

# Existence of long-time solutions to dynamic problems of viscoelasticity with rate-and-state friction

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## Funding information

Deutsche Forschungsgemeinschaft

\*Before the author passed away, he entrusted Alexander Mielke to submit his paper and act as a corresponding author in place of him.

We establish existence of global solutions to a dynamic problem of bilateral contact between a rigid surface and a viscoelastic body, subject to rate-and-state friction. The term *rate-and-state friction* describes friction laws where the friction is rate-dependent and depends on an additional internal state variable defined on the contact surface. Our mathematical conditions rule out certain slip laws, but do cover the ageing law, and thus at least one of the rate-and-state friction laws commonly used in the geoscience.

## KEYWORDS

ageing law, quasivariational evolutionary inequality, rate-and-state friction, surface friction, viscoelasticity

## MSC (2010)

74L10, 74M10, 35Q74

## 1 | INTRODUCTION

We consider here the dynamic motion of a viscoelastic body  $\Omega \subset \mathbb{R}^d$  in bilateral contact with a rigid foundation (on the boundary segment  $\Gamma_C$ ), undergoing infinitesimal displacement  $\mathbf{u}$  and linear viscoelastic total strain  $\boldsymbol{\sigma}$ . Following [2,18] the rate-and-state friction along the contact set  $\Gamma_C$  is given in terms of the sliding velocity  $\dot{\mathbf{u}}$  and a scalar internal state variable  $\alpha$  defined only on  $\Gamma_C$ , see below for more details.

To that end, we will derive a weak formulation of the following problem.

*Problem 1.1.* Find a displacement field  $\mathbf{u}$  on  $\Omega$  of the appropriate regularity that satisfies

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega \times I \quad (1.1)$$

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}} \quad \text{in } \Omega \times I \quad (1.2)$$

with the boundary conditions

$$\dot{\mathbf{u}} = 0 \quad \text{on } \Gamma_D \times I \quad (1.3)$$

$$\boldsymbol{\sigma} \mathbf{n} = 0 \quad \text{on } \Gamma_N \times I \quad (1.4)$$

$$\dot{\mathbf{u}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_C \times I \quad (1.5)$$

$$\left. \begin{aligned} b - \sigma_t &= \frac{\mu(|\dot{\mathbf{u}}|, \alpha) \bar{\sigma}_n + C}{|\dot{\mathbf{u}}|} \dot{\mathbf{u}} \quad \text{for } \dot{\mathbf{u}} \neq 0 \\ |\sigma_t| &\leq \mu(0, \alpha) \bar{\sigma}_n + C \quad \text{for } \dot{\mathbf{u}} = 0 \end{aligned} \right\} \quad \text{on } \Gamma_C \times I \quad (1.6)$$

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with prescribed  $\mathbf{u}(0, \cdot) : \Omega \rightarrow \mathbb{R}^d$  and  $\dot{\mathbf{u}}(0, \cdot) : \Omega \rightarrow \mathbb{R}^d$  as well as a scalar state field  $\alpha$  on  $\Gamma_C$  that satisfies

$$\dot{\alpha} + A(\alpha) = f(|\dot{\mathbf{u}}|) \quad \text{on } \Gamma_C \times I \quad (1.7)$$

with prescribed  $\alpha(0, \cdot) : \Gamma_C \rightarrow \mathbb{R}$ .

Here, the nonlinear functions  $A, f : \mathbb{R} \rightarrow \mathbb{R}$  in (1.7) define the flow law for the internal state variable  $\alpha$ . Moreover, we write  $\mathbf{u}$  for the displacement,  $\mathbf{b}$  for the body force,  $\boldsymbol{\sigma}$  for the stress tensor, and  $\boldsymbol{\sigma}_t$  for its tangential component where the tangential direction is computed from the outer normal  $\mathbf{n}$ . Linear Kelvin–Voigt viscoelasticity is prescribed in (1.1), formulated in terms of the linearized strain tensor  $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ , a viscosity tensor  $\mathcal{A}$  and an elasticity tensor  $\mathcal{B}$ . The unilateral contact conditions (1.5)–(1.7) model the unilateral contact with a rigid body in the small-strain idealization. This is the simplest theoretical or real experiment for testing friction in geophysics. Real-world models should use bilateral contact of two viscoelastic bodies along an interface (the fault). For notational simplicity and in accordance with the analog experiments in [14,15]. More specifically, the friction law (1.6) on  $\Gamma_C$  is made up of the friction coefficient  $\mu$ , the cohesion  $C \geq 0$  and a prescribed, constant quantity denoted by  $\bar{\sigma}_n > 0$ , meant to approximate the state-dependent normal stress  $\sigma_n$ . This approximation is usually called ‘‘Tresca-friction model’’. It simplifies the mathematics compared to true Coulomb friction (cf. [3]), and is well acceptable in geophysical applications, where  $\bar{\sigma}_n$  maybe understood as the difference between the lithostatic and the hydrostatic pressures, which nearly constant in fluid-saturated poroelastic rocks. Dirichlet and Neumann boundary conditions are, furthermore, imposed on the boundary segments  $\Gamma_D$  and  $\Gamma_N$ , respectively. The mass density is denoted by  $\rho$ .

The plan of the paper is as follows. In Section 2 we describe the background of the model and its applications in geoscience. In Section 3 we discuss several examples of rate-and-state friction laws  $(\dot{\mathbf{u}}, \alpha) \mapsto \mu(|\dot{\mathbf{u}}|, \alpha) + C$ . In Section 4 we provide the necessary mathematical assumptions for such laws and discuss their validity in specific cases. In Section 5 we start the analysis of the problem by reformulating it as an evolutionary variational inequality in weak form. The solution method relies on a splitting method where we first show that for fixed  $\alpha$  in a suitable function class we obtain a unique solution  $\mathbf{u} = R(\alpha)$ , and then we show that for fixed  $\mathbf{u}$  the problem for  $\alpha$  has a unique solution  $\alpha = S(\mathbf{u})$ , see Sections 6 and 7, respectively. In Section 8 the final global existence result of Theorem 8.2 is obtained by first showing that the mapping  $R \circ S$  is a contraction for small final times  $T$ , thus providing a local existence result. By concatenation of small time steps the global result is then deduced.

## 2 | BACKGROUND

Rate-and-state friction plays an important role in the modeling of faults, which in turn play an important role in earthquake nucleation, see [4,8,18,19] and the references therein. It expresses frictional resistance in terms of the sliding velocity or *slip rate*  $|\dot{\mathbf{u}}|$  and an internal *state variable*  $\alpha$  defined on the contact boundary  $\Gamma_C$ , which may have different physical interpretations depending on the context. In Geophysics, it is often called *ageing parameter* (see below); in engineering the name ‘interfacial damage variable’ is used; and the model in [6] allows for the interpretation that  $\alpha > 0$  is an interfacial temperature.

The evolution of  $\alpha$  is coupled to the sliding velocity  $|\dot{\mathbf{u}}|$  and in a specific way, as well as the friction coefficient  $\mu$  depends on  $|\dot{\mathbf{u}}|$  and  $\alpha$  is crucial. The dependence  $\mu = \mu(|\dot{\mathbf{u}}|, \alpha)$  should can be thought of as  $\mu$  depending on  $|\dot{\mathbf{u}}|$  in two different ways: firstly directly on  $|\dot{\mathbf{u}}|$  in a monotone fashion, and secondly indirectly, through  $\alpha$ , which reacts less immediately to changes in  $|\dot{\mathbf{u}}|$ , but generally in an anti-monotone fashion.

Although laws that go by this name have been derived from experiments [2,18], they could just as easily have been proposed as a regularization of slip rate dependent friction (in which the coefficient of friction is a function of the sliding rate only but the dependence is generally anti-monotone) due to the analytical and numerical difficulties that such ostensibly simpler stateless laws present, see [7].

The existence and uniqueness of solutions to (weak formulations of) dynamic problems of viscoelasticity and friction has been thoroughly studied. Rate-and-state friction falls outside the scope of these studies, however, because of the variable coupling between the rate and the state: Neither is typically known. The approach taken in this work is thus to consider the situation where  $\alpha$  is known a-priori, to then compute  $\dot{\mathbf{u}}$  under this assumption (such problems are covered by the current literature) and to then account for the actual lack of knowledge of  $\alpha$  through a fixed-point iteration.

Even though this model for rate-and-state dependent friction is very popular in geophysics, it does not have an underlying thermodynamical structure allowing for a proper balance of energies through external powers and dissipation. This mathematical drawback was already pointed out in [13], where two energies have been used to govern separately the momentum balance for  $\mathbf{u}$  and the flow rule for the internal variable  $\alpha$ , or in [17], where an energetic formulation was obtained for a slightly expanded model.

The quasistatic rate-and-state friction model (i.e. without the kinetic term  $\rho\ddot{u}$ ) but even with a Coulomb-friction variant has recently been analyzed in [11]. General dynamic friction models, even including true Coulomb friction, but without additional internal variables, are studied analytically in [3].

This work thus parallels earlier work from the author's dissertation in which the time-discrete setting was considered.<sup>[12]</sup>

### 3 | EXAMPLES

The following two rate-and-state friction laws are commonly used: the *ageing law* (also known as *slowness law*), which states

$$\mu = \mu_* + a \log \frac{r}{r_*} + b\alpha, \quad \dot{\alpha} = \frac{r_* e^{-\alpha} - r}{L}, \quad (3.1)$$

and the *slip law*, which states

$$\mu = \mu_* + a \log \frac{r}{r_*} + b\alpha, \quad \dot{\alpha} = -\frac{r}{L} \left( \log \frac{r}{r_*} + \alpha \right), \quad (3.2)$$

with positive constants  $a$  and  $b$ . When presented in this form, both laws use the same expression for  $\mu$ , so that their respective state variables  $\alpha$  can be identified; consequently, the names of these laws are typically used to refer to the associated state evolution equations for  $\alpha$  only.

The ageing law and the slip law as proposed by Dieterich and Ruina in [2] and [18], respectively, employ the term  $\log(r/r_*)$ , which becomes arbitrarily negative for sliding rates  $r$  close to zero; consequently, we have

$$\mu(r, \alpha) \rightarrow -\infty \quad \text{whenever } r \rightarrow 0$$

for fixed  $\alpha$ . They are thus unphysical for sufficiently small  $r$ , since they predict a negative coefficient of friction. If we introduce the quantity

$$r_\alpha = r_* \exp\left(-\frac{\mu_* + b\alpha}{a}\right),$$

this issue becomes even clearer, since now  $\mu$  can be written as

$$\mu(r, \alpha) = a \log \frac{r}{r_\alpha}, \quad (3.3)$$

so that  $r_\alpha$  denotes the rate at which the predicted coefficient of friction undergoes a sign change. In the literature, this undesirable behavior of the Dieterich–Ruina laws (cf. [2,18]) has been addressed by means of regularization, see [16]. To be precise, the logarithm on the right-hand side of (3.3) is replaced by the nonnegative function  $z \mapsto \operatorname{asinh}(z/2)$ , yielding the *regularized law*

$$\mu_r(r, \alpha) = a \operatorname{asinh}\left(\frac{r}{2r_\alpha}\right). \quad (3.4)$$

A different approach is to trust the original law as much as possible, and only modify it whenever it predicts a negative coefficient of friction. The requirement of monotonicity then leads to the *truncated law*

$$\mu_t(r, \alpha) = a \log^+ \frac{r}{r_\alpha} \quad \text{with} \quad \log^+ z = \log \max(1, z) \quad (3.5)$$

Both adjustments clearly guarantee nonnegativity of the friction coefficient.

In what follows, rather than consider such laws directly, we choose to work in an abstract setting where friction is described through the friction coefficient  $\mu : \mathbb{R}_0^+ \times \mathbb{R} \rightarrow \mathbb{R}_0^+$  and two functions  $A : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$  that govern the state evolution through the equation

$$\dot{\alpha} + A(\alpha) = f(r).$$

It is immediately clear that the slip law does not fall into this setting, unfortunately. The ageing law and potentially other laws of interest, however, do.

## 4 | ABSTRACT RATE-AND-STATE FRICTION

In working with  $\mu$ ,  $A$ , and  $f$ , we find it necessary to make the following assumptions.

The function  $\mu$  is nondecreasing and continuous in its first argument. (A1)

The function  $\mu$  is uniformly Lipschitz in its second argument, i.e. we have (A2)

$$|\mu(r, \alpha) - \mu(r, \beta)| \leq L_\mu |\alpha - \beta| \quad \text{for all } \alpha, \beta, \text{ and } r \geq 0.$$

The function  $\mu$  can be bounded as follows: (A3)

$$0 \leq \mu(r, \alpha) \leq C_\mu (1 + r + |\alpha|) \quad \text{for all } \alpha \text{ and } r \geq 0.$$

The function  $A$  is nondecreasing and continuous. (A4)

The function  $f$  is Lipschitz, so that we have (A5)

$$|f(r) - f(v)| \leq L_f |r - v| \quad \text{for all } r \text{ and } v.$$

Assumptions (A2) and (A3) are not independent; indeed, if we assume the former, the latter reduces to requiring  $\mu(r, 0) \leq C_\mu (1+r)$ .

As mentioned earlier, the slip law clearly does not fit into this framework because of the requirement that  $\dot{\alpha}$  can be written as a sum of two terms, one of which depends solely on  $\alpha$  with the other depending solely on  $r$ . The ageing law, in contrast, satisfies all of the assumptions made above.

**Proposition 4.1.** *Consider the ageing law (3.1), either regularized as per (3.4) or truncated as per (3.5). Then the resulting law satisfies (A1)–(A5).*

*Proof.* That  $\mu_r$  and  $\mu_t$  satisfy (A1) is clear. To show that  $\mu_r$  satisfies (A2), it suffices to prove

$$|\mu_r(r, \alpha) - \mu_r(r, \beta)| = a \left| \operatorname{asinh} \left( \frac{r}{2r_\alpha} \right) - \operatorname{asinh} \left( \frac{r}{2r_\beta} \right) \right| \leq a \left| \log \frac{r_\beta}{r_\alpha} \right|$$

for any  $\alpha, \beta$ , and  $r \geq 0$ , since the right-hand side equals  $b \cdot |\alpha - \beta|$ . For  $r = 0$ , this is immediate; for  $r > 0$ , it becomes clear once we prove the more general claim

$$|\operatorname{asinh}(x) - \operatorname{asinh}(y)| \leq |\log x - \log y|$$

for  $x, y > 0$ . Without loss of generality, assume  $x \geq y$ , so that we need to show

$$\operatorname{asinh}(x) - \operatorname{asinh}(y) \leq \log x - \log y.$$

From the logarithmic representation of the asinh function, we obtain that this is equivalent to

$$\log \frac{x + \sqrt{x^2 + 1}}{y + \sqrt{y^2 + 1}} \leq \log \frac{x}{y}$$

and thus

$$y\sqrt{x^2 + 1} \leq x\sqrt{y^2 + 1}$$

which is obviously true. For  $\mu_t$ , we proceed analogously and prove

$$|\mu_t(r, \alpha) - \mu_t(r, \beta)| = a \left| \log^+ \frac{r}{r_\alpha} - \log^+ \frac{r}{r_\beta} \right| \leq a \left| \log \frac{r_\beta}{r_\alpha} \right|.$$

Again, this is trivially true if  $r = 0$ . For  $r > 0$ , we have

$$\begin{aligned} \left| \log^+ \frac{r}{r_\alpha} - \log^+ \frac{r}{r_\beta} \right| &= \left| \log \max \left\{ \frac{r}{r_\alpha}, 1 \right\} - \log \max \left\{ \frac{r}{r_\beta}, 1 \right\} \right| \\ &= \left| \max \left\{ \log \frac{r}{r_\alpha}, 0 \right\} - \max \left\{ \log \frac{r}{r_\beta}, 0 \right\} \right| \leq \left| \log \left( \frac{r}{r_\alpha} \right) - \log \left( \frac{r}{r_\beta} \right) \right| \end{aligned}$$

since  $\max\{\cdot, 0\}$  is nonexpansive, so that the claim follows. To see that  $\mu_t$  and  $\mu_r$  satisfy (A3), observe only

$$\mu_t(r, \alpha) = a \log^+ \frac{r}{r_\alpha} \leq a \left( \log^+ \frac{r}{r_*} + \left| \log \frac{r_\alpha}{r_*} \right| \right) \leq a \frac{r}{r_*} + \mu_* + b|\alpha|.$$

and

$$\begin{aligned} \mu_r(r, \alpha) &= a \operatorname{asinh} \frac{r}{2r_\alpha} = a \log \left( \frac{r}{2r_\alpha} + \sqrt{\left( \frac{r}{2r_\alpha} \right)^2 + 1} \right) \leq a \log \left( \frac{r}{r_\alpha} + 1 \right) \\ &\leq a \log \left( 2 \max \left\{ 1, \frac{r}{r_\alpha} \right\} \right) = a \log 2 + \mu_t(r, \alpha). \end{aligned}$$

Finally, each law clearly satisfies the assumptions (A4) and (A5) with

$$A(\alpha) = -\frac{r_*}{L} e^{-\alpha}, \quad f(r) = r/L, \quad \text{and} \quad L_f = L. \quad \square$$

## 5 | WEAK FORMULATION

Here and in what follows, we will make the following typical assumptions on the domain  $\Omega$ , the viscoelastic parameters, the body force, and the normal stress that we prescribe on the frictional boundary  $\Gamma_C$ .

The domain  $\Omega$  is a bounded open subset of  $\mathbb{R}^d$  with a Lipschitz boundary such that the  $d$ -dimensional trace map  $\gamma$  is well-defined from  $H^1(\Omega)^d$  to  $L^2(\Gamma)^d$  with norm  $\|\gamma\|$ . (A6)

The viscosity tensor is symmetric as well as uniformly bounded from above and below through  $0 < m_A \leq M_A$ , so that for all  $\mathbf{v}, \mathbf{w} \in V$  we have (A7)

$$\begin{aligned} m_A \|\mathbf{v}\|_V^2 &\leq \langle \mathfrak{A}\mathbf{v}, \mathbf{v} \rangle = \int_\Omega \langle \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle \quad \text{and} \\ \int_\Omega \langle \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{w}) \rangle &= \langle \mathfrak{A}\mathbf{v}, \mathbf{w} \rangle \leq M_A \|\mathbf{v}\|_V \|\mathbf{w}\|_V. \end{aligned}$$

The elasticity tensor is symmetric as well as uniformly bounded from above and below through  $0 < m_B \leq M_B$ , so that for all  $\mathbf{v}, \mathbf{w} \in V$  we have (A8)

$$\begin{aligned} m_B \|\mathbf{v}\|_V^2 &\leq \langle \mathfrak{B}\mathbf{v}, \mathbf{v} \rangle = \int_\Omega \langle \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle \quad \text{and} \\ \int_\Omega \langle \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{w}) \rangle &= \langle \mathfrak{B}\mathbf{v}, \mathbf{w} \rangle \leq M_B \|\mathbf{v}\|_V \|\mathbf{w}\|_V. \end{aligned}$$

The body force  $\mathbf{b}$  satisfies  $\|\mathbf{b}\|_{L^2(0,T,V^*)} < \infty$ . (A9)

The prescribed normal stress  $\bar{\sigma}_n > 0$  satisfies  $\|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} < \infty$ . (A10)

We will work with the spaces

$$V = \{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = 0 \text{ on } \Gamma_D, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_C \}, \quad H = L^2(\Omega)^d, \text{ and } X = L^2(\Gamma_C),$$

In a standard fashion, by testing (1.2) with functions from  $V$  at fixed points in time, and putting (1.1) as well as (1.3)–(1.6) to use, we obtain the following weak rate problem in the form of an evolutionary variational inequality.

*Problem 5.1.* For given  $\alpha \in C(0, T, X)$ , find  $\mathbf{u} \in L^2(0, T, V)$  with  $\dot{\mathbf{u}} \in L^2(0, T, V)$  and  $\ddot{\mathbf{u}} \in L^2(0, T, V^*)$  such that

$$\begin{aligned} & \int_{\Omega} \rho \langle \ddot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle + \int_{\Omega} \langle \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)) \rangle + \int_{\Omega} \langle \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)) \rangle + \Phi_{\alpha}(t, \gamma\mathbf{v}) - \Phi_{\alpha}(t, \gamma\dot{\mathbf{u}}(t)) \\ & \geq \int_{\Omega} \langle \mathbf{b}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle \quad \text{for all } \mathbf{v} \in V \end{aligned} \quad (5.1)$$

for almost every  $t \in [0, T]$  with prescribed  $\mathbf{u}(0) = \mathbf{u}_0$ ,  $\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$  and the friction nonlinearities given by

$$\Phi_{\alpha}(t, \mathbf{v}) = \int_{\Gamma_C} \varphi_{\alpha}(t, x, |\mathbf{v}(x)|) dx \quad \text{and} \quad \varphi_{\alpha}(t, x, v) = \int_0^v (\mu(r, \alpha(t, x)) \bar{\sigma}_n + C) dr. \quad (5.2)$$

(To simplify notation we often do not make the  $x$ -dependence of integrand explicit, cf. (5.1) as an example, but we keep it in case of ambiguity as in (5.2).)

In contrast, we will stick to the strong formulation for the state variable  $\alpha : [0, T] \times \Gamma_C \rightarrow \mathbb{R}$  by requiring the following.

*Problem 5.2.* For given  $\dot{\mathbf{u}} \in L^2(0, T, V)$ , find  $\alpha \in C(0, T, X)$  such that

$$\dot{\alpha}(t) + A(\alpha(t)) = f(|\gamma\dot{\mathbf{u}}(t)|) \text{ almost everywhere on } \Gamma_C$$

for almost every  $t \in [0, T]$ , with prescribed  $\alpha(0) = \alpha_0$ .

The reformulation of the coupled Problem 1.1 we will work with from here on is thus the problem of finding a pair  $(\dot{\mathbf{u}}, \alpha) \in L^2(0, T, V) \times C(0, T, X)$  such that  $\mathbf{u}$  solves Problem 5.1 with state  $\alpha$  and  $\alpha$  solves Problem 5.2 with rate  $\dot{\mathbf{u}}$ . To analyze this problem coupling, we first consider each problem separately

## 6 | ANALYSIS OF THE RATE PROBLEM (THE MECHANICAL PROBLEM FOR $\mathbf{u}$ )

In operator notation, we can rewrite (5.1) as the variational inequality

$$\langle \rho\ddot{\mathbf{u}}(t) + \mathfrak{A}\dot{\mathbf{u}}(t) + \mathfrak{B}\mathbf{u}(t) - \mathbf{b}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle + \Phi_{\alpha}(t, \gamma\mathbf{v}) \geq \Phi_{\alpha}(t, \gamma\dot{\mathbf{u}}(t)) \quad \text{for all } \mathbf{v} \in V \quad (6.1)$$

or as the subdifferential inclusion

$$\mathbf{b}(t) - \rho\ddot{\mathbf{u}}(t) - \mathfrak{B}\mathbf{u}(t) \in \partial\mathcal{R}_{\alpha}(t, \dot{\mathbf{u}}(t)) \quad (6.2)$$

with the operators  $\mathfrak{A}, \mathfrak{B} : V \rightarrow V^*$  and the dissipation potential  $\mathcal{R}_{\alpha}$  given by

$$\mathfrak{A}\mathbf{v} = \int_{\Omega} \langle \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\cdot) \rangle, \quad \mathfrak{B}\mathbf{u} = \int_{\Omega} \langle \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\cdot) \rangle, \quad \text{and} \quad \mathcal{R}_{\alpha}(\mathbf{v}) = \frac{1}{2} \langle \mathfrak{A}\mathbf{v}, \mathbf{v} \rangle + \Phi_{\alpha}(t, \gamma\mathbf{v}).$$

In (6.2) the term  $\partial\mathcal{R}_{\alpha}(t, \mathbf{v})$  denotes the set-valued convex subdifferential with respect to  $\mathbf{v}$  (keeping  $t$  and  $\alpha$  fixed), namely  $\partial\mathcal{R}_{\alpha}(t, \mathbf{v}) = \mathfrak{A}\mathbf{v} + \gamma^* \partial\Phi_{\alpha}(t, \gamma\mathbf{v})$ , where  $\gamma^*$  is the adjoint of the trace operator  $\gamma$ .

The following existence result can be obtained by classical monotone operator theory for linear dynamic models with linear dissipation, see e.g. [5,20] or [3, Sect. 4.4.1]. However, none of the results covers our special situation with time-dependent friction. Hence, we use a result on second-order hemivariational inequalities now applies in particular to our variational setting.

**Proposition 6.1.** *Problem 5.1 has a unique solution for any  $\alpha \in C(0, T, X)$ ,  $\mathbf{u}_0 \in V$ , and  $\dot{\mathbf{u}}_0 \in H$ .*

*Proof.* The existence of a solution for the dynamic problem (6.1) will be derived from [9, Cor. 12]. The uniqueness follows in particular from Proposition 6.3 which is proved below.

A few comments are in order to justify why Theorem 8, and thus Corollary 12, of [9] can be applied: Assumptions (A7) and (A8) make  $\mathfrak{A}$  and  $\mathfrak{B}$  strongly monotone and symmetric bounded linear operators. Moreover, assumption (A1) makes  $\varphi_\alpha(t, x, \cdot)$  convex for almost every  $(t, x) \in [0, T] \times \Gamma_C$ , so that the Clarke subdifferential of  $\varphi_\alpha(t, x, \cdot)$  is actually a usual convex subdifferential. Finally, assumption (A3) guarantees

$$|\partial\varphi_\alpha(t, x, \cdot)(v)| \leq C_\mu(1 + |v| + |\alpha(t, x)|)\bar{\sigma}_n + C. \quad (6.3)$$

While [9, Thm. 8] requires (6.3) to hold without dependence on  $t$  or  $x$ , a look at the proof reveals that we are free to add any term from  $L^2(0, T, X)$ , and thus we can allow for arbitrary  $\alpha \in C(0, T, X)$ , see also [10].  $\square$

For given initial data  $\mathbf{u}_0 \in V$  and  $\dot{\mathbf{u}} \in H$ , the above proposition allows us to define a mapping

$$\mathbf{R} : C(0, T, X) \rightarrow L^2(0, T, V); \alpha \mapsto \dot{\mathbf{u}},$$

and the next result shows that this mapping is Lipschitz continuous. The crucial observation is the following lemma that uses the special form of the friction law. While for smooth  $\Phi$  such a product estimate for the double differences is to be expected, it is in general false for non-smooth functions.

**Lemma 6.2** (An estimate for double differences). *Let the function  $\Phi_\alpha$  be defined as in (5.2) with  $\mu$  satisfying (A2). Then, for all  $\alpha, \beta \in C(0, T, X)$ , all  $s \in [0, T]$ , and all  $v, w \in L^2(\Gamma_C)$  with  $v, w \geq 0$  we have*

$$|\Phi_\alpha(s, v) - \Phi_\beta(s, v) - \Phi_\alpha(s, w) + \Phi_\beta(s, w)| \leq L_\mu \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} \|\alpha(s, \cdot) - \beta(s, \cdot)\|_{L^2(\Gamma_C)} \|v - w\|_{L^2(\Gamma_C)}.$$

*Proof.* We first observe  $\varphi_\alpha(s, x, v) - \varphi_\beta(s, x, v) = \int_0^v (\mu(r, \alpha(s, x)) - \mu(r, \beta(s, x))) \bar{\sigma}_n(x) dr$ . Hence, we find

$$\begin{aligned} & |\varphi_\alpha(s, x, v(x)) - \varphi_\beta(s, x, v(x)) - \varphi_\alpha(s, x, w(x)) + \varphi_\beta(s, x, w(x))| \\ &= \left| \int_{v(x)}^{w(x)} (\mu(r, \alpha(s, x)) - \mu(r, \beta(s, x))) \bar{\sigma}_n(x) dr \right| \\ &\leq L_\mu \bar{\sigma}_n(x) |\alpha(s, x) - \beta(s, x)| |v(x) - w(x)|. \end{aligned}$$

Integration over  $\Omega$  and applying Hölder's inequality gives the desired result.  $\square$

With this we obtain the desirable Lipschitz continuity of  $\mathbf{R}$  by simply comparing the two different solutions.

**Proposition 6.3** (Lipschitz continuity of  $\mathbf{R}$ ). *For two solutions  $\mathbf{u}$  and  $\mathbf{w}$  of Problem 5.1 corresponding to  $\alpha$  and  $\beta$ , respectively, with identical initial conditions, we have*

$$\|\dot{\mathbf{w}} - \dot{\mathbf{u}}\|_{L^2(0, t, V)} \leq \sqrt{t} \frac{L_\mu \|\gamma\|}{m_{\mathcal{A}}} \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} \|\beta - \alpha\|_{C(0, t, X)} \quad \text{for all } t \in [0, T].$$

*In particular, the mapping  $\mathbf{R} : C(0, T, X) \rightarrow L^2(0, T, V)$  is single-valued and Lipschitz with the constant*

$$L_{\mathbf{R}} = \sqrt{T} \frac{L_\mu \|\gamma\|}{m_{\mathcal{A}}} \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)}.$$

*Proof.* We test (6.1) for  $\mathbf{u}$  with  $\dot{\mathbf{w}}$  and for  $\mathbf{w}$  with  $\dot{\mathbf{u}}$  to obtain

$$\begin{aligned} & \langle \rho(\dot{\mathbf{w}}(s) - \dot{\mathbf{u}}(s)) + \mathfrak{A}(\dot{\mathbf{w}}(s) - \dot{\mathbf{u}}(s)) + \mathfrak{B}(\mathbf{w}(s) - \mathbf{u}(s)), \dot{\mathbf{w}}(s) - \dot{\mathbf{u}}(s) \rangle \\ &\leq \Phi_\alpha(s, \gamma \dot{\mathbf{w}}(s)) - \Phi_\alpha(s, \gamma \dot{\mathbf{u}}(s)) + \Phi_\beta(s, \gamma \dot{\mathbf{u}}(s)) - \Phi_\beta(s, \gamma \dot{\mathbf{w}}(s)) \\ &\leq L_\mu \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} \|\gamma\| \|\dot{\mathbf{w}}(s) - \dot{\mathbf{u}}(s)\|_V \|\beta(s) - \alpha(s)\|_X \end{aligned}$$

for almost every  $s \in [0, T]$ , where we used Lemma 6.2 and (A6). Integrating this inequality over the time interval  $[0, t] \subset [0, T]$  and using (A7) and (A8) yields

$$\begin{aligned}
m_{\mathcal{A}} \|\dot{\mathbf{w}} - \dot{\mathbf{u}}\|_{L^2(0,t,V)}^2 &\leq \frac{\rho}{2} \|\dot{\mathbf{w}}(t) - \dot{\mathbf{u}}(t)\|_H^2 + \int_0^t \langle \mathfrak{A}(\dot{\mathbf{w}} - \dot{\mathbf{u}}), \dot{\mathbf{w}} - \dot{\mathbf{u}} \rangle ds + \frac{1}{2} \langle \mathfrak{B}(\mathbf{w}(t) - \mathbf{u}(t)), \mathbf{w}(t) - \mathbf{u}(t) \rangle \\
&\leq L_{\mu} \|\gamma\| \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} \int_0^t \|\dot{\mathbf{w}} - \dot{\mathbf{u}}\|_V \|\beta - \alpha\|_X ds \\
&\leq L_{\mu} \|\gamma\| \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} \|\dot{\mathbf{w}} - \dot{\mathbf{u}}\|_{L^2(0,t,V)} \sqrt{t} \|\beta - \alpha\|_{C(0,t,X)},
\end{aligned}$$

where we used Hölder's inequality in the last step. This provides the desired estimate.  $\square$

## 7 | ANALYSIS OF THE STATE PROBLEM (THE FLOW RULE FOR $\alpha$ )

In Problem 5.2, we view the monotone function  $A$  from (A4) as a monotone operator on the function space  $X$  and obtain a problem that has the structure of an evolution equation associated with a maximal monotone operator; in doing so, we do not put the superposition operator structure of  $A$  to use: To solve Problem 5.2 is to solve a family of ordinary differential equations at once. In what follows, we apply the first and second line of thinking, in this order.

**Proposition 7.1.** *Problem 5.2 has a unique solution for any  $\dot{\mathbf{u}} \in L^2(0, T, V)$  and  $\alpha_0 \in X$ .*

*Proof.* The desired result follows for example from [1, Thm. 1.3]. For this we remark that the requirement  $\alpha_0 \in \overline{\text{dom}(A)}$  is automatically fulfilled since we have  $L^\infty(\Gamma_C) \subset \text{dom}(A)$  and  $L^\infty(\Gamma_C)$  is dense in  $L^1(\Gamma_C)$ .  $\square$

For given initial value  $\alpha_0 \in X$ , the solution operator derived in Proposition 7.1 will be denoted by

$$\mathcal{S} : L^2(0, T, V) \rightarrow C(0, T, X); \dot{\mathbf{u}} \mapsto \alpha.$$

The following results states that  $\mathcal{S}$  is Lipschitz continuous.

**Proposition 7.2** (Lipschitz continuity of  $\mathcal{S}$ ). *For two solutions  $\alpha$  and  $\beta$  of Problem 5.2 corresponding to  $\dot{\mathbf{u}}$  and  $\dot{\mathbf{w}}$ , respectively, with identical initial conditions and  $t \in [0, T]$ , we have*

$$\|\alpha(\cdot, x) - \beta(\cdot, x)\|_{C(0,t)} \leq L_f \|\gamma \dot{\mathbf{u}}(\cdot, x) - \gamma \dot{\mathbf{w}}(\cdot, x)\|_{L^1(0,t,\mathbb{R}^d)} \quad (7.1)$$

for almost every  $x \in \Gamma_C$  and thus

$$\|\alpha - \beta\|_{C(0,T,X)} \leq \sqrt{T} L_f \|\gamma \dot{\mathbf{u}} - \gamma \dot{\mathbf{w}}\|_{L^2(0,T,X^d)}. \quad (7.2)$$

In particular, the solution operator  $\mathcal{S} : L^2(0, T, V) \rightarrow C(0, T, X)$  is Lipschitz with the constant

$$L_{\mathcal{S}} = \sqrt{T} \|\gamma\| L_f.$$

*Proof.* For almost every  $x \in \Gamma_C$  and  $s \in [0, T]$ , we have

$$\dot{\alpha}(s, x) + A(\alpha(s, x)) = f(|\gamma \dot{\mathbf{u}}(s, x)|) \quad \text{and} \quad \dot{\beta}(s, x) + A(\beta(s, x)) = f(|\gamma \dot{\mathbf{w}}(s, x)|).$$

Thus a pair of evolution equations that have the same structure as Problem 5.2 and are additionally one-dimensional. For each such pair we can derive

$$|\alpha(t, x) - \beta(t, x)| \leq \|f(|\gamma \dot{\mathbf{u}}(\cdot, x)|) - f(|\gamma \dot{\mathbf{w}}(\cdot, x)|)\|_{L^1(0,t,\mathbb{R}^n)}$$

for example from [1, Thm. 1.2(ii)]. Because of (A5), this implies (7.1).

To obtain (7.2), we apply Hölder's inequality which yields

$$|\alpha(t, x) - \beta(t, x)| \leq L_f \|\gamma \dot{\mathbf{u}}(\cdot, x) - \gamma \dot{\mathbf{w}}(\cdot, x)\|_{L^1(0,t,\mathbb{R}^d)} \leq \sqrt{t} L_f \|\gamma \dot{\mathbf{u}}(\cdot, x) - \gamma \dot{\mathbf{w}}(\cdot, x)\|_{L^2(0,t,\mathbb{R}^d)}$$

for almost every  $(t, x) \in [0, T] \times \Gamma_C$ . Hence, by integrating over  $\Gamma_C$  we find

$$\|\alpha(t, \cdot) - \beta(t, \cdot)\|_X \leq \sqrt{t} L_f \|\gamma \dot{\mathbf{u}} - \gamma \dot{\mathbf{w}}\|_{L^2(0,t,X^d)}.$$

Since  $t \in [0, T]$  was arbitrary, this proves (7.2).  $\square$

## 8 | ANALYSIS OF THE COUPLED PROBLEM

We first establish short-time existence and uniqueness of a solution.

**Proposition 8.1** (Local existence result). *For sufficiently small  $T > 0$ , the Problems 5.1 and 5.2 have a unique joint solution  $(\dot{\mathbf{u}}, \alpha) \in L^2(0, T, V) \times C(0, T, X)$  provided that  $\mathbf{u}_0 \in V$ ,  $\dot{\mathbf{u}}_0 \in H$ , and  $\alpha_0 \in X$ , i.e.,  $\alpha = \mathbf{S}(\dot{\mathbf{u}})$  and  $\dot{\mathbf{u}} = \mathbf{R}(\alpha)$ .*

*Proof.* By Propositions 6.3 and 7.2, the operator  $\mathbf{R} \circ \mathbf{S} : L^2(0, T, V) \rightarrow L^2(0, T, V)$  is Lipschitz with the constant  $L_{\mathbf{R} \circ \mathbf{S}} \leq L_{\mathbf{R}} L_{\mathbf{S}}$ , which satisfies  $L_{\mathbf{R} \circ \mathbf{S}} \leq C^* T < 1$  for sufficiently small  $T > 0$ . The claim now follows from Banach's fixed point theorem.  $\square$

We note that  $T$  is not constrained in any way by the values of the initial data  $\mathbf{u}_0$ ,  $\dot{\mathbf{u}}_0$  or  $\alpha_0$ . We can thus extend a solution provided by Proposition 8.1 to the interval  $[0, 2T]$  by applying the aforementioned proposition repeatedly. Indeed, the actual initial data gives a solution on the time interval  $[0, T]$  producing the *final data*  $\mathbf{u}(T)$ ,  $\dot{\mathbf{u}}(T)$ , and  $\alpha(T)$ . To be sure that we can continue the solution, we have to show  $(\mathbf{u}(T), \dot{\mathbf{u}}(T), \alpha(T)) \in V \times H \times X$ .

For this we use that with the space  $H^1(0, T, V, V^*) = \{\mathbf{v} \in L^2(0, T, V) : \dot{\mathbf{v}} \in L^2(0, T, V^*)\}$  equipped with the norm  $\|\mathbf{v}\|_{H^1(0,T,V)} = \|\mathbf{v}\|_{L^2(0,T,V)} + \|\dot{\mathbf{v}}\|_{L^2(0,T,V^*)}$  we have the embeddings

$$\mathbf{u} \in H^1(0, T, V) \subset C(0, T, V) \quad \text{and} \quad \dot{\mathbf{u}} \in H^1(0, T, V, V^*) \subset C(0, T, H).$$

Thus, we find  $\mathbf{u}(T) \in V$  and  $\dot{\mathbf{u}}(T) \in H$  in addition to  $\alpha(T) \in X$ . Now, the aforementioned continuation procedure can be repeated an arbitrary number of times, we can obtain solutions on  $[0, nT]$  for arbitrary  $n \in \mathbb{N}$  and thus intervals of arbitrary size. This provides the final global existence result.

**Theorem 8.2** (Global existence result). *For all  $T > 0$ , Problems 5.1 and 5.2 have a unique solution  $(\dot{\mathbf{u}}, \alpha) \in L^2(0, T, V) \times C(0, T, X)$  provided that  $\mathbf{u}_0 \in V$ ,  $\dot{\mathbf{u}}_0 \in H$ , and  $\alpha_0 \in X$ .*

## ACKNOWLEDGEMENTS

This work was supported by the Deutsche Forschungsgemeinschaft (DFG) via the project B01 *Fault networks and scaling properties of deformation accumulation* within the Collaborative Research Center SFB 1114 *Scaling Cascades in Complex Systems*. The author is grateful to Alexander Mielke and Ralf Kornhuber for continuing support and helpful discussions. A. Mielke is grateful to two anonymous referees for helpful comments that improved the readability of the paper.

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**How to cite this article:** E. Pipping. Existence of long-time solutions to dynamic problems of viscoelasticity with rate-and-state friction. *Z Angew Math Mech.* 2019;e201800263. <https://doi.org/10.1002/zamm.201800263>