Moduli of hypersurfaces in toric orbifolds

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Abstract

In this thesis we construct and study the moduli of hypersurfaces in toric orbifolds. A hypersurface in a variety X is an effective Weil divisor. Explicitly, we construct a quasi-projective coarse moduli space in the category of schemes of quasismooth hypersurfaces in certain toric orbifolds. Such a moduli space has the property that each geometric point represents a hypersurface of a given class up to change of Cox coordinates. Such schemes are constructed as quotients of algebraic group actions. We also examine the moduli spaces in low dimensions and degrees.

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Zusammensfassung

Das Thema dieser Dissertation ist die Modulntheorie der Hyperflächen in torischen Orbifaltigkeiten.

In Kapitel 1 geben wir eine kurze Einführung in die geometrische Invariantentheorie, die Modulntheorie und ihre koabhängige Beziehung. Wir führen die nicht-reduktive geometrische Invariantentheorie ein und führen entsprechende Vergleiche mit dem reduktiven GIT durch. In Kapitel 2 geben wir eine umfassende Einführung in die Theorie der torischen Varietäten, wie sie in dieser Arbeit benötigt wird. Kapitel 3 beinhaltet eine Übersicht über die Konstruktion der Automorphismusgruppe einer torischen Orbifaltigkeit. In Kapitel 4 haben wir das Modulnproblem formal aufgestellt. Dazu definieren wir den Modulnfunktor und beweisen die Existenz einer Familie mit der lokalen universellen Eigenschaft, so dass der Begriff der Äquivalenz durch die Wirkung einer algebraischen Gruppe gegeben ist. In Kapitel 5 stellen wir den A-Diskriminanten vor, der mit einer torischen Varietät X_A und der amplen Klasse $|\alpha|$ in Verbindung gebracht wird. Das A ist eine Ansammlung von Gitterpunkten des Polytops von (X_A, α); siehe Kapitel 2 Details. Wir beweisen, dass der A-Diskriminante, bezeichnet mit Δ_A, eine Semi-Invariante für die Gruppenaktion von $\mathrm{Aut}(X_A)$ auf dem kompletten Linearsystem $|\alpha|$ ist. In Kapitel 6 beweisen wir das Hauptergebnis, dass es einen groben Modulnraum von quasismooth Hyperflächen im gewichteten projektiven Raum gibt, in dem die $(\mathfrak{C}^*)$-Bedingung gilt. Wir beweisen auch die Existenz des Modulnraums von allgemeinen Hyperflächen in Produkten des projektiven Raums mit reduktivem GIT. In Kapitel 7 untersuchen wir das Modulnproblem von Hyperflächen in den gewichteten projektiven Linien.

Contents

Introduction

The aim of this thesis is to study the moduli of hypersurfaces in toric orbifolds. The main tool we use to do this is geometric invariant theory, both reductive and non-reductive. Reductive geometric invariant theory has been very successful in constructing and studying moduli spaces of various types of algebraic objects: both on the theoretical side, providing constructions and hence proving the existence of many key moduli spaces, such as curves and sheaves; and on the computational side, providing machinery to study the geometry of moduli spaces.

Points in a moduli space of hypersurfaces are given by equivalence classes of a relation on a fixed linear system, where the equivalence relation is defined by the action of the automorphism group of the ambient variety. For example, two hypersurfaces in projective space are equivalent if one is mapped to the other under a linear change of coordinates.

Constructing quotients by group actions in algebraic geometry is not as simple as merely taking the topological quotient with respect to the Zariski topology: one must check that the topological quotient, or *orbit space,* has the structure of a scheme and that the quotient map is a morphism of schemes. In practice, there is very rarely an algebraic structure on this orbit space. Mumford's geometric invariant theory (GIT), first presented in [MFK94] and subsequently referred to as reductive GIT, provides an answer to this problem when the acting group is reductive and one is given the extra data of a linearisation (Definition 1.2.2). We only give a brief summary of reductive GIT here as a detailed treatment can be found in Chapter 1; in particular, we suppress the role of the linearisation. Let R be a reductive group acting on a projective variety X. Reductive GIT determines an invariant open subset $X^{\mathrm{s}} \subset X$ for which the orbit space X^{s}/R admits the structure of an algebraic variety and also constructs a compactification of X^{s}/R. This open subset X^{s} is called the stable locus and, in the context of moduli theory, when the scheme X is a parameter space for some kind of object, the variety X^{s}/R is a *coarse moduli space* of stable objects. Thus, when using GIT to study a moduli problem, the task becomes twofold: *i*) which objects are stable (that is, can we phrase stability in geometric terms), and *ii*) what is the geometry of the moduli space itself?

To answer the first of these questions, Mumford introduced a numerical criterion to determine stability in certain situations, which is now known as the Hilbert-Mumford criterion (Theorem 1.2.15). The Hilbert-Mumford criterion comes in many guises and perhaps the most surprising and useful, when the acting group is a torus, is a discrete-geometric form, where the stability of a point is determined by its so-called *weight polytope* (Theorem 1.2.16). Note that even when the acting group in not a torus, one can still use these methods; see Section 1.2.5.

The moduli space of stable hypersurfaces in projective space was constructed by Mumford using reductive GIT. Consider the n-dimensional projective space $\mathbb{P}^n$ and let $d > 2$ be an integer. The linear system $Y_d = |\mathcal{O}_{\mathbb{P}^n}(d)| = \mathbb{P}(k[x_0, \dots, x_n]_d)$ is a parameter space for all hypersurfaces of degree d in $\mathbb{P}^n$ and the action of the reductive group $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}$ on $\mathbb{P}^n$ extends naturally to an action of PGL_{n+1} on Y_d. Mumford calls such actions *classical operations* (see Example 1.2.6) and indeed, the study of these actions goes back at least to Hilbert [Hil93]. It turns out to be hard to describe the open set of stable points in Y_d, let alone the actual quotient variety. However, what Mumford does prove is that if $d > 2$ (and $d > 3$ if $n = 1$), then a smooth hypersurface is stable. Thus reductive GIT constructs a moduli space of smooth hypersurfaces.

In further work [Mum77], Mumford uses the Newton polytope of a hypersurface to study stability and, restricting attention to curves and surfaces of low degrees, he provides a remarkable characterisation of stability in terms of singularity types. Although this type of characterisation is doubtless present and very much desirable in higher dimensions and degrees, it becomes technically more difficult to compute.

There are many similarities between the study of hypersurfaces in projective space and the study of hypersurfaces in toric orbifolds. However, one cannot mirror the construction of moduli spaces of hypersurfaces given above for toric orbifolds: the algebraic groups in question are in general non-reductive. Recent work of Kirwan, Bérczi, Doran and Hawes [BDHK15, BDHK18, BDHK16] develops a non-reductive GIT and allows one to construct such moduli spaces as non-reductive quotients. By a toric orbifold we mean a projective toric variety with at worst orbifold singularities. In toric geometry, such varieties are called simplicial. We refer to Chapter 2 for definitions.

The work of Cox [Cox95b, Cox95a] in the 90's shows that toric orbifolds can be viewed as natural generalisations of projective space. Much of the structure of projective space is also present for toric orbifolds. In his seminal paper [Cox95b], Cox proved that a simplicial toric variety X is a geometric quotient of an open subset of an affine space and that using this quotient, one can give the toric variety 'homogeneous coordinates' analogous to homogeneous coordinates on projective space. Moreover, Cox associates to a complete simplicial toric variety a graded polynomial ring which plays the role of the homogeneous coordinate ring of projective space. As for projective space, the homogeneous coordinate ring encodes all data about the sheaves on X and hence also all closed subschemes. This graded polynomial ring became known as the Cox ring and is the centre of much study, not only confined to toric geometry (see [ADHL15]).

In the same paper [Cox95b], Cox showed that the automorphism group of a complete simplicial toric variety can be calculated from graded automorphisms of the Cox ring. In particular, he proved the automorphism group is a linear algebraic group.

Using these results of Cox, we study the moduli of hypersurfaces in a toric orbifold as a generalisation of the construction for hypersurfaces in projective space. Suppose that X is a toric orbifold and fix an ample class $\alpha \in \mathrm{Cl}(X)$. Denote by $G = \mathrm{Aut}_\alpha(X)$ the subgroup of automorphisms of X which fix α. Cox and Batyrev note in [BC94, Section 13] that a moduli space of hypersurfaces should be constructed as quotient of the action of G on the linear system $|\alpha|$. However, as remarked above, the group G may not be reductive. For example, the Hirzebruch surface $\mathcal{H}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1})$ has non-reductive automorphism group

$$\mathrm{Aut}(\mathcal{H}_2) = \mathrm{GL}_2 \ltimes \mathbb{G}_a^3,$$

and for the weighted projective plane $\mathbb{P}(1,2,3)$, we have that

$$\mathrm{Aut}(\mathbb{P}(1,2,3)) \cong \left(\mathbb{G}_m^2/(\mu_2 \times \mu_3)\right) \ltimes \mathbb{G}_a^3.$$

Thus to construct such quotients, we need a theory of non-reductive GIT.

Non-reductive group actions are tricky for many reasons. They do not in general present any of the nice behaviour exhibited by reductive group actions, we mention two examples. Firstly, by a theorem of Nagata, rings of invariants of reductive group actions are finitely generated. This is not true for non-reductive groups; indeed, Nagata provided a counter example - answering Hilbert's 14th problem in the process - of a $\mathbb{G}_a^{13}$-action with an non-finitely generated invariant ring. Secondly, the induced quotient maps, given by the inclusion of the invariant subring, are not necessarily surjective.

There has been much work in search of GIT for non-reductive groups; see [DK07] for a comprehensive account of work undertaken in this area. In this thesis we use the non-reductive GIT (NRGIT) developed in [BDHK18, BDHK16, Haw15, BDHK15]. We recall the main idea behind NRGIT. Suppose that G is a linear algebraic group acting on a projective variety with respect to a very ample linearisation $\mathcal{L} \in \mathrm{Pic}^G(X)$ and fix a Levi decomposition $G \simeq R \ltimes U$, where U is the unipotent radical and R is the reductive Levi factor. Roughly, the idea is to take the quotient in two stages. First, take the quotient by the unipotent radical $X^{\mathrm{s},U} \to X^{\mathrm{s},U}/U$, which involves determining an open subset $X^{\mathrm{s},U} \subset X$ where this is possible, and then, using reductive GIT, take the quotient by the residual action of R on $X^{\mathrm{s},U}/U$.

To take the unipotent quotient, extra structure on U is required. In [Haw15, BK17] the notion of a graded unipotent group is introduced and provides a method of constructing a quotient by a unipotent group. A graded unipotent group is an extension of a unipotent group by a $\mathbb{G}_m$ with a positivity condition (Definition 1.3.12). The NRGIT theorems require an additional hypothesis, which can be regarded as a version of 'semistability coincides with stability' for the U-action (see Definition 1.3.18) and is refered to as the $(\mathfrak{C}^*)$ *condition.* However, in [BDHK16] a blow-up procedure is outlined which deals with the case where the $(\mathfrak{C}^*)$ condition is not satisfied. This blow-up procedure is based on the partial desingularisation construction of Kirwan [Kir85].

We show that the automorphism group of a toric orbifold does admit a graded unipotent radical (see Proposition 3.1.5 and also [BDHK18, Section 4]) and thus this theory of NRGIT is applicable to the problem of moduli of hypersurfaces in toric orbifolds.

NRGIT comes with its own notions of stability and so one may ask the question: what is the relationship between NRGIT stability of hypersurface in a toric orbifold and the geometry of these hypersurfaces? To do this we introduce a class of hypersurfaces in toric orbifolds: a quasismooth hypersurface in an orbifold is a suborbifold purely of codimension 1. Given that a toric orbifold can be singular, the quasismooth condition allows hypersurfaces to inherit the singularities of the ambient variety. Consequently, if the ambient variety is smooth, quasismoothness coincides with smoothness.

Let X be a weighted projective space where the condition $(\mathfrak{C}^*)$ is satisfied for the action of $\mathrm{Aut}(X)$. Theorem 6.3.12 proves that a Cartier quasismooth hypersurface in X is stable. Let $\mathcal{Y}_d$ be the parameter space of degree d hypersurfaces. The group $\mathrm{Aut}(X)$ acts on $\mathcal{Y}_d$ and we denote the stable locus by $\mathcal{Y}_d^{\mathrm{s}}$ and the quasismooth locus by $\mathcal{Y}_d^{\mathrm{QS}}$. Thus NRGIT constructs a quotient space of such hypersurfaces which is a coarse moduli space. In particular, this coarse moduli space is a scheme.

Theorem (Theorem 6.3.12)**.** *Let $X = \mathbb{P}(a_0, \ldots, a_n) = \mathrm{Proj}\, k[x_0, \ldots, x_n]$ be a well-formed weighted projective space and let $d >> \max\{a_0, \ldots, a_n\} + 2$. Suppose that the $(\mathfrak{C}^*)$ condition holds for the action of $G = \mathrm{Aut}(X)$ on $\mathcal{Y}_d = \mathbb{P}(k[x_0, \ldots, x_n]_d)$. Then there exists a linearisation such that a quasismooth hypersurface of degree d is a stable hypersurface. In other words, there is an inclusion of open subsets*

$$\mathcal{Y}_d^{QS} \subset \mathcal{Y}_d^{s}.$$

In particular, there exists a geometric quotient $\mathcal{Y}_d^{QS}/G$ and hence a coarse moduli space of quasismooth hypersurfaces of degree d in X. Moreover, the NRGIT quotient $\mathcal{Y}_d /\!\!/_{\mathcal{O}(1)^{\chi_\epsilon}} G$ is a compactification of $\mathcal{Y}_d^{QS}/G$.

The proof of the theorem relies on a discrete-geometric version of the Hilbert-Mumford criterion for NRGIT.

We provide an explicit construction for quasismooth hypersurfaces in $X = \mathbb{P}(1, \ldots, 1, r)$. The construction has two main ingredients; the finiteness of the stabilisers and the presence of the A-discriminant as an invariant section. The A-discriminant (Definition 5.1.7) is a hypergeometric function which can detect quasismoothness on a given locus of a hypersurface and is a generalisation of the classical notion of the discriminant.

To study the quasismooth locus in a given linear system, one uses the A-discriminant, defined and studied in [GKZ08] for general toric varieties and denoted by Δ_A. It follows that $\mathcal{Y}_d^{\mathrm{QS}} \subset (\mathcal{Y}_d)_{\Delta_A}$. We prove in Chapter 5 that the A-discriminant can be interpreted as an invariant section of an appropriate line bundle, just as for the classical discriminant.

Theorem (Corollary 5.2.8)**.** *Let X be the toric variety associated to a polytope P and let A be the lattice points of P. The A-discriminant Δ_A is a semi-invariant section for the G-action on $\mathcal{Y}_\alpha$ and a true U-invariant, where $U \subset G$ is the unipotent radical of G.*

Restricting our attention to a weighted projective space X, we prove in Chapter 3 that the stabiliser groups of the action of $G = \mathrm{Aut}(X)$ on $\mathcal{Y}_d^{\mathrm{QS}}$ is finite for $d \geq \max(a_0, \ldots, a_n) + 2$. The proof is a generalisation of the proof of Matsumura and Monsky [MM63] for hypersurfaces in projective space. In the same paper, they prove that, under mild conditions, the stabiliser groups coincide with the automorphism groups. Denote the stabiliser group by $\mathrm{Aut}(Y; X)$.

Theorem (Theorem 3.3.7)**.** *A quasismooth hypersurface in $X = \mathbb{P}(a_0, \ldots, a_n)$ of degree $d \geq \max\{a_0, \ldots, a_n\} + 2$ has only finitely many automorphisms coming from the automorphisms of the ambient weighted projective space. That is, the group $\mathrm{Aut}(Y; X)$ is finite for a quasismooth hypersurface $Y \subset \mathbb{P}(a_0, \ldots, a_n)$.*

A corollary of this theorem is the existence of a moduli space as an algebraic space. This is a direct consequence of the Keel-Mori theorem. Explicitly, let $X = \mathbb{P}(a_0, \ldots, a_n)$ be a weighted projective space $d > \max\{a_0, \ldots, a_n\} + 1$ be an integer. Then the quotient stack $\left[\mathcal{Y}_d^{\mathrm{QS}} / \mathrm{Aut}(X)\right]$ admits a coarse moduli space. The Keel-Mori theorem asserts the existence of a coarse moduli space as an algebraic space; however, Theorem 6.3.12 implies that this algebraic space is in fact a quasi-projective variety.

There are many different classes of varieties which present themselves as hypersurfaces in weighted projective spaces; for example, genus 2 curves are degree 6 curves in $\mathbb{P}(1,1,3)$, Petri special curves are degree 6 curves in $\mathbb{P}(1,1,2)$ and degree 2 del Pezzo surfaces are degree 4 surfaces in $\mathbb{P}(1,1,1,2)$ to name a few. Hence work in this thesis offers constructions of new moduli spaces or new constructions of well-known moduli spaces.

Layout

The layout of this thesis is as follows:

In Chapter 1, we give a brief introduction to geometric invariant theory, moduli theory and their codependent relationship. We introduce non-reductive geometric invariant theory and make appropriate comparisons to reductive GIT.

In Chapter 2, we provide a comprehensive introduction to the theory of toric varieties as it shall be needed in this thesis. Most notably, we discuss quasismooth hypersurfaces in toric orbifolds and their topology and geometry.

Chapter 3 contains a review of the construction of the automorphism group of a toric orbifold due to Cox [Cox14]. We recall his description explicitly so as to prove that these automorphism groups exhibit the structure required by NRGIT. We then prove that a quasismooth hypersurface in weighted projective space (omitting certain low degrees) has a finite stabiliser group and hence a finite automorphism group. We conclude the chapter with results pertaining to the connection between two equivalence relations on a linear system: one given by isomorphism and the other given by ambient automorphisms.

In Chapter 4 we formally set up the moduli problem. To do this we define the moduli functor and prove the existence of a family with the local universal property such that the notion of equivalence is given by the action of an algebraic group. To define the moduli functor we exhibit some structure results about Hilbert schemes of hypersurfaces in projective varieties. We prove that there exists a coarse moduli space (as an algebraic space) of quasismooth hypersurfaces in weighted projective space. We also show that the topological type of a smooth divisor in a toric orbifold is fixed by the value in the class group; this is proved for weighted projective space in Proposition 2.6.6 and for general projective toric orbifolds in Theorem 4.1.15.

In Chapter 5 we introduce the A-discriminant associated to a toric variety X_A and ample class α. The A is a collection of lattice points of the polytope of (X_A, α); see Chapter 2 for details. We prove that the A-discriminant, denoted Δ_A, is a semi-invariant for the group action of $\mathrm{Aut}(X_A)$ on the complete linear system $|\alpha|$. We also prove that the discriminant locus in the linear system is exactly the projective dual of the variety X_A. We also prove that $\mathcal{Y}_d^{\mathrm{QS}} \subset (\mathcal{Y}_d)_{\Delta_A}$ and semistability of quasismooth hypersurfaces follows.

In Chapter 6 we prove the main result; that there exists a coarse moduli space of quasismooth hypersurfaces in weighted projective space where the $(\mathfrak{C}^*)$ condition holds. We also prove the existence of the moduli space of general type hypersurfaces in products of projective spaces using reductive GIT.

In Chapter 7 we examine the moduli problem of hypersurfaces in the weighted projective lines $\mathbb{P}(1, r)$. In this case the automorphism groups are of the form $(\mathbb{G}_m)/\mu_r \ltimes \mathbb{G}_a$. We begin by recalling the equivalence of $\mathbb{G}_a$-actions with locally nilpotent derivations. We then use this equivalence in Section 7.2 to compute explicitly the ring of invariants for low degree hypersurfaces in a weighted projective line.

Example. (Example 7.2.4) Let $X = \mathbb{P}(1,2) = \operatorname{Proj} k[x, y]$ be the weighted projective line, where $\deg x = 1$ and $\deg y = 2$. Then $\operatorname{Aut}(X) = (\mathbb{G}_m)/\mu_2 \ltimes \mathbb{G}_a$ and

$$\mathbb{P}(k[x,y]_6) /\!\!/ \operatorname{Aut}(X) = \mathbb{P}(4,6).$$

For this example, the moduli space constructed can be interpreted as the moduli of four points on $\mathbb{P}^1$. This can be seen from the fact that a hypersurface consists of 3 points and that the stability condition forces you to miss the stacky point. Thus this gives an alternate construction of the moduli space of elliptic curves.

Notation and conventions

We work over an algebraically closed field k of characteristic 0. A lattice is a finitely generated free abelian group. A scheme is an algebraic scheme; that is, a scheme of finite type over k and we denote the category of such schemes by $\mathfrak{Sch}$. A variety shall be a separated integral scheme of finite type over k. In particular, a variety is irreducible. If X is a scheme, we denote its functor of points by

$$\underline{X} = \operatorname{Hom}(-, X) : \mathfrak{Sch}^{\mathrm{op}} \longrightarrow \mathfrak{Sets},$$

where $\mathfrak{Sets}$ is the category of sets. If a topological space satisfies the condition that every cover of it by open sets admits a finite subcover then we say it is 'quasi-compact'. By a 'point' in a scheme we will always mean a closed k-valued point and when we write $x \in X$

we mean a closed point. A compactification $X \to \overline{X}$ of a variety X is a dominant open immersion into a projective variety $\overline{X}$.

When we talk about actions of groups on varieties or vector spaces, we always mean a left action, unless stated otherwise.

Associated to a vector bundle V we understand the projective space $\mathbb{P}(V)$ to be the space whose points correspond to one-dimensional subbundles of V. Another way to say this is that $\mathbb{P}(V) = \operatorname{Proj}(\operatorname{Sym}(V^\vee))$, where $\operatorname{Sym}(V^\vee)$ is the symmetric algebra $\bigoplus_{m\geq 0} \operatorname{Sym}(V^\vee)$. With these conventions, if $L \to X$ is a very ample line bundle on a scheme X with a base-point-free linear system $V \subset H^0(X, L)$, then there is a canonical morphism $X \to \mathbb{P}(V^\vee)$.

For a scheme X, we denote the tangent space of X at a point $x \in X$ by $\mathbf{T}_x X$. For a line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ we denote the section ring by

$$A(X, \mathcal{L}) = \bigoplus_{r\geq 0} H^0(X, \mathcal{L}^{\otimes r}).$$

Chapter 1

Geometric invariant theory

In this chapter we review standard constructions in classical (reductive) geometric invariant theory along with modern constructions of non-reductive geometric invariant theory. We recall basic definitions and facts from moduli theory and their connection to GIT, both reductive and non-reductive.

In this thesis we shall only consider moduli problems as functors and search for schemes which (come close to) representing them. We largely avoid the language of stacks, though this is the most natural setting for this type of study.

1.1 Moduli problems

A moduli problem naively consists of two things. First of all, a class of objects together with a notion of what it means to have a family of these objects over a scheme. Second, a notion of equivalence between families of these objects. The word objects here is intentionally vague. Typically, the objects we are interested are algebro-geometric objects such as schemes, sheaves or morphisms or combinations of these. The definition of a moduli problem is somewhat vague; it is a presheaf whose objects have a higher meaning.

Definition 1.1.1. A *moduli problem* is a pair consisting of a presheaf on the category of schemes $\mathfrak{Sch}$ and a notion of equivalence. The k-points of the presheaf correspond to some fixed objects and the T-points for any $T \in \mathfrak{Sch}$ correspond to families of these objects parametrised by the scheme T. The notion of equivalence is given by an equivalence relation on every set of T-points, satisfying a compatibility relation.

More precisely, a moduli problem is a functor

$$\widetilde{\mathcal{M}} : \mathfrak{Sch}^{\mathrm{op}} \longrightarrow \mathfrak{Sets},$$

and a set of equivalence relations $\{\sim_T\}_{T \in \mathfrak{Sch}}$ such that $\widetilde{\mathcal{M}}(k)$ consists of objects and the following map

$$\mathcal{M} : \mathfrak{Sch}^{\mathrm{op}} \longrightarrow \mathfrak{Sets}, \quad T \mapsto \widetilde{\mathcal{M}}(T)/ \sim_T$$

is a functor.

For a scheme $T \in \mathfrak{Sch}$ and some equivalence class $[\mathcal{F}] \in \mathcal{M}(T)$, we refer to $\mathcal{F}$ as a *family parametrised by T,* where $\mathcal{F} \in \widetilde{\mathcal{M}}(T)$ is a representative of $[\mathcal{F}]$. We refer to $\mathcal{M}$ as the moduli functor.

Recall the two notions of solutions to moduli problems.

Definition 1.1.2. A scheme M representing a moduli functor $\mathcal{M}$ is a *fine moduli space* for $\mathcal{M}$. A scheme M *corepresents* a functor $F : \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$ if there is a natural transformation $\eta : F \to \underline{M}$ such that for all schemes N and natural transformations $\eta' : F \to \underline{N}$ there is a unique morphism $\phi : M \to N$ such that $\eta' = \underline{\phi} \circ \eta$.

We say that M is a *coarse moduli space* for a moduli functor $\mathcal{M}$ if it corepresents $\mathcal{M}$ and if for every algebraically closed field k', there is a bijection

$$\eta(k') : M(k') \xrightarrow{\simeq} \mathcal{M}(k').$$

Remark 1.1.3. A moduli problem is often defined by the richer notion of a moduli prestack[1]

$$\mathcal{M}^{\mathrm{stack}} : \mathfrak{Sch}^{\mathrm{op}} \longrightarrow \mathfrak{Grp}.$$

The key difference between a moduli problem and a moduli prestack is that the prestack takes values in the category of groupoids and hence remembers all the isomorphisms of families, whereas the moduli problem only recalls equivalence classes and so only recalls if two families are isomorphic or not. Given a moduli prestack, one can pass to a moduli problem.

[1] see Vistoli's chapter in [FGI+05, Chapter 1] for the definition of a prestack.

Let $\mathcal{M}$ be the moduli functor defined by $\mathcal{M}^{\text{stack}}$. If the prestack $\mathcal{M}^{\text{stack}}$ is indeed a stack, a coarse moduli space for the functor $\mathcal{M}$ will be a coarse moduli space for $\mathcal{M}^{\text{stack}}$ in the sense of [Ols16, Definition 11.1.1]. We note that the moduli functors considered in Chapter 4 will not necessarily define moduli stacks; thus we may have to stackify.

A moduli functor admitting a fine moduli space is the ideal situation; we have a scheme structure on the set of equivalences classes and moreover, there exists a so-called *universal family* U parametrised by M corresponding to the identity morphism $\mathrm{Id}_M : M \to M$ such that any family parametrised by any other scheme is the pullback of this family. Unfortunately, many moduli functors are not representable due to the presence of automorphisms. A coarse moduli space is the best approximation. Note that coarse moduli spaces also often do not exist, one often has to restrict to a subset of the objects considered. The objects for which a coarse moduli space exists are often called *stable* .

1.2 Reductive geometric invariant theory

1.2.1 Group quotients and linearisations

Let G be a linear algebraic group acting on a scheme X. We denote the action morphism by $\sigma : G \times X \to X$.

Definition 1.2.1. A *categorical quotient* for the G-action on X is a pair (Y, ϕ), where Y is a scheme and $\phi : X \to Y$ is a G-invariant morphism such that for every scheme Z and G-invariant morphism $\phi' : X \to Z$ there exists a unique morphism $\psi : Y \to Z$ such that $\phi' = \psi \circ \phi$. The scheme Y is called an *orbit space* if for every point $y \in Y$ the preimage $\phi^{-1}(y)$ is a single orbit.

A *geometric quotient* for the G-action on X is a pair (Y, ϕ), where Y is a scheme and $\phi : X \to Y$ is a G-invariant morphism satisfying the following properties:

1. the morphism ϕ is the topological quotient; ϕ is surjective, Y is an orbit space and $U \subset Y$ is open if and only if $\phi^{-1}(U) \subset X$ is open; and,

2. the morphism of sheaves $\phi^{\#} : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \simeq \phi_*(\mathcal{O}_X^G)$.

We say that $\phi : X \to Y$ is a *principal G-bundle* over a scheme Y if there is an étale covering $U \to Y$ such that there is a G-equivariant isomorphism $X \times_Y U \cong G \times U$ where G acts on $G \times U$ by only acting on the first factor. A principal G-bundle $X \to Y$ is a geometric quotient.

Note that a geometric quotient is a categorical quotient by [MFK94, Proposition 0.1]. A geometric quotient is the ideal situation as in this case the points of Y are in one-to-one correspondence with the orbits of the G-action, whereas for a categorical quotient some orbits may be identified.

Mumford's Geometric Invariant Theory (GIT), introduced in [MFK94], allows us to construct categorical quotients and geometric quotients of open subsets of X for reductive group actions. One characterisation of reductive groups is that these are the linear algebraic groups whose unipotent radical is trivial. See [Bor91] for the definition of reductive and unipotent groups.

Recall a G-equivariant sheaf is a pair $(\mathcal{F}, \Phi)$ where $\mathcal{F} \in \mathcal{O}_X$-Mod and $\Phi : \mathrm{pr}_X^* \mathcal{F} \xrightarrow{\simeq} \sigma^* \mathcal{F}$ is an isomorphism of sheaves on $G \times X$ satisfying a cocycle condition (see [MFK94, Definition 1.6] for the definition of the cocycle condition).

Definition 1.2.2. Let G be a linear algebraic group (not necessarily reductive) acting on a variety X. A *linearisation* of the G-action on X is an element $(\mathcal{L}, \Phi)$ of $\mathrm{Pic}^G(X)$. Where $\mathrm{Pic}^G(X)$ is the group of G-equivariant invertible sheaves. We say that G acts on X with respect to a linearisation $\mathcal{L}$. We say that a linearisation $(\mathcal{L}, \Phi)$ is (very) ample if the invertible sheaf $\mathcal{L}$ is (very) ample.

The data of a linearisation induces a dual action on $H^0(X, \mathcal{L})$ given by

$$H^0(X, \mathcal{L}) \xrightarrow{\sigma^*} H^0(G \times X, \sigma^* \mathcal{L}) \xrightarrow{\Phi} H^0(G \times X, \mathrm{pr}_X^* \mathcal{L}) \simeq H^0(G, \mathcal{O}_G) \otimes H^0(X, \mathcal{L}),$$

where the last isomorphism follows from the Künneth formula. Therefore, we may speak of *invariant sections* of $\mathcal{L}$.

Remark 1.2.3. Let $X = \mathbb{P}^n$ be a projective space and let G be an algebraic group acting on X. To define a linearisation on $\mathcal{O}_{\mathbb{P}^n}(d)$ it is enough to define an action on $H^0(X, \mathcal{O}_{\mathbb{P}^n}(d))$ by [MFK94, p.33], see also Example 1.2.6.

Example 1.2.4. Let $X = \mathbb{P}^n$ and consider the algebraic group $G = \operatorname{Aut}(X) = \mathrm{PGL}_{n+1}$. If $N = n^2 + 2n$, then we can see G as the principal open subset in

$$\mathbb{P}^N = \operatorname{Proj} k[a_{00}, \dots, a_{0n}; \dots ; a_{n0}, \dots, a_{nn}]$$

defined by $\det \neq 0$. Then we can define an action $G \times X \to X$ by the formula

$$\sigma((a_{ij}), (x_0 : \dots : x_n)) = \Big(\sum_{j=0}^{n} a_{0j} x_j : \dots : \sum_{j=0}^{n} a_{nj} x_j \Big).$$

Note that this is the restriction of the rational map $\sigma' : \mathbb{P}^N \times X \dashrightarrow X$ defined by sections

$$\Big(\sum_{j=0}^{n} a_{0j} x_j \Big), \dots, \Big(\sum_{j=0}^{n} a_{nj} x_j \Big) \in H^0(\mathbb{P}^N \times X, \mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{O}_X(1)).$$

The rational map σ' is not defined at any point $(A, x) \in \mathbb{P}^N \times X$ where $A \cdot x = 0$ since for any such pair (A, x) it holds that $\det A = 0$. Denote the locus of such points by

$$Z = \{(A, x) \in \mathbb{P}^N \times X \mid A \cdot x = 0\} \subset \mathbb{P}^N \times X.$$

It holds that $\mathbf{V}(\det) = \mathrm{pr}_1(Z) \subset \mathbb{P}^N$, where $\mathrm{pr}_1 : \mathbb{P}^N \times X \to \mathbb{P}^N$. Moreover, the restricted map $\mathrm{pr}_1 : Z \to \mathbf{V}(\det)$ is birational (since it is an isomorphism over matrices with rank n, which form an open set in $\mathbf{V}(\det)$). This shows that Z is of codimension greater than 2 in $\mathbb{P}^N \times X$ and hence, by Hartog's lemma, the line bundle $\sigma^* \mathcal{O}_X(1)$ on $G \times X$ is the restriction of the line bundle

$$\mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{O}_X(1) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathrm{pr}_X^* \mathcal{O}_X(1),$$

on $\mathbb{P}^N \times X$ so that

$$\sigma^* \mathcal{O}_X(1) = (\mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{O}_X(1))|_{G \times X}.$$

Thus, for every point $x \in X$, the sheaf $\sigma^* \mathcal{O}_X(1)$ restricted to $G \times \{x\}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^N}(1)$ restricted to $G = \mathbb{P}^N - \mathbf{V}(\det)$.

Suppose that $\mathcal{O}_X(1)$ admits a G-linearisation. Then we have an isomorphism of $\mathcal{O}_{G \times X}$-sheaves $\sigma^* \mathcal{O}_X(1) \simeq \mathrm{pr}_X^* \mathcal{O}_X(1)$ and hence $\mathcal{O}_{\mathbb{P}^N}(1)$ restricted to G must be trivial, as $\mathrm{pr}_X^* \mathcal{O}_X(1)|_{G \times \{x\}} \cong \mathcal{O}_G$. However, since $\mathbf{V}(\det)$ is a degree n hypersurface in $\mathbb{P}^N$, we know that $\operatorname{Pic}(G) = \mathbb{Z}/(n+1)\mathbb{Z}$ and this group is generated by $\mathcal{O}_{\mathbb{P}^N}(1)|_G$. This gives a contradiction and hence the action of G on X admits no $\mathcal{O}_X(1)$-linearisation. Note that higher powers of $\mathcal{O}_X(1)$ do admit PGL_{n+1}-linearisations, in particular, the anti-canonical bundle $\mathcal{O}_X(n+1)$ admits a linearisation (see [MFK94, p.34]).

Remark 1.2.5. Let X be a scheme with an action of a linear algebraic group G and $\mathcal{L} \in \mathrm{Pic}(X)$ an invertible sheaf. One can define a linearisation in terms of the corresponding line bundle. Denote the corresponding line bundle to $\mathcal{L}$ by $\pi : L \to X$. Consider an action of G on L such that:

1. the structure morphism of the bundle $\pi : L \to X$ is equivariant and,
2. for all $g \in G$ and $x \in X$ the map of fibres $L_x \to L_{g\cdot x}$ is a linear isomorphism.

The data of a G-action on L satisfying the above conditions is equivalent to the data of a linearisation. Let us describe how to construct such an action from a linearisation: let $(\mathcal{L}, \Phi)$ be a linearisation and consider $x \in X$ and $g \in G$. Consider the morphism of vector bundles Φ at the fibre at (g, x):

$$\Phi_{(g,x)} : (\mathrm{pr}_X^* L)_{(g,x)} = L_x \xrightarrow{\simeq} (\sigma^*)_{(g,x)} = L_{g\cdot x}\,,$$

where $\mathrm{pr}_X : G \times X \to X$. Then Φ_x is a linear isomorphism and this isomorphism defines an action on L. For the proof of the equivalence of the two definitions we refer to [Bri15, Lemma 3.2.4].

Example 1.2.6. Let V be an $(n+1)$-dimensional vector space and $d > 0$ a positive integer. Consider the natural representation

$$\mathrm{GL}(V) \longrightarrow \mathrm{GL}(\mathrm{Sym}^d(V)).$$

This representation defines an action of $\mathrm{PGL}(V)$ on $\mathbb{P}(\mathrm{Sym}^d(V))$. Such actions are defined by Mumford to be *classical operations* in [MFK94]. Note that for $d = 1$ this is precisely the action of Example 1.2.4. By the same arguments as in Example 1.2.4, the line bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^d(V))}(1)$ admits no linearisation. On the other hand, the action of $\mathrm{SL}(V)$ on $\mathbb{P}(\mathrm{Sym}^d(V))$ defined by the isogeny $\mathrm{SL}(V) \to \mathrm{PGL}(V)$ (which has the same orbits) does admit an $\mathcal{O}(1)$-linearisation: indeed, $\mathrm{SL}(V)$ acts canonically on $\mathrm{Sym}^d(V) - \{0\}$ so that the projection

$$(\mathrm{Sym}^d(V) - \{0\}) \longrightarrow \mathbb{P}(\mathrm{Sym}^d(V))$$

is $\mathrm{SL}(V)$-equivariant. This defines an action of $\mathrm{SL}(V)$ on $\mathrm{Tot}(\mathcal{O}(-1))$ since $\mathrm{Tot}(\mathcal{O}(-1))$ is obtained by blowing up $0 \in \mathrm{Sym}^d(V)$, which is a fixed point (see [Kir85] for equivariant blow-ups). By Remark 1.2.5, this defines a $\mathcal{O}(-1)$-linearisation. Since $\operatorname{Pic}^G(X)$ is a group, we obtain a $\mathcal{O}(1)$-linearisation as the dual to the $\mathcal{O}(-1)$-linearisation described above.

Example 1.2.7. Let X be a scheme and $L = X \times k$ be the trivial line bundle. We define an $\mathcal{O}_X$-linearisation of a G-action on X using a character $\chi : G \to \mathbb{G}_m$: consider a point $(x, z) \in L$, then

$$g \cdot (x, z) = (g \cdot x, \chi(g)z).$$

Definition 1.2.8. Let $\mathcal{L} \in \operatorname{Pic}^G(X)$ be an arbitrary linearisation and $\pi : L \to X$ be the corresponding line bundle with a G-action. We can modify the G-action of on L using a character $\chi : G \to \mathbb{G}_m$ by defining fibrewise

$$g \cdot_\chi (x, z) = (g \cdot x, \chi(g) g \cdot z)$$

for $x \in X$ and $g \in G$. This process of modification is called *twisting $\mathcal{L}$ by the character χ* and the corresponding linearisation is denoted $\mathcal{L}^\chi$.

1.2.2 Affine reductive GIT quotients

Let R be a reductive group acting on an affine scheme $\operatorname{Spec} A$, where A is a finitely generated k-algebra. This action induces an action of R on A which defines a subalgebra $A^R \subset A$. We call this algebra the *ring of invariants.* The induced morphism

$$\phi : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A^R$$

is called the affine GIT quotient. The following theorem of Nagata explains why the reductive property is so important in the theory of quotients in algebraic geometry.

Theorem 1.2.9 (Nagata's theorem)**.** [Nag63] *Let R be a reductive group acting on a finitely generated k-algebra A. Then the ring of invariants A^R is a finitely generated k-algebra.*

By Nagata's theorem, $\operatorname{Spec} A^R$ is again an affine scheme (of finite type over k), moreover, if $\operatorname{Spec} A$ is an affine variety, then $\operatorname{Spec} A^R$ will also be an affine variety, since $A^R \subset A$ cannot contain any nilpotent elements.

Theorem 1.2.10. [MFK94, Theorem 1.1] *Let R be a reductive group acting on an affine scheme* $\operatorname{Spec} A$*. Then the affine GIT quotient $\phi : \operatorname{Spec} A \to \operatorname{Spec} A^R$ is a categorical quotient.*

1.2.3 Projective quotients

Let X be a projective variety and R a reductive group acting on X. To construct a categorical quotient of the R-action on X, we need more data. This data is given by a linearisation.

We assume now that R is acting on X with respect to a very ample linearisation $\mathcal{L} \in \mathrm{Pic}^R(X)$. The linearisation induces an action on the vector space $H^0(X, \mathcal{L}^{\otimes r})$ for every $r \geq 0$ and the *ring of invariants*

$$\bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r})^R = A(X, \mathcal{L})^R \subset A(X, \mathcal{L}) \tag{$\star$}$$

forms a graded subalgebra. We define the *projective GIT quotient* to be the associated projective scheme

$$X /\!\!/_{\mathcal{L}} R = \mathrm{Proj}\, A(X, \mathcal{L})^R.$$

Note that the inclusion ($\star$) defines a rational morphism

$$q_R : X \dashrightarrow X /\!\!/_{\mathcal{L}} R.$$

Definition 1.2.11. Let X be a projective scheme with an R-action and $\mathcal{L}$ be an ample linearisation.

1. A point $x \in X$ is *semistable* with respect to $\mathcal{L}$ if there exists a non-zero invariant section $\sigma \in H^0(X, \mathcal{L}^{\otimes r})^R$ for $r > 0$ such that $x \in X_\sigma$. The open subset of semistable points is called the *semistable locus* and is denoted $X^{\mathrm{ss}}(\mathcal{L})$.

2. A point $x \in X$ is *stable* with respect to $\mathcal{L}$ if $\dim G \cdot x = \dim G$ and $x \in X_\sigma$ for some $\sigma \in H^0(X, \mathcal{L}^{\otimes r})^R$ for $r > 0$ such that the action on G on X_σ is closed. The open subset of stable points is called the *stable locus* and is denoted $X^{\mathrm{s}}(\mathcal{L})$.

3. We define the *unstable locus* to be the complement of the semistable locus and it is denoted by $X^{\mathrm{us}}(\mathcal{L}) = X - X^{\mathrm{ss}}(\mathcal{L})$.

Notation 1.2.12. We write $X^{(\mathrm{s})\mathrm{s}}$ for the (semi)stable locus when the linearisation is clear from context. Note that X^{ss} is the domain of definition of q_R.

Theorem 1.2.13. [MFK94, Theorem 1.10] *Let R be a reductive group acting on a projective scheme X with respect to an ample linearisation $\mathcal{L}$. Then the rational map q_R restricts to an affine morphism*

$$q_R : X^{ss} \longrightarrow X /\!\!/_{\mathcal{L}} R$$

which is a categorical quotient of the R-action on X^{ss}. Furthermore, $Y = X^s/R \subset X /\!\!/_{\mathcal{L}} R$ is an open subset such that $q_R^{-1}(Y) = X^s$ and $q_R : X^s \to Y$ is a geometric quotient.

Remark 1.2.14. Let $x, y \in X^{\mathrm{ss}}(\mathcal{L})$ be two semistable points. We say that x and y are S-equivalent if

$$\overline{R \cdot x} \cap \overline{R \cdot y} \cap X^{\mathrm{ss}}(\mathcal{L}) \neq \emptyset.$$

Then x and y are S-equivalent if and only if $q_R(x) = q_R(y)$; for example, see [Muk03].

1.2.4 The Hilbert-Mumford criterion

In this section we state a numerical criterion for stability due to Hilbert and Mumford. The significance of the existence of this criterion cannot be understated; determining from first principles whether or not a point is stable or semistable can be practically impossible, as computing the ring of invariants is a very hard problem. The Hilbert-Mumford criterion gives a way of identifying and studying the (semi)stable locus without computing invariant rings.

Let X be a projective scheme with an action of a reductive group R and let $\lambda : \mathbb{G}_m \to R$ be a 1-parameter subgroup. We have a natural embedding $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$ given by $t \mapsto [1 : t]$. For a fixed $x \in X$, the morphism $\lambda_x : \mathbb{G}_m \to X$ defined by $\lambda_x(t) = \lambda(t) \cdot x$ extends to a morphism $\hat{\lambda}_x : \mathbb{P}^1 \to X$ because X is proper. We define

$$\lim_{t \to 0} \lambda(t) \cdot x = \hat{\lambda}_x([1 : 0]) \quad \textit{and} \quad \lim_{t \to \infty} \lambda(t) \cdot x = \hat{\lambda}_x([0 : 1]).$$

The limit $\bar{x} = \lim_{t \to 0} \lambda(t) \cdot x$ is fixed by $\lambda(\mathbb{G}_m)$, as is the other limit. Thus $\mathbb{G}_m$ acts on the fibre $\mathcal{L}_{\bar{x}}$ via λ by a character $t \to t^r$. We define the *Hilbert-Mumford weight* to be

$$\mu^{\mathcal{L}}(x, \lambda) = r.$$

Theorem 1.2.15. [MFK94, Theorem 2.1] *Let R be a reductive group acting on a projective scheme X with respect to an ample linearisation $\mathcal{L}$. Then*

$$x \in X^{ss}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x,\lambda) \geq 0 \text{ for all 1-parameter subgroups } \lambda \text{ of } R \text{ and,}$$
$$x \in X^{s}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x,\lambda) > 0 \text{ for all 1-parameter subgroups } \lambda \text{ of } R.$$

1.2.5 The weight polytope

We can give a discrete-geometric description of the Hilbert-Mumford criterion. Suppose that the linearisation $\mathcal{L}$ is very ample, so that we have an R-equivariant embedding $X \hookrightarrow \mathbb{P}(H^0(X,\mathcal{L})^\vee)$, where $V = H^0(X,\mathcal{L})^\vee$ has an induced R-action. Fix a maximal torus $T \subset R$ and consider the T-weight space decomposition

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi,$$

where $X^*(T) = \mathrm{Hom}(T,k^*)$ is the character group and $V_\chi = \{v \in V \mid t \cdot v = \chi(t)v \ \forall t \in T\}$. The characters χ such that $V_\chi \neq 0$ are called the T-weights of V.

Consider $x \in X$ and some $v \in V$ lying over x and write $v = \sum v_\chi$. We define the T-weight set of x to be

$$\mathrm{wt}_T(x) = \{\chi \mid v_\chi \neq 0\},$$

and the associated weight polytope to be the convex hull of these weights:

$$\mathrm{Conv}_T(x) = \mathrm{Conv}(\chi \mid \chi \in \mathrm{wt}_T(x)) \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Theorem 1.2.16. [Dol03, Theorem 9.2 and 9.3] *Let R be a reductive group acting on a projective scheme X with linearisation $\mathcal{L}$. Then*

$$x \in X^{ss}(\mathcal{L}) \iff 0 \in \mathrm{Conv}_T(g \cdot x) \text{ for every } g \in R \text{ and,}$$
$$x \in X^{s}(\mathcal{L}) \iff 0 \in \mathrm{Conv}_T(g \cdot x)^\circ \text{ for every } g \in R,$$

where $\mathrm{Conv}_T(g \cdot x)^\circ$ *is the interior of the polytope.*

1.3 Non-reductive geometric invariant theory

There are many instances in algebraic geometry where one wishes to take a quotient by a non-reductive group and many efforts have been made to generalise GIT to the non-reductive case [GP93, Fau89]. The most successful generalisation, recovering many of the properties of reductive GIT began with the work of Kirwan and Doran in [DK07] and remains an active topic. Other than the moduli problem at hand, non-reductive group actions present themselves in the study of k-jets on a smooth variety [BK17] and when studying the moduli of noncommutative projective planes [AOU14] and many other examples.

In this section we introduce non-reductive GIT and present some recent results due to Bérczi, Doran, Hawes and Kirwan [BDHK16, BDHK18]. We call these results the $\hat{U}$-Theorems and they will be the main tools used in constructing the moduli spaces of hypersurfaces in complete simplicial toric varieties.

1.3.1 Non-reductive group actions

The most problematic issue with constructing quotients of actions by arbitrary, non-reductive groups is the following.

Theorem 1.3.1. [Nag59] *Let $X = \operatorname{Spec} A$ be an affine scheme with an action of a linear algebraic group G. The invariant ring A^G is not necessarily finitely generated.*

To prove the above theorem, Nagata constructed a counterexample which answered Hilbert's 14th problem concerning the finite generation of invariant rings in the negative. Thus trying to construct a quotient as in reductive GIT would lead to schemes not of finite type. In some cases, the ring of invariants will be finitely generated; for example, a theorem of Weitzenböck tells us that for linear $\mathbb{G}_a$-actions on affine varieties, the invariant ring is finitely generated.

Theorem 1.3.2. [Wei32] *Let $X = \operatorname{Spec} A$ be an affine variety and suppose that $\mathbb{G}_a$ acts on X linearly. Then $A^{\mathbb{G}_a}$ is finitely generated.*

In the paper [Ses61], Seshadri provides a modern proof in a slightly more general context. Note that Weitzenböck's theorem can fail in positive characteristic; see [Fau77].

Unfortunately, non-finite generation of the invariant ring is not the only issue one encounters. We now consider an example which compounds some other problems that can and do occur.

Example 1.3.3. Consider $X = \mathbb{A}^4$ so that $A = k[x, y, z, w]$. Let $\mathbb{G}_a$ act on X by

$$a \cdot (x, y, z, w) = (x, y + ax, z, w + az).$$

Then $A^{\mathbb{G}_a} = k[x, z, xw - zy] \subset A$ is finitely generated and we can consider the induced morphism of varieties $q : \mathbb{A}^4 \to \operatorname{Spec} k[x, z, xw - yz] = \mathbb{A}^3$ given by

$$q(x, y, z, w) = (x, z, xw - zy).$$

Then $q(\mathbb{A}^4) = \mathbb{A}^3 - \{(0, 0, t) \mid t \neq 0\}$, which is a constructible set, but not a scheme. This example shows that even if the invariant ring is finitely generated, the induced morphism of schemes is not necessarily surjective, and thus cannot be a categorical quotient. Moreover, the image of q is not a scheme, so we can also not take $q(X)$ to be the quotient. Worse still, there can exist no subscheme of $\mathbb{A}^3$ which will be a categorical quotient for the $\mathbb{G}_a$-action on $\mathbb{A}^4$; indeed, such a categorical quotient will factor through the constructible set $q(\mathbb{A}^4)$. However, consider the open $\mathbb{G}_a$-invariant subset $U = \{(x, y, z, w) \in X \mid (x, z) \neq (0, 0)\} \subset X$. Then we have that

$$q|_U : U \longrightarrow \mathbb{A}^3 - L$$

is surjective, where $L = \{(0, 0, t) \in \mathbb{A}^3 \mid t \in k\}$ and $\mathbb{A}^3 - L$ is a variety. Moreover, $q|_U$ is a geometric quotient for the action of $\mathbb{G}_a$ on U.

A theorem of Rosenlicht states that, as in the above example, there is always a non-empty invariant open subset which does admit a geometric quotient.

Theorem 1.3.4. [Ros63] *Let G be a linear algebraic group acting on an irreducible variety X. Then there is a nonempty G-invariant open subset $U \subset X$ admitting a quasi-projective geometric quotient for the G-action.*

For a short modern proof we refer the reader to [CDT87, Section 2]. The available proofs of this theorem are non-constructive and the question remains, how does one compute this open subset and study the resulting quotient.

1.3.2 Non-reductive GIT and the $\hat{U}$-theorem

The method adopted in [BDHK18, BDHK16] requires additional structure on the algebraic groups and the linearisations chosen. With this additional structure, many of the properties of reductive GIT can be recovered. In this section we now introduce and explore this additional structure and state the resulting theorems. It must be noted that we introduce definitions and state theorems in only as much generality as is required in this thesis. The definitions and results hold in greater generailty than is stated and we refer the reader to [BDHK16, Haw15] for statements in full generality.

Let X be a projective variety acted on by a linear algebraic group G with respect to a very ample linearisation $\mathcal{L}$, where G is not necessarily reductive.

Definition 1.3.5. We define the morphism of schemes associated to the inclusion of graded rings

$$A(X,\mathcal{L})^G \subset A(X,\mathcal{L})$$

to be the *enveloping quotient*

$$q_G : X \dashrightarrow X /\!\!/_{\mathcal{L}} G,$$

where $X /\!\!/_{\mathcal{L}} G = \operatorname{Proj} A(X,\mathcal{L})^G$ is a scheme, not necessarily of finite type.

Remark 1.3.6. Note that this is a different definition than in [Haw15, BDHK15] where they restrict the rational map to the finitely generated semistable locus (see definition below).

We define notions of semistability and stability for linear algebraic group actions. One motivation of this definition is to have a quotient locally of finite type.

Definition 1.3.7. We define

$$I^{\mathrm{fg}} = \{\sigma \in A(X,\mathcal{L})^G_+ \mid \mathcal{O}(X_\sigma) \text{ is finitely generated}\}$$

and the *finitely generated semistable locus* to be

$$X^{\mathrm{ss}} = \bigcup_{\sigma \in I^{\mathrm{fg}}} X_\sigma.$$

Further, we define $I^s \subset I^{\mathrm{fg}}$ to be G-invariant sections satisfying the following conditions:

- the action of G on X_σ is closed and all stabilisers are finite; and

- the restriction of the U-enveloping quotient map

$$q_U : X_\sigma \longrightarrow \operatorname{Spec}(\mathcal{O}(X)^U_{(\sigma)})$$

  is a principal U-bundle for the action of U on X_σ.

Then we define

$$X^{\mathrm{s}} = \bigcup_{\sigma \in I^{\mathrm{s}}} X_\sigma$$

to be the stable locus.

Notation 1.3.8. When there is a possibility of confusion, we write $X^{\mathrm{s},G}$ and $X^{\mathrm{ss},G}$ for X^{s} and X^{ss} respectively when we want to emphasise the group.

Since G is a linear algberaic group over an algebraically closed field of characteristic 0, a theorem of Mostow [Mos56] states that G admits a *Levi decomposition:*

$$G \simeq R \ltimes U,$$

where R is a reductive group and the so-called *Levi factor,* and U is the unipotent radical of G. We may turn our attention to unipotent group actions by the following lemma.

Lemma 1.3.9. [Haw15, Lemma 2.3.1] *Suppose G is a linear algebraic group, N is a normal subgroup of G and X is a scheme with a G-action. Suppose all the stabilisers for the restricted action of N on X are finite and this action has a geometric quotient $\pi : X \to X/N$. Note that G/N acts canonically on X/N. Then the following statements hold.*

1. *For all the G/N-orbits in X/N to be closed, it is necessary and sufficient that all the G-orbits in X are closed;*

2. *given $y \in X/N$, the stabiliser $\operatorname{Stab}_{G/N}(y)$ is finite if and only if $\operatorname{Stab}_G(x)$ is finite for some (and hence all) $x \in \pi^{-1}(y)$; and*

3. *if G/N is reductive and X/N is affine, then X/N has a geometric G/N-quotient if and only if all G-orbits in X are closed.*

Remark 1.3.10. Let $G \simeq R \ltimes U$ be the Levi decomposition. The lemma above details how we may study the quotient of an action of G on X in two stages. If we first deal with the action of U on X, then we may consider the action of R on X/U, provided it exists, using reductive GIT.

Using Lemma 1.3.9, we can construct a geometric quotient of the stable locus as defined in Definition 1.3.7.

Theorem 1.3.11. [Haw15, Theorem 2.4.2] *Let X be a projective variety and G a linear algebraic group acting on X with respect to a very ample line bundle. There is a commutative diagram*

$$\begin{tikzcd} X^s \arrow[r, hook] \arrow[d, "geo"'] & X^{ss} \arrow[r, hook] \arrow[d] & X \arrow[dl, dashed, "q_G"] \\ X^s/G \arrow[r, hook] & X /\!\!/_{\mathcal{L}} G & \end{tikzcd}$$

where the first arrow is a geometric quotient and all inclusions are open.

The question remains of how one can compute the stable and semistable locus. The following discussion aims to address this.

Let U be a unipotent group and $\lambda : \mathbb{G}_m \to \mathrm{Aut}(U)$ be a 1-parameter subgroup of automorphisms and let

$$\hat{U}_\lambda = \mathbb{G}_m \ltimes_\lambda U$$

be the semi-direct product, where multiplication is given as follows:

$$(u_1, t_1) \cdot (u_2, t_2) = (\lambda(t_2^{-1})(u_1) + u_2, t_1 t_2), \quad u_i \in U,\ t_i \in \mathbb{G}_m.$$

The pointwise derivation of λ defines a $\mathbb{G}_m$-action on $\mathrm{Lie}\, U$. This action defines a grading

$$\mathrm{Lie}\, U = \bigoplus_{i \in \mathbb{Z}} (\mathrm{Lie}\, U)_i$$

with respect to weights $i \in \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$.

Definition 1.3.12. We say that $\hat{U}_\lambda$ is *positively graded* if the induced action of $\mathbb{G}_m$ on $\mathrm{Lie}\, U$ has all positive weights. That is $(\mathrm{Lie}\, U)_i \neq 0$ implies that $i > 0$.

Let $G \simeq R \ltimes U$ be a linear algebraic group. We say that G has a *graded unipotent radical* if there exists a central 1-parameter subgroup $\eta : \mathbb{G}_m \to R$ such that $\lambda_g : \mathbb{G}_m \to \mathrm{Aut}(U)$ defined by

$$\lambda_g(t)(u) = \eta(t) \cdot u \cdot \eta(t)^{-1} \qquad \text{for } t \in \mathbb{G}_m,\ u \in U,$$

is such that $\hat{U}_{\lambda_g}$ is positively graded. We often drop the grading 1-parameter subgroup from the subscript and write $\hat{U}_{\lambda_g} = \hat{U}$. Note that $\lambda_g(t)$ is an automorphism of U since U is a normal subgroup of G.

Example 1.3.13. Consider $\hat{\mathbb{G}}_a = \mathbb{G}_m \ltimes \mathbb{G}_a \hookrightarrow \mathrm{SL}_2$ defined by

$$(t, a) \longmapsto \begin{pmatrix} t & ta \\ 0 & t^{-1} \end{pmatrix},$$

where $(t_1, a_2) \cdot (t_2, a_2) = (t_1 t_2, t_2^{-2} a_1 + a_2)$. Then define $\lambda_g : \mathbb{G}_m \to \mathrm{Aut}(\mathbb{G}_a)$ by

$$\lambda_g(t)(a) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 a \\ 0 & 1 \end{pmatrix} = t^2 a.$$

So $\lambda_g(\mathbb{G}_m)$ acts on $\mathrm{Lie}\,\mathbb{G}_a$ with weight 2 and thus $\hat{\mathbb{G}}_a$ is positively graded.

Let X be a projective variety and $\mathcal{L} \in \mathrm{Pic}(X)$ be a very ample line bundle. Suppose that $\hat{U}$ acts on X with respect to $\mathcal{L}$. By restricting the $\hat{U}$-action to $\mathbb{G}_m$, we have a $\mathbb{G}_m$-action on $V = H^0(X, \mathcal{L})^\vee$; let

$$\omega_{\min} = \text{ minimial weight in } \mathbb{Z} \text{ for the } \mathbb{G}_m\text{-action on } V$$

and

$$V_{\min} = \{v \in V \mid t \cdot v = t^{\omega_{\min}} v \text{ for all } t \in \mathbb{G}_m\}$$

the associated weight space. Then $\mathbb{P}(V_{\min})$ is a linear subspace of $\mathbb{P}(V)$.

Definition 1.3.14. Suppose that $X, \mathcal{L}$ and $\hat{U}$ are as above. We define

$$Z_{\min} = X \cap \mathbb{P}(V_{\min})$$

and

$$X^0_{\min} = \{x \in X \mid \lim_{t \to 0} t \cdot x \in Z_{\min}\} \quad \text{where } t \in \mathbb{G}_m \subset \hat{U}.$$

Remark 1.3.15. The subvarieties $Z_{\min}$ and $X^0_{\min}$ are unaffected by replacing the linearisation $\mathcal{L}$ by any element of the positive $\mathbb{Q}$-ray defined by $\mathcal{L}$ in $\mathrm{Pic}^{\hat{U}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Also note that $X^0_{\min}$ and the U-sweep $U \cdot Z_{\min}$ of $Z_{\min}$ are $\hat{U}$-invariant subsets.

As in Example 1.2.8, we may twist the linearisation $\mathcal{L}$ by a character $\chi : \hat{U} \to \mathbb{G}_m$. We denote the twisted linearisation by $\mathcal{L}^\chi$ and the minimum $\mathbb{G}_m$-weight of $V = H^0(X, \mathcal{L}^\chi)^\vee$ by $\omega_{\min}^\chi$.

Definition 1.3.16. For $\epsilon > 0$ we define an ϵ-*linearisation* to be a linearisation $\mathcal{L} \in \operatorname{Pic}^{\hat{U}}(X)_{\mathbb{Q}}$ such that we have the following inequality for the minimum $\mathbb{G}_m$-weight of $V = H^0(X, \mathcal{L})^\vee$

$$\omega_{\min} < 0 < \omega_{\min} + \epsilon.$$

Remark 1.3.17. Fix $\mathcal{L} \in \operatorname{Pic}^{\hat{U}}(X)$ a linearisation and note that every character of $\hat{U} = \mathbb{G}_m \ltimes U$ is of the form

$$\hat{U} \to \mathbb{G}_m ; \quad (t, u) \mapsto t^r,$$

for some $r \in \mathbb{Z}$. We identify the characters of $\hat{U}$ with $\mathbb{Z}$. Let $\epsilon > 0$ and consider the rational character $\chi = -\omega_{\min} - \frac{\epsilon}{2}$. Twist $\mathcal{L}$ by the character χ and denote this linearisation $\mathcal{L}^\chi \in \operatorname{Pic}^{\hat{U}}(X)_{\mathbb{Q}}$. Then $\mathcal{L}^\chi$ is an ϵ-linearisation: indeed, we have that

$$\omega_{\min}^\chi = \omega_{\min} + \chi = -\frac{\epsilon}{2} < 0 < \omega_{\min}^\chi + \epsilon.$$

Before we state the $\hat{U}$-theorem, there is a technical condition which we require.

Definition 1.3.18. The $\hat{U}$-action on X with respect to $\mathcal{L}$ is said to satisfy the *semistability equals stability* condition if

$$\operatorname{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\min}. \tag{$\mathfrak{C}^*$}$$

Theorem 1.3.19. [Haw15, Theorem 5.1.4] *Let X be a projective variety acted on by a graded unipotent group $\hat{U}$ with respect to a very ample linearisation $\mathcal{L}$. Suppose that the action satisfies the condition $(\mathfrak{C}^*)$. Then the following statement holds.*

1. *The restriction to $X_{\min}^0$ of the enveloping quotient for the U-action*

$$q_U : X_{\min}^0 \longrightarrow X_{\min}^0 / U$$

is a principal U-bundle, in particular, q_U is a geometric quotient.

Suppose furthermore that $X_{\min}^0 \neq U \cdot Z_{\min}$. For each $\epsilon > 0$ let χ_ϵ be a character such that the twisted linearisation $\mathcal{L}^{\chi_\epsilon}$ is an ϵ-linearisation. Then for sufficiently small $\epsilon > 0$ the following statements hold.

2. *There are equalities* $X^0_{\min} - U \cdot Z_{\min} = X^{s,\hat{U}}(\mathcal{L}^{\chi_\epsilon})$.

3. *The enveloping quotient* $X /\!/_{\mathcal{L}^{\chi_\epsilon}} \hat{U}$ *is a projective variety and*

$$q_{\hat{U}} : X^{s,\hat{U}}(\mathcal{L}^{\chi_\epsilon}) \longrightarrow X /\!/_{\mathcal{L}^{\chi_\epsilon}} \hat{U}$$

is a geometric quotient for the $\hat{U}$*-action. In particular, the ring of* $\hat{U}$*-invariants* $A(X, \mathcal{L}^{\chi_\epsilon})^{\hat{U}}$ *is finitely generated.*

Remark 1.3.20. It follows from the proof of the $\hat{U}$-theorem that if $Z_{\min}$ is a point (so that $\dim V_{\min} = 1$), then $X^0_{\min} = X_\sigma$ for some non-zero $\sigma \in (V_{\min})^\vee$. Thus $X^0_{\min}$ is an affine open subscheme of X. Moreover, when this is the case, the quotient

$$q_U : X^0_{\min} \longrightarrow X^0_{\min}/U$$

is a trivial U-bundle.

We now state the result for general linear algebraic groups. Let $G \cong R \ltimes U$ be a linear algebraic group with unipotent radical $U \subset G$. Suppose that there exists a 1-parameter subgroup $\lambda_g : \mathbb{G}_m \to R$ lying in the center of the Levi factor of G such that $\hat{U} = \lambda_g(\mathbb{G}_m) \ltimes U$ is a graded unipotent group.

Definition 1.3.21. A linearisation of the G-action is an *ϵ-linearisation* if its restriction to $\hat{U}$ is an ϵ-linearisation.

Theorem 1.3.22. [BDHK16, Theorem 0.1] *Let G be a linear algebraic group acting on a projective variety X with respect to $\mathcal{L}$. Assume that G has graded unipotent radical such that $(\mathfrak{C}^*)$ holds. Further, fix $\epsilon > 0$ and assume that $\mathcal{L}$ is an ϵ-linearisation. Then if $\epsilon > 0$ is sufficiently small, the following statements hold.*

1. *The G-invariants are finitely generated and the enveloping quotient*

$$X /\!/_{\mathcal{L}} G = \operatorname{Proj} A(X, \mathcal{L})^G$$

is a projective variety.

2. *The inclusion $A(X, \mathcal{L})^G \subset A(X, \mathcal{L})$ induces a categorical quotient of the semistable locus*

$$X^{ss,G} \longrightarrow X /\!/_{\mathcal{L}} G,$$

which restricts to a geometric quotient

$$X^{s,G} \longrightarrow X^{s,G}/G.$$

Remark 1.3.23. In the literature a linearisation which is an ϵ-linearisation for small enough $\epsilon > 0$ such that the above $\hat{U}$-theorems hold is called a *well-adapted* linearisation.

We now state a Hilbert-Mumford criteron, whose proof is outlined in [BDHK16].

Theorem 1.3.24. [BDHK16, Theorem 2.6] *Keep the notation and assumptions as in Theorem 1.3.22. The following Hilbert-Mumford criterion holds.*

$$X^{(s)s,\,G} = \bigcap_{g \in G} gX^{(s)s,\,T},$$

where $T \subset G$ is a maximal torus of G containing the grading $\mathbb{G}_m$.

Remark 1.3.25. The order in which we have stated the theorems is not representative of the order in which were proved. In the paper [BDHK16], it is proven that the set

$$X^{(\mathrm{s})s,G}_{\min,+} := \bigcap_{g \in G} gX^{(\mathrm{s})\mathrm{s},\,T}$$

admits a geometric quotient [BDHK16, Theorem 0.1] and then the equality

$$X^{(\mathrm{s})s,G}_{\min,+} = X^{(\mathrm{s})\mathrm{s},\,G},$$

is proven [BDHK16, Theorem 2.6].

As in the reductive GIT setting, we can also state the Hilbert-Mumford criterion in terms of weight polytopes.

Theorem 1.3.26. *Keep the notation and assumptions as in Theorem 1.3.22. The following Hilbert-Mumford criterion holds.*

$$x \in X^{ss,G} \iff 0 \in \mathrm{Conv}_T(g \cdot x) \text{ for every } g \in G,$$
$$x \in X^{s,G} \iff 0 \in \mathrm{Conv}_T(g \cdot x)^\circ \text{ for every } g \in G.$$

1.4 Constructing moduli spaces as quotients

Suppose that we have a moduli problem where the objects are parametrised by a scheme Y and the notion of equivalence of these objects is given by the action of an algebraic group G. To know that a categorical quotient of the G-action on Y is the moduli space for the given moduli problem, we need the local universal property.

Definition 1.4.1. Let $\mathcal{M} : \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$ be the moduli functor of a moduli problem. Suppose that $\mathcal{F}$ is a family parametrised by a scheme Y. We say that $\mathcal{F}$ has the *local universal property* if for any family $\mathcal{G}$ parametrised by a scheme S, there exists an open covering $\bigcup_i U_i = S$ and morphisms $\phi_i : U_i \to Y$ such that

$$[\mathcal{G}|_{U_i}] = [\phi_i^* \mathcal{F}] \in \mathcal{M}(U_i).$$

Proposition 1.4.2. [New78, Proposition 2.13] *Let $\mathcal{M} : \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$ be the moduli functor of a moduli problem. Suppose that $\mathcal{F}$ is a family parametrised by a scheme Y with the local universal property. Furthermore, suppose that there is an algebraic group G acting on Y such that two k-points $x, y \in Y$ are in the same G-orbit if and only if $\mathcal{F}_x \sim \mathcal{F}_y$. Then*

1. *any coarse moduli space is a categorical quotient of Y by G;*
2. *a categorical quotient of Y by G is a coarse moduli space if and only if it is an orbit space.*

Chapter 2

Toric varieties

In this chapter we review the basic theory of toric varieties and simplicial toric varieties to establish notation and conventions. All the material contained in this chapter is well-known and almost all results are contained in [CLS11, Cox95b, BC94]. Those which are not are easy consequences of results given in these references and proofs are given.

2.1 Fans, polytopes and toric varieties

In this section we introduce basic constructions and properties of toric varieties as can be found in [CLS11, Ful93].

Definition 2.1.1. A toric variety of dimension n is a normal variety X which contains a torus $T \cong (\mathbb{G}_m)^n$ as a dense open subset such that the natural action of T on itself extends to an action on X.

Let $T \cong (\mathbb{G}_m)^n$ be an algebraic torus and define $M = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ to be the character lattice and $N = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ to be the cocharacter lattice, where all the morphisms are homomorphisms of linear algebraic groups. Both M and N are lattices of rank n. There is a perfect pairing between M and N given by the composition

$$\langle\, , \,\rangle : M \times N \longrightarrow \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$$

where $\langle \chi_m, u \rangle = \chi_m \circ u$ and $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$ under the canonical isomorphism of groups

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \\ n &\longmapsto (t \mapsto t^n). \end{aligned}$$

Thus N and M are dual to one another.

Notation 2.1.2. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be the corresponding vector spaces. We denote elements of the lattice N by u and elements of M by m and the same for the corresponding vector spaces. However, when we wish to emphasise that an element of M is a character we shall denote it by χ_m. This is slightly different from the standard convention (see for example [CLS11, Section 1.1]) where the notation χ^m is used. We avoid this as it conflicts with the notation of taking powers of characters as appears later in this thesis.

2.1.1 Toric varieties and fans

Definition 2.1.3. A *convex rational polyhedral cone* is a subset $\sigma \subseteq N_{\mathbb{R}}$ such that $\sigma = \mathrm{Cone}(u_1, \ldots, u_r)$ with $u_i \in N$. We shall refer to a convex rational polyhedral cone simply as a cone. The *dual cone* is defined to be

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq M_{\mathbb{R}}.$$

A *face* τ of σ, is subset $\tau \subseteq \sigma$ such that $\tau = \sigma \cap H_m$ and $\sigma \subseteq H_m^+$, where $H_m \subset N_{\mathbb{R}}$ is a hyperplane defined by

$$H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\}$$

and $H_m^+ \subset N_{\mathbb{R}}$ is the half-space with H_m as a boundary such that elements pair positively with m. If τ is a face we write $\tau \preccurlyeq \sigma$. If $0 \preccurlyeq \sigma$ then σ is called *strongly convex* . From now on, when we say cone, we mean strongly convex cone.

By Gordon's Lemma [Ful93, Proposition 1], the semigroup $\sigma^{\vee} \cap M$ is finitely generated and thus the associated k-algebra $k[\sigma^{\vee} \cap M]$ is also finitely generated.

Definition 2.1.4. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex cone. The *affine toric variety associated to* σ is

$$U_{\sigma,N} = \operatorname{Spec} k[\sigma^{\vee} \cap M].$$

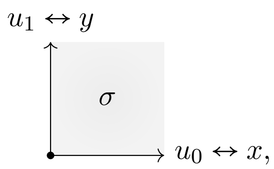
In general, we omit the lattice in the subscript unless there is a risk of confusion. By [CLS11, Theorem 1.2.18], U_σ is a toric variety and $\dim U_\sigma = n$ if and only if σ is strongly convex.

Example 2.1.5. Let $\sigma = \{0\} = \mathrm{Cone}(\varnothing)$. Then $\sigma^\vee = M_\mathbb{R}$, so $\sigma^\vee \cap M = M$ and

$$k[M] \cong k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}].$$

Thus $U_{\{0\}} = T$.

Example 2.1.6. Let $T = (\mathbb{G}_m)^2$ and consider $\sigma = \mathrm{Cone}((1,0),(0,1)) \subset \mathbb{R}^2 = N_\mathbb{R}$



so that $\sigma^\vee = \mathrm{Cone}((1,0),(0,1)) \subset \mathbb{R}^2 = M_\mathbb{R}$. Then $k[\sigma^\vee \cap M] \cong k[x,y]$ and hence

$$U_\sigma \cong \mathbb{A}^2.$$

Lemma 2.1.7. [CLS11, Proposition 1.3.16] *Let $\sigma \subseteq N_\mathbb{R}$ be a cone and $\tau \preccurlyeq \sigma$ such that $\tau = \sigma \cap H_m$ for some $m \in M$. Then*

$$k[\tau^\vee \cap M] = k[\sigma^\vee \cap M]_{\chi_m}.$$

Thus

$$U_\tau \subset U_\sigma$$

is an open affine subset.

Definition 2.1.8. A *fan* $\Sigma \subseteq N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that the following conditions are satisfied.

1. The origin is a face of every cone in Σ.

2. For all $\sigma \in \Sigma$, every face of σ is also in Σ.

3. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2 and hence also in Σ.

Furthermore, if Σ is a fan, then the *support* of Σ is

$$|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subset N_{\mathbb{R}}.$$

The set of l-dimensional cones of Σ is denoted by $\Sigma(l)$, where the dimension of a cone is the dimension of the smallest vector subspace of $N_{\mathbb{R}}$ containing it. We call elements of $\Sigma(1)$, i.e. one-dimensional faces, *rays.*

Given a fan $\Sigma \subseteq N_{\mathbb{R}}$ we can construct a variety X_Σ from the affine toric varieties $\{U_\sigma\}_{\sigma\in\Sigma}$. For every $\sigma_1, \sigma_2 \in \Sigma$ we have that $\tau = \sigma_1 \cap \sigma_2$ is a face of both and also in Σ. By Lemma 2.1.7, we have an open immersion $U_\tau \hookrightarrow \mathrm{U}_{\sigma_i}$. These open immersions form the data to glue $\{U_\sigma\}_{\sigma\in\Sigma}$ to a variety X_Σ, see [CLS11, Section 3.1].

Theorem 2.1.9. [CLS11, Theorem 3.1.5] *Let Σ be a fan in $N_{\mathbb{R}}$. Then the variety X_Σ is a toric variety of dimension n.*

In general all toric varieties arise as the toric variety associated to a fan [CLS11, Corollary 3.1.8].

Remark 2.1.10. We have used a restricted definition of toric variety. In the literature the normality condition often is removed and with this more general definition, it is no longer the case that every toric variety comes from a fan. For example the nodal cubic curve $\mathbf{V}(y^2z - x^2(x+z)) \subset \mathbb{P}^2$ is a toric variety with torus $C - \{(0:0:1)\} \cong \mathbb{G}_m$, see [CLS11, Section 3.A.1] for details.

Many geometric properties of toric varieties can be characterised simply by properties of the fan.

Definition 2.1.11. Let Σ be a fan and σ a cone.

1. σ is *smooth* if its minimal generators form part of a $\mathbb{Z}$-basis of N.
2. σ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$.
3. Σ is smooth (resp. simplicial) if every cone in Σ is smooth (resp. simplicial).
4. A toric variety is called simplicial if its associated fan is simplicial.

Theorem 2.1.12. [CLS11, Theorem 3.1.19] *Let $X = X_\Sigma$ be a toric variety associated to a fan Σ. Then the following statements hold.*

1. *X is a smooth variety if and only if Σ is a smooth fan.*

2. *X is an orbifold if and only if Σ is simplicial.*

3. *X is complete if and only if Σ is complete, that is, $|\Sigma| = N_\mathbb{R}$.*

See Section 2.4 for a proof of part (2) of the theorem.

We remark on an unimportant technical condition on a toric variety which comes up when studying divisors on toric varieties.

Definition 2.1.13. A toric variety has *torus factors* if it is equivariantly isomorphic to the product of a non-trivial torus and a toric variety of smaller dimension.

Proposition 2.1.14. [CLS11, Proposition 3.3.9] *A toric variety X has no torus factors if and only if there are no non-constant morphisms $X \to k^*$, i.e. $H^0(X, \mathcal{O}_X)^* = k^*$. In particular, if X is complete, then X has no torus factors.*

2.1.2 Lattice polytopes and projective toric varieties

Let $\mathcal{A} = \{m_0, \dots, m_s\} \subset M$ be a finite set of lattice points. We can define an affine and a projective toric variety using this finite set.

Consider the map defined by

$$\widetilde{\Phi}_\mathcal{A} : T \longrightarrow \mathbb{A}^{s+1}, \quad t \longmapsto (\chi_{m_0}(t), \dots, \chi_{m_s}(t))$$

and define $Y_\mathcal{A}$ to be the Zariski closure of the image of $\widetilde{\Phi}_\mathcal{A}$ in $\mathbb{A}^{s+1}$. If $M = \mathbb{Z}^n$, then χ_{m_i} is the Laurent monomial t^{m_i} and $Y_\mathcal{A}$ is the Zariski closure of the map

$$T \longrightarrow \mathbb{A}^{s+1}, \quad t \longmapsto (t^{m_0}, \dots, t^{m_s}).$$

Proposition 2.1.15. [CLS11, 1.1.8] *Let $\mathcal{A} \subset M$ be a finite set of lattice points and let $\mathbb{Z}\mathcal{A} \subset M$ be the sublattice generated by $\mathcal{A}$. Then $Y_\mathcal{A}$ is an affine toric variety with character lattice $\mathbb{Z}\mathcal{A}$. In particular, the dimension of $Y_\mathcal{A}$ is equal to the rank of $\mathbb{Z}\mathcal{A}$.*

We can analogously construct a projective variety. Consider the map defined by

$$\Phi_{\mathcal{A}} : T \longrightarrow \mathbb{P}^s, \quad t \longmapsto (\chi_{m_0}(t) : \cdots : \chi_{m_s}(t))$$

and define $X_{\mathcal{A}}$ to be the Zariski closure of the image of $\Phi_{\mathcal{A}}$. If $M = \mathbb{Z}^n$, then χ_{m_i} is the Laurent monomial t^{m_i} and $X_{\mathcal{A}}$ is the Zariski closure of the map

$$T \longrightarrow \mathbb{P}^s, \quad t \longmapsto (t^{m_0} : \cdots : t^{m_s}).$$

In this case, we will often write $\mathcal{A} \subset \mathbb{Z}^n$ as an $(n \times (s+1))$-matrix A with integer entries, so that the columns of A are the elements of $\mathcal{A}$. We denote $X_A = X_{\mathcal{A}}$ and $Y_A = Y_{\mathcal{A}}$.

Proposition 2.1.16. [CLS11, 2.1.2] *Let $\mathcal{A} = \{m_0, \dots, m_s\} \subset M$ be a finite set of lattice points. Then $X_{\mathcal{A}}$ is a projective toric variety of dimension equal to the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathcal{A}$.*

Remark 2.1.17. [CLS11, Proposition 2.1.4] Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set corresponding to the matrix A. Then Y_A is the affine cone over X_A if and only if the vector $(1, \dots, 1)$ is in the row space over $\mathbb{R}$ of the matrix A.

Definition 2.1.18. A *lattice polytope* $P \subset M_{\mathbb{R}}$ is the convex hull of a finite set $S \subset M$.

In light of Proposition 2.1.16, a lattice polytope $P \subset M_{\mathbb{R}}$ defines a projective toric variety $X_{P \cap M}$.

Example 2.1.19. Let $T = \mathbb{G}_m^2$ and hence $M = \mathbb{Z}^2$. Consider $P = \mathrm{Conv}((d,0),(0,d))$ so that

$$A = P \cap M = \begin{pmatrix} d & d-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & d-1 & d \end{pmatrix}.$$

The matrix A defines the map

$$\Phi_{\mathcal{A}} : (\mathbb{G}_m)^2 \longrightarrow \mathbb{P}^d, \quad (t,s) \longmapsto (t^d : t^{d-1}s : \cdots : s^d)$$

and hence $X_{\mathcal{A}}$ is the rational normal curve of degree d.

Definition 2.1.20. A lattice polytope $P \subset M_{\mathbb{R}}$ is *very ample* if for every vertex $m \in P$, the semigroup $S = S_{P,m} = \mathbb{N}(P \cap M - m)$ is saturated (that is, for every $k \in \mathbb{N}$ and $m' \in M$ we have that if $km' \in S$ then $m' \in S$).

Remark 2.1.21. Note that by [CLS11, Corollary 2.2.19], for any full-dimensional polytope $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ such that $n \geq 2$, it holds that for every $k \geq n-1$ we have that kP is very ample.

Definition 2.1.22. Suppose that $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ is a full-dimensional polytope. Let $\mathcal{A}_P = kP \cap M$ be the set of lattice points of kP for some integer $k > 0$ such that kP is very ample. We define the *projective toric variety associated to* P to be

$$X_P = X_{\mathcal{A}_P}.$$

This is independent of k by [CLS11, Proposition 2.3.9].

Remark 2.1.23. Note that if P is already a very ample polytope, then $X_P = X_{P \cap M}$.

2.2 Divisors on toric varieties

In this section we present essential facts about divisors in toric varieties. As before, the main reference is [CLS11].

Let us fix a toric variety $X = X_\Sigma$ associated to a fan Σ in $N_{\mathbb{R}} \simeq \mathbb{R}^n$. As before, we denote the set of rays of Σ by $\Sigma(1)$. The *orbit-cone correspondence* (see [CLS11, Theorem 3.2.6]) assigns to every ray $\rho \in \Sigma(1)$ in Σ a torus-invariant divisor $D_\rho \in \mathrm{Div}(X)$. The divisors D_ρ generate the subgroup of torus-invariant divisors $\mathrm{Div}_T(X) \subset \mathrm{Div}(X)$ so that

$$\mathrm{Div}_T(X) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho.$$

We have the following remarkable theorem.

Theorem 2.2.1. [CLS11, Theorem 4.1.3] *Suppose that X has no torus factors. We have the short exact sequence*

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_T(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where the first map is $m \mapsto \mathrm{div}(\chi_m)$ and the second sends a torus-invariant divisor to its class. In particular, $\mathrm{Cl}(X)$ *is a finitely generated abelian group, generated by torus-invariant divisors.*

Remark 2.2.2. Note that if X does have torus factors, we still have an exact sequence; however, the first map $M \to \mathrm{Div}_T(X)$ is not injective. In particular, $\mathrm{Cl}(X)$ is always a finitely generated abelian group. There is an analogous statement for torus-invariant Cartier divisors and the Picard group. In particular, the Picard group of a toric variety is finitely generated.

It is possible for the Picard group of a toric variety to have torsion, but under very light restrictions on the fan, the Picard group is torsion free.

Theorem 2.2.3. [CLS11, Proposition 4.2.5] *Suppose that $\Sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ contains a cone of dimension n. Then* $\mathrm{Pic}(X)$ *is a free abelian group.*

Remark 2.2.4. All toric varieties which we study in this thesis contain a cone of maximal dimension. The orbit-cone correspondence gives us that maximal cones $\Sigma(n)$ are in bijection with T-fixed points. This gives an alternate way of seeing (aside from the definition) that complete fans will always contain a cone of maximum dimension. Additionally, there is an equality $|\Sigma(n)| = \chi(X)$, where $\chi(X)$ is the topological Euler characteristic; see [CLS11, Theorem 12.3.9].

Note that we have that $\mathrm{Pic}(X) \subset \mathrm{Cl}(X)$. The following proposition gives a necessary and sufficient condition for equality.

Proposition 2.2.5. [CLS11, Proposition 4.2.6] *Let X be the toric variety associated to the fan Σ. Then the following statements hold:*

1. *X is smooth if and only if* $\mathrm{Pic}(X) = \mathrm{Cl}(X)$.
2. *X is simplicial if and only if* $\mathrm{Pic}(X)$ *has finite index in* $\mathrm{Cl}(X)$.

Proposition 2.2.5 characterises simplicial toric varieties as those for which every Weil divisor is $\mathbb{Q}$-Cartier.

Theorem 2.2.6. [CLS11, Theorem 6.1.15] *On a smooth complete toric variety, every ample divisor is very ample.*

Remark 2.2.7. Note that the above theorem does not hold for simplicial toric varieties. For example there exist weighted projective spaces who posses Cartier ample divisors which are not very ample.

Definition 2.2.8. Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space and define $l = \operatorname{lcm}(a_0, \dots, a_n)$, then $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(l)$. Let d be a positive integer such that l divides d so that hypersurfaces of degree d in X are Cartier divisors, we call such an integer a *Cartier degree.*

Let us turn our attention to divisors on projective toric varieties.

Definition 2.2.9. Let $X = X_\Sigma$ be a projective toric variety associated to a fan $\Sigma \subset N_\mathbb{R} \simeq \mathbb{R}^n$. Suppose that $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is a T-invariant divisor. We define the polytope of D to be

$$P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\},$$

where $u_\rho \in N$ is the primitive lattice point on the ray $\rho \in \Sigma(1)$. See [CLS11, Theorem 6.1.14] for a proof that P_D is indeed a polytope.

Proposition 2.2.10. [CLS11, Theorem 6.2.1] *Let X be a projective toric variety and $\alpha \in \operatorname{Pic}(X)$ a very ample class with D a torus-invariant divisor of class α. Then the polytope P_D is very ample and*

$$X_{P_D} \simeq X.$$

Moreover, the embedding $X_{P_D} \subset \mathbb{P}^d$ is precisely the embedding $X \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*)$, where d is the number of lattice points of P.

Remark 2.2.11. Proposition 2.2.10 implies that every projective toric variety arises from a polytope.

2.2.1 Weighted projective space as a toric variety

Weighted projective spaces are ubiquitous in algebraic geometry. In this section we recall the different constructions of weighted projective space. We refer to [Dol82] for a comprehensive study of weighted projective space and its subvarieties.

Definition 2.2.12. Let $a_0, \dots, a_n$ be positive integers. Define the weighted projective space

$$\mathbb{P}(a_0, \dots, a_n) = \left(\mathbb{A}^{n+1} - \{0\}\right) / \mathbb{G}_m$$

to be the geometric quotient of $\mathbb{A}^{n+1} - \{0\}$ by the action of $\mathbb{G}_m$ defined by $t \mapsto (t^{a_0}, \dots, t^{a_n})$. It follows from GIT that

$$\mathbb{P}(a_0, \dots, a_n) = \operatorname{Proj} k[x_0, \dots, x_n]$$

where $\deg x_i = a_i$.

Lemma 2.2.13. [IF00, Lemma 5.5] *For all positive integers $q \in \mathbb{Z}$ we have that*

$$\mathbb{P}(qa_0, \dots, qa_n) \simeq \mathbb{P}(a_0, \dots, a_n).$$

Lemma 2.2.14. [IF00, Lemma 5.6] *Suppose that $q = \gcd(a_1, \dots, a_n)$. Then*

$$\mathbb{P}\left(a_0, \frac{a_1}{q}, \dots, \frac{a_n}{q}\right) \cong \mathbb{P}(a_0, \dots, a_n).$$

Corollary 2.2.15. *Suppose that $X = \mathbb{P}(a, b)$ is a weighted projective line. Then $X \simeq \mathbb{P}^1$.*

Definition 2.2.16. We say that a weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is *well-formed* if for every $0 \le i \le n$ it holds that

$$\gcd(a_0, \dots, \hat{a_i}, \dots, a_n) = 1.$$

Remark 2.2.17. It follows from Lemma 2.2.14 that every weighted projective space is isomorphic to a well-formed weighted projective space as a scheme. However, one can define the weighted projective stack

$$\mathbf{P}(a_0, \dots, a_n) = \left[(\mathbb{A}^{n+1} - \{0\})/\mathbb{G}_m\right]$$

to be the quotient stack of the action of Definition 2.2.12. Weighted projective stacks are always non-isomorphic for different weight vectors.

We will almost always work with well-formed weighted projective spaces. The only exception is when we study weighted projective lines in Chapter 7.

Weighted projective spaces are singular simplicial toric varieties. Let us give the definition of weighted projective space as a toric variety in terms of a fan. To see that the two definitions are equivalent we refer to Example 2.3.12.

Definition 2.2.18. [CLS11, Example 3.1.17] Consider positive integers $a_0, \dots, a_n$ such that $\gcd(a_0, \dots, a_n) = 1$. Define the lattice $N = \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (a_0, \dots, a_n)$ and let u_i for $i = 0, \dots, i_n$ be the images in N of the standard basis of $\mathbb{Z}^{n+1}$, so the relation

$$a_0 u_0 + \dots + u_n u_n = 0$$

holds in N. Let $\Sigma \subset N_{\mathbb{R}}$ be the fan made up of the cones generated by all the proper subsets of $\{u_0, \dots, u_n\}$. Then define the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ to be the toric variety associated to this fan.

We may also give the definition in terms of a polytope.

Definition 2.2.19. Consider positive integers $a_0, \dots, a_n$ such that $\gcd(a_0, \dots, a_n) = 1$ and let $l = \operatorname{lcm}(a_0, \dots, a_n)$. Define integers $a'_i = \frac{l}{a_i}$ and define the *weighted simplex* to be

$$\Delta(\underline{a}) = \operatorname{Conv}(a'_0 e_0, \dots, a'_n e_n) \subset \mathbb{R}^{n+1}.$$

Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}/\mathbb{R} \cdot (a_0, \dots, a_n)$ be the quotient morphism and define the polytope $P(\underline{a}) = \varphi(\Delta(\underline{a}))$. Then we may also define weighted projective space to be the toric variety associated to $P(\underline{a})$

$$\mathbb{P}(a_0, \dots, a_n) = X_{P(\underline{a})}.$$

A direct proof of the equivalence of this definition to Definition 2.2.12 can be found in the preprint [RT11]. Alternatively, it follows from the fact that $\Delta(\underline{a})$ is the section polytope of the line bundle $\mathcal{O}_{\mathbb{P}(\underline{a})}(l)$ and Proposition 2.2.10. See Chapter 6 for the definition of the section polytope.

2.3 The Cox ring of a toric variety

Let $X = X_\Sigma$ be a toric variety associated to a fan Σ. In this section we introduce the Cox ring of X. It was first introduced by Cox as the homogeneous coordinate ring in [Cox95b]; it generalises the homogeneous coordinate ring of $\mathbb{P}^n$ and plays an analogous role in the study of both quasi-coherent sheaves on X and the automorphism group $\operatorname{Aut}(X_\Sigma)$.

Definition 2.3.1. Suppose that $X = X_\Sigma$ is a toric variety associated to a fan Σ. The one dimensional cones in Σ are called rays and the set of rays is denoted $\Sigma(1)$. Let

$$S = k[x_\rho \,|\, \rho \in \Sigma(1)]$$

be the polynomial ring in $|\Sigma(1)|$ variables. Every monomial $\prod x_\rho^{a_\rho} \in S$ defines an effective torus-invariant divisor $D = \sum a_\rho D_\rho$, we write this monomial as x^D. In this way, we define the following notion of degree:

$$\deg(x^D) = [D] \in \mathrm{Cl}(X).$$

Thus we have

$$S = \bigoplus_{\alpha \in \mathrm{Cl}(X)} S_\alpha,$$

where $S_\alpha = \{f \in S \mid \text{ all monomials of } f \text{ have degree } \alpha\}$. Then $S_\alpha \cdot S_\beta \subset S_{\alpha+\beta}$ and we define the *Cox ring of* X to be S with this grading.

Remark 2.3.2. For $\alpha \in \mathrm{Cl}(X)$, we have that

$$S_\alpha = \bigoplus_D \mathbb{C} \cdot x^D,$$

where the sum is taken over effective torus-invariant divisors D such that $[D] = \alpha$.

Example 2.3.3. Consider the weighted projective space $X = \mathbb{P}(a_0, \dots, a_n)$ with $n > 1$, then $\mathrm{Cl}(X) \cong \mathbb{Z}$. We have that $S = k[x_0, \dots, x_n]$ such that $\deg x_i = a_i$.

Proposition 2.3.4. [Cox95b, Proposition 1.1] *Let $X = X_\Sigma$ be the toric variety associated to the fan Σ. Then the following statements hold.*

1. *For each effective divisor D with $[D] = \alpha \in \mathrm{Cl}(X)$, we have*

$$\phi_D : S_\alpha \xrightarrow{\cong} H^0(X, \mathcal{O}_X(D)).$$

2. *Let $\alpha = [D]$ and $\beta = [E]$ for D, E effective divisors, then there is a commutative diagram*

$$\begin{tikzcd} S_\alpha \otimes S_\beta \arrow[r] \arrow[d, "\phi_D \otimes \phi_E"'] & S_{\alpha+\beta} \arrow[d, "\phi_{D+E}"] \\ H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(E)) \arrow[r] & H^0(X, \mathcal{O}_X(D+E)), \end{tikzcd}$$

where the arrow on the bottom is the multiplication map.

Definition 2.3.5. The *irrelevant ideal* of the Cox ring S of X is defined as

$$B_\Sigma = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle,$$

where $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. This ideal describes a closed subvariety of $\mathbb{A}^{\Sigma(1)}$, which we denote by

$$Z_\Sigma = \operatorname{Spec}(S/B_\Sigma) \subset \mathbb{A}^{\Sigma(1)}.$$

Remark 2.3.6. Note that B_Σ is a monomial ideal and that $Z_\Sigma \subset \mathbb{A}^{\Sigma(1)}$ has codimension at least 2. To see the second claim consider $I = \langle x^{\hat{\rho}} \mid \rho \in \Sigma(1) \rangle \subset B$, hence $Z_\Sigma = \mathbf{V}(B) \subset \mathbf{V}(I))$. Then as $\mathbf{V}(I)$ is the union of codimension 2 coordinate subspaces, Z_Σ must have codimension at least 2.

2.3.1 Quotient construction of a projective toric variety

In this section we provide a description of a projective toric variety as a GIT-quotient. Let us fix the notation for the following chapter. Let $X = X_\Sigma$ be a projective toric variety associated to a fan Σ. Define $\mathbf{D} := \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cl}(X), \mathbb{G}_m)$ to be the character group of $\operatorname{Cl}(X)$. Note that $\operatorname{Cl}(X)$ is a finitely generated abelian group, so that $\mathbf{D}$ is a diagonalisable group [M+11, Theorem 14.12]. In particular, $\mathbf{D}$ is reductive. The $\operatorname{Cl}(X)$-grading on S is equivalent to a $\mathbf{D}$-action on $\operatorname{Spec} S = \mathbb{A}^{\Sigma(1)}$; for example see [Cra08, Theorem 2.12].

Notation 2.3.7. Define $r = |\Sigma(1)| - 1$, so that $\operatorname{Spec} S = \mathbb{A}^{r+1}$. The group of characters of $\mathbf{D}$ is by definition given by $\operatorname{Cl}(X)$. Fix a character $\chi : \mathbf{D} \to \mathbb{G}_m$ corresponding to an ample Cartier class, such an ample class exists by the projectivity assumption on X. Consider the action of $\mathbf{D}$ on $\mathbb{A}^{r+1}$ linearised by $\mathcal{O}_{\mathbb{A}^{r+1}}$ twisted by χ as defined by King in [Kin94]. We shall denote the semistable locus of the $\mathbf{D}$-action by $\hat{X} = (\mathbb{A}^{r+1})^{(\mathbf{D},\chi)-\mathrm{s}}$. We denote the quotient morphism by $q : \hat{X} \to X$. Note that the semistable locus is independent of the choice of ample class.

Lemma 2.3.8. [CLS11, Proposition 14.1.9] *The unstable locus $(\mathbb{A}^{r+1})^{(\boldsymbol{D},\chi)-us}$ is equal to the vanishing locus of the irrelevant ideal as defined in Definition 2.3.5. That is,*

$$Z_\Sigma = (\mathbb{A}^{r+1})^{(\boldsymbol{D},\chi)-us}.$$

In particular, the codimension of the unstable locus is at least 2.

Theorem 2.3.9. [Cox95b, Theorem 2.1] *Let X be as above; then the following statements hold.*

1. *X is naturally isomorphic to the categorical quotient of $\hat{X}$ with respect to the $\boldsymbol{D}$-action.*

2. *The quotient morphism $q : \hat{X} \to X$ is a geometric quotient if and only if X is simplicial.*

Proof. Following the construction in [Cra08, Theorem 2.12], we have that for every character $\eta : \mathbf{D} \to \mathbb{G}_m$ and corresponding divisor class α there is an equality of graded subrings of S

$$\bigoplus_{j \in \mathbb{Z}} S^{(\mathbf{D}, \eta^j)} = \bigoplus_{j \in \mathbb{Z}} S_{j \cdot \alpha},$$

where $S^{(\mathbf{D},\eta^j)}$ is the ring of semi-invariants for the character η^j. Indeed, this is how the grading is defined. Thus, if we fix α to be the class of very ample Cartier divisor D with corresponding character $\chi : \mathbf{D} \to \mathbb{G}_m$ as above, then

$$\mathbb{A}^{r+1} /\!\!/_{\chi} \mathbf{D} = \operatorname{Proj} \bigoplus_{j \in \mathbb{Z}} S^{\mathbf{D} - \chi^j} = \operatorname{Proj} \bigoplus_{j \in \mathbb{Z}} S_{j \cdot \alpha} \cong \operatorname{Proj} \bigoplus_{j \in \mathbb{Z}} H^0(X, \mathcal{O}_X(D)^{\otimes j}) \cong X.$$

Let us prove one direction of the second statement. Suppose that q is a geometric quotient. Since $\mathbf{D}$ is an abelian group, X has at worst abelian quotient singularities. Thus X is an orbifold and hence simplicial by Theorem 2.1.12. For the converse we refer the reader to the original proof [Cox95b, Theorem 2.1]. □

Remark 2.3.10. It follows from the Luna étale slice theorem that a simplicial toric variety is smooth if the action of $\mathbf{D}$ on $\hat{X}$ is free.

Remark 2.3.11. The original paper of Cox [Cox95b] gives the construction of a normal toric variety as a GIT-quotient of an affine space. The paper of Craw [Cra08] considers normal semi-projective toric varieties. The projectivity assumption allows us to give a condensed version of the proof given by Craw; see [Cra08, Theorem 3.23].

Example 2.3.12. Suppose that $X = \mathbb{P}(a_0, \ldots, a_n)$, then $Z = \{0\} \subset \mathbb{A}^{\Sigma(1)} = \mathbb{A}^{n+1}$ and $\mathrm{Cl}(X) \cong \mathbb{Z}$. Then $G = \mathbb{G}_m$ acts on $\mathbb{A}^{n+1} - \{0\}$ as follows: for $t \in \mathbb{G}_m$

$$t \cdot (x_0, \ldots, x_n) = (t^{a_0} x_0, \ldots, t^{a_n} x_n).$$

We recover the standard definition of weighted projective space, that is

$$\mathbb{P}(a_0, \dots, a_n) = (\mathbb{A}^{n+1} - \{0\})/\mathbb{G}_m.$$

Example 2.3.13. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, then $S = k[x, y; z, w]$ with the usual bi-grading and $B = \langle xy, yz, zw, xw \rangle$. Thus Z is two planes: Explicitly, $\mathbb{A}^4 - Z = (\mathbb{A}^2 - \{0\}) \times (\mathbb{A}^2 - \{0\})$. The character group of $\mathrm{Cl}(X) = \mathbb{Z} \times \mathbb{Z}$ is $\mathbb{G}_m \times \mathbb{G}_m$; hence we have $\mathbb{G}_m \times \mathbb{G}_m$ acting on $\mathbb{A}^2 \times \mathbb{A}^2$ by each copy of $\mathbb{G}_m$ acting on $\mathbb{A}^2$ by scalar multiplication, giving

$$\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{A}^2 \times \mathbb{A}^2 - Z)/(\mathbb{G}_m \times \mathbb{G}_m).$$

2.4 Simplicial toric varieties

The definition of a smooth cone is derived from their corresponding affine toric variety; a cone is smooth if and only if the toric variety is smooth by Theorem 2.1.12. The notion of a simplicial toric variety is somehow the 'next best case'; that is, a cone is simplicial if and only if its corresponding toric variety is an orbifold. Many results for smooth toric varieties also hold for simplicial toric varieties and they form an important class of toric varieties, appearing in many other areas. In this section we study simplicial toric varieties with particular focus on weighted projective space.

2.4.1 Simplicial toric varieties as orbifolds

Lemma 2.4.1. *Let N be a rank n lattice. Suppose that $\sigma \subset N_{\mathbb{R}}$ is a smooth, full-dimensional cone. Then U_σ is isomorphic to affine space.*

Proof. As σ is smooth and full-dimensional, the minimal generators give an isomorphism from $N_{\mathbb{R}}$ to $\mathbb{R}^n$, where the minimal generators are mapped to the standard basis. Under such an isomorphism, σ is mapped to $\mathbb{R}^n_{\geq 0}$ isomorphically. Since $\mathbb{R}^n_{\geq 0}$ is exactly the cone of affine space we have that $U_\sigma \cong \mathbb{A}^n$. $\square$

Given a finite index sublattice $N' \subset N$, any cone $\sigma \subset N_{\mathbb{R}}$ (and hence fan) can be considered inside $N'_{\mathbb{R}}$. To distinguish between the two corresponding toric varieties we shall specify the lattice in the subscript. We study the relationship between the varieties $U_{\sigma,N} = \operatorname{Spec} k[\sigma^\vee \cap M]$ and $U_{\sigma,N'} = \operatorname{Spec} k[\sigma^\vee \cap M']$, as can be found in Section 1.3 of [CLS11].

Proposition 2.4.2. [CLS11, Proposition 1.3.18] *Suppose that $N' \subset N$ is a finite index sublattice. Then*

$$U_{\sigma,N} = U_{\sigma,N'}/\mu,$$

where $\mu = N/N'$ and is a finite abelian group.

We include the proof so as to see the working of the action of the group.

Proof. Let M' be the dual of the sublattice N'. Dualising reverses the inclusion, thus $M \hookrightarrow M'$ and in particular $k[\sigma^\vee \cap M] \subset k[\sigma^\vee \cap M']$. We claim that $k[\sigma^\vee \cap M]$ is the ring of invariants for a μ-action on $U_{\sigma,N'}$. Consider the exact sequence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'/M \longrightarrow 0.$$

As $T_N = \mathrm{Hom}_{\mathbb{Z}}(N, k^*)$, where T_N is the torus associated to N, applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, k^*)$ yields the sequence

$$1 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M'/M, k^*) \longrightarrow T_{N'} \longrightarrow T_N \longrightarrow 1.$$

Note that $\mu = N/N' = \mathrm{Hom}_{\mathbb{Z}}(M'/M, k^*)$. We define an action of μ on $U_{\sigma,N'}$. On the level of rings this action is as follows, for $g \in \mu \cong \mathrm{Hom}_{\mathbb{Z}}(M'/M, k^*)$ and for $\chi_{m'} \in k[\sigma^\vee \cap M']$ define

$$g \cdot \chi_{m'} = g([m'])^{-1}\chi_{m'}.$$

Therefore if $g \cdot \chi_{m'} = \chi_{m'}$ for every $g \in \mu$, we must have that $m' \in M$. This gives

$$k[\sigma^\vee \cap M] = k[\sigma^\vee \cap M']^\mu.$$

Therefore $U_{\sigma,N} \cong U_{\sigma,N'}/\mu$. □

This result is easily generalized to normal toric varieties by patching together the affine open covering given by the cones in the fan.

Corollary 2.4.3. [CLS11, Proposition 3.3.7] *Let $\Sigma \subset N_{\mathbb{R}}$ be a simplicial fan and $N' \subset N$ a finite index sublattice. Let $\mu = N/N'$. Then the toric morphism*

$$\phi : X_{\Sigma,N'} \to X_{\Sigma,N}$$

induced by the inclusion $N' \hookrightarrow N$ is a geometric quotient for the μ-action on $X_{\Sigma,N'}$.

Figure 2.1: The lattices $N = \mathbb{Z}^2$ and $N' = \mathbb{Z} \times 2\mathbb{Z}$

Example 2.4.4. Consider the weighted projective plane $X_\Sigma = \mathbb{P}(1,1,2)$. In this case $N = \mathbb{Z}^2$ and the minimal generators of the cones are $u_1 = e_1, u_2 = e_2$ and $u_0 = -e_1 - 2e_2$. Then consider $N' := \mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z}^2$. Then $\mu_2 = \mathbb{Z}/2\mathbb{Z}$ and $X_{\Sigma,N'} = \mathbb{P}^2$. The corresponding toric morphism

$$\phi : \mathbb{P}^2 \to \mathbb{P}(1,1,2)$$

is the geometric quotient of the action of μ_2 on $\mathbb{P}^2$ acting by $\xi \cdot (x : y : z) = (x : y : \xi z)$ for $\xi \in \mu_2$. See Figure 2.1. This description holds for a general weighted projective space $X_\Sigma = \mathbb{P}(a_0, \dots, a_n)$. Indeed, let $N = \mathbb{Z}^{n+1}/(a_0 e_0 + \cdots a_n e_n)$ be the character lattice and consider the finite index sublattice $N' \subset N$ given by the image of $a_0\mathbb{Z} \times \cdots \times a_n\mathbb{Z} \subset \mathbb{Z}^{n+1}$ in N. Then $X_{\Sigma,N'} \cong \mathbb{P}^n$, and $N/N' \cong \mu_{a_0} \times \cdots \times \mu_{a_n}$, hence

$$\mathbb{P}(a_0, \dots, a_n) \cong \mathbb{P}^n/\mu_{a_0} \times \cdots \times \mu_{a_n}.$$

Proposition 2.4.5. *Let $\sigma \subset N_\mathbb{R}$ be a simplicial cone. There exists a finite index sublattice $N' \subset N$, such that $\sigma \subset N'_\mathbb{R}$ is smooth. In particular, every affine simplicial toric variety is the quotient of a smooth affine toric variety by a finite abelian group.*

Proof. Let $u_1, \dots, u_r$ be the minimal generators of $\sigma \subset N_\mathbb{R}$. As they are linearly independent over $\mathbb{R}$, we can extend them to a $\mathbb{Z}$-basis of a finite index sublattice, denote it by $N' \subset N$. Then $\sigma \subset N'_\mathbb{R}$ is a smooth cone and by Proposition 2.4.2 we have that

$$U_{\sigma,N} \cong U_{\sigma,N'}/\mu,$$

where $\mu = N/N'$. $\square$

Note that if, in addition, $\sigma \subset N_{\mathbb{R}}$ is full-dimensional, the above result shows that $U_{\sigma,N} \cong \mathbb{A}^n/\mu$. This follows from Lemma 2.4.1, which tells us that $U_{\sigma,N'} \cong \mathbb{A}^n$.

Corollary 2.4.6. [CLS11, Theorem 3.1.19] *Let $X = X_\Sigma$ be a simplicial toric variety of dimension n. Then X is an orbifold.*

Proof. Without loss of generality, assume that Σ contains at least one full-dimensional cone. Since X is simplicial, every cone $\sigma \in \Sigma$ is simplicial. Moreover, the sets $U_{\sigma,N}$, with σ full-dimensional, form an open affine covering. Thus we have an open covering of X given by

$$U_\sigma/\mu_\sigma,$$

where σ is a full-dimensional cone, and $\mu_\sigma = (N/N_\sigma)$ with $N_\sigma \subset N$ the finite index sublattice such that U_{σ,N_σ} is smooth. Since $U_\sigma = \mathbb{A}^n$, by Proposition 2.4.5, we have that X is an orbifold. □

2.4.2 Finite index sublattices and the Cox ring

We wish to investigate the finite index sublattices and the quotient presentation of a simplicial toric variety. The best tool for this task is the homogeneous coordinate ring. We prove that for simplicial toric varieties the quotient presentations associated to finite index sublattices, as discussed above, are compatible in a very natural sense to the quotient construction of the toric variety.

The aim of this section is to generalise the presentation of weighted projective space as a quotient of projective space by a finite group action (see for example [Dol82]) to any simplicial toric variety of Picard rank 1. In the following all lattices and therefore toric varieties will be n-dimensional.

Consider a simplicial fan $\Sigma \subset N_{\mathbb{R}}$ and a finite index sublattice $N' \subset N$. The homogeneous coordinate rings of the two toric varieties $X_{\Sigma,N}$ and $X_{\Sigma,N'}$ are the same: this is due to the fact

$$S = k[x_\rho \,|\, \rho \in \Sigma(1)]$$

and that $\Sigma(1)$ remains the same irrespective of the lattice. Similarly the irrelevant ideal $B_\Sigma = \langle x^{\hat{\sigma}}, \sigma \in \Sigma \rangle$ also remains unchanged, where $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. However, crucially the gradings are not the same.

Example 2.4.7. Consider the weighted projective plane $\mathbb{P}(1,1,2)$ then $S = k[x,y,z]$ and $B_\Sigma = \langle x,y,z,xy,xz,yz\rangle = \langle x,y,z\rangle$. Thus $\mathbb{P}(1,1,2)$ is the quotient of $\mathbb{A}^3 - \{0\}$. Moreover, we have a commutative diagram of geometric quotients

$$\begin{tikzcd} \mathbb{A}^3 - \{0\} \arrow[r] \arrow[dr] & \mathbb{P}^2 \arrow[d] \\ & \mathbb{P}(1,1,2), \end{tikzcd}$$

where the vertical map is the from Example 2.4.4. This holds for a general weighted projective space using the same sublattice as in Example 2.4.4.

We can now give a modified version of [BC94, Lemma 2.11].

Theorem 2.4.8. *Let $\Sigma \subset N_\mathbb{R}$ be a complete simplicial fan such that $|\Sigma(1)| = n+1$. Then the following statements hold.*

1. *$X_{\Sigma,N}$ is a weighted projective space if and only if $\mathrm{Cl}(X_{\Sigma,N}) \cong \mathbb{Z}$,*
2. *there exists a finite index sublattice $N' \subset N$ such that $X_{\Sigma,N'}$ is a projective space,*
3. *there is a commuting diagram*

$$\begin{tikzcd} \mathbb{A}^{\Sigma(1)} - Z_\Sigma \arrow[r, "q'"] \arrow[dr, "q"'] & X_{\Sigma,N'} \arrow[d, "\phi"] \\ & X_{\Sigma,N}, \end{tikzcd}$$

where $N' \subset N$ is any finite index sublattice, q and q' are the quotients as in Theorem 2.3.9 and ϕ is the quotient by the finite group N/N'.

Proof. Part (i) of the theorem is [BC94, Lemma 2.11]. Suppose that $\Sigma(1) = \{\rho_0,\dots,\rho_n\}$ and that $e_i \in N$ is the primitive lattice point on ρ_i. Let N' be the sublattice generated by the e_i's. Since e_i generate N' as a $\mathbb{Z}$-module, $\Sigma \subset N'_\mathbb{R}$ is the fan of a weighted projective space, thus $X_{\Sigma,N'} \cong \mathbb{P}(a_0,\dots,a_n)$. Hence, by Corollary 2.4.3, $X_{\Sigma,N}$ is a quotient of a weighted projective space. By Example 2.4.4, we can again take a finite index sublattice giving the weighted projective space as a finite quotient of projective space. The commutativity of the diagram follows from the quotient construction of Theorem 2.3.9. $\square$

2.5 Hypersurfaces in toric varieties

A hypersurface in a variety is an equidimensional codimension 1 closed subscheme; that is, all irreducible components are codimension 1. Hypersurfaces are also known as effective Weil divisors. In this section, we prove that every hypersurface in a simplicial toric variety will come from a homogeneous element of the Cox ring, generalising the case for hypersurfaces in projective space [Har77, Chapter II.7]. We then give some further results on quasismooth hypersurfaces.

We begin by considering quasi-coherent sheaves and then restrict our attention to ideal sheaves.

2.5.1 Quasi-coherent sheaves on a toric variety

Let $X = X_\Sigma$ be a simplicial toric variety and $S = \mathrm{Cox}(X)$. An S-module F is graded if there is a direct sum decomposition

$$F = \bigoplus_{\alpha \in \mathrm{Cl}(X)} F_\alpha$$

such that $S_\alpha \cdot F_\beta \subset F_{\alpha+\beta}$. We recall the correspondence between graded modules and quasi-coherent sheaves on X, generalising the theory of quasi-coherent sheaves on projective space; see Section 3 of [Cox95b]. Given a graded S-module F, we construct a quasi-coherent sheaf $\tilde{F}$ over X. For each $\sigma \in \Sigma$, denote by S_σ the localisation of S by the monomial $x^{\hat{\sigma}}$. Then $F_\sigma := F \otimes_S S_\sigma$ is a graded S_σ-module and as $X_\sigma := \mathrm{Spec}((S_\sigma)_0)$ is an affine open subvariety, F_σ defines a quasi-coherent sheaf $\tilde{F_\sigma}$ on X_σ. As in the construction of toric varieties, we glue these sheaves along the common faces of the cones in the fan.

Theorem 2.5.1. [Cox95b, Proposition 3.2] *Suppose that $X = X_\Sigma$ is a simplicial toric variety, then the functor*

$$\sim : \mathrm{Mod}_{gr} S \longrightarrow \mathrm{Qcoh}(X), \quad F \longmapsto \tilde{F}$$

is exact and essentially surjective, where $\mathrm{Mod}_{gr} S$ is the category of graded S-modules. Moreover, if we restrict the functor to finitely generated modules (the category of which we denote by $\mathrm{mod}_{gr} S$), we have an exact essentially surjective functor

$$\mathrm{mod}_{gr} S \longrightarrow \mathrm{Coh}(X).$$

Remark 2.5.2. This means studying sheaves on a simplicial toric variety is related to studying graded modules of the homogeneous coordinate ring. However, the functor described above is far from being injective. As in the case for projective space, there is a notion of saturated graded modules. When we refine the category of graded S-modules to the category of so called 'saturated modules', the functors in Lemma 2.5.1 become equivalences. For details we refer the reader to the appendix to Chapter 6 in [CLS11].

2.5.2 Hypersurfaces in toric varieties

We restrict our attention to coherent ideal sheaves. To obtain a closed subscheme from a graded ideal $I \subset S$ note that as the functor taking graded S-modules to quasi-coherent sheaves is exact, $\mathcal{I} := \tilde{I} \subset \mathcal{O}_X$ is an ideal sheaf and thus defines a closed subscheme. We denote the subscheme $\mathbf{V}(I) \subset X$.

Proposition 2.5.3. [Cox95b, Theorem 3.7] *Let X be a simplicial toric variety with Cox ring S and irrelevant ideal $B \subset S$. Then the following statements hold.*

1. *Every closed subscheme of X has ideal sheaf determined by a graded ideal $I \subset S$,*
2. *two graded ideals I and J correspond to the same subscheme if and only if $(I : B^\infty)_\alpha = (J : B^\infty)_\alpha$ for every $\alpha \in \operatorname{Pic}(X)$.*

Example 2.5.4. Suppose that X is a simplicial toric variety and that $\alpha \in \operatorname{Cl}(X)$. Analogously to the Serre twisting sheaves, we define $\mathcal{O}_X(\alpha) = \widetilde{S(\alpha)}$, where $S(\alpha)$ is the Cox ring S with the grading shifted by α.

We now wish to restrict the above correspondence to hypersurfaces. First we discuss the saturation of a principal ideal.

Lemma 2.5.5. *Let $S = \bigoplus_{\alpha\in\mathbb{Z}} k[x_0, \dots, x_n]_\alpha$ be a polynomial ring with a $\mathbb{Z}$-grading, such that $\deg x_i \neq 0$ for each i. Let $B = \langle x_0, \dots, x_n\rangle$ be the irrelevant ideal. Then for any homogeneous element $f \in S$ we have*

$$(\langle f\rangle : B)^\infty = \langle f\rangle,$$

where $(\langle f\rangle : B)^\infty = \bigcup_{k\in\mathbb{Z}}(\langle f\rangle : B^k)$ is the saturation of the ideal $\langle f\rangle$.

Proof. Consider a homogeneous element $g \in (\langle f \rangle : B)^{\infty}$. Then $g \in (\langle f \rangle : B^N)$ for some $N \geq 1$. Thus $g \cdot B^N \subset \langle f \rangle$. In particular, we can find polynomials $b_i \in S$ for $i = 0, \ldots, n$ such that

$$g \cdot x_i^N = f \cdot b_i.$$

Now suppose for a contradiction there exists some j with $1 \leq j \leq N$ such that $x_0^j \mid f$ but $x_0^j \nmid b_0$. We define the following multiplicative function $\deg_{x_0} : S \to \mathbb{Z}_{\geq 0}$ with

$$\deg_{x_0}(p) = \min\{\text{degree of } x_0 \text{ in monomials of } p\}.$$

Then $\deg_{x_0}(g \cdot x_0^N) = \deg_{x_0}(f \cdot b_0)$, and thus $\deg_{x_0}(g) = \deg_{x_0}(f) + (\deg_{x_0}(b_0) - N)$. Since $x_0^j \nmid b_0$ for some $1 \leq j \leq N$, we have that $\deg_{x_0}(b_0) - N < 0$ and thus

$$\deg_{x_0}(g) < \deg_{x_0}(f).$$

Now consider $\deg_{x_0}(g \cdot x_1^N) = \deg_{x_0}(f \cdot b_1)$. Then

$$\deg_{x_0}(f) \leq \deg_{x_0}(f) + \deg_{x_0}(b_1) = \deg_{x_0}(g),$$

which is a contradiction. This means that if x_0^j divides f, then it must also divide b_0. Then as $g \cdot x_0^{N-1} \cdot x_0 = f \cdot b_0$ we know that $x_0 \mid b_0$. Define $b_0' := \frac{b_0}{x_0}$. We have that

$$g \cdot x^{N-1} = f \cdot b_0'.$$

Repeating this argument for each of the x_i, we obtain a new set of equations $g \cdot x_i^{N-1} = f \cdot b_i'$. As we may do the same procedure again, we end up with

$$g = f \cdot \frac{b_0}{x_0^N},$$

with $\frac{b_0}{x_0^N} \in S$, which completes the proof. □

Remark 2.5.6. Note that the above lemma applies to any weighted projective space and moreover any simplicial toric variety of Picard rank 1.

Remark 2.5.7. We give a more general proof of the above lemma. Applying to any graded polynomial ring: Suppose now that $S = \oplus_{\alpha} S_{\alpha}$ is a Cox ring of a toric variety and B is the irrelevant ideal.

We shall need the following result from [CLO07, p.195]. Suppose that $I, J \subset k[x_0, \ldots, x_n]$ are ideals in the polynomial ring. Then $V((I : J)^\infty) = \overline{V(I) - V(J)} \subset \mathbb{A}^{n+1}$ as algebraic sets.

Applying this result to $\langle f \rangle$ and B we have that

$$V(f) = V((\langle f \rangle : B)^\infty),$$

since $V(f)$ is codimension 1 and $V(B)$ is codimension 2. Thus $(\langle f \rangle, B)^\infty$ is a principal ideal containing f, so

$$\langle f \rangle \subset (\langle f \rangle, B)^\infty = \langle g \rangle$$

for some $g \in S$. However, since $V(f) = V(g) \subset \mathbb{A}^{n+1}$, we have that $\langle f \rangle = \langle g \rangle$.

Proposition 2.5.8. *Let $X = X_\Sigma$ be a simplicial toric variety and suppose that $Y \subset X$ is a hypersurface such that $\alpha = [Y] \in \mathrm{Cl}(X)$ is its value in the class group. Then there exists a homogeneous element $f \in S_\alpha$ such that $Y = \mathbf{V}(f)$. Conversely, given any homogeneous element $f \in S_\alpha$ the corresponding closed subscheme $\mathbf{V}(f)$ is a hypersurface of class α.*

Proof. Suppose that Y is a hypersurface of class α. This is equivalent to saying $\mathcal{I}_Y \cong \mathcal{O}_X(-\alpha)$. Since Y is codimension one, $\overline{q^{-1}(Y)}$ is also codimension 1 and so $\overline{q^{-1}(Y)} = V(f) \subset \mathbb{A}^{\Sigma(1)}$, where q is the geometric quotient of Theorem 2.3.9 and $f \in S_\alpha$. It then follows that $\widetilde{\langle f \rangle} \cong \mathcal{O}_X(-\alpha)$. Hence $f \in H^0(X, \mathcal{O}_X(\alpha)) = S_\alpha$.

Conversely, if $f \in S_\alpha$ then $\widetilde{\langle f \rangle} \cong \mathcal{O}_X(-\alpha)$ and so $\mathbf{V}(f)$ defines a hypersurface of class α. □

Proposition 2.5.9. *Suppose X is a weighted projective space. Then there is a bijection between hypersurfaces of degree α and $\mathbb{P}(S_\alpha)$. That is $|\alpha| = \mathbb{P}(S_\alpha)$ is the complete linear system of α.*

Proof. The result follows by combining Proposition 2.5.3, Lemma 2.5.5 and Proposition 2.5.8. □

Example 2.5.10. Let $X = \mathbb{P}(a_0, \ldots, a_n) = \operatorname{Proj} k[x_0, \ldots, x_n]$ be a weighted projective space. Then $|\mathcal{O}_X(d)| = \mathbb{P}(k[x_0, \ldots, x_n]_d)$ is the complete linear system of degree d hypersurfaces.

Example 2.5.11. Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_l}$ be a product of projective spaces. The Cox ring of X is

$$S = k[x_{1,1}, \ldots, x_{1,n_1}; \ldots; x_{l,1}, \ldots, x_{l,n_l}],$$

where each $x_{i,j}$ has degree $(0, \ldots, 1, \ldots 0)$, with the 1 in the i^{th} position. Then the hypersurfaces of degree $d = (d_1, \ldots, d_l) \in \mathbb{Z}^l$ are parametrised by $\mathbb{P}(S_d)$.

We state an Euler formula for hypersurfaces in simplicial projective toric varieties.

Lemma 2.5.12. [BC94, Lemma 3.8] *Let $X = X_\Sigma$ be a simplicial projective toric variety and let $\Sigma(1) = \{u_1, \ldots, u_r\}$ with $e_i \in u_i$ a primitive lattice point of each ray. Suppose there are elements $\phi_1, \ldots \phi_r \in k$ such that $\phi_1 e_1 + \cdots + \phi_r e_r = 0$ in $N \otimes_{\mathbb{Z}} \mathbb{C}$. Then for every $\alpha \in \mathrm{Cl}(X)$ there exists a constant $\phi(\alpha) \in k$ such that for every $f \in S_\alpha$, we have*

$$\phi(\alpha) f = \sum_{i=1}^{r} \phi_i x_i \frac{\partial f}{\partial x_i}.$$

In particular, if $X = \mathbb{P}(a_0, \ldots, a_n)$ is a weighted projective space and $f \in k[x_0, \ldots, x_n]_d$ is a weighted homogeneous polynomial, we have that

$$d \cdot f = \sum_{i=0}^{n} a_i x_i \frac{\partial f}{\partial x_i}.$$

We now state a generalised version of the Grothendieck-Lefschetz lemma, proved by Ravindra and Srinivas, which we will need in the sequel, involving hypersurfaces in a general projective variety.

Theorem 2.5.13. [RS06, Theorem 1] *Let X be an irreducible projective variety which is regular in codimension 1 and $\mathcal{L}$ an ample line bundle over X. Suppose that $V \subset H^0(X, \mathcal{L})$ gives a base point free linear system $|V|$. Then for a general element $Y \in |V|$ the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

is an isomorphism provided $\dim X \geq 4$.

Remark 2.5.14. In particular the above theorem holds when X is a projective toric variety of dimension $n \geq 4$. Then for a general hypersurface $Y \subset X$ the restriction map above is an isomorphism.

2.6 Quasismooth hypersurfaces

Let $X = X_\Sigma$ be a complete simplicial toric variety and $q : \hat{X} \to X$ the quotient from Theorem 2.3.9. A subvariety of projective space is smooth if and only if its affine cone is a smooth affine variety away from the vertex. Using this as the focus of generalisation to complete simplicial toric varieties, we define a class of subvarieties, called *quasismooth subvarieties.* A subvariety of X is quasismooth if and only if its inverse image in $\hat{X}$ is smooth (Definition 2.6.1). These subvarieties are only midly singular; singularities arise from non-trivial stabilisers in the action of Theorem 2.3.9, and so the singularties are inherited from the ambient variety. In particular, if X is smooth, then the quasismoothness coincides with smoothness. These were studied in at first in the case of weighted projective space in [Dol82] and in more detail in [IF00] and in general simplicial toric varieties in [BC94].

Definition 2.6.1. Let $Y \subset X$ be a hypersurface defined by $f \in S_\alpha$. We say that Y is *quasismooth* if $V(f) \subset \mathbb{A}^{\Sigma(1)}$ is smooth outside of $Z_\Sigma = V(B_\Sigma)$.

Remark 2.6.2. We can check quasismoothness using a Jacobian criterion; $Y = V(f)$ is quasismooth if and only if the equations $\frac{\partial f}{\partial x_\rho}$ for $\rho \in \Sigma(1)$ have no common zeros in $\hat{X}$.

Remark 2.6.3. A hypersurface is quasismooth if and only if it is a suborbifold [BC94, Proposition 3.5]. An immediate consequence of this is that if X is smooth, then a hypersurface is quasismooth if and only if it is smooth.

Definition 2.6.4. Let X be a simplicial toric variety and fix a class $\alpha \in \mathrm{Cl}(X)$. Let $\mathcal{Y}_\alpha = \mathbb{P}(S_\alpha)$. We define

$$\mathcal{Y}_\alpha^{\mathrm{QS}} = \{Y \subset X \mid Y \text{ is a quasismooth hypersurface of class } \alpha\} \subset \mathcal{Y}_\alpha.$$

We denote its complement, the non-quasismooth locus, by

$$\mathcal{Y}_\alpha^{\mathrm{NQS}} = \mathcal{Y}_\alpha - \mathcal{Y}_\alpha^{\mathrm{QS}}.$$

We shall drop the α from the subscript when the class is clear from the context.

Proposition 2.6.5. *Let X be a complete toric variety and $\alpha \in \mathrm{Cl}(X)$. The quasismooth locus $\mathcal{Y}^{QS} \subset \mathcal{Y}$ is open.*

Proof. For every hypersurface $Y \subset X$, we write f_Y for its corresponding homogeneous polynomial. Define

$$W = \left\{(x, Y) \in X \times \mathcal{Y} \mid f_Y(y) = \frac{\partial f_Y}{\partial x_\rho}(y) = 0 \text{ for } y \in q^{-1}(x)\right\}.$$

W is an algebraic set and hence closed. Let $\pi : X \times \mathcal{Y} \to \mathcal{Y}$ be the projection. For a hypersurface $Y \in \mathcal{Y}$, the fibre $\pi^{-1}(Y)$ is empty if and only if Y is quasismooth. Note that π is closed as X is complete and that $\mathcal{Y}^{\mathrm{NQS}} = \pi(W)$, thus we have that the set of quasismooth hypersurfaces is open. □

We mention a fact about the topological type of smooth hypersurfaces in weighted projective spaces over the complex numbers. See Theorem 4.1.15 for the proof of a more general statement.

Proposition 2.6.6. *Let $k = \mathbb{C}$ and $X = \mathbb{P}(a_0, \ldots, a_n)$ be a weighted projective space and let $d > 0$ be a positive integer. Then quasismooth hypersurfaces which are all smooth of the same degree d are diffeomorphic.*

Remark 2.6.7. This proposition allows us to take a slightly different perspective on hypersurfaces in complex weighted projective space: suppose that $Y(\mathbb{C})^{\mathrm{an}}$ is the (analytic) topological space underlying a hypersurface of degree d in a weighted projective space. Then Proposition 2.6.6 tells us that the locus $\mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d)^{\mathrm{SM}}$ of smooth hypersurfaces is a parameter space for algebraic structures on $Y(\mathbb{C})^{\mathrm{an}}$. This is the point of view often taken when working in the context of mirror symmetry, for example see [CK99].

We could go further and say that two hypersurfaces are homeomorphic in the analytic topology if and only if they are of the same degree. This is easily seen by considering the cohomology of the hypersurfaces, which is fixed by the degree; see [BC94].

Remark 2.6.8. The question of when two hypersurfaces in the same linear system are homeomorphic is an interesting one. We discuss analogous results in a more general context in Chapter 4. For an an exact statement and proof see Theorem 4.1.15.

Proposition 2.6.9. *Let X be a simplicial toric variety. If $Y \subset X$ is a quasismooth hypersurface, then Y is normal.*

Proof. Since X is simplicial, we have a geometric quotient $q : \hat{X} \to X$ by Theorem 2.3.9. Then Y is quasismooth, so $\hat{Y} = q^{-1}(Y)$ is smooth and hence normal. Hence $q|_{\hat{Y}} : \hat{Y} \to Y$ gives Y as a geometric quotient of a smooth variety and is hence normal, as the GIT quotient of a normal variety is normal by [Dol03, Proposition 3.1]. □

Remark 2.6.10. An immediate consequence is that quasismooth curves on a toric surface are smooth. Note that the reverse implication does not hold, that is, not every smooth curve is quasismooth. Let $X = \mathbb{P}(2,3,5)$ with coordinates x, y and z. Consider the degree 5 curve $C = \mathbf{V}(x^3 - y^2) \subset X$. Then all the partial derivatives vanish at the point $(0:0:1) \in C$. However, for the chart $U_z = (z \neq 0) = \frac{1}{5}(2,-2)$, we have

$$U_z = \operatorname{Spec} \mathbb{C}\Big[\frac{x^5}{z^2}, \frac{xy}{z}, \frac{y^5}{z^3}\Big] = \operatorname{Spec} \mathbb{C}[a,b,c]/(ac - b^5),$$

and the restriction to $C \cap U_z = \operatorname{Spec} \mathbb{C}[b]$ since $x^2(x^3 - y^2) = 0$ implies $a - b^2 = 0$ and $y^3(x^3 - y^2) = 0$ implies $b^3 - 1 = 0$. This implies that $(0:0:1)$ is a smooth point of C. For the other chart $U_x = \frac{1}{2}(1,1)$, we have that

$$U_x = \operatorname{Spec} \mathbb{C}\Big[\frac{y^2}{x^2}, \frac{yz}{x^4}, \frac{z^2}{x^5}\Big] = \operatorname{Spec} \mathbb{C}[a,b,c]/(ac - b^2)$$

and that

$$U_x \cap C = \operatorname{Spec} \mathbb{C}[b],$$

since $x^3 - y^2 = 0$ implies that $a = 1$ and thus $c = b^2$. We conclude that C is a smooth curve which is not quasismooth.

Theorem 2.6.11. [Dol82, Theorem 3.3.4] *Suppose that $X = \mathbb{P}(a_0, \dots, a_n)$ is a weighted projective space and that $Z \subset X$ is a degree d quasismooth hypersurface. Then*

$$\omega_Z \cong \mathcal{O}_Z(d - a_0 - \cdots - a_n).$$

Example 2.6.12. Let $X = \mathbb{P}(1,1,2)$ and consider $C \subset X$ a degree 4 quasismooth curve. Then combining Remark 2.6.10 and the above theorem, we have that

$$\omega_C \cong \mathcal{O}_C,$$

and thus C is an elliptic curve.

We conclude this chapter by giving a result of Iano-Fletcher and an easy consequence thereof, characterising quasismooth hypersurfaces in weighted projective space.

Theorem 2.6.13. [IF00, Theorem 8.1] *The general hypersurface of degree d in $\mathbb{P}(a_0, \dots a_n)$ is quasismooth if and only if*

either (1) there exists a variable x_i of degree d,

or (2) for every non-empty subset $I = \{i_0, \dots, i_{k-1}\}$ of $\{0, \dots, n\}$,
either (a) there exists a monomial $x_I^M = x_{i_0}^{m_0} \cdots x_{i_{k-1}}^{m_{k-1}}$ of degree d
or (b) for $\mu = 1, \dots, k$ there exists monomials

$$x_I^{M_\mu} x_{e_\mu} = x_{i_0}^{m_{0,\mu}} \cdots x_{i_{k-1}}^{m_{k-1,\mu}} x_{e_\mu}$$

of degree d, where $\{e_\mu\}$ are k distinct elements.

The above proposition shows that the monomials appearing in a weighted homogeneous form must be sufficiently generic.

The following lemma will be needed for the proof of the main result in Chapter 6. It follows from arguments used in the proof of Theorem 2.6.13 in [IF00]. We include a proof for completeness.

Lemma 2.6.14. *Suppose that $X = \mathbb{P}(a_0, \dots, a_n)$ is well formed and d a Cartier degree and denote $d_i = \frac{d}{a_i}$. Consider an $f \in k[x_0, \dots, x_n]_d$ which is quasismooth. Then for every variable x_i, either $x_i^{d_i}$ is a monomial of f or $x_i^{d_i - \frac{a_j}{a_i}} x_j$ is a monomial of f for some $j \neq i$ where $a_i | a_j$.*

Note that if for a fixed a_i, there exists no a_j such that $a_i | a_j$, then we must have that $x_i^{d_i}$ is a monomial of f.

Remark 2.6.15. Let $f \in k[x_0, \dots, x_n]_d$ such that $d > \max(a_0, \dots, a_n)$ and assume that condition (2) doesn't hold, it follows from [IF00, Theorem 8.1] that f is not quasismooth; that is, we can remove the general hypothesis from Theorem 2.6.13, as was done in Lemma 2.6.14.

Proof. Suppose that for the variable x_0 both of the above conditions fail. Then every monomial of f has the form $x_0^M \underline{x}^I$ where $M < d_0$ and $\underline{x}^I \in k[x_1, \dots x_n]$ has total degree greater than 2. Thus $(1:0:\dots:0) \in \mathbf{V}(f)$ and $(1:0:\dots:0) \in V(\frac{\partial f}{\partial x_i})$ for every i since each monomial in every $\frac{\partial f}{\partial x_i}$ contains a variable not equal to x_0. Thus f is not quasismooth. □

Remark 2.6.16. For a monomial of the form $x_i^{d_i}$, we call the monomials $x_i^{d_i - \frac{a_j}{a_i}} x_j$ the neighbours of $x_i^{d_i}$.

Chapter 3

Automorphisms and toric varieties

The construction of the automorphism group of a complete simplicial toric variety X is a generalisation of the construction of the automorphism group of projective space. The generalisation is to be seen as follows. The Cox ring of projective space with the grading of the class group is the standard homogeneous coordinate ring; that is, the polynomial ring with the usual $\mathbb{Z}$-grading given by the total degree. The group of graded automorphisms of this ring is GL_{n+1} which fits into the following short exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_{n+1} \longrightarrow \mathrm{PGL}_{n+1} \longrightarrow 0,$$

where $\mathrm{PGL}_{n+1} = \mathrm{Aut}(\mathbb{P}^n)$. More generally, let X be a complete toric variety associated to a fan Σ. Let $S = k[x_\rho \,|\, \rho \in \Sigma(1)]$ be the Cox ring of X and let $q : \hat{X} \to \hat{X}/\mathbf{D} = X$ be the geometric quotient of Theorem 2.3.9. When we refer to the degree of an element of S, we mean the degree with respect to the class group and by total degree we mean the degree with respect to the usual $\mathbb{Z}$-grading of the polynomial ring. We obtain a short exact sequence

$$0 \longrightarrow \mathbf{D} \longrightarrow \mathrm{Aut}_{\mathrm{g}}(S) \longrightarrow \mathrm{Aut}^0(X) \longrightarrow 0.$$

where $\mathrm{Aut}_{\mathrm{g}}(S)$ is the group of graded automorphisms of S. We first study the group $\mathrm{Aut}_{\mathrm{g}}(S)$.

3.1 Graded algebra automorphisms

We introduce some notation in order to study the structure of the group $\mathrm{Aut}_{\mathrm{g}}(S)$. There is an equivalence relation on $\Sigma(1)$ given by

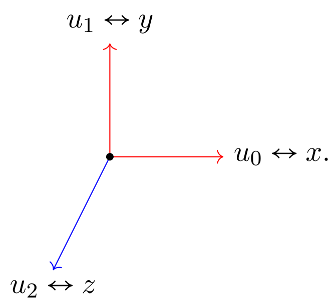
$$\rho \sim \rho' \iff \deg x_\rho = \deg x_{\rho'} \in \mathrm{Cl}(X).$$

This partitions the rays into equivalence classes $\Sigma(1) = \Sigma_1 \sqcup \cdots \sqcup \Sigma_l$ where

$$\Sigma_i = \{\rho \in \Sigma(1) \,|\, \deg \rho = \alpha_i\}$$

and $\alpha_i \in \mathrm{Cl}(X)$ are the degrees of the variables. We call the degrees of the variables the *generating degrees,* so in the above notation our generating degrees are $\alpha_1, \ldots, \alpha_l$ and are distinct. For each generating degree we write $S_i = S_{\alpha_i}$. Note that we have the decomposition $S_i = S_i' \oplus S_i''$, where $S_i' = \mathrm{Span}(x_\rho \,|\, \rho \in \Sigma_i)$ is the subspace given by single variables and S_i'' is the subspace spanned by the remaining monomials, all of which have total degree at least 2.

Example 3.1.1. Consider $X = \mathbb{P}(1,1,2)$. Then $\mathrm{Cl}(X) \cong \mathbb{Z}$ and the corresponding fan is:



Then $\Sigma(1) = \{u_0, u_1, u_2\}$ and $S = \mathbb{C}[x,y,z]$ and the colours indicate the equivalence classes with $\deg x = \deg y = 1$ and $\deg z = 2$. The generating degrees are 1 and 2 and we have

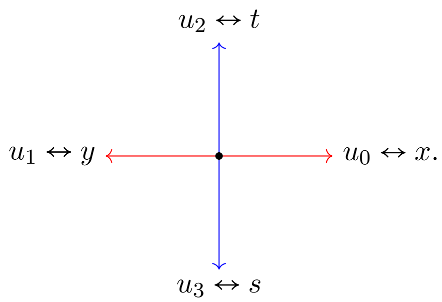
$$\Sigma(1) = \Sigma_1 \sqcup \Sigma_2 = \{u_0, u_1\} \sqcup \{u_2\}.$$

The decompositions are as follows

$$S_1 = \mathbb{C}[x,y,z]_1 = \mathbb{C} \cdot \{x, y\} \oplus 0, \quad S_2 = \mathbb{C}[x,y,z]_2 = \mathbb{C} \cdot \{z\} \oplus \mathbb{C} \cdot \{x^2, xy, y^2\}.$$

Explicitly, in the second generating graded piece $S_2' = \mathbb{C} \cdot \{z\}$ and $S_2'' = \mathbb{C} \cdot \{x^2, xy, y^2\}$.

Example 3.1.2. Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$. In this case we have $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ and fan:



In this example $S = \mathbb{C}[x, y; s, t]$ with the usual bigrading, the generating degrees are $(1, 0)$ and $(0, 1)$ and we have

$$S_{(1,0)} = \mathbb{C} \cdot \{x, y\} \oplus 0, \quad S_{(0,1)} = \mathbb{C} \cdot \{s, t\} \oplus 0.$$

Note that

$$\mathrm{Aut}_{\mathrm{g}}(S) \cong \mathrm{GL}_2 \times \mathrm{GL}_2$$

is a reductive group. Moreover, $\mathrm{Aut}(X) \cong \mu_2 \ltimes (\mathrm{PGL}_2 \times \mathrm{PGL}_2)$ is also a reductive group. This is always the case when for every generating degree it holds that $S_i'' = 0$; see the characterisation of the unipotent radical in Theorem 3.1.3.

The following theorem is taken from a paper of Cox [Cox95b, Proposition 4.3]. Note that the original proof in [Cox95b] contained an error in the construction of $\mathrm{Aut}_{\mathrm{g}}(S)$ and was later corrected in [Cox14].

Theorem 3.1.3. *Let X be a complete toric variety and let $S = \mathrm{Cox}(X)$ be its Cox ring. Then the following statements hold.*

1. *The group of graded algebra automorphisms $\mathrm{Aut}_{\mathrm{g}}(S)$ is a connected affine algebraic group of dimension $\sum_{i=1}^{l} |\Sigma_i| \dim_{\mathbb{C}} S_i$.*

2. *The unipotent radical U of $\mathrm{Aut}_{\mathrm{g}}(S)$ is of dimension $\sum_{i=1}^{l} |\Sigma_i|(\dim_{\mathbb{C}} S_i - |\Sigma_i|)$.*

3. *We have the following isomorphism*

$$\mathrm{Aut}_{\mathrm{g}}(S) \cong \prod_{i=1}^{l} \mathrm{GL}(S_i') \ltimes U.$$

We refer the reader to [Cox14] for a proof. We consider the group of graded automorphisms as a matrix group via the following lemma, whose proof is taken from the proof of the corrected version of [Cox14, Proposition 4.3]. We include the proof since we will need the explicit matrix description of the automorphism group.

Lemma 3.1.4. *The endomorphism algebra of S is a linear algebraic monoid with unit group $\operatorname{Aut}_{\mathrm{g}}(S)$ and there is an inclusion of linear algebraic monoids*

$$\begin{aligned}\operatorname{End}_{\mathrm{g}}(S) &\longrightarrow \prod_{i=1}^{l} \operatorname{End}_{\mathbb{C}}(S_i)\\ \phi &\longmapsto (\phi|_{S_i} : S_i \to S_i)_{i=1}^{l}.\end{aligned}$$

In particular, $\operatorname{Aut}_{\mathrm{g}}(S)$ *is a linear algebraic group.*

Proof. We show that the map

$$\operatorname{End}_{\mathrm{g}}(S) \longrightarrow \prod_{i=1}^{l} \operatorname{End}_{\mathbb{C}}(S_i)$$

is a closed immersion and hence $\operatorname{End}_g(S)$ is an affine submonoid. Since S is generated as an algebra by elements in $S_1, ..., S_l$, an endomorphism is completely determined by the above restrictions and hence the map is injective. The fact that the map respects composition (and is well-defined) is immediate since we consider only graded endomorphisms. Thus $\operatorname{End}_{\mathrm{g}}(S)$ is a submonoid and it only remains to show that it is a closed subset; that is, cut out by polynomials.

To do this, we write down the corresponding collection of matrices with respect to the basis of each $S_i = S_i' \oplus S_i''$ given by monomials of degree α_i:

$$\phi \longleftrightarrow \left(\left(\begin{array}{c|c} A_i & 0 \\ \hline B_i & C_i \end{array} \right) \right)_{i=1}^{l}, \qquad (\star)$$

where $B_i \in \operatorname{Hom}_{\mathbb{C}}(S_i', S_i'')$. We shall often suppress the brackets in this notation.

The matrices A_i and B_i come from evaluating the single variables in S_i'. The C_i come from evaluating monomials in S_i'' which are products of 2 or more variables in S_j' with $j \neq i$ and hence C_i is completely determined by A_j and B_j for $j \neq i$. We claim that the elements of C_i are polynomials in elements of A_j and B_j for $j \neq i$.

Let us prove this claim. Consider monomials $x^D, x^E \in S_i''$, where D and E are effective non-prime divisors with class α_i. Both x^D and x^E are elements of the monomial basis of S_i'' so that for $\phi \in \mathrm{End}_{\mathrm{g}}(S)$

$$\phi(x^D) = \cdots + c_i^{DE} x^E + \cdots,$$

where c_i^{DE} is the corresponding entry in C_i. Then $x^D = x_{\rho_1} \cdots x_{\rho_s}$ is a product of variables allowing duplications with $x_{\rho_i} \notin S_i'$. Thus

$$\phi(x_{\rho_1}) \cdots \phi(x_{\rho_s}) = \cdots + c_i^{DE} x^E + \cdots.$$

But each $\phi(x_{\rho_k})$ is a linear combination of monomials with coefficients given by elements of A_j and B_j with $j \neq i$. Thus the elements of the C_i are given by polynomials in the elements of A_j, B_j and we are done.

On the other hand, the A_i and B_i are chosen completely arbitrarily. In other words we have a bijection of sets

$$\begin{aligned} \mathrm{End}_{\mathrm{g}}(S) &\longleftrightarrow \prod_{i=1}^{l} \mathrm{Hom}_{\mathbb{C}}(S_i', S_i) \\ \phi &\longleftrightarrow \begin{pmatrix} A_i \\ B_i \end{pmatrix}. \end{aligned}$$

The 0 in the top right hand corner of the matrices $(\star)$ comes from the fact that

$$\phi(S_i'') \cap S_i' = 0$$

since $S_0 = \mathbb{C}$, and monomials in S_i'' contain more than one variable.

It remains to remark that $\mathrm{Aut}_{\mathrm{g}}(S)$ is the group of invertible elements in a linear algebraic monoid. It follows from [Put88, Corollary 3.26] that $\mathrm{Aut}_{\mathrm{g}}(S)$ is a linear algebraic group.

□

Proposition 3.1.5. *The unipotent radical U of $\mathrm{Aut}_{\mathrm{g}}(S)$ is given by matrices of the form*

$$\left(\begin{array}{c|c} I_i & 0 \\ \hline B_i & C_i \end{array}\right)$$

under the correspondence in $(\star)$, where C_i are lower triangular matrices with 1's on the diagonal.

Moreover, the 1-parameter subgroup given by

$$\begin{aligned} \lambda_g : \mathbb{G}_m &\longrightarrow \mathrm{Aut_g}(S) \\ t &\longmapsto (\phi_t : x_\rho \mapsto t^{-1} x_\rho) \end{aligned}$$

gives U a positive grading. We refer to λ_g as the distinguished $\mathbb{G}_m$.

Remark 3.1.6. Note that this result was already given in the paper [BDHK18]. The proof uses the original incorrect construction of the automorphism group given in the paper [Cox95b]. We present a proof using the corrected construction given in [Cox14].

Proof. It is clear that the matrices above form a unipotent subgroup and we refer the reader to [Cox14, Theorem 4.2] for a proof that it is in fact the unipotent radical. We prove that it is positively graded by the distinguished $\mathbb{G}_m$.

Under $(\star)$ we have

$$\lambda_g(t) \longleftrightarrow \left(\begin{array}{c|c} t^{-1} I_i & 0 \\ \hline 0 & Q_i(t) \end{array}\right)$$

where

$$Q_i(t) = \mathrm{diag}(t^{-l_1^i}, \ldots, t^{-l_k^i})$$

are diagonal matrices with $l_j^i \geq 2$. To see this, consider $x^D = x_{\rho_1} \cdots x_{\rho_l} \in S_i''$ again allowing duplications. Then

$$\lambda_g(t)(x^D) = \lambda_g(t)(x_{\rho_1}) \cdots \lambda_g(t)(x_{\rho_l}) = t^{-l} x^D$$

where l has to be greater than 2 since D was a non-prime divisor.

To calculate the weights on the Lie algebra of U consider the conjugation action

$$\lambda_g(t^{-1}) \begin{pmatrix} I_i & 0 \\ B_i & C_i \end{pmatrix} \lambda_g(t) = \left(\begin{array}{c|c} I_i & 0 \\ \hline tQ_i(t^{-1})B_i & Q_i(t) C_i Q_i(t) \end{array}\right)$$

of an arbitrary element of U by $\lambda_g(t)$. Then the matrix in the bottom left hand corner is given by

$$\begin{pmatrix} t^{l_1 - 1} & & \\ & \ddots & \\ & & t^{l_k - 1} \end{pmatrix} B_i$$

and since each $l_j \geq 2$, the exponents here are strictly positive. This suffices to show that the group is graded unipotent since the matrices B_i describe the Lie algebra. □

Example 3.1.7. Let $X = \mathbb{P}(1,1,2)$; then from Example 3.1.1 we have generating graded pieces S_1 and S_2. Thus the Levi factor of the $\mathrm{Aut}_g(\mathbb{C}[x,y,z])$ is given by $\mathrm{GL}_2 \times \mathbb{G}_m$, which are $\mathrm{GL}(\mathrm{Span}\{z\})$ and $\mathrm{GL}(\mathrm{Span}\{x,y\})$ respectively. The unipotent part is generated by three copies of $\mathbb{G}_a$

$$\begin{aligned} x &\longmapsto x \\ y &\longmapsto y \\ z &\longmapsto z + Ax^2 + Bxy + Cy^2, \end{aligned}$$

all three of which commute with each other giving that $U \cong (\mathbb{G}_a)^3$. Thus

$$\mathrm{Aut}_{\mathrm{g}}(\mathbb{C}[x,y,z]) = (\mathrm{GL}_2 \times \mathbb{G}_m) \ltimes (\mathbb{G}_a)^3.$$

and the distinguished $\mathbb{G}_m$ is

$$\lambda_g : t \longmapsto (t^{-1}I_2, t^{-1}, (0,0,0)).$$

If $X = \mathbb{P}(1,...,1,r)$ is a weighted projective space space of dimension n, then

$$\mathrm{Aut}_{\mathrm{g}}(S) = (\mathrm{GL}_n \times \mathbb{G}_m) \ltimes (\mathbb{G}_a)^N,$$

where $N = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$.

Remark 3.1.8. For a general weighted projective space $X = \mathbb{P}(a_0,\dots,a_n) = \operatorname{Proj} S$, where $S = k[x_0,\dots,x_n]$, we describe the Levi factor of the group $G = \mathrm{Aut}_{\mathrm{g}}(S)$ explicitly. First, partition the variables x_i into distinct weights $\Sigma_j = \{x_i \mid \deg x_i = a_j\}$ and set $n_i = |\Sigma_i|$. Then the Levi factor of G is equal to

$$\prod_{\Sigma_i} \mathrm{GL}_{n_i} \subset G,$$

where the product is taken over the distinct Σ_i. Thus the Levi factor contains all linear automorphisms: that is, automorphisms which take variables to linear combinations of other variables. As an automorphism must respect the grading, these linear combinations only contain variables of the same weight.

The unipotent radical of G is given by 'non-linear' automorphisms: that is, automorphisms which involve a monomial of total degree higher than 1 (see Example 3.1.7).

Let $X = \mathbb{P}(a_0, \dots, a_n) = \operatorname{Proj} k[x_1, \dots, x_{n'}, y_1, \dots, y_{n_l}]$ be a weighted projective space and let $G = \operatorname{Aut}_{\mathrm{g}}(S)$ be as above. Assume that the weights are in ascending order (so that $a_i \le a_{i+1}$) and label the distinct weights $b_1 < \dots < b_l$ where each b_j occurs exactly n_i times (the n_i coincide with the n_i in Remark 3.1.8). For weighted projective space we define another 1-parameter subgroup which grades the unipotent radical $U \subset G$ positively depending on a parameter $N \in \mathbb{Z}$.

Proposition 3.1.9. *Let $N > 0$ be a positive integer. The 1-parameter subgroup $\lambda_{g,N} : \mathbb{G}_m \to G$ defined by*

$$\lambda_{g,N} : t \longmapsto \left((t^{-N} I_{n_i})_{i=1}^{l-1}, tI_{n_l}, 0 \right)$$

gives $U \subset G$ a positive grading.

Proof. Let $X = \mathbb{P}(a_0, \dots a_n) = \operatorname{Proj} S$ where $S = k[x_0, \dots, x_{n'}, y_0, \dots, y_{n_l}]$ so that the y_i have the maximum weight $b_l = a_n$. Then $\lambda_{g,N}(\mathbb{G}_m) \subset G = \operatorname{Aut}_{\mathrm{g}}(S)$ acts on X as follows:

$$\lambda_{g,N}(t) \cdot (0 : \dots : x_i : \dots : 0) = (0 : \dots : t^{-N} x_i : 0 : \dots : 0)$$
$$\lambda_{g,N}(t) \cdot (0 : \dots : y_j : \dots : 0) = (0 : \dots : 0 : t y_j : \dots : 0)$$

for $0 \le i \le n'$ and $0 \le j \le n_l$. Let $u \in U \subset G$ be an element of the unipotent radical. By Remark 3.1.8, u acts on S as follows

$$u \cdot x_i = x_i + p_i(x_0, \dots, x_{n'})$$
$$u \cdot y_j = y_j + q_j(x_0, \dots, x_{n'}),$$

for $0 \le i \le n'$ and $0 \le j \le n_l$, where $p_i, q_j \in k[x_0, \dots, x_{n'}]$ are weighted homogeneous polynomials (possibly 0) of degree a_i and a_n respectively. Note that $p_i = 0$ for those i such that $a_i = b_1$ is the minimum weight and that p_i and q_j do not contain any factors of y_j, since the y_j all have the same maximal weight. In particular, if $p_i \ne 0$, then $\deg p_i > 1$.

Consider the action by conjugation of $\lambda_{g,N}(\mathbb{G}_m)$ on U, first on the x_i:

$$\begin{aligned}
\left(\lambda_{g,N}(t) \cdot u \cdot \lambda_{g,N}(t^{-1}) \right) \cdot x_i &= (\lambda_{g,N}(t) \cdot u) \cdot t^N x_i \\
&= \lambda_{g,N}(t) \cdot (t^N x_i + p_i(t^N x_0, \dots, t^N x_{n'})) \\
&= x_i + t^{-N} p_i(t^N x_0, \dots, t^N x_{n'}).
\end{aligned}$$

Those p_i's which are non-zero have degree $a_i > 1$ and hence if u is a weight vector for the $\lambda_{g,N}(\mathbb{G}_m)$-action, it has weights $a_iN - N > 0$. The argument for the y_i is identical and is omitted. □

3.2 Automorphisms of toric varieties

We continue our exposition of the automorphism group and detail how to obtain the automorphism group of a toric variety from the graded automorphisms of its Cox ring. In the following we restrict ourselves to complete simplicial toric varieties. Let $X = \hat{X}/\mathbf{D}$ be a complete simplicial toric variety presented as in Theorem 2.3.9.

Notation 3.2.1.

1. Let $\widetilde{\mathrm{Aut}}(X)$ be the normaliser of $\mathbf{D}$ in the automorphism group of $\hat{X}$.
2. Let $\widetilde{\mathrm{Aut}}^0(X)$ be the centraliser of $\mathbf{D}$ in the automorphism group of $\hat{X}$.

Lemma 3.2.2. *Every element of $\widetilde{\mathrm{Aut}}(X)$ sends a $\mathbf{D}$-orbit to a $\mathbf{D}$-orbit. Hence $\phi \in \widetilde{\mathrm{Aut}}(X)$ descends to a morphism $\bar{\phi} \in \mathrm{Aut}(X)$ and $\phi \mapsto \bar{\phi}$ defines a homomorphism $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$.*

Proof. Suppose that $\phi \in \widetilde{\mathrm{Aut}}(X)$ and take a point $x \in \hat{X}$. We consider $\mathbf{D} \subset \mathrm{Aut}(\hat{X})$ as a subgroup. For any $g \in \mathbf{D}$ we have $\phi(g \cdot x) = \phi(g(x)) = g'(\phi(x)) = g' \cdot \phi(x)$ for some $g' \in \mathbf{D}$ by the definition of the normaliser and hence ϕ preserves $\mathbf{D}$-orbits.

To see that the morphism descends, consider the following diagram

$$\begin{tikzcd} \hat{X} \arrow[r, "\phi"] \arrow[d, "q"] \arrow[dr] & \hat{X} \arrow[d, "q"] \\ X & X. \end{tikzcd}$$

Since ϕ sends $\mathbf{D}$-orbits to $\mathbf{D}$-orbits, the morphism $q \circ \phi$ is $\mathbf{D}$-invariant, thus by the universal property of q we get a morphism $\bar{\phi} : X \to X$ such that $q \circ \phi = \bar{\phi} \circ q$ and with inverse $\overline{\phi^{-1}}$. It is easy to check that the assignment $\phi \mapsto \bar{\phi}$ is a group homomorphism. □

Theorem 3.2.3. [Cox14, Theorem 4.2] *Let $X = X_\Sigma$ be a complete simplicial toric variety with Cox ring S such that $\Sigma(1) = \{\rho_1, \dots, \rho_l\}$ and with distinct generating degrees α_i for $i = 1, \dots, s$. Then the following statements hold.*

1. $\widetilde{\mathrm{Aut}}(X)$ *is a linear algebraic group of dimension*

$$\sum_{i=1}^{s} |\Sigma_i| \dim S_i$$

with connected component at the identity $\widetilde{\mathrm{Aut}}^0(X)$.

2. *The map* $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$ *described above induces an exact sequence*

$$0 \longrightarrow \boldsymbol{D} \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 0.$$

3. *There is an isomorphism of algebraic groups*

$$\mathrm{Aut}_{\mathrm{g}}(S) \longrightarrow \widetilde{\mathrm{Aut}}^0(X).$$

Let us describe the morphism given in part (3) of the theorem. Given $\phi \in \mathrm{Aut}_g(S)$, by taking the spectrum, we have a morphism $\phi_* : \mathbb{A}^{|\Sigma(1)|} \to \mathbb{A}^{|\Sigma(1)|}$. Since ϕ respects the grading of the ring S, it also leaves the unstable locus of the resulting GIT quotient invariant, thus we can restrict ϕ_* to a morphism

$$\phi_*|_{\hat{X}} : \hat{X} \longmapsto \hat{X}.$$

Since Spec is contravariant, we must take the inverse to get a group homomorphism. Define a homomorphism $\mathrm{Aut}_{\mathrm{g}}(S) \to \widetilde{\mathrm{Aut}}(X)$

$$\phi \longmapsto (\phi_*|_{\hat{X}})^{-1}.$$

It requires substantially more work to prove that the defined morphism lies in the centraliser.

Corollary 3.2.4. *Suppose we are in the same setting as Theorem 3.2.3 above and recall that* N *is the lattice such that* $\Sigma \subset N_{\mathbb{R}}$. *Then the following statements hold.*

1. $\mathrm{Aut}(X)$ *is an algebraic group.*

2. *Let* $\mathrm{Aut}^0(X) = (\mathrm{Aut}(X))^0$ *be the connected component at the identity. Then*

$$\mathrm{Aut}^0(X) \cong \mathrm{Aut}_{\mathrm{g}}(S)/\,\boldsymbol{D}.$$

3. *There is an isomorphism* $\pi_0(\mathrm{Aut}^0(X)) \cong \mathrm{Aut}(N,\Sigma)/\mathbb{S}$*, where* $\mathrm{Aut}(N,\Sigma)$ *is the group of isomorphisms of* N *which preserve* Σ *and*

$$\mathbb{S} = \prod_{i=1}^{s} \mathbb{S}_{\Sigma_i}$$

with $\mathbb{S}_{\Sigma_i}$ *the symmetric group on* Σ_i*.*

The fact that $\mathrm{Aut}(X)$ is a linear algebraic group was orginally proved in the case of smooth toric varieties by Demazure in [Dem70] and extended to the simplicial case by Cox. However, recent work of Brion [Bri18] allows us to extend this to all normal projective toric varieties.

Example 3.2.5. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathrm{Aut}_{\mathrm{g}}(S) = \mathrm{GL}_2 \times \mathrm{GL}_2$ and

$$\mathrm{Aut}(Q) = \mu_2 \ltimes (\mathrm{PGL}_2 \times \mathrm{PGL}_2).$$

We shall present a few select lemmas taken from the proof of Theorem 3.2.3 and suitable minor generalisations which will be required later in this thesis.

Lemma 3.2.6. *There is an exact sequence*

$$0 \longrightarrow \boldsymbol{D} \longrightarrow \widetilde{\mathrm{Aut}}^0(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathrm{Cl}(X)),$$

where the last map is given by $\phi \mapsto (\phi_* : [D] \mapsto [\phi(D)])$*; the direct image of a divisor.*

An immediate consequence of this lemma is the connectedness of the automorphism groups of weighted projective spaces.

Proposition 3.2.7. *The automorphism group of weighted projective space is connected.*

Proof. Let X be a weighted projective space, then $\mathrm{Cl}(X) \cong \mathbb{Z}$, So the final map in the exact sequence above is

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathrm{Cl}(X)) = \{1, -1\}.$$

Since the direct image of an effective divisor is again an effective divisor, the image of this map is $\{1\}$ and hence $\widetilde{\mathrm{Aut}}^0(X) \to \mathrm{Aut}(X)$ is surjective. Then as $\widetilde{\mathrm{Aut}}^0(X)$ is connected, the image of this group homomorphism is connected and we conclude that $\mathrm{Aut}(X)$ is connected. $\square$

Example 3.2.8. Continuing from Example 3.1.1 we have that

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) \cong ((\mathrm{GL}_2 \times \mathbb{G}_m) \ltimes (\mathbb{G}_a)^3)/\mathbb{G}_m,$$

where the quotient is by the $\mathbb{G}_m$ given by $t \mapsto (tI_2, t^2, (0,0,0))$, which gives $\mathbb{P}(1,1,2)$ as a GIT-quotient of $\mathbb{A}^3$.

We give another corollary of the lemma.

Corollary 3.2.9. *Suppose that X is a complete simplicial toric variety and $\alpha \in \mathrm{Cl}(X)$. The subgroup*

$$\mathrm{Aut}_\alpha(X) = \{\phi \in \mathrm{Aut}(X) \mid \phi^*\alpha = \alpha\} \subset \mathrm{Aut}(X)$$

is a finite index subgroup.

Proof. By Lemma 3.2.6, we have that $\mathrm{Aut}^0(X) \subset \mathrm{Aut}_\alpha(X)$ and the result follows. □

The following proposition demonstrates the usefulness of the automorphism group in the study of the variety itself. The action of the automorphism group on the variety will play an important role in studying the quasismooth locus of a linear system.

Proposition 3.2.10. *Let X be a complete toric variety and $G = \mathrm{Aut}(X)$ its automorphism group. The natural G-action on X is transitive if and only if X is a product of projective spaces.*

The 'if' direction is clear; the automorphism group of a product of projective spaces acts transitive. For the converse we give a proof in the case when X has Picard rank 1. Clearly X is smooth and hence simplicial. By Theorem 2.4.8, X is a weighted projective space and weighted projective spaces are smooth if and only if they are isomorphic to standard projective space [Dol82].

Remark 3.2.11. For the general case, see [Baz13].

We end this section with a result characterising when a class of weighted projective spaces have a reductive automorphism group.

Proposition 3.2.12. *Let $X = \mathbb{P}(a,\dots,a,b,\dots,b) = \mathrm{Proj}\, k[x_0,\dots,x_n,y_0,\dots,y_m]$ be a well-formed weighted projective space with $\deg x_i = a$ and $\deg y_i = b$. Then $\mathrm{Aut}(X)$ is reductive if and only if $a \nmid b$.*

Proof. Consider a graded automorphism

$$\phi : k[x_0,\dots,x_n,y_0,\dots,y_m] \longrightarrow k[x_0,\dots,x_n,y_0,\dots,y_m].$$

The automorphism ϕ is determined by where it sends the x_i and the y_j. However, the unipotent part of the group $\mathrm{Aut}_{\mathrm{g}}(S)$ is generated by automorphisms

$$y_j \longmapsto y_j + p_j(x_0,\dots,x_n),$$

where p_j is a homogeneous polynomial of degree $\frac{b}{a}$. We refer the reader to [Cox95b, Proposition 4.5] for a proof that the unipotent radical is generated by such automorphisms. In the language of the paper [Cox95b]; these automorphisms are the unipotent 'roots' of the automorphism group. $\square$

Example 3.2.13. Let $X = \mathbb{P}(2,2,3,3)$. Then $\mathrm{Aut}(X) = (\mathrm{GL}_2 \times \mathrm{GL}_2)/\mathbb{G}_m$.

Remark 3.2.14. This result can be generalised in the following way. Suppose that $X = \mathbb{P}(a_0,...,a_n)$ is a well-formed weighted projective space. Then $\mathrm{Aut}(X)$ is reductive if and only if there does not exist a subset of weights $\{a_{i_1},\dots,a_{i_s}\}$ such that $a_{i_1}+\dots+a_{i_s}$ divides any weight a_j. The argument is the same as Proposition 3.2.12, we omit it as it is notationally ugly.

3.3 Automorphism groups of hypersurfaces

Suppose that X is a complete toric variety and $Y \subset X$ a hypersurface. We call automorphisms of Y which come from automorphisms of X *homogeneous automorphisms* and denote them by

$$\mathrm{Aut}(Y;X) \subset \mathrm{Aut}(X).$$

In this section we shall prove that subgroup of automorphisms which fix a quasismooth hypersurface of sufficiently high degree in weighted projective space is finite. The other aim of this section is to show that for a generic hypersurface Y we have

$$\mathrm{Aut}(Y;X) = \mathrm{Aut}(Y).$$

We first observe some generalities on automorphism groups.

Theorem 3.3.1. [Bri18, Theorem 2.17] *Let X be a projective variety and $\mathcal{L} \in \mathrm{Pic}(X)$ be an ample line bundle. Then*

$$\mathrm{Aut}_{\mathcal{L}}(X) = \{\phi \in \mathrm{Aut}(X) \mid \phi^*\mathcal{L} \cong \mathcal{L}\}$$

is a linear algebraic group.

This theorem implies that when a projective variety has Picard rank 1, its automorphism group is linear algebraic. Combining this with Theorem 2.5.13, we know that a generic hypersurface will have a linear algebraic automorphism group.

3.3.1 Regular sequences and Nakayama's Lemma

Definition 3.3.2. Let R be a ring, then a sequence of elements $r_1, ..., r_n$ is called *regular* if they generate a proper ideal and for $i = 1, ..., n$ the canonical image $\overline{r_i} \in R/(r_1, ..., r_{i-1})$ is a non-zero-divisor (where $r_0 = 0$).

We shall need the following lemmas. The first is a graded version of Nakayama's lemma and the second shows that in a graded ring, any permutation of a regular sequence is regular.

Lemma 3.3.3. *Suppose that $R = \bigoplus_{i\geq 0} R_i$ is a graded ring and that M is a finitely generated graded R-module. If there exists $y \in R_+$ such that $yM = M$, then $M = 0$.*

Proof. Define $i_0 = \max\{i \mid M_i \neq 0\}$. Then $yM_{i_0} \subset \bigoplus_{i>i_0} M_i$ and since $yM = M$, it must be the case that $M = 0$. □

Lemma 3.3.4. *Suppose that $R = \bigoplus_{i\geq 0} R_i$ is a graded ring and that $x_1, ..., x_n$ is a regular sequence such that each r_i is homogeneous of degree greater than 0. Then any permutation of the sequence remains regular.*

Proof. It suffices to show if x, y is a regular sequence then y, x is also a regular sequence. All we need is the graded version of Nakayama's lemma.

Let $I = (0 : y)$ and take some $u \in I$. Then as $yu = 0$, certainly $\overline{yu} = 0 \in R/(x)$ and since $\overline{y}$ is a nonzero divisor, $u = xv$ for some $v \in R$. Hence $0 = yu = yxv = x(yv)$. Since x is a nonzero divisor, $yv = 0$ and thus $v \in I$. So for any $u \in I$ there exists some $v \in I$ such that $u = yv$, in other words $I = yI$. By Nakayama's lemma, $I = 0$ and so y is a nonzero divisor.

It remains to check that $\overline{x} \in R/(y)$ is a nonzero divisor. Suppose that $xv = yu$ for some $v, u \in R$. Since $\overline{y} \in R/(x)$ is a nonzero divisor, $u = xw$ for some $w \in R$. So that $xv = yxw$ and $x(v - yw) = 0$. Then as x is a nonzero divisor, $v = yw$ and we are done. □

Lemma 3.3.5. [Bou17, Chapter 5, p147] *Suppose that $f_0, ..., f_n \in k[x_0, ..., x_n]$ are homogeneous elements (with respect to some grading). Then they form a regular sequence if and only if $V(f_0, ..., f_n) = \{0\}$.*

Lemma 3.3.6. *Let R be a graded Cohen-Macauley ring, and $f_1, ..., f_n$ be a sequence of homogeneous elements in R. If the codimension of the ideal generated by $f_1, ..., f_n$ is n, then these elements form a regular sequence.*

Proof. This is an easy corollary of the following result [Eis95, Corollary 17.7]: If R is a graded ring and $\langle f_1, ..., f_n \rangle \subset R$ is a proper ideal containing a regular sequence of length n, then $f_1, ..., f_n$ is a regular sequence.

Since R is Cohen-Macauley, the depth of the ideal $\langle f_1, ..., f_n \rangle$ is n, and thus this ideal contains a regular sequence of length n and the lemma follows from the above result. □

3.3.2 Trivial Stabilisers of quasismooth weighted hypersurfaces

We now prove the main result of this section.

Let $S = \mathrm{Cox}(X) = k[x_0, ..., x_n]$ be the Cox ring of the weighted projective space $X = \mathbb{P}(a_0, ..., a_n)$ and assume that $a_0 \leq a_1 \leq \cdots \leq a_n$. Label the distinct values of the a_i's by $b_1, ..., b_l$ such that $b_1 < \cdots < b_l$. Define numbers $n_1, ..., n_l$ such that each of the b_j occur exactly n_j times, so that the n_j sum to $n + 1$. Recall from Theorem 3.1.3 that

$$\mathrm{Aut}_{\mathrm{g}}(S) = \prod_{j=1}^{l} \mathrm{GL}_{n_j} \ltimes U,$$

where U is the unipotent radical. Let $G = \mathrm{Aut}_{\mathrm{g}}(S)$ and denote the 1-parameter subgroup of G by

$$\lambda_{\underline{a}} : t \longmapsto ((t^{b_j} I_{n_j})_{j=1}^{l}, 0).$$

By Proposition 3.2.7 we have that

$$\mathrm{Aut}(\mathbb{P}(a_0,...,a_n)) \cong \mathrm{Aut}_{\mathrm{g}}(S)/\lambda_{\underline{a}}(\mathbb{G}_m).$$

Theorem 3.3.7. *Let* $S = k[x_0,...,x_n]$ *be the polynomial ring with the weighted grading* $\deg x_i = a_i$ *and let* $f \in k[x_0,...,x_n]_d$ *define a quasismooth hypersurface* $\boldsymbol{V}(f) \subset \mathbb{P}(a_0,\ldots,a_n)$ *where* $d \geq \max\{a_i\} + 2$*. Define the subgroup* $\mathrm{Aut}(f) \subset \mathrm{Aut}_{\mathrm{g}}(S)$ *as follows*

$$\mathrm{Aut}(f) = \{\phi \in \mathrm{Aut}_{\mathrm{g}}(S) \mid \boldsymbol{V}(\phi(f)) = \boldsymbol{V}(f)\}.$$

Then $\mathrm{Aut}(f) = \mu \ltimes \lambda_{\underline{a}}(\mathbb{G}_m)$*, with* $\lambda_{\underline{a}} : \mathbb{G}_m \to G$ *defined as above and* μ *is a finite group.*

Proof. We write $G = \mathrm{Aut}_{\mathrm{g}}(S) = \prod_{j=1}^{l} \mathrm{GL}_{n_j} \ltimes U$ and denote $\mathrm{Lie}\, G$ by

$$\mathfrak{g} = \prod_{j=1}^{l} \mathfrak{gl}_{n_j} \ltimes \mathfrak{u}.$$

It is clear that $\lambda_{\underline{a}}(\mathbb{G}_m) \subset \mathrm{Aut}(f)$. To obtain the desired result it suffices to show that the Lie algebras of $\mathrm{Aut}(f)$ and $\lambda_{\underline{a}}(\mathbb{G}_m)$ agree as sub-Lie algebras of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ acts on S by derivation: let $\xi \in \mathfrak{g}$ and $F \in S$ be arbitrary elements of $\mathfrak{g}$ and S respectively, then

$$\xi(F) = \sum_{i=0}^{n} F_i\, \xi(x_i),$$

where $F_i = \frac{\partial F}{\partial x_i}$. Suppose that $\xi \in \mathrm{Lie}(\mathrm{Aut}(f)) \subset \mathfrak{g}$. Then since f is semi-invariant under the action of $\mathrm{Aut}(f)$, it is also a semi-invariant for the action of $\mathrm{Lie}(\mathrm{Aut}(f))$; that is, $\xi(f) = \tilde{\alpha} f$ for some $\tilde{\alpha} \in k$. The weighted Euler formula tells us that $f = \frac{1}{d}\sum_{i=0}^{n} a_i f_i$ and so

$$\sum_{i=0}^{n} f_i(\xi(x_i) - \alpha a_i x_i) = 0,$$

where $\alpha = \frac{\tilde{\alpha}}{d}$.

Rearranging, for each i we get an equation

$$p_i f_i = -(p_0 f_0 + \cdots + p_{i-1} f_{i-1} + p_{i+1} f_{i+1} + \cdots + p_n f_n),$$

where $p_j = \xi(x_j) - \alpha a_j x_j$. Thus $p_i f_i \in (f_0,...,f_{i-1}, f_{i+1},...,f_n)$.

Since f is quasismooth, its partial derivatives $f_0,...,f_n$ form a regular sequence by Lemma 3.3.6. Moreover, by Lemma 3.3.4, any permutation of the f_i is a regular sequence. Thus f_i is a non-zero divisor in the ring $S/(f_0,...,f_{i-1}, f_{i+1},...,f_n)$ and hence

$p_i \in (f_0, ..., f_{i-1}, f_{i+1}, ..., f_n)$. However, $\deg p_i = a_j$ and since we assumed $\deg f \geq \max\{a_j\}+2$, this forces $p_i = 0$ and $\xi(x_i) = \alpha a_i x_i$. Thus α is the only parameter and we have shown that $\mathrm{Lie}(\mathrm{Aut}(f))$ is one dimensional and hence agrees with that of $\mathrm{Lie}\,\lambda_{\underline{a}}(\mathbb{G}_m)$.

Moreover, we can see explicitly that

$$\mathrm{Lie}(\mathrm{Aut}(f)) = \{((\alpha b_j I_{n_j})_{j=1}^{l}, 0) \mid \alpha \in k\} \subset \mathfrak{g},$$

which is precisely the Lie algebra of $\lambda_{\underline{a}}(\mathbb{G}_m)$. □

Corollary 3.3.8. *Keeping the assumptions of Theorem 3.3.7, the groups* $\mathrm{Aut}(Y; \mathbb{P}(a_0, \ldots, a_n))$ *are finite for quasismooth* Y.

Proof. The result follows from the fact that $\mathrm{Aut}(Y; \mathbb{P}(a_0, \ldots, a_n)) = \mathrm{Aut}(f)/\lambda_{\underline{a}}(\mathbb{G}_m)$. □

Theorem 3.3.9. *Furthermore, if $n \geq 4$ and $d \geq 3$ is Cartier we have an equality* $\mathrm{Aut}(Y) = \mathrm{Aut}(Y; \mathbb{P}(a_0, \ldots, a_n))$ *for a quasismooth hypersurface Y. Moreover, if Y_1 and Y_2 are quasismooth hypersurfaces which are isomorphic as abstract varieties, then there is an automorphism of $\mathbb{P}(a_0, \ldots, a_n)$ which brings Y_1 to Y_2.*

Proof. Note that the first statement follows from the second.

Let $l = \mathrm{lcm}(a_0, \ldots, a_n)$ and let $d' = \frac{d}{l}$ which is an integer since we assumed that d is Cartier. Let $X = \mathbb{P}(a_0, \ldots, a_n)$. Let $\varphi : Y_1 \to Y_2$ be an isomorphism. Since $n \geq 4$, we can apply the Grothendieck-Lefschetz hyperplane theorem (Theorem 2.5.13) and hence $\mathrm{Pic}(Y_1) \simeq \mathrm{Pic}(Y_2) \cong \mathbb{Z}$ where both groups are generated by the restriction of $\mathcal{O}_{\mathbb{P}}(l)$ and thus $\varphi^* \mathcal{O}_{Y_2}(l) \simeq \mathcal{O}_{Y_1}(l)$ and we have an isomorphisms

$$\varphi_r^* : H^0(Y_1, \mathcal{O}_{Y_1}(rl)) \xrightarrow{\simeq} H^0(Y_2, \mathcal{O}_{Y_2}(rl))$$

for $r \in \mathbb{Z}$. Consider the twisted short exact sequences

$$0 \longrightarrow \mathcal{I}_{Y_i}(rl) \longrightarrow \mathcal{O}_X(rl) \longrightarrow \mathcal{O}_{Y_i}(rl) \longrightarrow 0,$$

where $\mathcal{I}_i$ is the ideal sheaf for Y_i with $i = 1, 2,$. Note that for all $r \in \mathbb{Z}$ we have $H^1(X, \mathcal{I}_{Y_i}(rl)) = 0$ by Demazure vanishing (see [CLS11, Theorem 9.2.3]) and hence taking global sections, we get the short exact sequences

$$0 \longrightarrow H^0(X, \mathcal{I}_i(rl)) \longrightarrow H^0(X, \mathcal{O}_X(rl)) \longrightarrow H^0(Y_i, \mathcal{O}_{Y_i}(rl)) \longrightarrow 0.$$

Since $\mathcal{I}_i \simeq \mathcal{O}_X(-d)$ it follows that for $r < d'$ we have that $H^0(X, \mathcal{I}_i(rl)) = 0$ and hence there are isomorphisms $\phi_{i,r} : H^0(X, \mathcal{O}_X(rl)) \to H^0(Y_i, \mathcal{O}_{Y_i}(rl))$. Recall that

$$H^0(X, \mathcal{O}_X(rl)) = k[x_0, \dots, x_n]_{rl}$$

and hence the isomorphisms

$$(\phi_{2,r})^{-1} \circ \varphi_r^* \circ \phi_{1,r} : k[x_0, \dots, x_n]_{rl} \longrightarrow k[x_0, \dots, x_n]_{rl}$$

generate an isomorphism of vector spaces

$$\tilde{\varphi}^* : \bigoplus_{r \geq 0} k[x_0, \dots, x_n]_{rl} \longrightarrow \bigoplus_{r \geq 0} k[x_0, \dots, x_n]_{rl}.$$

The map $\tilde{\varphi}^*$ is a homomorphism of graded rings by the commutativity of the following diagram:

$$\begin{tikzcd} H^0(Y_1, \mathcal{O}_{Y_1}(rl)) \otimes H^0(Y_1, \mathcal{O}_{Y_1}(sl)) \arrow[r] \arrow[d, "\varphi_r^* \otimes \varphi_s^*"'] & H^0(Y_1, \mathcal{O}_{Y_1}((r+s)l)) \arrow[d, "\varphi_{r+s}^*"] \\ H^0(Y_2, \mathcal{O}_{Y_2}(rl)) \otimes H^0(Y_2, \mathcal{O}_{Y_2}(sl)) \arrow[r] & H^0(Y_2, \mathcal{O}_{Y_2}((r+s)l)). \end{tikzcd}$$

Note that the isomorphism takes the ideal $\oplus_r \mathcal{I}_1(rl)$ to the ideal $\oplus_r \mathcal{I}_2(rl)$ and hence taking the projective spectrum we get an isomorphism $\tilde{\varphi} : X \to X$ which takes Y_1 to Y_2. □

Remark 3.3.10. For projective space this was proven in [MM63]. The argument as given above is outlined in the paper [Fau99], where it is claimed that the same statement should hold for complete intersections in complete simplicial toric varieties.

Chapter 4

Families of hypersurfaces and the moduli problem

The aim of this chapter is to define the moduli functor for hypersurfaces in a complete simplicial toric variety of a fixed class up to automorphisms. To define a moduli functor is to define two things: first, families of the objects one wishes to classify; and second, a notion of equivalence of these families. The second is impossible without first having a definition of family. We explore several different notions of families and the functors they define.

When studying families of varieties one naturally encounters Hilbert schemes. Indeed, the starting place for most moduli problems concerning varieties is the Hilbert scheme: the moduli space of genus g curves is defined as a quotient of a component of a Hilbert scheme [Mum62], as is the moduli space of hypersurfaces in projective space. We begin the chapter by first studying Hilbert schemes of hypersurfaces and prove that there are always reasonable components, given by linear systems, which parameterise the hypersurfaces we want to consider. The key observation here is that for a toric variety X, it holds that $H^1(X, \mathcal{O}_X) = 0$. This implies that the Picard scheme is discrete, which, via the Abel map, enables us to piece together the components of the Hilbert scheme we are interested in; see Theorem 4.1.12.

Afterwards, we discuss the notion of equivalence between families which is defined by the automorphism group of the ambient variety.

4.1 Relative effective divisors

Grothendieck defined relative effective divisors when constructing the Picard scheme. The definition of relative effective divisors is very closely related to the definition of families we will use when defining the moduli functor of hypersurfaces in a toric variety. The main reference for the Picard scheme and relative effective divisors is [FGI+05]. Many of the results presented by Kleiman in [FGI+05] (originally due to Grothendieck) will be essential in understanding the moduli space of hypersurfaces in a given toric variety.

We begin by naively writing down a class of families of hypersurfaces. Fix a variety X and let S be a scheme and consider

$$\begin{tikzcd} D \arrow[dr] & \subset & X_S \arrow[d] \\ & & S \end{tikzcd}, \qquad (\star)$$

where $X_S = X \times S$. Let us assume that both arrows in this diagram represent flat morphisms and that for every point $s \in S$, the fibre $D_s \subset X_s$ is a hypersurface. Indeed, the object $(\star)$ above will be a family of hypersurfaces. However, there remains a question of whether this is a broad enough definition. For example, one asks: is the functor defined by this notion of family a Zariski (or étale, or fppf) sheaf?

Objects such as $(\star)$ appear in the Hilbert functor. We recall the definition.

Definition 4.1.1. Let T be a scheme and X be a projective T-scheme. Define the *Hilbert scheme functor* to be the presheaf defined as follows:

$$\underline{\mathrm{Hilb}}_{X/T}(S) = \{\mathcal{X} \subset X_S \mid \mathcal{X} \text{ is } S\text{-flat}\},$$

where S is a T-scheme and $X_S = X \times_T S$.

In the case where X is projective over a noetherian base T, Grothendieck proved that the Hilbert scheme functor is represented by the *Hilbert scheme.* The Hilbert scheme is a projective scheme over T, denoted $\mathrm{Hilb}_{X/T}$, however, it is not of finite type over T. If we fix a relatively ample line bundle $\mathcal{O}_X(1)$ on X/T, then the Hilbert scheme decomposes into open and closed subschemes of finite type over T:

$$\mathrm{Hilb}_{X/T} = \bigsqcup_P \mathrm{Hilb}^P_{X/T},$$

where $P \in \mathbb{Q}[t]$ and $\mathrm{Hilb}^P_{X/T}$ is a projective scheme representing the functor

$$\underline{\mathrm{Hilb}}^P_{X/T}(S) = \{\mathcal{X} \subset X_S \mid \mathcal{X} \text{ is } S\text{-flat, and } P = P_{\mathcal{X}_s} \forall s \in S\},$$

where $P_{\mathcal{X}_s}$ is the Hilbert polynomial of $\mathcal{X}_s$ with respect to $\mathcal{O}(1)$.

Remark 4.1.2. One would hope that the Hilbert polynomial detects if a subscheme is a hypersurface or not, so that one can specify a Hilbert polynomial such that every subscheme which occurs in the corresponding Hilbert scheme is a hypersurface. This is easily proven in the case of projective space (see Proposition 4.2.1) but is not the case in general. Explicitly, if we consider the Hilbert scheme associated to the Hilbert polynomial P of a hypersurface, then every closed point of $\mathrm{Hilb}^P_{X/T}$ must correspond to a subvariety of codimension 1, however all subvarieties are not necessarily purely codimension 1 as shown in Remark 4.2.8.

Recall that we have the following characterisation of Cartier divisors.

Lemma 4.1.3. [FGI+05, Section 9.3] *Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. There is a canonical isomorphism*

$$H^0(X,\mathcal{L})_{\mathrm{reg}}/H^0(X,\mathcal{O}_X^*) \to |\mathcal{L}|,$$

where $H^0(X,\mathcal{L})_{\mathrm{reg}}$ are regular sections (that is, sections which induce injections $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$) and $|\mathcal{L}|$ is the complete linear system associated to $\mathcal{L}$.

Definition 4.1.4. Let T be a scheme and X be a T-scheme. Suppose that $D \subset X$ is a closed subscheme. We say that D is an *effective (Cartier) divisor* if the ideal sheaf $\mathcal{I}_D$ is invertible, and we say D is a *relative effective divisor* on X/T if in addition D is T-flat.

Lemma 4.1.5. [FGI+05, Lemma 9.3.4] *Let X be a T-scheme and $D \subset X$ be a closed subscheme. Then the following are equivalent:*

1. *The subscheme D is a relative effective divisor on X/T.*
2. *The schemes X and D are T-flat and each fibre D_t is an effective divisor on X_t for every $t \in T$.*
3. *The scheme X is T-flat and each fibre D_t is is cut out by one element that is regular (a non-zerodivisor) on the fiber X_t.*

The equivalence of (1) and (2) says that a relative effective divisor is a family of hypersurfaces. We show that the Hilbert scheme parametrises relative effective divisors.

Definition 4.1.6. Let T be a scheme and X be a T-scheme. Define the presheaf

$$\underline{\mathrm{Div}}_{X/T}(S) = \{D \subset X_S \mid D \text{ is a relative effective divisor on } X_S/S\},$$

for a T-scheme S.

Theorem 4.1.7. [FGI+05, Theorem 9.3.7] *Assume that X is a projective T-scheme which is flat over T. Then $\underline{\mathrm{Div}}_{X/T}$ is represented by an open subscheme of the Hilbert scheme* $\mathrm{Hilb}_{X/T}$.

Indeed, it is certainly a subfunctor of the Hilbert scheme functor. However, that it is an open subfunctor is not immediately obvious. We denote the representing scheme by $\mathrm{Div}_{X/T}$.

Remark 4.1.8. Note that $\mathrm{Div}_{X/T}$ may not be of finite type, but it will admit a decomposition into open and closed subschemes

$$\mathrm{Div}_{X/T} = \bigsqcup_P \mathrm{Div}^P_{X/T},$$

where the union is taken over polynomials $P \in \mathbb{Q}[t]$ and $\mathrm{Div}^P_{X/T} = \mathrm{Div}_{X/T} \cap \mathrm{Hilb}^P_{X/T}$ is a scheme of finite type over T. The scheme $\mathrm{Div}^P_{X/T}$ consist of hypersurfaces with Hilbert polynomial P and is an open subset of $\mathrm{Hilb}^P_{X/T}$. Note that both $\mathrm{Hilb}^P_{X/T}$ and $\mathrm{Div}^P_{X/T}$ may not be connected (see Example 4.2.3).

We further describe $\mathrm{Div}^P_{X/T}$.

Definition 4.1.9. Let X and T be as above and $f : X \to T$ be the structure map. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle. Define the functor

$$\mathrm{LinSys}^{\mathcal{L}}_{X/T}(S) = \{D \in \mathrm{Div}_{X/T}(S) \mid \mathcal{O}_{X_S}(D) \cong \mathcal{L}_S \otimes f_S^*\mathcal{N} \text{ for some } \mathcal{N} \in \mathrm{Pic}(S)\}.$$

For every $\mathcal{L} \in \mathrm{Pic}(X)$ the functor $\mathrm{LinSys}^{\mathcal{L}}_{X/T}$ is a subfunctor of $\mathrm{Div}_{X/T}$. If each of these subfunctors is representable, then the subschemes $\mathrm{LinSys}^{\mathcal{L}}_{X/T}$ for varying $\mathcal{L}$ will cover the scheme $\mathrm{Div}_{X/T}$.

Theorem 4.1.10. [FGI+05, Theorem 9.3.13] *Assume that $X \to T$ is proper, flat and that the geometric fibres are integral. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Then there exists a coherent $\mathcal{O}_T$-module $\mathcal{Q}$ (depending on $\mathcal{L}$) such that $\mathbb{P}(\mathcal{Q})$ represents $\mathrm{LinSys}^{\mathcal{L}}_{X/T}$.*

The $\mathcal{O}_T$-module $\mathcal{Q}$ appearing in the theorem above is the sheaf appearing in the following theorem, when $\mathcal{F}$ is taken to be $\mathcal{L}$.

Theorem 4.1.11. [GD60, III.2, 7.7.6] *Let $f : X \to T$ be proper and $\mathcal{F}$ a coherent sheaf on X which is flat over T. Then there exists a coherent sheaf $\mathcal{Q}$ on T and an isomorphism of functors*

$$\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{Q}, \_) \cong f_*(\mathcal{F} \otimes f^*(\_)) : \mathrm{Qcoh}(T) \to \mathrm{Qcoh}(T).$$

This isomorphism is unique up to unique isomorphism.

Let us reduce to a simpler case. Let $T = \mathrm{Spec}\, k$, where k is an algebraically closed field and let X be a projective variety over k. Then the conditions of the above theorems are satisfied. Suppose that $\mathcal{L} \in \mathrm{Pic}(X)$ and let $\mathcal{N} \in \mathrm{Qcoh}(k)$ be a k-vector space. By the projection formula

$$f_*(\mathcal{L} \otimes f^*(\mathcal{N})) \cong f_*\mathcal{L} \otimes \mathcal{N} = H^0(X, \mathcal{L}) \otimes \mathcal{N}.$$

Thus $\mathcal{Q} = H^0(X, \mathcal{L})^*$. This implies that

$$\mathrm{LinSys}^{\mathcal{L}}_X = \mathbb{P}(H^0(X, \mathcal{L})^*),$$

which generalises [Har77, Proposition II.7.7] to singular projective varieties.

Theorem 4.1.12. *Suppose that X is a projective variety over k such that $H^1(X, \mathcal{O}_X) = 0$ and let $\mathcal{O}_X(1)$ be an ample line bundle on X. Let $Z \subset X$ be a Cartier divisor and $P = P_Z$ be its Hilbert polynomial with respect to $\mathcal{O}_X(1)$. Then*

$$\mathrm{Div}^P_X = \mathbb{P}(Q_1) \sqcup \cdots \sqcup \mathbb{P}(Q_r)$$

is a disjoint union of projective spaces, where each $Q_i = H^0(X, \mathcal{L}_i)^*$ is a finite dimensional vector space, such that the $\mathcal{L}_i \in \mathrm{Pic}(X)$ are not isomorphic and all share the same Hilbert polynomial P. Moreover, Div^P_X is a disjoint union of connected components of the Hilbert scheme Hilb^P_X.

Proof. Consider the set $\{\mathcal{L}_i\}_{i\in I} \subset \mathrm{Pic}(X)$ of line bundles with Hilbert polynomial P. Set $Q_i = H^0(X, \mathcal{L}_i)$ for each $i \in I$. It is clear that the subvarieties $\mathbb{P}(Q_i)$ cover Div_X^P. Let us describe the topology of these subsets and prove that I is a finite set.

Suppose that there exists $x \in \mathbb{P}(Q_i) \cap \mathbb{P}(Q_j)$, then x corresponds to a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ which is isomorphic to the associated line bundles of both Q_i and Q_j and so $Q_i = Q_j$. This proves that the projective spaces $\mathbb{P}(Q_i)$ are disjoint subsets of $\mathrm{Div}(X)^P$.

Next we must show that each $\mathbb{P}(Q_i)$ is a connected component of Div_X^P. Consider the Abel map [FGI+05, Definition 9.4.6]

$$A : \mathrm{Div}_X \longrightarrow \mathrm{Pic}(X),$$

which sends an effective divisor to the dual of its ideal sheaf. Since $H^1(X, \mathcal{O}_X) = 0$ and

$$H^1(X, \mathcal{O}_X) = \mathbf{T}_0 \,\mathrm{Pic}(X),$$

the Picard scheme is a disjoint union of points. Thus $A^{-1}(\mathcal{L}_i) = \mathbb{P}(Q_i)$ is open and closed. Hence $\mathbb{P}(Q_i)$ are connected components of Div_X^P. Since we have proven in Theorem 4.1.7 that $\mathrm{Div}_X^P \subset \mathrm{Hilb}_X^P$ is an open subset, it follows that I is a finite set, since Div_X^P can only have finitely many connected components.

It remains to prove that the $\mathbb{P}(Q_i)$ are connected components of the Hilbert scheme Hilb_X^P. As we have just proven, Div_X^P is also a disjoint union of projective spaces and thus is proper. Since Hilb_X^P is separated, Div_X^P is also closed in Hilb_X^P, by Lemma 4.1.13 below. So we conclude that Div_X^P is the disjoint union of connected components of Hilb_X^P. □

Lemma 4.1.13. *Let X be a separated scheme and $f : Z \hookrightarrow X$ an open immersion. If Z is complete, then f is also a closed immersion.*

Proof. Consider the following diagram

$$\begin{tikzcd} Z \arrow[dr, "\Gamma_f"'] \arrow[drr, "f"] & & \\ & Z \times X \arrow[r, "\mathrm{pr}_X"'] & X, \end{tikzcd}$$

where Γ_f is the graph of f. Since X is separated we have that Γ_f is a closed immersion. Moreover, since Z is complete, the projection pr_X is proper and hence f is a closed immersion. □

Remark 4.1.14. We have proved in Theorem 4.1.12 that there are only finitely many isomorphism classes of line bundles with the same Hilbert polynomial. This is a basic form of a deep finiteness result involving the Picard scheme; compare with [FGI+05, Theorem 9.6.20]. Note that this is not the case if $H^1(X, \mathcal{O}_X) \neq 0$; for example, if X is an elliptic curve, we do not have such a finiteness result.

In [Tot00], Totaro discusses a question of Fulton on the topology of linearly equivalent divisors. Fulton asks if the Betti numbers of homologous divisors are necessarily the same. Totaro shows that the answer is no in general and discusses some cases where it is true.

Assume that X is a projective normal variety and that $Y \subset X$ and $Y' \subset X$ are two hypersurfaces which are varieties. If $[Y] = [Y'] \in \mathrm{Cl}(X)$, are they homeomorphic in the analytic topology? The answer in general is no, for example the singular cubic surfaces in $\mathbb{P}^3$ provide a counterexample. The question of interest then becomes what extra conditions must we put on the hypersurfaces Y and Y' to force this to be true. In the case of cubics, both surfaces must be assumed to be smooth. In the case of hypersurfaces in weighted projective space, the result holds if again we assume the hypersurfaces to be smooth as mentioned in Proposition 2.6.6.

Theorem 4.1.15. *Suppose that X is a smooth projective variety over $k = \mathbb{C}$. Then all smooth hypersurfaces of the same class in* $\mathrm{Pic}(X)$ *are diffeomorphic.*

If X is a projective simplicial toric variety. Then all smooth hypersurfaces $Y \subset X$ of class $\alpha \in \mathrm{Pic}(X)$ are diffeomorphic.

Proof. The main idea is to apply Ehresmann's fibration theorem [Dim92, Proposition 3.1] to the universal family for the functor $\mathrm{LinSys}_X^{\mathcal{L}}$ from Theorem 4.1.10, which is represented by the scheme $\mathbb{P}(Q)$. Suppose first that X is smooth. Let $\mathcal{U}' \subset X \times \mathbb{P}(Q)$ be the universal family for $\mathrm{LinSys}_X^{\mathcal{L}}$. Consider the open set

$$\mathcal{U} = \{(x, [f]) \in \mathcal{U}' \mid \mathbf{V}(f) \subset X \text{ is smooth and } x \in \mathbf{V}(f)\} \subset \mathcal{U}'$$

of smooth divisors. Then the map $\mathcal{U} \to \mathbb{P}(Q)^{\mathrm{SM}}$ is a submersion by [Har77, Theorem III.10.2], where $\mathbb{P}(Q)^{\mathrm{SM}}$ consists of smooth hypersurfaces. Applying Ehresmann's fibration theorem to the map $\mathcal{U} \to \mathbb{P}(Q)^{\mathrm{SM}}$, gives that $\mathcal{U} \to \mathbb{P}(Q)^{\mathrm{SM}}$ is a locally trivial fibration.

Since the fibres are exactly the smooth divisors, every smooth divisor is homeomorphic in the analytic topology.

Suppose now that X is a projective toric orbifold and again consider the universal family $\mathcal{U}' \subset X \times \mathbb{P}(Q)$ for $\mathrm{LinSys}_X^{\mathcal{L}}$, where $\mathcal{L}$ is some line bundle such that $[\mathcal{L}] = \alpha$ in $\mathrm{Pic}(X)$. Recall that $Q = H^0(X, \mathcal{L})^* = S_\alpha^*$. Consider the restriction of $\mathcal{U}'$ to the smooth locus and denote it by $\varphi : \mathcal{U} \to \mathbb{P}(S_\alpha)^{\mathrm{SM}}$. Then $\mathcal{U} = \{(x,[f]) \in \mathcal{U}' \mid x \in \mathbf{V}(f) \text{ and } \mathbf{V}(f) \text{ is smooth}\} \subset \mathcal{U}'$ is an open set. Since the fibres are smooth, we can apply [Har77, Theorem III.10.2] which proves that φ is a submersion and hence the result follows by Ehresmann's fibration theorem. □

4.2 Components of Hilbert schemes of products of projective space

In this section we study the components of Hilbert schemes of hypersurfaces in products of projective spaces.

First, let $X = \mathbb{P}^n$ and let P be the Hilbert polynomial associated to a hypersurface of degree d. Then by Theorem 4.1.12

$$\mathrm{Div}_X^P = \mathbb{P}(k[x_0, \ldots, x_n]_d),$$

as $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = k[x_0, \ldots, x_n]_d$ and $\mathcal{O}_{\mathbb{P}^n}(d)$ is the only line bundle with the Hilbert polynomial P. Let us show that $\mathbb{P}(k[x_0, \ldots, x_n]_d)$ is the entire Hilbert scheme.

Proposition 4.2.1. [Ser06, Section 4.3.2] *Suppose that $X \subset \mathbb{P}^n$ is a closed subscheme. Then X is a hypersurface of degree d if and only if*

$$P_X(t) = \binom{t+n}{n} - \binom{t+n-d}{n}.$$

Remark 4.2.2. Combining this proposition with Theorem 4.1.7, we have that when P is the Hilbert polynomial of a hypersurface, then

$$\mathrm{Hilb}_{\mathbb{P}^n}^P = \mathrm{Div}_{\mathbb{P}^n}^P.$$

Example 4.2.3. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be the quadric surface in $\mathbb{P}^3$, that is $\mathbb{P}^1 \times \mathbb{P}^1$ polarised by $\mathcal{O}_Q(1,1) = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. Note that hypersurfaces in Q are curves. Fix $P \in \mathbb{Q}[t]$, the

Hilbert polynomial of a curve $C \subset Q$ of type (a,b), that is, the ideal sheaf of C is isomorphic to $\mathcal{O}_Q(-a,-b)$. Note that the Hilbert polynomial of a curve of type (a,b) will be the same as the Hilbert polynomial of a curve of type (b,a), however it is, a priori, possible to have other pairs (a',b') with the same Hilbert polynomial.

Let us calculate the Hilbert polynomial of a curve $C \subset Q$ of type (a,b) with respect to the bundle $\mathcal{O}_Q(1,1)$. For every $t \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_Q(t-a,t-b) \longrightarrow \mathcal{O}_Q(t,t) \longrightarrow \mathcal{O}_C(t,t) \longrightarrow 0.$$

Since Hilbert polynomials are additive over short exact sequences, we have

$$P_C(t) = h^0(Q,\mathcal{O}_Q(t,t)) - h^0(Q,\mathcal{O}_Q(t-a,t-b)),$$

for large enough $t \in \mathbb{Z}$. By the Künneth formula, we have that for any two integers r and s;

$$h^0(Q,\mathcal{O}_Q(r,s)) = h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(r)) \cdot h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(s)),$$

and so we have that

$$P_C(t) = h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(t))^2 - h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(t-a))h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(t-b))$$

and thus

$$P_C(t) = \binom{1+t}{t}^2 - \binom{1+t-a}{t-a}\binom{1+t-b}{t-b} = (a+b)t + (a+b-ab).$$

Note that this implies that the arithmetic genus of the curve is $g = 1-(a+b-ab) = (a-1)(b-1)$.

Lemma 4.2.4. *Let S_2 be the symmetric group acting canonically on $\mathbb{Z}^2$. Then the map*

$$(\mathbb{Z}_{\geq 0})^2/S_2 \longrightarrow (\mathbb{Z}_{\geq 0})^2$$

defined by $[a,b] \mapsto (a+b,ab)$ is injective.

Proof. Fix $n = a+b$ and consider the function $f(a) = ab = na - a^2$. For every c with $0 \leq c \leq n$ we have $f^{-1}(f(c)) = \{c, n-c\}$. □

Remark 4.2.5. Note that the unique maximum of the above function f is $a = \frac{n}{2}$ and hence (for $n = a+b$ fixed) the expression $a+b-ab = n-a(n-a)$ achieves a unique minimum if n is even and $a = b = \frac{n}{2}$.

The next corollary follows directly from Lemma 4.2.4 and Example 4.2.3.

Corollary 4.2.6. *Let $C \subset Q$ be a curve of type (a,b) in the quadric surface with Hilbert polynomial*

$$P(t) = (a+b)t + (a+b-ab).$$

Suppose that $C' \subset Q$ is a curve with the same Hilbert polynomial. Then C' is of type (a,b) or (b,a). In particular,

$$\mathrm{Div}_Q^P = \mathbb{P}(k[x,y;u,v])_{(a,b)} \sqcup \mathbb{P}(k[x,y;u,v])_{(b,a)}$$

if $a \neq b$ and

$$\mathrm{Div}_Q^P = \mathbb{P}(k[x,y;u,v])_{(a,a)}$$

otherwise.

Suppose that $C \subset Q$ is a curve of type (a,a). By the above calculation we know that $P_C(t) = 2at + (2a - a^2)$. We claim that the only subschemes with this Hilbert polynomial are divisors of type (a,a).

Proposition 4.2.7. *Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and $P(t) = 2at + 2a - a^2$ for some positive $a \in \mathbb{Z}$. Then*

$$\mathrm{Hilb}_Q^P = \mathbb{P}(k[x,y;u,v]_{(a,a)}).$$

Proof. The proof naturally falls into two cases: where the subscheme considered is a curve and where the subscheme considered is not pure dimensional and hence, not a curve. First the case where the subscheme is a divisor. This is precisely the statement of Corollary 4.2.6.

The other possibility is that there is a subscheme which is the union of a curve and collection of points (possibly with multiplicity). Suppose that $C' = \overline{C} \cup \{P_i\}_{i=1}^r$ is such a subscheme, where $\overline{C}$ is a curve of type (a',b'). Since $P_{C'}(t) = 2at + (2a - a^2)$ we know that

$$P_{\overline{C}}(t) = 2at + (2a - a^2) - r.$$

But then (a',b') is a pair such that $a' + b' = 2a$ and $a' + b' - a'b' = 2a - a^2 - r \leq 2a - a^2$. However, as stated in Remark 4.2.5, $2a - a^2$ is the unique minimum for such pairs. Thus $r = 0$ and $(a',b') = (a,a)$. Hence $\mathrm{Hilb}_Q^P = \mathbb{P}(k[x,y;u,v]_{(a,a)})$. □

Remark 4.2.8. Proposition 4.2.7 already breaks down for curves of type (a,b) such that $a \neq b$. For example, consider $(a,b) = (4,2)$. Then we know that $P(t) = 6t - 2$ and that

$$\mathrm{Div}_Q^P = \mathbb{P}(k[x,y;u,v]_{(4,2)}) \sqcup \mathbb{P}(k[x,y;u,v]_{(2,4)}).$$

Consider a subscheme $C = \tilde{C} \cup \{P\}$ with $P \in Q$, where $\tilde{C}$ is a curve of type $(3,3)$. Then $P_C(t) = 6t - 3 + 1 = P(t)$. This is the only other type of subscheme with this Hilbert polynomial. Indeed, suppose that $C = \tilde{C} \cup Z$, where Z is a union of more than one point, $\tilde{C}$ is a curve and $P_C(t) = P(t)$. Then $P_{\tilde{C}}(t) = 6t - b$ where $b \geq 4$. However, this contradicts the fact that b can be at most 3 by Remark 4.2.5 and in this case $\tilde{C}$ is of type $(3,3)$. Thus we can deduce that as sets

$$\mathrm{Hilb}_Q^P = \mathbb{P}(k[x,y;u,v]_{(4,2)}) \sqcup \mathbb{P}(k[x,y;u,v]_{(2,4)}) \sqcup (\mathbb{P}(k[x,y;u,v]_{(3,3)}) \times Q).$$

However, it is not clear what the scheme structure on the component $\mathbb{P}(k[x,y;u,v]_{(3,3)}) \times Q$ is. To study the scheme structure, one would calculate the Plücker embedding explicitly and compute the tangent space. This kind of analysis can easily be generalised to curves of all types.

Remark 4.2.9. The above results generalise to higher dimensions. However, the Hilbert polynomials are much more complicated: the numerical functions coming from the coefficients of the Hilbert polynomial are more complicated, but since there are more of them the required injectivity of a map analogous to that in Lemma 4.2.4 is satisfied. The result is that for $X = \mathbb{P}^n \times \mathbb{P}^n$ we have that $\mathrm{Hilb}_X^P = \mathbb{P}(k[x_0, \ldots, x_n; y_0, \ldots, y_n]_{(d,d)})$, when P corresponds to a symmetric type (d,d).

4.3 The moduli problem

In this section we define the moduli functor for hypersurfaces in toric varieties. We begin by introducing the case of hypersurfaces in projective space as motivation for the more general case of hypersurfaces in complete simplicial toric varieties.

4.3.1 Moduli of hypersurfaces in projective space

The study of the moduli of hypersurfaces in projective space has its roots in the classical invariant theory of Hilbert [Hil93]. Mumford was the first to approach this problem with a modern viewpoint in [MFK94, Chapter 4.2]. Other standard references are [Dol03, Muk03].

Explicitly, we consider the moduli problem of hypersurfaces of a fixed degree d in $\mathbb{P}^n$ up to linear change of coordinates which is given by the natural action of $\mathrm{Aut}(\mathbb{P}^n)$ on $\mathbb{P}(k[x_0,\dots,x_n]_d)$. Fix $d > 0$ and note that

$$\mathrm{Hilb}^P_{\mathbb{P}^n} = \mathbb{P}(k[x_0,\dots,x_n]_d) \cong \mathbb{P}^N,$$

where

$$N = \binom{n+d}{d} - 1$$

and P is the Hilbert polynomial associated with hypersurfaces of degree d with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$ (see Proposition 4.2.1). Then $\mathbb{P}(k[x_0,\dots,x_n]_d)$ is a parameter space for hypersurfaces of degree d, which we denote by

$$\mathcal{Y}_d = \mathbb{P}(k[x_0,\dots,x_n]_d).$$

We denote by $(\mathcal{Y}_d)^{\mathrm{SM}}$ the open subset of smooth hypersurfaces of degree d. The variety $\mathcal{Y}_d$ admits a natural action of $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}$ as defined in Example 1.2.6. The action is described on closed points as follows: given $F \in k[x_0,\dots,x_n]_d$ and $g \in \mathrm{GL}_{n+1}$ then

$$g \cdot F(-) = F(g^{-1} \cdot -). \tag{†}$$

This descends to an action of PGL_{n+1} on $\mathcal{Y}_d$.

There is a surjection $\mathrm{SL}_{n+1} \to \mathrm{PGL}_{n+1}$ with a finite kernel. Such a group homomorphism is called an *isogeny*. The action of SL_{n+1} on $\mathcal{Y}_d$ defined by this isogeny has the same orbits as the PGL_{n+1}-action and the only difference is that there is a global finite stabiliser group. However, from the perspective of GIT, finite stabilisers are unimportant: the quotient variety is the same. Working with the action of SL_{n+1} is more convenient as this action admits an $\mathcal{O}(1)$-linearisation where $\mathcal{O}(1) = \mathcal{O}_{\mathcal{Y}_d}(1)$ (see Example 1.2.6).

We construct the moduli space of hypersurfaces of degree d as a GIT quotient of $\mathcal{Y}_d$ by this action of SL_{n+1}. To do this we first define the moduli problem rigorously as a moduli functor and then show the existence of a local universal family for this functor. We can then apply Proposition 1.4.2 to show that a GIT quotient is indeed the moduli space we desire.

A natural way of constructing a family of hypersurfaces parametrised by a scheme S is to consider functions of the following form:

$$F_{(a_{i_0\dots i_n})}(s) = \sum_{i_0\dots i_n} a_{i_0\dots i_n}(s) x_0^{i_0} \cdots x_n^{i_n},$$

where the sum is taken over tuples of indices $(i_0, \dots, i_n)$ such that $\sum_{j=0}^n i_j = d$ and where $a_{i_0\dots i_n} \in \mathcal{O}_S(S)$ are regular functions on S. Furthermore, we require that for each point $s \in S$ the polynomial $F_{(a_{i_0\dots i_n})}(s) \in k[x_0, \dots, x_n]_d$ is non-zero.

However, not all families over S appear in this way. Indeed, we can define a functor using this definition of families:

$$\widetilde{\mathrm{Hyp}}_{n,d} : S \longmapsto \{(a_{i_0\dots i_n})_{i_0\dots i_n} \subset \mathcal{O}_S(S) \mid F_{(a_{i_0\dots i_n})}(s) \neq 0 \, \forall \, s \in S\} / \sim,$$

where $\sim$ is given by multiplication by an element of $H^0(S, \mathcal{O}_S^*)$. This functor is not a Zariski sheaf [New78, Example 1.1], implying that it cannot admit a coarse moduli space. We give the following definition of families as given in [New78, p.17].

Definition 4.3.1. A *family of hypersurfaces of degree d in $\mathbb{P}^n$ over a scheme S* is a pair $(\mathcal{L}, \sigma)$, where $\mathcal{L}$ is a line bundle over S and

$$\sigma = (\sigma_{i_0\dots i_n} \,|\, i_j \geq 0, \sum_{j=0}^n i_j = d)_{i_0\dots i_n} \subset H^0(X, \mathcal{L})$$

is a tuple of sections such that for each $s \in S$ the polynomial

$$F_s(\mathcal{L}, \sigma) = \sum_{i_0\dots i_n} \sigma_{i_0\dots i_n}(s) x_0^{i_0} \cdots x_n^{i_n} \in k[x_0, \dots, x_n]_d$$

is non-zero. We define the functor $\mathrm{Hyp}_{n,d} : \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$ by

$$\mathrm{Hyp}_{n,d}(S) = \{(\mathcal{L}, \sigma) \mid (\mathcal{L}, \sigma) \text{ is a family}\} / \sim,$$

and $\mathrm{Hyp}_{n,d}(f : T \to T')((\mathcal{L}, \sigma)) = (f^*\mathcal{L}, f^*\sigma)$, where $(\mathcal{L}, \sigma) \sim (\mathcal{L}, \lambda\sigma)$ for every $\lambda \in H^0(X, \mathcal{O}_S^*)$. Since two equivalent families $(\mathcal{L}, \sigma)$ and $(\mathcal{L}, \lambda\sigma)$ define the same subscheme, we refer to an equivalence class $[\mathcal{L}, \sigma]$ as a family of hypersurfaces.

Lemma 4.3.2. *There is a natural isomorphism of functors*

$$\mathrm{Hyp}_{n,d} \simeq \mathrm{Hilb}^{P}_{\mathbb{P}^n}.$$

Proof. Given a family $[\mathcal{L},\sigma] \in \mathrm{Hyp}_{n,d}(S)$, the associated polynomial (defined up to a non-zero multiple)

$$F_\sigma = \sum_{i_0\dots i_n} \sigma_{i_0\dots i_n} x_0^{i_0}\cdots x_n^{i_n}$$

is a regular section in $H^0(\mathbb{P}^n_S, \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^n}(d))$ and hence $\mathbf{V}(F_\sigma) \subset \mathbb{P}^n_S$ is a divisor. Moreover, it is an effective divisor by the non-vanishing condition on F_σ. Thus by Lemma 4.1.3, the map $\mathrm{Hyp}_{n,d}(S) \to \mathrm{Hilb}^P_{\mathbb{P}^n}(S)$ defined by

$$[\mathcal{L},\sigma] \longmapsto \mathbf{V}(F_\sigma)$$

is a bijection. Functoriality follows from the fact that for a morphism $f : T' \to T$, we have that $\mathbf{V}(f^*F_\sigma) = \mathbf{V}(F_{f^*\sigma})$. □

Remark 4.3.3. From the above result we deduce that the functor $\mathrm{Hyp}_{n,d}$ is the Zariski sheafification of the functor $\widetilde{\mathrm{Hyp}}_{n,d}$.

Definition 4.3.4. Define the moduli functor for hypersurfaces of degree d of n-dimensional projective space

$$\mathcal{M}_{n,d} : \mathfrak{Sch}^{\mathrm{op}} \longrightarrow \mathfrak{Sets}, \quad S \longmapsto \mathrm{Hyp}_{n,d}(S)/\sim.$$

Where two families $(\mathcal{L}_1,\sigma_1)$ and $(\mathcal{L}_2,\sigma_2)$ are said to be equivalent if there exists an isomorphism of line bundles $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\phi\circ\sigma = S^d(A)\cdot\sigma$ for some $A \in \mathrm{SL}_{n+1}$, where S^d is the d^{th}-symmetric power, that is, the image under the standard representation $\mathrm{GL}(V) \to \mathrm{GL}(\mathrm{Sym}^d V)$.

Define the quotient of a functor by an algebraic group as in [HL10, p.92]. Let the functor $\underline{\mathrm{SL}}_{n+1}$ act on the functor $\underline{\mathrm{Hilb}}^P_{\mathbb{P}^n}$ as defined by the map of schemes

$$\sigma : \mathrm{SL}_{n+1} \times \mathrm{Hilb}^P_{\mathbb{P}^n} \longrightarrow \mathrm{Hilb}^P_{\mathbb{P}^n}.$$

The next result follows immediately from Lemma 4.3.2.

Proposition 4.3.5. *There is an isomorphism of functors*

$$\mathcal{M}_{n,d} \cong \underline{\mathrm{Hilb}}^P_{\mathbb{P}^n}/\underline{\mathrm{SL}}_{n+1}.$$

Proposition 4.3.6. *Let d and n be positive integers and let SL_{n+1} act on $\mathcal{Y}_d = \mathbb{P}(k[x_0,\dots,x_n]_d)$ as in (†). Then the following statements hold.*

1. *The family $(\mathcal{O}_{\mathcal{Y}_d}(1),\sigma)$, where σ is the tuple given by the monomials of degree d, is a family with the local universal property for the functor $\mathcal{M}_{n,d}$.*

2. *Two points $F,G \in \mathcal{Y}_d$ are in the same SL_{n+1}-orbit if and only if*

$$\mathcal{O}_{\mathcal{Y}_d}(1))|_F \sim \mathcal{O}_{\mathcal{Y}_d}(1))|_G.$$

We omit the proof, as we shall give a proof in a more general context shortly; see Proposition 4.3.22.

Corollary 4.3.7. *Let $G = \mathrm{SL}_{n+1}$ act on $\mathcal{Y}_d$ as above. The GIT quotient $(\mathcal{Y}_d)^s/G$ of the stable set is a coarse moduli space for the functor $\mathcal{M}^s_{n,d}$, where $\mathcal{M}^s_{n,d}$ is the restriction of the functor to stable hypersurfaces and $(\mathcal{Y}_d)^s$ denotes the stable locus for the G-action linearised by $\mathcal{O}(1)$.*

Proof. Note that the GIT quotient of the stable locus is an orbit space by Theorem 1.2.13 and the result follows immediately from Proposition 1.4.2. □

Theorem 4.3.8 ([MFK94] Proposition 4.2)**.** *Let $d \geq 3$ and $n > 1$, if $n = 1$ assume $d \geq 4$. The open subset $(\mathcal{Y}_d)^{SM} \subset \mathcal{Y}_d$ of smooth hypersurfaces is invariant under the action of $G = \mathrm{SL}_{n+1}$. Furthermore, any smooth hypersurface is stable. That is, there is an inclusion of open subsets*

$$(\mathcal{Y}_d)^{SM} \subset (\mathcal{Y}_d)^s.$$

Proof. Semistability of smooth hypersurfaces is a consequence of the existence of the discriminant[1] and holds for all $n \geq 1$ and $d \geq 2$. Indeed for $[F] \in \mathcal{Y}_d$, there is a homogeneous polynomial Δ in the coefficients of F such that Δ is 0 if and only if F defines a singular hypersurface. Note that Δ is unique up to scalar multiplication. The discriminant Δ can be interpreted as a form in the homogeneous coordinates of $\mathcal{Y}_d$: we consider $\Delta \in H^0(\mathcal{Y},\mathcal{O}_{\mathcal{Y}}(l))$ for some $l > 0$ so that

$$(\mathcal{Y}_d)_\Delta = (\mathcal{Y}_d)^{\mathrm{SM}}.$$

[1]See Chapter 5 for a detailed discussion.

The divisor $\mathbf{V}(\Delta) \subset \mathcal{Y}_d$ is G-invariant since G acts by automorphisms of $\mathbb{P}^n$, and the property of being smooth is preserved under automorphism. Since the divisor $\mathbf{V}(\Delta)$ is invariant it holds that for any $g \in \mathrm{SL}_{n+1}$ then $g \cdot \Delta = \chi(g)\Delta$, where $\chi(g) \in k^*$. The assignment $\chi : g \mapsto \chi(g)$ is a group homomorphism by the group action property. However, SL_{n+1} does not have any non-trivial characters and so the form Δ is an invariant. We conclude that

$$(\mathcal{Y}_d)_\Delta \subset (\mathcal{Y}_d)^{\mathrm{ss}}$$

and thus any smooth hypersurface is semistable. Moreover, if $n > 1$ by Theorem 3.3.7, any smooth hypersurface of degree at least 3 has finite stabilisers. If $n = 1$ and $d \geq 4$, then a smooth hypersurface consists of d distinct points and hence the stabilisers are finite. Therefore

$$(\mathcal{Y}_d)_\Delta \subset (\mathcal{Y}_d)^{\mathrm{s}},$$

since the action of G on $(\mathcal{Y}_d)_\Delta$ is closed. □

Remark 4.3.9. We can consider the GIT quotient $\mathcal{Y}_d /\!/ G$ as a compactification of the moduli space of smooth hypersurfaces of degree d.

Remark 4.3.10. Note that the only choice of linearisation giving a non-empty quotient is the $\mathcal{O}(1)$-linearisation defined in Example 1.2.6 (we consider tensor powers of this linearisaion as the same since they define the same quotient). This is due to the fact that SL_{n+1} has no characters and the Picard group of $\mathcal{Y}_d$ is $\mathbb{Z}$ and hence there is no variation of GIT picture in the sense of [Tha96].

4.3.2 Moduli of hypersurfaces in toric varieties

Now let us adapt the method used above to the situation of ample hypersurfaces in a complete simplicial toric variety. We define the moduli functor and show the existence of a family with the local universal property for a given group action.

Let $X = X_\Sigma$ be a toric variety and S its Cox ring. Given some ample class $\alpha \in \mathrm{Cl}(X)$, write

$$N_{\Sigma,\alpha} = \dim S_\alpha - 1.$$

In other words, $N_{\Sigma,\alpha}+1$ is the number of monomials in the Cox ring of degree α or, equivalently, $N_{\Sigma,\alpha}+1$ is the number of torus invariant effective divisors with class α. As in Section 2.2, we write monomials as

$$x^D = \prod_{\rho\in\Sigma(1)} x_\rho^{a_\rho},$$

where $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ is a torus invariant divisor.

Notation 4.3.11. Let $X = X_\Sigma$ be a complete simplicial toric variety and $\alpha \in \mathrm{Cl}(X)$. We write $N = N_{\Sigma,\alpha}$ and denote the parameter space of degree α hypersurfaces by

$$\mathcal{Y}_\alpha = \mathbb{P}(S_\alpha) \cong \mathbb{P}^N.$$

Remark 4.3.12. By Proposition 2.3.4 we have that $S_\alpha = H^0(X, \mathcal{O}_X(D))$, where D is a hypersurface with class α. We may apply Theorem 4.1.10 to show that the the projective space $\mathbb{P}(S_\alpha)$ represents the functor LinSys_X^α. Since $H^1(X,\mathcal{O}_X) = 0$, we conclude by Theorem 4.1.12 that $\mathbb{P}(S_\alpha)$ is a connected component of the Hilbert scheme of X.

Suppose that $g \in G = \mathrm{Aut}_\alpha(X)$ (Definition 3.2.9). When we want to emphasise that g is an automorphism we write $\phi_g : X \to X$. The automorphism ϕ_g induces an automorphism of the Cox ring by pulling back sections:

$$\phi_g^* : S \longrightarrow S.$$

This automorphism is not necessarily graded, however $\phi_g^* S_\alpha = S_\alpha$. Then let G act on $\mathcal{Y}_\alpha$ as follows: for $g \in G$ and $[F] \in \mathcal{Y}_\alpha$ let

$$g \cdot [F] = [(\phi_g^{-1})^* F]. \tag{$\star\star$}$$

Remark 4.3.13. In [BC94, Lemma 13.4], Cox and Batyrev noted that there is a natural action of $\widetilde{\mathrm{Aut}}_\alpha(X)$ on $\mathbb{P}(S_\alpha)$, where $\widetilde{\mathrm{Aut}}(X)$ is as defined in Notation 3.2.1 and $\widetilde{\mathrm{Aut}}_\alpha(X) \subset \widetilde{\mathrm{Aut}}(X)$ is the subgroup which preserves α. The action ($\star\star$) defined above induces the action defined by Cox and Batyrev via the following exact sequence

$$0 \longrightarrow \mathbf{D} \longrightarrow \widetilde{\mathrm{Aut}}_\alpha(X) \longrightarrow \mathrm{Aut}_\alpha(X) \longrightarrow 0$$

which also appears in [BC94, Lemma 13.4].

Lemma 4.3.14. *Let X be a complete simplicial toric variety and $\alpha \in \mathrm{Cl}(X)$ an ample class. Then the quasismooth locus $\mathcal{Y}_\alpha^{QS} \subset \mathcal{Y}_\alpha$ is invariant under the action of $G = \mathrm{Aut}_{\mathrm{g}}(S)$.*

Proof. Suppose that $F \in S_\alpha$ is a quasismooth form of degree α and that $g \in G$ is an arbitrary graded automorphism. The lemma then follows from the fact that for every $\tilde{x} \in \hat{X}$, it holds that $g \cdot \tilde{x}$ is a non-quasismooth point of $g \cdot F$ if and only if $\tilde{x}$ is a non-quasismooth point of F. □

Example 4.3.15. Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space and suppose that $d > 0$. Note that $\mathrm{Aut}_d(X) = \mathrm{Aut}(X)$ and for $g \in \mathrm{Aut}(X)$, let $\tilde{g} \in \mathrm{Aut}_{\mathrm{g}}(S)$ be a lift of g under the surjective homomorphism $\mathrm{Aut}_{\mathrm{g}}(S) \to \mathrm{Aut}(X)$. Consider the isomorphism

$$\tilde{\phi}_{\tilde{g}} : (\mathbb{A}^{n+1} - \{0\}) \longrightarrow (\mathbb{A}^{n+1} - \{0\}),$$

corresponding to $\tilde{g} \in \mathrm{Aut}_{\mathrm{g}}(S)$. Then we may reformulate the action $(\star\star)$ as follows: let $g \in \mathrm{Aut}(X)$ and $F \in k[x_0, \dots x_n]_d$, then

$$g \cdot [F(-)] = [F((\tilde{\phi}_{\tilde{g}})^{-1}(-))],$$

where $\tilde{g} \in \mathrm{Aut}_{\mathrm{g}}(S)$ is any lift of g. In particular, if $X = \mathbb{P}^n$, the action is the same as in the previous section.

Remark 4.3.16. When X is a weighted projective space, it is more convenient to work with the action of $\mathrm{Aut}_{\mathrm{g}}(S)$ on $\mathcal{Y}_d$, as we shall see in Section 6.1. The orbits of the action are the same, however the presence of a global stabiliser coming from the 1-parameter subgroup $t \mapsto (t^{a_0}, \dots, t^{a_n})$ of $\mathrm{Aut}_{\mathrm{g}}(S)$ means that we must tweak the definition of stability; see Definition 6.1.4.

Example 4.3.17. Let $X = \mathbb{P}^n \times \mathbb{P}^n$ with coordinates $((x_0 : \dots : x_n), (y_0 : \dots : y_n))$ and let $\alpha = (a, a) \in \mathrm{Pic}(X)$ so that $\mathrm{Aut}(X) = \mathrm{Aut}_\alpha(X) = \mu_2 \ltimes (\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1})$. Consider some element $\phi = (\eta, g, h) \in \mathrm{Aut}(X)$ with $\eta \in \mu_2$ and $g, h \in \mathrm{PGL}_{n+1}$. If $\tilde{g}, \tilde{h} \in \mathrm{GL}_{n+1}$ are lifts of g and h respectively, we can define the following isomorphism

$$\phi^* : (\mathbb{A}^{n+1} - \{0\}) \times (\mathbb{A}^{n+1} - \{0\}) \longrightarrow (\mathbb{A}^{n+1} - \{0\}) \times (\mathbb{A}^{n+1} - \{0\})$$

by

$$\phi^*(\underline{x}, \underline{y}) = \eta \cdot ((\tilde{g} \cdot \underline{x}), (\tilde{h} \cdot \underline{y})),$$

where η acts by swapping $\underline{x}$ and $\underline{y}$ if it is primitive. Then we can reformulate the action $(\star\star)$ as follows: for $F \in k[x_0, \dots, x_n; y_0, \dots y_n]_{(a,a)}$ and $\phi = (\eta, g, h)$, we have

$$\phi \cdot [F(-)] = [F((\phi^{-1})^*(-))].$$

Theorem 4.3.18. *Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space $d > \max(a_0, \dots, a_n) + 1$ be an integer. Then the quotient stack $\left[\mathcal{Y}_d^{QS} / \operatorname{Aut}(X)\right]$ admits a coarse moduli space.*

Proof. Theorem 3.3.7 proves that for hypersurfaces of degree $d > \max(a_0, \dots, a_n) + 1$, the stabiliser group is finite. The theorem is then an immediate consequence of the Keel-Mori Theorem [KM97, Corollary 1.2]. □

Definition 4.3.19. Let $X = X_\Sigma$ be a toric variety and $\alpha \in \operatorname{Cl}(X)$.

1. A family of hypersurfaces in X of degree $\alpha \in \operatorname{Cl}(X)$ over a scheme T is a pair $(\mathcal{L}, \sigma)$, where $\mathcal{L}$ is a line bundle over T and

$$\sigma = \left(\sigma_D \in H^0(T, \mathcal{L}) \mid D \text{ torus-invariant and effective, } [D] = \alpha\right)$$

   is an $(N_{\Sigma,\alpha} + 1)$-tuple of sections such that for each $t \in T$ the polynomial

$$F_t(\mathcal{L}, \sigma) = \sum_\sigma \sigma_D(t) x^D.$$

   is non-zero. This defines the functor $\operatorname{Hyp}_{X,\alpha} : \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$, where

$$\operatorname{Hyp}_{X,\alpha}(T) = \{\text{families over } T\} / \approx,$$

   and $\operatorname{Hyp}_{\Sigma,\alpha}(f : T \to T')((\mathcal{L}, \sigma)) = (f^*\mathcal{L}, f^*\sigma)$ and $(\mathcal{L}, \sigma) \approx (\mathcal{L}, \lambda\sigma)$ for $\lambda \in H^0(T, \mathcal{O}_T^*)$.

2. Suppose that $(\mathcal{L}_1, \sigma_1)$ and $(\mathcal{L}_2, \sigma_2)$ are families over a scheme T. Then we say that the two families are equivalent $(\mathcal{L}_1, \sigma_1) \sim_T (\mathcal{L}_2, \sigma_2)$ if there exists an isomorphism $\Phi : \mathcal{L}_1 \to \mathcal{L}_2$ and an element $\phi \in \operatorname{Aut}_\alpha(X)$ such that $\phi^*\sigma_1 = \phi \cdot \sigma_2$. Let us describe what is meant by $\phi \cdot \sigma$: fix a numbering of toric invariant effective divisors $D_0, \dots, D_N$ of class α and consider the sum

$$\sigma_{D_0} x^{D_0} + \dots + \sigma_{D_N} x^{D_N}.$$

   Then we can write $\sigma_{D_0}\phi^*(x^{D_0}) + \dots + \sigma_{D_N}\phi^*(x^{D_N}) = \sigma'_{D_0} x^{D_0} + \dots + \sigma'_{D_N} x^{D_N}$. We define $\phi \cdot \sigma = (\sigma'_D)$.

3. We define the moduli functor $\mathcal{M}_{X,\alpha} : (\mathfrak{Sch})^{\mathrm{op}} \longrightarrow \mathfrak{Sets}$ by

$$\mathcal{M}_{X,\alpha}(T) = \mathrm{Hyp}_{X,\alpha}(T)/\sim_T,$$

and

$$\mathcal{M}_{X,\alpha}(f : T' \to T)([(\mathcal{L},\sigma)]) = [(f^*\mathcal{L}, f^*\sigma)]$$

The following lemma follows from the same argument as Lemma 4.3.2.

Lemma 4.3.20. *There is a natural isomorphism of functors*

$$\mathrm{Hyp}_{X,\alpha} \simeq \mathrm{LinSys}_X^{\alpha}.$$

Remark 4.3.21. Let $X = \mathbb{P}^n$ and $\alpha = d$. Then

$$\mathcal{M}_{X,\alpha} = \mathcal{M}_{n,d}.$$

Proposition 4.3.22. *Let $X = X_\Sigma$ be a complete toric variety and $\alpha \in \mathrm{Pic}(X)$ be a Cartier class. The family over $\mathcal{Y}_\alpha$ given by the line bundle $\mathcal{O}_{\mathcal{Y}_\alpha}(1)$ and sections given by the monomial basis of $H^0(\mathcal{Y}_\alpha, \mathcal{O}_{\mathcal{Y}_\alpha}(1))$ possesses the local universal property for $\mathcal{M}_{X,\alpha}$. We denote this family $\mathcal{U}$. Furthermore, two points $F, G \in \mathcal{Y}_\alpha$ are in the same $\mathrm{Aut}_\alpha(X)$-orbit for the action described in ($\star\star$) if and only if*

$$\mathcal{U}|_F \sim_T \mathcal{U}|_G,$$

where $T = \mathrm{Spec}\, k$.

Proof. Let $x_0, \dots, x_N$ be the monomial basis of $H^0(\mathcal{Y}_\alpha, \mathcal{O}_{\mathcal{Y}_\alpha}(1))$. Suppose that $(\mathcal{L},\sigma)$ is a family over T. Since the polynomial F_t defined by the family is non-zero for every $t \in T$, the sections $\sigma_D \in \sigma$ define a base-point-free linear system. Since $\mathcal{Y}_d$ is a projective space, the base-point-free linear system defines a morphism $\varphi : T \to \mathcal{Y}_\alpha$ such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathcal{Y}_\alpha}(1)$ and such that $\varphi^* x_i = \sigma_{D_i}$ (see [Har77, Theorem II.7.1]).

The second statement follows from the definition of equivalence: first note that $\mathcal{Y}_\alpha = \mathrm{Hyp}_{X,\alpha}(k)$ since a family $[\mathcal{L},\sigma] \in \mathrm{Hyp}_{X,\alpha}(k)$ is just the data of a polynomial in S_α, up to scalar multiple. Moreover, for $F \in \mathcal{Y}_\alpha$, the restricted family $\mathcal{U}|_F$ is equal to the polynomial defining F.

Then if $T = \operatorname{Spec} k$, the equivalence relation $\sim_T$ is given by the action of the group $\mathrm{Aut}_\alpha(X)$, thus it holds that $\mathcal{U}|_F \sim_T \mathcal{U}|_G$ if and only if the associated polynomials are in the same $\mathrm{Aut}_\alpha(X)$-orbit.

□

Remark 4.3.23. Note that in the above proof we do not consider an open covering of the scheme parametrising the family. However, $\mathcal{U}$ is not a universal family: the morphism defined depends on a choice of coordinates on $\mathcal{Y}_\alpha$ and so the morphism $\varphi : T \to \mathcal{Y}_\alpha$ is given up to the group action of PGL_{N+1} and thus not unique.

By Proposition 4.3.22, to construct a coarse moduli space for the functor $\mathcal{M}_{\Sigma,\alpha}$, we must construct a geometric quotient of an open subset of $\mathcal{Y}_\alpha$ by the action of $\mathrm{Aut}_\alpha(X)$. The automorphism group of a complete simplicial toric variety is in general non-reductive and so we must use the techniques of non-reductive GIT as discussed in Chapter 1. However, for those complete simplicial toric varieties with a reductive automorphism group, we may apply classical GIT.

Example 4.3.24. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and fix $(d,e) \in \mathrm{Cl}(Q) = \mathbb{Z}^2$. Hypersurfaces of degree (d,e) are given by bihomogeneous forms $F(x,y;w,v) \in k[x,y;w,v]_{(d,e)}$. Define $G = \mathrm{Aut}_{(d,e)}(Q)$ and note that if $d \neq e$ then

$$G = \mathrm{PGL}_2 \times \mathrm{PGL}_2,$$

and if $d = e$ then

$$G = \mathrm{Aut}(Q) = \mu_2 \ltimes (\mathrm{PGL}_2 \times \mathrm{PGL}_2).$$

The GIT quotient of the stable locus with respect to the linearisation $\mathcal{O}(1)$

$$\mathcal{Y}^{\mathrm{s}}_{(d,e)}/G \subset \mathcal{Y}_{(d,e)} /\!\!/ G$$

is a coarse moduli space for the functor $\mathcal{M}^{\mathrm{s}}_{Q,(d,e)}$, where $Y_{(d,e)} = \mathbb{P}(k[x,y;w,v]_{(d,e)})$. We show later in Example 6.4.5 that this moduli space is non-empty and contains smooth curves.

Example 4.3.25. Let $X = \mathbb{P}(2,2,5,5)$ and $d \in \mathbb{Z}_{>0}$ be a multiple of 10. Then

$$G = \mathrm{Aut}_d(X) = \mathrm{Aut}(X) = (\mathrm{GL}_2 \times \mathrm{GL}_2)/\mathbb{G}_m,$$

and $\mathcal{Y}_d = \mathbb{P}(k[x_1, x_2, y_1, y_2]_d)$ so that by Theorem 1.2.13 the quotient (with respect to the linearisation $\mathcal{O}(1)$) of the stable locus is $\mathcal{Y}_d^{\mathrm{s}}/G$ is a coarse moduli space of stable hypersurfaces. We prove in Theorem 6.3.12 that quasismooth hypersurfaces are stable.

Chapter 5

The A-discriminant of a toric variety

In this chapter we shall study the A-discriminant associated to a toric variety. We prove that the A-discriminant is an invariant for the action of the automorphism group of the toric variety on the linear system of a fixed class. From now on we shall work over the field of complex numbers to keep in line with the literature, however, all results hold over an arbitrary algebraically closed field of characteristic 0.

5.1 Dual varieties and discriminants

We follows the conventions of [Tev03] and [GKZ08]. All material in this section is found in [GKZ08] with the exception of Proposition 5.1.9.

5.1.1 The dual variety

The discriminant of a polynomial is a polynomial function of its coefficients, whose vanishing gives information about the roots of the polynomial without computing them. The term *discriminant* was coined by Sylvester in 1851 and is a well-studied notion in algebra. The most basic and well-known example is the discriminant of a quadratic equation $f = ax^2 + bx + c$, where the discriminant is defined by $\Delta(f) = b^2 - 4ac$. In this example, Δ vanishes if and only if f has a double root.

Let X be a projective toric variety and fix $\alpha \in \mathrm{Cl}(X)$. The so-called A-discriminant associated to α is a hypergeometric function on the parameter space of hypersurfaces in X of degree α which vanishes if and only if the hypersurface is not quasismooth on a certain locus. In the context of projective toric varieties, the discriminant associated to a fixed very ample class naturally arises as the defining polynomial of the dual variety to the original toric variety with respect to the embedding defined by the class. We recall the basic definitions of dual varieties and their connection to general A-discriminants.

Let V be an $(n+1)$-dimensional complex vector space and $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a projective variety. Recall that we regard $\mathbb{P}(V)$ as the set of 1-dimensional vector subspaces of V and $V^\vee$ is the space of linear forms on V. The points of the dual projective space $\mathbb{P}(V^\vee) = \mathbb{P}(V)^\vee$ are in one-to-one correspondence with hyperplanes in $\mathbb{P}(V)$. For a point $x \in X$, let $\widehat{\mathbf{T}_x X} \in \mathbb{P}(V)$ be the projectivisation of the tangent space.

Definition 5.1.1. Suppose that we are in the setting described above with $X \subset \mathbb{P}(V)$. Let

$$I_X^0 = \{(x, H) \in \mathbb{P}(V) \times \mathbb{P}(V)^\vee \mid x \in X^{\mathrm{sm}}, \widehat{T_x X} \subset H\}$$

and we define the *conormal variety* $I_X = \overline{I_X^0} \subset \mathbb{P}(V) \times \mathbb{P}(V)^\vee$ as the closure of I_X^0.

Let $\mathrm{pr}_1 : I_X^0 \to X^{\mathrm{sm}}$ and $\mathrm{pr}_2 : I_X \to \mathbb{P}(V)^\vee$ be the projections from the respective subvarieties. Define the projective dual of $X \subset \mathbb{P}(V)$ as

$$X^\vee = \mathrm{pr}_2(I_X) \subset \mathbb{P}(V)^\vee.$$

Note that the dual depends on the embedding $X \subset \mathbb{P}(V)$.

Let us explain why the projective dual is an irreducible variety. Let $x \in X^{\mathrm{sm}}$ and consider $\mathrm{pr}_1^{-1}(x) = \{(x, H) \mid \widehat{\mathbf{T}_x X} \subset H\} \cong \mathbb{P}(V/\mathbf{T}_x X)^\vee$. Thus pr_1 is a fibre bundle with fibres equal to a projective space of dimension $n - \dim X - 1$. By considering a trivialising cover of the tangent bundle on X^{sm}, we see that pr_1 is a locally trivial fibre bundle[1] and thus I_X^0 and I_X are irreducible varieties. This in turn implies that $X^\vee = \mathrm{pr}_2(I_X)$ is an irreducible variety. Moreover, as $\dim I_X = n - 1$, we expect that generically $\dim X^\vee = n - 1$ and thus $X^\vee$ is a hypersurface in $\mathbb{P}(V)^\vee$. This motivates the following definition.

[1]In fact, it is the bundle $\mathbb{P}(\mathcal{N}_{X^{\mathrm{sm}}|\mathbb{P}(V)})$.

Definition 5.1.2. We define the *defect* of $X \subset \mathbb{P}(V)$ to be $\operatorname{Def} X = \operatorname{codim}_{\mathbb{P}(V)^\vee}(X^\vee) - 1$. When $\operatorname{Def} X = 0$, we define the *discriminant* of X to be the homogeneous polynomial defining $X^\vee$ in $\mathbb{P}(V)^\vee$ and we denote it by Δ_X, so that

$$\mathbf{V}(\Delta_X) = X^\vee \subset \mathbb{P}(V)^\vee.$$

Note that the discriminant Δ_X is unique up to a scalar multiple.

Remark 5.1.3. Note that $\operatorname{Def} X$ is defined with respect to a projective embedding $X \subset \mathbb{P}(V)$. If $\operatorname{Def} X > 0$ we say that X is *dual defect* . See [DN10] for a combinatorial condition for a smooth polarised toric variety to be dual defect.

Assumption 5.1.4. We assume from now on that any variety we consider is not dual defect, that is $\operatorname{Def} X = 0$. Weighted projective space and products of projective spaces are *not* dual defect. This is a consequence of [GKZ08, Corollary 1.2].

5.1.2 Singular hypersurfaces

We keep the above notation and let $\dim X = r$. Let $x_0, \ldots, x_r$ be local parameters on $Y \subset V$ the affine cone over $X \subset \mathbb{P}(V)$ at a nonsingular point. For the definition of local parameters we refer to [Sha13, Section 2.1]. Consider a linear functional $f \in V^\vee$. Then $f : V \to k$ restricted to Y is an algebraic function on Y; that is, $f|_Y$ is a polynomial in $x_0, \ldots, x_r$.

Let $Y^\vee \subset V^\vee$ be the cone over $X^\vee$. Then, by definition we have that $f \in Y^\vee$ if

$$\mathbf{T}_{\tilde{x}} Y \subset V(f) \subset V,$$

for some $\tilde{x} \in Y$ where $\tilde{x}$ lies over a nonsingular point. However, $\mathbf{T}_{\tilde{x}} Y \subset V(f)$ if and only if $f(\tilde{x}) = 0$ and $\frac{\partial f}{\partial x_i}(\tilde{x}) = 0$ for every i, where $x_0, ..., x_r$ are some local parameters at $\tilde{x} \in Y$. Thus $X^\vee$ contains all singular hyperplane sections. Moreover, the singular hyperplane sections form an open dense subset of $X^\vee$. For more details on this construction we refer the reader to [GKZ08, Section 9.2].

Example 5.1.5. Let $X = \mathbb{P}^1$ be embedded in $\mathbb{P}^d = \mathbb{P}(\mathbb{C}^{d+1})$ via the Veronese embedding. Then local parameters on Y are given by

$$(x, y) \mapsto (x^d, x^{d-1}y, \ldots, y^d).$$

A general element $f \in (\mathbb{C}^{d+1})^\vee$ can be written as $f(z) = \sum_{i=0}^d a_i z_i$, where the z_i are the canonical basis of $(\mathbb{C}^{d+1})^\vee$. Restricting f to Y we get

$$f(x,y) = \sum_{i=0}^{d} a_i x^{d-i} y^i.$$

Setting $y = 1$, we have a non-homogeneous polynomial $f(x) = \sum_{i=1}^d a_{d-i} x^i$. Let H be the hyperplane in $\mathbb{P}^d$ corresponding to f. Then $H \in X^\vee$ if and only if $f(x)$ has a multiple root and thus Δ_X is the classical discriminant.

5.1.3 A-discriminants and toric varieties

Consider a torus $(\mathbb{C}^*)^{r+1}$ with coordinates $(x_0, \dots, x_r)$ and consider a matrix

$$A = (\omega^{(0)} | \cdots | \omega^{(N)}) \in \mathbb{Z}^{(r+1)\times(N+1)},$$

where $\omega^{(j)} \in \mathbb{Z}_{\geq 0}^{r+1}$ is a column vector for $0 \leq j \leq N$. Define the vector space of Laurent functions on $(\mathbb{C}^*)^{r+1}$ associated to A by

$$\mathbb{C}^A := \Big\{ \sum_{i=0}^{N} a_i x^{\omega^{(i)}} \mid a_i \in \mathbb{C} \Big\}.$$

Here $x^{\omega^{(i)}} = x_0^{\omega_0^{(i)}} \cdots x_n^{\omega_r^{(i)}}$, where $\omega^{(i)}$ is a column vector defined to be the transpose of $(\omega_0^{(i)}, \dots, \omega_r^{(i)})$.

Definition 5.1.6. Consider the following subset of $\mathbb{P}(\mathbb{C}^A)$ consisting of Laurent functions (up to scalar multiple) with a singular point on the torus

$$\nabla_A^\circ = \Big\{ f \in \mathbb{P}(\mathbb{C}^A) \mid \exists\, x \in (\mathbb{C}^*)^{r+1} \text{ s.t. } f(x) = \frac{\partial f}{\partial x_i}(x) = 0 \text{ for all } i = 0, \dots, n \Big\}.$$

Then define the *A-discriminant locus* to be

$$\nabla_A = \overline{\nabla_A^\circ} \subset \mathbb{P}(\mathbb{C}^A).$$

As before, we define

$$\operatorname{Def} A = \operatorname{codim}_{\mathbb{P}(\mathbb{C}^A)}(\nabla_A) - 1.$$

If $\operatorname{Def} A = 0$, then define the *A-discriminant* Δ_A as the polynomial defining ∇_A which is well defined and unique up to a scalar multiple. If the codimension is greater than 1, we set $\Delta_A = 1$. Note that ∇_A is irreducible [GKZ08, Proposition 9.1.1].

We now apply this theory in the context of toric varieties.

Definition 5.1.7. Let $X = X_\Sigma$ be a toric variety and $\alpha \in \mathrm{Cl}(X)$ and $S = \mathbb{C}[x_0, \dots, x_r]$ be the Cox ring of X. Let $N = \dim S_\alpha - 1$. We define a matrix $A_{\Sigma,\alpha} \in \mathbb{Z}^{(r+1)\times(N+1)}$ by collecting the exponents of the monomial basis of S_α as columns of this matrix with respect to some ordering of the monomials. We define the *A-discriminant associated to X and α* to be $\Delta_{A_{\Sigma,\alpha}}$. When it is clear from context, we shall drop the Σ and α from the subscript and write simply $A = A_{\Sigma,\alpha}$.

Remark 5.1.8. Let X and α be as above, then $\mathbb{C}^A = S_\alpha$ and hence

$$\nabla_A \subset \mathbb{P}(S_\alpha).$$

Let X be a simplicial projective toric variety and suppose that α is a very ample class. The corresponding A-discriminant is a special case of the discriminant as defined in Definition 5.1.2, as the following proposition shows.

Proposition 5.1.9. *Let X be a projective toric variety, $\alpha \in \mathrm{Cl}(X)$ be a very ample class and $A = A_{\Sigma,\alpha} \in \mathbb{Z}^{(r+1)\times(N+1)}$ be the associated matrix of exponents of the monomial basis of S_α. Then*

$$\nabla_A = X^{\vee,\alpha}$$

as subvarieties of $\mathbb{P}(S_\alpha)$, where $X^{\vee,\alpha}$ denotes the projective dual of X with respect to the embedding given by α. Moreover, it holds that $\operatorname{Def} X = 0$ *if and only if* $\operatorname{Def} A = 0$.

Proof. Since α is very ample and A corresponds to the monomial basis of $S_\alpha = \mathbb{C}^A$, the toric variety $X_A \subset \mathbb{P}(\mathbb{C}^A)^\vee$ is the toric variety X with the embedding defined by α, as in Theorem 2.2.10. Thus $X^{\vee,\alpha} = X_A^\vee$.

It remains to see that $X_A^\vee = \nabla_A$. To see this, we consider the map from Section 2.1.2, $\widetilde{\Phi}_A : (\mathbb{C}^*)^{n+1} \to \mathbb{C}^A$ as local parameters on the torus in the cone $Y_A \subset (\mathbb{C}^A)^\vee$ over X_A. It holds that $T_{Y_A} = \widetilde{\Phi}_A((\mathbb{C}^*)^{n+1})$. Note that since α is very ample, Y_A is an affine toric variety and that $Y_A - \{0\} \to X_A$ is a toric morphism.

We shall show that $\nabla_A^\circ \subset X_A^\vee \subset \mathbb{P}(\mathbb{C}^A)$. Let $\widetilde{\nabla}_A^\circ \subset \mathbb{C}^A$ be the cone over ∇_A°. Consider some $[f] \in \nabla_A^\circ$ and a representative $f \in \widetilde{\nabla}_A^\circ$. Thus there exists $y \in T_{Y_A}$ such that all the partial derivatives of f vanish at y. As in Section 5.1.2, it follows that $\mathbf{T}_y Y_A \subset V(f) \subset (\mathbb{C}^A)$. Thus $[f] \in X_A^\vee \subset \mathbb{P}(\mathbb{C}^A)$.

Thus we have identified ∇_A° with a non-empty open (and hence dense) subset of $X_A^\vee$ given by hyperplanes containing the tangent space to points on the torus. Note that $\nabla_A^\circ \subset \nabla_A$ is a dense subset by definition. Since both ∇_A and $X_A^\vee$ are irreducible hypersurfaces in $\mathbb{P}(\mathbb{C}^A)$, we must have $X_A^\vee = \nabla_A$. □

Remark 5.1.10. This proposition has a very nice geometric meaning. It tells us that for projective toric varieties, the locus of non-quasismooth hypersurfaces in a given complete linear system associated to a very ample class contains (as an irreducible component) the dual to the variety, where the dual is taken with respect to the embedding defined by the very ample line bundle. That is,

$$X_\Sigma^{\vee,\alpha} = \nabla_{A_{\Sigma,\alpha}} \subset \mathcal{Y}_\alpha^{\mathrm{NQS}} \subset \mathcal{Y}_\alpha = \mathbb{P}(S_\alpha) = \mathbb{P}(\mathbb{C}^{A_{\Sigma,\alpha}}),$$

where the first containment is as an irreducible component and the second containment is closed.

Example 5.1.11. Let $X = \mathbb{P}(1,1,2)$ and $\alpha = 4 \in \mathrm{Cl}(X) \simeq \mathbb{Z}$. Then $S = \mathbb{C}[x,y,z]$ where $\deg x = \deg y = 1$ and $\deg z = 2$. The monomial basis of $S_\alpha = \mathbb{C}[x,y,z]_4$ gives the following matrix

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

For example, the first and second column correspond to the monomials x^4 and x^3y respectively. An element $F \in \mathbb{C}^A$ is given by $F(x,y,z) = a_0x^4 + a_1x^3y + \cdots + a_4y^4 + a_5x^2z + a_6xyz + a_7y^2z + a_8z^2$ where $a_i \in \mathbb{C}$. Note that $S_\alpha = \mathbb{C}^A$ is a parameter space for degree 4 hypersurfaces in $\mathbb{P}(1,1,2)$.

Then $X_A \simeq \mathbb{P}(1,1,2)$ and $\mathcal{O}_X(4)$ is very ample. Explicitly,

$$X_A = \overline{\{[x^4 : x^3y : x^2y^2 : xy^3 : y^3 : x^2z : xyz : y^2 : z^2] \mid x,y,z \in \mathbb{C}^*\}} \subset \mathbb{P}^8.$$

We finish this section with a result on how to calculate the degree of the A-discriminant.

Theorem 5.1.12. [GKZ08, Theorem 9.2.8] *Suppose that $X = X_P$ is a smooth projective toric variety associated to a polytope P. Then the degree of the A-discriminant locus as a hypersurface in $\mathbb{P}(\mathbb{C}^A)$ is given by*

$$\deg \nabla_A = \sum_{\Gamma \subset P} (-1)^{\operatorname{codim} \Gamma} (\dim \Gamma + 1) \cdot \operatorname{Vol}_\Gamma(\Gamma),$$

where A corresponds to the polarisation defined by the polytope P and the sum is taken over faces $\Gamma \subset P$.

Example 5.1.13. The degree of the discriminant of cubic surfaces in $\mathbb{P}^2$ is 32; this was first calculated by Salmon in 1865 [Sal58].

Remark 5.1.14. In specific examples, Theorem 5.1.12 has been extended to singular toric varieties; see [MT11].

5.2 Invariance of the A-discriminant

In this section we prove that the A-discriminant of a toric variety X is a semi-invariant for the action of the automorphism group of X on $\mathbb{P}(S_\alpha)$. Since the unipotent radical U of $\operatorname{Aut}(X)$ admits no characters, it follows that the discriminant is a true U-invariant. To prove this, and to put ourselves in a better position to study the moduli spaces we shall construct in Section 6, we prove some results on the geometry of the discriminant locus.

Let $X = X_\Sigma$ be a projective toric variety and $\alpha \in \operatorname{Cl}(X)$ an effective class. By effective class, we mean a class such that the linear system $|\alpha|$ is non-empty. Let us fix some notation: let $G = \operatorname{Aut}_\alpha(X)$ and let $T \subset X$ be the torus in X. By the definition of toric varieties, the action of T on itself extends to an action on X. Thus, we have a map $T \hookrightarrow \operatorname{Aut}(X)$, which is injective since T acts faithfully on itself. In fact we have a morphism $T \hookrightarrow \operatorname{Aut}^0(X) \subset \operatorname{Aut}_\alpha(X)$ since T is connected.

Recall that $|\alpha| = \mathbb{P}(S_\alpha)$ and we write $\mathcal{Y} = \mathcal{Y}_\alpha = |\alpha|$. Consider the projection maps

$$\begin{tikzcd} & X \times \mathcal{Y} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ X & & \mathcal{Y}. \end{tikzcd}$$

As in Lemma 2.6.5, we define the closed set

$$W = \{(x,[f]) \in X \times \mathcal{Y} \mid f_i(x) = 0 \text{ for } 0 \le i \le r\} \subset X \times \mathcal{Y},$$

where $f_i = \frac{\partial f}{\partial x_i}$ and $r = |\Sigma(1)| - 1$. We define the restrictions of the projection maps

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \mathcal{Y}. \end{tikzcd}$$

Clearly we have a G-action on $X \times \mathcal{Y}$ given by $g \cdot (x,[f]) = (g \cdot x, g \cdot [f])$. With respect to this action W is an invariant subscheme. Indeed, for $g \in G$ and $(x,[f]) \in W$ we have $\frac{\partial (g \cdot f)}{\partial x_i}(g \cdot x) = f_i(x) = 0$ for every i, and hence $g \cdot (x,[f]) = (g \cdot x, g \cdot [f]) \in W$. We prove the following result describing the flattening stratification of the morphism p_1.

Proposition 5.2.1. *For any point $x_0 \in X$, the fibre $p_1^{-1}(x_0) \subset \mathcal{Y}$ is a linear subspace. Moreover, suppose that $x, y \in X$ are in the same G-orbit, then $p_1^{-1}(x) \cong p_1^{-1}(y)$.*

Proof. We can describe the fibre explicitly:

$$\begin{aligned} p_1^{-1}(x_0) &= \left\{(x_0,[f]) \mid \frac{\partial f}{\partial x_i}(x_0) = 0 \ \ 0 \le i \le r\right\} \\ &= \bigcap_{i=0}^{r} \left\{(x_0,[f]) \mid \frac{\partial f}{\partial x_i}(x_0) = 0\right\}. \end{aligned}$$

Each of the sets $\{(x_0,[f]) \mid \frac{\partial f}{\partial x_i}(x_0) = 0\}$ is the vanishing of a linear polynomial in the coefficients of the polynomial f. It follows that $p_1^{-1}(x_0)$ is the intersection of hyperplanes and thus a linear subspace.

For $g \in G$, we have that

$$g \cdot p_1^{-1}(x_0) = p_1^{-1}(g \cdot x_0),$$

as $g \cdot (x_0,[f]) = (g \cdot x_0, g \cdot [f])$. In particular, they are all linear subspaces of the same dimension. □

We need the following result on a characterisation of flat morphisms.

Theorem 5.2.2. [Har77, Theorem III.9.9] *Let B be an integral noetherian scheme and $X \subset \mathbb{P}^n_B$ a closed subscheme. For every $b \in B$, consider the Hilbert polynomial $P_b(t) \in \mathbb{Q}[t]$ of $X_b \subset \mathbb{P}^n_{k(b)}$. Then X is B-flat if and only if the Hilbert polynomial P_b is independent of b.*

Applying this theorem to the situation above we have the following corollary.

Corollary 5.2.3. *Define* $W' = p_1^{-1}(G \cdot T) \subset W$*. Then the map* $p_1|_{W'} : W' \to G \cdot T$ *is flat.*

Proof. Since

$$B = G \cdot T = \bigcup_{g \in G} g \cdot T \subset X$$

is an open subset of X, it is an integral noetherian scheme. Then

$$B \times \mathcal{Y} = \mathcal{Y}_B \subset \mathcal{Y}_X = X \times \mathcal{Y}$$

is open and $W \subset \mathcal{Y}_X$ is closed, so $W' \subset \mathcal{Y}_B$ is a closed subscheme. To see that all fibres over points in B have the same Hilbert polynomial, we observe that, since the torus acts transitively on itself, $G \cdot x = G \cdot T = B$ for all $x \in T$. So applying Lemma 5.2.1, we have that all the fibres over B are linear subspaces of the same dimension and thus have the same Hilbert polynomial. Indeed, the Hilbert polynomial of a linear subspace of dimension i is

$$\frac{1}{i!}(t+1)\cdots(t+1),$$

by [Ser55, Proposition 3]. □

By [GD60, IV.2, Corollaire 2.3.5 (iii)], we know that a flat map to an irreducible variety with irreducible generic fibre has an irreducible source. The result holds more generally for open maps, as detailed by the following lemma.

Lemma 5.2.4. [Sta18, Tag 004Z] *Let* $f : X \to Y$ *be a continuous map of topological spaces. If*

1. Y *is irreducible,*
2. f *is open and*
3. *there exists a dense subset* $V \subset Y$ *such that* $f^{-1}(y)$ *is irreducible for every* $y \in V$*,*

then X *is also irreducible.*

Proposition 5.2.5. *Let* W' *be defined as in Corollary 5.2.3. Then* W' *is irreducible.*

Proof. We must check that the conditions of Lemma 5.2.4 are satisfied for the map $p_1|_{W'} : W' \to G \cdot T$.

1. Since X is irreducible and $G \cdot T$ is open, $G \cdot T$ is also irreducible.
2. By Corollary 5.2.3, $p_1|_{W'}$ is flat and hence open.
3. By Lemma 5.2.1, every fibre is isomorphic to the same projective space, and hence all fibres are irreducible.

Hence we can apply Lemma 5.2.4 to $p_1|_{W'}$ and conclude that W' is irreducible. □

We are now in position to prove the main theorem of this section.

Theorem 5.2.6. *Suppose that $X = X_\Sigma$ is a complete toric variety and that $\alpha \in \mathrm{Cl}(X)$ is a class such that $|\alpha|$ is non-empty. Let $G = \mathrm{Aut}_\alpha(X)$ be the automorphism group preserving α. Let $A = A_{\Sigma,\alpha} \in \mathbb{Z}^{r\times N}$ be defined as in Definition 5.1.7. Then the discriminant locus $\nabla_A \subset \mathcal{Y}$ has the following description.*

$$\nabla_A = \overline{\{[f] \in \mathcal{Y} \mid \exists\, x_0 \in G\cdot T \textit{ such that } f_i(x_0) = 0 \ \textit{ for all } i\}}.$$

In particular, ∇_A is a G-invariant subvariety of $\mathcal{Y}$.

Proof. Note that by the definition of ∇

$$\nabla_A = \overline{\{[f] \in \mathcal{Y} \mid \exists x \in T \text{ such that } f_i(x) = 0 \text{ for all } i\}} = \overline{p_2(p_1^{-1}(T))},$$

where $T \subset X$ is the torus. Then since $T \subset G\cdot T$, it holds that $p_1^{-1}(T) \subset p_1^{-1}(G\cdot T)$. Thus

$$\overline{p_2(p_1^{-1}(T))} \subset \overline{p_2(p_1^{-1}(G\cdot T))}.$$

Applying the definition of W' and the observation that $\overline{p_2(p_1^{-1}(T))} = \nabla_A$, we conclude

$$\nabla_A \subset \overline{p_2(W')}.$$

Then since W' is irreducible, $\overline{p_2(W')}$ is irreducible. Also note that $\operatorname{codim} p_2(W') \geq 1$, since the quasismooth locus in $\mathcal{Y}$ is open by Proposition 2.6.5 and W' is disjoint from $\mathcal{Y}^{\mathrm{QS}}$. Hence $\overline{p_2(W')}$ is an irreducible closed subvariety of codimension 1. Then as ∇_A is an irreducible subvariety of codimension 1, we conclude that

$$\nabla_A = \overline{p_2(W')},$$

which completes the first part of the theorem.

Now we prove that ∇_A is G-invariant. Note that both maps p_1 and p_2 are G-equivariant since they are restrictions of projections. Then as $G \cdot T$ is G-invariant it follows that $W' = p_1^{-1}(G \cdot T)$ is G-invariant and thus $\nabla_A = \overline{p_2(W')}$ is also G-invariant. □

Remark 5.2.7. This means that the A-discriminant will check for hypersurfaces with singularities on the G-sweep of the torus in X. Note that by Remark 5.1.10 we have the inclusion

$$\mathcal{Y}_\alpha^{\mathrm{QS}} \subseteq (\mathcal{Y}_\alpha)_{\Delta_A}.$$

In general these subvarieties do not coincide.

Corollary 5.2.8. *Keep the notation of Theorem 5.2.6. The A-discriminant Δ_A is a semi-invariant section for the G-action on $\mathcal{Y}_\alpha$ and a true U-invariant, where $U \subset G$ is the unipotent radical of G.*

Proof. By definition, $\nabla_A = \mathbf{V}(\Delta_A) \subset \mathcal{Y}_\alpha = \mathcal{Y}$. The automorphism group G acts on $H^0(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(\deg \Delta_A))$. Since ∇_A is G-invariant, for every $g \in G$ we have that $\mathbf{V}(g \cdot \Delta_A) = \mathbf{V}(\Delta_A)$. Thus $g \cdot \Delta_A = \chi(g)\Delta_A$ for some $\chi(g) \in k^*$. It follows from the group action laws that $\chi(g'g) = \chi(g')\chi(g)$ and thus $\chi : g \mapsto \chi(g)$ is a character. This proves the result. □

Definition 5.2.9. Keep the notation of Theorem 5.2.6. Define

$$\chi_A : G \longrightarrow \mathbb{G}_m$$

to be the character for which Δ_A is a semi-invariant section. For every $F \in S_\alpha = H^0(\mathcal{Y}_\alpha, \mathcal{O}(1))^\vee$ and $g \in G$ it holds that

$$\Delta_A(g \cdot F) = \chi_A(g)\Delta_A(F).$$

Remark 5.2.10. Recall that $G = \prod_{i=1}^l \mathrm{GL}_{n_i} \ltimes U$ and that any character $\chi : G \to \mathbb{G}_m$ is of the form

$$\chi(g) = \prod_{i=1} (\det A_i)^{m_i},$$

where $g = ((A_i)_i, u)$ for matrices $A_i \in \mathrm{GL}_{n_i}$ and $m_i \in \mathbb{Z}$. Suppose that χ_A corresponds to $(m_1, \dots, m_l)$, note that each $m_i \geq 0$ since Δ_A is a polynomial and at least one $m_i > 0$, as Δ_A is non-trivial.

Example 5.2.11. Let $X = \mathbb{P}^n$ be standard projective space and $d > 0$ a positive integer. In this case $G \cdot T = \mathbb{P}^n$, since the action of $G = \mathrm{GL}_{n+1}$ on $\mathbb{P}^n$ is transitive. In this case we have that $\nabla_A = \nabla$ is the classical discriminant and that quasismoothness is equivalent to smoothness since X is smooth. Thus

$$\mathcal{Y}_d^{\mathrm{QS}} = \mathbb{P}(k[x_0, \dots, x_n]_d)^{\mathrm{SM}} = \mathbb{P}(k[x_0, \dots, x_n]_d) - \nabla.$$

This is the ideal situation. The quasismooth locus is given by the vanishing of one invariant section. In general this won't be true. However, we can generalise a little: for an arbitrary complete toric variety X, we have that $G \cdot T = X$ if and only if the action of G on X is transitive, and thus by Proposition 3.2.10, X is a product of projective spaces.

Example 5.2.12. Let $X = \mathbb{P}(1, \dots, 1, r) = \mathrm{Proj}\, k[x_0, \dots, x_{n-1}, y]$ be the rational cone of dimension n, let G the automorphism group of X and let $d = d'r > 1$ an integer divisible by r. Then X has a single isolated singularity at $(0 : \cdots : 0 : 1)$. Let $S_d = k[x_0, \dots, x_{n-1}, y]_d$, where $\deg x_i = 1$ and $\deg y = r$. Suppose that $F \in S_d$ is a weighted homogeneous polynomial; then

$$F(x_0, \dots, x_{n-1}, y) = \sum_{j=0}^{d'} F_j(x_0, \dots, x_{n-1}) y^j,$$

where the $F_j \in k[x_0, \dots, x_{n-1}]_{d-rj}$ are homogeneous (possibly 0) polynomials of degree $d - jr$. Note that $F_{d'} \in k$ is a constant, write $F_{d'} = c \in k$, then $F(0, \dots, 0, 1) = c$. Thus $(0 : \cdots : 0 : 1) \in \mathbf{V}(F)$ if and only if $c = 0$. Moreover, if $c = 0$ then $(0 : \cdots : 0 : 1)$ is a singular point of $\mathbf{V}(F)$. Indeed, the derivatives are given by

$$\frac{\partial F}{\partial x_i}(x_0, \dots, x_{n-1}, y) = \sum_{j=0}^{d'} \frac{\partial F_j}{\partial x_i}(x_0, \dots, x_{n-1}) y^j$$
$$\frac{\partial F}{\partial y}(x_0, \dots, x_{n-1}, y) = \sum_{j=1}^{d'} j F_j(x_0, \dots, x_{n-1}) y^{j-1}.$$

Since $d > 1$, the $\frac{\partial F_j}{\partial x_i}$ are either 0 or non-constant homogeneous polynomials in the x_i. Thus $\frac{\partial F}{\partial x_i}(0, \dots, 0, 1) = 0$ for every i and $\frac{\partial F}{\partial y}(0, \dots, 0, 1) = d'c$. Thus the point $(0 : \cdots : 0 : 1)$ is a singular point if and only if $c = 0$. Note that this means for hypersurfaces in X, quasismooth is equivalent to being smooth.

We can write down explicitly the non-quasismooth locus:

$$\mathcal{Y}^{\mathrm{NQS}} = \mathcal{Y} - \mathcal{Y}^{\mathrm{QS}} = \nabla_A \cup \mathbf{V}(c) = \mathbf{V}(\Delta_A \cdot c),$$

where $\nabla_A = \mathbb{P}(1,\dots,1,r)^{\vee,d}$ and we are considering c as a coordinate on $\mathcal{Y}$. In this example $G\cdot T = X - \{(0:\dots:0:1)\}$. To see this, note that

$$G = \mathrm{Aut}(X) = ((\mathbb{G}_m \times \mathrm{GL}_n) \ltimes \mathbb{G}_a^M)/\mathbb{G}_m,$$

and that $\mathrm{GL}_n \hookrightarrow \mathrm{Aut}(X)$ acts transitively on the set $\{(x_0:\dots:x_{n-1}:1) \mid x_i \neq 0 \text{ for some } i\} \subset X$. We prove in Proposition 6.2.9 that the unipotent radical is abelian and that

$$M = \binom{n-1+r}{r}.$$

Remark 5.2.13. Note that in the above example, the coefficient c is a section of the line bundle $H^0(\mathcal{Y}_d, \mathcal{O}_{\mathcal{Y}_d}(1))$. Hence, for all $N > 0$, the section $\Delta_A \cdot c^N \in H^0(\mathcal{Y}_d, \mathcal{O}_{\mathcal{Y}_d}(r+N))$, with $r = \deg \Delta_A$, is an invariant for the action of $\mathrm{Aut}_{\mathrm{g}}(S)$ on $\mathcal{Y}_d$ linearised with respect to $\mathcal{O}_{\mathcal{Y}_d}(1)$. Moreover, it holds that

$$\mathcal{Y}_d^{\mathrm{QS}} = (\mathcal{Y}_d)_{\Delta_a \cdot c^N}.$$

Example 5.2.14. Let us consider an example where the situation is slightly more complicated. Let $X = \mathbb{P}(1,2,3)$ with coordinates x,y,z with weights 1,2 and 3 respectively. Then $\mathrm{Aut}(X) = \left((\mathbb{G}_m)^3 \ltimes (\mathbb{G}_a)^3\right)/\mathbb{G}_m$.

Let $G = \mathbb{G}_m^2 \ltimes \mathbb{G}_a^3$, then there is a surjective morphism $G \to \mathrm{Aut}(X)$ with a finite kernel such that the orbits in $\mathcal{Y}_d = \mathbb{P}(k[x,y,z]_d)$ are the same. The group G acts on X as follows. Let $(\lambda,\lambda',(A,B,C)) \in G$; then for $(x:y:z) \in X$,

$$(\lambda,\lambda',(A,B,C))\cdot(x:y:z) = (x:\lambda(y+Ax^2):\lambda'(z+Bx^3+Cxy)).$$

This action has four orbits. Firstly, $G\cdot(1:0:0) = G\cdot T$ is an open orbit, then

$$\mathbf{V}(x) - \{(0:1:0),(0:0:1)\}$$

is an orbit which is neither open nor closed and the two singular points $\{(0:1:0)\}$ and $\{(0:0:1)\}$ are G-fixed points.

Now take $d = 6$ (although the situation will be the same for any d of the form $d = 6d'$). Then an arbitrary polynomial $F \in k[x,y,z]_6$ will have the following form:

$$F(x,y,z) = a_6x^6 + a_4x^4y + a_3x^3z + a_2x^2y^2 + a_1xyz + by^3 + cz^2,$$

where the enumeration of the coefficients reflects the power of x in the monomial (this notation will be useful in the more detailed study of this example in Chapter 6). It follows immediately that

$$F(0:1:0) = b,$$
$$F(0:0:1) = c.$$

Thus $(0:1:0) \in \mathbf{V}(F)$ if and only if $b = 0$ and $(0:0:1) \in \mathbf{V}(F)$ if and only if $c = 0$. Moreover, a quick calculation shows that

$$\frac{\partial F}{\partial x}(0:1:0) = 0,$$
$$\frac{\partial F}{\partial y}(0:1:0) = 3b,$$
$$\frac{\partial F}{\partial z}(0:1:0) = 0.$$

This means that if $(0:1:0) \in \mathbf{V}(F)$, then it is a singular point of $\mathbf{V}(F)$ which prevents $\mathbf{V}(F)$ from being quasismooth and the same holds true for $(0:0:1)$. So we have the containment

$$\mathbf{V}(b) \cup \mathbf{V}(c) \cup \nabla_A \subset \mathcal{Y}^{\mathrm{NQS}},$$

where $\nabla_A = \mathbb{P}(1,2,3)^{\vee,6}$.

We claim that this is in fact an equality. Suppose $(0:y_0:z_0)$ is a singular point of $\mathbf{V}(F)$ in $\mathbf{V}(x) - \{(0:1:0),(0:0:1)\}$. Then $F(0:y_0:z_0) = by_0^3 + cz_0^2 = 0$ and moreover

$$\frac{\partial F}{\partial z}(0:y_0:z_0) = 2cz_0^2 = 0.$$

Hence $b = c = 0$ and $[F] \in \mathbf{V}(b) \cap \mathbf{V}(c)$. Thus any degree 6 hypersurface with a singularity in $\mathbf{V}(x) - \{(0:1:0),(0:0:1)\}$, which will certainly not be quasismooth, must also have a singularity at both $(0:1:0)$ and $(0:0:1)$, yielding

$$\mathcal{Y}^{\mathrm{NQS}} = \nabla_A \cup \mathbf{V}(b) \cup \mathbf{V}(c).$$

Remark 5.2.15. Example 5.2.14 generalises to all Cartier weights in weighted projective spaces of the form $X = \mathbb{P}(1,\dots,1,2,3)$. Give X the coordinates $(x_1 : \dots : x_{n-1} : y : z)$ and consider hypersurfaces of degree $d = 6d'$. Let b and c be the coefficients of the monomials $y^{3d'}$ and $z^{2d'}$ respectively. Then

$$\mathcal{Y}^{\mathrm{NQS}} = \nabla_A \cup \mathbf{V}(b) \cup \mathbf{V}(c).$$

As in Remark 5.2.13, we have that $\Delta_A \cdot b \cdot c$ is an invariant section of $\mathcal{O}_{\mathcal{Y}}(r+2)$, where $r = \deg \Delta_A$, which defines the quasismooth locus.

Question 5.2.16. *Suppose that $X = \mathbb{P}(a_0, \ldots, a_n)$ is a well-formed weighted projective space and that $d > 0$ is a Cartier degree. Suppose further that each of the a_i which are different to 1 are distinct. Then does the following equality hold?*

$$\mathcal{Y}^{NQS} = \nabla_A \bigcup (\cup\, \boldsymbol{V}(c_i)) \subset \mathbb{P}(k[x_0, \ldots, x_n]_d),$$

where c_i is the coefficient of the monomial $x_i^{l_i}$, with $a_i l_i = d$ and the union is taken over all weights greater than 1.

This question is motivated by the following observation.

Observation 5.2.17. Suppose that $X = \mathbb{P}(a_0, \ldots, a_n)$ is a well-formed weighted projective space and that $d > 0$ is a Cartier degree. Suppose that $a_i > 1$ is a weight which appears once and for $F \in S_d$ let $c_i(F)$ be the coefficient of the monomial $x_i^{l_i}$ in F. Then the point $(0 : \cdots : 0 : 1 : 0 : \cdots : 0)$ with a 1 in the i^{th} position is contained in $\mathbf{V}(F)$ if and only if it is a 'non-quasismooth' point of $\mathbf{V}(F)$ if and only if $c_i(F) = 0$.

An even more basic question is the following.

Question 5.2.18. *Suppose that $X = \mathbb{P}(a_0, \ldots, a_n)$ is a well-formed weighted projective space and that d is a Cartier degree. Is the non-quasismooth locus a hypersurface?*

Chapter 6

Stability of quasismooth hypersurfaces

In this chapter we construct coarse moduli spaces of quasismooth hypersurfaces of fixed degree in certain toric orbifolds. We prove that quasismooth hypersurfaces of weighted projective space (excluding some low degrees) are stable when the $(\mathfrak{C}^*)$ condition is satisfied for the action of a grading of the unipotent radical of the automorphism group of this weighted projective space. Once stability is established, we apply the non-reductive GIT Theorem (Theorem 1.3.22) to conclude that a coarse moduli space of quasismooth hypersurfaces exists as a quasi-projective variety. Moreover, Theorem 1.3.22 provides a compactification of this moduli space. We also discuss the $(\mathfrak{C}^*)$ condition and show that it holds for certain weighted projective spaces. We give examples when it does not hold; in this case, one should be able to construct moduli spaces of quasismooth hypersurfaces using the blow-up procedure in [BDHK16].

We also consider smooth hypersurfaces in products of projective spaces and prove that smoothness implies semistability. If we suppose further that the degree is such that the hypersurfaces are of general type, then we prove that smoothness implies stability. Hence we construct a coarse moduli space of such hypersurfaces.

6.1 The linearisation of the action

In this section we discuss the linearisation of the group action on the parameter space for hypersurfaces defined in Section 4.3.2.

Let $X = \mathbb{P}(a_0, \ldots, a_n)$ be a well-formed weighted projective space and $d > 0$ be a positive integer. Consider the action of $\mathrm{Aut}_g(S)$ on $\mathcal{Y}_d = \mathbb{P}(k[x_0, \ldots x_n]_d)$ as defined in Example 4.3.15. As noted in Remark 4.3.16, this action has a global stabiliser coming from the 1-parameter subgroup defined by $\lambda_{\underline{a}} : t \mapsto (t^{a_0}, \ldots, t^{a_n})$. Using the standard definitions of stability (c.f. Definition 1.3.7) we find that the stable locus is empty. In order to take this into account we modify the definition of stability after King [Kin94, Definition 2.1] (see Definition 6.1.4).

6.1.1 Characters of $\mathrm{Aut}_{\mathrm{g}}(S)$

Let us describe the characters of the group $G = \mathrm{Aut}_{\mathrm{g}}(S)$. Label the distinct weights a_i by $b_1 < \cdots < b_l$ and suppose that each b_i occurs n_i times. By Theorem 3.1.3 we have

$$G = \prod_{i=1}^{l} \mathrm{GL}_{n_i} \ltimes U,$$

where U is the unipotent radical and so $X^*(G) = X^*(\prod \mathrm{GL}_{n_i})$. Then any character $\chi \in X^*(G)$ is of the form

$$\begin{aligned} \chi : G &\longrightarrow \mathbb{G}_m \\ \big((A_i)_{i=1}^{l}, u\big) &\longmapsto \prod_{i=1}^{l} (\det A_i)^{m_i} \end{aligned}$$

for $m_i \in \mathbb{Z}$. In this way we identify $X^*(G)$ with the lattice $\mathbb{Z}^l$. The following lemma is immediate, since $\mathrm{Aut}(X) = G/\lambda_{\underline{a}}(\mathbb{G}_m)$.

Lemma 6.1.1. *Keep the above notation. A character χ of G defined by the lattice point $(m_1, \ldots, m_l) \in \mathbb{Z}^l$ descends to a character of $\mathrm{Aut}(X)$ if and only if $\sum_{i=1}^{l} b_i n_i m_i = 0$.*

Let $\mathcal{O}(1) = \mathcal{O}_{\mathcal{Y}_d}(1)$ and consider the basis of $H^0(\mathcal{Y}_d, \mathcal{O}(1)) = (k[x_0, \ldots, x_n]_d)^\vee$ given by the dual elements of the monomials $x_0^{i_0} \cdots x_n^{i_n}$ where the i_j are non-negative integers such that $a_0 i_0 + \cdots + a_n i_n = d$.

Definition 6.1.2. We denote the $\mathcal{O}(1)$-linearisation defined by the action of G on $k[x_0,\dots,x_n]_d$ by $\mathcal{O}(1)$, by abuse of notation. See Example 1.2.6 for details.

Note that by [Dol03, Theorem 7.2] we have that

$$\operatorname{Pic}^G(\mathcal{Y}_d)_{\mathbb{Q}} = (\operatorname{Pic}(\mathcal{Y}_d) \times X^*(G))_{\mathbb{Q}} \simeq \mathbb{Q}^{l+1}.$$

Furthermore, since $\mathcal{O}(1)$ admits a G-linearisation, [Dol03, Theorem 7.2] implies that $\operatorname{Pic}^G(\mathcal{Y}_d) = \operatorname{Pic}(\mathcal{Y}_d) \times X^*(G) \cong \mathbb{Z}^{l+1}$.

We give an example in order to clear up any confusion concerning sign conventions.

Example 6.1.3. Consider $X = \mathbb{P}(1,1,2)$ and $d = 4$ so that $\operatorname{Aut}_{\mathrm{g}}(S) = (\mathrm{GL}_2 \times \mathbb{G}_m) \ltimes (\mathbb{G}_a)^3$. Then

$$\mathcal{Y}_4 = \mathbb{P}(k[x,y,z]_4)$$

and $V = H^0(\mathcal{Y}_4, \mathcal{O}(1))^\vee = \operatorname{Span}(x^4, x^3y, x^2y^2, xy^3, y^4, x^2z, xyz, y^2z, z^2)$. Consider the 1-parameter subgroup

$$\begin{aligned} \lambda : \mathbb{G}_m &\longrightarrow (\mathrm{GL}_2 \times \mathbb{G}_m) \ltimes (\mathbb{G}_a)^3 \\ t &\longmapsto \left(t^{-1}\operatorname{Id}_2, t^{-1}, 0\right) \end{aligned}$$

acting on V. Then $\lambda(t) \cdot x^i y^j z^k = t^{i+j+k} x^i y^j z^k$, where $i + j + 2k = 4$. In particular, there are three $\lambda(\mathbb{G}_m)$-weight spaces corresponding to weights $2, 3$ and 4:

$$\begin{aligned} V_4 &= \operatorname{Span}(x^4, x^3y, x^2y^2, xy^3, y^4) \\ V_3 &= \operatorname{Span}(x^2z, xyz, y^2z) \\ V_2 &= \operatorname{Span}(z^2). \end{aligned}$$

Note that this λ is the grading 1-parameter subgroup λ_g from Proposition 3.1.5.

6.1.2 Stability in the presence of a global stabiliser

Due to the presence of the diagonal weighted $\mathbb{G}_m$ acting trivially on $\mathcal{Y}_d$, we need our definition of stability to allow for a positive dimensional stabiliser, so we adopt a variant of Definition 1.3.7. We do this as it is easier to work a modified definition of stability rather than with the group resulting from quotienting out by this global stabiliser. Note that for weighted projective space the global stabiliser is a 1-parameter subgroup, however in general it will be the diagonalisable group appearing in the quotient construction of a toric variety (see Theorem 2.3.9).

Definition 6.1.4. Let G be a linear algebraic group acting on a projective variety X with respect to a linearisation $\mathcal{L}$. Suppose that there is a global stabiliser $D \subset G$ for the action, where D is a diagonalisable group. We define $I^s \subset I^{\mathrm{fg}}$ to be G-invariant sections satisfying the following conditions:

- the action of G on X_f is closed and for every $x \in X_f$ we have $D \subset \mathrm{Stab}_G(x)$ with finite index; and
- the restriction of the U-enveloping quotient map
$$q_U : X_f \longrightarrow \mathrm{Spec}(\mathcal{O}(X)^U_{(f)})$$
is a principal U-bundle for the action of U on X_f.

Then we define
$$X^{\mathrm{s}} = \bigcup_{f \in I^{\mathrm{s}}} X_f$$
to be the stable locus.

In Chapter 6, stability is always meant in the sense of Definition 6.1.4 and the locus X^{s} is as defined in Definition 6.1.4.

Remark 6.1.5. In the case where G is a reductive group, the notion of stability as defined in Definition 6.1.4 also broadens the reductive GIT notion of stability given in Definition 1.2.11. Moreover, since the global stabiliser D is diagonalisable and hence reductive, D must lie in the Levi factor of G. It follows that Theorem 1.3.11 holds with this definition of stability; that is, there exists a geometric quotient of the stable locus X^{s}. Note that the semistable locus is left unchanged by the presence of a global stabiliser.

The discrete-geometric version of the Hilbert-Mumford criterion for a torus described in Theorem 1.2.16 must be adapted to work in this situation. Suppose that a torus T is acting on a projective space $X = \mathbb{P}^n$ with respect to a very ample linearisation $\mathcal{O}(1)$. Note that $X = \mathbb{P}(V)$, where $V = H^0(X, \mathcal{O}(1))^\vee$. Suppose further that there exists a 1-parameter subgroup

$$\lambda_{\underline{a}} : \mathbb{G}_m \longrightarrow T$$

such that $\lambda_{\underline{a}}(\mathbb{G}_m) \subset \mathrm{Stab}_T(x)$ for every $x \in X$. Consider the weight space decomposition

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi,$$

where $X^*(T) = \mathrm{Hom}(T, k^*)$ is the character group and $V_\chi = \{v \in V \mid t \cdot v = \chi(t)v \ \forall t \in T\}$. The 1-parameter subgroup $\lambda_{\underline{a}}$ defines a point in $W = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, denote this point by $\underline{a} \in W$. Define the quotient vector space $H_{\underline{a}} = W/\mathbb{Q} \cdot \underline{a}$ and write $\overline{w} \in H_{\underline{a}}$ for the image of an element $w \in W$ in $H_{\underline{a}}$.

Let $\overline{T} = T/\lambda_{\underline{a}}(\mathbb{G}_m)$ and consider $x \in X$ and some $v \in V$ lying over x and write $v = \sum v_\chi$. Note that we can equivalently construct $H_{\underline{a}} = X^*(\overline{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We define the $\overline{T}$-weight set of x to be

$$\mathrm{wt}_{\overline{T}}(x) = \{\overline{\chi} \mid v_\chi \neq 0\} \subset H_{\underline{a}},$$

and the associated weight polytope to be the convex hull of these weights:

$$\mathrm{Conv}_{\overline{T}}(x) = \mathrm{Conv}(\chi \mid \chi \in \mathrm{wt}_T(x)) \subset H_{\underline{a}}.$$

Note that the new weight polytope is the image of the weight polytope from Section 1.2.5 under the quotient map $Q : W \to H_{\underline{a}}$.

With this new definition of torus weights, we get the same discrete-geometric Hilbert-Mumford Criterion for (semi)stability with respect to the torus.

Theorem 6.1.6 (Reductive Hilbert-Mumford criterion)**.** *Let T be a torus acting on a projective scheme X with linearisation $\mathcal{L}$ such that there is a global stabiliser $\lambda_{\underline{a}} : \mathbb{G}_m \to T$ acting trivially on X. Then*

$$x \in X^{ss,T}(\mathcal{L}) \iff 0 \in \mathrm{Conv}_{\overline{T}}(x),$$
$$x \in X^{s,T}(\mathcal{L}) \iff 0 \in \mathrm{Conv}_{\overline{T}}(x)^\circ,$$

where $\mathrm{Conv}_{\overline{T}}(x)^\circ$ is the interior of the polytope.

Combining this with Theorem 1.3.24, we have a non-reductive Hilbert-Mumford criterion.

Theorem 6.1.7 (Non-reductive Hilbert-Mumford criterion)**.** *Let G be a linear algebraic group acting on a projective variety X with respect to $\mathcal{L}$. Additionally suppose that there is a global stabiliser $\lambda_{\underline{a}} : \mathbb{G}_m \to T \subset G$ acting trivially on X, where T is a maximal torus of G. Assume that G has graded unipotent radical such that $(\mathfrak{C}^*)$ holds. The following Hilbert-Mumford criterion holds.*

$$x \in X^{ss,G} \iff 0 \in \mathrm{Conv}_T(g \cdot x) \text{ for every } g \in G,$$
$$x \in X^{s,G} \iff 0 \in \mathrm{Conv}_T(g \cdot x)^\circ \text{ for every } g \in G.$$

Example 6.1.8. We consider degree 4 curves in $\mathbb{P}(1,1,2)$ as in Example 6.1.3. Fix the maximal torus of $\mathrm{GL}_2 \times \mathbb{G}_m$ defined by

$$T = \{(\mathrm{diag}(t_1, t_2), s) \in \mathrm{GL}_2 \times \mathbb{G}_m \mid t_1, t_2, s \in k^*\}.$$

Let T act on $\mathcal{Y}_4 = \mathbb{P}(k[x,y,z]_4)$. Then for a general monomial $x^i y^j z^k \in k[x,y,z]_4 = V$ with $i + j + 2k = 4$, we have that

$$(t_1, t_2, s) \cdot x^i y^j z^k = t_1^i t_2^j s^k x^i y^j z^k.$$

Denote such a weight by $(i,j,k) \in X^*(T) \cong \mathbb{Z}^3$. Note that by collecting all possible weights as columns in a matrix, one gets exactly the matrix A from Example 5.1.11

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

Define the *section polytope* to be the convex hull of all torus weights in either W or $H_{\underline{a}}$ (see Figure 6.1). The weight polytope of any element $[f] \in \mathcal{Y}_4$ will be a subpolytope of the section polytope. As shown in figure one, the section polytope considered in W is not full-dimensional, and is in fact a 2-simplex in $W \cong \mathbb{Q}^3$. Hence 0 will not be in any subpolytope and hence by Theorem 1.2.16 implies that both the stable and semistable locus, as defined in Definition 1.2.11, are empty.

If we twist the linearisation of the action by a character of G, this shifts the weight picture and we can arrange for the origin to lie in the section polytope; hence we can have

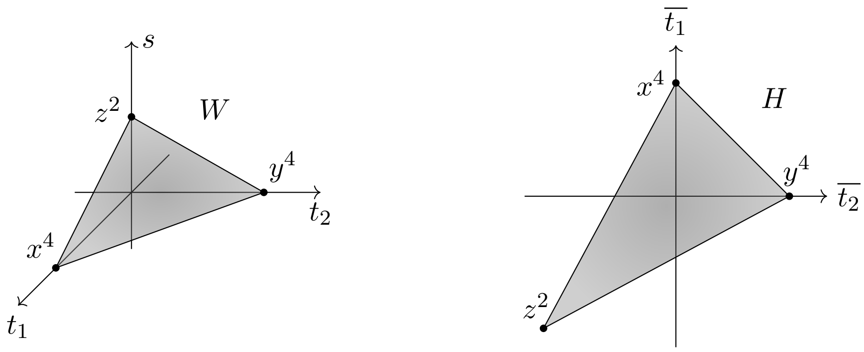


Figure 6.1: Section polytope of $\mathcal{O}(4)$ in W and H.

semistable points for such a twisted linearisation, but we cannot have stable points in the sense of Definition 1.2.11 even after twisting, as 0 is never in the interior of the weight polytope in W.

However, when working in H, the section polytope is full-dimensional and the origin is indeed contained in the interior. Thus the stable set in the sense of Definition 6.1.4 can be non-empty (and indeed we'll see that this is the case).

6.2 $\hat{U}$-stability for weighted hypersurfaces

Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space and assume that $a_0 \leq \dots \leq a_n$. Let us new give coordinates on X as follows. Let

$$X = \operatorname{Proj} k[x_1, \dots, x_{n'}, y_1, \dots, y_{n_l}],$$

such that $n' + n_l = n + 1$ and $\deg x_i < a_n$ and $\deg y_j = a_n$. Note that the y_j are variables with maximum weight. If all the weights coincide then $X = \mathbb{P}^n$, so we disregard this case.

Let d be a positive integer such that $\operatorname{lcm}(a_j)$ divides d so that hypersurfaces of degree d of X are Cartier divisors; recall that we call such an integer a *Cartier degree.*

Notation 6.2.1. We give the parameter space

$$\mathcal{Y}_d = \operatorname{Div}_X^d = \mathbb{P}(k[x_1, \dots, y_{n_l}]_d)$$

the following coordinates of the coefficients of the monomials: $(u_0 : \dots : u_{M'} : v_0 : \dots : v_M) \in \mathcal{Y}_d$, where the v_j correspond to monomials in the y_j and all have the same total degree, and the u_i are the coefficients of monomials containing an x_i for some $1 \leq i \leq n'$. The integer M is defined by

$$M = \begin{pmatrix} n_l + d' \\ d' \end{pmatrix},$$

where $d' = \frac{d}{a_n}$ and M' is computed in terms of the $a_i's$, but its exact value is not required for the subsequent discussion.

Recall $G = \mathrm{Aut}_{\mathrm{g}}(S)$ and that

$$G \simeq \prod_{i=1}^{l} \mathrm{GL}_{n_i} \ltimes U,$$

where U is the unipotent radical and that $b_1 < \cdots < b_l$ are the distinct values of $a_0, \ldots, a_n$ with each b_i occurring with multiplicity n_i. By Proposition 3.1.9, the 1-parameter subgroup of G given by

$$\lambda_{g,N} : t \longmapsto \big((t^{-N} \, \mathrm{Id}_{n_i})_{i=1}^{l-1}, t \, \mathrm{Id}_{n_l}, 0 \big),$$

for $N > 0$ defines a positive grading of the unipotent radical of G and we define the graded unipotent group $\hat{U}_N = \lambda_{g,N}(\mathbb{G}_m) \ltimes U$.

Remark 6.2.2. Note that $\hat{U}_N$ depends on the integer $N > 0$: that is, for different values of N, the subgroups $\lambda_{g,N}(\mathbb{G}_m) \ltimes U$ are different. However, we shall see (see Remark 6.2.4), that the semistable and stable locus for $\hat{U}_N$ is the same for all $N >> 0$.

Let G act on $\mathcal{Y}_d$ with respect to the linearisation $\mathcal{O}(1)$ as in Definition 6.1.2. Suppose that $\underline{x}^I$ is a monomial in $k[x_1, \ldots, x_{n'}, y_1, \ldots, y_{n_l}]_d$. Then

$$\lambda_{g,N}(t) \cdot \underline{x}^I = t^{r(N,I)} \underline{x}^I,$$

where $r(N,I) \in \mathbb{Z}$ is an integer depending on N and the monomial. Note that for $\underline{y}^I \in k[y_1, \ldots, y_{n_l}]_d$ we have that $r(N,I) = -d' = -\frac{d}{a_n}$ is independent of N and

$$\lambda_{g,N}(t) \cdot \underline{y}^I = t^{-d'} \underline{y}^I.$$

Recall the definition of $Z_{\min}$ and $(\mathcal{Y}_d)^0_{\min}$ from Definition 1.3.14. Both subsets are defined with respect to a $\hat{U}_N$-action.

Lemma 6.2.3. *Let* $X = \mathbb{P}(a_0, \ldots, a_n)$ *and* $d \in \mathrm{Pic}(X)$ *be a Cartier degree. Fix* $\hat{U}_N = \lambda_{g,N}(\mathbb{G}_m) \ltimes U$*. Then*

$$Z_{\min} = \{(0 : \cdots : 0 : v_0 : \cdots : v_M) \mid \exists j : \ v_j \neq 0\} \subset \mathcal{Y}_d$$

and the minimum weight of the $\lambda_{g,N}(\mathbb{G}_m)$*-action on* $V = H^0(\mathcal{Y}_d, \mathcal{O}(1))^\vee$ *is* $\omega_{\min} = -\frac{d}{a_n}$*.*

Note that both $Z_{\min}$ and $\omega_{\min}$ are independent of N.

Proof. As noted above, $\lambda_{g,N}$ acts on monomials containing only variables y_i with weight $-d' = -\frac{d}{a_n}$. Suppose that $\underline{x}^I \in V = k[x_1, \ldots, y_{n_l}]_d$ is another monomial containing at least one x_i variable. Then $\lambda_{g,N}(t) \cdot \underline{x}^I = t^{r(N,I)} \underline{x}^I$ and since $\lambda_{g,N}(t) \cdot x_i = t^N x_i$ we have that $r(N, I) > -d'$. Hence $V_{\min} = k[y_1, \ldots, y_{n_l}]_d$ and thus

$$Z_{\min} = \mathbb{P}(V_{\min}) = \{(0 : \cdots : 0 : v_0 : \cdots : v_M) \mid \exists j : \ v_j \neq 0\},$$

using Notation 6.2.1. □

Remark 6.2.4. It follows from the lemma that

$$(\mathcal{Y}_d)^0_{\min} = \{(u_0 : \cdots : u_{M'} : v_0 : \cdots : v_M) \mid \exists j : \ v_j \neq 0\} \subset \mathcal{Y}_d.$$

Furthermore, if we take $N > d'$ then we have the weight diagram shown in Figure 6.2, where $\omega_{\min+1}$ is the next biggest weight and $r(N)$ grows linearly with N, see Example 6.2.6.[1] To see that $\omega_{\min+1}$ is indeed positive, note that for any monomial not in $V_{\min}$, the weight will have a positive summand of $N > d'$ and a negative summand of strictly less that d', thus will be positive.

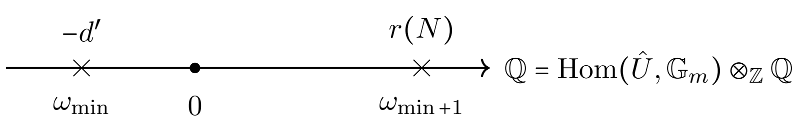


Figure 6.2: The weight diagram for $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \ltimes U$ for $N >> 0$.

The following lemma follows immediately from Lemma 6.2.3.

[1] This choice of lower bound $N > d'$ is not optimal; we could take a smaller N.

Lemma 6.2.5. *Let $X = \mathbb{P}(a_0, \dots, a_n)$ such that $a_n > a_{n-1} \geq \cdots \geq a_0$. Then for every Cartier degree $d \in \mathbb{Z}$, we have*

$$Z_{\min} = \{(0 : \cdots : 0 : 1)\} \subset \mathcal{Y}_d$$

is a point.

Example 6.2.6. Let $X = \mathbb{P}(1,1,2) = \operatorname{Proj} k[x_1, x_2, y]$ and $d = 2$. Then the monomial basis for $V = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1))^\vee$ is $(x_1^2, x_1x_2, x_2^2, y)$. Writing down an arbitrary polynomial

$$[f(x_1, x_2, y)] = [u_1x_1^2 + u_2x_1x_2 + u_3x_2^2 + vy] \in \mathcal{Y}_2$$

in coordinates gives $(u_1 : u_2 : u_3 : v) \in \mathcal{Y}_2$. Now let us calculate the weights for the grading $\mathbb{G}_m$-action defined by $\lambda_{g,N}$, with $N > 0$. For positive integers i and j such that $i + j = 2$ we have

$$\lambda_{g,N}(t) \cdot x_1^i x_2^j = \left(t^N x_1\right)^i \left(t^N x_2\right)^j = t^{2N} x_1^i x_2^j \quad \text{and} \quad \lambda_{g,N}(t) \cdot y = t^{-1} y.$$

Hence we have two distinct weights $2N$ and -1 and the decomposition into weight spaces is given by

$$V = V_{2N} \oplus V_{-1} = \operatorname{Span}(x_1^2, x_1x_2, x_2^2) \oplus \operatorname{Span}(y).$$

Thus $Z_{\min} = \mathbb{P}(V_{-1}) = \{(0:0:0:1)\}$.

Notation 6.2.7. For the rest of this section we fix $N > 0$ and $\hat{U} = \lambda_{g,N} \ltimes U$.

Proposition 6.2.8. *Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space and d be a Cartier degree. Denote $\mathcal{Y} = \mathcal{Y}_d$. Then we have the following inclusion:*

$$\mathcal{Y}^{QS} \subset \mathcal{Y}^0_{\min} - U \cdot Z_{\min}.$$

Proof. We begin by observing that

$$\mathcal{Y} - \mathcal{Y}^0_{\min} = \{(u_0 : \cdots : u_{M'} : 0 : \cdots : 0)\}.$$

Take some $f \in \mathcal{Y} - \mathcal{Y}^0_{\min}$. We know that f contains no monomials made up of only the y_i. Thus $(0 : \cdots : 0 : 1) \in X$ will be a common zero for all $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial y_i}$ since d is a Cartier degree (as monomials of the form $y_{n_l}^{d'-1} x_i$ can never be homogeneous of degree d). It follows that f is not quasismooth by Remark 2.6.2 and hence

$$\mathcal{Y}^{\mathrm{QS}} \subset \mathcal{Y}^0_{\min}.$$

Suppose that $f \in Z_{\min}$. Then f is a polynomial in the y_i and so $(1:0:\cdots:0) \in X$ will be a common zero for all $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial y_i}$. Thus f is not quasismooth and we have that

$$Z_{\min} \subset \mathcal{Y}^{\mathrm{NQS}}.$$

Since $\mathcal{Y}^{\mathrm{NQS}}$ is a G-invariant subset by Lemma 4.3.14, it follows that

$$U \cdot Z_{\min} \subset \mathcal{Y}^{\mathrm{NQS}}$$

and so we conclude that

$$\mathcal{Y}^{\mathrm{QS}} \subset \mathcal{Y}^{0}_{\min} - U \cdot Z_{\min}.$$

□

We show that the condition $(\mathfrak{C}^*)$ for the action of $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \ltimes U$ on $\mathcal{Y}_d$ linearised by $\mathcal{O}(1)$ (see Definition 1.3.18) holds for weighted projective spaces of the form $\mathbb{P}(1,\ldots,1,r)$.

Proposition 6.2.9. *Let $X = \mathbb{P}(1,\ldots,1,r) = \operatorname{Proj} k[x_0,\ldots,x_{n-1},y]$ and $d > 0$ be a Cartier degree (so that r divides d). Then the graded automorphism group of $S = k[x_0,\ldots,x_{n-1},y]$ is of the following form*

$$\mathrm{Aut}_{\mathrm{g}}(S) = (\mathrm{GL}_n \times \mathbb{G}_m) \ltimes \mathbb{G}_a^L,$$

where $L = \begin{pmatrix} n-1+r \\ r \end{pmatrix}$. In particular, the unipotent radical $U = \mathbb{G}_a^L$ is abelian. Moreover, the action of $\hat{U}$ on $\mathcal{Y}_d$ with respect to $\mathcal{O}(1)$ satisfies the condition $(\mathfrak{C}^*)$; that is, the stabiliser group is trivial

$$\mathrm{Stab}_U([f]) = \{e\}$$

for every $[f] \in Z_{\min} \subset \mathcal{Y}_d$.

Proof. Let $G = \mathrm{Aut}_{\mathrm{g}}(S)$; then a general automorphism in the unipotent radical $\phi \in U \subset G$ is given by

$$\phi : \begin{cases} x_i \longmapsto x_i & \text{for } \ 0 \le i \le n-1 \\ y \longmapsto y + p_\phi(x_0,\ldots,x_{n-1}) \end{cases}$$

for $p_\phi \in k[x_0,\ldots,x_{n-1}]_r$. Composing two such elements $\phi, \psi \in U$ gives

$$\phi \circ \psi : \begin{cases} x_i \longmapsto x_i & \text{for } \ 0 \le i \le n-1 \\ y \longmapsto y + p_\phi(x_0,\ldots,x_{n-1}) + p_\psi(x_0,\ldots,x_{n-1}). \end{cases}$$

It follows that any two automorphisms commute and hence U is abelian and thus

$$U \simeq \mathbb{G}_a^L,$$

where $L = \begin{pmatrix} n-1+r \\ r \end{pmatrix} = \dim k[x_0, \dots, x_{n-1}]_r$.

Let us prove the second statement. Fix coordinates on $\mathcal{Y}_d$ given by $(a_0 : \cdots : a_{M'} : b)$ where b is the coefficient of $y^{d'}$ for $d' = \frac{d}{r}$. Note that

$$Z_{\min} = \{(0 : \cdots : 0 : 1)\} = \left\{\left[y^{d'}\right]\right\}$$

by Lemma 6.2.5. Then for any $\phi \in U$ we have that

$$\phi \cdot \left[y^{d'}\right] = \left[(y + p_\phi(x_0, \dots, x_{n-1}))^{d'}\right].$$

It follows that $\phi \cdot [y^{d'}] = [y^{d'}]$ if and only if $p_\phi = 0$. Hence $\operatorname{Stab}_U\left(\left[y^{d'}\right]\right) = \{e\}$. □

Remark 6.2.10. The condition $(\mathfrak{C}^*)$ is not satisfied for every weighted projective space; for example, consider $X = \mathbb{P}(1,2,3) = \operatorname{Proj} k[x,y,z]$ and $d = 6$. Then

$$Z_{\min} = \{(0 : \cdots : 0 : 1)\} = \left\{\left[z^2\right]\right\}$$

is a point by Lemma 6.2.5 and corresponds to the hypersurface defined by z^2. However, the additive 1-parameter subgroup of U

$$a(u) : y \longmapsto y + ux^2$$

acts trivially on $Z_{\min}$.

There are other examples of weighted projective space for which the condition $(\mathfrak{C}^*)$ is satisfied. For example, let $X = \mathbb{P}(2,2,3,3,5)$ with coordinates x_1, x_2, y_1, y_2, z such that $\deg x_i = 2$, $\deg y_i = 3$ and $\deg z = 5$. Let $d = 20$ and note that

$$\operatorname{Aut}(X) = ((\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathbb{G}_m) \ltimes (\mathbb{G}_a)^4)/\lambda_{\underline{a}}(\mathbb{G}_m).$$

Again we have that $Z_{\min}$ is a point corresponding to the hypersurface z^4. Then the action of $(\mathbb{G}_a)^4$ is trivial on coordinates x_1, x_2, y_1, y_2 and on z the action is defined by

$$(A_1, A_2, A_3, A_4) \cdot z = z + A_1 x_1 y_1 + A_2 x_1 y_2 + A_3 x_2 y_1 + A_4 x_2 y_2,$$

where $(A_1, A_2, A_3, A_4) \in (\mathbb{G}_a)^4$. It follows that the $(\mathbb{G}_a)^4$-stabiliser of $\left[z^4\right]$ is trivial.

Remark 6.2.11. Suppose $X = \mathbb{P}(1,\dots,1,b,\dots,b) = \operatorname{Proj} k[x_1,\dots x_n, y_1,\dots,y_m]$ is a weighted projective space such that $b > 1$, then the unipotent radical of G is abelian and isomorphic to $\mathbb{G}_a^L$, where

$$L = m \cdot \binom{n-1+b}{b},$$

however for $m > 1$, the condition $(\mathfrak{C}^*)$ is not satisfied; see the following example.

Example 6.2.12. Let $X = \mathbb{P}(1,1,2,2)$ with coordinates as in Remark 6.2.11. Then $\operatorname{Aut}(X) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/\mathbb{G}_m^2 \ltimes \mathbb{G}_a^6$ and define an additive subgroup $\{\phi_A \in \operatorname{Aut}(X) \mid A \in k\}$ defined by

$$\phi_A : y_2 \mapsto y_2 + Ax_1^2$$

and the identity elsewhere. Consider hypersurfaces of even degree d. Then the hypersurface defined by y_1^2 lies in $V_{\min}$ and is always stabilised by automorphisms ϕ_A. Thus the condition $\mathfrak{C}^*$ is not satisfied.

Remark 6.2.13. Let $X = \mathbb{P}(a_0,\dots,a_n)$ be a weighted projective space and $d = \operatorname{lcm}(a_0,\dots,a_n)$. Suppose that there exist variables $x_{j_1},\dots,x_{j_h}$ with $h \geq 2$ (with repetitions allowed) such that $\deg x_{j_i} = a_{k_i}$ and

$$a_{k_1} + \cdots + a_{k_h} = a_k < a_n$$

for some $a_k < a_n$. Then $Z_{\min} \subset \mathcal{Y}_{ld}$ for every $l > 0$ has a non-trivial global stabiliser. Indeed, the following automorphism generates an additive 1-parameter subgroup in said stabiliser:

$$\phi : x_k \longmapsto x_k + x_{j_1} \cdots x_{j_h}.$$

Additionally, if one can show that the condition $(\mathfrak{C}^*)$ does not hold for d, it follows that it does not hold for any ld with $l > 0$. Indeed, suppose $F \in k[y_1,\dots,y_m]$ is homogeneous of total degree 1 and has a non-trivial stabiliser for the U action. Then $F^{ld} \in V_{\min}$ will also have non-trivial stabiliser for every $l > 0$.

Corollary 6.2.14. *Let $X = \mathbb{P}(a_0,\dots,a_n)$ be a weighted projective space and d be a Cartier degree. Assume that X satisfies the condition $(\mathfrak{C}^*)$ for the action of $\hat{U}$ on $\mathcal{Y}_d$ with respect to $\mathcal{O}(1)$, where $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \ltimes U \subset G = \operatorname{Aut}_{\mathrm{g}}(S)$ for $N > 0$. Then the following statements hold.*

1. The quotient morphism

$$q_U : \mathcal{Y}^0_{\min} \to \mathcal{Y}^0_{\min}/U$$

is a principal U-bundle.

Let $\epsilon > 0$ a rational number. Suppose further that χ_ϵ is a character of $\hat{U}$ such that $\mathcal{O}(1)^{\chi_\epsilon}$ is an ϵ-linearisation (in the sense of Definition 1.3.16). For $\epsilon > 0$ sufficiently small, we have the following statements.

2. The quotient morphism

$$q_{\hat{U}} : \mathcal{Y}^{s,\hat{U}}_d \longrightarrow \mathcal{Y}_d /\!\!/_{\mathcal{O}(1)^{\chi_\epsilon}} \hat{U}$$

is a projective geometric quotient, where $\mathcal{Y}^{s,\hat{U}}_d = \mathcal{Y}^0_{\min} - U \cdot Z_{\min}$

3. The subset

$$\mathcal{Y}^{QS}/\hat{U} = q_{\hat{U}}(\mathcal{Y}^{QS}) \subseteq \mathcal{Y}_d /\!\!/_{\mathcal{O}(1)^{\chi_\epsilon}} \hat{U}$$

is open, and thus $\mathcal{Y}^{QS}/\hat{U}$ is quasi-projective.

Proof. The first statement follows directly from the $\hat{U}$-Theorem (Theorem 1.3.19) and the second statement follows from the fact that a geometric quotient is an open map (since it is a topological quotient) and that by Proposition 6.2.8 we have that $\mathcal{Y}^{\mathrm{QS}} \subset (\mathcal{Y}^{s,\hat{U}})$ is an open subset. □

Remark 6.2.15. Note that for every $\epsilon > 0$ characters of $\hat{U}$ such as the χ_ϵ appearing in Corollary 6.2.14 certainly exist; we can take $\chi_\epsilon = -\omega_{\min} - \frac{\epsilon}{2}$.

6.3 Stability and quasismooth weighted hypersurfaces

6.3.1 Coarse moduli spaces of quasismooth hypersurfaces in $\mathbb{P}(1,\ldots,r)$

We provide an explicit constructions of a coarse moduli spaces as projective over affine varieties of quasismooth hypersurfaces in the case where $X = \mathbb{P}(1,\ldots,1,r) = \operatorname{Proj} k[x_1,\ldots,x_n,y]$ and $d = d' \cdot r$ with $d' > 0$ and $n > 1$.

We give a direct construction of these coarse moduli space using Lemma 1.3.9.

Theorem 6.3.1. *Let $X = \mathbb{P}(1,\dots,1,r)$ and $d = d'\cdot r$ be a Cartier degree such that $d \geq r+2$. Let $\mathcal{Y} = \mathbb{P}(k[x_1,\dots,x_n,y]_d)$ be the parameter space of degree d hypersurfaces. Then there exists a geometric quotient for the G-action on $\mathcal{Y}^{QS}$*

$$\mathcal{Y}^{QS} \longrightarrow \mathcal{Y}^{QS}/G$$

which is coarse moduli space and a projective over affine variety.

Proof. Let $c \in H^0(\mathcal{Y}, \mathcal{O}(1)) = (k[x_1,\dots,x_n,y]_d)^\vee$ be the section corresponding to the coefficient of the monomial $y^{d'}$. By Remark 5.2.13, we have that $\mathcal{Y}^{\mathrm{QS}} = \mathcal{Y}_{c\cdot\Delta_A}$ and hence $\mathcal{Y}^{\mathrm{QS}}$ is an affine variety. Note that we have the inclusion $\mathcal{Y}^{\mathrm{QS}} \subset \mathcal{Y}_c$. In this case, $Z_{\min} = \{(0:\dots:0:c) \mid c \neq 0\}$ is a point, and so by Remark 1.3.20 the quotient

$$q_U : \mathcal{Y}^0_{\min} \longrightarrow \mathcal{Y}^0_{\min}/U$$

from Corollary 6.2.14 is a trivial U-bundle. Hence $\mathcal{Y}^0_{\min}/U$ is affine by [AD09, Theorem 3.14]. Thus $\mathcal{Y}^{\mathrm{QS}} \to \mathcal{Y}^{\mathrm{QS}}/U$ is a trivial bundle and $Q = \mathcal{Y}^{\mathrm{QS}}/U$ is an affine variety.

Consider the action of $R = G/U$ on Q. Since $\mathcal{Y}^{\mathrm{QS}}$ is affine, Lemma 1.3.9 implies that Q admits a geometric quotient by R if and only if all the G-orbits are closed in $\mathcal{Y}^{\mathrm{QS}}$. Then as $d \geq r+2$, Theorem 3.3.7 implies that all stabiliser groups are finite giving that the action on $\mathcal{Y}^{\mathrm{QS}}$ is closed. Hence we have a geometric quotient

$$\mathcal{Y}^{\mathrm{QS}}/G = Q/R.$$

Since Q is an affine variety and Q/R is a reductive quotient, we conclude that $\mathcal{Y}^{\mathrm{QS}}/G$ is a projective over affine variety. □

Example 6.3.2. Suppose that $X = \mathbb{P}(1,1,2)$ and $d = 6$. Then quasismooth hypersurfaces are exactly Petri special curves of genus 4 in X. Thus $(\mathcal{Y}_6)^{\mathrm{QS}}/G$ is an projective over affine coarse moduli space of Petri special curves. This moduli space is a divisor on the moduli space of genus 4 curves (see [Tom05]).

Example 6.3.3. Let $X = \mathbb{P}(1,1,1,2)$ and consider $d = 4$. Then quasismooth hypersurfaces are exactly degree 2 del Pezzo surfaces. Hence $(\mathcal{Y}_4)^{\mathrm{QS}}/G$ is a projective over affine coarse moduli space of degree 2 del Pezzo surfaces.

6.3.2 The Newton polytope and stability

We present a proof that quasismooth hypersurfaces are stable using the non-reductive Hilbert-Mumford criteria of Theorem 6.1.7. The Hilbert-Mumford criterion says that if G is a linear algebraic group with graded unipotent radical acting on a projective variety Y, then a point $y \in Y$ is stable if and only if every G translate $g \cdot y$ is stable for a maximal torus $T \subset G$ containing the grading $\mathbb{G}_m$. We shall prove stability of quasismooth hypersurfaces for a maximal torus T and then use the fact that the quasismooth locus is invariant under the action of the automorphism group and the NRGIT Hilbert-Mumford to deduce stability for G. The proof of T-stability uses the Newton polytope of a hypersurface, which we define as the weight polytope for the canonical maximal torus.

Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space such that $a_i \leq a_{i+1}$ and d be a Cartier degree. Suppose that $T \subset G = \mathrm{Aut}_{\mathrm{g}}(S)$ is the maximal torus of G given by diagonal matrices and define $\overline{T} = T/\lambda_{\underline{a}}(\mathbb{G}_m)$ to be the quotient by the 1-parameter subgroup $\lambda_{\underline{a}}$. Recall from Section 6.1.2 that the stability of a hypersurface with respect to T is determined by its weight polytope considered inside $H = X^*\left(\overline{T}\right) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 6.3.4. For a degree d hypersurface $Y \subset X$, we define the *Newton polytope of* Y by

$$\mathrm{NP}(Y) = \mathrm{Conv}(\mathrm{wt}_{\overline{T}}(Y)) \subset H = X^*\left(\overline{T}\right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Note that $\mathrm{NP}(Y)$ is a subpolytope of the section polytope of $\mathcal{O}(d)$ (which is a weighted simplex).

We begin by giving an example which illustrates the idea behind the proof of the general case.

Example 6.3.5. Let $X = \mathbb{P}(1,1,2) = \mathrm{Proj}\, k[x_1, x_2, y]$ and $d = 4$ and fix $N > 1$. Then the monomial basis of $k[x_1, x_2, y]_4$ is given by

$$\begin{array}{ccccc} x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \\ & x_1^2y & x_1x_2y & x_2^2y & \\ & & y^2. & & \end{array}$$

Recall from Example 6.1.8 and Figure 6.1 the section polytope corresponding to the diagonal maximal torus in H. To apply Theorem 1.3.24, we must twist our linearisation by a rational character such that it is an ϵ-linearisation. This corresponds to shifting the section polytope such that the origin is at most ϵ distance from the face corresponding to $Z_{\min}$, which is in this example the vertex corresponding to y^2; see Figure 6.3. We denote this face by $F_{\min}$. Note that in the shifted section polytope in Figure 6.3, we have also indicated the positions of the other monomials.

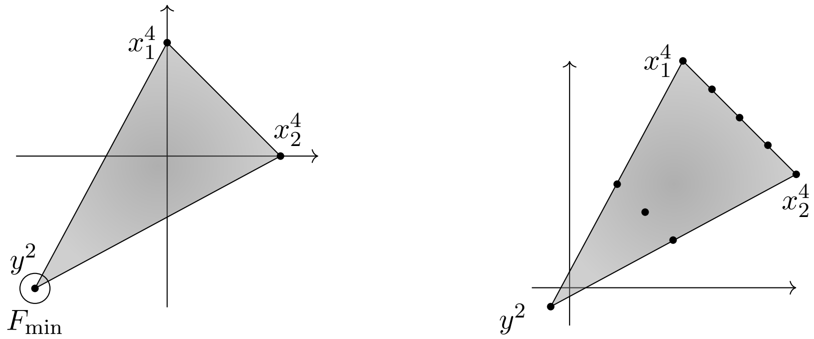


Figure 6.3: Shifting the section polytope by a character.

To check T-stability of a weighted hypersurface, we check if the origin lies inside the shifted Newton polytope. Note that the $\hat{U}$-stability condition forces the y^2 weight to appear (since its coefficient must be non-zero). Thus any Newton polytope of a $\hat{U}$-stable hypersurface will contain the vertex corresponding to y^2. It is clear from Figure 6.4 that the origin lies inside a Newton polytope if and only if it contains points on both sides of the red dividing line. Note that having points on the dividing line does not suffice.

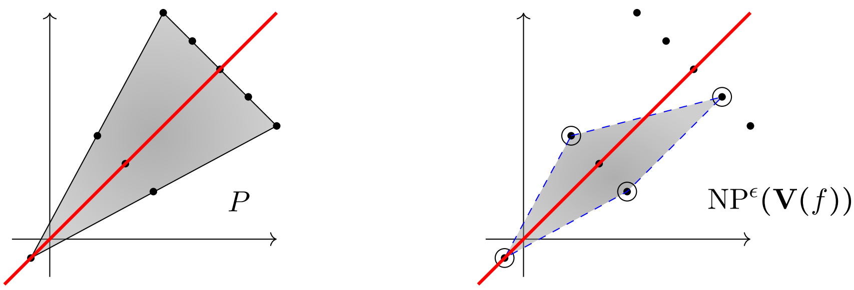


Figure 6.4: On the left is the section polytope P of $\mathcal{O}(4)$. On the right is the shifted Newton polytope of $\mathbf{V}(f)$ where $f = xy^3 + x^2z + y^2z + z^2$.

A quick calculation shows that if a curve in X is quasismooth, then the corresponding Newton polytope must contain monomials from either side of the dividing line (see also

Theorem 6.3.12 for a proof in a more general setting). Hence every quasismooth curve is stable for the maximal torus T: that is, we have the inclusion $\mathcal{Y}^{\mathrm{QS}} \subset \mathcal{Y}^{\mathrm{s},T}$. However, since $\mathcal{Y}^{\mathrm{QS}}$ is G-invariant, we have that $\mathcal{Y}^{\mathrm{QS}} \subset g \cdot \mathcal{Y}^{\mathrm{s},T}$ for every $g \in G$ and hence

$$\mathcal{Y}^{\mathrm{QS}} \subset \bigcap_{g \in G} g\mathcal{Y}^{\mathrm{s},T} = \mathcal{Y}^{\mathrm{s},G}$$

by Theorem 6.1.7. This proves that every quasismooth curve is stable and that there exists a geometric quotient of the quasismooth locus. As remarked in Example 5.2.12, curves in such weighted projective spaces are quasismooth if and only if they are smooth. Combining this with Example 2.6.12, we conclude that $\mathcal{Y}^{\mathrm{QS}}/G$ is a moduli space of elliptic curves.

We now prove in the general case that the quasismooth locus lies inside the stable locus of a torus with respect to an ϵ-linearisation. We shall need the following two lemmas.

Lemma 6.3.6. *Suppose that $Y \subset X = \mathbb{P}(a_0, \ldots, a_n)$ is a quasismooth hypersurface of degree $d >> \max\{a_0, \ldots a_n\} + 2$ defined by a weighted polynomial $f \in k[x_0, \ldots, x_n]_d$. Then $\mathrm{NP}(Y)^\circ \subset H$ contains the origin.*

Proof. Let $P \subset H$ be the section polytope of $\mathcal{O}_X(d)$. Recall that H is the quotient space of $W \cong \mathbb{R}^{n+1}$ with coordinates $e_0, \ldots, e_n$ by the vector $a_0e_0 + \cdots + a_ne_n$, we denote the quotient map by $Q : W \to H$. Then P is a weighted simplex as in Definition 2.2.19 and contains the origin. Indeed, one can check that

$$\frac{d}{a_0^2 + \cdots + a_n^2} Q(\sum_{i=0}^{n} a_ie_i) \in P.$$

By Lemma 2.6.14, since f is quasismooth, the Newton polytope $\mathrm{NP}(Y)$ contains at least one neighbour of each vertex or the vertex itself. That is, for every $i = 0, \ldots n$, either $x_i^{d_i} \in \mathrm{NP}(Y)$ or there exists a $j \neq i$ such that $x_i^{d_i - \frac{a_j}{a_i}} x_j \in \mathrm{NP}(Y)$.

Suppose that $\mathrm{NP}(Y)^\circ$ does not contain the origin. Then $\mathrm{NP}(Y)$ is contained in a closed half-space of H defined by a hyperplane passing through the origin. Thus all the monomials of f lie in this half space.

Then since $d \geq \max(a_0, \ldots, a_n) + 2$, there will exist a vertex which has the greatest distance from this half space. For $d >> \max\{a_0, \ldots, a_n\} + 2$, this vertex and its neighbours will lie in the complementary open half-space and thus outside of the Newton polytope. This is a contradiction.

□

Lemma 6.3.7. *Suppose that $Y \subset X$ is a quasismooth hypersurface defined by weighted polynomial f. Let $F_{\min} \subset P$ be the face corresponding to $Z_{\min}$. Then $F_{\min} \cap \mathrm{NP}(Y)$ contains the barycentre of the face $F_{\min}$.*

Proof. Recall that $X = \operatorname{Proj} k[x_1, \dots, x_{n'}, y_1, \dots, y_{n_l}]$ such that $\deg y_i = a_n$. Let $\tilde{f} \in k[y_1, \dots, y_{n_l}]_d$ be the part of f containing only monomials in $Z_{\min}$ that is $\tilde{f} = f(0, \dots, 0, y_1, \dots, y_{n_l})$. Then $F_{\min} \cap \mathrm{NP}(Y) \simeq \mathrm{NP}(\mathbf{V}(\tilde{f}))$ as abstract polytopes, however, they are embedded in different spaces. We want to show that $\mathrm{NP}(\mathbf{V}(\tilde{f})) \subset \mathbb{R}^{n_l-1}$ contains the origin.

Note that $\tilde{f}$ is a homogeneous polynomial of degree $d' = \frac{d}{a_n}$ in $y_1, \dots, y_{n_l}$. Thus if $\tilde{f}$ is smooth, by the previous lemma, the origin is contained inside $\mathrm{NP}(\mathbf{V}(\tilde{f}))$. Let us prove that $\tilde{f}$ is smooth. Suppose that $\tilde{f}$ has a singular point $\tilde{p} = (y_1 : \dots : y_{n_l}) \in \mathbb{P}^{n_l-1}$; then we claim that $p = (0 : \dots : 0 : y_1 : \dots : y_{n_l}) \in X$ is a non-quasismooth point of Y. Indeed, the partial differentials satisfy $\frac{\partial f}{\partial y_j}(p) = 0$ for $1 \le j \le n_l$ since $\frac{\partial \tilde{f}}{\partial y_j}(\tilde{p}) = 0$. To see that $\frac{\partial f}{\partial x_i}(p) = 0$ for $1 \le i \le n'$, note that there can exist no monomial of f of the form $\underline{y}^I x_i$ where $\underline{y}^I \in k[y_1, \dots, y_{n_l}]_{d-a_i}$ because $\deg(\underline{y}^I x_i) = \deg \underline{y}^I + a_i \le d - a_n + a_i < d$. This implies that $\frac{\partial f}{\partial x_i}(p) = 0$. Hence if $\tilde{f}$ has a singular point, then so must f. Hence $\mathbf{V}(\tilde{f}) \subset \mathbb{P}^{n_l-1}$ is smooth and $F_{\min} \cap \mathrm{NP}(Y)$ contains the barycentre of the face $F_{\min}$. □

Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space such that $n > 1$ which satisfies the condition $(\mathfrak{C}^*)$ and let $d \ge \max\{a_0, \dots, a_n\} + 2$ be a Cartier degree. Let $G = \mathrm{Aut}_{\mathrm{g}}(S)$ be the graded automorphism group of the Cox ring and consider the action of G on $\mathcal{Y} = \mathcal{Y}_d$. Define the graded unipotent radical $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \ltimes U \subset G$ for some fixed $N > d$. Recall from Section 6.1.1 that $X^*(G) \simeq \mathbb{Z}^l$.

Definition 6.3.8. Let $\epsilon \in \mathbb{Q}$ such that $0 < \epsilon \le 2d'$. We define

$$\chi_\epsilon = (0, \dots, 0, m_l^\epsilon) \in X^*(G) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $m_l^\epsilon = \frac{2d'-\epsilon}{2n_l} \in \mathbb{Q}$. Now consider the twisted linearisation $\mathcal{O}(1)^{\chi_\epsilon}$. Let $V = H^0(\mathcal{Y}, \mathcal{O}(1))$ and $V_{\min}$ the minimal weight space of the $\lambda_{g,N}(\mathbb{G}_m)$-action. For an arbitrary monomial

$\underline{y}^I \in V_{\min}$ we have

$$\begin{aligned}\lambda_{g,N}(t)\cdot\underline{y}^I &= \chi_\epsilon\left(\lambda_{g,N}(t)\right)t^{-d'}\underline{y}^I\\ &= t^{-m_l^\epsilon n_l - d'}\underline{y}^I\\ &= t^{-\frac{\epsilon}{2}}\underline{y}^I.\end{aligned}$$

Hence, $\mathcal{O}(1)^{\chi_\epsilon}$ is an ϵ-linearisation.

To compute the Newton polytope after twisting by a character, one adds the weight corresponding to the character. To compute the corresponding weight to the character χ_ϵ one considers the map $X^*(G) \to X^*(T)$ induced by the restriction of characters to the torus. Thus we see that χ_ϵ corresponds to $(0,\dots,0,m_l^\epsilon,\dots,m_l^\epsilon) \in X^*(\overline{T})\otimes\mathbb{Q}$ where we have a m_l^ϵ for every variable of maximal weight. Thus twisting by this character shifts the Newton polytope along the line defined by this vector. This line is precisely the line connecting the origin and the barycentre of the face $F_{\min}$. See Figure 6.3 for an example. Thus we have proved the following lemma.

Lemma 6.3.9. *Keep the above notation. Twisting the linearisation by the character χ_ϵ has the effect of shifting the section polytope and all Newton polytopes along the line $\overline{OQ}$, where Q is the barycentre of the face $F_{\min}$.*

Definition 6.3.10. Let $Y \subset X$ be a hypersurface. Define the *ϵ-shifted Newton polytope* $\mathrm{NP}^\epsilon(Y)$ to be the convex hull of the weights with respect to the rational linearisation $\mathcal{O}(1)^{\chi_\epsilon}$.

Figure 6.3 gives the example of $\mathrm{NP}(Y)$ and $\mathrm{NP}^\epsilon(Y)$, where $Y = \mathbf{V}(x_1^4+x_2^4+y^2)\subset\mathbb{P}(1,1,2)$.

Remark 6.3.11. Note that twisting the linearisation by χ_ϵ shifts the section polytope P along the line $\overline{OQ}$ by Lemma 6.3.9. In particular, the point Q moves closer to the origin.

We are now in a position to prove the main theorem of the section.

Theorem 6.3.12. *Let $X = \mathbb{P}(a_0,\dots,a_n)$ be a well-formed weighted projective space such that $n > 1$ which satisfies the condition $(\mathfrak{C}^*)$ and let $d >> \max\{a_0,\dots,a_n\}+2$ be a Cartier degree much greater than $\max\{a_0,\dots,a_n\}$. Let $G = \mathrm{Aut}_{\mathrm{g}}(S)$ be the graded automorphism* *group of the Cox ring and consider the action of G on $\mathcal{Y} = \mathcal{Y}_d$. Define the graded unipotent radical $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \ltimes U \subset G$ for some fixed $N > d$. Fix $\epsilon > 0$ and let χ_ϵ be as in Definition 6.3.8. Then for every $\epsilon \in \mathbb{Q}$ such that $0 < \epsilon \leq 2\frac{d}{a_n}$, we have the inclusion*

$$\mathcal{Y}^{QS} \subset \mathcal{Y}^{s,G}(\mathcal{O}(1)^{\chi_\epsilon}).$$

In particular, there exists a geometric quotient $\mathcal{Y}^{QS}/G$ and hence a coarse moduli space of quasismooth hypersurfaces as a quasi-projective variety. Moreover, $\mathcal{Y} /\!\!/_{\mathcal{O}(1)^{\chi_\epsilon}} G$ is a compactification of $\mathcal{Y}^{QS}/G$.

Proof. Note that it suffices to prove that $\mathcal{Y}^{\mathrm{QS}} \subset \mathcal{Y}^{\mathrm{s},T}(\mathcal{O}(1)^{\chi_\epsilon})$, since by the non-reductive Hilbert-Mumford criterion of Theorem 6.1.7 we get $\mathcal{Y}^{\mathrm{QS}} \subset \mathcal{Y}^{\mathrm{s},G}(\mathcal{O}(1)^{\chi_\epsilon})$. Indeed, as $\mathcal{Y}^{\mathrm{QS}}$ is G-invariant, we have that $\mathcal{Y}^{\mathrm{QS}} \subset g \cdot \mathcal{Y}^{\mathrm{s},T}(\mathcal{O}(1)^{\chi_\epsilon})$ and hence

$$\mathcal{Y}^{\mathrm{QS}} \subset \bigcap_{g \in G} g \cdot \mathcal{Y}^{\mathrm{s},T}(\mathcal{O}(1)^{\chi_\epsilon}) = \mathcal{Y}^{\mathrm{s},G}(\mathcal{O}(1)^{\chi_\epsilon}),$$

by Theorem 6.1.7.

Let us prove that $\mathcal{Y}^{\mathrm{QS}} \subset \mathcal{Y}^{\mathrm{s},T}(\mathcal{O}(1)^{\chi_\epsilon})$. Suppose that $Y \subset X$ is a quasismooth hypersurface of degree d. Then by Lemma 6.3.6 and Lemma 6.3.7, the the polytope $\mathrm{NP}(Y)$ contains the centre point Q of $F_{\min}$ (where $F_{\min}$ is the face corresponding to $Z_{\min}$ of the section polytope) and the interior contains the origin, that is $O \in \mathrm{NP}(Y)^\circ$.

Consider the line $L = \overline{OQ} - Q$. Since $\mathrm{NP}(Y)^\circ$ is convex, L is contained in $\mathrm{NP}(Y)^\circ$. Twisting the linearisation by χ_ϵ shifts $NP(Y)$ such that Q is shifted towards O along the line L (see Remark 6.3.11). Hence $\mathrm{NP}^\epsilon(Y)^\circ$ will contain the origin of H for all $\epsilon > 0$ such that $m_l^\epsilon > 0$, that is $\epsilon < 2d'$.

We can choose $1 >> \epsilon > 0$ such that the linearisation will be well adapted. Thus by the Hilbert-Mumford criterion of Theorem 6.1.6 quasismooth hypersurfaces are T-stable for the twisted linearisation $\mathcal{O}(1)^{\chi_\epsilon}$. □

Remark 6.3.13. In the case where the condition $(\mathfrak{C}^*)$ is not satisfied, there is a blow-up procedure outlined in [BDHK16] where one performs a sequence of blow-ups of the locus in $\mathcal{Y}$ where there is a positive dimensional U-stabiliser. Using this procedure it is expected that we can remove the requirement that the $(\mathfrak{C}^*)$ condition holds.

Remark 6.3.14. The reason for defining the grading 1-parameter subgroup $\lambda_{g,N}$ with respect to a variable N, is that for N sufficiently large, we have that $\mathcal{O}(1)$ is an ϵ-linearisation, so we do not need to twist by a rational character; see Figure 6.2. This perspective is taken in [Bun19].

6.4 Stability in the case where G is reductive

6.4.1 Products of projective space

Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and $(d,e) \in \mathbb{Z}$. Then $\mathrm{Aut}_{\mathrm{g}}(S) = \mathrm{GL}_{n+1} \times \mathrm{GL}_{m+1}$ and we consider the action of G on the projective space $\mathcal{Y} = \mathbb{P}(k[x_0, \dots, x_n; y_0, \dots, y_m]_{(d,e)})$. Note that every stabiliser contains the subgroup

$$\begin{aligned} \lambda : \mathbb{G}_m^2 &\longrightarrow \mathrm{Aut}_{\mathrm{g}}(S) \\ (t_1, t_2) &\longmapsto (t_1 I_{n+1}, t_2 I_{m+1}). \end{aligned}$$

We may replace the action of $\mathrm{Aut}_{\mathrm{g}}(S)$ with the action of the subgroup $G = \mathrm{SL}_{n+1} \times \mathrm{SL}_{m+1} \subset \mathrm{Aut}_{\mathrm{g}}(S)$, as the orbits are the same. By doing this, we remove the global stabiliser and moreover, we are now in the situation where G has no non-trivial characters and so Δ_A is a true invariant for the G-action.

Proposition 6.4.1. *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ be a product of projective spaces and $Y \subset X$ be a smooth hypersurface of degree $(d,e) \in \mathbb{Z}^2$. If $d > n+1$ and $e > m+1$, then* $\dim \mathrm{Stab}_G(Y) = 0$.

Proof. Since Y is a proper algebraic scheme, by [MO67, Lemma 3.4], we can identify the Lie algebra of the automorphism group of Y with the vector space $H^0(Y, \mathcal{T}_Y)$ of global vector fields on Y, where $\mathcal{T}_Y$ is the tangent sheaf. If we show that $H^0(Y, \mathcal{T}_Y) = 0$, then we can conclude that $\dim \mathrm{Aut}(Y) = 0$ and since $\mathrm{Stab}_G(Y) \subset \mathrm{Aut}(Y)$ we have that $\dim \mathrm{Stab}_G(Y) = 0$.

Let us prove that $H^0(Y, \mathcal{T}_Y) = 0$. Let $N = \dim Y = n + m - 1$; then by Serre duality $H^0(Y, \mathcal{T}_Y) \simeq H^N(Y, \Omega_Y \otimes \omega_Y)^\vee$, where ω_Y is the canonical line bundle of Y. Let $\mathcal{O}_Y(1,1)$ be the restriction of $\mathcal{O}_X(1,1) = \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}^m}(1)$ to Y. By the adjunction formula we have that $\omega_Y \cong \mathcal{O}_Y(d-n-1, e-m-1)$ and hence by our assumption we have that ω_Y is very ample. Then by Kodiara-Nakano vanishing (see for example [EV92, Theorem 1.3]) we have that $H^N(Y, \Omega_Y(d-n-1, e-m-1)) = 0$. Hence

$$H^0(Y, \mathcal{T}_Y) \simeq H^N(Y, \Omega_Y(d-n-1, e-m-1))^\vee = 0.$$

We conclude that $\dim \operatorname{Aut}(Y) = 0$. □

Remark 6.4.2. Note that $\operatorname{Aut}(Y)$ may not be a linear algebraic group, and in general is only locally linear algebraic.

Theorem 6.4.3. *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ be a product of projective spaces and $Y \subset X$ be a smooth hypersurface of degree $(d, e) \in \mathbb{Z}^2$. Consider the action of $G = \mathrm{SL}_{n+1} \times \mathrm{SL}_{m+1}$ on $\mathcal{Y} = \mathbb{P}(k[x_0, \dots, x_n; y_0, \dots, y_m]_{(d,e)})$ with linearisation given by $\mathcal{O}_{\mathcal{Y}}(1)$. Then we have the open inclusion*

$$\mathcal{Y}^{SM} \subset \mathcal{Y}^{ss}(\mathcal{O}(1)),$$

where $\mathcal{Y}^{SM}$ is the of smooth hypersurfaces. If $d > n+1$ and $e > m+1$ then we have the open inclusion

$$\mathcal{Y}^{SM} \subset \mathcal{Y}^{s}(\mathcal{O}(1)).$$

In particular, there exists a coarse moduli space of smooth hypersurfaces of degree (d, e).

Proof. First, note that the discriminant Δ_A is a true invariant for the G-action since G has no non-trivial characters. Then by Theorem 5.2.6, we have that

$$\mathcal{Y}^{\mathrm{SM}} = \mathcal{Y}_{\Delta_A},$$

since G acts transitively on X and hence $\mathcal{Y}^{\mathrm{SM}} \subset \mathcal{Y}^{\mathrm{ss}}$. Finally, if $d > n+1$ and $e > m+1$, by Proposition 6.4.1, we have that the stabiliser for every point in $\mathcal{Y}^{\mathrm{SM}}$ is finite and hence all the orbits are closed. It follows from the definition of the stable locus (Definition 1.2.11) that $\mathcal{Y}^{\mathrm{SM}} \subset \mathcal{Y}^{\mathrm{s}}$. □

Remark 6.4.4. Note that if $d \leq n+1$ or $e \leq m+1$ it should be possible to prove that $\mathcal{Y}^{\mathrm{SM}} \subset \mathcal{Y}^{\mathrm{s}}$ using a Newton polytope argument.

Example 6.4.5. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and fix $(d, e) \in \operatorname{Pic}(Q)$. Let $\mathcal{Y} = \mathcal{Y}_{(d,e)} = \mathbb{P}(k[x, y; u, v]_{(d,e)})$ and consider the action defined above. Recall that in this example $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ and, as in Theorem 6.4.3, since the action of the group G is transitive, it follows from the proof of Theorem 5.2.6 that $\mathcal{Y}^{\mathrm{SM}} = \mathcal{Y}_{\Delta_A}$.

Recall from Example 4.2.3 that the genus of such curves is $(d-1)(e-1)$. Thus for pairs $(d,e) \neq (2,2)$ where at least one of d or e is greater than 2, the automorphism groups of the smooth curves in $\mathcal{Y}_{\Delta_A}$ are finite by a famous result of Hurwitz [Hur92]. Since the stabiliser group of such a smooth curve is a subgroup of the automorphism group, it follows that the stabiliser groups must also be finite. This proves that

$$\mathcal{Y}^{\mathrm{SM}} \subseteq \mathcal{Y}^{\mathrm{s},G}$$

and hence there exists a geometric quotient and a compactification given by the reductive GIT quotient

$$\mathcal{Y}^{\mathrm{SM}}/G \subset \mathcal{Y} /\!\!/ G.$$

Chapter 7

Weighted projective lines

7.1 $\mathbb{G}_a$-actions and locally nilpotent derivations

Let $X = \operatorname{Spec} k[x_1, \dots x_n]$ be the n-dimensional affine space. There is a one-to-one correspondence between $\mathbb{G}_a$-actions on X and locally nilpotent derivations on $A = k[x_1, \dots, x_n]$. Under this correspondence the ring of invariants $A^{\mathbb{G}_a}$ is equal to the kernel of the corresponding derivation.

Definition 7.1.1. A derivation $D : A \to A$ is *locally nilpotent* if for every $f \in A$ there exists an integer $n_f \geq 0$ such that $D^{n_f}(f) = 0$.

Let the action be given by $\sigma : \mathbb{G}_a \times X \to X$ and denote by $\sigma^* : A \to A \otimes_k k[T] = A[T]$. We define a map $D : A \to A$ associated to σ as follows:

$$D_\sigma = \left(\frac{\sigma^*(f) - f}{T} \right)_{|T=0}.$$

Proposition 7.1.2. [VdE12, Proposition 9.5.2] *Let $X = \operatorname{Spec} A$ be an affine space and $\sigma : \mathbb{G}_a \times X \to X$ be a $\mathbb{G}_a$-action on X, where $A = k[x_1 \dots, x_n]$. Then D_σ is a locally nilpotent derivation on A. Moreover, there is an equality*

$$A^{\mathbb{G}_a} = \ker(D)$$

and the fix points are given by the vanishing locus of the ideal $\langle D(x_1), \dots, D(x_n) \rangle$.

The kernel of a locally nilpotent derivation may not be finitely generated. If the kernel is known to be finitely generated Van den Essen provied an algorithm to compute the ring of invariants in [vdE93]. Thus by Weitzenböck's theorem (Theorem 1.3.2), we can apply Van den Essen's algorithm to compute the invariants of linear $\mathbb{G}_a$-actions. Pfister and Greuel implemented this algorithm in the computer algebra system Singular. We use their package "ainvar.lib" to compute invariants for some specific examples.

Example 7.1.3. Let $X = \operatorname{Spec} k[x,y,z,w]$ and recall the action defined in Example 1.3.3:

$$\sigma(t,(x,y,z,w)) = (x, y+tx, z, w+tz).$$

The coaction $\sigma^* : k[x,y,z,w] \to k[x,y,z,w][T]$ is given by

$$\sigma^*(f(x,y,z,w)) = f(x, y+Tx, z, w+Tz).$$

Every derivation has the following form

$$D = p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} + p_w \frac{\partial}{\partial w},$$

for polynomials $p_x, p_y, p_z, p_w \in k[x,y,z,w]$ and so to compute each polynomial appearing in this expression, we consider where D sends each variable. Let us apply this to D_σ. For example,

$$D_\sigma(y) = \left(\frac{\sigma^*(y) - y}{T}\right)_{|T=0} = \left(\frac{y + Tx - y}{T}\right)_{|T=0} = x,$$

and hence $p_y = x$. Similarly, $D_\sigma(w) = z$ and $D_\sigma(x) = \mathbf{D}_\sigma(z) = 0$. Hence we have computed

$$D_\sigma = x\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}.$$

Using the Singular program we find that

$$\ker D_\sigma = k[x,y,z,w]^{\mathbb{G}_a} = k[x,z,xw-yz].$$

Note that, as shown in Example 1.3.3, the morphism of schemes associated to the inclusion $k[x,y,z,w]^{\mathbb{G}_a} \subset k[x,y,z,w]$ is not surjective

7.2 Weighted projective lines

In this section we consider the hypersurfaces of the quotient stack $\mathbb{P}(a,b) = [\mathbb{A}^2 - \{0\} / \mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^2 - \{0\}$ as follows: $t \cdot (x,y) = (t^a x, t^b y)$. The stack $\mathbb{P}(a,b)$ is a smooth toric Deligne-Mumford (DM) stack [FMN10]. One can define the notion of the Cox ring for such stacks which play an analogous role as in the case of toric varieties. We refer the reader to [HM15] for the definition and discussion of the Cox ring of a toric stack. For the stack $\mathbb{P}(a,b)$, the Cox ring is $S = k[x,y]$ where $\deg x = a$ and $\deg y = b$ and we shall consider the action of the automorphism group of the stack on graded pieces of S.

Let us assume that $\mathrm{hcf}(a,b) = 1$, otherwise there will exist a generic stabiliser. The coarse moduli space of $\mathbb{P}(a,b)$ is given by $\mathbb{P}^1$ (even if there exists a generic stabiliser) and one can think of $\mathbb{P}(a,b)$ as $\mathbb{P}^1$ together with two distinct marked points one with weight a and the other with weight b. If $\mathrm{hcf}(a,b) = 1$, this procedure happens via the root stack construction; see [AGV08, Appendix B]. Furthermore, the automorphism group of the stack $\mathbb{P}(a,b)$ is the subgroup of the automorphism group of $\mathbb{P}^1$ which fixes the aforementioned marked points.

Lemma 7.2.1. *Let $\mathbb{P}(a,b)$ be a weighted projective line. If $a,b > 1$ are such that $\mathrm{hcf}(a,b) = 1$ then*

$$\mathrm{Aut}(\mathbb{P}(a,b)) = \mathbb{G}_m.$$

If a and b are distinct such that $a \mid b$ then

$$\mathrm{Aut}(\mathbb{P}(a,b)) = \mathbb{G}_m \ltimes \mathbb{G}_a.$$

Finally, if $a = b$ then

$$\mathrm{Aut}(\mathbb{P}(a,b)) = \mathrm{PGL}_2.$$

The proof of the lemma is the same as for weighted projective spaces of higher dimension: one computes the graded automorphisms of the Cox ring. For the construction of smooth toric DM stacks in [FMN10] and [HM15] for their Cox rings.

Let $X = \mathbb{P}(1,r)$ and fix an integer d such that $d = rd'$ with $d' > 0$. Let $\mathcal{Y} = \mathbb{P}(k[x,y]_d) \cong \mathbb{P}^{d'}$ with the following coordinates:

$$[a_0 x^d + a_1 x^{d-2} y + \cdots + a_{d'-1} x^2 y^{d'-1} + a_{d'} y^{d'}] = (a_0 : a_1 : \cdots : a_{d'-1} : a_{d'}).$$

Consider the action of $\mathrm{Aut}(X) = \mathbb{G}_m \ltimes \mathbb{G}_a = \hat{\mathbb{G}}_a$ on $\mathcal{Y}$ defined by the action on $\mathbb{P}(1,r)$ given by

$$(t,a) \cdot (x : y) = (t^{-N} x : t(y + a x^r)), \tag{$\star$}$$

where $N > 0$. Note that the orbits of the action are the same for all $N > 0$. For N large enough, the linearisation will be well-adapted as in Remark 6.2.4. To apply the $\hat{U}$-theorem, note that the subgroup

$$\begin{aligned} \mathbb{G}_m &\longrightarrow \hat{\mathbb{G}}_a \\ t &\longmapsto (t, 0) \end{aligned}$$

gives $\mathbb{G}_a$ a positive grading and that the $(\mathfrak{C}^*)$ condition is satisfied. Indeed,

$$Z_{\min} = \left\{[y^{d'}]\right\} \subset Y$$

and has trivial stabiliser, we deduce that $Y^0_{\min} \cong \mathbb{A}^{d'}$. Moreover,

$$\mathbb{G}_a \cdot Z_{\min} = \left\{[(y + ax^r)^{d'}] \mid a \in k\right\} \cong \mathbb{A}^1,$$

and hence

$$Y^0_{\min} - \mathbb{G}_a \cdot Z_{\min} \cong \mathbb{A}^{d'} - \mathbb{A}^1.$$

Note that the line we have removed here is a twisted $\mathbb{A}^1$.

Remark 7.2.2. The $\hat{U}$-theorem implies that

$$q_U : Y^0_{\min} \longrightarrow Y^0_{\min}/\mathbb{G}_a$$

is a $\mathbb{G}_a$-bundle, and moreover, by Remark 1.3.20, q_U is a trivial $\mathbb{G}_a$-bundle; that is,

$$(Y^0_{\min} \,/\, \mathbb{G}_a) \times \mathbb{G}_a \simeq Y^0_{\min}.$$

Theorem 7.2.3. *Let $X = \mathbb{P}(1, r)$ and $d = d' \cdot r$ and consider the action of $\hat{\mathbb{G}}_a$ on $\mathcal{Y} = \mathbb{P}(k[x, y]_d)$ as defined in $(\star)$ for $N >> 0$. Then semistability coincides with stability and $[F] \in \mathcal{Y}$ is stable if and only if $(0 : 1) \notin \boldsymbol{V}(F) \subset \mathbb{P}(1, r)$ and all points of $\boldsymbol{V}(F)$ do not coincide.*

Proof. The fact that semistability coincides with stability follows from the $\hat{U}$-theorem, which we can apply as the linearisation is well-adapted since N is large enough. The point $(0:1) \in \mathbf{V}(F)$ if and only if $a_{d'} = 0$ and since $Y^0_{\min} = Y_{a_{d'}}$ we have that no stable hypersurface can contain the stacky point $(0 : 1)$. The orbit of the point of $\mathbb{G}_a \cdot Z_{\min}$ is the subset of all hypersurfaces corresponding to equations of the form $(y + ax^r)^{d'}$. Thus $Y^0_{\min} - \mathbb{G}_a \cdot Z_{\min}$ consists of all hypersurfaces which consist of at least two distinct points and which do not contain the point $(0:1)$. $\square$

The above theorem has a rather nice application for actual computation, we shall take a different approach. We utilise the algorithm of Van den Essen to actually compute invariants.

Example 7.2.4. Consider degree 6 hypersurfaces in the weighted projective line $\mathbb{P}(1,2)$. Geometrically this corresponds to 3 points on $\mathbb{P}(1,2)$. We compute the quotient of the $\mathbb{G}_m \ltimes \mathbb{G}_a$-action on $Y = \mathbb{P}(k[x,y]_6)$ for $N = 2$ in the notation of $(\star)$. Let $(a_0 : a_1 : a_2 : a_3) \in Y$ correspond to the polynomial $a_0x^6 + a_1x^4y + a_2x^2y^2 + a_3y^3$. Then the action of $\mathbb{G}_a$ is given by the following matrix

$$c \longmapsto \begin{pmatrix} 1 & c & c^2 & c^3 \\ 0 & 1 & 2c & 3c^2 \\ 0 & 0 & 1 & 3c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This corresponds to the derivation on $k[a_0, a_1, a_2, a_3]$ given by

$$D = a_1\frac{\partial}{\partial a_0} + 2a_2\frac{\partial}{\partial a_1} + 3a_3\frac{\partial}{\partial a_2}.$$

Using the Singular package "ainvar.lib", we compute the generators of the invariant ring of the $\mathbb{G}_a$-action to be

$$r_0 = a_3$$
$$r_1 = a_2^2 - 3a_1a_3$$
$$r_2 = 2a_2^3 - 9a_1a_2a_3 + 27a_0a_3^2$$
$$r_3 = a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 + 18a_0a_1a_2a_3 - 27a_0^2a_3^2.$$

Next we consider the $\mathbb{G}_m$-action. The coordinates $a_0, \dots, a_3$ are weight vectors for the $\mathbb{G}_m$-action and by direct calculation, one observes that the invariants $r_0, \dots, r_3$ are also weight vectors with weights displayed in the table displayed in Figure 7.1.

Since we picked the linearisation corresponding to the linearisation of the positive character of $\mathbb{G}_m$, the weight r_0 is not a semi-invariant. At this point, we compute the quotient

Element	a_0	a_1	a_2	a_3	r_0	r_1	r_2	r_3
$\mathbb{G}_m$-weight	12	7	2	−3	−3	4	6	18

Figure 7.1: $\mathbb{G}_m$-weights for the action on $\mathbb{P}(k[x,y]_6)$.

to be

$$\mathcal{Y} /\!\!/ \hat{\mathbb{G}}_a = \operatorname{Proj} k[r_1, r_2, r_3]/I,$$

where I is the ideal of relations and the $\mathbb{N}$-grading is given by Figure 7.1. However, I may be non-trivial, that is, there may be relations between the generators.

An alternate construction, where the geometry of the resulting quotient is a little more clear, is taking the quotient in stages. Consider the $\mathbb{G}_a$-action on $Y^0_{\min} = \mathbb{A}^3 = \operatorname{Spec} k[y_1, y_2, y_3]$ where $y_1 = \frac{a_0}{a_3}$, $y_2 = \frac{a_1}{a_3}$ and $y_3 = \frac{a_2}{a_3}$. One can compute either by hand or using singular that the invariant ring is generated by

$$\begin{aligned} r_1 &= y_3^2 - y_1 \\ r_2 &= 2y_3^3 - 9y_2y_3 + 27y_1. \end{aligned}$$

Hence

$$Y^0_{\min}/\mathbb{G}_a = \mathbb{A}^2 = \operatorname{Spec} k[r_1, r_2].$$

We then note that r_1 has weight 4 and r_2 has weight 6. Thus taking the residual $\mathbb{G}_m$-quotient produces a weighted projective line

$$Y /\!\!/ \hat{\mathbb{G}}_a = \mathbb{P}(4, 6).$$

Remark 7.2.5. One should compare this to the computation [Dol03, p.149], where Dolgachev computes the moduli space of four points on $\mathbb{P}^1$. In Dolgachev's notation, it is shown that

$$\operatorname{Hyp}_4(1) /\!\!/ \operatorname{SL}_2 = \mathbb{P}(2, 3),$$

and moreover, that $\operatorname{Hyp}_4(1) /\!\!/ \operatorname{SL}_2$ is the moduli space of elliptic curves, since an elliptic curve is determined by its double cover to $\mathbb{P}^1$ branched at four points. As remarked in Example 7.2.4, points of the space $Y /\!\!/ \hat{\mathbb{G}}_a$ represent 3 points on $\mathbb{P}(1, 2)$. However, given that the stability condition requires us to avoid the stacky point $(0 : 1)$ and that all automorphisms fix this point, one can consider $Y /\!\!/ \hat{\mathbb{G}}_a$ as representing 4 points in $\mathbb{P}^1$.

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