set we associate a nonnegative integer; to the second set a polynomial in one variable:

$$\begin{aligned} e_A(i_1, \ldots, i_b) &= \sum_{i=1}^{i_b - b} a_{i,j_i} \\ \mathrm{hci}_A(i_1, \ldots, i_b) &= \mathrm{HP}^{(\mathrm{g}_A(i_1,\ldots,i_b))}(z) \end{aligned}$$

Proposition 6.3. *The h-polynomial of any degree matrix $A = (a_{i,j}) \in \mathbb{Z}^{t \times (t+c-1)}$ is given by*

$$\mathrm{HP}^A(z) = \sum_{0 < i_1 < \cdots < i_{c-1} < t+c-1} z^{e_A(i_1,\ldots,i_{c-1})} \cdot \mathrm{hci}_A(i_1, \ldots, i_{c-1}).$$

Proof. For $t = 1$ and any $c \geq 1$ we obtain only one summand, and the equality clearly holds. We will use induction on t and on c. The recursive formula of Lemma 6.1 gives us

$$\mathrm{HP}^A(z) = z^{a_{1,1}} \mathrm{HP}^{A^{(1,1)}}(z) + (1 + \cdots + z^{a_{1,1}-1}) \mathrm{HP}^{A^{(0,1)}}(z).$$

Let us denote the entries of the matrix $A^{(1,1)}$ by $(a'_{i,j})$ and the entries of $A^{(0,1)}$ by $(a''_{i,j})$. By definition $a'_{i,j} = a_{i+1,j+1}$ and $a''_{i,j} = a_{i,j+1}$. By the inductive hypothesis on t we have

$$\mathrm{HP}^{A^{(1,1)}}(z) = \sum_{0 < i_1 < \cdots < i_{c-1} < t+c-2} z^{e_{A^{(1,1)}}(i_1,\ldots,i_{c-1})} \cdot \mathrm{hci}_{A^{(1,1)}}(i_1, \ldots, i_{c-1}).$$

For any sequence $0 < i_1 < \cdots < i_{c-1} < t+c-2$ we have

$$e_{A^{(1,1)}}(i_1,\ldots,i_{c-1}) = \sum_{i=1}^{i_{c-1}-(c-1)} a'_{i,j_i} = \sum_{i=2}^{i_{c-1}+1-(c-1)} a_{i,j_i} = e_A(i_1+1,\ldots,i_{c-1}+1) - a_{1,1}.$$

It is easy to check that that this implies

$$\mathrm{HP}^{A^{(1,1)}}(z) = \sum_{1<i_1<\cdots<i_{c-1}<t+c-1} z^{e_A(i_1,\ldots,i_{c-1})-a_{1,1}} \cdot \mathrm{hci}_A(i_1,\ldots,i_{c-1}).$$

By the inductive hypothesis on c we obtain

$$\mathrm{HP}^{A^{(0,1)}}(z) = \sum_{0<i_1<\cdots<i_{c-2}<t+c-2} z^{e_{A^{(0,1)}}(i_1,\ldots,i_{c-2})} \cdot \mathrm{hci}_{A^{(0,1)}}(i_1,\ldots,i_{c-2}).$$

It is easy to check as above that $\mathrm{g}_{A^{(0,1)}}(i_1,\ldots,i_{c-2}) = \mathrm{g}_A(1,i_1+1,\ldots,i_{c-1}+1)\setminus\{a_{1,1}\}$, and that $e_{A^{(0,1)}}(i_1,\ldots,i_{c-2}) = e_A(1,i_1+1,\ldots,i_{c-2}+1)$. This implies that

$$\mathrm{HP}^{A^{(0,1)}}(z) = \sum_{1<i_2<\cdots<i_{c-1}<t+c-1} z^{e_A(1,i_2,\ldots,i_{c-1})} \cdot \frac{\mathrm{hci}_A(1,i_2,\ldots,i_{c-1})}{1+\cdots+z^{a_{1,1}-1}},$$

and by Lemma 6.1 we conclude.

□

We now focus on the degree and the leading coefficient of the h-polynomial.

Lemma 6.4. *Let $A = (a_{i,j}) \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and let $h^A = (h_0,\ldots,h_{\tau(h^A)})$. Then:*

(i) $\tau(h^A) = a_{1,1}+\cdots+a_{1,c}+a_{2,c+1}+\cdots+a_{t,t+c-1}-c.$

(ii) $h_{\tau(h^A)} = \binom{r+c-2}{c-1}$, *where $r = \max\{i \ : \ a_{1,1} = \cdots = a_{i,1}\}$.*

Proof. We will prove the claim by induction on t and c. For $t,c=1$, both statements are clear, so let $t,c>1$. Comparing the degrees of the h-polynomials in Lemma 6.1, with $(k,l) = (t,t+c-1)$ we obtain

$$\tau(h^A) = \max\{\tau(h^{A^{(t,t+c-1)}}) + a_{t,t+c-1}, \tau(h^{A^{(0,t+c-1)}}) + a_{t,t+c-1} - 1\} \tag{6.3}$$

and (i) follows by induction.

From (6.3) and (i) we deduce that, if $a_{t,t+c-2} < a_{t-1,t+c-1}$, then the leading coefficients of $\mathrm{HP}^A(z)$ and $\mathrm{HP}^{A^{(t,t+c-1)}}(z)$ are equal. Thus it is enough to prove the second statement for matrices with equal rows (i.e. with $r=t$). If we denote by $A' = A^{(t,t+c-1)}$ and $A'' = A^{(0,t+c-1)}$, and apply Lemma 6.1 for $(k,l) = (t,t+c-1)$ we obtain

$$h_{\tau(h^A)} = h^{A'}_{\tau(h^{A'})} + h^{A''}_{\tau(h^{A''})} = \binom{t+c-3}{c-1} + \binom{t+c-3}{c-2} = \binom{t+c-2}{c-1}$$

□

From now on, r will denote the *number of maximal equal rows* in a degree matrix. That is

$$r = \max\{i \ : \ a_{1,1} = \cdots = a_{i,1}\}$$

Remark 6.5. Let $A \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix, and let $h^A = (h_0,\dots,h_s)$. We denote by $h' = (h'_0,\dots,h'_{s'})$ the h-vector of $A^{(t,t+c-1)}$ and by $h'' = (h''_0,\dots,h''_{s''})$ the vector given by

$$h''_i = \sum_{k=0}^{a_{t,t+c-1}-1} h^{A^{(0,t+c-1)}}_{i-k},$$

where $h^{A^{(0,t+c-1)}}_{i-k} = 0$ if $i<k$. Lemma 6.1 states that h^A is computed by component-wise addition:

$$\begin{array}{cccccccccc}
0 & \dots & 0 & h'_0 & \dots & h'_{s'-a_{1,1}+a_{t,1}} & h'_{s'-a_{1,1}+a_{t,1}+1} & \dots & h'_{s'} & + \\
h''_0 & \dots & h''_{a_{t,t+c-1}-1} & h''_{a_{t,t+c-1}} & \dots & h''_{\tau(h'')} & 0 & \dots & 0 & \\
\hline
h^A_0 & \dots & h^A_{a_{t,t+c-1}-1} & h^A_{a_{t,t+c-1}} & \dots & h^A_{s-a_{1,1}+a_{t,1}} & h^A_{s-a_{1,1}+a_{t,1}+1} & \dots & h^A_s &
\end{array}$$

By Lemma 6.4 we have $s'-s'' = a_{1,1}-a_{t,1}$. In particular, as $a_{1,1} = a_{r,1} > a_{r+1,1} \geq \dots \geq a_{t,1}$, the last $a_{1,1}-a_{r+1,1}$ entries of h^A are equal to the last $a_{1,1}-a_{r+1,1}$ entries of $h^{\bar{A}}$, where $\bar{A}$ is the $r\times(r+c-1)$ upper-left block of A.

The following proposition describes the last part of the h-vector of a degree matrix with equal rows. By the above remark these values provide lower bounds for the last entries of the h-vector of any degree matrix. In what follows, we use the convention that $\binom{a}{b} = 0$, if $b<0$ or $a<b$.

Proposition 6.6. *Let $A \in \mathbb{Z}^{r\times(r+c-1)}$ be a degree matrix with equal rows. Denote by $s = \tau(h^A)$ and by $a_j = a_{l,j}$, $\forall\, l,j$. For any $i = 0,\dots,a_{r+1}-1$ we have:*

$$\begin{aligned}
h^A_{s-i} = &\binom{r+c-2}{c-1}\cdot\binom{c+i-1}{c-1} + \\
&+\sum_{\alpha=1}^{c-1}(-1)^{\alpha}\binom{r-\alpha+c-2}{c-1-\alpha}\sum_{1\leq j_1<\dots<j_\alpha\leq r}\binom{c+i-1-a_{j_1}-\dots-a_{j_\alpha}}{c-1}.
\end{aligned}$$

Proof. We will prove the claim by induction on r and c using the binomial formula

$$\sum_{i=0}^{a-1}\binom{d-i}{b} = \binom{d+1}{b+1} - \binom{d-a+1}{b+1}. \tag{6.4}$$

The case $c=1$ corresponds to a hypersurface, and the claim clearly holds. When $r=1$, denote by $h' = (h'_0,\dots,h'_{s'})$ the h-vector of a complete intersection of type $(a_2,\dots,a_c)$. For $i = 0,\dots,a_2-1$, using (6.2), induction on c, and (6.4) we obtain:

$$\begin{aligned}
h_{s-i} &= \sum_{k=0}^{a_1-1} h'_{s'-(i-k)} \\
&= \sum_{k=0}^{a_1-1}\binom{c-2+i-k}{c-2} + \sum_{k=0}^{a_1-1}\binom{c-2+i-k-a_2}{c-2} \\
&= \binom{c-1+i}{c-2} - \binom{c-1+i-a_1}{c-1}.
\end{aligned}$$

Let now $r,c>1$. We will write for shortness $h^{A^{(1,1)}} = (h'_0,\dots,h'_{s'})$ and $h^{A^{(0,1)}} = (h''_0,\dots,h''_{s''})$, By Lemma 6.4 we have $s = s'+a_1$ and $s = s''+(a_1-1)$. By Remark 6.2 we have in this notation that, for any $i = 0,\dots,a_{r+1}-1$,

$$h^A_{s-i} = h'_{s'-i} + \sum_{j=0}^{a_1-1} h''_{s''-(i-j)}. \tag{6.5}$$

For the following computation we use induction on c and r, the formula (6.4) and the correspondence between the indices in A, A' and A''. We also take into account that, if $j_\alpha = a_{r+1}$, then for any $i = 0, \dots, a_{r+1}-1$ we have $\binom{c+i-1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1} = 0$.

$$
\begin{aligned}
h'_{s'-i} &= \binom{r+c-3}{c-1}\binom{c-1+i}{c-1} + \\
&\quad + \sum_{\alpha=1}^{c-1}(-1)^\alpha \binom{r-\alpha+c-3}{c-1-\alpha} \sum_{2\le j_1<\cdots<j_\alpha\le r} \binom{c+i-1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1}. \\
\sum_{j=0}^{a_1-1} h''_{s''-(i-j)} &= \binom{r+c-3}{c-2}\binom{c-1+i}{c-1} - \binom{r+c-3}{c-2}\binom{c-1+i-a_1}{c-1} + \\
&\quad + \sum_{\alpha=1}^{c-1}(-1)^\alpha \binom{r-\alpha+c-3}{c-2-\alpha} \sum_{2\le j_1<\cdots<j_\alpha\le r} \binom{c+i-1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1} - \\
&\quad - \sum_{\alpha=1}^{c-2}(-1)^\alpha \binom{r-\alpha+c-3}{c-2-\alpha} \sum_{2\le j_1<\cdots<j_\alpha\le r} \binom{c+i-1-a_1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1}.
\end{aligned}
$$

Substituting these formulae in (6.5), grouping the summands, and applying (6.4) we obtain

$$
\begin{aligned}
h^A_{s-i} &= \binom{r+c-2}{c-1}\binom{c-1+i}{c-1} \\
&\quad + \sum_{\alpha=1}^{c-1}(-1)^\alpha \binom{r-\alpha+c-2}{c-1-\alpha} \sum_{2\le j_1<\cdots<j_\alpha\le r} \binom{c+i-1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1} + \\
&\quad + \sum_{\alpha=2}^{c-1}(-1)^\alpha \binom{r-\alpha+c-2}{c-1-\alpha} \sum_{2\le j_1<\cdots<j_{\alpha-1}\le r} \binom{c+i-1-a_1-a_{j_1}-\cdots-a_{j_{\alpha-1}}}{c-1} - \\
&\quad - \binom{r+c-3}{c-2}\binom{c-1+i-a_1}{c-1},
\end{aligned}
$$

and the claim follows by straight forward rewriting of this formula. □

6.3 Standard determinantal ideals and pure O-sequences

In this section we prove the main result of this paper (Theorem 6.8), which states that if all the rows in a degree matrix are equal, then its h-vector is a log-concave pure O-sequence. By Proposition 6.9 such matrices correspond exactly to level standard determinantal ideals. We conjecture the converse of the main theorem to hold (Conjecture 6.10). Among the support we bring for this statement are the validity for codimension two, for the last entry of the h-vector equal to one, and for matrices with all entries positive.

In codimension one, all h-vectors are finite sequences of 1s, thus pure O-sequences. We will assume throughout this section that the codimension c is greater than one.

An *O-sequence* is a finite vector of integers which is the Hilbert function of some standard graded Artinian algebra, that is it satisfies the numerical conditions in Macaulay's theorem [Mac27]. An O-sequence is called *pure* if it is the Hilbert function of a level, monomial Artinian algebra. A Cohen-Macaulay, standard graded quotient of the polynomial ring S is called *level* if the last S-module in its minimal free resolution is of the form $S(-s)^a$, where s and a are positive integers.

Pure O-sequences have also a purely combinatorial interpretation as follows. We will write $\mathrm{Mon}(\mathbf{y})$ for the collection of all monomials in $\mathbf{y} = (y_1, \dots, y_m)$. An *order ideal* of $\mathrm{Mon}(\mathbf{y})$ is a finite subset $\Gamma \subset \mathrm{Mon}(\mathbf{y})$ closed under division, i.e. if $M \in \Gamma$ and N divides M, then $N \in \Gamma$. An order ideal is called *pure* if all maximal monomials have the same degree. We write

$$\Gamma = \langle M \in \Gamma \ : \ M \text{ is maximal with respect to division}\rangle.$$

The *f-vector* of an order ideal Γ is $f(\Gamma) = (f_0, \dots, f_s)$, where $f_i(\Gamma) = |\{M \in \Gamma : \deg(M) = i\}|$. It is not difficult to check that a vector $h = (h_0, \dots, h_s)$ is a pure O-sequence if and only if it is the f-vector of some pure order ideal. The vector h is called *log-concave* if

$$h_i^2 \geq h_{i-1} \cdot h_{i+1}, \ \forall\, i = 1, \dots, s-1.$$

We use matroids to obtain a connection between h-vectors of standard determinantal ideal and pure O-sequences. A *simplicial complex* on the vertex set $[n] = \{1, \dots, n\}$ is a collection of subsets $\Delta \subseteq 2^{[n]}$ closed under taking subsets, i.e. if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. A *matroid* is a simplicial complex with the extra property that if $F, G \in \Delta$, with $|G| < |F|$, then there exists $i \in F$ such that $G \cup \{i\} \in \Delta$. For $v \in \Delta$, the *link of v in* Δ, respectively the *deletion of v in* Δ are the simplicial complexes

$$\begin{array}{rcl} \mathrm{link}_\Delta(v) & = & \{F \in \Delta \ : \ v \notin F \,\mathrm{and}\, F \cup \{v\} \in \Delta\}, \\ \Delta \setminus v & = & \{F \in \Delta \ : \ v \notin F\}. \end{array}$$

When Δ is a matroid, then both $\mathrm{link}_\Delta(v)$ and $\Delta \setminus v$ are matroids as well. The maximal faces under inclusion are called *facets*; they determine the simplicial complex. We denote the set of facets of Δ by $\mathscr{F}(\Delta)$. A vertex $v \in \Delta$ with $v \in F$ for any $F \in \mathscr{F}(\Delta)$ is called a *cone point* of Δ.

For any simplicial complex Δ, the *cover ideal* is the square-free monomial ideal of the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$ defined as

$$J(\Delta) = \bigcap_{F \in \Delta} (x_i \ : \ i \in F).$$

We will denote by h^Δ the h-vector of $S/J(\Delta)$. According to [CV15, Remark 1.7], if Δ is a matroid and $v \in \Delta$ not a cone point, then

$$h_i^\Delta = h_{i-1}^{\Delta \setminus v} + h_i^{\mathrm{link}_\Delta(v)}. \tag{6.6}$$

Remark 6.7. For any simplicial complex Δ, the *dual* (or complement) of Δ is the simplicial complex Δ^{c} with

$$\mathscr{F}(\Delta^{\mathrm{c}}) = \{[n] \setminus F \ : \ F \in \mathscr{F}(\Delta)\}.$$

A classical matroid theory result states that Δ is a matroid if and only if Δ^{c} is a matroid [Oxl11].

In common matroid terminology, the vector h^Δ we defined above is the "classical" h-vector of the dual matroid. This choice was made in order to keep a coherent notation with the main result of [CV15], which we use to prove the following theorem.

Theorem 6.8. *Let $X \subseteq \mathbb{P}^n$ be a codimension c standard determinantal scheme. If the corresponding degree matrix A has equal rows, then h^A is a log-concave pure O-sequence.*

In particular, if $A \in \mathbb{Z}^{t \times (t+c-1)}$, with rows $(a_1, \dots, a_{t+c-1})$, then $h^A = f(\Gamma)$ where Γ is the order ideal of $\mathrm{Mon}(y_1, \dots, y_c)$ *given by*

$$\Gamma = \left\langle y_1^{(\sum_{i=1}^{l_1-1} a_i)-1} \cdot y_2^{(\sum_{i=l_1}^{l_2-1} a_i)-1} \cdots y_c^{(\sum_{i=l_{c-1}}^{t+c-1} a_i)-1} \ : \ \forall 1 = l_0 < l_1 < \cdots < l_{c-1} \leq t+c-1 \right\rangle.$$

Proof. We will write $m = t+c-1$ for short. For $i = 1,\dots,m$, let A_i be a set of vertices of cardinality a_i. As in [CV15], we define the simplicial complex $\Delta_0(c,m,(a_1,\dots,a_m))$ on $\sqcup_{i=1}^m A_i$ as

$$\langle\{v_{i_1},\dots,v_{i_c}\} \ : \ 1 \le i_1 < \cdots < i_c \le m \text{ and } v_{i_j} \in A_{i_j} \text{ for every } i_j\rangle.$$

One can easily check that $\Delta_0(c,m,(a_1,\dots,a_m))$ is a matroid. We will show by induction on c and t that the h-vectors h^A and $h^{\Delta_0(c,m,(a_1,\dots,a_m))}$ coincide. For $t=1$ or $c=1$ the claim is straight forward. Let $t,c > 1$. By Lemma 6.1 applied for a_m we have

$$h = h^{A^{(t,m)}}_{i-a_m} + \sum_{k=0}^{a_m-1} h^{A^{(0,m)}}_{i-k}.$$

On the other hand, applying a_m times the formula (6.6), once for every vertex in A_m, we obtain

$$h_i^{\Delta_0(c,m,(a_1,\dots,a_m))} = h_{i-a_m}^{\Delta_0(c,m-1,(a_1,\dots,a_{m-1}))} + \sum_{k=0}^{a_m-1} h_{i-k}^{\Delta_0(c-1,m-1,(a_1,\dots,a_{m-1}))},$$

and we conclude by induction. In particular, by [CV15, Theorem 3.5], h^A is the pure O-sequence given by Γ as claimed.

Furthermore $\Delta_0(c,m,(a_1,\dots,a_m))$ is representable over any infinite field $\mathbb{F}$ of characteristic zero. A presentation matrix D can be constructed as follows: choose m generic vectors $w_1,\dots,w_m \in \mathbb{F}^c$, that is any c of them are linearly independent. Let the first a_1 columns of D be w_1, the next a_2 be equal to w_2 and so on. Clearly D represents the matroid $\Delta_0(c,m,(a_1,\dots,a_m))$. A matroid is representable over $\mathbb{F}$ if and only if its dual is representable over $\mathbb{F}$ (see [Oxl11, Corollary 2.2.9]). So, by Remark 6.7, we may use J. Huh's result on h-vectors of matroids which are representable over fields of characteristic zero ([Huh12, Theorem 3]) and conclude that h^A is log-concave. □

The result which we used to conclude ([Huh12, Theorem 3]) has been in the meantime generalized by Huh and E. Katz in [HK12]. However, we find the weaker version which we cite in the proof better adapted to our setting.

The next result shows that, not only is the h-vector of a degree matrix with equal rows the Hilbert function of some level algebra, but that the standard determinantal schemes having such a degree matrix are exactly the level ones.

Proposition 6.9. *Let $X \subseteq \mathbb{P}^n$ be a standard determinantal scheme of codimension c, with degree matrix $A = (a_{i,j}) \in \mathbb{Z}^{t\times(t+c-1)}$. Then X is level if and only if A has equal rows.*

Proof. Let $M = (f_{i,j})$ be the homogeneous matrix whose maximal minors generate the defining ideal I_X of X. Let $a_j - b_i = a_{i,j} = \deg f_{i,j}$, so M defines a graded homomorphism of degree zero

$$\varphi : F = \bigoplus_{i=1}^{t} S(b_i) \longrightarrow \bigoplus_{j=1}^{t+c-1} S(a_j) = G.$$

The minimal free resolution of S/I_X is given by the Eagon-Northcott complex with respect to φ (see [BV88; Mir08]). Therefore, the last free module in it is of the form

$$F_c = \bigwedge^{t+c-1} G^* \otimes S_{c-1}(F) \otimes \bigwedge^{t} F,$$

where

$$\begin{aligned} G^* &= \bigoplus_{j=1}^{t+c-1} S(-a_j) & \textstyle\bigwedge^{t+c-1} G^* &= S\left(-\textstyle\sum_{j=1}^{t+c-1} a_j\right) \\ S_{c-1}(F) &= \bigoplus_{1\le k_1\le\cdots\le k_{c-1}\le t} S\left(\textstyle\sum_{j=1}^{c-1} b_{k_j}\right) & \textstyle\bigwedge^t F &= S\left(\textstyle\sum_{i=1}^{t} b_i\right). \end{aligned}$$

We can rewrite the shifts in F_c in terms of the entries of A as follows

$$\begin{aligned} F_c &= \bigoplus_{1\le k_1\le\cdots\le k_{c-1}\le t} S(-a_1-\cdots-a_{t+c-1}+b_1+\cdots+b_t+b_{k_1}+\cdots+b_{k_{c-1}}) \\ &= \bigoplus_{1\le k_1\le\cdots\le k_{c-1}\le t} S(-a_{k_1,1}-\cdots-a_{k_{c-1},c-1}-a_{1,c}-\cdots-a_{t,t+c-1}). \end{aligned}$$

The scheme X is level if and only if $F_c = S^b(-d)$. In particular, the shifts corresponding to the summation indices $(1,t,\ldots,t),\ldots,(t,t,\ldots,t)$ are all equal, that is

$$a_{1,1}+a_{t,2}+\cdots+a_{t,c-1} = \cdots = a_{t,1}+a_{t,2}+\cdots+a_{t,c-1}.$$

This implies that $a_{1,1}=\cdots=a_{t,1}$, which is equivalent to the rows of A being equal. □

We believe that, just as in Proposition 6.9, an equivalence holds also in Theorem 6.8.

Conjecture 6.10. *If A is a degree matrix without zeros, then h^A is a pure O-sequence if and only if A has equal rows.*

The last part of this section is dedicated to bringing evidence in support of this statement. We first prove that that Conjecture 6.10 holds in codimension two.

Proposition 6.11. *If $X \subseteq \mathbb{P}^n$ be a codimension 2 standard determinantal scheme, whose degree matrix $A \in \mathbb{Z}^{t\times t+1}$ has no zeros, then h^A is a pure O-sequence if and only if A has equal rows.*

Proof. Assume that h^A is a pure O-sequence and let B be the Artinian reduction of S/I_X. Then, there exists an Artinian monomial level algebra R/J, where $R=K[x_1,x_2]$, such that $h^A = HF_B = HF_{R/J}$. Since A has no zeros, and we are in codimension two, by the Hilbert-Burch theorem (see for instance [Eis95, Theorem 20.15]), the Hilbert function of B determines uniquely its minimal free resolution, and also the one of S/I_X. Thus R/J being level implies that also B is level. The claim follows now from Proposition 6.9. □

Remark 6.12. If $h=(1,c,h_2,\ldots,h_s)$ is a pure O-sequence, then by counting monomials and divisors of monomials in each degree, one easily obtains that

$$h_{s-i} \le \min\left\{ \binom{c-1+s-i}{c-1},\ h_s\cdot\binom{c-1+i}{c-1} \right\}, \quad \forall\, i=0,\ldots,s.$$

The next result shows that our conjecture holds when the second-largest entry in the first column of the degree matrix is positive. In particular, it holds for matrices with all entries positive.

Proposition 6.13. *Let $X \subseteq \mathbb{P}^n$ be codimension c standard determinantal scheme, whose degree matrix $A=(a_{i,j}) \in \mathbb{Z}^{t\times(t+c-1)}$ has r equal maximal rows, with $r<t$, and no zeros. If $a_{r+1,1}>0$, then h^A is not a pure O-sequence.*

Proof. As $a_{r+1,1}>0$, by Remark 6.5 and by Lemma 6.4 we have

$$h^A_{s-a_{1,1}+a_{r+1,1}} \ge \binom{r+c-2}{c-1}\cdot\binom{c-1+i}{c-1}+\binom{r+c-3}{c-2}.$$

By Lemma 6.4 the last entry of h^A is $h_s=\binom{r+c-2}{c-1}$. At the beginning of this section we assumed that $c\ge 2$, so Remark 6.12 implies that h^A is not a pure O-sequence. □

Hibi proved in [Hib89] that all pure O-sequences are flawless, i.e. $h_i \le h_{s-i}$ for $i=0,\ldots,\lfloor s/2\rfloor$.

Proposition 6.14. *Let $A = (a_{i,j}) \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and $h^A = (h_0,\dots,h_s)$ be the corresponding h-vector. If $a_{2,1} < 0$, then there exists an integer i_0 such that $h_{i_0} > h_{s-i_0}$. In particular, h^A is not a pure O-sequence.*

Proof. According to Remark 6.2, $h^A_{s-i} = h^{(a_{1,1},\dots,a_{1,c})}_{s-i}$, for $i = 0,\dots,(a_{1,1}-a_{2,1}-1)$. By Proposition 6.6, for all $i = 0,\dots,a_{1,2}-1$ we have

$$h^{(a_{1,1},\dots,a_{1,c})}_{s-i} = \binom{c-1+i}{c-1} - \binom{c-1+i-a_{1,1}}{c-1}.$$

In particular, as $a_{1,2}-1 = a_{1,1}+a_{2,2}-a_{2,1}-1 > a_{1,1}-a_{2,1}-1 \geq a_{1,1}$, we obtain

$$h^{(a_{1,1},\dots,a_{1,c})}_{s-i} < \binom{c-1+i}{c-1}, \qquad \text{for every } i = a_{1,1},\dots,(a_{1,1}-a_{2,1}-1).$$

Thus, as $h^A_i = \binom{c-1+i}{c-1}$ for all $i = 0,\dots,\sum_{j=0}^{t} a_{j,j}-1$, every index $i_0 \in \{a_{1,1},\dots,a_{1,1}-a_{2,1}-1\}$ satisfies $h_{i_0} > h_{s-i_0}$. □

Propositions 6.13 and 6.14 have the following direct consequence.

Corollary 6.15. *Conjecture 6.10 holds for any degree matrix with only one maximal row.*

The following examples show that Proposition 6.14 has no easy generalization to matrices with two or more maximal rows.

Example 6.16. The matrices A, B and their upper left 3×4 submatrices $A^{(4,5)}, B^{(4,5)}$ show that the conditions $a_{r+1,r} < 0$ and $a_{t,t-1} < 0$ do not influence flawlessness. Clearly h^A and $h^{A^{(4,5)}}$ are flawless, while h^B and $h^{B^{(4,5)}}$ are not. A quick exhaustive computer search shows that none of the four is a pure O-sequence.

$$A = \begin{pmatrix} 2 & 2 & 5 & 5 & 5 \\ 2 & 2 & 5 & 5 & 5 \\ -2 & -2 & 1 & 1 & 1 \\ -2 & -2 & 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 5 & 5 & 5 \\ 1 & 2 & 5 & 5 & 5 \\ -3 & -2 & 1 & 1 & 1 \\ -3 & -2 & 1 & 1 & 1 \end{pmatrix}$$

$$h^A = (1,2,3,4,5,6,4,4,4,2) \qquad h^B = (1,2,3,4,5,3,3,3,2)$$

$$h^{A^{(4,5)}} = (1,2,3,4,5,4,4,4,2) \qquad h^{B^{(4,5)}} = (1,2,3,4,3,3,3,2)$$

The matrices C and D below show that for one maximal row and all entries positive both situations may appear, namely h^C does not satisfy Hibi's inequalities, while h^D does. By Proposition 6.13 none of them is a pure O-sequence.

$$C = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$h^C = (1,3,6,10,9,7,3,1) \qquad h^D = (1,3,6,4,1)$$

6.4 Computational aspects

The computer experiments we performed while dealing with Conjecture 6.10 are presented here. The most important consequence of these computations is that an "exhaustive search approach" to finding counterexamples has small chances of succeeding. This is essentially due to the fact that the number of socles to check when computing pure O-sequences grows doubly exponentially in the number of variables and the length, and exponentially in the type. We checked all "reasonable" cases, i.e. those where enumerating all vectors of fixed length, type and codimension takes no longer than a couple of weeks on a standard server, and found no counterexamples. While performing these computations, we built a data base of both pure and non-pure O-sequences, which is available online at `https://github.com/alexconstantinescu/PureOSequences` and licensed under the GPL. The computer code we used can also be found at the above address. Our search for a counterexample to Conjecture 6.10 runs as follows.

`Step 1.` Fix three positive integers c, τ and s.

`Step 2.` Compute the list $H_{c,\tau,s}$ of candidate h-vectors of length s and type τ, arising from codimension c degree matrices.

`Step 3.` Compute the list $P_{c,\tau,s}$ of all pure O-sequences with the same parameters.

`Step 4.` Check that the intersection of the two lists is void.

`Step 1.` By Proposition 6.11 we need to choose $c \geq 3$. By Lemma 6.4, τ has to be of the form $\binom{c+r-2}{c-1}$, where r counts the number of maximal rows. By Corollary 6.15 we must choose $r \geq 2$. From Lemma 6.4, together with the fact that the degree matrix should not contain zeros, and that the first entry of the $(r+1)$-th row has to be negative (otherwise, the conjecture holds by Proposition 6.13), we obtain that $s \geq r+2c-2$.

`Step 2.` To determine $H_{c,\tau,s}$, we first construct all degree matrices for which the conjecture is not known, and then compute the corresponding h-vector. To compute all such matrices $(a_{i,j})_{0<i<t+1,\ 0<j<t+c}$, we run a loop over all possible $a_{1,1}+\cdots+a_{1,c}+a_{2,c+1}+\cdots+a_{t,t+c-1}$ which sum up to $c+s$. Notice that t, the total number of rows, varies between $r+1$ and $s-2c+3$. The first $c+r-1$ summands are the first entries of the maximal row, thus are in increasing order. The remaining ones are not necessarily in increasing order. Thus, for each $i = r+3c-1, \ldots, s+c-1$ we have to compute the sets:

$$\begin{aligned} P_{c+r-1}(i) &= \{\text{partitions of } i \text{, of length } c+r-1\}, \\ \sigma P(s+c-i) &= \{\text{all permutations of all the partitions of } s+c-i\}. \end{aligned}$$

The loop runs over $D_i = P_{c+r-1}(i) \times \sigma P(s+c-i)$. An element of D_i does not uniquely determine a degree matrix. For each element of D_i, we also have to run a loop over all admissible vectors with positive entries $\delta = (\delta_1, \ldots, \delta_{t-r})$, where δ_j will be $a_{r+j,1} - a_{1,1}$ in the degree matrix. Using homogeneity, positivity on the diagonal, and the increasing order of the entries in each row, we obtain upper bounds for the δ_js, so the number of admissible matrices is actually finite. For each $\mathbf{a} \in D_i$ and each δ, the first r rows of the degree matrix will be

$$(a_{1,1}, \ldots, a_{1,c}, \ldots, a_{r,r+c-1}, a_{r+1,r+c} + \delta_1, \ldots, a_{t,t+c-1} + \delta_{t-r}).$$

After checking that the above integers are in weakly increasing order, we compute the other rows of the matrix by subtracting the appropriate δ_j from each entry. We then check that each new vector contains no zeros, and that its last r$+j$ entries are positive. Finally, the computation of the h-vector is done in a straightforward fashion, using Proposition 6.3.

In our implementation, the sets of partitions and their permutations are precomputed using `Polymake` [GJ00].

`Step 3.` This step is generally the most expensive one. It runs over all $\binom{\alpha(s,c)}{\tau}$ sets of τ monomials in c variables of degree s, where $\alpha(s,c) = \binom{s+c-1}{s}$. For this computation, we use the `C++`-library

$s =$	6	7	8	9	10	11	12	13	14	15
$\lvert H_{3,3,s}\rvert$	1	2	6	12	24	43	76	126	208	340
$\lvert P_{3,3,s}\rvert$	51	102	202	375	676	1138	1938	3054	4835	7248
$\lvert H_{3,6,s}\rvert$	–	1	2	6	12	25	46	82	$*$	$*$
$\lvert P_{3,6,s}\rvert$	$*$	233	662	1855	5050	13125	33555	83798	$*$	$*$
$\lvert H_{4,4,s}\rvert$	–	–	1	2	5	10	19	$*$	$*$	$*$
$\lvert P_{4,4,s}\rvert$	$*$	$*$	5506	18045	61071	178336	549410	$*$	$*$	$*$

$s =$	16	17	18	19	20	21	22
$\lvert H_{3,3,s}\rvert$	552	903	1473	2418	3955	6508	10658
$\lvert P_{3,3,s}\rvert$	10874	15608	22427	30975	42911	57617	77323

Figure 6.1: Cases in which Conjecture 6.10 holds.

developed by the first author, together with Kahle and Varbaro for [CKV14], which is available at `https://github.com/tom111/GraphBinomials` and is licensed under the GPL.

`Step 4.` This step is fast compared to the previous two. We used the `Intersection` command in CoCoA.

In Figure 6.1 we show the cardinalities of the sets $H_{c,\tau,s}$ and $P_{c,\tau,s}$ for which we checked Conjecture 6.10. With a computing time estimated between one and six months, we expect to add to the online database the cases $(c,\tau,s) = (3,6,14),(4,4,13)$. The codimension 3, type 3 cases are those for which computing $H_{3,3,s}$ is more expensive than computing $P_{3,3,s}$ (for $s \leq 30$ at least). With the algorithm presented in `Step 2.`, we estimate the computing time for $H_{3,3,23}$ at roughly 2.93 CPU years. The estimation is based on the computation time for smaller values.

For all other cases, the more expensive part is computing $P_{c,\tau,s}$. In Figure 6.2 we roughly estimate the time it would take to compute these sets in the next interesting cases. The number of socles processed per second is based on the previous smaller cases.

Set to compute	socles to check	socles/second	estimated CPU time
$P_{3,6,15}$	$7.85 \cdot 10^9$	350	259 CPU days
$P_{4,4,14}$	$8.83 \cdot 10^9$	200	1.4 CPU years
$P_{5,5,10}$	$8.29 \cdot 10^{12}$	3200	821 CPU years

Figure 6.2: First large cases in codimensions 3,4 and 5.

Remark 6.17. While computing $H_{c,\tau,s}$, we noticed that different degree matrices, not containing zeros, produce different h-vectors. We have systematically checked this for all admissible triples (c,τ,s) with $s \leq 20$, and for $H_{3,3,21}$. We would like to know if this is true in general.

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