The Artin Defect in Algebraic K-Theory

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Contents

1 Introduction

Computing the algebraic K-theory of group rings is of great importance in geometric topology. Given a topological space X many problems can be reduced to statements about the K-theory of the integral group ring $\mathbb{Z}\pi_1(X)$ where $\pi_1(X)$ denotes the fundamental group of X. For example, the s-cobordism theorem states that an h-cobordism over a manifold M is trivial, if and only if its Whitehead torsion vanishes. The Whitehead torsion is an element in the Whitehead group of M, which is defined as a quotient of the K-group $K_1(\mathbb{Z}\pi_1(M))$.

In general, for a group G and a ring R with unit one would like to be able to compute the K-groups $K_m(RG)$ with $m \in \mathbb{Z}$ using information about the group G and the K-theory of the ring R separately. An approach to the computation of the K-theory of group rings in this way is given by the Farrell-Jones conjecture. It predicts that the so called assembly map

$$H_*^G(E_{\mathcal{VCYC}}G, \mathbb{K}_R) \to H_*^G(pt, \mathbb{K}_R) = K_*(RG)$$

is an isomorphism where $H_*^G(-, \mathbb{K}_R)$ denotes the equivariant homology theory satisfying $H_m^G(G/H; \mathbb{K}_R) \cong H_m^H(pt; \mathbb{K}_R) \cong K_m(RH)$ for a subgroup H of G. The space $E_{\mathcal{VCYC}}G$ is a model for the classifying G-CW-complex for the family $\mathcal{VCYC}$ of virtually cyclic subgroups of G. A family of subgroups of G is a set of subgroups of G which is closed under taking subgroups and conjugation. The G-CW-complex $E_{\mathcal{VCYC}}G$ has the property that all isotropy groups belong to $\mathcal{VCYC}$. The assembly map stated in the conjecture is induced by the projection $E_{\mathcal{VCYC}}G \to pt$. For a comprehensive report on the Farrell-Jones conjecture the reader is referred to [LR05].

Although the left-hand side of the above assembly map is more accessible, it is not easy to compute either. By making the following assumptions we can simplify it further. We take the ring R to be the integers $\mathbb{Z}$ and tensor both sides with $\mathbb{Q}$. In this case we have an additional tool at hand, the equivariant chern character introduced by Lück in [Lüc02]. For a finite cyclic subgroup C of G denote by Z_GC and N_GC the centralizer and normalizer of C in G, respectively. The quotient $W_GC = N_GC/Z_GC$ is called the Weyl group of C in G. Further, let $\mathcal{FCY}$ be the set of conjugacy classes of finite cyclic subgroups of G. The equivariant chern character is an isomorphism

$$H_q^G(E_{\mathcal{VCYC}}G, \mathbb{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \bigoplus_{k+m=q} \bigoplus_{(C)\in\mathcal{FCY}} H_k(BZ_GC; \mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} S_m(C)$$

where

$$S_m(C) = \operatorname{coker}\left(\bigoplus_{D\lneq C} K_m(\mathbb{Z}D)\otimes_{\mathbb{Z}}\mathbb{Q} \to K_m(\mathbb{Z}C)\otimes_{\mathbb{Z}}\mathbb{Q}\right)$$

is called the Artin defect of C in degree $m \in \mathbb{Z}$. Since a finite cyclic group is unique up to isomorphism, we will write $S_m(n)$ for the Artin defect where n is the order of C. The inclusion of a subgroup D of C defines an inclusion of rings $\mathbb{Z}D \to \mathbb{Z}C$ and the above maps $K_m(\mathbb{Z}D)\otimes_{\mathbb{Z}}\mathbb{Q} \to K_m(\mathbb{Z}C)\otimes_{\mathbb{Z}}\mathbb{Q}$ are the ones induced from these inclusions. The Weyl group acts on the Artin defect in the following way. Denote by $\operatorname{Aut}(C)$ the automorphism group of C. There is an injective group homomorphism

$$W_GC \to \operatorname{Aut}(C), \quad [g] \mapsto (c \mapsto gcg^{-1})$$

and so we can consider the Weyl group as a subgroup of $\operatorname{Aut}(C)$. Since the K-groups are functors, the action of $\operatorname{Aut}(C)$ on C induces an action on $K_m(\mathbb{Z}C)\otimes_{\mathbb{Z}}\mathbb{Q}$. This action commutes with the maps induced from inclusion of subgroups of C and hence gives a well-defined action on the Artin defect.

The purpose of this thesis is to compute the dimension of the Artin defect as a $\mathbb{Q}$-vector space for all finite cyclic groups in all degrees and to describe the action of the automorphism group on it. We will use different methods to achieve this in the different degrees of K-theory. In negative K-theory degrees our main tool will be the localization sequence for lower K-theory. In degree 1 we have a more explicit description of the Artin defect using bases of the rationalized units of the ring of integers in cyclotomic extension of $\mathbb{Q}$. In higher degrees we apply Borel's results on the computation of the ranks of the K-groups of the ring of integers in a an algebraic number field to our situation. Denote by φ Euler's phi function, which is defined as $\varphi(n) = |\{1 \leq k \leq n \mid \gcd(k,n) = 1\}|$ for $n \in \mathbb{N}$ and by μ_n the set of n-th roots of unity in $\mathbb{C}$. The main result can be stated as follows.

Theorem. *Let C be a finite cyclic group of order $n > 1$. For $m \in \mathbb{Z}$ denote by $S_m(n)$ the cokernel of the map*

$$\bigoplus_{D \lneq C} K_m(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by the inclusion of proper subgroups D of C. For a prime p denote by G_p the Galois group of the cyclotomic extension $\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p$ of the p-adic numbers and by I the subgroup of $\operatorname{Aut}(C)$ generated by the automorphism $\phi(c) = c^{-1}$. As an element in $K_0(\mathbb{Q}\operatorname{Aut}(C))$ we have

$$[S_m(n)] = \begin{cases} [\bigoplus_{p|n} \mathbb{Q}[\operatorname{Aut}(C)/G_p]] - [\mathbb{Q}] & \text{if } m = -1 \\ [\mathbb{Q}[\operatorname{Aut}(C)/I]] - [\mathbb{Q}] & \text{if } m = 1 \\ [\mathbb{Q}[\operatorname{Aut}(C)/I]] & \text{if } m > 1, m \equiv 1 \bmod 4 \\ [\mathbb{Q}[\operatorname{Aut}(C)]] - [\mathbb{Q}[\operatorname{Aut}(C)/I]] & \text{if } m > 1, m \equiv 3 \bmod 4 \\ [0] & \text{else} \end{cases}$$

and in particular,

$$\dim_{\mathbb{Q}}(S_m(n)) = \begin{cases} \varphi(n)/2 & \text{if } m > 1, m \equiv 1 \bmod 2 \text{ and } n > 2 \\ \varphi(n)/2 - 1 & \text{if } m = 1 \text{ and } n > 2 \\ s(n) - 1 & \text{if } m = -1 \\ 1 & \text{if } m > 1, m \equiv 1 \bmod 4 \text{ and } n = 2 \\ 0 & \text{else} \end{cases}$$

with

$$s(n) = \sum_{i=1}^{s} \varphi(n/p_i^{e_i})/f_{p_i}$$

where f_{p_i} is the smallest number such that $p_i^{f_{p_i}} \equiv 1 \bmod n/p^{e_i}$ and $n = \prod_{i=1}^{s} p_i^{e_i}$ is the prime factorization of n.

In some of the cases we give an even more explicit description of the defect as a $\mathbb{Q}\operatorname{Aut}(C)$-module. In the course of proving the main result we also compute an analogous defect for any field of characteristic 0 in K-theory degree 0.

Theorem. *Let C be a finite cyclic group of order $n \in \mathbb{N}$ and F a field of characteristic* 0*. Denote by G_F the Galois group of the cyclotomic extension $F(\mu_n)/F$. There is an isomorphism of $\mathbb{Q}\operatorname{Aut}(C)$-modules*

$$\operatorname{coker}\left(\bigoplus_{D \lneq C} K_0(FD) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FC) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \cong \mathbb{Q}[\operatorname{Aut}(C)/G_F]$$

where G_F can be identified in a canonical way as a subgroup of $\operatorname{Aut}(C)$*.*

Additionally, we also get a description of higher K-theory of the integral group ring of a finite cyclic group C tensored with $\mathbb{R}$ as a functor from the subgroup category of C. The subgroup category has as objects the subgroups of C and as morphisms the inclusion of subgroups.

Theorem. *Let C be a finite cyclic group and $m \in \mathbb{N}$ with $m > 1$. There is an isomorphism*

$$K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R} \cong \begin{cases} \mathbb{R}[C]^+ & \text{if } m \equiv 1 \bmod 4 \\ \mathbb{R}[C]^- & \text{if } m \equiv 3 \bmod 4 \\ 0 & \text{else,} \end{cases}$$

which defines a natural transformation of functors from the subgroup category of C with

$$\mathbb{R}[C]^+ = \{x \in \mathbb{R}[C] \mid \phi(x) = x\},$$
$$\mathbb{R}[C]^- = \{x \in \mathbb{R}[C] \mid \phi(x) = -x\}$$

and ϕ is the automorphism of $\mathbb{R}[C]$ induced by $\phi(c) = c^{-1}$ for $c \in C$.

Organization

The thesis is divided into the different degrees of K-theory for which the computation of the Artin defect can be treated simultaneously. These are put into four chapters corresponding to the degrees lower than 0, degree 0, degree 1 and degrees greater than 1. In each chapter we start with setting up the necessary preliminaries to describe the corresponding rational K-groups of the integral group ring of a finite cyclic group with emphasis on naturality as functors from the subgroup category. This provides the basis for the computation of the Artin defect. At the end of each chapter we determine the dimension of the Artin defect and its structure as a module over the automorphism group. We assume that the reader is familiar with the definition and basic properties of algebraic K-theory as treated for example in [Ros94].

2 The Artin Defect in Degrees $m < 0$

In this chapter we treat the Artin defect in negative K-theory degrees. In the first section we explain the localization sequence for lower K-theory. This sequence is our main tool to approach the relevant negative K-groups. We will conclude that the only case where the Artin defect is not trivial is in degree $m = -1$. In this case, the defect can be expressed in terms of K_0 of group rings over the rational and the p-adic numbers. Therefore, in the second section we examine K_0 of group rings of finite cyclic groups over fields of characteristic 0 with emphasis on functoriality with respect to inclusion of finite cyclic groups and field extensions. In the last section of this chapter we apply these results to compute the Artin defect as a $\mathbb{Q}$-vector space and describe the corresponding action of the automorphism group.

2.1 The Localization Sequence in Lower K-Theory

A computational approach to negative K-theory of group rings has been introduced by Carter in [Car80] where he proves a localization sequence for lower K-Theory. In this section we give an outline of his construction. The starting point is the localization sequence in degrees 1 and 0 introduced by Bass in [Bas68, Chapter IX, Theorem 6.3]. For a commutative ring R, an R-algebra Λ and a subset $S \subseteq R$ such that multiplication by an element $s \in S$ is injective Bass constructed an exact sequence

$$K_1(\Lambda) \longrightarrow K_1(S^{-1}\Lambda) \xrightarrow{\delta} K_0(H_S(\Lambda)) \longrightarrow K_0(\Lambda) \longrightarrow K_0(S^{-1}\Lambda)$$

where $H_S(\Lambda)$ denotes the category of S-torsion left Λ-modules which admit a finite resolution by finitely generated projective left Λ-modules. For $i \in \{0, 1\}$ the map $K_i(\Lambda) \to K_i(S^{-1}\Lambda)$ is the one induced from the inclusion $\Lambda \subseteq S^{-1}\Lambda$. Hence, the main work goes into constructing the map δ. Carter uses this sequence to prove the exactness of the analog sequences

$$K_{m+1}(\Lambda) \longrightarrow K_{m+1}(S^{-1}\Lambda) \longrightarrow K_m(H_S(\Lambda)) \longrightarrow K_m(\Lambda) \longrightarrow K_m(S^{-1}\Lambda)$$

for all $m \le -1$. He proceeds with considering noetherian algebras over Dedekind rings. For these one can get rid of the groups $K_m(H_S(\Lambda))$ with the help of P-adic completions in the following way. Let R be a Dedekind ring, S the set of nonzero elements of R and K its field of fractions. Further, let Λ be a noetherian R-algebra that is R-torsion free. For a prime ideal P of R we set $\Lambda_P = R_P \otimes_R \Lambda$ where R_P denotes the P-adic completion of R at P. Carter shows that the induced map

$$H_S(\Lambda) \to \coprod_P H_S(\Lambda_P)$$

is an equivalence of categories. Splitting the long localization sequence into short exact sequences and mapping them into the corresponding local versions yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K}_{m+1}(\Lambda) & \longrightarrow & K_m(H_S(\Lambda)) & \longrightarrow & \widetilde{K}_m(\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_P \overline{K}_{m+1}(\Lambda_P) & \longrightarrow & \bigoplus_P K_m(H_S(\Lambda_P)) & \longrightarrow & \bigoplus_P \widetilde{K}_m(\Lambda_P) & \longrightarrow & 0
\end{array}$$

where $\widetilde{K}_m(\Lambda) = \ker(K_m(\Lambda) \to K_m(A))$, $\overline{K}_m(\Lambda) = K_m(A)/\mathrm{Im}(K_m(\Lambda))$ and $A = S^{-1}\Lambda \cong K \otimes_R \Lambda$. The vertical map in the middle is an isomorphism and so applying the snake lemma gives an exact sequence

$$0 \longrightarrow \overline{K}_{m+1}(\Lambda) \longrightarrow \bigoplus_P \overline{K}_{m+1}(\Lambda_P) \longrightarrow \widetilde{K}_m(\Lambda) \longrightarrow \bigoplus_P \widetilde{K}_m(\Lambda_P) \longrightarrow 0\,.$$

Last, Carter takes $\Lambda = RG$ to be the group ring of a finite group G over a Dedekind ring R, for which further simplifications are possible. This is also the situation we are interested in. Again, a result by Bass (see [Bas68, Chapter XII, Prop. 10.1]) shows that the negative K-groups of $A = KG$ and $A_P = K_PG$ vanish. Therefore, the above exact sequences simplify to

$$0 \longrightarrow \overline{K}_0(RG) \longrightarrow \bigoplus_P \overline{K}_0(R_PG) \longrightarrow K_{-1}(RG) \longrightarrow \bigoplus_P K_{-1}(R_PG) \longrightarrow 0$$

and

$$0 \longrightarrow K_{-m}(RG) \longrightarrow \bigoplus_P K_{-m}(R_PG) \longrightarrow 0$$

for $m > 1$. Under the condition that the field of fractions of R has characteristic zero one can show that the groups $K_{-m}(RG)$ for $m > 1$ and $K_{-1}(R_PG)$ are also trivial leaving us with an exact sequence

$$0 \longrightarrow \overline{K}_0(RG) \longrightarrow \bigoplus_P \overline{K}_0(R_PG) \longrightarrow K_{-1}(RG) \longrightarrow 0\ .$$

This sequence can be rearranged into the exact sequence

$$K_0(RG) \longrightarrow K_0(KG) \oplus \bigoplus_P K_0(R_PG) \longrightarrow \bigoplus_P K_0(K_PG) \longrightarrow K_{-1}(RG) \longrightarrow 0$$

where the first map is given by the maps induced by the inclusions $R \to R_P$ and by the negative of the map induced by the inclusion $R \to K$. The second map in the diagram is given by the maps induced by the inclusions $K \to K_P$ and $R_P \to K_P$. We apply this to $R = \mathbb{Z}$ and note that by results of Swan (see [Swa70, Theorem 4.2 and 2.21]) it follows that the image of the first map of the above sequence is equal to the image generated by free RG-modules. Since these are all distinct we get the exact sequence

$$0 \longrightarrow K_0(\mathbb{Z}) \longrightarrow K_0(\mathbb{Q}G) \oplus \bigoplus_p K_0(\mathbb{Z}_pG) \longrightarrow \bigoplus_p K_0(\mathbb{Q}_pG) \longrightarrow K_{-1}(\mathbb{Z}G) \longrightarrow 0$$

where $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the p-adic numbers and integers, respectively and the direct sums are taken over all primes p in $\mathbb{Z}$. Further, it is known from representation theory that the naturally induced map $K_0(\mathbb{Z}_pG) \to K_0(\mathbb{Q}_pG)$ is an isomorphism, if G is a finite group of order prime to p (see [Ser77, Section 15.5]). This means that we can exclude the summands in the above sequence corresponding to primes not dividing the order n of G without harming the exactness. Now we consider the following maps induced from subgroups of G. In general, for a ring R with unit denote by ind_R the map

$$\bigoplus_{H \lneq G} K_0(RH) \to K_0(RG)$$

induced from the inclusions $H \to G$ for subgroups H of G. These commute with the localization sequence and so for a finite cyclic group C of order n we get a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \bigoplus\limits_{D \subsetneq C} K_0(\mathbb{Z}) & \to & \bigoplus\limits_{D \subsetneq C} \big(K_0(\mathbb{Q}D) \oplus \bigoplus\limits_{p|n} K_0(\mathbb{Z}_p D)\big) & \to & \bigoplus\limits_{D \subsetneq C} \bigoplus\limits_{p|n} K_0(\mathbb{Q}_p D) & \to & \bigoplus\limits_{D \subsetneq C} K_{-1}(\mathbb{Z}D) & \to 0 \\
 & \downarrow & & \downarrow {\scriptstyle \mathrm{ind}_{\mathbb{Q}} \oplus \bigoplus_p \mathrm{ind}_{\mathbb{Z}_p}} & & \downarrow {\scriptstyle \bigoplus_p \mathrm{ind}_{\mathbb{Q}_p}} & & \downarrow & \\
0 \longrightarrow & K_0(\mathbb{Z}) & \longrightarrow & K_0(\mathbb{Q}C) \oplus \bigoplus\limits_{p|n} K_0(\mathbb{Z}_p C) & \longrightarrow & \bigoplus\limits_{p|n} K_0(\mathbb{Q}_p C) & \longrightarrow & K_{-1}(\mathbb{Z}C) & \longrightarrow 0
\end{array} \tag{2.1}$$

where we excluded all summands corresponding to primes not dividing the order of C according to the previous comment. This diagram will play a crucial role in determining the Artin defect in degree -1.

2.2 Functoriality of K_0 of Group Rings Over Fields

The purpose of this section is to get a better understanding of the groups $K_0(FC) \otimes_{\mathbb{Z}} \mathbb{Q}$ as functors in the finite cyclic group C for a field F of characteristic 0. This is important, because the Artin defect in degree -1 can be expressed in terms of above K-groups as we will see. Let us make this precise by introducing the subgroup category of a group.

Definition 2.1. *Let G be a group. Denote by $Sub(G)$ the category whose objects are the subgroups of G. Further, for subgroups H and K of G define the set*

$$\mathrm{con}_G(H, K) = \{f \colon H \to K \mid \exists\, g \in G : f(h) = ghg^{-1}\ \forall\, h \in H\}.$$

The group of inner automorphisms $\mathrm{Inn}(K)$ acts from the left on $\mathrm{con}_G(H, K)$ by composition and we define the set of morphisms from H to K in $Sub(G)$ to be the set $\mathrm{Inn}(K) \backslash \mathrm{con}_G(H, K)$.

Note that if G is abelian, then $Sub(G)$ boils down to the category of subgroups of G and inclusions of groups as morphisms. Next, we introduce dual groups.

Definition 2.2. *Let G be a finite abelian group and let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, then the set*

$$\widehat{G} = \mathrm{Hom}(G, S^1)$$

of group homomorphisms from G to S^1 is called the dual group of G. The group multiplication is given by pointwise multiplication of maps.

A first observation is that a finite cyclic group is isomorphic to its dual group, although not in a canonical way.

Lemma 2.3. *Let C be a finite cyclic group. The choice of a generator c of C defines an isomorphism $\widehat{C} \to C$ by $\sigma \mapsto \sigma(c)$.*

Proof. Let $n \in \mathbb{N}$ be the order of C. The map $\sigma \mapsto \sigma(c)$ is injective, because c is a generator and a group homomorphism from a cyclic group is determined by the image of a generator. Further, the order $\widehat{C}$ is also n, because c can be mapped to any of the n different n-th roots of unity and so the map must also be bijective. □

This result holds more general for finite abelian groups, but for our purposes it suffices to have it for finite cyclic groups. Note that a different choice of a generator defines a different isomorphism and there is no canonical isomorphism. For the remaining part of this section let F be a field of characteristic 0, let C be a finite cyclic group of order $n \in \mathbb{N}$ and denote by μ_n the set of n-th roots of unity in $\mathbb{C}$. We consider the two $F(\mu_n)$-algebras

$$F(\mu_n)C \qquad \text{and} \qquad \operatorname{map}(\widehat{C}, F(\mu_n))$$

which denote the group ring of C over $F(\mu_n)$ and the space of all maps in the category of sets from the dual group $\widehat{C}$ to $F(\mu_n)$, respectively. The ring structure of the latter is given by pointwise multiplication of maps. The Galois group $G(F(\mu_n)/F)$ of the extension $F(\mu_n)/F$ acts on μ_n by automorphisms. Since the n-th roots of unity form a cyclic group of order n, there is an isomorphism $\mu_n \cong C$. This induces an action of $G(F(\mu_n)/F)$ on C. Although the identification $\mu_n \cong C$ depends on choices of generators in each group, the induced action of the Galois group does not depend on these choices. More precisely, an automorphism $\phi \in G(F(\mu_n)/F)$ is uniquely determined by the fact that it maps a and then any primitive n-th root of unity ζ_n to ζ_n^t for some $t \in \{1, \ldots, n\}$ coprime to n. The action on C is defined similarly by $\phi(c) = c^t$ for $c \in C$. Hence, the Galois group $G(F(\mu_n)/F)$ can be considered in a canonical way as a subgroup of the automorphism group of C, which we denote by $\operatorname{Aut}(C)$. Note that this action commutes with elements in the dual group in the sense that $\phi(\sigma(c)) = \sigma(\phi(c))$ for all $\sigma \in \widehat{C}$ and $c \in C$. Further, the action of $G(F(\mu_n)/F)$ on C restricts to an action on every subgroup D of C, because it preserves the order of elements. The inclusion $D \to C$ is $G(F(\mu_n)/F)$-equivariant. With this in mind we have the following group actions on the algebras.

Lemma 2.4. *Let C be a finite cyclic group and $n \in \mathbb{N}$ a multiple of the order of C. Denote by $G = G(F(\mu_n)/F)$ the Galois group of the extension $F(\mu_n)/F$, then $G \times G$ acts on*

$$F(\mu_n)C \qquad \text{by} \qquad (\phi, \psi) \cdot (\sum_{c \in C} \lambda_c c) = \sum_{c \in C} \phi(\lambda_c)\psi^{-1}(\phi(c))$$

and on

$$\operatorname{map}(\widehat{C}, F(\mu_n)) \qquad \text{by} \qquad ((\phi, \psi) \cdot f)(\sigma) = \phi(f(\psi^{-1} \circ \sigma)).$$

Further, we have fixed point F-vector subspaces

$$(F(\mu_n)C)^{G \times G} = (FC)^{1 \times G} \qquad \text{and} \qquad \operatorname{map}(\widehat{C}, F(\mu_n))^{G \times G} = \operatorname{map}(\widehat{C}, F)^{1 \times G}.$$

Proof. Knowing that $G \times G$ is abelian it is straight forward to show that both actions are well-defined. Now let us determine the fixed point subspaces. An element in $(F(\mu_n)C)^{G \times G}$ must satisfy in particular

$$\sum_{c \in C} \lambda_c c = (\phi, \phi) \cdot (\sum_{c \in C} \lambda_c c) = \sum_{c \in C} \phi(\lambda_c) c$$

for all $\phi \in G$. This means that the coefficients λ_c are invariant under automorphisms of the Galois group and so $\lambda_c \in F(\mu_n)^G = F$ for all $c \in C$. Additionally, we have the condition

$$\sum_{c \in C} \lambda_c c = (id, \psi^{-1}) \cdot (\sum_{c \in C} \lambda_c c) = \sum_{c \in C} \lambda_c \psi(c)$$

for all $\psi \in G$ and so $\lambda_c = \lambda_{\psi(c)}$, which implies that coefficients belonging to the same orbit under the G-action must be equal. Both conditions together yield $(F(\mu_n)C)^{G \times G} \subseteq (FC)^{1 \times G}$. On the other hand, given an element in $(FC)^{1 \times G}$ we check that

$$(\phi, \psi) \cdot (\sum_{c \in C} \lambda_c c) = \sum_{c \in C} \phi(\lambda_c)\phi(c) = \sum_{c \in C} \lambda_c \phi(c) = (id, \phi) \cdot (\sum_{c \in C} \lambda_c c) = \sum_{c \in C} \lambda_c c$$

which shows equality. Considering the fixed point space $\operatorname{map}(\widehat{C}, F(\mu_n))^{G \times G}$ we see that being fixed under the subgroup $G \times 1$ implies that such a map must have values in F and being fixed under $1 \times G$ means that a map is constant on

G-orbits of $\widehat{C}$. Any element in $G\times G$ can be written as a product of elements in $1\times G$ and $G\times 1$ and so combining both conditions yields the desired results. □

Since $\widehat{C}$ and C have the same order, it is obvious that the two algebras introduced above are isomorphic as vector spaces over $F(\mu_n)$. But we will show that they are even equivariantly isomorphic as algebras and functors. For a field F denote by $\mathcal{VS}_F$ the category of finite dimensional F-vector spaces and F-linear maps. We have the following result.

Lemma 2.5. *Keep the notations of the previous lemma. The map*

$$F(\mu_n)C \to \mathrm{map}(\widehat{C}, F(\mu_n)), \quad c \mapsto f_c$$

with $f_c(\sigma) = \sigma(c)$ for $c \in C$ and $\sigma \in \widehat{C}$ is a $G \times G$-equivariant isomorphism of $F(\mu_n)$-algebras. Further, it is a natural transformation of functors from $Sub(C)$ to $\mathcal{VS}_{F(\mu_n)}$ and natural with respect to automorphisms of C.

Proof. It is straight forward to check that this map defines a homomorphism of algebras, so we proceed with showing that it is an isomorphism. Since both spaces have $F(\mu_n)$-dimension $m = |C|$ it is sufficient to show injectivity. We have an inner product on $\mathrm{map}(\widehat{C}, F(\mu_n))$ defined by $\langle f, g\rangle = \frac{1}{m}\sum_{\sigma\in\widehat{C}} f(\sigma)\overline{g(\sigma)}$. We will use this to show that the vectors f_c for $c \in C$ are orthogonal, which implies injectivity. Fix a generator x of C, then for $k, l \in \{0, \ldots, m-1\}$ with $k \neq l$ and $d = \gcd(m, k-l)$ we compute

$$\langle f_{x^k}, f_{x^l}\rangle = \frac{1}{m}\sum_{\sigma\in\widehat{C}} \sigma(x^{k-l}) = \frac{1}{m}\sum_{\zeta\in\mu_m} \zeta^{k-l} = \frac{d}{m}\sum_{\xi\in\mu_{m/d}} \xi^{(k-l)/d} = 0.$$

In the last step we used the fact that the sum of all m/d-th roots of unity is 0. We proceed with proving equivariance. Since the action commutes with taking sums, it is enough to check equivariance on an element of the form $\lambda_c c$ for $c \in C$ and $\lambda_c \in F(\mu_n)$. Let $(\phi, \psi) \in G\times G$, then the element $\phi(\lambda_c)\psi^{-1}(\phi(c))$ is mapped to $\phi(\lambda_c)f_{\psi^{-1}(\phi(c))}$. Evaluation on $\sigma \in \widehat{C}$ gives

$$\phi(\lambda_c)f_{\psi^{-1}(\phi(c))}(\sigma) = \phi(\lambda_c)\sigma(\psi^{-1}(\phi(c))).$$

On the other hand, we have

$$((\phi, \psi) \cdot (\lambda_c f_c))(\sigma) = \phi(\lambda_c f_c(\psi^{-1} \circ \sigma)) = \phi(\lambda_c)\phi(\psi^{-1}(\sigma(c))).$$

Since G is abelian and the action of G on C commutes with all $\sigma \in \widehat{C}$ the map is indeed equivariant. Last, we have to show naturality. We will show slightly more than stated. Let C, C' be finite cyclic groups and $\beta\colon C \to C'$ any group homomorphism. We claim that the induced diagram

$$\begin{array}{ccc} F(\mu_n)C & \xrightarrow{\cong} & \operatorname{map}(\widehat{C}, F(\mu_n)) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} \\ F(\mu_n)C' & \xrightarrow{\cong} & \operatorname{map}(\widehat{C'}, F(\mu_n)) \end{array}$$

commutes. The vertical map β_* on the right maps $f \in \operatorname{map}(\widehat{C}, F(\mu_n))$ to $\beta_*(f)(\sigma) = f(\sigma \circ \beta)$ for $\sigma \in \widehat{C'}$. For $c \in C$ we see that $f_c(\sigma \circ \beta) = \sigma(\beta(c)) = f_{\beta(c)}(\sigma)$. This finishes the proof, because β_* is in both cases a homomorphism of algebras and so it is enough to check naturality on a basis. □

Now the rational K-groups in degree 0 come into play. The next lemma shows that they are compatible with a certain action of a Galois group.

Lemma 2.6. *Let L/K be a Galois extension of fields of characteristic 0 and H a finite group. The Galois group G of the extension L/K acts on the group ring LH by acting on L as automorphisms. This induces an action of G on $K_0(LH)$ and there is an isomorphism of $\mathbb{Q}$-vector spaces*

$$K_0(LH)^G \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_0(KH) \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad [M] \otimes q \mapsto [\operatorname{res}^{LH}_{KH} M] \otimes q$$

where $\operatorname{res}^{LH}_{KH} M$ denotes the KH-module we obtain by restricting scalar multiplication from LH to KH.

Proof. Denote by $\operatorname{ind}^{LH}_{KH}\colon K_0(KH) \to K_0(LH)$ the map $[M] \mapsto [M \otimes_{KH} LH]$ and consider the map

$$K_0(KH) \to K_0(KH), \quad [M] \mapsto [\operatorname{res}^{LH}_{KH}(\operatorname{ind}^{LH}_{KH}(M))].$$

For a KH-module M we compute

$$\operatorname{res}^{LH}_{KH}(LH \otimes_{KH} M) \cong \operatorname{res}^{LH}_{KH}(L \otimes_K M) \cong K^{[L:K]} \otimes_K M \cong M^{\oplus [L:K]}$$

and see that it is multiplication with $[L : K]$. Similarly, we also have the map

$$K_0(LH) \to K_0(LH), \quad [M] \mapsto [\operatorname{ind}^{LH}_{KH}(\operatorname{res}^{LH}_{KH}(M))]$$

and for an LH-module M we get

$$LH \otimes_{KH} M \cong L \otimes_K M \cong (L \otimes_K L) \otimes_L M \cong \bigoplus_{\sigma \in G} \sigma L \otimes_L M \cong \bigoplus_{\sigma \in G} \sigma M.$$

Further, for a KH-module M we have $\sigma L \otimes_K M \cong L \otimes_K M$ as LH-modules. In particular, the image of the induction map is contained in $K_0(LH)^G$. In summary, we get a commutative diagram

$$\begin{array}{ccccc} & & K_0(LH)^G & \xrightarrow{\sum_{\sigma \in G} \sigma_*} & K_0(LH)^G \\ & {\scriptstyle \mathrm{ind}} \nearrow & & {\scriptstyle \mathrm{res}} \searrow \quad {\scriptstyle \mathrm{ind}} \nearrow & \\ K_0(KH) & & \xrightarrow{\cdot [L:K]} & & K_0(KH) \end{array}$$

The lower horizontal map is rationally an isomorphism which implies that restriction is rationally surjective. Since we restricted to $K_0(LH)^G$ the upper horizontal map is again multiplication with $[L : K]$ and thus also rationally an isomorphism. Hence, restriction is rationally also injective yielding the desired isomorphism. Note that if we don't tensor with $\mathbb{Q}$ we still get an injective map $K_0(KH) \to K_0(LH)^G$, because these $\mathbb{Z}$-modules are free. ☐

We get the following corollary which will be of use in a later chapter.

Corollary 2.7. *Let K and L be fields with $K \subseteq L \subseteq \mathbb{C}$ and H a finite group. The group homomorphism*

$$K_0(KH) \to K_0(LH), \quad [P] \mapsto [P \otimes_K L]$$

is injective.

Proof. Let n be the order of H and denote by μ_n the set of n-th roots of unity in $\mathbb{C}$. Consider the diagram

$$\begin{array}{ccccc} K_0(KH) & \longrightarrow & K_0(LH) & \longrightarrow & K_0(\mathbb{C}H) \\ & \searrow & & & \uparrow \cong \\ & & & & K_0(K(\mu_n)H), \end{array}$$

where each map is the obvious one induced by inclusion of fields. The fact that the vertical map is an isomorphism is well known from representation theory (see [Ser77, Chapter 12.3, Theorem 24]). The diagonal map is injective according to the proof of the previous lemma, because the extension $K(\mu_n)/K$ is Galois. It follows that the composition of the horizontal maps is injective and hence also the map stated in the corollary. □

We can finally determine the object we were initially interested in, namely the rational K-group $K_0(FC) \otimes_{\mathbb{Z}} \mathbb{Q}$ as functor from the subgroup category of a finite cyclic group.

Theorem 2.8. *Let F be a field of characteristic 0 and C a finite cyclic group of order $n \in \mathbb{N}$. Denote by G_F and $G_{\mathbb{Q}}$ the Galois groups of the extensions $F(\mu_n)/F$ and $\mathbb{Q}(\mu_n)/\mathbb{Q}$, respectively. For every subgroup D of C there is an isomorphism*

$$K_0(FD) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\mathbb{Q}(\mu_n)D)^{G_{\mathbb{Q}} \times G_F}.$$

This defines a natural transformation of functors from $Sub(C)$ to $\mathcal{VS}_{\mathbb{Q}}$. Additionally, it is natural with respect to automorphisms of C.

Proof. We have isomorphisms

$$\begin{aligned}
K_0(FD) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong K_0(F(\mu_n)D)^{G_F} \otimes_{\mathbb{Z}} \mathbb{Q} && (1)\\
&\cong K_0(\mathrm{map}(\widehat{D}, F(\mu_n)))^{G_F} \otimes_{\mathbb{Z}} \mathbb{Q} && (2)\\
&\cong \mathrm{map}(\widehat{D}, K_0(F(\mu_n)))^{G_F} \otimes_{\mathbb{Z}} \mathbb{Q} && (3)\\
&\cong \mathrm{map}(\widehat{D}, \mathbb{Z})^{G_F} \otimes_{\mathbb{Z}} \mathbb{Q} && (4)\\
&\cong \mathrm{map}(\widehat{D}, \mathbb{Q})^{G_F} && (5)\\
&\cong \mathrm{map}(\widehat{D}, \mathbb{Q}(\mu_n))^{G_{\mathbb{Q}} \times G_F} && (6)\\
&\cong (\mathbb{Q}(\mu_n)D)^{G_{\mathbb{Q}} \times G_F} && (7)
\end{aligned}$$

In (1) we applied Lemma 2.6. Isomorphisms (2) and (7) are a direct consequence of Lemma 2.5 and they are natural as functors from the subgroup category. In (3) and (4) we make use of the fact that K_0 respects products. Also, K_0 of a field is isomorphic to $\mathbb{Z}$ and injective field homomorphisms induce the identity on K_0. Step (5) is obvious and for (6) we used Lemma 2.4. The crucial steps to check for compatibility with an automorphism of C are (2) and (7). But in the proof of Lemma 2.5 we showed that the isomorphism is compatible with any group homomorphism between finite cyclic groups and this obviously includes automorphisms. □

Remark 2.9. *Let F and F' be fields of characteristic 0 with $F \subseteq F'$ and denote by G_F and $G_{F'}$ the Galois groups of the $F(\mu_n)/F$ and $F'(\mu_n)/F'$, respectively. The Galois group $G_{F'}$ can be canonically identified with a subgroup of G_F and it is straight forward to check that the isomorphism from the previous theorem is natural with respect to inclusion of fields in the sense that the map*

$$K_0(FC) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(F'C) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad [P] \mapsto [P \otimes_F F']$$

induces the obvious inclusion of fixed point subspaces

$$(\mathbb{Q}(\mu_n)C)^{G_{\mathbb{Q}} \times G_F} \to (\mathbb{Q}(\mu_n)C)^{G_{\mathbb{Q}} \times G_{F'}}.$$

2.3 The Artin Defect and the Action of the Automorphism Group

In this last section we will apply the previous results to compute the Artin defect in negative degrees and understand the action of the automorphism group. As usual, C denotes a finite cyclic group of order $n \in \mathbb{N}$. Recall that the Artin defect is defined as

$$S_m(n) = \operatorname{coker}\left(\bigoplus_{D \lneq C} K_m(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \right)$$

and in this chapter we consider the degrees $m < 0$. We have seen in the first section that $K_m(\mathbb{Z}C) = 0$ for all $m < -1$, because $\mathbb{Z}$ is a Dedekind ring. This holds for every finite cyclic group and so the maps on K-theory induced from inclusion of subgroups are the zero maps. Consequently, the Artin defect vanishes in this case. Thus, it remains to treat the defect in degree $m = -1$. Our main tool is the diagram

$$\begin{array}{ccccccccc}
0 \to & \bigoplus\limits_{D \lneq C} K_0(\mathbb{Z}) & \to & \bigoplus\limits_{D \lneq C} \Big(K_0(\mathbb{Q}D) \oplus \bigoplus\limits_{p \mid n} K_0(\mathbb{Z}_p D)\Big) & \to & \bigoplus\limits_{D \lneq C} \bigoplus\limits_{p \mid n} K_0(\mathbb{Q}_p D) & \to & \bigoplus\limits_{D \lneq C} K_{-1}(\mathbb{Z}D) & \to 0 \\
 & \downarrow & & \downarrow{\scriptstyle \operatorname{ind}_{\mathbb{Q}} \oplus \bigoplus_p \operatorname{ind}_{\mathbb{Z}_p}} & & \downarrow{\scriptstyle \bigoplus_p \operatorname{ind}_{\mathbb{Q}_p}} & & \downarrow & \\
0 \to & K_0(\mathbb{Z}) & \to & K_0(\mathbb{Q}C) \oplus \bigoplus\limits_{p \mid n} K_0(\mathbb{Z}_p C) & \to & \bigoplus\limits_{p \mid n} K_0(\mathbb{Q}_p C) & \to & K_{-1}(\mathbb{Z}C) & \to 0
\end{array}$$

we constructed in the first section. We see that the main ingredients in determining the Artin defect in degree -1 are the torsion-free abelian groups $K_0(\mathbb{Q}C)$, $K_0(\mathbb{Z}_pC)$, $K_0(\mathbb{Q}_pC)$ and the corresponding induction maps. Note that the vertical map on the very left is surjective, because the free $\mathbb{Z}D$-module $\mathbb{Z}D$ is mapped to the free $\mathbb{Z}C$-module $\mathbb{Z}C$ under the map on K_0 induced by $\mathbb{Z}D \to \mathbb{Z}C$. We proceed with computing the cokernels of the induction maps $\mathrm{ind}_{\mathbb{Q}}$, $\mathrm{ind}_{\mathbb{Q}_p}$ and $\mathrm{ind}_{\mathbb{Z}_p}$ for each prime divisor p of n. It is sufficient to consider the rational versions. We start with the K-groups of the group ring over $\mathbb{Z}_p$, for which we will use the following result. It can be found in [Ser77, Section 14.4, Corollary 3].

Theorem 2.10. *Let G be a finite group and K a discrete valuation field of characteristic 0. Denote by A the valuation ring, $\mathfrak{m}$ the maximal ideal and $k = A/\mathfrak{m}$ the corresponding residue field of characteristic $p > 0$. The quotient map $A \to A/\mathfrak{m}$ induces an isomorphism $K_0(AG) \cong K_0(kG)$.*

We apply this to $K = \mathbb{Q}_p$, $A = \mathbb{Z}_p$ and $k = \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ to get an isomorphism $K_0(\mathbb{Z}_pC) \cong K_0(\mathbb{F}_pC)$ induced by the projection $\mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p$. We use it to show that the induction map $\mathrm{ind}_{\mathbb{Z}_p}$ is surjective.

Lemma 2.11. *Let C be a finite cyclic group of order $n \in \mathbb{N}$ and p a prime dividing n, then the map*

$$\mathrm{ind}_{\mathbb{F}_p}\colon \bigoplus_{D \lneq C} K_0(\mathbb{F}_pD) \longrightarrow K_0(\mathbb{F}_pC)$$

is surjective.

Proof. Write $n = p^l m$ with $p \nmid m$ and $l \in \mathbb{N}$. We have a decomposition $C \cong C_{p^l} \times C_m$ where C_{p^l} and C_m are the unique subgroups of C of order p^l and m, respectively. It is known from modular representation theory (see [Ser77, Example 15.7]) that a finitely generated $\mathbb{F}_pC$-module E is projective, if and only if $E \cong F \otimes_{\mathbb{F}_p} \mathbb{F}_pC_{p^l}$ for some $\mathbb{F}_pC_m$-module F. Since $F \otimes_{\mathbb{F}_p} \mathbb{F}_pC_{p^l} \cong F \otimes_{\mathbb{F}_pC_m} \mathbb{F}_pC$ this means that E is in the image of the induction map. Note that since $p \nmid m$ the ring $\mathbb{F}_pC_m$ is semisimple due to Maschke's Theorem and so every module is projective. Now p divides n and so C_m is always a proper subgroup of C. Combining this with the previous statement that every projective $\mathbb{F}_pC$-module is induced from an $\mathbb{F}_pC_m$-module shows that the induction map is indeed surjective. □

In particular, using the isomorphism from Theorem 2.10, which commutes with the maps induced from subgroups, we see that $\operatorname{coker}(\operatorname{ind}_{\mathbb{Z}_p}) = 0$ for every prime p dividing the order of C. Now we turn to K_0 of the algebras $\mathbb{Q}C$ and $\mathbb{Q}_pC$. More general, we consider the rational induction map

$$\bigoplus_{D \lneq C} K_0(FD) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_0(FC) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for any field F of characteristic 0. We have treated these groups in the previous section and we will use the results to compute the cokernel of the above induction maps as modules over $\mathbb{Q}\operatorname{Aut}(C)$. But before let us make one remark. We have canonically identified the Galois group $G(F(\mu_n)/F)$ of the extension $F(\mu_n)/F$ with a subgroup of $\operatorname{Aut}(C)$. Therefore, an action of $\operatorname{Aut}(C)$ on a set M can be restricted to an action of $G(F(\mu_n)/F)$ on M. Since $\operatorname{Aut}(C)$ is abelian, both actions commute and we get a well-defined action of $\operatorname{Aut}(C)$ on the fixed point set $M^{G(F(\mu_n)/F)}$.

Theorem 2.12. *Let C be a finite cyclic group of order $n \in \mathbb{N}$ and F a field of characteristic 0. Denote by μ_n the set of n-th roots of unity in $\mathbb{C}$ and by G the Galois group of the extension $F(\mu_n)/F$. Further, let $\operatorname{ind}_F \otimes \mathbb{Q}$ be the induction map $\bigoplus_{D \lneq C} K_0(FD) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FC) \otimes_{\mathbb{Z}} \mathbb{Q}$. The choice of a generator of C defines an isomorphism of $\mathbb{Q}\operatorname{Aut}(C)$-modules*

$$\operatorname{coker}(\operatorname{ind}_F \otimes \mathbb{Q}) \cong \mathbb{Q}[\operatorname{Aut}(C)]^{G_F}.$$

In particular, $\dim_{\mathbb{Q}}(\operatorname{coker}(\operatorname{ind}_F \otimes \mathbb{Q})) = \varphi(n)/|G_F|$ where φ is Euler's phi function.

Proof. We use the isomorphism from Theorem 2.8 to get the corresponding map

$$\bigoplus_{D \lneq C} (\mathbb{Q}(\mu_n)D)^{G_{\mathbb{Q}} \times G_F} \longrightarrow (\mathbb{Q}(\mu_n)C)^{G_{\mathbb{Q}} \times G_F}$$

induced by the inclusion of subgroups D of C. The group C is the disjoint union

$$C = \bigcup_{d|n} E_d(C)$$

of the sets $E_d(C) = \{c \in C \mid \operatorname{ord}(c) = d\}$. Since the action of the Galois group $G_{\mathbb{Q}}$ on C preserves the order of an element, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$(\mathbb{Q}(\mu_n)C)^{G_\mathbb{Q}\times G_F} \cong \bigoplus_{d|n} \mathbb{Q}(\mu_n)[E_d(C)]^{G_\mathbb{Q}\times G_F}.$$

For each divisor $d \neq n$ there exists a proper subgroup D of C such that $E_d(C) = E_d(D)$. Therefore, the image of the induction map is precisely the sum over all $d \neq n$ and so the cokernel is isomorphic to $\mathbb{Q}(\mu_n)[E_n(C)]^{G_\mathbb{Q}\times G_F}$. We claim that a choice of a generator c' of C defines an isomorphism

$$\mathbb{Q}(\mu_n)[E_n(C)]^{G_\mathbb{Q}\times G_F} \to \mathbb{Q}(\mu_n)^{G_F}, \qquad \sum_{c\in E_n(C)} \lambda_c c \mapsto \lambda_{c'}.$$

We have $\mathbb{Q}(\mu_n)[E_n(C)]^{G_\mathbb{Q}\times G_F} = \mathbb{Q}(\mu_n)[E_n(C)]^{G_\mathbb{Q}\times 1} \cap \mathbb{Q}(\mu_n)[E_n(C)]^{1\times G_F}$ and so we determine the fixed point set of each action separately. The Galois group $G_\mathbb{Q}$ acts transitively on the generators of C. This means that for all $c \in E_n(C)$ there is a $\phi \in G_\mathbb{Q}$, such that $\phi(c') = c$. The fixed point condition

$$\sum_{c\in E_n(C)} \lambda_c c = \sum_{c\in E_n(C)} \phi(\lambda_c)\phi(c)$$

thus implies that $\lambda_c = \phi(\lambda_{c'})$. Hence, the choice of $\lambda_{c'} \in \mathbb{Q}(\mu_n)$ determines all other coefficients and the map defined above is injective. On the other hand, we have the fixed point condition under G_F which acts on the basis given by the generators of C. This means that coefficients belonging to the same orbit of $E_n(C)/G_F$ must be equal. Therefore, an element being fixed under both actions must additionally satisfy $\lambda_{c'} = \phi(\lambda_{c'})$ for all $\phi \in G_F$ which means that $\lambda_{c'}$ lies in $\mathbb{Q}(\mu_n)^{G_F}$ and this implies surjectivity. Now since $\mathbb{Q}(\mu_n)/\mathbb{Q}$ is a finite Galois extension, there exists a normal basis of the $\mathbb{Q}$-vector space $\mathbb{Q}(\mu_n)$ (see [Nar04, Theorem 4.28]). More specifically, there is an element $\alpha \in \mathbb{Q}(\mu_n)$, such that the set $B_\alpha = \{\phi(\alpha) \mid \phi \in G_\mathbb{Q}\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}(\mu_n)$. Via the canonical isomorphism $G_\mathbb{Q} \cong \mathrm{Aut}(C)$ described previously we get an isomorphism of $\mathbb{Q}$-vector spaces $\mathbb{Q}(\mu_n)^{G_F} \cong \mathbb{Q}[\mathrm{Aut}(C)]^{G_F}$ defined by $\phi(\alpha) \mapsto \phi$ on the basis. □

If n is square-free, then a normal basis of $\mathbb{Q}(\mu_n)$ is given by all Galois conjugates of a primitive n-th root of unity. This does not hold in general for arbitrary n. To see that assume there is a prime p with $p^2 \mid n$. Let ζ_n be a primitive n-th root of unity and set $\zeta_p = \zeta_n^{n/p}$. We have

$$\sum_{i=0}^{p-1} \zeta_n \cdot \zeta_p^i = \zeta_n \cdot \sum_{i=0}^{p-1} \zeta_p^i = \zeta_n \cdot 0 = 0,$$

but for all $i = 1, \dots, p-1$ the product $\zeta_n \cdot \zeta_p^i = \zeta_n^{1+i(n/p)}$ is a primitive n-th root of unity, because $\gcd(n, 1 + i(n/p)) = 1$ since n/p is divisible by all prime divisors of n.

The following lemma shows how the combination of all previous results can be used to determine the Artin defect in degree -1.

Lemma 2.13. *Let A_i and B_i, $i = 1, \dots, 4$ be abelian groups and let the diagram*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \xrightarrow{f} & B_3 & \longrightarrow & B_4 & \longrightarrow & 0
\end{array}$$

be commutative with exact rows. If f_1 is surjective and $\overline{f}$ denotes the map $\mathrm{coker}(f_2) \to \mathrm{coker}(f_3)$ *induced by f, then*

$$\mathrm{coker}(f_4) \cong \mathrm{coker}(\overline{f}).$$

Proof. Define A' and B' to be the cokernels of the maps $A_1 \to A_2$ and $B_1 \to B_2$, respectively. We split above diagram into the two commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f'} & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B' & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \\
0 & \longrightarrow & B' & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & 0
\end{array}$$

where the rows are again exact. Since f_1 is surjective, applying the snake lemma to the first diagram yields $\mathrm{coker}(f_2) \cong \mathrm{coker}(f')$. Using this and applying the snake lemma once more to the second diagram gives

$$\operatorname{coker}(f_4) \cong \operatorname{coker}(f_3) \Big/ \operatorname{Im}(\operatorname{coker}(f') \to \operatorname{coker}(f_3)) \cong \operatorname{coker}(f_3) \Big/ \operatorname{Im}(\overline{f}).$$

□

Now we have all ingredients to determine the Artin defect in degree -1.

Theorem 2.14. *Let C be a finite cyclic group of order $n \in \mathbb{N}$ and denote by $S_{-1}(n)$ the cokernel of the map*

$$\bigoplus_{D \lneq C} K_{-1}(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_{-1}(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by the inclusion of subgroups D of C. Further, for a prime p dividing n denote by G_p the Galois group of the extension $\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p$. There is an isomorphism of $\mathbb{Q}\operatorname{Aut}(C)$-modules

$$S_{-1}(n) \cong \operatorname{coker}(\mathbb{Q} \xrightarrow{\Delta} \bigoplus_{p|n} \mathbb{Q}[\operatorname{Aut}(C)]^{G_p})$$

where $\mathbb{Q}$ is the trivial $\mathbb{Q}\operatorname{Aut}(C)$-module and Δ denotes the diagonal map on the sum and on each summand. Here, $\operatorname{Aut}(C)$ acts on each summand by acting on the basis given by $\operatorname{Aut}(C)$ itself. In particular,

$$\dim_{\mathbb{Q}} S_{-1}(n) = \sum_{p|n} n_p - 1$$

with $n_p = \varphi(n/p^{\nu_p})/f_p$ where ν_p denotes the p-adic valuation on $\mathbb{Z}$ and f_p is the order of p in the multiplicative group of units $(\mathbb{Z}/(n/p^{\nu_p}))^{\times}$ of the ring of integers modulo n/p^{ν_p}.

Proof. We apply Lemma 2.13 to the rational version of diagram 2.1. Hence, the square of interest is

$$\begin{array}{ccc}
\bigoplus_{D \lneq C} (K_0^{\mathbb{Q}}(\mathbb{Q}D) \oplus \bigoplus_{p|n} K_0^{\mathbb{Q}}(\mathbb{Z}_p D)) & \longrightarrow & \bigoplus_{D \lneq C} \bigoplus_{p|n} K_0^{\mathbb{Q}}(\mathbb{Q}_p D) \\
\Big\downarrow {\scriptstyle (\operatorname{ind}_{\mathbb{Q}} \otimes \mathbb{Q}) \oplus \bigoplus_p (\operatorname{ind}_{\mathbb{Z}_p} \otimes \mathbb{Q})} & & \Big\downarrow {\scriptstyle \bigoplus_p \operatorname{ind}_{\mathbb{Q}_p} \otimes \mathbb{Q}} \\
K_0^{\mathbb{Q}}(\mathbb{Q}C) \oplus \bigoplus_{p|n} K_0^{\mathbb{Q}}(\mathbb{Z}_p C) & \longrightarrow & \bigoplus_{p|n} K_0^{\mathbb{Q}}(\mathbb{Q}_p C)
\end{array}$$

where $K_0^{\mathbb{Q}}$ denotes the rational K-groups. First, note that according to Theorem 2.10 and Lemma 2.11 the induction map $\mathrm{ind}_{\mathbb{Z}_p}$ is surjective for all primes dividing n and so the corresponding cokernel is 0. It remains to determine the cokernels of the maps $\mathrm{ind}_{\mathbb{Q}} \otimes \mathbb{Q}$ and $\mathrm{ind}_{\mathbb{Q}_p} \otimes \mathbb{Q}$. To do that we apply Lemma 2.12. In the first case it is well known that the Galois group $G_{\mathbb{Q}}$ of the extension $\mathbb{Q}(\mu_n)/\mathbb{Q}$ can be identified with the full group of units $(\mathbb{Z}/n)^\times$ of the ring of integers modulo n, which has order $\varphi(n)$. Therefore, we have

$$\mathrm{coker}(\mathrm{ind}_{\mathbb{Q}} \otimes \mathbb{Q}) \cong \mathbb{Q}[\mathrm{Aut}(C)]^{G_{\mathbb{Q}}} \cong \mathbb{Q},$$

since $G_{\mathbb{Q}}$ acts transitively on $\mathrm{Aut}(C)$. In the second case we have

$$\mathrm{coker}(\mathrm{ind}_{\mathbb{Q}_p} \otimes \mathbb{Q}) \cong \mathbb{Q}[\mathrm{Aut}(C)]^{G_p}.$$

Cyclotomic extensions of $\mathbb{Q}_p$ are treated for example in [Ser79, Chapter IV, §4]. We just give the results. Write $|C| = n = p^l \cdot m$ with $p \nmid m$ and $l \in \mathbb{N}$, then

$$\begin{aligned} G(\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p) &\cong G(\mathbb{Q}_p(\mu_{p^l})/\mathbb{Q}_p) \times G(\mathbb{Q}_p(\mu_m)/\mathbb{Q}_p) \\ &\cong G(\mathbb{Q}(\mu_{p^l})/\mathbb{Q}) \times G(\mathbb{F}_p(\mu_m)/\mathbb{F}_p) \\ &\cong (\mathbb{Z}/p^l)^\times \times \langle \zeta_m \mapsto \zeta_m^p \rangle \\ &\leq (\mathbb{Z}/p^l)^\times \times (\mathbb{Z}/m)^\times \cong (\mathbb{Z}/n)^\times, \end{aligned}$$

where ζ_m is a primitive m-th root of unity. So in general this Galois group is not the full group of units and thus does not act transitively on the set of generators. More specifically, denote by f_p the order of p in the multiplicative group $(\mathbb{Z}/m)^\times$, then we have

$$\dim_{\mathbb{Q}}(\mathrm{coker}(\mathrm{ind}_{\mathbb{Q}_p} \otimes \mathbb{Q})) = \frac{\varphi(n)}{\varphi(p^l) \cdot f_p} = \frac{\varphi(m)}{f_p}.$$

Denote by $\overline{f}$ the map $\mathrm{coker}(\mathrm{ind}_{\mathbb{Q}} \otimes \mathbb{Q}) \to \bigoplus_{p|n} \mathrm{coker}(\mathrm{ind}_{\mathbb{Q}_p} \otimes \mathbb{Q})$ induced by the natural map $K_0(\mathbb{Q}C) \to \bigoplus_{p|n} K_0(\mathbb{Q}_p C)$, then applying Lemma 2.13 yields

$$\begin{aligned} S_{-1}(n) &\cong \left(\bigoplus_p \mathrm{coker}(\mathrm{ind}_{\mathbb{Q}_p} \otimes \mathbb{Q}) \right) \Big/ \overline{f}(\mathrm{coker}(\mathrm{ind}_{\mathbb{Q}} \otimes \mathbb{Q})) \\ &\cong \left(\bigoplus_{p|n} \mathbb{Q}[\mathrm{Aut}(C)]^{G_p} \right) \Big/ \overline{f}(\mathbb{Q}). \end{aligned}$$

According to Remark 2.9, on $\mathbb{Q}$ the map $\overline{f}$ is given by the inclusion

$$\mathbb{Q} \cong \mathbb{Q}[\mathrm{Aut}(C)]^{G_{\mathbb{Q}}} \to \mathbb{Q}[\mathrm{Aut}(C)]^{G_p}$$

for each p, thus inducing the diagonal map on each summand. In particular, we have $\dim_{\mathbb{Q}}(S_{-1}(n)) = \sum_{p|n} n_p - 1$. Finally, the $\mathrm{Aut}(C)$-action is indeed the one induced by the action on the basis $\mathrm{Aut}(C)$, because we proved in Theorem 2.8 that the isomorphisms used to determine the cokernels of the different induction maps are natural with respect to automorphisms of C. □

The number n_p appears in the study of the factorization of prime ideals of $\mathbb{Z}$ in the ring of integers of $\mathbb{Q}(\mu_n)$. Since the extension $\mathbb{Q}(\mu_n)/\mathbb{Q}$ is Galois, the fundamental identity of extensions of Dedekind domains simplifies to

$$[\mathbb{Q}(\mu_n) : \mathbb{Q}] = efr,$$

where e is the ramification index, f the inertia degree and r the number of distinct primes in the factorization of the ideal (p) in $\mathbb{Q}(\mu_n)$. Using the previous notation we have $[\mathbb{Q}(\mu_n) : \mathbb{Q}] = \varphi(n)$, $e = \varphi(p^{\nu_p})$ and $f = f_p$ and so $n_p = r$ is the number of distinct prime ideals appearing in the factorization of (p). It is also the number of distinct extensions of the p-adic valuation of $\mathbb{Q}$ to the cyclotomic field $\mathbb{Q}(\mu_n)$. For more details see [Neu99, Chapter II, §8].

3 The Artin Defect in Degree $m = 0$

This case can be answered quite briefly. Let C be a finite cyclic group. Since finitely generated projective $\mathbb{Z}$-modules are free and up to isomorphism determined by their rank, we have $K_0(\mathbb{Z}) \cong \mathbb{Z}$. The natural inclusion $i\colon \mathbb{Z} \to \mathbb{Z}C$ splits via the augmentation map, which maps an element in the group ring to the coefficient of the identity element of C. Hence, the induced map $i_*\colon \mathbb{Z} \to K_0(\mathbb{Z}C)$ is also a split injection. The reduced K_0-group of $\mathbb{Z}C$ is defined as the quotient

$$\widetilde{K}_0(\mathbb{Z}C) = K_0(\mathbb{Z}C)/i_*(\mathbb{Z})$$

and according to the previous comments it measures the part of $K_0(\mathbb{Z}C)$ which does not come from finitely generated free $\mathbb{Z}C$-modules. Swan proved in [Swa60, Prop. 9.1] that $\widetilde{K}_0(RG)$ is finite for R a ring of algebraic integers and any finite group G. Thus, tensoring with $\mathbb{Q}$ yields

$$K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$$

for all finite cyclic groups C. In particular, given a subgroup D of C the map

$$\mathbb{Q} \cong K_0(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$$

induced from the inclusion $D \to C$ is an isomorphism, since $[M] \to [M \otimes_{\mathbb{Z}D} \mathbb{Z}C]$ is obviously not the zero map. It follows that the induction map

$$\bigoplus_{D \lneq C} K_0(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective if the order of C is greater than 1 and the Artin defect in degree 0, which is defined as the cokernel of this map, vanishes. If C is the trivial group there are no proper subgroups of C and hence the cokernel of the above induction map is isomorphic to $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

4 The Artin Defect in Degree $m = 1$

The goal of this chapter is to determine the Artin defect in degree 1. We will see that K_1 of the integral group ring of a finite cyclic group is isomorphic to the units of the group ring. This yields an explicit description of the K-groups we are interested in. In order to compute the dimension of the Artin defect in degree 1 we construct a filtered basis of the rationalized units $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$. That is a basis which contains a basis of the subspace $(\mathbb{Z}D)^\times \otimes_\mathbb{Z} \mathbb{Q}$ for each subgroup D of C. This is done in the first three sections. In the last section we determine the structure of the Artin defect as a module over $\mathbb{Q}\operatorname{Aut}(C)$ with the help of a well-known basis of the rationalized cyclotomic units. The cyclotomic units are a finite index subgroup of the full group of units of the ring of integers in a cyclotomic field.

4.1 A Filtered Basis of the Cyclotomic Units

In order to construct a filtered basis of the rationalized units of the integral group ring of a finite cyclic group we use the existence of a filtered basis of the cyclotomic units. Such a basis was constructed by M. Conrad in [Con00] and we want to describe it in this section. We keep the description as brief as possible, but with enough details so that one can comprehend the construction. For further details the reader is referred to the original article. We start with the definition of cyclotomic integers and cyclotomic units.

Definition 4.1. *For $n \in \mathbb{N}$ denote by μ_n the set of n-th roots of unity in $\mathbb{C}$ and let $\zeta_n \in \mu_n$ be a primitive n-th root of unity. The n-th cyclotomic numbers are defined as the multiplicative group generated by elements of the form $1 - \zeta_n^k$ with $k \in \{1, \ldots, n-1\}$ inside $\mathbb{Q}[\mu_n]$ and denoted by $W^{(n)}$. The n-th cyclotomic units are defined as $U^{(n)} = W^{(n)} \cap \mathbb{Z}[\mu_n]^\times$. Further, define the relative n-th cyclotomic numbers and relative n-th cyclotomic units as*

$$\widehat{W}^{(n)} = \operatorname{coker}\left(\prod_{d|n, d\neq n} W^{(d)} \to W^{(n)} \right)$$

and

$$\widehat{U}^{(n)} = \operatorname{coker}\left(\prod_{d|n, d\neq n} U^{(d)} \to U^{(n)}\right),$$

respectively using the inclusions of subsets $W^{(d)} \to W^{(n)}$ *and* $U^{(d)} \to U^{(n)}$ *for a divisor* d *of* n.

The cyclotomic numbers are introduced as a tool to construct a basis for the cyclotomic units. These two objects are connected as follows.

Lemma 4.2. *For* $n \in \mathbb{N}$ *the inclusion of* $U^{(n)}$ *into* $W^{(n)}$ *induces isomorphisms*

$$\widehat{U}^{(n)} \cong \begin{cases} \widehat{W}^{(n)} & \text{if } n \neq p^\alpha \\ \Delta\widehat{W}^{(n)} & \text{if } n = p^\alpha \end{cases}$$

where $\Delta\widehat{W}^{(n)}$ *denotes the subgroup of* $\widehat{W}^{(n)}$ *generated by elements of the form* $(1-\zeta_n^a)/(1-\zeta_n)$ *with* $a \in G_n = \{1 \leq a < n \mid \gcd(a,n) = 1\}$.

Conrad constructs a basis B_n of $U^{(n)}$ which contains a basis of B_d of $U^{(d)}$ for every divisor d of n. He does it by constructing a basis of the relative cyclotomic units $\widehat{U}^{(d)}$ for each divisor d of n and proving that the union of these form the desired basis. In order to do that he introduces weak σ-bases and the cyclotomic module. Let us see what these are precisely. Throughout this section every module is a $\mathbb{Z}$-module together with an involution σ. We start with the definition of weak σ-bases.

Definition 4.3. *A weak* σ*-basis of a module* M *is a triple* $[E^0, E^+, E^-]$ *of subsets of* M *such that the union* $B = E^0 \cup \sigma E^0 \cup E^+ \cup E^-$ *is disjoint and a basis of* M *with*

(i) $\sigma e - e \in \langle E^0 \cup \sigma E^0\rangle$ *for* $e \in E^+$

(ii) $\sigma e + e \in \langle E^0 \cup \sigma E^0\rangle$ *for* $e \in E^-$.

Next, we define the cyclotomic module, a basis of which will lead to a basis of the cyclotomic units. For a subset S of a module M we set $\Sigma(S) = \sum_{s\in S} s$.

Definition 4.4. *For* $n > 1$ *define the cyclotomic module* $Z(n)$ *as follows. Let* $G_d = \{1 \leq a < d \mid \gcd(a,d) = 1\}$ *and for a prime* p *let* $A_p = \{0, \ldots, p-1\}$.

We define

$$Z(n) = \begin{cases} \mathbb{Z}[G_p] \Big/ \langle \Sigma(G_p) \rangle_{\mathbb{Z}} & \text{if } n = p \text{ prime} \\ \mathbb{Z}[G_{q/p}] \otimes_{\mathbb{Z}} \mathbb{Z}[A_p] \Big/ \langle \Sigma(A_p) \rangle_{\mathbb{Z}} & \text{if } n = q = p^\alpha,\ \alpha > 1. \end{cases}$$

If $n = q_1 \cdot \ldots \cdot q_r$ where q_i are powers of distinct primes, then we define $Z(n) = Z(q_1) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} Z(q_r)$. A σ-operation is defined on G_d by $\sigma b = d - b$ for $b \in G_d$ and on A_p by $\sigma a = p - 1 - a$ for $a \in A_p$, which defines a σ-operation on the cyclotomic module.

Conrad also shows that the module $Z(n)$ is isomorphic to $\mathbb{Z}[G_n]/R_n$ for some submodule R_n. We proceed with describing weak σ-bases for the modules $Z(n)$ where n is a power of a prime. Such a basis B_n is given by

$$B_n = \begin{cases} [\varnothing, \varnothing, \varnothing] & \text{if } n = 2 \\ [\varnothing, \varnothing, \{(1,0)\}] & \text{if } n = 4 \\ [\{2, \ldots, (p-1)/2\}, \varnothing, \{1\}] & \text{if } 2 \neq n = p \text{ prime} \\ [\{(b,a) \mid 1 \leq b < \frac{1}{2}n/p,\ p \nmid b,\ 1 \leq a < p\}, \varnothing, \varnothing] & \text{if } 4 \neq n = p^\alpha, \alpha > 1. \end{cases}$$

In order to get a weak σ-basis of the cyclotomic module $Z(n)$ for arbitrary n one can use the following lemma. It also tells us how to get a $\mathbb{Z}$-basis for the module $M/\ker(1+\sigma)$ from a weak σ-basis of M.

Lemma 4.5. *Let $B = [E^0, E^+, E^-]$ and $C = [F^0, F^+, F^-]$ be weak σ-bases of modules M and L, respectively. Then $[G^0, G^+, G^-] \subseteq M \times L$ with*

$$\begin{aligned} G^0 &= (E^0 \times (F^0 \cup \sigma F^0 \cup F^+ \cup F^-)) \cup (E^+ \times F^0) \cup (E^- \times F^0), \\ G^+ &= (E^+ \times F^+) \cup (E^- \times F^-), \\ G^- &= (E^+ \times F^-) \cup (E^- \times F^+) \end{aligned}$$

induces a weak σ-basis of $M \otimes_{\mathbb{Z}} L$. Additionally, $E^0 \cup E^+$ induces a basis of $M/\ker(1+\sigma)$.

The next theorem shows the connection between cyclotomic modules and cyclotomic numbers.

Theorem 4.6. *For $n \in \mathbb{N}$ with $n \neq 4$ the map $G_n \to W^{(n)}$, $a \mapsto 1 - \zeta_n^a$ induces an isomorphism*

$$Y_n \Big/ \ker(1+\sigma) \longrightarrow \widehat{W}^{(n)}$$

where

$$Y_n = \begin{cases} 0 & \text{if } n = 1 \\ \mathbb{Z}[G_n] & \text{if } n = p \text{ prime} \\ Z(n) & \text{else} \end{cases}$$

The last ingredient for constructing a filtered basis of the cyclotomic units is the following result by Conrad.

Theorem 4.7. *Let $n \in \mathbb{N}$. If $\widehat{B}_d \subseteq U^{(n)}$ induces a basis of $\widehat{U}^{(d)}$ for all divisors d of n, then $B_n = \bigcup_{d|n} \widehat{B}_d$ is a basis of $U^{(n)}$.*

Let us summarize the construction process for a basis of $U^{(n)}$. According to the last result, once we have basis of $\widehat{U}^{(d)}$ for each divisor d of n, we just have to take the union of all these to get a filtered basis of $U^{(n)}$. We get a basis of $\widehat{U}^{(d)}$ in the following way. If d is not a power of a prime, we can use Theorem 4.6 in combination with Lemma 4.2 to get isomorphisms

$$\widehat{U}^{(d)} \cong \widehat{W}^{(d)} \cong Z(d)/\ker(1+\sigma).$$

Lemma 4.5 explains how to construct a basis of $Z(d)/\ker(1+\sigma)$, which we use to get a basis of $\widehat{U}^{(d)}$ via above isomorphism. On the other hand, if d is a prime we have $\widehat{U}^{(d)} \cong U^{(d)}$ and in this case there is a well-known basis for the cyclotomic units given by

$$\left\{ \frac{1-\zeta_n^a}{1-\zeta_n} \;\middle|\; 1 < a < \frac{p}{2} \right\}.$$

Finally, if d is a power of a prime p^α with $\alpha > 1$ one uses the isomorphism

$$\widehat{U}^{(d)} \cong \Delta\widehat{W}^{(d)} \cong \Delta(Z(d)/\ker(1+\sigma))$$

to deduce a basis, where $\Delta M = \langle e - e' \mid e \in E\rangle_{\mathbb{Z}}$ for a module M with basis E. It is explicitly given by

$$\left\{ \frac{1-\zeta_d^{ap^{\alpha-1}+b}}{1-\zeta_d} \;\middle|\; (b,a) \in \{b \in G_{d/p} \mid 1 \le b \le \frac{d}{2p}\} \times \{1,\dots,p-1\} \right\}$$

and this completes the construction.

4.2 A Filtered Basis of the Units of the Integral Group Ring of a Finite Cyclic Group

We will use the filtered basis of the cyclotomic units described in the previous section to construct a filtered basis of the rationalized units of the integral group ring $\mathbb{Z}C$. By that we mean a basis B_n of $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$ which contains a basis of the subspace $(\mathbb{Z}D)^\times \otimes_\mathbb{Z} \mathbb{Q}$ for any subgroup D of C. This is related to $K_1(\mathbb{Z}C) \otimes_\mathbb{Z} \mathbb{Q}$ and thus to the Artin defect in degree 1 in the following way.

For a commutative ring R with unit we set

$$GL(R) = \bigcup_{n\in\mathbb{N}} GL_n(R) \quad \text{and} \quad SL(R) = \bigcup_{n\in\mathbb{N}} SL_n(R)$$

where $GL_n(R)$ and $SL_n(R)$ are the general and special linear group of degree n, respectively. Further, we denote by $E_n(R)$ the subgroup of $GL_n(R)$ generated by matrices which have 1's on the diagonal and at most one non-zero off-diagonal entry. A matrix of this form is called an elementary matrix and we set $E(R) = \cup_{n\in\mathbb{N}} E_n(R)$. Note that $E(R)$ is a subgroup of $SL(R)$. Therefore, we can define

$$SK_1(R) = SL(R)/E(R),$$

which is a subgroup of $K_1(R) = GL(R)/E(R)$. Consequently, we obtain a short exact sequence

$$1 \longrightarrow SK_1(R) \longrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1$$

where det denotes the determinant map. This sequence splits via the identification $R^\times = GL_1(R)$ and we have a decomposition

$$K_1(R) \cong R^\times \times SK_1(R).$$

We are interested in $R = \mathbb{Z}C$, the integral group ring of a finite cyclic group. Bass, Milnor and Serre proved in [BMS67] that the group $SK_1(\mathbb{Z}C)$ is trivial in this case. Hence, the determinant map yields an isomorphism

$$K_1(\mathbb{Z}C) \cong (\mathbb{Z}C)^\times.$$

This holds more general for the group ring of a finite cyclic group over the ring of integers in an algebraic number field. For a proof and a complete treatment of K_1 of integral group rings of finite groups the reader is referred to [Oli88].

Next, we want to see what the map on K_1 induced from the inclusion of subgroups of C yields on the corresponding units via the above isomorphism. In general, for a subring S of R the induced map $K_1(S) \to K_1(R)$ is given by considering matrices with entries in S as matrices with entries in R. Thus, for a subgroup D of C the induced inclusion of rings $\mathbb{Z}D \to \mathbb{Z}C$ yields a commutative diagram

$$\begin{array}{ccc} K_1(\mathbb{Z}D) & \xrightarrow{\det} & (\mathbb{Z}D)^\times \\ \downarrow & & \downarrow \\ K_1(\mathbb{Z}C) & \xrightarrow{\det} & (\mathbb{Z}C)^\times \end{array}$$

and we see that the vertical map on the right is just the inclusion of units as a subset. Remember that we are only interested in rational K-theory, so let us summarize what is known about the rank of the group of units of $\mathbb{Z}C$. We consider $\mathbb{Z}C$ as a subring of $\mathbb{Q}C$, for which we have the following decomposition.

Lemma 4.8. *Let C be a finite cyclic group of order $n \in \mathbb{N}$. For a divisor d of n denote by μ_d the set of d-th roots of unity in $\mathbb{C}$. The choice of a generator c of C and a primitive d-th root of unity ζ_d for each divisor d of n defines an isomorphism of $\mathbb{Q}$-algebras*

$$\mathbb{Q}C \to \prod_{d|n} \mathbb{Q}(\mu_d), \quad c \mapsto (\zeta_d)_{d|n}.$$

Proof. Denote by Φ_d the d-th cyclotomic polynomial. We have isomorphisms

$$\mathbb{Q}C \cong \mathbb{Q}[X]/(X^n - 1) \cong \prod_{d|n} \mathbb{Q}[X]/\Phi_d(X) \cong \prod_{d|n} \mathbb{Q}(\mu_d)$$

defined by $c \mapsto X + (X^n - 1) \mapsto (X + \Phi_d(X))_{d|n} \mapsto (\zeta_d)_{d|n}$. The second map is an isomorphism by the chinese remainder theorem and the last isomorphism depends on the choice of a primitive d-th root of unity for each d. □

Restriction of this decomposition to the subring $\mathbb{Z}C$ of $\mathbb{Q}C$ yields an injective map

$$\mathbb{Z}C \to \prod_{d|n} \mathbb{Z}[\mu_d]$$

and Higman showed in [Hig40, Theorem 5] that the rank of $\prod_{d|n} \mathbb{Z}[\mu_d]^\times$ is equal to the rank of $(\mathbb{Z}C)^\times$. As a consequence, we get an isomorphism of $\mathbb{Q}$-vector spaces

$$(\mathbb{Z}C)^\times \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{d|n} \mathbb{Z}[\mu_d]^\times \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since our goal is to determine the Artin defect, we have to understand the maps

$$K_1(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_1(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced from the inclusion of subgroups D of C. Given such a subgroup of order $|D| = d$ and identifying K_1 of the corresponding group rings with the units yields a commutative diagram

$$\begin{array}{ccc} (\mathbb{Z}D)^\times \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & (\mathbb{Z}C)^\times \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{k|d} \mathbb{Z}[\mu_k]^\times \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{i_d^n} & \bigoplus_{k|n} \mathbb{Z}[\mu_k]^\times \otimes_{\mathbb{Z}} \mathbb{Q}. \end{array}$$

and we want to describe the map i_d^n next. We have mentioned that the vertical isomorphisms depend on choices and for the remaining part of this chapter we make the following ones. Fix a generator c of C, then a generator of the subgroup D of C of order d is given by $c^{n/d}$. Further, choose a primitive n-th root of unity ζ_n, then a primitive d-th root of unity is given by $\zeta_n^{n/d}$ for a divisor d of n and we set $\zeta_d = \zeta_n^{n/d}$. With these choices the above isomorphisms are defined by $c \mapsto (\zeta_k)_{k|n}$ and $c^{n/d} \mapsto (\zeta_k)_{k|d}$. All other possible isomorphisms are obtained by composition with Galois automorphisms of cyclotomic fields. For reasons of convenience we abbreviate

$$Z_n = \mathbb{Z}[\mu_n]^\times \otimes_{\mathbb{Z}} \mathbb{Q}$$

for $n \in \mathbb{N}$. Further, for $t \in \mathbb{N}$ with $\gcd(n,t) = 1$ we define

$$f_t \colon Z_n \to Z_n$$

to be the automorphism induced by $\zeta_n \mapsto \zeta_n^t$, which does not depend on our choice of ζ_n. Also, note that for a divisor d of n we have an inclusion

$$\mathrm{inc}_d^n \colon Z_d \to Z_n$$

induced by the inclusion of sets $\mu_d \subseteq \mu_n$. It is sufficient to understand the map i_d^n for $d = n/p$ where p is a prime, because all other such maps can be written as a composition of these.

Lemma 4.9. *Let d be a divisor of n/p with $n \in \mathbb{N}$ and p a prime dividing n. Further, let x_d be an element in the summand Z_d of $\bigoplus_{k|\frac{n}{p}} Z_k$. We have*

$$i_{n/p}^n(x_d) = \begin{cases} \mathrm{inc}_d^{dp}(x_d) \in Z_{dp} & \text{if } p \mid d \\ (f_p(x_d), \mathrm{inc}_d^{dp}(x_d)) \in Z_d \oplus Z_{dp} & \text{if } p \nmid d. \end{cases}$$

Proof. The map $i_{n/p}^n$ is $\mathbb{Q}$-linear and so it is enough to consider an element of the form $x_d = g(\zeta_d) \otimes 1$ in the summand Z_d where g is an integer polynomial such that $g(\zeta_d)$ is a unit. Denote by i the isomorphism from Lemma 4.8. Since the map

$$(\mathbb{Z}D)^\times \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{i \otimes \mathbb{Q}} \bigoplus_{k|\frac{n}{p}} \mathbb{Z}[\mu_k]^\times \otimes_\mathbb{Z} \mathbb{Q}$$

is an isomorphism, there exists an integer m such that $g(\zeta_d)^m$ lies in the image of i and a polynomial $h \in \mathbb{Z}[X]$ such that $h(\zeta_d) = g(\zeta_d)^m$ and $h(\zeta_k) = 1$ for all divisors k of n/p with $k \neq d$. In particular, we have

$$(i \otimes \mathbb{Q})(h(c^p) \otimes m^{-1}) = g(\zeta_d)^m \otimes m^{-1} = x_d$$

where $c^p \in C$ is a generator of the subgroup D of order n/p. We consider $h(c^p) \otimes m^{-1}$ as an element in $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$ using the natural inclusion $(\mathbb{Z}D)^\times \otimes_\mathbb{Z} \mathbb{Q} \subseteq (\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$. It remains to determine its image under the isomorphism $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q} \to \bigoplus_{k|n} \mathbb{Z}[\mu_k]^\times \otimes_\mathbb{Z} \mathbb{Q}$. It is given by $(h(\zeta_k^p))_{k|n} \otimes m^{-1}$ and if $p \mid d$ we have

$$h(\zeta_k^p) = \begin{cases} g(\zeta_d)^m & \text{if } k = dp \\ 1 & \text{else.} \end{cases}$$

On the other hand, if $p \nmid d$ we get

$$h(\zeta_k^p) = \begin{cases} g(\zeta_d)^m & \text{if } k = dp \\ g(\zeta_d^p)^m & \text{if } k = d \\ 1 & \text{else} \end{cases}$$

and this implies the desired result. □

Now that we have understood the map induced by the inclusion on the product decomposition we proceed with constructing a filtered basis of the $\mathbb{Q}$-vector space $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$. In other words, for all $n \in \mathbb{N}$ we will construct a basis E_n of $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$ which contains a basis for each subspace $(\mathbb{Z}D)^\times \otimes_\mathbb{Z} \mathbb{Q}$ where D is a subgroup of C. In order to do that we need some more notation. For a prime p let $\nu_p \colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be the p-adic valuation on $\mathbb{Z}$. Further, for $n \in \mathbb{N}$ and a divisor d of n we define the set

$$T_d^n = \{d \cdot l \mid \gcd(d,l) = 1 \text{ and } l \text{ divides } n\}$$

and the number n_d to be the biggest divisor of n which is coprime to d. Additionally, if S is a subset of the set of all divisors of n, we denote by $(x_s)_{s\in S}$ the element in $\bigoplus_{d|n} Z_d$ with $x_s \in Z_s$ and $x_d = 1$ for all $d \notin S$. Now we can construct the filtered basis.

Theorem 4.10. *Let C be a finite cyclic group of order $n \in \mathbb{N}$. For a divisor d of n denote by B_d the filtered basis of Z_d introduced in the first section of this chapter. We define the set*

$$N_d = \left\{ (f_{\frac{n_d}{k_d}}(v))_{k\in T_d^n} \;\middle|\; v \in B_d \right\} \subseteq \bigoplus_{l|n} Z_l \cong (\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q},$$

then $E_n = \bigcup_{d|n} N_d$ is a filtered basis of $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$.

Proof. We consider E_n a subset of $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$ via the above isomorphism and for a subgroup D of order d we have $E_d \subseteq (\mathbb{Z}D)^\times \otimes_\mathbb{Z} \mathbb{Q} \subseteq (\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$. We want to show that E_d is contained in E_n. It suffices to show that for all n and all primes p dividing n we have $E_{\frac{n}{p}} \subseteq E_n$, then the statement follows inductively. Let $w \in E_{\frac{n}{p}}$ so that $w \in N_k$ for some divisor k of n/p and we can write

$$w = \left(f_{\frac{(n/p)_k}{l_k}}(v) \right)_{l\in T_k^{n/p}}$$

for some $v \in B_k$. We distinguish between the two cases $p \mid k$ and $p \nmid k$. First, assume $p \mid k$. Applying Lemma 4.9 yields

$$i_{n/p}^n(w) = \left(f_{\frac{(n/p)_k}{(l/p)_k}}(v) \right)_{l\in pT_k^{n/p}} = \left(f_{\frac{n_{kp}}{l_{kp}}}(v) \right)_{l\in T_{kp}^n} \in N_{kp} \subseteq E_n$$

where we used $(n/p)_k = n_{kp}$, $(l/p)_k = l_{kp}$ and $T^n_{kp} = \{pm \mid m \in T^{n/p}_k\}$. These equalities hold, because p divides k. Also, note that $i^n_{n/p}(N_k)$ is contained in $N_{kp} \subseteq E_n$ since $B_k \subseteq B_{kp}$. Now assume $p \nmid k$. For $i \in \mathbb{N}$ we define the sets

$$\begin{aligned} T^n_k(p^i) &= \{l \in T^n_k \mid p^i \text{ divides } l\} \\ T^n_k(\overline{p}) &= \{l \in T^n_k \mid p \text{ does not divide } l\}. \end{aligned}$$

and apply once again Lemma 4.9 to get

$$\begin{aligned} i^n_{n/p}(w) &= i^n_{n/p}\left((w_l)_{l \in T^{n/p}_k(p)}\right) + i^n_{n/p}\left((w_l)_{l \in T^{n/p}_k(\overline{p})}\right) \\ &= \left(f_{\frac{(n/p)_k}{(l/p)_k}}(v)\right)_{l \in pT^{n/p}_k(p)} + \left(f_p(f_{\frac{(n/p)_k}{l_k}}(v))\right)_{l \in T^{n/p}_k(\overline{p})} + \left(f_{\frac{(n/p)_k}{(l/p)_k}}(v)\right)_{l \in pT^{n/p}_k(\overline{p})} \\ &= \left(f_{\frac{n_k}{l_k}}(v)\right)_{l \in T^n_k(p^2)} + \left(f_{\frac{n_k}{l_k}}(v)\right)_{l \in T^{n/p}_k(\overline{p})} + \left(f_{\frac{n_k}{l_k}}(v)\right)_{l \in pT^{n/p}_k(\overline{p})} \\ &= \left(f_{\frac{n_k}{l_k}}(v)\right)_{l \in T^n_k} \in N_k \subseteq E_n. \end{aligned}$$

We used the fact that T^n_k is the disjoint union of the sets $T^n_k(p^2)$, $T^{n/p}_k(\overline{p})$ and $pT^{n/p}_k(\overline{p})$ and we also have $(n/p)_k/(l/p)_k = n_k/l_k$ and $p \cdot (n/p)_k/l_k = n_k/l_k$. Further, we see that in this case $i^n_{n/p}(N_k) = N_k \subseteq E_n$ holds and this completes the proof. □

4.3 The Dimension of the Artin Defect

In this section we compute the dimension of the Artin defect in degree 1 as a $\mathbb{Q}$-vector space. Recall that the Artin defect $S_1(n)$ in degree 1 is defined as the cokernel of the induction map

$$\bigoplus_{D \lneq C} K_1(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_1(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where C is a finite cyclic group of order $n \in \mathbb{N}$. Using the results of the previous sections we have

$$S_1(n) \cong \operatorname{coker}\left(\bigoplus_{D \lneq C} (\mathbb{Z}D)^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to (\mathbb{Z}C)^\times \otimes_{\mathbb{Z}} \mathbb{Q}\right)$$

with the obvious inclusions $(\mathbb{Z}D)^\times \otimes_\mathbb{Z} \mathbb{Q} \to (\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$. The advantage of the filtered basis E_n of $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$ we constructed is that a basis of the above cokernel is easily given by $E_n \setminus \bigcup_{d|n, d\neq n} E_d$. Let us describe this basis precisely.

Lemma 4.11. *For $n \in \mathbb{N}$ let B_n be the previously introduced filtered basis of $Z[\mu_n]^\times \otimes_\mathbb{Z} \mathbb{Q}$ and E_n the filtered basis of $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$. There is a bijection of sets*

$$E_n \setminus \bigcup_{d|n, d\neq n} E_d \cong B_n.$$

Proof. Since E_n is a filtered basis we have $\bigcup_{d|n,d\neq n} E_d = \bigcup_{p|n} E_{\frac{n}{p}}$ where p is prime. In the proof of Theorem 4.10 we noted that for a prime p dividing n and a divisor l of n/p the inclusion

$$\bigcup_{d|\frac{n}{p}} N_d = E_{\frac{n}{p}} \subseteq E_n = \bigcup_{k|n} N_k$$

is given by

$$i^n_{n/p}(N_l) = \begin{cases} N_l \subseteq N_{lp} & \text{if } p \mid l \\ N_l & \text{if } p \nmid l. \end{cases}$$

We define the set $F_n = \{d \mid n : \nu_p(d) \geq 1 \ \forall \text{ primes } p \mid n\}$ and write

$$E_n \setminus \bigcup_{p|n} E_{\frac{n}{p}} = \bigcup_{d\in F_n} \left(N_d \setminus \bigcup_{p^2|d} N_{\frac{d}{p}} \right) = \bigcup_{d\in F_n} \left(B_d \setminus \bigcup_{p^2|d} B_{\frac{d}{p}} \right),$$

where the union over all $d \in F_n$ is a disjoint union, because the corresponding basis B_d is contained in the summand Z_d. The last equality is a direct consequence of the definition of the sets N_d, because for $d \in F_n$ we have $N_d = B_d \subseteq Z_d$. Denote by M the right hand side of the above equalities. It is obviously a subset of the disjoint union $\bigcup_{d\in F_n} B_d$ and since every set B_d is contained in B_n we can define the map $M \to B_n$, $(x, d) \mapsto x$ and we will show that this is a bijection. We start with injectivity. Assume the map is not injective, then there exists an $x \in B_n$ such that for $d, d' \in F_n$ with $d \neq d'$ the element x is contained in the intersection $(B_d \setminus \bigcup_{p^2|d} B_{d/p}) \cap (B_{d'} \setminus \bigcup_{p^2|d'} B_{d'/p})$. It follows that $x \in B_d \cap B_{d'} = B_{\gcd(d,d')}$, since the basis is filtered. Now either $\gcd(d, d') < d$ or $\gcd(d, d') < d'$ and hence there exists a prime p such that $x \in B_{d/p}$ with $p^2 \mid d$ or $x \in B_{d'/p}$ with $p^2 \mid d'$. In both cases, x cannot lie in the above intersection and this is a contradiction. Last, we show surjectivity. Let $x \in B_n$ and choose the smallest $d \in F_n$ such that $x \in B_d$, then x lies in $B_d \setminus \bigcup_{p^2|d} B_{\frac{d}{p}}$ and (x, d) is mapped to x. □

As a consequence we get a description of the Artin defect in degree 1 as a $\mathbb{Q}$-vector space. We can express the dimension using Euler's phi function.

Theorem 4.12. *Let C be a finite cyclic group of order $n \in \mathbb{N}$ and denote by $S_1(n)$ the cokernel the induction map*

$$\bigoplus_{D \lneq C} K_1(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_1(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

There is an isomorphism of $\mathbb{Q}$-vector spaces

$$S_1(n) \cong \mathbb{Z}[\mu_n]^\times \otimes_{\mathbb{Z}} \mathbb{Q}$$

and in particular

$$\dim_{\mathbb{Q}}(S_1(n)) = \begin{cases} \frac{\varphi(n)}{2} - 1 & \textit{if } n > 2 \\ 0 & \textit{if } n \in \{1, 2\} \end{cases}$$

Proof. The isomorphism as $\mathbb{Q}$-vector spaces is defined by the bijection of bases from Lemma 4.11. The dimension is given by Dirichlet's unit theorem, which states that the ring of integers in an algebraic number field F has rank $r+s-1$, where r is the number of embeddings $F \to \mathbb{R}$ and s is the number of pairs of embeddings $F \to \mathbb{C}$ which are not already contained in $\mathbb{R}$ (see [Neu99] for a proof). Obviously, for $n = 1, 2$ there is exactly one real embedding of $\mathbb{Q}(\mu_n)$ and no complex ones. For $n > 2$ there are exactly $\varphi(n)$ complex embeddings, one for each primitive n-th root of unity and no real ones. This yields the desired dimension formula. □

4.4 The Action of the Automorphism Group

In this last section we describe the action of the automorphism group of a finite cyclic group on the Artin defect in degree 1. The induced $\mathrm{Aut}(C)$-action on $K_1(\mathbb{Z}C) \cong (\mathbb{Z}C)^\times$ is given by restriction of the induced action on $\mathbb{Z}C$ to the units of the group ring. Since an automorphism of a cyclic group restricts to an automorphism of each subgroup D of C we also get an action on the Artin defect. Further, if n is the order of C, then there is an isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^\times \to \mathrm{Aut}(C), \quad t \mapsto \phi_t$$

where ϕ_t denotes the automorphism $c \mapsto c^t$ for $c \in C$. Thus, we can think of the automorphisms of C as elements indexed over the units of the ring $\mathbb{Z}/n\mathbb{Z}$. Note that the induced action on each summand of the decomposition

$$(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q} \cong \bigoplus_{d|n} \mathbb{Z}[\mu_d]^\times \otimes_\mathbb{Z} \mathbb{Q}$$

is given by $\phi_t(\zeta_d) = \zeta_d^t$. Working with a filtered basis of $\mathbb{Z}[\mu_n]^\times \otimes_\mathbb{Z} \mathbb{Q}$ was very helpful in previous computations, but it is not apparent where the basis elements are mapped under the above automorphisms. Therefore, in this section we will use a different basis for which we can better understand the $\mathrm{Aut}(C)$-action. More precisely, we will prove that there exist $\mathrm{Aut}(C)$-sets T, S and a short exact sequence

$$0 \longrightarrow \mathbb{Q}[T] \longrightarrow \mathbb{Q}[S] \longrightarrow S_1(n) \longrightarrow 0$$

so that the Artin defect is the quotient of permutation modules. The basis we introduce is well-known and for more details see [Was97].

Theorem 4.13. *Let $n \in \mathbb{N}$ with prime factorization $n = \prod_{i=1}^s p_i^{e_i}$, ζ_n a primitive n-th root of unity and $a \in \mathbb{Z}$. Define*

$$\xi_a = \prod_I \frac{1-\zeta_n^{an_I}}{1-\zeta_n^{n_I}}$$

where I runs over all subsets of $\{1, \dots, s\}$ except $\{1, \dots, s\}$ and $n_i = \prod_{i \in I} p_i^{e_i}$. The set $D_n = \{\xi_a \mid 1 < a < \frac{n}{2}, \gcd(a,n) = 1\}$ is a $\mathbb{Q}$-basis for $\mathbb{Z}[\mu_n]^\times \otimes_\mathbb{Z} \mathbb{Q}$.

The advantage of this particular basis is that we can understand where an element is mapped under an automorphism as the next lemma shows.

Lemma 4.14. *Let $\phi_t \in \mathrm{Aut}(C)$ and $a \in \mathbb{Z}$, then*

(1) $\xi_1 = 1$

(2) $\xi_a = \xi_{-a}$

(3) $\phi_t(\xi_a) = \xi_{at} \cdot \xi_t^{-1}$.

Proof. The first statement follows directly from the definition. For the second one we use the fact that elements of finite order are equal to 1 in $\mathbb{Z}[\mu_n]^\times \otimes_\mathbb{Z} \mathbb{Q}$ and hence

$$1 - \zeta^{-an_I} = -\zeta^{-an_I}(1 - \zeta^{an_I}) = (1 - \zeta^{an_I}).$$

Finally, for the last statement we compute

$$\phi_t \cdot \xi_a = \prod_I \frac{1 - \zeta^{tan_I}}{1 - \zeta^{tn_I}} = \prod_I \frac{1 - \zeta^{tan_I}}{1 - \zeta^{n_I}} \cdot \frac{1 - \zeta^{n_I}}{1 - \zeta^{tn_I}} = \xi_{at} \cdot \xi_t^{-1}.$$

□

In particular, we have $\phi_{-1}(\xi_a) = \xi_a$ for any $\xi_a \in D_n$ and so the automorphism ϕ_{-1} fixes all basis elements. We have seen in the previous section that the Artin defect is isomorphic to the rationalized group of units of $\mathbb{Z}[\mu_n]$. Therefore, our strategy is to understand $Z_n = \mathbb{Z}[\mu_n]^\times \otimes_\mathbb{Z} \mathbb{Q}$ as an $\mathbb{Q}\operatorname{Aut}(C)$-module first and then show that the same holds for the defect. But first, let us recall some facts about the units of $\mathbb{Z}/n$, the ring of integers modulo some $n \in \mathbb{N}$. Let n have prime factorization $n = \prod_{p|n} p^{e_p}$, then there is a decomposition

$$(\mathbb{Z}/n)^\times \cong \prod_{p|n} (\mathbb{Z}/p^{e_p})^\times,$$

where the groups $(\mathbb{Z}/p^{e_p})^\times$ are cyclic of order $\varphi(p^{e_p})$, except if $p^{e_p} = 2^k$ with $k \geq 3$. In the latter case we have $(\mathbb{Z}/2^k)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{k-2}$ and we can choose $-1 \bmod 2^k$ and $5 \bmod 2^k$ as generators of the factors $\mathbb{Z}/2$ and $\mathbb{Z}/2^{k-2}$, respectively. For a divisor d of n denote by C_d the unique subgroup of C order d. We have mentioned before that an automorphism of C_n restricts to an automorphism of C_d. This defines a surjective group homomorphism $\operatorname{Aut}(C_n) \to \operatorname{Aut}(C_d)$. Hence, $\operatorname{Aut}(C_d)$ becomes a transitive $\operatorname{Aut}(C_n)$-set. If d satisfies $\nu_p(d) = \nu_p(n)$ for all primes dividing d, then $\operatorname{Aut}(C_d)$ can be considered in a canonical way as a subgroup of $\operatorname{Aut}(C_n)$ and there is an $\operatorname{Aut}(C_n)$-equivariant group isomorphism $\operatorname{Aut}(C_n)/\operatorname{Aut}(C_{n/d}) \cong \operatorname{Aut}(C_d)$. Next, we construct the short exact sequence announced before.

Lemma 4.15. *Let $n \in \mathbb{N}$ with prime factorization $n = \prod_{i=1}^s p_i^{e_i}$ and let C be a finite cyclic group of order n. Let I be the subgroup of $\operatorname{Aut}(C)$ generated by the automorphism $c \mapsto c^{-1}$ for $c \in C$. Further, for $i = 1, \ldots, s$ let g_i be a generator of the maximal cyclic subgroup of $(\mathbb{Z}/p_i^{e_i})^\times$ and define $n_i = \prod_{j=i}^s p_j^{e_j}$.*

The $\mathbb{Q}$-linear map

$$\pi\colon \bigoplus_{i=1}^{s} \mathbb{Q}[\mathrm{Aut}(C_{n_i})/I] \longrightarrow Z_n, \quad [\phi_{t_i}] \mapsto \phi_{t_i}(\xi_{g_i}) \otimes 1, [\phi_{t_i}] \in \mathrm{Aut}(C_{n_i})/I$$

is an $\mathrm{Aut}(C)$*-equivariant surjective map and there is an isomorphism of* $\mathbb{Q}\,\mathrm{Aut}(C)$*-modules*

$$\ker(\pi) \cong \mathbb{Q} \oplus \bigoplus_{i=2}^{s} \mathbb{Q}[\mathrm{Aut}(C_{n_i})/I].$$

Proof. Without loss of generality we can assume $p_1 < \ldots < p_s$. The map π is well-defined, because statements 2 and 3 from the previous lemma ensure that $\phi_{t_i}(\xi_{g_i}) = \phi_{-t_i}(\xi_{g_i})$. If $n \in \{1, 2\}$, then π is the map $\mathbb{Q} \to 0$, which obviously has kernel isomorphic to $\mathbb{Q}$ and so we can assume $n > 2$. We will show that every $\xi_a \in D_n$ lies in the image of π, where D_n is the basis of Z_n introduced previously. Let $1 < a < n/2$ with $\gcd(a, n) = 1$, then for $i = 1, \ldots, s$ there exist $r_i \in \mathbb{N}_0$, such that either a or $-a$ is congruent $g_1^{r_1} \cdot \ldots \cdot g_s^{r_s} \bmod n$, because the g_i's generate $(\mathbb{Z}/n)^\times \cong \prod_{i=1}^{s} (\mathbb{Z}/p_i^{e_i})^\times$ modulo the subgroup generated by $-1 \bmod n$. Since $\xi_a = \xi_{-a}$ it is sufficient to show that on of these lies in the image. For fixed $i \in \{1, \ldots, s\}$ we set $\overline{g}_i = g_i^{r_i} \cdot \ldots \cdot g_s^{r_s}$ and $\overline{g}_{s+1} = 1$, then π maps the element

$$x_i = \sum_{l=0}^{r_i - 1} \phi_{g_i{}^l \overline{g}_{i+1}} \in \mathbb{Q}[\mathrm{Aut}(C_{n_i})/I]$$

to

$$\prod_{l=0}^{r_i-1} \phi_{g_i{}^l \overline{g}_{i+1}}(\xi_{g_i}) = \prod_{l=0}^{r_i-1} \xi_{g_i{}^{l+1} \overline{g}_{i+1}} \cdot \xi^{-1}_{g_i{}^l \overline{g}_{i+1}} = \xi_{g_i^{r_i} \overline{g}_{i+1}} \cdot \xi^{-1}_{\overline{g}_{i+1}} = \xi_{\overline{g}_i} \cdot \xi^{-1}_{\overline{g}_{i+1}}$$

and so $(x_i)_{i=1,\ldots,s}$ is mapped to

$$\prod_{i=1}^{s} \xi_{\overline{g}_i} \cdot \xi^{-1}_{\overline{g}_{i+1}} = \xi_{\overline{g}_1} = \xi_a$$

and this shows surjectivity. Now let us determine the kernel of π. Again, fix $i \in \{1, \ldots, s\}$ and let $t \in (\mathbb{Z}/n_{i+1})^\times \leq (\mathbb{Z}/n_i)^\times$. Denote by m_i the multiplicative order of g_i modulo n, then the element

$$x_{i,t} = \sum_{l=0}^{m_i - 1} \phi_{g_i^l t} \in \mathbb{Q}[\mathrm{Aut}(C_{n_i})/I]$$

is mapped to

$$\prod_{l=0}^{m_i-1} \phi_{g_i^l t}(\xi_{g_i}) = \prod_{l=0}^{m_i-1} \xi_{g_i^{l+1}t} \cdot \xi_{g_i{}^l t}^{-1} = 1$$

and therefore lies in the kernel. We define the set

$$T_i = \{x_{i,t} \mid t \in (\mathbb{Z}/n_i)^\times\} \subseteq \mathbb{Q}[\mathrm{Aut}(C_{n_i})/I]$$

and note that $x_{i,t} = x_{i,t'}$, if and only if $t' = \pm g_i^k t$ for some $k \in \mathbb{N}_0$, which means that t and t' are congruent modulo n_{i+1}. We get an isomorphism of $\mathrm{Aut}(C_n)$-sets $T_i \cong \mathrm{Aut}(C_{n_{i+1}})/I$ defined by $x_{i,t} \mapsto [\phi_t]$, where we set $n_{s+1} = 1$. Additionally, the set T_i is linear independent, because cosets are disjoint. The space $K_i = \langle T_i \rangle_{\mathbb{Q}}$ lies in the kernel of π and $\dim_{\mathbb{Q}}(K_i) = |T_i| = \varphi(n_{i+1})/2$, if $i < s$ and $\dim_{\mathbb{Q}}(K_s) = 1$. Consequently, we have

$$\bigoplus_{i=1}^{s} K_i \subseteq \ker(\pi)$$

and $\dim_{\mathbb{Q}}(\bigoplus_{i=1}^{s} K_i) = 1 + \sum_{i=1}^{s-1} \varphi(n_{i+1})/2$. On the other hand, since π is surjective we have $\dim_{\mathbb{Q}}(\ker(\pi)) = (\sum_{i=1}^{s} \varphi(n_i)/2) - (\varphi(n)/2 - 1) = 1 + \sum_{i=2}^{s} \varphi(n_i)/2$ and so equality must hold. In summary, we have

$$\ker(\pi) \cong \bigoplus_{i=1}^{s} \mathbb{Q}[\mathrm{Aut}(C_{n_{i+1}})/I].$$

□

We will use this result describe the Artin Defect in degree 1 as a $\mathbb{Q}\,\mathrm{Aut}(C)$-module.

Theorem 4.16. *Let C be a cyclic group of order $n \in \mathbb{N}$ and denote by $S_1(n)$ the Artin defect in degree 1. As an element in $K_0(\mathbb{Q}\,\mathrm{Aut}(C))$ we have*

$$[S_1(n)] = [\mathbb{Q}[\mathrm{Aut}(C)/I]] - [\mathbb{Q}],$$

where $\mathbb{Q}$ is the trivial $\mathbb{Q}\,\mathrm{Aut}(C)$-module and I is the subgroup generated by the automorphism $c \mapsto c^{-1}$ for $c \in C$.

Proof. We will show that the isomorphism from Theorem 4.12 is equivariant with respect to the $\mathrm{Aut}(C)$-action and apply Lemma 4.15. Using the definitions introduced previously, we recall that the isomorphism was defined by the bijection of bases

$$\bigcup_{d \in F_n} \left(B_d \setminus \bigcup_{p^2 \mid d} B_{\frac{d}{p}} \right) \to B_n, \quad (x, d) \mapsto x.$$

We remarked in the beginning of this section that the action on the product decomposition of $(\mathbb{Z}C)^\times \otimes_\mathbb{Z} \mathbb{Q}$ is given by $\phi_t \cdot (x, d) = (\phi_t \cdot x, d)$ for an automorphism ϕ_t and so the above projection is indeed $\mathrm{Aut}(C)$-equivariant. Finally, applying the previous lemma yields

$$\begin{aligned} [S_1(n)] &= \left[\bigoplus_{i=1}^{s} \mathbb{Q}[\mathrm{Aut}(C_{n_i})/I] \right] - \left[\mathbb{Q} \oplus \bigoplus_{i=2}^{s} \mathbb{Q}[\mathrm{Aut}(C_{n_i})/I] \right] \\ &= [\mathbb{Q}[\mathrm{Aut}(C)/I]] - [\mathbb{Q}] \end{aligned}$$

as an element in $K_0(\mathbb{Q}\,\mathrm{Aut}(C))$. □

5 The Artin Defect in Degrees $m > 1$

In this last chapter we treat the Artin defect in degrees greater than 1. As before, our goal is to determine its dimension as a $\mathbb{Q}$-vector space and describe the action of the automorphism group of the corresponding finite cyclic group. The ranks of the groups $K_m(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Q}$ where $\mathcal{O}$ is the ring of integers of an algebraic number field have been computed by Borel in [Bor74]. His methods can be used to get similar results for $K_m(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ for any finite group G as described in [Jah09].

In the first five sections of this chapter we establish the necessary preliminaries in order to be able to describe the computation of the ranks of higher rational K-groups of the group ring $\mathbb{Z}C$ where C is a finite cyclic group. Afterwards, we describe the decomposition of the real algebra $\mathbb{R}C$ and induced maps by subgroups of C, which will play an important part in describing the higher K-groups we are interested in as functors from the subgroup category of C in a more accessible way. This is done in sections six and seven. This will enable us to compute the dimension of the Artin defect as a $\mathbb{Q}$-vector space and determine its structure as a $\mathbb{Q}\,\mathrm{Aut}(C)$-module in the last section. We assume that the reader is familiar with the definition of higher K-Theory which can be found for example in [Ros94].

5.1 Classifying Spaces and Fibrations

In this section we recall the definition of classifying spaces and fibrations and show that a short exact sequence of groups induces a homotopy fibration of the associated classifying spaces. Further, we state a condition under which a fibration remains a fibration after applying the plus construction. We will assume basic knowledge in topology as can be found for example in [Ros94] in the context of higher K-theory, although we will state some definitions explicitly.

Definition 5.1. *Let G be a group and X a contractible Hausdorff topological space such that G acts freely and properly discontinuously by homeomorphisms on X and such that X/G is paracompact. We write EG for X and call $BG = X/G$ a classifying space of G.*

One can show that a classifying space always exists and that it is well defined up to homotopy equivalence (see [Ros94, Theorem 5.1.15]). Therefore, we will sometimes say that a space is a model of BG. Also, there is a construction by Eilenberg and Mac Lane such that EG and BG are CW-complexes (see [EML86, p. 369]) and such that B defines a functor from the category of groups and group homomorphisms to the category of topological spaces and homotopy classes of continuous maps. Throughout this section we will always use this particular construction when we refer to classifying spaces. We proceed with the definition of a fibration.

Definition 5.2. *A fibration is a continuous map of topological spaces $p\colon E \to B$ which satisfies the homotopy lifting property. This means that for any space X, any homotopy $h\colon X \times [0,1] \to B$ and any continuous map $H_0\colon X \to E$ such that $h_0 = p \circ H_0$, there is a continuous map $H\colon X \times [0,1] \to E$ with $H(x,0) = H_0(x)$ and $p \circ H = h$. For $b \in B$, $p^{-1}(b)$ is called the fibre of p over b, B is called the base space and E the total space of the fibration.*

Next, we will see that a short exact sequence of groups induces a fibration of the associated classifying spaces up to homotopy.

Lemma 5.3. *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} G/N \longrightarrow 1$$

be an exact sequence of groups, where N is a normal subgroup of G and i is the inclusion map. Then, up to homotopy, there is a fibration of classifying spaces $BG \to B(G/N)$ with fibre BN.

Proof. Since $E(G/N)$ is contractible by definition, the space $EG \times_G E(G/N)$ is a model for BG and the map

$$EG \times_G E(G/N) \longrightarrow E(G/N)/(G/N) = B(G/N)$$

induced from $EG \to pt$ is a model for Bp. We take EG/N as a model for BN and use $EG/G = (EG/N)/(G/N)$ to identify the model for Bp with the fibre bundle

$$BN \times_{G/N} E(G/N) \longrightarrow B(G/N),$$

which has homotopy fibre BN. □

We would also like to know when a fibrations remains a fibration after applying the plus construction. Given a connected CW-complex X with a fixed base point x_0, the plus construction is a way to construct a new CW-complex X^+ containing X and having the same homology and cohomology groups, but changing the fundamental group to $\pi_1(X^+, x_0) = \pi_1(X, x_0)/\pi$ where π is a perfect maximal subgroup of $\pi_1(X, x_0)$. For all details see [Ros94, Theorem 5.2.2]. In [Ber83] Berrick proves the following condition under which the plus construction respects fibrations.

Theorem 5.4. *Let*

$$F \longrightarrow E \longrightarrow B$$

be a fibration, where F denotes the fibre, E the total space and B the base space. Then $F^+ \longrightarrow E^+ \longrightarrow B^+$ is again a fibration, if and only if $\mathcal{P}\pi_1(B)$ acts on F^+ by maps homotopic to the identity. Here, $\mathcal{P}\pi_1(B)$ denotes the maximal perfect subgroup of $\pi_1(B)$ and its action on F^+ is induced by the one on F.

5.2 H-Spaces, Hopf Algebras and the Hurewicz Map

In this section we mention some basic facts about H-spaces and Hopf algebras. We start with the definitions, which can also be found in [Hat01] together with a broader discussion of these topics.

Definition 5.5. *An H-space is a topological space X together with a continuous map $\mu : X \times X \to X$ and an identity element $e \in X$ such that the two maps $X \to X$ given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity.*

One specific H-space we will be particularly interested in is the following. For more details see [Cor11].

Example 5.6. Let R be a ring with unit and denote by $GL(R) = \cup_{n\in\mathbb{N}} GL_n(R)$ the infinite general linear group of R. We want to see that $BGL(R)^+$ is an H-space. First, there is a sum operation

$$\oplus\colon GL(R) \times GL(R) \to GL(R)$$

defined by

$$M \oplus N = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

for $M, N \in GL(R)$. Up to conjugation with permutation matrices the sum is associative, commutative and the identity matrix is an identity element. Further, the projections of $GL(R) \times GL(R)$ onto the first and second factor induce a homotopy equivalence

$$\phi\colon B(GL(R) \times GL(R))^+ \to BGL(R)^+ \times BGL(R)^+$$

and using this we can define a continuous map

$$\mu\colon BGL(R)^+ \times BGL(R)^+ \to BGL(R)^+$$

by $\mu = BGL(\oplus)^+ \circ \phi^{-1}$ where ϕ^{-1} is the homotopy inverse of ϕ. One can show that conjugation with permutation matrices induces a map on $BGL(R)^+$ which is homotopic to the identity. It follows that $BGL(R)^+$ is a commutative and associative H-space. The same argument can be used to show that $BSL(R)$ is also an H-space.

Having this kind of multiplication on a space X gives additional structure to its cohomology. We will see below that if X satisfies some additional conditions its cohomology becomes a Hopf algebra. But first, let us define what a Hopf algebra is.

Definition 5.7. *A graded algebra $A = \oplus_{n\geq 0} A^n$ over a commutative ring R is called a Hopf algebra, if*

(i) there is an identity $1 \in A^0$ such that the map $R \to A^0$, $r \mapsto r \cdot 1$ is an isomorphism,

(ii) there is a coproduct $\Delta\colon A \to A \otimes_R A$ which is a homomorphism of graded algebras satisfying $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_i \alpha_i' \otimes \alpha_i''$ where $|\alpha_i'| > 0$ and $|\alpha_i''| > 0$ for all α with $|\alpha| > 0$. The multiplication on $A \otimes_{\mathbb{R}} A$ is given by $(\alpha \otimes \beta)(\gamma \otimes \delta) = (-1)^{|\beta||\gamma|}(\alpha\gamma \otimes \beta\delta)$.

Further, set $Q(A) = \oplus_{n\geq 1} A^n$ and define the primitive and indecomposable elements of a Hopf algebra as

$$P(A) = \{\alpha \in A \mid \Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha\},$$
$$I(A) = Q(A)/(Q(A))^2$$

respectively. Note that the primitives are a subspace of A, whereas the indecomposables are a quotient.

One can show that if A^n is a finitely generated projective R-module for all n, then $A^* = \mathrm{Hom}_R(A, R)$ is also a Hopf algebra called the dual Hopf algebra of A. We have the following connection between primitives and indecomposables.

Lemma 5.8. *Let $A = \oplus_{n\geq 0} A^n$ be a Hopf algebra such that A^n is a finitely generated free R-module for all n, then $P(A^*) \cong I(A)^*$.*

Again, we consider an example, which is of particular interest to us.

Example 5.9. For an H-space X the singular homology $H_*(X;\mathbb{R})$ is a Hopf algebra. If the homology groups are finite dimensional, then its dual Hopf algebra is $H^*(X;\mathbb{R})$. The coproduct is induced from the diagonal map $X \to X \times X$. Hence, the primitive elements of $H_m(X;\mathbb{R})$ are given by

$$PH_m(X;\mathbb{R}) = \{x \in H_m(X;\mathbb{R}) \mid \Delta_*(x) = x \otimes 1 + 1 \otimes x\}$$

where

$$\Delta_*\colon H_m(X;\mathbb{R}) \to H_m(X \times X;\mathbb{R}) \cong \bigoplus_{s+t=m} H_s(X;\mathbb{R}) \otimes_{\mathbb{R}} H_t(X;\mathbb{R})$$

is the map induced from the diagonal map. The space of indecomposables of $H^m(X;\mathbb{R})$ is the quotient

$$IH^m(X;\mathbb{R}) = H^m(X;\mathbb{R})/\{x \smile y \mid \deg(x), \deg(y) > 0\}.$$

If X is path-connected and $H^n(X;\mathbb{R})$ is finite dimensional for each n, we have $PH_m(X;\mathbb{R}) \cong (IH^m(X;\mathbb{R}))^*$ according to Lemma 5.8 where $V^* = \mathrm{hom}_{\mathbb{R}}(V, \mathbb{R})$ is the dual vector space.

One tool to examine the relationship between homotopy and homology groups is the Hurewicz homomorphism, which connects these two objects.

Definition 5.10. *Let X be a topological space and $x_0 \in X$. For $n \in \mathbb{N}$ denote by S^n the n-sphere and let x_n be a generator of $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. The Hurewicz map is the group homomorphism*

$$h\colon \pi_n(X, x_0) \to H_n(X;\mathbb{Z}), \quad f \mapsto f_*(x_n)$$

where $f_*\colon H_n(S^n;\mathbb{Z}) \to H_n(X;\mathbb{Z})$ denotes the induced map on singular homology.

The Hurewicz map is natural with respect to inclusion of spaces. We are interested in the precise image, which is given by the following Theorem. For a proof see [MM65, Appendix].

Theorem 5.11. *Let X be an H-space which is also a CW-complex. The Hurewicz map defines an isomorphism $\pi_m(X) \otimes_{\mathbb{Z}} \mathbb{R} \to PH_m(X;\mathbb{R})$.*

5.3 Lie Algebra Cohomology

In this section we recall the definition of Lie algebra cohomology and explain its connection to singular cohomology. Lie algebra cohomology is motivated by considering the de Rham cohomology of compact connected Lie groups, because in this case these two cohomology theories coincide and that is where Lie algebra cohomology originates from.

We assume that the reader is familiar with the very basic definitions of Lie group theory. During this section $\mathfrak{g}$ denotes a real Lie algebra and $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ the dual vector space. We will define Lie algebra cohomology of $\mathfrak{g}$ with coefficients in $\mathbb{R}$. There is a definition using more general coefficients, but for our purposes using the field of real numbers will suffice.

Definition 5.12. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. For $n \in \mathbb{N}$ define cochains*

$$C^n(\mathfrak{g}) = \Lambda^n \mathfrak{g}^*$$

where $\Lambda^n \mathfrak{g}^*$ denotes the n-th exterior power of $\mathfrak{g}^*$. Further, define coboundary maps

$$d^n \colon \Lambda^n \mathfrak{g}^* \to \Lambda^{n+1} \mathfrak{g}^*$$

by

$$d^n \omega(X_0, \ldots, X_n) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n).$$

One checks that $d^{n+1} \circ d^n = 0$ and so $(C^*(\mathfrak{g}), d)$ is a differential graded algebra. The Lie algebra cohomology of $\mathfrak{g}$ is defined as the cohomology of the complex $(C^*(\mathfrak{g}), d^*)$ and denoted by $H^*(\mathfrak{g})$.

Note that $\Lambda^n \mathfrak{g}^*$ can be identified with $\mathrm{Alt}^n(\mathfrak{g}; \mathbb{R})$, the space of alternating multilinear forms of n variables on $\mathfrak{g}$. To return to our comment in the beginning of this section let us see how this is connected to de Rham cohomology. Let M be a smooth manifold and for $n \in \mathbb{N}_0$ denote by $\Omega^n(M)$ the space of real valued n-forms on M. The de Rham cohomology of M is defined as the cohomology of the complex

$$0 \to \mathbb{R} \to \Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \dots$$

with the exterior differential as boundary map. Given a compact connected Lie group G which acts smoothly on M integration on G can be used to show that the inclusion of the G-invariant forms $\Omega^n(M)^G \subseteq \Omega^n(M)$ induces an isomorphism in cohomology. Hence, as graded algebras the de Rham cohomology is isomorphic to the cohomology of the complex

$$0 \to \mathbb{R} \to \Omega^0(M)^G \to \Omega^1(M)^G \to \Omega^2(M)^G \to \dots$$

of G-invariant forms (see [FOT08, Theorem 1.28]). The action of G on $\Omega^n(M)$ is induced from the action on M. In the special case $M = G$ we have an action of G on itself by left multiplication. Let $e \in G$ be the identity, then the restriction of differentials to the identity given by $\Omega^n(G)^G \to \Lambda^n\mathfrak{g}^*$, $\omega \mapsto \omega_e$ defines an isomorphism of chain complexes yielding the same cohomology.

Given a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra, there is also a relative version of Lie algebra cohomology. Before making the definition we have to introduce actions of Lie algebras.

Definition 5.13. *Let $\mathfrak{g}$ be a Lie algebra and V a vector space over $\mathbb{R}$. A Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(V)$ is called an action of $\mathfrak{g}$ on V. We also say that $\mathfrak{g}$ acts on V. The space $\mathrm{End}(V)$ of endomorphisms of V is a Lie algebra via $[f, g] = f \circ g - g \circ f$.*

We consider the following example of a Lie algebra action. Given a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we define the following action of $\mathfrak{h}$ on $\Lambda^n(\mathfrak{g}/\mathfrak{h})^*$. First, $\mathfrak{h}$ acts on the vector space $\mathfrak{g}/\mathfrak{h}$ by the adjoint action

$$(X, Y + \mathfrak{h}) \mapsto \mathrm{ad}_X(Y) + \mathfrak{h} = [X, Y] + \mathfrak{h}$$

for $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$. It is well-defined, because $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. This action can be extended to an action on $\Lambda^n(\mathfrak{g}/\mathfrak{h})^* = \mathrm{Alt}^n(\mathfrak{g}/\mathfrak{h}; \mathbb{R})$ by setting

$$X \cdot \omega(X_1 + \mathfrak{h}, \dots, X_n + \mathfrak{h}) = \sum_{i=1}^{n} \omega(X_1 + \mathfrak{h}, \dots, [X_i, X] + \mathfrak{h}, \dots, X_n + \mathfrak{h})$$

for $X \in \mathfrak{h}$. This action is used in the definition of relative Lie algebra cohomology.

Definition 5.14. *For $n \in \mathbb{N}$ define cochains*

$$C^n(\mathfrak{g}, \mathfrak{h}) = (\Lambda^n(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} = \{\omega \in \Lambda^n(\mathfrak{g}/\mathfrak{h}) \mid X \cdot \omega = 0 \; \forall \; X \in \mathfrak{h}\}.$$

The projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ induces an inclusion $C^n(\mathfrak{g}, \mathfrak{h}) \subseteq C^n(\mathfrak{g})$ and the previously defined coboundary maps d^n respect the action of $\mathfrak{h}$, so the complex $(C^*(\mathfrak{g}, \mathfrak{h}), d^*)$ is a subcomplex of $(C^*(\mathfrak{g}), d^*)$. We define the relative Lie algebra cohomology of a pair $(\mathfrak{g}, \mathfrak{h})$ to be the cohomology of the complex $(C^*(\mathfrak{g}, \mathfrak{h}), d^*)$ and denote it by $H^*(\mathfrak{g}, \mathfrak{h})$.

The following theorem by Chevalley and Eilenberg (see [CE48]) connects relative Lie algebra cohomology and singular cohomology.

Theorem 5.15. *Let G be a compact connected Lie Group and $K \subseteq G$ a connected Lie subgroup with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. There is an isomorphism of graded algebras*

$$H^*(\mathfrak{g}, \mathfrak{k}) \cong H^*(G/K; \mathbb{R}).$$

Proof. We will give a sketch of the proof with enough details for our purposes. Note that since G is a Lie group and K is a Lie subgroup the space G/K is a smooth manifold (see [BtD85, p.33]). It is known that for smooth manifolds the cohomology of the de Rham complex

$$0 \to \mathbb{R} \to \Omega^0(G/K) \to \Omega^1(G/K) \to \Omega^2(G/K) \to \dots$$

is naturally isomorphic to singular cohomology (see [Lee03, Theorem 11.34]). The group G acts on G/K by left multiplication $xK \mapsto gxK$ for $x, g \in G$ and thus induces an action on $\Omega^n(G/K)$. Since G is compact and connected the de Rham cohomology can be computed using G-invariant forms only. Last, restricting a differential to the origin defines an isomorphism

$$\Omega^n(G/K)^G \cong (\Lambda^n(\mathfrak{g}/\mathfrak{k})^*)^{\mathfrak{k}}$$

which induces an isomorphism of complexes. Note that we have to take the forms which are also invariant under the $\mathfrak{k}$ action, because although the action of G restricted to the subgroup K fixes the identity $eK \in G/K$, it does not induce the identity on the tangent space $\mathfrak{g}/\mathfrak{k}$. $\square$

If G is not compact there is still a way to compute relative Lie algebra cohomology. One has to replace G with some compact group G_u without changing the Lie algebra cohomology. This is often referred to as the unitarian trick. In the remaining part of this section we will describe how to find the compact twin. In order to do that we have to introduce Cartan involutions first.

Definition 5.16. *Let $\mathfrak{g}$ be a real Lie algebra and denote by $B(-,-)$ the Killing form on $\mathfrak{g}$. An involution on $\mathfrak{g}$ is a Lie algebra automorphism $\theta\colon \mathfrak{g} \to \mathfrak{g}$ such that $\theta^2 = \mathrm{Id}_{\mathfrak{g}}$. Further, an involution $\theta\colon \mathfrak{g} \to \mathfrak{g}$ is called a Cartan involution, if*

$$B_\theta(X, Y) = -B(X, \theta(Y))$$

is a positive definite bilinear form. Two involutions are said to be equivalent, if they only differ by an inner automorphism.

One can show that any real semisimple Lie algebra has a Cartan involution and any two such involutions are equivalent (see [Kna02, Chapter VI, Section 2]). We will use this fact to define the Cartan decomposition for such a Lie algebra.

Definition 5.17. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\theta\colon \mathfrak{g} \to \mathfrak{g}$ a Cartan involution and denote by*

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$$
$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$$

the ± 1 eigenspaces of θ. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the Cartan decomposition of $\mathfrak{g}$.

Note that $\mathfrak{k}$ is a sub Lie algebra and $\mathfrak{p}$ is just a subspace of $\mathfrak{g}$ satisfying

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \quad \text{and} \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Further, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal complements of each other with respect to the Killing form on $\mathfrak{g}$. Remember that we want to associate a compact Lie group G_u to a given semisimple Lie group G without changing the Lie algebra too much. More precisely, we will define G_u in such a way that $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}_u \otimes_{\mathbb{R}} \mathbb{C}$, where

$\mathfrak{g}_u$ is the Lie algebra of G_u. Under certain conditions, which hold in the cases we will consider in the following chapters, there is a complexification $G_{\mathbb{C}}$ of the real Lie algebra G. This is used to define the compact twin.

Definition 5.18. *Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Denote by $G_{\mathbb{C}}$ a complex Lie group containing G, such that its Lie algebra satisfies $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Define the Lie algebra*

$$\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$$

and let G_u be the real Lie subgroup of $G_{\mathbb{C}}$ having $\mathfrak{g}_u$ as Lie algebra. We call G_u the compact twin of G.

The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. It follows that it is negative definite on $\mathfrak{g}_u$ and so the compact twin is indeed compact. The following lemma shows that we can substitute G by G_u without changing the corresponding relative Lie algebra cohomology.

Lemma 5.19. *Let G be a semisimple connected Lie Group and G_u its compact twin. Denote by $\mathfrak{g}$ and $\mathfrak{g}_u$ their corresponding Lie algebras. There is an isomorphism of graded algebras*

$$H^*(\mathfrak{g}, \mathfrak{k}) \cong H^*(\mathfrak{g}_u, \mathfrak{k}).$$

Proof. By definition of the compact twin we have decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$, so $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ and $\mathfrak{g}_u/\mathfrak{k} \cong i\mathfrak{p}$. There is an isomorphism

$$\iota_n \colon (\Lambda^n(i\mathfrak{p})^*)^{\mathfrak{k}} \to (\Lambda^n(\mathfrak{p})^*)^{\mathfrak{k}}$$

defined by $\iota_n\omega(X_1, \ldots, X_n) = \omega(iX_1, \ldots, iX_n)$ for $X_i \in \mathfrak{p}$. This is compatible with the boundary maps and thus defines an isomorphism of complexes. □

5.4 Continuous Cohomology

In order to formulate a result by Borel, which plays a key role in understanding higher rational K-theory of $\mathbb{Z}C$, we need to introduce continuous cohomology. Given a topological group there are several ways to incorporate the topological structure into a cohomology theory. One is continuous cohomology, which was introduced by Hochschild and Mostow in [HM62]. In this section we give the definition where we strictly follow [HM62] and at the end compute some examples with the help of the results of the previous section. The general idea is to mimic the definition of group cohomology as the right derived functor of taking fixed points, but for a category which includes the topological structure. Throughout this section G will denote a locally compact topological group. We start with the basic notions.

Definition 5.20. *A continuous G-module V is a Hausdorff topological vector space over $\mathbb{R}$ equipped with a continuous group action of G. This is a continuous map $G \times V \to V$, $(g, v) \mapsto g \cdot v$ such that $v \mapsto g \cdot v$ is a linear automorphism of V for all $g \in G$.*

We need the following stronger definition of an exact sequence.

Definition 5.21. *An exact sequence $\ldots \longrightarrow V_i \xrightarrow{\alpha_i} V_{i+1} \longrightarrow \ldots$ of continuous G-module homomorphisms is strongly exact, if there is a sequence of continuous linear maps $\gamma_i \colon V_i \to V_{i-1}$ such that for each i we have $\gamma_{i+1} \circ \alpha_i + \alpha_{i-1} \circ \gamma_i = \mathrm{id}_{V_i}$. The sequence $(\gamma_i)_i$ is called a continuous contracting homotopy.*

Very similar to the definition of group cohomology we need to define what the injective objects are.

Definition 5.22. *A continuous G-module V is continuously injective, if for every strongly exact sequence $0 \longrightarrow U \xrightarrow{\rho} A \longrightarrow W \longrightarrow 0$ of continuous G-modules and every continuous G-module homomorphism $\alpha \colon U \to V$ there is a continuous G-module homomorphism $\beta \colon A \to V$ such that $\beta \circ \rho = \alpha$.*

Finally, in order to define resolutions we need a suiting notion of an embedding.

Definition 5.23. *If V and W are continuous G-modules, a strong embedding of V into W is a continuous G-module homomorphism $\alpha \colon V \to W$ such that there is a continuous linear map $\beta \colon W \to V$ with $\beta \circ \alpha = \mathrm{id}_V$.*

One can show that every continuous G-module has a strong embedding in a continuously injective G-module. In particular, the category of continuous G-modules has enough injectives. Hence, for every continuous G-module V there is a strongly exact sequence of continuous G-module homomorphisms

$$0 \longrightarrow V \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \ldots$$

where each X_i is continuously injective. Such a sequence is called a continuously injective resolution of V.

Given such a resolution we obtain a complex of topological vector spaces

$$0 \longrightarrow X_0^G \longrightarrow X_1^G \longrightarrow \dots,$$

where X_i^G denotes the G-fixed points of X_i. The homology space of this complex is up to natural isomorphism independent of the choice of the continuously injective resolution of V. We denote it by $H_c(G;V)$ and call it the continuous cohomology of G with coefficients in V.

Similar to group cohomology there is a standard resolution which always exists. It is the same as in group cohomology with the only difference that we want cochains to consist of just the continuous maps $G \times \cdots \times G \to V$ instead of all such maps. The boundary operator is defined exactly like in group cohomology. It follows that continuous cohomology of a group equipped with the discrete topology is just usual group cohomology.

The greatest success in computing continuous cohomology has been done for Lie groups, because in this case continuous cohomology coincides with its corresponding Lie algebra cohomology. More precisely, Van Est proved the following result relating these two theories.

Theorem 5.24. *Let G be a connected Lie Group and $K \subset G$ a maximal compact subgroup with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, then there is an isomorphism of graded algebras*

$$H_c^*(G;\mathbb{R}) \cong H^*(\mathfrak{g},\mathfrak{k}).$$

Proof. We will not explain all details, but enough for our purposes. Denote by $\Omega^n(G/K)$ the space of real-valued n-forms on the symmetric space G/K. One can show that the de Rham complex

$$0 \to \mathbb{R} \to \Omega^0(G/K) \to \Omega^1(G/K) \to \Omega^2(G/K) \to \dots$$

is a continuously injective resolution of the trivial G-module $\mathbb{R}$. The boundary maps are given by the exterior derivative and G acts on forms by the pullback of left multiplication with an element of G. Thus, the continuous cohomology of G is the cohomology of the complex of G-invariant forms denoted by $\Omega^*(G/K)^G$ and we have identified this with relative Lie algebra cohomology previously. □

Combining the results of this and the previous section we conclude that for a connected Lie group G and its maximal compact subgroup $K \subset G$ we have an algebra isomorphism $H_c^*(G;\mathbb{R}) \cong H^*(G_u/K;\mathbb{R})$, where G_u denoted the compact twin of G. It is natural in the following sense. Given a Lie group homomorphism $G \to G'$, then a maximal compact subgroup K of G is mapped into a maximal compact subgroup K' of G', because the continuous image of a compact set is compact. This induces a well-defined map $G_u/K \to G'_u/K'$ and thus on singular cohomology. Further, using the above isomorphism a Künneth formula for continuous cohomology of connected Lie groups can be derived from the analog formula for singular cohomology. The singular cohomology of symmetric spaces G_u/K for compact connected Lie groups is known in many cases and we will present a computation for two examples, which will be of importance later on.

Example 5.25 ($G = SL_n(\mathbb{R})$)**.** The group $SL_n(\mathbb{R})$ is a non-compact, connected Lie group with Lie algebra

$$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) \mid \operatorname{tr}(M) = 0\},$$

where $M_n(\mathbb{R})$ is the vector space of $n \times n$-matrices with real entries and $\operatorname{tr}(M)$ denotes the trace of M. A Cartan involution on $\mathfrak{sl}_n(\mathbb{R})$ is given by the negative transpose $\theta(M) = -M^\top$. We get the eigenspaces

$$\mathfrak{k} = \{M \in \mathfrak{sl}_n(\mathbb{R}) \mid M = -M^\top\},$$
$$\mathfrak{p} = \{M \in \mathfrak{sl}_n(\mathbb{R}) \mid M = M^\top\}.$$

Note that $\mathfrak{k} = \mathfrak{so}(n)$ is the Lie algebra of the special orthogonal group $SO(n)$, which is a maximal compact subgroup of $SL_n(\mathbb{R})$. The complexification of $\mathfrak{g}$ is

$$\mathfrak{sl}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_n(\mathbb{C}) = \{M \in M_n(\mathbb{C}) \mid \operatorname{tr}(M) = 0\},$$

where the isomorphism is given by $M \otimes z \mapsto zM$ and the complex structure is the usual complex conjugation of matrices. The compact twin is given by

$$\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} = \{M \in \mathfrak{sl}_n(\mathbb{C}) \mid M = -\overline{M}^\top\} \cong \mathfrak{su}(n)$$

and so $G_u = SU(n)$ is the special unitary group. Finally, applying the previous results we get

$$H_c^*(SL_n(\mathbb{R});\mathbb{R}) \cong H^*(SU(n)/SO(n);\mathbb{R}).$$

The singular homology is known (see [MT91, Chapter 3, Theorem 6.7]) and for odd n given by

$$H^*(SU(n)/SO(n);\mathbb{R}) \cong \Lambda^*_{\mathbb{R}}(e_5, e_9, e_{13}, \dots, e_{4k+1})$$

where $k = \lfloor \frac{n-1}{2} \rfloor$ and the right hand-side is the exterior algebra on generators $e_q \in H^q(SU(n)/SO(n);\mathbb{Z})$. The proof uses the Serre spectral sequence on the homotopy fibration $SU(n) \to SU(n)/SO(n) \to BSO(n)$.

Example 5.26 ($G = SL_n(\mathbb{C})$)**.** As a second example we take $G = SL_n(\mathbb{C})$ considered as a real Lie group. Its Lie algebra is

$$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{M \in M_n(\mathbb{C}) \mid \operatorname{tr}(M) = 0\}$$

and a Cartan involution is given by $\theta(M) = -\overline{M}^\top$, the negative complex conjugate transpose of M. As previously, we have the ± 1 eigenspaces

$$\mathfrak{k} = \{M \in \mathfrak{sl}_n(\mathbb{C}) \mid M = -\overline{M}^\top\},$$
$$\mathfrak{p} = \{M \in \mathfrak{sl}_n(\mathbb{C}) \mid M = \overline{M}^\top\}.$$

We see that $\mathfrak{k} = \mathfrak{su}(n)$ is the Lie algebra of the special unitary group $K = \mathrm{SU}(n)$. The complexification of $\mathfrak{g}$ is given by the isomorphism

$$\mathfrak{sl}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C}), \quad M \otimes z \mapsto z(\overline{M}, M).$$

Hence, the inclusion $\mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$ is given by $M \mapsto (\overline{M}, M)$. Note that we have two different complex structures on the complexification. First, there is the usual complex conjugation of matrices in $\mathfrak{sl}_n(\mathbb{C})$. We ignored it until now, because we considered $SL_n(\mathbb{C})$ as a real Lie group. It induces the involution $(M,N) \mapsto (N,M)$ on $\mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$. Second, there is the complex conjugation on the factor $\mathbb{C}$, which induces $(M,N) \mapsto (\overline{N}, \overline{M})$ on the product. As the compact twin we get

$$\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \cong \mathfrak{su}(n) \times \mathfrak{su}(n)$$

and $G_u = SU(n) \times SU(n)$. The map $G_u/K \to SU(n)$, $[(M,N)] \mapsto M\overline{N}^{-1}$ is a homeomorphism and so we can compute the continuous cohomology as previously getting

$$H_c^*(SL_n(\mathbb{C});\mathbb{R}) \cong H^*(SU(n);\mathbb{R}).$$

It remains to compute the singular cohomology of $SU(n)$ and we want to describe this in detail. It is done by induction on n using the fibration

$$SU(n-1) \to SU(n) \to S^{2n-1},$$

where the inclusion of the fibre $SU(n-1)$ into $SU(n)$ is given by adding 1 in the lower right corner and zeros elsewhere. The projection onto S^{2n-1} is defined by mapping a matrix in $SU(n)$ to its last column. The last column is a vector of length 1 in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and can thus be considered as an element in the $(2n-1)$-sphere. This fibration yields a long exact sequence

$$\ldots \longrightarrow \pi_i(SU(n-1)) \longrightarrow \pi_i(SU(n)) \longrightarrow \pi_i(S^{2n-1}) \longrightarrow \pi_{i-1}(SU(n-1)) \longrightarrow \ldots$$

and we know that $\pi_i(S^{2n-1}) = 0$ for $i < 2n-1$ and $\pi_{2n-1}(S^{2n-1}) \cong \mathbb{Z}$. In particular, the map $\pi_i(SU(n-1)) \to \pi_i(SU(n))$ induced from the inclusion is surjective and so $\pi_i(SU(n), SU(n-1)) = 0$ for $i < 2n-1$. This means that the pair $(SU(n), SU(n-1))$ is $(2n-2)$-connected. By the Hurewicz theorem the relative cohomology of the pair $(SU(n), SU(n-1))$ vanishes in degrees $i < 2n-1$ and so the long exact sequence of cohomology groups of a pair

$$\ldots \longrightarrow H^i(X,A) \longrightarrow H^i(X) \longrightarrow H^i(A) \longrightarrow H^{i+1}(X,A) \longrightarrow \ldots$$

implies that the induced map $H^i(SU(n);\mathbb{Z}) \to H^i(SU(n-1);\mathbb{Z})$ is an isomorphism for all $i \leq 2n-3$. By the induction hypothesis we have $H^*(SU(n-1);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(\varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{2n-3})$ and there exist $c_1, \ldots, c_{2n-3} \in H^*(SU(n);\mathbb{Z})$ which restrict to the generators ε_i. Since products of distinct ε_i form a basis of the exterior algebra and these products are also restrictions from products of the $c_i' s$, the Leray-Hirsch Theorem applies (see [Hat01, Theorem 4D.1]) which yields isomorphisms

$$\begin{aligned} H^*(SU(n);\mathbb{Z}) &\cong H^*(SU(n-1);\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^{2n-1};\mathbb{Z}) \\ &\cong \Lambda^*_{\mathbb{Z}}(\varepsilon_3, \varepsilon_5, \varepsilon_7, \ldots, \varepsilon_{2n-3}) \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}(\varepsilon_{2n-1}) \\ &\cong \Lambda^*_{\mathbb{Z}}(\varepsilon_3, \varepsilon_5, \varepsilon_7, \ldots, \varepsilon_{2n-1}). \end{aligned}$$

We used the fact that the cohomology of S^n with coefficients in $\mathbb{Z}$ vanishes except in degrees 0 and n, where it is $\mathbb{Z}$.

Besides computing continuous cohomology of these two examples we also want to determine what complex conjugation of matrices in $SL_n(\mathbb{C})$ induces on continuous cohomology. In general, an involution T on a Lie Group G defines an involution on its Lie algebra $\mathfrak{g}$ by taking the derivative $\tau = d\mathrm{T}$. This can be extended to an involution $\tau \otimes id$ on the complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Under the assumption that the original involution T commutes with a given Cartan involution on G, the restriction of $\tau \otimes id$ to the compact twin $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ defines an involution on $\mathfrak{g}_u$, which preserves $\mathfrak{k}$. Consequently, we get a well-defined involution on $\mathfrak{g}_u/\mathfrak{k}$ and G_u/K. We want to apply this to $G = SL_n(\mathbb{C})$, so T is complex conjugation of matrices and $G_u/K = SU(n)$. Note that the induced involution on $SU(n)$ is not again complex conjugation of matrices.

Lemma 5.27. *Let $G = SL_n(\mathbb{C})$ with maximal compact subgroup $K = SU(n)$ and compact twin $G_u = SU(n) \times SU(n)$. The involution $\mathrm{T}\colon G \to G$, $\mathrm{T}(M) = \overline{M}$ induces the involution $G_u/K \to G_u/K$, $M \mapsto M^\top$ via the identification $G_u/K \cong SU(n)$.*

Proof. The derivation $\tau = d\mathrm{T}$ is again complex conjugation on the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We already saw that the complexification of $\mathfrak{sl}_n(\mathbb{C})$ is $\mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$, where the inclusion is given by $M \mapsto (\overline{M}, M)$ and the induced involution $\tau \otimes id$ on the complexification is given by $(M, N) \mapsto (N, M)$. Complex conjugation commutes with the given Cartan involution, so $\tau \otimes id$ preserves $\mathfrak{g}_u = \mathfrak{su}(n) \times \mathfrak{su}(n)$ and $\mathfrak{k} = \mathfrak{su}(n)$ and we get a well-defined involution on $\mathfrak{su}(n) \times \mathfrak{su}(n)/\mathfrak{su}(n)$. The map

$$\mathfrak{su}(n) \cong (\mathfrak{su}(n) \times \mathfrak{su}(n))/\mathfrak{su}(n), \quad M \mapsto [(0, M)]$$

is a homeomorphism and so the involution on $\mathfrak{g}_u/\mathfrak{k} = \mathfrak{su}(n)$ under this identification is given by $M \mapsto -\overline{M}$. Finally, on the Lie group level the involution on $G_u/K = SU(n)$ is given by $M \mapsto \overline{M}^{-1} = M^\top$. □

Since the isomorphism $H_c^*(SL_n(\mathbb{C}); \mathbb{R}) \cong H^*(\mathrm{SU}(n); \mathbb{R})$ is natural in the sense we explained previously, it remains to compute what transposing induces on singular cohomology. We will apply the following theorem, which follows from the fact that for connected orientable closed manifolds the choice of a generator of the top homology with integer coefficients is an orientation.

Theorem 5.28. *Let G be a connected Lie group of dimension $n \in \mathbb{N}$ and $\tau\colon G \to G$ an involution with derivative $d\tau\colon \mathfrak{g} \to \mathfrak{g}$, then $H^n(G;\mathbb{Z}) \cong \mathbb{Z}$ and the induced map*

$$\tau^*\colon H^n(G;\mathbb{Z}) \to H^n(G;\mathbb{Z})$$

is multiplication by 1 or -1. More precisely, it is multiplication by -1, if and only if $\det(d\tau) < 0$.

We use this to compute the map induced on continuous cohomology.

Lemma 5.29. *Complex conjugation of matrices in $SL_n(\mathbb{C})$ induces on $H_c^*(SL_n(\mathbb{C});\mathbb{R}) \cong \Lambda^*_{\mathbb{R}}(\varepsilon_{2k-1}, k = 2, \dots, n)$ the map given by $\varepsilon_{2k-1} \mapsto (-1)^{k+1}\varepsilon_{2k-1}$.*

Proof. According to Lemma 5.27 we have to compute what transposing of matrices induces on $H^*(SU(n);\mathbb{R}) \cong \Lambda^*_{\mathbb{R}}(\varepsilon_3, \varepsilon_5, \varepsilon_7, \dots, \varepsilon_{2n-1})$. The dimension of $SU(n)$ is $n^2 - 1$ and so according to Theorem 5.28 we have

$$H^{n^2-1}(SU(n);\mathbb{R}) = \langle \varepsilon_3 \cdot \varepsilon_5 \cdot \varepsilon_7 \cdot \dots \cdot \varepsilon_{2n-1} \rangle_{\mathbb{R}} \cong \mathbb{R}.$$

An $\mathbb{R}$-basis of $\mathfrak{su}(n)$ is given by $\{iE^k, iA^{p,q}, B^{p,q} \in M_n(\mathbb{C}) \mid 1 \le k \le n-1, 1 \le p < q \le n\}$ where

$$E^k_{i,j} = \begin{cases} 1 & \text{if } i = j = k \\ -1 & \text{if } i = j = n \\ 0 & \text{else} \end{cases}$$

$$A^{p,q}_{i,j} = \begin{cases} 1 & \text{if } (i,j) = (p,q) \text{ or } (i,j) = (q,p) \\ 0 & \text{else} \end{cases}$$

$$B^{p,q}_{i,j} = \begin{cases} 1 & \text{if } (i,j) = (p,q) \\ -1 & \text{if } (i,j) = (q,p) \\ 0 & \text{else.} \end{cases}$$

Hence, the only basis elements which are not fixed by transposing are $B^{p,q}$ for $1 \le p < q \le n$, of which there are exactly $n(n-1)/2$. These are mapped to their negative, so

$$\det(d\tau) = (-1)^{n(n-1)/2} = \begin{cases} 1 & n \equiv 0, 1 \bmod 4 \\ -1 & n \equiv 2, 3 \bmod 4. \end{cases}$$

where $d\tau$ denotes transposing of matrices. Again, we conclude from Theorem 5.28 that the map induced from transposing maps the element $\varepsilon_3 \cdot \varepsilon_5 \cdot \ldots \cdot \varepsilon_{2n-1}$ to $(-1)^{\frac{n(n-1)}{2}} \cdot \varepsilon_3 \cdot \varepsilon_5 \cdot \ldots \cdot \varepsilon_{2n-1}$ and thus a generator ε_{2k-1} of the cohomology ring must be mapped to $(-1)^{k+1}\varepsilon_{2k-1}$. We used the fact that the inclusion $SU(n) \to SU(n+1)$ induces on $H^*(SU(n+1);\mathbb{R}) \to H^*(SU(n);\mathbb{R})$ the projection given by $\varepsilon_i \mapsto \varepsilon_i$ for $i \neq 2n+1$ and $\varepsilon_{2(n+1)+1} \mapsto 0$. □

5.5 Algebraic Groups and Arithmetic Subgroups

Besides continuous cohomology we also need the very basic definitions of algebraic and arithmetic groups. There are several different ways to define algebraic groups. They can be defined as functors (see [Jan07]), as algebraic varieties which have a compatible group structure (see [Hum75]) or in a more concrete way as described for example by Borel in [Bor69]. We will introduce the latter version, since it is sufficient for the groups we will consider.

Roughly speaking, an algebraic group is a group defined by zeros of a set of polynomials. For $n \in \mathbb{N}$ we denote by $M_n(\mathbb{C})$ the group of all $n \times n$ matrices with entries in $\mathbb{C}$ and by $GL_n(\mathbb{C})$ the group of all invertible $n \times n$ matrices. The Zariski topology defines a topology on the set $M_n(\mathbb{C}) = \mathbb{C}^{n^2}$, which induces a topology on the open subset $GL_n(\mathbb{C})$. If not stated otherwise topological properties always refer to this topology. The precise definition of an algebraic group is given as follows.

Definition 5.30. *An algebraic group G is a closed subgroup of $GL_n(\mathbb{C})$. In other words, G is an algebraic group if there exist polynomials $f_\alpha \in \mathbb{C}[X_{ij} \mid 1 \leq i,j \leq n]$ with α in some index set I, such that*

$$G = \{(x_{ij}) \in GL_n(\mathbb{C}) \mid f_\alpha(x_{ij}) = 0, \alpha \in I\}.$$

Further, for a subring R of $\mathbb{C}$ we define $G(R) = G \cap GL_n(R)$, where $GL_n(R)$ is the set of $n \times n$-matrices with entries in R, such that the determinant is a unit in R. For a subfield k of $\mathbb{C}$ we say that G is defined over k or G is a k-group, if G is defined by a set of polynomials with coefficients in k. A closed subgroup H of G is called a k-subgroup if it is defined over k as an algebraic group.

We also want to define what a semisimple algebraic group is.

Definition 5.31. *An algebraic group is called solvable if it is solvable as an* *abstract group and it is called semisimple if it has no non-trivial normal solvable connected closed subgroup.*

Besides the Zariski topology on $GL_n(\mathbb{C})$ we have considered up to now, there is also the topology induced by the usual topology on $\mathbb{C}^{n^2}$ and it can be shown that an algebraic group is connected if and only if it is connected with respect to the usual topology on $GL_n(\mathbb{C})$ (see [Bor69, 7.4]). Let us consider some examples of algebraic groups.

Example 5.32. The group $G = GL_n(\mathbb{C})$ is an algebraic group defined over $\mathbb{Q}$. This is obvious, because we can choose the set of defining polynomials to be empty.

Example 5.33. The group $G = SL_n(\mathbb{C})$ of invertible $n \times n$ matrices with entries in $\mathbb{C}$ and determinant 1 is an algebraic group defined over $\mathbb{Q}$. Denote by $\det \in \mathbb{Q}[X_{ij} \mid 1 \leq i, j \leq n]$ the polynomial defined by the determinant of a matrix, then we have

$$SL_n(\mathbb{C}) = \{(x_{ij}) \in GL_n(\mathbb{C}) \mid \det(x_{ij}) - 1 = 0\}$$

and it is known that det has coefficients in $\mathbb{Z}$. Further, we have $G(k) = SL_n(\mathbb{C}) \cap GL_n(k) = SL_n(k)$ for a subfield k of $\mathbb{C}$.

Since we are interested in group rings the following examples will be of great importance.

Example 5.34. Let H be a finite abelian group, then $G = GL_n(\mathbb{C}H)$ is an algebraic group defined over $\mathbb{Q}$. Denote by m the order of H, then an isomorphism of $\mathbb{C}$-vector spaces $\mathbb{C}H \cong \mathbb{C}^m$ defines an injective map $GL_n(\mathbb{C}H) \to GL_N(\mathbb{C})$ with $N = nm$ and so $GL_n(\mathbb{C}H)$ can be considered a subgroup of $GL_N(\mathbb{C})$. The conditions imposed on the coefficients of a matrix in $GL_N(\mathbb{C})$ which is in the image of the above inclusion can be expressed as polynomials in $\mathbb{Q}[X_{ij} \mid 1 \leq i, j \leq N]$ and we denote these by f_α with $\alpha \in I$ for some finite index set I. Hence, we can write

$$GL_n(\mathbb{C}H) = \{(x_{ij}) \in GL_N(\mathbb{C}) \mid f_\alpha(x_{ij}) = 0, \alpha \in I\}$$

and we have $G(k) = GL_n(\mathbb{C}H) \cap GL_N(k) = GL_n(kH)$. A similar argument can be used to show that $SL_n(\mathbb{C}H)$ is also an algebraic group defined over $\mathbb{Q}$. We want to see that $SL_n(\mathbb{C}H)$ is semisimple. For our purposes it suffices to take

$H = C$ to be finite cyclic. Using the isomorphism of $\mathbb{C}$-algebras $\mathbb{C}C \cong \prod_{\sigma \in \widehat{C}} \mathbb{C}$ defined by $c \mapsto \sigma(c)$ for $c \in C$ we write

$$SL_n(\mathbb{C}C) \cong \prod_{\sigma \in \widehat{C}} SL_n(\mathbb{C})$$

and note that $SL_n(\mathbb{C})$ has no non-trivial normal solvable connected closed subgroups. Hence, $SL_n(\mathbb{C}C)$ is semisimple.

We proceed with the definition of arithmetic groups.

Definition 5.35. *Let G be an algebraic group defined over $\mathbb{Q}$. A subgroup Γ of $G(\mathbb{Q})$ is called arithmetic if it is commensurable with $G(\mathbb{Z}) = G \cap GL_n(\mathbb{Z})$, which means that $\Gamma \cap G(\mathbb{Z})$ has finite index both in Γ and in $G(\mathbb{Z})$. Given any group Γ, it is called arithmetic if it can be embedded as an arithmetic subgroup in $G(\mathbb{Q})$ for some algebraic group G defined over $\mathbb{Q}$.*

We are interested in the following example.

Example 5.36. Let H be a finite abelian group and let $G = GL_n(\mathbb{C}H)$ be the algebraic group from the previous example. The subgroup $\Gamma = GL_n(\mathbb{Z}H)$ of $G(\mathbb{Q}) = GL_n(\mathbb{Q}H)$ is an arithmetic group, since $GL_n(\mathbb{Z}H) = G \cap GL_N(\mathbb{Z})$ with $N = n \cdot |C|$. Similarly, $SL_n(\mathbb{Z}H)$ is an arithmetic subgroup of $SL_n(\mathbb{Q}H)$.

5.6 The Group Algebra $\mathbb{R}C$

One crucial step in determining higher rational K-theory of the integral group ring of a finite cyclic group C will be considering the map on continuous cohomology induced from the inclusion $\mathbb{Z}C \to \mathbb{R}C$. The advantage of the $\mathbb{R}$-algebra $\mathbb{R}C$ is that it decomposes into a product of fields. Since we are particularly interested in K-theory as a functor from the subgroup category of C and in the action of $\mathrm{Aut}(C)$ on the Artin defect, we want to examine what inclusion of subgroups and automorphisms induce on the decomposition of $\mathbb{R}C$. We start with introducing some notation. If we denote by $\gamma\colon \mathbb{C} \to \mathbb{C}$ complex conjugation in $\mathbb{C}$, then the Galois group $G(\mathbb{C}/\mathbb{R})$ of the extension $\mathbb{C}/\mathbb{R}$ is generated by γ. The group $G(\mathbb{C}/\mathbb{R})$ acts on the dual group $\widehat{C}$ by $\sigma \mapsto \gamma \circ \sigma = \overline{\sigma}$ for $\sigma \in \widehat{C}$

and we split $\widehat{C}/\gamma$ into the disjoint subsets

$$\widehat{C}^+ = \{[\sigma] \in \widehat{C}/\gamma \mid \sigma = \overline{\sigma}\},$$
$$\widehat{C}^- = \{[\sigma] \in \widehat{C}/\gamma \mid \sigma \neq \overline{\sigma}\}.$$

Note that $\widehat{C}^+$ is the fixed point set under the $G(\mathbb{C}/\mathbb{R})$-action and the fixed points are exactly those group homomorphisms $C \to S^1$ whose image is contained in $\{\pm 1\}$. Further, for an element $\sigma \in \widehat{C}$ we set

$$\mathbb{K}_\sigma = \begin{cases} \mathbb{R} & \text{if } [\sigma] \in \widehat{C}^+ \\ \mathbb{C} & \text{if } [\sigma] \in \widehat{C}^-. \end{cases}$$

Now we can describe the decomposition of the real algebra $\mathbb{R}C$ into a product of fields.

Lemma 5.37. *Choose one representative for each orbit of $\widehat{C}/\gamma$ and denote by $\mathcal{C}$ the set of the representatives. The map*

$$\mathbb{R}C \to \prod_{\sigma \in \mathcal{C}} \mathbb{K}_\sigma, \quad c \mapsto (\sigma(c))_{\sigma \in \mathcal{C}}$$

is an isomorphism of $\mathbb{R}$-algebras.

Proof. Denote by μ_n the set of n-th roots of unity in $\mathbb{C}$. There are isomorphisms of $\mathbb{C}$-algebras

$$\mathbb{C}C \cong \mathbb{C}[X]/(X^n - 1) \cong \prod_{\sigma \in \widehat{C}} \mathbb{C}[X]/(X - \sigma(c)) \cong \prod_{\sigma \in \widehat{C}} \mathbb{C}$$

given by $c \mapsto X + (X^n - 1) \mapsto (X + (X - \sigma(c))_{\sigma \in \widehat{C}} \mapsto (\sigma(c))_{\sigma \in \widehat{C}}$ for a generator $c \in C$. Now $\mathbb{R}C$ is the fixed point set of $\mathbb{C}C$ under the $G(\mathbb{C}/\mathbb{R})$-action on the coefficients. This induces a $G(\mathbb{C}/\mathbb{R})$-action $(z_\sigma)_{\sigma \in \widehat{C}} \mapsto (\overline{z}_{\overline{\sigma}})_{\sigma \in \widehat{C}}$ on the product decomposition and we get the desired isomorphism

$$\mathbb{R}C \cong (\prod_{\sigma \in \widehat{C}} \mathbb{C})^{G(\mathbb{C}/\mathbb{R})} \cong \prod_{\sigma \in \widehat{C}^+} \mathbb{R} \times \prod_{\sigma \in \widehat{C}^-} \mathbb{C},$$

where we identified $(\mathbb{C} \times \mathbb{C})^{G(\mathbb{C}/\mathbb{R})} = \{(z, \overline{z}) \mid z \in \mathbb{C}\} \cong \mathbb{C}$. This depends on a choice of a representative for each pair $\{\sigma, \overline{\sigma}\} \in \widehat{C}/\gamma$ and yields the desired isomorphism $\mathbb{R}C \cong \prod_{\sigma \in \mathcal{C}} \mathbb{K}_\sigma$. □

Next, we want to determine the map on the product decomposition induced by the inclusion of a subgroup D of C.

Lemma 5.38. *Let D be a subgroup of a finite cyclic group C and denote by $\mathcal{D}$ and $\mathcal{C}$ a system of representatives of $\widehat{C}/\gamma$ and $\widehat{D}/\gamma$, respectively. The inclusion $D \to C$ induces an inclusion $\mathbb{R}D \to \mathbb{R}C$ and a commutative diagram*

$$\begin{array}{ccc} \mathbb{R}C & \xrightarrow{\cong} & \prod_{\sigma \in \mathcal{C}} \mathbb{K}_\sigma \\ \uparrow & & \uparrow{\scriptstyle i} \\ \mathbb{R}D & \xrightarrow{\cong} & \prod_{\tau \in \mathcal{D}} \mathbb{K}_\tau \end{array}$$

where i is given by

$$i(z)_\sigma = \begin{cases} z_{\sigma \circ \mathrm{inc}} & \text{if } \sigma \circ \mathrm{inc} \in \mathcal{D} \\ \overline{z}_{\overline{\sigma} \circ \mathrm{inc}} & \text{else} \end{cases}$$

for an element $z = (z_\tau)_{\tau \in \mathcal{D}}$.

Proof. Write $z = (\tau(x))_{\tau \in \mathcal{D}}$ for some $x \in \mathbb{R}D$. This is possible, because the horizontal maps are isomorphisms. Applying the inclusion $\mathrm{inc}\colon \mathbb{R}D \to \mathbb{R}C$ and the upper isomorphism maps x to $(\sigma(\mathrm{inc}(x)))_{\sigma \in \mathcal{C}}$. For those $\sigma \in \mathcal{C}$ with $\sigma \circ \mathrm{inc} \in \mathcal{D}$ we have $\sigma(\mathrm{inc}(x)) = z_{\sigma \circ \mathrm{inc}}$. If $\sigma \circ \mathrm{inc} \notin \mathcal{D}$, then $\overline{\sigma} \circ \mathrm{inc} \in \mathcal{D}$ and so $\sigma(\mathrm{inc}(x)) = \overline{z}_{\overline{\sigma} \circ \mathrm{inc}}$. □

We want to note that in general it is not possible to choose the representatives in such a way that $\sigma \circ \mathrm{inc} \in \mathcal{D}$ holds whenever $\sigma \in \mathcal{C}$ for all subgroups of C simultaneously. Last, we describe the map on the decomposition induced by automorphisms of C. Similarly, we get the following result.

Lemma 5.39. *Let C be a finite cyclic group and $\mathcal{C}$ a system of representatives for $\widehat{C}/\gamma$. An automorphism $\phi \in \mathrm{Aut}(C)$ defines an automorphism of $\mathbb{R}C$, which induces the map*

$$\Phi\colon \prod_{\sigma \in \mathcal{C}} \mathbb{K}_\sigma \longrightarrow \prod_{\sigma \in \mathcal{C}} \mathbb{K}_\sigma$$

given by

$$\Phi(z)_\sigma = \begin{cases} z_{\sigma \circ \phi} & \text{if } \sigma \circ \phi \in \mathcal{C} \\ \overline{z}_{\overline{\sigma} \circ \phi} & \text{else} \end{cases}$$

for an element $z = (z_\sigma)_{\sigma \in \mathcal{C}}$ in the product.

Proof. As before we write $z = (\sigma(x))_{\sigma\in\mathcal{C}}$ for some $x \in \mathbb{R}C$. For those σ with $\sigma \circ \phi \in \mathcal{C}$ we can write $\Phi(z)_\sigma = \sigma(\phi(x)) = z_{\sigma\circ\phi}$. But if $\sigma \circ \phi \notin \mathcal{C}$, then $\overline{\sigma \circ \phi} = \overline{\sigma} \circ \phi$ must lie in $\mathcal{C}$ and so we have $\Phi(z)_\sigma = \sigma(\phi(x)) = \overline{\overline{\sigma}(\phi(x))} = \overline{z}_{\overline{\sigma}\circ\phi}$, which finishes the proof. □

5.7 Higher Rational K-Theory of $\mathbb{Z}C$ and Functoriality

The purpose of this section is to determine the K-groups $K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R}$ in degrees $m > 1$ as functors from the subgroup category of a finite cyclic group C as well as describe the action of the automorphism group on them. This will be the basis for computing the Artin defect in higher degrees. The rank of $K_m(\mathbb{Z}G)$ for $m > 1$ can be computed for any finite group G using the following result by Borel on the cohomology of arithmetic groups (see [Jah09]). For the proof of Borel's theorem see [Bor74].

Theorem 5.40. *Let G be a semisimple algebraic group defined over $\mathbb{Q}$ such that $G(\mathbb{R})$ is connected and let $\Gamma \leq G(\mathbb{Q})$ be an arithmetic group. If $q + 1 \leq$ $\mathrm{rank}_{\mathbb{Q}}(\mathrm{G})/4$, then the corestriction map $H^q_c(G(\mathbb{R}); \mathbb{R}) \to H^q(\Gamma; \mathbb{R})$ is an isomorphism.*

It will be necessary to consider the K-groups tensored with $\mathbb{R}$ in order to be able to apply this result. Afterwards we explain what can be derived for the rational version. Our approach is to check naturality with respect to the cyclic group in each step of the computation of the groups $K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R}$ as a vector space. This will allow us to determine functorial properties.

As we will see, computing the above K-groups can be reduced to computing the group cohomology of $SL_n(\mathbb{Z}C)$. Therefore, we have to apply Borel's result to $\Gamma = SL_n(\mathbb{Z}C)$ and $G = SL_n(\mathbb{C}C)$. We showed that $SL_n(\mathbb{C}C)$ is a semisimple algebraic group and we can choose n to be large enough such that the rank condition is satisfied. It turns out that $SL_n(\mathbb{Z}C)$ is cohomologically stable. The isomorphism on cohomology is induced from the inclusion $SL_n(\mathbb{Z}C) \to SL_n(\mathbb{R}C)$ and thus is natural. More precisely, we have the following result.

Lemma 5.41. *Let $n, m \in \mathbb{N}$ with $n \gg 0$. There is an isomorphism*

$$H^m(SL_n(\mathbb{Z}C); \mathbb{R}) \cong \Big(\bigotimes_{\sigma\in\widehat{C}^+} \Lambda^*_{\mathbb{R}}(e_5, e_9, \ldots, e_{4k+1}) \otimes_{\mathbb{R}} \bigotimes_{\sigma\in\widehat{C}^-} \Lambda^*_{\mathbb{R}}(\varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{2n-1})\Big)^m$$

where $k = \lfloor\frac{(n-1)}{2}\rfloor$. In particular, the cohomology stabilizes.

Proof. First, we note that SL_n respects products of rings. We already saw that $SL_n(\mathbb{Z}C)$ is an arithmetic subgroup of the semisimple algebraic group SL_n^C. Since we can choose n to be arbitrary large the rank condition from Theorem 5.40 is also satisfied and thus in combination with Lemma 5.37 we get

$$H^m(SL_n(\mathbb{Z}C);\mathbb{R}) \cong H_c^m(SL_n(\mathbb{R}C);\mathbb{R}) \cong (\bigotimes_{\sigma\in\mathcal{C}} H_c^*(SL_n(\mathbb{K}_\sigma);\mathbb{R}))^m,$$

where we used the Künneth formula for continuous cohomology in the last step. The continuous cohomology of $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$ has been computed in examples 5.25 and 5.26, which we apply to finish the proof. □

No we can describe the computation of $K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R}$ as a vector space with emphasis on naturality with respect to inclusion of subgroups of C.

Theorem 5.42. *Let $m > 1$ and C a finite cyclic group. There is an isomorphism of $\mathbb{R}$-vector spaces*

$$K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R} \cong \begin{cases} \bigoplus_{\widehat{C}/\gamma} \mathbb{R} & \text{if } m \equiv 1 \bmod 4 \\ \bigoplus_{\widehat{C}^-} \mathbb{R} & \text{if } m \equiv 3 \bmod 4 \\ 0 & \text{if } m \equiv 0, 2 \bmod 4. \end{cases}$$

Proof. For n sufficiently large we have isomorphisms

$$\begin{aligned} K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R} &= \pi_m(BGL(\mathbb{Z}C)^+) \otimes_{\mathbb{Z}} \mathbb{R} && (1)\\ &\cong \pi_m(BSL(\mathbb{Z}C)^+) \otimes_{\mathbb{Z}} \mathbb{R} && (2)\\ &\cong PH_m(BSL(\mathbb{Z}C)^+;\mathbb{R}) && (3)\\ &\cong IH^m(BSL(\mathbb{Z}C)^+;\mathbb{R})^* && (4)\\ &\cong IH^m(BSL(\mathbb{Z}C);\mathbb{R})^* && (5)\\ &\cong IH^m(SL(\mathbb{Z}C);\mathbb{R})^* && (6)\\ &\cong IH^m(SL_n(\mathbb{Z}C);\mathbb{R})^* && (7)\\ &\cong \bigoplus_{\widehat{C}^+} I\Lambda_{\mathbb{R}}^m(e_5, e_9, \ldots, e_{4k+1})^* \oplus \bigoplus_{\widehat{C}^-} I\Lambda_{\mathbb{R}}^m(\varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{2n-1})^*. && (8) \end{aligned}$$

(1) This is just the definition of higher K-Theory using Quillen's plus construction. The explicit construction of higher K-groups and further properties can be found in [Ros94].

(2) The short exact sequence

$$1 \longrightarrow SL(\mathbb{Z}C) \longrightarrow GL(\mathbb{Z}C) \xrightarrow{\det} (\mathbb{Z}C)^{\times} \longrightarrow 1$$

induces a fibration $BSL(\mathbb{Z}C) \longrightarrow BGL(\mathbb{Z}C) \longrightarrow B(\mathbb{Z}C)^{\times}$ according to Lemma 5.3. Now $\pi_1(B(\mathbb{Z}C)^{\times}) = (\mathbb{Z}C)^{\times}$ is abelian and so its perfect maximal subgroup is the trivial group, which can only act trivially on the fibre. Hence, Theorem 5.4 shows that

$$BSL(\mathbb{Z}C)^{+} \longrightarrow BGL(\mathbb{Z}C)^{+} \longrightarrow (B(\mathbb{Z}C)^{\times})^{+}$$

is again a fibration. As mentioned, $\pi_1(B(\mathbb{Z}C)^{\times})$ is abelian and so it is equal to $\pi_1((B(\mathbb{Z}C)^{\times})^{+})$. Further, since the classifying space is a CW-complex and the plus construction induces an isomorphism on cohomology, there is a homotopy equivalence $(B(\mathbb{Z}C)^{\times})^{+} \simeq B(\mathbb{Z}C)^{\times}$. In particular, the higher homotopy groups of $(B(\mathbb{Z}C)^{\times})^{+}$ are trivial and so the long exact sequence of a fibration yields $\pi_m(BGL(\mathbb{Z}C)^{+}) \cong \pi_m(BSL(\mathbb{Z}C)^{+})$ for $m > 1$. The isomorphism is natural, since it is induced by the inclusion $SL(\mathbb{Z}C) \to GL(\mathbb{Z}C)$, which commutes with maps induced from ring homomorphisms. In particular, the ones induced from inclusion of subgroups or automorphisms of C. We used the fact that taking homotopy groups, the plus construction, the classifying space and tensoring with $\mathbb{R}$ are all covariant functors.

(3) We saw in example 5.6 that $BSL(\mathbb{Z}C)^{+}$ is an H-space and so Lemma 5.11 applies. Also, the Hurewicz map is natural.

(4) Here we apply Lemma 5.8.

(5) The plus construction induces an isomorphism on cohomology. Since this isomorphism is induced by the inclusion $BSL(\mathbb{Z}C) \to BSL(\mathbb{Z}C)^{+}$ it is natural.

(6) The group cohomology of a group is naturally isomorphic to the singular cohomology of its classifying space (see [Bro82, Chapter III, Section 1]).

(7) By definition we have $SL(\mathbb{Z}C) = \varinjlim SL_n(\mathbb{Z}C)$. It is known that direct limits commute with group homology (see [Bro82, p.121]). Since we have coefficients in a field the universal coefficient theorem can be used to see

that $H^m(\varinjlim SL_n(\mathbb{Z}C); \mathbb{R}) \cong \varprojlim H^m(SL_n(\mathbb{Z}C); \mathbb{R})$. Lemma 5.41 shows that $SL_n(\mathbb{Z}C)$ is cohomologically stable yielding the isomorphism for n sufficiently large. Again, it is induced by the inclusion $SL_n(\mathbb{Z}C) \to SL(\mathbb{Z}C)$ and thus natural.

(8) In this last step we apply the result of Lemma 5.41 and note that the indecomposables of the tensor product is the direct sum of the indecomposable elements.

The indecomposables in degree m of an exterior algebra are the generators in degree m. We see that if $m \equiv 1 \bmod 4$, then in both exterior algebras there is exactly one generator in that degree and we get one $\mathbb{R}$ summand for each element in $\widehat{C}^+ \cup \widehat{C}^- = \widehat{C}/\gamma$. If $m \equiv 3 \bmod 4$ there is no generator in the exterior algebra coming from the cohomology of $SL_n(\mathbb{R})$ and we just get one $\mathbb{R}$ summand for each element in $\widehat{C}^-$. □

As a next step we want to describe these K-groups as a functor from the subgroup category of C in a more accessible way. In order to do that we can apply results from section 2.2. Using the notation we introduced there we consider $F = \mathbb{R}$, so $F(\mu_n) = \mathbb{C}$ and $G = G(\mathbb{C}/\mathbb{R}) = \langle\gamma\rangle$ where γ is complex conjugation. The product $G \times G$ acts on the $\mathbb{C}$-algebras $\mathbb{C}C$ and $\mathrm{map}(\widehat{C}, \mathbb{C})$ as described in Lemma 2.4. Additionally, for a vector space V and an automorphism $f\colon V \to V$ with $f^2 = id_V$ we define

$$V^+ = \{v \in V \mid f(v) = v\},$$
$$V^- = \{v \in V \mid f(v) = -v\}$$

to be the eigenspaces to the eigenvalues 1 and -1, respectively. This yields a decomposition $V = V^+ \oplus V^-$. We are in particularly interested in the $\mathbb{R}$-vector spaces

$$V = (\mathbb{C}C)^{G\times 1} \qquad \text{and} \qquad V = \mathrm{map}(\widehat{C}, \mathbb{C})^{G\times 1}.$$

In each case we have an automorphism $V \to V$ defined by $v \mapsto (1, \gamma) \cdot v$. Note that $\gamma^2 = id_{\mathbb{C}}$, so it has eigenvalues 1 or -1. Since $\mathrm{map}(\widehat{C}, \mathbb{C})^{G\times 1} = \mathrm{map}(\widehat{C}, \mathbb{R})$ we have

$$\mathrm{map}(\widehat{C}, \mathbb{R})^+ = \{f \in \mathrm{map}(\widehat{C}, \mathbb{R}) \mid f(\sigma) = f(\overline{\sigma})\},$$
$$\mathrm{map}(\widehat{C}, \mathbb{R})^- = \{f \in \mathrm{map}(\widehat{C}, \mathbb{R}) \mid f(\sigma) = -f(\overline{\sigma})\}.$$

On the other hand we see that $(\mathbb{C}C)^{G\times 1} = \{\sum_{c\in C} \lambda_c c \in \mathbb{C}C \mid \lambda_c = \overline{\lambda}_{c^{-1}}\}$ and so

$$((\mathbb{C}C)^{G\times 1})^+ = (\mathbb{C}C)^{G\times G} = \mathbb{R}[C]^+$$
$$(\mathbb{C}[C]^{G\times 1})^- = \{\sum_{c\in C} \lambda_c c \mid \lambda_c = \lambda_{c^{-1}}, \lambda_c \in \mathbb{R}i\} = \mathbb{R}i[C]^- \cong \mathbb{R}[C]^-$$

where $\mathbb{R}[C]^+$ and $\mathbb{R}i[C]^-$ are the 1 and -1 eigenspaces of the automorphism defined by $c \mapsto c^{-1}$ on the $\mathbb{R}$-vector spaces $\mathbb{R}[C]$ and $\mathbb{R}i[C]$, respectively. We have constructed the following functors.

Lemma 5.43. *Let C be a finite cyclic group and $\varepsilon \in \{+,-\}$. The assignments*

$$D \mapsto \mathrm{map}(\widehat{D}, \mathbb{R})^\varepsilon,$$
$$D \mapsto \mathbb{R}[D]^\varepsilon$$

define covariant functors from the category $Sub(C)$ to the category $\mathcal{VS}_\mathbb{R}$. Further, we have natural isomorphisms

$$\mathbb{R}[D]^\varepsilon \cong \mathrm{map}(\widehat{D}, \mathbb{R})^\varepsilon.$$

Proof. This is a direct consequence of Lemma 2.5. □

Using these functors we can formulate the main result of this section.

Theorem 5.44. *Let C be a finite cyclic group and $m > 1$. There is an isomorphism of $\mathbb{R}$-vector spaces*

$$K_m(\mathbb{Z}C) \otimes_\mathbb{Z} \mathbb{R} \cong \begin{cases} \mathbb{R}[C]^+ & \text{if } m \equiv 1 \bmod 4 \\ \mathbb{R}[C]^- & \text{if } m \equiv 3 \bmod 4 \\ 0 & \text{if } m \equiv 0 \bmod 2 \end{cases}$$

which defines a natural transformation of functors from the category $Sub(C)$ to $\mathcal{VS}_\mathbb{R}$. Additionally, it is natural with respect to automorphisms of C. The isomorphism in the case $m \equiv 3 \bmod 4$ depends on a choice of representatives of $\widehat{C}/\gamma$.

Proof. We will show that the K-groups are isomorphic to $\mathrm{map}(\widehat{C}, \mathbb{R})^\pm$ as a functor in the corresponding cases and then apply the previous lemma. The case $m \equiv 0 \bmod 2$ is obvious, because we have seen that in this case the K-groups are 0. We proceed with the case $m \equiv 1 \bmod 4$. According to Theorem

5.42 as $\mathbb{R}$-vector spaces we have $K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R} \cong \bigoplus_{\widehat{C}/\gamma} \mathbb{R}$. The right-hand side can be identified with $\operatorname{map}(\widehat{C}, \mathbb{R})^+$ via the isomorphism

$$\operatorname{map}(\widehat{C}, \mathbb{R})^+ \to \bigoplus_{\widehat{C}/\gamma} \mathbb{R}, \quad f \mapsto (f(\sigma))_{[\sigma] \in \widehat{C}/\gamma}. \tag{5.1}$$

It remains to prove functoriality. Let D be a subgroup of C and denote by inc the inclusion $D \to C$. Consider the diagram

$$\begin{array}{ccc} \operatorname{map}(\widehat{C}, \mathbb{R})^+ & \xrightarrow{\cong} & \bigoplus_{\widehat{C}/\gamma} \mathbb{R} \\ \alpha \uparrow & & \uparrow \beta \\ \operatorname{map}(\widehat{D}, \mathbb{R})^+ & \xrightarrow{\cong} & \bigoplus_{\widehat{D}/\gamma} \mathbb{R} \end{array}$$

where α is the naturally induced map given by $\alpha(f)(\sigma) = f(\sigma \circ \mathrm{inc})$. We claim that β is induced from the map $\widehat{C}/\gamma \to \widehat{D}/\gamma$ on the index sets defined by $[\sigma] \mapsto [\sigma \circ \mathrm{inc}]$. In order to see that and show that the diagram commutes, we have to take a closer look at the proof of Theorem 5.42. We argued that the isomorphisms involved are natural with respect to inclusion of subgroups and automorphisms of C up to where we reached the continuous cohomology $H_c^m(SL_n(\mathbb{R}C); \mathbb{R})$. The inclusion $\mathbb{R}D \to \mathbb{R}C$ defines a map on the decomposition of these algebras, which we described in Lemma 5.38. Using the notation we introduced there, for a fixed $\tau' \in \mathcal{D}$ and $\sigma' \in \mathcal{C}$ with $\sigma' \circ \mathrm{inc} = \tau'$ the composition

$$\mathbb{K}_{\tau'} \to \prod_{\tau \in \mathcal{D}} \mathbb{K}_\tau \to \prod_{\sigma \in \mathcal{C}} \mathbb{K}_\sigma \to \mathbb{K}_{\sigma'}$$

is either the identity on $\mathbb{R}$ or $\mathbb{C}$, the inclusion $\mathbb{R} \to \mathbb{C}$ or complex conjugation in $\mathbb{C}$. The covariant functor $IH^m(SL_n(-))^*$ respects products and we apply it to the composition to get induced maps

$$IH_c^m(SL_n(\mathbb{K}_{\tau'}); \mathbb{R})^* \to IH_c^m(SL_n(\mathbb{K}_{\sigma'}); \mathbb{R})^*.$$

We know from the computation of continuous cohomology that both are 1-dimensional $\mathbb{R}$-vector spaces. The identity induces again the identity after applying the above functor. According to Lemma 5.29 complex conjugation also induces the identity on the indecomposables in degrees congruent 1 modulo 4. As for the inclusion $\mathbb{R} \to \mathbb{C}$ we can fix a basis element of $IH_c^m(SL_n(\mathbb{C}); \mathbb{R})^*$ and choose the preimage of this element as a basis of $IH_c^m(SL_n(\mathbb{R});\mathbb{R})^*$. With these choices the map $\mathbb{R} \to \mathbb{R}$ after identification of the indecomposables with $\mathbb{R}$ on both sides is also the identity. In summary, β is indeed the map induced from the one on the index sets. We check that for $f \in \mathrm{map}(\widehat{D}, \mathbb{R})^+$

$$(\alpha(f)(\sigma))_{[\sigma]\in\widehat{C}/\gamma} = (f(\sigma \circ \mathrm{inc}))_{[\sigma]\in\widehat{C}/\gamma} = \beta(f(\tau)_{[\tau]\in\widehat{D}/\gamma})$$

and so the diagram commutes. As for automorphisms of C we use Lemma 5.39 to see that the induced $\mathrm{Aut}(C)$-action on $\bigoplus_{\widehat{C}/\gamma}\mathbb{R}$ is given by the action on the index set $\widehat{C}/\gamma$ and note that the isomorphism 5.1 is $\mathrm{Aut}(C)$-equivariant, where an automorphism acts on the left-hand side by acting on $\widehat{C}$. Now assume $m \equiv 3 \bmod 4$. The choice of representatives of $\widehat{C}^-$ denoted by $\mathcal{C}^-$ defines an isomorphism

$$\mathrm{map}(\widehat{C}, \mathbb{R})^- \to \bigoplus_{\widehat{C}^-}\mathbb{R}, \quad f \mapsto (f(\sigma))_{\sigma\in\mathcal{C}^-}. \tag{5.2}$$

Again, we consider the diagram

$$\begin{array}{ccc} \mathrm{map}(\widehat{C}, \mathbb{R})^- & \xrightarrow{\cong} & \bigoplus_{\widehat{C}^-}\mathbb{R} \\ \alpha\uparrow & & \uparrow\beta \\ \mathrm{map}(\widehat{D}, \mathbb{R})^- & \xrightarrow{\cong} & \bigoplus_{\widehat{D}^-}\mathbb{R} \end{array}$$

induced by a subgroup D of C and denote by $\mathcal{D}^-$ a set of representatives of $\widehat{D}^-$. As before, α is the naturally induced map, but in order to determine β we have to take into consideration that complex conjugation in $\mathbb{C}$ induces multiplication with -1 on $IH^m(SL_n(\mathbb{C});\mathbb{R})^*$ according to Lemma 5.29. This occurs whenever $\sigma \circ \mathrm{inc} \notin \mathcal{D}^-$ for $\sigma \in \mathcal{C}^-$ and therefore β is given by

$$\beta(x)_\sigma = \begin{cases} x_{\sigma\circ\mathrm{inc}} & \text{if } \sigma\circ\mathrm{inc} \in \mathcal{D}^- \\ -x_{\overline{\sigma}\circ\mathrm{inc}} & \text{else} \end{cases}$$

for $x = (x_\tau)_{\tau\in\mathcal{D}^-}$. For $f \in \mathrm{map}(\widehat{D}, \mathbb{R})^-$ and $\sigma \in \mathcal{C}^-$ we have $f(\overline{\sigma}\circ\mathrm{inc}) = -f(\sigma\circ\mathrm{inc})$ and using this we compute

$$\beta(f(\tau)_{\tau\in\mathcal{D}^-})_\sigma = \begin{cases} f(\sigma\circ\mathrm{inc}) & \text{if } \sigma\circ\mathrm{inc} \in \mathcal{D}^- \\ -f(\overline{\sigma}\circ\mathrm{inc}) & \text{else} \end{cases} = f(\sigma\circ\mathrm{inc}) = \alpha(f)(\sigma)$$

and thus the diagram commutes. Considering the naturality with respect to automorphisms of C we use again Lemma 5.39 to see that the induced action on $\bigoplus_{\widehat{C}^-} \mathbb{R}$ is given by

$$\phi(x)_\sigma = \begin{cases} x_{\sigma\circ\phi} & \text{if } \sigma\circ\phi \in \mathcal{C}^- \\ -x_{\overline{\sigma}\circ\phi} & \text{else} \end{cases}$$

for $\phi \in \mathrm{Aut}(C)$. But isomorphism 5.2 is equivariant, where the action on $\mathrm{map}(\widehat{C}, \mathbb{R})^-$ is defined by the action on $\widehat{C}$. Last, we apply Lemma 5.43 in both cases to finish the proof. □

5.8 The Artin Defect and the Action of the Automorphism Group

Having determined the groups $K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{R}$ as functors from the subgroup category of C provides the basis for computing the Artin defect in higher degrees. Although we are interested in rational K-theory, we treated the K-groups tensored with $\mathbb{R}$ until now. This was necessary in order to apply Borel's theorem on continuous cohomology. In this section we will formulate the results concerning the Artin defect over $\mathbb{R}$ first and explain afterwards what can be derived for the rational version.

Theorem 5.45. *Let C be finite cyclic group of order $n \in \mathbb{N}$ and for $m > 1$ denote by $S_m(n)$ the cokernel of the map*

$$\bigoplus_{D \lneq C} K_m(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced from inclusions of subgroups D of C, then there is an isomorphism of $\mathbb{R}$-vector spaces

$$S_m(n) \otimes_{\mathbb{Q}} \mathbb{R} \cong \begin{cases} \mathbb{R}[E(C)]^+ & \textit{if } m \equiv 1 \bmod 4 \\ \mathbb{R}[E(C)]^- & \textit{if } m \equiv 3 \bmod 4 \\ 0 & \textit{if } m \equiv 0 \bmod 2 \end{cases}$$

where $E(C)$ is the set of generators of C. In particular,

$$\dim_{\mathbb{R}}(S_m(n) \otimes_{\mathbb{Q}} \mathbb{R}) = \begin{cases} \frac{\varphi(n)}{2} & \text{if } n > 2 \text{ and } m \equiv 1 \bmod 2 \\ 1 & \text{if } n \in \{1,2\} \text{ and } m \equiv 1 \bmod 4 \\ 0 & \text{else} \end{cases}$$

where φ is Euler's phi function.

Proof. We apply Theorem 5.44 from the previous section. If $m \equiv 0 \bmod 2$, then the K-groups are 0 and so is the cokernel of the induction map. We treat the case $m \equiv 1 \bmod 4$ next and write the induction map tensored with $\mathbb{R}$ as

$$\bigoplus_{D \lneq C} \mathbb{R}[D]^+ \to \mathbb{R}[C]^+$$

and recall that + denotes the subspace of all elements whose coefficients satisfy $\lambda_c = \lambda_{c^{-1}}$ for $c \in C$. Therefore, a basis of $\mathbb{R}[D]^+$ for any subgroup D of C is given by the set $\{e_c = \frac{1}{2}(c + c^{-1}) \mid c \in D\}$. Of course, this also holds for the choice $D = C$. The above map is induced by inclusions $D \to C$ and so the only basis elements which do not lie in the image are those e_c where c is a generator of C. Hence, a basis of the cokernel is given by

$$E^+ = \{e_c \mid c \in E(C)\}.$$

Now assume $m \equiv 3 \bmod 4$. We want to make the same argument, but have to describe a basis of $\mathbb{R}[D]^-$ differently. Choose one $c \in \{x, x^{-1}\}$ for each set of pairs $\{x, x^{-1}\} \subseteq D$ with $x \neq x^{-1}$ and denote by $\mathcal{D}$ the set of those, then a basis is given by

$$\{e_c = 1/2(c - c^{-1}) \mid c \in \mathcal{D}\}.$$

We denote by $\mathcal{C}$ the choice corresponding to the group C itself. Again, the basis elements which do not lie in the image of the induction map are those corresponding to generators of C and thus a basis of the cokernel in this case is given by $E^- = \{e_c \mid c \in \mathcal{C}, c \in E(C)\}$. In order to determine the dimension of the cokernel we just have to count the elements in the bases E^+ and E^- of the defect in the two different cases. If C is of order 1 or 2, we have $|E^+| = 1$ and for $n > 2$ we have $|E^+| = \varphi(n)/2$. On the other hand, if C is of order 1 or 2, then $|E^-| = 0$ and for $n > 2$ we have $|E^-| = \varphi(n)/2$. □

The next step is to understand the action of the automorphism group of C on the Artin defect. As mentioned we will once again consider the defect tensored with $\mathbb{R}$ and treat the rational version afterwards. We start with the case $m \equiv 1 \bmod 4$, where the Artin defect is a permutation module.

Theorem 5.46. *Let C be a finite cyclic group and $m > 1$ with $m \equiv 1 \bmod 4$. Denote by I the subgroup of $\operatorname{Aut}(C)$ generated by the automorphism $c \mapsto c^{-1}$ for $c \in C$. There is an isomorphism of $\mathbb{R}\operatorname{Aut}(C)$-modules*

$$S_m(n) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}[\operatorname{Aut}(C)/I].$$

Proof. The isomorphism from Theorem 5.45 ist $\operatorname{Aut}(C)$-equivariant. Further, the basis $E^+ = \{e_c \mid c \in E(C)\}$ of $\mathbb{R}[C]^+$ introduced before is $\operatorname{Aut}(C)$-invariant, because

$$\phi \cdot e_c = 1/2(\phi(c) + \phi(c)^{-1}) = e_{\phi(c)}$$

holds for an automorphism ϕ of C and $c \in E(C)$. Last, the choice of a generator c of C defines an isomorphism of $\operatorname{Aut}(C)$-sets

$$\operatorname{Aut}(C)/I \to E^+, \quad \phi \mapsto e_{\phi(c)}.$$

□

Now we turn to the case $m \equiv 3 \bmod 4$. In this case the Artin defect cannot be a permutation module. To see that assume there is an $\operatorname{Aut}(C)$-invariant basis T of $S_m(n) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}[C]^-$. For every element $e_c = 1/2(c - c^{-1}) \in E^-$ we have

$$\phi_{-1} \cdot e_c = 1/2(c^{-1} - c) = -e_c$$

where $\phi_{-1}(c) = c^{-1}$. It follows that for each $x \in T$ we have $\phi_{-1} \cdot x = -x$. But this contradicts the fact that T is a basis, because x and $-x$ cannot be both elements of a basis. Nonetheless, we still have the following result.

Theorem 5.47. *Let C be a finite cyclic group of order $n \in \mathbb{N}$ and $m > 1$ with $m \equiv 3 \bmod 4$. Denote by I the subgroup of $\operatorname{Aut}(C)$ generated by the automorphism $c \mapsto c^{-1}$ for $c \in C$. There is a short exact sequence of $\mathbb{R}\operatorname{Aut}(C)$-modules*

$$0 \longrightarrow \mathbb{R}[\operatorname{Aut}(C)/I] \xrightarrow{i} \mathbb{R}[\operatorname{Aut}(C)] \xrightarrow{p} S_m(n) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow 0$$

with $i([\phi]) = \phi + \phi \circ \phi_{-1}$ and $p(\phi) = e_{\phi(c)}$ where c is a generator of C.

Proof. The map p is surjective, because the basis E^- lies in the image of p. Further, elements of the form $\phi + \phi \circ \phi_{-1} \in \mathbb{R}[\mathrm{Aut}(C)]$ are contained in the kernel of p, since $p(\phi + \phi \circ \phi_{-1}) = e_{\phi(c)} + e_{\phi(c)^{-1}} = e_{\phi(c)} - e_{\phi(c)} = 0$. They are linear independent and if $n > 2$ there are $\varphi(n)/2$ of those. We know that the dimension of the Artin defect is $\varphi(n)/2$ and so they form a basis of the kernel of p and the above sequence is indeed exact. If $n \in \{1, 2\}$, then the Artin defect is trivial and i is an isomorphism. □

Now we want to state results for rational K-theory, instead of tensored with $\mathbb{R}$. It was necessary to consider the latter, because one crucial step in determining higher K-groups of $\mathbb{Z}C$ as functors was Borel's result stating that for an arithmetic subgroup Γ of a semisimple algebraic group G the induced map

$$H_c^q(G(\mathbb{R}); \mathbb{R}) \to H^q(\Gamma; \mathbb{R})$$

is an isomorphism in certain degrees q. The definition of continuous cohomology relies on having $\mathbb{R}$ as the coefficient field. Despite the fact that in our situation the continuous cohomology of the group $G(\mathbb{R})$ was isomorphic to singular cohomology of the symmetric space G_u/K where G_u is the compact twin, it is not clear whether the resulting isomorphism $H^q(G_u/K; \mathbb{R}) \to H^q(\Gamma; \mathbb{R})$ maps the obvious rational subspace $H^q(G_u/K; \mathbb{Q})$ to $H^q(\Gamma; \mathbb{Q})$. In a subsequent paper to his initial result Borel investigates this question and computes the determinant of the induced map on the indecomposables of the cohomology with real coefficients up to multiplication with a rational number (see [Bor77]). In general, it is not rational and this implies that the natural rational subspaces are not preserved. Nonetheless, we can still deduce results for the Artin defect from the version tensored with $\mathbb{R}$. We start with the dimension as a $\mathbb{Q}$-vector space.

Lemma 5.48. *We have* $\dim_{\mathbb{Q}}(S_m(n)) = \dim_{\mathbb{R}}(S_m(n) \otimes_{\mathbb{Q}} \mathbb{R})$ *for* $n \in \mathbb{N}$ *and* $m > 1$.

Proof. This follows from the fact that the functor $- \otimes_{\mathbb{Q}} \mathbb{R}$ is exact. □

Last, we describe the Artin defect as an element in $K_0(\mathbb{Q}\,\mathrm{Aut}(C))$ using the previous results for $S_m(n) \otimes_{\mathbb{Q}} \mathbb{R}$.

Theorem 5.49. *Let C be a finite cyclic group of order $n \in \mathbb{N}$ and $m > 1$. Denote by I the subgroup of* $\mathrm{Aut}(C)$ *generated by the automorphism $c \mapsto c^{-1}$ for*

$c \in C$. *If* $m \equiv 1 \bmod 4$, *we have* $S_m(n) \cong \mathbb{Q}[\mathrm{Aut}(C)/I]$ *and if* $m \equiv 3 \bmod 4$, *there is an exact sequence of* $\mathbb{Q}\,\mathrm{Aut}(C)$*-modules*

$$0 \longrightarrow \mathbb{Q}[\mathrm{Aut}(C)/I] \longrightarrow \mathbb{Q}[\mathrm{Aut}(C)] \longrightarrow S_m(n) \longrightarrow 0\,.$$

Proof. According to Corollary 2.7 the map

$$K_0(\mathbb{Q}\,\mathrm{Aut}(C)) \to K_0(\mathbb{R}\,\mathrm{Aut}(C)), \quad [M] \mapsto [M \otimes_{\mathbb{Q}} \mathbb{R}]$$

is injective. For $m \equiv 1 \bmod 4$ we showed that $S_m(n) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}[\mathrm{Aut}(C)/I]$ and so $[S_m(n) \otimes_{\mathbb{Q}} \mathbb{R}] = [\mathbb{R}[\mathrm{Aut}(C)/I]] \in K_0(\mathbb{R}\,\mathrm{Aut}(C))$. Since the above map is injective we have $[S_m(n)] = [\mathbb{Q}[\mathrm{Aut}(C)/I]] \in K_0(\mathbb{Q}\,\mathrm{Aut}(C))$ and thus $S_m(n) \cong \mathbb{Q}[\mathrm{Aut}(C)/I]$, because $\mathbb{Q}\,\mathrm{Aut}(C)$ is semisimple. The same argument in the case $m \equiv 3 \bmod 4$ shows that $[S_m(n)] = [\mathbb{Q}[\mathrm{Aut}(C)]] - [\mathbb{Q}[\mathrm{Aut}(C)/I]] \in K_0(\mathbb{Q}\,\mathrm{Aut}(C))$, which implies the existence of the exact sequence stated above. □

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Abstract

The purpose of this thesis is to compute the Artin defect in algebraic K-theory. For $m \in \mathbb{Z}$ the Artin defect of a finite cyclic group C in degree m is defined as the cokernel of the naturally induced map

$$\bigoplus_{D \lneq C} K_m(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $\mathbb{Z}C$ denotes the integral group ring of C. We compute its dimension as a $\mathbb{Q}$-vector space and describe the natural action of the automorphism group of C on the defect. The Artin defect is of importance when computing rational K-theory of integral group rings of any group using the Farrell-Jones conjecture. The Farrell-Jones conjecture is an approach to compute algebraic K-theory of group rings. It is known to be true for a large class of groups.

Zusammenfassung

Gegenstand dieser Dissertation ist die Berechnung des Artin Defektes in der algebraischen K-Theorie. Der Artin Defekt einer endlichen zyklischen Gruppe C im Grad $m \in \mathbb{Z}$ ist definiert als der Kokern der natürlich induzierten Abbildung

$$\bigoplus_{D \lneq C} K_m(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_m(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

wobei $\mathbb{Z}C$ den ganzzahligen Gruppenring von C bezeichnet. Wir berechnen die Dimension des Artin Defektes als $\mathbb{Q}$-Vektorraum und beschreiben die natürliche Wirkung der Automorphismengruppe von C auf dem Defekt. Der Artin Defekt tritt bei der Berechnung der rationalen K-Theorie ganzzahliger Gruppenringe beliebiger Gruppen mittels der Farrell-Jones Vermutung auf. Die Farrell-Jones Vermutung ist ein Ansatz zur Berechnung der algebraischen K-Theorie von Gruppenringen. Sie wurde bereits für eine große Klasse von Gruppen bewiesen.

Eigenständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne unerlaubte Hilfe angefertigt habe. Alle verwendeten Hilfsmittel und Quellen sind im Literaturverzeichnis vollständig aufgeführt und die aus den benutzten Quellen wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

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