COMBINATORIAL AND GEOMETRIC ASPECTS OF FEYNMAN GRAPHS AND FEYNMAN INTEGRALS

DISSERTATION

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PREFACE

The integrals associated to Feynman graphs must have been a source of frustration for particle physicists ever since. Indeed there is a delicate difference between being able to draw a Feynman graph and being able to compute the associated Feynman integral. Although perturbation theory has brought enormous breakthroughs, many physicists turned to more abstract developments in quantum field theory, looked for other ways to produce perturbational results, or left the field entirely. Nonetheless there is a significant number of physicists, computational and theoretical, who pursue the quest for concepts and algorithms to compute and understand those integrals to higher and higher orders. Their motivation is to help test the validity of the underlying physical theory.

For a mathematician, Feynman graphs and their integrals provide a rich subject in their own right, independent of their computability. It was only recently though that the work of Bloch, Esnault and Kreimer has brought a growing interest of mathematicians from various disciplines to the subject. In fact it opened up a completely new direction of research: a motivic interpretation of Feynman graphs that unites their combinatorial, geometric and arithmetic aspects. This idea had been in the air for a while, based on computational results of Broadhurst and Kreimer, and on a theorem of Belkale and Brosnan related to a conjecture of Kontsevich about the generality of the underlying motives.

A prerequisite for the motivic approach is a profound understanding of renormalization that was established less recently in a modern language by Connes and Kreimer. This dissertation studies the renormalization of Feynman graphs in position space using an adapted resolution of singularities, and makes two other contributions of mostly combinatorial nature to the subject. I hope this may serve as a reference for somebody who feels comfortable with the traditional position space literature and looks for a transition to the research of Bloch and Kreimer.

I want to thank D. Kreimer for sharing his wealth of experience and ideas with me and E. Vogt for helping me make things more precise; and both for dedicating a great deal of time and effort to advising this research. I am grateful to R. Brunetti for pointing me to a reference that was crucial for my understanding, and for his collaboration. I also thank R. Schader for his support over many years. While carrying out this research, I benefitted from discussions with R. Brunetti, K. Fredenhagen, F. Brown, S. Bloch, H. Esnault, S. Rosenberg, F. Vignes-Tourneret, T. Gracey, H. Hauser, A. Rej,

M. Marcolli, H. Gangl, and S. Müller-Stach. My research was funded by the DFG. During the past three years I was able to visit several times the Institut des Hautes Études Scientifiques in Bures-sur-Yvette, the Erwin-Schrödinger Institut in Vienna, the Boston University Center for Mathematical Physics, the Max-Planck-Institut für Mathematik in Bonn, the Fields Institute in Toronto, the University of Mainz, the University of Trento, and the DESY in Hamburg. I am grateful to the people who invited me.

CONTENTS

1. INTRODUCTION

1.1. **Introduction for mathematicians.** A *Feynman graph* Γ is a finite graph[1] with set of vertices $V(\Gamma)$ and set of edges $E(\Gamma)$. Associated to a Feynman graph Γ there are several types of "*Feynman integrals*" which do not necessarily converge right away. It is therefore better at this stage to think of an "integral" $\int_A u(x)dx$ as a pair (A, u) consisting of a subset A of some $\mathbb{R}^n$, and a distribution u on A minus certain subsets of positive codimension. This helps us postpone the question of convergence of the integrals for a while.

Three important examples of Feynman integrals associated to Γ are

$$I_{\rm pos}(\Gamma) = \int_{R^{4|V(\Gamma)|}} \prod_{i<j} \frac{1}{(x_i - x_j)^{2n_{ij}}} dx \tag{1}$$

where for each $i \in V(\Gamma)$ the x_i is a vector in $\mathbb{R}^4$, and n_{ij} is the number of edges between i and j;

$$I_{\rm mom}(\Gamma) = \int_{R^{4|E(\Gamma)|}} \prod_{v\in V(\Gamma)} \delta_0\left(\sum_{v\in\partial e} \pm k_e\right) \prod_{e\in E(\Gamma)} \frac{dk_e}{k_e^2}, \tag{2}$$

where the k_e are again vectors in $\mathbb{R}^4$, the sign $\pm$ depends on an orientation of the edges, and δ_0 is the Dirac measure; and

$$I_{\rm param}(\Gamma) = \int_{R_{\geq 0}^{|E(\Gamma)|}} \frac{da}{\Psi_\Gamma^2(a)} \tag{3}$$

where Ψ_Γ is the (Kirchhoff) graph polynomial associated to Γ. All of these notions will be properly defined later.

Such pairs and the question of their integrability are at the origin of our studies. The maps $\Gamma \to I_{\rm pos}(\Gamma)$, $I_{\rm mom}(\Gamma)$, $I_{\rm param}(\Gamma)$ may be called position space, momentum space, and parametric *Feynman rules*, respectively. The three Feynman integrals introduced above are related. For example the first one is in a way the Fourier transform of the second one, and the third gotten from the second using the trick $\frac{1}{k^2} = \int_0^\infty e^{-ak^2} da$. Depending on what one is after, each of the three integral representations will have advantages and disadvantages.

A map transforming pairs (A, u) into *convergent* integrals $\int_A u_R(x)dx$ is

[1] where multiple (parallel) edges between the same pair of vertices are allowed. Some call this a *multigraph*.

called a *renormalization*, if it satisfies certain consistency conditions inspired from physics.

Indeed, the Feynman integrals arise in the quantum field theoretic description of elementary particle physics as single terms of the perturbative expansion of an interacting field theory in a neighborhood of a free field theory. The interacting theory determines a special class of Feynman graphs, for example graphs with vertices of fixed degree k. Intuitively, the edges of a graph describe particles, and vertices describe interactions between particles. Very roughly, the infinite sum of all such renormalized Feynman integrals, once made convergent in a way to be defined, is related to the outcome of scattering experiments in high energy physics.

1.2. **Introduction for mathematical physicists.** The subject of perturbative renormalization in four-dimensional interacting quantum field theories looks back to a successful history. Thanks to the achievements of Bogoliubov, Hepp, Zimmermann, Epstein, Glaser, 't Hooft, Veltman, Polchinski, Wilson – to mention just some of the most prominent contributors –, the concept seems in principle well-understood; and the predictions made using the renormalized perturbative expansion match the physics observed in the accelerators with tremendous accuracy. However, several decades later, our understanding of realistic interacting quantum field theories is still everything but satisfying. Not only is it extremely difficult to perform computations beyond the very lowest orders, but also the transition to a non-perturbative framework and the incorporation of gravity pose enormous conceptual challenges.

Over the past fifteen years, progress has been made, among others, in the following three directions. In the algebraic approach to quantum field theory, perturbation theory was generalized to generic (curved) space-times by Brunetti and Fredenhagen [23], see also [47]. On the other hand, Connes and Kreimer introduced infinite-dimensional Hopf- and Lie algebras [28, 58] providing a deeper conceptual understanding of the combinatorial and algebraic aspects of renormalization, also beyond perturbation theory. More recently, a conjecture concerning the appearance of a very special class of periods [4, 19, 20] in all Feynman integrals computed so far, has initiated a new area of research [14, 15, 17] which studies the perturbative expansion from a motivic point of view. The main purpose of this dissertation is to contribute to the three approaches mentioned, by giving a new description of perturbative renormalization of short-distance divergences using a resolution of singularities. For future applications to curved spacetimes it is most appropriate to do this in the position space framework of Epstein and

Glaser [23, 35]. However the combinatorial features of the resolution allow for a convenient transition to the momentum space picture of the Connes-Kreimer Hopf algebras, and to the residues of [14, 15] in the parametric representation. Both notions are not immediately obvious in the original Epstein-Glaser literature.

1.3. **Basic ideas.** Let us present some of the basic ideas in a nutshell. Consider, in euclidean space-time $M = \mathbb{R}^4$, the following Feynman graph

$\Gamma =$ 

.

The Feynman rules, in position space, associate to Γ a distribution

$$u_\Gamma(x_1, x_2) = u_0^2(x_1 - x_2).$$

where $u_0(x)$ is the Feynman propagator, in the massless case $u_0(x) = 1/x^2$, the x are 4-vectors with coordinates $x^0, \ldots, x^3$, and x^2 the euclidean square $x^2 = (x^0)^2 + \ldots + (x^3)^2$. Note that since u_Γ depends only on the difference vector $x_1 - x_2$, we may equally well consider $\underline{u}_\Gamma(x) = u_\Gamma(x, 0)$. Because of the singular nature of u_0 at $x = 0$, the distribution u_Γ is only well-defined outside of the diagonal $D_{12} = \{x_1 = x_2\} \subset M^2$. In order to extend u_Γ from being a distribution on $M^2 - D_{12}$ onto all of M^2, one can introduce an analytic regularization, say

$$\underline{u}_\Gamma^s(x) = u_0^{2s}(x).$$

Viewing this as a Laurent series in s, we find, in this simple case,

$$\underline{u}_\Gamma^s(x) \;=\; \frac{1}{x^{4s}} = \frac{c\delta_0(x)}{s-1} + R_s(x)$$

with $c \in \mathbb{R}$, δ_0 the Dirac measure at 0, and $s \mapsto R_s$ a distribution-valued function holomorphic in a complex neighborhood of $s = 1$, the important point being that the distribution R_s is defined *everywhere* on M^2. The usual way of renormalizing $\underline{u}_\Gamma$ is to subtract from it a distribution which is equally singular at $x = 0$ and cancels the pole, for example

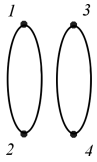
$$\underline{u}_{\Gamma,R} = \left(u_\Gamma^s - u_\Gamma^s[w_0]\delta_0\right)\big|_{s=1}.$$

Here w_0 is any test function which satisfies $w_0(0) = 1$ for then $\frac{\delta_0}{s-1}[w_0] = \frac{1}{s-1}$. Consequently

$$\underline{u}_{\Gamma,R} = R_1 - R_1[w_0]\delta_0$$

which is well-defined also at 0. The distribution $\underline{u}_{\Gamma,R}$ is considered the solution to the renormalization problem for Γ, and different choices of w_0 give

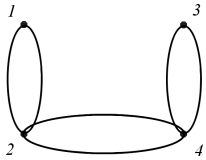
rise to the renormalization group. Once the graph Γ is renormalized, there is a canonical way to renormalize the graph

$\Gamma' =$ 

which is simply a disjoint union of two copies of Γ. Indeed,

$$u_{\Gamma'}(x_1, x_2, x_3, x_4) = u_0^2(x_1 - x_2)u_0^2(x_3 - x_4) = \underline{u}_\Gamma \otimes \underline{u}_\Gamma)(x_1 - x_2, x_3 - x_4).$$

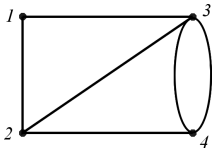
In other words, $u_{\Gamma'}$ is a cartesian product, and one simply renormalizes each factor of it separately: $(u_{\Gamma',R})(x_1, \ldots, x_4) = \underline{u}_{\Gamma,R}^{\otimes 2}(x_1 - x_2, x_3 - x_4)$. This works not only for disconnected graphs but for instance also for

$\Gamma'' =$ 

which is connected but (one-vertex-) reducible, to be defined later. Indeed,

$$\begin{aligned} u_{\Gamma''}(x_1, x_2, x_3, x_4) &= u_0^2(x_1 - x_2)u_0^2(x_2 - x_4)u_0^2(x_3 - x_4) \\ &= \underline{u}_\Gamma^{\otimes 3}(x_1 - x_2, x_2 - x_4, x_3 - x_4) \end{aligned}$$

Again, one simply renormalizes every factor of $u_{\Gamma''}$ on its respective diagonal. This is possible because the diagonals D_{12}, D_{24} and D_{34} are pairwise perpendicular in M^4. Consider now a graph which is not of this kind:

$\Gamma''' =$ 

$$u_{\Gamma'''}(x_1, \ldots, x_4) = u_0(x_1 - x_2)u_0(x_1 - x_3)u_0(x_2 - x_3)u_0(x_2 - x_4)u_0^2(x_3 - x_4).$$

By the usual power counting we see that $u_{\Gamma'''}$ has non-integrable singularities at $D_{34} = \{x_3 = x_4\}$, at $D_{234} = \{x_2 = x_3 = x_4\}$ and at $D_{1234} = \{x_1 = x_2 = x_3 = x_4\}$. These three linear subspaces of M^4 are nested ($D_{1234} \subset D_{234} \subset D_{34}$) instead of pairwise perpendicular. In the geometry of M^4 it does not seem possible to perform the three necessary subtractions separately and independently one of another. For if a test function has support on some of say D_{1234}, its support intersects also $D_{234} - D_{1234}$ and $D_{34} - D_{234}$. This is one of the reasons why much literature on renormalization is based on recursive or step-by-step methods. If one instead transforms M^4 to another smooth manifold $\beta : Y \to M^4$ such that the preimages under β of the three linear spaces $D_{34}, D_{234}, D_{1234}$ look locally like cartesian coordinate hyperplanes $y_1 y_2 y_3 = 0$, one can again perform

the three renormalizations separately, and project the result back down to M^4. For this procedure there is no recursive recipe needed – the geometry of Y encodes all the combinatorial information. The result is the same as from the Epstein-Glaser, BPHZ or Hopf algebra methods, and much of our approach just a careful geometric rediscovery of existing ideas.

1.4. **Main results and a short outline.** The material presented in this dissertation corresponds to a subset of the contents of the three papers [6, 9, 10] (the first of which is about to be finished , the second is published, and the third is unfinished) where sections written by the coauthors are omitted. The material presented in this dissertation is my own research and worked out by myself, ideas contributed by others than the advisors are acknowledged as such in the text. The dissertation is divided into the first sections (2-6), which study renormalization and resolution of singularities, and two further sections (7 and 8) containing related research about non-perturbative Dyson-Schwinger equations and graph polynomials, in the momentum space and parametric representation.

(1) In the first sections (2-6), which correspond to the paper [6], I define two subspace arrangements associated to a Feynman graph. I describe a variety of adapted resolutions of singularities for those arrangements, using a more general construction of De Concini and Procesi. On these smooth models, I study the pullback of the Feynman integrand and show how to renormalize the Feynman integral, according to the physical principle of locality. The proofs use an analytic regularization, which was kindly proposed by R. Brunetti who also showed me the reference [2].
(2) In section 7, see also [9], I show the correspondence between solutions of certain combinatorial Dyson-Schwinger equations and Hopf subalgebras of the Connes-Kreimer Hopf algebra of rooted trees, generalizing an earlier result of Kreimer. The proof uses only the Hochschild cocycles of that Hopf algebra.
(3) I give an expression for the Kirchhoff graph polynomial as graphs are inserted one into another (section 8). This is a more abstract version of a result that I contributed, among other things, to a joint project with A. Rej [10], answering a question of M. Marcolli.

It follows a more detailed outline of 2-6. In section 2 the two subspace arrangements associated to a Feynman graph are defined, describing the locus of singularities, and the locus of non-integrable singularities, respectively. In section 3 an analytic regularization for the propagator is introduced. Some necessary technical prerequisites for dealing with distributions and birational transformations are made, and the important notion of residue density for a primitive graph is defined. The rest of the paper is devoted to a

more systematic development. Section 4 describes the De Concini-Procesi "wonderful" models for the subspace arrangements and provides an explicit atlas and stratification for them in terms of nested sets. Different models are obtained by varying the so-called building set, and we are especially interested in the minimal and maximal building set/model in this class. Section 5 examines the pullback of the regularized Feynman distribution onto the smooth model and studies relations between its Laurent coefficients wrt. the regulator. In section 6 it is shown that the proposed renormalization on the smooth model satisfies the physical constraint of locality: the subtractions made can be packaged as local counterterms into the Lagrangian. For the model constructed from the minimal building set, this is satisfied by construction. From the geometric features of the smooth models one arrives quickly at an analogy with the Hopf algebras of Feynman graphs, and a section relating the two approaches concludes the exposition. As a technical simplication in the main part of the paper only massless scalar euclidean theories are considered, and only Feynman graphs with at most logarithmic singularities. The general case is briefly discussed in section 6.4.

This research is motivated by a careful analysis of Atiyah's paper [2] – see also [13]; and [5] for a first application to Feynman integrals in the parametric representation – the similarity of the Fulton-MacPherson stratification with the Hopf algebras of perturbative renormalization observed in [11, 61], and recent results on residues of primitive graphs and periods of mixed Hodge structures [14, 17]. Kontsevich has pointed out the relevance of the Fulton-MacPherson compactification for renormalization long ago [52], and a real (spherical) version had been independently developed by him (and again independently by Axelrod and Singer [3]) in the context of Chern-Simons theory, see for example [54]. In the parametric representation, many related results have been obtained independently in the recent paper [15], which provides also a description of renormalization in terms of limiting mixed Hodge structures. That is beyond our scope.

2. Subspace arrangements associated to Feynman graphs

Let $U \subseteq \mathbb{R}^k$ be an open set. By $\mathcal{D}(U)$ we denote the space of test functions with compact support in U, with the usual topology. $\mathcal{D}'(U)$ is the space of distributions in U. See [49] for a general reference on distributions. We work in Euclidean spacetime $M = \mathbb{R}^d$ where $d \in 2 + 2\mathbb{N} = \{4, 6, 8, \ldots\}$ and use the (massless) propagator distribution

$$u_0(x) = \frac{1}{x^{d-2}} = \frac{1}{((x^0)^2 + \ldots + (x^{d-1})^2)^{\frac{d-2}{2}}} \tag{4}$$

which has the properties

$$u_0(\lambda x) = \lambda^{2-d} u_0(x), \quad \lambda \in \mathbb{R} \setminus \{0\} \tag{5}$$

and

$$\operatorname{sing\,supp} u_0 = \{0\}. \tag{6}$$

The singular support of a distribution u is the set of points having no open neighborhood where u is given by a smooth function.

Let now Γ be a Feynman graph, that is a finite graph, with set of vertices $V(\Gamma)$ and set of edges $E(\Gamma)$. We assume that Γ has no loops (a loop is an edge that connects to one and the same vertex at both ends). The Feynman distribution is given by the distribution

$$u_\Gamma(x_1, \ldots, x_n) = \prod_{i<j} u_0(x_i - x_j)^{n_{ij}} \tag{7}$$

on $M^n \setminus \cup_{i<j} D_{ij}$ where D_{ij} is the diagonal defined by $x_i = x_j$ and n_{ij} is the number of edges between the vertices i and j (For this equation we assume that the vertices are numbered $V(\Gamma) = \{1, \ldots, n\}$). A basic observation is that u_Γ may be rewritten as the restriction of the distribution $u_0^{\otimes |E(\Gamma)|} \in \mathcal{D}'(M^{|E(\Gamma)|})$ to the complement of a subspace arrangement, contained in $M^{|E(\Gamma)|}$, as follows.

2.1. **Configurations and subspace arrangements of singularities.** It is convenient to adopt a more abstract point of view as in [14]. Let k be an infinite field, E a finite set and k^E the k-vector space spanned by E. An inclusion of a linear subspace $i_W : W \hookrightarrow k^E$ is called a *configuration*. Since k^E comes with a canonical basis, a configuration defines an arrangement of up to $|E|$ linear hyperplanes in W : namely for each $e \in E$ the subspace annihilated by the linear form $e^\vee i_W$, unless this linear form equals zero. Note that different basis vectors $e \in E$ may give one and the same hyperplane.

Given a connected graph Γ, temporarily impose an orientation of the edges (all results will be independent of this orientation). This defines for a vertex $v \in V(\Gamma)$ and an edge $e \in E(\Gamma)$ the integer $(v : e) = \pm 1$ if v is the final/initial vertex of e, and $(v : e) = 0$ otherwise. The (simplicial) cohomology of Γ is encoded in the sequence

$$0 \longrightarrow k \xrightarrow{c} k^{V(\Gamma)} \xrightarrow{\delta} k^{E(\Gamma)} \longrightarrow H^1(\Gamma, k) \longrightarrow 0 \tag{8}$$

with $c(1) = \sum_{v \in V(\Gamma)} v$, $\delta(v) = \sum_{e \in E(\Gamma)} (v : e) e$. This sequence defines two configurations: the inclusion of $\operatorname{coker} c$ into $k^{E(\Gamma)}$, and dually the inclusion of $H_1(\Gamma, k)$ into $k^{E(\Gamma)^\vee}$. We are presently interested in the first one, which corresponds to the position space picture.

It will be convenient to fix a basis V_0 of $\operatorname{coker} c$. For example, the choice of a vertex $v_0 \in V(\Gamma)$ (write $V_0 = V(\Gamma) \setminus \{v_0\}$) provides an isomorphism $\phi : k^{V_0} \to \operatorname{coker} c$ sending the basis element $v \in V_0$ to $v + \operatorname{im} c$. We then have a configuration

$$i_\Gamma = \delta\phi : k^{V_0} \hookrightarrow k^{E(\Gamma)}. \tag{9}$$

Each $e \in E(\Gamma)$ defines a linear form $e^\vee i_\Gamma \in (k^{V_0})^\vee$. It is non-zero since Γ has no loops. Consider instead of $(k^{V_0})^\vee$ the vector space $(M^{V_0})^\vee$ where $M = \mathbb{R}^d$. For each $e \in E(\Gamma)$ there is a d-dimensional subspace

$$A_e = (\operatorname{span} e^\vee i_\Gamma)^{\oplus d} \tag{10}$$

of $(M^{V_0})^\vee$. We denote this collection of d-dimensional subspaces of $(M^{V_0})^\vee$ by

$$\mathcal{C}(\Gamma) = \{A_e : e \in E(\Gamma)\}. \tag{11}$$

Note that the A_e need not be pairwise distinct nor linearly independent. By duality $\mathcal{C}(\Gamma)$ defines an arrangement of codimension d subspaces in M^{V_0}

$$(M^{V_0})_{sing}(\Gamma) = \bigcup_{e \in E(\Gamma)} A_e^\perp \tag{12}$$

where $A_e^\perp$ is the linear subspace annihilated by A_e. The image of $c^{\oplus d}$ in $M^{V(\Gamma)}$ is the thin diagonal Δ. It is in the kernel of all the $e^\vee i_\Gamma$, and therefore it suffices for us to work in the quotient space $\operatorname{coker} c$. By construction $A_e^\perp = D_{jl} + \Delta$ where j and l are the boundaries of e. In particular, if $\Gamma = K_n$ is the complete graph on n vertices, then it is clear that $(M^{V_0})_{sing}(K_n)$ is the large diagonal $\bigcup_{j<l} D_{jl} + \Delta$. The composition $\Phi : M^{V(\Gamma)} \to M^{V(\Gamma)}/\Delta \to M^{V_0}$ is given by $\Phi(x_1, \ldots, x_n) = (x_1 - x_n, \ldots, x_{n-1} - x_n)$, $x_i \in M$, where a numbering $V(\Gamma) = \{1, \ldots, n\}$, $v_0 = n$, of the vertices is assumed.

For a distribution u on M^V constant along Δ we write $\underline{u} = \Phi_* u$ for the pushforward onto M^{V_0}. We usually write $(x_1, \ldots, x_n)$ for a point in $M^{\{1,\ldots,n\}}$, where x_i is a d-tuple of coordinates $x_i^0, \ldots, x_i^{d-1}$ for M. Similarly, if $f \in (k^{V_0})^\vee$ then $f^0, \ldots, f^{d-1}$ are the obvious functionals on M^{V_0} such that $f^{\oplus d} = (f^0, \ldots, f^{d-1})$.

2.2. **Subspace arrangements of divergences.** Now we seek a refinement of the collection $\mathcal{C}(\Gamma)$ in order to sort out singularities where u_Γ is locally integrable and does not require an extension. In a first step we stabilize the collection $\mathcal{C}(\Gamma)$ with respect to sums. Write

$$\mathcal{C}_{sing}(\Gamma) = \left\{ \sum_{e \in E'} A_e;\ \emptyset \subsetneq E' \subseteq E(\Gamma) \right\}. \tag{13}$$

This is again a collection of non-zero subspaces of $(M^{V_0})^\vee$. A subset E' of $E(\Gamma)$ defines a unique subgraph γ of Γ (not necessarily connected) with $E(\gamma) = E'$ and $V(\gamma) = V(\Gamma)$. Each subgraph γ of Γ determines an element

$$A_\gamma = \sum_{e \in E(\gamma)} A_e \tag{14}$$

of $\mathcal{C}_{sing}(\Gamma)$. The map $\gamma \mapsto A_\gamma$ is in general not one-to-one.

Definition 2.1. *A subgraph $\gamma \subseteq \Gamma$ is called* saturated *if $A_\gamma \subsetneq A_{\gamma'}$ for all subgraphs $E(\gamma') \subseteq E(\Gamma)$ such that $E(\gamma) \subsetneq E(\gamma')$.*

It is obvious that for any given γ there is always a saturated subgraph, denoted γ_s, with $A_\gamma = A_{\gamma_s}$. Also, $A_e \cap A_{\gamma_s} = \{0\}$ for all $e \in E(\Gamma) \setminus E(\gamma_s)$.

Definition 2.2. *A graph Γ is called* at most logarithmic *if all subgraphs $\gamma \subseteq \Gamma$ satisfy the condition $d \dim H_1(\gamma) - 2|E(\gamma)| \leq 0$.*

Definition 2.3. *A subgraph $\gamma \subseteq \Gamma$ is called* divergent *if $d \dim H_1(\gamma) = 2|E(\gamma)|$.*

Proposition 2.1. *Let Γ be at most logarithmic. If $\gamma \subseteq \Gamma$ is divergent then it is saturated.*

Proof. Assume that γ satisfies the equality and is not saturated. Then there is an $e \in E(\gamma_s) \setminus E(\gamma)$. Since γ and $\gamma \cup \{e\}$ have the same number of components but $\gamma \cup \{e\}$ one more edge, it follows from (8) that $\dim H_1(\gamma \cup \{e\}) = \dim H_1(\gamma) + 1$. Consequently, $d \dim H_1(\gamma \cup \{e\}) = 2|E(\gamma \cup \{e\})| + 2$ in contradiction to Γ being at most logarithmic. □

Let Γ be at most logarithmic. We define

$$\mathcal{C}_{div}(\Gamma) = \{A_\gamma;\, \emptyset \subsetneq \gamma \subseteq \Gamma,\, \gamma \text{ divergent}\} \tag{15}$$

as a subcollection of $\mathcal{C}_{sing}(\Gamma)$. It is closed under sum (because $\dim H_1(\gamma_1 \cup \gamma_2) \geq \dim H_1(\gamma_1) + \dim H_1(\gamma_2)$). It does not contain the space $\{0\}$. In the dual, the arrangement

$$(M^{V_0})_{div}(\Gamma) = \bigcup_{\substack{\emptyset \subsetneq \gamma \subseteq \Gamma \\ d \dim H_1(\gamma) = 2|E(\gamma)|}} A_\gamma^\perp \tag{16}$$

in M^{V_0} describes the locus where extension is necessary:

Proposition 2.2. *Let Γ be at most logarithmic. Then the largest open subset of M^{V_0} to which $u_0^{\otimes |E(\Gamma)|}$ can be restricted is the complement of $(M^{V_0})_{div}(\Gamma)$. The restriction equals $\underline{u}_\Gamma$ there, and the singular support of $\underline{u}_\Gamma$ is the complement of $(M^{V_0})_{div}(\Gamma)$ in $(M^{V_0})_{sing}(\Gamma)$.*

Proof. Recall the map i_Γ defining the configuration (9). It provides an inclusion $i_\Gamma^{\oplus d} : M^{V_0} \hookrightarrow M^{E(\Gamma)}$. Wherever defined, $\underline{u}_\Gamma$ may be written $\underline{u}_\Gamma(x_1, \ldots, x_{n-1}) = \prod_{e\in E(\Gamma)} u_0\left(\sum_v (v:e)x_v\right)$ with $V_0 = \{1, \ldots, n-1\}$. Since $i_\Gamma(v) = \sum_e (v:e)e$, in coordinates $i_\Gamma(\xi_1, \ldots, \xi_{n-1}) = \left(\sum_v (v:e)\xi_v\right)_{e\in E(\Gamma)}$, it is clear that $\underline{u}_\Gamma = (i_\Gamma^{\oplus d})^* u_0^{\otimes |E(\Gamma)|}$ wherever it is defined. As by (6), $\operatorname{sing\,supp} u_0 = \{0\}$, the singular support of $u_0^{\otimes|E(\Gamma)|}$ is the locus where at least one d-tuple of coordinates vanishes: $x_e^0 = \ldots = x_e^{d-1} = 0$ for some $e \in E(\Gamma)$. Its preimage under $i_\Gamma^{\oplus d}$ is the locus annihilated by one of the A_e, whence the last statement. For the first statement, we have to show that for a compact subset $K \subset M^{V_0}$ the integral $\underline{u}_\Gamma|_K[\underline{1}] = \int_K \underline{u}_\Gamma(x)dx$ converges if and only if K is disjoint from all the $A_\gamma^\perp$, for $\gamma \subseteq \Gamma$ such that $d \dim H_1(\gamma) = 2|E(\gamma)|$. Assume that $K \cap \left(A_\gamma^\perp \setminus \bigcup_{\gamma_s \subsetneq \gamma'} A_{\gamma'}^\perp\right) \neq \emptyset$ for some γ. Write $\underline{u}_\Gamma = \prod_{e\in E(\gamma_s)} u_0(\sum_v (v:e)x_v) f$ where $f = \prod_{e\in E(\Gamma)\setminus E(\gamma_s)} u_0(\sum_v (v:e)x_v)$. The distribution f is smooth on $A_{\gamma_s}^\perp \setminus \bigcup_{\gamma_s\subsetneq\gamma'} A_{\gamma'}^\perp$ since $A_e \cap A_{\gamma_s} = \{0\}$ for all $e \in E(\Gamma)\setminus E(\gamma_s)$. The integral $\int_K \underline{u}_\Gamma(x)dx$ is over a $d(n-1)$-dimensional space. The subspace $A_{\gamma_s}^\perp$ is given by $\dim A_{\gamma_s}$ equations. Each single $u_0(x)$ is of order $o(x^{2-d})$ as $x \to 0$, and there are $|E(\gamma_s)|$ of them in the first factor of u_Γ. Hence the integral is convergent only if $\dim A_{\gamma_s} > (d-2)|E(\gamma_s)|$, which is the same as $2|E(\gamma_s)| > d \dim H_1(\gamma_s)$. Conversely if this is the case for all $\gamma_s' \subseteq \gamma_s$ then the integral is convergent. Our restriction to saturated subgraphs γ_s is justified by Proposition 2.1. □

From now on we will assume that Γ is at most logarithmic. The general case where linear, quadratic, etc. divergences occur is discussed in section 6.4.

2.3. **Subspaces and polydiagonals.** Let again $\gamma \subseteq \Gamma$, that is $E(\gamma) \subseteq E(\Gamma)$ and $V(\gamma) = V(\Gamma)$. Recall from the end of section 2.1 that

$$\Phi^{-1}(A_\gamma^\perp) = \bigcap_{e\in E(\gamma)} D_e \tag{17}$$

with the diagonals $D_e = D_{jl}$ for j and l boundaries of e. An intersection $\bigcap_{e\in E(\gamma)} D_e$ of diagonals is called a *polydiagonal*.

Just as in (8) we have an exact sequence

$$0 \longrightarrow H^0(\gamma, k) \xrightarrow{c_\gamma} k^{V(\Gamma)} \xrightarrow{\delta_\gamma} k^{E(\gamma)} \longrightarrow H^1(\gamma, k) \longrightarrow 0 \tag{18}$$

with c_γ sending each generator of $H^0(\gamma, k)$ (i. e. , a connected component of γ) to the sum of vertices in this component, $1_C \mapsto \sum_{v\in C} v$ and $\delta_\gamma(v) = \sum_{e\in E(\gamma)} (v:e)e$. It is then a matter of notation to verify

Proposition 2.3.

$$\Phi^{-1}(A_\gamma^\perp) = \ker \delta_\gamma^{\oplus d}. \tag{19}$$

□

A polydiagonal $\Phi^{-1}(A_\gamma^\perp)$ corresponds therefore to a partition $cc(\gamma)$ on the vertex set $V(\Gamma)$ as follows: $cc(\gamma) = \{Q_1, \dots, Q_k\}$ with pairwise disjoint *cells* $Q_1, \dots, Q_k \subseteq V(\Gamma)$ such that the vectors

$$\sum_{v \in Q_i} v, \quad i = 1, \dots, k, \tag{20}$$

generate $\ker \delta_\gamma$.

In other words, $cc(\gamma)$ is the equivalence relation/partition "connected by γ" on the set $V(\Gamma)$. If $\Gamma = K_n$ is the complete graph on n vertices, this correspondence is clearly a bijection

$$\{A_\gamma^\perp : \gamma \subseteq K_n\} \xrightarrow{\cong} \{\text{ Partitions of } V(K_n)\}. \tag{21}$$

The next proposition refines this statement. Recall our index notation from the end of section 2.1.

Proposition 2.4. *Let $\gamma, t \subseteq \Gamma$. Then the set*

$$\mathcal{B} = \left\{(e^\vee i_\Gamma)^j : e \in E(t), j = 0, \dots, d-1\right\} \tag{22}$$

is a basis of A_γ if and only if t is a spanning forest for $cc(\gamma)$,

where a spanning forest is defined as follows.

Definition 2.4. *Let $\gamma, t \subseteq \Gamma$. Then t is a* spanning forest for $cc(\gamma)$ *if the map $\delta_t : k^{V(\Gamma)} \to k^{E(t)}$ as in (18) is surjective and* $\ker \delta_t = \ker \delta_\gamma$.

Definition 2.5. *Let $\gamma, t \subseteq \Gamma$ and t be a spanning forest for $cc(\gamma)$. If $t \subseteq \gamma$ then t is a* spanning forest of γ. *If γ is connected (then so is t) then t is called a* spanning tree *of γ.*

In other words, a spanning forest of γ is a subgraph of γ without cycles that has the same connected components. A spanning forest for $cc(\gamma)$ has the same property but needs not be a subgraph of γ.

Proof of Proposition 2.4. By Proposition 2.3, $A_\gamma = A_t$ if and only if $\ker \delta_\gamma = \ker \delta_t$. It remains to show that the set (22) is linearly independent if and only if δ_t is surjective. Since $\ker \delta_\Gamma \subseteq \ker \delta_t$ the map δ_t is surjective if and only if $i_t = \delta_t \phi : k^{V_0} \to k^{E(t)}$ (see (9)) is surjective, which in turn is equivalent to (22) having full rank. □

We also note two simple consequences for future use.

Proposition 2.5. *Let* $\gamma_1, \gamma_2 \subseteq \Gamma$. *Then*

$$A_{\gamma_1} \cap A_{\gamma_2} = A_\gamma \tag{23}$$

where γ *is any subgraph of* Γ *with*

$$\mathrm{cc}(\gamma_1) \cap \mathrm{cc}(\gamma_2) = \mathrm{cc}(\gamma). \tag{24}$$

The intersection $P_1 \cap P_2$ of partitions P_1, P_2 on the same set $V(\Gamma)$ is defined by $P_1 \cap P_2 = \{Q_1 \cap Q_2 : Q_1 \in P_1, Q_2 \in P_2\}$. It is easily seen that this is a partition on $V(\Gamma)$ again. We write 0 for the full partition $\{\{v\} : v \in V(\Gamma)\}$.

Proof. It is clear from Proposition 2.3 that

$$\Phi^{-1}((A_{\gamma_1} \cap A_{\gamma_2})^\perp) = \ker \delta_{\gamma_1}^{\oplus d} + \ker \delta_{\gamma_2}^{\oplus d},$$

and one needs a partition $cc(\gamma)$ whose cells provide a system of generators as in (20) but now for the space $\ker \delta_{\gamma_1} + \ker \delta_{\gamma_2}$. Let $cc(\gamma_i) = \{Q_1^i, \ldots, Q_{l_i}^i\}$. Since

$$\sum_{v \in Q_k^1} v \in \mathrm{span}\Big(\sum_{v \in Q_k^1 \cap Q_1^2} v, \ldots, \sum_{v \in Q_k^1 \cap Q_{l_2}^2} v \Big),$$

and similarly for 1 and 2 interchanged, the vectors $\sum_{v \in Q_k^1 \cap Q_m^2} v$ generate $\ker \delta_{\gamma_1} + \ker \delta_{\gamma_2}$. □

Apart from the intersection of partitions as defined above, it is useful to have the notion of a union of partitions. Let $cc(\gamma_1), cc(\gamma_2)$ be partitions on $V(\Gamma)$. One defines most conveniently

$$cc(\gamma_1) \cup cc(\gamma_2) = cc(\gamma_1 \cup \gamma_2). \tag{25}$$

From the description before (21) it is clear that this definition depends only on $cc(\gamma_1)$ and $cc(\gamma_2)$ but not on γ_1 and γ_2 themselves. We immediately have

Proposition 2.6. *Let* $\gamma_1, \gamma_2, \gamma \subseteq \Gamma$. *Then*

$$A_{\gamma_1} + A_{\gamma_2} = A_\gamma \tag{26}$$

if and only if

$$\mathrm{cc}(\gamma_1) \cup \mathrm{cc}(\gamma_2) = \mathrm{cc}(\gamma). \tag{27}$$

□

It will be convenient later to have an explicit description of the dual basis $\mathcal{B}^\vee$, for $\mathcal{B}$ as in Proposition 2.4, that is the corresponding basis of M^{V_0}. Recall our choice (above equation (9)) of a vertex v_0 in order to work modulo the thin diagonal. Recall also that the edges are oriented. Given a spanning tree t of Γ, we say $e \in E(t)$ *points to* v_0 if the final vertex of e is closer to v_0 in t than the initial vertex of e. Otherwise we say that e *points away*

from v_0. Furthermore, erasing the edge e from t separates t into two connected components. The one *not* containing v_0 is denoted t_1, and we write $V_1 = V_{\text{eff}}(t_1)$ for the set of its vertices.

Proposition 2.7. *Let* $\mathcal{B}^\vee = \{b_e^j : e \in E(t), j = 0, \dots, d-1\}$ *be the basis of* M^{V_0} *dual to a basis* $\mathcal{B}$ *of* $(M^{V_0})^\vee$ *as in Proposition 2.4 , that is* $(e^\vee i_\Gamma)^j(b_{e'}^k) = \delta_{e,e'}\delta_{j,k}$. *Then*

$$b_e = (-1)^{Q_e} \sum_{v \in V_1} v.$$

(V_1*, being a subset of the basis* V_0 *of* k^{V_0}*, is also contained in* k^{V_0}*). We define* $Q_e = \pm 1$ *if* e *points to/away from* v_0.

Proof. Write $b_{e'} = \sum_{v \in V_0} \beta_v^{e'} v$. We require

$$\delta_{e,e'} = (e^\vee i_\Gamma)(b_{e'}) = (e^\vee \delta\phi)(b_{e'}) = \sum_{v \in V_0} \beta_v^{e'}(v : e).$$

Now fix an e. Write $v_{in}(e), v_{out}(e)$ for the initial and final vertex of e, respectively. We have $\beta^e_{v_{in}(e)} - \beta^e_{v_{out}(e)} = 1$ and $\beta^e_{v_{in}(e')} = \beta^e_{v_{out}(e')}$ for the other edges e' except the one e'_0 leading to v_0, for which $\beta^e_{v_{in}(e'_0)} = 0$ or $\beta^e_{v_{out}(e'_0)} = 0$, depending on the direction of e'_0. Thus starting from v_0 and working one's way along the tree t in order to determine the β_v^e, all the $\beta_v^e = 0$ until one reaches the edge e, where β_v^e jumps up or down to 1 or -1, depending on the orientation of e, and stays constant then all beyond e. □

Let us now describe the map $i_\Gamma^{\oplus d} : M^{V_0} \to M^{E(\Gamma)}$ in such a dual basis $\mathcal{B}^\vee$. Let $x \in k^{V_0}$, write $x = \sum_{e \in E(t)} x_e b_e$ with $b_e = (-1)^{Q_e} \sum_{v \in V_1} v$ as in Proposition 2.7. Write $[v_i, v_j] \subseteq E(t)$ for the unique path in t connecting the vertices v_i and v_j. It follows that

$$i_\Gamma(x) = \sum_{e \in E(\Gamma)} \sum_{v \in V_0} \sum_{e' \in [v_0, v]} (-1)^{Q_{e'}} x_{e'}(v : e)e.$$

For a given e, only two vertices v contribute to the sum, namely the boundaries $v_{in}(e)$ and $v_{out}(e)$ of e. All the terms $(-1)^{Q_{e'}} x_{e'}$ for e' on the path from v_0 to $v_{in}(e)$ cancel since they appear twice, once with a negative sign $(v_{in}(e) : e)$, once with a positive sign $(v_{out}(e) : e)$. What remains are the terms on the path in t from $v_{in}(e)$ to $v_{out}(e)$. We write $e' \rightsquigarrow e$ if $e' \in [v_{in}(e), v_{out}(e)] \subset E(t)$. Then

$$i_\Gamma(x) = \sum_{e \in E(\Gamma)} \sum_{e' \rightsquigarrow e} x_{e'} e = \sum_{e \in E(t)} x_e e + \sum_{e \in E(\Gamma) \setminus E(t)} \sum_{e' \rightsquigarrow e} x_{e'} e. \tag{28}$$

Note that in the second sum there may be terms with only one $x_{e'}$ contributing, namely when $A_e = A_{e'}$.

3. Regularization, blowing up, and residues of primitive graphs

The purpose of this section is first to review a few standard facts about distributions and simple birational transformations. See [49] for a general reference on distributions. In the second part, the important notion of residue of a primitive Feynman graph is introduced by raising u_Γ to a complex power s in the neighborhood of $s = 1$ and considering the residue at $s = 1$ as a distribution supported on the exceptional divisor of a blowup.

3.1. **Distributions and densities on manifolds.** We recall basic notions that can be looked up, for example, in [49, Section 6.3]. When one wants to define the notion of distributions on a manifold one has two choices: The first is to model a distribution locally according to the idea that distributions are supposed to generalize smooth functions, so they should transform like $u_i = (\psi_j\psi_i^{-1})^* u_j$ where ψ_i, ψ_j are two charts. On the other hand, distributions are supposed to be measures, that is one wants them to transform like $\tilde{u}_i = |\det \operatorname{Jac} \psi_j\psi_i^{-1}|(\psi_j\psi_i^{-1})^*\tilde{u}_j$. The latter concept is called a distribution density.

By a manifold we mean a paracompact connected smooth manifold throughout the paper. Let $\mathcal{M}$ be a manifold of dimension n with an atlas (ψ_i, U_i) of local charts $\psi_i : M_i \to U_i \subset \mathbb{R}^n$.

Definition 3.1. *A* distribution u *on* $\mathcal{M}$ *is a collection* $u = \{u_i\}$ *of distributions* $u_i \in \mathcal{D}'(U_i)$ *satisfying*

$$u_i = (\psi_j\psi_i^{-1})^* u_j$$

in $\psi_j(U_i \cap U_j)$. *The set of distributions on* $\mathcal{M}$ *is denoted* $\mathcal{D}'(\mathcal{M})$.

Definition 3.2. *A* distribution density $\tilde{u}$ *on* $\mathcal{M}$ *is a collection* $\tilde{u} = \{\tilde{u}_i\}$ *of distributions* $\tilde{u}_i \in \mathcal{D}'(U_i)$ *satisfying*

$$\tilde{u}_i = |\det \operatorname{Jac} \psi_j\psi_i^{-1}|(\psi_j\psi_i^{-1})^*\tilde{u}_j$$

in $\psi_j(U_i \cap U_j)$. *The set of distribution densities on* $\mathcal{M}$ *is denoted* $\tilde{\mathcal{D}}'(M)$. *A density is called* smooth *if all* $\tilde{u}_i$ *are smooth. The set of smooth densities with compact support is denoted* $\tilde{C}_0^\infty$.

Proposition 3.1.

(i) $C_0^{\infty\prime}(\mathcal{M}) = \tilde{\mathcal{D}}'(\mathcal{M})$.
(ii) $\tilde{C}_0^{\infty\prime}(\mathcal{M}) = \mathcal{D}'(\mathcal{M})$.
(iii) *Any strictly positive or strictly negative smooth density* α *(i. e. an orientation) provides isomorphisms* $u \mapsto u\alpha$ *between* $\mathcal{D}'(\mathcal{M})$ *and* $\tilde{\mathcal{D}}'(\mathcal{M})$, *and* $C_0^\infty(\mathcal{M})$ *and* $\tilde{C}_0^\infty(\mathcal{M})$, *respectively.*

□

Smooth densities are also called *pseudo n-forms*. If the manifold is oriented, every pseudo n-form is also a regular n-form. On the other hand, then an n-form ω gives rise to two pseudo n-forms: ω and $-\omega$. In a nonorientable situation we want to work with distribution densities and write them like pseudo forms $u(x)|dx|$.

3.2. **Distributions and birational transformations.** Let $\mathcal{M}$ be a smooth manifold of dimension n and $x \in \mathcal{M}$ a point in it. We work in local coordinates and may assume $\mathcal{M} = \mathbb{R}^n$ and $x = 0$. Blowing up 0 means replacing 0 by a real projective space $\mathcal{E} = \mathbb{P}^{n-1}(\mathbb{R})$ of codimension 1. The result is again a smooth manifold as follows.

Let $Y = (\mathcal{M} \setminus \{0\}) \sqcup \mathcal{E}$ as a set. Tangent directions at 0 shall be identified with elements of $\mathcal{E}$. Let therefore Y' be the subset of $\mathcal{M} \times \mathcal{E}$ defined by $x_i u_j = x_j u_i$, $1 \leq i, j \leq n$ where $x_1, \ldots, x_n$ are the affine coordinates of $\mathbb{R}^n$ and $u_1, \ldots, u_n$ are homogeneous coordinates of $\mathbb{P}^{n-1}$. The set Y' is a smooth submanifold of $\mathcal{M} \times \mathcal{E}$. On the other hand, there is an obvious bijection $\lambda : Y \to Y'$ whose restriction onto $\mathcal{M} \setminus \{0\} \subset Y$ is a diffeomorphism onto its image. Pulling back along λ the differentiable structure induced on Y' defines a differentiable structure on all of Y. The latter is called *blowup of* $\mathcal{M}$ at $\{0\}$. The submanifold $\mathcal{E}$ of Y is called the *exceptional divisor*. There is a smooth proper map $\beta : Y \to \mathcal{M}$ which is the identity on $\mathcal{M} \setminus \{0\}$ and sends $\mathcal{E}$ to 0. Viewed as a map from $Y' \subset \mathcal{M} \times \mathcal{E}$, β is simply the projection onto the first factor.

Note that if n is even (which is the case throughout the paper) then Y is not orientable but $\mathcal{E}$ is. If n is odd then Y is orientable but $\mathcal{E}$ is not. Indeed Y can be seen as a bundle $\tau : Y \to \mathcal{E}$ over $\mathcal{E}$ with fiber $\mathbb{R}$ – the tautological bundle. For example, for $n = 2$, Y is the open Möbius strip.

In our case we work with distributions on open subspaces of $\mathcal{M}$. $\mathcal{M}$ being orientable, distributions can be identified with distribution densities, see Proposition 3.1 (iii). These densities can be pulled back along β, one can work with them there and push the result forward again along β. The image, a density on $\mathcal{M}$, can again be identified with a distribution on $\mathcal{M}$.

Let n be even from now on. For $U_i = \mathbb{R}^n$, $i = 1, \ldots, n$, one defines maps

$\rho_i : U_i \to \mathcal{M} \times \mathcal{E}$,

$$\begin{aligned} (y_1, \ldots, y_n) &\mapsto ((x_1, \ldots, x_n), [x_1, \ldots, x_n]) \\ & \quad x_i = (-1)^i y_i, \\ & \quad x_k = y_i y_k,\ k \neq i \end{aligned} \tag{29}$$

where x_i are coordinates on $\mathcal{M}$ and at the same time homogeneous coordinates for $\mathcal{E}$. Clearly ρ_i maps into Y and onto the affine chart of $\mathcal{E}$ where $x_i \neq 0$. Let $\psi_i = \rho_i^{-1}$ on $\rho_i(U_i)$. Then (ψ_i, U_i) furnish an atlas for Y. We note for future reference the transition maps

$$\begin{aligned} \psi_j \psi_i^{-1} : & \quad U_i - \{y_j = 0\} \to U_j \setminus \{y_i' = 0\} \\ (y_1, \ldots, y_n) &\mapsto (y_1', \ldots, y_n') \\ & \quad y_i' = (-1)^{i+j}/y_j, \\ & \quad y_j' = (-1)^j y_i y_j, \\ & \quad y_k' = (-1)^j y_k / y_j,\ k \neq i, j \end{aligned} \tag{30}$$

and the determinants of their derivatives

$$\det \operatorname{Jac} \psi_j \psi_i^{-1} = (-1)^{j+1} y_j^{1-d}. \tag{31}$$

Note that the atlas (ψ_i, U_i) is therefore not even oriented on the open set $Y \setminus \mathcal{E}$ diffeomorphic to $\mathcal{M} \setminus \{0\}$. For the exceptional divisor $\mathcal{E} = \mathbb{P}^{n-1}$, which is given in U_i by the equation $y_i = 0$, we use induced charts (V_i, ϕ_i) with coordinates $y_1, \ldots, \widehat{y_i}, \ldots, y_n$ (in this very order) where $\widehat{y_i}$ means omission. The transition map

$$\begin{aligned} \phi_j \phi_i^{-1} : & \quad V_i \setminus \{y_j = 0\} \to V_j \setminus \{y_i' = 0\} \\ (y_1, \ldots, \widehat{y_i}, \ldots, y_n) &\mapsto (y_1', \ldots, \widehat{y_i'}, \ldots, y_n') \\ & \quad y_i' = (-1)^{i+j}/y_j, \\ & \quad y_k' = (-1)^j y_k / y_j,\ k \neq i, j \end{aligned} \tag{32}$$

has Jacobian determinant

$$\det \operatorname{Jac} \phi_j \phi_i^{-1} = y_j^{-d} > 0. \tag{33}$$

The induced atlas (V_i, ϕ_i) is therefore an oriented one. The tautological bundle τ is given in local coordinates by $\tau : (y_1, \ldots, y_n) \mapsto (y_1, \ldots, \widehat{y_i}, \ldots, y_n)$.

Similarly one defines blowing up along a smooth submanifold: The submanifold is replaced by its projectivized normal bundle. Assume the submanifold is given in local coordinates by $x_1 = \ldots = x_k = 0$. Then a natural choice of coordinates for the blowup is given again by (29), applied only to the subset of coordinates $x_1, \ldots, x_k$. See for instance [65, Section 3] for details.

The map $\beta : Y \to \mathcal{M}$ is surjective, proper and smooth everywhere but open (i. e. has surjective differential) only away from the exceptional divisor. It is called the blowdown map. It will be useful to be able to push distributions forward and to pull them back along this map.

In general, let $f : U \to V$ be a surjective proper smooth map between open sets U of $\mathbb{R}^n$ and V of $\mathbb{R}^m$. Let u be a distribution on U. The pushforward of u by f, denoted f_*u, is the distribution on V defined by $(f_*u)[\phi] = u[f^*\phi]$ where ϕ is a test function on V and $f^*\phi$ is its pullback along f : $f^*\phi = \phi \circ f$. If u has compact support the requirement that f be proper can be dropped. Similarly, for $f : \mathcal{M} \to \mathcal{N}$ a surjective proper smooth map between manifolds $\mathcal{M}$ and $\mathcal{N}$ with atlases (ψ_i, U_i) and (θ_i, V_i), let u be a distribution density on $\mathcal{M}$. Then f_*u defined by

$$(f_*u)_i = (\theta_i f \psi_k^{-1})_* u_k$$

on $V_i \cap (\theta_i f \psi_k^{-1})(U_k)$, is a distribution density on $\mathcal{N}$. Let now $f : \mathcal{M} \to \mathcal{N}$ a surjective smooth map between manifolds $\mathcal{M}$ and $\mathcal{N}$. It need not be proper. Let $u \in \tilde{\mathcal{D}}(\mathcal{M})$ and $\phi \in \mathcal{D}(\mathcal{M})$. The density $u[\phi]_f \in \tilde{\mathcal{D}}'(\mathcal{N})$ is defined by

$$u[\phi]_f = f_*(\phi u). \tag{34}$$

Note that ϕu has compact support so the pushforward is well-defined although f is not necessarily proper. If u is given by a locally integrable function $u(x)$ on $\mathcal{M} = \mathbb{R}^n$ and $\mathcal{N} = \{y_{i+1}, \ldots, y_n = 0\} \subseteq \mathbb{R}^n$, $i < n$, this notion corresponds to integrating out the orthogonal complement $\{y_1, \ldots, y_i = 0\}$ of $\mathcal{N}$ in $\mathbb{R}^n$:

$$u[\phi]_f(y_{i+1}, \ldots, y_n) = \int u\phi(y_1, \ldots, y_n) dy_1, \ldots, dy_i.$$

The reverse operation of pulling back distributions along smooth maps is only possible under certain conditions, see [49, Sections 6.1, 8.2, etc.] for a general exposition. Here we only need the following: Let $U_1, U_2 \subseteq \mathbb{R}^n$ open and $f : U_1 \to U_2$ a smooth and everywhere open map. Then there is a unique continuous linear map $f^* : \mathcal{D}'(U_2) \to \mathcal{D}'(U_1)$ such that $f^*u = u \circ f$ if $u \in C^0(U_2)$. See [49, Theorem 6.1.2] for a proof of this statement. It can obviously be generalized to the case of a submersion $f : \mathcal{M} \to \mathcal{N}$ where $\mathcal{M}$ is a manifold of dimension n, by collecting pullbacks in the chart domains: $(f^*u)_i = (f\psi_i^{-1})^* u$ where (ψ_i, U_i) is an atlas for $\mathcal{M}$.

If β is the blowdown map, by the pullback $\beta^*\tilde{u}$ of a distribution density $\tilde{u} \in \tilde{\mathcal{D}}'(\mathcal{M} \setminus \{0\})$ obviously the pullback along the diffeomorphism $\beta|_{Y \setminus \mathcal{E}}$ is understood. The result is a distribution density on $Y \setminus \mathcal{E}$.

3.3. **Analytic regularization.** As a first step toward understanding u_Γ^s as a distribution-valued meromorphic function of s in a neighborhood of $s = 1$, we study distributions u on $\mathbb{R} \setminus 0$ of the form $u = |x|^{-a}$ where $a \in \mathbb{Z}$. Clearly if $a < 1$ then $u \in L^1_{loc}(\mathbb{R})$. The case $a \geq 1$ can be handled in a canonical way using analytic continuation with respect to the exponent. Let $a \in \mathbb{N}$ be fixed. We extend $u^s = |x|^{-as}$ meromorphically to the area $\Re s > 1$ as follows. Let $n = \lfloor a/2 \rfloor$.

$$\begin{aligned} u^s[\phi] &= \int_0^1 x^{-as}(\phi(x) + \phi(-x))dx + \int_{\mathbb{R}\setminus[-1,1]} |x|^{-as}\phi(x)dx \\ &= \int_0^1 x^{-as}\left(\phi(x) + \phi(-x) - 2\left(\phi(0) + \ldots + \frac{x^{2n}\phi^{(2n)}(0)}{(2n)!}\right)\right)dx \qquad (35) \\ &\quad + \int_{\mathbb{R}\setminus[-1,1]} |x|^{-as}\phi(x)dx + 2\sum_{k=0}^{n} \frac{\phi^{(2k)}(0)}{(2k)!((2k+1)-as)}. \end{aligned}$$

This holds for $\Re s < 1 + \frac{1}{a}$. See [41, Section I.3] for the complete argument. There will be more poles beyond the half-plane $\Re s < 1 + \frac{1}{a}$ but they are not relevant for our purposes.

Definition 3.3. *The* canonical regularization *of $|x|^{-a}$ is the distribution-valued meromorphic function in $s \in (-\infty, 1 + \frac{1}{a}) + i\mathbb{R}$ given by*

$$|x|^{-as}_{ext} = 2\sum_{k=0}^{n} \frac{\delta_0^{(2k)}}{(2k)!((2k+1)-as)} + |x|^{-as}_{fin} \qquad (36)$$

where $n = \lfloor a/2 \rfloor$ and

$$\begin{aligned} |x|^{-as}_{fin}[\phi] &= \int_0^1 x^{-as}\left(\phi(x) + \phi(-x) - 2\left(\phi(0) + \ldots + \frac{x^{2n}\phi^{(2n)}(0)}{(2n)!}\right)\right)dx \\ &\quad + \int_{\mathbb{R}\setminus[-1,1]} |x|^{-as}\phi(x)dx. \qquad (37) \end{aligned}$$

The function $s \mapsto |x|^{-as}_{fin}$ is holomorphic in $(-\infty, 1 + \frac{1}{a}) + i\mathbb{R}$. When the context allows, we simply write $|x|^{-as}$ for $|x|^{-as}_{ext}$ again. Let $f \in C^\infty(\mathbb{R})$. Since $s \mapsto f^s[\phi]$ is holomorphic, it makes sense to define the canonical regularization for $|x|^{-a}f$ also:

$$(|x|^{-a}f)^s_{ext} = |x|^{-as}_{ext} \cdot f^s. \qquad (38)$$

This does not work for $f \in L^1_{loc}(\mathbb{R})$. For example, $|x|^{-(a+b)s}_{ext} \neq |x|^{-as}_{ext}|x|^{-bs}_{ext}$.

Unfortunately, the term "regularization" is used for two different notions in the mathematics and physics literature that need to be carefully distinguished. While in the mathematics literature, the "regularized" distribution

is usually understood to be $|x|^{-a}_{fin}$, a physicist calls this the "renormalized" distribution, and refers to the mapping $s \mapsto |x|^{-as}$ as a regularization (in fact, one out of many possible regularizations). The latter is also our convention.

We finally note the special case $a = 1$,

$$|x|^{-s}_{ext} = -\frac{2\delta_0}{s-1} + |x|^{-s}_{fin}, \tag{39}$$

$$|x|^{-s}_{fin}[\phi] = \int_{-1}^{1} |x|^{-s}(\phi(x) - \phi(0))dx + \int_{\mathbb{R}\setminus[-1,1]} |x|^{-s}\phi(x)dx. \tag{40}$$

And, for future reference, in the area $\Re s < \frac{2+(D-1)}{D}$,

$$|x|^{D-Ds-1}_{ext} = -\frac{2}{D}\frac{\delta_0}{s-1} + |x|^{D-Ds-1}_{fin} \tag{41}$$

where $D \in 2\mathbb{N}$.

3.4. **Primitive graphs, their residues and renormalization.** We consider the blowup $\beta : Y \to \mathcal{M}$ as in section 3.2 where now $\mathcal{M} = M^{V_0}$ for a Feynman graph Γ (see section 2 for notation). In this section we continue to use the coordinates $x_1, \dots, x_{d(n-1)}$ on M^{V_0} and $y_1, \dots, y_{d(n-1)}$ on the charts U_i for Y. Note that n is now the number of vertices of Γ. Recall that since Y is not orientable (and the induced atlas on $Y \setminus \mathcal{E}$ is not oriented), top degree forms and densities can not be identified, in particular pulling back (along a diffeomorphism) a form is different from pulling back a density. We only use forms on the oriented submanifold $\mathcal{E}$, where the two notions coincide. We write $|dx|$ for the Lebesgue measure on $\mathcal{M}$.

Definition 3.4. *A connected Feynman graph Γ is called* primitive *if $\mathcal{C}_{div}(\Gamma) = \{A_\Gamma\}$.*

Lemma 3.1. *Let Γ be primitive. Let t be a spanning tree for Γ and t' a subforest of t. Then*

$$d|E(t')| \leq (d-2)(|E(\Gamma)| \setminus |E((t-t')_s)|)$$

and equality holds if and only if $t' = t$.

Proof. Clearly $\dim A_t = \dim A_{t'} + \dim A_{t\setminus t'}$ and $\dim A_{t'} = d|E(t')|$. Since Γ is divergent, $(d-2)|E(\Gamma)| = \dim A_t$. Since Γ has no divergent subgraphs, $(d-2)|E((t\setminus t')_s)| < \dim A_{(t\setminus t')_s} = \dim A_{t\setminus t'}$ for all subforests t' of t. □

Lemma 3.2. *Let $\delta_{\mathcal{E}}$ (resp. $\frac{1}{|y_{\mathcal{E}}|}$) be collections of distributions[2] in the U_i given by $(\delta_{\mathcal{E}})_i = \delta_0(y_i)$ and $(1/|y_{\mathcal{E}}|)_i = \frac{1}{|y_i|}$ in U_i. Let ω be a locally integrable volume form ω on $\mathcal{E}$. Then $\omega\delta_{\mathcal{E}}$ and $\omega/|y_{\mathcal{E}}|$, locally*

$$\begin{aligned}(\omega\delta_{\mathcal{E}})_i &= \omega_i(\delta_{\mathcal{E}})_i = \omega_i(y_1,\dots,\widehat{y_i},\dots,y_n)\delta_0(y_i),\\ (\omega/|y_{\mathcal{E}}|)_i &= \omega_i/|y_{\mathcal{E}}|_i = \omega_i(y_1,\dots,\widehat{y_i},\dots,y_n)/|y_i|\end{aligned}$$

define densities on Y.

Proof. By (31) and (33) $|\det \mathrm{Jac}\,\psi_j\psi_i^{-1}| = \det \mathrm{Jac}\,\phi_j\phi_i^{-1}\cdot|1/y_j|$ and both δ_0 and $1/|y_i|$ transform with the factor $|1/y_j|$ under transition $U_i \to U_j$. □

Theorem 3.1. *Let Γ be primitive. Write $d_\Gamma = d|V_0|$.*

(i) *By pullback along the diffeomorphism $\beta|_{Y\setminus\mathcal{E}}$, the distribution density $\tilde{u}_\Gamma = \underline{u}_\Gamma|dx|$ furnishes a strictly positive density $\tilde{w}_\Gamma$ on $Y\setminus\mathcal{E}$, given in local coordinates of U_i by*

$$(\tilde{w}_\Gamma)_i|dy| = \frac{1}{|y_i|}(f_\Gamma)_i(y_1,\dots,\widehat{y_i},\dots,y_n)|dy| \tag{42}$$

where $(f_\Gamma)_i \in L^1(V_i)$. The $(f_\Gamma)_i\, dy_1\wedge\ldots\wedge\widehat{dy_i}\wedge\ldots\wedge dy_n$ in each V_i determine an integrable volume form f_Γ on $\mathcal{E}$. We may therefore write $\tilde{w}_\Gamma = f_\Gamma/|y_{\mathcal{E}}|$.

(ii) *The meromorphic density-valued function $s\mapsto\tilde{w}_\Gamma^s = \beta^*\tilde{u}_\Gamma^s$,*

$$(\tilde{w}_\Gamma^s)_i|dy| = \frac{(f_\Gamma)_i^s|dy|}{|y_i|^{d_\Gamma s-(d_\Gamma-1)}}$$

has a simple pole at $s=1$. Its residue is the density

$$\mathrm{res}_{s=1}\,\tilde{w}_\Gamma^s = -\frac{2}{d_\Gamma}\delta_{\mathcal{E}}f_\Gamma, \tag{43}$$

supported on the exceptional divisor. Pushing forward along β amounts to integrating a projective integral over the exceptional divisor:

(44)

$$\beta_*(\mathrm{res}_{s=1}\,\tilde{w}_\Gamma^s) = -\frac{2}{d_\Gamma}\delta_0|dx|\int_{\mathcal{E}} f_\Gamma = -\frac{2}{d_\Gamma}\delta_0\int_{V_i}(f_\Gamma)_i dy_1\ldots\widehat{dy_i}\ldots dy_n$$

for any i.

(iii) *Let $\mu\in\mathcal{D}(\mathbb{R}^d)$ with $\mu(0)=1$, and $\nu=\beta^*\mu$. Let $\tau: Y\to\mathcal{E}$ be the tautological bundle. Write $\tilde{u}_\Gamma^s = \beta_*(\tilde{w}_\Gamma^s)$. Then*

$$\tilde{w}_{\Gamma,R}^s = \tilde{w}_\Gamma^s - \tilde{w}_\Gamma^s[\nu]_\tau\delta_{\mathcal{E}} \tag{45}$$

[2]We do not claim that they are distribution or densities on Y themselves as they do not transform correctly.

defines a density-valued function on Y holomorphic in a neighborhood of $s = 1$. Also $\beta_* \tilde{w}^s_{\Gamma,R} = \tilde{u}^s_{\Gamma,R} = (\tilde{u}^s_\Gamma - \tilde{u}^s_\Gamma[\mu])\delta_0|dx|$.

The density (43) is called *residue density*, the volume form f_Γ *residue form*, and the complex number

$$\operatorname{res}\Gamma = -\frac{2}{d_\Gamma}\int_{\mathcal{E}} f_\Gamma \tag{46}$$

residue of Γ. The distribution $\underline{u}_{\Gamma,R}$ is defined on all of M^{V_0} and said to be the *renormalized distribution.*

Proof of Theorem 3.1. (i) For (42) observe that in U_i the map β is given by ρ, see (29). The Lebesgue measure $|dx|$ on M^{V_0} pulls back to $|y_i|^{d_\Gamma - 1}|dy|$ on U_i. By (5), $(\beta^*\tilde{u}_\Gamma)_i$ scales like $\lambda^{(2-d)|E(\Gamma)|}$ as $y_i \to \lambda y_i$. Since Γ is divergent, $d_\Gamma = (2-d)|E(\Gamma)|$, which explains the factor $1/|y_i|$ in (42). Furthermore f_Γ clearly does not depend on y_i. That $f_\Gamma \in L^1_{loc}(V_i)$ follows from Proposition 2.2, where $M^{V_0}_{sing} = A^\perp_\Gamma = \{0\}$, and $\beta|_{Y\setminus\mathcal{E}}$ being a diffeomorphism. In order to show that $f_\Gamma \in L^1(V_i)$ one uses Lemma 3.1 as follows: Choose a spanning tree t for Γ such that the coordinate x_i equals $(e^\vee i_\Gamma)^0$ for some $e \in E(t)$ (see Proposition 2.4). Write $x^j_e = (e^\vee i_\Gamma)^j$ for $e \in E(t), j = 0, \dots, d-1$. In this basis $\underline{u}_\Gamma$ is given by $u_\Gamma(\{x^j_e\}) = \prod_{e\in E(\Gamma)} u_0(\sum_{e'\rightsquigarrow e} x^j_{e'})$ (see (28)). Therefore, if the coordinates y^j_e, $e \in E(t')$ defined by t' a proper subforest of t, go to ∞, then there are exactly $E(t_s) \setminus E((t \setminus t')_s)$ factors of u_0 the argument of which goes to ∞. Lemma 3.1 shows that the integration over that subspace converges. One verifies that all subspaces susceptible to infrared divergences are of this form. Therefore $(f_\Gamma)_i \in L^1(V_i)$. Finally, $(f_\Gamma)_i$ transform like y_i^{-d} under transition between charts. By (33) this makes f_Γ a density on $\mathcal{E}$. Since $\mathcal{E}$ is oriented, a strictly positive density is also a strictly positive (L^1_{loc})- volume form.
(ii) The simple pole and (43) follow from (42) by (39), the local expressions matched together using Lemma 3.2. For (44) let $\phi \in \mathcal{D}(M^{V_0})$. Then $\beta_*(\operatorname{res}_{s=1}\tilde{w}_\Gamma)[\phi] = \operatorname{res}_{s=1}\tilde{w}_\Gamma[\beta^*\phi]$. The distribution $\operatorname{res}_{s=1}\tilde{w}_\Gamma$, being supported on $\mathcal{E}$, depends only on $\beta^*\phi|_{\mathcal{E}} = \phi(0)$. By the results of (i), $\int_{\mathcal{E}} f_\Gamma$ is a projective integral and it suffices to integrate inside one chart, say U_i. There $\operatorname{res}_{s=1}\tilde{w}_\Gamma[\beta^*\phi] = -\frac{2}{d}\int_{U_i}\delta_0(y_i)f_\Gamma(y)\phi(\rho(y))dy = -\frac{2}{d}\phi(0)\int_{V_i} f_\Gamma(y)dy = -\frac{2}{d}\phi(0)\int_{\mathcal{E}} f_\Gamma$, where again integration in one chart suffices by the previous argument.
(iii) There is no pole at $s = 1$ since $\nu|_{\mathcal{E}} = 1$. The $(\tilde{w}^s_{\Gamma,R})_i$ furnish a density by Lemma 3.2: The Jacobian of $\delta_{\mathcal{E}}$ cancels the one of $[\dots]_\tau$. For the last statement, let again $(\psi_i, U_i)_{i=1,\dots,d(n-1)}$ be the chosen atlas for Y and $(\phi_i, V_i)_{i=1,\dots,d(n-1)}$ the induced atlas for $\mathcal{E}$. Since $\mathcal{E}$ is compact, there exists a partition of unity $(\xi_i\phi_i)_{i=1,\dots,d(n-1)}$ on $\mathcal{E}$ subordinate to the V_i such

that $\xi_i \in \mathcal{D}(V_i)$, $\xi_i \geq 0$ and $\sum_i (\xi_i \phi_i)(x) = 1$ for all $x \in \mathcal{E}$. Let $\tau : Y \to \mathcal{E}$. Then $(\xi_i \phi_i \tau)_{i=1,\dots,d(n-1)}$ is a partition of unity on Y subordinate to $(\psi_i, U_i)_{i=1,\dots,d(n-1)}$ (however not compactly supported). We fix such a partition of unity (ξ_i). In U_i we write y for $(y_1, \dots, y_n)$ and $\widehat{y_i}$ for $(y_1, \dots, \widehat{y_i}, \dots, y_n)$, for example $\xi_i(y) = \xi_i(\widehat{y_i})$ since it is constant along y_i. We also write $u(y_i, y_i\widehat{y_i}) = u(y_i y_1, \dots, y_i, \dots, y_i y_n)$ for convenience. Let $f \in \mathcal{D}(M^{V_0})$.

$$\begin{aligned}
\beta_*(\tilde{w}^s_{\Gamma,R})[f] &= \beta_*(\tilde{w}^s_\Gamma - \tilde{w}^s_\Gamma[\nu]_\tau \delta_\mathcal{E})[f] \\
&= \sum_i (\tilde{w}^s_\Gamma - \tilde{w}^s_\Gamma[\nu]_\tau \delta_\mathcal{E})_i [\xi_i \beta^* f] \\
&= \sum_i \int_{U_i} (\tilde{w}^s_\Gamma(y) - \int_\mathbb{R} \tilde{w}^s_\Gamma(z_i, \widehat{y_i}) \mu(z_i, z_i\widehat{y_i}) dz_i \delta_0(y_i)) \\
&\quad \times \xi_i(y) f(y_i, y_i\widehat{y_i}) dy \\
&= \sum_i \int_{U_i} \tilde{w}^s_\Gamma(y) \xi_i(y) f(y_i, y_i\widehat{y_i}) \\
&\quad - \tilde{w}^s_\Gamma(y) \mu(y_i, y_i\widehat{y_i}) \xi_i(0, \widehat{y_i}) f(0) dy \\
&= \sum_i (\beta_* \tilde{w}^s_\Gamma - \beta_* \tilde{w}^s_\Gamma[\xi_i \nu] \delta_0)[f].
\end{aligned}$$

□

The following corollary concerns infrared divergences of a graph Γ. Those are divergences which do not occur at the $A^\perp_\gamma$ but as the coordinates x_i of M^{V_0} approach ∞, in other words, if one attempts to integrate u_Γ against a function which is not compactly supported.

Corollary 3.1. *Let Γ be at most logarithmic and primitive. Then u_Γ is not (globally) integrable on $M^{V_0} \setminus M^{V_0}_{div}(\Gamma)$. However $(\chi u_\Gamma)[\underline{1}_L \otimes \mu]$ is well-defined, if μ is a test function on a non-zero subspace of M^{V_0}, $\underline{1}_L$ the constant function on the orthogonal complement L, and χ the characteristic function of the complement of an open neighborhood of $M^{V_0}_{div}(\Gamma)$ in M^{V_0}.*

Proof. This follows from part (i) of Theorem 3.1. □

The renormalized distribution $u_{\Gamma,R} = u^s_{\Gamma,R}|_{s=1}$ obtained from the theorem depends of course on μ. Write $u_{\Gamma,R}$ for one using μ and $u'_{\Gamma,R}$ for another one using μ', then the difference $u_{\Gamma,R} - u'_{\Gamma,R}$ is supported on 0 and of the form $c\delta_0$ with $c \in \mathbb{R}$. This one-dimensional space of possible extensions represents the renormalization ambiguity.

Here is an example. Let $M = \mathbb{R}^4$. For

$\Gamma =$ 

$$u_\Gamma(x) = u_0^2(x) = 1/x^4,$$

the latter a distribution on $M^{V_0} \setminus \{0\} = M \setminus \{0\}$. Pulling back along β,

$$(\beta^* \tilde{u}_\Gamma)_i |dy| = (\psi_i^{-1})^* \beta^* \tilde{u}_\Gamma |dy| = \frac{|dy|}{|y_i|(1+\sum_{j\neq i} y_j^2)^2}$$

in $U_i - \{y_i = 0\}$, $i = 0, \ldots, 3$. As $\tilde{u}_\Gamma$ was not defined at 0, $(\beta^* \tilde{u}_\Gamma)_i$ is not defined at $\mathcal{E}$, given locally by $\{y_i = 0\}$. Raising to the power s gives

$$\begin{aligned}(\beta^* \tilde{u}_\Gamma^s)_i |dy| &= \frac{|dy|}{|y_i|^{4s-3}(1+\sum_{j\neq i} y_j^2)^{2s}} \\ &= \left(\frac{-\delta_0(y_i)}{2(s-1)} + o(s-1)^0\right) \frac{|dy|}{(1+\sum_{j\neq i} y_j^2)^{2s}}\end{aligned}$$

Therefore the residue density at $s = 1$ is given, in this chart, by

$$\mathrm{res}_{s=1}(\beta^* \tilde{u}_\Gamma)_i^s |dy| = -\frac{1}{2}\delta_0(y_i) \frac{1}{(1+\sum_{j\neq i} y_j^2)^2} |dy|.$$

The residue is given as a projective integral by

$$\mathrm{res}\,\Gamma = -\frac{1}{2}\int_{\mathcal{E}} \frac{\sum_i (-1)^i Y_i dY_1 \wedge \ldots \wedge \widehat{dY_i} \wedge \ldots \wedge dY_4}{Y^4}$$

where $Y_1, \ldots, Y_4$ are homogeneous coordinates. In any of the charts V_i, and for the integration one chart suffices,

$$\mathrm{res}\,\Gamma = -\frac{1}{2}\int_{V_i} \frac{dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_n}{(1+\sum_{j\neq i} y_j^2)^2}.$$

As mentioned before, there is a 1-dimensional space of possible extensions $u_{\Gamma,R}$ due to the choice of μ that needs to be made. There is no canonical μ. However from practice in momentum space the following choice is useful. In momentum space, the ill-defined Fourier transform of u_0^2 is

$$(\mathcal{F}u_0)^{*2}: \quad p \mapsto \int \frac{d^4k}{k^2(k-p)^2}.$$

A regularization or cutoff is now being understood in the integral. This can be renormalized, for example, by subtracting the value at $p^2 = m^2$ where

$m > 0$ has the meaning of an energy scale.

$$(\mathcal{F}u_0)^{*2}_R : \quad p \mapsto \int \frac{d^4k}{k^2(k-p)^2} - \int \frac{d^4k}{k^2(k-p)^2}\bigg|_{p^2=m^2}$$

This prescription has the advantage that it is useful for calculations beyond perturbation theory. The Fourier transform of the distribution $\delta(p^2 - m^2)$ is a Bessel function $\mu(x)$ (with noncompact support), which can be approximated by a sequence $\mu_n \to \mu$ of test functions μ_n with compact support. Since $m > 0$, $\mu \neq \underline{1}$, and infrared divergences do not occur.

In the case of primitive graphs, the renormalization operation described above can be performed, and the residue be defined, while on M^{V_0}, without blowing up. For general graphs however blowing up provides an advantage, as will be shown in section 6: All divergences can be removed at the same time while observing the physical principle of locality. This concludes our discussion of primitive divergences, and we start with the general theory for arbitrary graphs.

4. Models for the complements of subspace arrangements

In section 2 a description of the singular support of $\underline{u}_\Gamma$ and of the locus where $\underline{u}_\Gamma$ fails to be locally integrable was given as subspace arrangements in a vector space. In general both $(M^{V_0})_{sing}(\Gamma)$ and $(M^{V_0})_{div}(\Gamma)$ will not be cartesian products of simpler arrangements. In this section we describe birational models for M^{V_0} where the two subspace arrangements are transformed into normal crossing divisors. For this purpose it is convenient to use results of De Concini and Procesi [33] on more general subspace arrangements. See also the recent book [32] for a general introduction to the subject. Although for the results of the present paper only the smooth models for the divergent arrangements $(M^{V_0})_{div}(\Gamma)$ are needed, it is very instructive, free of cost, and useful for future application to primitive graphs, to develop the smooth models for the singular arrangements $(M^{V_0})_{sing}(\Gamma)$ at the same time.

4.1. **Smooth models and normal crossing divisors.** Consider for a finite dimensional real vector space V a collection $\mathcal{C} = \{A_1, \ldots, A_m\}$ of subspaces A_i of $V^\vee$ and the corresponding arrangement $V_\mathcal{C} = \bigcup_{A\in\mathcal{C}} A^\perp$ in V. The problem is to find a smooth manifold $Y_\mathcal{C}$ and a smooth proper surjective morphism $\beta : Y_\mathcal{C} \to V$ such that

(1) β is an isomorphism outside of $\beta^{-1}(V_\mathcal{C})$.
(2) The preimage $\mathcal{E}$ of $V_\mathcal{C}$ is a divisor with normal crossings, i. e. there are local coordinates $z_1, \ldots, z_n$ for $Y_\mathcal{C}$ such that $\beta^{-1}(V_\mathcal{C})$ is given in the chart by the equation $z_1 \cdot \ldots \cdot z_k = 0$.

(3) β is a composition of blowups along smooth centers.

Such a map $\beta : Y_{\mathcal{C}} \to V$ is called a *smooth model for* $V_{\mathcal{C}}$. Since β is a composition of blowups, it is a birational equivalence. By the classical result of Hironaka it is clear that for much more general algebraic sets $V_{\mathcal{C}}$ such a model always exists in characteristic 0. For the special case of subspace arrangements $V_{\mathcal{C}}$ a comprehensive and very useful treatment is given in [33]. It will be instructive to not only consider one smooth model, but a family of smooth models constructed below along the lines of [33]. By abuse of language, a smooth model may be seen as a "compactification" of the complement of the arrangement, for if $K \subset V$ is compact, then $\beta|_{\beta^{-1}(K)}$ is a compactification of $(V \setminus V_{\mathcal{C}}) \cap K$ since β is proper.

In the following we construct the smooth models of De Concini and Procesi for the special case of $V = M^{V_0}$ and $\mathcal{C} = \mathcal{C}_{sing}(\Gamma)$ or $\mathcal{C} = \mathcal{C}_{div}(\Gamma)$.

4.2. **The Wonderful Models.** For a vector space V write $\mathbb{P}(V)$ for the projective space of lines in V. For any subspace U of V there is an obvious map $V \setminus U \to V/U \to \mathbb{P}(V/U)$. The smooth models of De Concini and Procesi, called "wonderful models", are defined as the closure $Y_{\mathcal{P}}$ of the graph of the map

$$V \setminus V_{\mathcal{C}} \to \prod_{A \in \mathcal{P}} \mathbb{P}(V/A^{\perp}) \tag{47}$$

(the closure taken in $V \times \prod_{A \in \mathcal{P}} \mathbb{P}(V/A^{\perp})$) where $\mathcal{P}$ is a subset of $\mathcal{C}$, subject to certain conditions, to be defined below. The set $\mathcal{P}$ controls what the irreducible components of the divisor $\mathcal{E}$ are, and how they intersect. In other words, one gets different smooth models as one varies the subset $\mathcal{P}$. We assume that the collection $\mathcal{C}$ is closed under sum. The following definition describes the most basic combinatorial idea for the wonderful models.

Definition 4.1. *A subset $\mathcal{P}$ of $\mathcal{C}$ is a* building set *if every $A \in \mathcal{C}$ is the direct sum $A = \bigoplus_i B_i$ of the maximal elements B_i of $\mathcal{P}$ that are contained in A, such that, in addition, for every $C \in \mathcal{C}$ with $C \subseteq A$ also $C = \bigoplus_i (C \cap B_i)$. Elements of a building set are called* building blocks.

Our definition is a slight specialization of the one in [33, Theorem (2) in 2.3]. In their notation, our building sets $\mathcal{P}$ are those for which $\mathcal{C} = \mathcal{C}_{\mathcal{P}}$ (see [33, 2.3]). Note that a building set is not in general closed under sum again. Definition 4.1 singles out subsets $\mathcal{P}$ of $\mathcal{C}$ for which taking the closure of (47) makes sense. Indeed one has

Theorem 4.1 (De Concini, Procesi)**.** *If $\mathcal{P}$ is a building set, then the closure $Y_{\mathcal{P}}$ of the graph of (47) provides a smooth model for the arrangement $V_{\mathcal{C}}$. Its*

divisor $\mathcal{E}$ is the union of smooth irreducible components $\mathcal{E}_A$, one for each $A \in \mathcal{P}$. □

4.3. **Irreducibility and building sets.** Let us now turn toward the building sets and the wonderful models for $V = M^{V_0}$ and $\mathcal{C} = \mathcal{C}_{sing}(\Gamma)$ or $\mathcal{C}_{div}(\Gamma)$. We review some basic notions from [33] and apply them to the special case of graph arrangements.

Definition 4.2. *For an $A \in \mathcal{C}$ a* decomposition *of A is a family of non-zero $A_1, \ldots, A_k \in \mathcal{C}$ such that $A = A_1 \oplus \ldots \oplus A_k$ and, for every $B \subseteq A, B \in \mathcal{C}$, also $B \cap A_1, \ldots, B \cap A_k \in \mathcal{C}$ and $B = (B \cap A_1) \oplus \ldots \oplus (B \cap A_k)$. If A admits only the trivial decomposition it is called* irreducible*. The set of irreducible elements is denoted $\mathcal{F}(\mathcal{C})$.*

It is easily seen that A is irreducible if and only if there are no $A_1, A_2 \in \mathcal{C}$ such that $A = A_1 \oplus A_2$ and $B = (B \cap A_1) + (B \cap A_2)$ for all $B \subseteq A, B \in \mathcal{C}$. For if $A = A_1 \oplus A_2 \oplus A_2'$ is a decomposition of A, then $A = A_1 \oplus (A_2 \oplus A_2')$ is a decomposition of A into two terms since $(B \cap A_2) \oplus (B \cap A_2') \subseteq B \cap (A_2 \oplus A_2')$.

We now describe the irreducible elements of $\mathcal{C}_{sing}(\Gamma), \mathcal{C}_{div}(\Gamma)$. Recall our definition of a subgraph γ of Γ : If Γ is a graph with set of vertices $V(\Gamma)$ and set of edges $E(\Gamma)$, a subgraph γ is given by a subset $E(\gamma) \subseteq E(\Gamma)$ of edges. By definition $V(\gamma) = V(\Gamma)$. However, we define $V_{\text{eff}}(\gamma)$ to be the subset of vertices in $V(\gamma)$ which are not isolated – a vertex v is not isolated if it is connected to another vertex through γ. We say γ is connected if it is connected with respect to $V_{\text{eff}}(\gamma)$ and $E(\gamma)$. In other words, the connected components of γ exclude by definition the isolated vertices. For two partitions P_1, P_2 write $P_1 \leq P_2$ if $\{i, j\} \subseteq Q \in P_1$ implies $\{i, j\} \subseteq Q' \in P_2$ for some Q'. Write $P_1 < P_2$ if $P_1 \leq P_2$ and $P_1 \neq P_2$.

Definition 4.3. *Let $\mathcal{G}$ be a collection of subgraphs of Γ. A subgraph γ of Γ is called* irreducible wrt. *$\mathcal{G}$ if for all subgraphs $\gamma_1, \gamma_2 \in \mathcal{G}$ – defining partitions $P_1 = cc(\gamma_1), P_2 = cc(\gamma_2)$ on $V(\gamma)$ – such that $P_1 \cup P_2 = \mathrm{cc}(\gamma)$ and $P_1 \cap P_2 = 0$ there exists a subgraph $g \in \mathcal{G}$ with $\mathrm{cc}(g) \leq \mathrm{cc}(\gamma)$ which is not the union of a subgraph in P_1 with a subgraph in P_2. (A subgraph in P_i is a subgraph g_i of Γ such that $cc(g_i) \cap P_i = cc(g_i)$.)*

It follows from the definition that all subgraphs with only two vertices ($|V_{\text{eff}}(\gamma)| = 2$) are irreducible (because there are no such P_1 and P_2 at all). Also, every irreducible graph is connected. Indeed, let γ be irreducible wrt. $\mathcal{G}$ and γ have two components $\gamma = \gamma_1 \sqcup \gamma_2$. Taking $P_1 = \mathrm{cc}(\gamma_1)$ and $P_2 = \mathrm{cc}(\gamma_2)$ one arrives at a contradiction. Note also that the notion of irreducibility of γ wrt. $\mathcal{G}$ depends only on $\mathrm{cc}(\gamma)$ and $\mathcal{G}$.

It turns out that the irreducible graphs are exactly those which provide irreducible subspaces:

Proposition 4.1.

$$\mathcal{F}(\mathcal{C}_{sing}(\Gamma)) = \{A_\gamma \in \mathcal{C}_{sing}(\Gamma) : \gamma \textit{ irred. wrt. all subgraphs of } \Gamma\}, \tag{48}$$

$$\mathcal{F}(\mathcal{C}_{div}(\Gamma)) = \{A_\gamma \in \mathcal{C}_{div}(\Gamma) : \gamma \textit{ divergent and irreducible wrt. all divergent subgraphs of } \Gamma\}, \tag{49}$$

$$\mathcal{F}(\mathcal{C}_{sing}(K_n)) = \{A_\gamma \in \mathcal{C}_{sing}(K_n) : \gamma \textit{ connected }\}. \tag{50}$$

Proof. (48)-(49): By the remark after the definition, A_γ is reducible in $\mathcal{C}_{sing}(\Gamma)$ ($\mathcal{C}_{div}(\Gamma)$) if and only if there are (divergent) subgraphs γ_1, γ_2 of Γ such that $A_\gamma = A_{\gamma_1} \oplus A_{\gamma_2}$ and $A_g = A_g \cap A_{\gamma_1} + A_g \cap A_{\gamma_2}$ for all (divergent) subgraphs g of Γ with $A_g \subseteq A_\gamma$ (which means $cc(g) \leq cc(\gamma)$). Using Proposition 2.5 and Proposition 2.6, this is equivalent to saying that $\mathrm{cc}(\gamma) = \mathrm{cc}(\gamma_1) \cup \mathrm{cc}(\gamma_2)$, $\mathrm{cc}(\gamma_1) \cap \mathrm{cc}(\gamma_2) = 0$ and $\mathrm{cc}(g) = (\mathrm{cc}(g) \cap \mathrm{cc}(\gamma_1)) \cup (\mathrm{cc}(g) \cap \mathrm{cc}(\gamma_2))$, whence the statement.
(50): Since the connectedness of γ is necessary for A_γ to be irreducible (see the remark after Definition 4.3), we only need to show sufficiency. Let therefore $\gamma, \gamma_1, \gamma_2$ be connected subgraphs of K_n such that $\mathrm{cc}(\gamma) = \mathrm{cc}(\gamma_1) \cup \mathrm{cc}(\gamma_2)$ and $\mathrm{cc}(\gamma_1) \cap \mathrm{cc}(\gamma_2) = 0$. Pick an edge $e \in E(K_n)$ which joins a vertex in $V_{\text{eff}}(\gamma_1)$ with one in $V_{\text{eff}}(\gamma_2)$. This gives an $A_e \in \mathcal{C}_{sing}(K_n)$ such that $A_e \cap A_{\gamma_1} = A_e \cap A_{\gamma_2} = \{0\}$. Consequently A_γ is irreducible. □

Recall the definition of a building set, Definition 4.1, which we can now rephrase as follows: All $A \in \mathcal{C}$ have a decomposition (in the sense of Definition 4.2) into the *maximal* building blocks contained in A.

The irreducible elements $\mathcal{F}(\mathcal{C})$ of a collection $\mathcal{C}$ are the minimal building set for the compactification of $V \setminus \bigcup_{A \in \mathcal{C}} A^\perp$.

Proposition 4.2. *The irreducible elements $\mathcal{F}(\mathcal{C})$, and $\mathcal{C}$ itself, form building sets in $\mathcal{C}$, and $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{P} \subseteq \mathcal{C}$ for every building set $\mathcal{P}$ in $\mathcal{C}$.*

Proof. (see also [33][Proposition 2.1 and Theorem 2.3 (3)]) It is obvious that every $A \in \mathcal{C}$ has a decomposition into irreducible elements B_i. Assume one of them is not maximal, say $A = \bigoplus_i B_i$ with $B_1 \subsetneq B \in \mathcal{F}(\mathcal{C})$. Let $C \in \mathcal{C}$, $C \subset B$, then $B = \bigoplus_i (B \cap B_i)$ with $C = \bigoplus_i (C \cap B_i) = \bigoplus_i C \cap (B \cap B_i)$ would be a nontrivial decomposition of B. Therefore $\mathcal{F}(\mathcal{C})$ is a building set. Let now $\mathcal{P}$ be an arbitrary building set, and $A \in \mathcal{F}(\mathcal{C})$. There is a decomposition of A into maximal building blocks, but since A is irreducible the decomposition is trivial and A is a building block itself. Consequently $\mathcal{F}(C) \subseteq \mathcal{P}$. The remaining statements are obvious. □

We conclude this section with a short remark about reducible divergent graphs.

Proposition 4.3. *Let $\gamma \subseteq \Gamma$ be divergent, and let $A_\gamma = A_{\gamma_1} \oplus \ldots \oplus A_{\gamma_k}$ be a decomposition in $\mathcal{C}_{div}(\Gamma)$. We may assume that the γ_i are saturated, that is $\gamma_i = (\gamma_i)_s$. Then all γ_i are divergent themselves.*

Proof. Using (18), we need to conclude $(d-2)|E(\gamma_i)| = \dim A_{\gamma_i}$ from $(d-2)|E(\gamma)| = \dim A_\gamma$. Since the γ_i decompose γ and are saturated, we have a disjoint union $E(\gamma) = E(\gamma_1) \sqcup \ldots \sqcup E(\gamma_k)$. Also $\dim A_\gamma = \sum_i \dim A_{\gamma_i}$. Consequently, if we had an i such that $(d-2)|E(\gamma_i)| \lneq \dim A_{\gamma_i}$, then there would be a j such that $(d-2)|E(\gamma_j)| \gneq \dim A_{\gamma_j}$, in contradiction to Γ being at most logarithmic (see Definition 2.2). □

4.4. **Nested sets.** Let $\mathcal{P}$ be a building set in $\mathcal{C}$. We are now ready to describe the wonderful models $Y_\mathcal{P}$. Note that $V_\mathcal{C} = V_{\mathcal{F}(\mathcal{C})}$ since $(A_1 \oplus A_2)^\perp = A_1^\perp \cap A_2^\perp$. Consequently, using Proposition 4.2, $V_\mathcal{C} = V_\mathcal{P}$. The charts for $Y_\mathcal{P}$ are assembled from *nested* sets of subspaces, defined as follows (see also [33, Section 2.4])

Definition 4.4. *A subset $\mathcal{N}$ of $\mathcal{P}$ is nested wrt. $\mathcal{P}$ if for any $A_1, \ldots, A_k \in \mathcal{N}$ pairwise non-comparable we have $\sum_{i=1}^k A_i \notin \mathcal{P}$ (unless $k = 1$).*

Note that in particular the $\mathcal{F}(\mathcal{C})$-nested sets are sets of *irreducible* subspaces. We now determine the $\mathcal{P}$-nested sets of $\mathcal{C} = \mathcal{C}_{sing}(\Gamma)$, $\mathcal{C}_{div}(\Gamma)$, $\mathcal{C}_{sing}(K_n)$ for the minimal and maximal building sets $\mathcal{P} = \mathcal{F}(\mathcal{C})$ and $\mathcal{P} = \mathcal{C}$, respectively. Let γ be a subgraph of Γ. Recall from section 2.3 that A_γ depends only on the partition $cc(\gamma)$ of the vertex set $V(\Gamma)$.

Proposition 4.4. *A subset $\mathcal{N} = \{A_{\gamma_1}, \ldots, A_{\gamma_k}\}$ is nested in $\mathcal{C} = \mathcal{C}_{sing}(\Gamma)$ (resp. $\mathcal{C}_{div}(\Gamma)$)*

(i) *wrt. $\mathcal{P} = \mathcal{C}$ if and only if the set $\{cc(\gamma_1), \ldots, cc(\gamma_k)\}$ is linearly ordered by the strict order $<$ of partitions,*
(ii) *wrt. $\mathcal{P} = \mathcal{F}(\mathcal{C})$ if and only if the γ_i are irreducible wrt. all (divergent) subgraphs of Γ, and for all $I \subseteq \{1, \ldots, k\}$, $|I| \geq 2$, the graph $\bigcup_{i \in I} \gamma_i$ is reducible wrt. (divergent) subgraphs, unless $cc(\gamma_i) < cc(\gamma_j)$ for some $i, j \in I$.*

Proof. Straightforward from the definitions. □

Proposition 4.5. *A subset $\mathcal{N} = \{A_{\gamma_1}, \ldots, A_{\gamma_k}\}$ is nested in $\mathcal{C}_{sing}(K_n)$ wrt. the minimal building set if and only if the γ_i are connected and for $i \neq j$ if either $V_{\text{eff}}(\gamma_i) \subset V_{\text{eff}}(\gamma_j)$, $V_{\text{eff}}(\gamma_j) \subset V_{\text{eff}}(\gamma_i)$, or $V_{\text{eff}}(\gamma_i) \cap V_{\text{eff}}(\gamma_j) = \emptyset$.*

Proof. Straightforward from (50). □

We recall further notions from [33, Section 2]. Let $\mathcal{P}$ be a building set and $\mathcal{N}$ a $\mathcal{P}$-nested set for $\mathcal{C}$. For every $x \in V^\vee \setminus \{0\}$ the set of subspaces in $\mathcal{N}' = \mathcal{N} \cup \{V^\vee\}$ containing x is linearly ordered and non-empty. Write $p(x)$ for the minimal element in $\mathcal{N}'$. This defines a map $p : V^\vee \setminus \{0\} \to \mathcal{N}'$.

Definition 4.5. *A basis $\mathcal{B}$ of $V^\vee$ is* adapted *to $\mathcal{N}$ if, for all $A \in \mathcal{N}$ the set $\mathcal{B} \cap A$ generates A. A* marking *of $\mathcal{B}$ is, for all $A \in \mathcal{N}$, the choice of an element $x_A \in \mathcal{B}$ with $p(x_A) = A$.*

In the case of arrangements coming from graphs, $\mathcal{C} = \mathcal{C}_{sing}(\Gamma), \mathcal{C}_{div}(\Gamma)$, particular bases are obtained from spanning forests, cf. Proposition 2.4.

Proposition 4.6. *Let t be a spanning tree*[3] *of Γ. Then the basis $\mathcal{B} = \{(e^\vee i_\Gamma)^j : e \in E(t),\ j = 0, \dots, d-1\}$ of $(M^{V_0})^\vee$ is adapted to $\mathcal{N} = \{A_{\gamma_1}, \dots, A_{\gamma_k}\}$ if and only if the graph with edges $\{e \in E(t) : e \leq cc(\gamma_i)\}$ is a spanning forest for $cc(\gamma_i)$ for all $i = 1, \dots, k$.*

Proof. Straightforward from Proposition 2.4. □

We call such a spanning forest an *adapted spanning forest*. Also, a marking of the basis corresponds to a certain subforest $E(t_M) \subseteq E(t)$ with $k+1$ edges, and a choice of one out of d copies for each edge.

Proposition 4.7. *Let $\mathcal{N}$ be a $\mathcal{P}$-nested set for $\mathcal{C} = \mathcal{C}_{sing}(\Gamma)$ or $\mathcal{C}_{div}(\Gamma)$. Then there exists an adapted spanning tree.*

Proof. By induction on the dimension: Let $A_{\gamma_1}, \dots, A_{\gamma_h}$ be the maximal elements in $\mathcal{N}$ contained in a given A_γ. Assume an adapted spanning forest (see Proposition 4.6) for each of the A_{γ_i} is chosen. The union of these bases is then a basis $\mathcal{B}'$ for $\bigoplus_i A_{\gamma_i}$ (the sum is direct because $\mathcal{N}$ is nested and the A_{γ_i} maximal). The set $\{(e^\vee i_\Gamma)^j : e \in E(\gamma)\}$ is a generating set for A_γ. Extending the basis $\mathcal{B}'$ to a basis for A_γ using this generating set provides, by Proposition 2.4, an adapted spanning forest for γ. □

Let us now return to marked bases in general. A marking of an adapted basis $\mathcal{B}$ provides a partial order on $\mathcal{B}$: $y_1 \preceq y_2$ if $p(y_1) \subseteq p(y_2)$ and y_2 is marked. This partial order determines a map $\rho : V \to V$ as follows. Consider the elements of $\mathcal{B} = \{y_1, \dots, y_k\}$ as (linear) coordinates on the source V. The (nonlinear) coordinates $(x_1, \dots, x_k)$ of the image $\rho(y_1, \dots, y_k)$ are

[3] Recall that Γ is connected.

given by

$$x_i = \prod_{y_i \preceq y_j} y_j = \begin{cases} y_i \prod_{p(y_i)\subset A} y_A & \text{if } y_i \text{ is not marked,} \\ \prod_{p(y_i)\subset A} y_A & \text{if } y_i \text{ is marked.} \end{cases} \tag{51}$$

The map ρ, and already the partial order $\preceq$, determine implicitly a sequence of blowups. Indeed

Proposition 4.8. *(see* [33, Lemma 3.1]*)*

(i) *ρ is a birational morphism,*
(ii) *$\rho(\{y_A = 0\}) = A^\perp$ and*
(iii) *ρ restricts to an isomorphism $V \setminus \bigcup_{A\in\mathcal{N}}\{x_A = 0\} \cong V \setminus \bigcup_{A\in\mathcal{N}} A^\perp$.*
(iv) *Let $x \in V^\vee \setminus \{0\}$ and $p(x) = A \in \mathcal{N}$. Then $x = x_A P_x(y_i)$, where $x_A = \prod_{y_A \preceq y_i} y_i$ and P_x is a polynomial depending on the variables $y_i < x_A$, and linear in each variable, that is $\partial^2 P_x/\partial y_i^2 = 0$.*

□

4.5. **Properties of the Wonderful Models.** Recall the definition (47) of the wondeful models: $Y_\mathcal{P}$ is the closure of $V \setminus V_\mathcal{P}$ embedded into $V \times \prod_{A\in\mathcal{P}} \mathbb{P}(V/A^\perp)$. The birational map $\beta : Y_\mathcal{P} \to V$ is simply the projection onto the first factor V. Let $\mathcal{N}$ be a $\mathcal{P}$-nested set in $\mathcal{C}$, and $\mathcal{B}$ an adapted, marked basis of $V^\vee$. Both determine a birational map $\rho : V \to V$ as defined in (51). For a given building block $B \in \mathcal{P}$ set $Z_B = \{P_x = 0, x \in B\} \subset V$. The composition of ρ with the rational map $V \to V/A^\perp \to \mathbb{P}(V/A^\perp)$ is then defined as a regular morphism outside of Z_B. Doing this for every factor in $\prod_{A\in\mathcal{P}} \mathbb{P}(V/A^\perp)$, one gets an open embedding $j^\mathcal{B}_\mathcal{N} : U^\mathcal{B}_\mathcal{N} = V \setminus \bigcup_{B\in\mathcal{P}} Z_B \hookrightarrow Y_\mathcal{P}$ [33, Theorem 3.1]. Write $Y^\mathcal{B}_\mathcal{N} = j^\mathcal{B}_\mathcal{N}(U^\mathcal{B}_\mathcal{N})$. As $\mathcal{N}$ and the marking of $\mathcal{B}$ vary, one obtains an atlas $(Y^\mathcal{B}_\mathcal{N}, (j^\mathcal{B}_\mathcal{N})^{-1})$ for $Y_\mathcal{P}$. It is also shown in [33, Theorem 3.1] that the divisor $\mathcal{E} = \beta^{-1}(V_\mathcal{P})$ is given locally by

$$(j^\mathcal{B}_\mathcal{N})^{-1}(\mathcal{E} \cap Y^\mathcal{B}_\mathcal{N}) = \left\{ \prod_{A\in\mathcal{N}} y_A = 0 \right\}. \tag{52}$$

Remarks. In the case of the full graph K_n, the minimal wonderful model $Y_{\mathcal{F}(\mathcal{C}_{sing}(K_n))}$ is known as the Fulton-MacPherson compactification [40], while the maximal wonderful model $Y_{\mathcal{C}_{sing}(K_n)}$ has been described in detail by Ulyanov [73]. For any graph, the benefit of the minimal model is that the divisor is small in the sense that it has only a minimal number of irreducible components, whereas the actual construction by a sequence of blowups is less canonical. On the other hand, for the maximal model, which has a larger number of irreducible components, one can proceed in the obvious way blowing up strict transforms by increasing dimension. See figures 1, 2, 3 for an example. Also the resolution of projective arrangements described

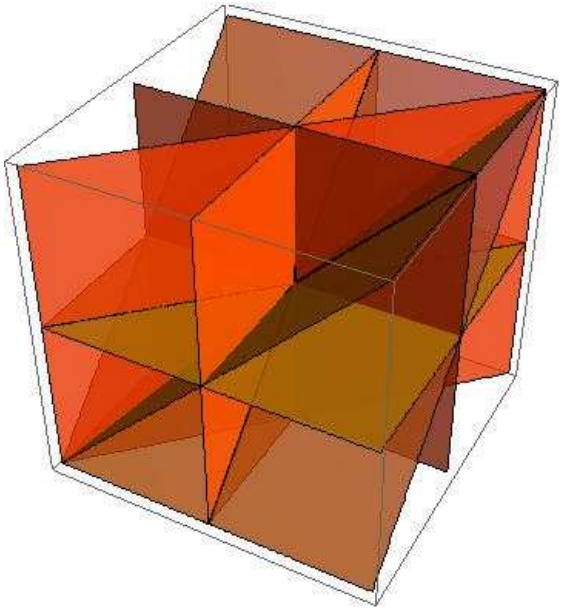
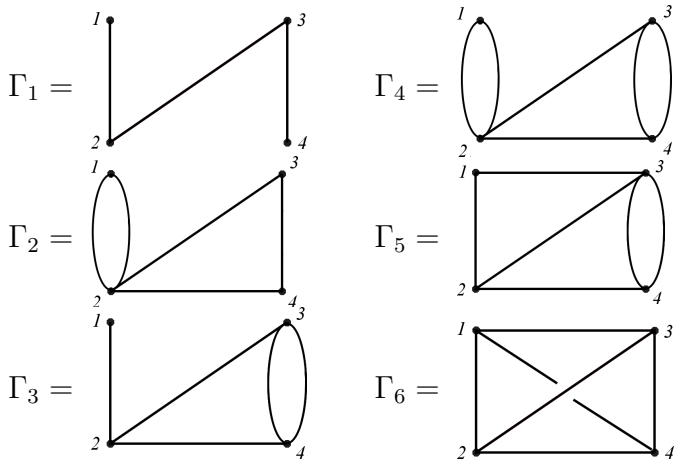


FIGURE 1. A picture of $\mathbb{R}^{V_0}_{sing}(K_4)$.

in [36] and referred to in [14, Lemma 5.1] proceeds by increasing dimension and corresponds to the maximal wonderful model.

4.6. **Examples.** For the fixed vertex set $V = \{1, 2, 3, 4\}$ we consider a series of graphs on V with increasing complexity. Only some of them are relevant for renormalization.



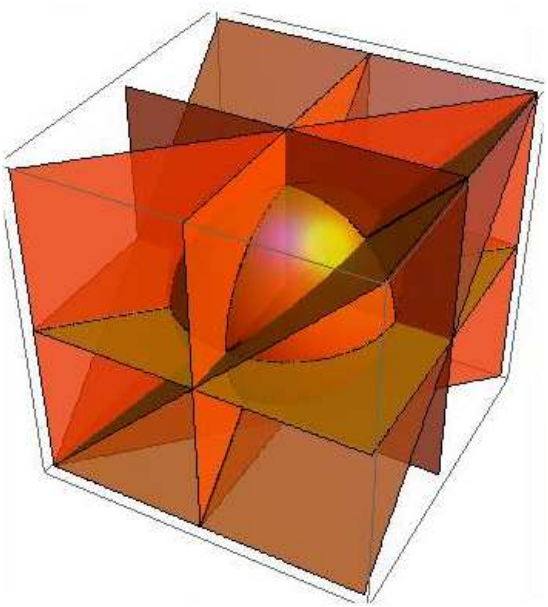


FIGURE 2. (Spherical) blowup of the origin in $\mathbb{R}^{V_0}_{sing}(K_4)$, where projective spaces are replaced by spheres. The maximal wonderful model would proceed by blowing up all strict transforms of lines incident to the exceptional divisor, and finally the strict transforms of the planes.

For these graphs, we examine the arrangements $M^{V_0}_{sing}$ and $M^{V_0}_{div}$, the irreducible subspaces and nested sets for the minimal and maximal building set, respectively. Write A_{ij} for A_e with e an edge connecting the vertices i and j.

$$
\begin{aligned}
\mathcal{C}_{sing}(\Gamma_1) &= \{A_{12}, A_{23}, A_{34}, \text{ and their sums}\} \\
\left.\begin{matrix}\mathcal{C}_{sing}(\Gamma_2)\\ \mathcal{C}_{sing}(\Gamma_3)\\ \mathcal{C}_{sing}(\Gamma_4)\end{matrix}\right\} &= \{A_{12}, A_{23}, A_{24}, A_{34}, \text{ and their sums}\} \\
\mathcal{C}_{sing}(\Gamma_5) &= \{A_{12}, A_{13}, A_{23}, A_{24}, A_{34}, \text{ and their sums}\} \\
\mathcal{C}_{sing}(\Gamma_6) &= \{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}, \text{ and their sums}\}
\end{aligned}
$$

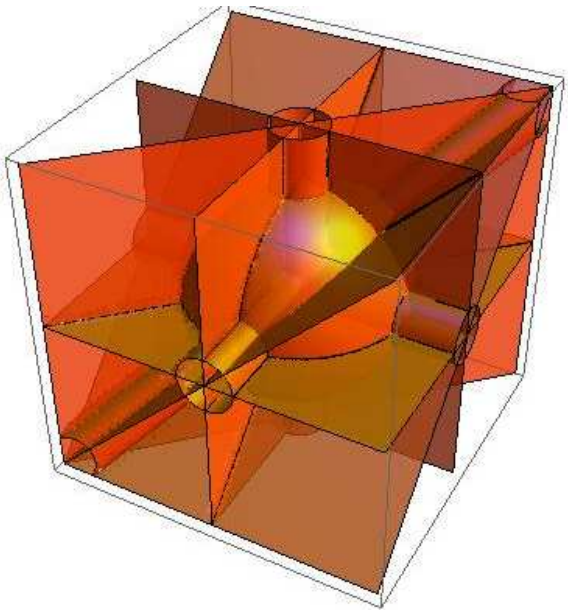


FIGURE 3. Minimal (spherical) model of $\mathbb{R}^{V_0}_{sing}(K_4)$, corresponding to the Fulton-MacPherson compactification of the configuration space of 4 points in $\mathbb{R}$. After the central blowup, only those strict transforms of lines are blown up which are not a normal crossing intersection in the first place.

The divergent arrangements are determined by the collections of dual spaces:

$$
\begin{aligned}
\mathcal{C}_{div}(\Gamma_1) &= \emptyset \\
\mathcal{C}_{div}(\Gamma_2) &= \{A_{12}\} \\
\mathcal{C}_{div}(\Gamma_3) &= \{A_{34}, A_{23}+A_{34}\} \\
\mathcal{C}_{div}(\Gamma_4) &= \{A_{12}, A_{34}, A_{23}+A_{34}, A_{12}+A_{34}, A_{12}+A_{23}+A_{34}\} \\
\mathcal{C}_{div}(\Gamma_5) &= \{A_{34}, A_{23}+A_{34}, A_{12}+A_{23}+A_{34}\} \\
\mathcal{C}_{div}(\Gamma_6) &= \{A_{12}+A_{23}+A_{34}\}
\end{aligned}
$$

The irreducible singular subspace collections are

$$\begin{aligned}
\mathcal{F}(\mathcal{C}_{sing}(\Gamma_1)) &= \{A_{12}, A_{23}, A_{34}\} \\
\left.\begin{array}{l}\mathcal{F}(\mathcal{C}_{sing}(\Gamma_2)) \\ \mathcal{F}(\mathcal{C}_{sing}(\Gamma_3)) \\ \mathcal{F}(\mathcal{C}_{sing}(\Gamma_4))\end{array}\right\} &= \{A_{12}, A_{23}, A_{24}, A_{34}, A_{23}+A_{34}\} \\
\mathcal{F}(\mathcal{C}_{sing}(\Gamma_5)) &= \{A_{12}, A_{13}, A_{23}, A_{24}, A_{34}, \\
&\quad A_{12}+A_{13}, A_{23}+A_{24}, A_{12}+A_{23}+A_{34}\} \\
\mathcal{F}(\mathcal{C}_{sing}(\Gamma_6)) &= \{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}, \\
&\quad A_{12}+A_{13}, A_{12}+A_{14}, A_{13}+A_{14}, A_{23}+A_{34}, \\
&\quad A_{12}+A_{23}+A_{34}\}
\end{aligned}$$

The irreducible divergent subspace collections are

$$\begin{aligned}
\mathcal{F}(\mathcal{C}_{div}(\Gamma_1)) &= \emptyset \\
\mathcal{F}(\mathcal{C}_{div}(\Gamma_2)) &= \{A_{12}\} \\
\mathcal{F}(\mathcal{C}_{div}(\Gamma_3)) &= \{A_{34}, A_{23}+A_{34}\} \\
\mathcal{F}(\mathcal{C}_{div}(\Gamma_4)) &= \{A_{12}, A_{34}, A_{23}+A_{34}\} \\
\mathcal{F}(\mathcal{C}_{div}(\Gamma_5)) &= \{A_{34}, A_{23}+A_{34}, A_{12}+A_{23}+A_{34}\} \\
\mathcal{F}(\mathcal{C}_{div}(\Gamma_6)) &= \{A_{12}+A_{23}+A_{34}\}
\end{aligned}$$

The maximal nested sets of the divergent collection wrt. the minimal building set:

$$\begin{aligned}
&\text{for } \Gamma_1: && \emptyset \\
&\text{for } \Gamma_2: && \{A_{12}\} \\
&\text{for } \Gamma_3: && \{A_{234}, A_{34}\} \\
&\text{for } \Gamma_4: && \{A_{12}, A_{234}, A_{34}\} \\
&\text{for } \Gamma_5: && \{A_{1234}, A_{234}, A_{34}\} \\
&\text{for } \Gamma_6: && \{A_{1234}\}
\end{aligned}$$

The maximal nested sets of the divergent collection wrt. the maximal building set:

$$\begin{aligned}
&\text{for } \Gamma_1: && \emptyset\\
&\text{for } \Gamma_2: && \{A_{12}\}\\
&\text{for } \Gamma_3: && \{A_{234}, A_{34}\}\\
&\text{for } \Gamma_4: && \{A_{1234}, A_{12}\oplus A_{34}, A_{12}\},\\
& && \{A_{1234}, A_{12}\oplus A_{34}, A_{34}\},\\
& && \{A_{1234}, A_{234}, A_{34}\}\\
&\text{for } \Gamma_5: && \{A_{1234}, A_{234}, A_{34}\}\\
&\text{for } \Gamma_6: && \{A_{1234}\}
\end{aligned}$$

5. Laurent coefficients of the meromorphic extension

5.1. **The Feynman distribution pulled back onto the wonderful model.** Recall the definition (7) of the Feynman distribution $u_\Gamma = \prod_{i<j} u_0(x_i - x_j)^{n_{ij}}$. We write $\underline{u}_\Gamma = \Phi_* u_\Gamma$ where Φ is the projection along the thin diagonal defined at the end of section 2.1. It is clear from the discussion in section 2 that $\underline{u}_\Gamma = (i_\Gamma^{\oplus d})^* u_0^{\otimes |E(\Gamma)|}$. Let $\beta : Y_{\mathcal{P}} \to M^{V_0}$ be a wonderful model for the arrangement $(M^{V_0})_{div}(\Gamma)$ or $(M^{V_0})_{sing}(\Gamma)$. The purpose of this section is to study the regularized pullback $\beta^* \tilde{\underline{u}}_\Gamma^s$ (as a density-valued meromorphic function of s) of $\tilde{\underline{u}}_\Gamma^s$ onto $Y_{\mathcal{P}} \setminus \mathcal{E}$.

Theorem 5.1. *Let $\mathcal{N}$ be a $\mathcal{P}$-nested set in $\mathcal{C}_{div}(\Gamma)$ ($\mathcal{C}_{sing}(\Gamma)$), and $\mathcal{B} = \{y_e^i : e \in E(t), i = 0, \dots, d-1\}$ an adapted basis with marked elements $y_A^{i_A}$, $A \in \mathcal{N}$. Then, in the chart $U_{\mathcal{N}}^{\mathcal{B}}$,*

$$\beta^* \underline{u}_\Gamma(\{y_e^i\}) = f_\Gamma(\{y_e^i\}) \prod_{A\in\mathcal{N}} (y_A^{i_A})^{n_A} \tag{53}$$

where $f_\Gamma \in L^1_{loc}(U_{\mathcal{N}}^{\mathcal{B}})$ ($C^\infty(U_{\mathcal{N}}^{\mathcal{B}})$), and $n_A \in -2\mathbb{N} \cup \{0\}$. More precisely

$$n_{A_\gamma} = (2-d)|E(\gamma_s)|. \tag{54}$$

In addition, f_Γ is smooth in the variables $y_A^{i_A}$, $A \in \mathcal{N}$.

Note: γ_s is the subgraph defined in Definition 2.1. Divergent subgraphs are saturated (Proposition 2.1).

Proof. Recall from the last paragraph of section 4.5 that the map β is given in the chart $U_{\mathcal{N}}^{\mathcal{B}}$ by ρ (see (51)):

$$\rho : \sum_{j=0}^{d-1} \sum_{e\in E(t)} y_e^j b_e^j \mapsto \sum_{j=0}^{d-1} \sum_{e\in E(t)} \prod_{y_e^j \preceq y_{e'}^k} y_{e'}^k b_e^j$$

where $\preceq$ is the partial order on the basis $\mathcal{B} = \{y_e^j\}$ of $(M^{V_0})^\vee$ adapted to $\mathcal{N}$. Consequently, using (28),

$$\begin{aligned} \beta^* \underline{u}_\Gamma(\{y_e^j\}) &= u_0^{\otimes E(\Gamma)} i_\Gamma^{\oplus d} \rho(\{y_e^j\}) \\ &= \prod_{e \in E(\Gamma)} u_0 \left(\left\{ \Sigma_{e' \rightsquigarrow e} \Pi_{y_{e'}^j \preceq y_{e''}^k} y_{e''}^k \right\}_{j=0}^{d-1} \right). \end{aligned} \tag{55}$$

By Proposition 4.8 (iv), each $\xi_e^j = \sum_{e' \rightsquigarrow e} x_{e'}^j$ is a product $x_A^{i_A} P_{\xi_e^j}(\{y_j^i\})$ where $A = p(\xi_e^j) \in \mathcal{N}$. As u_0 is homogeneous (5), the factor $x_A^{i_A} = \prod_{A \subseteq B \in \mathcal{N}} {y_B}^{i_B}$, can be pulled out, supplied with an exponent $2-d$. Since $x_A^{i_A} = \prod_{A \subseteq B} {y_B}^{i_B}$, the factor $(y_{A_\gamma}^{i_A})^{2-d}$ appears once for each $e \in E(\Gamma)$ such that $A_e \subseteq A_\gamma$, in other words for each $e \leq cc(\gamma)$. Hence (54). We finally show that the remaining factor

$$f_\Gamma(\{y_i^j\}) = \prod_{e \in E(\Gamma)} u_0(\{P_{\xi_e^j}(\{y_i^k\})\}_{j=0}^{d-1}) \tag{56}$$

of $\beta^* \underline{u}_\Gamma$ satisfies $f_\Gamma \in L^1_{loc}(U_{\mathcal{N}}^{\mathcal{B}})$ if the divergent arrangement was resolved or $f_\Gamma \in C^\infty(U_{\mathcal{N}}^{\mathcal{B}})$ if the singular arrangement was resolved, respectively. The set $U_{\mathcal{N}}^{\mathcal{B}}$ contains by definition no point with coordinates y_i^j such that for any building block $B \in \mathcal{P}$ all $P_x(\{y_i^j\}) = 0, x \in B$. In the case of $\mathcal{C}_{sing}(\Gamma)$, all $A_e \in \mathcal{P}$, $(e \in E(\Gamma))$, since they are irreducible, see Proposition 4.2. On the other hand, A_e is spanned by the ξ_e^j, $j = 0, \ldots, d-1$. Therefore none of the $P_{\xi_e^j}$ in (56) vanishes on $U_{\mathcal{N}}^{\mathcal{B}}$. Hence, using (6), $f_\Gamma \in C^\infty(U_{\mathcal{N}}^{\mathcal{B}})$. In the case of $\mathcal{C}_{div}(\Gamma)$, let γ be divergent. By Proposition 4.3 we may assume without loss that A_γ is irreducible. Therefore $A_\gamma \in \mathcal{P}$ as in the first case. By the same argument as above, not all the $P_{\xi_e^j}$ in the arguments of $\prod_{e \in E(\gamma)} u_0$ can vanish at the same time on $U_{\mathcal{N}}^{\mathcal{B}}$, whence this product is now locally integrable. In order to see that f_Γ is smooth in the $y_A^{i_A}$, it suffices to show that not all d of the $P_{\xi_e^j}(\{y_i^k\}) \to 0$ (for $j = 0, \ldots, d-1$) as the $y_A^{i_A} \to 0$ while the other coordinates are fixed. From Proposition 4.8 (iv) we know that every P_x is linear in the $y_A^{i_A}$, if therefore all $P_{\xi_e^j}$ vanished at some $y_A^{i_A} = 0$ they would have $y_A^{i_A}$ as a common factor. This contradicts Proposition 4.8 as then $p(\xi_e) \subseteq A$. □

In the preceding theorem, $\underline{u}_\Gamma$ was pulled back along β as a distribution. The next corollary clarifies the situation for the density $\beta^* \tilde{u}_\Gamma = \beta^*(\underline{u}_\Gamma |dx|)$. We write $|dy|$ for $|dy_{e_1}^0 \wedge \ldots \wedge dy_{e_k}^{d-1}|$.

Corollary 5.1. *Under the assumptions of Theorem 5.1,*

$$\beta^* \tilde{u}_\Gamma(\{y_e^i\})|dy| = f_\Gamma(\{y_e^i\}) \prod_{A \in \mathcal{N}} |y_A^{i_A}|^{m_A} |dy| \tag{57}$$

where

$$m_{A_\gamma} = 2|E(\gamma_s)| - d\dim H_1(\gamma_s) - 1 \geq -1. \tag{58}$$

In the case of the divergent arrangement $\mathcal{C}_{div}(\Gamma)$*, all* $m_{A_\gamma} = -1$*, and moreover*

$$\beta^*\tilde{u}^s_\Gamma(\{y^i_e\})|dy| = f^s_\Gamma(\{y^i_e\}) \prod_{A\in\mathcal{N}} |y^{i_A}_A|^{-d_A s + d_A - 1}|dy| \tag{59}$$

where $d_A = \dim A$.

We also write $d_\gamma = d_{A_\gamma}$.

Proof. Formally,

$$\begin{aligned} |dx| &= |\bigwedge_{e\in E(t), j=0\ldots,d-1} dx^j_e| = |\bigwedge d \prod_{y^j_e \preceq y^k_{e'}} y^k_{e'}| \\ &= \prod_{A\in\mathcal{N}} |y^{i_A}_A|^{q_A}| \bigwedge dy^j_e| \end{aligned}$$

where the q_A are determined as follows. Since the x^j_e, $(j = 0, \ldots, d-1)$ span A_e, the factor $y^{i_{A_\gamma}}_{A_\gamma}$ appears from all dx^j_e such that $e \leq cc(\gamma)$, except one, namely $dx^{i_{A_\gamma}}_{A_\gamma}$ itself which corresponds to the marking. Since t is an adapted spanning tree, the set $\{e \in E(t) : e \leq cc(\gamma)\}$ defines a spanning forest of γ, and one concludes using Proposition 2.4 that $q_{A_\gamma} = d_\gamma - 1$. Finally note that $\dim H_1(\gamma_s) = |E(\gamma_s)| - d_\gamma/d$ and Γ is at most logarithmic. □

5.2. **Combinatorial description of the Laurent coefficients.** Let $V = V(\Gamma)$, $E = E(\Gamma)$ and $p : V \to V'$ a map of sets which is not injective. In the dual this defines a map $p^\vee : k^{V'} \to k^V$ sending $\sum_{v\in V'} \alpha_{v'}v'$ to $\sum_{v\in V} \alpha_{p(v)}v$. Let $E(\gamma) \subseteq E(\Gamma)$. Then the graph γ_p with vertex set $V(\gamma_p) = V'$ and set of edges $E(\gamma_p) = E(\gamma)$ such that $\delta_{\gamma_p} = \delta_\gamma \circ p^\vee : k^{V(\gamma_p)} \to k^{E(\gamma_p)}$ (see (18)) is called *the graph* γ *contracted along* p.

Note: The graph contracted along p may have loops. It is not necessarily a subgraph of Γ anymore.

We assume, as in (9), a distinguished vertex $v_0 \in V(\Gamma)$ such that $V_0 = V(\Gamma) \setminus \{v_0\}$. Let now t be a spanning tree of Γ and $s \subseteq t$ a subforest of t. This defines a map $p_{t,s} : V(\Gamma) \to V(\Gamma)$ as follows: Let $v \in V(\Gamma)$ be given. Since t is a spanning tree of Γ, there is a unique path t_v in t from v_0 to v. Let $p_{t,s}(v)$ be the unique vertex which is connected to v by edges of s only and is nearest to v_0 on the path t_v. See figure 4 for an example. This gives

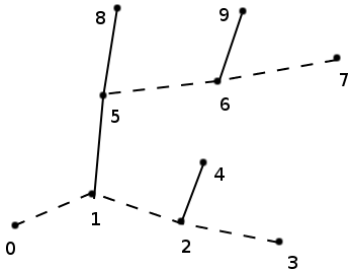


FIGURE 4. The edges of s are broken lines, the edges of $t \setminus s$ full lines. $p_{t,s}(\{v_0, v_1, v_2, v_3\}) = v_0$, $p_{t,s}(v_4) = v_4$, $p_{t,s}(\{v_5, v_6, v_7\}) = v_5$, $p_{t,s}(v_8) = v_8$, $p_{t,s}(v_9) = v_9$.

us a graph $\Gamma_{p_{t,s}}$. It is obvious from the construction that $t \setminus s$ is a spanning forest of $\Gamma_{p_{t,s}}$ whereas all edges of s are transformed into loops.

Let $\mathcal{N} = \{A_{\gamma_1}, \ldots, A_{\gamma_k}\}$ be a $\mathcal{P}$-nested set in $\mathcal{C}_{sing}(\Gamma)$ or $\mathcal{C}_{div}(\Gamma)$. Let t be an adapted spanning tree. All γ_i are assumed saturated. We define the graph $\gamma_i//\mathcal{N}$ as follows. Let $A_{\gamma_{j_1}}, \ldots, A_{\gamma_{j_l}}$ be the maximal elements $\subseteq A_{\gamma_i}$. Let s be the forest defined by $E(s) = E(t) \cap (E(\gamma_{j_1}) \cup \ldots \cup E(\gamma_{j_l}))$. Then $\gamma_i//\mathcal{N}$ is the graph with edges $E(\gamma_i) \setminus \bigcup_{m=1}^{l} E(\gamma_{j_m})$ contracted along the map $p_{t,s}$.

The graph $\gamma_i//\mathcal{N}$ obviously depends on t, although only up to a permutation of the vertices, as is easily verified.

Lemma 5.1. *Under the assumptions above:*

(i) *The graph $\gamma_i//\mathcal{N}$ has no loops.*
(ii) *If γ_i is connected, so is $\gamma_i//\mathcal{N}$ (wrt. $V_{\text{eff}}(\gamma_i//\mathcal{N})$).*
(iii) *In the case of the divergent collection $\mathcal{C}_{div}(\Gamma)$, let $\mathcal{N}$ be a maximal nested set. If γ_i is connected, $\gamma_i//\mathcal{N}$ is at most logarithmic and primitive. Therefore $\operatorname{res}(\gamma_i//\mathcal{N})$ is defined (see (46)).*
(iv) *In this case $\operatorname{res}(\gamma_i//\mathcal{N})$ does not depend upon the choice of an adapted spanning tree t.*

Note that for $\mathcal{P} = \mathcal{F}(\mathcal{C})$ every γ_i is connected (as it is irreducible). For non-connected γ_i, the statements hold for each component.

Proof. (i) Suppose e were a loop in $\gamma_i//\mathcal{N}$ at the vertex v. Since γ_i has no loops, $|p_{t,s}^{-1}(v)| > 1$. However, $p_{t,s}$ moves only the vertices adjacent to edges of s. We conclude $e \in E(\gamma_{j_m})$ as the γ_j are saturated, and have a contradiction.
(ii) By construction $p^{\vee}(\sum_{v' \in V_{\text{eff}}(\gamma_i//\mathcal{N})} v') = p^{\vee}(\sum_{v' \in V(\gamma_i//\mathcal{N})} v') = \sum_{v \in V(\gamma_i)} v$

since the sum is over *all* vertices of $V_{\text{eff}}(\gamma_i)$ (the vertices not in V_{eff} map to 0). On the other hand, $p^\vee(x)$ of a sum $x = \sum_{v' \in U} v'$ where $U \subsetneq V_{\text{eff}}(\gamma_i//\mathcal{N})$, is not contained in $\operatorname{span} \sum_{v \in V(\gamma_i)} v$. Write $\delta = \delta_{\gamma_i}$ and $\delta_p = \delta_{(\gamma_i)_p}$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\gamma_i) & \longrightarrow & k^{V(\gamma_i)} & \xrightarrow{\delta} & k^{E(\gamma_i)} \\ & & & & \uparrow{\scriptstyle p^\vee} & & \uparrow \\ 0 & \longrightarrow & H^0(\gamma_i//\mathcal{N}) & \longrightarrow & k^{V_{\text{eff}}(\gamma_i//\mathcal{N})} & \xrightarrow{\delta_p} & k^{E(\gamma_i)\setminus\cup_{m=1}^{l} E(\gamma_{j_m})} \end{array}$$

Note that δ_p as a map into $k^{E((\gamma_i)_p)}$ is the same as as a map into $k^{E(\gamma_i//\mathcal{N})}$ since the missing edges are all loops. Consequently, if $x \in \ker \delta_p$, then $p^\vee(x) \in \ker \delta$, by definition of $(\gamma_i)_p$. However, because γ_i is connected, $\ker \delta = \operatorname{span} \sum_{v \in V(\gamma_i)} v$. Therefore $\dim \ker \delta_p = 1$, if δ_p is restricted to $V_{\text{eff}}(\gamma_i//\mathcal{N})$, and hence $\gamma_i//\mathcal{N}$ connected.
(iii) By definition, a graph γ on $V(\Gamma)$ is divergent if and only if $\dim A_\gamma = (d-2)|E(\gamma)|$. It is convergent if $\dim A_\gamma > (d-2)|E(\gamma)|$. We may restrict ourselves to saturated subgraphs because the number of edges increases the susceptibility to divergences, and every divergent graph is saturated. Let $\gamma_p \subseteq \gamma_i//\mathcal{N}$ be saturated as a subgraph of $\gamma_i//\mathcal{N}$. Therefore $E(\gamma_p) \subseteq E(\gamma_i)\setminus\bigcup_{m=1}^{l} E(\gamma_{j_m})$. Let now γ_s be the saturated graph for γ_p as a subgraph of γ_i. Since p maps each component of γ_{j_m} to a single vertex, $\gamma_i//\mathcal{N}$ has $\sum_{m=1}^{l} \dim A_{\gamma_{j_m}}$ components more than γ_i. More generally,

$$\dim A_{\gamma_p} = \dim A_{\gamma_s} - \dim A_{s \cap \gamma_s}.$$

On the other hand,

$$|E(\gamma_p)| = |E(\gamma_s)| - |E((s \cap \gamma_s)_s)|.$$

Therefore $(d-2)|E(\gamma_p)| \leq \dim A_{\gamma_p}$, and equality only if $\gamma_s = \gamma_i$ (equivalently $\gamma_p = \gamma_i//\mathcal{N}$) by the maximality of $\mathcal{N}$. It follows that $\gamma_i//\mathcal{N}$ is divergent, and proper subgraphs γ_p of $\gamma_i//\mathcal{N}$ are convergent, divergent, worse than logarithmically divergent if and only if they are as subgraphs of γ_i; whence $\gamma_i//\mathcal{N}$ is also at most logarithmic and primitive.
(iv) Let t, t' be two choices of an adapted spanning tree. Then $t \setminus s$ and $t' \setminus s'$ are spanning trees of $\gamma_i//\mathcal{N}$, and by the argument in the proof of Theorem 3.1 (ii) $\operatorname{res} \gamma_i//\mathcal{N}$ is independent of the basis chosen. □

We will shortly use this lemma in connection with the following theorem, which helps understand the geometry of the divisor $\mathcal{E}$ in $Y_{\mathcal{P}}$.

Theorem 5.2. *(see* [33, Theorem 3.2]*) Let $\beta : Y_{\mathcal{P}} \to M$ be a wonderful model.*

(i) *The divisor is* $\mathcal{E} = \bigcup_{P\in\mathcal{P}} \mathcal{E}_P$ *with* $\mathcal{E}_P$ *smooth irreducible and* $\beta(\mathcal{E}_P) = P^\perp$.

(ii) *The components* $\mathcal{E}_{P_1}, \dots, \mathcal{E}_{P_k}$ *have nonempty intersection if and only if* $\{P_1, \dots, P_k\}$ *is* $\mathcal{P}$*-nested. In this case the intersection is transversal.*

□

We consider only the divergent case $\mathcal{C}_{div}(\Gamma)$ with arbitrary building set $\mathcal{P}$ and conclude for the Laurent expansion at $s = 1$:

Theorem 5.3. *Let* $\tilde{w}^s_\Gamma = \beta^* \tilde{u}^s_\Gamma$ *as a density.*

(i) *The density* $\tilde{w}^s_\Gamma$ *has a pole of order* N_{max} *at* $s = 1$, *where* N_{max} *is the cardinality of the largest nested set*[4].

(ii) *Let*

$$\tilde{w}^s_\Gamma = \sum_{k=-N_{max}}^{\infty} \tilde{a}_{\Gamma,k}(s-1)^k. \tag{60}$$

Then, for $k \le -1$,

$$\operatorname{supp} \tilde{a}_{\Gamma,k} = \bigcup_{|\mathcal{N}|=-k} \bigcap_{A_\gamma\in\mathcal{N}} \mathcal{E}_\gamma,$$

which is a subset of codimension $-k$. *The union is over* $\mathcal{P}$*-nested sets* $\mathcal{N}$.

(iii) *Let* $\mathcal{P} = \mathcal{F}(\mathcal{C}_{div}(\Gamma))$. *Let* $\mathcal{N}$ *be a nested set such that* $|\mathcal{N}| = N_{max}$. *Then*

$$\tilde{a}_{\Gamma,-N_{max}}[\underline{1}] = \sum_{|\mathcal{N}|=N_{max}} \prod_{A_\gamma\in\mathcal{N}} \operatorname{res}(\gamma//\mathcal{N}). \tag{61}$$

where all γ *are assumed saturated.*

Recall from Theorem 5.1 that f_Γ is smooth in the $y_A^{i_A}$. Therefore the canonical regularization can be used consistently (see (38)). The identity (61) is known as a consequence of the scattering formula in [29] in a momentum space context. More general identities for the higher coefficients can be obtained but are not necessary for the purpose of this paper.

Proof. (i) From (59), $\tilde{w}^s_\Gamma|dy| = f^s_\Gamma \prod_{A\in\mathcal{N}} |y_A^{i_A}|^{(d_A-1)-d_A s}|dy|$ in local coordinates. By the results of section 3.3, in particular (41),

$$\tilde{w}^s_\Gamma|dy| = f^s_\Gamma \prod_{A\in\mathcal{N}} \left(-\frac{2\delta_0(y_A^{i_A})}{d_A(s-1)} + |y_A^{i_A}|^{(d_A-1)-d_A s}_{fin} \right) |dy|, \tag{62}$$

[4]We suspect, but this is not needed here, that in the divergent arrangement all maximal nested sets have (equal) cardinality N_{max}.

whence the first statement.
(ii) This follows from (62), using that $\mathcal{E}_\gamma$ is locally given by $y_{A_\gamma}^{i_{A_\gamma}} = 0$. Theorem 5.2 (ii) shows that the codimension is k.
(iii) Throughout this proof we assume all γ defining the nested set are saturated. By Theorem 5.2 (ii), for $|\mathcal{N}| = N_{max}$, the set $\cap_{\gamma\in\mathcal{N}}\mathcal{E}_\gamma$ intersects no other $\mathcal{E}_{\gamma'}$, $\gamma' \notin \mathcal{N}$. Using (ii), $\tilde{a}_{\Gamma,-N_{max}}$ is in fact supported on a disjoint union subsets of codimension k, and we may compute $\tilde{a}_{\Gamma,-N_{max}}[\underline{1}]$ on each of them and sum the results up. It suffices, therefore, to show
(63)

$$(-2)^{N_{max}} \int f_\Gamma \prod_{A_\gamma\in\mathcal{N}} \delta_0(y_{A_\gamma}^{i_{A_\gamma}})/d_\gamma|dy| = \prod_{A_\gamma\in\mathcal{N}} \mathrm{res}(\gamma//\mathcal{N}) \quad (\text{in } U_{\mathcal{N}}^{\mathcal{B}})$$

for all maximal nested sets $\mathcal{N}$. Integration inside one chart suffices since there is no other nested set $\mathcal{N}'$ such that $j(U_{\mathcal{N}'})$ covers $\cap_{A_\gamma\in\mathcal{N}}\mathcal{E}_\gamma$ and charts from another choice of marked basis need not be considered, see the argument in the proof of Theorem 3.1 (ii). Recall (28) on M^{V_0} and (55)

$$w_\Gamma(\{y_e^j\}) = (\beta^*\underline{u}_\Gamma)(\{y_e^j\}) = \prod_{e\in E(\Gamma)} u_0(\{\sum_{e'\rightsquigarrow e} \prod_{y_{e'}^j \preceq y_{e''}^k} y_{e''}^k\}_{j=0}^{d-1}).$$

in $U_{\mathcal{N}}^{\mathcal{B}}$. In order to study $f_\Gamma|_{y_{A_\gamma}^{i_{A_\gamma}}=0}$ one observes that all products $\prod_{y_{e'}^j\preceq y_{e''}^k} y_{e''}^k$ vanish at $y_{A_\gamma}^{i_{A_\gamma}} = 0$, once $e' \in E(\gamma)$. If all d components $x_{e'}^0, \dots, x_{e'}^{d-1}$ of all $e' \rightsquigarrow e$ vanish at the same time, this does not affect f_Γ, as it is taken care of by a power of $y_A^{i_A}$ pulled out of u_Γ in (53). Consequently, for a fixed $e \in E(\Gamma)$,

$$u_0(\{\sum_{e':e'\rightsquigarrow e} \prod_{y_{e'}^j\preceq y_{e''}^k} y_{e''}^k\}_{j=0}^{d-1}) \prod_{A_\gamma\in\mathcal{N}, e\in E(\gamma)} (y_{A_\gamma}^{i_{A_\gamma}})^{d-2} \prod_{A_\gamma\in\mathcal{N}} \delta_0(y_{A_\gamma}^{i_{A_\gamma}})$$

$$= u_0(\{\sum_{\substack{e':e'\rightsquigarrow e \text{ and } \forall A_\gamma\in\mathcal{N} \\ e'\in E(\gamma)\Rightarrow e\in E(\gamma)}} \prod_{y_{e'}^j\preceq y_{e''}^k} y_{e''}^k\}_{j=0}^{d-1}) \prod_{A_\gamma\in\mathcal{N}, e\in E(\gamma)} (y_{A_\gamma}^{i_{A_\gamma}})^{d-2}.$$

On the other hand, consider the graph $\gamma//\mathcal{N}$ where $\gamma \in \mathcal{N}$. Write $p = p_{t_\gamma, s_\gamma}$ where $E(t_\gamma) = E(t)\cap E(\gamma)$, t is the chosen adapted spanning tree for Γ and s_γ the subforest defined by the maximal elements of the nested set contained in γ. Since γ is connected, t_γ is a spanning tree of γ. A vertex $v_{0,\gamma} \in V_{\text{eff}}(t_\gamma)$ is chosen. For each component c of s_γ there is a unique element $v_c \in V_{\text{eff}}(c)$ which is nearest to $v_{0,\gamma}$ in t_γ. By definition,

$$p^\vee(v) = \begin{cases} \sum_{v'\in V_{\text{eff}}(c)} v' & \text{if } v = v_c, \\ 0 & \text{if } v \in V_{\text{eff}}(s_\gamma) \setminus \bigcup\{v_c\}, \\ v & \text{if } v \in V(\Gamma) \setminus V_{\text{eff}}(s_\gamma). \end{cases}$$

Let $x = \sum_{e\in E(t_\gamma)} x_e b_e$ with $b_e = (-1)^{Q_e}\sum_{v\in V_1} v$ as in Proposition 2.7. One finds $p^\vee(b_e) = (-1)^{Q_e}\sum_{v\in V_1\setminus V_1\cap V_{\mathrm{eff}}(c)} v$ where c is the component of s_γ which contains e, and $c = \emptyset$ if $e \in E(t_\gamma \setminus s_\gamma)$. In particular $p^\vee(b_e) = b_e$ if $e \in E(t_\gamma \setminus s_\gamma)$. Consequently

$$\begin{aligned} i_{\gamma//\mathcal{N}}(x) &= \delta p^\vee(x) \\ &= \sum_{e\in E(\gamma//\mathcal{N})}\sum_{e'\in E(t_\gamma)} (-1)^{Q_{e'}} x_{e'} \sum_{v\in V_1\setminus V_1\cap V_{\mathrm{eff}}(c)} (v:e)e \\ &= \sum_{e\in E(\gamma//\mathcal{N})}\sum_{\substack{e'\rightsquigarrow e\\ e'\in E(t_\gamma\setminus s_\gamma)}} x_{e'}e \end{aligned}$$

where $t_\gamma \setminus s_\gamma$ is a spanning tree for $\gamma//\mathcal{N}$. Therefore

$$\begin{aligned} \tilde{a}_{\gamma//\mathcal{N},-1} &= \prod_{e\in E(\gamma//\mathcal{N})} u_0(\{\sum_{\substack{e'\rightsquigarrow e\\ e'\in E(t_\gamma\setminus s_\gamma)}} \prod y^k_{e''}\}) \\ &\qquad \times \prod_{\gamma\subseteq\gamma'\in\mathcal{N}} (y^{i_{A_{\gamma'}}}_{A_{\gamma'}})^{(d-2)|E(\gamma//\mathcal{N})|}|dy|. \end{aligned}$$

In a final step, define for each $e \in E(\Gamma)$ the minimal element $A_{\gamma_e} \in \mathcal{N}$ such that $e \in E(\gamma_e)$. We have $E(\Gamma) = \bigsqcup_{A_\gamma\in\mathcal{N}}\{e \in E(\Gamma) : \gamma_e = \gamma\}$ $= \bigsqcup_{A_\gamma\in\mathcal{N}} E(\gamma//\mathcal{N})$ as is shown by a simple induction. Similarly $E(t) = \bigsqcup_{A_\gamma\in\mathcal{N}}\{e \in E(t) : \gamma_e = \gamma\} = \bigsqcup_{A_\gamma\in\mathcal{N}} E(t_\gamma) \setminus E(s_\gamma)$ is a decomposition into spanning trees since t is adapted. Write $|dy| = |\bigwedge_{\substack{e\in E(t)\\ j=0,\dots,d-1}} dy^j_e|$ and $|d\hat{y}| = |\bigwedge_{\substack{e\in E(t),j=0,\dots,d-1\\ y^j_e\neq y^{i_A}_A}} dy^j_e|$. Then, in $U^{\mathcal{B}}_{\mathcal{N}}$,

$$\begin{aligned} \tilde{a}_{\Gamma,-N_{max}} &= \tilde{w}_\Gamma(\{y^j_e\}) \prod_{A_\gamma\in\mathcal{N}} |y^{i_{A_\gamma}}_{A_\gamma}|\delta_0(y^{i_{A_\gamma}}_{A_\gamma})|dy| \\ &= \prod_{e\in E(\Gamma)} u_0(\{\sum_{\substack{e':e'\rightsquigarrow e \text{ and } \forall A_\gamma\in\mathcal{N}\\ e'\in E(\gamma)\Rightarrow e\in E(\gamma)}} \prod y^k_{e''}\}) \prod_{\substack{A_\gamma\in\mathcal{N}\\ e\in E(\gamma)}} (y^{i_{A_\gamma}}_{A_\gamma})^{d-2}|d\hat{y}| \\ &= \prod_{A_\gamma\in\mathcal{N}} (y^{i_{A_\gamma}}_{A_\gamma})^{(d-2)|E(\gamma)|} \prod_{\substack{e\in E(\Gamma)\\ \gamma_e=\gamma}} u_0(\{\sum_{\substack{e'\rightsquigarrow e\\ \gamma_{e'}=\gamma_e}} \prod y^k_{e''}\}_{j=0}^{d-1})|d\hat{y}| \qquad (64) \\ &= \bigotimes_{A_\gamma\in\mathcal{N}} \tilde{a}_{\gamma//\mathcal{N},-1} \end{aligned}$$

Consequently (64) integrates to the product of residues as claimed. □

Theorem 5.2 and Theorem 5.3 (ii) implicitly describe a stratification of $Y_{\mathcal{P}}$.

In the next section we will show that all the information relevant for renormalization is encoded in the geometry of $Y_{\mathcal{P}}$.

6. Renormalization on the wonderful model

In this section we describe a map that transforms $\tilde{w}^s_\Gamma = \beta^* \tilde{u}^s_\Gamma$ into a renormalized distribution density $\tilde{w}^s_{\Gamma,R}$, holomorphic at $s = 1$, such that $\tilde{u}_{\Gamma,R} = \beta_* \tilde{w}^s_{\Gamma,R}|_{s=1}$ is defined on all of M^{V_0} and satisfies the following (equivalent) physical requirements:

(i) The terms subtracted from u_Γ in order to get $u_{\Gamma,R}$ can be rewritten as counterterms in a renormalized local Lagrangian.
(ii) The $u_{\Gamma,R}$ satisfy the Epstein-Glaser recursion (renormalized equations of motion, Dyson-Schwinger equations).

One might be tempted to simply define $u_{\Gamma,R}$ by discarding the pole part in the Laurent expansion of $u^s_{\Gamma,R}$ at $s = 1$. However, unless Γ is primitive, this would not provide an extension satisfying those requirements, and the resulting "counterterms" would violate the locality principle. See [26, Section 5.2] for a simple example in momentum space.

The equivalence between (i) and (ii) is adressed in the original work of Epstein and Glaser [35], see also [18, 23, 68]. We circumvent a number of technical issues by restricting ourselves to logarithmic divergences of massless graphs on Euclidean space-time throughout the paper.

6.1. **Conditions for physical extensions.** In this section we suppose as given the unrenormalized distributions $\underline{u}_\Gamma \in \mathcal{D}'(M^{V_0} \setminus (M^{V_0})_{div}(\Gamma))$, and examine what the physical condition (ii) implies for the renormalized distribution $\underline{u}_{\Gamma,R} \in \mathcal{D}'(M^{V_0})$ to be constructed.

Let $V = \{1, \dots, n\}$ be the vertex set of all graphs under consideration. The degree of a vertex is the number of adjacent edges. In the previous sections, Γ was always supposed to be connected. Here we need disconnected graphs and sums of graphs. Therefore all graphs are supposed to be subgraphs of the N-fold complete graph K^N_n on n vertices with N edges between each pair of vertices. N can always be chosen large enough as to accomodate any graph, in a finite collection of graphs Γ on V, as one of its subgraphs.

We write $l_V = (l_1, \dots, l_n)$ for an $\mathbb{N}_0$- multiindex satisfying $\sum l_i \in 2\mathbb{N}_0$. Also $l_V - k_V = (l_1 - k_1, \dots, l_n - k_n)$, $\binom{l_V}{k_V} = \binom{l_1}{k_1} \dots \binom{l_n}{k_n}$ etc. Let $V = I \sqcup J$. Let $\mathrm{Bip}(k_I, k_J)$ be the set of (I, J)-bipartite graphs on V, where the degree of the vertex i is given by k_i. Finally, let $(p_{I,J})_{\emptyset \subsetneq I \subsetneq V}$ be a partition of unity

subordinate to the open cover $\bigcup_{\emptyset \subsetneq I \subsetneq V} C_I$ of $M^{V_0} \setminus \{0\}$ with

$$C_I = M^{V_0} \setminus (M^{V_0})_{sing}(K_{I,V\setminus I})$$

where $K_{I,J}$ is the complete (I,J)-bipartite graph (i. e. the graph with exactly one edge between each $i \in I$ and each $j \in J$). The set $(M^{V_0})_{sing}(K_{I,J})$ is therefore the locus where at least one $x_i - x_j = 0$ for $i \in I$, $j \in J$.

The Epstein-Glaser recursion for vacuum expectation values of time-ordered products (see [23, Equation (31)]) is given, in a euclidean version, by the equality

(65)

$$t_V^{l_V} = \sum_{V=I\sqcup J} \Phi^* p_{I,J} \sum_{\substack{k_V=0 \\ \sum_{i\in I} l_i - k_i = \sum_{j\in J} l_j - v_j}}^{l_V} \binom{l_V}{k_V} t_I^{k_I} t_J^{k_J} \sum_{\Gamma \in \mathrm{Bip}(l_I - k_I, l_J - k_J)} u_\Gamma$$

on $M^V \setminus \Delta = \Phi^{-1}(M^{V_0} \setminus \{0\})$. The distributions $t_V^{l_V}$ therein, vaccuum expectation values of time-ordered Wick products, relate to the single graph distributions u_Γ and their renormalizations $u_{\Gamma,R}$ as follows:

$$t_V^{l_V} = \sum_{\Gamma \in \mathrm{Gr}(l_V)} c_\Gamma u_\Gamma \quad \text{on } \Phi^{-1}(M^{V_0} \setminus (M^{V_0})_{sing}(K_n))$$

(66)
$$t_V^{l_V} = \sum_{\Gamma \in \mathrm{Gr}(l_V)} c_\Gamma u_{\Gamma,R} \quad \text{on } M^V$$

$\mathrm{Gr}(l_V)$ is the set of all graphs Γ with given vertex set $V(\Gamma)$ such that the degree of the vertex i is l_i. There are no external edges and no loops (edges connecting to the same vertex at both ends). The combinatorial constants $c_\Gamma = \frac{\prod_{i=1}^n l_i!}{\prod_{i<j} l_{ij}!}$ where l_{ij} is the number of edges between i and j, are not needed in the following. See [51, Appendix B] for the complete argument.

Proposition 6.1. *On the level of single graphs, a sufficient condition for equation (65) to hold is, for any* Γ,

(67) $u_{\Gamma,R} = u_{\gamma_1,R} \cdot u_{\gamma_2,R} \cdot u_{\Gamma\setminus(\gamma_1\sqcup\gamma_2)}$ *on* $\Phi^{-1}(M^{V_0} \setminus (M^{V_0})_{sing}(\Gamma \setminus (\gamma_1 \sqcup \gamma_2)))$

whenever γ_1, γ_2 *are connected saturated subgraphs of* Γ, *such that* $V_{\text{eff}}(\gamma_1) \cap V_{\text{eff}}(\gamma_2) = \emptyset$.

Note that $u_{\gamma_1,R} \cdot u_{\gamma_2,R}$ is in fact a tensor product since $cc(\gamma_1) \cap cc(\gamma_2) = 0$. The locus where the remaining factor $u_{\Gamma\setminus(\gamma_1\sqcup\gamma_2)}$ is not smooth is excluded by restriction to $M^{V_0} \setminus (M^{V_0})_{sing}(\Gamma \setminus (\gamma_1 \sqcup \gamma_2))$. The product is therefore well-defined. Note also that (67) trivially holds on $M^{V_0} \setminus (M^{V_0})_{div}(\Gamma)$ by the very definition (7) of u_Γ. Proposition 6.1 implies, in particular, that if Γ is a disjoint union ($\Gamma = \gamma_1 \sqcup \gamma_2$ and $V_{\text{eff}}(\gamma_1) \cap V_{\text{eff}}(\gamma_2) = \emptyset$), then

$u_{\Gamma,R} = u_{\gamma_1,R} \otimes u_{\gamma_2,R}$ everywhere.

The system of equations (67) is called the Epstein-Glaser recursion for $u_{\Gamma,R}$. Recursive equations of this kind are also referred to as renormalized Dyson-Schwinger equations (equations of motion) in a momentum space context [9, 63].

Proof of Proposition 6.1. Let all $u_{\Gamma,R}$ satisfy the requirement of (67). We only need the case where $\{I, J\}$ with $I = V_{\text{eff}}(\gamma_1)$, $J = V_{\text{eff}}(\gamma_2)$ is a partition, i. e. $I \sqcup J = V$. Since $(M^{V_0})_{sing}(\Gamma \setminus (\gamma_1 \sqcup \gamma_2)) \subseteq (M^{V_0})_{sing}(K_{I,J})$, (67) is valid in particular on $C_I \supseteq \operatorname{supp} p_{I,J}$. Furthermore, since γ_1 and γ_2 are saturated, $\Gamma \setminus (\gamma_1 \sqcup \gamma_2)$ is (I, J)-bipartite. Therefore, $t_V^{l_V}$ as in (66) with (67) inserted, provides one of the terms on the right hand side of (65). Conversely, every graph Γ with prescribed vertex degrees can be obtained by chosing a partition $I \sqcup J = V$, taking the saturated subgraphs γ_i for I and γ_j for J, respectively, and supplying the missing edges from the (I, J)-bipartite graph. □

6.2. **Renormalization prescriptions.** We consider the divergent arrangement $\mathcal{C} = \mathcal{C}_{div}(\Gamma)$ only, with building set $\mathcal{P}$ minimal or maximal, that is $\mathcal{P} = \mathcal{F}(\mathcal{C})$ or $\mathcal{C}$. Let $\mathcal{N}$ be a nested set which, together with an adapted spanning tree t and a marking of the corresponding basis $\mathcal{B}$, provide for a chart $U_{\mathcal{N}}^{\mathcal{B}}$ for $Y_{\mathcal{P}}$.

By Theorem 5.3 (ii) the subset of codimension 1 where $\tilde{w}_\Gamma^s$ has only a simple pole at $s = 1$ is covered by those charts $U_{\mathcal{N}}^{\mathcal{B}}$ where $\mathcal{N} = \{A_\gamma\}$ with γ any divergent (and irreducible if $\mathcal{P} = \mathcal{F}(\mathcal{C})$) graph. From (62) one has

$$\tilde{w}_\Gamma^s |dy| = f_\Gamma^s \left(-\frac{2\delta_0(y_{A_\gamma}^{i_{A_\gamma}})}{d_\gamma(s-1)} + |y_A^{i_{A_\gamma}}|_{fin}^{(d_\gamma-1)-d_\gamma s} \right) |dy|$$

In these charts, one performs one of the following subtractions in order to get a renormalized distribution. In the first case, only the pole is removed

$$\tilde{w}_\Gamma^s |dy| \mapsto \tilde{w}_{\Gamma,R_0}^s |dy| = f_\Gamma^s |y_{A_\gamma}^{i_{A_\gamma}}|_{fin}^{d_\gamma s-(d_\gamma-1)} |dy| \tag{68}$$

One might call this *local minimal subtraction*.

For $A_\gamma \in \mathcal{N}$ let $A_{\gamma_1}, \ldots, A_{\gamma_k} \in \mathcal{N}$ be the maximal elements contained in A_γ where all graphs are assumed saturated. For each $A_\gamma \in \mathcal{N}$ choose a $\nu_{A_\gamma} \in C^\infty(Y_{\mathcal{P}})$ such that $\nu_{A_\gamma}|_{y_{A_\gamma}^{i_{A_\gamma}}=0} = 1$ and ν_{A_γ} depends only on the coordinates y_e^j, $e \in E(t) \cap (E(\gamma) \setminus E(\cup_{j=1}^k \gamma_j))$ in $U_{\mathcal{B}}^{\mathcal{N}}$, and has compact support

in the associated linear coordinates x_e^j, $e \in E(t) \cap (E(\gamma) \setminus E(\cup_{j=1}^k \gamma_j))$. The ν_{A_γ} are called *renormalization conditions*. In practice, the ν_{A_γ} will be chosen as described at the end of section 3.4.

The second renormalization prescription is then

$$\begin{aligned} \tilde{w}^s_\Gamma |dy| &\mapsto \tilde{w}^s_{\Gamma,R_\mu} |dy| \\ &= \tilde{w}^s_\Gamma - |y^{i_{A_\gamma}}_{A_\gamma}|^{d_\gamma s-(d_\gamma-1)} [\nu_A]_{p_A} \delta_0(y^{i_{A_\gamma}}_{A_\gamma}) f^s_\Gamma |dy|, \end{aligned} \tag{69}$$

which is called *subtraction at fixed conditions*. The notation $[\nu_A]_{p_A}$ means integration along the fiber of the projection

$$p_A : (y^0_{e_1}, \dots, y^{d-1}_{e_{|E(t)|}}) \mapsto (y^0_{e_1}, \dots, \widehat{y^{i_A}_A}, \dots, y^{d-1}_{e_{|E(t)|}})$$

defined in (34). Both prescriptions provide us local expressions holomorphic at $s = 1$ in all charts $U^{\mathcal{B}}_{\mathcal{N}}$ where $\mathcal{N}$ contains a single element.

In the charts $U^{\mathcal{B}}_{\mathcal{N}}$, for a general nested set $\mathcal{N}$, where

$$\tilde{w}^s_\Gamma |dy| = f^s_\Gamma \prod_{A \in \mathcal{N}} \frac{1}{|y^{i_A}_A|^{d_A s-(d_A-1)}} |dy|$$

one applies the subtraction (68) in every factor (local minimal subtraction)

$$\tilde{w}^s_{\Gamma,R_0} |dy| = f^s_\Gamma \prod_{A \in \mathcal{N}} |y^{i_A}_A|^{(d_A-1)-d_A s}_{fin} |dy|. \tag{70}$$

Similarly, by abuse of notation, in the same chart,

$$\tilde{w}^s_{\Gamma,R_\mu} |dy| = \tilde{w}^s_\Gamma \prod_{A \in \mathcal{N}} \left(1 - \dots [\nu_A]_{p_A} \delta_0(y^{i_A}_A)\right) |dy| \tag{71}$$

generalizing the subtraction at fixed conditions (69). A precise notation for (71) – which disguises however the multiplicative nature of this operation – is

$$\begin{aligned} \tilde{w}^s_{\Gamma,R_\mu} |dy| &= \sum_{\{A_1,\dots,A_k\} \subseteq \mathcal{N}} (-1)^k \prod_{A \in \mathcal{N}} \frac{1}{|y^{i_A}_A|^{d_A s-(d_A-1)}} \left[\Pi^k_{j=1} \nu_{A_j}\right]_{p_{A_1,\dots,A_k}} \\ &\times \prod_{j=1}^k \delta_0(y^{i_{A_j}}_{A_j}) f^s_\Gamma |dy| \end{aligned} \tag{72}$$

where $p_{A_1,\dots,A_k}$ is the projection omitting the coordinates $y^{i_{A_j}}_{A_j}$, $j = 1, \dots, k$. Corollary 3.1 shows that there are no infrared divergences when pushing forward along β.

Note that $\tilde{w}^s_{\Gamma,R_0}|_{s=1} |dy|$ defines a density on $Y_{\mathcal{P}}$, but this is not true for

general s. One needs a moment to verify that $\tilde{w}^s_{\Gamma,R_\mu}|dy|$ is a globally well-defined density for all s in a neighborhood of $s = 1$.

Proposition 6.2. *The local expressions $\tilde{w}^s_{\Gamma,R_0}|_{s=1}|dy|$ given by (70) define a density on $Y_{\mathcal{P}}$. The $\tilde{w}^s_{\Gamma,R_\mu}$ given by (71,72) define a density-valued function on $Y_{\mathcal{P}}$, holomorphic in a neighborhood of $s = 1$.*

Proof. Note that $\tilde{w}^s_\Gamma$ is by construction a density for all s. Local minimal subtraction: The $|y^{i_A}_A|^{-1}_{fin}$ transform like $|y^{i_A}_A|^{-1}$ under transition between charts. Subtraction at fixed conditions: Each term in the sum (72) differs from $\tilde{w}^s_\Gamma$ by a number of integrations in the $y^{i_{A_j}}_{A_j}$ and a product of delta distributions in the same $y^{i_{A_j}}_{A_j}$. Under transition between charts, the contribution to the Jacobian from the integrations cancels the one from the delta distributions. It remains to show that $\tilde{w}^s_{\Gamma,R_\mu}$ has no pole at $s = 1$: Using that $\nu_A\big|_{y^{i_A}_A} = 1$, we have in local coordinates

$$\begin{aligned}\tilde{w}^s_{\Gamma,R_\mu} \quad = \quad & \sum_{\{A_1,\dots,A_k\}\subseteq\mathcal{N}} (-1)^k \prod_{j=1}^{k} \left(\frac{-2\delta_0(y^{i_{A_j}}_{A_j})}{d_\Gamma(s-1)} + |y^{i_{A_j}}_{A_j}|^{d_\gamma-1-d_\gamma s}_{fin} [\nu_{A_j}]_{p_{A_j}} \right. \\ & \left. \cdot\, \delta_0(y^{i_{A_j}}_{A_j}) \right) \prod_{A\in\mathcal{N}\setminus\{A_1,\dots,A_k\}} \left(\frac{-2\delta_0(y^{i_A}_A)}{d_\Gamma(s-1)} + |y^{i_A}_A|^{d_\gamma-1-d_\gamma s}_{fin} \right) f^s_\Gamma.\end{aligned}$$

Combining this to a binomial power finishes the proof. □

Theorem 6.1. *Let $\mathcal{P} = \mathcal{F}(\mathcal{C}_{div})$. Then both assignments*

$$\begin{aligned}\Gamma \quad &\mapsto \quad \tilde{u}_{\Gamma,R_0} = \beta_* \tilde{w}^s_{\Gamma,R_0}|_{s=1}, \\ \Gamma \quad &\mapsto \quad \tilde{u}_{\Gamma,R_\mu} = \beta_* \tilde{w}^s_{\Gamma,R_\mu}|_{s=1}\end{aligned}$$

(with consistent choice of the μ_A) satisfy the locality condition (67) for graphs.

The proof is based on the following lemmata. If $A_\gamma \in \mathcal{P}$ then γ is supposed saturated. Recall that an atlas for $Y_{\mathcal{P}}$ is provided by the $U^{\mathcal{B}}_{\mathcal{N}}$.

Lemma 6.1. *Under the assumptions of Proposition 6.1, let $A_\gamma \in \mathcal{P}$ and $cc(\gamma) \not\leq cc(\gamma_1 \sqcup \gamma_2)$. Then*

$$\mathcal{E}_\gamma \subseteq \beta^{-1}(M^{V_0}_{sing}(\Gamma \setminus (\gamma_1 \sqcup \gamma_2))).$$

Proof. If $cc(\gamma) \not\leq cc(\gamma_1 \cup \gamma_2)$, then γ contains an edge $e \in E(\Gamma\setminus(\gamma_1 \sqcup \gamma_2))$. Consequently $A^\perp_\gamma = \bigcap_{e\in E(\gamma)} A^\perp_e \subseteq \bigcup_{e\in E(\Gamma\setminus(\gamma_1\sqcup\gamma_2))} A^\perp_e = M^{V_0}_{sing}(\Gamma \setminus (\gamma_1 \sqcup \gamma_2))$. Since $\beta^{-1}(A^\perp_\gamma) \supseteq \mathcal{E}_\gamma$, the result follows. □

Lemma 6.2. *A subset $\mathcal{N} \subseteq \mathcal{G}$ is nested wrt. the minimal building set if and only if $\mathcal{N} = \mathcal{N}_1 \sqcup \mathcal{N}_2$, where $\mathcal{N}_i$ is a nested set wrt. the minimal building set for the connected graph γ_i with vertex set $V_{\text{eff}}(\gamma_i)$.*

Proof. Let $\mathcal{P}(G) = \mathcal{F}(\mathcal{C}_{div}(G))$ for a graph G. First, since $V_{\text{eff}}(\gamma_1) \cap V_{\text{eff}}(\gamma_2) = \emptyset$, every connected subgraph γ of $\gamma_1 \sqcup \gamma_2$ is either contained in γ_1 or in γ_2. Let now $\mathcal{N} \subseteq \mathcal{G}$ be nested wrt. $\mathcal{P}(\Gamma)$. All irreducible graphs are connected. We can therefore write $\mathcal{N} = \mathcal{N}_1 \sqcup \mathcal{N}_2$ where the elements of $\mathcal{N}_i$ are contained in γ_i. Since γ_i is saturated, a subgraph of γ_i is irreducible as a subgraph of γ_i if and only if it is as a subgraph of Γ. Consequently the $\mathcal{N}_i$ are $\mathcal{P}(\gamma_i)$-nested because $\mathcal{P}(\gamma_i) \subseteq \mathcal{P}(\Gamma)$. Conversely, suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are given. Let some $\gamma_{i_1}, \dots, \gamma_{i_l} \subseteq \gamma_1$ and $\gamma_{j_1}, \dots, \gamma_{j_m} \subseteq \gamma_2$ be pairwise noncomparable. Then the sum $\sum_{k=1}^{l} A_{\gamma_{i_k}} + \sum_{n=1}^{m} A_{\gamma_n}$ is in fact a decomposition into two terms and therefore not contained in $\mathcal{P}(\Gamma)$, unless one of the two terms is zero. But in this case, the other term is a nontrivial decomposition itself, for it is not contained in $\mathcal{P}(\gamma_i)$. Therefore it is not contained in $\mathcal{P}(\Gamma)$, and $\mathcal{N}_1 \sqcup \mathcal{N}_2$ is nested wrt. $\mathcal{P}(\Gamma)$. □

Proof of Theorem 6.1. Let $\Gamma, \gamma_1, \gamma_2$ as in Proposition 6.1. Let $\phi \in \mathcal{D}(M^{V_0})$ such that $\operatorname{supp}\phi \cap M^{V_0}_{sing}(\Gamma \setminus (\gamma_1 \sqcup \gamma_2)) = \emptyset$. In a first step, we study the compact set $X = \operatorname{supp}\psi$ where $\psi = \beta^*\phi$. We say γ has property $(*)$ if it satisfies

$$(*) \quad \gamma \subseteq \Gamma \text{ divergent and } cc(\gamma) \not\leq cc(\gamma_1 \sqcup \gamma_2).$$

Let $\mathcal{G} = \{A_\gamma \in \mathcal{P} : \gamma \text{ has not property } (*)\} \subseteq \mathcal{P}$. By Lemma 6.1, X does not intersect any $\mathcal{E}_\gamma$ where γ has property (*). Therefore

$$X \cap j^{\mathcal{B}}_{\mathcal{N}}(U^{\mathcal{B}}_{\mathcal{N}}) \subseteq j^{\mathcal{B}}_{\mathcal{N}\cap\mathcal{G}}(U^{\mathcal{B}}_{\mathcal{N}\cap\mathcal{G}})$$

(where at the right hand side the marking of $\mathcal{B}$ is restricted to $\mathcal{N} \cap \mathcal{G}$). In a second step, consider the map $\beta_{1,2} : Y_{\mathcal{P}(\gamma_1)} \times Y_{\mathcal{P}(\gamma_2)} \to M^{V_0}$ which is the cartesian product of two wonderful models (with two minimal building sets). If $U^{\mathcal{B}_i}_{\mathcal{N}_i}$ is a chart for $Y_{\mathcal{P}(\gamma_i)}$, then $U^{\mathcal{B}_1}_{\mathcal{N}_1} \times U^{\mathcal{B}_2}_{\mathcal{N}_2}$ is a chart for the product. As the nested sets $\mathcal{N}_1$ and $\mathcal{N}_2$ and the marking $\mathcal{B}_1$ and $\mathcal{B}_2$ of the basis vary, one obtains an atlas for $Y_{\mathcal{P}(\gamma_1)} \times Y_{\mathcal{P}(\gamma_2)}$. Similarly, let $q^{\mathcal{B}_1,\mathcal{B}_2}_{\mathcal{N}_1,\mathcal{N}_2} = q^{\mathcal{B}_1}_{\mathcal{N}_1} \otimes q^{\mathcal{B}_2}_{\mathcal{N}_2}$ be a subordinate partition of unity with compact support for the compact set $X' = \operatorname{supp}\beta^*_{1,2}\phi$ in $Y_{\mathcal{P}(\gamma_1)} \times Y_{\mathcal{P}(\gamma_2)}$.
In a third step, we use Lemma 6.2 to identify $\mathcal{P}(\Gamma)$-nested sets $\mathcal{N} \subseteq \mathcal{G}$ with $\mathcal{N}_1 \sqcup \mathcal{N}_2$, and to show that there is a partition of unity $p^{\mathcal{B}}_{\mathcal{N}}$ for $X \subset Y_{\mathcal{P}}$ subordinate to the atlas $U^{\mathcal{B}}_{\mathcal{N}}$, which looks locally like $q^{\mathcal{B}_1,\mathcal{B}_2}_{\mathcal{N}_1,\mathcal{N}_2}$. Since $U^{\mathcal{B}}_{\mathcal{N}} = U^{\mathcal{B}_1}_{\mathcal{N}_1} \times U^{\mathcal{B}_2}_{\mathcal{N}_2} \setminus \cup_{A\in\mathcal{P}\setminus\mathcal{G}} Z_A$, (see section 4.5), with $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ and $j^{\mathcal{B}}_{\mathcal{N}} = j^{\mathcal{B}_1}_{\mathcal{N}_1} \times j^{\mathcal{B}_2}_{\mathcal{N}_2}$, the $q^{\mathcal{B}_1,\mathcal{B}_2}_{\mathcal{N}_1,\mathcal{N}_2}$ provide indeed such a partition of unity with compact support, because a small enough neighborhood of X does not intersect the

strict transforms Z_A, $A \notin \mathcal{G}$.
Finally in a chart $U^{\mathcal{B}}_{\mathcal{N}}$, identified with $U^{\mathcal{B}_1}_{\mathcal{N}_1} \times U^{\mathcal{B}_2}_{\mathcal{N}_2}$, by definition (70,71), the renormalized distributions satisfy

$$\tilde{w}_{\Gamma,R}(y)|dy| = \tilde{w}_{\gamma_1,R}\tilde{w}_{\gamma_2,R}\tilde{w}_{\Gamma\setminus(\gamma_1\sqcup\gamma_2)}(y)|dy|$$

where $\tilde{w}_{\gamma_i,R} \otimes \tilde{w}_{\gamma_2,R} = \beta^*_{1,2}(\tilde{u}_{\gamma_i,R} \otimes \tilde{u}_{\gamma_2,R})$ and $\tilde{w}_{\Gamma\setminus(\gamma_1\sqcup\gamma_2)} = \beta^*_{1,2}\tilde{u}_{\Gamma\setminus(\gamma_1\sqcup\gamma_2)}$.
Let $\psi_{1,2} = \beta^*_{1,2}\phi$. Since also $\beta = \beta_{1,2}$ in this chart, we have $\psi = \psi_{1,2}$ in local coordinates. This finishes the proof. □

Remarks. Local minimal subtraction is easily defined, but depends on the choice of regularization in a crucial way. The subtraction at fixed conditions is independent of the regularization and therefore the method of choice for the renormalization of amplitudes and non-perturbative computations.
If one extends the requirement (67) to general decompositions $A_\Gamma = A_{\gamma_1} \oplus A_{\gamma_2}$ into connected saturated subgraphs, then it is obvious that the minimal model ($\mathcal{P} = \mathcal{F}(\mathcal{C}_{div}(\Gamma))$) provides exactly the right framework for renormalization. On the other hand, the maximal model ($\mathcal{P} = \mathcal{C}_{div}(\Gamma)$) requires unnecessary subtractions if there are disjoint or, more generally, reducible divergent subgraphs. Locality must then be imposed by additional conditions. It can be shown that local renormalization schemes such as local minimal subtraction can also be applied on the maximal (and all intermediate) models, as will be reported elsewhere.

6.3. **Hopf algebras of Feynman graphs.** In this section we relate our previous results to the Hopf algebras introduced for renormalization by Connes and Kreimer [28, 58], and generalized in [15]. This is not entirely straightforward, see also the remarks at the end of this section. In summary, as long as worse than logarithmic divergences are avoided, the Hopf algebras for renormalization in momentum space [15] and position space are the same.

Only the divergent collection $\mathcal{C}_{div}(\Gamma)$ and the minimal building set $\mathcal{P} = \mathcal{F}(\mathcal{C}_{div}(\Gamma))$ is considered at this stage, and *irreducible* and *nested* refer to this setting.

Definition 6.1. *Two Feynman graphs Γ_1, Γ_2 are* isomorphic *if there is an isomorphism between their exact sequences (18) for a suitable orientation of edges.*

Lemma 6.3. *Let $\gamma \subsetneq \Gamma$ be divergent graphs where Γ is connected and at most logarithmic. Let t be an adapted spanning tree for the nested set $\mathcal{N} = \{\Gamma, \gamma\}$. Then the isomorphism class of $\Gamma//\mathcal{N}$ is independent of t and $\Gamma//\mathcal{N}$ connected, divergent and at most logarithmic.*

In this case we write $\Gamma//\gamma$ for the isomorphism class of $\Gamma//\mathcal{N}$.

Proof. Follows from Lemma 5.1 (ii),(iii) and the definition of the quotient graph using $p_{t,s}$. □

Let $\mathcal{H}_{FG}$ be the polynomial algebra over $\mathbb{Q}$ generated by the empty graph (which serves as unit) and isomorphism classes of connected, at most logarithmic, divergent graphs. There is no need to restrict to graphs of a specific interaction, but this can obviously be done by introducing external (half-) edges and fixing the degree of the vertices. All subgraphs are now understood to have vertex set V_{eff}. Products of linear generators of $\mathcal{H}_{FG}$ are identified with disjoint unions of graphs. One defines

$$\Delta(\Gamma) = \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma//\gamma \tag{73}$$

where in the sum only divergent subgraphs γ are understood, including the empty graph. The quotient graph $\Gamma//\gamma$ is well-defined and a generator of $\mathcal{H}_{FG}$ by Lemma 6.3. One extends Δ as an algebra homomorphism onto all of $\mathcal{H}_{FG}$.

By the analysis of [15, Section 2.2], the map $\Delta : \mathcal{H}_{FG} \to \mathcal{H}_{FG} \otimes \mathcal{H}_{FG}$ is coassociative. Note that divergent and at most logarithmic implies one-particle-irreducible (core) as in [15]:

Definition 6.2. *A graph Γ is called* core (one-particle irreducible) *if* $\dim H_1(\Gamma \setminus e) < \dim H_1(\Gamma)$ *for any* $e \in E(\Gamma)$.

Proposition 6.3. *A divergent, at most logarithmic graph Γ is core.*

Proof. If $\dim H_1(\Gamma \setminus e) = \dim H_1(\Gamma)$ for some $e \in E(\Gamma)$ then $\Gamma \setminus e$ would be worse than logarithmically divergent. □

One can divide $\mathcal{H}_{FG}$ by the ideal $\mathcal{I}$ generated by all polynomials $\gamma - \prod \gamma_j$ where $A_\gamma = A_{\gamma_1} \oplus \ldots \oplus A_{\gamma_j}$ is an irreducible decomposition, as in [15, Equation (2.5)]. Indeed, if γ is connected and $A_\gamma = A_{\gamma_1} \oplus A_{\gamma_2}$ a decomposition then γ is a join: $E(\gamma) = E(\gamma_1) \cup E(\gamma_2)$ and $V_{\text{eff}}(\gamma_1) \cap V_{\text{eff}}(\gamma_2) = \{v\}$. We refer then to [15, Equation (2.5)] for the complete argument that $\mathcal{I}$ is a coideal. The quotient Hopf algebra is denoted $\overline{\mathcal{H}}_{FG} = \mathcal{H}_{FG}/\mathcal{I}$, and we will use only this Hopf algebra in the following. Its corresponds to the minimal building set. The antipode is denoted S and the convolution product of linear endomorphisms $f \star g = m(f \otimes g)\Delta$. Note that a connected divergent graph Γ is primitive in the sense of Definition 3.4 if and only if $\Delta(\Gamma) = \emptyset \otimes \Gamma + \Gamma \otimes \emptyset$.

Theorem 6.2. *If Γ is irreducible,*

$$S(\Gamma) \;=\; \sum_{A_\Gamma \in \mathcal{N}} (-1)^{|\mathcal{N}|} \prod_{A_\gamma \in \mathcal{N}} \gamma//\mathcal{N},$$

where the sum is over nested sets $\mathcal{N}$ wrt. $\mathcal{F}(\mathcal{C}_{div}(\Gamma))$.

Proof. Since the antipode satisfies $S(\emptyset) = \emptyset$ and

$$S(\Gamma) = -\sum_{\gamma \subsetneq \Gamma} S(\gamma)\Gamma//\gamma,$$

for Γ irreducible, γ divergent, one has $S(\Gamma) = -\Gamma$ if Γ is primitive. Let now Γ be general irreducible. The sum over nested sets $\mathcal{N}$ wrt. $\mathcal{F}(\mathcal{C}_{div}(\Gamma))$ containing A_Γ can be written as a sum over proper divergent subgraphs γ of Γ and nested sets $\mathcal{N}'$ wrt. $\mathcal{F}(\mathcal{C}_{div}(\gamma))$ containing the irreducible components of A_γ such that $\mathcal{N} = \mathcal{N}' \cup \{A_\Gamma\}$. By Lemma 6.3, $\Gamma//\gamma = \Gamma//\mathcal{N}$, and the statement follows by induction. □

By Theorem 5.3 (ii)-(iii), the antipode S describes thus the stratification of the divisor $\mathcal{E}$ of $Y_{\mathcal{P}}$. A similar (but weighted) sum is given by $S \star Y$ where Y is the algebra homomorphism $Y : \mathcal{H}_{FG} \to \mathcal{H}_{FG}$, $Y(\Gamma) = \dim H_1(\Gamma)\Gamma$, see for example [29]. This provides the link between the scattering formula of [29] and Theorem 5.3 (iii), and we refer to future work for the details.

In the case of dimensional regularization and minimal subtraction, one considers algebra homomorphisms from $\mathcal{H}_{FG}$ into an algebra of Laurent series in the regulator, and a projector onto the finite part of the series, in order to describe the renormalization process [28, 29, 58], see also sections 7.1-7.5. In our framework, the Hopf algebra is encoded in the geometry of the divisor. The renormalization process is simply to approach the divisor and perform the simple subtraction along the irreducible components, and to take the product of the subtracted factors where the components intersect. Therefore the renormalization schemes studied here (70)-(72) can again be described by the antipode twisted with a subtraction operator. The latter depends however on local information as opposed to global minimal subtraction. A comprehensive discussion of the difference between local renormalization schemes as described here and (global) minimal subtraction is reserved for future work.

Remarks. The role of the Connes-Kreimer Hopf algebras in Epstein-Glaser renormalization was previously discussed in [43], [67] and [8]. The third paper, which is about entire amplitudes and uses rooted trees, relies on a quite symbolic notation which is now justified by the results of the previous

sections. A general flaw in the first paper [43] is reveiled in the introduction of [67]. On the other hand the coproduct in the second paper [67] is not coassociative the way it is defined. As a counterexample consider the cycle on four vertices plus two additional edges between a pair of vertices. This can be repaired by introducing irreducible, core or at most logarithmic and saturated subgraphs as it is done here. See [15, Section 2.2] for a general discussion for which classes $\mathcal{P}$ of graphs the map $\Delta(\Gamma) = \sum_{\substack{\gamma\subseteq\Gamma\\ \gamma\in\mathcal{P}}} \gamma\otimes\Gamma//\gamma$ has a chance of being coassociative.

6.4. **Amplitudes, non-logarithmic divergences and regulators.** In this section we briefly sketch ideas how to extend our previous results, which are so far confined to single graphs with at most logarithmic divergences, to a more general class of graphs. Indeed, if one considers amplitudes, or vacuum expectation values of time-ordered products in the Epstein-Glaser framework, one wants to regularize and renormalize sums of Feynman distributions simultaneously, and some of them will obviously have worse than logarithmic singularities.

For an introductory discussion of non-logarithmic divergences the reader is referred to [15, Section 7.4], [26, Section 5]. The general philosophy is to reduce seemingly non-logarithmic (quadratic etc.) divergences to logarithmic ones by isolating contributions to different terms in the Lagrangian (such as wave function renormalization, mass renormalization); and by projecting onto a subspace of distribution-valued meromorphic functions where local terms with infrared divergences are discarded. This shall only be sketched at the example of the primitive graph

$$\Gamma = \quad , \qquad \underline{u}_\Gamma(x)|d^6x| = \frac{|d^6x|}{x^8}$$

in $d = 6$ dimensions, which is quadratically divergent. By (36), $\underline{u}^s_\Gamma$ has relevant poles[5] at $s = \frac{3}{4}$ and $s = 1$. Indeed, by (36),

(74)
$$\tilde{w}^s_\Gamma|dy| = \frac{f^s_\Gamma(y)|dy|}{|y^0|^{8s-5}} = -\left(\frac{\delta_0(y^0)}{4s-3} + \frac{\delta_0''(y^0)}{8(s-1)} - |y^0|^{5-8s}_{fin}\right) f^s_\Gamma(y)|dy|.$$

[5]Just as in dimensional regularization, the (linear) divergence at $s = 7/8$ is not detected by the regulator.

Note that neither the residue at $s = \frac{3}{4}$ nor $|y^0|_{fin}^{5-8s} f_\Gamma^s$ is globally defined as a distribution density. One would like to work in a space of distributions where w_Γ is equivalent to a linear combination of distribution densities with at most logarithmic singularities, having only a pole at $s = 1$. If one disposes of an infrared regulation such that the so-called adiabatic limit vanishes

$$u_\Gamma^s[\underline{1}] = 0 \tag{75}$$

one can subtract $u_\Gamma^s[\underline{1}]\delta_0$ from (74) without changing it:

$$\begin{aligned} \tilde{w}_\Gamma^s|dy| &= w_\Gamma^s - \delta_0(y^0)\int_{\mathcal{E}} \tilde{w}_\Gamma^s(z)|dz| \\ &= -\left(\frac{\delta_0(y^0)}{4s-3} + \frac{\delta_0''(y^0)}{8(s-1)} - |y^0|_{fin}^{5-8s}\right) f_\Gamma^s(y)|dy| \\ &\quad -\delta_0(y^0)\left(-\frac{1}{4s-3} + \text{holomorphic terms}\right), \end{aligned}$$

which kills the pole at $s = \frac{3}{4}$ and leaves a linear ultraviolet divergence. Using similar subtractions of zero the linear divergence may then be reduced to logarithmic ones and convergent terms, again at the expense of introducing infrared divergent integrals which vanish however in a quotient space where $u_\Gamma^s[\underline{1}] = 0$ for all Γ. We have not worked out the general case, but dimensional regularization suggests that it can be done consistently. Indeed, the idea (75) can be traced back to the "identity"

$$\int d^d k k^{2\alpha} = 0, \quad \alpha \text{ arbitrary} \tag{76}$$

in momentum space dimensional regularization, see also [26, Sections 4.2, 4.3], [15, Remark 7.6]. Equation (76) is a consequence of the fact that dimensional regularization balances ultraviolet and infrared divergences, using only one regulator d.

A complete treatment of non-logarithmic singularities and entire amplitudes is reserved for future work, as well as a more general study of regularization methods, such as dimensional regularization, in position space.

6.5. **Final remarks.** Pulling back the Feynman distribution onto a smooth model with normal crossing divisor seems an obvious thing to do for an algebraic geometer. Less obvious is maybe the question which kind of smooth model is useful. In the recent paper [15], which studies the parametric representation though, a toric compactification is used. Back in coordinate space, there are recent approaches [48],[66] which seem to implicitly use

the Fulton-MacPherson compactification, a special case of the De Concini-Procesi models, but in a spherical version.

Apart from the open problems already mentioned and some incomplete material omitted in this dissertation, there arise two immediate questions. The first is to find the right analytic framework in order to generalize these results to arbitrary propagators on manifolds, with a more versatile regularization than the ad-hoc analytic regularization used here. The second question is how the motivic description in [15] is related to our approach.

7. Dyson-Schwinger equations and Hopf algebras

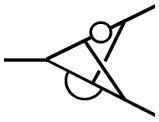
In the previous sections, the Connes-Kreimer Hopf algebras were briefly introduced in the context of the wonderful models. The purpose of this section is to sketch the algebraic features of solutions of *Dyson-Schwinger equations* in this framework. Dyson-Schwinger equations are sometimes called quantum equation of motion, and their solutions are the Green's function of the quantum field theory. See [31, 50] for two (quite different) introductions to the subject. Usually, a Dyson-Schwinger equation is an integral equation, obtained from repeatedly inserting Feynman graphs one into another. A basic idea in the work of Kreimer is that the algebraic aspects of these equations can be separated from the analytic ones. One is therefore lead to define the notion of *combinatorial Dyson-Schwinger equations*, equations in formal power series with coefficients in a combinatorial Hopf algebra. The main result, published in a joint paper with D. Kreimer [9], is Theorem 7.1 where for a quite general class of combinatorial Dyson-Schwinger equations it is shown that the coefficients of their solution generate a Hopf subalgebra. The exposition follows mostly the first sections of [9].

7.1. **Motivation: Renormalization and the Connes-Kreimer Hopf algebras.** We briefly review the theory of Connes and Kreimer, with special emphasize on the Hopf algebra of rooted trees. Throughout this section, we are now in momentum space. The Feynman graphs are as defined previously but may have half-edges attached to some vertices, such that the degree of each vertex (counting both edges and half-edges) is fixed, and determined by the theory. For example, in the ϕ^3 theory in six dimensions, there are only three-valent vertices. The Feynman integrals are given by a generalization of (2) to graphs with half-edges:

(77)

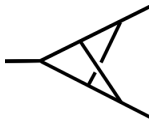
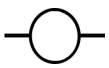
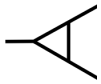
$$I_{\mathrm{mom}}(\Gamma)(p_1,\dots,p_n) = \int_{\mathbb{R}^{d|E(\Gamma)|}} \prod_{v\in V(\Gamma)} \delta_0\left(\sum_{e\in E(\Gamma)} (v:e)k_e + p_v\right) \prod_{e\in E(\Gamma)} \frac{d^d k_e}{k_e^2},$$

where for each vertex v there is an external momentum vector $p_v \in \mathbb{R}^d$. If there is no half-edge attached to v then $p_v = 0$. A useful regularization is *dimensional regularization* where the measure $d^d k$, $d \in 2 + 2\mathbb{N}$, is replaced by $d^{d+2\epsilon}k$, $\epsilon \in \mathbb{C}$, see for example [26]. Rooted trees store information about nested and disjoint subdivergences of Feynman graphs in a natural way. For instance, the subdivergences of the ϕ^3 diagram in six spacetime dimensions

$\Gamma =$ 

can be represented by the decorated tree

where

$\gamma_1 =$  , $\gamma_2 =$  , $\gamma_3 =$  .

Additional labeling is understood in order to keep track of the actual insertion places.

In a moment we will need the trees

 and 

which represent the graph γ_1 for which γ_2 or γ_3, respectively, is suitably inserted.

According to the Bogoliubov recursion [45, 75], the renormalized Feynman integral for Γ is given by

$$\begin{aligned} \phi_R\left(\;\right) \;=\; & (id - R)\left(\phi\left(\;\right) - R\phi(\bullet_{\gamma_2})\phi\left(\;\right)\right. \\ & -R\phi(\bullet_{\gamma_3})\phi\left(\;\right) - R\left(\phi(\bullet_{\gamma_2}\bullet_{\gamma_3})\right. \\ & \left.\left. -\phi(\bullet_{\gamma_2})R\phi(\bullet_{\gamma_3}) - \phi(\bullet_{\gamma_3})R\phi(\bullet_{\gamma_2})\right)\phi(\bullet_{\gamma_1})\right) \end{aligned} \tag{78}$$

where ϕ denotes the regularized but unrenormalized contribution of the graph which a given tree represents. In dimensional regularization with minimal subtraction, ϕ is a map into the algebra $V = \mathbb{C}[\epsilon^{-1}, \epsilon]]$ of Laurent series with finite pole part. The map $R : V \to V$ is the projection $R(\epsilon^k) = \epsilon^k$ if $k < 0$ and $R(\epsilon^k) = 0$ otherwise.

7.2. **Basic definitions and notation.** Let k be a field of characteristic zero. We consider k-bialgebras $(A, m, \mathbb{I}, \Delta, \epsilon)$ that are graded connected, that is

$$A = \bigoplus_{n=0}^{\infty} A_n, \quad A_0 \cong k, \quad A_m A_n \subseteq A_{m+n}, \quad \Delta(A_n) \subseteq \bigoplus_{l+m=n} A_l \otimes A_m.$$

By abuse of notation, we write $\mathbb{I}$ both for the unit and the unit map. Also, we sometimes consider ϵ as a map $A \to A_0$. We assume that $\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$. It follows that $\epsilon(\mathbb{I}) = 1$ while $\epsilon(A_n) = 0$ for $n \neq 0$. The kernel of ϵ is called the augmentation ideal, and the map $P : A \to A$, $P = id - \epsilon$, is called the projection onto the augmentation ideal. The coproduct Δ gives rise to another coassociative map: $\tilde{\Delta}$, defined by

$$\tilde{\Delta}(x) = \Delta(x) - \mathbb{I} \otimes x - x \otimes \mathbb{I}.$$

Recall that elements in the kernel of $\tilde{\Delta}$ are called *primitive*. We will occasionally use Sweedler's notation $\Delta(x) = \sum x' \otimes x''$ and also $\tilde{\Delta}(x) = \tilde{\sum} x' \otimes x''$.

It is a well known fact that connected graded bialgebras are Hopf algebras. Indeed, the sequence defined by the recursive relation

$$S(x) = -x - \tilde{\sum} S(x')x'' \text{ for } x \notin A_0, \quad S(\mathbb{I}) = \mathbb{I} \tag{79}$$

converges in $\mathrm{End}_k(A)$.

For a coalgebra (A, Δ) and an algebra (B, m), the vector space $\mathrm{Hom}_k(A, B)$ of linear maps $A \to B$ is equipped with a convolution product $\star$ by $(f, g) \mapsto f \star g = m(f \otimes g)\Delta$. Thus $(f \star g)(x) = \sum f(x')g(x'')$. Using the modified product $\star_P : (f, g) \mapsto f \star_P g = m(f \otimes g)(id \otimes P)\Delta$, equations (79) can be rewritten

$$S(x) = -(S \star_P id)(x) \text{ for } x \notin A_0, \quad S(\mathbb{I}) = \mathbb{I}$$

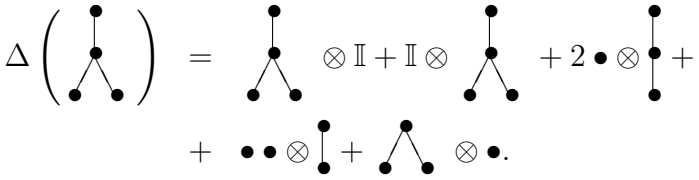
which will be convenient later on.

7.3. **The Hopf algebra of rooted trees and some variants.** Now we give a more detailed construction of the Hopf algebra $\mathcal{H}$ of rooted trees [27, 58]. A (non-planar) rooted tree is a connected contractible finite graph with a distinguished vertex called the root. By convention, we will draw the root on top. We are only interested in rooted trees up to isomorphism (an isomorphism of rooted trees being an isomorphism of graphs which maps the root to the root). As a graded algebra, $\mathcal{H}$ is the polynomial algebra generated by rooted trees (including the empty tree which we consider the unit $\mathbb{I}$) with the weight grading: the weight of a tree is the number of its vertices. A product of rooted trees is called a forest, and the weight of

a forest is the sum of the weights of its trees. On $\mathcal{H}$ a coproduct Δ is introduced by

$$\Delta(\tau) = \mathbb{I} \otimes \tau + \tau \otimes \mathbb{I} + \sum_{adm.c} P_c(\tau) \otimes R_c(\tau) \tag{80}$$

where the sum goes over all *admissible cuts* of the tree τ. By a cut of τ we mean a nonempty subset of the edges of τ that are to be removed. The product of subtrees which "fall down" upon removal of those edges is called the *pruned part* and is denoted $P_c(\tau)$, the part which remains connected with the root is denoted $R_c(\tau)$. This makes sense only for certain "admissible" cuts: by definition, a cut $c(\tau)$ is admissible, if for each leaf l of τ it contains at most one edge on the unique path from l to the root. For instance,



The coassociativity of Δ is shown in [58]. $\mathcal{H}$ is obviously not cocommutative. Since the coproduct is compatible with the grading, $\mathcal{H}$ is a Hopf algebra. There is an important linear endomorphism of $\mathcal{H}$, the grafting operator B_+ defined as follows:

$$\begin{aligned} B_+(\mathbb{I}) &= \bullet \\ B_+(\tau_1 \dots \tau_n) &= \underbrace{\quad\quad}_{\tau_1 \ \dots \ \tau_n} \quad \text{for trees } \tau_i \end{aligned} \tag{81}$$

In words: B_+ creates a new root and connects it with each root of its argument. The special importance of B_+ will become evident in section 7.4: $B_+ : \mathcal{H} \to \mathcal{H}$ is a closed but not exact Hochschild 1-cochain.

The Hopf algebra $\mathcal{H}$ is the dual of a Hopf algebra considered earlier by Grossman and Larson [44], see [39].

From the Hopf algebra $\mathcal{H}$, defined in the previous section, several generalizations can be constructed: Hopf algebras of decorated trees, of planar trees, etc. This can be phrased most elegantly from a general point of view in terms of *tree-like structures*, as for example introduced in in [72]: Consider the category of rooted trees and embeddings (an embedding $\tau' \to \tau$ is an isomorphism from τ' to a subtree of τ). A rooted tree-structure is then defined to be a contravariant functor from this category to the category of sets. For example, decorated (labelled) trees can be described by the functor ϕ which maps a tree onto a certain set its vertices and/or edges are

decorated with. Being contravariant, ϕ maps embeddings of trees to the respective restrictions of decorations. Similarly, a planar structure is provided by a functor ϕ mapping a tree to the set of its topological embeddings into the real plane modulo orientation-preserving homeomorphisms of $\mathbb{R}^2$ onto itself. Now let ϕ be a rooted tree-structure. A rooted ϕ-tree is a pair (τ, s) where τ is a tree and s is an element of $\phi(\tau)$. The notions of isomorphisms and subtrees of rooted ϕ-trees are immediate.

Using this framework, there are immediately other Hopf algebras at hand: Let S be a set. The Hopf algebra $\mathcal{H}(S)$ is defined as in the previous section, replacing the word tree by S-decorated tree (for our purposes, we only decorate vertices, not edges). Similarly, $\mathcal{H}_{pl}$ is the (noncommutative) Hopf algebra of planar rooted trees. In particular, for these Hopf algebras, the proofs of the coassociativity of Δ are verbatim the same. The planar Hopf algebra and its decorated versions $H_{pl}(S)$ were extensively studied by Foissy [39]. He showed that they are self-dual and constructed isomorphisms to several other Hopf algebras on trees that have appeared in the literature.

While rooted trees describe nested divergences in an obvious manner, the resolution of *overlapping* divergences into trees requires some care. This problem exists only in momentum space. By basing a Hopf algebra directly on Feynman graphs instead of trees, these issues can be avoided [28, 59]. As an algebra, let $\mathcal{H}_{CK}$ be the free commutative algebra on 1PI Feynman graphs (of a given theory; the case of a non-scalar theory requires to take form factors (external structures) into account which we avoid here). The empty graph serves as a unit $\mathbb{I}$. In the following, a product of graphs is identified with the disjoint union of these graphs. On a graph, a coproduct is given [28] by

$$\Delta(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$

where the sum is over all 1PI superficially divergent proper subgraphs γ of Γ.

7.4. **Hochschild cohomology of bialgebras.** Let A be a bialgebra. We consider linear maps $L : A \to A^{\otimes n}$ as n-cochains and define a coboundary operator b by

$$bL := (id \otimes L)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i L + (-1)^{n+1} L \otimes \mathbb{I} \tag{82}$$

where Δ denotes the coproduct and Δ_i the coproduct Δ applied to the i-th factor in $A^{\otimes n}$. The map $L \otimes \mathbb{I}$ is given by $x \mapsto L(x) \otimes \mathbb{I}$. It is essentially

due to the coassociativity of Δ that b squares to zero, which gives rise to a cochain complex (C, b). Clearly (C, b) captures only information about the coalgebra structure of A. The cohomology of (C, b), denoted $HH^\bullet_\epsilon(A)$, is easily seen to be the dual (A considered as a bi*co*module rather than a bimodule over itself) notion of the Hochschild cohomology of algebras. Note that the right bicomodule action is here $(id \otimes \epsilon)\Delta$ which explains the last summand in (82) and the subscript in $HH^\bullet_\epsilon$.

For $n = 1$, the cocycle condition $bL = 0$ reduces to, for $L : A \to A$,

$$\Delta L = (id \otimes L)\Delta + L \otimes \mathbb{I}. \tag{83}$$

Sometimes the following equivalent statement, using the map $\tilde{\Delta}$, is more convenient:

$$\tilde{\Delta} L = (id \otimes L)\tilde{\Delta} + id \otimes L(\mathbb{I}). \tag{84}$$

Let us now try to understand the space $HH^1_\epsilon(\mathcal{H})$ of "outer coderivations on $\mathcal{H}$." We first describe the 0-coboundaries ("inner coderivations"). They are of the form

$$L(\tau) = \sum \alpha_{\tau''} \tau' - \alpha_\tau \mathbb{I}$$

in Sweedler's notation, where α_τ is an element of k for each forest τ. For example, $L : \tau \mapsto \sum \tau' - \mathbb{I}$ is a 0-coboundary. Note that $\mathbb{I}$ is in the kernel of any 0-coboundary.

It is a crucial fact that the grafting operator B_+, introduced in section 7.3, is a 1-cocycle [27]:

$$\Delta B_+ = (id \otimes B_+)\Delta + B_+ \otimes \mathbb{I}. \tag{85}$$

$$\tilde{\Delta} B_+ = (id \otimes B_+)\tilde{\Delta} + id \otimes \bullet. \tag{86}$$

The statement follows from (86): Let τ be a forest. The first term at the right side of (86) refers to cuts of $B_+(\tau)$ which affect at most all but one of the edges connecting the new root of $B_+(\tau)$ to the roots of τ, while the second summand takes care of the cut which completely separates the root of $B_+(\tau)$ from all its children. □

Since B_+ is a homogeneous linear endomorphism of degree 1, it is not a 0-coboundary – note that the coboundaries have no chance to increase the degree. Thus B_+ is a generator (among others) of $HH^1_\epsilon(\mathcal{H})$.

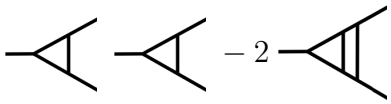
When looking for other generators L of $HH^1_\epsilon(\mathcal{H})$, the cocycle conditions (83,84) immediately yield the requirement that $L(\mathbb{I})$ be a primitive element (and zero if L is exact). While $\bullet$ is up to scalar factors obviously the only

primitive element in degree 1, there are plenty of primitives in higher degrees. For example,

(87)

is a primitive element in degree 2. Foissy [39] showed that $L \mapsto L(\mathbb{I})$ is a surjective map $HH^1_\epsilon(\mathcal{H}) \to$Prim$(\mathcal{H})$ onto the set of primitive elements of $\mathcal{H}$. In the case of Hopf algebras of decorated rooted trees $\mathcal{H}(S)$ obviously any element $s \in S$ yields a homogeneous cocycle of degree 1 denoted B^s_+ which, applied to a forest, connects its roots to a new root decorated by s.

It should be clear that each 1PI Feynman graph which is free of subdivergences is a primitive element of $\mathcal{H}_{CK}$. In general, there are primitive elements in higher degrees too, for example, cf. (87), the linear combination



in ϕ^3 theory in six dimensions.

The category of objects (A, L) consisting of a commutative bialgebra A and a Hochschild 1-cocycle L on A with morphisms bialgebra morphisms commuting with the cocycles has the initial object $(\mathcal{H}, B_+)$. This is a result of [27]. Indeed, let (A, L) be such a pair. The map $\rho : \mathcal{H} \to A$ is simply defined by $\rho(\mathbb{I}) = \mathbb{I}$ and pushing forward along B_+ (and L) and the multiplication. The fact that ρ is a morphism of coalgebras is an easy consequence of (85).

Also it was shown in [8] that, conversely, the coproduct Δ of $\mathcal{H}$ is determined if one requires the map B_+ to be a 1-cocycle. This may serve to find different presentations of $\mathcal{H}$.

For any $\mathcal{H}$-bicomodule B, the higher Hochschild cohomology $HH^n(\mathcal{H}, B)$, $n \geq 2$, is trivial [39], thus in particular $HH^n_\epsilon(\mathcal{H}) = 0$.

7.5. **Convergence and locality from the Hopf algebra.** Given a specific quantum field theory, Hopf algebras $\mathcal{H}(S)$ and $\mathcal{H}_{CK}$ are determined by the perturbative expansion into Feynman graphs. We denote this Hopf algebra generically by $\mathcal{H}$. The next step is to define a target algebra V and regularized Feynman rules $\phi : \mathcal{H} \to V$, and a renormalization scheme $R : V \to V$. The map ϕ is supposed to be a (unital) algebra homomorphism. We stick to the example $(V = \mathbb{C}[\epsilon^{-1}, \epsilon]], \phi)$ of dimensional regularization and minimal

subtraction as in section 7.1. The map R satisfies the Rota-Baxter equation

$$R(xy) + R(x)R(y) = R(xR(y)) + R(R(x)y) \tag{88}$$

which is key to the Birkhoff decomposition of [28], see for example [34]. It also guarantees that the renormalized Feynman rules are again an algebra homomorphism [60] as are the unrenormalized rules ϕ. The twisted antipode is defined by

$$S_R^\phi(\tau) = -R(S_R^\phi \star_P \phi)(\tau) \text{ for } \tau \notin \mathcal{H}_0, \quad S_R^\phi(\mathbb{I}) = 1, \tag{89}$$

equivalently, in Sweedler's notation

$$S_R^\phi(\tau) = -R\left(\phi(\tau) + \tilde{\sum} S_R^\phi(\tau')\phi(\tau'')\right) \text{ for } \tau \notin \mathcal{H}_0, \quad S_R^\phi(\mathbb{I}) = 1$$

where the term "twisted antipode" refers to the recursive expression (79) for the regular antipode. The map S_R^ϕ yields the counterterm for ϕ. The renormalized Feynman rules are then given by

$$\phi_R = S_R^\phi \star \phi. \tag{90}$$

One can find a non-recursive description of ϕ_R [27, 58] which shows the equivalence with Zimmermann's forest formula [75].

In order to understand the twisted antipode, we come back to the example of section 7.1. On the relevant trees, the coproduct acts as follows:

$$\begin{aligned} \Delta\left(\begin{array}{ccc} & \bullet^{\gamma_1} & \\ \bullet_{\gamma_2} & & \bullet_{\gamma_3} \end{array}\right) &= \mathbb{I} \otimes \begin{array}{ccc} & \bullet^{\gamma_1} & \\ \bullet_{\gamma_2} & & \bullet_{\gamma_3} \end{array} + \begin{array}{ccc} & \bullet^{\gamma_1} & \\ \bullet_{\gamma_2} & & \bullet_{\gamma_3} \end{array} \otimes \mathbb{I} + \\ &\quad + \bullet_{\gamma_2} \otimes \begin{array}{c} \bullet^{\gamma_1} \\ \vdots \\ \bullet_{\gamma_3} \end{array} + \bullet_{\gamma_3} \otimes \begin{array}{c} \bullet^{\gamma_1} \\ \vdots \\ \bullet_{\gamma_2} \end{array} + \bullet_{\gamma_2} \bullet_{\gamma_3} \otimes \bullet_{\gamma_1}, \\ \Delta(\bullet_{\gamma_i}) &= \mathbb{I} \otimes \bullet_{\gamma_i} + \bullet_{\gamma_i} \otimes \mathbb{I}, \end{aligned} \tag{91}$$

According to (89) and (90), the algorithm for ϕ_R consists of the following steps:

- (F) Apply the coproduct Δ to the tree under consideration
- (C_n) apply the map $(id \otimes P)\Delta \otimes id^{\otimes n}$ (for $n = 1 \ldots$) until each summand is of the form $\mathbb{I} \otimes \ldots$.
- (M) apply $\phi^{\otimes n}$ to go into $V^{\otimes n}$. As $\phi(\mathbb{I}) = S_R^\phi(\mathbb{I}) = 1$, the first factor $\mathbb{I}$ of each term is mapped to 1.
- (C_n') (for $n = \ldots 1$) apply the map $-Rm \otimes id^{\otimes n}$ until we end up in $V^{\otimes 2}$
- (F') apply the map m to get into V.

For the tree $\begin{array}{ccc} & \bullet^{\gamma_1} & \\ \bullet_{\gamma_2} & & \bullet_{\gamma_3} \end{array}$ this algorithm is performed in Figures 1–3.

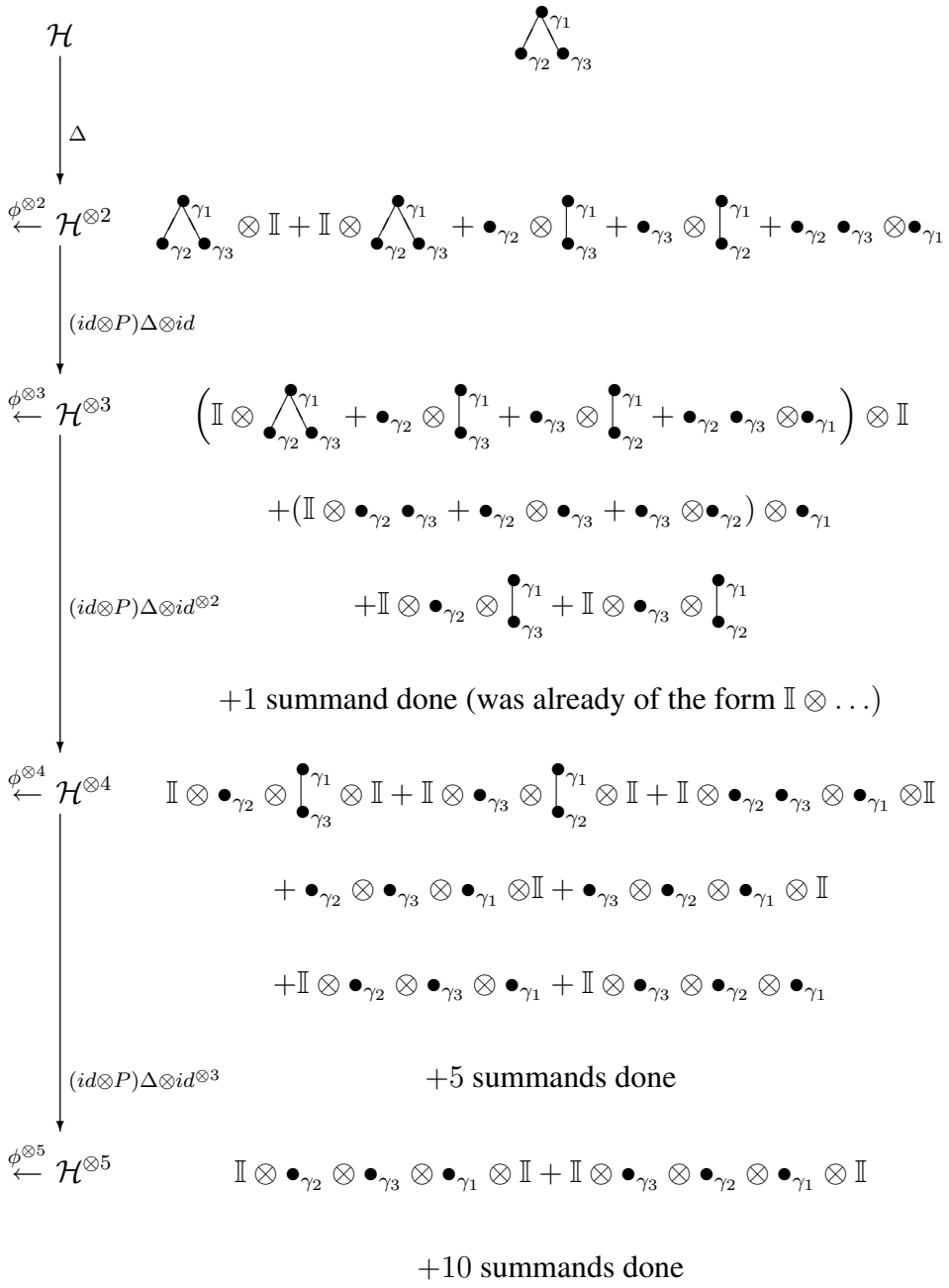


FIGURE 5. First part of the calculation of ϕ_R. Apply Δ and then $(id \otimes P)\Delta \otimes id^{\otimes n}$ until each summand is of the form $\mathbb{I} \otimes \ldots$.

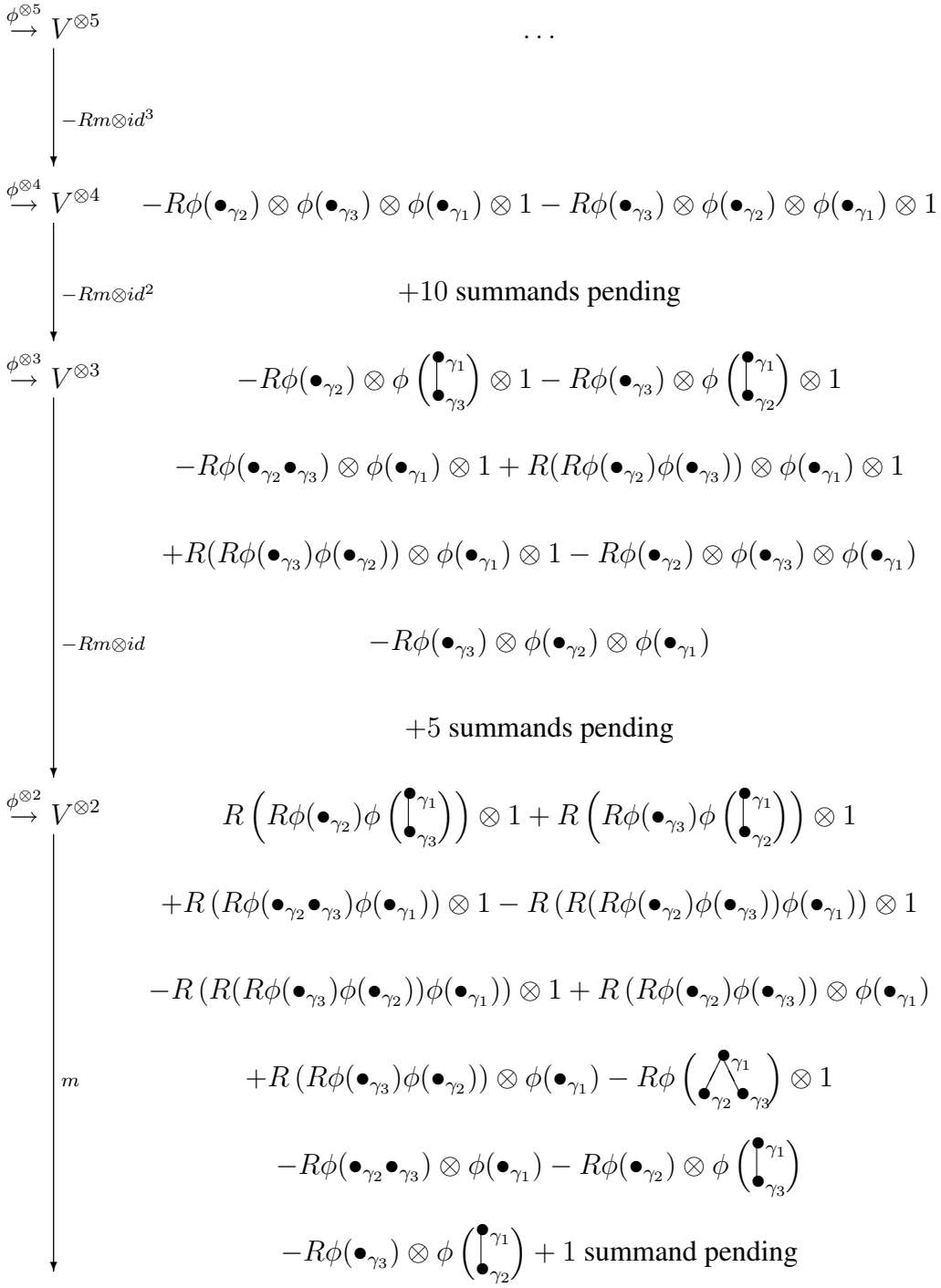


FIGURE 6. Second part of the calculation of ϕ_R. Apply $\phi^{\otimes n}$ and then $-Rm\otimes id^{\otimes n}$ until arrival in $V^{\otimes 2}$. Then apply m to get into V.

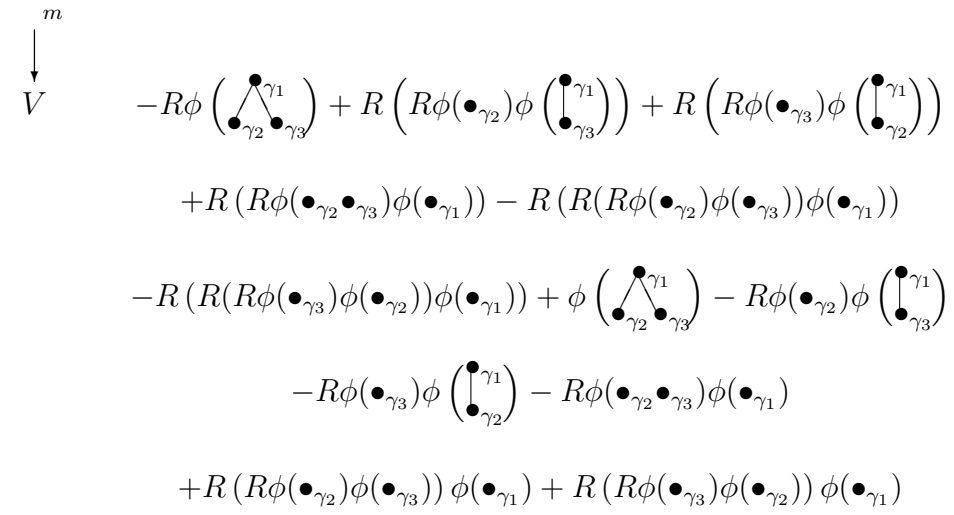


FIGURE 7. Third part of the calculation of ϕ_R, to be compared with (78). Using the fact that S^ϕ_R is an algebra homomorphism (if R is a Rota-Baxter map), the last step (C3) in Figure 1 and the first step (C3') in Figure 2 could have been avoided.

7.6. **Dyson-Schwinger equations and Hopf subalgebras.** Hopf subalgebras of the Hopf algebras of (decorated) rooted trees or Feynman graphs are in close relationship with Dyson-Schwinger equations. Indeed, any Dyson-Schwinger equation (to be defined below) gives rise to a Hopf subalgebra. It will turn out in Theorem 7.1 that all Hopf subalgebras coming from a reasonably general class of Dyson-Schwinger equations are in fact isomorphic.

Hopf subalgebras of decorated rooted trees. For simplicity, we start our considerations in the Hopf algebra $\mathcal{H}$ of undecorated rooted trees. A full classification of their Hopf subalgebras is beyond reach. However, we give a few examples the last of which will be directly related to Dyson-Schwinger equations.

Bounded fertility, finite parts, primitive elements. For $n \in \mathbb{N}$ let $\mathcal{H}_n$ be the subalgebra of $\mathcal{H}$ generated by trees whose vertices have fertility bounded from above by n. A glance at the definition of the coproduct (80) suffices to see that $\mathcal{H}_n$ is a Hopf subalgebra of $\mathcal{H}$. In particular, the Hopf algebra $\mathcal{H}_1$ with one generator in each degree is known as the Hopf algebra of *ladders*. It is closely related to iterated integrals [25, 60].

Similarly, the polynomial algebra generated by trees of degree $\leq n$ forms

a Hopf subalgebra for any n since the coproduct respects the grading. Another example where there is nothing to check are subalgebras generated by an arbitrary collection of primitive elements of $\mathcal{H}$.

The Connes-Moscovici Hopf subalgebra. A less trivial example of a Hopf subalgebra of $\mathcal{H}$ arose in the work of Connes and Moscovici on local index formulas for transversally hypoelliptic operators on foliations [27, 30]. In the case of a foliation of codimension 1, the relevant Hopf algebra $\mathcal{H}_T$ is defined by the generators X, Y, δ_n for $n \in \mathbb{N}$, the relations

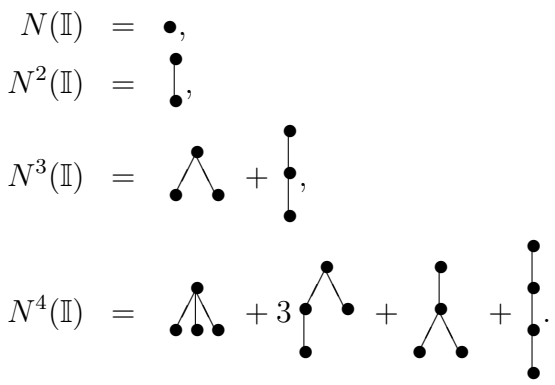
$$[X, Y] = -X, \quad [X, \delta_n] = \delta_{n+1}, \quad [Y, \delta_n] = n\delta_n, \quad [\delta_n, \delta_m] = 0,$$

and the coproduct

$$\Delta(X) = X \otimes \mathbb{I} + \mathbb{I} \otimes X + \delta_1, \quad \Delta(Y) = Y \otimes \mathbb{I} + \mathbb{I} \otimes Y, \quad \Delta(\delta_1) = \delta_1 \otimes \mathbb{I} + \mathbb{I} \otimes \delta_1.$$

Note that the relations above and the requirement that Δ be an algebra homomorphism determine Δ on the generators δ_n for $n \geq 2$ as well. Let N be the linear operator, called *natural growth operator*, on $\mathcal{H}$, defined on a tree τ by adding a branch to each vertex of τ and summing up the resulting trees, extended as a derivation onto all of $\mathcal{H}$. For example,

(92)



Now identifying δ_1 with $\bullet$, and generally δ_n with $N^n(\mathbb{I})$, the commutative Hopf subalgebra of $\mathcal{H}_T$ generated by the δ_n can be embedded into $\mathcal{H}$ [27]. The resulting Hopf subalgebra is denoted $\mathcal{H}_{CM}$. For example,

$$\begin{aligned} \tilde{\Delta}(\delta_1) &= 0, \\ \tilde{\Delta}(\delta_2) &= \delta_1 \otimes \delta_1, \\ \tilde{\Delta}(\delta_3) &= 3\delta_1 \otimes \delta_2 + (\delta_2 + \delta_1^2) \otimes \delta_1. \end{aligned}$$

The δ_n can be specified in a non-recursive manner:

$$\delta_n = \sum_{\tau \in \mathcal{T}_n} c_\tau \tau.$$

Here $\mathcal{T}_n$ is the set of trees of weight n. The integers c_τ, called *Connes-Moscovici weights*, have been computed in [39, 60] using the tree factorial

$$c_\tau = \frac{n!}{\tau! \operatorname{Sym}(\tau)}$$

where $\operatorname{Sym}(\tau)$ is the order of the group of symmetries of τ.

A quadratic Dyson-Schwinger equation. Now we turn to the study of another source of Hopf subalgebras, the combinatorial Dyson-Schwinger equations. As a first example, we consider the equation

$$X = \mathbb{I} + \alpha B_+(X^2) \tag{93}$$

in $\mathcal{H}[[\alpha]]$. Using the ansatz

$$X = \sum_{n=0}^{\infty} \alpha^n c_n$$

one easily finds $c_0 = \mathbb{I}$ and

$$c_{n+1} = \sum_{k=0}^{n} B_+(c_k c_{n-k}) \tag{94}$$

which determine X by induction. The first few c_n are easily calculated:

$c_0 = \mathbb{I},$

$c_1 = \bullet,$

$c_2 = 2$ 

$c_3 =$  $+ 4$ 

$c_4 = 4$  $+ 2$  $+ 8$ 

We observe that c_n is a weighted sum of trees with vertex fertility bounded by 2 – this is due to the square of X in the Dyson-Schwinger equation (93). The recursive nature of (93) makes one suspect that the c_n generate a Hopf subalgebra of $\mathcal{H}$. Indeed, for each $n \geq 0$ and $k \leq n$ there is a polynomial P_k^n in the c_l for $l \leq n$ such that

$$\Delta c_n = \sum_{k=0}^{n} P_k^n \otimes c_k. \tag{95}$$

They are inductively determined by

$$P_{k+1}^{n+1} = \sum_{l=0}^{n-k} P_0^l P_k^{n-l} \tag{96}$$

and $P_0^{n+1} = c_{n+1}$. For a proof of this statement, see the more general Theorem 7.1 in the next section. For the moment, we merely display the first P_k^n in an upper triangular matrix where columns are indexed by $n = 0 \dots 5$ and rows by $k = 0 \dots n$.

$$\begin{bmatrix} \mathbb{I} & c_1 & c_2 & c_3 & c_4 & c_5 \\ & \mathbb{I} & 2c_1 & 2c_2 + c_1^2 & 2c_3 + 2c_1c_2 & 2c_4 + 2c_1c_3 + c_2^2 \\ & & \mathbb{I} & 3c_1 & 3c_2 + 3c_1^2 & 6c_1c_2 + c_1^3 + 3c_3 \\ & & & \mathbb{I} & 4c_1 & 6c_1^2 + 4c_2 \\ & & & & \mathbb{I} & 5c_1 \\ & & & & & \mathbb{I} \end{bmatrix}$$

The coefficients are basically multinomial coefficients as will become clear in the next section.

Combinatorial Dyson-Schwinger equations. Let A be any connected graded Hopf algebra which is free or free commutative as an algebra, and $(B_+^{d_n})_{n\in\mathbb{N}}$ a collection of Hochschild 1-cocycles on it (not necessarily pairwise distinct). The most general Dyson-Schwinger equation we consider here is

$$X = \mathbb{I} + \sum_{n=1}^{\infty} \alpha^n w_n B_+^{d_n}(X^{n+1}) \tag{97}$$

in $A[[\alpha]]$. The parameter α plays the role of a coupling constant. The w_n are scalars in k. Again we decompose the solution

$$X = \sum_{n=0}^{\infty} \alpha^n c_n$$

with $c_n \in A$.

Lemma 7.1. *The Dyson-Schwinger equation (97) has a unique solution described by* $c_0 = \mathbb{I}$ *and*

$$c_n = \sum_{m=1}^{n} w_m B_+^{d_m} \left(\sum_{k_1+\dots+k_{m+1}=n-m,\, k_i\geq 0} c_{k_1} \dots c_{k_{m+1}} \right). \tag{98}$$

Proof. Inserting the ansatz into (97) and sorting by powers of α yields the result. Uniqueness is obvious. □

Theorem 7.1. *The elements c_n generate a Hopf subalgebra of A :*

$$\Delta(c_n) = \sum_{k=0}^{n} P_k^n \otimes c_k$$

where the P_k^n are homogeneous polynomials of degree $n-k$ in the $c_l, l \leq n$:

$$P_k^n = \sum_{l_1+\ldots+l_{k+1}=n-k} c_{l_1} \ldots c_{l_{k+1}}. \tag{99}$$

In particular, the P_k^n are independent of the w_n and $B_+^{d_n}$.

We emphasize that the main ingredient for the proof of this theorem is the fact that the $B_+^{d_n}$ are Hochschild 1-cocycles, the rest being a cumbersome but straightforward calculation.

Proof. We proceed by proving inductively the following statements:

- (α_n) The theorem holds up to order n.
- (β_n) For a given $m \in \{1 \ldots n\}$ let $l_1 + \ldots + l_{m+1} =: p \in \{0 \ldots n-m\}$, $l_i \geq 0$. Then the right hand sum

$$P(n-m, m, p) := \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq l_i} P_{l_1}^{k_1} \ldots P_{l_{m+1}}^{k_{m+1}} \tag{100}$$

  does not depend on the single l_i but only on p, $n-m$ and m, justifying the notation $P(n-m, m, p)$.
- (γ_n) In the above notation and for any $q \in \{1 \ldots n\}$, the term $P(n-m, m, q-m)$ does not depend on $m \in \{1 \ldots q\}$.

To start the induction, we note that (α_0) is obvious. (β_1) is trivial as $m = 1$ enforces $l_1 = l_2 = 0$. Similarly, for (γ_1) only one m is in range and the statement thus trivially satisfied. We proceed to (α_n). By definition, and using (83) for the $B_+^{d_n}$,

$$\begin{aligned} \Delta(c_n) &= \sum_{m=1}^{n} w_m((id \otimes B_+^{d_m})\Delta + B_+^{d_m} \otimes \mathbb{I}) \\ &\qquad \left(\sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq 0} c_{k_1} \ldots c_{k_{m+1}} \right) \end{aligned}$$

(using the induction hypothesis (α_{n-1}))

$$\begin{aligned} &= c_n \otimes \mathbb{I} + \sum_{m=1}^{n} w_m(id \otimes B_+^{d_m}) \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq 0} \sum_{l_1 \ldots l_{m+1}=0}^{k_1 \ldots k_{m+1}} \\ &\quad P_{l_1}^{k_1} \ldots P_{l_{m+1}}^{k_{m+1}} \otimes c_{l_1} \ldots c_{l_{m+1}} = \end{aligned}$$

(by rearranging indices)

$$= \quad c_n \otimes \mathbb{I} + \sum_{m=1}^{n} w_m \sum_{p=0}^{n-m} \sum_{l_1+\ldots+l_{m+1}=p} \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq l_i} P_{l_1}^{k_1} \ldots P_{l_{m+1}}^{k_{m+1}} \otimes B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}) =$$

(by the induction hypothesis (β_n) and using the notation of (100))

$$= \quad c_n \otimes \mathbb{I} + \sum_{m=1}^{n} w_m \sum_{p=0}^{n-m} P(n-m,m,p) \otimes \sum_{l_1+\ldots+l_{m+1}=p} B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}) =$$

(rearranging indices (q replaces $m+p$) and using (γ_n))

$$\begin{aligned} &= \quad c_n \otimes \mathbb{I} + \sum_{q=1}^{n} \sum_{m=1}^{q} w_m P(n-m,m,q-m) \otimes \\ &\quad \otimes \sum_{l_1+\ldots+l_{m+1}=q-m} B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}) = \\ &= \quad c_n \otimes \mathbb{I} + \sum_{q=1}^{n} P(n-q,q,0) \otimes \sum_{m=1}^{q} w_q \sum_{l_1+\ldots+l_{m+1}=q-m} B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}). \end{aligned}$$

Since the right hand tensor factor is c_q, a glance at (100), using that $P_0^k = c_k$, verifies (α_n).

The items (β_n) and (γ_n) follow from (α_{n-1}) :

$$\begin{aligned} P(n-m,m,p) \quad &= \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq l_i} P_{l_1}^{k_1} \ldots P_{l_{m+1}}^{k_{m+1}} = \\ &= \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq l_i} \sum_{r_1^1+\ldots+r_{l_1+1}^1=k_1-l_1} \ldots \\ &\quad \ldots \sum_{r_1^{m+1}+\ldots+r_{l_{m+1}+1}^{m+1}=k_{m+1}-l_{m+1}} c_{r_1^1} \ldots c_{r_{l_{m+1}+1}^{m+1}} = \\ &= \sum_{r_1+\ldots+r_{m+p+1}=n-m-p} c_{r_1} \ldots c_{r_{m+p+1}}, \end{aligned}$$

which is independent of any l_i whence (β_n). Substituting $p = q-m$ shows (γ_n). □

At first sight the fact that the coproduct on the c_i does not depend on the w_k and hence that all Dyson-Schwinger equations of this kind yield isomorphic Hopf subalgebras (provided there are no relations among the c_n) might well

come as a surprise. The deeper reason for this is the recursiveness of (97) as will become more apparent in the next paragraphs.

Description using trees. Now we specialize to the case $A = \mathcal{H}(S)$ where $(S = \sqcup S_n, |\cdot|)$ is an arbitrary graded set of decorations such that $|d_n| = n$ for all n (one can even allow $d_n \subset S_n$ and define $B_+^{d_n} := \sum_{\delta \in d_n} B_+^\delta$). The maps $B_+^{d_n}$ are defined as in (81) where the newly created vertex is decorated by d_n.

Lemma 7.2. *The solution of (97) satisfies* $c_0 = \mathbb{I}$ *and*

$$c_n = \sum_{\tau \in \mathcal{T}(S),\, |\tau| = n} \frac{\tau}{\mathrm{Sym}(\tau)} \prod_{v \in \tau^{[0]}} \gamma_v \tag{101}$$

where

$$\gamma_v = \begin{cases} w_{|dec(v)|} \frac{(|dec(v)|+1)!}{(|dec(v)|+1-\mathrm{fert}(v))!} & \text{if } \mathrm{fert}(v) \leq |dec(v)| + 1 \\ 0 & \text{else.} \end{cases}$$

Here $\mathcal{T}(S)$ *denotes the set of* S*-decorated trees,* $\tau^{[0]}$ *the set of vertices of* τ*,* $dec(v)$ *the decoration (in* S*) of* v*,* $|\tau|$ *the decoration weight of* τ*, i. e.* $|\tau| = \sum_{v \in \tau^{[0]}} |dec(v)|$*, and* $\mathrm{fert}(v)$ *the fertility (number of outgoing edges) of the vertex* v*.*

Proof. This is an easy induction using the following argument: Let τ be a given tree in c_n and let its root o be decorated by something in degree m. According to (98), $\tau = B_+^{d_m}(\mathbb{I}^{k_0} \tau_1 \dots \tau_{m+1-k_0})$ where the τ_i are trees different from $\mathbb{I}$. The fertility of the root is thus $m + 1 - k_0$. We assume $\tau_1 \dots \tau_{m+1-k_0} = \sigma_1^{k_1} \dots \sigma_p^{k_p}$ where the σ_i are pairwise different trees. In (98), there are $C := \frac{(m+1)!}{k_0! \dots k_p!}$ choices to make which yield the tree τ. Since the γ_v are simply multiplied for all vertices v of a tree, it remains to see that for the only new vertex o in τ, we have

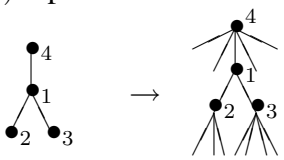
$$\gamma_o / w_m = \frac{(m+1)!}{k_0!} = C \frac{\mathrm{Sym}(\tau)}{\mathrm{Sym}(\tau_1) \dots \mathrm{Sym}(\tau_{m+1-k_0})}.$$

This however follows immediately from the definition of Sym. □

As emphasized by Kreimer, the coefficients

$$\prod_v \gamma_v \tag{102}$$

can be interpreted as follows: Consider each tree as an "operadic" object with $|dec(v)| + 1 - \mathrm{fert}(v)$ inputs at each vertex v. For example,



Clearly, the total number of inputs is $n+1$ for any tree of decoration weight n. The coefficient (102) is the number of planar embeddings of this operadic tree (where the trunk, i. e. the original tree is kept fixed). In other words, (102) counts the number of ways that the input edges can sway around the trunk. The coefficients (102) arise thus due to the transition from a non-commutative (planar) to a commutative (non-planar) setting.

In general, the combinatorial Dyson-Schwinger equations relate to the integral equations in [31, 50] by a simple application of the Feynman rules, mapping decorated rooted trees or Feynman graphs to integrals. In particular, the pushforward of $B_+^{d_n}$ along the Feynman rules is an integral operator whose kernel is determined by the Feynman rules for d_n, where each d_n is a primitive Feynman graph and $|d_n| = \mathrm{rank}\, H_1(d_n)$. See the second part of [9] for details.

For a special subclass of the combinatorial Dyson-Schwinger equations (97) a quite similar description of the Hopf subalgebras was later given by Hoffman [46], who seems to have been unaware of our result. Foissy showed [38] several related results about Dyson-Schwinger equations and Hopf subalgebras within the undecorated Hopf algebra, see also [37] for an isomorphism between the Connes-Moscovici Hopf algebra and Faa-di-Bruno Hopf algebras. On the physics side, a comprehensive discussion of Dyson-Schwinger equations and systems thereof is given in Yeats' thesis [74], see also [56, 57].

Closely related to Hopf subalgebras are Hopf ideals, studied for the Feynman graph Hopf algebras $\mathcal{H}_{CK}$ in [55, 62, 64, 69–71]. They help show that the renormalized Feynman rules are compatible with additional structures such as gauge symmetries.

8. A RESULT ABOUT GRAPH POLYNOMIALS

Beyond the pure position space (sections 2-6) and momentum space (section 7) representations of the Feynman rules, there exist two so-called *parametric representations*, one of which was already mentioned in the introduction. It is especially useful for the algebro-geometric approach to Feynman

integrals [14, 15, 17]. The basic object is a homogeneous polynomial in the edge variables which captures all the relevant data from the graph. It is called graph polynomial or Kirchhoff polynomial. Bloch, Esnault and Kreimer study the mixed Hodge structure on the cohomology of the projective hypersurface defined by this polynomial [14], called graph hypersurface. Of special interest is an understanding of the geometry of graph hypersurfaces as graphs are inserted one into another. The main result here is Theorem 8.1 which expresses the graph polynomial of an inserted graph in terms of graph polynomials of the sub- and cograph, and related graphs. I contributed a version of this result, among other things, to a joint project with A. Rej [10], answering a question of M. Marcolli.

8.1. **The parametric representation.** Let Γ be a connected Feynman graph. The linear algebra of singularities of the Feynman distribution in position space studied in section 2 relates to the momentum space picture simply by dualizing the exact sequence (8):

$$0 \to H_1(\Gamma, k) \to k^{E(\Gamma)} \xrightarrow{\partial} k^{V(\Gamma)} \xrightarrow{deg} k \to 0.$$

One considers the dual configuration $j_\Gamma : H_1(\Gamma, k) \hookrightarrow k^{E(\Gamma)}$ [14, Sections 1-2]. For a fixed basis of $H_1(\Gamma)$ let M_e be the matrix of the quadratic form $(e^\vee j_\Gamma)^2$. Associated to a configuration one defines a polynomial in the set of variables $E(\Gamma)$ by

$$\Psi_\Gamma = \Psi_{j_\Gamma} = \det\left(\sum_{e \in E(\Gamma)} M_e e\right).$$

The position space polynomial Ψ_{i_Γ} with i_Γ as defined in (9) can also be considered. It is known to coincide with the momentum space polynomial $\Psi_{j_{\Gamma^\vee}}$ of a dual graph, if one exists [17], and in general of the dual matroid.

The polynomial Ψ_Γ is called *graph polynomial* or *Kirchhoff polynomial*. It has been known and used in the physics literature for a long time. We recall from [14] a few properties of Ψ_Γ.

Proposition 8.1. [14, Section 2]

(i) Ψ_Γ *is a homogeneous sum of monomials with coefficient* $+1$, *and of degree* ≤ 1 *in each variable* e,

(ii) Ψ_Γ *is given by the spanning trees of* Γ :

$$\Psi_\Gamma = \sum_{T \,\text{sp.tree of}\, \Gamma} \prod_{e \notin T} e,$$

where a spanning tree T of Γ is defined as in section 2.3: a subgraph T (given by a subset of edges $E(T) \subseteq E(\Gamma)$) such that the composition

$$\beta : k^{E(T)} \hookrightarrow k^{E(\Gamma)} \xrightarrow{\partial} \ker \deg$$

is an isomorphism.

In our context, the interest in the graph polynomial results from the following fact. Let Γ be at most logarithmic and primitive. Recall from section (2.2) that a divergent graph has an even number of edges. Then the projective integral

$$\int_{\mathbb{P}^{d|E(\Gamma)|-1}(\mathbb{R})} \prod_{v \in V(\Gamma)} \delta_0 \left(\sum_{e \in E(\Gamma)} (v:e) k_e \right) \prod_{e \in E(\Gamma)} \frac{d^d k_e}{k_e^2}, \tag{103}$$

(cf. (2,77)) converges and is interpreted as the momentum space residue of the distribution (77), in analogy with section 3.4. Using the "Schwinger trick" $\frac{1}{k_e^2} = \int_0^\infty \exp(-e{k_e}^2) de$ for each edge $e \in E(\Gamma)$, one shows that (103) equals the projective integral

$$c \int_\sigma \frac{\Omega}{\Psi_\Gamma^{d/2}} \tag{104}$$

where $\Omega = \sum_{i=1}^{2m} (-1)^i e_i de_1 \wedge \ldots \wedge \widehat{de_i} \wedge \ldots \wedge de_{2m}$ in homogeneous coordinates where the edges are numbered $E(\Gamma) = \{e_1, \ldots, e_{2m}\}$, $\sigma \subset \mathbb{P}^{|E(\Gamma)|-1}(\mathbb{R})$ is the locus where all homogeneous coordinates can be chosen ≥ 0, and c is a generic constant in $\mathbb{Q}^\times / \pi^{4m}$ [14, Section 6]. If Γ and all its subgraphs are convergent, an analogous result holds for the two affine integrals. In any case, an integral of an inverse product of d-dimensional quadrics is reduced to an integral of an inverse power of the graph polynomial, where the new domain of integration is now a simplex.

8.2. **The work of Bloch, Esnault and Kreimer.** The integrals (103,104) are *periods* in the sense of [53]: integrals of algebraic forms over semi-algebraic sets defined over $\mathbb{Q}$. The integrals (103,104) are called (momentum space, parametric) *Feynman periods*. All Feynman periods computed so far (see [19, 20] for such results) are, up to the generic factor c, rational linear combinations of *multiple zeta values*

$$\zeta(n_1, \ldots, n_k) = \sum_{i_1 > \ldots > i_k \geq 1} \frac{1}{i_1^{n_1} \cdots i_k^{n_k}}, \quad n_i \in \mathbb{N}, n_1 \geq 2$$

which generalize the Riemann zeta function $\zeta(n) = \sum_{i \geq 1} \frac{1}{i^n}$ at integer arguments $n \geq 2$. It was shown by Goncharov and Manin [42] that every multiple zeta value is the period of a mixed Tate motive obtained from the

moduli space $\overline{\mathcal{M}}_{0,n}$ of stable curves of genus 0. A mixed Tate motive is a particularly simple kind of mixed motive [1].

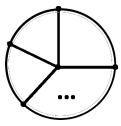
Kontsevich conjectured that the function $q \mapsto$ number of solutions of $\Psi_\Gamma = 0$ in $\mathbb{F}_q$ is a polynomial in q for all Feynman graphs Γ, which would be sensible to assume if all Feynman periods were multiple zeta values. This conjecture was disproved by Belkale and Brosnan [4], but as of today no particular example of a Feynman period different from a multiple zeta value is known.

Let X_Γ be the complex projective hypersurface, called *graph hypersurface*, defined by the homogeneous polynomial Ψ_Γ. Let $\Delta = \bigcup_{e \in E(\Gamma)} \{e = 0\}$. A sequence of blowups $P \to \ldots \to \mathbb{P}^{2m-1}$ is performed in order to separate Δ, which contains the boundary of the chain of integration σ, from the singularities X_Γ of the integrand. Let Y_Γ be the strict transform of X_Γ and B the inverse image of Δ. The Feynman period is then a period of the mixed Hodge structure [24] on the middle-dimensional cohomology

$$H^{2m-1}(P \setminus Y_\Gamma, B \setminus B \cap Y_\Gamma).$$

This is in principle similar to the situation in the moduli space [21,42] where the periods are well-understood.

In [14] Bloch, Esnault and Kreimer study a special class of primitive graphs whose periods are known to be rational multiples of Riemann zeta values: The wheel with n spokes, $n \geq 3$,

$\Gamma_n :=$  ,

which is primitive in $d = 4$ dimensions, has a (parametric) period contained in $\zeta(2n-3)\mathbb{Q}$. The main result of the paper [14] is that, by a computation which is in principle supposed to be valid also for motivic cohomology, $H_c^{2n-1}(\mathbb{P}^{2n-1} \setminus X_{\Gamma_n}) \cong \mathbb{Q}(-2)$ and $H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_{\Gamma_n}) \cong \mathbb{Q}(-2n+3)$ for the wheel graphs Γ_n. The interest in this result is that $\zeta(2n+3)$ is known to be the period, up to a rational factor, of any non-split extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(2n+3)$ in the category of mixed Hodge structures, see [16],[14, Section 9].

See [15, 17, 22] for the most recent results in this direction. In general, one is interested in the geometry of the varieties X_Γ and in particular in their intersection with Δ.

8.3. **Insertion of graphs.** A possible approach toward understanding X_Γ is obviously by studying the contribution to X_Γ of X_γ for subgraphs γ of Γ. In a first step, one is interested in the graph polynomial as graphs are inserted one into another. Let Γ be a connected Feynman graph with set of edges $E(\Gamma)$ and set of vertices $V(\Gamma)$, and consider the maps

$$\mathbb{Z}^{E(\Gamma)} \xrightarrow{\partial} \mathbb{Z}^{V(\Gamma)} \xrightarrow{deg} \mathbb{Z} \to 0$$

By definition, a spanning tree of Γ is a subgraph T (given by a subset of edges $E(T) \subseteq E(\Gamma)$) such that the composition

$$\beta : \mathbb{Z}^{E(T)} \hookrightarrow \mathbb{Z}^{E(\Gamma)} \xrightarrow{\partial} \ker \deg$$

is an isomorphism. Let now γ and Γ be connected Feynman graphs. Let $v \in V(\Gamma)$. Let E_v be the set of edges in $E(\Gamma) \cup E^{ext}(\Gamma)$ adjacent to v. Fix a one-to-one map $s : E_v \to E^{ext}(\gamma)$. This map is the gluing map. Let $\Gamma \leftarrow_{v,s} \gamma$ be the graph obtained by removing v and identifying external (open) edges via s. When v and s are clear from the context, we simply write $\Gamma \leftarrow \gamma$. The goal is to describe $\Psi_{\Gamma \leftarrow \gamma}$ in terms of graph polynomials of graphs related to Γ and γ. For $v \in V(\Gamma)$ denote by $\overline{\Gamma \setminus v}$ the graph Γ where v has been removed, and the resulting $|E_v|$ open edges have been supplied with one new ending vertex each. In other words, $\overline{\Gamma \setminus v}$ is the graph Γ where v has been split into $|E_v|$ disjoint pieces. The set of edges is to remain the same. For this it may be useful not to think of E_v as a set of ordered pairs of vertices but rather to fix a bijection $E_v = \{1, \dots, |E_v|\}$ once and for all. Then even when we perform operations on the graph like merging or splitting vertices, the numbering of the edges is supposed to remain the same. Before we turn to the graph polynomials, we need a few more definitions.

Let S be a set. A partition P of S is a set of subsets of S, called cells of P, such that every $s \in S$ is contained in exactly one element of P. A partition P' of S is said to be subordinate to P, $P' \leq P$, if for all $p' \in P'$ there is a $p \in P$ such that $p' \subseteq p$. Those p are then unique. The full partition $P = \{\{s\}, s \in S\}$ is denoted $P = 0$. Recall that partitions of S are the same as equivalence relations on S. Indeed, the equivalence classes of a given equivalence relation form the cells of a partition, and for a given partition its cells define the classes of an equivalence relation. The empty equivalence relation corresponds to the full partition. Given a graph Γ and $v \in V_\Gamma$, consider now on E_v the following equivalence relation: $e_1 \sim e_2$ iff e_1 and e_2 are connected in $\overline{\Gamma \setminus v}$. We write P_v for the resulting partition of E_v. Note that for $e_1 \sim e_2$ it is necessary (but not sufficient) that there is a cycle $c \in H_1(\Gamma)$ with $e_1 \cup e_2 \subset |c|$.

Let $P \leq P_v$. For an external line $e \in E^{ext}(\gamma)$ denote by ∂e the unique (external) vertex of γ which meets e. Now for all $\{q_1 \dots q_n\} \in P$ identify (merge) the vertices $\partial s(q_1) \dots \partial s(q_n)$ of γ respectively. The resulting graph is denoted γ_P. Note that $E(\gamma_P) = E(\gamma)$ in our conventions.

Recall the complete graph F_n where $n = |E_v|$. Fix a bijection $b : E_v \to E^{ext}(F_n)$. The partition P of E_v then determines a partition of $V(F_n)$ which we denote by P'. Let now t be a spanning tree of F_n such that all restrictions of t to the full subgraphs of F_n defined by any $Q' \in P'$ are connected. It is easy to see that such a t always exists. Let D be the subforest of t that consists of all edges connecting vertices from one and the same cell of P'. Let d be the subforest of t that consists of edges connecting different cells, such that $E(t) = E(D) \sqcup E(d)$. Consider the graph $\Gamma \leftarrow t$ (as a subgraph of $\Gamma \leftarrow_{v,b} F_n$) and remove all edges of D from it. Then shrink (collapse) all edges of d, identifying their boundary vertices. The resulting graph $(\Gamma \leftarrow d)/d$ will depend in general on the choice of t and is denoted $\Gamma^P(t)$. It is connected since $P \leq P_v$. Note that $E(\Gamma^P(t)) = E(\Gamma)$ in our convention. In particular, subgraphs of $\Gamma^P(t)$ may be identified with subgraphs of Γ. In the following it is convenient to also consider Γ, D, d, $\Gamma \leftarrow d$ etc. as subgraphs of $\Gamma \leftarrow F_n$.

Definition 8.1. *A subgraph T of $\Gamma^P(t)$ is called a* direct spanning tree *if*

$$\beta : \mathbb{Z}^{E(T \leftarrow d)} \hookrightarrow \mathbb{Z}^{E(\Gamma \leftarrow F_n)} \xrightarrow{\partial} \ker \deg$$

is an isomorphism and $\partial(\mathbb{Z}^{E(D)}) \subseteq \beta(\mathbb{Z}^{E(T)})$.

In other words, a direct spanning tree is a spanning tree which connects the vertices of each cell of P directly within $\overline{\Gamma \setminus v}$, without using the connections through d.

Proposition 8.2. *Let t, t' be two choices of spanning trees for F_n. Then T is a direct spanning tree of $\Gamma^P(t)$ if and only if it is for $\Gamma^P(t')$.*

As a consequence, the *direct graph polynomial*

$$\tilde{\Psi}_{\Gamma^P(t)} = \sum_{T \text{ dir st}} \prod_{e \in E(\Gamma) \setminus E(T)} e$$

is independent of t, and we simply write $\tilde{\Psi}_{\Gamma^P}$.

Proof. It is obvious that d and d' have equal number of edges and that $\partial(\mathbb{Z}^{E(D)}) = \partial(\mathbb{Z}^{E(D')})$. It suffices, therefore, to show $\beta(\mathbb{Z}^{E(T \leftarrow d)}) = \beta(\mathbb{Z}^{E(T \leftarrow d')})$. This follows since $\beta(\mathbb{Z}^{E(d)}) \subseteq \partial(\mathbb{Z}^{E(t)}) = \partial(\mathbb{Z}^{E(t')}) = \partial(\mathbb{Z}^{E(d' \cup D')}) \subseteq \partial(\mathbb{Z}^{E(d')}) + \beta(\mathbb{Z}^{E(T)})$, and similarly for d and d' interchanged. $\square$

Theorem 8.1.

$$\Psi_{\Gamma\leftarrow\gamma} = \sum_{0\neq P\leq P_v} \Psi_{\gamma_P}\tilde{\Psi}_{\Gamma^P}. \tag{105}$$

Remarks. In [14, Section 3] it is already shown, in a slightly different notation, that

$$\Psi_{\Gamma\leftarrow\gamma} = \Psi_\gamma\Psi_\Gamma + \tilde{\Psi}_{\gamma,\Gamma}$$

where the remainder $\tilde{\Psi}_{\gamma,\Gamma}$, unless vanishing, is of degree higher (lower) in the edge variables of γ (of Γ). Since $\Psi_{\gamma^0} = \Psi_\gamma$ and $\tilde{\Psi}_{\Gamma_0} = \Psi_\Gamma$, equation (105) provides now an explicit expression for this remainder term.

Proof. Since $E(\Gamma \leftarrow \gamma) = E(\Gamma_P) \sqcup E(\gamma^P)$, it suffices to prove that T is a spanning tree of $\Gamma \leftarrow \gamma$ if and only if T_Γ is a direct spanning tree of Γ_P and T_γ a spanning tree of γ^P (for some P) where T_Γ is the subgraph of Γ_P such that $E(T_\Gamma) = E(T) \cap E(\Gamma)$ and T_γ the subgraph of γ^P such that $E(T_\gamma) = E(T) \cap E(\gamma)$ (consequently $T = T_\Gamma \leftarrow T_\gamma$). Note that, since Γ_P has $|V(\Gamma)| + |P| - 1$ vertices and γ^P has $|V(\gamma)| - |P| + 1$ vertices, a spanning tree of Γ_P has $|V(\Gamma)| + |P| - 2$ edges and one of γ^P has $|V(\Gamma)| - |P|$ edges. Recall that $V(\Gamma \leftarrow \gamma)$ has $|V(\Gamma)| + |V(\gamma)| - 1$ vertices, and its spanning trees $|V(\Gamma)| + |V(\gamma)| - 2$ edges. The numbers of edges of T, T_Γ, T_γ, fit thus together in the first place. We write

$$\begin{aligned}
\mathbb{Z}^{E(\Gamma\leftarrow\gamma)} &\xrightarrow{\partial} \mathbb{Z}^{V(\Gamma\leftarrow\gamma)} \xrightarrow{deg} \mathbb{Z}, \qquad & V_0(\Gamma \leftarrow \gamma) &= \ker \deg \\
\mathbb{Z}^{E(\Gamma\leftarrow F_n)} &\xrightarrow{\partial} \mathbb{Z}^{V(\overline{\Gamma\setminus v})} \xrightarrow{deg_\Gamma} \mathbb{Z}, \qquad & V_0(\overline{\Gamma \setminus v}) &= \ker \deg_\Gamma \\
\mathbb{Z}^{E(\gamma)} &\xrightarrow{\partial} \mathbb{Z}^{V(\gamma)} \xrightarrow{deg_\gamma} \mathbb{Z}, \qquad & V_0(\gamma) &= \ker \deg_\gamma
\end{aligned}$$

and may consider all of the V_0 as subspaces of $V(\Gamma \leftarrow \gamma)$ such that

$$V_0(\Gamma \leftarrow \gamma) = V_0(\overline{\Gamma \setminus v}) + V_0(\gamma)$$

whereas

$$V_0(\overline{\Gamma \setminus v}) \cap V_0(\gamma) = V_0(F_n) = V_0(D) + V_0(d)$$

with $t = D \cup d$ as before a spanning tree of F_n adapted to P. Let now T_Γ be a direct spanning tree and T_γ a spanning tree of Γ_P and γ^P, respectively. By definition, $\partial(\mathbb{Z}^{E(T_\Gamma\cup d)}) + \partial(\mathbb{Z}^{E(T_\Gamma\cup D)}) = V_0(\overline{\Gamma \setminus v}) + V_0(\gamma)$. Since $\partial(\mathbb{Z}^{E(D)}) \subseteq \partial(\mathbb{Z}^{E(T_\Gamma)})$ and $\partial(\mathbb{Z}^{E(d)}) \subseteq \partial(\mathbb{Z}^{E(T_\gamma)})$, even $\partial : \mathbb{Z}^{E(T_\Gamma\cup T_\gamma)} \to V_0(\overline{\Gamma \setminus v}) + V_0(\gamma) = V_0(\Gamma \leftarrow \gamma)$ is surjective. It is then also injective because domain and range have equal dimension. Consequently T is a spanning tree of Γ. Suppose now that T is a spanning tree of Γ, and let T_Γ and T_γ be defined as above. We have $\partial(\mathbb{Z}^{E(T_\Gamma)}) \cap V_0(K_n) = V_0(D)$ for some subforest D of K_n. The connected components of D determine a partition $P \leq P_v$. We complete D to a spanning tree $t = D \cup d$ of K_n.

Obviously $\partial(\mathbb{Z}^{E(T_\Gamma)}) \subseteq V_0(\overline{\Gamma \setminus v})$. Since T is a spanning tree for $\Gamma \leftarrow \gamma$ and $\mathbb{Z}^{E(T_\gamma)} \cap V_0(\overline{\Gamma \setminus v}) \subseteq V_0(K_n)$, the forest T_Γ reaches all vertices of $\overline{\Gamma \setminus v}$ except possibly some at K_n. Using $V_0(D) \subseteq \partial(\mathbb{Z}^{E(T_\Gamma)})$, only the connections $V_0(d)$ are missing for T_Γ to be connected, and we have $\partial(\mathbb{Z}^{E(T_\Gamma \cup d)}) = V_0(\overline{\Gamma \setminus v})$. Injectivity follows again from matching dimensions. T_Γ is therefore a spanning tree of Γ^P. It is direct since $V_0(D) \subseteq \partial(\mathbb{Z}^{E(T_\Gamma)})$ by definition of D. The same argument shows that T_γ is a spanning tree of γ^P. □

Consequences for the geometry of graph hypersurfaces, in particular their singular loci, will be discussed elsewhere.

Let us not turn to what physicists call propagator corrections. Insertion of a subgraph γ into an edge of Γ means

(i) Insert a vertex in the middle of this edge.
(ii) Insert γ into this vertex.

Having Theorem 8.1 it remains to study step (i). Let $e \in E(\Gamma)$. Let Γ^e be the graph obtained from Γ by inserting a new vertex v in the middle of e. We call e_1 and e_2 the two edges of Γ^e adjacent to v.

Proposition 8.3.

$$\Psi_{\Gamma^e} = \Psi_\Gamma|_{e=e_1+e_2}$$

Proof. Let t be a spanning tree of Γ. Either t includes e or it does not. If it does, $t \setminus e \cup e_1 \cup e_2$ defines a spanning tree of Γ^e. If t does not include e, then $t \setminus e \cup e_1$ and $t \setminus e \cup e_2$ define spanning trees of Γ^e. Conversely, every spanning tree of Γ^e is obtained this way. □

From Proposition 8.3 one concludes that $X_{\Gamma^e} \cong \overline{C} X_\Gamma$ where $\overline{C}$ denotes the projective cone (projective closure of the affine cone).

INDEX OF NOTATION

Sections 2 - 6:

$\underline{1}$	the constant function $x \mapsto 1$
A	linear subspace of $V^\vee$, see section 4.1
A_e	see (10)
$A^\perp$	linear subspace annihiliated by A
A_γ	see (14)
Bip	set of bipartite graphs, see section 6
β	birational transformation $Y_{\mathcal{C}} \to V$, see section 4.1
c	see (8)
$\mathcal{C}$	collection of linear subspaces, see section 4.1
$\mathcal{C}(\Gamma)$	see (11)
$\mathcal{C}_{div}(\Gamma)$	collection of divergent subspaces, see (15)
$\mathcal{C}_{sing}(\Gamma)$	collection of singular subspaces, see (13)
$\mathcal{D}'(\mathcal{M})$	space of distributions on $\mathcal{M}$, see section 3.1
$\tilde{\mathcal{D}}'(\mathcal{M})$	space of distribution densities on $\mathcal{M}$, see section 3.1
δ	see (8)
δ_0	Dirac-measure at 0
Δ	thin diagonal, see last paragraph of section 2.1
$\Delta(\Gamma)$	coproduct, see section 6.3
d	dimension of space-time
d_A	$\dim A$
d_Γ	$\dim A_\gamma$
$\lvert dx\rvert$	Lebesgue measure on $\mathbb{R}^n$
D_{ij}	diagonal in M^n defined by $x_i = x_j$, see beginning of section 2
D_e	diagonal in M^n defined as D_{jl} where $(j:e) = 1$, $(l:e) = -1$
$\mathcal{E}_A$	irreducible component of the exceptional divisor, see section 4.2
e	edge in $E(\Gamma)$, basis element of k^E, see beginning of section 2.1
$e^\vee$	dual of e
E	finite set
$\mathcal{E}$	exceptional divisor of the smooth model, see section 4.1
$E(\Gamma)$	set of edges of a Feynman graph, see beginning of section 2.1
$\mathcal{F}$	Fourier transform
f^*u	pullback of u along f
f_Γ	see (46) and Theorem 5.1
γ	subgraph of Γ, see beginning of section 2.2
$\gamma//\mathcal{N}$	contracted graph, see section 5.2
γ_p	contracted graph, see section 5.2
γ_s	saturated subgraph, see section 2
Γ	Feynman graph, see beginning of section 2
$\Gamma//\gamma$	contracted graph, see section 6.3
$H_1(\gamma)$	first homology of γ
$H^1(\Gamma, k)$	see (8)
$\mathcal{H}_{FG}, \overline{\mathcal{H}_{FG}}$	see section 6.3
k	ground field, here $k = \mathbb{R}$
k^E	k-vector space spanned by E
i_W	configuration, see beginning of section 2.1
$j^{\mathcal{B}}_{\mathcal{N}}$	see section 4.5
K	a compact subspace
K_n	complete graph on n vertices
$L^1_{loc}(\mathbb{R}^n)$	space of locally integrable functions
M	space-time, here $M = \mathbb{R}^d$
$\mathcal{M}$	manifold, see section 3.1
$(M^{V_0})_{div}(\Gamma)$	arrangement of divergent subspaces, see (16)
$(M^{V_0})_{sing}(\Gamma)$	arrangement of singular subspaces, see (12)
μ	see Theorem 3.1
$\mathcal{N}$	nested set, see section 4.4

N_{max}	see Theorem 5.3
n_{ij}	number of edges between i and j
ν_{A_γ}	renormalization condition, see section 6.2
$\mathcal{P}$	building set, see section 4.2
$\mathbb{P}(V)$	projective space of lines in V, see section 4.2
$p(x)$	see section 4.4
$p^\vee$	see section 5.2
P_x	see Proposition 4.8 (iii)
$p_{t,s}$	see section 5.2
$p_{I,J}$	see section 6
Φ	map $M^V \to M^{V_0}$, see last paragraph of section 2.1
$\Phi_* u$	pushforward of u along Φ
ϕ	see middle of section 2.1
ρ	see (51)
$\Re s$	real part of s
$\operatorname{res}\Gamma$	residue of Γ, see (46)
S	antipode, see section 6.3
s	complex regulator, see section 3
supp	support
sing supp	singular support, see section 2
τ	tautological bundle, see section 3.1
$\tilde{u}$	distribution density, see section 3.1
$U^{\mathcal{B}}_{\mathcal{N}}$	see section 4.5
$u[\phi]$	distribution u applied to the test function ϕ
$u[\phi]_f$	see section 3.1
$u(x)$	distribution u evaluated at x (if u is continuous at x)
u_0	Feynman propagator, see (4)
u_Γ	Feynman graph distribution, see (7)
$\underline{u}_\Gamma$	$\Phi_* u_\Gamma$
u_{Γ,R_0}	see Theorem 6.1
u_{Γ,R_μ}	see Theorem 6.1
v	vertex in $V(\Gamma)$, basis element of $k^{V(\Gamma)}$, see beginning of section 2.1
v_0	distinguished vertex, see section 2.1
$(v:e)$	see beginning of section 2.1
V	finite-dimensional real vector space, see 4.1
$V_{\mathcal{C}}$	arrangement in V, see section 4.1
$V(\Gamma)$	vertex set of a Feynman graph, see beginning of section 2
V_{eff}	see section 4.3
V_0	see section 2.1
$\tilde{w}_{\Gamma,R_0}$	see section 6.2
$\tilde{w}_{\Gamma,R_\mu}$	see section 6.2
$\lvert x\rvert^{-as}_{ext}$	see (36)
$\lvert x\rvert^{-as}_{fin}$	see (37)
x_i	d-tuple of coordinates $x_i^1, \ldots, x_i^d$ on some M^n, $1 \le i \le n$
x_A	marking of a basis, see section 4.4
ξ_i	coordinates on some k^n, see end of section 2.1
$Y_{\mathcal{C}}$	smooth model for $V_{\mathcal{C}}$, see section 4.1
$Y_{\mathcal{P}}$	wonderful model for $V_{\mathcal{C}}$, see section 4.2
$y_A^{i_A}$	marking of a basis, see section 4.4
z_i	local coordinates
Z_B	see section 4.5
$\sqcup$	disjoint union
$\le$	(for partitions), see section 4.3
$\preceq$	partial order on a basis, see section 4.4

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ZUSAMMENFASSUNG

Die vorliegende Dissertation beschäftigt sich mit Feynman-Graphen und zugeordneten Feynman-Integralen, die in der störungstheoretischen Quantenfeldtheorie von Bedeutung sind. Im ersten Teil der Arbeit definiere ich zu einem Feynman-Graphen zwei Arrangements von linearen Teilräumen, die jeweils den singulären Ort des Feynman-Integranden und den Ort, wo dieser nicht einmal lokal integrierbar ist, beschreiben. Ich studiere mehrere geeignete Auflösungen von Singularitäten, die die Arrangements in einen Divisor mit normalen Überkreuzungen verwandeln, unter Benützung eines allgemeineren Ergebnisses von De Concini und Procesi. Der Feynman-Integrand lässt sich nun auf das Komplement des Divisors zurückziehen, und mittels einer analytischen Regularisierung als meromorphe Distributionswertige Funktion auf das glatten Modell fortsetzen. Ich beweise physikalisch relevante Relationen zwischen den Laurent-Koeffizienten, und studiere lokalitätserhaltende Renormierungsverfahren auf dem glatten Modell. Im Gegensatz zu den in der Literatur vorhandenen rekursiven Renormierungsverfahren für den Ortsraum sind hier die kombinatorischen Einzelheiten in der Geometrie des glatten Modells kodiert, und eine einzige Subtraktion entlang dem Divisor genügt. Hierfür sind auch die von Connes und Kreimer eingeführten Hopfalgebren hilfreich.
Im zweiten Teil der Arbeit beweise ich den Zusammenhang zwischen kombinatorischen Dyson-Schwinger-Gleichungen und Hopf-Teilalgebren der Connes-Kreimer-Hopfalgebren. Eine gewisse Rolle spielt dabei die erste Hochschild-Kohomologie dieser Hopfalgebren.
Der dritte Teil der Arbeit leistet einen Beitrag zur parametrischen Darstellung von Feynman-Integralen, die unter anderem von Bloch, Esnault und Kreimer benutzt wird, um die motivische Kohomologie von Feynmangraph-Hyperflächen zu verstehen. Das Ergebnis bezieht sich auf das Verhalten von Graphpolynomen, wenn Graphen ineinander eingesetzt werden.

ERKLÄRUNG

Hiermit versichere ich, dass ich in meiner Dissertation sämtliche Hilfsmittel und Hilfen angegeben und die Arbeit selbständig verfasst habe. Die Arbeit ist nicht bereits in einem früheren Promotionsverfahren eingereicht worden.

Berlin, den 14.04.09

Aus Gründen des Datenschutzes enthält die elektronische Version keinen Lebenslauf.