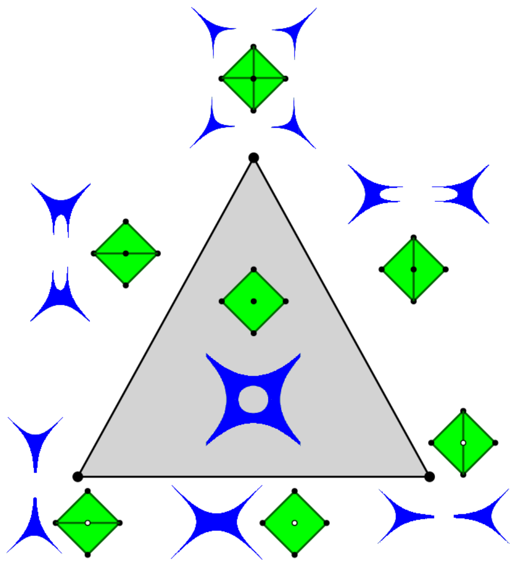
Polytopal subdivisions in Grassmannians, tropical geometry and algebraic curves



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Abstract

This thesis studies three particular types polytopal subdivisions with concrete applications to other mathematical objects, particularly in algebraic geometry.

The first type of polytopal subdivision consists in hypersimplicial subdivisions. These are subdivisions induced by linear projections of hypersimplices. In the case where the projection sends the canonical bases to the vertices of a convex polygon, hypersimplicial subdivisions are in bijection with Grassmannian graphs, a type of planar graph that appears in work of Postnikov to parametrize the positive Grassmannian [Pos19]. We show that for these cases, the poset of hypersimplical subdivions, and hence the poset of Grassmannian graphs of a given type, is homotopic to a sphere, solving a question of Postnikov. For more general projections of hypersimplices, we study the fiber polytope and show that in some cases it is normally equivalent to the Minkowski sum of some the faces of the corresponding secondary polytope.

The second type of polytopal subdivision is regular matroid subdivisions. These objects are polytopal complexes dual to tropical linear spaces. Given any matroid polytope, functions from the set of vertices to the reals that induce a matroid subdivision are the tropical analog of a Plücker vector [Spe08]. We study the Dressian of that matroid, which is the space of all such tropical Plücker vectors. It is a subfan of the secondary fan of the matroid polytope. We show that matroid subdivisions are determined by its 3-skeleton. We study tropical linear spaces arising from matrices with tropical entries, called Stiefel tropical linear spaces in [FR15]. We show that these are a valuated analog of transversal matroids, generalizing much of the theory of transversal matroids to the valuated case. In particular, we concretely describe the space of all tropical matrices with the same tropical Plücer vector. In the process, we show that transversality is a 'local' property.

The third type of polytopal subdivisions is regular lattice polygon subdivisions. We make use of them to study Harnack curves. We generalize to arbitrary toric surfaces work of Kenyon and Okounkov [KO06], who computed the moduli space of Harnack curves in the projective plane with a given degree. Then we use the fact that Harnack curves can be constructed using regular lattice polygon subdivisions via Viro's patchworking method to construct a meaningful compactification of the moduli space of Harnack curves. In the process we also make use of abstract tropical curves, which exhibits again the duality between polygon subdivisions and tropical varieties. The result is a compact moduli space of Harnack curves that has a cell complex structure with the same poset as the secondary polytope of the Newton polygon.

Zusammenfassung

Diese Arbeit untersucht drei Arten der Polytopen-Unterteilungen mit konkreten Anwendungen auf andere mathematische Objekte, insbesondere in der algebraischen Geometrie.

Die erste Art der Polytopen-Unterteilung besteht aus hypersimplizialen Unterteilungen. Diese sind Unterteilungen, die durch lineare Projektionen von Hypersimplizes induziert werden. In dem Fall, in welchem die Projektion die kanonische Basis auf die Ecken eines konvexen Polygons abbildet, sind die hypersimplizialen Unterteilungen in Bijektion mit Grassmannschen Graphen. Letztere sind eine Art von Graphen, die in der Arbeit von Postnikov auftauchen, um den Grassmannschen zu parametrisieren. Wir zeigen, dass in diesem Fall das Poset der hypersimplizialen Unterteilung und damit das Poset der Grassmannschen Graphen eines bestimmten Typs den Homotopie-Typ einer Kugel hat. Für allgemeinere Projektionen von Hypersimplizes untersuchen wir das Faserpolytop und zeigen, dass es in einigen Fällen normaläquivalent zu der Minkowski-Summe einiger Seiten des entsprechenden Sekundärpolytops ist.

Die zweite Art der Polytopen-Unterteliung ist die reguläre Unterteilung der Matroiden. Diese Objekte sind polytopale Komplexe, die zu tropischen linearen Räumen dual sind. Für jedes Matroid-Polytop sind Funktionen von der Menge der Eckpunkte zu den reellen Zahlen, die eine Matroid-Unterteilung induzieren, das tropische Analogon eines Plücker Vektors [Spe08]. Wir untersuchen den Dressschen dieses Matroids, der der Raum von allen solchen tropischen Plückervektoren ist. Es ist ein Unterfächer des Sekundärpolytops des Matroidpolytops. Wir zeigen, dass die Unterteilungen der Matroiden durch das 3-Skelett bestimmt werden. Wir untersuchen tropische lineare Räume, die aus Matrizen mit tropischen Einträgen hervorgehen. In [FR15] sind dies sogenannte Stiefel tropische lineare Räume. Wir zeigen, dass diese ein Analogon der transversalen Matroiden für bewertete Matroiden sind, wobei viel der Theorie der transversalen Matroiden auf den bewerteten Fall verallgemeinert wird. Insbesondere beschreiben wir konkret den Raum aller tropischen Matrizen mit demselben tropischen Plücker Vektor. Dabei zeigen wir, dass Transversalität eine "lokale" Eigenschaft ist.

Die dritte Art von Polytopen-Unterteilungen sind reguläre Gitterpolygonunterteilungen. Wir verwenden sie, um Harnack-Kurven zu studieren. Wir verallgemeinern die Arbeit von Kenyon und Okounkov [KO06], die den Modulraum von Harnack-Kurven in der projektiven Ebene mit einem bestimmten Grad berechnet, auf beliebige torische Flächen. Wir nutzen die Tatsache, dass Harnack-Kurven unter Verwendung von regulären Gitterpolygonunterteilungen mit Viros Patchwork-Methode konstruiert werden können, um eine bedeutsame Kompaktifizierung des Modulraums von Harnack-Kurven zu konstruieren. Dabei bedienen wir uns auch abstrakter tropischer Kurven, mit denen die Dualität zwischen Polytopen-Unterteilungen und tropischen Varietäten weiter aufgezeigt wird. Das Ergebnis ist ein Zellkomplex, der ein kompakter Modulraum von Harnack Kurven und isomorph zu dem Sekundärpolytops des Newton-Polygons ist.

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Contents

Chapter 1

Introduction and overview.

This thesis is about three specific types of polytopal subdivisions, all of which have a concrete connection to mathematical objects that lie in the intersection of algebraic geometry and combinatorics:

1. **Hypersimplicial subdivisions.** These are subdivisions that are induced by a linear projection of a hypersimplex. The motivation to study them comes from the *positive Grassmannian*, since these subdivisions are in bijection with *Grassmannian graphs* in the case where the projection sends the cannonical basis to a convex polygon.

2. **Matroid subdivisions.** There is a particular class of polytopes that is cryptomorphic to matroids. Regular subdivisions of matroid polytopes into smaller matroid polytopes are known to be the dual objects to tropical linear spaces and constitute the theory of *valuated matroids.*

3. **Lattice polygon subdivisions.** In other words, regular subdivisions consisting of 2-dimensional polytopes whose vertices have integer coordinates. They are an ingredient in the process known as *patchworking* of real algebraic curves, which we use to construct the compactified moduli space of Harnack curves.

Since polytopal subdivisions are central objects throughout the thesis, in Chapter 2 we provide a brief review of their basic theory. A particular emphasis will be given to *regular subdivisions*, since most of the subdivisions concerning this thesis are regular. This chapter also includes a brief introduction to tropical geometry, showing some of the many ways in which regular subdivisions are essential in this field of mathematics.

After Chapter 2, the thesis is divided into three parts, one for each specific type of polytopal subdivision we are considering together with their specific application we are interested in. After settling down some notation that is going to be used throughout this thesis, we provide a summary of the main results of each of the parts.

After the summary, we end the introductory chapter with a section about the positive Grassmannian. In that section we show that all three parts of this thesis are related to each other by having concrete applications to the same setting. Some of the results stated there are new. Even though these new results follow from the work done in the rest of the thesis, they were not explicitly included in any of the research articles on which this thesis is built on. So this section is the only place so far where these results can be found.

1.1 Notation

- We write $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{T}$ for the integers, reals, complex and tropical numbers respectively.
- We write $[n]$ for the set of integers $\{1, \ldots, n\}$.
- We write $\binom{[n]}{k}$ for the set of subsets of $[n]$ of cardinality k.
- Given a set A we write $|A|$ for cardinality.
- Given a complex number $z \in \mathbb{C}$ we write $|z|$ for the absolute value of z.
- We write $\mathbb{R}_{\geq 0}$ for the set of non negative real numbers.
- Given a field $\mathbb{K}$ we write $\mathbb{K}^*$ for the multiplicative group $\mathbb{K} \setminus \{0\}$.
- We write $\mathbb{PK}^n$ for the n-dimensional projective space over $\mathbb{K}$.
- We write $\mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ for the ring of Laurent polynomials over $\mathbb{K}$
- For any $\alpha \in \mathbb{Z}^n$ we write x^α for $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.
- Given a vector space $\mathbb{K}^n$, we write $e_1, \ldots, e_n$ for the canonical basis.
- For $a, b \in \mathbb{K}^n$ we use $a \cdot b$ or $\langle a, b \rangle$ for the dot product of a and b.
- Given a subset A of a topological space, we write $\overline{A}$ for its closure.
- To simplify exposure, we sometimes write 123 for the set $\{1, 2, 3\}$ and $A \cup i$ for $A \cup \{i\}$.
- Given sets $A \subseteq B$ we write $[A, B]$ for the set of sets that contain A but are contained in B (an interval in the boolean lattice).

- For multisets, we use double brackets $\{\!\{\_\}\!\}$ to list their elements.
- However, we use $\mathbb{K}\{\!\{t\}\!\}$ for the field of Puiseux series.

1.2 Hypersimplicial subdivisions

This part is based on a joint paper with Francisco Santos, [OS19].

The main object of study in this part are *hypersimplicial subdivisions.* These are subdivisions that are induced by linear projections of the hypersimplex $\Delta_{k,n}$.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be a linear projection. Let A be the set of n (possibly repeated) points to which the canonical basis of $\mathbb{R}^n$ is mapped to. We write $A^{(k)}$ for the image of the vertices of the hypersimplex $\Delta_{k,n}$ under π. A *hypersimplicial subdivision* of $A^{(k)}$ is a polytopal subdivision of $\operatorname{conv}(A^{(k)})$ such that every face of the subdivision is the image of a face of $\Delta_n^{(k)}$ under π. These are the π-induced subdivisions as introduced in [BS92, BKS94].

The main motivation to study them comes from the case where $A \subseteq \mathbb{R}^2$ is the set of vertices of a convex polygon. In this case, the poset of hypersimplicial subdivisions by refinement is isomorphic to the poset of *Grassmannian graphs* [Pos19]. In particular, triangulations of this form are in bijection with complete reduced plabic graphs [Gal18]. This relationship is explained further in Section 1.5.

In [Pos19, Problem 10.3] Postnikov asks the *generalized Baues problem* for the case when A is the set of vertices of the d-dimensional cyclic polytope $\mathbf{C}(n, d)$. In other words, he asks whether the poset of hypersimplicial subdivisions of $\mathbf{C}(n, d)^{(k)}$ has the homotopy type of a $(n - d - 2)$-sphere. For $k = 1$ this was shown to have a positive answer by Rambau and Santos [RS00]. For $d = 2$, Balitskiy and Wellman show the poset to be simply connected and again ask the Baues question for it ([BW19, Theorem 6.4 and Question 6.1]). In Chapter 3 we answer this question:

Theorem 1.1. *(Theorem 3.55) Let $\mathbf{P}_n$ be the vertices of any convex n-gon. The poset of hypersimplicial subdivisions $\mathcal{B}(\Delta_n^{(k)} \to \mathbf{P}_n^{(k)})$ retracts onto the poset of* coherent *hypersimplicial subdivisions. In particular, it has the homotopy type of an $(n-4)$-sphere.*

[Pos19, Problem 10.3] also asks for which values of the parameters can all hypersimplicial subdivisions of $\mathbf{C}(n, d)^{(k)}$ be lifted to zonotopal tilings of the cyclic zonotope. Galashin [Gal18] showed that in dimension two all *hypertriangulations* can be lifted, a result that was generalized to all hypersimplicial subdivisions by Balitskiy and Wellman [BW19, Lemma 6.3]. On the other hand, it was already known that for $d = 1$ [Pos19, Example

10.4] there are hypersimplicial subdivisions that do not lift to zonotopal tilings. We generalize the counterexamples to every odd dimension:

Theorem 1.2. *(Theorem 3.40) Consider the cyclic polytope* $\mathbf{C}(n,d) \subset \mathbb{R}^d$ *for odd* d *and* $n \geq d+3$*. Then, for every* $k \in [2, n-2]$ *there exist hypersimplicial subdivisions of* $\mathbf{C}(n,d)^{(k)}$ *that do not extend to zonotopal tilings of the cyclic zonotope* $Z(\mathbf{C}(n,d))$*.*

The poset of coherent hypersimplicial subdivisions of $\pi : \Delta_{k,n} \to A^{(k)}$ is isomorphic to the face poset of a polytope which we call the *k-th hypersecondary polytope of A*. This is a particular case of a fiber polytope and when $k = 1$ this is just the secondary polytope of A. We study hypersecondary polytopes for any $A \subset \mathbb{R}^d$ and $k \leq d+1$. Specifically, we show that for any $A \subset \mathbb{R}^d$ and $k \leq d+1$, hypersecondary polytopes are normally equivalent to the Minkowski sum of certain faces of the secondary polytope of A. By symmetry, an analogue statement holds for $n-d-1 \leq k < n$.

Theorem 1.3. *(Theorem 3.24) Let* $A \subseteq \mathbb{R}^d$ *be a configuration of size* n *and* $k \in [d+1]$*. Let* $s = \max(n-k+1, d+2)$*. The hypersecondary polytope* $\mathcal{F}^{(k)}(A)$ *is normally equivalent to the Minkowski sum of the secondary polytopes of all subsets of* A *of size* s*.*

At the end of this part the problem of enumerating hypertriangulations of a convex polygon is discussed.

1.3 Tropical linear spaces

These part is divided into three chapters. In Chapter 4 we give an introduction to tropical linear spaces. This includes the basic theory of matroids and more generally valuated matroids, as well as the main constructions regarding them. Chapter 5 is based on joint work with Benajmin Schröter and Marta Panizzut [OPS]. Chapter 6 is based on joint work with Alex Fink [FO19].

Recall that the tropical semiring is $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ with addition 'min' and multiplication '+' (see Theorem 2.9). We study the tropical analog of a linear space, that is, tropical varieties given by equations of degree 1. A tropical linear space is given by tropical Plücker vectors, better known as *valuated matroids*, which are vectors in the tropical projective space $V \in \mathbb{PT}^{\binom{n}{d}-1}$, where $0 \leq d \leq n$ are integers (when talking about a rank d subspace of an n dimensional space). The coordinates of V are labelled by subsets of $\binom{[n]}{d}$. They are required to satisfy the tropical Plücker relations: for every $S \in \binom{n}{d-1}$ and $T \in \binom{n}{d-1}$ the minimum

$$\min\left(\sum_{j \in T\setminus S} V_{S\cup j} + V_{T\setminus j}\right) \tag{1.1}$$

is attained at least twice. For each valuated matroid $V \in \mathbb{PT}^{\binom{n}{d}-1}$ there is a corresponding tropical linear space $\mathcal{L}(V) \subseteq \mathbb{T}^n$ of dimension d.

The support (the set of coordinates with finite entry) of a valuated matroid V is the set of basis of a matroid called the underlying matroid $\underline{V}$. The space of all valuated matroids of rank d on $[n]$ is known as the Dressian $\mathrm{Dr}(d, n)$. Given a matroid M, we call the space of all valuated matroids with underlying matroid M the *local Dressian* of $\mathrm{Dr}(M)$. This is the main object of study in Chapter 5. Valuated matroid V are characterized by inducing a regular subdivision of the matroid polytope of $\underline{V}$ into smaller matroid polytopes (see Theorem 4.13).

Theorem 1.4. *(Theorem 5.2) Let V be a valuated matroid with underlying matroid M. The regular subdivision of P_M induced by V is determined by its 3-skeleton (the 3-dimensional cells).*

This gives as a corollary a new and simpler proof of the following statement originally proved by Dress and Wenzel [DW92, Theorem 5.11]. Binary matroids (matroids representable over some field of characteristic 2) do not admit non-trivial matroid subdivisions. We also prove the following, which was independently proven by López de Medrano, Rincón and Shaw [LdMRS17, Lemma 4.7 and Corollary 4.8]:

Theorem 1.5. *(Theorem 5.14) Let M_1 and M_2 be two matroids and let $M_1 \oplus M_2$ be their direct sum. then $\mathrm{Dr}(M_1 \oplus M_2) \cong \mathrm{Dr}(M_1) \times \mathrm{Dr}(M_2)$.*

One way of obtaining valuated matroids is by taking the maximal tropical minors of a matrix with tropical entries. This is known as the tropical Stiefel map, $\pi : \mathbb{T}^{d\times n} \to \mathrm{Dr}(d, n)$. Given a matrix $A \in \mathbb{T}^{d\times n}$, the tropical linear space $\pi(A)$ can be thought of as a tropical analog of 'the span of the rows of A'. However, not all valuated matroids arise this way (see Theorem 6.19). Fink and Rincón showed that if a valuated matroid is in the image of the tropical Stiefel map then the facets of the regular subdivision are transversal matroids [FR15] (see Theorem 6.13). We prove the converse:

Theorem 1.6. *(Theorem 6.47) A valuated matroid is in the image of the tropical Stiefel map if and only if the facets of the matroid subdivision are transversal matroids.*

Therefore we call any such valuated matroid V a *transversal valuated matroid* and any matrix in $A \in \pi^{-1}(V)$ a *presentation* of V. The name presentation, comes from classical transversal matroid theory. Transversal matroids arise from matchings in bipartite graphs, which can be represented by a $\{0, 1\}$-matrix, which in the tropical world becomes a $\{0, \infty\}$-matrix. In analogy with the work of Brualdi and Dinolt [BD72] on classical transversal matroids, we give an explicit description of the space of presentations of a given transversal valuated matroid and in particular we show the following:

Theorem 1.7. *(Theorem 6.40) The space of presentations of a transversal valuated matroid is the orbit of an affine fan in $\mathbb{T}^{d\times n}$ under permuting rows.*

Theorem 6.40 is actually much more concrete and provides a very precise description of the space of presentations of a transversal valuated matroids.

In [Mas72] Mason introduced a class of matroids called *gammoids*, which are the minor and dual closure of the class of transversal matroids. We develop a valuated version of this class of matroids. In particular we define *valuated strict gammoids* using tropical flows on directed graphs. Ingleton and Piff proved that strict gammoids are exactly the duals of transversal matroids [IP73]. We generalize this to valuated matroids:

Theorem 1.8. *(Theorem 6.23) Valuated strict gammoids are exactly the duals of valuated transversal matroids.*

This in particular characterizes tropical linear spaces that are stable intersections of tropical hyperplanes. Theorems 1.6 and 1.7 can be dualized to provide statements for this scenario. More concretely, valuated strict gammoids characterize tropical linear spaces that are the stable intersection of tropical hyperplanes. The space of tropical hyperplanes with a given stable intersection is described by Theorem 6.40.

1.4 Harnack curves

This part is based on [Ola17].

Harnack curves are real algebraic plane curves with several remarkable properties. They are named after Axel Harnack [Har76], who first discovered them to show that his upper bound on the number of connected components that a plane curve can have is tight. For projective toric surfaces this bound is $g+1$. Part of Hilbert's 16th problem is to classify all topological types of curves that achieve such maximum number of components.

Much later, Mikhalkin defined them for any projective toric surface as follows.

Let Δ be a lattice polygon with m sides, g interior points and n boundary points. Let X_Δ be the real part of the toric projective surface associated to the normal fan of Δ, Let $L_1, \dots, L_m \subseteq X_\Delta$ be the axes of X_Δ, that is, the 1-dimensional torus invariant closed sets, ordered according to the clockwise order of the sides of Δ. Let $d_1, \dots, d_m$ the integer lengths of the respective sides of Δ.

Definition 1.9. [Mik00, MR01] (Definitions 7.3 and 7.4) Let $C \subseteq X_\Delta$ be a smooth real algberaic curve with Newton polygon Δ. It is called a *Harnack* curve if the following conditions hold:

- The number of connected components of C is $g+1$.
- Only one component of C intersects $L_1 \cup \cdots \cup L_m$, and this can be divided into m arcs, $\theta_1 \dots \theta_m$, that appear in this order along C, such that $\theta_j \cap L_j$ consists of d_j points (counted with multiplicity) and $\theta_j \cap L_k = \emptyset$ for $j \neq k$.

If C is not smooth then it is a Harnack curve if its only singularities are isolated double points and when all of these double points are replaced by a small circle around it the result has the same topological type of a smooth Harnack curve.

Mikhalkin made progress on Hilbert's 16th problem by proving that for a fixed Δ, the topological type of smooth Harnack curves is unique [Mik00, Theorem 3]. Shortly after, several remarkable properties about Harnack curves, specially about their *amoebas* (see Theorem 7.7) were found. For example, Harnack curves are precisely the curves whose amoebas have the maximum area [MR01]. Amoebas of Harnack curves also appear in physics, where the dimer model is used to study crystal surfaces (see [KOS06] for details). In this model, the limit of the shape of a crystal surface is given by the amoeba of a Harnack curve.

Motivated by the dimer model, Kenyon and Okounkov [KO06] studied the space of projective Harnack curves $C \subseteq \mathbb{RP}^2$ of degree d modulo the torus action. Equivalently, this is the space of amoebas of Harnack curves modulo translation. They show that this moduli space has global coordinates given by the areas of holes of the amoeba and the distances between consecutive tentacles. Therefore it is diffeomorphic to $\mathbb{R}_{\geq 0}^{(d+4)(d-1)/2}$. Crétois and Lang [CL18] generalized some of the techniques used in [KO06] to Harnack curves in any projective toric surface. They showed that given a lattice polygon Δ, the moduli space $\mathcal{H}_\Delta$ of Harnack curves with Newton polygon Δ is path connected and conjectured that it is also contractible. In Chapter 7 we prove this conjecture. Moreover, we show that $\mathcal{H}_\Delta$ has a set of global coordinates similar to [KO06], generalizing their computation of $\mathcal{H}_\Delta$:

Theorem 1.10. *(Theorem 7.22) Let Δ be a lattice m-gon with g interior lattice points and n boundary lattice points. Then the moduli space $\mathcal{H}_\Delta$ of Harnack curves with Newton polytope Δ is diffeomorphic to $\mathbb{R}^{m-3} \times \mathbb{R}_{\geq 0}^{n+g-m}$.*

In Chapter 8 we use tropical curves to construct a compactification of $\mathcal{H}_\Delta$ similar in spirit to the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$. This compactification consists of 'Harnack meshes', that is, collections of Harnack curves that can be patchworked using Viro's method to produce a curve in $\mathcal{H}_\Delta$ (see Section 7.2.4 for a brief summary on patchworking or [Vir06] for a detailed description of the method). A Harnack mesh consists of a regular subdivision of Δ and a Harnack curve with Newton polytope Δ_i for each facet Δ_i of the subdivision, with some gluing conditions. The space of Harnack meshes is naturally stratified in cells according to which regular subdivision is used in the patchworking recipe. The above can be summed up in the following:

Theorem 1.11. *(Theorem 8.11) The space $\mathcal{H}_\Delta$ has a compactification $\overline{\mathcal{H}_\Delta}$ consisting of all Harnack meshes over Δ. Moreover, $\overline{\mathcal{H}_\Delta}$ has a cell complex structure whose poset is isomorphic to the face poset of the secondary polytope* $\mathrm{Sec}(\Delta \cap \mathbb{Z}^2)$*.*

1.5 Applications to the positive Grassmannian

We now show how all three parts of this thesis relate to each other through a common application: the positive Grassmannian. This is a particular case of the general phenomenon known as *total positivity* studied by Fomin and Zelevinsky, which resulted in the development of the theory of cluster algebras [FZ02]. Postnikov found that the positive Grassmannian enjoys very rich combinatorial structures [Pos06]. Due to a direct connection with scattering amplitutes in quantum field theory, the positive Grassmannian has gained a lot of attention from the physics community [AHBC$^+$16]. Polyhedral subdivisions have recently taken an important role in the study of the positive Grassmannian [Pos19]. We discuss this connection, while showing that several results in this thesis have direct consequences in this theory.

Definition 1.12. Let $(\mathbb{K}, \leq)$ be an ordered field. Let $A \in \mathbb{K}^{d\times n}$ be a matrix. We say A is *totally positive* if all of its maximal minors are positive. Similarly we say A is *totally non-negative* if all of its maximal minors are non-negative. The *totally positive (respectively totally non-negative) Grassmannian* $\mathrm{Gr}^+(d, \mathbb{K}^n)$ ($\mathrm{Gr}^{\geq 0}(d, \mathbb{K}^n)$) is the set of all linear spaces that admit as a basis the rows of a totally positive (respectively totally non-negative) matrix. A *positroid* is a matroid of the form $M(L)$ (see Theorem 4.7) for $L \in \mathrm{Gr}^{\geq 0}(d, \mathbb{K}^n)$.

In other words, a linear space is in the totally positive Grassmannian if all of its Plücker coordinates are non-zero and of the same sign (see Section 4.1). From the definition it is clear that $\mathrm{Gr}^{\geq 0}(d, \mathbb{K}^n)$ is the closure of $\mathrm{Gr}^+(d, \mathbb{K}^n)$ within $\mathrm{Gr}(d, \mathbb{K}^n)$. Observe that being a positroid is not a property of the matroid alone; it depends on the order of the element set $[n]$. However, being a positroid is invariant under cyclic permutations of $[n]$.

Example 1.13. The rank 2 matroid M on [4] with bases $\mathcal{B}(M) = \binom{[4]}{2} \setminus \{34\}$ is a positroid. However, the matroid M' with bases $\mathcal{B}(M') = \binom{[4]}{2} \setminus \{24\}$ is not a positroid, even though M and M' are isomorphic.

Positroids were introduced by Postnikov [Pos06] to study the totally non-negative Grassmannian through its matroid decomposition. Many combinatorial objects have been found with positroids associated to them and with nice bijections between them.

A *plabic graph* (short for planar bicolored graph), is a planar graph embedded in a closed disc such that each interior vertex is colored either black or white and all the vertices in the boundary are of degree 1. Monochromatic edges are allowed. We say that a plabic graph is of *type* (k, n) if it has n boundary vertices and the number of black vertices minus the number of white vertices is $2k - n$. The boundary vertices are labelled cyclically by $[n]$.

Every plabic graph has a postroid associated to it. One way to obtain the positroid associated to the plabic graph is the following. A *perfect orientation* $\mathcal{O}$ of a plabic graph is an orientation of the edges such that for every black vertex there is exactly one outgoing edge and for every white vertex there is exactly one ingoing edge. Assume G is a plabic graph with a perfect orientation. Let $B_0 \subseteq [n]$ be the set of boundary vertices that are targets (the edge incident to the vertex is oriented towards it). The matroid $M(G)$ associated to G is the strict gammoid of the directed graph $(G, \mathcal{O})$ with targets B_0, restricted to the vertices in the boundary (see Section 6.2.5).

Proposition 1.14. *[Pos06, Lemma 11.10] Let G be a plabic graph with a perfect orientation. The matroid $M(G)$ does not depend on the perfect orientation $\mathcal{O}$ and it is a positroid. Furthermore, every positroid arises this way.*

Remark 1.15. There are other equivalent ways of defining $M(G)$ for plabic graphs, even if they do not admit a perfect orientation.

Recall that a *gammoid* is any matroid which is the restriction of a strict gammoid. Alternatively, it is the contraction of a transversal matroid. Gammoids form a class of matroids closed under restriction, contraction and duality (see [Mas72]). A consequence of the above proposition is that positroids are gammoids.

A generalization of plabic graphs was introduced in [Pos19], called *Grassmannian graphs*. These are also planar graphs with boundary vertices labelled cyclically, but now internal vertices v have an integer number $h(v)$ from 1 to $\deg(v) - 1$ assigned called *helicity*. The *helicity* of the Grassmannian graph G is given by

$$h(G) - \frac{n}{2} = \sum_v h(v) - \frac{\deg(v)}{2}$$

where the sum goes over all internal vertices. The type of a vertex v is $(h(v), \deg(v))$ and we say a Grassmanian graph is of *type* (k, n) if it has helicity k and there are n boundary vertices. Plabic graphs of type (k, n) are Grassmannian graphs of the same type by assigning helicity 1 to white vertices and helicity $\deg(v)-1$ to each black vertex v.

We restrict our attention to Grassmannian and plabic graphs that are *reduced*, which means you do not get isolated components, double edges or loops when performing certain moves that we will not describe (see [Pos06, Section 12]). Just like plabic graphs, each Grassmannian graph has a positroid assigned to it. Reduced Grassmannian graphs can always be given a perfect orientation.

We call a Grassmannian graph *complete* if its corresponding positroid is the uniform matroid. We say that a reduced Grassmannian graph G coarsens G' if G can be obtained from G' by replacing vertices v of type (a, b) with a complete reduced Grassmannian graphs of the same type; we write $G \geq G'$. Coarsening does not change the positroid of the graph, so it is a partial order on the set of reduced Grassmannian graphs of a given positroid. The following theorem connects hypersimplicial subdivisions of Part I with the positive Grassmannian.

Theorem 1.16. *[Pos19, Theorem 11.1] The poset of complete reduced Grassmannian graphs of type (k, n) is isomorphic to the baues poset $\mathcal{B}(\Delta_{k,n} \to \mathbf{P}_n)$ of the projection from the hypersimplex $\Delta_{k,n}$ to a convex polygon P_n (see Section 3.2.2). Trivalent plabic graphs are the minimal elements of this poset and this isomorphism extends the bijection given in [Gal18] between them and hypertriangulations.*

The isomorphism consists of replacing vertices of type (a, b) with tiles $[X, Y]^{(k)}$ with $|X| = k-a$ and $|Y| = k-a+b$. The moves on plabic graphs we mentioned correspond to flips of zonotopal tilings of the zonotope $Z(P_n)$. The following corollary is a consequence of Theorem 3.55:

Corollary 1.17. *The poset of complete reduced Grassmannian graphs of type (k, n) has the homotopy type of an $(n-4)$-dimensional sphere.*

Plabic graphs allow us to describe the matroid cells of the totally non-negative Grassmannian in the following way. Given a positroid M, let

$$S_M := \{L \in \mathrm{Gr}^{\geq 0}(d, \mathbb{K}^n) \;\; M(L) = M\}$$

be the *matroid cell* of M intersected with the non-negative Grassmannian.

Now consider a reduced plabic graph G of type (d, n) such that $M(G) = M$ and consider a perfect orientation $\mathcal{O}$ on G where $B_0 \subseteq [n]$ are the target vertices. A *plabic network* $\mathcal{N}$ on $(G, \mathcal{O})$ is an assignment of a positive real x_e to each edge e of G. For simplicity, we assume that $(G, \mathcal{O})$ does not have directed cycles. For every $B \in \binom{[n]}{d}$ let

$$w_B(\mathcal{N}) := \sum_{\Theta} \prod_{e \in \Theta} x_e^{\pm 1}$$

where the sum is over all linkings from B to B_0 (see Theorem 6.21) in the undirected graph G and the exponent x_e is 1 if the orientation of e agrees with the direction of the linking and -1 if it does not. By convention, the sum above is taken to be 0 if no such linking exists. The vector $(w_B)_{B \in \binom{[n]}{d}}$ satisfies the Plüker relations, so we have a well defined map from the set of all plabic networks on G to the Grassmannian $\mathrm{Gr}(d, \mathbb{R}^n)$ called the *measurement map*. The image of the map from the set of all planar networks on G to $\mathrm{Gr}(d, \mathbb{K}^n)$ which sends $\mathcal{N} \mapsto [w_B(\mathcal{N})]_{B \in \binom{[n]}{d}}$ is exaclty S_M [Pos06, Corollary 16.5].

This parametrization can be tropicalized. Williams and Speyer defined the tropical positive Grassmannian [SW05] and the tropical non-negative Grassmannian can be defined similarly. Recall the field of real Puiseux series $\mathbb{R}\{\{t\}\}$ (see Section 2.3). It is an ordered field in which an element $f \in \mathbb{R}\{\{t\}\}$ is positive if the coefficient of the leading term is positive. The positive Grassmannian $\mathrm{Gr}^+(d, \mathbb{R}\{\{t\}\}^n)$ can be defined in the same way as $\mathrm{Gr}^+(d, \mathbb{R}^n)$.

Definition 1.18. The *tropical positive Grassmannian* is

$$\mathrm{TGr}^+ := \overline{\nu(\mathrm{Gr}^+(d, \mathbb{R}\{\{t\}\}^n))} \subseteq \mathrm{TGr}_0(d, n)$$

and the *tropical non-negative Grassmanian* is

$$\mathrm{TGr}^{\geq 0} := \overline{\nu(\mathrm{Gr}^{\geq 0}(d, \mathbb{R}\{\{t\}\}^n))} \subseteq \mathrm{TGr}_0(d, n),$$

(see Theorem 4.1). We call V a *valuated positroid* if $V \in \mathrm{TGr}^{\geq 0}(d, n)$.

In [SW05] the tropical positive Grassmannian is parametrized. This is essentially tropicalizing the parametrization of the positive Grassmannian $\mathrm{Gr}^{\geq 0}(d, \mathbb{R}\{\{t\}\}^n)$ by flows in a particular complete plabic network with positive Puiseux series as weights. Since we are restricting ourselves to positive Puiseux series, there is never cancellation when computing the measurement maps, so the measurement map and the valuation commute. This works for any plabic graph. The tropical measurement map is exactly what in Section 6.2.5 we call *the weight of a linking*, which we use to define valuated strict gammoids.

Definition 1.19. A valuated gammoid is the restriction of a valuated strict gammoid. Alternatively, it is the contraction of a valuated transversal matroid.

The discussion above implies the following:

Proposition 1.20. *Valuated positroids are valuated gammoids.*

Proof. Let V be a valuated positroid. Consider a plabic network without loops whose tropical measurement maps give V. Such a plabic network exists by taking the web plabic graph (see [SW05, Figure 1]) and by [Pos06, Corollary 16.5]. Then V is the restriction to the boundary points of the valuated gammoid given by that the plabic network. □

This result together with Theorem 6.22 and the interpretation of valuated matroid minors and duality in terms of the tropical linear space discussed in Section 4.5 imply the following:

Corollary 1.21. *Let L be the tropicalization of a linear space in the totally non-negative Grassmannian, i.e., $L = \mathcal{L}(V)$ for $V \in \mathrm{TGr}^{\geq 0}(d, n)$. Then:*

- *L is the projection of a stable intersection of tropical hyerplanes.*
- *L is the intersection of a stable sum of points with $\{x_j = \infty \; j \in J\}$ for some subset of coordinates j.*

In Chapter 6 we show that transversality is a local property, i.e., a valuated matroid is transversal if and only if it all the matroids in its subdivision are transversal (see Theorem 6.47). Similarly, it is easy to show that all matroids in $\mathcal{M}(V)$ for a valuated positroid V are positroids. The converse is an open problem that has been asked by Felipe Rincón and is related to Problem 7.10 in [RVY19].

Conjecture 1.22. *A valuated matroid V is a valuated positroid if and only if every matroid in $\mathcal{M}(V)$ is a positroid.*

Recall that $\mathrm{TGr}(2, n)$ (which does not depend on the charateristic) is isomorphic to the space of phylogenetic trees, see Theorem 4.2.

Proposition 1.23. *[SW05, Proposition 5.3] The phylogenetic trees that correspond to $\mathrm{TGr}^{+}(2, n)/\mathbb{R}^n$ are those which are dual to triangulations of a polygon $\mathbf{P}_n$ whose edges are labeled by $[n]$ in cyclical order.*

The way in which total positivity relies on the cyclic order is strikingly similar to Harnack curves (see Theorem 4.2). One way in which this is reflected is that the $3 \times n$ matrix used in the proof of Theorem 7.19 is totally non-negative. We can make a more explicit connection. Let Δ be a lattice polygon with n sides and no lattice points in the boundary other than vertices. Consider a rational Harnack curve C in $\mathcal{H}_{0,\Delta}$ and let $\Upsilon'(C) \in \mathcal{M}_{0,n}^{\text{trop}}$ be the amoeba of C (we here mean the usual spine that does not take into account contracted components of the complement of the amoeba, not the expanded spine that we introduce in Theorem 7.9). Let us identify $\text{TGr}^+(2,n)$ with the corresponding subsapce of $\mathcal{M}_{0,n}^{\text{trop}}$ by Theorem 1.23.

Corollary 1.24. *The map $\Upsilon' : \mathcal{H}_{0,\Delta} \to \mathcal{M}_{0,n}^{trop}$ is a homeomorphism between $\mathcal{H}_{0,\Delta}$ and* $\text{TGr}^+(2,n)$.

One could expect then to associate the closure $\overline{\mathcal{H}_{0,\Delta}}$ with the tropical non-negative Grassmannian. However modding out the lineality space makes the corresponding valuated matroid ambiguous. Instead, similar as what we do in Chapter 8, the reasonable object to use as boundary in a compactification of $\text{TGr}^+(2,n)$ in $\mathcal{M}_{0,n}^{\text{trop}}$ is collections of valuated positroids, one for each facet of a matroid subdivision.

This same idea can be used to construct a compactification of the local Dressian of a given matroid (see Chapter 5) modulo the lineality space, in a similar way as in Theorem 1.11.

Proposition 1.25. *Let M be a matroid on $[n]$ and L be the lineality space of its local Dressian $\text{Dr}(M)$. Then $\text{Dr}(M)/L$ has a compactification given by considering all regular matroid subdivisions S of P_M and all collections of valuated matroids $V_1, \dots V_s$, one for each facet of S, such that that the linear spaces agree in the common intersection. More precisely, if M is the intersection of the undelying matroids of V_i and V_j, i.e. $P_M = P_{\underline{V_i}} \cap P_{\underline{V_j}}$, then*

$$V_i|_M = V_j|_M \in \text{Dr}(M).$$

Chapter 2

Review on polytopal subdivisions and tropical geometry.

2.1 Polyhedral complexes

We start by introducing the basic concepts of polyhedral theory. The main references regarding polyhedra and polyhedral complexes are [Zie95, Grü03] and for subdivisions and triangulations we recommend [DLRS10].

A *polyhedron* P is the feasible set of a system of linear inequalities, that is

$$P = \{x \in \mathbb{R}^d \mid Ax \leq b\}$$

for a matrix $A \in \mathbb{R}^{d \times m}$ and $b \in \mathbb{R}^m$. If P is a bounded polyhedron we call it a *polytope*. Its *dimension* is the dimension of its affine span.

Given a functional $w : \mathbb{R}^d \to \mathbb{R}$, the *face* of P defined by w is

$$P^w := \{x \in P \mid w \cdot x \leq w \cdot y, \ \ \forall y \in P\}.$$

Every face of a polyhedron is a polyhedron and the face of a polytope is a polytope. Notice that P itself is a face (defined by the 0 functional) and, by convention, the emptyset is also considered a face. We call *vertices* the 0-dimensional faces, *edges* the (bounded) 1-dimensional faces, and *facets* the codimension 1 faces. The faces of a polytope form a lattice (as in poset) by inclusion.

Any polytope is the convex hull of its vertices, that is, if $v_1, \dots, v_m$ are the vertices of P then

$$\begin{aligned}P &= \operatorname{conv}(v_1, \dots, v_n)\\ &:= \{\lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0},\ \ \lambda_1 + \cdots + \lambda_n = 1\}.\end{aligned}$$

Example 2.1.

- *The cube* $[0,1]^n$ is the feasible set of the system $\{0 \leq x_i \leq 1 \mid i \in [n]\}$. It is the convex hull of all $\{0,1\}$-vectors in $\mathbb{R}^n$.
- *The symplex* Δ_{n-1} is the convex hull of $\{e_i \mid i \in [n]\}$. It is of dimension $n-1$, since it is the intersection of the cube with the plane $\{x_1 + \cdots + x_n = 1\}$.
- *The* k,n*-hypersimplex* is the convex hull of all $\{0,1\}$-vectors with exactly k ones. It is the intersection of the cube with the plane $\{x_1 + \cdots + x_n = k\}$. The vertices of the hypersimplex are in correspondence with subsets in $\binom{\{[n]}{k\}}$, by taking the indicator vector of the subset. The edges are always parallel to $e_i - e_j$ for $i, j \in [n]$. Hypersimplices are possibly the most important polytopes in this thesis. They are the main characters of Part I and they also play a major role in Part II (see for example Theorem 4.10).

Definition 2.2. A polyhedral complex $\mathcal{C}$ is a collection of polyhedra satisfying the following axioms:

1. If $P \in \mathcal{C}$ and F is a face of P then $F \in \mathcal{C}$ *(Closure Proeperty)*.
2. For any $P, Q \in \mathcal{C}$ the intersection $P \cap Q$ is a (possibly empty) face of both P and Q *(Intersection Property)*.

The *support* of a polyhedral complex $\mathcal{C}$ is the set

$$\bigcup_{P \in \mathcal{C}} P$$

We call elements of a polyhedral complex *cells*. We say that a polyhedral complex is *pure* if all the maximal cells are of the same dimension. A *(polyhedral) cone* σ is a polyhedron such that for any $x \in \sigma$ and $\lambda \in \mathbb{R}_{\geq 0}$ we have that $\lambda x \in \sigma$. The *lineality space* of a cone is the largest linear space contained in it. Equivalently, the lineality space of a cone is its minimal non empty face. A *(polyhedral) fan* is a polyhedral complex consisting of cones. All cones in a fan have the same lineality space, so we call that space the lineality space of the fan. If the support of a fan is all of the ambient space $\mathbb{R}^d$ we say that the fan is *complete*. Given a face F of a polytope P, the closure of the set of vectors that defines that face forms a cone:

$$\sigma(F) = \overline{\{w \in \mathbb{R}^d \mid P^w = F\}}$$

The *(inner) normal fan* of a polytope is the fan

$$\{\sigma(F) \mid F \text{ face of } P\}.$$

The normal fan is a complete fan. The dimension of the lineality space of the normal fan of P is the codimension of P; more precisely, the lineality space is the orthogonal complement of the linear space parallel to the affine span of P. Equivalently, the lineality space of the normal fan is the space of functionals constant over P.

Remark 2.3. One can also define the normal fan of an unbounded polyhedron. In such case, we only consider the linear functionals w for which P achieves its minimum. So in this case the normal fan is not complete.

Definition 2.4. Let $A \subseteq \mathbb{R}^d$ a point configuration, i.e., a finite set of points. A *polytopal subdivision* of A is a collection of subsets of A, $\mathcal{S} = \{S_1, \ldots, S_s\}$, such that

$$\{\text{conv}(S_1), \ldots, \text{conv}(S_s)\}$$

is a polyhedral complex with support $\text{conv}(A)$. For technical reasons, we require that for every polytope P there is at most one set S_i with $S_i \in P$ and if P is a face of $\text{conv}(S_j)$ then $S_j \cap P = S_i$. As with polyhedral complexes, we call the elements of a polytopal subdivision *cells*.

Remark 2.5. Sometimes we want to allow A to have repeated points. In other words, points may have different labels. We consider different subdivisions if the points they use have different labels (actually, the definition in [DLRS10] uses labels). In this thesis, this matters in Chapter 3, where different vertices of the hypersimplex may be projected to the same point.

Given two subdivisions $\mathcal{S}_1$ and $\mathcal{S}_2$ of A we say that $\mathcal{S}_1$ *refines* $\mathcal{S}_2$ if every cell in $\mathcal{S}_1$ is contained in a cell of $\mathcal{S}_2$. It is easy to see that refinement is a partial order; it is transitive and antisymmetric. Hence the set of all subdivisions of A is a poset. We write $\mathcal{S}_1 \leq \mathcal{S}_2$ if $\mathcal{S}_1$ refines $\mathcal{S}_2$. The maximal element of this poset is the subdivision $\{A\}$. Minimal elements of this poset are called *triangulations* of A. The name comes from the fact that in a triangulation every cell is a simplex.

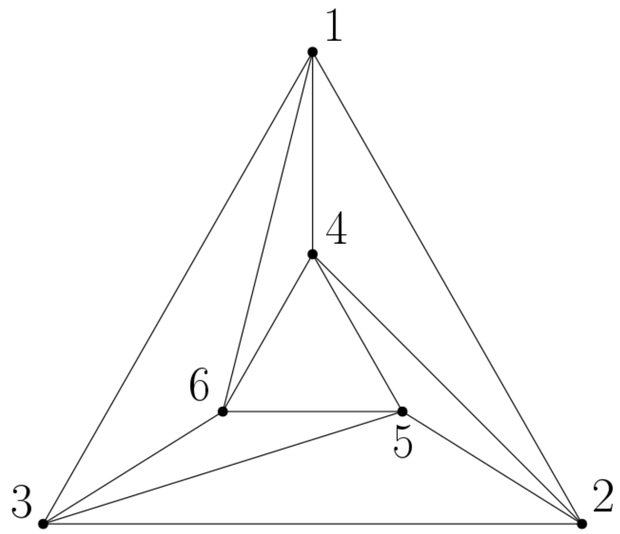


FIGURE 2.1: The mother of all examples

2.2 Regular subdivisions

We now show one of the main recipes for constructing polytopal subdivisions. Let $A \subseteq \mathbb{R}^d$ be a point configuration. Given a function $h : A \to \mathbb{R}$, consider the polyhedron

$$P_h = \mathrm{conv}(\{(a,t) \mid a \in A,\ t \geq h(a)\}).$$

For any bounded face F of P_h, let $A_F := \{a \in A \mid (a, h(a)) \in F\}$. Then the regular subdivision of A induced by h is

$$\mathcal{S}_h(A) := \{A_F \mid F \text{ bounded face of } P_h\}.$$

Colloquially, the regular subdivision induced by h consists of lifting the points of A at height h, and projecting back the 'downward looking' faces of the convex hull. Not all subdivisions are regular, as the following examples shows:

Example 2.6 (The mother of all examples)**.** Consider the point configuration of Figure 2.1 where triangle $\{1,2,3\}$ is homothetic to $\{4,5,6\}$ by a factor of $\lambda > 1$. Suppose the triangulation displayed is regular. Then it exists a height function $h : [6] \to \mathbb{R}$. The existence of the line $\{1,6\}$ implies that $h(1) + \lambda h(6) < h(3) + \lambda h(4)$, the existence of the line $\{2,4\}$ implies that $h(2) + \lambda h(4) < h(1) + \lambda h(5)$ and the existence of the line $\{3,5\}$ implies that $h(3) + \lambda h(5) < h(2) + \lambda h(6)$, all together producing a contradiction.

Let $h : A \to \mathbb{R}$ be any function. The following facts are easy to verify:

1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be any affine linear function and let $h' = h + f|_A$. Then $\mathcal{S}_h(A) = \mathcal{S}_{h'}(A)$.

2. Let $\lambda \in \mathbb{R}$ be a positive scalar. Then $\mathcal{S}_h(A) = \mathcal{S}_{\lambda h}(A)$.

3. Let $h'A \to \mathbb{R}$ be a function such that $\mathcal{S}_h(A) = \mathcal{S}_{h'}(A)$. Then $\mathcal{S}_h(A) = \mathcal{S}_{h+h'}(A)$.

Properties 2 and 3 imply that given a regular subdivision $\mathcal{S}$, the set

$$\sigma_{\mathcal{S}}(A) := \{h \in \mathbb{R}^A \mid \mathcal{S}_h(A) = \mathcal{S}\}$$

is a cone. Property 1 implies that this cone contains the linear subspace given by the restrictions to A of all affine functions $f : \mathbb{R}^d \to \mathbb{R}$. The dimension of this linear space is the affine dimension $\dim(A)$ of A plus 1.

Definition 2.7. The *secondary fan* of A is the fan given by

$$\{\overline{\sigma_{\mathcal{S}}(A)} \mid \mathcal{S} \text{ regular subdivion of } A\}.$$

This fan is complete and has a lineality space of dimension $\dim(A) + 1$.

Theorem 2.8. *(Gelfand, Kapranov, Zelevinsky [GKZ94, Chapter 7, Theorem 1.17]) There exists a polytope* $\mathrm{Sec}(A)$*, called* the secondary polytope of A*, whose inner normal fan equals the secondary fan of* A*.*

This polytope lives in $\mathbb{R}^n$. However, its codimension equals the dimension of the linearity space of the secondary fan of A, that is, $\dim(A) + 1$. By Theorem 2.8, faces of $\mathrm{Sec}(A)$ are in bijection with regular subdivisions of A. This bijection is order preserving, so the face poset of $\mathrm{Sec}(A)$ is isomorphic to the poset of regular subdivisions of $\mathrm{Sec}(A)$. In particular, vertices of $\mathrm{Sec}(A)$ are in bijection with regular triangulations of A.

2.3 Tropical geometry

Tropical geometry is the study of tropical varieties, which can be regarded as ‘combinatorial shadows’ of algebraic varieties. A general philosophy in tropical geometry is to take objects and theorems of algebraic geometry and find tropical analogues of them. For example, there is an analogue of Bezout’s theorem for tropical curves (see [MS15]). Tropical varieties retain much of the information of their algebraic counterparts, such as the degree. However, the combinatorial nature of tropical varieties allow for simpler computations on them. A great example of this is Mikhalkin’s celebrated correspondence theorem [Mik05]. Our main reference for tropical geometry is [MS15].

Definition 2.9. The semiring of *tropical numbers* consists of $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ with the following operations:

$$a \oplus b = \min(a, b) \quad \text{and } a \odot b = a + b$$

where $a \oplus \infty = a$ and $a \odot \infty = \infty$.

So, for example, in the tropical semiring

$$2 \oplus 3 = 2 \quad \text{and } 2 \odot 3 = 5$$

. In this semiring, ∞ is the additive identity element and 0 is the multiplicative identity element.

As a topological space, $\mathbb{T}$ is a partial compactification of the Euclidean topology on $\mathbb{R}$ by the point ∞. We take infinity to be positive, i.e. as a basis of neighbourhoods it has the compactified rays $(h, \infty]$ for $h \in \mathbb{R}$.

Just as we have polynomials over a ring, we have that a tropical polynomial f in d variables is of the form

$$f(x) = \bigoplus_{a \in A} c_a x^{\odot a} = \min(x \cdot a + c_v \mid a \in A)$$

where $A \subseteq \mathbb{N}^d$ is a finte set and $c_a \in \mathbb{R}$. We define the zero set of F as

$$\mathbf{V}(f) = \{x \in \mathbb{T}^d \mid \text{ the minimum in } f(x) \text{ is achieved at least twice}\}.$$

We call any $\mathbf{V}(F)$ a *tropical hypersurface.*

Example 2.10. Consider the tropical polynomial $f(x, y) = x \oplus 1 \odot y \oplus 3$. The tropical hypersurface defined by f consists of 3 rays, one for $x = 1 + y \leq 3$, another for $x = 3 \leq 1 + y$ and the other one for $1 + y = 3 \leq x$. The three rays meet at the vertex $(3, 2)$.

In general, for any subset of monomials of f the region where those terms are exactly the ones achieving the minimum is a polyhedron. So $V(f)$ is always the support set of a polyhedral complex.

To connect algebraic geometry with tropical geometry there is a process called tropicalization, which takes polynomials and algebraic varieties to tropical varieties and tropical hypersurfaces. This process requires a field with a valuation:

Definition 2.11. Let $\mathbb{K}$ be a field. A *valuation* ν over a $\mathbb{K}$ is a function $\nu : \mathbb{K} \to \mathbb{T}$ such that:

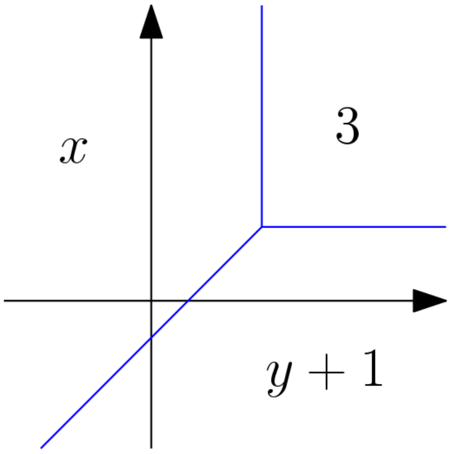


FIGURE 2.2: The tropical hypersurface $V(x \oplus 1 \odot y \oplus 3)$

- For all a and b in $\mathbb{K}$, $\nu(a+b) \geq \nu(a) \oplus \nu(b)$ with equality if $\nu(a) \neq \nu(b)$.
- For all a and b in $\mathbb{K}$, $\nu(ab) \geq \nu(a) \odot \nu(b)$.
- $\nu(a) = \infty$ if and only if $a = 0$.

Example 2.12. For any field $\mathbb{K}$, the function

$$\nu(a) := \begin{cases} \infty, & \text{if } a = 0 \\ 0, & \text{if } a \neq 0 \end{cases}$$

is always a valuation called the *trivial* valuation.

Example 2.13. The field of Puiseux series over a field $\mathbb{K}$, written $\mathbb{K}\{\{t\}\}$ is the field of formal power series of the form

$$\sum_{i>m}^{\infty} a_{i/n} t^{i/n}$$

where $n \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}$, $a_i \in \mathbb{K}$ for every $i \in \mathbb{Z}_{\geq m}$ and t is just a formal variable. The sum and multiplication are the expect ones which extend that of polynomials. If $\mathbb{K}$ is algebraically closed then $\mathbb{K}\{\{t\}\}$ is also algebraically closed. It has a valuation given by

$$\sum_{i>m}^{\infty} a_{i/n} t^{i/n} \mapsto \min(\{q \in \mathbb{Q} \mid a_q \neq 0\}).$$

with the convention $\nu(\emptyset) = \infty$. The image of this valuation is $\mathbb{Q} \cup \{\infty\}$. For more on Puiseux series see [Mar10]

Let f be any Laurent polynomial in $\mathbb{K}[x_1^{\pm}, \ldots, x_d^{\pm}]$. We can write it as

$$f = \sum_{a \in A} c_a x^a$$

for a finite set $A \subseteq \mathbb{Z}^d$. The *tropicalization* of f is the tropical polynomial

$$\begin{aligned}\operatorname{trop}(f) &:= \bigoplus_{a\in A} \nu(c_a)x^{\odot a}\\ &= \min_{a\in A}(\nu(c_a) + a\cdot x)\end{aligned}$$

If ν is the trivial valuation, then the coefficients of $\operatorname{trop}(f)$ are all equal to 0. In this case, the tropical hypersurface $V(\operatorname{trop}(f))$ is a fan.

For a vector $x = (x_1, \ldots, x_d) \in \mathbb{K}^d$ we write $\nu(x)$ for $(\nu(x_1), \ldots, \nu(x_d)) \in \mathbb{T}^d$. The following is one of the earliest theorems in tropical geometry, stating the link between tropical hypersurfaces and algebraic hypersurfaces (see [MS15, Theorem 3.1.3]):

Theorem 2.14. *Kapranov's Theorem [EKL06] Let $\mathbb{K}$ be an algebraically closed field with a non-trivial valuation ν. Let $f \in \mathbb{K}[x_1^{\pm}, \ldots, x_d^{\pm}]$ be a Laurent polynomial and $\mathbf{V}(f) \subseteq \mathbb{K}^*$ be its zero set. Then*

$$\mathbf{V}(\operatorname{trop}(f)) = \overline{\nu(\mathbf{V}(f))}.$$

Informally, Kapranov's theorem says that taking zeros and tropicalizing commute.

Tropical hypersurfaces can be understood from the perspective of regular subdivisions. Let f be a tropical polynomial as above. The coefficients c_a can be seen as a height vector $c : A \to \mathbb{R}$, so they induce a regular subdivision $\mathcal{S}_c$ of A, given by projections of the bounded faces of the polytope

$$P = \operatorname{conv}(\{(a,t) \mid a \in A,\ \ t \geq h(a)\}.$$

Every bounded face Q of P is of the form $P^{(w,1)}$ for a vector $(w, 1) \in \mathbb{R}^{d+1}$. But $(w, 1) \cdot (a, c_a) = c_a \odot w^{\odot a}$, so A_Q consists of all the points a whose corresponding term in $f(w)$ achieve the minimum. Hence $\operatorname{trop}(f)$ is the set of all vectors w such that A_Q is not a singleton, that is, such that $P^{(w,1)}$ is not a vertex.

Given a bounded face Q of P, the of set of vector w such that $Q = P^{(w,1)}$ is the intersection of the normal cone $\sigma(Q)$ and the hyperplane $\{x_{d+1} = 1\}$. So the support of $\operatorname{trop}(f)$ is the support of a polytopal complex, which is the union of all w for which $P^{(w,1)}$ is not a vertex. To see that it is indeed a complex, notice that it is the intersection of all the cones of positive codimension of the normal fan of P with the hyperplane $\{x_{d+1} = 1\}$. This shows that $\operatorname{trop}(f)$ is 'dual' to the subdivision $\mathcal{S}_c(A)$; for every cell S of dimension $k \geq 1$ in $\mathcal{S}_c(A)$, there is a polytope of codimension k in $\operatorname{trop}(f)$ which is orthogonal to S.

Algebraic varieties of higher codimension can also be tropicalized. Let $\mathbb{K}$ be a field with valuation and consider $X = \mathbf{V}(I)$ an algebraic variety in $(\mathbb{K}^*)^d$. Then the *tropicalization* of X is

$$\operatorname{trop}(X) := \bigcap_{f \in I} V(\operatorname{trop}(f)).$$

Once again, trop(X) is the support of a polyhedral complex. The following is a generalization of Kapranov's theorem.

Theorem 2.15. *The Fundamental Theorem of Tropical Geometry [MS15, Theorem 3.2.5] Let $\mathbb{K}$ be an algebraically closed field with a non-trivial valuation ν. Let I be an ideal of $\mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$ and $X = \mathbf{V}(I)$. Then*

$$\operatorname{trop}(X) = \overline{\nu(X)}$$

Remark 2.16. If $\{f_1, \dots, f_k\}$ is a basis of I, in general trop(X) may not be the intersection of the tropical hypersurfaces $\mathbf{V}(\operatorname{trop}(f_1)) \cap \cdots \cap \mathbf{V}(\operatorname{trop}(f_1))$. Any intersection of tropical hypersurfaces is called a *tropical prevariety.* Whenever $\operatorname{trop}(X) = \mathbf{V}(\operatorname{trop}(f_1)) \cap \cdots \cap \mathbf{V}(\operatorname{trop}(f_1))$ we call $\{f_1, \dots, f_k\}$ a *tropical basis I*. Every ideal has a finite tropical basis [MS15, Theorem 2.6.5].

If X is an irreduible variety, trop(X) is a pure polyhedral complex of the same dimension.

As a direct analogy from the classical projective space, we can define the *tropical linear projective* space $\mathbb{TP}^{d-1}$ as

$$(\mathbb{T}^d \setminus \{(\infty, \dots, \infty)\})/\mathbb{R}(1, \dots, 1)$$

where $\mathbb{R}(1, \dots, 1)$ is tropical scaling (adding a constant to all coordinates). If the tropical polynomial f is homogeneous, i.e. all monomials have the same degree, then $\mathbf{V}(f)$ is closed under the $\mathbb{R}$ action. So there is a well defined vanishing locus

$$(\mathbb{T}^d \setminus \{(\infty, \dots, \infty)\})/\mathbb{R}(1, \dots, 1) \subset \mathbb{TP}^{d-1}$$

which we call a *tropical projective variety.*

In this thesis we study two particular types of tropical varieties:

- Those of degree 1, that is, tropical linear spaces. They are the main object of study in Part II. We take tropical linear spaces as projective tropical varieties.

- Hypersurfaces of dimension 1, that is, plane tropical curves. In Part III we associate to each Harnack curve a plane tropical curve that we call the expanded spine (see Theorem 7.9)

Part I

Hypersimplicial Subdivisions

Chapter 3

Hypersimplicial Subdivisions

3.1 Introduction

The main object of study in this chapter are hypersimplicial subdivisions, defined as follows. Let A be a set of n points affinely spanning $\mathbb{R}^d$. Let Δ_n be the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$. Consider the linear projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ sending the vertices of Δ^n to the points in A. (We implicitly consider the points in A labelled by $[n]$, so that π sends e_i to the point labelled by i). Let $\Delta_n^{(k)} := k\Delta_n \cap [0,1]^n$ be the standard hypersimplex and $A^{(k)}$ the image of the vertices of $\Delta_n^{(k)}$ under π (so that points in $A^{(k)}$ are labelled by k-subsets of $[n]$). A *hypersimplicial subdivision* of $A^{(k)}$ is a polyhedral subdivision of $\mathrm{conv}(A^{(k)})$ such that every face of the subdivision is the image of a face of $\Delta_n^{(k)}$ under π. Put differently, we call hypersimplicial subdivisions the π-induced subdivisions of the projection $\pi : \Delta_n^{(k)} \to \mathrm{conv}(A^{(k)})$, as introduced in [BS92, BKS94] (see also [Rei99, DLRS10]). See more details in Section 3.2.

One reason to study such subdivisions comes from the case where $A \subset \mathbb{R}^2$ are the vertices of a convex polygon. Galashin [Gal18] shows that in this case fine hypersimplicial subdivisions, which we call *hypertriangulations*, are in bijection with maximal collections of chord-separated k-sets. These, in turn, correspond to reduced plabic graphs, [OPS15] which are a fundamental tool in the study of the positive Grassmannian [Pos06, Pos19].

More generally, it is of interest the case where A are the vertices of a cyclic polytope $\mathbf{C}(n,d) \subset \mathbb{R}^d$. (The n-gon is the case $d = 2$). In [Pos19, Problem 10.3] Postnikov asks the *generalized Baues problem* for this scenario; that is, he asks whether the poset of hypersimplicial subdivisions of $\mathbf{C}(n,d)^{(k)}$ has the homotopy type of a $(n-d-2)$-sphere. For $k = 1$ this was shown to have a positive answer by Rambau and Santos [RS00]. For $d = 2$, Balitskiy and Wellman show the poset to be simply connected and again ask the

Baues question for it ([BW19, Theorem 6.4 and Question 6.1]). We here give the answer to this:

Theorem 3.1. *Let* $\mathbf{P}_n$ *be the vertices of any convex* n*-gon. The poset of hypersimplicial subdivisions* $\mathcal{B}(\Delta_n^{(k)} \to \mathbf{P}_n^{(k)})$ *retracts onto the poset of* coherent *hypersimplicial subdivisions. In particular, it has the homotopy type of an* $(n-4)$*-sphere.*

[Pos19, Problem 10.3] also asks for which values of the parameters can all hypersimplicial subdivisions of $\mathbf{C}(n,d)^{(k)}$ be lifted to zonotopal tilings of the cyclic zonotope. This was already known to be false for $d = 1$ [Pos19, Example 10.4] and we generalize the counterexamples to every odd dimension:

Theorem 3.2. *Consider the cyclic polytope* $\mathbf{C}(n,d) \subset \mathbb{R}^d$ *for odd* d *and* $n \geq d+3$*. Then, for every* $k \in [2, n-2]$ *there exist hypersimplicial subdivisions of* $\mathbf{C}(n,d)^{(k)}$ *that do not extend to zonotopal tilings of the cyclic zonotope* $Z(\mathbf{C}(n,d))$*.*

In contrast, Galashin [Gal18] showed that the answer to Postnikov's question is positive in dimension two for *hypertriangulations*, a result that was generalized to all hypersimplicial *subdivisions* by Balitskiy and Wellman [BW19, Lemma 6.3].

The poset of coherent hypersimplicial subdivisions of any A is isomorphic to the face poset of a polytope, a particular case of a fiber polytope. When $k = 1$ this is just the secondary polytope of A, so for $k > 1$ we call it the *k-th hypersecondary polytope of* A. We study hypersecondary polytopes for any $A \subset \mathbb{R}^d$ and $k \leq d+1$. Specifically, we show that these polytopes are normally equivalent to the Minkowski sum of certain faces of the secondary polytope of A. By symmetry, an analogue statement holds for $n - d - 1 \leq k < n$.

Theorem 3.3. *Let* $A \subseteq \mathbb{R}^d$ *be a configuration of size* n *and* $k \in [d+1]$*. Let* $s = \max(n-k+1, d+2)$*. The hypersecondary polytope* $\mathcal{F}^{(k)}(A)$ *is normally equivalent to the Minkowski sum of the secondary polytopes of all subsets of* A *of size* s*.*

The chapter is organized as follows: Section 3.2 introduces notation and basic background on induced subdivisions in general, and hypersimplicial subdivisions in particular. In Section 3.3 we look at coherent hypersimplicial subdivisions and hypersecondary polytopes as Minkowski sums and prove Theorem 3.3, among other results. In Section 3.4 we study the connection of hypersimplicial subdivisions with zonotopal tilings. In particular, we extend to tiles of positive dimension the concept of *A-separated sets* introduced in [GP17]. With this machinery we show that if all hypertriangulations of A are separated then all hypersubdivisions are separated too (Theorem 3.36). In Section 3.5 and Section 3.6 we prove Theorem 3.2 and Theorem 3.1 respectively. Finally, we briefly discuss the enumeration of hypersimplicial subdivisions of $\mathbf{P}_n^{(2)}$ in Section 3.7.

3.2 Preliminaries and notation

3.2.1 Fiber polytopes

We here briefly recall the main concepts and results on fiber polytopes. See [BS92] or [Rei99] for more details.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be a linear projection map. Let $Q \subset \mathbb{R}^n$ be a polytope and let $A = \pi(\text{vertices}(Q))$. A π-induced subdivision of A is a polyhedral subdivision S (in the sense of, for example, [DLRS10]), such that every face of S is the image under π of a face F of Q.

Given a vector $w \in (\mathbb{R}^n)^*$ the face Q^w of Q selected by w is the convex hull of all vertices of Q which minimize w. A π-coherent subdivison is a π-induced subdivision in which the faces of Q are chosen "coherently" via a vector $w \in (\mathbb{R}^n)^*$. More precisely, we define the *π-coherent subdivision of A given by w* to be

$$S(Q \xrightarrow{\pi} A, w) := \left\{\pi(F) : \exists \tilde{w} \in (\mathbb{R}^n)^* \text{ s.t. } \tilde{w}|_{\ker(\pi)} = w|_{\ker(\pi)} \text{ and } Q^{\tilde{w}} = F\right\}.$$

The *fiber fan* of the projection $Q \xrightarrow{\pi} A$ is the stratification of $(\mathbb{R}^n)^*$ according to what π-coherent subdivision is produced. It is a polyhedral fan with linearity space equal to

$$\{w \in (\mathbb{R}^n)^* : \ker(\pi) \subset \ker(w)\} + \{w \in (\mathbb{R}^n)^* : w|_Q = \text{constant}\}.$$

As we will see below, it is the normal fan of a certain polytope $\mathcal{F}(Q \xrightarrow{\pi} A)$ of dimension $\dim(Q) - \dim(A)$.

To define $\mathcal{F}(Q \xrightarrow{\pi} A)$, we look at fine π-induced subdivisions. A π-induced subdivision S is *fine* if $\dim(F) = \dim(\pi(F))$ for each of the faces $F \leq Q$ whose images are cells in S Put differently, a fine π-induced subdivision is the image of a subcomplex of Q that is a section of $\pi : Q \to \text{conv}(A)$. To each fine π-induced subdivision S we associate the following point:

$$\text{GKZ}(S) := \sum_{\substack{F \leq Q \\ \pi(F) \in S}} \frac{\text{vol}(\pi(F))}{\text{vol}(A)} \mathbf{c}(F) \in \mathbb{R}^n,$$

where $\mathbf{c}(F)$ denotes the centroid of F.

Definition 3.4. The *fiber polytope* of the projection $\pi : Q \to \text{conv}(A)$ is the convex hull of the vectors $\text{GKZ}(S)$ for all fine π-induced subdivisions. We denote it $\mathcal{F}(Q \to A)$.

The main property of the fiber polytope is the following result of Billera and Sturmfels. In fact, for the purposes of this chapter this theorem can be taken as a *definition* of the fiber polytope, since our results are mostly not about the polytope but about its normal fan (see, eg Section 3.3).

Theorem 3.5 (Billera and Sturmfels [BS92]). *$\mathcal{F}(Q \to A)$ is a polytope of dimension $\dim(Q) - \dim(A)$ whose normal fan equals the fiber fan.*

In particular, the face lattice of $\mathcal{F}(Q \to A)$ is isomorphic to the poset of π-coherent subdivisions ordered by refinement. For example, vertices of $\mathcal{F}(Q \to A)$ correspond bijectively to fine π-coherent subdivisions.

Two cases of this construction are of particular importance. Let $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ be a configuration of n points. Then:

1. If we let $\pi : \Delta_n \to \mathrm{conv}(A)$ be the affine map $e_i \mapsto a_i$ bijecting vertices of Δ_n to A, then all the polyhedral subdivisions of A are π-induced, and the coherent ones are usually called *regular* subdivisions of A. The corresponding fiber polytope is the *secondary polytope* of A and we denote it $\mathcal{F}^{(1)}(A)$ (in the next sections we define $\mathcal{F}^{(k)}(A)$ for other values of k).

2. Let
$$Z(A) = \sum_i \mathrm{conv}\{0, (a_i, 1)\} \subset \mathbb{R}^{d+1}$$
be the zonotope generated by the *vector* configuration $A \times \{1\} \subset \mathbb{R}^{d+1}$. The π in the previous case extends to a linear map $\pi : [0,1]^n \to Z(A)$ still sending $e_i \mapsto a_i$. Then the π-induced subdivisions are precisely the *zonotopal tilings* of $Z(A)$. The corresponding fiber polytope is the *fiber zonotope* of $Z(A)$ (or of A) and we denote it $\mathcal{F}^Z(A)$.

3.2.2 The Baues problem

The poset of all π-induced subdivisions (excluding the trivial subdivision for technical reasons) is called the *Baues poset* of the projection and we denote it $\mathcal{B}(Q \to A)$. The subposet of π-coherent subdivisions is denoted $\mathcal{B}_{\mathrm{coh}}(Q \to A)$. The *Baues problem* is, loosely speaking, the question of how similar are $\mathcal{B}(Q \to A)$ and $\mathcal{B}_{\mathrm{coh}}(Q \to A)$, formalized as follows:

To every poset $\mathcal{P}$ one can associate a simplicial complex called the *order complex of* $\mathcal{P}$ by using the elements of $\mathcal{P}$ as elements and chains in the poset as simplices. In particular, one can speak of the *homotopy type* of $\mathcal{P}$ meaning that of its order complex. Similarly, an order preserving map of posets

$$f : \mathcal{P}_1 \to \mathcal{P}_2$$

induces a simplicial map between the corresponding order complexes, and one can speak of the homotopy type of f.

The prototypical example is the following: if $\mathcal{P}$ is the face poset of a polyhedral complex $\mathcal{C}$, then the order complex of $\mathcal{P}$ is (isomorphic to) the barycentric subdivision of $\mathcal{C}$. In particular, since $\mathcal{B}_{\text{coh}}(Q \to A)$ is the face poset of the polytope $\mathcal{F}(Q \to A)$, it is homotopy equivalent (in fact, homeomorphic) to a sphere of dimension $\dim(Q) - \dim(A) - 1$.

Question 3.6 (Baues Problem)**.** *Under what conditions is the inclusion* $\mathcal{B}_{\text{coh}}(Q \to A) \hookrightarrow \mathcal{B}(Q \to A)$ *a homotopy equivalence?*

See [Rei99] for a (not-so-recent) survey about this question, and [San06, Liu17] for examples where the answer is no and having Q a simplex and a cube, respectively.

3.2.3 Cyclic polytopes

Cyclic polytopes are a family of polytopes of particular interest for this manuscript and are defined as follows. The trigonometric moment curve (also known as the Carathéodory curve), is parametrized by

$$\phi_d : \; t \to (\sin(t), \cos(t), \sin(2t), \cos(2t), \dots) \in \mathbb{R}^d.$$

Let $t_1, \dots, t_n$ be n cyclically equidistant numbers in $[0, 2\pi)$, for example, $t_i = \frac{2\pi(i-1)}{n}$. The *cyclic polytope* $\mathbf{C}(n, d)$ is the convex hull of $\phi(t_1), \dots, \phi(t_n)$.

The combinatorics of the cyclic polytope can be nicely described in terms of the circuits of the corresponding oriented matroid. Namely, all circuits are of the form

$$(\{a_1, a_3, \dots\}, \{a_2, a_4, \dots\})$$

such that $a_1 < a_2 < \dots < a_{d+2}$ and their opposites (giving the label i to the vertex $\phi(t_i)$).

Cyclic polytopes can also be defined by using the polynomial moment curve $t \to (t, t^2, \dots, t^d)$ instead of the trigonometric moment curve and the combinatorial type remains the same. However, the coherence of subdivisions and hence fiber polytopes depend also on the embedding (see Theorem 3.25). When using the trigonometric moment curve in even dimension the cyclic polytope has more symmetry. That is, it is invariant under the cyclic group action on the vertices. When $d = 2$ the cyclic polytope $\mathbf{C}(n, 2)$ is a is a regular polygon and we abbreviate it by $\mathbf{P}_n$.

The Baues problem is known to have positive answer for cyclic polytopes in the following two cases:

Theorem 3.7 ([RS00, SZ93]). *Let $n > d \in \mathbb{N}$. Then, the following two cases of the Baues question have a positive answer:*

- *When $Q = \Delta_n$ and $A = \mathbf{C}(n,d)$ is the* cyclic polytope *of dimension d with n vertices [RS00].*
- *When $Q = [0,1]^n$ and $A = Z(\mathbf{C}(n,d))$ is the* cyclic zonotope *of dimension $d+1$ with n generators [SZ93].*

3.2.4 Hypersecondary polytopes.

Let $A = \{a_1, \dots, a_n\} \in \mathbb{R}^d$ be a point configuration. For each $k = 1, \dots, n-1$ we consider the following k-th deleted (Minkowski) sum of A with itself, which we denote $A^{(k)}$:

$$A^{(k)} := \left\{ a_{i_1} + \cdots + a_{i_k} \in \mathbb{R}^d : \{i_1, \dots, i_k\} \in \binom{[n]}{k} \right\}.$$

The k-th deleted sum of the standard $(n-1)$-simplex $\Delta_n := \mathrm{conv}(e_1, \dots, e_n)$ equals the k-th hypersimplex of dimension $n-1$:

$$\Delta_n^{(k)} := \mathrm{conv}\left\{ \sum_{i \in B} e_i : B \in \binom{[n]}{k} \right\} = [0,1]^n \cap \left\{ x : \sum_{i=1}^{n} x_i = k \right\}.$$

(Observe that the notation $\Delta_n^{(k)}$ here is an abbreviation of $\mathrm{conv}(\mathrm{vertices}(\Delta_n)^{(k)})$).

As mentioned above, the projection $\mathbb{R}^n \to \mathbb{R}^d \times \{1\}$ that sends the vertices of Δ_n to A extends to a linear map $\mathbb{R}^n \to \mathbb{R}^{d+1}$ that sends the unit cube $[0,1]^n$ to the zonotope $Z(A)$. In turn, this linear map restricts to an affine map sending each $\Delta_n^{(k)} \subset \mathbb{R}^n$ to $A^{(k)} \subset \mathbb{R}^d \times \{k\}$. We use the same letter π for all these projections.

Definition 3.8. The π-induced subdivisions of the projection $\pi : \Delta_n^{(k)} \to A^{(k)}$ are called *hypersimplicial subdivisions of level k* of A, or just *hypersimplicial subdivisions* of $A^{(k)}$. Fine hypersimplicial subdivisions are called *hypertriangulations*. We denote $\mathcal{B}^{(k)}(A)$ and $\mathcal{F}^{(k)}(A)$ the corresponding Baues poset and fiber polytope, and call the latter the *k-th hypersecondary polytope* of A. We denote $\mathcal{B}^{(k)}_{\mathrm{coh}}(A)$ for the coherent subdivisions in $\mathcal{B}^{(k)}(A)$.

Remark 3.9. The Baues poset $\mathcal{B}^{(k)}(A)$ only depends on the oriented matroid of A while $\mathcal{B}^{(k)}_{\mathrm{coh}}(A)$ does depend on the embedding of the oriented matroid.

3.2.5 Lifting subdivisions

By construction, the intersection of any zonotopal tiling of $Z(A)$ with the hyperplane $\sum x_i = k$ is a hypersimplicial subdivision of $A^{(k)}$. But the converse is in general not true. Not every hypersimplicial subdivision of $A^{(k)}$ "extends" to a zonotopal tiling of $Z(A)$. Following [BLVS$^+$99, Pos19, San02] the ones that extend are called *lifting* hypersimplicial subdivisions. The following are examples of them:

- For a cyclic polytope $\mathbf{C}(n, d)$, all triangulations in the standard sense (that is, all hypertriangulations of $\mathbf{C}(n, d)^{(1)}$) are lifting [RS00]. The same is not known for non-simplicial subdivisions.
- For arbitrary k and a convex n-gon $\mathbf{P}_n$, all hypertriangulations of $\mathbf{P}_n^{(k)}$ are lifting [Gal18]. The same result for all hypersimplicial subdivisions has recently been provedin [BW19].

Non-lifting triangulations of $A^{(1)}$ are not known in dimension two but easy to construct in dimension three or higher. For example, if a subdivision S of A has the property that its restriction to some subset B of A cannot be extended to a subdivision of B, then S is non-lifting. Such subdivisions (and triangulations) exist when A is the vertex set of a triangular prism together with any point in the interior of it, the vertex set of a 4-cube, or the vertex set of $\Delta_4 \times \Delta_4$, among other cases (see, e.g., [San02, Chapter 5], or [DLRS10, Proof (10) in Sect. 7.1.2,]).

To better understand lifting subdivisions, let us look at zonotopal tilings of $Z(A)$. We denote $\mathcal{B}^Z(A)$, $\mathcal{B}^Z_{\text{coh}}(A)$ and $\mathcal{F}^Z(A)$ for the poset of zonotopal tilings, its subposet of coherent tilings and the secondary zonotope of $Z(A)$ respectively. We call any subset of $[n]$ a *point*, since it represents an element of the point configuration $\sum_{i\in[n]}\{0, a_i\}$. A *tile* is a poset interval $[X, Y]$ of the boolean poset $2^{[n]}$, where $X \subseteq Y$. To be precise, $[X, Y] := \{I \subseteq [n] \mid X \subseteq I \subseteq Y\}$. Geometrically, we think of $[X, Y]$ as the zonotope $X + Z(Y \setminus X)$, but we prefer the combinatorial notation where the tile is described as the set of vertices of $[0, 1]^n$ of which it is the projection.

Every tile is a cell in a coherent zonotopal tiling of $Z(A)$, by letting $w(j)$ be -1, 0 or 1 depending on whether j is in X, $Y \setminus X$, or none of them. Indeed, this w gives value at least $-|X|$ to every point in $Z(A)$, with equality if and only if the point belongs to $[X, Y]$.

Turning our attention to hypersimplices, observe that every face of the hypersimplex $\Delta_n^{(k)}$ is the intersection of a face of $[0, 1]^n$ with the hyperplane $\{x : \sum_{i=1}^n x_i = k\}$. Therefore we can denote the projection under π of any face of $\Delta_n^{(k)}$ by

$$[X,Y]^{(k)} := [X,Y] \cap \left(\mathbb{R}^d \times \{k\}\right) = \{B \mid X \subseteq B \subseteq Y \quad |B| = k\}.$$

By definition, a subdivision of $A^{(k)}$ is hypersimiplicial if and only if all of its cells are of the form $[X,Y]^{(k)}$. A hypersimplicial subdivision is fine if for every cell $[X,Y]^k$ we have that Y/X is an affine basis in A. This spells out the following relation with zonotopal tilings:

Proposition 3.10. *For every configuration A of n points and every $k \in [n-1]$:*

1. *Intersection of zonotopal tilings with the hyperplane at level k induces an order-preserving map*
$$r^{(k)} : \mathcal{B}^Z(A) \to \mathcal{B}^{(k)}(A).$$

2. *The normal fan of $\mathcal{F}^Z(A)$ refines the normal fan of $\mathcal{F}^{(k)}(A)$.*

Proof. For the first claim, notice that the intersection of a zonotopal tiling $S = \{[X_i, Y_i] \mid i \in I\}$ with the hyperplane $\mathbb{R}^d \times k$ gives the subdivision

$$r^{(k)}(S) := \left\{[X_i, Y_i]^{(k)} \mid i \in I \ \ |X_i| < k < |Y_i|\right\} \cup \left\{X \in \binom{n}{k} \cap S\right\}$$

of $A^{(k)}$, which clearly is hypersimplicial. We denote $r^{(k)}(S)$ as $S^{(k)}$ for simplicity. The second claim follows from the fact that $S(Z(A), w)^{(k)} = S(A^{(k)}, w)$ for every $w \in (\mathbb{R}^n)^*$. □

We say that a tile $[X,Y]$ *covers level* k, if $|X| < k < |Y|$. In other words, $[X,Y]$ covers level k if $[X,Y]^{(k)}$ is of positive dimension.

Example 3.11. Consider the regular hexagon $\mathbf{P}_6$. Figure 3.1 shows a hypersimplicial subdivision of $\mathbf{P}_6^{(2)}$ whose set of facets are the triangles $[\emptyset, 123]^{(2)}$, $[\emptyset, 135]^{(2)}$, $[\emptyset, 156]^{(2)}$, $[\emptyset, 345]^{(2)}$, $[1, 1236]^{(2)}$, $[1, 1356]^{(2)}$, $[3, 1235]^{(2)}$, $[3, 2345]^{(2)}$, $[5, 1345]^{(2)}$ and $[5, 1456]^{(2)}$. The colour of the triangle $[X,Y]^{(2)}$ is dark gray if $X = \emptyset$ and white if $|X| = 1$, which agrees with the colouring of vertices of the corresponding plabic graph (see [Gal18]).

This subdivision is not coherent. To see this, suppose there is a lifting vector $w \in (\mathbb{R}^*)^6$ whose regular subdivision is this. Then notice that the presence of the edge $[1, 136]^{(2)}$ implies $w_3 + w_6 < w_2 + w_5$, the presence of the edge $[3, 235]^{(2)}$ implies $w_2 + w_5 < w_1 + w_4$ and the presence of the edge $[5, 145]^{(2)}$ implies $w_1 + w_4 < w_3 + w_6$, together forming a contradiction. This contrasts the fact that every subdivision of a convex polygon is regular.

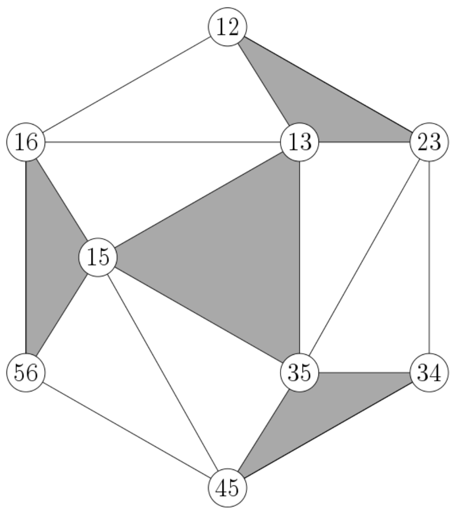


FIGURE 3.1: A non-coherent hypersimplicial subdivision of $\mathbf{P}_6^{(2)}$.

3.2.6 Lifting subdivisions via Gale transforms. The Bohne-Dress Theorem

As a general reference for the contents of this section we recommend the book [DLRS10], more specifically Chapters 4, 5 and 9.

A *Gale transform* of a point configuration $A = \{a_1, \ldots, a_n\}$ is a vector configuration $\mathbf{G}_A = \{a_1^*, \ldots, a_n^*\}$ with the property that a vector $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ is the coefficient vector of an affine dependence in A if and only if it is the vector of values of a linear functional on $\mathbf{G}_A$. The definition implicitly assumes a bijection between A and $\mathbf{G}_A$ given by the labels $1, \ldots, n$.

Gale duality is an involution: the Gale duals of a Gale dual of A are linearly isomorphic to A when considering A as a vector configuration via *homogenization*, by which we mean looking at affine geometry on the *points* $a_1, \ldots, a_n$ as linear algebra on the *vectors* $(a_1, 1), \ldots, (a_n, 1)$. In fact, if A and B are Gale duals to one another then their oriented matroids are dual, which implies that their ranks add up to n. In our setting where A has affine dimension d and hence rank $d+1$, its Gale duals have rank $n-d-1$.

The normal fan of the secondary polytope $\mathcal{F}^{(1)}(A)$ of A lives naturally in the ambient space of $\mathbf{G}_A$: it equals the common refinement of all the complete fans with rays taken from $\mathbf{G}_A$. Put differently, vectors $w \in \operatorname{span}(\mathbf{G}_A)$ are in natural bijection to lifting functions $A \to \mathbb{R}$ (where the latter, which forms a linear space isomorphic to $\mathbb{R}^n$, is considered modulo the linear subspace of affine functions restricted to A). Under this identification, w_1 and w_2 define the same coherent subdivision of A if and only if they lie in exactly the same family of cones among the finitely many cones spanned by subsets of B. The precise combinatorial rule to construct the coherent subdivision $S = S(\Delta_n \xrightarrow{\pi}$

$A, w)$ of A induced by a $w \in \text{span}(\mathbf{G}_A)$ is: *a subset $Y \subset [n]$ is a cell in S if and only if w lies in the relative interior of $[n]\backslash Y$.*

This rule can be made purely combinatorial as follows. Instead of starting with a vector $w \in \text{span}(\mathbf{G}_A)$, let $\mathcal{M}^*(A)$ be the oriented matroid of $\mathbf{G}_A$ and let $\mathcal{M}'$ be a *single-element extension* of $\mathcal{M}^*(A)$. That is, $\mathcal{M}'$ is an oriented matroid of the same rank as $\mathcal{M}$ on the ground set $[n] \cup \{w\}$ and such that $\mathcal{M}'$ restricted to $[n]$ equals $\mathcal{M}^*(A)$. Any vector $w \in \text{span}(\mathbf{G}_A)$ induces such an extension, but the definition is more general since $\mathcal{M}'$ needs not be realizable, or it may be realizable but not extend the given realization $\mathbf{G}_A$ of $\mathcal{M}^*(A)$. Yet, any such extension w allows to define a subdivision $S(w)$ of A as follows.

Proposition 3.12. *With the notation above, the following rules define, respectively, a polyhedral subdivision $S^{(1)}(A, w)$ of A and a zonotopal tiling $S^{(Z)}(A, w)$ of $Z(A)$:*

1. *A subset $Y \subset [n]$ is a cell in $S^{(1)}(A, w)$ if and only if $([n] \setminus Y, \{w\})$ is a vector in the oriented matroid $\mathcal{M}'$.*

2. *An interval $[X, Y]$ is a tile in $S^{(Z)}(A, w)$ if and only if $([n] \setminus Y, X \cup \{w\})$ is a vector in the oriented matroid $\mathcal{M}'$.*

By construction, $S^{(1)}(A, w)$ is the slice at height 1 of $S^{(Z)}(A, w)$. In fact:

Theorem 3.13 (Bohne-Dress Theorem)**.** *The construction of Theorem 3.12(2) is a bijection (and a poset isomorphism, with the weak map order on extensions of $\mathcal{M}^*(A)$) between one-element extensions of $\mathcal{M}^*(A)$ and zonotopal tilings of $Z(A)$. In particular, lifting subdivisions of $A^{(1)}$ are precisely the ones that can be obtained by the construction in Theorem 3.12(1).*

3.3 Normal fans of hypersecondary polytopes

The goal of this section is to study hypersecondary polytopes, and the relations between them and the secondary zonotope. Most of such relations say that the normal fan of one of the polytopes refines that of another one. We introduce the following definition to this effect:

Definition 3.14. Let $P, Q \in \mathbb{R}^d$ be two polytopes. We say that Q is a *Minkowski summand* of P, and write $Q \leq P$, if any of the following equivalent conditions holds:

1. The normal fan of P refines that of Q.

2. $P + Q$ is combinatorially isomorphic to P.

If P and Q are Minkowski summands of one another then they are *normally equivalent* and we write $P \cong Q$.

Remark 3.15. The equivalence of these two conditions follows from the fact that the normal fan of $P + Q$ is the common refinement of the normal fans of P and Q. It can be shown $Q \leq P$ is also equivalent to the existence of a polytope Q' and an $\varepsilon > 0$ such that $P = Q' + \varepsilon Q$, hence the name "Minkowski summand".

Throughout this section we will assume that $A \subseteq \mathbb{R}^d$ is a point configuration that spans affinely $\mathbb{R}^d$. As a first example, it follows from Theorem 3.10 that:

Proposition 3.16. *For every configuration $A \subset \mathbb{R}^d$ of size n:*

1. $\mathcal{F}^{(k)}(A) \leq \mathcal{F}^Z(A)$.
2. *Let $k_0 = 0 < k_1 < \cdots < k_p = n$ be a sequence of integers with $k_{i+1} - k_i \leq d+1$ for all i. Then,*
$$\mathcal{F}^Z(A) \cong \sum_{i=0}^{p} \mathcal{F}^{(k_i)}(A).$$

In particular:

Corollary 3.17. *For every configuration $A \subset \mathbb{R}^d$ of size n,*

1. *If $n \leq 2d+2$ then*
$$\mathcal{F}^Z(A) \cong \mathcal{F}^{(k)}(A), \qquad \forall k \in [n-d-1, d+1].$$
2. *If $n \geq 2d+2$ then*
$$\sum_{k=d+1}^{n-d-1} \mathcal{F}^{(k)}(A) \cong \mathcal{F}^Z(A).$$

Lemma 3.18. *Let S be coherent zonotopal subdivision of A and let $B \subseteq A$ be a spanning subset. Then there is at most one $X \subseteq A \backslash B$, such that $[X, X \cup B] \in S$.*

Proof. Let $w \in (\mathbb{R}^*)^n$ such that $S = S(Z(A), w)$. Since B is of maximal dimension, there is at most one $\tilde{w}$ such that $\tilde{w}|_{\ker(\pi)} = w|_{\ker(\pi)}$ and $w \cdot b = 0$ for every $b \in B$. If such $\tilde{w}$ exists then the only tile of the form $[X, X \cup B]$ that is in S is the one where $X = \{x \in A \mid \tilde{w} \cdot x < 0\}$. If no such $\tilde{w}$ exists then there is no tile of that form in the subdivision. □

In the following result and in the rest of this section we denote by A_J the subset of A labelled by J, for any $J \subset [n]$.

Lemma 3.19. *Fix $k \geq 1$ and a lifting vector $w \in (\mathbb{R}^n)^*$, for a point configuration A of size n. For each tile $[X,Y] \subset 2^{[n]}$ such that $Y\backslash X$ a basis of A, the following are equivalent:*

1. *$[X,Y]^{(k+1)}$ is a cell in $S^{(k+1)}(A,w)$.*
2. *There is an $x \in X$ such that $[X\backslash x, Y\backslash x]^{(k)}$ is a cell in $S^{(k)}(A_{[n]\backslash x}, w)$ but not in $S^{(k)}(A,w)$.*
3. *For every $x \in X$, $[X\backslash x, Y\backslash x]^{(k)}$ is a cell in $S^{(k)}(A_{[n]\backslash x}, w)$ but not in $S^{(k)}(A,w)$.*

If, moreover, $k > 1$, then they are also equivalent to:

(4) There are $x_1, x_2 \in X$ such that $[X\backslash x_i, Y\backslash x_i]^{(k)}$ is a cell in $S^{(k)}(A_{[n]\backslash x_i}, w)$ for $i = 1,2$.

(5) For every $x \in X$, $[X\backslash x, Y\backslash x]^{(k)}$ is a cell in $S^{(k)}(A_{[n]\backslash x}, w)$.

Proof. The implication (3)⇒(2) is obvious.

To show that (2)⇒(1), consider an x such that the cell $[X\backslash x, Y\backslash x]^{(k)}$ is a cell in $S((A_{[n]\backslash x})^{(k)}, w)$. Then by Theorem 3.10, $[X\backslash x, Y\backslash x]$ is a cell of $S(Z(A_{[n]\backslash x}), w)$. Therefore either $[X\backslash x, Y\backslash x] \in S(Z(A), w)$ or $[X,Y] \in S(Z(A), w)$ but not both by Theorem 3.18. In other words, either $[X\backslash x, Y\backslash x]^{(k)} \in S(A^{(k)}, w)$ or $[X,Y]^{(k+1)} \in S(A^{(k+1)}, w)$ but not both. Since we assumed $[X\backslash x, Y\backslash x]^{(k)} \notin S(A^{(k)}, w)$, we are done.

To see that (1)⇒(3), notice that if $[X,Y]^{(k+1)} \in S(A^{(k+1)}, w)$ then $[X,Y] \in S(Z(A), w)$. So for all $x \in X$ we have that the tile $[X\backslash x, Y\backslash x]$ is a cell of $S(Z(A_{[n]\backslash x}), w)$ and in particular $[X\backslash x, Y\backslash x]^{(k)} \in S((A_{[n]\backslash h})^{(k)}, w)$. But as $[X,Y] \in S(Z(A), w)$ then by Theorem 3.18 $[X\backslash x, Y\backslash x]$ can not be a cell of $S(Z(A), w)$, so $[X\backslash x, Y\backslash x]^{(k)}$ can not be a cell of $S(A^{(k)}, w)$.

Now assume that $k > 1$. It is clear that (3)⇒(5)⇒(4). To see that (4)⇒(2) notice that it if $[X\backslash x_i, Y\backslash x_i]^{(k)} \in S(A^{(k)}, w)$ holds for $i = 1,2$, then the two zonotopes $[X\backslash x_1, Y\backslash x_1]$ and $[X\backslash x_2, Y\backslash x_2]$ are in $S(Z(A), w)$, which can not happen by Theorem 3.18. □

Proposition 3.20. *For every configuration A of size n and every $k \in [n-1]$ we have that $\mathcal{F}^{(k+1)}(A)$ is a Minkowski summand of*

$$\mathcal{F}^{(k)}(A) + \sum_{i \in [n]} \mathcal{F}^{(k)}(A_{[n]\backslash i}).$$

Proof. Saying that $\mathcal{F}^{(k+1)}(A)$ is a Minkowski summand of $\mathcal{F}^{(k)}(A) + \sum_{i\in[n]} \mathcal{F}^{(k)}(A_{[n]\setminus i})$ is equivalent to saying that if, for a given w we know the subdivisions that w induces in $A^{(k)}$ and in $A\setminus x^{(k)}$ for every x then we also know the subdivision induced in $A^{(k+1)}$. For a cell $[X, Y]^{(k+1)}$ with $|X| = k$, Theorem 3.19 says that its presence in $S(A^{(k+1)}, w)$ is determined by its presence in $S(A^{(k)}, w)$ and $S(A\setminus x^{(k)}, w)$. Cells $[X, Y]^{(k+1)}$ with $|X| < k$ are in $S(A^{(k+1)}, w)$ if and only if $[X, Y]^{(k)} \in S(A^{(k)}, w)$. □

The converse is only true for small k:

Proposition 3.21. *For every configuration $A \subseteq \mathbb{R}^d$ of size n and every $k \in [d]$ we have that*

$$\mathcal{F}^{(k+1)}(A) \cong \mathcal{F}^{(k)}(A) + \sum_{i\in[n]} \mathcal{F}^{(k)}(A_{[n]\setminus i}).$$

Proof. One direction is Theorem 3.20. For the other direction we have that by Theorem 3.19 then $S(A^{(k+1)}, w)$ determines $S(A^{(k+1)}_{[n]\setminus i}, w)$ for all $i \in [n]$. Any maximal cell in $[X, Y]^{(k)} \in S(A^{(k)}, w)$ must satisfy $|Y\setminus X| \geq d + 1$, in particular $|Y| \geq d + 1 > k$, so $[X, Y]^{(k+1)}$ is also a cell in $S(A^{(k+1)}, w)$. This implies that $S(A^{(k+1)}, w)$ determines $S(A^{(k)}, w)$. □

Proposition 3.22. *For every configuration $A \subseteq \mathbb{R}^d$ of size $n > d + 2$ and every $k \in [d]$ we have that*

$$\mathcal{F}^{(k)}(A) \leq \sum_{i=1}^{n} \mathcal{F}^{(k)}(A_{[n]\setminus i})$$

Proof. We need to prove that for every $w \in \mathbb{R}^d$, knowing $S(A^{(k)}_{[n]\setminus i}, w)$ for every i determines $S(A^{(k)}, w)$. It is enough to prove it for a generic w, so we can assume the subdivisions are fine. Let $[X, Y]$ be a tile such that $Y\setminus X$ is an affine basis. We claim that $[X, Y]^{(k)} \in S(A^{(k)}_{[n]\setminus i}, w)$ if and only if $[X\setminus i, Y\setminus i]^{(k)} \in S(A^{(k)}_{[n]\setminus i}, w)$ for every $i \in [n]\setminus(Y\setminus X)$.

There is exactly one $\tilde{w}$ that agrees with w in $\ker(\pi)$ and such that $\tilde{w} \cdot x = 0$ for every $x \in Y\setminus X$. We have that $[X, Y]^{(k)} \in S(A^{(k)}_{[n]\setminus i}, w)$ if and only if $\tilde{w} \cdot x < 0$ for every $x \in X$ and $\tilde{w} \cdot x > 0$ for every $x \in [n]\setminus Y$. Notice that as $n > d + 2$, $|[n]\setminus(Y\setminus X)| > 2$. Let $i \in [n]\setminus(Y\setminus X)$. As $k \leq d$ and $|Y\setminus X| = d + 1$, then for $Y\setminus i > k$ so $[X\setminus i, Y\setminus i]^{(k)}$ is a full dimensional cell in the level k. So it is in $S(A^{(k)}_{[n]\setminus i}, w)$ if and only if $\tilde{w} \cdot x < 0$ for every $x \in X\setminus i$ for all $x \in X\setminus i$ and $\tilde{w} \cdot x > 0$ for every $x \in [n]\setminus(Y \cup i)$. As $|[n]\setminus(Y\setminus X)| > 2$, we can do this for two different elements in $[n]\setminus(Y\setminus X)$ so we can verify the sign of $\tilde{w} \cdot i$ for every $i \in [n]\setminus(Y\setminus X)$. □

A consequence of this is that Theorem 3.21 can be strengthened as follows:

Proposition 3.23. *For every configuration $A \subseteq \mathbb{R}^d$ of size $n > d+2$ and every $k \in [d]$ we have that*

$$\mathcal{F}^{(k+1)}(A) \cong \sum_{i \in [n]} \mathcal{F}^{(k)}(A_{[n]\setminus i}).$$

Notice that if $n = d+1$ then the fiber polytopes are just points and if $n = d+2$ they are just segments and in particular $\mathcal{F}^{(k+1)}(A) \cong \mathcal{F}^{(k)}(A)$. Now we are ready to prove the main result of this section:

Theorem 3.24. *Let $A \subseteq \mathbb{R}^d$ be a configuration of size n and $k \in [d+1]$. Let $s = \max(n-k+1, d+2)$. Then*

$$\mathcal{F}^{(k)}(A) \cong \sum_{J \in \binom{[n]}{s}} \mathcal{F}(A_J)$$

Proof. We prove this by iterating Theorem 3.23 several times. At each iteration, for $1 < i \leq k$, we replace each $\mathcal{F}^{(i+1)}(A_J)$ by $\sum_{j \in [n]} \mathcal{F}^{(i)}(A_{J\setminus j})$ if $|J| > d+2$ or by $\mathcal{F}^{(i)}(A_J)$ if $|J| = d+2$. The iteration stops at level 1 with the desired result (notice that Minkowski sum is idempotent with respect to normal equivalence). □

Example 3.25. Consider the regular hexagon $\mathbf{P}_6$. The secondary polytope $\mathcal{F}^{(1)}(\mathbf{P}_6)$ is the 3-dimensional associahedron, as seen in Figure 3.2. Its border consists of 6 pentagons and 3 squares. By Theorem 3.24, the hypersecondary polytope $\mathcal{F}^{(2)}(\mathbf{P}_6)$ is normally equivalent to the Minkowski sum of those 6 pentagons, see Figure 3.3. It has 66 vertices and the facets consist of 27 quadrilaterals (18 rectangles, 6 rhombi and 3 squares), 6 pentagons, 2 hexagons and 6 decagons. The short edges correspond to flips which do not change the set of vertices of the triangulation and the long edges correspond to those flips that do change the set of vertices.

The GKZ vector corresponding to the triangulation from Theorem 3.11 is in the center of one of the hexagons. There are 4 non-coherent hypertriangulations of $\mathbf{P}_6^{(2)}$, which come in pairs with the same GKZ-vector, each in the center of one of the two hexagons. If instead of a regular hexagon we had a hexagon where the three long diagonals do not intersect in the same point, two of those subdivisions would become coherent and the hypersecondary polytope would have instead of each hexagon a triple of rhombi around the new vertex.

The order complex of the Baues poset $\mathcal{B}^{(2)}(\mathbf{P}_6)$ is the (barycentric subdivision of the border of the) hyperassociahedron $\mathcal{F}^{(2)}(\mathbf{P}_6)$ where the hexagons are replaced by cubes. In particular it satisfies the Baues problem, that is, $\mathcal{B}^{(2)}(\mathbf{P}_6)$ retracts onto $\mathcal{F}^{(2)}(\mathbf{P}_6)$. We will generalize this in Section 3.6.

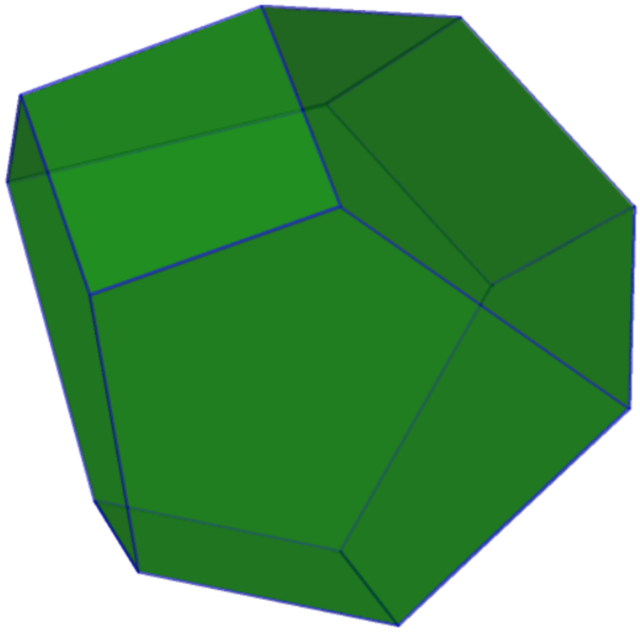


FIGURE 3.2: The associahedron $\mathcal{F}^{(1)}(\mathbf{P}_6)$.

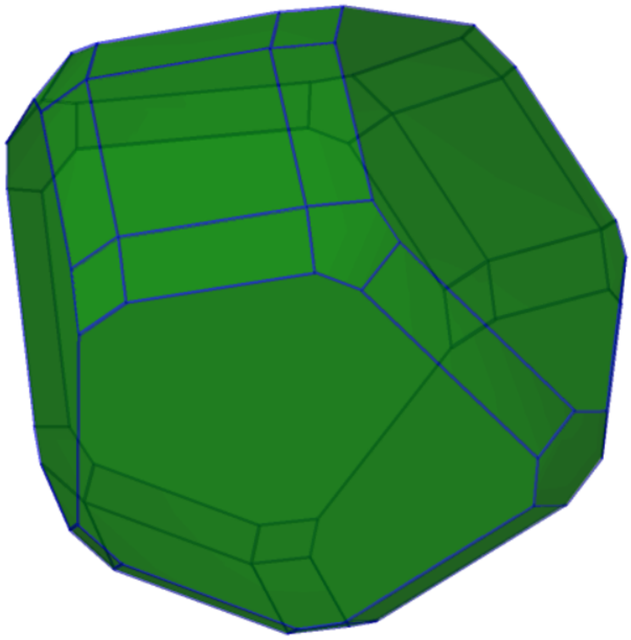


FIGURE 3.3: The hyperassociahedron $\mathcal{F}^{(2)}(\mathbf{P}_6)$.

3.4 Separation and lifting subdivisions

Throughout this section let $A \subset \mathbb{R}^d$ be a point configuration labelled by $[n]$, and let $Z(A) \subset \mathbb{R}^{d+1}$ be the zonotope generated by the vector configuration $A \times \{1\} \subset \mathbb{R}^d \times \{1\}$. Recall that a *point* in $Z(A)$ is a subset $X \subset [n]$ and a *tile* is an interval $[X, Y] \subset 2^{[n]}$, where $X \subset Y \subset [n]$.

Following [GP17], we say that two points $X_1, X_2 \subset [n]$ are *separated with respect to* A or A-*separated* for short if there is an affine functional positive on $A_{X_1 \setminus X_2}$ and negative on $A_{X_2 \setminus X_1}$. Equivalently, if there is no oriented circuit (C^+, C^-) in A with $C^+ \subset X_1 \setminus X_2$ and $C^- \subset X_2 \setminus X_1$. Their motivation is that the notions of *strongly separated* and *chord separated* that were introduced in [LZ98] and [Gal18, OPS15] are equivalent to "$\mathbf{C}(n,1)$-separated" and "$\mathbf{C}(n,2)$-separated" respectively ([GP17, Lemmas 3.7 and 3.10]).[1] One of their main results is as follows (their statement is a bit more general, since it is stated for arbitrary oriented matroids, rather then "point configurations"):

Theorem 3.26 ([GP17, Theorems 2.7 and 7.2])**.** *Let A be a point configuration and let m be the number of affinely independent subsets of A. Then:*

1. *No family of A-separated points in A has size larger than m.*
2. *The map sending each zonotopal tiling to its set of vertices gives a bijection*

$$\{\text{fine zonotopal tilings of } Z(A)\} \leftrightarrow \{S \subset 2^{[n]} : S \text{ is } A\text{-separated and } |S| = m\}.$$

We here extend their definition to separation of tiles. In the rest of the chapter we omit A and write "separated" instead of A-separated:

Definition 3.27. Let $[X_1, Y_1]$ and $[X_2, Y_2]$ be two tiles. We say they are separated if there is no circuit (C^+, C^-) such that $C^+ \subset Y_1 \setminus X_2$, $C^- \subset Y_2 \setminus X_1$ and $C^+ \cup C^- \not\subseteq (Y_1 \cap Y_2) \setminus (X_1 \cup X_2)$.

The following diagram illustrates the circuits forbidden by the first two conditions in this definition. The third condition forbids circuits with support fully contained in the middle cell:

	X_2	$Y_2 \setminus X_2$	$[n] \setminus Y_2$
X_1	0	≥ 0	≥ 0
$Y_1 \setminus X_1$	≤ 0	$*$	≥ 0
$[n] \setminus Y_1$	≤ 0	≤ 0	0

By the orthogonality between circuits and covectors in an oriented matroid [BLVS+99, Proposition 3.7.12], and the fact that covectors of a realized oriented matroid are the sign vectors of affine functionals this definition is equivalent to:

Proposition 3.28. *Two tiles $[X_1, Y_1]$ and $[X_2, Y_2]$ are separated if there is a covector (that is, an affine functional) that is positive on $(X_1 \setminus X_2) \cup (Y_1 \setminus Y_2)$, negative on $(X_2 \setminus X_1) \cup (Y_2 \setminus Y_1)$, and zero on $(Y_1 \cap Y_2) \setminus \{X_1 \cup X_2\}$.*

[1] Observe that [OPS15] uses the expression "weakly separated" for "chord separated", but "weakly separated" had a different meaning in [LZ98]

The following diagram illustrates the sign-patterns of covectors witnessing that two tiles are separated:

	X_2	$Y_2 \setminus X_2$	$[n] \setminus Y_2$
X_1	$*$	$+$	$+$
$Y_1 \setminus X_1$	$-$	0	$+$
$[n] \setminus Y_1$	$-$	$-$	$*$

Proof. Consider the subset $I = (Y_1 \cup Y_2) \setminus (X_1 \cap X_2)$ of A, and let A' be the restriction of A to I. Remember that the circuits of A' are the circuits of A with support contained in A', while the covectors of A' are the covectors of A (all of them) restricted to A'. In particular, the characterization of covectors of A' as the sign vectors orthogonal to all circuits says that

$$((X_1 \setminus X_2) \cup (Y_1 \setminus Y_2), (X_2 \setminus X_1) \cup (Y_2 \setminus Y_1))$$

is a covector in A' if and only if a circuit as in the definition of separation does not exist. □

Example 3.29. Two "singleton tiles" (that is, $X_1 = Y_1$ and $X_2 = Y_2$) are separated as tiles if and only if they are separated as points in the sense of Galashin and Postnikov. Two tiles containing the origin, that is with $X_1 = X_2 = \emptyset$, are separated if and only if Y_1 and Y_2 *intersect properly* in the usual sense, as *cells* in A. Finally, the whole zonotope $2^{[n]} = [\emptyset, [n]]$ is separated from a tile $[X, Y]$ if and only if the cells Y and $[n] \setminus X$ intersect properly; this is equivalent to $[X, Y]$ being a face of the zonotope $Z(A)$.

The following result clarifies the relation between separation of points and tiles. In it, we say that a tile $[X, Y]$ is *fine* if $Y \setminus X$ is an independent set. Fine tiles are the ones that can be used in fine zonotopal tilings of $Z(A)$.

Proposition 3.30. *Let $[X_1, Y_1]$ and $[X_2, Y_2]$ be two tiles. If every point $B_1 \in [X_1, Y_1]$ is separated from every point $B_2 \in [X_2, Y_2]$, then $[X_1, Y_1]$ and $[X_2, Y_2]$ are separated. The converse holds if the tiles are fine.*

Proof. For the first direction, by induction on $|Y_1 \setminus X_1| + |Y_2 \setminus X_2|$, we can assume that $[X_1, Y_1]$ is not a singleton and that every tile properly contained in it is separated from $[X_2, Y_2]$. In particular, taking any element $i \in Y_1 \setminus X_1$ we have that both $[X_1 \cup i, Y_1]$ and $[X_1, Y_1 \setminus i]$ are separated from $[X_2, Y_2]$. By Theorem 3.28, that implies the following two covectors:

	X_2	$Y_2 \backslash X_2$	$[n] \backslash Y_2$
X_1	$*$	$+$	$+$
i	$*$	$+$	$+$
$Y_1 \backslash X_1 \backslash i$	$-$	0	$+$
$[n] \setminus Y_1$	$-$	$-$	$*$

	X_2	$Y_2 \backslash X_2$	$[n] \backslash Y_2$
X_1	$*$	$+$	$+$
$Y_1 \backslash X_1 \backslash i$	$-$	0	$+$
i	$-$	$-$	$*$
$[n] \setminus Y_1$	$-$	$-$	$*$

If $i \in X_2$ or $i \in [n] \backslash Y_2$ then the first or the second covector, respectively, show that $[X_1, Y_1]$ and $[X_2, Y_2]$ are separated. If $i \in Y_2 \backslash X_2$ then elimination of i in these two covectors gives a covector with values

	X_2	$Y_2 \backslash X_2$	$[n] \backslash Y_2$
X_1	$*$	$+$	$+$
i		0	
$Y_1 \backslash X_1 \backslash i$	$-$	0	$+$
$[n] \setminus Y_1$	$-$	$-$	$*$

,

which again shows that $[X_1, Y_1]$ and $[X_2, Y_2]$ are separated.

For the converse, suppose first that $[X_1, Y_1]$ and $[X_2, Y_2]$ are separated and let V be the covector showing it. Let B_1 and B_2 be points in them. Since the set $C := (Y_1 \setminus X_1) \cap (Y_2 \setminus X_1)$ is independent and is contained in the zero-set of V, no matter what signs we prescribe for its elements there is a covector V' that agrees with V where V is not zero and has the prescribed signs on C. This implies the points B_1 and B_2 are separated. □

Theorem 3.31. *Let $[X_1, Y_1]$ and $[X_2, Y_2]$ be two tiles. Then, the following conditions are equivalent:*

1. *The tiles are separated.*
2. *There is a zonotopal tiling of $Z(A)$ using both.*
3. *There is a coherent zonotopal tiling of $Z(A)$ using both.*
4. *There is a polyhedral subdivision of A using $Y_1 \backslash X_2$ and $Y_2 \backslash X_1$ as cells.*
5. *There is a coherent polyhedral subdivision of A using $Y_1 \backslash X_2$ and $Y_2 \backslash X_1$ as cells.*

Proof. Throughout the proof, let $A = \{a_1, \ldots, a_n\}$ and denote $\tilde{a}_i = (a_i, 1)$ the corresponding generator of $Z(A)$.

- $1 \Rightarrow 3$. Suppose the tiles are separated. By Theorem 3.28 this implies there is a linear functional $v \in (\mathbb{R}^{d+1})^*$ such that $v \cdot \tilde{a}_i$ takes the following values on the generators of $Z(A)$:

	X_2	$Y_2 \setminus X_2$	$[n] \setminus Y_2$
X_1	$*$	>0	>0
$Y_1 \setminus X_1$	<0	0	>0
$[n] \setminus Y_1$	<0	<0	$*$

  Let $w \in (\mathbb{R}^n)^*$ be defined as follows on each $i \in [n]$:

	X_2	$Y_2 \setminus X_2$	$[n] \setminus Y_2$
X_1	$-N$	$-2\,v \cdot \tilde{a}_i$	$-v \cdot \tilde{a}_i$
$Y_1 \setminus X_1$	0	0	0
$[n] \setminus Y_1$	$-v \cdot \tilde{a}_i$	$-2\,v \cdot \tilde{a}_i$	$+N$

  where N is a very large positive number. Since w is negative in X_1, positive in $[n]\backslash Y_1$, and zero in $Y_1 \backslash X_1$, the tile selected by w in the subdivision $S(Z(A), w)$ is $[X_1, Y_1]$. Similarly, the vector $w' \in (\mathbb{R}^n)^*$ defined by $w'_i = w_i + 2v \cdot \tilde{a}_i$ has the following values

	X_2	$Y_2 \setminus X_2$	$[n] \setminus Y_2$
X_1	<0	0	$v \cdot \tilde{a}_i$
$Y_1 \setminus X_1$	$2\,v \cdot \tilde{a}_i$	0	$2\,v \cdot \tilde{a}_i$
$[n] \setminus Y_1$	$v \cdot \tilde{a}_i$	0	>0

  ,

  which shows that $[X_2, Y_2]$ is also in $S(Z(A), w)$, since the difference between w and w' is a linear function.

- $2 \Rightarrow 1$. By the Bohne-Dress Theorem, zonotopal tilings of $Z(A)$ correspond to *lifts* of the oriented matroid of $Z(A)$. Here, a lift is an oriented matroid $\mathcal{M}$ of rank $d+2$ on the ground set $[n+1]$ and such that $\mathcal{M}/(n+1) = \mathcal{M}(A)$. The tiles of the subdivision defined by the lift $\mathcal{M}$ are the intervals $[X, Y] \subset 2^{[n]}$ such that $\mathcal{M}$ has a covector that is negative on X, zero on $Y \setminus X$, and positive on $[n+1]\backslash Y$.

  That is, our hypothesis is that there is a lift $\mathcal{M}$ of A that contains the covectors

  $$([n+1] \setminus Y_1\,,\, X_1) \qquad \text{and} \qquad (X_2\,,\, [n+1] \setminus Y_2).$$

  Elimination of the element $n+1$ among these covectors gives us a covector of Theorem 3.28.

- $1 \Rightarrow 5$. Let v as in the proof of $1 \Rightarrow 3$, and define $w \in (\mathbb{R}^n)^*$ as follows:

	X_2	$Y_2 \setminus X_2$	$[n] \setminus Y_2$
X_1	N	0	0
$Y_1 \setminus X_1$	$-v \cdot \tilde{a}_i$	0	0
$[n] \setminus Y_1$	$-v \cdot \tilde{a}_i$	$-v \cdot \tilde{a}_i$	N

.

  Then w and the w' defined by $w'_i = w_i + v \cdot \tilde{a}_i$ show that $Y_1 \setminus X_2$ and $Y_2 \setminus X_1$ are cells in $S(A, w)$.

- $4 \Rightarrow 1$ For $C_1 := Y_1 \setminus X_2$ and $C_2 := Y_2 \setminus X_1$ to be cells in a subdivision it is necessary that their convex hulls intersect in a common face. That is, there must be a covector in A that is zero in $C_1 \cap C_2$, negative on $C_1 \setminus C_2$, and positive on $C_2 \setminus C_1$. These are precisely the same conditions as required in Theorem 3.28.

- $3 \Rightarrow 2$ and $5 \Rightarrow 4$ are obvious. □

Remark 3.32. With this theorem, it is now easy to see that Theorem 3.18 also holds for non-coherent subdivisions. If $Y_1 \setminus X_1 = Y_2 \setminus X_2$ is a spanning set then there can not be a linear functional vanishing on it, so $[X_1, Y_1]$ and $[X_2, Y_2]$ are not separated (unless $X_1 = X_2$, in which case they are the same cell).

Remark 3.33. The definition of separated points and tiles makes sense for an arbitrary oriented matroid $\mathcal{M}$, since it uses only the notion of circuits, and Theorem 3.28 still holds in tis more general setting.

The notions of zonotopal tiling and of subdivision also make sense for arbitrary oriented matroids: the former is interpreted as "extension of the dual oriented matroid" via Theorem 3.13 and the latter is studied in detail in [San02]. In this setting the implications (2) $\Rightarrow$ (4) $\Rightarrow$ (1) of Theorem 3.31 still hold, the first one as a consequence of the oriented matroid analogue of Theorem 3.12 and the second one because our proof above works at the level of oriented matroids. Yet:

1. The notion of *coherent* subdivisions needs a realization of the oriented matroid be given. Not only the notion does not make sense for nonrealizable oriented matroids. Also, different realizations of the same oriented matroid may have different sets of coherent subdivisions, and non-isomorphic secondary polytopes/zonotopes.

2. The implication (4) $\Rightarrow$ (2) fails in the example of [San02, Section 5.2] (see Proposition 5.6(i) in that section), and the implication (1) $\Rightarrow$ (4) fails in the *Lawrence polytope* that one can construct from that example.

Corollary 3.34. *Let $[X_1, Y_1]$ and $[X_2, Y_2]$ be two separated tiles. Then any pair of subtiles $[\tilde{X_1}, \tilde{Y_1}] \subseteq [X_1, Y_1]$ and $[\tilde{X_2}, \tilde{Y_2}] \subseteq [X_2, Y_2]$ are separated.*

Proof. By Theorem 3.31, there is a zonotopal tiling using $[X_1, Y_1]$ and $[X_2, Y_2]$ and such tiling uses $[\tilde{X}_1, \tilde{Y}_1]$ and $[\tilde{X}_2, \tilde{Y}_2]$. □

Proposition 3.35. *Let A be a configuration of n pairwise independent points. Let $k \in [n-1]$. Let $[X_1, Y_1]$ and $[X_2, Y_2]$ be two tiles that cover level k (that is, $|X_i| < k < |Y_i|$). Suppose that $[X_1, Y_1]$ and $[X_2, Y_2]$ are not-separated and that one of them is not fine.*

Then, there are fine tiles $[X_1', Y_1']$ and $[X_2', Y_2']$ contained in $[X_1, Y_1]$ and $[X_2, Y_2]$, still covering level k and still not separated.

Proof. By induction on the dependence rank of the tiles we only need to show that if $[X_1, Y_1]$ is dependent then there is a tile $[X_1', Y_1']$ properly contained in $[X_1, Y_1]$, covering level k, and non-separated from $[X_2, Y_2]$.

Let (C_+, C_-) be a circuit showing that $[X_1, Y_1]$ and $[X_2, Y_2]$ are not-separated. Let $C = C_+ \cup C_-$ be its support.

If there is an element $a \in (Y_1 \backslash X_1) \backslash C$ then both $[X_1 \cup a, Y_1]$ and $[X_1, Y_1 \backslash a]$ are not separated from $[X_2, Y_2]$, and one of them still covers level k, since dependent sets are of size at least 3.

If there is no such an a, then $Y_1 \backslash X_1 \subset C$. Since C is a circuit we conclude that $Y_1 \backslash X_1 = C$. By definition, we have that $C_- \subset Y_2$ and $C_+ \subset [n] \backslash X_2$. Again, we take as new tile $[X_1 \cup a, Y_1]$ or $[X_1, Y_1 \backslash b]$, depending on which of the two still covers level k, where $a \in C_+$ and $b \in C_-$. □

Corollary 3.36. *Let A be a point configuration in general position ("uniform") and let $k \in [n-1]$. If no hypertriangulation of $A^{(k)}$ contains two non-separated tiles, then no hypersimplicial subdivision of $A^{(k)}$ contains them either.*

Proof. Suppose that a subdivision S has two non-separated tiles $[X_1, Y_1]$ and $[X_2, Y_2]$. Let $[X_1', Y_1']$ and $[X_2', Y_2']$ be the tiles guaranteed by Theorem 3.35. Then, we can refine $[X_1, Y_1]$ and $[X_2, Y_2]$ to fine subdivisions using $[X_1', Y_1']$ and $[X_2', Y_2']$. By general position this extends to a hypertriangulation refining S and with two non-separated tiles. □

3.5 Non-separated subdivisions

We call a subdivision S of $A^{(k)}$ non-separated if it contains two non-separated cells. Non-separated subdivisions are certainly non-lifting.

Example 3.37. We here construct a non-separated subdivision in dimension two, which contrasts the fact that for $\mathbf{P}_n$ such things do not exist [BW19]. Let A be the configuration of the following 5 points in the plane: $p_1 = (1,2)$, $p_2 = (0,4)$, $p_3 = (4,4)$, $p_4 = (4,0)$ and $p_5 = (0,0)$. Figure 3.4 on the right shows a hypertriangulation of $A^{(2)}$ consisting of the triangles:

$$[\emptyset, 234]^{(2)}, [\emptyset, 245]^{(2)},$$
$$[2, 1235]^{(2)}, [2, 2345]^{(2)}, [4, 1234]^{(2)}, [4, 1245]^{(2)}, [4, 1345]^{(2)}, [5, 1245]^{(2)}.$$

The circuit $(14, 35)$ shows that the cell $[2, 2345]^{(2)}$ is not separated from the cells $[4, 1234]^{(2)}$ and $[4, 1245]^{(2)}$.

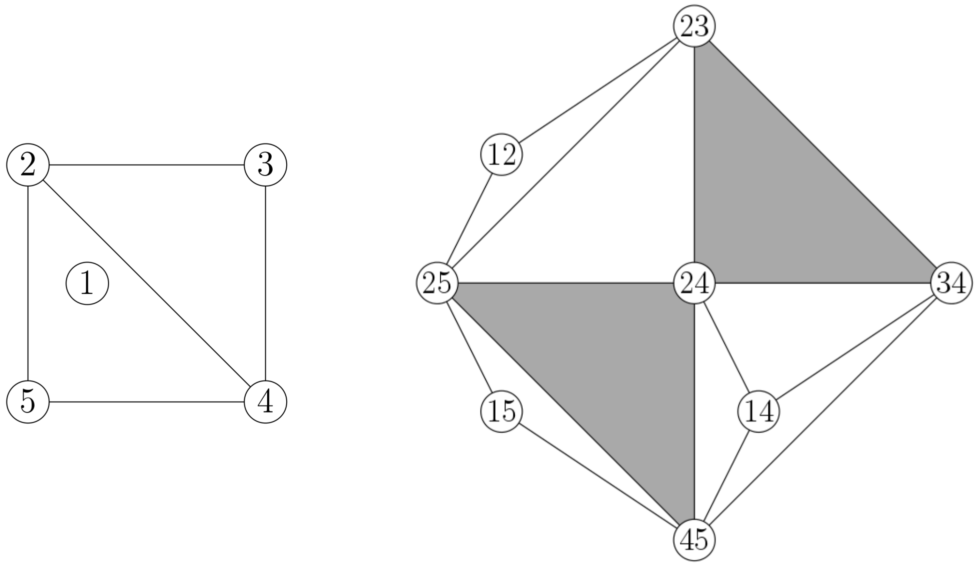


FIGURE 3.4: A not separated hypertriangulation in the plane.

The following non-separated subdivision of $\mathbf{C}(4,1)^{(2)}$ appears in [Pos19, Exm. 10.4]:

$$S = \left\{ [1, 123]^{(2)}, [1, 134]^{(2)}, [4, 124]^{(2)}, [4, 234]^{(2)} \right\}.$$

Here we generalize it to

Lemma 3.38. *For every odd d and every $k \in [2, d-2]$ there is a non-separated hypertriangulation of $\mathbf{C}(d+3, d)^{(k)}$.*

Proof. A hypertriangulation of a configuration A with $n = d + 3$ has all its full-dimensional cells of one of the following forms, where $a < b \in [n]$ and we omit the superscript (k), which will be clear from the context:

$$[\,\emptyset,\ [n]\backslash ab\,], \quad [\,a,\ [n]\backslash b\,], \quad [\,b,\ [n]\backslash a\,], \quad [\,ab,\ [n]\,].$$

To simplify notation, we denote these four cells simply as ab, $\overline{a}b$, $a\overline{b}$ and $\overline{a}\overline{b}$, respectively (observe that we always write the indices a and b in increasing order). For example, in this notation the subdivision S of $\mathbf{C}(4,1)^{(2)}$ mentioned above becomes

$$S = \{\overline{1}4, \overline{1}2, 3\overline{4}, 1\overline{4}\}^{(2)}.$$

One reason for this notation is that via the correspondence in Theorem 3.12 the tile $[X, Y]$ corresponds in $\mathbf{G}_A$ to the cone spanned by $X \cup \overline{[n] \setminus Y}$, where we use $\overline{B}$ to denote the set of vectors opposite to B, for $B \subset [n]$.

With this notation, Theorem 3.12(2) gives us that the following is a (coherent) zonotopal tiling of $Z(\mathbf{C}(d+3,d))$ (Figure 3.5 shows the case of $\mathbf{C}(6,3)$):

$$S_0 := \{\overline{a}b : a \text{ odd}, b \text{ odd}\} \cup \{\overline{a}\overline{b} : a \text{ odd}, b \text{ even}\} \cup \{ab : a \text{ even}, b \text{ odd}\} \cup \{a\overline{b} : a \text{ even}, b \text{ even}\}.$$

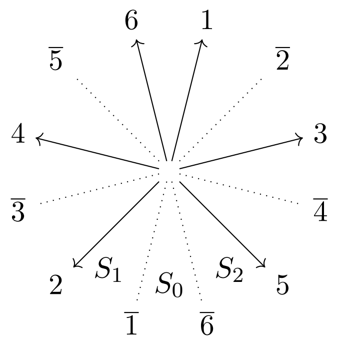


FIGURE 3.5: The Gale transform of $\mathbf{C}(6,3)$, with the regions corresponding to the zonotopal tilings S_0, S_1 and S_2 marked in it.

S_0 admits the following cubical flips:

- Flip 1: negate the other symbol in every cell containing $\overline{1}$. That is, remove

$$\{\overline{1}b : b > 1 \text{ odd}\} \cup \{\overline{1}\overline{b} : b \text{ even}\}$$

  and insert

$$\{\overline{1}\overline{b} : b > 1 \text{ odd}\} \cup \{\overline{1}b : b \text{ even}\}.$$

- Flip 2: negate the other element in every cell containing $\overline{n}$. That is, remove

$$\{a\overline{n} : a < n \text{ even}\} \cup \{\overline{a}\overline{n} : a \text{ odd}\}$$

  and insert

$$\{\overline{a}\overline{n} : a < n \text{ even}\} \cup \{a\overline{n} : a \text{ odd}\}.$$

These flips transform S_0 into two new coherent tilings S_1 and S_2, also shown in Figure 3.5. The two flips are not compatible, since both want to remove the tile $\overline{1}\overline{n}$ from S_0, and we can only remove it once. But $\overline{1}\overline{n}$ only affects level 1 of the tiling, which means that in any $S_0^{(k)}$ with $k \geq 2$ we can do these two flips one after the other. After performing them we get a subdivision that contains (for $k \in [2, d-2]$) the non-separated cells

$$\overline{1}2 \qquad \text{and} \qquad n-1\overline{n}. \qquad \square$$

To further generalize this construction we need the following easy lemma:

Lemma 3.39. *Let A be a d-dimensional configuration of size n in general position. If $A^{(k)}_{[n]\setminus i}$ has a non-separated subdivision S for some $i \in [n]$ then $A^{(k)}$ and $A^{(k+1)}$ have non-separated subdivisions too.*

Proof. For $A^{(k)}$ do the following: Extend S to a subdivision S' of A by adding all the cells of the form $[X, Y \cup i]^{(k)}$ with $[X, Y] \subset 2^{[n]}$ such that $[X, Y \cup i]$ is separated from $[\emptyset, [n]\setminus i]$. (The latter is equivalent to saying that $[X, Y]$ is contained in a facet of $Z(A_{[n]\setminus i})$ whose normal vector has positive scalar product with i). S' is non-separated since it contains S.

For $A^{(k+1)}$ apply the same construction upside-down. That is, consider the non-separated subdivision $\overline{S}$ of $A^{(n-k-1)}_{[n]\setminus i}$ obtained from S via the map $[X, Y] \to [[n]\setminus Y, [n]\setminus X]$. From $\overline{S}$ construct a non-separated subdivision $\overline{S'}$ of $A^{(n-k-1)}$ as above, then turn $\overline{S'}$ upside-down to get a non-separated subdivision of $A^{(k+1)}$. $\square$

Corollary 3.40. *For every odd d, every $n \geq d+3$, and every $k \in [2, n-2]$, there is a non-separated hypertriangulation of $\mathbf{C}(n, d)^{(k)}$.*

Question 3.41. *Are there non-separated hypertriangulations of $\mathbf{C}(n, d)^{(k)}$ for $d \geq 4$ even? The case of $\mathbf{C}(n, 2)$ suggests that the answer is no.*

3.6 Baues posets for $A = \mathbf{P}_n$

In this section we will restrict ourselves to the case when A is a convex polygon $\mathbf{P}_n$.

Definition 3.42. Let $S = \{[X_i, Y_i]^{(k_i)}\}_{i \in I}$ be a subdivision of $\mathbf{P}_n^{(k)}$. We define

$$S^+ := \{[X_i, Y_i] \mid i \in I \quad |Y_i| > k+1\}$$
$$S^- := \{[X_i, Y_i] \mid i \in I \quad |X_i| < k-1\}$$

Proposition 3.43. *Let S be a zonotopal tiling of $Z(\mathbf{P}_n)$. Then*

$$S^{(k)+} = S^{(k+1)-}$$

Proof. It is straightforward to check that both sets equal

$$\{[X,Y] \in S \mid |X| < k \quad |Y| > k+1\}$$

□

The proposition suggests we use the notation

$$S^{(k+\frac{1}{2})} := S^{(k+1)-} = S^{(k)+}$$

and define the following poset:

Definition 3.44. We define $\mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$ to be the poset on the set

$$\{S^{(k+\frac{1}{2})} \mid S \in \mathcal{B}^Z(\mathbf{P}_n)\}$$

where the order is refinement, as in subdivisions: $S_1 < S_2$ if and only if $\forall \sigma \in S_1 \ \exists \tau \in S_2 : \ \sigma \subseteq \tau$. We have two natural order-preserving maps $\mathcal{U} : \mathcal{B}^{(k)}(\mathbf{P}_n) \to \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$ and $\mathcal{D} : \mathcal{B}^{(k+1)}(\mathbf{P}_n) \to \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$ such that for every $S \in \mathcal{B}^Z(\mathbf{P}_n)$ we have

$$\mathcal{U}(S^{(k)}) = \mathcal{D}(S^{(k+1)}) = S^{(k+\frac{1}{2})}.$$

Remark 3.45. The maps $\mathcal{U}$ and $\mathcal{D}$ are well defined thanks to the fact that all hypersimplicial subdivisions of $\mathbf{P}_n$ are lifting ([BW19]). For more general configurations the definitions above would only make sense restricted to lifting subdivisions.

Example 3.46. Consider the subdivision $T \in \mathcal{B}^{(2)}(\mathbf{P}_6)$ in Figure 3.6 whose maximal cells are

$$\left\{[\emptyset, 124]^{(2)}, [\emptyset, 234]^{(2)}, [\emptyset, 1456]^{(2)}, [1, 1246]^{(2)}, [2, 1234]^{(2)}, [4, 1345]^{(2)}, [4, 2345]^{(2)}\right\}$$

The gray cells of T in the figure give $\mathcal{D}(T)$; that is:

$$\mathcal{D}(T) = \{[\emptyset, 124], [\emptyset, 234], [\emptyset, 1456]\}.$$

As seen in the right part of the figure, the cells in $\mathcal{D}(T)$ are precisely the ones that have a full-dimensional intersection with the first level.

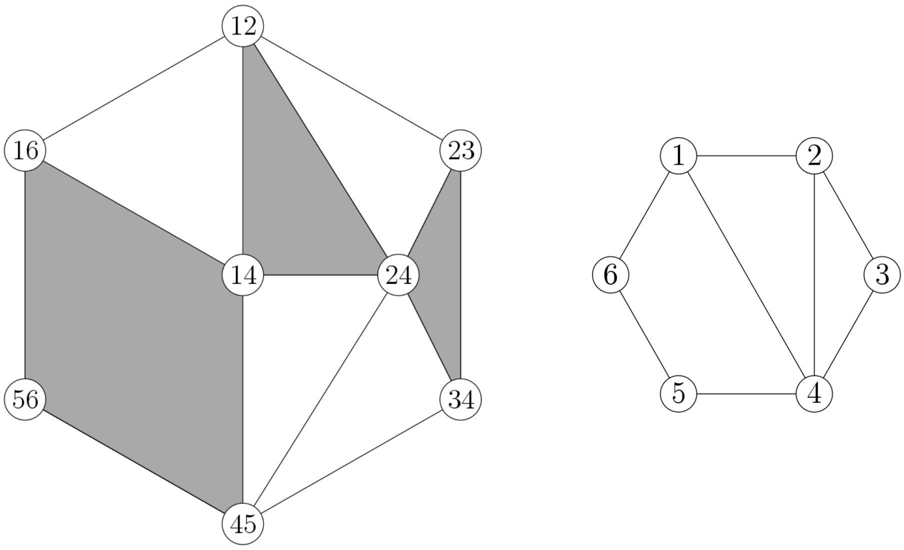


FIGURE 3.6: The subdivision $T \in \mathcal{B}^{(2)}(\mathbf{P}_6)$ of Theorem 3.46 (left) and $\mathcal{D}(T)^{(1)}$ (right).

The main result in this section is that $\mathcal{U}$ and $\mathcal{D}$ induce homotopy equivalences of the corresponding order complexes (Theorem 3.54). To prove this we use the following criterion, originally proved by Babson [Bab93]. Another proof can be found in [SZ93] and some generalizations appear in [BWW05]:

Lemma 3.47 (Babson's Lemma)**.** *Let $f : \mathcal{P} \to \mathcal{Q}$ be an order preserving map between two posets. Suppose that for every $q \in \mathcal{Q}$ we have that*

1. *$f^{-1}(q)$ is contractible, and*
2. *$f^{-1}(q) \cap \mathcal{P}_{\leq p}$ is contractible, for every $p \in f^{-1}(\mathcal{Q}_{\geq q})$.*

Then f is a homotopy equivalence.

For a collection S of subzonotopes of $Z(A)$, let $\mathrm{vertices}^{(k)}(S)$ be the set of vertices of cardinality k of all zonotopes in S. We only consider a point B in $[X, Y]$ to be a vertex if it is a face; that is, if $[X, Y]$ is separated from $\{B\}$.

Proposition 3.48. *Let $S \in \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$. Consider a point $X \in \mathrm{vertices}^{(k)}(S)$. Define*

$$\mathrm{uh}_S(X) := X \cup \{i \in [n] \mid X \cup i \in \mathrm{vertices}^{(k+1)}(S)\}.$$

*(Here "*uh*" stands for "upper hole"). Then $[X, \mathrm{uh}_S(X)]$ is separated from every cell in S.*

Proof. Observe that $\mathrm{uh}_S(X)$ equals

$$\{i \in [n] \mid \exists j \in X \quad [X \setminus j, X \cup i] \text{ is a face of a cell in } S\}.$$

Suppose there exists $X \in \text{vertices}^{(k)}(S)$ and $[I,J] \in S$ such that $[X, \cup \text{uh}_S(X)]$ and $[I,J]$ are not separated. Since $d = 2$ we may assume that $|J \setminus I| \leq 2$ and there is $Y \in [X, \text{uh}_S(X)]^{(k+2)}$ such that $[X,Y]$ is not separated from $[I,J]$. So we have a circuit (C^+, C^-) such that $C^+ \in Y \setminus I$ and $C^- \in J \setminus X$ Further, since $S \in \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$ we can also assume $|I| \leq k-1$. Let $y \in Y \setminus X$. Since $y \in \text{uh}_S(X) \setminus X$ we have that there is $x \in X$ such that $[X \setminus x, X \cup y]$ is a face of a cell in S. Then by Theorem 3.34 and the fact that S is pairwise separated we have that $[X \setminus x, X \cup y]$ is separated from $[I,J]$. So C^+ can not be contained in $X \cup y$. This means that $C^+ = Y \setminus X$. Notice that for every $i \in [n] \setminus C$ there is $y \in C^+$ such that $(C^+ \setminus y \cup i, C^-)$ is a circuit. So if there is an $i \in X \setminus I$, this circuit would imply that $[X, Y \setminus y]$ is not separated from $[I,J]$, which can not be as $[X, Y \setminus y]$ is a face of some cell in S. But this means $X \setminus I = \emptyset$ which is a contradiction since $|X| = k > k-1 = |I|$. □

Corollary 3.49. *Let $S \in \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$. Then*

$$S^{(k+1)} \cup \{[X, \text{uh}_S(X)]^{(k+1)} \mid X \in \text{vertices}^{(k)}(S)\},$$

together with all their faces, form the unique coarsest subdivision in the fibre $\mathcal{D}^{-1}(S)$.

Proof. We need to show that for $X_1, X_2 \in \text{vertices}^{(k)}(S)$, $[X_1, \text{uh}_S(X_1)]$ and $[X_2, \text{uh}_S(X_2)]$ are separated. If not, we can again assume there are subsets $Y_1 \subseteq \text{uh}_S(X_1)$ and $Y_2 \subseteq \text{uh}_S(X_2)$ of cardinality $k+2$ such that $[X_1, Y_1]$ and $[X_2, Y_2]$ are separated. As any subtile of them are faces of S, we have that there is a circuit $C^+ = Y_1 \setminus X_1$ and $C^- = Y_2 \setminus X_2$. Similarly as the proof of 3.48, this implies that $X_1 = X_2$. The corollary follows from the fact that every cell in a subdivision in $\mathcal{D}^{-1}(S)$ not coming from S is of type 1 and hence it is contained in $[X, \text{uh}_S(X)]$ for some X. □

Example 3.50. Consider the subdivision $S \in \mathcal{B}^{(1)}(\mathbf{P}_6)$ in Figure 3.6 whose maximal cells are

$$\left\{[\emptyset, 124]^{(1)}, [\emptyset, 234]^{(1)}, [\emptyset, 145]^{(1)}, [\emptyset, 156]^{(1)}\right\}.$$

We have that

$$\begin{aligned} &\text{uh}(1) = 12456, \quad \text{uh}(2) = 1234, \quad \text{uh}(3) = 234, \\ &\text{uh}(4) = 12345, \quad \text{uh}(5) = 156, \quad \text{uh}(6) = 156, \end{aligned}$$

so that the coarsest subdivision $\hat{S}$ of $\mathcal{D}^{-1}(S)$ has maximal cells

$$\Big\{[\emptyset, 124]^{(2)}, [\emptyset, 234]^{(2)}, [\emptyset, 145]^{(2)}, [\emptyset, 156]^{(2)}, \\ [1, 12456]^{(2)}, [2, 1234]^{(2)}, [4, 12345]^{(2)}, [5, 1456]^{(2)}\Big\}.$$

The two cells

$$[3, \mathrm{uh}(3)]^{(2)} = [3, 234]^{(2)} \subset [\emptyset, 234]^{(2)}, \qquad \text{and}$$
$$[6, \mathrm{uh}(6)]^{(2)} = [6, 156]^{(2)} \subset [\emptyset, 1456]^{(2)}$$

are also in $\hat{S}$, but they are not maximal: they are edges.

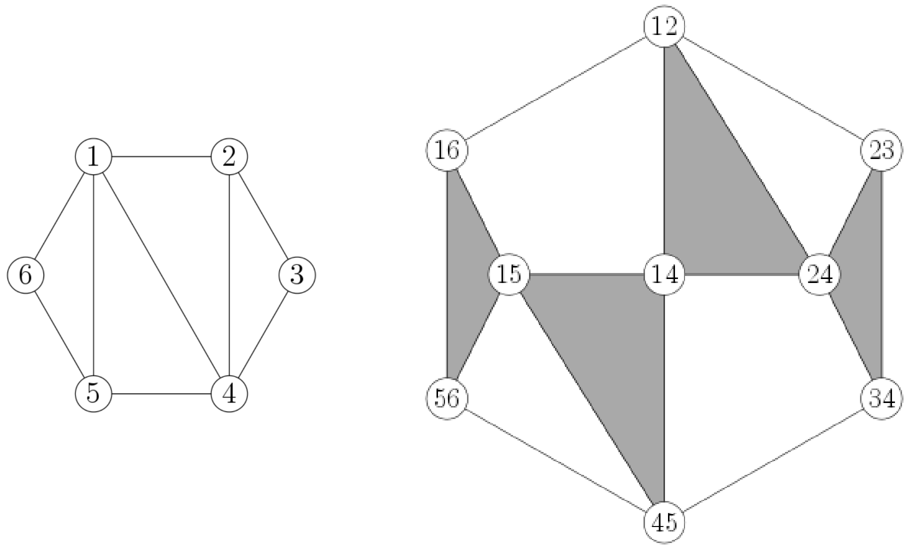


FIGURE 3.7: The subdivision $S \in \mathcal{B}^{(1)}(\mathbf{P}_6)$ of Theorem 3.50 (left) and $\hat{S} \in \mathcal{B}^{(2)}(\mathbf{P}_6)$ (right).

Lemma 3.51. *Let $S \in \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$ and let $T \in \mathcal{B}^{(k+1)}(\mathbf{P}_n)$ be such that $S \leq \mathcal{D}(T)$. Then, the poset $\mathcal{D}^{-1}(S) \cap \mathcal{B}^{(k+1)}(\mathbf{P}_n)_{\leq T}$ has a unique maximal element.*

Proof. Let $\hat{S}$ be the maximal element of $\mathcal{D}^{-1}(S)$, as described in Theorem 3.49.

Let $T' \in \mathcal{D}^{-1}(S)$, which is a refinement of $\hat{S}$. If a cell $[X, Y]^{(k+1)} \in T'$ is such that $|X| < k$, then $[X, Y] \in S$ which implies that it is contained in a cell of $\mathcal{D}(T)$. Then, $[X, Y]^{(k+1)}$ is contained in a cell of T. Thus, for T' to be a refinement of T, it is enough that $[X, Y]^{(k+1)} \in T'$ is contained in a cell of T for every $[X, Y] \in T'$ with $|X| = k$.

For every such X, the cells $[X, Y']^{(k+1)} \in T$ are a subdivision of $[X, \mathrm{uh}_{\mathcal{D}(T)}(X)]^{(k+1)}$. Let $[X, Y_1]^{(k+1)}, \ldots, [X, Y_l]^{(k+1)}$ be such subdivision. For each Y there are two possibilities:

- If $Y \subseteq \mathrm{uh}_{\mathcal{D}(T)}(X)$, then $[X, Y]^{(k+1)}$ is contained in a cell of T if and only if there is some $i \in [l]$ such that $Y \subseteq Y_i$.
- If Y is not contained in $\mathrm{uh}_{\mathcal{D}(T)}(X)$, then $[X, Y]^{(k+1)}$ is contained in a cell of T if and only if $[X, Y]^{(k+1)}$ does not intersect the interior of $[X, \mathrm{uh}_{\mathcal{D}(T)}(X)]^{(k+1)}$. To see this, notice that if $[X, Y]^{(k+1)}$ does not intersect the interior of $[X, \mathrm{uh}_{\mathcal{D}(T)}(X)]^{(k+1)}$,

then all vertices of $[X, Y]^{(k+1)}$ correspond to edges of $S^{(k)}$ contained in the same cell of $\mathcal{D}(T)$. If this cell is $[X', Y']$, then $[X', Y']^{(k+1)} \in T$ contains $[X, Y]$.

The discussion above implies that: a $T' \in \mathcal{D}^{-1}(S)$ is a refinement of T if and only if all edges of T are also edges in T'. This follows from the fact that the only edges in T' not in $\hat{S}$ are of the form $[X, Y]$ with $|X| = k$ and $Y \subseteq \mathrm{uh}_S(X)$. For each X, there is a unique coarsest subdivision of the polygon $[X, \mathrm{uh}_S(X)]^{(k+1)}$ that uses those edges. The subdivision that does that for each X is the unique coarsest refinement of T in $\mathcal{D}^{-1}(S)$. □

Example 3.52. Consider the subdivisions T from Theorem 3.46 and S from Theorem 3.50. We have that S refines $\mathcal{D}(T)$. The unique minimal, (actually, the only) subdivision in $\mathcal{D}^{-1}(S) \cap \mathcal{B}^{(k+1)}(\mathbf{P}_n)_{\leq T}$ is T' as depicted in Figure 3.8.

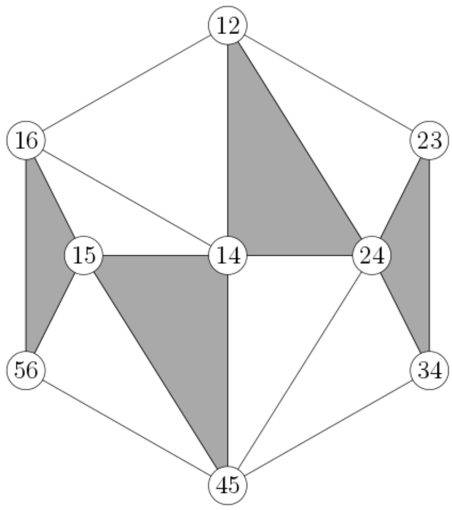


FIGURE 3.8: The only subdivision T' in $\mathcal{D}^{-1}(S) \cap \mathcal{B}^{(k+1)}(\mathbf{P}_6)_{\leq T}$

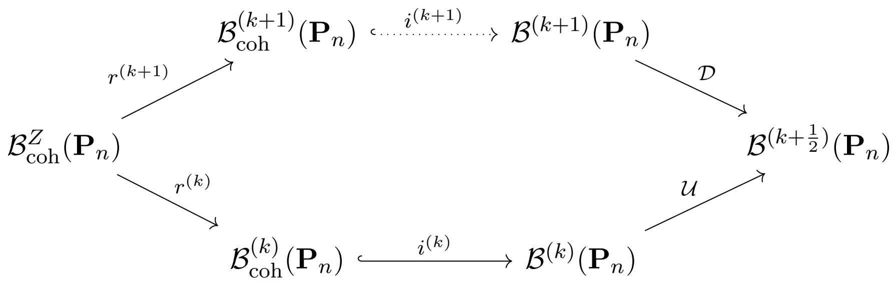
Remark 3.53. One could expect the unique maximal element stated in Theorem 3.51 to coincide with the maximal element $\widehat{\mathcal{D}(T)}$ in $\mathcal{D}^{-1}(\mathcal{D}(T))$. That is not the case in Theorem 3.52. In fact, in that example $\widehat{\mathcal{D}(T)}$ (whose picture would be as the picture of T in Figure 3.6 without the edge $\{45, 24\}$) does not refine $\hat{S}$.

Corollary 3.54. *The maps $\mathcal{D} : \mathcal{B}^{(k+1)}(\mathbf{P}_n) \to \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$ and $\mathcal{U} : \mathcal{B}^{(k)}(\mathbf{P}_n) \to \mathcal{B}^{(k+\frac{1}{2})}(\mathbf{P}_n)$ are homotopy equivalences.*

Proof. For $\mathcal{D}$, conditions (1) and (2) in Babson's Lemma follow from Theorem 3.49 and Theorem 3.51, respectively, since a poset with a unique maximal element is clearly contractible. For $\mathcal{U}$ the proof is completely symmetric. □

Theorem 3.55. *Let A be the vertex set of a convex n-gon. The inclusion $\mathcal{B}^{(k)}_{\mathrm{coh}}(A) \to \mathcal{B}^{(k)}(A)$ is a homotopy equivalence, for $k = 1, \ldots, n-1$.*

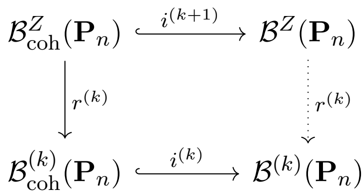
Proof. The proof is by induction on k. The base case, $k = 1$, is the main result of Rambau and Santos in [RS00]. Now let us suppose that $\mathcal{B}^{(k)}_{\text{coh}}(A) \to \mathcal{B}^{(k)}(\mathbf{P}_n)$ is a homotopy equivalence and we will prove that $\mathcal{B}^{(k+1)}_{\text{coh}}(\mathbf{P}_n) \to \mathcal{B}^{(k+1)}(\mathbf{P}_n)$ is also a homotopy equivalence. Consider the following diagram, which commutes by Theorem 3.43:



The maps $i^{(k)}$ and $i^{(k+1)}$ are the inclusions of coherent subdivisions into all subdivisions. The maps $r^{(k)}$ and $r^{(k+1)}$ are the restriction of each zonotopal tiling to its k and $k+1$ levels; that is, $S \mapsto S^{(k)}$ and $S \mapsto S^{(k+1)}$ respectively. They are homotopy equivalences since they can be geometrically realized as the identity maps among the normal fans of $\mathcal{F}^Z(\mathbf{P}_n)$, $\mathcal{F}^{(k)}(\mathbf{P}_n)$ and $\mathcal{F}^{(k+1)}(\mathbf{P}_n)$. Since $\mathcal{D}$ and $\mathcal{U}$ are homotopy equivalences by Theorem 3.54, and $i^{(k)}$ is a homotopy equivalence by inductive hypothesis, the dotted arrow $i^{(k+1)}$ must also be a homotopy equivalence. □

Corollary 3.56. *The restriction map* $r^{(k)} : \mathcal{B}^Z(\mathbf{P}_n) \to \mathcal{B}^{(k)}(\mathbf{P}_n)$ *is a homotopy equivalence.*

Proof. We now use the following commutative diagram:



The top arrow is a homotopy equivalence by [SZ93] and the bottom arrow by Theorem 3.55. The left arrow is also a homotopy equivalence, as mentioned in the proof of Theorem 3.55, so the right arrow is a homotopy equivalence too. □

3.7 Hypercatalan numbers

Let $C_n^{(k)}$ be the number of hypertriangulations of $\mathbf{P}_n^{(k)}$, which we will call hypercatalan number. When $k = 1$ these are the usual Catalan numbers C_n. In this section we look at the case $k = 2$. For a triangulation T of $\mathbf{P}_n$ and a vertex $i \in [n]$ we write $\deg_T(i)$ for the number of diagonals (edges excluding the sides of $\mathbf{P}_n$) in T incident to i and we call it the *degree* of i.

Lemma 3.57.

$$C_n^{(2)} = \sum_T \prod_{i \in [n]} C_{\deg_T(i)},$$

where the sum runs over all trinangulations T of $\mathbf{P}_n$.

Proof. Let T be a triangulation of $\mathbf{P}_n$. To get a hypertriangulation of $\mathbf{P}_n^{(2)}$ that agrees with T we need to triangulate $[i, \mathcal{U}_T(i)]^{(2)}$ for every i. As $[i, \mathcal{U}_T(i)]^{(2)}$ is a polygon with $\deg_T(i) + 2$ vertices, the number of ways to triangulate it is $C_{\deg_T(i)}$. So for each triangulation T there are $\prod_{i \in [n]} C_{\deg_T(i)}$ hypertriangulations of $\mathbf{P}_n^{(2)}$. Summing over all triangulations gives the desired result. ☐

Example 3.58. For $n = 3, \dots, 10$ we have computed this formula to give the following values:

n	3	4	5	6	7	8	9	10
	1	2	10	70	574	5176	49656	497640

The computation for $n = 6$ is as follows. Triangulations of the hexagon fall into three symmetry classes:

- Two triangulations with degree sequence 020202, each contributing $1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 = 8$ to the sum.
- Six triangulations with degree sequence 012012, each contributing $1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 = 4$ to the sum.
- Six triangulations with degree sequence 011103, each contributing $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 5 = 5$ to the sum.

This gives a total of $2 \cdot 8 + 6 \cdot 4 + 6 \cdot 5 = 70$ fine subdivisions in $\mathcal{B}^{(2)}(\mathbf{P}_6)$.

Lemma 3.59. *Let T be a triangulation of an $(n+2)$-gon with $n \geq 4$. Then*

$$2^{n-2} \leq \prod_{i \in [n+2]} C_{\deg_T(i)}, \leq 2^{\frac{5}{2}n-7}.$$

Proof. Let $k_1 \dots k_j$ be the sequence of the degrees of the vertices of T which are positive. The terms of this sequence add up to $2n-2$. The contribution of T to the sum is $\prod_{i=1}^{j} C_{k_i}$. Observe that the number $(n+2)-j$ is the number of ears in T, which lies between 2 and $\frac{n}{2}+1$. Thus, j lies between $\frac{n}{2}+1$ and n.

For the lower bound, take into account that for every $k \geq 1$ one has $2^{k-1} \leq C_k$, we deduce the contribution of T to be at least 2^{2n-2-j}. Plugging in that $j \leq n$, we get the desired lower bound.

For the upper bound, let l be number of degree 1 vertices. Reorder the k_i so that the last l are equal to 1. We have that $\sum_{i=1}^{j-l} k_i = 2n-2-l$. Now take into account that for $k \geq 2$ we have that $C_k \leq 2^{2k-3}$, so

$$\prod_{i \in [n+2]} C_{\deg_T(i)} = \prod_{i=1}^{j-l} C_{k_j} \leq 2^{2(2n-2-l)-3(j-l)} = 2^{n-10+3e+l}$$

where $e = n+2-j$ is the number of ears. So to prove the upper bound we need to show that $3e + l \leq \frac{3n+6}{2}$.

Suppose T is the triangulation that maximizes $3e+l$. If there was a vertex of inner degree 1 such that it is not adjacent to an ear, flipping this edge would not decrease the number $3e+l$. So we can assume every degree 1 vertex is next to an ear. But then the vertex of degree 1 can not be neighbour to two ears, otherwise $n=2$, and it can not be neighbour to another vertex of degree 1, otherwise $n=3$. Also, an ear can not be neighbour to two degree 1 vertices, otherwise $n=2$. So the other neighbours of a pair of consecutive vertices (ear,degree 1) must have degree at least 2. Let e' the number of ears not adjacent to any degree 1 vertex. Then $e-e'=l$ is the number of pairs (ear,degree 1) and we have:

$$\begin{aligned} l + 2e &= 3l + 2e' \leq n+2 \\ l + 3e &\leq n+2+e \leq \frac{3(n+2)}{2} \end{aligned}$$

□

Corollary 3.60. *For $n \geq 6$,*

$$2^{n-2} \leq \frac{C_n^{(2)}}{C_n} \leq 2^{\frac{5}{2}n-7}.$$

□

Remark 3.61. The lower bound of 2^{n-2} of Theorem 3.59 for the contribution of a single triangulation T is attained by a zigzag triangulation, in which all degrees are 2 except for two 1s and two 0s. When T is a star triangulation in which a vertex is joined to all others, the contribution of T is $C_{n-1} \sim 4^n$ (neglecting a polynomial factor). A higher contribution is obtained by the following procedure: start with any triangulation T_0 (e.g. a zig-zag or a star). Let T_1 be obtained by adding an ear at each boundary edge of T_0, let T_2 be obtained from T_1 in the same way, etcetera. This method produces triangulations that contribute about 4.133^n (according to our computations) for n large.

Remark 3.62. By [Gal18, Theorem 1.2], hypercatalan numbers are bounded from above by the number of fine zonotopal tilings of $Z(\mathbf{P}_n)$, which is sequence A060595 in the Online Encyclopedia of Integer Sequences. The known terms are

n	3	4	5	6	7
	1	2	10	148	7686

Part II

Matroid subdivisions: Tropical Linear Spaces

Chapter 4

Introduction to tropical linear spaces

In this chapter we provide an overview of tropical linear spaces. Tropical linear spaces, in the more general sense, are cryptomorphic to valuated matroids, originally defined in [DW92]. We recommend [MS15, chap. 4] as a general reference for this material.

4.1 Plücker Coordinates

We begin by reviewing some notions from classical linear algebra which are mimicked in tropical geometry. Let $\mathbb{K}$ be any field. *The Grassmanian* $\mathrm{Gr}(d, \mathbb{K}^n)$ is the moduli space of all d-dimensional linear subspaces of $\mathbb{K}^n$. As a set, $\mathrm{Gr}(d, \mathbb{K}^n)$ can be indentified with the space of matrices with coefficients in $\mathbb{K}$ modulo row transformations, i.e., $\mathbb{K}^{d\times n}/\,\mathrm{GL}(d, \mathbb{K}^n)$. If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ it is a compact manifold. In general, it is a smooth algebraic variety via the Plücker embedding which we now explain.

Let L be linear subspace of $\mathbb{K}^n$ of dimension d. Consider a matrix $A \in \mathbb{K}^{d\times n}$ whose rows are a basis of L. A *maximal minor* of A is the determinant of a $d \times d$ submatrix. For each subset $B \in \binom{[n]}{d}$, there is a corresponding maximal minor A_B by taking the submatrix of A whose columns are those indexed by B. All the maximal minors together form a vector in $\mathbb{K}^{\binom{[n]}{d}}$ (which can be thought of as the k-th exterior space of $\mathbb{K}^n$). Row operations on A only affect the vector of maximal minors as multiplication by a scalar. So, by projectivizing, we get that the vector

$$[A_B]_{B\in\binom{[n]}{d}} \in \mathbb{P}\left(\mathbb{K}^{\binom{[n]}{d}}\right)$$

is independent of the choice of A for a fixed L. This implies there is a well-defined map

$$\pi : \mathrm{Gr}(d, \mathbb{K}^n) \to \mathbb{P}\left(\mathbb{K}^{\binom{[n]}{d}}\right)$$

that sends a linear space L to the vector of maximal minors of any matrix whose rows are a basis of L. This map is called the *Stiefel map*. The image of the Stiefel map is called the *Plücker embedding* of $\mathrm{Gr}(d, \mathbb{K}^n)$ and the vector $\pi(L)$ the *Plücker coordinates* of L.

The Plücker embedding is a smooth projective variety, whose ideal is given by the Plücker relations: for any two sets $S \in \binom{[n]}{d-1}$ and $T \in \binom{[n]}{d+1}$ where the elements of T are $t_1 < \cdots < t_{d+1}$ we have that

$$\sum_{t_i \in T \setminus S} (-1)^i A_{B \cup t_i} A_{T \setminus t_i} = 0.$$

The linear space L can be recovered from its Plücker coordinates $\pi(L)$ in the following way:

$$L = \left\{ x \in \mathbb{K}^n \mid \forall T \in \binom{[n]}{d+1}, \ \sum_{t_i \in T} (-1)\pi(L)_{T \setminus t_i} \cdot x_{t_i} = 0 \right\}$$

4.2 The tropical Grassmanian

Recall the tropical semiring $\mathbb{T}$ from Theorem 2.9.

Definition 4.1 ([SS04]). Let $\mathbb{K}$ be an algebraically closed field of characteristic p with non trivial valuation. The tropical Grassmannian $\mathrm{TGr}_p(d, n) \subseteq \mathbb{P}\left(\mathbb{T}^{\binom{n}{d}}\right)$ is the tropicalization of $\mathrm{Gr}(d, \mathbb{K}^n)$ as a tropical projective variety.

As the notation suggests, the tropical Grassmannian depends on the charateristic of the field, but it does not depend on anything else about $\mathbb{K}$. Even though the Plücker relations are the same for every characteristic, the ideal generated by them has different tropical bases for different characteristics. The tropical Grassmannian is a pure $d(n-d)$-dimensional fan for every characteristic.

The study of the tropical Grassmannian was initiated by Speyer and Sturmfels [SS04]. The authors focused on $\mathrm{Gr}_p(2, n)$ and the special case $\mathrm{Gr}_p(3, 6)$. The fan structure and the homology of the tropical Grassmannian $\mathrm{Gr}_p(3, 7)$ is studied in [HJJS09].

Let $\mathrm{Gr}_p^\circ(2, n)$ be the restriction of the tropical Grassmannian $\mathrm{Gr}_p^\circ(2, n)$to its finite part $\mathbb{R}^{\binom{[n]}{2}}$ and consider its linearity space $\mathbb{R}^n$.

Theorem 4.2 ([SS04, Theorem 3.4 and Corollary 4.5])**.** *The Grassmannian* $\mathrm{Gr}_p^(2, n)$ *is characteristic-free and* $\mathrm{Gr}_p^\circ(2, n)/\mathbb{R}^n$ *coincides with the space of phylogenetic trees with* n *labeled leaves.*

Definition 4.3. A *tropical linear space realizable over* $\mathbb{K}$ or *tropicalized linear space* is a subset of $\mathbb{T}^n$ of the form $\mathrm{trop}(L)$ where $L \in \mathrm{Gr}(d, \mathbb{K}^n)$.

Let V be a vector in the tropical Grassmanian $\mathrm{TGr}_p(d, n)$. For each subset $S \in \binom{[n]}{d+1}$, consider the tropical linear polynomial

$$f_S(v) := \bigoplus_{i \in S} v_{S \setminus i} \odot x_i \tag{4.1}$$

and define $\mathcal{L}(V)^\circ$ as the intersection of the tropical hyperplanes $V(f_S)$, as S varies over all elements in $\binom{[n]}{d+1}$. Then, $\mathcal{L}(V)^\circ$ is the tropicalization of a classical d-dimensional linear space.

Speyer in [Spe08, Proposition 4.5.1] showed that every tropicalization of a linear space arises this way. By Theorem 2.15, $\mathrm{TGr}_p(d, n)$ consists of the valuations of Plücker coordinates of classical linear spaces. Given a linear space $L \in \mathrm{Gr}(d, \mathbb{K}^n)$, let $V \in \mathrm{TGr}_p(d, n)$ be the valuation of $\pi(L)$. Then L_V coincides with the tropicalization of L.

Theorem 4.4 ([SS04, Theorem 3.8])**.** *The bijection between the classical Grassmannian* $Gr(d, n)$ *and the set of* d*-planes in* $\mathbb{K}^n$ *induces a unique bijection* $V \mapsto L_V$ *between the tropical Grassmannian and the set of realizable tropical linear spaces of dimension* d *in* $\mathbb{T}^n$.

4.3 Matroids

The concept of tropical linear space is more general than the realizable ones introduced in the previous section. In order to introduce it, we first need to speak about matroids. Matroids are classical objects in discrete mathematics. They are an abstraction of the concept of linear independence. Nakasawa and Whitney introduced them independently in the 1930s. There are many cryptomorphic ways to define a matroid. We will present just one definition and focus on their relation to polyhedral structures. We recommend Oxley [Oxl11] and White [Whi86] as main references for matroid theory.

Definition 4.5. A *matroid* $M = (E, \mathcal{B})$ consists of a ground set E and a set $\mathcal{B}$ of subsets of E such that the following axioms are satisfied:

1. $\mathcal{B} \neq \emptyset$.

2. For any $B_1, B_2 \in \mathcal{B}$ and $i \in B_1 \backslash B_2$ there exists $j \in B_2 \backslash B_1$ such that $B_1 \cup j \backslash i \in \mathcal{B}$ *(Exchange axiom).*

We call the elements of $\mathcal{B}$ the *bases* of M, which we write $\mathcal{B}(M)$ for clarity. All elements of $\mathcal{B}(M)$ have the same cardinality, called the *rank* of M, denoted by $\text{rk}(M)$. Unless stated otherwise, we take $E = [n]$. Any subset $I \subseteq [n]$ such that there is a basis $B \in \mathcal{B}(M)$ containing it is called *independent.* Otherwise it is called *dependent.*

Example 4.6. Given $0 \leq d \leq n$, the *uniform matroid* $U_{d,n}$ is the matroid on $[n]$ such that $\mathcal{B}(U_{d,n}) = \binom{[n]}{d}$.

Example 4.7. Let $L \subseteq \mathbb{K}^n$ be a linear subspace. Then the collection of subsets $B \in \binom{[n]}{k}$ for which $\pi(L)_B \neq 0$ forms the bases of a matroid $M(L)$ on $[n]$. Any matroid that arises this way is called *representable over* $\mathbb{K}$. The rank of $M(L)$ equals the dimension of L. Matroids representable over characteristic 2 are called *binary* and matroids representable over characteristic 3 are called *ternary.*

Example 4.8. Consider the matroid F_7 on $[7]$ of rank 3 represented by all non zero vectors of $\mathcal{F}_2^3$, called the *Fano* matroid (see Figure 4.1). This matroid is binary, but it is only representable over fields of characteristic. Now consider the *non-Fano* matroid $\overline{F_7}$ also on $[7]$ where $\mathcal{B}(\overline{F_7}) = \mathcal{B}(F_7) \cup \{456\}$. The non-Fano matroid is not binary, but it is representable over every other characteristic. The direct sum, defined shortly ahead, $F_7 \oplus \overline{F_7}$ is not representable over any field.

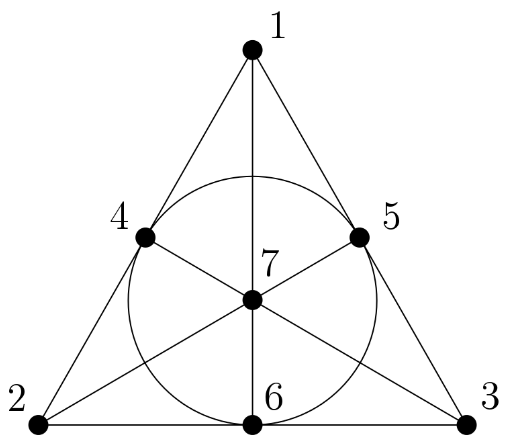


FIGURE 4.1: The Fano matroid F_7. Lines (and the circle) represent triples of dependent elements.

A *connected component* of M is a minimal non-empty subset $A \subseteq [n]$ such that $|A \cap B|$ is the same for all $B \in \mathcal{B}(M)$. The connected components of any matroid partition $[n]$, and a matroid is called *connected* if its only connected component is $[n]$. A *loop* of M is an element $j \in [n]$ not contained in any basis $B \in \mathcal{B}(M)$. A *coloop* is an element $j \in [n]$

contained in every basis $B \in \mathcal{B}(M)$. Both loops and coloops are connected components of size 1.

The *matroid polytope* of M is

$$P_M := \operatorname{conv}\{e_B \mid B \in \mathcal{B}(M)\} \subseteq \mathbb{R}^n$$

The dimension of P_M is equal to $\operatorname{rk}(M)$ minus the number of connected components of M.

Example 4.9. The matroid polytope $P_{U_{d,n}}$ of the uniform matroid $U_{d,n}$ of Theorem 4.6 is the hypersimplex $\Delta_{d,n}$ from Theorem 2.1.

Proposition 4.10 (Edmonds [Edm70]. See also [GGMS87])**.** *A polytope P is a matroid polytope $P = P_M$ for a matroid M of rank d on $[n]$ if and only if all of its vertices and all of its edges are also vertices and edges of the hypersimplex $\Delta(d,n)$. In other words, it is a polytope whose vertices are $\{0,1\}$-vectors of length n with d ones and its edges are parallel to $e_i - e_j$.*

Remark 4.11. Consider the Grassmannian $\operatorname{Gr}(d, \mathbb{C}^n)$ over the complex numbers $\mathbb{C}$. The algebraic torus $(\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1x_1, \ldots, t_nx_n)$. The action is linear so it maps subspaces to subspaces. Therefore, it induces an action on the Grassmannian $\operatorname{Gr}(d, \mathbb{C}^n)$. Given a point $L \in \operatorname{Gr}(d, \mathbb{C}^n)$, the closure of the orbit $T \cdot L$ is a toric variety. If $M = M(L)$ is the matroid associated to L, then the toric variety $\overline{T \cdot L}$ is isomorphic to X_{P_M}. See [GGMS87] for details.

The *rank function* of matroid M on ground set $[n]$, is defined for every subset of $[n]$ as

$$\operatorname{rk}(J) := \max_{B \in \mathcal{B}(M)} |B \cap J|$$

. Given a rank d matroid M with ground set $[n]$ and a subset $J \subseteq [n]$ with $k = \operatorname{rk}(J)$. The *restriction* of M to J is the matroid $M|J$ on the ground set J of rank k whose bases are

$$\mathcal{B}(M|J) := \{B \cap J \mid B \in \mathcal{B}(M),\ \ |B \cap J| = k\}.$$

The *contraction* of J in M is the matroid M/J on the ground set $[n] \setminus J$ of rank $d - k$ whose bases are

$$\mathcal{B}(M/J) := \{B \setminus J \mid B \in \mathcal{B}(M),\ \ |B \cap J| = k\}.$$

The *dual* of M is the matroid M^* on the ground set $[n]$ of rank $n - d$ whose bases are

$$\mathcal{B}(M^*) := \{[n] \setminus B \mid B \in \mathcal{B}(M)\}$$

One can easily verify that $(M|J)^* = M^*/([n] \setminus J)$ and $(M/J)^* = M^*|([n] \setminus J)$.

Given two matroids M_1 and M_2, of ranks d_1 and d_2 on the disjoint ground sets E_1 and E_2 respectively, the *direct sum* $M_1 \oplus M_2$ is a matroid on the ground set $E_1 \cup E_2$ where

$$\mathcal{B}(M_1 \oplus M_2) := \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}(M_1),\ \ B_2 \in \mathcal{B}(M_2)\}.$$

If $J_1, \ldots, J_c$ are the connected components of M, then $M = M|J_1 \oplus \cdots \oplus M|J_c$. It is easy to see that $(M_1 \oplus M_2)^* = M_1^* \oplus M_2^*$.

In terms of polytopes we have that $P_{M_1 \oplus M_2} = P_{M_1} \times P_{M_2}$. The polytope P_{M^*} is the image of P_M under the affine map

$$(x_1, \ldots, x_n) \mapsto (1 - x_1, \ldots, 1 - x_n).$$

If $j \in [n]$ is not a loop then $P_{M/j}$ is the same as the intersection of P_M with the hyperplane $\{x_j = 1\}$. If j is not a coloop then $P_{M|[n]\setminus j}$ is the same as the intersection of P_M with the hyperplane $\{x_j = 0\}$.

The *flats* of a matroid M are the subsets $F \subseteq [n]$ such that for every $j \in [n] \setminus F$,

$$\operatorname{rk}(F \cup \{j\}) > \operatorname{rk}(F)$$

. We denote the set of flats of M by $\mathcal{F}(M)$.

The set $\mathcal{F}(M)$ is a ranked lattice of rank d under set inclusion. The intersection of two flats is also a flat. Hence, for an arbitrary set $J \subseteq [n]$, not necessarily a flat, there is a unique smallest flat of M containing it, called the *closure* of J, and denoted $\operatorname{cl}(J)$.

Dually, a *cyclic set* of M is a set C such that

$$\operatorname{rk}(C \setminus \{j\}) = \operatorname{rk}(C)$$

for all $j \in C$. For each $J \subseteq [n]$ there is a unique largest cyclic set of M contained in it, called the *coclosure* of J, and denoted $\operatorname{cocl}(J)$. A *cyclic flat* is a cyclic set which is a flat. We denote the set of cyclic flats of M by $\mathcal{CF}(M)$.

Observe that J is cyclic if and only if $M|J$ is coloop-free and J is a flat if and only if M/J is loop-free. For any $F \in \mathcal{F}(M)$ the intersection of P_M with the hyperplane $\left\{ \sum_{j \in F} x_j = \operatorname{rk}(F) \right\}$ is a face of P_M and it is the polytope of the matroid $M|F \oplus M/F$. Any facet of P_M which intersects the interior of $\Delta(d, n)$ is of this form for a cyclic flat $F \in \mathcal{CF}(M)$, and all the other facets are also of this form for some singleton F.

Given a matroid M of rank d on $[n]$, the *matroid cell* S_M is the subset of $\mathrm{Gr}(d, \mathbb{K}^n)$ of all linear subspaces L such that $M(L) = M$. If M is not representable over $\mathbb{K}$ then the corresponding cell is empty. Matroid cells provide the Grassmannian a stratification by representable matroids. However the strata or not as nice as one could hope for. In fact, it follows from Mnëv's Universality theorem that matroid cells can be as complicated as any semialgebraic set [Mnë88].

4.4 Valuated matroids

Valuated matroids are the tropical analogue of a Plücker vectors. They were first defined by Dress and Wenzel in [DW92] before the theory of tropical linear spaces (and most of tropical geometry) was developed.

Definition 4.12. A *valuated matroid* V on the ground set $[n]$ of *rank* $\mathrm{rk}(V) = d$ with $0 \leq d \leq n$, is a vector in $\mathbb{P}\left(\mathbb{T}^{\binom{n}{d}}\right)$ that satisfies the tropical Plücker relations: for any sets $S \in \binom{[n]}{d-1}$ and $T \in \binom{[n]}{d+1}$, the minimum of

$$\bigoplus_{i \in T \setminus S} V_{S \cup \{i\}} \odot V_{T \setminus \{i\}} \tag{4.2}$$

is achieved at least twice. The space of all valuated matroids $V \in \mathbb{P}(\mathbb{T}^{\binom{n}{d}})$ is called the *Dressian* $\mathrm{Dr}(d, n)$.

Since the tropical Grassmannian $\mathrm{TGr}_p(d, n)$ satisfies the tropical Plücker relations, we have the containment $\mathrm{TGr}_p(d, n) \subseteq \mathrm{Dr}(d, n)$. However, this containment is strict for $d > 2$ (see Theorem 4.16). This happens because, on one hand, the tropical Grassmannian $\mathrm{TGr}_p(d, n)$ is the vanishing of the tropicalization of the ideal $I_{d,n}$ generated by the Plücker relations, while the Dressian is only the vanishing of the tropicalization of the generators of $I_{d,n}$, which is not alway a tropical basis. In other words, the tropical Grassmannian is a tropical variety while the Dressian is only a tropical prevariety (see Theorem 2.16). Note that while $\mathrm{TGr}_p(d, n)$ depends on the characteristic p, the Dressian $\mathrm{Dr}(d, n)$ does not. The difference between the tropical Grassmannian and the Dressian can also be observe from their dimensions; the dimension of $\mathrm{Dr}(d, n)$ is of order n^{d-1} for fixed d, while the dimension of $\mathrm{Gr}_p(d, n)$ grows linear in n, see [JS17, Corollary 32].

With this language, a matroid M can be thought of as a valuated matroid with trivial valuation, that is, the vector in $\mathbb{P}\left(\mathbb{T}^{\binom{n}{d}}\right)$ where

$$M_B \begin{cases} 0 \text{ if } B \in \mathcal{B}(M) \\ \infty \text{ if } B \notin \mathcal{B}(M) \end{cases}$$

The Plücker relations restricted to $\{0, \infty\}$ are equivalent to the exchange axiom. This also implies that given a valuated matroid V, the set of B such that V_B is finite form the basis of a matroid which we call *underlying matroid* $\underline{V}$ of V. We identify a valuated matroid with values in $\{0, \infty\}$ with its underlying matroid.

A valuated matroid can be regarded as an enrichment of a matroid by assigning a real number to each of its bases. In other words, it is a height vector on the vertices of the matroid polytope $P_{\underline{V}}$. Thus, it induces a regular subdivision of $P_{\underline{V}}$ which we denote $\mathcal{S}_V$.

Theorem 4.13. *[Spe08, Proposition 2.2] Let M be a matroid and $V : \mathcal{B}(M) \to \mathbb{R}$. The following are equivalent:*

1. *V is a valuated matroid.*
2. *V satisfies the 3-term Plücker relations, that is, Equation* (4.2) *only for $S \subseteq T$.*
3. *All the cells of the regular subdivision $\mathcal{S}_V$ are matroid polytopes.*

Given a vector $x \in \mathbb{R}^n$, by definition of regular subdivision (see Section 2.2), the set

$$\{B \in \mathcal{B}(\underline{V}) \mid \forall B' \in \mathcal{B}(\underline{V}) \;\; e_B \cdot x \leq e_{B'} \cdot x\}$$

is an element of $\mathcal{S}_V$. Hence, this set of bases of a matroid V^x known as the *initial matroid* of V at x. This definition can be extended to any $x \in \mathbb{TP}^{n-1}$. We write $\mathcal{M}(V)$ for the set of all initial matroids of V all of whose loops are loops in V, i.e. matroids whose polytopes are cells of $\mathcal{S}_V$ not contained in any hyperplane $\{x_j = 0\}$.

Example 4.14. Consider the uniform matroid $U_{2,4}$. Its matroid polytope is the hypersimplex $\Delta_{2,4}$, which is an octahedron. Now consider the valuated matroid V where $V_{34} = 1$ and $V_B = 0$ for every $B \in \binom{4}{2} \setminus \{3, 4\}$. The matroid subdivision induced by V divides the octahedron into two square pyramids, one with apex e_{12} and the other one with apex e_{34}. The only x that selects the pyramid with apex e_{12} is $[0 : 0 : 0 : 0]$ while the only x that selects the pyramid with apex e_{34} is $[0 : 0 : 1 : 1]$. The initial matroids contained in $\mathcal{M}(V)$ are those whose polytopes are the two square pyramids, their common square face, and four of the triangular faces, namely $\mathrm{conv}(\{e_{12}, e_{13}, e_{14}\})$ and its S_4-images.

Given a valuated matroid V on the ground set $[n]$ and of rank d underlying matroid $M = \underline{V}$, we say an element $j \in [n]$ is a *(co)loop* of V if it is a (co)loop of the underlying matroid $\underline{V}$. The *dual* of V is the valuated matroid V^* of rank $n-d$ given by $V^*_B := V_{[n]\setminus B}$. Notice that $(V^*)^* = V$.

Let B_c be any basis of M/J. Then the *restriction* of V to J is the valuated matroid $V|J$ on the ground set J of rank k such that $V|J_B = V_{B\cup B_c}$ for any $B \in \binom{J}{k}$. This definition does not depend on the choice of $B_c \in \mathcal{B}(M/J)$, as choosing a different basis means tropically scaling all Plücker coordinates by the same factor (see, for example, Lemma 4.1.11 in [Fre13]). In particular $\underline{V}|J = \underline{V|J}$ and, if $k = d$, that is, $[n] \setminus J$ consists of only loops, then $V|J_B = V_B$. The *contraction* of J in V can be defined as $V/J := (V^*|([n] \setminus J))^*$.

4.5 Tropical linear spaces

Now we disscus tropical linear spaces in its full generality:

Definition 4.15. The *affine tropical linear space* associated to a valuated matroid V is

$$\mathcal{L}(V)^\circ := \bigcap_{T\in\binom{[n]}{d+1}} \mathbf{V}\left(\bigoplus_{j\in C} V_{T\setminus j}\odot x_j\right)$$

and the *projective tropical linear space* is the tropical projective variety

$$\mathcal{L}(V) := (\mathcal{L}(V)^\circ\setminus\{(\infty,\dots,\infty)\})/\mathbb{R}(1,\dots,1) \subseteq \mathbb{TP}^{n-1}.$$

When unspecified, by tropical linear space we usually mean a projective one. Let $V = \nu(w)$ for a (classical) Plücker vector $w \in \mathrm{Gr}(d,\mathbb{K}^n)$. The space $\mathcal{L}(V)^\circ$ is the tropicalization of the linear space L_w. The matroids $\mathcal{M}(V)$ obtained this way are all representable over $\mathbb{K}$. Since not all matroids are representable, not all tropical linear spaces are tropicalizations of classical linear spaces.

Example 4.16. The matroid of $F_7 \oplus \overline{F_7}$ from Theorem 4.8 with trivial valuation is in the Dressian $\mathrm{Dr}(3,7)$ but not in the tropical Grasmannian over any characteristic. Notice that also F_7 with trivial valuation is in $\mathrm{TGr}_2(3,7)$ but not in $\mathrm{TGr}_3(3,7)$, for example, so the tropical Grassmannian does depend on the characteristic.

The set $\mathcal{L}(V)^\circ$ can be given the structure of a polyhedral complex in several ways. When V is a matroid, two of these have been much discussed in the literature. They are distinguished in [FS05] as the *Bergman fan* and the *nested set complex*, and in [AK06] as the "coarse subdivision" and the "fine subdivision" respectively. Our arguments use the valuated generalization of the Bergman fan structure, which we introduce in this section. The name of the Bergman fan recognizes Bergman's work [Ber71]. We will explain the nested set complex after Theorem 4.19.

Let $\iota_J : \mathbb{R}^J \to \mathbb{T}^n$ be the inclusion of $\mathbb{R}^J$ in $\mathbb{T}^n$ by filling ∞ in the $[n] \setminus J$ coordinates.

Proposition 4.17 ([Spe08, Prop 2.3]; implicit in [KSZ92])**.** *Let V be a valuated matroid where J is the set of non loops. Then*

$$\mathcal{L}(V)^\circ = \iota_J(\{x \in \mathbb{R}^J \mid (V|J)^x \text{ has no loops}\}).$$

For simplicity, assume V is a valuated matroid with no loops. The interiors of the cells in the Bergmann fan structure of $L = \mathcal{L}(V)$ are the sets of points $x \in \mathbb{R}^n$ such that the matroid V^x is constant. The complex $\mathcal{L}(V)$ is pure of dimension $d-1$. For any initial matroid M of V which has the same set of loops as V, we write L_M for its corresponding cell, that is:

$$L_M := \overline{(\{x \in L^\circ \mid V^x = M\})}.$$

When this cell is 0-dimensional, we call it v^L_M (pedantically, v^L_M is the point which is the single element of L_M).

Lemma 4.1.11 of [Fre13] describes the effects of restriction and contraction on $\mathcal{L}(V)$. Given a subset $A \subseteq [n]$ we have that

$$\mathcal{L}(V/A) = \{x \in \mathbb{P}(\mathbb{T}^{[n]\setminus A}) \mid \hat{x} \in L\}$$

where $\hat{x} \in \mathbb{TP}^{n-1}$ is the extension of x by setting the coordinates indexed by A to be ∞. Let $\mathbb{TP}^{n-1}_A := \{x \in \mathbb{TP}^{n-1} \mid \exists i \in A \ x_i \neq \infty\}$ and let $\pi_A : \mathbb{TP}^{n-1}_A \to \mathbb{P}(\mathbb{T}^A)$ be the projection of x to the coordinates indexed by A. Then

$$\mathcal{L}(V|A) = \pi_A(L \cap \mathbb{TP}^{n-1}_A).$$

If M is a matroid, the polyhedral complex structure we have just placed on the tropical linear space $\mathcal{L}(M)$ is the Bergman fan. If $L = \mathcal{L}(V)$ is a tropical linear space and $x \in \mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ is in the relative interior of L_M, then $\mathcal{L}(M)$ equals the set of vectors y such that $x+\varepsilon y \in L$ for all sufficiently small $\varepsilon > 0$. That is, L looks like the translation $\mathcal{L}(M) + x$ locally near x.

Example 4.18. Consider the valuated matroid V from Theorem 4.14. The polytopes in the subdivision induced by V that correspond to loopless matroids are the two square pyramids, the square separating the pyramids and the four triangles which are inside each of the hyperplanes $x_i = 1$ for $i \in [4]$. Figure 6.1 shows a picture of the associated linear space.

We will use a construction of the set $\mathcal{L}(M)^\circ$ in terms of flats. For simplicity we state it in the loopless case.

Proposition 4.19 ([MS15], Theorem 4.2.6)**.** *Let M be a matroid with no loops. Then*

$$\mathcal{L}(M)^\circ = \left\{ \lambda e_{[n]} + \sum_{i=1}^{s} a_{F_i} e_{F_i} \mid \lambda \in \mathbb{R}, a_{F_i} \geq 0, F_1 \subset \cdots \subset F_s \in \mathcal{F}(M) \right\}.$$

The *nested set complex* of M is the order complex of the lattice of flats, interpreted as a polyhedral complex structure on $\mathcal{L}(M)$ by the above proposition. For further details see [FS05], [AK06] and [MS15, Chapter 4]. Moreover, the introduction of Hampe [Ham15] gives a broad overview about properties and developments of tropical linear spaces.

Chapter 5

Local Dressians

5.1 Introduction

The Grassmannian $\mathrm{Gr}(d, \mathbb{K}^n)$ can be stratified by representable matroids, where the strata S_M of M consists of all linear spaces L whose corresponding matroid is M. As remarked in [HJJS09], a similar stratification can be considered in the tropical setting. The strata in the tropical setting are the main object of study in this chapter and they are defined as follows:

Definition 5.1. [HJJS09] Let M be a rank d matroid on $[n]$. The *Dressian* on M, denoted by $\mathrm{Dr}(M)$, is the subset of $\mathrm{Dr}(d, n)$ consisting of all valuated matroids with underlying matroid is M.

When talking about the Dressian of a matroid we also call it *local*, to distinguish it from $\mathrm{Dr}(d, n)$. The Dressian of matroid was introduced by Herrmann, Jensen, Joswig and Sturmfels in [HJJS09], where they studied the fan structure of the local Dressian for the Pappus matroid. They also studied the fan of finite tropical Plücker vectors, which naturally corresponds to the Dressian $\mathrm{Dr}(U_{d,n})$ of the uniform matroid.

Local Dressians can be endowed with two fan structures: one coming from the Plücker relations and one as a subfan of the secondary fan. By Theorem 4.13 the Dressian is defined by the 3-term Plücker relations. More concretely, the 3-term Plücker relations say that for every $S \in \binom{[n]}{d}$, the minimum

$$\min(V_{Sij} + V_{Skl}, V_{Sik} + V_{Sjl}, V_{Sil} + V_{Sjk})$$

is achieved twice for every $i, j, k, l \in [n] \setminus S$. These relations induce the *Plücker fan structure* on $\mathrm{Dr}(d, n)$, given by the set of terms where V achieves the minimum for each 3-term Plücker relation.

On the other hand, Theorem 4.13 also implies that the Dressian $\mathrm{Dr}(M)$ is a subfan of the secondary fan of of the matroid polytope P_M. This endows $\mathrm{Dr}(M)$ with a *secondary fan structure* as subfan of the secondary fan of P_M. In [HJJS09, Theorem 4.4], the authors proved that for the uniform matroid $U_{3,n}$ the two fan structures coincide.

In Section 5.2 we prove the main result of this chapter is Theorem 5.2, which states that for any matroid M the two fan structures on $\mathrm{Dr}(M)$ coincide. The proof is based on a careful analysis of the subdivision induced by a point in the local Dressian on the 3-dimensional skeleton of the matroid polytope. From our study it follows that a matroid subdivision is completely determined by its restriction to the 3-skeleton.

We then focus on local Dressians of disconnected matroids. We show that the local Dressian of the direct sum of two matroids is the product of their local Dressians. Again, the key step in the proof is to look at the 3-dimensional skeleton of the matroid polytope.

Finally, in Section 5.3 we move our attention to rigid matroids, i.e., matroids which do not admit matroid subdivisions of their matroid polytopes. The local Dressians of such matroids are linear spaces. We give a new proof that binary matroids are rigid. We discuss some questions regarding refinement of matroid subdivisions, such as whether finest matroid subdivisions are always composed of rigid matroids as cells. We show that this is the case for rank 2 matroids.

5.2 The fan structure of local Dressians

Since we already discussed all the prelimanaries in Chapter 4, we can start right away with the main result:

Theorem 5.2. *Let M be a matroid of rank d on n elements. The Plücker fan structure coincides with the secondary fan structure on* $\mathrm{Dr}(M)$.

Proof. First, we take valuated matroids V and W on M lying in the same cone of the secondary fan. They induce the same subdivision of the matroid polytope P_M, in particular of the 3-dimensional skeleton. Therefore V and W satisfy the same three term Plücker relations and lie in the same cone of the local Dressian equipped with the Plücker structure.

Now we focus on the viceversa. We take V and W lying in the same Plücker cone σ. This means that they satisfy the same equations and inequalities coming from the three term Plücker relations. By Theorem 4.13, they induce two matroid subdivisions $\mathcal{S}_V$ and $\mathcal{S}_W$ of P_M. We want to show that $\mathcal{S}_V = \mathcal{S}_W$. This will imply that V, W are in the same secondary cone. By the fact that they satisfy the same Plücker relations, we know that $\mathcal{S}_V|_{3-\text{skeleton}} = \mathcal{S}_W|_{3-\text{skeleton}}$ as the 3-faces are either tetrahedra or octahedra. We pick S_V a maximal dimensional cell in $\mathcal{S}_V$. We suppose that S_V is not in $\mathcal{S}_W$. It means without loss of generality there are vertices q_1 and q_k in the cell S_V such that q_1 and q_k do not lie in a maximal dimensional cell of $\mathcal{S}_W$. Let $q_1\, q_2 \dots q_k$ be a path in the vertex-edge graph of the cell S_V. We pick a cell S_W in $\mathcal{S}_W$ that contains $q_1\, \dots\, q_i$ for some $i \le k$ and there is no cell in $\mathcal{S}_W$ containing $q_1\, \dots\, q_{i+1}$.

Now we have that q_{i-1} and q_{i+1} are at most of distance two. So we can use the base exchange axiom in the definition of a matroid to construct up to six points giving the unique face F of S_V spanned by q_{i-1} and q_{i+1}. The following situations may arise.

- Either F is a octahedron, then F is subdivided in $\mathcal{S}_W$ as q_{i-1}, q_i are in S_W and q_{i+1} is not. This is a contradiction to the fact that the subdivisions agree on the 3-skeleton.
- If F is a pyramid, it cannot be subdivided, therefore F is a face of S_W and hence q_{i+1} is a vertex of S_W, and that contradicts our assumption.
- Similarly if F is 2-dimensional, i.e., a square or a triangle.

Hence we conclude that both points q_1 and q_k are in S_W and hence the subdivisions $\mathcal{S}_V$ and $\mathcal{S}_W$ agree. □

Corollary 5.3. *The Plücker fan structure on the Dressian* $\mathrm{Dr}(d, n)$ *as a fan in* $\mathbb{R}^{\binom{n}{d}-1}$ *coincides with the secondary fan structure.*

Proof. It is enough to consider the uniform matroid $U_{d,n}$ in the previous statement. □

Corollary 5.4. *Let* $d \ge 2$*, and* $\mathcal{S}$ *and* $\mathcal{S}'$ *be two matroid subdivisions of the hypersimplex* $\Delta(d, n)$*. If they induce the same subdivision on the* 3*-skeleton, or equivalently on the octahedral faces of* $\Delta(d, n)$*, then* $\mathcal{S}$ *and* $\mathcal{S}'$ *coincide.*

Remark 5.5. The above statement extends Proposition 4.3 and Theorem 4.4 by Herrmann et al. [HJJS09] and is the key in the algorithm in Section 6 of Herrmann et al. [HJS12] for computing (local) Dressians. Note that the abstract tree arrangements in Section 4 of Herrmann et al. [HJJS09] are a cover of the 3-skeleton of the hypersimplex $\Delta(3, n)$ for $n \ge 6$ and the metric condition guarantees that the height functions agree on all three maps that contain a given vertex.

We derive the following characterization of the lineality space which follows from the characterization of the dimension of a matroid polytope in terms of connected components by Edmonds [Edm70] or Feichtner and Sturmfels [FS05]. Together with the fact that the secondary fan of a set of vertices has a lineality space of the same dimension as the affine dimension of the set of vertices.

Corollary 5.6. *Let b be the number of bases of a matroid M on n elements and with c connected components. The lineality space of the Dressian* $\mathrm{Dr}(M)$ *in* $\mathbb{R}^b/\mathbb{R}\mathbf{1}$ *is of dimension* $dim P_M = n - c$.

Proof. Adding a linear functions to the height function of a regular subdivision does not change the subdivision. Therefore the linealty space is the image of the map $\mathbb{R}^n \to \mathbb{R}^b$ with $e_i \mapsto \sum_{B \ni i} e_B$. □

Example 5.7. The local Dressian of the uniform matroid $U_{2,4}$ coinsides with the Dressian $\mathrm{Dr}(2,4)$. This is a 5-dimensional pure balanced fan in $\mathbb{R}^6/\mathbb{R}\mathbf{1}$ consisting of three maximal cells and a 3-dimensional lineality space.

Example 5.8. The local Dressian of the matroid $U_{1,2} \oplus U_{1,2}$ is a 2-dimensional linear space in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ spanned by $e_{13}+e_{14}$ and $e_{13}+e_{23}$. The corresponding matroid polytope $P_{U_{1,2}} \times P_{U_{1,2}}$ is a square, which has no finer matroidal subdivision.

Let us discuss two examples of local Dressians of non-regular connected ternary $(3,6)$-matroids. These are matroids that are representable over the field with three elements, but are not representable over the field with two elements.

Example 5.9. Let M be the matroid on 6 elements and rank 3 whose bases are $\binom{[6]}{3} \setminus \{123, 145, 356\}$, see Figure 5.1. The polytope P_M is full dimensional so the local Dressian $\mathrm{Dr}(M)$ has a lineality space of dimension 5 in $\mathbb{R}^{16} = \mathbb{R}^{17}/\mathbb{R}\mathbf{1}$. The local Dressian is 6-dimensional and consists of three maximal cones. These cones correspond to the vertex split with the hyperplane $x_2+x_4+x_6 = 0$ and two 3-splits, i.e., a subdivision into three maximal cells that intersect in a common cell of codimension 2. The three maximal cells of one of those 3-splits is illustrated in Figure 5.2.

Example 5.10. Let M be the connected matroid given by the 14 bases:

$$135,\ 136,\ 145,\ 146,\ 156,\ 235,\ 236,\ 245,\ 246,\ 256,\ 345,\ 346,\ 356,\ 456\ .$$

The local Dressian $\mathrm{Dr}(M)$ consists of three maximal cones of dimension 6 and a 5-dimensional lineality space in $\mathbb{R}^{13}$. In other words the polytope P_M has four matroidal subdivisions. The trivial subdivision and three splits with respect to the hyperplanes $x_4+x_5+x_6 = 2$, $x_3+x_5+x_6 = 2$ or $x_3+x_4 = 1$.

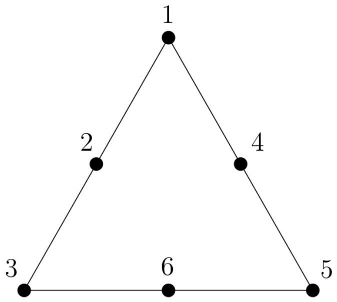


FIGURE 5.1: The terrnary matroid of Example 5.9.

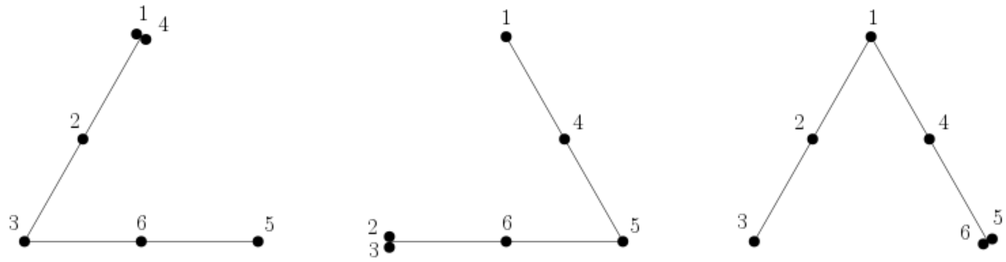


FIGURE 5.2: The three matroids of one of the subdivisions of Example 5.9.

Remark 5.11. For any point W in the local Dressian $\mathrm{Dr}(M)$ we can construct a tropical linear space L_W, by taking the intersection over $S \in M$ of the tropical hyperplanes defined by

$$f_S(W) = \bigoplus_{i \in S} W_{S \setminus i} \odot x_i \quad .$$

Proposition 5.12. *Let M and M' be matroids such that P_M is combinatorially isomorphic to $P_{M'}$. Then,*

$$Dr(M) \cong Dr(M').$$

Proof. A matroid subdivision of the polytope P_M does not impose new edges. The isomorphism between the polytopes P_M and $P_{M'}$ induces a subdivision of $P_{M'}$ as images of cells. Moreover, this subdivision is matroidal as the 1-cells are edges of $P_{M'}$. This subdivision is regular, as the map between P_M and $P_{M'}$ is a concatenation of a coordinate permutation, an embedding and a reflection. This follows from the explicit description in Remark 5.13. □

Remark 5.13. It can be shown that the two matroid polytopes of M and M' are combinatorially isomorphic if and only if the matroids are isomorphic up to loops, coloops or dual connected components. This is part of the work by Pineda-Villavicencio and Schröter [PVS].

The following statement deals with Dressians of direct sums of matroids. It was independently proven by Lopez de Medrano, Rincón and Shaw [LdMRS17, Lemma 4.7 and Corollary 4.8]

Theorem 5.14. *Let M_1 and M_2 be matroids with disjoint element sets. Then*

$$\mathrm{Dr}(M_1 \oplus M_2) = \mathrm{Dr}(M_1) \times \mathrm{Dr}(M_2).$$

Proof. We have the map

$$\begin{aligned} \otimes: \ \mathrm{Dr}(M_1) \times \mathrm{Dr}(M_2) &\to \mathrm{Dr}(M_1 \oplus M_2) \\ (W, V) &\mapsto W \otimes V \end{aligned}$$

where $(W \otimes V)_{B_1 \sqcup B_2} := W_{B_1} + V_{B_2}$ for any $B_1 \in M_1$ and $B_2 \in M_2$. To check that $W \otimes V$ satisfies the tropical Plücker relations notice the following: any octahedron contained in $P_{M_1 \oplus M_2}$ must be of the form $\{e_{B_1}\} \times O_2$, with $B_1 \in M_1$ and O_2 octahedron contained in P_{M_2}, or $O_1 \times \{e_{B_2}\}$, with $B_2 \in M_2$ and O_1 octahedron contained in P_{M_1}. Then the Plücker relations follow from those of $\mathrm{Dr}(M_1)$ and $\mathrm{Dr}(M_2)$. In particular, the cone where $W \otimes V$ lies is determined by the cones where W and V lie, so $\otimes$ maps cones into cones.

To construct the inverse of $\otimes$, we fix a basis $B_1 \sqcup B_2 \in M_1 \oplus M_2$ and we define the map

$$\begin{aligned} \phi: \ \mathrm{Dr}(M_1 \oplus M_2) &\to \mathrm{Dr}(M_1) \times \mathrm{Dr}(M_2) \\ W &\mapsto (\phi_1(W), \phi_2(W)) \end{aligned}$$

where $\phi_1(W)_{A_1} := W_{A_1 \sqcup B_2}$ and $\phi_2(W)_{A_2} := W_{B_1 \sqcup A_2}$ for any $A_1 \in M_1$ and any $A_2 \in M_2$. It is straight forward to verify that the Plücker relations satisfied by W imply that the projections $\phi_1(W)$ and $\phi_2(W)$ satisfy them as well. In particular, ϕ maps cones to cones.

Now we prove that ϕ is independent of the choice of basis $B_1 \sqcup B_2$. We do this by contradiction. Suppose it is not, without loss of generality we can assume there exist $B_1 \sqcup B_2$ and $B_1 \sqcup B_2'$, with B_2 and B_2' of distance 1 such that ϕ does not agree for these two choices. Clearly ϕ_2 is the same for both choices, so we look at ϕ_1. Let $A, A' \in M_1$ be bases at distance 1. We have that the points $e_{A \sqcup B_2}, e_{A \sqcup B_2'}, e_{A' \sqcup B_2'}, e_{A' \sqcup B_2}$ form a square face of $P_{M_1 \oplus M_2}$. This square can not be subdivided, so

$$W_{A \sqcup B_2} - W_{A' \sqcup B_2} = W_{A \sqcup B_2'} - W_{A' \sqcup B_2'} \ .$$

But this means that the difference of ϕ_1 for A and A' is independent of the choice of B_2. By connectivity of the graph of P_{M_1}, we can conclude that ϕ_1 is independent of the choice of B_2.

We are left with proving that ϕ is the inverse of $\otimes$. First we check that for any $(V, W) \in \mathrm{Dr}(M_1) \times \mathrm{Dr}(M_2)$ we have that $\phi(W \otimes V) = (W, V)$. To see this, notice that $\phi_1(W \otimes$

$V)_A = (W \otimes V)_{A \sqcup B_2} = W_A + V_{B_2}$ for any $A \in M_1$. But V_{B_2} is a constant independent of A, so $\phi_1(W \otimes V) = W$ in the tropical torus. Analogously, we get that $\phi_2(W \otimes V) = V$.

Now we check the other direction, that is, for any $W \in \mathrm{Dr}(M_1 \oplus M_2)$ we have $W = \phi_1(W) \otimes \phi_2(W)$. Consider two bases of $(M_1 \oplus M_2)$ at distance 1. Without loss of generality let them be $A_1 \sqcup A_2$ and $A_1 \sqcup A_2'$. We have that

$$
\begin{aligned}
(\phi_1(W) \otimes \phi_2(W))_{A_1 \sqcup A_2} - (\phi_1(W)\otimes\phi_2(W))_{A_1 \sqcup A_2'} & \\
&= \phi_1(W)_{A_1} + \phi_2(W)_{A_2} - \phi_1(W)_{A_1} - \phi_2(W)_{A_1'} \\
&= W_{A_1 \sqcup B_2} + W_{B_1 \sqcup A_2} - W_{A_1 \sqcup B_2} - W_{B_1 \sqcup A_2'} \\
&= W_{B_1 \sqcup A_2} - W_{B_1 \sqcup A_2'} \ .
\end{aligned}
$$

We have already shown that ϕ is independent of the choice of B_1, so we may assume $B_1 = A_1$. Hence, the above equals $W_{A_1 \sqcup A_2} - W_{A_1 \sqcup A_2'}$. By connectivity of the graph of $P_{M_1 \oplus M_2}$, we get $W = \phi_1(W) \otimes \phi_2(W)$ as we wanted.

Therefore, the maps ϕ and $\otimes$ are bijective linear maps which send cones to cones, which implies $\mathrm{Dr}(M_1 \oplus M_2) = \mathrm{Dr}(M_1) \times \mathrm{Dr}(M_2)$. □

Remark 5.15. The statement above generalizes Theorem 4 by Chatelain and Ramírez [CRA14] which deals with sequences of weakly compatible hyperplane splits. While the article by Joswig and Schröter [JS17] provides the case of sequences of strongly compatible hyperplane splits and the matroid polytopes that occur in these matroid subdivisions. We refer to Herrmann and Joswig [HJ08] for the definitions.

Let M be a matroid $(E, \mathcal{B})$. Two elements e and e' in E are *parallel* if $\mathrm{rk}(\{e, e'\}) = 1$. We denote this by $e \parallel e'$. Remark that this implies that $M \backslash e = M \backslash e'$.

Theorem 5.16. *Let M be a matroid and $e \parallel e'$ in M. Then*

$$\mathrm{Dr}(M) \Big/ \mathrm{lin}\,\mathrm{Dr}(M) \cong \mathrm{Dr}(M \setminus e') \Big/ \mathrm{lin}\,\mathrm{Dr}(M \setminus e')$$

and $\dim \mathrm{lin}\,\mathrm{Dr}(M) = \dim \mathrm{lin}\,\mathrm{Dr}(M \setminus e') + 1$.

Proof. Clearly, M contains the circuit $\{e, e'\}$. Hence, the number of connected components of M is the same as the number of connected components of $M \setminus e'$. It follows that $\dim \mathrm{lin}\,\mathrm{Dr}(M) = \dim \mathrm{lin}\,\mathrm{Dr}(M \setminus e') + 1$.

The projection $\mathrm{Dr}(M) \to \mathrm{Dr}(M \setminus e')$ that forgets the coordinates that correspond to bases that contain e' is surjective. Our goal is to show that this projection is injective if we quoten by the lineality spaces. Let $W \in \mathrm{Dr}(M)$ and B_e be a basis of M that contains

e and $B_{e'} = B_e \setminus \{e\} \cup \{e'\}$. We may assume that $W_{B_e} = W_{B_{e'}}$ as the lineality space of $\mathrm{Dr}(M)$ contains $\sum_{B \ni e'} e_B$. Let $B'_{e'}$ be a basis of M and $B'_e = B'_{e'} \setminus \{e'\} \cup \{e\}$ of distance $\# B'_e \setminus B_e = 1$. That is e_{B_e}, $e_{B'_e}$, $e_{B_{e'}}$, $e_{B'_{e'}}$, form a square in the vertex-edge graph of P_M. The set $B_e \cap B'_{e'} \cup \{e, e'\}$ is a non-basis of distance 1 to those four bases. Therefore, the square is not subdivided by the regular subdivision induced by W. We conclude that $W_{B_e} + W_{B'_{e'}} = W_{B_{e'}} + W_{B'_e}$ and by our assumption $W_{B'_{e'}} = W_{B'_e}$. Iterating our argument shows that $W_B = W_{B \setminus \{e\} \cup \{e'\}}$ for any basis B that contains e. As the basis exchange graph of a matroid is connected. Therefore, we derive that the projection is injective up to lineality and therefore the desired isomorphism. □

The combination of Theorem 5.14 and Theorem 5.16 allows to deduce the local Dressian $\mathrm{Dr}(M)$ of an arbitary matroid M from the simplifications of its connected componenets.

5.3 Rigid matroids

In this section, we study the notion of rigidity, first introduced by Dress and Wenzel:

Definition 5.17. [DW92, Definition 2.2] A matroid is said to be rigid if and only if its polytope does not allow a non-trivial matroid subdivision.

Notice that if M is a rigid matroid, then the local $\mathrm{Dr}(M)$ is just a linear space.

Recall that any matroid obtained from successive deletions and contractions form a matroid M is a minor of M. Theorem 3 by Chatelain and Ramírez [CRA11] which states that a matroid polytope of a binary matroid can not be splited into two matroid polytopes. However this is a weaker statement than that of Dress and Wenzel in 1992:

Theorem 5.18. *[DW92, Theorem 5.11] The following matroids are rigid:*

1. *Binary matroids.*
2. *Matroids of finite projective spaces of dimension at least 2. That is, the matroid with ground set $\mathbb{PK}^{d-1}$ where $d \geq 3$ and $\mathbb{K}$ is a finite field and with bases given by the bases of $\mathbb{K}^d$.*

We show a new proof of rigidity of binary matroids, which is essentially a corollary from Theorem 5.2. We make use of the following characterization of binary matroids in terms of forbidden minors.

Proposition 5.19 (Tutte[Tut58])**.** *A matroid is binary if and only if it has no minor isomorphic to the uniform matroid* $U_{2,4}$.

Corollary 5.20. *Binary matroids are rigid.*

Proof. Let M be a binary matroid and P_M its matroid polytope. The 3-skeleton of the polytope P_M does not contain a octahedral face as such a face corresponds to a minor isomorphic to the uniform matroid $U_{2,4}$. From Corollary 5.4 we deduce that P_M only has a trivial matroid subdivision. That is the Dressian is a linear space and M is rigid. □

If a matroid subdivision contains only polytopes of rigid matroids, then it clearly can not be subdivided any further while staying a matroid subdivision. Then it is natural to ask whether all matroid subdivisions are of this form.

Question 5.21. *Are all cells in a finest matroid subdivision polytopes of rigid matroids?*

We now show that this is true for $U_{2,n}$:

Proposition 5.22. *The cells of a finest matroid subdivision of the hypersimplex* $\Delta(2, n)$ *correspond to binary matroids. In particular, they are rigid.*

Proof. By Theorem 4.2 we have that $\mathrm{Dr}(2, n) = \mathrm{Gr}(2, n)$ and that rank 2 valuated matroids are in correspondence to the space of phylogenetic trees. In this correspondence, nodes of the phylogenetic tree correspond to facets of the subdivision induced by the valuated matroid, which in turn correspond to connected matroids. The valency of each node is at least 3 and correspond to the number of rank 1 flats of the corresponding matroid. Using these correspondence, we see that finest matroid subdivisions correspond to stable trees, that is, trees where all nodes have valency at most 3. This means that all matroids have at most three rank 1 flats hence they are binary by Theorem 5.19.

□

One way to determine that matroid is rigid, is by showing that there it has no connected submatroid. This is the case for binary matroids. However, it is unclear whether this is always the case.

Question 5.23. *Does there exist two connected matroids* M *and* M' *such that* $P_{M'}$ *is strictly contained in* P_M *but no matroid subdivision of* P_M *has* $P_{M'}$ *as a cell?*

Notice that when M is a uniform matroid then the corank subdivision has $P_{M'}$ as a cell. But the corank function of M' does not necessarily satisfy local Plücker relations.

Example 5.24. Let M be a matroid with bases 12, 13, 14, 23 and 24 and M' be the matroid With the two bases 12 and 13. Then the local corank lifting is $V = (0, 0, 1, 1, 1)$ and this vector is not in the local Dressian $\mathrm{Dr}(\mathcal{M})$ as it subdivides the square pyramid P_M into two tetrahedra, hence creating a new edge.

Further evidence for a positive answer of Question 5.23 is the ternary projective plane. This was already knwon to be rigid Theorem 5.18

Proposition 5.25. *Consider the simple matroid $\mathcal{P}$ on 13 elements and rank 3 given by the ternary projective plane. There are no connected submatroids of $\mathcal{P}$.*

Proof. Assume that M is a proper connected submatroid of $\mathcal{P}$. Being a submatroid means that every basis of M is a basis of $\mathcal{P}$.

The proof consists of five steps:

- There are two parallel elements in M.
- Two lines of $\mathcal{P}$ collapse to a line in M.
- A quadrilateral in $\mathcal{P}$ collapses to a point in M.
- Three concurrent lines of $\mathcal{P}$ collapse into a line in M.
- Contradiction.

After each step, for improving the exposition, we reset the labeling. We make sure to clarify the new assigned labels. We do this in order to assure that there is no loss of generality. Keep in mind that for M to be connected there is no line such that its complement is a single point. In particular there must be a least four points, i.e., four parallelism classes.

Our first step is to show that M contains a pair of parallel elements. Suppose that the set 123 is a basis of $\mathcal{P}$ but it is dependent in M. Either 123 contains a parallel pair or $\mathrm{cl}_M(123)$ is of rank 2 as M is loop free. In the latter case, let 4 be not in the rank 2 flat $\mathrm{cl}_M(123)$. This implies that the intersection of the lines $\mathrm{cl}_M(14) \cap \mathrm{cl}_M(123)$ is of rank 1 in M. As 2 is not parallel to 3 in M, then $\mathrm{cl}_{\mathcal{P}}(23) \subseteq \mathrm{cl}_M(23) = \mathrm{cl}_M(123)$ and, as 123 is independent in $\mathcal{P}$, there is an element 5 with $5 \in \mathrm{cl}_{\mathcal{P}}(23) \cap \mathrm{cl}_{\mathcal{P}}(14)$. This means that $5 \in \mathrm{cl}_M(14) \cap \mathrm{cl}_M(123)$ and hence it is parallel to 1 in M.

Suppose now that 1 and 2 are two parallel elements in M. Notice that there are at lest three elements not in $\mathrm{cl}_M(12)$. Moreover, $\mathrm{cl}_{\mathcal{P}}(12)$ has four elements, at least two of which are in $\mathrm{cl}_M(12)$. Then there exists an element 3 such that 3 is not in $\mathrm{cl}_M(12) \cup$

$\mathrm{cl}_{\mathcal{P}}(12)$. Therefore, $\mathrm{cl}_{\mathcal{P}}(13)$ and $\mathrm{cl}_{\mathcal{P}}(23)$ are two different lines in $\mathcal{P}$ which are contained in $\mathrm{cl}_M(13) = \mathrm{cl}_M(23)$.

Suppose that the eight points on the two lines 1234 and 1567 in $\mathcal{P}$ span a line in M. There must be at least two points 8 and 9 outside this line in the connected matroid M. Each of the three lines $\mathrm{cl}_{\mathcal{P}}(28)$, $\mathrm{cl}_{\mathcal{P}}(38)$ and $\mathrm{cl}_{\mathcal{P}}(48)$ intersects the line 1567 in a different point in the projective geometry $\mathcal{P}$. This induces a bijection between 234 and 567 where elements are mapped to parallel elements in M. Similarly, a bijection can be constructed by considering the lines from 9. These bijections do not agree and hence, there are at least four parallel elements in M that span a quadrilateral in $\mathcal{P}$.

Suppose that 1234 is a quadrilateral in $\mathcal{P}$ which collapses to a point in M. Let $5 \in \mathrm{cl}_{\mathcal{P}}(12) \cap \mathrm{cl}_{\mathcal{P}}(34)$, and $6 \in \mathrm{cl}_{\mathcal{P}}(13) \cap \mathrm{cl}_{\mathcal{P}}(24)$, and $7 \in \mathrm{cl}_{\mathcal{P}}(14) \cap \mathrm{cl}_{\mathcal{P}}(23)$. As M is connected, there are at least three elements outside $\mathrm{cl}_M(1234)$. Suppose that these points are exactly 5, 6 and 7. Then $\mathrm{cl}_{\mathcal{P}}(56) \cap \mathrm{cl}_M(1234) \neq \emptyset$ forcing $\mathrm{cl}_M(1234)$, 5 and 6 to be colinear in M, and M disconnected. So there is another point 8 outside $\mathrm{cl}_M(1234)$. In particular, three of the lines in $\mathcal{P}$ passing through 8 also pass through at least one point in the quadrilateral 1234. Therefore they collapse in a single line in M.

Suppose three concurrent lines passing through 1 in $\mathcal{P}$ collapse to a single line in M. Let S be the set of elements different from 1 forming these three lines. As M is connected there must be at least two elements outside $\mathrm{cl}_M(S)$. For each point, the lines passing through it and not 1 induces a partition of S in three subsets of size three, such that the elements in each subsets belong to the same parallelism class. The two partitions are transversal, therefore S is in the same parallelism class. As the complement of S is a line in $\mathcal{P}$, then M is disconnected and we obtain a contradiction. $\square$

Chapter 6

Transversal Valuated matroids

6.1 Introduction

A linear subspace of a vector space can be described in several ways: as a row space of a matrix A; as the set of solutions of a system of linear equations; by its Plücker coordinates, which are the maximal minors of A; or others besides. In tropical mathematics, the objects defined in each of these ways no longer coincide. It is the Plücker perspective that has become accepted as defining *tropical linear spaces.* The vectors that serve as Plücker coordinate vectors in tropical geometry were introduced by Dress and Wenzel [DW92], who named them *valuated matroids.*

Using the operations of the tropical semifield, the counterpart of the row space, i.e. the set of all $(\mathbb{R}\cup\{\infty\})$-linear combinations of a set of tropical vectors, is the *tropical convex hull*, an object which has been intensely studied from many points of view [AD09, DS04, JL16, AGG12, GK07, Ser09, But10]. Tropical convex hulls are usually not tropical linear spaces at all: [YY, Theorem 16] describes when they are. But there is a bridge between the two classes of object, the *tropical Stiefel map* [FR15]. The tropical Stiefel map provides a tropical linear space *containing* a given tropical convex hull (Theorem 6.15). If the tropical convex hull is r-dimensional and defined by $r+1$ points, then the tropical Stiefel map provides an r-dimensional tropical linear space, which is smallest possible.

The combinatorics of the tropical Stiefel map is governed by *transversal matroids.* Let $\mathcal{A} = \{\!\{A_1, \ldots, A_d\}\!\}$ be a multiset of subsets of a finite set E. Edmonds and Fulkerson [EF65] observed that the set of subsets $J \subseteq E$ which form a *transversal* of $\mathcal{A}$, i.e. such that there is an injection $f : J \to \{1, \ldots, d\}$ with $j \in A_{f(j)}$ for each $j \in J$, are the independent sets of a matroid. A matroid M arising in this way is called a tranversal matroid, and $\mathcal{A}$ is called a *presentation* of M. If M is transversal then it has a presentation where d equals the rank of M; we will restrict our attention to presentations of this size.

As Edmonds explained [Edm67], the set system $\mathcal{A}$ can be represented as a boolean $d \times n$ matrix A, where $n = |E|$, and then M is obtained from A by formally computing its maximal minors within the boolean semifield, where addition is OR and multiplication is AND. Encoding true as 0 and false as ∞ makes this a (min-plus) tropical computation, which can then be extended by allowing any tropical numbers as matrix entries. We define a *transversal valuated matroid* V to be the vector of tropical maximal minors of a $d \times n$ matrix A of tropical numbers, and we call the multiset of rows of A a presentation of V. The function that computes V from A is the tropical Stiefel map, and the tropical linear space dual to V is called a *Stiefel tropical linear space* in [FR15]. The name "Stiefel" reflects the above matrix analogy: the space of tropical matrices maps to the space of valuated matroids just as the *non-compact Stiefel manifold* of $d \times n$ matrices of rank d (i.e. not necessarily orthogonal *d-frames*) maps to the Grassmannian of d-planes in n-space.

Mason [Mas71] gave a characterization of transversal matroids (Proposition 6.42). Shortly thereafter, Brualdi and Dinolt [BD72, Theorem 5.2.6] described all presentations of a given transversal matroid M as follows (we reformulate it as Theorem 6.17). There is a unique maximal presentation of M, which consists of $\tau_M(F)$ copies of $E \setminus F$ for each flat F of M, where $\tau_M(F)$ is computed by a recurrence (6.2) on the lattice of flats. Any presentation $\{\!\{E \setminus F_1, \ldots, E \setminus F_d\}\!\}$ is obtained from the maximal one by deleting relative coloops in a way that doesn't contravene Hall's theorem, i.e. that satisfies

$$\operatorname{cork}\Big(\bigcap_{i \in I} F_i\Big) \leq |I| \tag{6.1}$$

for every $I \subseteq \{1, \ldots, d\}$, where $\operatorname{cork}(J) = d - \operatorname{rk}(J)$ is the corank function.

In this work we give an explicit description of the fibers of the tropical Stiefel map, which is a direct generalization of Brualdi and Dinolt's result.

Theorem 6.1 (Synopsis of Theorem 6.40). *Each nonempty fiber of the tropical Stiefel map is the orbit of a fan in the space of $d \times n$ tropical matrices under the action of S_d permuting the rows.*

The apex of the fan corresponds to the unique maximal presentation of Brualdi and Dinolt. Apart from a d-dimensional lineality space spanned by the zero-one indicator vectors of rows, all rays of the fan are in coordinate directions, and the sets of rays that form faces are governed by "local" counterparts of (6.1).

Theorem 6.1 mirrors the classical fact that the non-compact Stiefel manifold is a principal GL_d bundle over the Grassmannian. The only invertible matrices of tropical numbers are the generalized permutation matrices, those which have exactly one finite entry in every row and column, forming a group isomorphic to $\mathbb{R} \wr S_d$. Our theorem implies that the space of $d \times n$ tropical matrices without too many infinities (Theorem 6.4) has a deformation retract onto the Minkowski sum of the set of apices and the lineality space, which is a *ramified* $\mathbb{R} \wr S_d$ bundle over its image. The ramification arises because an apex can have equal rows. It remains an open question to describe the topology of the image of the tropical Stiefel map; the above bundle perspective suggests a possible approach.

In [FR15] a necessary condition for a valuated matroid V to be transversal was given (Theorem 6.8). Assuming for convenience that V is connected, the condition is that if V is transversal, all connected initial matroids of V must be transversal. The initial matroids are those whose polytopes appear in the subdivision induced by V. We obtain a converse.

Theorem 6.2 (Theorem 6.47)**.** *A connected valuated matroid is transversal if and only if all of its connected initial matroids are transversal.*

What about solutions to systems of linear equations? The solutions of a single tropical linear equation form a tropical hyperplane. Set-theoretic intersections of two or more tropical hyperplanes need not be tropical linear spaces. Minimal collections of tropical hyperplanes whose intersection is a given tropical linear space are studied by Yu and Yuster [YY]. However, an altered version of intersection which always produces linear spaces, *stable intersection*, is well known in tropical geometry. Its first appearance was as the *fan displacement rule* of Fulton and Sturmfels [FS97]. Stable intersection is dual to the tropical Stiefel map; in the language of matroids, we dualize the transversal matroids to strict gammoids (Section 6.2.5). Their valuated counterparts are presented by weighted directed graphs akin to the graphs Speyer and Williams use to parametrize the tropical positive Grassmannian [SW05]. Therefore our main results all have counterparts for stable intersections. Theorem 6.40 explicitly describes the space of all d-tuples of tropical hyperplanes whose stable intersection is a given tropical linear space, and Theorem 6.47 gives a “local” criterion for a tropical linear space to be a tropical stable complete intersection.

Section 6.2 introduces transversality and the Stiefel map, and interprets the former as the $\{0, \infty\}$-valued case of the latter. The end of the section describes the dual picture. We begin to characterize presentations in Section 6.3, by bounds on the number of rows chosen from certain regions of the tropical linear space. Section 6.4 proves the main theorems.

6.2 Transversality

6.2.1 The tropical Stiefel map

The fibers of the following map π are our main subject.

Definition 6.3 ([FR15])**.** Let $A \in \mathbb{T}^{d\times n}$ be a tropical matrix. The *tropical Stiefel map* is the map $\pi : \mathbb{T}^{d\times n} \dashrightarrow \mathbb{P}(\mathbb{T}^{\binom{n}{d}})$ defined by

$$\pi(A)_B = \min\left\{\sum_{i=1}^{d} A_{i,j_i} : B = \{j_1, \ldots, j_d\}\right\}.$$

Remark 6.4. The domain of π is the subset of $\mathbb{T}^{d\times n}$ where at least one injective function $j : [d] \to [n]$ achieves $A_{i,j(i)} \neq \infty$ for all $i \in [d]$. By Hall's theorem, the only matrices excluded from the domain are those that have a $k \times (n + 1 - k)$ submatrix all of whose entries are ∞ for some $1 \le k \le d$.

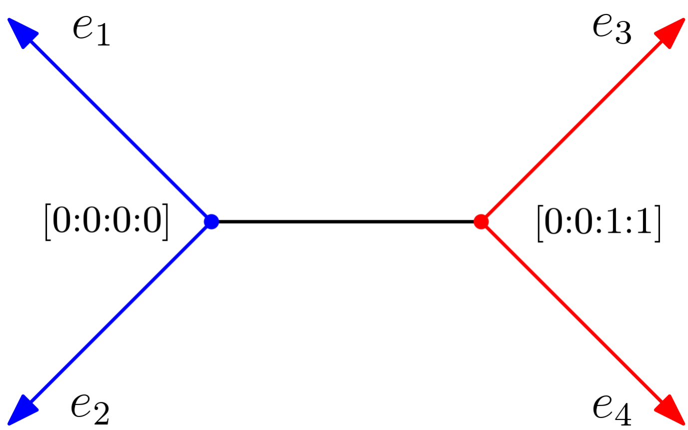


FIGURE 6.1: The tropical linear space $\mathcal{L}(\pi(A)) \subseteq \mathbb{TP}^3$ of Theorem 6.5.

Example 6.5. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

in $\mathbb{T}^{2\times 4}$. Computing the tropical minors gives $\pi(A)_B = 0$ for any $B \in \binom{[4]}{2} \setminus \{3,4\}$ and $\pi(A)_{34} = 1$, which is the same valuated matroid as in Theorems 4.14 and 4.18. Notice that replacing either $A_{1,1}$ or $A_{1,2}$ (but not both at the same time) by any tropical number larger than 0 does not change any of the minors, so the resulting matrix would be mapped to the same valuated matroid. Similarly, replacing either $A_{2,3}$ or $A_{2,4}$ by a number larger than 1 also does not change $\pi(A)$. Figure 6.1 shows the tropical linear space of $\pi(A)$. Any matrix A' with $\pi(A') = \pi(A)$ must have one row giving projective coordinates for a point in the blue subcomplex of the figure, and the other row doing the same for the red subcomplex. Later, we will show how all fibers of π have a similar behavior.

Permuting the rows of A, or adding a scalar to any row, does not change $\pi(A)$, and therefore neither does left multiplication by any invertible tropical matrix. The first invariance implies that $\pi(A)$ is determined by the list of the projectivization (lying in $\mathbb{TP}^{n-1}$) of each row of A, and the second invariance means that $\pi(A)$ is determined by the *unordered* list, i.e. the multiset, of these projectivizations. So we will normally discuss fibers of π in terms of such multisets.

Definition 6.6. A (*transversal*) *presentation* of a valuated matroid V of rank d is a multiset $\mathcal{A}$ of d points in $\mathbb{TP}^{n-1}$ such that $V = \pi(A)$, where A is a matrix whose rows are coordinate vectors for the elements of $\mathcal{A}$.

If we say that a multiset $\mathcal{A}$ is a presentation of a tropical linear space $\mathcal{L}(V)$, we mean that it is a presentation of V.

The tropical Stiefel map is not surjective onto the space of valuated matroids. In [FR15] the name *Stiefel tropical linear space* was given to tropical linear spaces of the form $\mathcal{L}(\pi(A))$. In view of Section 6.2.2, we grant the valuated matroids another name.

Definition 6.7. A valuated matroid $V \in \mathbb{P}(\mathbb{T}^{\binom{n}{d}})$ is *transversal* if it is in the image of π.

Here is a necessary condition for transversality.

Proposition 6.8 (Fink, Rincón [FR15])**.** *Let V be a transversal valuated matroid. Then every matroid $M \in \mathcal{M}(V)$ such that P_M is a facet of $\mathcal{S}_V$ is transversal.*

In Theorem 6.47 we show that this condition is also sufficient.

Remark 6.9. The image of π is always contained in the tropical Grassmanian $\mathrm{TropGr}(d, n)$, the tropicalization of the Grassmannian over a field in its Plücker embedding [SS04]. The matroid of Theorem 6.19 lies in the tropical Grassmannian for any field, so π does not surject onto $\mathrm{TropGr}(d, n)$.

Remark 6.10. A family of presentations that have been the focus of much previous work are the *pointed* presentations, where A has a tropical identity matrix as a maximal submatrix [HJS12, Rin13b, JL16]. The unvaluated matroids with pointed presentations are called *fundamental transversal matroids* [Bon10, Section 3.1] (see also [Bix77, RI80]); by Theorem 6.27, these presentations can be taken to be by $\{0, \infty\}$ matrices. If V has a pointed presentation A, then all facets of $\mathcal{S}_V$ share the vertex e_J where A_J is the identity submatrix. The converse is false: for example, non-fundamental transversal matroids exist. In other words, whereas the Grassmannian $\mathrm{Gr}(d, \mathbb{K}^n)$ over a field $\mathbb{K}$ has an atlas of charts isomorphic to $\mathbb{A}_{\mathbb{K}}^{d(n-d)}$, one for each position of the identity submatrix, the corresponding maps from $\mathbb{T}^{d(n-d)}$ fail even to cover the image of π.

Remark 6.11. If V and V' are valuated matroids on $[n]$ of respective ranks d and d', their *stable sum* $V + V'$ is the valuated matroid of rank $d + d'$ defined by

$$(V + V')_J = \min\{V_B + V'_{B'} \mid B \in \tbinom{[n]}{d}, B' \in \tbinom{[n]}{d'}, B \cup B' = J\}$$

for each $J \in \binom{[n]}{d+d'}$, provided that $(V + V')_J < \infty$ for some J. Stable sum generalizes matroid union in the special case that the matroid union is additive in rank, for which reason Frenk [Fre13, Section 4.1] calls it the "valuated matroid union". In this language, presentations are decompositions of a valuated matroid as a stable sum of rank 1 valuated matroids.

Remark 6.12. A way of looking at the tropical Stiefel map which we do not take up here is in terms of the semimodule theory of $\mathbb{T}$. This viewpoint is adopted in [CGM], and is generalized in [Mun] to the valuated version of Perfect's "induction" of a matroid across a directed graph [Per69].

6.2.2 Transversal matroids

We recommend [Bru87] as a general reference for transversal matroids.

Definition 6.13. A matroid is *transversal* if it is of the form $\pi(A)$ where $A \in \{0, \infty\}^{d \times n}$.

Let us unpack this and see that it agrees with the usual definition. The matrix A determines a multiset of nonempty subsets of $[n]$, i.e. a *set system*, whose members are the sets $A_i = \{j \in [n] \mid A_{ij} = 0\}$ for $i \in [d]$. Then the bases of the transversal matroid $\pi(A)$ are the sets $B \in \binom{[n]}{d}$ whose elements can be labelled $B = \{j_1, \ldots, j_d\}$ in such a way that $j_i \in A_i$ for each $i \in [d]$, which is what is necessary for the tropical B-minor of A not to be infinite. This data $(j_1, \ldots, j_d)$ is the classical notion of transversal of a set system.

Theorem 6.27 will imply that we could have allowed arbitrary $A \in \mathbb{T}^{d \times n}$ in Theorem 6.13, not just $A \in \{0, \infty\}^{d \times n}$. The same set of (unvaluated) matroids will be obtained.

We caution readers of the literature on transversal matroids that most authors allow the set system presenting a rank d matroid to contain more than d sets. These authors would say that all our presentations are "of rank d".

6.2.3 Points in presentations

The search for transversal presentations of a tropical linear space L is helpfully delimited by the fact that all elements of a presentation must lie in L. This is essentially the tropical Cramer rule [AGG14, RGST05], but the proof is short so we include it for convenience.

Lemma 6.14. *Let $\{\!\{A_1, \dots, A_d\}\!\}$ be a transversal presentation of a valuated matroid V. Then $A_i \in \mathcal{L}(V)$ for each $i \in [d]$.*

Proof. Write the presentation as a matrix $A \in \mathbb{T}^{d\times n}$. Define an expanded matrix $A^{(i)}$ whose first d rows agree with A and whose $(d+1)$st row equals its ith row. Given a set $C \in \binom{[n]}{d+1}$, let $(j(i') \mid i' \in [d+1])$ be a transversal from $[d+1]$ to C in $A^{(i)}$ so that $\sum_{i'} A^{(i)}_{i',j(i')}$ is minimal. By construction of $A^{(i)}$, swapping the ith and $(d+1)$th entries of the transversal preserves this sum. This implies that both $j = j(i)$ and $jf = j(d+1)$ minimize the quantity $A_{i,j} + L_{C\setminus\{j\}}$, because in each case $L_{C\setminus\{j\}}$ is the sum of the matrix entries in the transversal other than the entry in the $(d+1)$th row, which contributes $A_{i,j}$. Therefore the tropical equations in the definition of $\mathcal{L}(V)$ hold at A_i. □

Because tropical linear spaces are *tropically convex*, i.e. closed under tropical linear combinations [DS04, Theorem 7], Theorem 6.14 implies the following.

Corollary 6.15 ([FR15, Theorem 6.3])**.** *The Stiefel tropical linear space $\mathcal{L}(\pi(A))$ contains the tropical convex hull $\mathbb{T}^d \cdot A$.*

Every bounded cell of $\mathcal{L}(\pi(A))$ is contained in the tropical convex hull $\mathbb{T}^d \cdot A$ [FR15, Theorem 6.8]. More generally, $\mathbb{T}^d \cdot A$ contains the cells of $\mathcal{L}(\pi(A))$ dual to coloop-free matroids, which is exactly the bounded part of $\mathcal{L}(\pi(A))$ if $\underline{V} = U_{d,n}$.

6.2.4 The set of presentations of a matroid

In the matroid case, Lemma 6.14 asserts that any point in an presentation of a transversal matroid M by points with $\{0, \infty\}$ coordinates has the form $\overline{e}_F$ defined as

$$(e_F)_j := \begin{cases} \infty & \text{if} \quad j \in F \\ 0 & \text{if} \quad j \notin F \end{cases}$$

where F is a flat of M. In terms of set systems, the sets which may appear are the complements $[n] \setminus F$ of the flats. Given this, to characterize the presentations of M is to determine when a multiset of d flats of M constitutes the complements of a presentation of M. This problem was solved by Brualdi and Dinolt [BD72] who proved that every transversal matroid M has a unique maximal presentation and showed how to derive all other presentations from it. To describe the unique maximal presentation they use an algorithm which we now discuss.

Let μ be the Möbius function on the lattice of cyclic flats $\mathcal{CF}(M)$. For $F \in \mathcal{CF}$ define

$$\tau(F) := \sum_{F' \in \mathcal{CF}(M),\ F \subseteq F'} \mu(F', 1)\,\mathrm{cork}(F'). \tag{6.2}$$

If τ is non-negative, we can consider the multiset of cyclic flats $\mathcal{DF}(M)$ where each $F \in \mathcal{CF}(M)$ has multiplicity $\tau(F)$. Brualdi calls this the *distinguished family of cyclic flats* [Bru87, p. 77].

Proposition 6.16 (Brualdi and Dinolt [BD72])**.** *Let M be a transversal matroid. Then τ is non-negative, and the complements of the distinguished family of cyclic flats make up the unique maximal presentation of M. Moreover, $\mathcal{A} = \{\!\{A_1, \ldots, A_d\}\!\}$ is a presentation if and only if the complements are flats $F_i = [n] \setminus A_i$ such that*

$$\{\!\{\mathrm{cocl}(F_1), \ldots, \mathrm{cocl}(F_d)\}\!\} = \mathcal{DF}(M)$$

and for every $I \subseteq [d]$

$$\mathrm{cork}(\bigcap_{i \in I} F_i) \geq |I|.$$

At the heart of this chapter is the idea of generalizing the above result to valuated matroids.

The literature contains several statements similar or equivalent to the above. Below we describe another reformulation of Theorem 6.16 as a precise bijection between integer vectors and presentations. See for example Bonin [Bon10] for more detail on the equivalence.

Proposition 6.17. *Let M be a matroid, and $\beta : \mathcal{F}(M) \to \mathbb{Z}$. Then M has a transversal presentation consisting of $\beta(F)$ copies of $\overline{e}_F$ for each $F \in \mathcal{F}(M)$ if and only if β satisfies the following inequalities:*

$$\beta(F) \geq 0 \quad \textit{for all } F \in \mathcal{F}(M) \tag{6.3}$$

$$\sum_{G \geq F} \beta(G) \leq \mathrm{cork}(F) \quad \textit{for all } F \in \mathcal{F}(M) \tag{6.4}$$

$$\sum_{G \geq F} \beta(G) = \mathrm{cork}(F) \quad \textit{for all } F \in \mathcal{CF}(M). \tag{6.5}$$

Notice that if M is a transversal matroid, extending τ to be 0 for every non-cyclic flat yields a solution of the integer program in Theorem 6.17. This is the minimal such function in the following sense: if β is a solution of this system for some matroid M, then by Theorem 6.16 we have that for every $F \in \mathcal{CF}(M)$

$$\sum_{\mathrm{cocl}(G)=F} \beta(G) = \tau(F).$$

Testing if M is transversal can be done by checking whether τ satisfies inequalities (6.3) and (6.4). Another test for transversality, Theorem 6.42, was provided by Mason and Ingleton.

The above discussion shows that every unvaluated presentation of M can be obtained from the maximal presentation by replacing some elements $\overline{e}_F$ with $\overline{e}_G$ where $\mathrm{cocl}(G) = F$. Coclosure is a decreasing operation, so the maximal presentation is the one with a maximum set of zero coordinates, i.e. it is maximum in the usual sense when viewed as a set system or a graph. For any flat G of M we have $\mathrm{cocl}(G) = G \setminus J$ where J is the set of coloops of $M|G$. Therefore, every unvaluated presentation of M is obtained from the maximal presentation by adding relative coloops to the flats chosen.

Example 6.18. The work [FR15] focusses on presentations of valuated matroids V with no $V_B = \infty$, so in the matroid case its concern is with presentations of the uniform matroid $U_{d,n}$ (see Theorem 4.6). The only cyclic flats of $U_{d,n}$ are $\emptyset$ and $[n]$, so we get $\tau([n]) = 0$ (as is the case for all matroids) and $\tau(\emptyset) = d$. Hence the maximal presentation of $U_{d,n}$ is $\{\!\!\{\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{d}\}\!\!\}$, where $\mathbf{0} \in \mathbb{T}^n$ is the vector with all entries 0.

The other unvaluated presentations are obtained by changing some zeroes to infinities. The non-cyclic flats of $U_{d,n}$ are all sets F such that $0 < |F| < d$. So since the vectors in the presentation must be of the form $\overline{e}_F$, we cannot put d or more infinities into any single vector; and inequality (6.4) says that for $F \subseteq [n]$ one of these flats, we cannot change all the positions indexed by F to infinity in more than $\mathrm{cork}(F) = d - |F|$ of the vectors. But any multiset of $\{0, \infty\}$ vectors where we avoid making such concentrated changes is a presentation of $U_{d,n}$. This is the statement (c)$\Leftrightarrow$(d) of [FR15, Proposition 8]. The reader may check that when $n = d$ we have recovered Philip Hall's marriage theorem, and when $n = d + 1$, the dragon marriage theorem of Postnikov [Pos09].

Example 6.19. Consider the matroid M on 6 elements of rank 2 given by $\mathcal{B}(M) := \binom{6}{2} \setminus \{12, 34, 56\}$. For M to have a transversal presentation, β would have to satisfy $\beta(12) = \beta(34) = \beta(56) = 1$, as all of the sets 12, 34, 56 are cyclic flats of corank 1. But this means that $\sum_{F \geq \emptyset} \beta(F) \geq 3 > \mathrm{cork}(\emptyset) = 2$, which is a violation of condition 6.5. In consequence, no valuated matroid V such that $M \in \mathcal{M}(V)$ can be in the image of the Stiefel map.

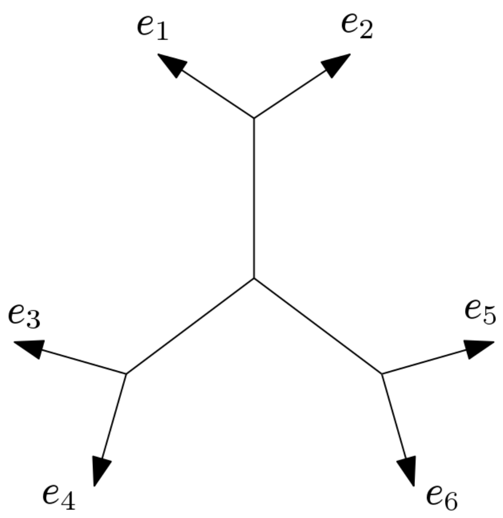


FIGURE 6.2: The 'snowflake' tropical linear space, where $V_{12} = V_{34} = V_{56} = 1$ and $V_B = 0$ for $B \in \binom{6}{2} \setminus \{12, 34, 56\}$, does not correspond to a transversal valuated matroid.

Similar reasoning shows that no rank 2 matroid with three or more nontrivial parallel classes has a transversal presentation. This provides one proof that the tree formed by the bounded faces of a Stiefel tropical linear space of rank 2 is a path. Figure 6.2 exhibits an example of a tropical linear space without this property.

6.2.5 Strict gammoids and stable intersections

Speyer [Spe08, Section 3] described the *stable intersection* of tropical linear spaces in terms of Plücker coordinates.

Definition 6.20. Let V and V' be valuated matroids on $[n]$ of respective ranks d and d'. Their stable intersection $V \underset{\text{stable}}{\cap} V'$ is the valuated matroid of rank $d + d' - n$ defined by

$$(V \underset{\text{stable}}{\cap} V')_J = \min\{V_B + V'_{B'} \mid B \in \tbinom{[n]}{d}, B' \in \tbinom{[n]}{d'}, B \cap B' = J\}$$

for each $J \in \binom{[n]}{d+d'-n}$, provided that there exists some J for which the above formula yields $(V \underset{\text{stable}}{\cap} V')_J < \infty$.

In particular, for such a valuated matroid to exist we must have $d+d' \geq n$. By comparing this definition to Theorem 6.11, we see that stable intersection is dual to stable sum, in the sense that

$$(V \underset{\text{stable}}{\cap} V')^* = V^* + V'^* \quad \text{and} \quad (V + V')^* = V^* \underset{\text{stable}}{\cap} V'^*.$$

The linear space $L(V \underset{\text{stable}}{\cap} V')$ is contained inside $L(V) \cap L(V')$ but in general this containment can be strict (for example, whenever $V = V'$).

In terms of matroids, the dual of a transversal matroid is commonly known as a *strict gammoid.*

Definition 6.21. Let $\Gamma = ([n], E)$ be a directed graph with vertices $[n]$ and directed edges $E \subset [n]^2$, and let $J \subseteq [n]$ be a subset of size d. A *linking* from a set $B \subseteq [n]$ to J is a collection of vertex-disjoint directed paths such that each path starts from a vertex in N and ends in J, and each vertex of B is the start of exactly one path.

We allow a path to be zero edges long.

Proposition 6.22. *The collection of all sets B of size d such that there is a linking from B to J forms the bases of a matroid. A matroid arises this way if and only if it is the dual of a transversal matroid.*

The first sentence of Theorem 6.22 is due to Mason [Mas72], the second to Ingleton and Piff [IP73].

Our work provides a valuated version of strict gammoids. Consider a weighted directed graph $\Gamma = ([n], E)$ where E is now a weight function $E : [n]^2 \to \mathbb{T}$ which is 0 on the diagonal, and let $J \subseteq [n]$ be a subset of size d. Given a linking from a set B to J, the *weight* of that linking is the sum of the weights of all of the edges used in that linking.

Proposition 6.23. *Let Γ be a weighted directed graph with no negative cycles. Let $V \in \mathbb{P}(\mathbb{T}^{\binom{n}{d}})$ be the vector such that for every subset $B \in \binom{[n]}{d}$, V_B is the minimum weight among all linkings from B to J. Then V is a valuated matroid. Moreover, a valuated matroid arises this way if and only if it is the dual of a transversal valuated matroid.*

Proof. Consider $A \in \mathbb{T}^{(n-d)\times n}$ to be the matrix where the rows are indexed by $I = [n] \setminus J$ and $A_{i,j}$ is the weight of the edge from i to j. In particular, $A_{i,i}$ is 0 for every $i \in I$. Let $B \in \binom{[n]}{d}$ and consider the tropical minor of A corresponding to the columns $[n] \setminus B$. A matching from those columns to the rows corresponds to picking edges such that every vertex in $[n] \setminus B$ has exactly one edge coming in and all vertices in I have exactly one edge coming out. Taken together this is exactly a linking from B to J plus possibly some cycles in $I \setminus B$. The value of the term of that matching in the tropical minor corresponds to weight of the linking corresponds plus the weight of the cycles. However, as there are no negative cycles, removing the cycles (choosing the matching where for every vertex i in a cycle is matched with itself instead) the value of the corresponding term can only decrease. So the corresponding minor is equal to the minimum weight of a matching for B to J, that is, V_B. So V is exactly the dual of $\pi(A)$.

Now if V is dual to to a transversal valuated matroid $\pi(A)$ with $A \in \mathbb{T}^{(n-d)\times n}$, to construct the corresponding weighted graph Γ, let I be any basis of $\underline{\pi(A)}$ and let $\sigma : [n-d] \to I$ be a matching that achieves the minimum of $\pi(A)_I$. Let Γ be the weighted directed graph where for every $(i,j) \in I \times [n]$ there is an edge from i to j with weight $A_{\sigma^{-1}(i),j} - A_{\sigma^{-1}(i),i}$. As σ achieves the minimum among matchings $[n-d] \to I$ there cannot be any negative cycles in Γ. So when the matrix A' is constructed from Γ as described above, then A' is obtained from A by subtracting $A_{\sigma^{-1}(i),\sigma(i)}$ from each entry of the row $\sigma^{-1}(i)$. In particular $\pi(A') = \pi(A)$, so V is the valuated matroid associated to Γ. □

Theorem 6.47 implies that these are exactly the valuated matroids that correspond to stable intersections of tropical hyperplanes.

6.3 Characterizing presentations by regions

In this section, we characterize presentations of a valuated matroid V in terms of bounds on the number of points which may lie in certain regions of $\mathcal{L}(V)$.

Our first step is to generalize Proposition 6.17, which characterizes unvaluated presentations of matroids, to describe presentations of unvaluated matroids by points with unrestricted tropical coordinates. In this case, the regions we invoke can be seen as generalizing the ranges of summation in inequalities (6.4) and (6.5).

We begin by defining *relative support.* This is essentially the same notion as covectors in the theory of tropical hyperplane arrangements [AD09, Section 3]. The covector of a point is the list of complements of its relative supports with respect to the apex of each tropical hyperplane.

Definition 6.24. Let x and y be two points in $\mathbb{TP}^{n-1}$. The *relative support* $\mathrm{rs}_x(y) \subseteq [n]$ of y with respect to x is the set indexing the coordinates where $y - x$ does not attain its minimum.

Note that addition of a scalar multiple of $(1,\ldots,1)$ to the coordinates of a point does not affect its relative support, so the relative support is well defined. If x has a fixed vector of affine coordinates $(x_1,\ldots,x_n) \in \mathbb{T}^n$, then we say that the *supportive* choice of affine coordinates $(y_1,\ldots,y_n)$ for y, with respect to $(x_1,\ldots,x_n)$, is the one which achieves $\min_j(y_j - x_j) = 0$. In terms of supportive coordinates, Theorem 6.24 becomes

$$\mathrm{rs}_x(y) = \{j \in [n] \mid y_j > x_j\}.$$

Let $L = \mathcal{L}(M)$ where M is a matroid of rank d on $[n]$. By definition of L, we have that $\mathrm{rs}_0(y) \in \mathcal{F}(M)$ for every $y \in L$. So for each flat $F \in \mathcal{F}(M)$ we define the region

$$R_0(F, L) := \{y \in L \mid F \subseteq \mathrm{rs}_0(y)\}.$$

In supportive coordinates with respect to the zero vector, $R_0(F, L)$ consists of all the points which have positive entries in the coordinates indexed by F. Similarly, for each cyclic flat $F \in \mathcal{CF}(M)$ we define another region

$$R_\infty(F, L) := \{y \in L \mid \forall j \in F,\ \ y_j = \infty\}.$$

Clearly $R_0(F, L) \subseteq R_\infty(F, L)$. Given a multiset of d points in L, $\mathcal{A} = \{\!\{A_1, \dots, A_d\}\!\}$, we define the numbers

$$\sigma_0(\mathcal{A}, F) := |\{i \in [d] \mid A_i \in R_0(F, L)\}|$$
$$\sigma_\infty(\mathcal{A}, F) := |\{i \in [d] \mid A_i \in R_\infty(F, L)\}|$$

where F is a flat in the first line, and a cyclic flat in the second.

Proposition 6.25. *Let M be a transversal matroid, $L = \mathcal{L}(M)$ and $A_1, \dots, A_d \in L$. Then $\mathcal{A} = \{\!\{A_1, \dots, A_d\}\!\}$ is a presentation of M if and only if the following conditions hold:*

1. $\forall F \in \mathcal{F}(M),\ \ \sigma_0(\mathcal{A}, F) \leq \mathrm{cork}(F)$.
2. $\forall F \in \mathcal{CF}(M),\ \ \sigma_\infty(\mathcal{A}, F) = \mathrm{cork}(F)$.

Proof. Let $A \in \mathbb{T}^{d\times n}$ be the matrix whose rows are the supportive coordinates for $A_1, \dots, A_d$ with respect to 0, so all entries are nonnegative and each row contains a zero. First we assume that $\{\!\{A_1, \dots, A_d\}\!\}$ is a presentation of M, that is $\pi(A) = M$. Let $F \in \mathcal{F}$ and suppose that condition (1) is not satisfied for F. Let $k = \mathrm{cork}(F)$. Let $B \in \mathcal{B}(M)$ such that $|F \cap B| = d - k$. There are $k + 1$ rows with positive coordinates in all of the columns indexed by F. This means that in the square $d \times d$ submatrix given by the columns of B, there is a $(k+1) \times (d-k)$ submatrix whose entries are all positive. Then the tropical minor corresponding to B must be positive, which is a contradiction as $M_B = 0$.

Now suppose there is a cyclic flat $F \in \mathcal{CF}(M)$ that violates condition (2). As we already proved condition (1) is satisfied, we can assume $\sigma_\infty(\mathcal{A}, F) < \mathrm{cork}(F) = k$. Then there are $d - k + 1$ rows with finite entries in the columns corresponding to F. Assume there is a matching of the submatrix of F with these rows. Then any matching of the whole matrix can be used to get a matching that uses the columns of F in all of those $d-k+1$ rows by exchanging the entries. This is a contradiction to the rank of F; so no such matching exists, and there must be a violation of Hall's condition. Let I be the violating subset of rows of size m, so that there are at most $m-1$ columns with which elements of I can be matched. Let j be one of those columns. Because F is cyclic there should be a matching of $d-k$ rows to $F-j$. So there is a row i corresponding to a point in $R_\infty(F,L)$ which is not used in this matching. Then $I-i$ has access to at most $\leq m-2$ columns of $F-j$, which is a contradiction to the matching.

We now do the other direction. Assume conditions (1) and (2) are satisfied. Because $A_i \in L$, we have $\mathrm{rs}_0(A_i) \in \mathcal{F}(M)$. Consider the initial matroid $M' = \pi(A)^{\mathbf{0}}$, that is, the matroid whose bases are given by the entries where $\pi(A)$ is 0. This M' is transversal, and Condition (1) implies that all independent sets in M are also independent sets in M' (see Lemma 4.4 in [BD72]). This means that for each $B \in \mathcal{B}(M)$ there is a matching on the 0 entries of A, so that $B \in M'$.

Now let $B \in \binom{[n]}{d} \setminus \mathcal{B}(M)$. Then there exists $F \in \mathcal{CF}(M)$ of rank k such that $|B \cap F| > k$. By condition (2) there are $d-k$ rows with infinity entries at the columns of F. This means that in the square submatrix of A with columns indexed by B, there is a $(k+1) \times (d-k)$ submatrix with all entries infinity. So $\pi(A)_B = \infty$. Altogether, this shows $\pi(A) = M$. □

We now turn our attention to the more general case $L = \mathcal{L}(V)$ where V is any valuated matroid. The following definition helps us pass to the Bergman fan case.

Definition 6.26. Let $L = \mathcal{L}(V)$ be a tropical linear space, $M \in \mathcal{M}(V)$ and $x \in \mathrm{relint}(L_M)$. The *zoom* map of L to x is the map $Z_x : L \to \mathcal{L}(M)$ such that

$$Z_x(y)_j := \begin{cases} 0 & \text{when } j \notin \mathrm{rs}_x(y) \\ \infty & \text{when } j \in \mathrm{rs}_x(y) \end{cases}$$

Proposition 6.27. *Let $M \in \mathcal{M}(V)$ be a coloop-free matroid, not necesarily connected, and let x be a point in the relative interior of L_M. Suppose $\mathcal{A} = \{\!\{A_1, \dots, A_d\}\!\}$ is a Stiefel presentation of the tropical linear space L. Then $Z_x(\mathcal{A}) = \{\!\{Z_x(A_1), \dots, Z_x(A_d)\}\!\}$ is a presentation of M.*

The basic idea of the proof is a primal-dual technique as used in the Hungarian method for the assignment problem [Kuh55]. The corresponding arguments in [FR15] are Propositions 5.5 and 5.9.

Proof. Let $A \in \mathbb{T}^{d\times n}$ be the matrix whose ith row consists of A_i written in supportive coordinates with respect to x. We have that $\mathcal{L}(\pi(A))$ equals $L - x$, the tropical linear space L translated so that x is at the origin. Tropically exponentiating (i.e. classically multiplying) each entry of A by t transforms $L - x$ by a classical homothety centered at the origin of factor t, so $\mathcal{L}(\pi(A^t)) = t(L - x)$. When $t \to \infty$, we have that $A^t \to Z_x(A)$ where $Z_x(A)$ is the matrix where the row i is given by $Z_x(A_i)$. It also attains when $t \to \infty$ that $t(L - x) \to \mathcal{L}(M)$, because L coincides with $\mathcal{L}(M) + x$ near x. Because π is a continuous map in its domain, these two limits imply that $\pi(Z_x(A)) = \mathcal{L}(M)$ as long as $Z_x(A)$ is still in the domain of π. So the only thing left to prove is that this is the case, namely, that there is a set B for which $\pi_B(A) = 0$.

If there were no maximal minor of A equal to 0, then there would be an $a \times b$ submatrix A' of A consisting of strictly positive entries such that $a+b > n$. Among such matrices A' select one where b is maximal, i.e. with the most columns. Let I be the set of rows taken by A' and J be the set of columns not taken by A'. Notice that $|I| = a > n - b = |J|$. Consider a bipartite graph G whose vertices are $I \amalg J$ and containing the edge (i, j) just if $A_{i,j} = 0$. If Γ is disconnected, then there is a connected component with vertices $I' \subseteq I$ and $J' \subseteq J$ with $|I'| < |J'|$. So the submatrix of A given by rows I and columns $[n] \setminus J'$ is strictly positive and has more columns than A', which is a contradiction. So G is connected.

Let $j \in J$. As M has no coloops, then there is a basis $B \in \mathcal{B}(M)$ such that $j \notin B$. Because $0 \in \mathcal{L}(\pi(A))_M$, then $\pi(A)_B$ is minimal among all maximal minors of A. The value of $\pi(A)_B$ is achieved by a matching $\sigma : B \to [d]$. This matching must use an entry of A', meaning that there is an element $j' \in J$ such that $\sigma(j') \in I$. Let G' be the graph where you add to G the vertex j' and the edge $(\sigma(j'), j')$. As G' is connected, then there is a path Γ from j' to j. The matching given by σ does not use consecutive edges. By replacing each edge used by σ in Γ by the edge that follows it, we get a matching σ' from $B - i \cup j$ to $[d]$. But the weight of this matching is less than that of σ as we replaced a strictly positive entry $A_{\sigma(j'),j'}$ by zero. This contradicts the minimality of $\pi(A)_B$. □

When we look at general tropical linear spaces, we have to define the regions R_0 and R_∞ more carefully. They will now have three parameters: the tropical linear space $L = \mathcal{L}(V)$, a point $x \in L$ and a flat $F \in \mathcal{F}(M)$ such that the relative interior of L_M contains x. Before we define these regions, we provide the following lemma which explains why it still makes sense to take flats as parameters.

Proposition 6.28. *Let $L = \mathcal{L}(V)$ be a tropical linear space, $M \in \mathcal{M}(V)$ and x be a point in the relative interior of L_M. Then $\mathrm{rs}_x(y) \in \mathcal{F}(M)$ for any $y \in L$.*

Proof. Without loss of generality we can translate L so that x is the origin. In this case, we may assume that $V_B = 0$ if and only if $B \in \mathcal{B}(M)$. Now suppose that there exists $y \in L$ such that $\mathrm{rs}_x(y) \notin \mathcal{F}(M)$. This means there is an element $i \in [n] \setminus \mathrm{rs}_x(y)$ such that $i \in \mathrm{cl}_M(\mathrm{rs}_x(y))$. Let $B \in \mathcal{B}(M)$ be such that $B \cap \mathrm{rs}_x(y) = \mathrm{rk}_M(\mathrm{rs}_x(y))$. Then $i \notin B$, and $B \cup \{i\} \setminus \{j\} \notin \mathcal{B}(M)$ for any $j \in B \setminus \mathrm{rs}_x(y)$. By the tropical Plücker equation corresponding to $B \cup \{i\}$, the minimum in

$$\min_{B' \cup \{j\} = B \cup \{i\}} V_{B'} + y_j$$

is achieved twice. We have that $V_B + y_i = 0$. But for any other $B' \cup \{j\}$, if $j \in \mathrm{rs}_x(y)$ then $y_j > 0$ and if $j \notin \mathrm{rs}_x(y)$ then $V_{B'} > 0$. So the minimum is only attained once, which is a contradiction. $\square$

Given a tropical linear space $L = \mathcal{L}(V)$, a matroid $M \in \mathcal{M}(V)$, a flat $F \in \mathcal{F}(M)$ and a point $x \in \mathrm{relint}(L_M)$, we define the regions:

$$R_0(F, x, L) := \{y \in L \mid F \subseteq \mathrm{rs}_x(y)\},$$

and, whenever $F \in \mathcal{CF}(M)$,

$$R_\infty(F, x, L) := \bigcap_{y \in L_{M|F \oplus M/F}} R_0(F, y, L).$$

Observe that when M is a matroid, the regions $R_0(F, 0, \mathcal{L}(M))$ and $R_\infty(F, 0, \mathcal{L}(M))$ are the sets $R_0(F, \mathcal{L}(M))$ and $R_\infty(F, \mathcal{L}(M))$ defined earlier. Indeed, to see that $R_\infty(F, 0, \mathcal{L}(M)) = R_\infty(F, \mathcal{L}(M))$ note that every $x \in R_0(F, \mathcal{L}(M))$ has positive entries in F when written in supportive coordinates with respect to 0 and any $y \in R_0(F, x, \mathcal{L}(M))$ must have coordinates larger than x in F. As x can have arbitrarily large coordinates in F, any $y \in R_\infty(F, 0, \mathcal{L}(M))$ must have infinite entries at F, so $R_\infty(F, 0, \mathcal{L}(M)) \subseteq R_\infty(F, \mathcal{L}(M))$. But clearly also $R_\infty(F, \mathcal{L}(M)) \supseteq R_0(F, x, \mathcal{L}(M))$ for every $x \in R_0(F, \mathcal{L}(M))$, so the equality holds.

Given a multiset $\mathcal{A} = \{\!\{A_1, \ldots, A_d\}\!\}$ of d points in L we can define σ as in the unsubdivided case. For $x \in \mathrm{relint}(L_M)$,

$$\sigma_0(\mathcal{A}, F, x) := |\{i \in [d] \mid A_i \in R_0(F, x, L)\}|$$
$$\sigma_\infty(\mathcal{A}, F, x) := |\{i \in [d] \mid A_i \in R_\infty(F, x, L)\}|$$

where F is a flat of M in the first line, and a cyclic flat of M in the second.

Lemma 6.29. *Let $M \in \mathcal{M}(V)$ be a coloop-free matroid, $x \in \operatorname{relint}(L_M)$, $F \in \mathcal{CF}(M)$ and $y \in L_{M|F \oplus M/F}$. Then $R_0(F, y, L) \subseteq R_0(F, x, L)$.*

Proof. If $y \in L_{M|F \oplus M/F}$, then y is of the form $c_1 e_{F_1} + \cdots + c_k e_{F_k}$ for a flag $F_1 \subset \ldots F_k$ containing F and such that $c_i \geq 0$ for every i. This is the same form as points have in the cone $\mathcal{L}(M)_F$ of the Bergman fan of M. This means in particular that for any $j \notin F$ and $j' \in F$ we have $y_j \leq y_{j'}$ when written in the supportive coordinates with respect to (fixed coordinates for) x. So if $z \in R_0(F, y, L)$, then there is a $j \notin F$ such that $j \notin \operatorname{rs}_y(z)$. For every $j' \in F$ it follows that $(z-y)_{j'} > (z-y)_j$, and $(y-x)_{j'} \geq (y-x)_j$, so $(z-x)_{j'} > (z-x)_j$ which means that $z \in R_0(F, x, L)$. □

We will need the following lemma whose proof is straightforward from the definition of Z_x.

Lemma 6.30. *Let $M \in \mathcal{M}(V)$ be a coloop-free matroid and let $x \in L_M$ lie in a coloop-free face M. For $F \in \mathcal{F}(M)$ we have that*

$$Z_x^{-1}(R_\infty(F, 0, \mathcal{L}(M))) = R_0(F, x, L).$$

Proposition 6.31. *Let $\mathcal{A}$ be a presentation of V. Then for any coloop-free matroid $M \in \mathcal{M}(V)$ and $x \in \operatorname{relint}(L_M)$ we have that $\sigma_0(\mathcal{A}, F, x) \leq \operatorname{cork}_M(F)$ for $F \in \mathcal{F}(M)$, with equality if $F \in \mathcal{CF}(M)$.*

Proof. By Theorem 6.27 we have that $Z_x(\mathcal{A})$ is a presentation of $\mathcal{L}(M)$. Then by Theorem 6.25 there are at most $\operatorname{cork}_M(F)$ elements of $Z_X(\mathcal{A})$ in $R_0(F, 0, \mathcal{L}(M))$. By Theorem 6.30,

$$Z_x(R_0(F, x, L)) \subseteq R_\infty(F, 0, \mathcal{L}(M)) \subseteq R_0(F, 0, \mathcal{L}(M))$$

so there at most $\operatorname{cork}_M(F)$ elements of $\mathcal{A}$ in $R_0(F, x, L)$. If $F \in \mathcal{CF}(M)$ then there are exactly $\operatorname{cork}_M(F)$ elements of $Z_x(\mathcal{A})$ in $R_\infty(F, 0, \mathcal{L}(M))$ so there are exactly $\operatorname{cork}_M(F)$ elements of $\mathcal{A}$ in $R_0(F, x, L)$. □

Theorem 6.32. *Let $L = \mathcal{L}(V)$ be a tropical linear space and $A_1, \ldots, A_d \in L$. Then $\mathcal{A} = \{\!\{A_1, \ldots, A_d\}\!\}$ is a Stiefel presentation of L if and only if for every connected matroid $M \in \mathcal{M}(V)$ the following hold:*

(1) $\sigma_0(\mathcal{A}, F, v_M^L) \leq \operatorname{cork}_M(F)$ *for all* $F \in \mathcal{F}(M)$*; and*

(2) $\sigma_\infty(\mathcal{A}, F, v_M^L) = \operatorname{cork}_M(F)$ *for all* $F \in \mathcal{CF}(M)$.

Proof. Let A be a Stiefel presentation of a tropical linear space L. Applying Theorem 6.31 for every vertex v_M^L of L gives us condition (1). For any connected matroid M and every $F \in \mathcal{CF}(M)$ we have that there are exactly $\mathrm{cork}(F)$ elements of A in $R_0(F, v_M^L, L) = Z_{v_M^L}^{-1}(R_\infty(F, 0, \mathcal{L}(M)))$. If condition (2) is not satisfied, it means that one of those points is in $R_0(F, v_M^L, L) \setminus R_\infty(F, v_M^L, L)$. Let A_i be that point.

Then there exists $y \in L_{M|F \oplus M/F}$ such that $A_i \notin R_0(F, y, L)$. From $F \in \mathcal{CF}(M)$ we see that $M|F \oplus M/F$ is coloop-free and $F \in \mathcal{CF}(M|F \oplus F)$, so by Theorem 6.31 we have that $\mathrm{cork}_{M/F \oplus M|F}(F) = \sigma_0(\mathcal{A}, F, y)$. Notice also that $\mathrm{cork}_M(F) = \mathrm{cork}_{M|F \oplus M/F}(F)$. However by Theorem 6.29 we have that $R_0(F, y, L) \subseteq R_0(F, v_M^L, L)$ so

$$\begin{aligned}
\sigma_0(\mathcal{A}, F, y) &\leq \sigma_0(\mathcal{A}, F, v_M^L) - 1 \\
&= \mathrm{cork}_M(F) - 1 \\
&= \mathrm{cork}_{M/F \oplus M|F}(F) - 1 \\
&= \sigma_0(\mathcal{A}, F, y) - 1
\end{aligned}$$

which is a contradiction.

Conversely, suppose $\mathcal{A}$ satisfies conditions (1) and (2). Let A be the matrix which has $\mathcal{A}$ as its rows, so what we have to prove is that $\pi(A) = V$. For any connected matroid M, we have that $Z_{v_M^L}(\mathcal{A})$ satisfies (1) and (2) for $\mathcal{L}(M)$, so it is a Stiefel presentation of $\mathcal{L}(M)$. By adding v_M^L to each element of $Z_{v_M^L}(\mathcal{A})$ we get a presentation of $\mathcal{L}(M) + v_M^L$. The matrix we obtain by concatenating all of these presentations coincides in its finite entries with A. As the finite Plücker coordinates of $\mathcal{L}(M) + v_M^L$ agree with V up to adding a scalar, the difference between any pair of Plücker coordinates of $\pi(A)$ both indexed by elements of $\mathcal{B}(M)$ has the value called for by V. Because the incidence graph of edges and maximal cells in $\mathcal{S}_V$ is connected, we conclude that all finite Plücker coordinates of $\pi(A)$ agree with V up to a single global scalar.

Let B be a nonbasis of $\underline{V}$. Consider a facet Q of $P_{\underline{V}}$ such that e_B fails to satisfy its defining inequality. Let P_M be one of the maximal cells of $\mathcal{S}_V$ which have a facet contained in Q, and let F be the cyclic flat that defines that facet. Then $|B \cap F| > \mathrm{rk}_M(F)$. As the polytope of $P_{M/F \oplus M|F}$ is in the boundary of $P_{\underline{V}}$, we have $\sup\{z_j \mid z \in L_{M/F \oplus M|F}\} = \infty$ for all $j \in F$. This implies that points in $R_\infty(F, v_M^L, L)$ have ∞ entries in the coordinates corresponding to F. Because of (2) for M and F, there are $\mathrm{cork}(F)$ elements of $\mathcal{A}$ in $R_\infty(F, v_M^L, L)$. So at most $\mathrm{rk}_M(F)$ of the rows of A contain a finite entry in a column indexed by $B \cap F$. This is a violation of Hall's condition, so there is no matching for B using finite entries of A. So $\pi(A)_B = \infty$. $\square$

We end this section by using the previous theorem to understand how presentations behave under contractions.

Proposition 6.33. *Let $\mathcal{A}$ be a presentation of V and $F \in \mathcal{CF}(\underline{V})$ a cyclic flat of rank k. Then there are exactly $d-k$ points in $\mathcal{A}$ all of whose coordinates indexed by elements of F are ∞. The projection of these points to the $[n] \setminus F$ coordinates form a presentation of V/F.*

Proof. As $F \in \mathcal{CF}(\underline{V})$, there are coloop-free matroids in $\mathcal{M}(V)$ such that their polytopes are contained in the hyperplane

$$H_F := \left\{ \sum_{j \in F} x_j = k \right\}.$$

Condition (2) of Theorem 6.32 applied to any of these matroids implies that there are exactly $d-k$ points of $\mathcal{A}$ with ∞ in the F coordinates, because the cells of L corresponding to these cells extend to infinity in the e_F direction. Let $\mathcal{A}_F \subseteq \mathcal{A}$ be the multiset of those points.

For every coloop-free matroid in $M' \in \mathcal{M}(V/F)$ there is a coloop-free matroid $M \in \mathcal{M}(V)$ such that $M/F = M'$ and $P_M \subseteq H_F$. In particular, $F \in \mathcal{CF}(M)$. For every point $x' \in \mathcal{L}(V/F)_{M'}$ there is a point $x \in L_M$ which coincides with x' in the $[n]/F$ coordinates and is arbitrarily large in the F coordinates. For such points and for any flat $F \subseteq F' \in \mathcal{F}(M)$ we have that

$$R_0(F', x, L) \cap \{x_j = \infty \mid j \in F\} = \iota_F(R_0(F', x', \mathcal{L}(V/F))$$

where ι_F again means the inclusion $\mathcal{L}(V/F) \to L$ which sets the F coordinates to ∞. As the lattice of flats of M' is isomorphic to the interval above F in lattice of flats of M, the conditions that Theorem 6.32 imposes on $\mathcal{A}_F$ when applied to V are exactly the same as its conditions for presentations of V/F. □

6.4 The presentation space of L

We begin this section by recalling the τ function as defined in Equation (6.2). A *matroid valuation* is a valuation in the sense of convex geometry that is defined on the class of matroid polytopes. For more detail, in particular the equivalence of the different ways the last sentence might be interpreted, see [DF10, Section 3].

Lemma 6.34. *The function $M \mapsto \tau_M(\emptyset)$ is a matroid valuation.*

Proof. Let $X : X_0 \subseteq \cdots \subseteq X_k$ be a chain of subsets of $[n]$, and $r : r_0 \leq \cdots \leq r_k$ nonnegative integers. The matroid function $s_{X,r}$ which takes value 1 on M if $\mathrm{rk}_M(X_i) = r_i$ for each i, and 0 otherwise is known to be a matroid valuation [DF10, Proposition 5.3].

We prove the lemma by way of an auxiliary function. Let X and r be as above, and let $c_{X,r}$ be the 0-1-valued matroid function which takes value 1 on M if each X_i is a *cyclic flat* of M with $\mathrm{rk}_M(X_i) = r_i$ and 0 otherwise, To prove that $c_{X,r}$ is a valuation, it will suffice to write it as a linear combination of functions $s_{X',r'}$.

A set J is a cyclic flat of M if and only if there is no $j \in [n] \setminus J$ such that $\mathrm{rk}(J \cup \{j\}) = \mathrm{rk}(J)$ and no $j \in J$ such that $\mathrm{rk}(J \setminus \{j\}) = \mathrm{rk}(J) - 1$. If $K \supseteq J$, then the assertion $\mathrm{rk}(K) = \mathrm{rk}(J)$ is equivalent to $\mathrm{rk}(J \cup \{k\}) = \mathrm{rk}(J)$ for each $k \in K \setminus J$. Therefore the indicator function of the predication "J is a flat of rank r", i.e. "$\mathrm{rk}(J) = r$ and there is no $j \in [n] \setminus J$ such that $\mathrm{rk}(J \cup \{j\}) = r$", can be written by inclusion-exclusion as

$$\sum_{K \supseteq J} (-1)^{|K \setminus J|} s_{(J,K),(r,r)}.$$

Repeating the same argument in the dual allows $c_{(J),(r)}$ (where the two indices are lists of length one) to be written as an alternating sum of terms $s_{(I,J,K),(r-|J|+|I|,r,r)}$. We thus have

$$\begin{aligned} c_{X,r}(M) &= \prod_{i=0}^{k} c_{(X_i),(r_i)}(M) \\ &= \sum \prod_{i=0}^{k} (-1)^{|K_i \setminus I_i|} \, s_{(I_i,X_i,K_i),(r_i-|X_i|+|I_i|,r_i,r_i)}(M) \end{aligned} \tag{6.6}$$

where the sum is over choices of sets $I_i \subseteq X_i$ and $K_i \supseteq X_i$ for each i.

Submodularity implies that if $\mathrm{rk}(K) = \mathrm{rk}(J)$ for some $K \subseteq J$, then also $\mathrm{rk}(K \cup L) = \mathrm{rk}(J \cup L)$ for every L disjoint from K. Therefore, for any term of (6.6) in which $K_i \not\subseteq X_{i+1}$ for some $i < k$, with $j \in X_{i+1} \setminus K_i$, inserting j into or removing it from K_k gives another term which is equal with opposite sign. So we may cancel these terms, and by repeating the argument in the dual we may impose on the index set of the sum (6.6) the further conditions $K_i \subseteq X_{i+1}$ and $I_i \supseteq X_{i-1}$. We have furthermore that any term with $K_i \not\subseteq I_{i+1}$ is zero, because if $j \in K_i \setminus I_{i+1}$, submodularity is violated at $X_i \cup \{j\}$ and $X_{i+1} \setminus \{j\}$. Thus we can impose the condition $K_i \subseteq I_{i+1}$ on (6.6) as well. Under this condition all the sets in the indices form a single chain and we have

$$\prod_{i=0}^{k} s_{(I_i,X_i,K_i),(r_i-|X_i|+|I_i|,r_i,r_i)}(M)$$
$$= s_{(I_0,X_0,K_0,I_1,\dots,K_k),(r_0-|X_0|+|I_0|,\dots,r_k)}(M)$$

which is a valuation. This establishes that $c_{X,r}(M)$ is a valuation.

We can now prove the lemma. By Philip Hall's theorem, the Möbius function $\mu(F',1)$ is a sum over the chains of cyclic flats from F' to 1 in $\mathcal{CF}$, with a chain of length i weighted $(-1)^i$. Therefore $\mu(F',1)\,\mathrm{cork}(F')$ can be written as a linear combination of the $c_{X,r}$ running over all chains of sets $X = (X_0 = F', \dots, X_k = [n])$ and all tuples $r = (r_0, \dots, r_k)$, the coefficient of $c_{X,r}$ being $r_0(-1)^k$. We conclude $M \mapsto \tau_M(\emptyset)$ is a valuation. □

We want to prove the converse of Theorem 6.8, so we say that V *has transversal facets* if it satisfies that proposition's conclusion, that is, if all of its facets $\mathcal{S}_V$ correspond to polytopes of transversal matroids. Define

$$\overline{\mathcal{M}}(V) := \bigcup_{F \in \mathcal{CF}(\underline{V})} \mathcal{M}(V/F).$$

All of the matroids in this set index cells of $\mathcal{S}_V$.

Definition 6.35. Let V be a valuated matroid with transversal facets. The *distinguished multiset of matroids* $\mathcal{DM}(V)$ of V contains each matroid $M \in \overline{\mathcal{M}}(V)$ with multiplicity $\tau_M(\emptyset)$. For any connected matroid $M \in \mathcal{M}(V/F)$ with $F \in \mathcal{CF}(\underline{V})$, let $v_M^L \in L = \mathcal{L}(V)$ be the point in $\mathbb{TP}^{n-1}$ whose coordinate vector extends $v_M^{\mathcal{L}(V/F)}$ by setting the coordinates corresponding to F to be ∞. The *distinguished multiset of apices* $\mathcal{DA}(L)$ of L consists of v_M^L for every $M \in \mathcal{DM}(V)$, with the same multiplicities.

If V has transversal facets, then all elements of $\overline{\mathcal{M}}(V)$ are transversal, because contraction of cyclic flats preserves transversality. Therefore $\tau_M(\emptyset)$ only takes non-negative values for any $M \in \overline{\mathcal{M}}(V)$.

Corollary 6.36. *Let V be a valuated matroid of rank d with transversal facets. Then $|\mathcal{DM}(V)| = d$.*

Proof. For each distinguished cyclic flat F of $\underline{V}$, $\tau_{\underline{V/F}}(\emptyset) = \tau_{\underline{V}}(F)$. By Theorem 6.34 there are exactly $\tau_{\underline{V}}(F)$ matroids from $\mathcal{M}(V/F)$ in $\mathcal{DA}(L)$ counted with multiplicities. So there are as many distinguished matroids as distinguished cyclic flats of $\underline{V}$, which is exactly d. □

Definition 6.37. Let M be a transversal matroid and let $t = \tau_M(\emptyset)$. The *presentation fan* ϕ_M of M consists of all tuples of points $(p_1, \dots, p_t) \in \mathcal{L}(M)^t$ such that $\mathrm{rs}_0(p_i)$ are independent flats and there is a presentation $\mathcal{A} = \{\!\{A_1, \dots, A_d\}\!\}$ of M such that $A_i = [n] \setminus \mathrm{rs}_0(p_i)$ for $i \in [t]$. If V is a valuated matroid with transversal facets and $L = \mathcal{L}(V)$, then for every $M \in \mathcal{DM}(V)$ we define

$$\phi_L(M) := \phi(M) + v_M^L$$

Finally we define the *presentation space* $\Pi(L)$ of L to be the orbit of

$$\prod_{M \in \mathcal{DM}(V)} \phi_L(M)$$

under the action of S_d by permuting points.

In the product $\phi_L(M)$ is only taken once, regardless of the multiplicity of M in $\mathcal{DM}(V)$; multiplicities are already accounted for in the definition of $\phi(M)$. Notice that $\phi(M)$ and therefore $\phi_L(M)$ are invariant under the S_t action, and $\Pi(L)$ is invariant under the S_d action.

Example 6.38. Consider the valuated matroid V from Theorems 4.14, 4.18 and 6.5 and let M_1 and M_2 be the connected matroids in $\mathcal{M}(V)$ corresponding to the square pyramids where $\mathcal{B}(M_1) = \binom{[4]}{2} \setminus \{3,4\}$ and $\mathcal{B}(M_2) = \binom{[4]}{2} \setminus \{1,2\}$. We have that $\mathcal{DF}(M_1) = \{\!\{\emptyset, \{3,4\}\}\!\}$ and $\mathcal{DF}(M_2) = \{\!\{\emptyset, \{1,2\}\}\!\}$ so $\tau(M_1) = \tau(M_2) = 1$ and $\mathcal{DM}(V) = \{\!\{M_1, M_2\}\!\}$. The distinguished apices are $\mathcal{DA}(L) = \{\!\{v_{M_1}^L, v_{M_2}^L\}\!\} = \{\!\{[0:0:0:0], [0:0:1:1]\}\!\}$. The presentation fan $\phi(M_1)$ consists of two rays, one in direction e_1 and the other in direction e_2 while $\phi(M_2)$ has its rays going in direction e_3 and e_4. Figure 6.1 shows $\phi_L(M_1)$ in blue and $\phi_L(M_2)$ in red. The presentation space $\Pi(L)$ consists of the S_2 orbit of the product of these fans, in other words,

$$\Pi(L) = \phi_L(M_1) \times \phi_L(M_2) \cup \phi_L(M_2) \times \phi_L(M_1).$$

Example 6.39. The uniform matroid $M = U_{d,n}$ is the unique rank d matroid such that $\tau_M(\emptyset) = d$. The presentation fan of the uniform matroid is an S_d-invariant subset of $\mathbb{T}^{d \times n}$ where $(A_1, \dots, A_d) \in \phi(U_{d,n})$ if and only if for every non-empty subset $I \subseteq [d]$,

$$\left| \bigcap_{i \in I} \mathrm{rs}_0(A_i) \right| \leq d - |I|.$$

The $\{0, \infty\}$-vectors within $\phi(U_{d,n})$ give the unvaluated presentations from Theorem 6.18.

The reason for calling $\Pi(L)$ a presentation space is the following theorem.

Theorem 6.40. *Let V be a transversal valuated matroid. Then $\mathcal{A} = \{\!\{A_1, \dots, A_d\}\!\}$ is a presentation of V if and only if $(A_1, \dots, A_d) \in \Pi(\mathcal{L}(V))$.*

In other words, the theorem asserts that $\Pi(\mathcal{L}(V)) \subseteq (\mathbb{TP}^{n-1})^d$ equals the row-wise projectivization of $\pi^{-1}(V)$. Notice that if $L = \mathcal{L}(M)$ is the Bergman fan of a matroid M, then the distinguished set of apices $\mathcal{DA}(L)$ consists of $\mathcal{DA}(L) = \{\!\{\overline{e}_F : F \in \mathcal{DF}(M)\}\!\}$ (where $\overline{e}_F$ is as defined in Section 6.2.4). So the distinguished set of apices $\mathcal{DA}(L)$ are the valuated generalization of the unique maximal presentation of a transversal matroid.

We begin the proof of Theorem 6.40 with the easier inclusion.

Proposition 6.41. *Let V be a transversal valuated matroid. If $\mathcal{A} = \{\!\{A_1, \dots, A_d\}\!\}$ is a presentation of V then $(A_1, \dots, A_d) \in \Pi(\mathcal{L}(V))$.*

Proof. Let $\mathcal{A}$ be a presentation of V and let $M \in \mathcal{DM}(V)$. First assume $M \in \mathcal{M}(V)$. Then by Theorem 6.27 we have that $Z_{v_M^L}(\mathcal{A})$ is a presentation of $\mathcal{L}(M)$. By Theorem 6.25(2) applied to every cyclic flat in the inclusion-exclusion manner, there are exactly $\tau_M(\emptyset)$ points of $Z_{v_M^L}(\mathcal{A})$ whose relative support with respect to 0 is an independent set of M. The tuple formed from the corresponding points in $\mathcal{A}$ will then be in $\phi_L(M)$.

Now if M is not in $\mathcal{M}(V)$ but in $\mathcal{M}(V/F)$ for some $F \in \mathcal{CF}(\underline{V})$, then by Theorem 6.33 there is $\mathcal{A}_F \subseteq \mathcal{A}$ such that its projection to the $[n]/F$ coordinates is a presentation of V/F. Then by the same argument as above, there are $\tau_M(\emptyset)$ of those points in $\phi_{\mathcal{L}(V/F)}(M)$ which proves the desired result as $\iota_F(\phi_{\mathcal{L}(V/F)}(M)) = \phi_L(M)$. ☐

For the other direction of Theorem 6.40, we begin by recalling the following characterization of transversal matroids in its form due to Ingleton [Ing77]. Essentially the same characterization, but quantifying over all cyclic sets, was given earlier by Mason [Mas71].

Proposition 6.42. *A matroid M is transversal if and only if for every collection of cyclic flats $F_1, \dots, F_k$ the following inequality is satisfied:*

$$\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} \operatorname{rk}\left(\bigcup_{i \in I} F_i\right) \leq -\operatorname{rk}\left(\bigcap_{i=1}^{k} F_i\right).$$

Notice that for $k = 2$, this is the submodularity axiom of the rank function. We also remark that on substituting $\operatorname{rk}(J) = d - \operatorname{cork}(J)$ in the above inequality, the d terms cancel out, and therefore a formally identical inequality is true where rk is replaced by cork and $\leq$ by $\geq$.

Definition 6.43. Let M be a transversal matroid of rank d. We say that a collection $G_1, \dots G_d$ of flats of M is a *pseudopresentation* if

$$\{\!\{\operatorname{cocl}(G_1), \dots \operatorname{cocl}(G_d)\}\!\} = \mathcal{DF}(M).$$

To motivate this definition, note that it is a necessary condition for a presentation of M that the complements of its members be a pseudopresentation (see Theorem 6.16). The following lemma says that if a pseudopresentation fails to be the complements of a presentation, then the failure is "local", that is, there is a distinguished cyclic flat F such that the G_i which extend F were poorly chosen.

Lemma 6.44. *Let M be a transversal matroid with $\mathcal{DF}(M) = \{\!\{F_1, \dots, F_d\}\!\}$ and let $G_1, \dots, G_d \in \mathcal{F}(M)$ be a pseudopresentation. Suppose that $G_1, \dots, G_d$ are not the complements of a presentation. Then there exists $F \in \mathcal{DF}(M)$ and $I, J \subseteq [d]$, such that:*

- $\operatorname{cocl}(G_i) = F$ *for every* $i \in I$
- $F \subsetneq F_j$ *for every* $j \in J$.
- $\operatorname{cork}\left(\bigcap_{i\in I} G_i \cap \bigcap_{j\in J} F_j\right) < |I| + |J|$

Proof. Suppose that such F does not exist but $G_1, \dots, G_d$ are not the complements of a presentation. Then there is a set of indices $I \subseteq [d]$ such that

$$\operatorname{cork}\left(\bigcap_{i\in I} G_i\right) < |I|.$$

Let k be the number of different elements of $\{\operatorname{cocl}(G_i) \mid i \in I\}$ and without loss of generality let that set be $\{F_1, \dots, F_k\}$. For $j \in [k]$ let $I_j = \{i \in I \mid \operatorname{cocl}(G_i) = F_j\}$ and let $m_j = |I_j|$. The I_j clearly partition I so we have that

$$\sum_{j=1}^{k} m_j = |I|$$

Let $K = \bigcap_{i\in I} G_i$. For any proper subset $J \subseteq [k]$ let

$$a_J = \left| K \cap \left(\bigcap_{j\in J} F_j\right) \cap \left(\bigcap_{j\notin J} [n] \setminus F_j\right) \right|$$

and let $a_{[k]} = \operatorname{rk}\left(\bigcap_{i\in I} F_i\right)$. Notice that for any element $x \in K \setminus \bigcap_{i\in I} F_i$, x is a coloop of some G_i, so in particular it is a coloop in K. Therefore we have that

$$\operatorname{rk}(K) = \sum_{J\subseteq[k]} a_J.$$

For every $i \in I_1$ we have that $K \subset G_i$, so

$$\operatorname{rk}\left(\bigcap_{i\in I_1} G_i\right) \geq \operatorname{rk}(F_1) + \sum_{J\subseteq[d]\setminus\{1\}} a_J.$$

As we assume $(F_1, I_1, \emptyset)$ do not satisfy the conditions of the lemma (for (F, I, J) in the statement), we have that

$$m_1 \leq \operatorname{cork}\left(\bigcap_{i\in I_1} G_i\right) \leq \operatorname{cork}(F_1) - \sum_{J\subseteq[d]\setminus\{1\}} a_J.$$

Now for any $2 \leq j \leq k$, let

$$J_j = \left\{ j' \in [d] \mid F_j \cup \left(\bigcap_{j''<j} F_{j''}\right) \subseteq F_{j'} \text{ and } F_j \neq F_{j'} \right\}.$$

Similarly as before, we assume the conditions of the lemma are not satisfied for (F_j, I_j, J_j). By inclusion-exclusion, we have that

$$|J_j| \geq \sum_{\emptyset\neq J\subseteq[j-1]} (-1)^{|J|-1} \operatorname{cork}\left(\bigcup_{j'\in J\cup\{j\}} F_{j'}\right).$$

On the other hand we have

$$\operatorname{rk}\left(\bigcap_{i\in I_j} G_i \cap \bigcap_{j'\in J_j} F_{j'}\right) \geq \operatorname{rk}(F_j) + \sum_{[j-1]\subseteq J\subseteq[d]\setminus\{j\}} a_J.$$

So by assumption we get

$$\begin{aligned} m_j &\leq \operatorname{cork}\left(\bigcap_{i\in I_j} G_i \cap \bigcap_{j'=1}^{j-1} F_{j'}\right) - |J_j| \\ &\leq \sum_{J\subseteq[j-1]} (-1)^{|J|} \operatorname{cork}\left(\bigcup_{j'\in J\cup\{j\}} F_{j'}\right) - \sum_{[j-1]\subseteq J\subseteq[d]\setminus\{j\}} a_J. \end{aligned}$$

Adding all bounds for the m_j and using Theorem 6.42 we get:

$$\begin{aligned}\sum_{j=1}^{k} m_j = |I| &\leq \sum_{\emptyset \neq J \subseteq [k]} (-1)^{|J|} \operatorname{cork}\left(\bigcup_{j \in J} F_j\right) - \sum_{J \subsetneq [d]} a_J \\ &\leq \operatorname{cork}(\bigcap_{j=1}^{k} F_j) - \sum_{J \subsetneq [d]} a_J \\ &= d - \sum_{J \subseteq [d]} a_J \\ &= \operatorname{cork}(K)\end{aligned}$$

which is a contradiction, as we assumed $|I| > \operatorname{cork}(K)$. □

Proposition 6.45. *Let $L = \mathcal{L}(V)$ be a tropical linear space such that V has transversal facets and let $\mathcal{DA}(L) = \{\!\{A_1, \ldots, A_d\}\!\}$ be its distinguished multiset of apices. Then for every $M \in \mathcal{M}(V)$ coloop-free and $x \in L_M$ the multiset $\{\!\{\operatorname{rs}_x(A_1), \ldots, \operatorname{rs}_x(A_d)\}\!\}$ is a pseudopresentation of M.*

Proof. Let x be any point inside a coloopless cell L_M. If $\tau_M(\emptyset) > 0$, then $L_M = \{x\}$ and x has multiplicity exactly $\tau_M(\emptyset)$ in $\mathcal{DA}(L)$ and $\operatorname{cocl}_M(\operatorname{rs}_x(x)) = \emptyset$.

Now fix a coloopless cell L_{M_0} of V and a flat $F \in \mathcal{DF}(M_0)$, and let $t = \tau_{M_0}(F)$. We claim that

$$\text{there are } t \text{ distinguished apices } A_i \text{ with } \operatorname{cocl}_M(\operatorname{rs}_x(A_i)) = F. \qquad (6.7)$$

We will prove this as follows. For each distinguished apex A_i we construct a finite sequence of matroids $M_0, M_1, \ldots, M_k$ which form a "path" from x to A_i, and sets $F = F^0, F^1, \ldots, F^k$ where F^j is a flat of M_j. We reduce claim (6.7) about (M_j, F^j) to the same claim about (M_{j+1}, F^{j+1}) for each $j < k$. The last matroid M_k satisfies $L_{M_k} = \{A_i\}$, for which claim (6.7) is proved by the previous paragraph.

The construction of the path is iterative, starting with M_0. Let (M, F) be the last matroid and flat (M_j, F^j) constructed, and let H_F be the hyperplane

$$H_F := \left\{\sum_{i \in F} z_i = \operatorname{rk}_M(F)\right\}.$$

Consider three cases.

Case 1. P_M is not contained in H_F. This means that the affine span of L_M does not contain the vector e_F. So let $M' = M|F \oplus M/F$. We have that $P_{M'}$ is the proper face of P_M maximized by e_F. As F is cyclic in M, it follows that M' is coloop-free. The lattice of flats of M' decomposes as $(\mathcal{F}(M'),\subset) = (\mathcal{F}(M|F),\subset) \times (\mathcal{F}(M/F),\subset)$ where $(\mathcal{F}(M|F),\subset)$ is isomorphic to the sublattice of $(\mathcal{F}(M'),\subset)$ below F and $(\mathcal{F}(M/F),\subset)$ is isomorphic to the sublattice above F. In particular, $\tau_{M'}(F) = t$. This means that, letting $y = x + \varepsilon e_F \in L_{M'}$ for small $\varepsilon > 0$, if there are $\tau_{M'}(F) = t$ distinguished apices A_i with $\mathrm{cocl}_{M'}(\mathrm{rs}_y(A_i)) = F$, then those same points satisfy $\mathrm{rs}_x(A_i) = \mathrm{rs}_y(A_i)$ and $\mathrm{cocl}_M(\mathrm{rs}_x(A_i)) = \mathrm{cocl}_{M'}(\mathrm{rs}_y(A_i)) = F$. Therefore, we have the necessary reduction with the choice $M_{j+1} := M'$ and $F^{j+1} = F$.

Case 2. P_M is contained in H_F and $\mathrm{rk}_{\underline{V}}(F) > \mathrm{rk}_M(F)$. Let $y = x + \lambda e_F$, where λ is maximum subject to $y \in L$. Then $y \in L_{M'}$ where $L_{M'}$ is a face of L_M. Dually, P_M is the proper face of $P_{M'}$ maximized by $[n] \setminus F$. This case is the reverse of the first case, in that $(\mathcal{F}(M),\subset) = (\mathcal{F}(M'|([n] \setminus F)),\subset) \times (\mathcal{F}(M'/([n] \setminus F)),\subset)$.

If $\mathrm{rk}_M(F) = r$, then $\mathcal{DF}(M)$ contains exactly r supersets (possibly not strict) of $[n] \setminus F$, which will also be in $\mathcal{DF}(M')$ because the upper intervals above $[n] \setminus F$ are identical in $\mathcal{F}(M)$ and $\mathcal{F}(M')$. For $F' \in \mathcal{CF}(M')$ a proper subset of $[n] \setminus F$, we have

$$\tau_M(F \cup F') = |\{\!\{G \in \mathcal{DF}(M') \mid F' = \mathrm{cocl}_{M'}(G \setminus F)\}\!\}|. \tag{6.8}$$

To see this, compare the use of the recursion (6.2) to compute τ_M on the interval $[F, [n]]$ and τ'_M on the interval $[\emptyset, [n] \setminus F]$. Note that these two intervals are isomorphic. The coranks in the latter interval exceed those in the former by r; this is accounted for by the r distinguished flats of M' above $[n] \setminus F$. The other difference is the presence of flats G not comparable with $[n] \setminus F$ in M'. Because $\mathcal{CF}(M')$ is a lattice, it contains a greatest lower bound of G and $[n] \setminus F$, namely $\mathrm{cocl}_{M'}(G \setminus F)$. This is the maximal element of $[\emptyset, [n] \setminus F]$ contained in G. Therefore, terms $\tau(G)$ behave in the recursion as if they were terms $\tau(\mathrm{cocl}_{M'}(G \setminus F))$, and this is the fact expressed by (6.8).

The case $f' = \emptyset$ of the above means that if $t = \tau_M(F)$ there are exactly t elements $\{\!\{F_1, \ldots, F_t\}\!\} \subseteq \mathcal{DF}(M')$ such that $F_i \setminus F$ is an independent set. In particular, $\mathrm{cocl}_M(F_i \cup F) = F$ for every $i \in [t]$. So if there are exactly t distinguished apices such that, up to reordering, $\mathrm{cocl}_{M'}(\mathrm{rs}_y(A_i)) = F_i$, then we have that $\mathrm{rs}_x(A_i) = \mathrm{rs}_y(A_i) \cup F$ and $\mathrm{cocl}_M(\mathrm{rs}_x(A_i)) = \mathrm{cocl}_M(F_i \cup F) = F$. So $M_{j+1} := M'$ and $F^{j+1} := F_i$ provides our reduction.

Case 3. P_M is contained in H_F but $\mathrm{rk}_{\underline{V}}(F) = \mathrm{rk}_M(F)$. This means that P_M is in the boundary of $P_{\underline{V}}$ and that the affine span of L_M contains e_F. In particular L_M is unbounded in the e_F direction. But then $M' = M/F$ is a coloopless matroid with $\tau_{M'}(\emptyset) = \tau_M(\emptyset)$ and $L_{M'}$ consists of a vertex v with infinity in the coordinates corresponding to F. In particular, the multiplicity of v in $\mathcal{DA}$ is $\tau(M)$, and we have $\mathrm{rs}_x(v) = F$ and in particular $\mathrm{cocl}_M(\mathrm{rs}_x(v)) = F$. So in this case our path terminates.

Finally, we argue that the path $M_0, \dots, M_k$ terminates. Each member M_{j+1} except possibly the last is of the form $V^{x_{j+1}}$ where $x_{j+1} = x_j + \lambda e_{F^j}$ where $\lambda > 0$. We have that $S = F^{j+1} \setminus F^j$ consists of coloops of the restriction $M_{j+1}|(S \cup F^{j+1})$. In Case 2, this implies that S is independent in M_{j+1}, and therefore that S is a set of coloops of $M_j|(S \cup F^j)$. This property also propagates backwards in Case 1. Therefore, the union $U = \cup_{j=0}^{k} F^j$ consists of F^0 together with coloops of $M_0|U$. Because M_0 is coloop-free, U is a subset of some hyperplane H containing F^0, excluding an element ℓ of the ground set. Therefore, in supportive coordinates with respect to x_0, the x_ℓ coordinate is constant and the coordinates in H are nondecreasing. □

Theorem 6.46. *Let $L = \mathcal{L}(V)$ be a tropical linear space such that V has transversal facets. Then the distinguished multiset of apices is a Stiefel presentation of L.*

Proof. The proof has the same structure as the proof of Theorem 6.45. If the distinguished multiset of apices $\mathcal{DA}(L) = \{\!\{A_1, \dots, A_d\}\!\}$ is not a Stiefel presentation of L, then by Theorem 6.32 there exists $x \in L_{M_0}$ where M_0 is a coloopless matroid such that $\mathrm{rs}_x(\mathcal{DA}(L)) = \{\!\{\mathrm{rs}_x(A_1), \dots, \mathrm{rs}_x(A_d)\}\!\}$ is not a presentation. By Theorem 6.45, $\mathrm{rs}_x(\mathcal{DA}(L))$ is indeed a pseudopresentation, so by Theorem 6.44 we know there is a flat $F \in \mathcal{F}(M_0)$, a set I such that $\mathrm{cocl}_{M_0}(\mathrm{rs}(A_i)) = F$ for every $i \in I$ and distinguished flats $\{\!\{F_1, \dots, F_k\}\!\} \subseteq \mathcal{DF}(M_0)$ such that $F_j \supseteq F$ for every $j \in [k]$ and

$$\mathrm{cork}_{M_0}\left(\bigcap_{i \in I} \mathrm{rs}_x(A_i) \cap \bigcap_{j=1}^{k} F_j\right) < |I| + k.$$

We show that this failure of presentation implies a sequence of such failures

$$(M_0, F^0 := F), (M_1, F^1), \dots, (M_k, F^k)$$

such that we can argue a contradiction for (M_k, F^k). The M_j will be the same matroids as in Theorem 6.45, so this proof will consider the exact same three cases. The sequence terminates either in Case 3 or at $(M_k, \emptyset)$; the latter is an immediate contradiction because $\mathrm{cork}(\emptyset) = d$.

Otherwise, let $(M, F) = (M_j, F^j)$, and construct $(M_{j+1}, F^{j+1}) = (M', F')$ as follows.

Case 1. In the first case, recall that we defined $y = x + \varepsilon e_F \in L_{M'}$ where L_M is a face of $L_{M'}$. We have that $\mathrm{rs}_y(A_i) = \mathrm{rs}_x(A_i)$ for any $i \in I$, so

$$\mathrm{cork}_{M'}\left(\bigcap_{i \in I} \mathrm{rs}_y(A_i) \cap \bigcap_{j=1}^{k} F_j\right) < |I| + k.$$

Case 2. In the second case, recall that $y = x + \lambda e_F \in L_{M'}$, where $L_{M'}$ is a face of L_M. Here, $\mathrm{rs}_y(A_i) \supseteq \mathrm{rs}_x(A_i) \setminus F$. For every $j \in [k]$, $F_j \setminus F$ is a cyclic flat in M'. However, it may be the case that $\tau_{M'}(F_j \setminus F) \leq \tau_M(F_j)$. This happens when there is a cyclic flat F'_j such that $F'_j \setminus F = F_j \setminus F$. In any case, we can find distinguished flats $\{\!\{F'_1, \ldots, F'_k\}\!\} \subseteq \mathcal{DF}(M')$ such that for every $j \in [k]$ we have $F'_j \setminus F = F_j \setminus F$. Moreover, there are another $r = \mathrm{cork}_{M'}([n] \setminus F) = \mathrm{rk}_M(F)$ distinguished flats $F'_{k+1}, \ldots, F'_{k+r}$ such that $F'_{k+j} \supseteq [n] \setminus F$ for every $j \in [r]$. In total we have that

$$\bigcap_{i\in I} \mathrm{rs}_y(A_i) \cap \bigcap_{j=1}^{k+r} F'_j \supseteq \left(\bigcap_{i\in I} \mathrm{rs}_x(A_i) \cap \bigcap_{j=1}^{k} F_j\right) \setminus F$$

$$\begin{aligned}\mathrm{cork}_{M'}\left(\bigcap_{i\in I} \mathrm{rs}_y(A_i) \cap \bigcap_{j=1}^{k+r} F'_j\right) &\leq \mathrm{cork}_M\left(\bigcap_{i\in I} \mathrm{rs}_x(A_i) \cap \bigcap_{j=1}^{k} F_j\right) + \mathrm{rk}(F)\\ &< |I| + j + r.\end{aligned}$$

Then $\mathrm{rs}_y(\mathcal{DA}(L))$ is not a presentation of M. So we can use Theorem 6.44 again to find $F' \in \mathcal{DF}(M')$ and $I' \subseteq I$ such that $\mathrm{cocl}_{M'}(\mathrm{rs}_y(A_i)) = F'$ where the conditions for presentation fail.

Case 3. In the third case, we have that $v = A_i$ for every $i \in I$. In particular, $\mathrm{rs}_x(v) = F$. But then

$$\mathrm{cork}_M\left(F \cap \bigcap_{j=1}^{k} F_j\right) < |I| + k$$

is a contradiction to the fact that M is a transversal matroid with $\mathcal{DF}(M) \supseteq \{F^{|I|}, F_1, \ldots F_k\}$.

□

We get three important results as corollaries. The first corollary is Theorem 6.40:

Proof of Theorem 6.40. One of the directions is Theorem 6.41. For the other direction, let $(A_1, \ldots, A_d) \in \Pi(L)$ and $\mathcal{A} = \{\!\{A_1, \ldots, A_d\}\!\}$. Let $M \in \mathcal{M}(V)$ be coloop-free and $x \in L_M$. By Theorem 6.45, $\mathrm{rs}_x(\mathcal{DA}(L))$ is a pseudopresentation of M. Now, the argument used in proving Theorem 6.46 can be strengthened to show that $\mathcal{A}$ is actually a presentation of V. Let M and M' be successive matroids in the path, and label the distinguished apices for M_i as $\{\!\{A'_1, \ldots, A'_d\}\!\}$ so that the apex of M corresponding to A'_i is A_i. As all the cones in $\phi(M')$ are generated by rays going in the direction of a single coordinate e_j, the difference $\mathrm{rs}_x(A'_i) \setminus \mathrm{rs}_x(A_i)$ must consist of coloops of $\mathrm{rs}_x(A_i)$. So

$$\mathrm{cocl}_M(\mathrm{rs}_x(\mathcal{A})) = \mathrm{cocl}_M(\mathrm{rs}_x(\mathcal{DA}(L)))$$

which means $\mathcal{A}$ is still a pseudopresentation of M. □

The second corollary is the converse of Theorem 6.8.

Theorem 6.47. *A tropical linear space is in the Stiefel image if and only if all the facets in its dual subdivision are transversal.*

Since the class of transversal matroids is closed under contractions of cyclic sets [BM72, Theorem 5.4] and arbitrary deletions, if V is transversal then so is any initial matroid V^x which has no new coloops. Thus Theorem 6.47 can be sloganized: *transversality is a local property of a tropical linear space.*

Corollary 6.48. *Let M be a matroid and suppose P_M has a regular subdivision such that all facets in the subdivision are transversal. Then M is transversal.*

Proof. Let L be a tropical linear space dual to such a regular subdivision. By Theorem 6.47, L is in the Stiefel image so it has presentation A. Consider the matrix $\tilde{A}$ that replaces all finite entries of A by 0. Then $\pi(L)$ is a tropical linear space dual to M, so M is transversal. □

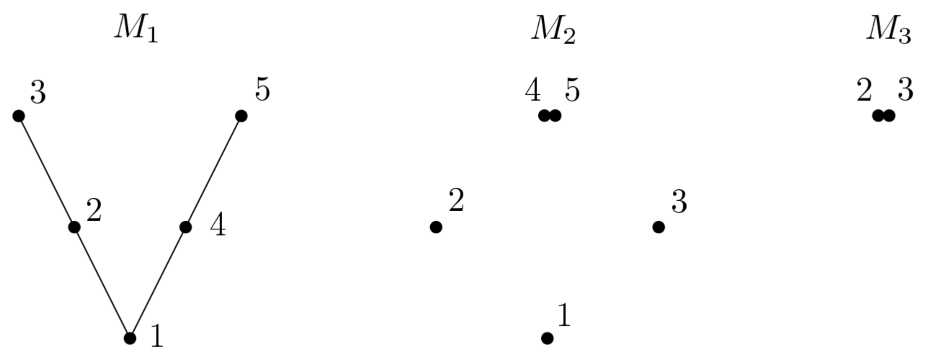


FIGURE 6.3: The distinguished matroids of V in Theorem 6.49.

Example 6.49. Let V be the valuated matroid of rank 3 on 5 elements such that $V_{123} = 1$, $V_{145} = \infty$, and $V_B = 0$ for any $B \in \binom{[5]}{3}$ other than these two. The three distinguished matroids M_1, M_2 and M_3 of V are shown in Figure 6.3. The respective distinguished apices of $\mathcal{L}(V)$ are $x_1 = [0 : 0 : 0 : 0 : 0]$, $x_2 = [1 : 1 : 1 : 0 : 0]$ and $x_3 = [\infty : 0 : 0 : \infty : \infty]$. Figure 6.4 shows the presentation fan of each distinguished matroid: the fan from x_1 is the cone over the boundary of a square and the fan from x_2 is the cone over the boundary of a triangle, while the fan from x_3 is the single point x_3. So any matrix $A \in \pi^{-1}(V)$ must have one row in the red zone, another row in the blue zone and a third row lying exactly at the green point.

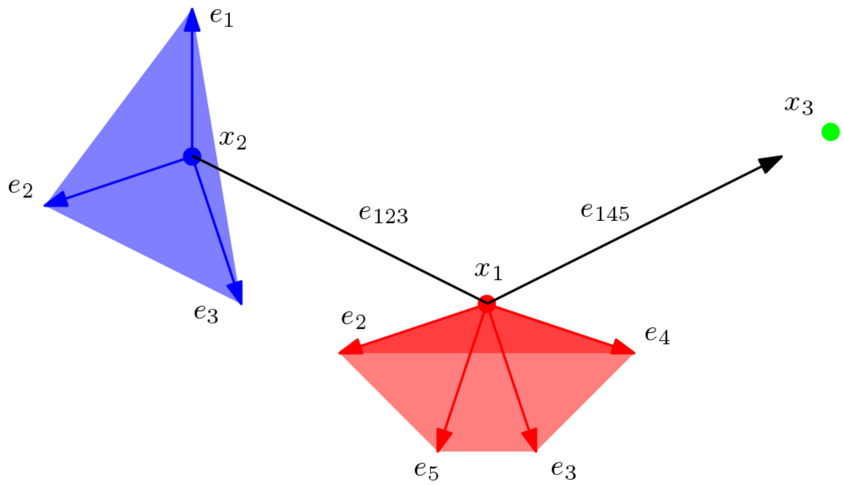


FIGURE 6.4: The presentation fan ϕ_{M_i} of each of the distinguished matroids M_i in Theorem 6.49, as they appear together in $\mathbb{TP}^4$. Labels e_J on rays and edges indicate their directions.

Part III

Lattice polygon subdivisions: Harnack Curves

Chapter 7

The moduli space of Harnack curves

7.1 Introduction

Harnack curves are real algebraic plane curves with several remarkable properties. By definition, they have the maximum possible number of connected components and these components are arranged in a unique particular way [Mik00] (see Definition 7.3). Their amoebas are particularly special, since they are precisely the ones with maximal area for a fixed Newton polygon [MR01]. Because of this, they have found applications in physics, where the dimer model is used to study crystal surfaces (see [KOS06] for details). In this model, the limit of the shape of a crystal surface is given by the amoeba of a Harnack curve.

To better understand the dimer model, Kenyon and Okounkov [KO06] studied the space of Harnack curves of degree d in the projective plane modulo the action of the torus $(\mathbb{C}^*)^2 \subseteq \mathbb{CP}^2$. Equivalently, this is the space of amoebas of Harnack curves modulo translation. They show that this moduli space has global coordinates given by the areas of holes of the amoeba and the distances between consecutive tentacles. Therefore it is diffeomorphic to $\mathbb{R}_{\geq 0}^{(d+4)(d-1)/2}$. Crétois and Lang [CL18] generalized some of the techniques used in [KO06] to Harnack curves in any projective toric surface. They showed that given a lattice polygon Δ, the moduli space $\mathcal{H}_\Delta$ of Harnack curves with Newton polygon Δ is path connected and conjectured that it is also contractible. We confirm this belief and further generalize the results of [KO06] to compute $\mathcal{H}_\Delta$:

Theorem 7.1. *Let Δ be a lattice m-gon with g interior lattice points and n boundary lattice points. Then the moduli space $\mathcal{H}_\Delta$ of Harnack curves of Newton polytope Δ is diffeomorphic to $\mathbb{R}^{m-3} \times \mathbb{R}_{\geq 0}^{n+g-m}$.*

The interior of $\mathcal{H}_\Delta$ corresponds to the smooth Harnack curves with transversal intersections with the axes of X_Δ.

We further show that $\mathcal{H}_\Delta$ admits a compactification similar in spirit to the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$. This compactification consists of 'Harnack meshes', that is, collections of Harnack curves that can be patchworked (using Viro's method, see [Vir06]) to produce a curve in $\mathcal{H}_\Delta$. A Harnack mesh consist of a regular subdivision of Δ and aa Harnack curve with Newton polytope Δ_i for each facet Δ_i of the subdivision with some gluing conditions. The space of Harnack meshes is naturally stratified in cells according to which regular subdivision is used in the patchworking recipe. The above can be summed up in the following:

Theorem 7.2. *The space $\mathcal{H}_\Delta$ has a compactification $\overline{\mathcal{H}_\Delta}$ consisting of all Harnack meshes over Δ. Moreover, $\overline{\mathcal{H}_\Delta}$ has a cell complex structure whose poset is isomorphic to the face poset of the secondary polytope* $\operatorname{Sec}(\Delta \cap \mathbb{Z}^2)$.

In Section 7.2 we set notation and recall some background results on Harnack curves. Sections 7.3 and 7.4 are dedicated to proving Theorem 7.1 (Theorem 7.22) and Chapter 8 is dedicated to prove Theorem 7.2 (Theorem 8.11). Most of the proofs consist in showing that there are different parameters that can be taken as global coordinates for Harnack curves. In Section 7.3 we consider the following diagram:

$$\left\{\begin{matrix}\text{Rational}\\ \text{Harnack}\\ \text{curves}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{Roots of}\\ \text{rational}\\ \text{parametrization}\end{matrix}\right\}/\mathrm{PGL}(\mathbb{R},2) \xrightarrow{\tilde{\rho}} \left\{\begin{matrix}\text{Positions of}\\ \text{amoeba}\\ \text{tentacles}\end{matrix}\right\}/\mathbb{R}^2$$

In the left we have the moduli space of rational Harnack curves, which we denote $\mathcal{H}_{0,\Delta}$; in the middle we have parametrizations $\phi : \mathbb{CP}^1 \to X_\Delta$ of Harnack curves modulo the action of $\mathrm{PGL}(\mathbb{R}, 2)$ on $\mathbb{CP}^1$; and in the right we have the positions of the tentacles of the amoeba modulo translations of the amoeba. The main result of section 3 is that the map $\tilde{\rho}$ is a smooth embedding when restricted to the image of the first map.

In Section 7.4 we show that the following are diffeomorphisms:

$$\left\{\begin{matrix}\text{Harnack curves}\\ \text{with fixed}\\ \text{tentacle positions}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Bounded}\\ \text{Ronkin}\\ \text{intercepts}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Areas of}\\ \text{holes of}\\ \text{the amoeba}\end{matrix}\right\} \tag{7.1}$$

By putting together the two diagrams above we have:

$$\mathcal{H}_\Delta \hookrightarrow \begin{matrix}\{\text{Tentacle positions}\}/\mathbb{R}^2\\ \times\{\text{Bounded intercepts}\}\end{matrix} \to \left\{\begin{matrix}\text{All Ronkin}\\ \text{intercepts}\end{matrix}\right\}/\mathbb{R}^3 \overset{\Upsilon}{\hookrightarrow} \mathcal{M}^{\text{trop}}_{g,n}$$

where the $\mathbb{R}^3$ action in the the third space refers to translations of the graph of the Ronkin function. The first map is a smooth embedding by putting together the two previous diagrams.

In Chapter 8 we look at the last two maps. The second map is a linear bijection between tentacle positions and unbounded Ronkin intercepts. The last map is given by what we call the *expanded spine*. Since $\mathcal{M}_{g,n}^{\text{trop}}$ is not a manifold (it is a tropical variety), this map can no longer be a diffeomorphism. However we show that it is a piecewise linear embedding. We also show how that Harnack meshes can be smilarly embedded into the closure of the embedding $\mathcal{H}_\Delta \hookrightarrow \mathcal{M}_{g,n}^{\text{trop}}$. This allows us to construct the compactification $\overline{\mathcal{H}_\Delta}$. We end the thesis by suggesting some directions for future research. In particular we conjecture $\overline{\mathcal{H}_\Delta}$ to be a regular CW-complex and we suggest a possible smooth structure on $\overline{\mathcal{H}_\Delta}$ as a manifold with generalized corners (see [Joy16]).

7.2 Preliminaries

7.2.1 Notation

We fix the following notation for the rest of this part. As is usual in toric geometry, $M \cong N \cong \mathbb{Z}^2$ are the lattices of characters and one-parametric subgroups of the algebraic two-dimensional torus $(\mathbb{C}^*)^2$ respectively. See [CLS11] as a general reference for toric varieties.

Let $\Delta \subset M \otimes \mathbb{R}$ be a convex lattice polygon. We write $\partial\Delta$ for the boundary of Δ, $\text{int}(\Delta)$ for the interior of Δ. We write Δ_M for the lattice points in Δ, that is, $\Delta_M = \Delta \cap M$. We use n and g to denote the number of lattice points in $\partial\Delta$ and $\text{int}(\Delta)$, respectively, and m for the number of edges of Δ. For any positive integer k, $[k]$ denotes the set $\{1, \ldots, k\}$. We denote by Γ_i, $i = 1, \ldots, m$, the edges of Δ in cyclic anticlockwise order. Let $d_1, \ldots, d_m$ be their respective integer lengths (i.e. $d_i = |\Gamma_i \cap M| - 1$). Let $u_i \in N$ be the primitive inner normal vector of Γ_i. We have the following equation:

$$\sum_{i=1}^{m} d_k u_i = 0 \tag{7.2}$$

To each $v = (v_1, v_2) \in M$ there is an associated Laurent monomial $x^v := x_1^{v_1} x_2^{v_2}$. The *Newton polygon* of a Laurent polynomial $f(x) = \sum_{v \in M} c_v x^v$ is the convex hull of $\{v \in M \mid c_v \neq 0\}$. For any subset $\Delta' \subseteq \Delta$ we write $f|_{\Delta'}(x) := \sum_{v \in \Delta'_M} c_v x^v$.

Given a lattice polygon Δ there is an associated projective toric surface X_Δ whose geometry reflects the combinatorics of Δ. It contains a dense copy of the torus $(\mathbb{C}^*)^2$

where coordinate-wise multiplication extends to an action on all of X_Δ. For each edge Γ_i of Δ, there is a corresponding irreducible divisor L_i in X_Δ which is invariant under the action of the torus. We call these divisors the *axes* of X_Δ. Two axes intersect in a point if and only if they correspond to consecutive edges of Δ. We denote the real part of X_Δ as $\mathbb{R}X_\Delta$.

7.2.2 Harnack Curves

Let f be a Laurent polynomial with real coefficients and Newton polygon Δ. The zeros of f define a curve $C^\circ \subset (\mathbb{C}^*)^2$. The closure of C° in X_Δ is a compact algebraic curve C. If C is smooth its genus is equal to g [Hov77]. The intersection of C with $\mathbb{R}X_\Delta$ is a real algebraic curve $\mathbb{R}C$. The intersection of C with an axis L_i is given by the restriction of f to Γ_i, which is, after a suitable change of variable, a polynomial of degree d_i. Therefore $L_i \cap C$ consists of exactly d_i points counted with multiplicities.

Definition 7.3. [Mik00, Definition 2] Let Δ be a lattice polygon with g, m and the d_i's defined as above. A smooth real algebraic curve $\mathbb{R}C \subseteq \mathbb{R}X_\Delta$ is called a *smooth Harnack curve* if the following conditions hold:

- The number of connected components of $\mathbb{R}C$ is $g+1$.
- Only one component of $\mathbb{R}C$ intersects $L_1 \cup \cdots \cup L_m$. This component can be subdivided into m disjoint arcs, $\theta_1 \ldots \theta_m$, in that order, such that $C \cap L_i = \theta_i \cap L_i$.

The components that are disjoint from $L_1 \cup \cdots \cup L_m$ are called *ovals*.

Harnack curves were originally called in "cyclically maximal position" in [Mik00]. In the literature these curves are sometimes called "simple Harnack curves". However, following [MR01, KOS06, KO06] we omit the adjective 'simple' when referring to them (see [MO07, Remark 6.6]).

These curves are named after Axel Harnack because he showed in 1876 that smooth curves of genus g in the real projective plane have at most $g+1$ connected components. To show that the bound was tight, he constructed the eponymous curves [Har76]. Curves which attain the maximum number of components are called M-curves. These are the topic of the first part of Hilbert's 16^{th} problem, which asks to classify all possible topological types of M-curves. When $\mathbb{R}X_\Delta = \mathbb{R}\mathbb{P}^2$, Harnack curves are the M-curves such that only one component intersects the axes and it does so in order. Mikhalkin proved that, for any given Δ, if $\mathbb{R}C$ is a Harnack curve with Newton polygon Δ then the topological type of $(\mathbb{R}X_\Delta, \mathbb{R}C, \mathbb{R}L_1\cup, \ldots, \cup\mathbb{R}L_n)$ is unique [Mik00, Theorem 3].

Recall that a singular point in $\mathbb{R}C$ is an ordinary isolated double point if it is locally isomorphic to the singularity of $x_1^2 + x_2^2 = 0$.

Definition 7.4. [MR01, Definition 3] A (possibly singular) real algebraic curve $\mathbb{R}C \subseteq X_\Delta$ is a *Harnack curve* if

- The only singularities of $\mathbb{R}C$ are ordinary isolated double points away from the torus invariant divisors.
- Replacing each singular point of $\mathbb{R}C$ by a small oval around it yields a curve $\mathbb{R}C'$ such that $(\mathbb{R}X_\Delta, \mathbb{R}C', \mathbb{R}L_1\cup, \dots, \cup\mathbb{R}L_n)$ has the topological type of smooth Harnack curves.

Notice that any singular Harnack curve can be aproximated by smooth Harnack curves. To see this, let f be a polynomial that vanishes on $\mathbb{R}C$ and let $g(x, y) := f(\lambda x, \lambda y)$ for a real number λ close to 1 but different from 1, so that the singular points of f and g are close but do not coincide. Then $f - \epsilon g$ vanishes on a smooth Harnack curve which approaches $\mathbb{R}C$ when ϵ tends to 0.

Let $\mathbb{R}[\Delta_M]$ be the vector space of real polynomials with Newton polygon contained in Δ. Since scaling all coefficients of f by the same constant does not change the curve $\mathbb{R}C$, we can identify the space of real curves with Newton polygon contained in Δ with $\mathbb{P}(\mathbb{R}[\Delta_M])$. The action of the torus $(\mathbb{R}^*)^2$ on $\mathbb{R}X_\Delta$ induces an action on $\mathbb{P}(\mathbb{R}[\Delta_M])$ given by $f(x_1, x_2) \mapsto f(r_1^{-1}x_1, r_2^{-1}x_2)$.

Definition 7.5. The *moduli space* $\mathcal{H}_\Delta$ *of Harnack curves* is the subspace of $\mathbb{P}(\mathbb{R}[\Delta_M])/(\mathbb{R}^*)^2$ consisting of all (possibly singular) Harnack curves with Newton polygon Δ modulo the action of $(\mathbb{R}^*)^2$.

Given an element in $\mathbb{R}C \in \mathcal{H}_\Delta$, we say that a polynomial vanishes on $\mathbb{R}C$ if its zero locus is in the equivalence class given by $\mathbb{R}C$.

Remark 7.6. The notation $\mathcal{H}_\Delta$ was used in [CL18] to note the space of Harnack curves without taking them modulo the action of $(\mathbb{R}^*)^2$. They defined it with a more algebro-geometric language as follows: the space of curves with Newton polygon contained in Δ can be identified with the complete linear system $|D_\Delta|$ of the Cartier divisor D_Δ of X_Δ associated to Δ. Since X_Δ is a complete normal toric variety, $|D_\Delta|$ can be identified with the projectivization of the space of global sections of the line bundle associated to Δ. Therefore $\mathcal{H}_\Delta$ can be defined as the subspace of $|D_\Delta|$ of Harnack curves, modulo the action of the torus $(\mathbb{R}^*)^2$ on $\mathbb{R}X_\Delta$.

The case when Δ is the d-th dilation of the unimodular triangle corresponds to degree d curves in $\mathbb{RP}^2$ and the corresponding moduli space is diffeomorphic to the closed orthant $\mathbb{R}_{\geq 0}^{(d+4)(d-1)/2}$ [KO06, Corollary 11].

7.2.3 Amoebas and the Ronkin function

Amoebas, which are essential to understand Harnack curves, were defined in [GKZ94, Chapter 6] where details about them can be found.

Definition 7.7. Let $\text{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ be the map

$$\text{Log}(z_1, z_2) := (\log|z_1|, \log|z_2|)$$

The *amoeba* of an algeraic curve C is $\mathcal{A}(C) := \text{Log}(C^\circ)$.

The amoebas of Harnack curves are specially well-behaved:

Proposition 7.8. *[MR01] Let $\mathbb{R}C$ be a real algebraic curve with Newton polygon Δ and $\mathcal{A} = \mathcal{A}(C)$ its amoeba. The following are equivalent:*

1. *$\mathbb{R}C$ is Harnack curve*
2. *The map $\text{Log}|_{C^\circ}$ is at most 2-to-1.*
3. $\text{area}(\mathcal{A}) = \pi^2 \,\text{area}(\Delta)$

For arbitrary curves, $\text{area}(\mathcal{A}) \leq \pi^2 \,\text{area}(\Delta)$ [PRr04], so Harnack curves have the amoebas with maximal area. Smooth Harnack curves are also characterized by having maximal curvature, and by having totally real logarithmic Gauss map [PR11, Mik00]. However there are more general singular curves whose logarithmic Gauss map is also totally real [Lan15].

Each connected component of the complement of an amoeba is convex and has a point in Δ_M naturally associated to it, as we now show. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Laurent polynomial. The *Ronkin function* $R_f : \mathbb{R}^2 \to \mathbb{R}$ of f defined in [Ron74], is

$$R_f(x) := \frac{1}{(2\pi\sqrt{-1})^2} \int\limits_{\text{Log}^{-1}(x)} \frac{\log|f(z_1, z_2)|}{z_1 z_2} dz_1 dz_2.$$

The Ronkin function is convex, see [PRr04]. Its gradient vector $\nabla R_f = (\nu_1, \nu_2)$ is given by

$$\nu_i(x) = \frac{1}{(2\pi\sqrt{-1})^2} \int\limits_{\text{Log}^{-1}(x)} \frac{z_i \partial_{z_i} f(z_1, z_2)}{z_1 z_2 f(z_1, z_2)} dz_1 dz_2.$$

For any $x \in \mathbb{R}^2$ we have that $\nabla R_f(x) \in \Delta$. If two points are in the same connected component of $\mathbb{R}^2 \backslash \mathcal{A}$, then their preimages under Log are homologous cycles in $(\mathbb{C}^*)^2 \backslash C^\circ$. This implies that ∇R_f is constant in each component and it has integer coordinates by the residue theorem. Therefore $\nabla R_f(x)$ induces an injection from the components of $\mathbb{R}^2 \setminus \mathcal{A}$ to Δ_M. The value that ∇R_f takes in a component of $\mathbb{R}^2 \setminus \mathcal{A}$ is called the *order* of that component and we write E_v for the component of order v if it exists. For details of this construction see [FPT00].

To better understand amoebas, we review some facts about their behaviour, see [GKZ94, Section 6.1]. The component E_v is bounded if and only if v is in the interior of Δ. For each vertex v of Δ, E_v exists and contains a translation of $-\operatorname{cone}(u_i, u_{i+1})$ where u_i and u_{i+1} are the inner normal vectors of the edges adjacent to v. If v is a lattice point in the relative interior of an edge Γ_i, E_v is only unbounded in the direction $-u_i$. Parts of the amoeba extend to infinity in between the unbounded components of $\mathbb{R}^2 \setminus \mathcal{A}$, in direction u_i for some i. These are called the *tentacles* of the amoeba. Figure 7.1 serves as an illustration of how typical amoebas of Harnack curves look like.

For each $v \in \Delta_M$ such that E_v exists, let $F_v : \mathbb{R}^2 \to \mathbb{R}$ be the affine linear function that coincides with R_f in E_v. The *spine* of a curve C as defined in [PRr04] is the corner locus of $\max F_v$ where max is taken over all E_v that exist. Notice that scaling f by a a constant only changes R_f by an additive constant constant, so the spine of C is well defined.

The spine varies continuously for smooth curves. However, if E_v vanishes for $v \in \operatorname{int}(\Delta)$, then the spine changes abruptly. Fortunately, for Harnack curves there is an easy work around. By the definition of singular Harnack curves, for each $v \in \operatorname{int}(\Delta) \cap M$ such that $E_v = \emptyset$ there is an isolated double point p_v in $\mathbb{R}C$ such that there is a smooth Harnack curve $\mathbb{R}C'$ arbitrarily close to $\mathbb{R}C$ with a component near p_v with order v. Therefore $\nabla R_f(\operatorname{Log}(p_v)) = v$. Let F_v be the tangent plane of R_f at $\operatorname{Log}(p_v)$.

Definition 7.9. Let $\mathbb{R}C$ be a Harnack curve. We call *expanded spine* of C and denote $\Upsilon(C)$ the corner locus of the piecewise affine linear convex function $\max\limits_{v \in \Delta_M} F_v$.

The expanded spine and the usual spine coincide if and only if $\mathbb{R}C$ is a smooth Harnack curve. The expanded spine varies continuously for Harnack curves, even singular ones. It has a cycle for each $v \in \operatorname{int}(\Delta) \cap M$. The bounded part of the expanded spine is a planar graph of genus g. This definition will be crucial in Section 8.2.

By definition, the expanded spine is a tropical plane curve. The intercepts c_v of the affine functions F_v are the coefficients of the tropical polynomial that vanishes on the

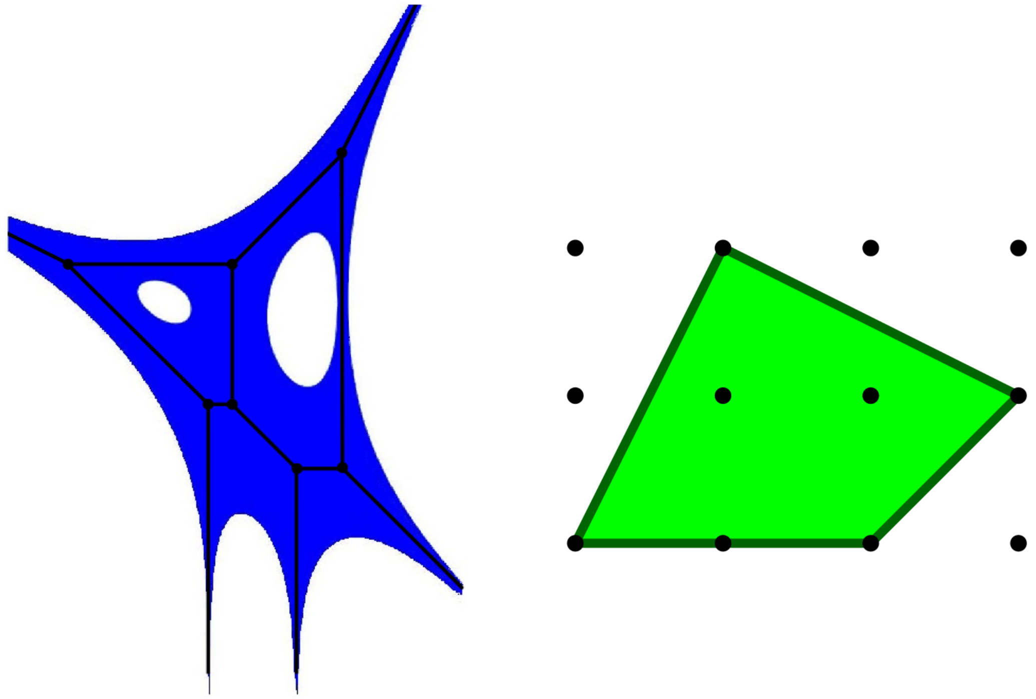


Figure 7.1: The amoeba (blue), the spine (black) and the Newton polygon (green) of a Harnack curve.

expanded spine, that is

$$\Upsilon(C) = \mathrm{trop}(\bigoplus_v c_v \odot x^{\odot v}).$$

We call the numbers c_v the *Ronkin intercepts*. We call a Ronkin intercept c_v *bounded* if $v \in \mathrm{int}(\Delta)$ and *unbounded* if $v \in \partial\Delta$. In other words, we say c_v is bounded if and only if whenever E_v exists it is bounded.

7.2.4 Patchworking

We now give a basic overview of patchworking of real algebraic curves, a powerful tool to construct curves with a prescribed topology, developed by Viro, see [Vir06] for details. The ingredients for (real) pathworking are a regular subdivision $\mathcal{S} = \mathcal{S}(h)$ of a polygon Δ and a real polynomial $f \in \mathbb{R}[\Delta_M]$. Let $\Delta_1, \ldots, \Delta_s$ be the facts of $\mathcal{S}$. Then the polynomial $f|_{\Delta_i}$ defines a real curve $\mathbb{R}C_i \subseteq X_{\Delta_i}$. Suppose that every curve $\mathbb{R}C_i$ is smooth and intersects transversally the axes of X_{Δ_i}, that is, $\mathbb{R}C_i$ intersects each axis in d different points, where d is the integer length of the corresponding edge of Δ_i. Let

$$f_t(x) := \sum_{v \in \Delta_M} t^{h(v)} a_v x^v$$

and let $\mathbb{R}C_t \subseteq X_\Delta$ be the vanishing locus of f_t. The Patchworking Theorem by Viro [Vir06] says that there exists $t_0 > 0$ small enough such that for every $t \in (0, t_0]$ the topological type of $\mathbb{R}C_t$ can be computed from the topological type of each $\mathbb{R}C_i$ by gluing them in a certain way. We say $\mathbb{R}C_t$ is the result of patchworking the curves $\mathbb{R}C_i$. Given the curves $\mathbb{R}C_i$, there exist different f such that $f|_{\Delta_i}$ vanishes on $\mathbb{R}C_i$. However, the topological type of the resulting curve $\mathbb{R}C_t$ only depends on the signs of each $f|_{\Delta_i}$ (real polynomials with the same zero locus differ only by scaling by a constant in $\mathbb{R}^*$).

We do not show in general how to do this computation, see [Vir06] for that purpose. We do however mention some important facts regarding Harnack curves. First, Mikhalkin showed that Harnack curves can be constructed using patchworking, [Mik00, Appendix]. There it is shown that Harnack curves are *T-curves*, that is, curves whose topological type can be obained from patchworking using regular unimodular triangulations as regular subdivision. In that case, the signs of each coefficient of f contain all the relevant information and this is known as combinatorial pathcworking [IV96]. Consider the sign configuration $\Delta_M \to \{-1, 1\}$ given by $v \mapsto (-1)^{v_1 v_2}$. No matter the triangulation chosen, the result from patchworking with this sign configuration will always be a Harnack curve [Mik00, Apendix]. Moreover, it is essentially (up to $\mathbb{Z}_2^2$) the only sign configuration whose patchwork is invariant under the chosen unimodular triangulation. These statements follow directly from the discussion in [GKZ94, Chapter 11 Section 5C].

Another important fact is that for any regular subdivision $\mathcal{S}$, if each curve $\mathbb{R}C_i$ is a Harnack curve, then there exists a choice of f_i such that the result from patchworking is a Harnack curve, see Theorem 8.6.

7.2.5 Cox coordinates

We now review Cox coordinates for toric surfaces, since it will help us understand the parametrizations of rational Harnack curves. For details see Chapter 5 of [CLS11]. They are a generalization of homogeneous coordinates in the projective space $\mathbb{CP}^d = (\mathbb{C}^{d+1} \setminus \{0\})/\mathbb{C}^*$.

Let Δ be a Newton polygon and recall $u_1, \dots, u_m$ to be the primitive inner normal vectors of Δ. Let $\alpha : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^2$ be the group homomorphism given:

$$(z_1, \dots, z_m) \mapsto (\prod_{i=1}^{m} z_i^{u_{i1}}, \prod_{i=1}^{m} z_i^{u_{i2}}).$$

We have that $\ker(\alpha)$ is a subgroup of $(\mathbb{C}^*)^m$. Let Z be the subset of $\mathbb{C}^m$ with at least three coordinates equal to 0 or at least two not-cyclically-consecutive coordinates equal to 0.

Proposition 7.10. *[CLS11, Theorem 5.1.11] Let Δ be a lattice polygon. All* $\ker(\alpha)$*-orbits of $\mathbb{C}^m \setminus Z$ are closed and the quotient $(\mathbb{C}^m \setminus Z)/\ker(\alpha)$ is isomorphic to X_Δ as an algebraic variety.*

We write $[z_1 : \cdots : z_m]_\Delta$ to denote the point in X_Δ corresponding to the orbit of $(z_1, \ldots, z_m) \in \mathbb{C}^m \backslash Z$ under the action of $\ker(\alpha)$. We have that

$$L_i = \{[z_1 : \cdots : z_m]_\Delta \in X_\Delta \mid z_i = 0\}$$

and

$$\mathbb{R}X_\Delta = \{[z_1 : \cdots : z_m]_\Delta \in X_\Delta \mid z_i \in \mathbb{R} \;\; \forall i\} .$$

Example 7.11. Let Δ be any rectangle with edges parallel to the $\mathbb{R}^2$ axes. The map $\alpha : M \to \mathbb{Z}^4$ is given by the matrix:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Then $\alpha : (\mathbb{C}^*)^4 \to (\mathbb{C}^*)^2$ is given by $(z_1, z_2, z_3, z_1) \mapsto (z_1 z_3^{-1}, z_2 z_1^{-1})$. The action of G consists of coordinatewise multiplications by vectors of the form $(\lambda_1, \lambda_2, \lambda_1, \lambda_2)$ where $\lambda_1, \lambda_2 \in \mathbb{C}^*$. The set Z consists of the points where $z_1 = z_3 = 0$ or $z_2 = z_1 = 0$. So in this case X_Δ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

7.3 Rational Harnack curves

7.3.1 Parametrizations of rational Harnack curves

We start this section by describing a parametrization of rational Harnack curves which was already used in [KO06] for $X_\Delta = \mathbb{CP}^2$ and more generally in [CL18]. We rewrite it using Cox homogenous coordinates. Real rational curves in X_Δ with Newton polygon Δ can be parametrized by $\phi = [p_1 : \cdots : p_m]_\Delta$, where each $p_i : \mathbb{CP}^1 \to \mathbb{C}$ is a homogenuous polynomial of degree d_i with real coefficients for $i \in [m]$ and no two polynomials have a common root. If the curve is Harnack, $\phi(\mathbb{RP}^1)$ is the 1-dimensional component of $\mathbb{R}C$. This implies that the roots of p_i are real and are ordered in the cyclic way according to Theorem 7.3. In fact, this condition is sufficient for $\mathbb{R}C$ to be a Harnack curve. This was shown in [KO06, Proposition 4] for $X_\Delta = \mathbb{CP}^2$ and it was noticed in [CL18, Equation (2)] that the same arguments work for any projective toric surface. So $\mathbb{R}\phi(\mathbb{CP}^1)$ is a

Harnack curve if and only if, for some chart of $\mathbb{CP}^1$, we have

$$\phi(t) = \left[b_1 \prod_{i=1}^{d_1} (t - a_{1,i}) : \quad \cdots \quad : b_m \prod_{i=1}^{d_m} (t - a_{m,i}) \right]_\Delta \tag{7.3}$$

where all $a_{i,j}$ are real, all b_i are real different from zero and

$$a_{1,1} \le \cdots \le a_{1,d_1} < a_{2,1} \le \cdots \le a_{2,d_n} < \cdots < a_{m,1} \le \cdots \le a_{m,d_m}. \tag{7.4}$$

We call the $a_{1,1} \dots a_{m,d_m}$ the *roots* of ϕ. Composing ϕ with α yields a parametrization for $C^\circ \subset (\mathbb{C}^*)^2$:

$$\alpha \circ \phi(t) = \left(\prod_{i=1}^{m} b_i^{u_{i1}} \prod_{j=1}^{d_i} (t - a_{i,j})^{u_{i1}}, \prod_{i=1}^{m} b_i^{u_{i2}} \prod_{j=1}^{d_i} (t - a_{i,j})^{u_{i2}} \right)$$

Let $\mathcal{H}_{0,\Delta}$ be the subspace of $\mathcal{H}_\Delta$ consisting of rational Harnack curves. The following generalizes [KO06, Corollary 5]:

Proposition 7.12. *Let Δ be a lattice polygon with m sides and n lattice points in its boundary. Then $\mathcal{H}_{0,\Delta}$ is diffeomorphic to $\mathbb{R}^{m-3} \times \mathbb{R}^{n-m}_{\geq 0}$.*

Proof. The parametrization above is unique up to the action of the projective special linear group $\mathrm{PGL}(\mathbb{R}, 2)$ on the parameter t. This induces an action of $\mathrm{PGL}(\mathbb{R}, 2)$ on the roots of ϕ. More explicitely, for $\psi \in \mathrm{PGL}(\mathbb{R}, 2)$ the function $\phi \circ \psi^{-1}$ also parametrizes C and has roots $\psi(a_{1,1}), \dots, \psi(a_{m,d_m})$. The roots $\psi(a_{1,1}), \dots, \psi(a_{m,d_m})$ are in the same cyclic order as in Equation (7.4).

The action of $(\mathbb{R}^*)^2$ in $\mathbb{R}X_\Delta$ affects ϕ by changing the constants $b_1, \dots, b_n$, but not the roots. The same is true for choosing different representatives of the Cox homogenous coordinates of X_Δ in eq. (7.3). So every rational Harnack curve is equivalent in $\mathcal{H}_\Delta$ to a curve with $b_1 = \cdots = b_d = 1$. Therefore, the elements of $\mathcal{H}_\Delta$ corresponding to rational curves are uniquely determined by the roots $a_{1,1} \dots a_{m,d_m}$, up to the action of $\mathrm{PGL}(\mathbb{R}, 2)$ on them.

The action of $\mathrm{PGL}(\mathbb{R}, 2)$ can be fixed, for example, by considering the unique Möbius transformation $\psi \in \mathrm{PGL}(\mathbb{R}, 2)$ such that $\psi(a_{1,1}) = 0$, $\psi(a_{2,1}) = 1$ and $\psi(a_{3,1}) = 2$. The map

$$\begin{aligned} \mathbb{R}C \mapsto \quad & (\psi(a_{4,1}), \dots, \psi(a_{m,1})) \\ & \times (\psi(a_{1,2}) - \psi(a_{1,1}), \dots, \psi(a_{m,d_m}) - \psi(a_{m,d_m-1})) \end{aligned}$$

is a diffeomorphism between $\mathcal{H}_{0,\Delta}$ and $\mathbb{R}^{m-3} \times \mathbb{R}^{n-m}_{\geq 0}$ (by identifying copies of $\mathbb{R}$ with $(\phi(a_{i,d_i}), 0)$ for $i \geq 3$). $\square$

The roots of ϕ are associated naturally to the segments of primitive segments of $\partial\Delta$. The proposition above says that we can take as global coordinates of $\mathcal{H}_{0,\Delta}$ the difference between two consecutive roots corresponding to the same edge of Δ together with the first root of each edge except the first 3 edges.

7.3.2 The positions of the tentacles

Following [KO06], we now make a useful change of global coordinates in $\mathcal{H}_{0,\Delta}$. Instead of the roots of ϕ, we will use the positions of the tentacles of the amoeba which correspond to boundary points $C \setminus C^\circ$.

Let $J : N \to M$ be the 2×2 matrix that rotates vectors $\pi/2$ clockwise, i.e. $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Observe that Ju_i is a character, which maps $[x_1, \ldots, x_m]_\Delta \mapsto \prod_{k=1}^m x_k^{u_k \wedge u_i}$ where $u_j \wedge u_i \in \mathbb{R}$ is the determinant of the 2×2 matrix whose columns are u_j and u_i in that order. In other words, $u_i \wedge u_j = \langle Ju_j, u_i \rangle$. This is not well-defined over all X_Δ. We take $0^0 = 1$ by convention, so Ju_i is well-defined over L_i except the torus invariant points. In fact, the ring of functions of L_i (without the torus invariant points) consists of Laurent polynomials on Ju_i.

Definition 7.13. Let $\mathbb{R}C$ be a rational Harnack curve parametrized by ϕ as in Equation (7.3). For $1 \leq i \leq m$ and $1 \leq j \leq d_i$, *the position of the (i,j) tentacle of the amoeba* is

$$\log |\phi(a_{i,j})^{Ju_i}|$$

Explicitly,

$$\log |\phi(a_{i,j})^{Ju_i}| = \sum_{k \neq i}^{m} \sum_{l=1}^{d_k} u_k \wedge u_i (\log |b_i| + \log |a_{i,j} - a_{k,l}|).$$

Since the curve does not change when acting the roots by $\mathrm{PGL}(\mathbb{R}, 2)$, the positions of the tentacles of the amoeba do not change either. However, the tentacle positions are not invariant under the action of $(\mathbb{R}^*)^2$ on $\mathbb{R}X_\Delta$. Concretely, multiplying $\mathbb{R}C$ by $r \in (\mathbb{R}^*)^2$ translates the amoeba by $\mathrm{Log}\,|r|$, thus changing the position of the (i,j) tentacle by $\langle Ju_i, \mathrm{Log}\,|r| \rangle$. Thus, the $(\mathbb{R}^*)^2$ action on $\mathbb{R}X_\Delta$ induces an $\mathbb{R}^2$ action on the position of the tentacles by translations of the amoeba.

Consider the maps $\rho_{i,j} : \mathbb{R}^n \dashrightarrow \mathbb{R}$ given by

$$\rho_{i,j} := \sum_{k \neq i}^{n} \sum_{l=1}^{d_k} u_k \wedge u_i \log |a_{i,j} - a_{k,l}|.$$

This is almost the position of the (i, j) tentacle except that we drop the $\log |b_i|$ terms. The maps $\rho_{i,j}$ are well-defined on the space of roots satisfying eq. (7.4). Together they form a map $\rho : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ invariant under the $\mathrm{PGL}(\mathbb{R}, 2)$ action on the roots. Additionally, consider the action of $\mathbb{R}^2$ on the target space $\mathbb{R}^n$ of ρ given by translations of the amoeba, that is:

$$r \cdot \rho_{i,j} := \rho_{i,j} + \langle J u_i, r \rangle \tag{7.5}$$

By construction, ρ descends to a map $\tilde{\rho}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{R}^n & \overset{\rho}{\dashrightarrow} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ \mathbb{R}^n/\mathrm{PGL}(\mathbb{R}, 2) & \overset{\tilde{\rho}}{\dashrightarrow} & \mathbb{R}^{n-2} \end{array}$$

The left downwards arrow is the quotient of the space of roots by $\mathrm{PGL}(\mathbb{R}, 2)$ and the right downward arrow is the quotient of $\mathbb{R}^n$ by the action of $\mathbb{R}^2$ described above. By Theorem 7.12, we can identify $\mathcal{H}_{0,\Delta}$ with the space of roots that satisfy eq. (7.4) modulo the $\mathrm{PGL}(\mathbb{R}, 2)$ action, so $\tilde{\rho}$ is a well defined map on $\mathcal{H}_{0,\Delta}$. The main objective of this section is to prove the following:

Theorem 7.14. *The restriction $\tilde{\rho}|_{\mathcal{H}_{0,\Delta}}$ is a smooth embedding.*

This is a generalization of Theorem 4 in [KO06], where the same statement is proven for the case where Δ is a dilated unit triangle. To prove that $\tilde{\rho}|_{\mathcal{H}_{0,\Delta}}$ is a diffeomorphism we show that it is proper (Section 7.3.3) and that the differential is injective (Section 7.3.4).

Before going to the proof, let us show a concrete diffeomorphism between the positions of the tentacles and $\mathbb{R}^{m-3} \times \mathbb{R}^{n-m}_{\geq 0}$. For the semi-bounded components, consider the distance between two parallel tentacles

$$\rho_{i,j+1} - \rho_{i,j} \quad \text{for } 1 \leq i \leq m, \;\; 1 \leq j \leq d_i - 1.$$

Notice that $\rho_{i,j+1} - \rho_{i,j}$ is invariant under the $\mathbb{R}^2$-action of translating the amoeba. For the unbounded components take as coordinates

$$\tilde{\rho}_{i,1} \quad \text{for } 4 \leq i \leq m,$$

where $\tilde{\rho}_{i,1}$ is the position of the $(i, 1)$-tentacle after translating the amoeba, so that the position of the $(1, 1)$ and $(2, 1)$ tentacles are both 0. It is straightforward to see that the image satisfies

$$\sum_{i=1}^{m}\sum_{j=1}^{d_i}\rho_{i,j}=0 \quad \text{and} \quad \rho_{i,j+1}-\rho_{i,j}\geq 0 \quad \text{for all } i,j. \tag{7.6}$$

In particular, $\tilde{\rho}_{3,1}$ is determined by the rest of the coordinates. The space described by the equation and the inequalities above is simply connected. This is important since we will use the following global diffeomorphism theorem, which was known by Hadamard. A proof of it can be found in [Gor72].

Proposition 7.15. *A local diffeomorphism between two manifolds which is proper and such that the image is simply connected is a diffeomorphism.*

7.3.3 Properness

To prove that $\tilde{\rho}|_{\mathcal{H}_{0,\Delta}}$ is proper, we make use of the following lemma:

Lemma 7.16. *Let $x_1,\dots x_n, y_1,\dots,y_n\in\mathbb{R}$ such that $x_1\leq\cdots\leq x_n$, $\sum_{i=1}^{n} y_i=0$ and there exists j such that $y_i<0$ for $i<j$ and $y_i\geq 0$ for $i>j$. Then*

$$\sum_{i=1}^{n} x_i y_i \geq 0.$$

Proof. We do induction on n. For $n=1$ we have $y_1=0$ so the above sum is 0. Let $n>1$ and suppose the inequality above holds for less than n terms. Subtract from the left-hand-side $y_n(x_n-x_{n-1})$, which is a non-negative number. The result is

$$\sum_{i=1}^{n-2} x_i y_i + x_{n-1}(y_{n-1}+y_n)$$

which is non-negative by applying the induction hypothesis to $x_1,\dots,x_{n-1},y_1,\dots,y_{n-2},y_{n-1}+y_n$. □

Proposition 7.17. *The map $\tilde{\rho}|_{\mathcal{H}_{0,\Delta}}$ is proper when restricting the codomain to the quotient of the space described by the equation and inequalities in (7.6).*

Proof. We want to show that the preimage of a compact set is a bounded set in $\mathcal{H}_{0,\Delta}\cong\mathbb{R}^{m-3}\times\mathbb{R}^{n-m}_{\geq 0}$. First we bound the parameters corresponding to the $\mathbb{R}^{n-m}_{\geq 0}$ part of $\mathcal{H}_{0,\Delta}$. These correspond to the difference between roots along the same edge of Δ, which are trivially bounded from below by 0. Without loss of generality consider the roots $a_{1,j-1}$ and $a_{1,j}$ for $1<j\leq d_1$. We fix the $\mathrm{PGL}(\mathbb{R},2)$ action on the space of roots by setting $a_{1,j-1}=-1$, $a_{2,1}=0$ and $a_{m,d_m}=1$. To show that $a_{1,j}-a_{1,j-1}$ attains its upper bound,

we show that $a_{1,j}$ can not be arbitrarily close to $a_{2,1} = 0$. By assumption, the difference between the position of the tentacles

$$\rho_{1,j} - \rho_{1,j-1} = \sum_{i=2}^{n} \sum_{k=1}^{d_i} u_i \wedge u_1 \left(\log|a_{i,k} - a_{1,j}| - \log|a_{i,k} + 1|\right) \tag{7.7}$$

is bounded. We have that $a_{1,j} \in [-1, 0)$ and $a_{i,k} \in [0, 1]$ if $i \geq 2$. The function $\log|x - a_{1,j}| - \log|x + 1|$ is increasing in $[0, 1]$. On the other hand,

$$\sum_{i=2}^{n} \sum_{k=1}^{d_i} u_i \wedge u_1 = \left(\sum_{i=1}^{n} d_i u_i\right) \wedge u_1 = 0$$

by Equation (7.2). Notice that for $i \in \{2, \ldots, m\}$, $u_i \wedge u_1$ is positive for the first numbers and then negative for the rest (with a 0 in between if and only if Δ has another edge parallel to Γ_1).

So, applying Theorem 7.16 to the sequences $\log|a_{i,j} - a_{1,j}| - \log|a_{i,j} + 1|$ and $u_i \wedge u_1$ (repeated d_i times) we get that $\rho_{1,j} - \rho_{1,j-1}$ is non-negative. More importantly, since $u_2 \wedge u_1 < 0 < u_n \wedge u_1$, we can subtract

$$\begin{aligned} &(\log|a_{m,d_m} - a_{1,j}| - \log|a_{m,d_m} + 1|) - (\log|a_{2,1} - a_{1,j}| - \log|a_{2,1} + 1|) \\ =& \log|1 - a_{1,j}| - \log|2| - \log|a_{1,j}| \\ \geq& -\log|2| - \log|a_{1,j}| \end{aligned}$$

from the right-hand-side in Equation (7.7) and again by Theorem 7.16 the result is still non-negative. This implies $\rho_{1,j} - \rho_{1,j-1} \geq -\log|2| - \log|a_{1,j}|$, which is arbitrarily large if $a_{1,j}$ is arbitrarily close to 0. Since we assumed $\rho_{1,j} - \rho_{1,j-1}$ is bounded, $a_{1,j}$ is not arbitrarily close to 0.

Now we turn our attention to the $\mathbb{R}^{m-3}$ component, assuming $m > 3$. We choose a different representative of the $\mathrm{PGL}(\mathbb{R}, 2)$-orbit on the roots by setting $a_{1,1} = 0$, $a_{3,1} = 1$ and $a_{m,d_m} = 2$. We will show that $a_{2,1}$ can not be arbitrarily close to $a_{1,1} = 0$. That implies that the paramater $a_{2,1}$ has a minimum in the preimage under $\tilde{\rho}$ of any compact set, and it attains it by continuity. Analogously, all other bounds regarding the $\mathbb{R}^{m-3}$ component are achieved and we conclude the preimage is compact.

To fix the $\mathbb{R}^2$ action on the positions of the tentacles, we assume that the position of the $(1, 1)$ tentacle and the (m, d_m) tentacle are both 0. This is translating the amoeba by the vector

$$w = \frac{-\rho_{m,d_m}}{u_1 \wedge u_m} u_1 + \frac{-\rho_{1,1}}{u_m \wedge u_1} u_m.$$

We show that if the position of the second tentacle after this translation,

$$\hat{\rho}_{2,1} = \rho_{2,1} - \rho_{m,d_m}\frac{u_1 \wedge u_2}{u_1 \wedge u_m} - \rho_{1,1}\frac{u_m \wedge u_2}{u_m \wedge u_1},$$

is bounded from below then $a_{2,1}$ is not arbitrarily close to $a_{1,1} = 0$.

By Theorem 7.16, ρ_{m,d_m} is non-positive, since $\log|x-2|$ is a decreasing function in $[0,2)$ and $u_m \wedge u_i$ is negative for the first values of i and positive for the later. Since $u_1 \wedge u_2 < 0 < u_1 \wedge u_m$, we have that $-\rho_n \dfrac{u_1 \wedge u_2}{u_1 \wedge u_m}$ is non-positive.

In both $\rho_{2,1}$ and $\rho_{1,1}$ there is a $\log|a_i|$ term which is arbitrarily large in absolute value if $a_{2,1}$ is close to 0. As $a_{i,j} > 1$ if $i \geq 3$, the only terms in $\rho_{1,1}$ and $\rho_{2,1}$ which can be arbitrarily large in absolute value are those corresponding to $a_{2,j}$ and $a_{1,j}$, respectively.

In other words, the part that could grow arbitrarily large in absolute value in $\rho_{1,1}$ is

$$\sum_{j=1}^{d_2} u_2 \wedge u_1 \log|a_{2,j}|$$

and in $\rho_{2,1}$ it is

$$\sum_{j=1}^{d_1} u_1 \wedge u_2 \log|a_{2,1} - a_{1,j}|.$$

Notice that

$$|\log|a_{2,1} - a_{1,j}|| \geq |\log|a_{2,1}|| \geq |\log|a_{2,j}|. \tag{7.8}$$

Let c_1 be the real number such that

$$\sum_{j=1}^{d_1} u_1 \wedge u_2 \log|a_{2,1} - a_{1,j}| = c_1 u_1 \wedge u_2 \log|a_{2,1}|.$$

By Equation (7.8) we have that $c_1 \geq d_1$. Similarly, if c_2 is such that

$$\sum_{j=1}^{d_2} u_2 \wedge u_1 \log|a_{2,j}| = c_2 u_2 \wedge u_1 \log|a_{2,1}|$$

then by Equation (7.8) $c_2 \leq d_2$. So the part of $\hat{\rho}_{2,1}$ which grows in absolute value is

$$\begin{aligned}
&c_1 u_1 \wedge u_2 \log|a_{2,1}| - c_2 u_2 \wedge u_1 \log|a_{2,1}|\frac{u_m \wedge u_2}{u_m \wedge u_1} \\
&= \log|a_{2,1}|\frac{u_1 \wedge u_2}{u_m \wedge u_1}\left(c_1 u_m \wedge u_1 + c_2 u_m \wedge u_2\right) \\
&= \log|a_{2,1}|\frac{u_1 \wedge u_2 \cdot u_m \wedge (c_1 u_1 + c_2 u_2)}{u_m \wedge u_1}
\end{aligned}$$

Notice that $-d_1u_1-d_2u_2$ is the inner normal vector of the third side of the triangle formed by Γ_1 and Γ_2, so $u_m \in \operatorname{cone}(-d_1u_1-d_2u_2, u_1)$ since $m > 3$. Thus, $u_m \wedge (d_1u_1+d_2u_2) > 0$. Because $c_1 \geq d_1$ and $c_2 \leq d_2$, we have that $c_1u_1 + c_2u_2 \in \operatorname{cone}(u_1, d_1u_1 + d_2u_2)$. As $u_m \wedge u_1$ is also positive, we have that $u_m \wedge (c_1u_1 + c_2u_2) > 0$. We conclude that

$$\frac{u_1 \wedge u_2 \cdot u_m \wedge (c_1u_1 + c_2u_2)}{u_m \wedge u_1} > 0,$$

which implies that $\hat{\rho}_{2,1}$ is negative and arbitrarily large in absolute value if $a_{2,1}$ is arbitrarily close to 0. □

7.3.4 The Jacobian of ρ

In this subsection we proof that $\tilde{\rho}|_{\mathcal{H}_{0,\Delta}}$ is a local diffeomorphism.

From now on, we sightly change the notation we have used so far to simplify the exposition and the computations. Instead of labelling the roots by pairs (i, j) with $i \in [m]$ and $j \in [d_i]$, we relabel them as $a_1, \ldots, a_n$ in the global cyclic order. Similarly, we relabel the u_i's to agree with the labelling of the roots; that is, we have vectors $u_1, \ldots, u_n$ where u_i is the primitive inner normal vector of the edge of Δ that corresponds to the axis in which $\phi(a_i)$ vanishes. With this notation, we have that

$$\alpha(\phi)(t) = \prod_{i=1}^{n} (t - a_i)^{u_i}$$

and that the a_i and the u_i correspond to a parametrization of a Harnack curve if

1. $\sum_{i=1}^{n} u_i = 0$.
2. $a_1 \leq \cdots \leq a_n$.
3. The $u_1, \ldots, u_n$ are ordered anticlockwise.

We write $a = (a_1, \ldots, a_n)$ and $U = (u_1, \ldots, u_n)$ for short and we say a and U are *cyclically ordered* if they satisfy the above conditions.

Now we consider the Jacobian matrix D of ρ at a given point a. We have that

$$D_{i,j} = \begin{cases} \dfrac{u_i \wedge u_j}{a_i - a_j} & \text{if } a_i \neq a_j \\ 0 & \text{if } a_i = a_j \text{ but } i \neq j \\ -\sum_{k \neq i} D_{i,k} & \text{if } i = j \end{cases}$$

In general, D is a matrix that depends on a and U, so we denote it as $D(a, U) = D(a_1, \dots, a_n; u_1, \dots, u_n)$.

Proposition 7.18. *The map $\tilde{\rho}|_{\mathcal{H}_{0,\Delta}}$ is a local diffeomorphism.*

Proof. Let a and U be cyclically ordered. Let $T_a\mathrm{PGL}(\mathbb{R}, 2)a$ be the tangent space of the orbit of a under the $\mathrm{PGL}(\mathbb{R}, 2)$ action at a and similarly let K be the kernel of the quotient $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^2$ by the $\mathbb{R}^2$ action defined in Equation (7.5). In other words,

$$K = \{(r \wedge u_1, \dots, r \wedge u_n) \mid r \in \mathbb{R}^2\}$$

which is a linear space. Let us look at the relation of these spaces with D.

To compute the tangent space $T_a\mathrm{PGL}(\mathbb{R}, 2)a$, recall that Möbius transformations are of the form

$$t \mapsto \frac{at + b}{ct + d}.$$

We see that

$$\frac{\partial}{\partial \epsilon} t + \epsilon \bigg|_{\epsilon=0} = 1, \quad \frac{\partial}{\partial \epsilon}(1 + \epsilon)t \bigg|_{\epsilon=0} = t, \quad \frac{\partial}{\partial \epsilon} \frac{t}{\epsilon t + 1} \bigg|_{\epsilon=0} = -t^2,$$

so $T_a\mathrm{PGL}(\mathbb{R}, 2)a$ is spanned by the vectors $(1, \dots, 1)$, $(a_1, \dots, a_n)$ and $(a_1^2, \dots, a_n^2)$. Since

$$\sum_{j=1}^{n} D_{i,j} = D_{i,i} - D_{i,i} = 0$$

and

$$\sum_{j=1}^{n} a_j D_{i,j} = \sum_{j=1}^{n} \frac{a_j \cdot u_i \wedge u_j}{a_i - a_j} - \sum_{j=1}^{n} \frac{a_i \cdot u_i \wedge u_j}{a_i - a_j} = -\sum_{j=1}^{n} u_i \wedge u_j = 0,$$

we have that both $(1, \dots, 1)$ and $(a_1, \dots, a_n)$ are in the kernel of D. In Theorem 7.19 we will proof that the kernel of D is 2 dimensional, so the vector $(a_1^2, \dots, a_n^2)$ is not in the kernel. However, since ρ descends to the map $\tilde{\rho}$, we have that $D \cdot (a_1^2, \dots, a_n^2)^\top \in K$.

Since D is symmetric, its image is the orthogonal complement of its kernel. So the image of D is orthogonal to $(1, \dots, 1)$ and $(a_1, \dots, a_n)$. On the other hand, K is always orthogonal to $(1, \dots, 1)$. By a similar argument as in the proof of Theorem 7.17, we have that $\sum_{i=1}^{n} a_i u_i \neq 0$, so $(a_1, \dots, a_n)$ is never orthogonal to K. Thus, the intersection of the image of D with K is exactly one dimensional so it is spanned by $(a_1^2, \dots, a_n^2)$. This implies that the Jacobian of $\tilde{\rho}$ is injective, since no vector outside $T_a\mathrm{PGL}(\mathbb{R}, 2)a$ vanishes under the composition of D and the quotient $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^2$. □

Now the only thing left to prove is the following:

Proposition 7.19. *If* a *and* U *are cyclically ordered then the rank of* $D(a, U)$ *is* $n-2$.

Proof. In [KO06, Theorem 4] the authors prove this for the case where Δ is the dilation of a the unit triangle. They do it by showing that D is a sum of 3×3-block semidefinite positive matrices of rank 1, each corresponding to the Jacobian of the unimodular triangle case. We here generalize this for any polygon Δ. Let e_1, e_2, e_3 be the primitive normal vector of the standard unimodular triangle in clockwise order and let $T(a_i, a_j, a_k) = D(a_i, a_j, a_k, e_1, e_2, e_3)$ (see Equation (4.8) in [KO06]). We have that $T(a_i, a_j, a_k)$ is a rank 1 matrix with kernel generated by $(1, 1, 1)$ and (a_i, a_j, a_k). We obtain thenon-zero eigenvalue by computing the image under D of the cross-product of the two vectors in the kernel, $(1, 1, 1) \times (a_i, a_j, a_k)$. We get that the eigenvalue is

$$\frac{(a_i - a_j)^2 + (a_j - a_k)^2 + (a_k - a_i)^2}{(a_i - a_j)(a_j - a_k)(a_k - a_i)},$$

which is always positive when $a_i < a_j < a_k$.

Let $T_{i,j,k}(a_i, a_j, a_k)$ be the $n \times n$ matrix that restricts to $T(a_i, a_j, a_k)$ in the 3×3 submatrix with indices $\{i, j, k\}$ and that is zero elsewhere. We will show that if a and U are cyclically ordered, then $D(a, U)$ is a positive sum of matrices of the form $T_{i,j,k}(a_i, a_j, a_k)$. In other words we want to show that:

$$D(a, U) \in \operatorname{cone}\left(\{T_{i,j,k}(a_i, a_j, a_k) \quad | \quad 1 \leq i < j < k \leq n\}\right).$$

To do so, we write each u_i in the unique way $u_i = x_i e_1 + y_i e_2 + z_i e_3$ where $x_i, y_i, z_i \geq 0$ and at most two are positive. With this notation we have that

$$u_i \wedge u_j = x_i y_j + y_i z_j + z_i x_j - x_i z_j - y_i x_j - z_i y_j$$

and that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i.$$

We call c the constant above. For $i < j < k$ let

$$q_{i,j,k} = \det \begin{pmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{pmatrix}.$$

For all $i < j < k$ we have that $q_{i,j,k} \geq 0$. To see that, notice that the vectors (x_i, y_i, z_i) are ordered cyclically along

$$\mathbb{R}^2 \times \{0\} \cup \mathbb{R} \times \{0\} \times \mathbb{R} \cup \{0\} \times \mathbb{R}^2,$$

since they project to U under $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$. Therefore, by the right-hand-rule, the determinant of that matrix is non-negative. Now we claim that

$$\sum_{i<j<k} q_{i,j,k} T_{i,j,k}(a_i, a_j, a_k) = cD(a, U).$$

To verify that claim, look at the coefficient of $\frac{1}{a_i - a_j}$ in the (i, j)-entry of the left hand side. We have that the coefficient is equal to

$$\begin{aligned} & x_i y_j \left(\sum_{k=1}^{n} z_k\right) + y_i z_j \left(\sum_{k=1}^{n} x_k\right) + z_i x_j \left(\sum_{k=1}^{n} y_k\right) \\ & -x_i z_j \left(\sum_{k=1}^{n} y_k\right) - y_i x_j \left(\sum_{k=1}^{n} z_k\right) - z_i y_j \left(\sum_{k=1}^{n} x_k\right) \qquad = c \cdot u_i \wedge u_j \end{aligned}$$

Therefore, $D(a, U)$ is the sum of positive semidefinite matrices, so its kernel is the intersection of the kernels of all of the summands. This implies that the kernel is the span of $(1, \ldots, 1)$ and $(a_1, \ldots, a_n)$. ☐

Proof of Theorem 7.14. By Theorems 7.17 and 7.18, $\tilde{\rho}$ is a proper local diffeomorphism whenever a and U are cyclically ordered. By Theorem 7.15, $\tilde{\rho}|_{\mathcal{H}_{0,\Delta}}$ is a diffeomorphism onto the space defined by Equation (7.6) modulo $\mathbb{R}^2$. ☐

7.4 From $\mathcal{H}_{0,\Delta}$ to $\mathcal{H}_\Delta$

The reason for the change of coordinates by $\tilde{\rho}$ is that fixing the position of the tentacles is fixing $C \setminus C^\circ$, which implies fixing $f|_{\partial\Delta}$ up to scaling by a constant. Polynomials using the remaining monomials $\mathrm{int}(\Delta) \cap M$ were shown to be in correspondence with holomorphic differentials, in [KO06, Section 2.2.4] for $\mathbb{CP}^2$ and in [CL18, Lemma 4.3] for smooth toric projective surfaces. The following lemma generalizes this to X_Δ for any Δ.

Lemma 7.20. *Let Δ be any lattice polygon and let $C \subseteq X_\Delta$ be the vanishing set of a polynomial f with Newton polygon Δ such that $\mathbb{R}C$ is a Harnack curve. Then the space of holomorphic differentials on C is isomorphic to the space of polynomials with coefficients in* $\mathrm{int}(\Delta) \cap M$*, via the map*

$$h(z_1, z_2) \mapsto \frac{h(z_1, z_2) dz_2}{\partial_{z_1} f(z_1, z_2) z_1 z_2} \tag{7.9}$$

Proof. The map is injective and both spaces have dimension g, so it remains to prove that the differentials from Equation (7.9) are holomorphic. If Δ has a vertex at the origin with incident edges given by the coordinate axes, then the differentials from Equation (7.9) are holomorphic over $\mathbb{C}^2 \cap C$ (see [CL18, Lemma 4.3]).

Given any lattice-preserving affine map $A : M \otimes \mathbb{R} \to M \otimes \mathbb{R}$, that sending a polygon Δ' to Δ, there is a dual map $A^\vee X_\Delta \to X_{\Delta'}$. In [CL18] it is shown that the pullback of $A^\vee$ sends differentials of the form of Equation (7.9) for Δ' to differentials of that form for Δ. For each vertex v of Δ, consider the lattice preserving affine map that sends the positive orthant to the cone spanned by Δ from v. Then the differentials from Equation (7.9) are holomorphic in the intersection of C with the affine chart corresponding to v. Doing that for every vertex we get that they are holomorphic in all of C.

If Δ is not a smooth polygon, then such a lattice-preserving map does not exist. However, given any vertex v of Δ, there is a lattice-preserving map that sends the cone spanned by $(1, 0)$ and (p, q) to the cone spanned by Δ from v, for some suitable $p, q \in \mathbb{N}$. Let Δ' be the preimage of Δ under such map. By the same arguments as in [CL18, Lemma 4.3], the differentials of the form of Equation (7.9) for Δ' are holomorphic in $(\mathbb{C} \times \mathbb{C}^*) \cap C$. This implies that the pullback is holomorphic in $((\mathbb{C}^*)^2 \cup L) \cap C$, where L is the axis that corresponds to the edge contained in the image under A of the coordinate axis $\{x_1 = 0\}$. This can be done for every edge of Δ. Since C does not contain the intersection any two axes, it follows that the differentials in Equation (7.9) are holomorphic over all C. □

Proposition 7.21. *The areas of the holes of the amoeba are global coordinates for the moduli space of Harnack curves with fixed Newton polygon Δ and fixed boundary points. Moreover, the moduli space of Harnack curves with fixed boundary is diffeomorphic to $\mathbb{R}^g_{\geq 0}$.*

Proof. Recall the diagram from Equation (7.1) in the introduction. The first map sends a Harnack curve with fixed boundary (that is, we fix $f|_{\partial\Delta}$) to the bounded Ronkin intercepts. By Theorem 7.20, the differential of that map is the period matrix of C (see [KO06, Proposition 6] and [CL18, Theorem 3]). The second map, from the bounded intercepts to the areas of the holes in the amoeba, is also a local diffeomorphism because its differential is diagonally dominant (see [KO06, Proposition 10]). The areas of the holes of the amoeba are non-negative and the composition of the two maps is proper over $\mathbb{R}^g_{\geq 0}$ (see [KO06, Theorem 6]). All of these facts are proven in [KO06] and none of the arguments used there require that $X_\Delta = \mathbb{CP}^2$. Again, by Theorem 7.15 the composition of the maps is a diffeomorphism onto $\mathbb{R}^g_{\geq 0}$. □

Notice that the positions of the tentacles are also well-defined numbers for non-rational Harnack curves: they are simply the evaluation of Ju_i on the points $C \cap L_i$. So by Theorem 7.14 and Theorems 7.12 and 7.21 we have that the positions of the tentacles of the amoeba modulo translation together with the areas of the holes of the amoeba are global coordinates for $\mathcal{H}_\Delta$. Hence, we have proved that.

Theorem 7.22. *Let Δ be a lattice polygon with m sides, g interior lattice points and n boundary lattice points. Then $\mathcal{H}_\Delta$ is diffeomorphic to $\mathbb{R}^{m-3} \times \mathbb{R}_{\geq 0}^{n+g-m}$.*

Chapter 8

The compactified moduli space of Harnack curves

The goal of this chapter is to construct a natural compactification $\overline{\mathcal{H}_\Delta}$ of $\mathcal{H}_\Delta$ by collections of 'patchworkable' Harnack curves.

8.1 Abstract tropical curves

We begin with a review of abstract tropical curves and of $\mathcal{M}_{g,n}^{\text{trop}}$, the moduli space of tropical curves with n legs and genus g. For details of this construction see [Cap13].

A *weighted graph with n legs* $\mathcal{G}$ is a triple (V, E, L, w) where

- (V, E) is a perhaps non-simple connected graph, that is, we allow multiple edges and loops.
- $L : [n] \to V$ is a function which we think of as attaching n labelled *legs* at vertices of the graph.
- w is a function $V \to \mathbb{N}$ which we call the *weights* of the vertices.

The *genus* of $\mathcal{G}$ is the usual genus of (V, E) plus the sum of the weights on all vertices; that is

$$\text{genus}(G) = \sum_{v \in V} w(v) - |V| + |E| + 1.$$

An *isomorphism* between two graphs with n legs $\mathcal{G}_1 = (V_1, E_1, L_1, w_1)$ and $\mathcal{G}_2 = (V_2, E_2, L_2, w_2)$ is a pair of bijections $\phi_V : V_1 \to V_2$ and $\phi_E : E_1 \to E_2$ such that

- For any edge $e \in E_1$ and any vertex $v \in V_1$, $\phi_E(e)$ is incident to $\phi_V(v)$ if and only if e is incident to v.
- $L_2 = \phi_V(L_1)$.
- $w_1(v) = w_2(\phi_V)$.

Let $\mathcal{G}/e$ denote the usual contraction of $\mathcal{G}$ over an edge e with the following change of weights: if we contract a non loop ab, then the contracted vertex gets weight $w(a)+w(b)$. If the contracted edge is a loop on a, then the weight of a is increased by 1. Observe that the genus is invariant under contraction.

We say that a weighted graph $\mathcal{G}$ is *stable* if every vertex with weight 0 has degree at least 3 and every vertex with weight 1 has positive degree. An *(abstract) tropical curve* is a pair $(\mathcal{G}, l)$ where $\mathcal{G}$ is a stable weighted graph and l is a function that assigns *lengths* to the edges of $\mathcal{G}$, in other words, l is a function $l : E(\mathcal{G}) \to \mathbb{R}_{\geq 0}^{|E(\mathcal{G})|}$. The *genus* of the tropical curve is the genus of $\mathcal{G}$. An isomorphism between two abstract tropical curves $(\mathcal{G}_1, l_1)$ and $(\mathcal{G}_2, l_2)$ is an isomorphism ϕ of the weighted graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $l_1 = l_2 \circ \phi_E$, or such that one is the result of contracting an edge of length 0 form the other, or the transitive closure of these two relations.

Given a weighted stable graph $\mathcal{G}$, one can identify the space of all tropical curves over $\mathcal{G}$ with $\mathbb{R}_{\geq 0}^{|E(\mathcal{G})|}$. We define $\mathcal{M}_{g,n}^{\text{trop}}(\mathcal{G}) := \mathbb{R}_{\geq 0}^{|E(\mathcal{G})|}$ and

$$\mathcal{M}_{g,n}^{\text{trop}} := \left(\bigsqcup_{\mathcal{G} \text{ stable}} \mathcal{M}_{g,n}^{\text{trop}}(\mathcal{G}) \right) \Big/ \sim ,$$

where $\sim$ denotes isomorphism. This is a connected Hausdorff topological space which parametrizes bijectively isomorphism classes of tropical curves. It is covered by $\mathcal{M}_{g,n}^{\text{trop}}(\mathcal{G})$ where $\mathcal{G}$ runs over all 3 valent graphs with all vertices of weight 0. For such graphs we have that $\mathcal{M}_{g,n}^{\text{trop}}(\mathcal{G})$ is just $\mathbb{R}^{3g+n-3}$. However $\mathcal{M}_{g,n}^{\text{trop}}$ is not a manifold as there are triples of graphs of this form glued along codimension 1.

To compactify this space we allow lengths to be infinite. Let $\mathbb{R}_\infty = \mathbb{R}_{\geq 0} \sqcup \{\infty\}$ be the one point compactification of $\mathbb{R}_{\geq 0}$. An *extended* tropical curve $(\mathcal{G}, l)$ consists of a stable weighted graph $\mathcal{G}$ and a length function $l : E(\mathcal{G}) \to \mathbb{R}_\infty^{|E(\mathcal{G})|}$. We define isomorphism classes of extended tropical curves in the same way as for tropical curves. This way we define $\overline{\mathcal{M}_{g,n}^{\text{trop}}(\mathcal{G})} := \mathbb{R}_\infty^{|E(\mathcal{G})|}$ and

$$\overline{\mathcal{M}_{g,n}^{\text{trop}}} := \left(\bigsqcup_{\mathcal{G} \text{ stable}} \overline{\mathcal{M}_{g,n}^{\text{trop}}(\mathcal{G})} \right) \Big/ \sim$$

This is a compact hausdorff space with $\mathcal{M}_{g,n}^{\text{trop}}$ as an open dense subspace.

8.2 Ronkin intercepts

From now on, lightly abusing notation, we call elements of $\mathcal{H}_\Delta$ curves and denote one of them C, even though by definition they are equivalence classes of Harnack curves. We say that a polynomial vanishes on C if its zero locus is in the equivalence class C.

The expanded spines of different Harnack curves in the same equivalence class in $\mathcal{H}_\Delta$ only differ by translations. In particular, the combinatorial type and the lengths of the bounded edges remain the same. So given a curve $C \in \mathcal{H}_\Delta$, we have a well defined abstract tropical curve structure for its expanded spine $\Upsilon(C) \in \mathcal{M}_{g,n}^{\text{trop}}$: fix a labelling of the boundary segments of Δ by $[n]$ in a cyclical way and let $L(\Upsilon(C))(k)$ be the vertex incident to the ray corresponding to the segment labelled k. This induces a map

$$\Upsilon : \mathcal{H}_\Delta \to \mathcal{M}_{g,n}^{\text{trop}}.$$

Recall that Ronkin intercepts are the coefficients of the tropical polynomial defining the expanded spine.

Proposition 8.1. *The Ronkin intercepts modulo translations of the graph of the Ronkin function can be taken as global coordinates for $\mathcal{H}_\Delta$.*

Notice that translations of the Ronkin function are the same as translations of the amoeba.

Proof. Since we proved in Theorem 7.21 that composition of the maps in Equation (7.1) is a diffeomorphism, each of the maps themselves are diffeomorphisms. This implies that the bounded Ronkin intercepts can be taken as global coordinates for Harnack curves with fixed boundary.

Now, Theorem 7.14 says that the positions of the amoeba tentacles can be taken as global coordinates for rational Harnack curves, and it is easy to recover the unbounded Ronkin intercepts from the positions of the tentacles as follows. Let ρ_i be the position of a tentacle. It corresponds to a segment in $\partial\Delta$ lying between two lattice points. Let c_i and c_{i+1} be the intercepts corresponding to those points. It is straightfoward that $\rho_i = c_i - c_{i+1}$. This implies that the map

$$\left\{\begin{matrix}\text{Positions of}\\ \text{amoeba}\\ \text{tentacles}\end{matrix}\right\}/\mathbb{R}^2 \to \left\{\begin{matrix}\text{Unbounded}\\ \text{Ronkin}\\ \text{intercepts}/\mathbb{R}^3\end{matrix}\right\}$$

is a linear bijection when restricted to $\left\{\sum_{i=1}^{n} \rho_i = 0\right\}$. □

Proposition 8.2. *The map $\mathcal{H}_\Delta \to \mathcal{M}_{g,n}^{trop}$ is a piecewise linear topological embedding.*

Proof. By Theorem 8.1, we can take the Ronkin intercepts as global coordinates of $\mathcal{H}_\Delta$. The Ronkin intercepts can be recovered from $\Upsilon(C)$ and Δ up to translations of the amoeba. Computing the lengths of the bounded edges of a tropical curve from the tropical polynomial is again solving a system of linear equations. So, over every component $\mathcal{M}_{g,n}^{\mathrm{trop}}((G))$ of the co-domain, the map Υ is linear. □

8.3 Harnack meshes

Definition 8.3. Let $B \subseteq M$ be a finite set of affine dimension 2. We define $\mathcal{H}_B$ the subset of $\mathcal{H}_{\mathrm{conv}(B)}$ consisting of curves C such that for every $v \in \mathrm{conv}(B)_M$ the corresponding component E_v in $\mathbb{R}^2 \setminus \mathcal{A}(C)$ exists if and only if $v \in B$.

This is well defined since the existence of E_v depends only on the equivalence class of a Harnack curve. By Theorem 7.22, $\mathcal{H}_B$ is diffeomorphic to $\mathbb{R}^{|B|-3}$.

Definition 8.4. Consider a regular subdivision $\mathcal{S}$ of Δ with facets $\{B_1, \ldots, B_s\}$ and let $\Delta_i = \mathrm{conv}(B_i)$. A *Harnack mesh over $\mathcal{S}$* is a collection of curves $(C_1, \ldots, C_s)$ with $C_i \in \mathcal{H}_{B_i}$ such that there exists a polynomial f with $f|_{\Delta_i}$ vanishing on C_i. We denote

$$\mathcal{H}_\Delta(\mathcal{S}) \subseteq \prod_{i=1}^{s} \mathcal{H}_{B_i}$$

for the space of all Harnack meshes over $\mathcal{S}$.

Notice that $\mathcal{H}_\Delta$ is equal to the disjoint union of all $\mathcal{H}_\Delta(\mathcal{S})$ where $\mathcal{S}$ has exactly one face, i.e., all $\mathcal{S}$ of the form $\{B\}$ with $\Delta = \mathrm{conv}(B)$.

The existence of such f in the definition above is equivalent to the C_i agreeing on their common boundary. That is, given Δ_i and Δ_j such that $\Gamma = \Delta_i \cap \Delta_j$, the distances between the tentacles of C_i corresponding to Γ are the same to the distances between the respective tentacles in C_j.

Given any Harnack mesh $(C_1, \ldots C_s) \in \mathcal{H}_\Delta(\mathcal{S})$ we can define its expanded spine as an extended tropical curve. Let Υ_i be the expanded spines of C_i. For each edge $\Delta_i \cap \Delta_j$, glue the expanded spines Υ_i and Υ_j by removing the legs corresponding to that edge and placing instead an edge of infinite length for each primitive segment in $\Delta_i \cap \Delta_j$

between the two vertices that were incident to the corresponding leg. The remaining legs are labelled by the boundary segments of Δ. This way we have a map

$$\Upsilon_{\mathcal{S}} : \mathcal{H}_\Delta(\mathcal{S}) \to \overline{\mathcal{M}_{g,n}^{\text{trop}}},$$

which is an embedding, by Theorem 8.1.

Definition 8.5. Let Δ be a lattice polygon. The *compactified moduli space of Harnack curves* is

$$\overline{\mathcal{H}_\Delta} := \bigsqcup_{\mathcal{S}} \mathcal{H}_\Delta(\mathcal{S}),$$

where the union runs over all regular subdivisions $\mathcal{S}$ of Δ_M. We give it the coarsest topology that makes the map

$$\Upsilon : \overline{\mathcal{H}_\Delta} \to \overline{\mathcal{M}_{g,n}^{\text{trop}}}$$

defined by $\Upsilon|_{\mathcal{H}_\Delta(\mathcal{S})} := \Upsilon_{\mathcal{S}}$, continuous.

We will prove that this in fact is a compactification of $\mathcal{H}_\Delta$, i.e. a compact space where $\mathcal{H}_\Delta$ is dense.

Harnack meshes are essentially collections of Harnack curves that can be patchworked into another Harnack curve, except that we allow singularities and non transversal intersection with the axes. We call a subdivision $\mathcal{S}$ of Δ is *full* if it uses all the points, that is $\bigcup_{B_i \in \mathcal{S}} B_i = \Delta_M$. So we can only do true patchworking with them whenever $\mathcal{S}$ full.

Proposition 8.6. *Let $\mathcal{S}$ be a full regular subdivision of Δ with lifting function h and let $(C_1, \ldots, C_s) \in \mathcal{H}_\Delta(\mathcal{S})$ be a Harnack mesh. Then there exist polynomials $f_1, \ldots, f_s$ vanishing on $C_1, \ldots, C_s$ such that the polynomial f_t obtained by patchworking them vanishes on a Harnack curve.*

Proof. $\mathcal{S}$ being full implies that each C_i is smooth and non transversal in the boundary, so that they can be patchworked together. The topological type of C_i can be obtained by patchworking using a unimodular triangulation of Δ_i and the 'Harnack' sign pattern $v \mapsto (-1)^{v_1 v_2}$. By taking a unimodular triangulation of Δ that refines $\mathcal{S}$ and the Harnack sign pattern, the result of patchworking is a Harnack curve. It must have the same topological type as the patchwork using $\mathcal{S}$ and the curves C_i when the polynomials f_i are chosen with the same sign as the polynomial that results from patchworking Δ_i with the Harnack sign configuration. □

Proposition 8.7. *Let $\mathcal{S}$ be a regular subdivision of Δ. Then $\mathcal{H}_\Delta(\mathcal{S})$ is diffeomorphic to $\mathbb{R}^{n+g-\dim(\sigma(\mathcal{S}))}$ where $\sigma(\mathcal{S})$ is the cone in the secondary fan of Δ corresponding to $\mathcal{S}$.*

Proof. First consider the case where $\mathcal{S}$ has a single facet B. By Theorem 7.22, $\mathcal{H}_\Delta(\mathcal{S}) = \mathcal{H}_B \cong \mathbb{R}^{|B|-3}$ and the result follows from $\dim(\sigma(\mathcal{S})) = n + g + 3 - |B|$.

Now let $\mathcal{S}$ be any full regular subdivision with facets $B_1, \ldots, B_s$ and let $\Delta_i = \mathrm{conv}(B_i)$. Let us compute $\dim(\sigma(\mathcal{S}))$. Note that for $h \in \sigma(\mathcal{S})$, fixing h for 3 affinely independent points of B_i fixes h on all of B_i. First, Δ_1 is fixed after fixing 3 affinely independent points. Suppose Δ_2 is adjacent to Δ_1 and let v_1 be any element of $B_2 \setminus B_1$. We have that $h(v_1)$ can take any positive real value. However, after fixing $h(v_1)$, all of $h|_{\Delta_2}$ is determined. Furthermore, if B_i shares sides with both B_1 and B_2 then f_i is also determined.

Let $I_1 \subseteq [s]$ is the minimum set containing 1 and 2 and such that $\bigcup\limits_{i \in I_1} \Delta_i$ is a convex set. Then if h is determined for B_1 and v_1, it is determined for all of $\bigcup\limits_{i \in I_1} B_i$. If $I_1 \neq [s]$, we can repeat last step; choose a vertex v_2 in a facet F_i with $i \neq I_1$ but adjacent to a facet F_j with $j \in I_1$. Further determining the value of h on v_2 determines h on a set of facets indexed by I_2. We can repeat this until $I_d = [s]$. By construction, $d + 3 = \dim(\sigma(\mathcal{S}))$. We can reorder $[s]$ so that without loss of generality we can assume that $I_i = [k_i]$ for some $1 \leq k_i \leq s$. If $\mathcal{S}$ is not full, then

$$\dim(\sigma(\mathcal{S})) = d + 3 + |\Delta_M \backslash \bigcup_{i=1}^{s} B_i|,$$

since the value of h for points in $\Delta_M \backslash \bigcup\limits_{i=1}^{s} B_i$ can be arbitrary as long as it is large enough.

Now let us compute $\mathcal{H}_\Delta(\mathcal{S})$. Recall that the global coordinates for $\mathcal{H}_\Delta$ that were taken in Theorem 7.22 consist of areas of ovals, distances between consecutive parallel tentacles, and the position of the first tentacle of every edge after the second. We start with B_1 and we have that $\mathcal{H}_{B_1} \cong \mathbb{R}^{|B_1|-3}$. The distances between the tentacles corresponding to the edge $\Gamma = \Delta_1 \cap \Delta_2$ are the same for C_1 and C_2. So, these $|\Gamma| - 2$ parameters of $\mathcal{H}_{B_2}$ are fixed if we fix C_1. The subspace of $\mathcal{H}_{b_2}$ of curves C_2 that agree with C_1 on the boundary is isomorphic to $\mathbb{R}^{|B_2|-|\Gamma|-1}$. Let

$$q_i = \left| B_i \cap \left(\bigcup_{j=1}^{i} B_i \right) \right|.$$

Similarly, we have that the subspace of Harnack curves $C_i \in \mathcal{H}_{B_i}$ compatible with $C_1, \ldots C_{i-1}$ is isomorphic to $\mathbb{R}^{|B_i|-q_i-1}$ if Δ_i only shares an edge with one of $\Delta_1, \ldots, \Delta_{i-1}$. If it shares edges with 2 of $\Delta_1, \ldots, \Delta_{i-1}$, then there are $q_i - 3$ distances between tentacles being fixed, so the subspace of Harnack curves $C_i \in \mathcal{H}_{B_i}$ compatible with $C_1, \ldots C_{i-1}$ is isomorphic to $\mathbb{R}^{|B_i|-|\Gamma|}$. The last statement also holds when Δ_i shares edges with 3 or more of $\Delta_1, \ldots, \Delta_{i-1}$, since the position of the first tentacle of each edge after the second gets fixed. By construction, B_i shares only one edge with the previous polygons if and only if $i = 2$ or $i = k_j + 1$, for $1 \leq j < d$. We have that

$$\begin{aligned}\dim(\mathcal{H}_\Delta(\mathcal{S})) &= \sum_{i=1}^{s} |B_i| - 3 - d \\ &= |\bigcup \mathcal{S}| - 3 - d \\ &= n + g - \dim(\sigma(\mathcal{S})).\end{aligned}$$

□

Proposition 8.8. *Let Δ be a lattice polygon and $\mathcal{S}$ be a regular subdivision of Δ_M. Then $\Upsilon_{\mathcal{S}}(\mathcal{H}_\Delta(\mathcal{S})) \subseteq \overline{\Upsilon(\mathcal{H}_\Delta)}$, where $\overline{\Upsilon(\mathcal{H}_\Delta)}$ is the closure of $\Upsilon(\mathcal{H}_\Delta)$ in $\overline{\mathcal{M}_{g,n}^{trop}}$.*

Proof. Suppose $\mathcal{S}$ is a full subdivision of Δ. Choose a height function $h \in \sigma(\mathcal{S})$. By Theorem 8.6, for any Harnack mesh $\mathcal{C} = (C_1, \ldots, C_s)$ in $\mathcal{H}_\Delta(\mathcal{S})$ there exists polynomials $f_1, \ldots, f_s$ and $t_0 > 0$ such that for any $0 < t < t_0$ the curve C_t obtained by patchworking is in $\mathcal{H}_\Delta$. So, we have a path $(0, t_0) \to \mathcal{H}_\Delta$. For each facet B_i of $\mathcal{S}$ there is a family of polynomials $\{f_t^i \mid t \in (0, t_0)\}$ with real coefficients such that f_t^i vanishes on C_t and every coefficient of f_t^i outside $\text{conv}(B_i)$ goes to 0 as t goes to 0. This follows from picking the height function h^i affinely equivalent to h such that points in B_i have height 0 and doing patchworking with h^i. The limit $\lim_{t\to\infty} f_t^i = f_i$ vanishes on C_i.

As t goes to 0, the lengths of the edges of $\Upsilon(C_t)$ that corresponds to the interior of B_i tend to the lengths of the edges of $\Upsilon(C_i)$. Doing this for every i we have that all the finite lengths of $\Upsilon_{\mathcal{S}}(\mathcal{C})$ agree with the lengths of $\lim_{t\to 0} \Upsilon(C_t)$. The edges going to infinity are precisely those dual to primitive segments of $\mathcal{S}$. Then, $\Upsilon(C_t)$ forms a path $(0, t_0) \to \Upsilon(\mathcal{H}_\Delta)$ such that the limit of this path when t goes to 0 is $\Upsilon(\mathcal{C})$. So, $\mathcal{C} \in \overline{\mathcal{H}_\Delta}$.

If $\mathcal{S}$ is not full, let $\mathcal{S}'$ be the subdivision whose facets are $B' = \text{conv}(B)_M$ for each facet $B \in \mathcal{S}$. That is, $\mathcal{S}'$ is the finest full subdivision that coarsens $\mathcal{S}$. It is regular, as we can take a height function a height function in $\sigma(\mathcal{S})$ and linearly extrapolate in each Δ_i to make it full. As the expanded spine is continuous, even when ovals contract, we have that

$$\Upsilon_{\mathcal{S}'}(\mathcal{H}_\Delta(\mathcal{S}')) \subseteq \overline{\Upsilon_{\mathcal{S}}(\mathcal{H}_\Delta(\mathcal{S}))} \subseteq \overline{\mathcal{H}_\Delta}.$$

□

Lemma 8.9. *Let Δ be a lattice polygon. Then $\overline{\Upsilon(\mathcal{H}_\Delta)} = \Upsilon(\overline{\mathcal{H}_\Delta})$.*

Proof. Proposition 8.8 implies that $\Upsilon(\overline{\mathcal{H}_\Delta}) \subset \overline{\Upsilon(\mathcal{H}_\Delta)}$.

For the other containment, let $C_1, C_2, \dots$ be a sequence of curves in $\mathcal{H}_\Delta$ such that their expanded spines converge to a point in $G \in \overline{\mathcal{M}_{g,n}^{\text{trop}}}$. We call connected components of G the components obtained by deleting from G all edges of infinite length. Notice that vertices of an expanded spine correspond to polygons inside Δ given by the regular subdivision dual to the expanded spine. This association is carried on to the limit, so the connected components $G_1, \dots, G_s$ induce a regular subdivision $\mathcal{S} = \{B_1, \dots, B_s\}$ of Δ. For a connected component G_i of G, we can choose polynomials $f_1^i, f_2^i, \dots$ vanishing on $C_1, C_2 \dots$ such that they converge to a polynomial f^i which vanishes on a curve whose expanded spine is G_i. This can be done, for example, by picking a vertex of G_i and fixing it to be in the origin, i.e., translating the amoebas of $C_1, C_2 \dots$ so that the corresponding vertex in the expanded spine is always at the origin. Since the limit of Harnack curves is a Harnack curve (see [MR01, Remark 2]), f^i vanishes on a Harnack curve $C^i \in \mathcal{H}_{B_i}$. The collection $\mathcal{C} = (C_1, \dots, C_s)$ is a Harnack mesh over $\mathcal{S}$ and we have that $\Upsilon(\mathcal{C}) = G$. □

Corollary 8.10. *Let Δ be a lattice polygon. Then,*

$$\overline{\mathcal{H}_\Delta(\mathcal{S})} = \bigcup_{\mathcal{T} \leq \mathcal{S}} \mathcal{H}_\Delta(\mathcal{T}),$$

where the union runs over all subdivisions $\mathcal{T}$ of Δ that refine $\mathcal{S}$.

Theorem 8.11. *Let Δ be a lattice polygon. The stratification of $\overline{\mathcal{H}_\Delta}$ by $\mathcal{H}_\Delta(\mathcal{S})$ is a cell complex with a poset isomorphic to the face poset of the secondary polytope* $\mathrm{Sec}(\Delta_M)$ *given by its faces.*

Proof. The faces of $\mathrm{Sec}(\Delta_M)$ are in correspondance with regular subdivisions. By Theorem 8.7, $\mathcal{H}_\Delta(\mathcal{S})$ has the same dimension as the face of $\mathrm{Sec}(\Delta_M)$ corresponding to $\mathcal{S}$. By Theorem 8.10, the boundary of $\overline{\mathcal{H}_\Delta(\mathcal{S})}$ consists of $\mathcal{H}_\Delta(\mathcal{T})$ for every subdivision $\mathcal{T}$ that refines $\mathcal{S}$. Similarly, the faces contained in the face of $\mathrm{Sec}(\Delta_M)$ corresponding to $\mathcal{S}$ are those corresponding to refinements of $\mathcal{S}$.

□

Example 8.12. Let $\Delta := \mathrm{conv}((1,0),(0,1),(-1,0),(0,-1))$. We have that $\mathrm{Sec}(\Delta)$ is a triangle. Figure 8.1 shows the space $\overline{\mathcal{H}_\Delta}$ together with the subdivisions of the corresponding face in $\mathrm{Sec}(\Delta)$ and the amoebas of the corresponding Harnack meshes. The horizontal coordinate represents the relative position of the tentacles. This is parametrized, for example, by $\rho_1 + \rho_3$. Going to the left stretches the amoeba vertically while going to the right stretches it horizontally. The vertical coordinate corresponds to the area of the oval, where going downwards decreases the area while going upwards increases it. The bottom open segment corresponds to $\mathcal{H}_{0,\Delta}$, and that segment together with the interior face corresponds to $\mathcal{H}_\Delta \cong \mathbb{R} \times \mathbb{R}_{\geq 0}$.

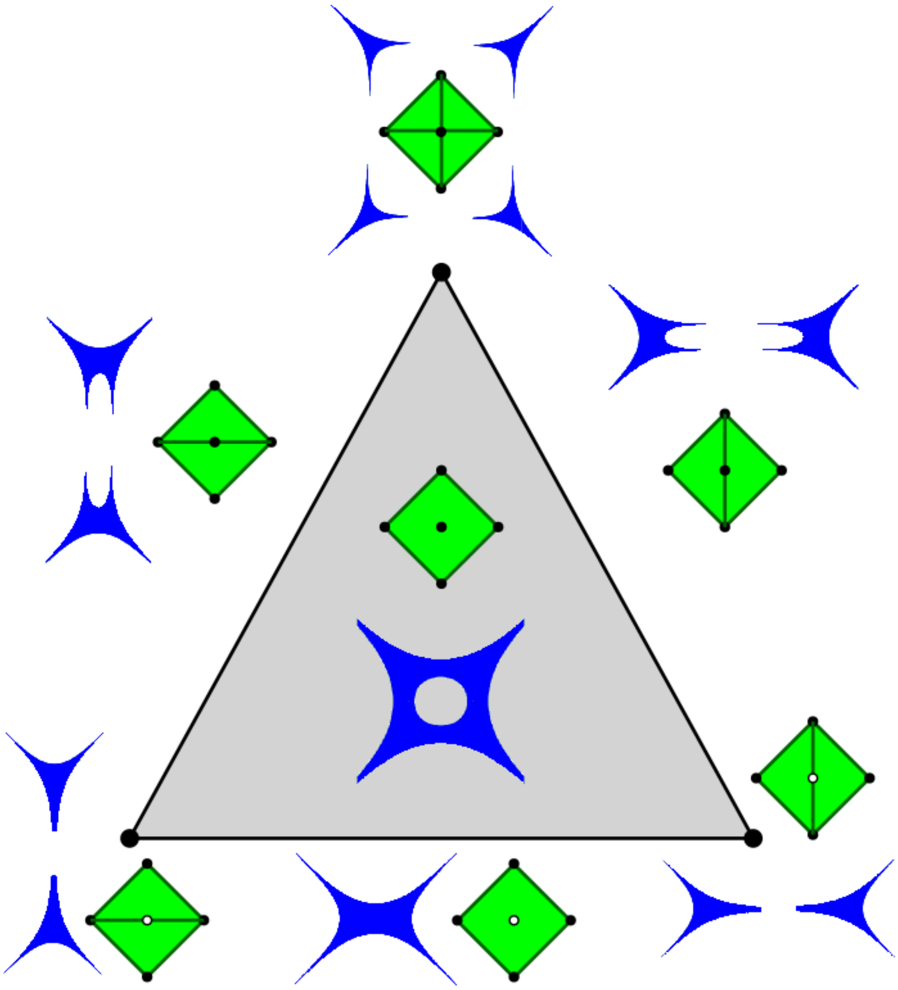


FIGURE 8.1: $\overline{\mathcal{H}_\Delta}$ for $\Delta = \mathrm{conv}((1,0),(0,1),(-1,0),(0,-1))$
.

8.4 Questions and future directions

8.4.1 $\overline{\mathcal{H}_\Delta}$ as a CW-complex

We begin by suggesting the following strengthening of Theorem 8.11:

Conjecture 8.13. *The cell decomposition of the compactified moduli space* $\overline{\mathcal{H}_\Delta} = \bigsqcup_{\mathcal{S}} \mathcal{H}_\Delta(\mathcal{S})$ *is a regular CW-complex.*

Two regular CW-complexes with isomorphic cell posets are isomorphic, that is, there is a homeomorphism that maps each cell to the corresponding cell given by the poset isomorphism. By Theorem 8.11, if Theorem 8.13 hold, $\overline{\mathcal{H}_\Delta}$ would be isomorphic to

$\mathrm{Sec}(\Delta_M)$. To show that $\overline{\mathcal{H}_\Delta}$ is a regular CW complex it is enough to show that for any regular subdivision $\mathcal{S}$ of Δ, $\Upsilon_{\mathcal{S}}(\mathcal{H}_\Delta(\mathcal{S}))$ is a closed ball. Moreover, it is enough to prove the following:

Conjecture 8.14. *Let $\mathcal{C} \in \mathcal{H}_\Delta(\mathcal{S})$ be a Harnack mesh and $\mathcal{S}'$ be a coarsening of $\mathcal{S}$. Then there is a neighborhood of $\mathcal{C}$ in $\overline{\mathcal{H}_\Delta(\mathcal{S}')}$ homeomorphic to a half space of dimension* $\dim(\mathcal{H}_\Delta(\mathcal{S}'))$.

Since the poset of $\overline{\mathcal{H}_\Delta}$ is Eulerian by Theorem 8.11, Theorem 8.14 implies that the closure of the cells of $\overline{\mathcal{H}_\Delta}$ are closed balls by (a reformulation of) Pointcaré's conjecture. This argument was recently used by Galashin Lam and Karp in order to prove that the positroid stratification of the totally non-negative Grassmannian is a CW-complex [GKL19]. It is worth remarking that Harnack curves enjoy several similarities with the total positivity phenomenon (see, for example, [KOS06, Section 5.2] or our proof of Theorem 7.19). In the next subsection we will see that Theorem 8.14 holds when $\mathcal{S}$ is full.

8.4.2 $\overline{\mathcal{H}_\Delta}$ as a manifold with generalized corners.

The above discussion suggests to study topological charts in $\mathcal{H}_\Delta$. We can be more ambitious and try to endow $\mathcal{H}_\Delta$ with a smooth structure. Theorem 7.22 is already a description of $\mathcal{H}_\Delta$ as a smooth manifold with corners. A natural question is whether we can extend this smooth structure to $\overline{\mathcal{H}_\Delta}$. A desirable trait of such a smooth structure (besides being compatible with the chart given by Theorem 7.22) is that the cell complex structure from Theorem 8.11 can be recovered from it. However, secondary polytopes are not always simple polytopes and manifolds with corners lack the capacity to describe non-simple vertices. To mend this, we turn our attention to a wider category, namely that of manifolds with *generalized corners*, or *gc-manifolds*, as defined in [Joy16].

Definition 8.15 ([Joy16])**.** A *g-chart* of a topological space X is a triple $(\phi, \mathbb{L}, U)$ such that:

- $\mathbb{L}$ is a weakly toric monoid, i.e. a semi-lattice of the form $\mathbb{L} = \mathbb{Z}^s \cap \sigma$ where s is a positive integer and $\sigma \subseteq \mathbb{R}^s$ is a rational polyhedral cone.
- U is an open subset of $\mathrm{Hom}(\mathbb{L}, \mathbb{R}_{\geq 0})$, i.e. the space of monoid morphisms from $\mathbb{L}$ to the monoid $(\mathbb{R}_{\geq 0}, \cdot)$ with the weakest topology that makes evaluation on a point $q \in \mathbb{L}$ continuous.
- $\phi : U \to X$ is a topological embedding to an open subset $\phi(U) \subseteq X$.

We call X a gc-manifold if it has a *g-atlas*, that is, a collection of g-charts covering X and satisfying certain compatibility conditions on the transition functions. These conditions depend on the monoids, but we restrain from explaining them in detail in this paper to avoid overextending us. Easy examples of gc-manifolds are $\mathrm{Hom}(\mathbb{N}, \mathbb{R}_{\geq 0}) \cong \mathbb{R}_{\geq 0}$ and $\mathrm{Hom}(\mathbb{Z}, \mathbb{R}_{\geq 0}) \cong \mathbb{R}$. So as a gc-manifold, $\mathcal{H}_\Delta \cong \mathrm{Hom}(\mathbb{N}^{m-3} \times \mathbb{Z}^{n+g-m}, \mathbb{R}_{\geq 0})$ by Theorem 7.22.

Proposition 8.16. *Let $\mathcal{C} \in \mathcal{H}_\Delta(\mathcal{S})$ be a Harnack mesh where $\mathcal{S}$ is a full subdivision. Then there exists a g-chart around $\mathcal{C}$. Moreover, all charts provided this way are compatible with each other.*

Proof. Since $(\mathbb{R} \cup \{\infty\}, +)$ is isomorphic as a monoid to $(\mathbb{R}_{\geq 0}, \cdot)$ with $x \mapsto e^{-x}$ as isomorphism, g-charts can be equivalently defined to be homeomorphisms from open subsets of 'affine tropical toric varieties', i.e. from open subsets $U \subseteq \mathrm{Hom}(\mathbb{L}, \mathbb{R} \cup \{\infty\})$. Consider a graph G embedded in $\mathbb{R}^2$ that is dual to the subdivision $\mathcal{S}$. In particular, the edges of G have a prescribed slope. The lengths of these edges satisfy linear equations with integer coefficients given by the circuits of G (two for each circuit). These equations are binomial relations under the isomorphism $(\mathbb{R} \cup \{\infty\}, +) \cong (\mathbb{R}_{\geq 0}, \cdot)$. Thus, the edges of G (which correspond to edges of $\mathcal{S}$ in the interior of Δ) generate a toric monoid $\mathbb{L}_\mathcal{S}$ under these relations.

Let $\mathcal{C}'$ be a Harnack mesh close enough to $\mathcal{C}$. The spine $\Upsilon(\mathcal{C}')$ has a subgraph G_i which is very close to $\Upsilon(C_i)$ for each curve $C_i \in \mathcal{C}$. These subgraphs are glued together with edges of very large (possibly infinite) length. Contracting these subgraphs results in the graph dual to $\mathcal{S}$, so the distances between these graphs induce a homomorphism $\phi_{\mathcal{C}'} : \mathbb{L}_\mathcal{S} \to \mathbb{R} \cup \{\infty\}$.

The coordinates of a Harnack mesh in $\mathcal{H}_\Delta(\mathcal{S})$ encode the same information as the spines of each curve in the Harnack mesh. Since $\mathcal{C}'$ is close enough to $\mathcal{C}$ there exists a mesh $\mathcal{H}_\Delta(\mathcal{S})$ that has a curve whose spine is isomorphic as metric graphs to G_i for each i (here we use that $\mathcal{S}$ is full). The coordinates of this mesh in $\mathcal{H}_\Delta(\mathcal{S})$ is a vector in $\mathbb{R}^d$, where $d = n + g - \dim(\sigma(\mathcal{S}))$, by Theorem 8.7. This vector induces a homomorphism $\psi_{\mathcal{C}'} : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}$.

The Harnack mesh $\mathcal{C}'$ is completely determined by $\phi_{\mathcal{C}'}$ and $\psi_{\mathcal{C}'}$, so we obtain an embedding from a neighborhood of $\mathcal{C}$ in $\overline{\mathcal{H}_\Delta}$ to $\mathrm{Hom}(\mathbb{L}_\mathcal{S} \times \mathbb{Z}^d, \mathbb{R}_{\geq 0})$ given by

$$\mathcal{C}' \mapsto \Big((x, y) \mapsto e^{-\phi_{\mathcal{C}'}(x)} \psi_{\mathcal{C}'}(y)\Big)$$

where $x \in \mathbb{L}_\mathcal{S}$ and $y \in \mathbb{Z}^d$. Since $\mathrm{Hom}(\mathbb{L}_\mathcal{S} \times \mathbb{Z}^d, \mathbb{R}_{\geq 0})$ is of the same dimension as $\overline{\mathcal{H}_\Delta}$, this mapping forms a g-chart.

That this g-chart is compatible with the g-chart of $\mathcal{H}_\Delta$ given by Theorem 7.22 is a consequence of Theorem 8.1. Similarly, charts constructed this way are compatible with each other.

□

Corollary 8.17. *Theorem 8.14 holds when $\mathcal{S}$ is full.*

Proof. For any $\mathbb{L}$, the space $\mathrm{Hom}(\mathbb{L}, \mathbb{R}_{\geq 0})$ is stratified by its support. All of the strata are again of the form $\mathrm{Hom}(\mathbb{L}', \mathbb{R}_{\geq 0})$ for some submonoid $\mathbb{L}' \leq \mathbb{L}$ and are topological manifolds with boundary. The g-charts constructed above respects the cell strata of $\mathrm{Hom}(\mathbb{L}_\mathcal{S} \times \mathbb{Z}^d, \mathbb{R}_{\geq 0})$ and $\overline{\mathcal{H}_\Delta}$, so the result follows. □

Unfortunately, we do not know of a good way of constructing g-charts for points in cells corresponding to non-full subdivisions. Length of edges is not a good parameter for the chart, since the edges of cycles in Γ that correspond to ovals contracting to a point are finite (that is, positive after applying $x \mapsto e^{-x}$) so the preimage in such chart would not be open. One could expect a g-chart covering $\mathcal{H}_\Delta(\mathcal{S})$ for a non-full subdivision $\mathcal{S}$ to be defined over an open subset of $\mathrm{Hom}(\mathbb{L}_\mathcal{S} \times \mathbb{Z}^d \times \mathbb{N}^k, \mathbb{R}_{\geq 0})$ where k is the number of missing points of $\mathcal{S}$. However, it is not clear what the coordinates corresponding to the copies of $\mathbb{N}$ should be.

For example, using the area of ovals as coordinates, as we did in Theorem 7.22, does not work either. Consider the bottom right corner of Theorem 8.12. If we take a continuous path along the interior of the triangle by stretching the amoeba horizontally but maintaining the area of the bounded component is constant. Since the square bounded by the expanded spine is contained in the union of the amoeba with the bounded component of the complement, its area is bounded. Stretching the amoeba horizontally causes the length of the vertical edges of the square to tend to 0, which means that the path ends in the bottom right corner. This coordinate should be 0 at this point, so the continuity is broken.

Question 8.18. *Is there a natural way of completing a g-atlas on $\overline{\mathcal{H}_\Delta}$ with g-charts respecting the cell strata?*

A positive answer to this question implies a positive answer to the conjectures in Section 8.4.1.

8.4.3 A cell complex for T-curves.

Harnack meshes can also be patchworked into non-Harnack curves by choosing polynomials with different sign patterns. The resulting curves are called *T-curves.* They can be thought of as the 'neighborhood' of $\overline{\mathcal{H}_\Delta}$, which suggests the following question.

Question 8.19. *Given a lattice polygon Δ, are there other topological types of curves in X_Δ such that their moduli space can be given a cell complex structure similar to $\overline{\mathcal{H}_\Delta}$? Can such moduli spaces be glued together to form a larger cell complex, or even a polytopal complex, where cells correspond to different topological types?*

Example 8.20. When Δ is the unit square, $\overline{\mathcal{H}_\Delta}$ is a segment. When the Harnack meshes of the extremes are patchworked in a non Harnack way, we get a curve whose amoeba has a pinching (see [Mik00, Example 1]). From one of the extremes the resulting expanded spine has a bounded edge parallel to $\{x_1 = x_2\}$ and from the other extreme the edge is parallel to $\{x_1 = -x_2\}$. When the length of the bounded edge goes to 0, both cases degenerate to a reducible curve (the union of two axis-parallel lines). In this case the complex of Question 8.19 exists and it is isomorphic to the boundary of a triangle.

Declaration of Authorship

I, Jorge Alberto Olarte, declare that this thesis titled, “Polytopal subdivisions in Grassmannians, tropical geometry and algebraic curves” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ______________________________

Date: ______________________________

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