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On the continuum limit of the entanglement Hamiltonian

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We consider the entanglement Hamiltonian for an interval in a chain of free fermions in its ground state and show that the lattice expression goes over into the conformal one if one includes the hopping to distant neighbours in the continuum limit. For an infinite chain, this can be done analytically for arbitrary fillings and is shown to be the consequence of the particular structure of the entanglement Hamiltonian, while for finite rings or temperatures the result is based on numerical calculations.

I. INTRODUCTION

It is well known that the entanglement properties of a quantum state are contained in the reduced density matrix ρ of one of the two subsystems into which the total system is divided, see e.g. [1, 2]. Writing $\rho = \exp(-\mathcal{H})/Z$, the features of ρ are then encoded in the operator $\mathcal{H}$, and for this reason it has been termed the entanglement Hamiltonian.

For the most studied quantity, the entanglement entropy, one only needs its eigenvalues, the so-called entanglement spectrum [3], which also gives information on topological features of the system [3–5]. However, for a general understanding one also would like to know its explicit form and structure. This will depend strongly on the system and on the way it is partitioned. If a ladder is divided into its two legs, $\mathcal{H}$ can be proportional to the physical Hamiltonian of the leg, see e.g. [6–9]. If it is divided into two half-ladders, $\mathcal{H}$ corresponds to an inhomogeneous system, and results for integrable chains [2] or from field theory [10] indicate that the terms in $\mathcal{H}$ increase linearly as one moves away from the dividing point.

In the present work, we consider such inhomogeneous entanglement Hamiltonians for critical systems because they can be expected to show universal features. We do this in one dimension and for the case of finite subsystems. Then there are two kinds of results.

For *continuous* critical systems, conformal field theory gives the following general expression for an interval of length ℓ in a larger chain, see [11–16]

$$\mathcal{H} = 2\pi\ell \int_0^\ell dx\, \beta(x)\, T_{00}(x)\,. \tag{1}$$

Here $T_{00}(x)$ is the energy density in the physical Hamiltonian and $\beta(x)$ is a weight factor arising from the conformal mapping which relates the path integral for ρ in the actual geometry to that for a strip. Together with the prefactor, it can be viewed either as a local inverse temperature [11, 12, 17–21] or as a local velocity of the particles described by T_{00}. For an interval in an infinite chain, it is a parabola

$$\beta(x) = \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right) \tag{2}$$

and vanishes linearly at the ends. It shares this property with the simpler case of a half-infinite interval where $\beta(x)$ is completely linear [10, 22, 23].

For a homogeneous fermionic hopping model in its ground state, which is a *discrete* critical system, the structure of $\mathcal{H}$ is also known. Because the ground state is a simple Fermi sea, $\mathcal{H}$ must be a free-particle lattice Hamiltonian [24]

$$\mathcal{H} = \sum_{i,j=1}^{N} H_{i,j}\, c_i^{\dagger} c_j \tag{3}$$

with N denoting the number of sites in the subsystem. Numerical calculations for small intervals then show that the hopping in $\mathcal{H}$ is predominantly to nearest neighbours as in the physical Hamiltonian, but hopping to more distant neighbours also exists, although with small amplitudes [2, 25]. For a large interval in an infinite half-filled chain, analytical expressions for the hopping amplitudes have been found recently [25]. From these one sees that the nearest-neighbour hopping follows *almost* the parabolic law (2), deviating from it only slightly in the centre of the interval, while the longer-range hoppings vary in space roughly like powers of that parabola. The same general feature is found for finite chains [26]. Thus the discrete $\mathcal{H}$ differs from the conformal result even if the subsystem is large and the lattice structure should perhaps not matter. This is somewhat surprising and also intriguing, because an operator $\mathcal{T}$ exists which commutes with $\mathcal{H}$ and *does* have the (discretized) conformal form, see [25–29].

The way out of this discrepancy has already been indicated in [20], namely one should include the longer-range hopping when taking the continuum limit and thereby obtain an effective $\beta(x)$. Doing that numerically, the authors obtained a rather good approximate parabola. In this communication, we want to go a step further and show analytically that the conformal parabola results. The key ingredient is a relation which expresses the matrix H in (3) as a power series of the tridiagonal matrix T commuting with it. While this relation leads to relatively complicated matrix elements for H, it turns out in the end that for the continuum limit at half filling only the lowest power of T contributes and that this can be understood from the particular structure of T. For general filling, the mechanism is not quite as simple, but the result is the same.

In the following Section II we revisit the continuum limit for homogeneous and inhomogeneous hopping models and derive a general expression for the resulting $\mathcal{H}$ and the quantity $\beta(x)$. In Section III, we evaluate $\beta(x)$ for the entanglement Hamiltonian one encounters in an infinite half-filled chain and give an interpretation of the mechanism. We also present numerical results for finite chains and finite temperatures. In Section IV, we consider chains with arbitrary filling and show that the same $\beta(x)$ results. Section V contains our conclusions and three appendices give details on the commuting operator, the summations needed for general filling and on higher derivatives in the continuum limit.

II. CONTINUUM LIMIT OF HOPPING MODELS

In this section, we consider hopping models with a bipartite structure where the hopping only takes place between even and odd lattice sites. This is the situation for the entanglement Hamiltonian if the system is half-filled. Setting $H_{i,i+2p+1} = -t_{2p+1}(i+p+1/2)$ in (3), where $i+p+1/2$ is the midpoint between initial and final site, the Hamiltonian then takes the form

$$\mathcal{H} = -\sum_i \sum_{p\geq 0} t_{2p+1}(i+p+1/2)\left(c_i^\dagger c_{i+2p+1} + c_{i+2p+1}^\dagger c_i\right) . \tag{4}$$

A. Homogeneous case

It is instructive to consider first the case where the t_{2p+1} do not depend on the position. The corresponding Hamiltonian will be denoted by $\mathcal{H}_h$. For a ring with N sites and lattice spacing s, a Fourier transformation then gives

$$\mathcal{H}_h = \sum_q \nu_q\, c_q^\dagger c_q\,, \quad \nu_q = -\sum_{p\geq 0} t_{2p+1}\, 2\cos[(2p+1)qs]\,, \quad q = \frac{2\pi}{Ns}\,k\,, \quad k = 0, \pm 1, \pm 2, \dots \tag{5}$$

and $\mathcal{H}_h$ has a half-filled ground state with Fermi momentum $q_F = \pm\pi/2s$ if t_1 dominates. To obtain its continuum limit, one first shifts the right and left parts of the dispersion relation in such a way that both Fermi points lie at the origin. This is done by introducing new Fermi operators for even and odd sites (see e.g. [30])

$$c_{2n} = \mathrm{i}^{2n} a_n\,, \quad c_{2n+1} = \mathrm{i}^{2n+1} b_n\,. \tag{6}$$

Then $\mathcal{H}_h$ contains only mixed terms $a^\dagger b$ and $b^\dagger a$ and is diagonalized by writing the Fourier-transformed quantities as

$$a_q = \frac{1}{\sqrt{2}}\, e^{-\mathrm{i}qs/2}(\alpha_q + \beta_q)\,, \quad b_q = \frac{1}{\sqrt{2}}\, e^{\mathrm{i}qs/2}(\alpha_q - \beta_q) \tag{7}$$

which leads to

$$\mathcal{H}_h = \sum_q \omega_q \left(\alpha_q^\dagger \alpha_q - \beta_q^\dagger \beta_q\right)\,, \qquad \omega_q = \sum_{p\geq 0} (-1)^p\, t_{2p+1}\, 2\sin[(2p+1)qs]\,. \tag{8}$$

The Brillouin zone is now limited by $\pm\pi/2s$ and the operators α_q (β_q) describe right (left) moving particles with energy ω_q and velocity

$$v_q = \frac{d\omega_q}{dq} = \sum_{p\geq 0} (-1)^p\, (2p+1)s\, t_{2p+1}\, 2\cos[(2p+1)qs]\,. \tag{9}$$

For $q = 0$, this becomes the Fermi velocity

$$v_F = 2s \sum_{p\geq 0} (-1)^p\, (2p+1)\, t_{2p+1} \tag{10}$$

where the hopping amplitudes are multiplied by the corresponding hopping distances. The factor $(-1)^p$ is best understood if one notes that the same result is obtained if one works in the initial formulation (5) and differentiates ν_q at $q = \pi/2s$. Then $(-1)^p$ appears because the slopes of the functions $\cos[(2p+1)qs]$ at the Fermi points alternate with p.

The form (8) for small q where $\omega_q = v_F q$ is the continuum limit of $\mathcal{H}_h$ in momentum space. In real space, one can obtain it directly by introducing field operators $\psi_1(x)$ and $\psi_2(x)$ for a_n and b_n which are attached to the points of the original lattice, i.e.

$$a_n \to \sqrt{2s}\, \psi_1(x)\ , \qquad b_n \to \sqrt{2s}\, \psi_2(x+s)\ , \qquad \sum_n \to \int \frac{dx}{2s}\,. \tag{11}$$

Here the lattice constant $2s$ of the sublattices has been used. In the limit $s \to 0$, one can expand the quantities for shifted sites as

$$a_{n+r} \to \sqrt{2s}\, \psi_1(x+2sr) \simeq \sqrt{2s}\, \big(\psi_1(x) + 2sr\, \psi_1'(x) + \dots\big) \tag{12}$$

and similarly for b_{n+r}. This leads to an expression for $\mathcal{H}_h$ where only terms with one derivative remain. Either by shifting summation indices or by partial integration, this derivative can be brought into the second place such that

$$\mathcal{H}_h = -\,\mathrm{i}\, v_F \int_0^\ell dx\, \left(\psi_1^\dagger(x)\psi_2'(x) + \psi_2^\dagger(x)\psi_1'(x)\right) \tag{13}$$

where $\ell = Ns$ and v_F is given by (10). The transformation to right- and left-movers corresponding to (7) for $qs \to 0$

$$\psi_1 = \frac{1}{\sqrt{2}}\,(\psi_{\rm R} + \psi_{\rm L})\,, \qquad \psi_2 = \frac{1}{\sqrt{2}}\,(\psi_{\rm R} - \psi_{\rm L}) \tag{14}$$

then gives the final result

$$\mathcal{H}_h \;=\; v_F \int_0^\ell dx\, \Big(\psi_{\rm R}^\dagger(x)\,(-{\rm i}\,\partial_x)\psi_{\rm R}(x) - \psi_{\rm L}^\dagger(x)\,(-{\rm i}\,\partial_x)\psi_{\rm L}(x)\Big)\,. \tag{15}$$

B. Inhomogeneous case

The procedure outlined above can easily be generalised to hopping amplitudes that vary slowly in space. To obtain the continuum limit, one writes

$$t_{2p+1}(i+p+1/2) \to t_{2p+1}(x+(p+1/2)s)\,. \tag{16}$$

Then the hopping processes between x and $x+s$ and their $(2p+1)$-th neighbours to the right lead to the following terms in $\mathcal{H}$

$$\begin{aligned} &t_{2p+1}(x+(p+1/2)s)\left[\psi_1^\dagger(x)\psi_2(x+(2p+1)s) - \psi_2^\dagger(x+(2p+1)s)\psi_1(x)\right] \\ &+t_{2p+1}(x+(p+3/2)s)\left[\psi_2^\dagger(x+s)\psi_1(x+(2p+2)s) - \psi_1^\dagger(x+(2p+2)s)\psi_2(x+s)\right]\,. \end{aligned} \tag{17}$$

Expanding all quantities as in (12) gives

$$t_{2p+1}s\left[2p(\psi_1^\dagger\psi_2' - \psi_2^{\dagger\prime}\psi_1) + (2p+2)(\psi_2^\dagger\psi_1' - \psi_1^{\dagger\prime}\psi_2)\right] - t_{2p+1}'s\left[\psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1\right] \tag{18}$$

where the argument is now x everywhere. For the complete Hamiltonian, the last term can be converted into one where t_{2p+1} appears by a partial integration. The boundary contributions vanish even for an open system since the fields and the hopping amplitudes are zero outside. Thus (18) becomes effectively

$$(2p+1)\,t_{2p+1}s\left[(\psi_1^\dagger\psi_2' + \psi_2^\dagger\psi_1') - (\psi_2^{\dagger\prime}\psi_1 + \psi_1^{\dagger\prime}\psi_2)\right]\,. \tag{19}$$

The second term in the bracket, which is the hermitean conjugate of the first one, now has to be kept as it is. The final expression therefore reads

$$\mathcal{H} = \int_0^\ell dx\, v_F(x)\, T_{00}(x) \tag{20}$$

where the operator of the energy density is now given by

$$T_{00}(x) = \frac{1}{2}\left[\psi_{\rm R}^\dagger(x)\,(-{\rm i}\,\partial_x)\psi_{\rm R}(x) - \psi_{\rm L}^\dagger(x)\,(-{\rm i}\,\partial_x)\psi_{\rm L}(x) + \text{ h.c.}\right] \tag{21}$$

and it generalizes (15) to a spatially varying local velocity $v_F(x)$ given by using $t_{2p+1}(x)$ in (10). This has the general conformal form (1) with $v_F(x)$ appearing in place of $2\pi\ell\beta(x)$. In the following section, we will calculate $v_F(x)$ explicitly for the entanglement Hamiltonian of an interval.

III. ENTANGLEMENT HAMILTONIAN FOR AN INTERVAL

With the result of the previous section, we can now determine the continuum limit of the entanglement Hamiltonian for an interval in a half-filled chain. If the total system is infinite and in its ground state, the calculation can be carried out analytically. The case of a finite chain or finite temperature will be considered by resorting to a numerical evaluation of the sums involved.

A. Infinite system

For an interval of N sites in an infinite chain, the entanglement Hamiltonian can be given in closed form for large N and was reported in [25]. It is extensive and the scaled matrix $h = -H/N$ has the representation

$$h = \sum_{m\geq 0} \alpha_m \beta_m T^{2m+1} \tag{22}$$

with a symmetric matrix T which commutes with H and is bidiagonal for half filling with elements $T_{i,i+1} = i/N(1 - i/N)$, i.e. it describes nearest-neighbour hopping with parabolically varying amplitudes. The coefficients in (22) are given by

$$\alpha_m = \frac{1}{\sqrt{\pi}} \frac{\Gamma(m+1/2)}{\Gamma(m+1)}\,, \qquad \beta_m = \sqrt{\pi}\; 2^{2m} \frac{\Gamma(2m+1/2)}{\Gamma(2m+2)}\,. \tag{23}$$

As a result, H contains only hopping to the $(2p+1)$-st neighbours as in section II, and in a proper scaling limit $i, N \to \infty$ with i/N kept fixed, the hopping amplitudes read [25]

$$\frac{1}{N}\, t_{2p+1}(z_p) = \sum_{m\geq p} \alpha_m \beta_m \binom{2m+1}{m-p} z_p^{2m+1} \tag{24}$$

where the variable z_p defined as

$$z_p = \frac{i+p}{N}\left(1 - \frac{i+p}{N}\right) \tag{25}$$

is the element $T_{i+p,i+p+1}$ of T and symmetric under the reflection $i+p \to N-(i+p)$. The infinite sum in (24) can be expressed in terms of generalized hypergeometric functions ${}_3F_2$ for all values of p, but this is not necessary for the following calculation.

Introducing now the lattice spacing s, the limit which has to be considered is $s \to 0$ and $N \to \infty$, keeping $x = is$ and $\ell = Ns$ fixed. Then for any fixed p the dependence of z_p on p can be treated perturbatively and introduces only a correction of order s/ℓ. Hence, for our purposes, we can ignore this effect and work in (24) with the variable $z = z_0$ or, in terms of x and ℓ and bearing in mind (2)

$$z(x) = \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right) \equiv \beta(x)\,. \tag{26}$$

With this simplification, the only p-dependent term in t_{2p+1} is the combinatorial factor and the expression for v_F can be written as follows

$$v_F(z) = 2\ell \sum_{m\geq 0} \alpha_m \beta_m \, S_m \, z^{2m+1}\,, \qquad S_m = \sum_{p=0}^{m} (-1)^p \,(2p+1) \binom{2m+1}{m-p}\,. \tag{27}$$

Note that we have exchanged the sums over m and p, and the sum over p in the definition of S_m now runs only up to m. In other words, the hopping t_{2p+1} with $p > m$ does not contribute at order z^{2m+1}. We now rewrite the sum in S_m by substituting $p \to m - p$ which leads to

$$S_m = (-1)^m \sum_{p=0}^{m} (-1)^p (2m+1-2p) \binom{2m+1}{p}\,. \tag{28}$$

The factor multiplying the binomial coefficient can be split as $(2m+1-p) - p$ and S_m can be written as a difference of two sums

$$S_m = (-1)^m \,(2m+1) \left[\sum_{p=0}^{m} (-1)^p \binom{2m}{p} - \sum_{p=1}^{m} (-1)^p \binom{2m}{p-1} \right]. \tag{29}$$

The second sum can be further transformed by the substitution $p \to 2m+1-p$ into

$$\sum_{p=1}^{m} (-1)^p \binom{2m}{p-1} = - \sum_{p=m+1}^{2m} (-1)^p \binom{2m}{p}\,. \tag{30}$$

Now one can combine the two sums and realize that the resulting expression is nothing else but the expansion of $(1-1)^{2m}$. Hence one arrives at the simple result

$$S_m = (-1)^m \,(2m+1) \sum_{p=0}^{2m} (-1)^p \binom{2m}{p} = \delta_{m,0} \tag{31}$$

and in (27) only the $m = 0$ term survives. Using $\alpha_0 \beta_0 = \pi$ and $z = \beta$, we obtain our main result

$$v_F(x) = 2\pi\ell\, \beta(x)\,. \tag{32}$$

This means that, for the continuum limit, one has effectively $h = \pi T$ and all the higher terms in the series (22) and their expansion coefficients are irrelevant. Since the elements in T vary parabolically, one recovers in this way the conformal formula for the quantity $\beta(x)$ and also its prefactor in (1).

The feature that only the first term in (22) contributes, can be discussed in the following way. In the considerations above, the matrix T has elements depending on position and given by $z(x)$ in the continuum description. But the form of $z(x)$ does not play a role in the calculation of $v_F(x)$, so the same result will be obtained for a constant $z(x) \equiv z$. In this case, T is the hopping matrix of a homogeneous open chain and one can calculate v_F directly as in section II. The eigenvalues of T are $2z\cos(qs)$ and those of H follow as

$$\nu_q = -N \sum_{m\geq 0} \alpha_m \beta_m \big(2z\cos(qs)\big)^{2m+1}\,. \tag{33}$$

Looking at $q = \pi/2s$ where ν_q vanishes, one sees that all terms with $m \neq 0$ have slope zero there and only $m = 0$ remains to give a velocity $v_F = 2\pi\ell z$ which is exactly (32). Thus, for the homogeneous case, the property $h \simeq \pi T$ can be found in a very simple way and is a direct consequence of the structure of T. The calculation above shows that this remains true in a system which is only locally homogeneous.

B. Finite system

The same approach can be used for the case of an interval with N sites in a finite ring of M sites, although there we do not have the analytical form of the entanglement Hamiltonian. Thus the matrix elements have to be determined numerically following [24] and their sums carried out in exactly the same way as it was done in [20] for the infinite chain case. Aiming directly at $\beta = v_F/2\pi\ell$, we calculate the truncated sum

$$\beta(i) = \frac{1}{\pi N} \sum_{p=0}^{P} (-1)^p (2p+1) H_{i-p,i+p+1} \tag{34}$$

where P denotes the cutoff value and the matrix element corresponds to $-t_{2p+1}(i+1/2)$ in the notation of section II. This ensures that the reflection symmetry on the lattice is respected and that the maxima of the hopping amplitudes for different values of p are aligned with each other. Due to the finite size of the subsystem, one can only obtain and plot $\beta(i)$ in the range $P+1 \leq i \leq N-(P+1)$.

The result is shown in Fig. 1 for two different ratios $r = N/M = \ell/L$ $(L = Ms)$ and for increasing cutoff values P in the sum. The numerical sums are compared to the CFT result of

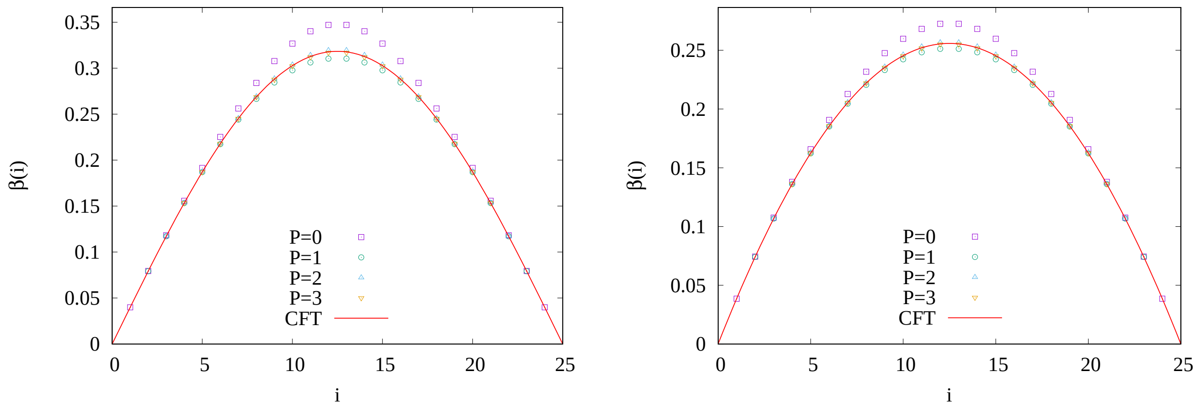

FIG. 1: The quantity $\beta(i)$ for an interval of $N = 25$ sites in a finite ring with two different subsystem ratios $r = 1/2$ (left) and $r = 1/6$ (right). The symbols correspond to different cutoffs P in (34), while the solid lines show the CFT result (35). Note the different vertical scales.

[12, 13]

$$\beta(x) = \frac{L}{\pi\,\ell}\,\frac{\sin(\pi\,x/L)\sin(\pi(\ell-x)/L)}{\sin(\pi\,\ell/L)} \tag{35}$$

using $x/L = i/M$ and $\ell/L = N/M$ on the right hand side. In both cases shown, one can see a clear convergence to the CFT formula. Due to the factor $(-1)^p$ in the sum, the approximations lie alternately above and below it and already four terms are sufficient to obtain a very good agreement. For $r = 1/6$, one is already relatively close to the case of an infinite system with the CFT result deviating only slightly from the parabola (2).

C. Finite temperature

If the chain is infinite but at finite inverse temperature β, one can proceed in the same way and calculate $\beta(i)$ via (34). The corresponding CFT formula is obtained from (35) by replacing the sine functions with hyperbolic sines and the length L of the ring with β. This gives [12, 13]

$$\beta(x) = \frac{\beta}{\pi\,\ell}\,\frac{\sinh(\pi\,x/\beta)\sinh(\pi(\ell-x)/\beta)}{\sinh(\pi\,\ell/\beta)} \tag{36}$$

with $x = i$ and $\ell = N$ on the right hand side. Note that for small β, i.e. high temperature, such that $\ell \gg \beta$, $i \gg \beta$ and $\ell - i \gg \beta$, the quotient of the hyperbolic sines is approximately $1/2$ and one has $2\pi\ell\beta(i) \simeq \beta$. Thus the local temperature in the bulk of the subsystem equals the global temperature and its profile shows a pronounced plateau.

The comparison with the finite sums is shown in Fig. 2, with again a very good agreement. In particular, one can see that for larger temperatures the contribution from the higher terms is very small. For $\beta = 10$ (right) it is essentially enough to add the third-neighbour ($p = 1$) contribution to recover the CFT result, while for $\beta = 20$ (left) the convergence is slower, similarly to the ground-state case.

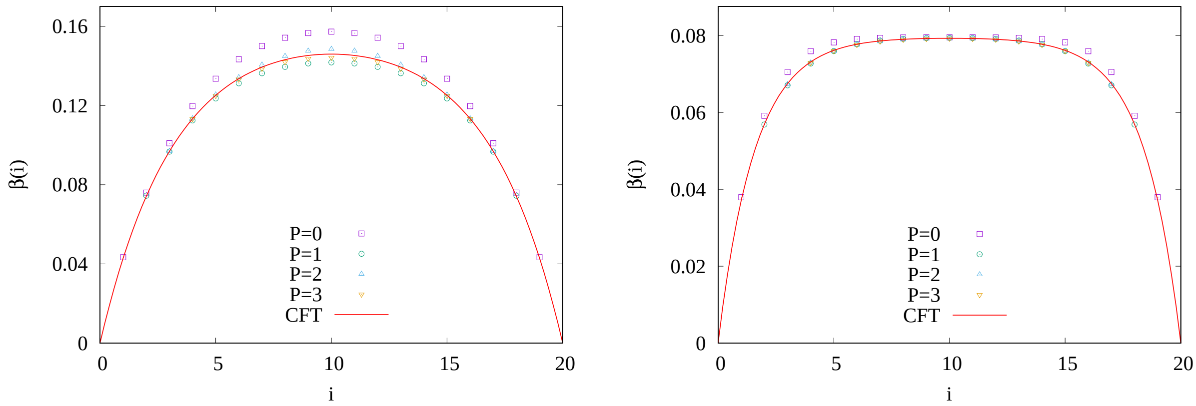

FIG. 2: The quantity $\beta(i)$ for an interval of $N = 20$ sites in an infinite chain at inverse temperature $\beta = 20$ (left) and $\beta = 10$ (right). The symbols correspond to different cutoffs P in (34), while the solid lines show the CFT result (36). Note the different vertical scales.

IV. INTERVAL IN A CHAIN WITH ARBITRARY FILLING

In this section we extend the analysis of section III to the case of an infinite chain with arbitrary filling. This filling will be measured by the Fermi wave number q_F which is related to the average site occupation via $\bar{n} = q_F s/\pi$. The scaled matrix $h = -H/N$ for large N then has the representation [25]

$$h = \sum_{m\geq 0} \sum_{n=0}^{2m+1} \alpha_m \beta_{m,n} \left(T + \frac{A}{2}\right)^{2m+1-n} \left(\frac{A}{2}\right)^n, \qquad A = \cos(q_F s) \tag{37}$$

where the matrix T now has also diagonal elements, see Appendix A. The coefficients α_m and $\beta_{m,n}$ are given in Appendix B (see (60)). In the previous notation, the entanglement Hamiltonian then has the form

$$\mathcal{H} = -\sum_i t_0(i)\, c_i^\dagger c_i - \sum_i \sum_{r\geq 1} t_r(i + r/2)\left(c_i^\dagger c_{i+r} + c_{i+r}^\dagger c_i\right) \tag{38}$$

with on-site terms, hopping to neighbours at arbitrary distances r and no sublattice structure.

To find the continuum limit, we therefore take a different path and introduce right- and left-movers directly as in bosonization [31]

$$
\begin{aligned}
c_i \;&\longrightarrow\; \sqrt{s}\,\Big(e^{\mathrm{i} q_F x}\, \psi_{\mathrm{R}}(x) + e^{-\mathrm{i} q_F x}\, \psi_{\mathrm{L}}(x) \Big) && (39)\\
c_{i+r} \;&\longrightarrow\; \sqrt{s}\,\Big(e^{\mathrm{i} q_F (x+rs)}\, \psi_{\mathrm{R}}(x+rs) + e^{-\mathrm{i} q_F (x+rs)}\, \psi_{\mathrm{L}}(x+rs) \Big)\,. && (40)
\end{aligned}
$$

Note that this representation with the factors $e^{\pm \mathrm{i} q_F x}$ refers to the full chain problem, although it will be used only in a finite interval.

For the quadratic Hermitean operators occurring in the sums of (38) one obtains

$$
c_i^\dagger c_i \;\longrightarrow\; s\,\Big(\psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x) + \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x) \Big) \tag{41}
$$

$$
\begin{aligned}
c_i^\dagger c_{i+r} + \text{h.c.} \;\longrightarrow\; s\, \Big\{ &\cos(r q_F s) \Big[\psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x+rs) + \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x+rs) + \text{h.c.} \Big] \\
&+ \sin(r q_F s) \Big[\mathrm{i} \Big(\psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x+rs) - \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x+rs) \Big) + \text{h.c.} \Big] \Big\}
\end{aligned} \tag{42}
$$

where the terms containing $e^{\pm 2 \mathrm{i} q_F x}$ have been neglected because they are rapidly oscillating.

To proceed, the $t_r(i+r/2)$ which follow from (37) are necessary. The explicit formula is complicated and not essential at this point (it is given in Appendix B, see (59),(60),(61)). Here we just need to observe that in the continuum limit $t_r(i+r/2)$ becomes a function of $x + rs/2$ (see (58)), hence it has to be expanded as $s \to 0$, finding that $t_r(i+r/2) \to t_r(x) + s\, r\, t_r'(x)/2 + O(s^2)$. Notice that $t_0(i) \longrightarrow t_0(x)$ and that the product $q_F s$ remains constant as $s \to 0$.

By expanding the expressions within the square brackets in (42) up to $O(s)$ terms included, we obtain for the continuum limit of (38)

$$
\begin{aligned}
\mathcal{H} \;=\; & -\int_0^\ell dx\, t_0(x) \Big\{ \psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x) + \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x) \Big\} - \int_0^\ell dx \sum_{r=1}^{\infty} \Big(t_r(x) + s\, \frac{r}{2}\, t_r'(x) \Big) \\
& \times \Big\{ \cos(r q_F s) \Big[2 \left(\psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x) + \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x) \right) + \; s\, r\, \partial_x \Big(\psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x) + \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x) \Big) \Big] \\
& \qquad + \sin(r q_F s) \Big[s\, r \Big(\mathrm{i} \left[\psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}'(x) - \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}'(x) \right] + \; \text{h.c.} \Big) \Big] \Big\}\,.
\end{aligned} \tag{43}
$$

We remark that an integration by parts leads to a crucial cancellation between the term containing $t_r'(x)$ and the term involving $\partial_x[\, \psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x) + \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x)\,]$. Thus, (43) can be written as

$$
\mathcal{H} \;=\; -\int_0^\ell dx\, F_0(x) \Big\{ \psi_{\mathrm{R}}^\dagger(x)\, \psi_{\mathrm{R}}(x) + \psi_{\mathrm{L}}^\dagger(x)\, \psi_{\mathrm{L}}(x) \Big\} + \int_0^\ell dx\, F_1(x)\, T_{00}(x) \tag{44}
$$

where T_{00} is given by (21) and the functions $F_0(x)$ and $F_1(x)$ are defined as

$$
F_0(x) \equiv t_0(x) + 2 \sum_{r=1}^{\infty} \cos(r q_F s)\, t_r(x)\,, \qquad F_1(x) \equiv\; 2s \sum_{r=1}^{\infty} r\, \sin(r q_F s)\, t_r(x)\,. \tag{45}
$$

In Appendix C we perform a systematic analysis of the higher order terms in s, which have been neglected in (44) and involve higher derivatives of the fields.

Compared to the result (20) in section II, the first term in (44) containing only densities is new, while in the second one the weight factor in the integral has changed. Performing these sums by inserting the analytic expressions for $t_r(x)$ is a non trivial task and the technical details are reported in Appendix B. The final result is very simple, however, namely

$$F_0(x) = 0\,, \qquad\qquad F_1(x) = \; 2\pi\ell\,\beta(x) \tag{46}$$

with $\beta(x)$ given by (2). This is completely independent of the filling, and by inserting (46) into (44) one recovers (20) with the expression (32) for v_F. In other words, the continuum limit of the entanglement Hamiltonian has always the same form as for half filling.

This result can be understood as in the previous section by considering a system which is homogeneous. The matrix T is now a hopping matrix with diagonal terms. If its elements are constant, it has eigenvalues $2z(\cos(qs) - A)$ and those of H follow as

$$\nu_q = -\,N \sum_{m\geq 0}\sum_{n=0}^{2m+1} \alpha_m \beta_{m,n} \left[2z(\cos(qs) - A) + \frac{A}{2}\right]^{2m+1-n} \left(\frac{A}{2}\right)^{n}. \tag{47}$$

This expression can be shown to vanish for $q = q_F$, and the velocity at this point is given by

$$v_F = 2\ell z \sin(q_F s) \sum_{m\geq 0}\sum_{n=0}^{2m+1} \alpha_m \beta_{m,n} (2m + 1 - n) \left(\frac{A}{2}\right)^{2m}. \tag{48}$$

Inserting the values of the coefficients given in Appendix B, one finds that the sums can be carried out as

$$\sum_{n=0}^{2m+1} \beta_{m,n} (2m + 1 - n) \left(\frac{1}{2}\right)^{2m} = \pi\,, \qquad \sum_{m\geq 0} \alpha_m A^{2m} = \frac{1}{\sqrt{1 - A^2}} = \frac{1}{\sin(q_F s)} \tag{49}$$

and the final result is $v_F = 2\pi\ell z$ as for half filling. The mechanism is somewhat different, however. While for $A = 0$ only the term $m = 0$ remains, one needs here the full sum over m, and this cancels the quantity $\sin(q_F s)$ which appears initially.

V. CONCLUSIONS

We have studied the question, how the results for the entanglement Hamiltonian in discrete hopping chains can be reconciled with the predictions of conformal field theory. For this, we first found out how the long-range couplings of the discrete system enter into the continuum limit and then evaluated the corresponding sums. This was quite transparent for a half-filled infinite chain, but considerably more involved for general filling. In both cases, the conformal expression could be reobtained analytically.

The main ingredient was a formula which expresses the hopping matrix H in $\mathcal{H}$ as a power series of a commuting, in general tridiagonal matrix T. For half filling, the mechanism found was that while the individual hopping amplitudes in H have contributions from various powers of T, almost all of these cancel in the superposition which is needed for the continuum limit and the local velocity $v_F(x)$ is completely determined by the first power. The desired continuum result is then obtained, because T has the conformal form already on the lattice. Turning the argument around, one could say that it is the continuum limit which demands this particular property of T.

One should mention that the relation $H \simeq -\pi NT$ has been encountered before in a related problem, namely for the low-lying eigenvalues of the two quantities [25, 28]. Assuming that these are also the relevant ones for the continuum limit, this is an anticipation of the result found here for the Hamiltonian itself. For general filling, the relation is changed to $H \simeq -\pi NT/\sin(q_F s)$, where the sine factor can be viewed as the Fermi velocity in the physical Hamiltonian. Indeed, as shown in Appendix A, the above relation becomes exact between the operators $\mathcal{H}$ and $\mathcal{T}$, if one considers both of them in the continuum limit.

We have considered also the case of a finite ring and found that the approach works equally well there and the conformal result is recovered numerically. In this case, no explicit formula like (22) has been derived, but a commuting matrix T is known and its elements have the conformal form. So the mechanism seems to be the same in this case. This is in line with the results for the low-lying eigenvalues. The same is expected to hold for open chains, where again a commuting T exists [26].

Taking all this together, our calculations both show how the discrete and the continuum results are connected in these free-fermion systems and how the particular commuting operator which exists here enters into the considerations. We think that this sheds additional light onto this algebraic structure.

We have limited ourselves here to chains which were homogeneous, but there are also conformal predictions for the entanglement Hamiltonian in inhomogeneous chains [32]. It would be interesting to see, how it can be obtained from the lattice result in such a situation, for example in the so called rainbow chain [33, 34], where the hopping decreases exponentially from the centre of the system. Whether a simple commuting operator exists in this case, is not known.

Finally, it would be interesting to consider also interacting lattice models, where analytical results on the entanglement Hamiltonians are still missing, and a direct numerical evaluation is limited to very small system sizes [35]. To overcome this barrier, various numerical approaches have been proposed recently [36–39]. The results for the low-lying eigenvalues suggest that the continuum limit derived here for free fermions should also apply to more complicated systems.

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Appendices

Appendix A CONTINUUM LIMIT OF THE OPERATOR $\mathcal{T}$

In this appendix we consider the continuum limit of the operator $\mathcal{T}$

$$\mathcal{T} = \sum_{i,j=1}^{N} T_{i,j} c_i^{\dagger} c_j \tag{50}$$

formed with the matrix T which occurred repeatedly in the main text. Because T commutes with H in (3), the operator $\mathcal{T}$ commutes with $\mathcal{H}$. Given the matrix elements of T and in the previous notation, for arbitrary filling it reads

$$\mathcal{T} = \sum_{i=1}^{N} d(i)\, c_i^{\dagger}\, c_i + \sum_{i=1}^{N-1} t(i) \big(c_i^{\dagger}\, c_{i+1} + c_{i+1}^{\dagger}\, c_i \big) \tag{51}$$

where

$$d(i) \equiv -2\cos(q_F s)\, \frac{2i-1}{2N} \left(1 - \frac{2i-1}{2N} \right), \qquad t(i) \equiv \frac{i}{N} \left(1 - \frac{i}{N} \right). \tag{52}$$

The continuum limit of $\mathcal{T}$ can be studied by following the same steps discussed for the entanglement Hamiltonian $\mathcal{H}$ in section IV. However, the analysis is simpler because the hopping term involves only the nearest neighbours and it is useful to present it separately.

If one uses $(i-1/2)s \to x$ in the continuum limit, one finds $d(i) \to -2\cos(q_F s)\, \beta(x)$, with $\beta(x)$ given by (2). The continuum limit of the coefficient $t(i)$ is also straightforward, once it is written in the following form

$$t(i) \;=\; \frac{i-1/2}{N}\left(1-\frac{i-1/2}{N}\right)+\frac{1/2}{N}\left(1-2\,\frac{i-1/2}{N}\right)-\left(\frac{1/2}{N}\right)^2 \tag{53}$$
$$\longrightarrow\;\; \beta(x)+\frac{\beta'(x)}{2}\,s\,-\left(\frac{1/2}{\ell}\right)^2 s^2 .$$

The continuum limit of the operators in (51) is obtained by using the above observations, by introducing the fields $\psi_{\mathrm{L}}(x)$ and $\psi_{\mathrm{R}}(x)$ as in (39), and by employing (40) with $r=1$. The result reads

$$N\,\mathcal{T} \;\longrightarrow\; -\,2\,N\sum_{n=1}^{N} s\,\cos(q_F s)\,\beta(x)\left\{\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}(x)+\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}(x)\right\} \tag{54}$$
$$+\,N\sum_{n=1}^{N} s\left(\beta(x)+\frac{\beta'(x)}{2}\,s\right)$$
$$\times\left\{\cos(q_F s)\left[\,2\left(\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}(x)+\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}(x)\right)+\,s\,\partial_x\Big(\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}(x)+\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}(x)\Big)\right]\right.$$
$$\left.+\,s\,\sin(q_F)\left[\,\mathrm{i}\left(\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}(x)'-\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}(x)'\right)+\,\mathrm{h.c.}\right]\right\}.$$

It is straightforward to notice that the two terms at leading order cancel; hence we are left with

$$N\,\mathcal{T} \;\longrightarrow\;\; \ell\int_0^{\ell}\bigg\{\,\beta'(x)\,\cos(q_F s)\left(\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}(x)+\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}(x)\right) \tag{55}$$
$$+\,\beta(x)\,\cos(q_F s)\,\partial_x\Big(\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}(x)+\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}(x)\Big)$$
$$+\,\beta(x)\,\sin(q_F s)\left[\,\mathrm{i}\left(\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}'(x)-\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}'(x)\right)+\,\mathrm{h.c.}\right]\bigg\}\,dx\,.$$

The first two lines in this expression cancel once an integration by parts is performed. Thus, the final result reads

$$N\,\mathcal{T} \;\;\longrightarrow\;\; \sin(q_F s)\;\ell\int_0^{\ell} dx\,\beta(x)\left[\,\mathrm{i}\left(\psi_{\mathrm{R}}^\dagger(x)\,\psi_{\mathrm{R}}'(x)-\psi_{\mathrm{L}}^\dagger(x)\,\psi_{\mathrm{L}}'(x)\right)+\,\mathrm{h.c.}\right]. \tag{56}$$

This shows that the rescaled operator $-\pi N\mathcal{T}/\sin(q_F s)$ has the same continuum limit as the entanglement Hamiltonian $\mathcal{H}$. Such a relation also exists for the low-lying eigenvalues of the corresponding matrices T and H [25].

Finally, let us observe that, by defining x in the continuum limit as $is \to x$ instead of $(i-1/2)s \to x$, one finds

$$d(i) \;\longrightarrow\; -\,2\cos(q_F s)\left[\beta(x)-\frac{\beta'(x)}{2}\,s\,-\left(\frac{1/2}{\ell}\right)^2 s^2\right], \qquad\qquad t(i)\;\longrightarrow\;\beta(x)\,. \tag{57}$$

Then, after some simplifications similar to the ones discussed above, the limit (56) is recovered, as expected.

Appendix B CALCULATION OF THE SUMS FOR ARBITRARY FILLING

In this appendix we prove the identities (46) for the sums defined in (45).

As noted in the main text, the r-th neighbour hopping amplitude $t_r(i) = Nh_{i,i+r}$ in the entanglement Hamiltonian is actually a function of the scaling variable

$$z_r = \frac{2i+r-1}{2N}\left(1-\frac{2i+r-1}{2N}\right). \tag{58}$$

The explicit analytic expression in this scaling limit was found in [25] and reads

$$\frac{1}{N}\,t_r(z_r) = \sum_{m=0}^{\infty}\sum_{n=0}^{2m+1}\alpha_m\beta_{m,n}\left(\frac{A}{2}\right)^n \tilde{t}_{2m+1-n,r}(z_r) \tag{59}$$

where the coefficients α_m and $\beta_{m,n}$ are given by

$$\alpha_m = \frac{1}{\sqrt{\pi}}\frac{\Gamma(m+1/2)}{\Gamma(m+1)}\,, \qquad \beta_{m,n} = 2^{2m}\frac{\Gamma(2m-n+1/2)\Gamma(n+1/2)}{\Gamma(2m-n+2)\Gamma(n+1)} \tag{60}$$

whereas $A = \cos(q_F s)$ and

$$\tilde{t}_{k,r}(z_r) = \sum_{\substack{\ell=0\\ k-r-\ell \text{ even}}}^{k-r}\binom{k-\ell}{\frac{k-r-\ell}{2}}\binom{k}{\ell}\left(\frac{A}{2}-2Az_r\right)^{\ell} z_r^{k-\ell}\,. \tag{61}$$

The appearance of the scaling variable z_r follows from simple symmetry reasons, by requiring that $h_{i,i+r}$ be invariant under the reflection $i+r \to N+1-i$. However, as explained in the main text, when carrying out the continuum limit it is easier to take these shifts into account via the expansion (43), and work instead with the scaling variable $z \equiv z_0$ for each r. Using $Ns = \ell$ and $z = \beta(x)$, the identities (46) we have to prove are

$$t_0(z) + 2\sum_{r=1}^{\infty}\cos(rq_F s)\,t_r(z) = 0\,, \qquad \sum_{r=1}^{\infty} r\sin(rq_F s)\,t_r(z) = N\pi z\,. \tag{62}$$

The strategy one should follow is essentially the same as for half filling, but the calculation is much more cumbersome. First one rewrites the hoppings as a power series

$$t_r(z) = N\sum_{p=r}^{\infty}\gamma_{p,r}z^p\,. \tag{63}$$

In fact, the lowest order is always given by the range r of the hopping, which becomes clear by rewriting the sum (61) after the substitution $\ell \to k-\ell$ as

$$\tilde{t}_{k,r}(z) = {\sum_{\ell=r}^{k}}'\binom{\ell}{\frac{\ell-r}{2}}\binom{k}{\ell}\left(\frac{A}{2}-2Az\right)^{k-\ell} z^{\ell}\,. \tag{64}$$

The prime over the sum denotes that the summation index ℓ must have the same parity as r. To extract the coefficient $\gamma_{p,r}$ of z^p, one should expand

$$\left(\frac{A}{2} - 2Az\right)^{k-\ell} = \sum_{j=0}^{k-\ell} \binom{k-\ell}{j} \left(\frac{A}{2}\right)^j (-2Az)^{k-\ell-j} . \tag{65}$$

Clearly, in order to produce the power z^p, one has to match the factor z^ℓ with a term of order $z^{p-\ell}$, such that we need only the term satisfying $k - j = p$. The prefactor of z^p in $\tilde{t}_{k,r}(z)$ is then

$$\sum_{\ell=r}^{p}{}' \binom{\ell}{\frac{\ell-r}{2}} \binom{k}{\ell} \binom{k-\ell}{k-p} \left(\frac{A}{2}\right)^{k-p} (-2A)^{p-\ell} . \tag{66}$$

Note that we need to have $k \geq p$. Since the z dependence of $t_r(z)$ is entirely due to the terms $\tilde{t}_{2m+1-n,r}(z)$, one can now set $k = 2m+1-n$ and plug the factors (66) into the sum (59) to get the prefactor $\gamma_{p,r}$. This yields

$$\gamma_{p,r} = \left[\sum_{m=0}^{\infty} \sum_{n=0}^{2m+1-p} \alpha_m \beta_{m,n} \left(\frac{A}{2}\right)^{2m+1-p} \frac{(2m+1-n)!}{(2m+1-n-p)!} \right] C_{p,r} = \sigma_p \, C_{p,r} \tag{67}$$

where the quantity σ_p in the brackets only depends on p and we have defined

$$C_{p,r} = \sum_{\ell=r}^{p}{}' \frac{(-2A)^{p-\ell}}{\left(\frac{\ell-r}{2}\right)! \left(\frac{\ell+r}{2}\right)! (p-\ell)!} = \sum_{k=0}^{\lfloor \frac{p-r}{2} \rfloor} \frac{(-1)^{p-r} (2A)^{p-r-2k}}{k!(k+r)!(p-r-2k)!} . \tag{68}$$

The second equality above follows by substituting $k = (\ell - r)/2$. Two special values of σ_p are

$$\sigma_0 = 0 , \qquad \sigma_1 = \frac{\pi}{\sin(q_F s)} \tag{69}$$

where the first one follows from the relation $\sum_{n=1}^{2m+1} \beta_{m,n} = 0$ and the second one from the equations (49).

Having the power series (63) at hand, the identities (62) now translate into

$$\gamma_{p,0} + 2 \sum_{r=1}^{p} \cos(r q_F s) \gamma_{p,r} = 0 , \qquad \sum_{r=1}^{p} r \sin(r q_F s) \, \gamma_{p,r} = \pi \, \delta_{p,1} . \tag{70}$$

In other words, we have to prove that the prefactor of each power z^p in (62) vanishes, except for the linear term $p = 1$ in the second sum. For $p = 0$, the sums are absent since then $p < r$. In this case, the left equation holds because $\sigma_0 = 0$ and the right one is trivial. Using (67), the relations (70) can be transformed into the following ones for the $C_{p,r}$

$$C_{p,0} + 2 \sum_{r=1}^{p} \cos(r q_F s) C_{p,r} = \delta_{p,0} , \qquad \sum_{r=1}^{p} \frac{\sin(r q_F s)}{\sin(q_F s)} \, r \, C_{p,r} = \delta_{p,1} , \tag{71}$$

where in the first equation $\delta_{p,0}$ is due to the fact that $C_{0,0} = 1$ and in the second one σ_1 from (69) has been used.

In order to prove (71), one should note that each factor $C_{p,r}(A)$ in (68) is given as a polynomial in terms of the parameter A. Thus, we will need a similar representation of the trigonometric factors as well. This is accomplished by recognizing them as the Chebyshev polynomials T_r and U_r [40], which for $r \neq 0$ can be written as

$$\cos(rq_F s) = T_r(A) = \frac{r}{2} \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^k \frac{(r-1-k)!}{k!(r-2k)!} (2A)^{r-2k} \tag{72}$$

and

$$\frac{\sin(rq_F s)}{\sin(q_F s)} = U_{r-1}(A) = \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \frac{(r-1-k)!}{k!(r-1-2k)!} (2A)^{r-1-2k} \,. \tag{73}$$

Dropping now the arguments A, we have to prove

$$C_{p,0} + 2 \sum_{r=1}^{p} T_r \, C_{p,r} = \delta_{p,0} \,, \qquad \sum_{r=1}^{p} r \, U_{r-1} \, C_{p,r} = \delta_{p,1} \,. \tag{74}$$

First we show that the second identity follows simply from the first one by taking the derivative with respect to A. Indeed, one has

$$C'_{p,0} + 2 \sum_{r=1}^{p-1} T_r \, C'_{p,r} + 2 \sum_{r=1}^{p} T'_r \, C_{p,r} = 0 \,, \tag{75}$$

Furthermore one has

$$T'_r = r \, U_{r-1} \,, \qquad C'_{p,r} = -2C_{p-1,r} \tag{76}$$

where the first one is a well-known identity between Chebyshev polynomials and the second one can be easily verified using the definition (68). Substituting into (75) and using the first (yet unproven) identity from (74) one gets

$$2 \sum_{r=1}^{p} r \, U_{r-1} \, C_{p,r} = 2(C_{p-1,0} + 2 \sum_{r=1}^{p-1} T_r \, C_{p-1,r}) = 2\delta_{p-1,0} = 2\delta_{p,1} \,. \tag{77}$$

Thus it remains to prove the first identity of (74), which essentially amounts to collect the various powers of A in the product of polynomials. Indeed, one should note that the $C_{p,r}$ are polynomials of order $p - r$, where only terms corresponding to that parity appear. Similarly, the Chebyshev polynomials T_r are of the order r, containing only terms with that parity. Thus, for each r, the highest power that appears in their product is p, corresponding to the $k = 0$ term in the sums (68) and (72). Collecting also the contribution from $C_{p,0}$ and summing over r, the p-th order term in the polynomial is given by

$$\left[\frac{1}{p!}+\sum_{r=1}^{p}\frac{(-1)^r}{r!(p-r)!}\right](-1)^p(2A)^p=\delta_{p,0}\,. \tag{78}$$

Hence, we see that the highest order terms already deliver the desired result.

The last step is to prove that each of the remaining powers sum up to zero. Let us consider the contribution of the $(p-2\ell)$-th power for a fixed $\ell>0$. This can combine as $(p-r-2k)+(r-2(\ell-k))$, i.e. each term k in the sum of $C_{p,r}$ must combine with $k'=\ell-k$ in that of T_r. The corresponding contribution of their product then reads

$$\frac{(-1)^{\ell-k+r}r\,(r-1-(\ell-k))!}{k!(k+r)!(p-r-2k)!(\ell-k)!(r-2(\ell-k))!}(-1)^p(2A)^{p-2\ell}\,. \tag{79}$$

When collecting the contributions from the various k and r, one has to take care of the limits. Here we would like to treat k as an independent summation variable. To ensure that $k'\geq 0$, one has to choose $0\leq k\leq \ell$. Note, however, that the original range of hopping $1\leq r\leq p$ must also be restricted. Indeed, one has to ensure that both powers in the product be positive, $p-r-2k\geq 0$ and $r-2(\ell-k)\geq 0$, thus the range of summation over r depends on the value of k. Collecting also the contribution of the term $C_{p,0}$, the overall prefactor of the term $(-1)^p(2A)^{p-2\ell}$ becomes

$$\frac{1}{(\ell!)^2(p-2\ell)!}+\sum_{k=0}^{\ell}\sum_{r=\max(2(\ell-k),1)}^{p-2k}\frac{(-1)^{\ell-k+r}r\,(r-1-(\ell-k))!}{k!(k+r)!(p-r-2k)!(\ell-k)!(r-2(\ell-k))!}\,. \tag{80}$$

One can see that in the above sum over r the $k=\ell$ term requires special attention, since for this value the lower limit $2(\ell-k)$ would give zero, instead of the requirement $r\geq 1$. Thus, we first consider $k<\ell$ and introduce the new variables $r'=r-2(\ell-k)$ and $k'=\ell-k$, such that the sum can be rewritten as

$$\sum_{k'=1}^{\ell}\sum_{r'=0}^{p-2\ell}\frac{(-1)^{k'+r'}(r'+2k')\,(r'+k'-1)!}{(l-k')!(r'+k'+\ell)!(p-r'-2\ell)!\,k'!\,r'!}\,. \tag{81}$$

Now, since the summation ranges are independent of each other, we can interchange them. It turns out that the k'-sum can then be carried out as

$$\sum_{k'=1}^{\ell}\frac{(-1)^{k'}(r'+2k')\,(r'+k'-1)!}{k'!(\ell-k')!(r'+k'+\ell)!}=-\frac{r'!}{\ell!(\ell+r')!}\,. \tag{82}$$

Including now also the $k=\ell$ term which was left out, the sums in (80) can be rewritten as

$$\sum_{r=1}^{p-2\ell}\frac{(-1)^r}{\ell!(\ell+r)!(p-r-2\ell)!}-\sum_{r'=0}^{p-2\ell}\frac{(-1)^{r'}}{\ell!(\ell+r')!(p-r'-2\ell)!}=-\frac{1}{(\ell!)^2(p-2\ell)!}\,. \tag{83}$$

Since this is exactly the opposite of the first term in (80), the prefactor vanishes for arbitrary p and $\ell>0$. This concludes our proof of (74).

Appendix C HIGHER DERIVATIVES

The result (44) for the continuum limit of (38) involves only the zero-th and first order derivatives of the fields. In this appendix we consider also the subleading terms, which involve higher order derivatives of the fields and have been neglected in (43) because they vanish in the continuum limit. They are interesting, however, since they lead to closely related expressions.

As for the coefficients in (38), we have that $t_0(i) \longrightarrow t_0(x)$, while $t_r(i + r/2)$ provides the following derivative expansion in the continuum limit

$$t_r(i + r/2) \;\longrightarrow\; t_r(x) + \sum_{l=1}^{\infty} \frac{(s\,r/2)^l}{l!}\, t_r^{(l)}(x) \qquad r \geq 1 \tag{84}$$

Furthermore, let us consider the expansion of the expression within the curly brackets in (42), which can be written as

$$-\,2\cos(rq_F s) \sum_{k=0}^{\infty} \frac{(rs)^k}{k!}\, \Psi_+^{(k)} - 2\sin(rq_F s) \sum_{k=0}^{\infty} \frac{(rs)^k}{k!}\, \Psi_-^{(k)} \tag{85}$$

where we have introduced the following quadratic expressions of the fields

$$\Psi_+^{(k)} \equiv -\frac{1}{2} \left(\psi_{\rm R}^\dagger\, \psi_{\rm R}^{(k)} + \psi_{\rm L}^\dagger\, \psi_{\rm L}^{(k)} \right) + \text{h.c.}\,, \qquad \Psi_-^{(k)} \equiv -\frac{\rm i}{2} \left(\psi_{\rm R}^\dagger\, \psi_{\rm R}^{(k)} - \psi_{\rm L}^\dagger\, \psi_{\rm L}^{(k)} \right) + \text{h.c.} \tag{86}$$

being $k \geq 0$ and $\psi^{(k)} \equiv \partial_x^{(k)} \psi(x)$. Notice that for $k = 0$ we have $\Psi_+^{(0)} = -(\psi_{\rm R}^\dagger\, \psi_{\rm R} + \psi_{\rm L}^\dagger\, \psi_{\rm L})$ and $\Psi_-^{(0)} = 0$. Instead, for $k = 1$ we recognise $\Psi_-^{(1)} = T_{00}$ defined in (21).

The continuum limit of the entanglement Hamiltonian (38) can be studied by employing (41), (84) and (85), finding that

$$\begin{aligned} \mathcal{H} \;=&\; \int_0^\ell dx\, t_0\, \Psi_+^{(0)} \\ &+ 2 \sum_{r=1}^{\infty} \int_0^\ell dx \left(\sum_{l=0}^{\infty} \frac{(s\,r)^l}{q!\, 2^l}\, t_r^{(l)} \right) \sum_{k=0}^{\infty} \frac{(rs)^k}{k!} \Big[\cos(rq_F s) \Psi_+^{(k)} + \sin(rq_F s)\, \Psi_-^{(k)} \Big] \end{aligned} \tag{87}$$

which reduces to (43) when the $o(s)$ terms are neglected. The r-th term of the sum in the r.h.s. in (87) can be written as follows

$$2 \int_0^\ell dx \sum_{m=0}^{\infty} \frac{(rs)^m}{m!\, 2^m} \sum_{k=0}^{m} 2^k \binom{m}{k}\, t_r^{(m-k)} \Big[\cos(rq_F s) \Psi_+^{(k)} + \sin(rq_F s)\, \Psi_-^{(k)} \Big]. \tag{88}$$

In each term of this expression, let us perform the proper number of integrations by parts in order to isolate the functions t_r and plug the result back into (87). Then, by taking into account also the sum in r in (87) and exchanging the order of the two sums in the second integral, we find that the entanglement Hamiltonian (87) can be written as follows

$$\mathcal{H} \;=\; \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^\ell dx \left\{ F_m^{(+)}(x)\, \mathcal{H}_+^{(m)}(x) + F_m^{(-)}(x)\, \mathcal{H}_-^{(m)}(x) \right\} \tag{89}$$

where we have introduced the operators

$$\mathcal{H}_{\pm}^{(m)} \equiv \sum_{k=0}^{m} \binom{m}{k} \left(-\frac{1}{2}\,\partial_x\right)^{m-k} \Psi_{\pm}^{(k)} \tag{90}$$

and the corresponding weight functions as

$$\begin{aligned} F_m^{(+)}(x) &\equiv \delta_{m,0}\, t_0(x) + 2\, s^m \sum_{r=1}^{\infty} r^m \cos(r q_F s)\, t_r(x)\,, \\ F_m^{(-)}(x) &\equiv 2\, s^m \sum_{r=1}^{\infty} r^m \sin(r q_F s)\, t_r(x)\,. \end{aligned} \tag{91}$$

Notice that the $F_m^{(\pm)}(x)$ contribute at order $O(s^m)$ and that the operators $\mathcal{H}_{\pm}^{(m)}$ are combinations of $\partial^a \psi^\dagger \partial^b \psi$ where $a + b = m$.

The results obtained in section IV correspond to the terms with $m = 0$ and $m = 1$ in (89). Indeed, the leading term has $m = 0$ and for the corresponding operators (90) one finds $\mathcal{H}_{+}^{(0)} = \Psi_{+}^{(0)}$ and $\mathcal{H}_{-}^{(0)} = 0$. Furthermore, $F_0^{(+)}(x) = F_0(x)$ defined in (45). Thus, the $m = 0$ term of (89) provides the first integral in (44). As for the $m = 1$ term, by employing the definitions (86) we find that the corresponding operators (90) are $\mathcal{H}_{+}^{(1)} = 0$ (this operator vanishes because $\Psi_{+}^{(1)} = \frac{1}{2}\partial_x \Psi_{+}^{(0)}$) and $\mathcal{H}_{-}^{(1)} = \Psi_{-}^{(1)} = T_{00}$. Then, since $F_1^{(-)}(x) = F_1(x)$ introduced in (45), we conclude that the $m = 1$ term of (89) can be written as the second integral in (44).

The $O(s^2)$ and $O(s^3)$ contributions correspond respectively to the $m = 2$ and $m = 3$ terms in the sum (89). For $m = 2$ the operators (90) become $\mathcal{H}_{+}^{(2)} = \Psi_{+}^{(2)} - \frac{1}{2}\,\partial_x \Psi_{+}^{(1)}$ and $\mathcal{H}_{-}^{(2)} = 0$. The latter identity is equivalent to $\Psi_{-}^{(2)} = \partial_x \Psi_{-}^{(1)} = \partial_x T_{00}$. The contribution of the third derivatives of the fields to the entanglement Hamiltonian (89) is due to the $m = 3$ term. In this case the operators (90) read $\mathcal{H}_{+}^{(3)} = \frac{1}{8}(8\Psi_{+}^{(3)} - 12\,\partial_x \Psi_{+}^{(2)} + 4\,\partial_x^2 \Psi_{+}^{(1)}) = 0$ and $\mathcal{H}_{-}^{(3)} = \Psi_{-}^{(3)} - \frac{3}{4}\,\partial_x \Psi_{-}^{(2)}$. It would be interesting to further explore the structure of the operators $\mathcal{H}_{\pm}^{(m)}$. We expect that $\mathcal{H}_{+}^{(2p+1)} = \mathcal{H}_{-}^{(2p)} = 0$ for any $p \geq 0$.

At half filling the functions (91) simplify significantly; indeed they reduce to

$$F_m^{(+)}(x) = 0\,, \qquad F_m^{(-)}(x) = 2\, s^m \sum_{p=0}^{\infty} (-1)^p\, (2p+1)^m\, t_{2p+1}\,. \tag{92}$$

Furthermore, for $m = 3$ we can perform the sum in (92) analytically by following a procedure similar to the one described in section III A. This leads us to write the expansion of the entanglement Hamiltonian (89) at half filling restricted to the terms up to the third derivatives included as follows

$$\mathcal{H} = 2\pi\,\ell \int_0^\ell dx\, \beta(x)\, T_{00} + 2\pi\,\ell\, \frac{s^2}{3!} \int_0^\ell dx\, \beta(x)\Big(1 - 6\,\beta(x)^2\Big)\left(\Psi_{-}^{(3)} - \frac{3}{4}\,\partial_x \Psi_{-}^{(2)}\right) + \dots \tag{93}$$

where $\beta(x)$ is the parabola defined in (2). The weight factor in the second integral varies as $\beta(x)$ near the ends of the interval, but the maximum at $x = 1/2$ is replaced by a plateau.

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