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A Simpson Correspondence for Abelian Varieties in Positive Characteristic

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Zusammenfassung/Summary

Sei X/k eine abelsche Varietät über einem algebraisch abgeschlossener Körper k der Charakteristik $p > 0$. Sei X'/k der Basiswechsel von X/k entlang dem Frobenius über k. Erinnern wir uns daran, dass die Garbe der kristalliner Differentialoperatoren eine Azumaya-Algebra ist. In dieser Arbeit zeigen wir unter Verwendung dieser Tatsache und der Morita-Äquivalenz, dass der Modulstack von Higgs-Bündeln über X' und der von lokalen Systemen über X lokal isomorph sind, für die étale Topologie auf der Hitchin-basis. Dieser Artikel folgt dem gleichen Ansatz in [Gro16].

Let X/k be an abelian variety over an algebraically closed field k of characteristic $p > 0$. In this paper, using the Azumaya property of the sheaf of crystalline differential operators and the Morita equivalence, we show that étale locally over the Hitchin base, the moduli stack of Higgs bundles on the Frobenius twist X', is equivalent to that of local systems on X. This article follows the same approach as that in [Gro16].

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Contents

1 Introduction

Let $X/\mathbf{C}$ be a smooth projective variety over the complex numbers. In [Sim92], Simpson established an equivalence between the category of local systems (vector bundles with integrable algebraic connections, equivalently[1], finite dimensional representations of the fundamental group $\pi_1(X^{\mathrm{an}})$) and that of the semi-stable Higgs bundles whose Chern class is zero. The correspondence between Higgs bundles and local systems can be viewed as a Hodge theorem for nonabelian cohomology. The theory is hence called the *non-abelian Hodge theory*, and sometimes is called the *Simpson correspondence*.

It has been a while to search for such a correspondence in positive characteristic. From now on, let X/S be a smooth scheme over a scheme S of characteristic $p > 0$.

In the work of [OV07], Ogus and Vologodsky established such a correspondence for nilpotent objects. More precisely, under the assumption that a lifting of X'/S modulo p^2 exists, where X' is the Frobenius twist of X, they construct a Cartier transform from the category of modules with flat connections nilpotent of exponent[2] $\leq p$ to the category of Higgs modules nilpotent of exponent $\leq p$. There is an alternative approach to this result in [LSZ15]. A generalisation of this work to higher level arithmetic differential operators (in the sense of [Ber96]) is given in [GLQ10]. Some other recent related works include [Shi15], [Oya17] and [Xu17].

In his work [Gro16], Groechenig gave a full version of this correspondence for (orbi)curves X over an algebraically closed field $S = \operatorname{Spec} k$. At the same time, Chen and Zhu [CZ15] further generalised this correspondence for curves but between the category of G-Higgs bundels and G-local systems, where G is a reductive group.

In the present work, following the approach of [Gro16], using the Azumaya property the sheaf of crystalline differential operators proved by Bezrukavnikov, Mirkovi, and Rumynin in [BMR08], we generalise the result from curves to abelian varieties. To be precise, we proved the following result.

1.0.1 Theorem (Theorem 7.3.1) Let X/k be an abelian variety over an algebraically closed field k of characteristic $p > 0$. Denote by $\mathbf{Higgs}_{X/k,r}$ the stack of rank r Higgs bundles on the Frobenius twist X' of X, and by $\mathbf{LocSys}_{X/k,r}$ the stack of rank r local systems on X. Then we have the following results.

[1] This is the *Riemann-Hilbert correspondence*.

[2] There is a tiny difference of conventions between the definition in [Kat70, Definition 5.6] and the definition used in [OV07]. In the latter, these modules are said to be of nilpotent $\leq p-1$, as they are supported on the $(p-1)$-st infinitesimal neighbourhood of the zero section of the cotangent bundle, see §4.3 for how this makes sense.

1. Each of the two stacks admits a natural map to a common base scheme

$$B'_r := \operatorname{Spec}\operatorname{Sym}^\bullet\left(\bigoplus_{i=1}^{r}\Gamma(X, \operatorname{Sym}^i\Omega^1_{X/k})\right)^\vee.$$

2. There is a closed subscheme $\widetilde{Z}$ of the cotangent bundle of $X'_{B'_r}/B'_r$, that is finite flat over $X'_{B'_r}$, and a $\mathbf{P}$-equivariant isomorphism

$$C^{-1}_{X/k} : \mathbf{S} \times^{\mathbf{P}} \mathbf{Higgs}_{X'/k,r} \longrightarrow \mathbf{LocSys}_{X/k,r}$$

over B'_r, where $\mathbf{P} = \mathbf{Pic}_{\widetilde{Z}/B'_r}/B'_r$ is the relative Picard stack and $\mathbf{S}/B'_r$ is a $\mathbf{P}$-torsor.
3. In particular, there is an étale surjective map $U \to B'_r$, such that

$$\mathbf{Higgs}_{X'/k,r} \times_{B'_r} U \simeq \mathbf{LocSys}_{X/k,r} \times_{B'_r} U.$$

The idea of the proof is as follows. We know from [BMR08] that the sheaf of crystalline differential operators defines an Azumaya algebra $\mathscr{D}_{X/k}$ over the cotangent bundle $\mathrm{T}^*(X'/k)$ of X' (Theorem 3.1.1). We can also view Higgs bundles on X' as quasi-coherent $\mathscr{O}$-modules on the cotangent bundle $\mathrm{T}^*(X'/k)$, supported on a closed subscheme called the *spectral cover* (Proposition 4.3.2). Meanwhile, local systems on X can be identified with certain $\mathscr{D}_{X/k}$-modules on the cotangent bundle $\mathrm{T}^*(X'/k)$, via their p-curvatures (Proposition 5.5.1). If there is a splitting of the Azumaya algebra $\mathscr{D}_{X/k}$ on the spectral cover, then the Morita theory (Proposition 2.2.1) will give an equivalence between these $\mathscr{O}$-modules and $\mathscr{D}_{X/S}$-modules on $\mathrm{T}^*(X'/k)$ (§6).

The existence of splittings in curve case is guaranteed by Tsen's theorem (see [Gro16, §3.4]). For abelian varieties, the existence of splittings of $\mathscr{D}_{X/S}$ over (the formal neighbourhood of) a spectral cover, is obtained by observing that the stack of splittings of $\mathscr{D}_{X/S}$ is equivalent to the stack $\mathbf{Pic}^\natural_{X/k}$ of line bundles with a flat connection on X (Proposition 7.1.2), and that $\mathbf{Pic}^\natural_{X/k}/B'_1$ is *smooth*.

One of the main differences between the curve case and the higher dimensional case is that in the latter case, the spectral cover can be empty and is *not flat* over X' in general (Proposition 4.2.8 and Remark 4.2.9). However, since the cotangent bundle of an abelian variety is trivial, for any given spectral cover, we can construct a larger cover, that is finite, flat, locally of finite presentation over X' (Example and Definition 4.2.4). This larger cover will play a role in the construction and the proof.

Actually the proof is rather formal. We will try to recall all the necessary definitions and some explicit examples are given.

2 Generalities

2.1 Frobenius

A careful treatment of Frobenius morphisms can be found in [SGA 5, Exposé XV, §1].

Fix a scheme S of characteristic $p > 0$ and let $f : X \longrightarrow S$ be an S-scheme. As usual, denote by Fr_S (resp. Fr_X) the *absolute Frobenius* on S (resp. X), by $X' := X^{(p)} := X \times_{S,\mathrm{Fr}_S} S$ the *Frobenius twist* of X over S, and by $F_{X/S} := \mathrm{Fr}_{X/S} = (\mathrm{Fr}_X, f) : X \longrightarrow X'$ the *relative Frobenius*. In other words, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "F_{X/S}"] \arrow[rr, bend left, "\mathrm{Fr}_X"] \arrow[dr, "f"'] & X' \arrow[r, "w"] \arrow[d] \arrow[rr, bend left, "\mathrm{Fr}_{X'}"] \arrow[dr, phantom, "\square"] & X \arrow[r, "F_{X/S}"] \arrow[d, "f"] & X' \arrow[dl, "f'"] \\
 & S \arrow[r, "\mathrm{Fr}_S"] & S. &
\end{tikzcd}$$

Recall that both the absolute Frobenius and the relative Frobenius are universal homeomorphisms (equivalently, integral, surjective and universally injective; [SP, Tag 0CC8 and Tag 0CCB]). It then follows that if X/S is étale, then the relative Frobenius $F_{X/S}$ is étale and universally injective (radicial) hence an open embedding hence an isomorphism ([SP, Tag 0EBS]).[3] Therefore, if f is *smooth of relative dimension* d, the relative Frobenius $F_{X/S}$ is *finite locally free* (i.e., $F_{X/S}$ is affine and $(F_{X/S})_*\mathcal{O}_X$ is a locally free $\mathcal{O}_{X'}$-module; or equivalently, $F_{X/S}$ is finite flat and locally of finite presentation, see, e.g., [SP, Tag 02K9]) of rank p^d. In fact, since X/S is smooth of relative dimension d, Zariski locally we have the following commutative diagram ([SP, Tag 054L])

$$\begin{tikzcd}
X \arrow[d, "f"'] & U \arrow[l, hook', "\circ" marking] \arrow[r, "\text{étale}", "g"'] \arrow[d, "f|_U"'] & \mathbf{A}^d_V. \arrow[dl] \\
S & V \arrow[l, hook', "\circ" marking] &
\end{tikzcd} \tag{2.1}$$

It is enough to show $F_{U/V}$ is locally free of rank p^d. It follows from definition that $F_{U/V} = g^*(F_{\mathbf{A}^d_V/V}) \circ F_{U/\mathbf{A}^d_V}$. Since g is étale, $F_{U/\mathbf{A}^d_V}$ is an isomorphism. So it suffices to show $F_{\mathbf{A}^d_V/V}$ is locally free of rank p^d, but this case reduces to local computations and it is

[3] The converse also holds, see [SGA 5, Exposé XI, §1, Proposition 2].

straightforward.

Recall [SP, Tag 0BD2], [EGA II, §6.5], or [GW10, Remark 12.25] that, for a finite locally free morphism $f : Y \to X$, there is a well defined *norm* map $\mathrm{Nm}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, whose formation commutes with arbitrary base change.

2.1.1 Lemma Let X/S be a smooth scheme of relative dimension d. Denote by

$$\mathrm{Nm}_{F_{X/S}} : (F_{X/S})_*\mathcal{O}_X \longrightarrow \mathcal{O}_{X'}$$

the norm map. Then the composition

$$(F_{X/S})_*\mathcal{O}_X \xrightarrow{\mathrm{Nm}_{F_{X/S}}} \mathcal{O}_{X'} \xrightarrow{F^\natural_{X/S}} (F_{X/S})_*\mathcal{O}_X$$

is the p^d-th power map, i.e., for any open subset $U \subseteq X'$ and any section $g \in (F_{X/S,*}\mathcal{O}_X)(U) = \mathcal{O}_X(F^{-1}_{X/S}U)$, we have $F^\natural_{X/S}\,\mathrm{Nm}_{F_{X/S}}(g) = g^{p^d} \in \mathcal{O}_X(F^{-1}_{X/S}U)$.

Proof This is a local question and via (2.1), we can further reduce the problem to the affine space case. First we consider the case $d = 1$. The relative Frobenius in this case corresponds to the ring homomorphism

$$F : A[t'] \longrightarrow A[t], \quad t' \longmapsto t^p,$$

where A is a unital commutative ring of characteristic p. Note that $1, t, t^2, \ldots, t^{p-1}$ form an $A[t']$-basis for $A[t]$. So for an arbitrary element $T \in A[t]$, we can write $T = a_0 + a_1 t + a_2 t^2 + \cdots + a_{p-1}t^{p-1} \in A[t]$, with $a_i \in A[t']$. The multiplication-by-T map has a matrix representation

$$\mathbf{T} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{p-1} \\ a_{p-1}t' & a_0 & a_1 & \cdots & a_{p-2} \\ a_{p-2}t' & a_{p-1}t' & a_0 & \cdots & a_{p-3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 t' & a_2 t' & a_3 t' & \cdots & a_0 \end{pmatrix} \in \mathrm{Mat}_{p\times p}(A[t']).$$

We have

$$\det(\mathbf{T}) = a_0^p + a_1^p t' + a_2^p (t')^2 + \cdots + a_{p-1}^p (t')^{p-1} \in A[t'],$$

because by definition (expansion) of the determinant, all the other terms have coefficients a multiple of $\binom{p}{i}$ with $0 < i < p$, hence vanish in characteristic $p > 0$. So clearly

$$\begin{aligned} F(\det(\mathbf{T})) &= a_0^p + a_1^p t^p + a_2^p t^{2p} + \cdots + a_{p-1}^p t^{(p-1)p} \\ &= (a_0 + a_1 t + a_2 t^2 + \cdots + a_{p-1}t^{p-1})^p = T^p \in A[t]. \end{aligned}$$

For $d \geq 1$, we can decompose the relative Frobenius map F into

$$R_0 := A[t'_1, t'_2, \dots, t'_{d-1}, t'_d] \xrightarrow{F_{d-1}} R_{d-1} := A[t_1, t_2, \dots, t_{d-1}, t'_d]$$
$$\xrightarrow{F_d} R_d := A[t_1, t_2, \dots, t_{d-1}, t_d],$$

where, as the notation suggests, F_{d-1} sends t'_i to t_i^p for $1 \leq i \leq p-1$ and t'_d to t'_d, meanwhile F_d sends t_i identically to t_i for all $1 \leq i \leq p-1$ and t'_d to t_d^p. For any $T = \sum_{i=0}^{p-1} a_i(t_1, t_2, \dots, t_{d-1}) \cdot t_d^i \in R_d$, with $a_i(t_1, \dots, t_{d-1}) \in R_{d-1}$, we can view the multiplication-by-T map m_T on R_d as an R_{d-1}-endomorphism as well as R_0-endomorphism. It then follows from the $d = 1$ case that

$$\begin{aligned} \det_{R_{d-1}/R_0}(m_T) &= \sum_{i=1}^{p-1} (a_i(t_1, \dots, t_{d-1}))^p \cdot (t'_d)^p \\ &= F_{d-1}\left(\sum_{i=1}^{p-1} a_i^{(p)}(t'_1, \dots, t'_{p-1}) \cdot (t'_d)^i\right) =: F_{d-1}T', \end{aligned}$$

where $a_i^{(p)}$ is the polynomial by raising the coefficients of a_i, which are in A, to the p-th power. Noting that that T' lies in R_0, we know $\mathrm{Nm}_{R_{d-1}/R_0} F_{d-1}T' = (T')^{p-1}$ as the multiplication-by-$F_{d-1}T'$ map on R_{d-1} is represented by a diagonal matrix with all diagonal entries T' of size $p^{d-1} \times p^{d-1}$. So it follows from the following Lemma 2.1.2 that

$$F(\det_{R_d/R_0}(m_T)) = F\left(\mathrm{Nm}_{R_{d-1}/R_0} \det_{R_{d-1}/R_0}(m_T)\right) = F\left((T')^{p^{d-1}}\right) = T^{p^d}.$$

This completes the proof.

2.1.2 Lemma ([Bou07, Chaptitre III, §9, n°4, Propostion 6]) Let A be a ring. Suppose that $\pi : A \longrightarrow B$ is an A-algebra and that B is free of rank n as an A-module. Let M be a B-module free of rank m and $\varphi \in \mathrm{End}_B(M)$ be a B-endomorphism of M. Then M is free of rank mn as an A-module and if we denote by ${}_A\varphi$ the induced A-endomorphism of M, then

$$\det_A({}_A\varphi) = \mathrm{Nm}_\pi \det_B(\varphi).$$

2.2 Azumaya algebras and the Morita equivalence

Fix a scheme $(X, \mathscr{O}_X)$. Unadorned tensor products are taken over $\mathscr{O}_X$. We use the curly $\mathscr{H}om$ to denote the internal Hom, and the straight Hom to denote the Hom set; the same convention applies to $\mathscr{E}nd$ and End.

An *Azumaya algebra* (or a *central separable algebra*) A over X is a sheaf of $\mathscr{O}_X$-algebras (not necessarily commutative), such that there is an étale surjective map (or equivalently, an fppf map; see [Mil80, Prop. IV.2.1]) $g : U \longrightarrow X$, such that

$$g^*A \simeq \mathscr{E}nd_{\mathscr{O}_U}(\mathscr{O}_U^{\oplus r}) =: \mathscr{M}at_{r\times r}(\mathscr{O}_U)$$

for some $r \in \mathbf{N}$. This is a generalisation of *central simple algebras* over a field k. Clearly, an Azumaya algebra is locally free of rank r^2 (see e.g., [SP, Tag 05B2]).

Let A be an Azumaya algebra over X. A *splitting* of A consists of of a morphism $g : T \longrightarrow X$, a locally free $\mathscr{O}_T$-module P, and an isomorphism $\alpha : g^*A \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_T}(P)$. Then to each Azumaya algebra A over X, we can define a fibred category over $(\mathbf{Sch}/X)$, whose objects are splittings $(g : T \longrightarrow X, P, \alpha)$, and a morphism from an object $(g' : T' \longrightarrow X, P', \alpha')$ to another $(g : T \longrightarrow X, P, \alpha)$ consists of a morphism $h : T' \longrightarrow T$ and an isomorphism $\mu : h^*P \xrightarrow{\sim} P'$ of locally free sheaves such that $g \circ h = g'$ and $\bar{\mu} \circ h^*(\alpha) = \alpha'$, where $\bar{\mu} : \mathscr{E}nd_{\mathscr{O}_{T'}}(h^*P) \longrightarrow \mathscr{E}nd_{\mathscr{O}_{T'}}(P')$, $\varphi \mapsto \mu \circ \varphi \circ \mu^{-1}$ is the one induced by μ. This indeed defines a $\mathbb{G}_m$-gerbe, and it is called the *gerbe of splittings of A* (see [Ols16, §§12.3.5, 12.3.6] or [Gir71, Chapitre V, §4.2, *gerbe des banalisations*]). If a splitting $(g : T \longrightarrow X, P, \alpha)$ exists, we will say that A splits over T.

The following proposition can be found in [Lam99, Thm. 17.25 & Thm. 18.11].

2.2.1 Proposition (Morita) Let A be an $\mathscr{O}_X$-algebra (not commutative in general) which is locally free as an (left) $\mathscr{O}_X$-module. Suppose that $A \simeq \mathscr{E}nd_{\mathscr{O}_X}(P)$, for some locally free $\mathscr{O}_X$-module P, then

$$\begin{aligned} \mathbf{Mod}_{\mathscr{O}_X} &\longleftrightarrow \mathbf{Mod}_A \\ \mathscr{H}om_A(P, F) &\longleftarrow\!\!\dashv F \\ E &\longmapsto P \otimes E, \end{aligned}$$

is an equivalence of categories.

2.3 Characteristic polynomial of a twisted endomorphism

As in the previous paragraph, let $(X, \mathbb{O}_X)$ be a fixed scheme. Unadorned tensor, symmetric, and exterior products are always taken over $\mathscr{O}_X$. A reference for this section is [EGA II, §6.4].

Let E, K be two *finite locally free* $\mathscr{O}_X$-modules and $\varphi : E \longrightarrow E \otimes K$ be a homomorphism of $\mathscr{O}_X$-modules. Suppose that $\operatorname{rk} E = r$ and $\operatorname{rk} K = d$. We will call φ a (K-)*twisted endomorphism* of E (over $\mathscr{O}_X$) and also view it an element in $\Gamma(X, E^\vee \otimes E \otimes K) = \Gamma(X, \mathscr{E}nd_{\mathscr{O}_X}(E) \otimes K)$. Any $\lambda \in \Gamma(X, K) = \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, K)$ determine a twisted endomorphism $\mathrm{id}_E \otimes \lambda \in \Gamma(X, \mathscr{E}nd_{\mathscr{O}_X}(E) \otimes K)$ of E, still denoted by λ, and called the *multiplication-by-λ map*.

We have the canonical inclusion to degree 1 map $K \longrightarrow \operatorname{Sym}^\bullet K$. So any twisted endomorphism φ gives rise to an $\mathbb{O}_X$-linear map $E \longrightarrow E \otimes \operatorname{Sym}^\bullet K$. Then using the adjoint pair

$$\mathrm{Hom}_{\mathcal{O}_X}(E, E \otimes \mathrm{Sym}^\bullet K) \simeq \mathrm{Hom}_{\mathrm{Sym}^\bullet K}(E \otimes \mathrm{Sym}^\bullet K, E \otimes \mathrm{Sym}^\bullet K),$$

we can view φ as an $(\mathrm{Sym}^\bullet K)$-endomorphism of the locally free module $E \otimes \mathrm{Sym}^\bullet K$. It has an obviously well-defined *characteristic polynomial* $\chi_\varphi \in \Gamma(X, \mathrm{Sym}^\bullet K[t])$ in the usual sense. We may describe it more concretely and explicitly in the following way.

First of all, the *trace* $\mathrm{tr}(\varphi) \in \Gamma(X, K)$ of $\varphi \in \Gamma(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes K)$ is just the one induced by the trace map $\mathrm{tr} : \mathscr{E}nd_{\mathcal{O}_X}(E) \to \mathcal{O}_X$ on $\mathscr{E}nd_{\mathcal{O}_X}(E)$. Observe that for any $1 \le i \le r$, the quotient map $(E \otimes K)^{\otimes i} \simeq E^{\otimes i} \otimes K^{\otimes i} \twoheadrightarrow (\wedge^i E) \otimes (\mathrm{Sym}^i K)$ factors through $\wedge^i(E \otimes K)$. So there is a natural map $\wedge^i E \to \wedge^i(E \otimes K) \to (\wedge^i E) \otimes (\mathrm{Sym}^i K)$, where the first map is the i-th exterior power of φ. Still denote this composition by $\wedge^i\varphi$. The *determinant* $\det\varphi$ of φ is defined as the trace $\mathrm{tr}(\wedge^r\varphi) \in \Gamma(X, \mathrm{Sym}^r K)$. Observe that $(\mathrm{Sym}^i K)$ is locally free of rank $\binom{d+i-1}{i}$. Also set $\mathrm{tr}(\wedge^0\varphi) := 1 \in \Gamma(X, \mathcal{O}_X)$. In case $d = 1$, i.e., when K is an invertible sheaf, we have $\mathrm{Sym}^i K = K^{\otimes i}$ canonically.

By embedding K into $\mathrm{Sym}^\bullet K$ as the degree 1 part, it makes sense to talk about multiplications of global sections of K and $\mathrm{Sym}^\bullet K$. Given a section $\lambda \in \Gamma(X, K)$, identified with the multiplication-by-λ map, then the determinant $\det(\lambda - \varphi) \in \Gamma(X, \mathrm{Sym}^r K)$ can be shown to be

$$\begin{aligned}\det(\lambda - \varphi) := \mathrm{tr}\left(\wedge^r(\lambda - \varphi)\right) &= \sum_{i=0}^{r} (-1)^i \,\mathrm{tr}(\wedge^i\varphi)\lambda^{r-i} \\ &= \lambda^r - \mathrm{tr}(\varphi)\lambda^{r-1} + \cdots + (-1)^r \det(\varphi) \in \Gamma(X, \mathrm{Sym}^r K),\end{aligned}$$

with $\mathrm{tr}(\wedge^i\varphi) \in \Gamma(X, \mathrm{Sym}^i K)$, $\lambda^{r-i} \in \Gamma(X, \mathrm{Sym}^{r-i} K)$. The *characteristic polynomial* of the K-endomorphism $\varphi : E \to E \otimes K$ is defined as the *homogeneous* element

$$\chi_\varphi := \chi_\varphi(t) = \sum_{i=0}^{r} (-1)^i \,\mathrm{tr}(\wedge^i\varphi)t^{r-i} = t^r - \mathrm{tr}(\varphi)t^{r-1} + \cdots + (-1)^r \det(\varphi) \tag{2.2}$$

of degree r in the graded ring $\Gamma\big(X, (\mathrm{Sym}^\bullet K)[t]\big)$, where t is a formal symbol (commuting with $\mathrm{Sym}^\bullet K$) *of degree* 1.

Moreover, for any *commutative* $\mathcal{O}_X$-algebra $\pi : \mathcal{O}_X \to R$, let $(\pi^*\varphi) := \varphi_R := 1 \otimes \varphi$ be the extension of scalars of φ by R. Identifying any section $\lambda \in \mathrm{Hom}_R(R, R \otimes K)$ with the multiplication-by-λ-map, we have $\det(\lambda - \varphi_R) \in \mathrm{Hom}_R(R, R \otimes \mathrm{Sym}^r K)$ is just the image of $1 \otimes \chi_\varphi \in \mathrm{Hom}_R\left(R, (\mathrm{Sym}^\bullet K)[t]\right)$ under the evaluation map (which is a ring homomorphism) $\mathrm{Hom}_R(R, R \otimes \mathrm{Sym}^\bullet K[t]) \to \mathrm{Hom}_R(R, R \otimes_{\mathcal{O}_X} \mathrm{Sym}^\bullet K)$ sending t to λ.

Finally, substituting t by φ in χ_φ, we get an $\mathcal{O}_X$-linear map

$$\chi_\varphi(\varphi) = \sum_{i=0}^{r} \mathrm{tr}(\wedge^i\varphi)\varphi^{r-i} : E \to E \otimes \mathrm{Sym}^r K,$$

where φ^i is the $\mathbb{O}_X$-linear map $E \to E \otimes \operatorname{Sym}^i K$ obtained by composition

$$E \xrightarrow{\varphi} E \otimes K \xrightarrow{\varphi\otimes\mathrm{id}} E \otimes K^{\otimes 2} \longrightarrow \cdots \longrightarrow E \otimes K^{\otimes i} \twoheadrightarrow E \otimes \operatorname{Sym}^i K.$$

and $\operatorname{tr}(\wedge^i\varphi)\varphi^{r-i}$ is then the $\mathbb{O}_X$-linear map $E \to E \otimes \operatorname{Sym}^r K$ is just multipliying φ^i by $\operatorname{tr}(\wedge^i\varphi)$ on the second factor. Now as in the classical case, the Cayley-Hamilton theorem reads as follows.

2.3.1 Proposition The map

$$\chi_\varphi(\varphi) : E \longrightarrow E \otimes \operatorname{Sym}^r K \tag{2.3}$$

is the *zero* map.

2.3.2 Remark One can define all the above locally using "coordinate", then using gluing argument to show everything glues and does not depend on any choice of basis nor choice of localizations. The above global and functorial description avoids this, and it guarantees that we can reduce proofs or computations to the local case.

2.4 Vanishing locus of a section of a vector bundle

Let $(X, \mathcal{O}_X)$ be a scheme, K a locally free sheaf of finite rank d and $s \in \Gamma(X, K)$ a global section of K. Then the vanishing locus $V(s)$ on X of s can be described in the following ways.

1. (local description) Since K is locally free, there is a open covering (U_i) of X together with isomorphisms $u_i : K|_{U_i} \to \mathcal{O}_U^{\oplus d}$ of $\mathcal{O}_U$-modules. So $u_i(s|_{U_i}) \in \Gamma(U, \mathcal{O}_U^{\oplus r})$ is represented by an d-tuple $(f_1, \ldots, f_d)$ with $f_i \in \Gamma(U, \mathcal{O}_U)$, $1 \le i \le d$. Then over U_i, the vanishing locus $V(s)$ is cut out by such local equations $f_i = 0$. In other words, $V(s) \times_X U$ is the closed subscheme of U defined by the quasi-coherent sheaf of ideals generated by $f_1, \ldots, f_d$.
2. (algebraically) A global section s is the same an $\mathcal{O}_X$-morphism $s : \mathcal{O}_X \to K$ with dual map $s^\vee : \mathrm{Hom}_{\mathcal{O}_X}(K, \mathcal{O}_X) := K^\vee \to \mathcal{O}_X$. Then $I := \mathrm{Im}(s^\vee) \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf of ideals and $\iota : V(s) := \mathcal{S}pec(\mathcal{O}_X/I) \to X$. We have a short exact sequence of $\mathcal{O}_X$-modules
$$0 \longrightarrow K^\vee \xrightarrow{s^\vee} \mathcal{O}_X \xrightarrow{\iota^\sharp} \iota_*\mathcal{O}_{V(s)} \longrightarrow 0. \tag{2.4}$$
3. (geometrically) A global section $s \in \Gamma(X, K)$ is the same as a section $s : X \to \mathbf{V}(K) := \mathcal{S}pec\operatorname{Sym}^\bullet K$ of the projection $\pi : \mathbf{V}(K) \to X$. Moreover, we have the zero section $0 : X \to \mathbf{V}(K)$ corresponding to $0 \in \Gamma(X, K)$. It's known that s and 0 are both closed embeddings as π is affine hence separated. Then $V(s)$ is the fibre product of of s and 0, i.e., $V(s) = s^{-1}(0)$.

4. (set-theoretically) The underlying closed set of $V(s)$ is the set

$$\{x \in X : s_x \in \mathfrak{m}_x \cdot K_x, i.e., s(x) = 0 \in K(x)\},$$

where $K(x) := K \otimes_{\mathscr{O}_{X,x}} \kappa(x)$.

2.4.1 Example Suppose that s and t are two global sections of K. They define a global section st of $\mathrm{Sym}^2 K$. Then we know that (set-theoretically) $V(st) \subseteq V(s) \cup V(t)$. Actually, for example, suppose that we have fix a local trivialization $u : K|_U \to \mathscr{O}_U^{\oplus d}$ with $u(s) = (f_1, \ldots, f_d)$ and $u(t) = (g_1, \ldots, g_d)$, with $f_i, g_i \in \Gamma(U, \mathscr{O}_U)$. Then $V(s)$ is cut out by $f_1, \ldots, f_d$, $V(t)$ is cut out by $g_1, \ldots, g_d$, and $V(st)$ is cut out by sections $f_i g_i$ and $f_i g_j + f_j g_i$ of $\mathrm{Sym}^2 K$, with $1 \leq i, j \leq d$ and $i \neq j$. Note that there are $d(d+1)/2$ many such local equations. This generalises to the product of arbitrarily many sections.

2.5 Affine morphisms and finite morphisms

Recall the following standard results that we will use repeatedly (see e.g., [EGA II, §§1.3–1.4, 1.7, 6.1]).

2.5.1 A morphism of schemes $f : Y \to X$ is affine if $Y \simeq \mathcal{S}pec\, A$ for some quasi-coherent $\mathscr{O}_X$-algebra A, i.e., A is an $\mathscr{O}_X$-algebra that is quasi-coherent as an $\mathscr{O}_X$-module. For any such affine morphism, there is an equivalence of categories

$$\begin{aligned} \mathbf{QCoh}(\mathscr{O}_X, A) &\xrightarrow{\sim} \mathbf{QCoh}(\mathscr{O}_Y) \\ M &\longmapsto \widetilde{M} \\ f_* N &\longleftarrow\!\shortmid N, \end{aligned} \tag{2.5}$$

where $\mathbf{QCoh}(\mathscr{O}_X, A)$ is the category A-modules which are $\mathscr{O}_X$-quasicoherent. We will be particularly interested in the following cases:

- f is the relative Frobenius, see §2.1,
- Y is $\mathbf{V}(E) := \mathcal{S}pec\, \mathrm{Sym}^\bullet(E^\vee)$ with $\mathscr{E}$ a quasi-coherent $\mathscr{O}_X$-module over X, and
- Y is a closed subscheme $\mathcal{S}pec(\mathscr{O}_X/I)$ defined by a quasi-coherent sheaf I of ideals of $\mathscr{O}_X$. In this case, an object M in $\mathbf{QCoh}(\mathscr{O}_X, \mathscr{O}_X/I)$ is just an $\mathscr{O}_X$-module that is annihilated by I, i.e., $I \cdot M = 0$. Moreover, $\widetilde{M}$ coincides with the f^*M, for the reason that $M \otimes_{\mathscr{O}_X} (\mathscr{O}_X/I) \simeq M$ as long as $I \cdot M = 0$.

The correspondence (2.5) also restricts to an equivalence of the corresponding categories of coherent sheaves, in case f_* sends coherent sheaves to coherent ones, e.g., when f is proper and X is locally noetherian ([EGA III_1, Thm. 3.2.1]).

2.5.2 Moreover, the map $f : \mathcal{S}pec\, A \longrightarrow X$ has the universal property that

$$\mathrm{Hom}_X(T, \mathcal{S}pec\, A) = \mathrm{Hom}_{\mathbf{CAlg}_{\mathcal{O}_X}}(A, f_*\mathcal{O}_T).$$

In particular, in case $A = \mathrm{Sym}^\bullet(E^\vee)$, where E is locally free of finite rank, using the adjoint pair[4] $\mathrm{Sym}^\bullet \dashv \texttt{forgetful} : \mathbf{CAlg}_{\mathcal{O}_X} \longrightarrow \mathbf{Mod}_{\mathcal{O}_X}$, we obtain that

$$\mathrm{Hom}_X(T, \mathbf{V}(E)) = \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(E^\vee, f_*\mathcal{O}_T) = \Gamma(T, f^*E). \tag{2.6}$$

We remark that our definition of $\mathbf{V}(E)$ differs from that in [EGA II, §1.7] by a dual.

2.5.3 Recall also that a morphism $f : Y \longrightarrow X$ is *finite* if $Y \simeq \mathcal{S}pec\, A$ for some integral $\mathcal{O}_X$-algebra A *of finite type*. In particular, such an f is affine. Then in this case, the equivalence (2.5) induces an equivalence of categories ([EGA II, Proposition 6.1.12] and [GW10, Proposition 12.13])

$$f_* : \mathbf{Vect}_m(Y) \longrightarrow \mathbf{Vect}_m(X, f_*\mathcal{O}_Y) \tag{2.7}$$

where $\mathbf{Vect}_m(X)$ stands for the category of locally free sheaves of rank m on X and $\mathbf{Vect}_m(X, f_*\mathcal{O}_Y)$ stands for that of locally free $f_*\mathcal{O}_Y$-modules on X. Take care that this result is *not* tautological but rests essentially on the assumption that f is affine. The latter category consists of objects that are $f_*\mathcal{O}_Y$-modules E on X such that there exists a Zariski cover $U_i \longrightarrow X$, such that $E|_{U_i} \simeq (\mathcal{O}_Y|_{f^{-1}(U_i)})^{\oplus m}$ for all i.

Therefore, if $f : Y \longrightarrow X$ is *finite locally free* of rank m (see [SP, Tag 02K9]), i.e., $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module of rank m, and if E is a finite locally free $\mathcal{O}_X$-module of rank r, then f_*E is finite locally free of rank mr as an $\mathcal{O}_X$-module. In fact, since f is locally free of rank m, there exists an Zariski cover $(U_i \longrightarrow X)$ such that each $(f_*\mathcal{O}_Y)|_{U_i} = \mathcal{O}_Y|_{f^{-1}(U_i)} \simeq \mathcal{O}_{U_i}^{\oplus m}$ and $E|_{U_i}$ is locally free over $f^{-1}(U_i)$. Then by (2.7), there exist Zariski covers $(V_{ij} \longrightarrow U_i)$, such that $E|_{f^{-1}(V_{ij})} = \big((f|_{U_i})_*(E|_{U_i})\big)\,|_{V_{ij}} \simeq (\mathcal{O}_Y|_{f^{-1}(V_{ij})})^{\oplus r} = (\mathcal{O}_{V_{ij}}^{\oplus m})^{\oplus r}$. Note that this description puts us in the situation of Lemma 2.1.2.

[4] The notation **CAlg** stands for the category of *commutative* algebras, to distinguish it from **Alg**, the category of not necessary commutative algebras, which will appear later.

3 Crystalline differential operators and its Azumaya property

Let $f : X \longrightarrow S$ be an S-scheme of characteristic $p > 0$. Berthelot, in his thesis [Ber74], using divided power techniques, constructed the sheaf of *divided power differential operators*[5] PD-$\underline{\mathrm{Diff}}_{X/S}(\mathscr{O}_X, \mathscr{O}_X)$ and developed the theory of crystalline cohomology with related machinery. This sheaf, in his later work [Ber96] of arithmetic $\mathscr{D}$-modules, becomes the sheaf $\mathscr{D}^{(0)}_{X/S}$ of *arithmetic differential operators of level* 0. See also [Ber02] for an introduction. In the present notes, we will use the terminology *sheaf of crystalline differential operators*, and denote it simply by $D_{X/S}$.

Berthelot's definition of the arithmetic differential operators is quite involved, as he developed a very general theory. However, in case that $f : X \longrightarrow S$ is *smooth*, the sheaf $D_{X/S}$ of rings has a rather simple, explicit and intuitive definition.[6]

Denote by $\mathscr{D}er(X/S)$ the S-derivations of $\mathscr{O}_X$ to itself. As an $\mathscr{O}_X$-module, we have that $\Theta_{X/S} \simeq \mathscr{D}er(X/S) \subseteq \mathscr{E}nd_{f^{-1}\mathscr{O}_S}(\mathscr{O}_X)$. Recall that for any local sections ∂ and ∂' of $\mathscr{D}er(X/S)$, we have $[\partial, \partial'] := \partial \circ \partial' - \partial' \circ \partial$ and $\partial^p := \partial \circ \cdots \circ \partial$ (p-times, also denoted by $\partial^{[p]}$) are again S-derivations.

Then $D_{X/S}$ is the sheaf of rings generated by $\Theta_{X/S}$ and $\mathscr{O}_X$ subject to the usual relations

- $f \cdot \partial = f\partial$,
- $\partial \cdot f - f \cdot \partial = \partial(f)$, and
- $\partial' \cdot \partial'' - \partial'' \cdot \partial' = [\partial', \partial'']$,

for local sections $\partial, \partial', \partial''$ of $\Theta_{X/S}$ and f of $\mathscr{O}_X$ (cf. [BMR08, §1.2] and [HKR62, §6]). In

[5]They are also called the PD differential operators for short, after the French *puissances divisées*. Other names appearing literatures are *crystalline differential operators* and *differential operators without divided power*. These names do cause some confusions. While the "divided power" in the name "divided power differential operators" means these operators are defined using divided power structures, while the "without divided powers" in the name "differential operators without divided powers", means that they are locally represented (generated) by ∂^n, where ∂ is a tangent vector, instead of (formally) $\partial^n/n!$, in contrast of the differential operators *à la* Grothendieck [EGA IV$_4$, §16]. So "divided power differential operators have no divided power".

[6]This description has already appeared in Berthelot's work, e.g., [Ber74, II, §4.2, Prop. 4.2.5], or more explicitly [Ber96, §2.2, Prop. 2.2.4, Cor. 2.2.5] and [Ber02, §1.2.3, (1.2.3.1), (1.2.3.2)]. Unfortunately, in these works, this handy description is viewed as a corollary of the general theory without any emphasis, so it's a little bit hard to catch on. A complete "proof" can be found in [HKR06, §1 Proposition]. Note also that the equivalence of definitions was also implicitly mentioned in [Kat70, (1.2)].

particular $D_{X/S}$ is a coherent left $\mathcal{O}_X$-modules over X. Denote by ι the natural map

$$\iota : \Theta_{X/S} := (\Omega^1_{X/S})^{\vee} \simeq \mathscr{D}er(X/S) \longrightarrow D_{X/S}. \tag{3.1}$$

We consider the morphism of additive abelian groups $\Theta_{X/S} \to D_{X/S}$, defined by $\partial \mapsto (\iota(\partial))^p - \iota(\partial^p)$ for local sections ∂ of $\Theta_{X/S}$. By local computations, one can check that the following facts hold.

- It is p-(semi-)linear (Fr_X-linear), hence defines a morphism

$$\mathrm{Fr}_X^* \Theta_{X/S} \simeq F_{X/S}^* \Theta_{X'/S} \longrightarrow D_{X/S}$$

  of sheaves (left) $\mathscr{O}_X$-modules; Adjointly, we get a map of $\mathscr{O}_{X'}$-modules

$$\Theta_{X'/S} \longrightarrow F_{X/S,*} D_{X/S}. \tag{3.2}$$

- The image of the obtained map lies in the center of $F_{X/S,*}D_{X/S}$. Hence it further induces a map

$$\pi_* \mathscr{O}_{\mathrm{T}^*_{X/S}} \simeq \mathrm{Sym}^{\bullet} \Theta_{X'/S} \xrightarrow{\psi} Z(F_{X/S,*} D_{X/S}) \hookrightarrow F_{X/S,*} D_{X/S}, \tag{3.3}$$

  where Z stands for taking the center of a sheaf of non-commutative rings.

Finally note that in case X/S is smooth, the formation of $D_{X/S}$ respects arbitrary base change: if $T \to S$ is morphism of schemes, then $D_{X_T/T} \simeq f_T^* D_{X/S}$, where $f_T : X_T \to T$ is the base change of f.

3.1 Azumaya property of the sheaf of crystalline differential operators

3.1.1 Theorem Assume that X/S is smooth of relative dimension d. Then we have the following facts.

1. ([BMR08, Lem. 1.3.2] and [GLQ10, Prop. 3.6]) The morphism

$$\psi : \mathrm{Sym}^{\bullet} \Theta_{X'/S} \longrightarrow Z(F_{X/S,*} D_{X/S}) \subseteq F_{X/S,*} D_{X/S}, \tag{3.4}$$

   described above in (3.3) is an isomorphism of $\mathscr{O}_{X'}$-algebras. This isomorphism defines (according to §2.5) a sheaf $\mathscr{D}_{X/S}$ of $\mathscr{O}_{\mathrm{T}^*(X'/S)}$-algebra on the cotangent bundle of X'/S.
2. ([BMR08, Lem. 2.1.1] and [GLQ10, Prop. 3.6]) The adjoint map of (3.4) also induces an isomorphism of $\mathscr{O}_X$-algebras

$$\psi' : F_{X/S}^* \mathrm{Sym}^{\bullet} \Theta_{X'/S} \longrightarrow Z_{D_{X/S}}(\mathscr{O}_X) \subseteq D_{X/S}. \tag{3.5}$$

   where $Z_{D_{X/S}}(\mathscr{O}_X)$ is the centralizer of $\mathscr{O}_X$ in $D_{X/S}$.

3. ([BMR08, Prop. 2.2.2 and Thm. 2.2.3] and [GLQ10, Thm. 3.7]) Moreover, $\mathscr{D}_{X/S}$ is an Azumaya algebra of rank p^{2d} over the cotangnet bundle . Actually, there is an isomorphism of $(F_{X/S}^* \operatorname{Sym}^\bullet \Theta_{X'/S})$-algebras

$$F_{X/S}^* \left(F_{X/S,*} D_{X/S}\right) \simeq \mathscr{E}nd_{F_{X/S}^*(\operatorname{Sym}^\bullet \Theta_{X'/S})}(D_{X/S}), \tag{3.6}$$

which defines an splitting of the pullback of $\mathscr{D}_{X/S}$ to $\mathrm{T}^*(X'/S) \times_{X', F_{X/S}} X$. This Azumaya algebra is nontrivial if the relative dimension $\dim(X/S)$ is more than 0.

4. ([BMR08, Rmk. 2.1.2] and [OV07, Prop. 2.3]) Let $\iota : Z \longrightarrow \mathrm{T}^*(X/S)$ be any morphism. Suppose that we have a splitting

$$\iota^* \mathscr{D}_{X/S} \simeq \mathscr{E}nd_{\mathcal{O}_Z}(P).$$

Then P is a direct image of a rank one locally free sheaf $\widetilde{P}$ on $Z \times_{X', F_{X/S}} X$.

5. ([BMR08, §2.2.5)]) Let $\iota : X' \longrightarrow \mathrm{T}^*(X'/S)$ be a section of the projection $\pi : \mathrm{T}^*(X'/S) \longrightarrow X'$. Then we have an canonical splitting

$$\iota^* \mathscr{D}_{X/S} \simeq \mathscr{E}nd_{\mathscr{O}_{X'}}(F_{X/S,*} \mathscr{O}_X).$$

Proof Actually this problem is local. Using the "étale coordinates" (2.1) again, we can reduce the problem to the case $\mathbf{A}_S^d \longrightarrow S$ with S affine. Then one can check it directly. See the cited references for proofs.

For clarity, we briefly explain why P in 4. is a direct image. Introduce notations as in the following commutative diagram, with all squares being cartesian:

$$\begin{tikzcd}
W := Z \times_{X'} X \arrow[r, "\varphi"] \arrow[d, "\tilde{\iota}"] & Z \arrow[d, "\iota"] \\
T' \arrow[r, "\sigma"] \arrow[d, "\tilde{\pi}"] & T^* := \mathrm{T}^*(X'/S) \arrow[d, "\pi"] \\
X \arrow[r, "F := F_{X/S}"] & X'.
\end{tikzcd}$$

Note also that all maps in the above diagram are affine. We know from (3.5) that we have a natural map

$$F_*(F^* \operatorname{Sym}^\bullet \Theta_{X'/S}) \simeq F_*(F^* \pi_* \mathscr{O}_{T^*}) \simeq F_*(\tilde{\pi}_* \mathscr{O}_{T'}) \simeq \pi_* \sigma_* \mathscr{O}_{T'} \longrightarrow F_* D_{X/S}$$

hence (applying (2.5) to π) a natural map

$$\sigma_* \mathscr{O}_{T'} \longrightarrow \mathscr{D}_{X/S}.$$

Therefore the $\iota^* \mathscr{D}_{X/S}$-module P naturally restricts to a $\iota^* \sigma_* \mathscr{O}_{T'} \simeq \varphi_* \mathscr{O}_W$-module. So applying (2.5) to φ we know that P is a direct image.

3.1.2 Remark For a generalisation of this result to smooth algebraic stacks, see [CZ17, Appendix B], and for a generalisation to higher level differential operators, see [GLQ10, Prop. 3.6 and Thm. 3.7].

3.2 Gerbe of splittings

According to Theorem 3.1.1, we have an Azumaya algebra $\mathscr{D}_{X/S}$ on $\mathrm{T}^*(X'/S)$, so we have the associated $\mathbb{G}_m$-gerbe $\mathbf{S}_{\mathscr{D}} := \mathbf{S}_{\mathscr{D}_{X/S}}$ of splittings over $\mathrm{T}^*(X'/S)$ as recalled in §2.2.

For later use, we introduce here some notations. Let $\mathbf{X}$ be a stack over S. Suppose that $g : S'' \to S$ and $h : S \to S'$ are two morphisms of schemes.

- Denote by $g^*\mathbf{X}$ or $\mathbf{X}|_{S''}$ the pull-back of $\mathbf{X}$, namely, $g^*\mathscr{X} : (\mathbf{Sch}/S'')^{\mathrm{op}} \to \mathbf{Grpd}$, $T \mapsto \mathbf{X}(T)$.
- Denote by $h_*\mathbf{X}$ or $\mathrm{Res}_{S/S'}\,\mathbf{X}$ the *restriction of scalars*, or the *Weil restriction*, from S to S', namely, $h_*\mathbf{X} : (\mathbf{Sch}/S'')^{\mathrm{op}} \to \mathbf{Grpd}$, $T \mapsto \mathbf{X}(S \times_{S'} T)$.

$$\begin{array}{ccccc} \mathbf{X}|_{S''} & \longrightarrow & \mathbf{X} & \longrightarrow & \mathrm{Res}_{S/S'}(\mathbf{X}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ S'' & \xrightarrow{g} & S & \xrightarrow{h} & S' \end{array}$$

We will later apply these two operations to the $\mathbb{G}_{\mathrm{m}}$-gerbe $\mathbf{S}_{\mathscr{D}}$.

4 Higgs bundles and the Hitchin base

Now let $f : X \longrightarrow S$ be a *smooth* scheme of relative dimension d. Then $\Omega^1_{X/S}$ is a locally free sheaf of rank d. Denote by $\Theta_{X/S} := (\Omega^1_{X/S})^\vee = \mathscr{H}om_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{O}_X)$ the sheaf of tangent vectors, or which is the same, the sheaf of S-derivations of $\mathscr{O}_X$ to $\mathscr{O}_X$. Denote by π the canonical projection $\mathrm{T}^*(X/S) = \mathscr{S}pec_X \operatorname{Sym}^\bullet(\Omega^1_{X/S})^\vee \longrightarrow X$ from the cotangent bundle of X/S to X.

Let E be an quasi-coherent $\mathscr{O}_X$-module. A *Higgs field* $\theta : E \longrightarrow E \otimes \Omega^1_{X/S}$ is an $\mathscr{O}_X$-module homomorphism such that $\theta \wedge \theta = 0$, in the sense that the composition

$$E \xrightarrow{\theta} E \otimes \Omega^1_{X/S} \xrightarrow{\theta\otimes\mathrm{id}_\Omega} E \otimes \Omega^1_{X/S} \otimes \Omega^1_{X/S} \xrightarrow{\mathrm{id}_E \otimes q} E \otimes \Omega^2_{X/S}$$

vanishes, where $q : \Omega^1_{X/S} \otimes \Omega^1_{X/S} \longrightarrow \Omega^2_{X/S}$ is the natural quotient map. A *Higgs module* is a pair (E, θ) consisting of a quasi-coherent $\mathscr{O}_X$-module E and an Higgs field θ. A *Higgs bundle* is a Higgs module (E, θ) where E is locally free of finite rank. The *rank* of the Higgs bundle is just the rank of E. For simplicity, we usually say E is a Higgs module/bundle.

To give a Higgs module (E, θ), is equivalent to give a map $\Theta_{X/S} \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(E)$ of $\mathscr{O}_X$-modules. Let $T^\bullet$ be the tensor algebra functor, so we have an adjoint pair $\mathrm{T}^\bullet \dashv \texttt{forgetful} : \mathscr{O}_X\text{-}\mathbf{Alg} \longrightarrow \mathscr{O}_X\text{-}\mathbf{Mod}$ (noting that $\mathscr{E}nd_{\mathscr{O}_X}(E)$ is in general non-commutative). Therefore, we get a map $\mathrm{T}^\bullet\Theta_{X/S} \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(E)$ of sheaves of $\mathscr{O}_X$-algebras. The condition $\theta \wedge \theta = 0$ is equivalent to say that this map factors through the quotient $\mathrm{T}^\bullet\Theta_{X/S} \longrightarrow \operatorname{Sym}^\bullet \Theta_{X/S}$, i.e., we get a morphism of sheaves of $\mathscr{O}_X$-algebras $\operatorname{Sym}^\bullet \Theta_{X/S} \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(E)$, which is equivalent to a $\operatorname{Sym}^\bullet \Theta_{X/S}$-module structure on E. Conversely, any $\operatorname{Sym}^\bullet \Theta_{X/S}$-module structure on E determines a Higgs field on E. Combining this fact and the discussion in §2.5, we have the following observation.

4.0.1 Proposition The category of Higgs modules on X is equivalent to the category $\mathbf{QCoh}(\mathscr{O}_X, \operatorname{Sym}^\bullet \Theta_{X/S})$, hence to the category $\mathbf{QCoh}(\mathrm{T}^*(X/S))$ of quasi-coherent sheaves on $\mathrm{T}^*(X/S)$ by §2.5. The category of Higgs bundles of rank r on X is equivalent to the fully faithful subcategory of $\mathbf{QCoh}(\mathrm{T}^*(X/S))$ consisting of objects whose direct image to X is locally free of rank r.

Finally, denote by $\mathbf{Higgs}_{X/S,r} := \mathbf{Higgs}_{X/S} := \mathbf{Higgs}_r \longrightarrow (\mathbf{Sch}/S)$ the stack of Higgs bundles of X/S of rank r: for each test scheme T/S, $\mathbf{Higgs}_{X/S,r}(T)$ is the category of Higgs bundles on $X_T := X \times_S T$ of rank r.

4.1 Spectral Cover

Now take an arbitrary Higgs bundle (E,θ) of rank r. Note in particular that a Higgs field $\theta : E \to E \otimes \Omega^1_{X/S}$ is a $\Omega^1_{X/S}$-*twisted endomorphism* of E in the sense of §2.3. Let $\pi : \mathrm{T}^*(X/S) \to X$ be the projection from the cotangent bundle of X/S to X. Recall that there is a global section $\lambda \in \Gamma(\mathrm{T}^*(X'/S), \pi^*\Omega^1_{X'/S})$, corresponding to $\mathrm{id} \in \mathrm{End}_X(\mathrm{T}^*(X/S))$, under the identification (2.6), which is usually called the *tautological 1-form* or the *contact form*. Then the determinant $\det(\lambda - \pi^*\theta) =: \chi_\theta(\lambda)$ is a global section of the locally free sheaf $\mathrm{Sym}^r_{\mathscr{O}_{\mathrm{T}^*(X/S)}}(\pi^*\Omega^1_{X/S}) = \pi^*\,\mathrm{Sym}^r_{\mathscr{O}_X}\Omega^1_{X/S}$ of rank $\binom{d+r-1}{r}$ over $\mathrm{T}^*(X/S)$, which is obtained from the characteristic polynomial

$$\chi_\theta(t) = t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r \in \Gamma\big(X, (\mathrm{Sym}^\bullet\,\Omega^1_{X/S})[t]\big), \quad \text{with } a_i \in \Gamma(X, \mathrm{Sym}^i\,\Omega^1_{X/S}),$$

of θ by pulling back along π then substituting t by the tautological section λ, as described in the general setting in §2.3. According to §2.4, the vanishing locus on $\mathrm{T}^*(X/S)$ of the section $\chi_\theta(\lambda) : \mathscr{O}_{\mathrm{T}^*(X/S)} \to \pi^*\,\mathrm{Sym}^r\,\Omega^1_{X/S}$ is defined by the quasi-coherent sheaf of ideals

$$I_\theta := I_{\chi_\theta} := \mathrm{Im}\left(\left(\pi^*(\mathrm{Sym}^r\,\Omega^1_{X/S})\right)^\vee \xrightarrow{(\chi_\theta(\lambda))^\vee} \mathscr{O}_{\mathrm{T}^*(X/S)} \right) \subseteq \mathscr{O}_{\mathrm{T}^*(X/S)}. \tag{4.1}$$

Denote the corresponding closed embedding by $\iota_\theta := \iota_{\chi_\theta} : Z_{\chi_\theta} \to \mathrm{T}^*(X/S)$. Besides, the morphism $\pi \circ \iota_\theta : Z_{\chi_\theta} \to X$, or just Z_{χ_θ}, is called the *spectral cover* of X associated to (E,θ).

4.2 The Hitchin base and Hitchin map for Higgs bundles

Define on the site $(\mathbf{Sch}/S)_{\mathrm{fppf}}$ the presheaf

$$\begin{aligned} \mathbf{B} := \mathbf{B}_r := \mathbf{B}_{X/S,r} : (\mathbf{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathtt{Set} \\ (\chi : T \to S) &\longmapsto \bigoplus_{i=1}^{r} \Gamma\big(X_T, \mathrm{Sym}^i\,\Omega^1_{X_T/T}\big) \end{aligned}$$

where $X_T := X \times_S T$. This is called the *Hitchin base*. We identify a T-point $\chi = (a_1, \dots, a_r) \in \mathbf{B}_{X/S,r}(T)$ as the polynomial

$$\chi(t) = t^r - a_1 t^{r-1} + \cdots + (-1)^i a_i t^{r-i} + \cdots + (-1)^r a_r \in \Gamma\big(X_T, (\mathrm{Sym}^\bullet\,\Omega^1_{X_T/T})[t]\big) \tag{4.2}$$

with $a_i \in \Gamma\big(X_T, \mathrm{Sym}^i\,\Omega^1_{X_T/T}\big)$. So by pulling back this polynomial to $\mathrm{T}^*(X_T/T)$ and substituting t by the tautological section λ, can we similarly obtain a closed subscheme Z_χ as we did for the characteristic polynomial of a Higgs field in §4.1.

4.2.1 Example In case $r = 1$, given a $\chi \in \mathbf{B}_{X'/S,1}(T)$, $Z_\chi \hookrightarrow \mathrm{T}^*(X'_T/T)$ is isomorphic to the section $X'_T \to \mathrm{T}^*(X'_T/T)^*$ of $\mathrm{T}^*(X'_T/T) \to X'_T$ corresponding to the global section ω determined by (2.6). Recall Theorem 3.1.1.5 that, the Azumaya algebra $\mathscr{D}_{X/S}$ splits over Z_χ.

4.2.2 Example In case $X/S = X/k$ is proper over a field k, $\Gamma(X, \Omega^1_{X/k})$ is a finite dimensional k-vector space, and any map $T \to \operatorname{Spec} k$ is flat. So flat base change implies that $\mathbf{B}$ is representable by the affine k-scheme

$$B := \operatorname{Spec} \operatorname{Sym}^\bullet \left(\bigoplus_{i=1}^{r} (\Gamma(X, \operatorname{Sym}^i \Omega^1_{X/k}) \right)^\vee$$

(see §2.5 and cf. [Sim95, p. 20]).

4.2.3 Example Here we give an explicit *local equation* for Z_χ. To this aim, we may assume that $\chi \in \mathbf{B}(S)$ is an S-point of $\mathbf{B} = \mathbf{B}_{X/S,r}$; otherwise, we may consider X_T/T instead of X/S. Moreover, we may assume that $X = \operatorname{Spec} R$ is an affine scheme and the sheaf $\Omega^1_{X/S}$ of Kähler differentials is free; otherwise, replace X by a small enough affine open subscheme $U = \operatorname{Spec} R$ over which $\Omega^1_{X/S}|_U$ is trivialized:

$$\Omega^1_{X/S} \simeq \bigoplus_{i=1}^{d} \mathscr{O}_X \cdot \omega_i,$$

where $\omega_i \in \Gamma(X, \Omega^1_{X/S})$, $1 \le i \le d$, form a basis for the free R-module $\Gamma(X, \Omega^1_{X/S})$. This fixed trivialization induces isomorphisms

$$\pi^*\Omega^1_{X/S} \simeq \bigoplus_{i=1}^{d} \mathscr{O}_{\mathrm{T}^*(X/S)} \cdot \pi^*\omega_i, \quad \text{and} \quad \pi_*\mathscr{O}_{\mathrm{T}^*(X/S)} \simeq \operatorname{Sym}^\bullet \left(\Omega^1_{X/S}\right)^\vee \simeq \mathscr{O}_X[\partial_1, \dots, \partial_d],$$

with $\partial_i \in \Gamma\big(X, (\Omega^1_{X/S})\big)^\vee = \Gamma\big(X, (\Omega^1_{X/S})^\vee\big) \subseteq \Gamma(\mathrm{T}^*(X/S), \mathscr{O}_{\mathrm{T}^*(X/S)})$ being the R-dual of ω_i. Moreover, for any $1 \le m \le r$,

$$\operatorname{Sym}^m \Omega^1_{X/S} \simeq \bigoplus_{|\underline{i}|=S(m,d)} \mathscr{O}_X \cdot \underline{\omega}^{\underline{i}}, \quad \text{and} \quad \pi^* \operatorname{Sym}^m \Omega^1_{X/S} \simeq \bigoplus_{|\underline{i}|=S(m,d)} \mathscr{O}_{\mathrm{T}^*(X/S)} \cdot \pi^*\underline{\omega}^{\underline{i}}, \tag{4.3}$$

where $S(m,d) := \binom{d+m-1}{m}$ and the usual multi-index convention is used; that is, for any multi-index $\underline{i} = (i_1, i_2, \dots, i_d)$, where $i_j \ge 0$ for all $1 \le j \le d$, we set $|\underline{i}| := \sum_{j=1}^d i_j$ and

$$\underline{\omega}^{\underline{i}} := \omega_1^{i_1} \cdot \omega_2^{i_2} \cdots \omega_d^{i_d} \in \Gamma(X, \operatorname{Sym}^r \Omega^1_{X/S}).$$

The tautological section in this case can be written as

$$\lambda = \sum_{j=1}^{d} \partial_j \cdot (\pi^* \omega_j) \in \Gamma(\mathrm{T}^*(X/S), \pi^* \Omega^1_{X/S}). \tag{4.4}$$

For any $1 \leq m \leq r$, and any $a_m \in \Gamma(X, \mathrm{Sym}^m \Omega^1_{X/S})$, we can write, according to (4.3),

$$a_m = a_{m1} \cdot \omega_1^m + a_{m2} \cdot \omega_2^m + \cdots + a_{md} \cdot \omega_d^m + (\text{other terms}),$$

where $a_{mj} \in \Gamma(X, \mathscr{O}_X) = R$, for each $1 \leq j \leq d$, and (other terms) is a sum of terms of the form $a_{m\underline{i}} \cdot \underline{\omega}^{\underline{i}}$, with $a_{m\underline{i}} \in R$ and with $|\underline{i}| \neq 1$. Similarly, the equation (4.4) gives that

$$\lambda^{r-m} = \partial_1^{r-m} \cdot \pi^* \omega_1^{r-m} + \partial_2^{r-m} \cdot \pi^* \omega_2^{r-m} + \cdots + \partial_d^{r-m} \cdot \pi^* \omega_d^{r-m} + (\text{other terms}).$$

Therefore, any χ of the form $\lambda^r - a_1 \lambda^{r-1} + \cdots + (-1)^r a_r$ of $\pi^* \mathrm{Sym}^r \Omega^1_{X/S}$ can be then written as

$$\begin{aligned}
\chi &= \lambda^r - a_1 \lambda^{r-1} + \cdots + (-1) a_r \\
&= \quad (\partial_1^r - a_{11} \partial_1^{r-1} + \cdots + (-1)^r a_{r1}) \cdot \pi^* \omega_1^r \\
&\quad + (\partial_2^r - a_{12} \partial_1^{r-1} + \cdots + (-1)^r a_{r2}) \cdot \pi^* \omega_2^r \\
&\quad + \cdots \\
&\quad + (\partial_d^r - a_{1d} \partial_1^{r-1} + \cdots + (-1)^r a_{rd}) \cdot \pi^* \omega_d^r \\
&\quad + (\text{other terms}) \\
&=: g_1 \cdot \pi^* \omega_1^r + g_2 \cdot \pi^* \omega_2^r + \cdots + g_d \cdot \pi^* \omega_d^r + (\text{other terms}).
\end{aligned} \tag{4.5}$$

In the above equation, (other terms) is a sum of terms of the form $g_{\underline{i}} \cdot \pi^* \underline{\omega}^{\underline{i}}$, where $g_{\underline{i}} \in R[\partial_1, \ldots, \partial_d]$ are polynomials of degree r, and for each $1 \leq m \leq d$, we write g_m instead of $g_{\underline{i}}$ when $|\underline{i}| = 1$ and $i_m = 1$.

So we can conclude that Z_χ is (Zariski locally over $\operatorname{Spec} R \subseteq X$, if you started with general X/S) defined by the ideal $(g_1, \ldots, g_d, \ldots, g_{\underline{i}}, \ldots)$ of $R[\partial_1, \ldots, \partial_d]$, generated by $S(r, d)$ polynomials; in other words, Z_χ is the spectrum of

$$\frac{R[\partial_1, \ldots, \partial_d]}{(g_1, \ldots, g_d, \ldots, g_{\underline{i}}, \ldots)} \tag{4.6}$$

We remark that $g_m \in R[\partial_m]$ with $1 \leq m \leq d$ are polynomials in only one variable of degree r.

4.2.4 Example and Definition Suppose that X/k is an abelian variety. Then $\mathbf{B}$ is representable by a k-scheme B as in Example 4.2.2. Moreover, we know that $\Gamma(X, \mathscr{O}_X) = k$ because X is proper geometrically reduced and geometrically connected, and that $\Gamma(X, \Omega^1_{X/k})$

is a k-vector space of dimension $d = \dim X$. Actually,

$$\Omega^1_{X/k} \simeq \bigoplus_{i=1}^{d} \mathscr{O}_X \cdot \omega_i,$$

where $\omega_i \in \Gamma(X, \Omega^1_{X/k})$, $i = 1, \ldots, d$, are the *invariant differentials*. In other words, the sheaf $\Omega^1_{X/S}$ is *globally* trivialized. Denote by ∂_i, $1 \leq i \leq d$ the k-dual of ω_i. These ω_i's (resp. ∂_i's) form not only a k-basis for the k-vector space $\Gamma(X, \Omega^1_{X/k})$ (resp. $(\Gamma(X, \Omega^1_{X/k}))^\vee = \Gamma(X, (\Omega^1_{X/k})^\vee)$), but also an $\mathscr{O}_X$-basis for the free $\mathscr{O}_X$-module $\Omega^1_{X/k}$ (resp. $(\Omega^1_{X/k})^\vee$).

Then apply the computations in Example 4.2.3, we know that for any $\chi \in B(T)$, we have

$$Z_\chi = \mathcal{S}pec_{\mathscr{O}_{X_T}} \frac{\mathscr{O}_{X_T}[\partial_1, \ldots, \partial_d]}{(g_1, \ldots, g_d, \ldots, g_{\underline{i}}, \ldots)}, \tag{4.7}$$

with $g_{\underline{i}}$ has coefficient in $\Gamma(X_T, \mathscr{O}_{X_T}) = \Gamma(T, \mathscr{O}_T)$ (note T/k is always flat). Now for each $\chi \in B(T)$, define

$$\widetilde{Z}_\chi = \mathcal{S}pec_{\mathscr{O}_{X_T}} \frac{\mathscr{O}_{X_T}[\partial_1, \ldots, \partial_d]}{(g_1, \ldots, g_d)} \subseteq Z_\chi. \tag{4.8}$$

It is clear that the association of χ to $\widetilde{Z}_\chi$ is functorial, i.e., if $\chi' = \chi \circ \psi$ with $\psi : T' \longrightarrow T$, then $\widetilde{Z}_{\chi'} = \widetilde{X}_\chi \times_T T'$.

4.2.5 The Hitchin map Now, the map sending each object $(T, (E, \theta))$ of $\mathbf{Higgs}_{X/S,r}$ over T in $(\mathbf{Sch}/S)$, to the coefficients of the characteristic polynomial $\chi_\theta(t)$ of θ, gives a morphism

$$c_{\mathrm{Dol}} : \mathbf{Higgs}_{X/S,r} \longrightarrow \mathbf{B}_{X/S,r} \tag{4.9}$$

of stacks over S.

4.2.6 The Universal Spectral cover In case $\mathbf{B}$ is representable by B, we have an *universal* element $\chi_{\mathrm{univ}} \in \bigoplus_{i=1}^r \Gamma\big(X_B, \mathrm{Sym}^i \Omega^1_{X_B/B}\big)$ corresponding to id_B. So there is a corresponding closed subscheme $Z := Z_{\mathrm{univ}}$ of $\mathrm{T}^*(X_B/B)$. We will call Z/X_B the *universal spectral cover*. It is universal in the sense that for any $\chi : T \longrightarrow B$ over S, Z_χ is the pullback of Z_{univ} along χ.

In case $X/S = X/k$ is an abelian variety over k, denote by $\widetilde{Z} = \widetilde{Z}_{\mathrm{univ}} \subseteq Z_{\mathrm{univ}}$ as in (4.8).

4.2.7 Example

- *Suppose that $X/S := X/k$ is a smooth proper scheme over a field k.* Hence Example 4.2.2 applies.
  Denote by $(\omega_j)_{j\in J}$, the k-basis for $\bigoplus_{i=1}^r \Gamma(X, \mathrm{Sym}^i \Omega^1_{X/k})$, and by $(\partial_j)_{j\in J}$ the corresponding k-dual basis for $\bigoplus_{i=1}^r \Gamma(X, \mathrm{Sym}^i \Omega^1_{X/k})^\vee$. Then χ_{univ} is actually the tautological

section

$$\chi_{\text{univ}} = \sum_{J\in j} \partial_j \otimes \omega_j \in \left(\mathrm{Sym}^\bullet \left(\bigoplus_{i=1}^{r} \Gamma(X, \mathrm{Sym}^i \Omega^1_{X/k}) \right)^\vee \right) \otimes_k \left(\bigoplus_{i=1}^{r} \Gamma(X, \mathrm{Sym}^i \Omega^1_{X/k}) \right).$$

- *Suppose moreover that X/k is a connected group scheme* (hence *geometrically connected*). In other words, X/k is an abelian variety. So Example 4.2.4 applies.
  Moreover, using notations in Example 4.2.3, for each m, $(\underline{\omega}^{\underline{i}})_{|i|=m}$ form a k-basis for the k-vector space $\Gamma(X, \mathrm{Sym}^m \Omega^1_{X/k}) = \mathrm{Sym}^m \Gamma(X, \Omega^1_{X/k})$, as well as the free $\mathscr{O}_X$-module $\mathrm{Sym}^m \Omega^1_{X/k}$; and $(\underline{\partial}^{\underline{i}})_{|i|=m}$ from a k-basis for the k-vector space $\Gamma(X, \mathrm{Sym}^m(\Omega^1_{X/k})^\vee) = \mathrm{Sym}^m \Gamma(X, (\Omega^1_{X/k})^\vee)$. Denote by $\underline{\partial}^{[\underline{i}]}$ the k-dual basis for $(\Gamma(X, \mathrm{Sym}^m \Omega^1_{X/k}))^\vee = \Gamma(X, (\mathrm{Sym}^m \Omega^1_{X/k})^\vee)$. Note that $\underline{\partial}^{\underline{i}}$ and $\underline{\partial}^{[\underline{i}]}$ are not the same unless $|i| = 1$.
- *Suppose further that the characteristic of k is larger than the fixed number r (hence $n! \neq 0$).*
  In this case, for each $1 \leq m \leq r$, we have the following "bad" isomorphism[7]

$$\begin{aligned} \Gamma\big(X, \mathrm{Sym}^m(\Omega^1_{X/k})^\vee\big) &\xrightarrow{\sim} \big(\Gamma(X, \mathrm{Sym}^m \Omega^1_{X/k})\big)^\vee \\ \underline{\partial}^{\underline{i}} &\longmapsto \frac{1}{\underline{i}!}\underline{\partial}^{\underline{i}} = \underline{\partial}^{[\underline{i}]} \end{aligned} \tag{4.10}$$

of k-vector spaces, and $\frac{1}{\underline{i}!}\underline{\partial}^{\underline{i}}$, $|\underline{i}| = m$, form a basis form the $\binom{m+d-1}{m}$-dimensional k-vector space on the right hand side. Observe that

$$\Gamma(X_B, \Omega^1_{X_B/B}) = \left(\mathrm{Sym}^\bullet \left(\bigoplus_{i=1}^{r} \Gamma(X, \mathrm{Sym}^i \Omega^1_{X/k}) \right)^\vee \right) \otimes_k \Gamma(X, \Omega^1_{X/k}).$$

Write $\mu := \partial_1 \otimes \omega_1 + \partial_2 \otimes \omega_2 + \cdots + \partial_d \otimes \omega_d \in \Gamma(X_B, \Omega^1_{X_B/B})$. Then using the "bad"

[7]The problem is that for any k-vector space V and and any natural number n, the natural paring

$$\mathrm{Sym}^n V \times \mathrm{Sym}^n(V^\vee) \longrightarrow k, \quad (v_1, \ldots, v_n, \ell_1, \ldots, \ell_n) \longmapsto \sum_{\sigma\in\mathfrak{S}_n} \prod_{i=1}^{n} \ell_i(v_{\sigma(i)}),$$

is perfect if the characteristic of k is strictly larger than n. Moreover, the *natural* map

$$\mathrm{Sym}^n(V^\vee) \longrightarrow (\mathrm{Sym}^n V)^\vee$$

is an isomorphism if and only if $n! \neq 0$ in k. To avoid this problem, the more natural way is to use divided power algebras instead of symmetric algebras.

isomorphism (4.10), we can write

$$\begin{aligned}\chi_{\mathrm{univ}} &= \sum_{m=1}^{r}\sum_{|\underline{i}|=m} \underline{\partial}^{[\underline{i}]} \otimes \underline{\omega}^{\underline{i}} \\ &= \partial_1 \otimes \omega_1 + \partial_2 \otimes \omega_2 + \cdots + \partial_d \otimes \omega_d \\ &\quad + \frac{1}{2}\partial_1^2 \otimes \omega_1^2 + \cdots + \frac{1}{2}\partial_d^r \otimes \omega_d^2 + \sum_{1\le i,j\le d, i\neq j} \partial_i\partial_j \otimes \omega_i\omega_j \\ &\quad + \cdots \\ &=: \mu + \frac{1}{2}\mu^2 + \cdots + \frac{1}{r!}\mu^r \in \bigoplus_{i=1}^{r} \Gamma(X_B, \operatorname{Sym}^i \Omega^1_{X_B/B}),\end{aligned}$$

The corresponding characteristic polynomial then can be written as

$$\begin{aligned}\chi_{\mathrm{univ}}(t) &= \sum_{i=0}^{r} \frac{(-1)^i}{i!}\mu^i t^{r-i} \\ &= t^r - \mu t^{r-1} + \frac{1}{2}\mu^2 t^{r-1} - \cdots + (-1)^r \frac{1}{r!}\mu^r \in \Gamma\big(X_B, (\operatorname{Sym}^\bullet \Omega^1_{X_B/B})[t]\big).\end{aligned}$$

- *Assume moreover that k is algebraically closed.*
  Now we can even factorize it as

$$\chi_{\mathrm{univ}}(t) = (t - c_1\mu)(t - c_2\mu)\cdots(t - c_r\mu),$$

  for some constants $c_i \in \Gamma(X, \mathscr{O}_X) = k$.

4.2.8 Proposition For any $\chi \in \mathbf{B}(T)$ with *non-empty* Z_χ, the natural map $Z_\chi \longrightarrow X_T$ is finite and locally of finite presentation, hence in particular is proper. Suppose that X/S is proper, and that $\mathbf{B}$ is representable by B, then Z/B is proper, where Z/X_B is the universal spectral cover.

Moreover, if $r = 1$ or $d = 1$, Z_χ is always non-empty and Z_χ/X_T is also flat so finite locally free. Hence if in addition X/S is flat, so is Z/B.

Proof Finiteness is a local question. According to Example 4.2.3, for any affine open $U = \operatorname{Spec} R \subseteq X_T$ where $\Omega^1_{X_T/T}$ is trivialized, $Z_\chi|_U$ is the spectrum of the R-algebra given in (4.6). One observe that for each $1 \le m \le d$, g_m has a single variable ∂_m. Therefore, (4.6) is a finite R-module, and Z_χ/X_T is finite.

In case X/S is proper and $\mathbf{B}$ is representable by B, we have that $Z \longrightarrow X_B \longrightarrow B$ is a composition of proper morphisms hence is also proper.

Flatness also follows from the local description (4.6).

4.2.9 Remark Note, the scheme Z_χ defined by an arbitrary $\chi \in \mathbf{B}(T)$ can be empty. This can be seen more obviously from the local description (4.6) in Example 4.2.3, because of the presence of the polynomials $g_{\underline{i}}(\partial_1, \dots, \partial_d)$, $|\underline{i}| \neq 1$. For the same reason, unless $d = 1$ or $r = 1$, Z_χ / X_T is in general *not* flat,

Nevertheless, in case $X/S = X/k$ is an abelian variety, $\widetilde{Z}_\chi$ is always non-empty and is finite and flat over X, and $\widetilde{Z}_\chi / B$ is proper, for any $\chi \in \mathbf{B}(T)$. However, the definition of $\widetilde{Z}_\chi$ only makes sense in case the cotangent bundle is trivial, even though it is defined Zariski locally for general X/S.

More generally, in [CN19, Thm. 5.1 & Conj. 5.2], it was proved that the Hitchin map (4.9) factors through a closed subscheme of the Hitchin base, and was conjectured that the resulting map is surjective. This phenomenon can already be seen form Example 4.2.3.

4.3 An equivalence (the BNR correspondence)

Fix an S-point χ of $\mathbf{B}_{X/S,r}$, i.e., fix a polynomial $\chi(t) = t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r \in \Gamma(X, \operatorname{Sym}^\bullet \Omega^1_{X/S}[t])$.

Suppose that (E, θ) is a Higgs bundle of rank r on X such that the characteristic polynomial $\chi_\theta(t)$ of the Higgs field θ equals to $\chi(t)$. Since E is a Higgs bundle, it gives a quasi-coherent sheaf $\widetilde{E}$ on $\mathrm{T}^*(X/S)$. Then the Cayley-Hamilton theorem (Proposition 2.3) implies that $\widetilde{E}$ is supported on the spectral cover Z_χ defined by χ. To see this, we only need to verify that $I_\theta \cdot \widetilde{E} = 0$, where I_θ is the sheaf of ideals (4.1) that defines Z_χ. In other words, it suffices to show that the morphism

$$\mathrm{id}_{\widetilde{E}} \otimes (\chi(\lambda))^\vee : \widetilde{E} \otimes_{\mathscr{O}_{\mathrm{T}^*(X/S)}} \pi^*(\operatorname{Sym}^r \Omega^1_{X/S})^\vee \longrightarrow \widetilde{E}$$

is zero, where λ is the tautological section of $\pi^*\Omega^1_{X/S}$ and $\chi(\lambda) = \det(\lambda - \pi^*\theta) : \mathscr{O}_{\mathrm{T}^*(X/S)} \to \pi^* \operatorname{Sym}^r \Omega^1_{X/S}$. By §2.5 and projection formula, it suffices to show that

$$\pi_*(\mathrm{id}_{\widetilde{E}} \otimes (\chi(\lambda)^\vee) : E \otimes (\operatorname{Sym}^r \Omega^1_{X/S})^\vee \longrightarrow E \tag{4.11}$$

is zero. One checks easily, for example by local computation, that (4.11) is exactly (2.3) composed with the evaluation map $\operatorname{Sym}^r \Omega^1_{X/S} \otimes (\operatorname{Sym}^r \Omega^1_{X/S})^\vee \to \mathscr{O}_X$, hence is zero. Therefore, the Higgs bundle (E, θ) with characteristic polynomial χ gives rise to a quasi-coherent sheaf on $Z_\chi \subseteq \mathrm{T}^*(X/S)$.

Conversely, suppose that M is a quasi-coherent module on Z_χ defined by χ, with direct image $E' := \pi_* \iota_* M$ to X a locally free $\mathscr{O}_X$-module of rank r. Then (E', θ') is a Higgs bundle on X, with the Higgs field θ' induced by the $\mathscr{O}_{\mathrm{T}^*(X/S)}$-module structure of $\iota_* M$ (see Proposition 4.3.2).

4.3.1 Remark However, it is not clear to us how the characteristic polynomial of θ' is related to χ. Besides, $\tilde{E}$ is supported on Z_χ, but its scheme-theoretic support can be "thinner" than Z_χ. For example, if $\theta \equiv 0$, then the characteristic polynomial is t^r. In this case, Z_χ is "defined by $t^n = 0$", but $\tilde{E}$ has scheme-theoretic support "defined by $t = 0$". In other words, the "minimal polynomial" may have lower degree than the characteristic polynomial.

The above discussion clearly holds for any $\chi \in \mathbf{B}_{X/S,r}(T)$ for any S-scheme T. Now we can summarized the above discussion into the following proposition (cf. [BNR89]).

4.3.2 Proposition ([Gro16, Theorem 3.2]) Given any T-point χ of $\mathbf{B}_{X/S,r}$ with $\iota : Z_\chi \hookrightarrow \mathrm{T}^*(X_T/T)$, there is an equivalence of categories between

(HA) the fully faithful subcategory $c_{\mathrm{Dol}}^{-1}(\chi)$ of $\mathbf{Higgs}_{X/S,r}$, and
(HB) the fully faithful subcategory of $\mathbf{QCoh}(Z_\chi)$ consisting of objects M such that

- $(\pi_T \circ \iota)_* M$ is locally free rank r on X_T, and
- the induced Higgs field has characteristic polynomial χ.

4.3.3 Remark We can repeat all the above discussion replacing $\Omega^1_{X/S}$ by an arbitrary locally free sheaf K of finite rank, accordingly replacing $\mathrm{T}^*(X/S)$ by the geometric vector bundle $\mathbf{V}(K)$ over X. One also accordingly define the Hitchin base $\mathbf{B}_{X/S,r;K}$ (as a sheaf, or $B_{X/S,r;K}$ as a scheme if representable) and the Hitchin morphism

$$\mathbf{Higgs}_{X/S,r;K} \longrightarrow \mathbf{B}_{X/S,r;K}. \tag{4.12}$$

In particular, for any scheme morphism $u : X \to Y$ over S, a u-Higgs bundle on X is a locally free sheaf E of finite rank over X, equipped with a u-Higgs field, i.e., an $\mathscr{O}_X$-homomorphism $\theta : E \to E \otimes u^*\Omega^1_{Y/S}$, satisfying $\theta \wedge \theta = 0$. Note that a u-Higgs module is equivalent to a quasicoherent sheaf on $\mathrm{T}^*(Y/S) \times_{Y,u} X$. We will later be interested in the case where $u = F_{X/S} : X \to X$ is the relative Frobenius morphism. We will denote the stack of $F_{X/S}$-Higgs bundle of rank r by

$$F\text{-}\mathbf{Higgs}_{X/S,r} := \mathbf{Higgs}_{X/S,r;F^*_{X/S}\Omega^1_{X'/S}}.$$

5 Flat connections

Again, let $f : X \longrightarrow S$ be a *smooth* morphism of relative dimension d.

Recall that an *S-connection* (or just a connection) on a quasi-coherent $\mathcal{O}_X$-module E is a $f^{-1}\mathcal{O}_X$-linear morphism $\nabla : E \longrightarrow E \otimes \Omega^1_{X/S}$ satisfying the Leibniz rule: $\nabla(fe) = e \otimes \mathrm{d}_{X/S}(f) + f\nabla(e)$, for any local sections f of $\mathcal{O}_X$ and e of E, where $\mathrm{d}_{X/S} : \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$ is the canonical S-derivation. A connection ∇ extends to a homomorphism of sheaf of abelian groups $\nabla_2 : E \otimes \Omega^1_{X/S} \longrightarrow E \otimes \Omega^2_{X/S}$ by

$$\nabla_2(e \otimes \omega) := e \otimes \mathrm{d}_{X/S,2}\omega + \nabla(e) \wedge \omega,$$

for local sections e of E and ω of $\Omega^1_{X/S}$, where $\mathrm{d}_{X/S,2}$ is the natural exterior derivation $\mathrm{d}_{X/S,2} : \Omega^1_{X/S} \longrightarrow \Omega^2_{X/S}$. A connection ∇ is *flat* or *integrable* if its *curvature* $K := K_\nabla := \nabla_2 \circ \nabla$ vanishes.

Suppose that E_1 and E_2 are quasi-coherent $\mathcal{O}_X$-modules with connections ∇_1 and ∇_2 respectively. An $\mathcal{O}_X$-morphism $\phi : E_1 \longrightarrow E_2$ is *flat* if the diagram

$$\begin{tikzcd}
E_1 \arrow[r, "\nabla_1"] \arrow[d, "\phi"] & E_1 \otimes \Omega^1_{X/S} \arrow[d, "\phi\otimes\mathrm{id}"] \\
E_2 \arrow[r, "\nabla_2"] & E_2 \otimes \Omega^1_{X/S}
\end{tikzcd}$$

of sheaves of abelian groups commute.

The category of pairs (E, ∇) with E a quasi-coherent $\mathcal{O}_X$-module and ∇ an S-connection on E, and flat morphisms between these pairs, is usually denote by **MIC**(X/S), with **MIC** standing for `M`odules with `I`ntegrable `C`onnection.

Given a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d] & X \arrow[d, "f"] \\
T \arrow[r] & S
\end{tikzcd}$$

and an quasi-coherent $\mathcal{O}_X$-module E with a S-connection ∇, there is a *pullback connection*

$u^*\nabla$ on u^*E defined by extending the $u^{-1}f^{-1}\mathscr{O}_S$-natural map

$$u^{-1}E \xrightarrow{u^{-1}\nabla} u^{-1}E \otimes_{u^{-1}\mathscr{O}_X} u^{-1}\Omega^1_{X/S} \longrightarrow u^*E \otimes_{\mathscr{O}_Y} u^*\Omega^1_{X/S} \longrightarrow u^*E \otimes_{\mathscr{O}_Y} \Omega^1_{Y/T}$$

using Leibniz rule.[8]

There are some special phenomenon in case S is of characteristic $p > 0$. First of all, the canonical S-derivation $\mathrm{d}_{X/S} : \mathscr{O}_X \to \Omega^1_{X/S}$ is $F^{-1}_{X/S}\mathscr{O}_{X'}$-linear. So for any quasi-coherent $\mathscr{O}_X$-module E on X', it makes sense to define

$$\nabla^{\mathrm{can}} := \mathrm{id}_{F^{-1}E} \otimes \mathrm{d}_{X/S} : F^*_{X/S}E = F^{-1}_{X/S}E \otimes_{F^{-1}_{X/S}\mathscr{O}_{X'}} \mathscr{O}_X \longrightarrow F^{-1}_{X/S}E \otimes_{F^{-1}\mathscr{O}_{X'}} \Omega^1_{X/S} \simeq F^*E \otimes \Omega^1_{X/S},$$

on the Frobenius pullback $F^*_{X/S}E$ of E. One easily see that it is a *flat S-connection* on E, and is called the *canonical connection*, as the notation indicates. Secondly, if E is a quasi-coherent $\mathscr{O}_X$-module with a flat connection ∇, then the sheaf *of abelian groups* $E^\nabla := \mathrm{Ker}(E)$ has a natural $F^{-1}_{X/S}\mathscr{O}_{X'}$-module structure. Put another way, $(F_{X/S})_*(E^\nabla)$ is an $\mathscr{O}_{X'}$-module. The relation between these two constructions will be clear in §5.2.

5.1 The p-curvature

Let S be of characteristic $p > 0$. Suppose also that X/S is smooth, so that $\Omega^1_{X/S}$ is locally free of finite rank.

Now suppose that E is a quasi-coherent $\mathscr{O}_X$-module with a flat connection ∇. The map $\nabla : E \to E \otimes \Omega^1_{X/S}$ is equivalent to an $\mathscr{O}_X$-morphism $\widetilde{\nabla} : (\Omega^1_{X/S})^\vee \simeq \mathscr{D}er(X/S) =: \Theta_{X/S} \to \mathscr{E}nd_{f^{-1}\mathscr{O}_S}(E)$, as $\Omega^1_{X/S}$ is locally free of finite rank. For any local section ∂ of $\Theta_{X/S}$, viewing ∂ as an S-derivation of $\mathscr{O}_X$ to itself, we know its p-th iteration ∂^p as a derivation is again an S-derivation by Leibniz rule. Moreover, we can check that the followings hold (see e.g., [Kat70, §5]).

- The difference $\widetilde{\psi}(\partial) := (\widetilde{\nabla}(\partial))^p - \widetilde{\nabla}(\partial^p)$, *a priori* just a local section of $\mathscr{E}nd_{f^{-1}\mathscr{O}_S}(E)$, is actually a local section of $\mathscr{E}nd_{\mathscr{O}_X}(E)$. Hence $\partial \mapsto \widetilde{\psi}(\partial)$ defines an map of sheaves of abelian groups $\widetilde{\psi} := \widetilde{\psi}_\nabla : \Theta_{X/S} \to \mathscr{E}nd_{\mathscr{O}_X}(E)$.
- The thus-obtained map $\widetilde{\psi}$ is p-linear, i.e., $\widetilde{\psi}(f\partial) = f^p\widetilde{\psi}(\partial)$. So it determines an $\mathscr{O}_X$-morphism
$$(\mathrm{Fr}_X)^*\Theta_{X/S} = (F_{X/S})^*\Theta_{X'/S} \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(E), \tag{5.1}$$
which further corresponds to an $\mathscr{O}_X$-morphism
$$\psi := \psi_\nabla : E \longrightarrow E \otimes F^*_{X/S}\Omega^1_{X'/S}.$$

This ψ_∇ is called the *p-curvature* of ∇ It has notably the following properties.

[8] That taking pullback of connections is functorial, can be easier seen using the language of crystals, see [Ber74].

- It is an $F_{X/S}$-Higgs field, in the sense of §4.3.3. So we have a map

$$\mathbf{LocSys}_{X/S,r} \longrightarrow F\text{-}\mathbf{Higgs}_{X/S,r} \tag{5.2}$$

  of stacks over $(\mathbf{Sch}/S)$.
- It is a flat morphism with respect to ∇ on E and $\nabla \otimes \nabla^{\mathrm{can}}$ on $E \otimes F^*_{X/S}\Omega^1_{X'/S}$ (see e.g., [Kat70, (5.2.3)]).

5.1.1 Remark We sometimes consider also the adjoint map

$$\Theta_{X'/S} \longrightarrow (F_{X/S})_* \mathscr{E}nd_{\mathscr{O}_X}(E), \tag{5.3}$$

of (5.1). Composed with $(F_{X/S})_* \mathscr{E}nd_{\mathscr{O}_X}(E) \simeq \mathscr{E}nd_{F_*\mathscr{O}_X}((F_{X/S})_* E) \to \mathscr{E}nd_{\mathscr{O}_X}((F_{X/S})_* E)$, it gives an $\mathscr{O}_{X'}$-module morphism

$$\psi' := \psi'_\nabla : (F_{X/S})_* E \longrightarrow (F_{X/S})_* E \otimes \Omega^1_{X'/S}$$

We also refer to this map as as the p-curvature of ∇. It can be also obtained by taking the direct image of ψ under $F_{X/S}$ and using the projection formula $F_{X/S,*}(E \otimes F^*_{X/S}\Omega^1_{X'/S}) \simeq F_{X/S,*}E \otimes \Omega^1_{X'/S}$. It is a Higgs field on the locally free sheaf $(F_{X/S})_* E$ of rank $p^d r$.Hence similar to (5.2), we have a map

$$\mathbf{LocSys}_{X/S,r} \longrightarrow \mathbf{Higgs}_{X'/S,p^d r}. \tag{5.4}$$

of stacks over S.

5.1.2 In terms of D-modules Recall [Ber74, Cor. 4.2.12] or [BO78, Thm. 4.8]) that the category of $D_{X/S}$-modules that are quasi-coherent as $\mathscr{O}_X$-modules is equivalent to the category $\mathbf{MIC}(X/S)$.[9] Basically, the equivalence is just given by extending (resp. restricting) the action of $\Theta_{X/S}$ (resp. $D_{X/S}$) on E via (3.1). Under this identification, given a flat connection (E, ∇), by restricting via (3.2) the direct image under $F_{X/S}$ of the $D_{X/S}$-module structure $D_{X/S} \to \mathscr{E}nd_{\mathscr{O}_X}(E)$, we obtain the map

$$\Theta_{X'/S} \xrightarrow{(3.2)} (F_{X/S})_* D_{X/S} \longrightarrow (F_{X/S})_* \mathscr{E}nd_{\mathscr{O}_X}(E),$$

which is exact the p-curvature map (5.3); or similarly, by restricting $D_{X/S} \to \mathscr{E}nd_{\mathcal{O}_X}(E)$ via (3.4), we get the (5.1). Therefore, we may also take this as the definition for the p-curvature of (E, ∇). Actually, the computations above in this section are exactly the same as those in §3. Besides, the p-curvature has another Crystalline interpretation according to S. Mochizuki, see for example [GLQ10, §3] and [OV07, Prop. 1.6 & 1.7]. But in this article, we do not need

[9] In contrast, if we denote by $\mathbb{D}_{X/S}$ the sheaf of rings of differential operators in the sense of Grothendieck [EGA IV$_4$, §16], then every $\mathbb{D}_{X/S}$-module that is coherent as $\mathcal{O}_X$-module is automatically locally free and has a Frobenius structure. See [Kat70].

that viewpoint.

Moreover, that Higgs fields are equivalent to $\mathrm{Sym}^\bullet \Omega^\vee_{X/S}$-modues and that flat connections are equivalent to $D_{X/S}$-modules, can be derived form a unified argument stating that that an L-module is the same as an $U(L)$-module, where L is a Lie algebroid,[10] and $U(L)$ is its universal enveloping algebroid, applying to the case $L = \mathscr{D}er(X/S)$ with the zero bracket and the standard bracket respectively. From this viewpoint, we may even study the flat λ-connections that were introduced by Deligne and developed in [Sim94; Sim97].

5.2 Cartier descent and the Cartier operator

We recall the classical Cartier descent and the Cartier operator.

5.2.1 Theorem Let X/S be a smooth of characteristic $p > 0$. Denote by $\mathbf{MIC}(X/S)_{\psi=0}$ the category consisting of objects quasi-coherent $\mathscr{O}_X$-modules with flat connections ∇ whose p-curvature vanish, and arrows flat morphisms. Then the assignment

$$\begin{aligned} \mathbf{MIC}(X/S)_{\psi=0} &\longrightarrow \mathbf{QCoh}(X') \\ (E,\nabla) &\longmapsto (F_{X/S})_*(E^\nabla) \\ (F^*_{X/S}E', \nabla^{\mathrm{can}}) &\longleftarrow\!\shortmid\; E' \end{aligned}$$

is an equivalence of categories.

Proof The standard reference is in [Kat70, §5.1]. A proof using Azumaya property of the sheaf of crystalline differential operators can be found in in [Gro16, Thm. 3.11] and [OV07, Rmk. 2.2].

5.2.2 It is easy to see that the complex $F_{X/S,*}\Omega^\bullet_{X/X}$ is a complex of $\mathbb{O}_{X'}$-modules. So we can form the sheaf $\mathscr{H}^i(F_*\Omega^\bullet_{X/S})$ of cohomology groups of this complex.

5.2.3 Theorem ([Kat70, Theorem 7.2]) There is a unique isomorphism of $\mathbb{O}_{X'}$-modules (called the *inverse Cartier operator*)

$$C^{-1}_{X/S} : \Omega^i_{X'/S} \longrightarrow \mathscr{H}^i(F_{X/S,*}\Omega^\bullet_{X/S})$$

such that for local sections f of $\mathscr{O}'_X$ and $\omega\tau$ of $\Omega^i_{X'/S}$.

- $C^{-1}_{X/S}(1) = 1$,

[10]By a *Lie algebroid*, we mean a left $\mathscr{O}_X$-module L that is an sheaf of Lie algbras over $f^{-1}\mathscr{O}_S$, together with an $\mathscr{O}_X$-module morphism $L \to \mathscr{D}er(X/S)$ which is also a homomorphism of Lie algebras over $f^{-1}\mathscr{O}_S$ such that $[X, fY] = f[X,Y] + \rho(X)(f)Y$, for all local sections X, Y of L and f of $\mathscr{O}_X$. It is also called a sheaf of Rinehart-Lie algebra, see [Rin63] and [Lan14].

- $C^{-1}_{X/S}(\omega \wedge \tau) = C^{-1}_{X/S}(\omega) \wedge C^{-1}_{X/S}(\tau)$, and
- $C^{-1}_{X/S}(\mathrm{d}(W^*(f))) = [f^{p-1}\mathrm{d}f] \in \mathscr{H}^1(F_{X/S,*}\Omega^\bullet_{X/S})$,

whose inverse, pre-composed with the quotient $F_{X/S,*}Z\Omega^i_{X/S} \to \mathscr{H}^i(F_{X/S,*}\Omega^\bullet_{X/S})$, is called the *Cartier operator*.

5.2.4 Example (Flat connections on $\mathscr{O}_X$ and their p-curvatures) It is easy to see that every flat S-connection on $\mathscr{O}_X$ has the form $\mathrm{d}+\omega$, where $\mathrm{d} : \mathscr{O}_X \to \Omega^1_{X/S}$ is the universal derivation and $\omega \in \Gamma(X, Z\Omega^1_{X/S})) = \Gamma(X', F_{X/S,*}(Z\Omega^1_{X/S}))$ is a *closed* one form, given by $\omega = \nabla(1)$. The p-curvature, $\psi_\omega : \mathscr{O}_X \to F^*_{X/S}\Omega^1_{X'/S}$ of the flat connection $(\mathscr{O}_X, \mathrm{d}+\omega)$, identified as a section of $F^*_{X/S}\Omega^1_{X'/S}$, is given by (see [Kat72, Proposition (7.1.2)], [Car58, 201, Lemma 4])

$$F^*_{X/S}\left((w^* - C_{X/S})(\omega)\right) \in \Gamma(X, F^*_{X/S}\Omega^1_{X'/S}).$$

Actually, there is an exact sequence of sheaves of abelian groups on $X'_{\text{ét}}$ ([Mil80, Proposition III.4.14]),

$$0 \longrightarrow \mathbb{O}^*_{X'} \xrightarrow{F^*_{X/S}} F_{X/S,*}\mathbb{O}^*_X \xrightarrow{\mathrm{d}\log} F_{X/S,*}(Z\Omega^1_{X/S}) \xrightarrow{w^*-C_{X/S}} \Omega^1_{X'/S} \longrightarrow 0,$$

where $w : X' := X \times_{X,\mathrm{Fr}_S} S \to X$ is the projection and $C_{X/S}$ is the Cartier operator in Theorem 5.2.3.

5.2.5 Remark [CZ15, A.7] gave a generalisation for the above example.

5.2.6 Example Continue with the previous example and assume that $S = \operatorname{Spec} k$ is the spectrum of an algebraically closed field of characteristic $p > 0$, and that X/k is an abelian variety. In this case, $w : X' \to X$ is an isomorphism, and every global 1-from is closed. So the assignment to flat connection on $\mathscr{O}_X$ to its p-curvature reduces to the map[11]

$$\mathrm{id} - (w^{-1})^* C_{X/k} : \Gamma\left(X, \Omega^1_{X/k}\right) \to \Gamma\left(X, \Omega^1_{X/k}\right).$$

It is then a classical result on p^{-1}-linear maps, see e.g., [Cha98, Exposé III, n°3, Lemma 3.3] and [SGA 7_{II}, Exposé XXII, n°1, Proposition 1.2], that this map is surjective. Note there that the map $(w^{-1})^*C_{X/k}$ is p^{-1}-linear, where in the mentioned references, p-linear maps are discussed; however, the proof in [Cha98] runs verbatim for p^{-1}-linear maps, as we assumed that k is algebraically closed.

[11] The map $(w^{-1})^*C_{X/k}$ is the original Cartier operator considered by Cartier in [Car58, §2.6], see [Kat72, Remark 7.1.4].

5.3 Local systems

In this article, a *local system of rank r* on X/S is just a synonym of an object (E, ∇) in $\mathbf{MIC}(X/S)$ with E a locally free $\mathscr{O}_X$-module E of rank r, or equivalently, an $D_{X/S}$-module that is locally free as $\mathscr{O}_X$-module of rank r. For simplicity, we usually say E is a local system without mentioning ∇.

Let $\mathbf{LocSys}_{X/S,r} := \mathbf{LocSys}_{X/S} := \mathbf{LocSys}_r \to (\mathbf{Sch}/S)$ be the stack of local systems of rank r. More precisely, $\mathbf{LocSys}_{X/S}$, as a fibred category over $(\mathbf{Sch}/S)$, has objects $(T, (E, \nabla))$, where T is an S-scheme and E is a locally free sheaf of rank r on X_T and ∇ is a flat T-connection. An arrow from $(T', (E', \nabla'))$ to $(T, (E, \nabla))$ in $\mathbf{LocSys}_{X/S}$ consists of an S-morphism $u : T' \to T$ and a flat morphism $(u^*E, u^*\nabla) \to (E', \nabla')$.

5.4 The Hitchin map for local systems

5.4.1 Proposition ([LP01, Prop. 3.2], [Gro16, Def. 3.16], and [CZ15, Prop. 3.1]) There is a map

$$c_{\mathrm{dR}} : \mathbf{LocSys}_{X/S,r} \to \mathbf{B}_{X'/S},$$

rendering the diagram

$$\begin{tikzcd}
 & \mathbf{Higgs}_{X'/S,p^d r} \arrow[rr, "(4.9)"] & & \mathbf{B}_{X'/S,p^d r} \arrow[dr, "F^*_{X/S}"] & \\
\mathbf{LocSys}_{X/S,r} \arrow[ur, "(5.4)"] \arrow[dr, "(5.2)"'] \arrow[rr, dotted, "\exists\, c_{\mathrm{dR}}"] & & \mathbf{B}_{X'/S,r} \arrow[ur, "(-)^{p^d}"'] \arrow[dr, "F^*_{X/S}"] & & \mathbf{B}_{X/S,p^d r;F^*_{X/S}\Omega^1_{X'/S}} \\
 & F\text{-}\mathbf{Higgs}_{X/S,r} \arrow[rr, "(4.12)"] & & \mathbf{B}_{X/S,r;F^*_{X/S}\Omega^1_{X'/S}} \arrow[ur, "(-)^{p^d}"] &
\end{tikzcd}$$

commutative.

Proof Given a local system (E, ∇) of rank r on X/S, we know its p-curvature $\psi : E \to E \otimes F^*_{X/S}\Omega^1_{X/S}$ is an F-Higgs bundle of rank r (resp. $\psi' : F_{X/S,*}E \to F_{X/S,*}E \otimes \Omega^1_{X'/S}$ is a Higgs bundle of rank $p^d r$). Denote by χ and χ' the characteristic polynomials of ψ and ψ' respectively. Then we have

$$F^*_{X/S}\chi' = \chi^{p^d} \in \mathbf{B}_{X/S,p^d r;F^*_{X/S}\Omega^1_{X'/S}}(S),$$

i.e., the outer hexagon of the diagram commutes. In fact, the characteristic polynomial χ is that of the $(F^*_{X/S}\operatorname{Sym}^\bullet \Omega^1_{X'/S})$-module $E \otimes F^*_{X/S}\operatorname{Sym}^\bullet \Omega^1_{X'/S}$. Meanwhile, the characteristic polynomial χ' is that of the $(\operatorname{Sym}\Omega^1_{X'/S})$-module $F_{X/S,*}E \otimes \operatorname{Sym}^\bullet \Omega^1_{X'/S} \simeq F_{X/S,*}(E \otimes F^*_{X/S}\operatorname{Sym}^\bullet \Omega^1_{X'/S})$ (using projection formula for each degree then taking direct sum), via the natural map

$$\mathrm{Sym}^{\bullet}\,\Omega^1_{X'/S} \longrightarrow F_{X/S,*}F^*_{X/S}\,\mathrm{Sym}^{\bullet}\,\Omega^*_{X'/S}.$$

Then the equality follows from Lemma 2.1.1 and Lemma 2.1.2 and the fact that Nm respects arbitrary base change ([SP, Tag 0BD2]).

On the other hand, ψ is a *flat* morphism with respect to the connection ∇ on E, the canonical connection ∇^{can} on $F^*_{X/S}\Omega^1_{X/S}$ and the tensor product connections they induced. Then according to Theorem 5.2.1, we have

$$\chi = F^*_{X/S}(\chi'') \in \mathbf{B}_{X/S,r;F^*_{X/S}\Omega^1_{X'/S}}(S),$$

for some $\chi'' \in \mathbf{B}_{X'/S,r}(S)$. See [CZ15, Prop. 3.1] and [LP01, Prop. 3.2] (for $\dim(X/S) = 1$ and $S = \operatorname{Spec} k$, due to J.-B. Bost) for details. It follows that

$$F^*_{X/S}\chi' = \chi^{p^d} = (F^*_{X/S}(\chi''))^d = F^*_{X/S}((\chi'')^d).$$

Moreover, we know the relative Frobenius $F_{X/S}$ is faithfully flat, hence the pullback map $F^*_{X/S}$ on sections is injective. So we can conclude that

$$\chi' = (\chi'')^{p^d} \in \mathbf{B}_{X'/S,p^d r}(S). \tag{5.5}$$

The above argument works functorially, i.e., it works for any local system (E,∇) over X_T/T for any T/S. The map c_{dR} is then defined by sending each (E,∇) to χ'', and the commutativity of the diagram are just the above equalities.

5.5 An equivalence (the BNR correspondence)

Fix a $\chi'' \in \mathbf{B}_{X'/S,r}(S)$. Let (E,∇) be a rank r local system on X, regarded as a $D_{X/S}$-module. Recall §5.1.2 that $F_{X/S,*}E$ is then an $(F_{X/S,*}D_{X/S})$-module, and this action restricted via the natural map $\Theta_{X'/S} \to F_{X/S,*}D_{X/S}$ in (3.2) gives the (the adjoint map of) the p-curvature map $\psi' : F_{X/S,*}E \to F_{X/S,*}E \otimes \Omega^1_{X'S}$, which is a Higgs field. As before, this ψ' realizes the $\mathcal{O}_{\mathrm{T}^*(X/S)}$-module $\widetilde{F_{X/S,*}E}$ as an $\mathscr{D}_{X/S}$-module over $\mathrm{T}^*(X'/S)$. Moreover, as in §4.3, it follows from Proposition 2.3.1 that $\widetilde{F_{X/S,*}E}$ is supported on the closed subscheme $Z_{\chi'} \subseteq \mathrm{T}^*(X'/S)$ defined by the ideal $I_{\chi'} \subseteq \mathcal{O}_{\mathrm{T}^*(X/S)}$, where $\chi' \in \mathbf{B}_{X'/S,p^d r}(S)$ is the characteristic polynomial of ψ'. We know from Proposition 5.4.1 that χ' is a p^d-th power. Assume that $\chi' = (\chi'')^{p^d}$ for the fixed χ'', i.e., (E,∇) is an object of $c^{-1}_{\mathrm{dR}}(\chi'')$. Note that

$$I_{\chi'} = I_{(\chi'')^{p^d}} \subseteq (I_{\chi''})^{p^d} \subseteq I_{\chi''}.$$

So $I_{\chi'} \cdot \widetilde{F_{X/S,*}E} = 0$ implies that $I_{\chi''} \cdot \widetilde{F_{X/S,*}E} = 0$. Therefore the $\mathscr{D}_{X/S}$-module $\widetilde{F_{X/S,*}E}$ is supported on $Z_{\chi''} \subseteq Z_{\chi'}$.

Conversely, suppose we have a $\iota^*\mathscr{D}_{X/S}$-module M on $Z_{\chi''}$, such that the $\mathcal{O}_X$-module $\widetilde{\pi_*\iota_*M}$ is locally free of rank r, where $\iota : Z_{\chi''} \longrightarrow \mathrm{T}^*(X'/S)$ is the inclusion. Then $\pi_*\iota_*M$ is also an $(F_{X/S})_*D_{X/S}$-module (via the natural map $(F_{X/S})_*D_{X/S} = \pi_*\mathscr{D}_{X/S} \longrightarrow \pi_*\iota_*\iota^*\mathscr{D}_{X/S}$). Therefore, $\widetilde{\pi_*\iota_*M}$ is a quasi-coherent $\mathcal{O}_X$-module as well as an $\widetilde{(F_{X/S})_*D_{X/S}} = D_{X/S}$-module. Note that our assumption implies that the $\mathcal{O}_{X'}$-module $\pi_*\iota_*M = F_{X/S,*}\left(\widetilde{\pi_*\iota_*M}\right)$ is locally free of rank $p^d r$.

Of course the above arguments work functorially for any $\chi'' \in \mathbf{B}_{X'/S,r}(T)$ for any T/S. So similar to the equivalence in Proposition 4.3.2, we obtain the following result.

5.5.1 Proposition (Groechenig) Given any $\chi \in \mathbf{B}_{X'/S,r}(T)$, with $\iota : Z_\chi \hookrightarrow \mathrm{T}^*(X'_T/T)$ being the corresponding closed embedding. Then we have an equivalence of categories between

(LA) the fully faithful subcategory $c_{\mathrm{dR}}^{-1}(\chi)$ of $\mathbf{LocSys}_{X/S,r}$, and

(LB) the fully faithful subcategory of $\mathbf{QCoh}(Z_\chi, \iota^*\mathscr{D}_{X_T/T})$ consisting of objects M such that

- the induced $\mathcal{O}_{X_T}$-module $\widetilde{\pi_*\iota_*M}$ is locally free of rank r, and
- the induced Higgs filed on $\pi_*\iota_*M$ has characteristic polynomial χ^{p^n}.

5.5.2 Remark In [Gro16, Prop. 3.15] and [EG17, Thm. 2.4], they use the Morita equivalence Proposition 2.2.1 and the equivalence 4.3.2 to show the existence of χ'' satisfying (5.5) and established the equivalence Proposition 5.5.1.

6 An equivalence (the splitting principle)

In this section, we recall the *Splitting Principle* from [Gro16, Lem. 3.27].

Again fix a $\chi \in \mathbf{B}_{X'/S}(S)$, we have the corresponding closed subscheme Z_χ of $\mathrm{T}^*(X'/S)$. Denote by $\pi : \mathrm{T}^*(X'/S) \to X'$ the projection and $\iota : Z_\chi \hookrightarrow \mathrm{T}^*(X'/S)$ the inclusion. Suppose we have a splitting

$$\iota^* \mathscr{D}_{X/S} \simeq \mathscr{E}nd_{\mathscr{O}_{Z_\chi}}(P).$$

Then, according to Proposition 2.2.1, we have an equivalence of categories

$$\begin{aligned} \mathbf{QCoh}(Z_\chi) &\xrightarrow{\sim} \mathbf{QCoh}(Z_\chi, \iota^* \mathscr{D}_{X/S}) \\ M &\longmapsto P \otimes_{\mathscr{O}_{Z_\chi}} M. \end{aligned}$$

We introduce some temporary notations as in the following Cartesian diagram

$$\begin{tikzcd} W_\chi := Z_\chi \times_{X'} X \arrow[r, "\varphi"] \arrow[d, "\tau"] & Z_\chi \arrow[d, "\pi\circ\iota"] \\ X \arrow[r, "F_{X/S}"] & X'. \end{tikzcd}$$

and fix an object M in $\mathbf{QCoh}(Z_\chi)$.

Recall Theorem 3.1.1.4 that the $\mathscr{O}_{W_\chi}$-module $\widetilde{P} =: L$ is locally free of rank 1. Then we have canonical isomorphisms of $\mathscr{O}_X$-modules

$$\begin{aligned} \widetilde{(\pi \circ \iota)_*(P \otimes_{\mathscr{O}_{Z_\chi}} M)} &\simeq \tau_*(\widetilde{P \otimes_{\mathscr{O}_{Z_\chi}} M}) \\ &\simeq \tau_*(L \otimes_{\mathscr{O}_{W_\chi}} \varphi^* M) \qquad\qquad ([\text{EGA I}_{\text{n}}, \text{Corollaire 9.3.9}]) \\ &\simeq \tau_* L \otimes_{\tau_* \mathscr{O}_{W_\chi}} \tau_* \varphi^* M \\ &\simeq \tau_* L \otimes_{\tau_* \mathscr{O}_{W_\chi}} F^*_{X/S} (\pi \circ \iota)_* M. \end{aligned}$$

Note that τ is finite because $(\tau \circ \iota)$ is (Proposition 4.2.8). Then, according to (2.7) in §2.5, $\tau_* L$ is an $\tau_* \mathscr{O}_{W_\chi}$-module locally free of rank 1 over X. Therefore, Zariski locally on X, $\widetilde{(\pi \circ \iota)_*(P \otimes_{\mathscr{O}_{Z_\chi}} M)}$ is isomorphic to $F^*_{X/S}(\pi \circ \iota)_* M$. Moreover, $F_{X/S}$ is faithfully flat and locally of finite presentation, hence $(\pi\circ\iota)_*\widetilde{(P\otimes_{\mathcal{O}_{Z_\chi}} M)}$ is locally free of rank r if and only if $(\pi\circ\iota)_*M$ is locally free of rank r.

Due to the same reason, the canonical isomorphism of $\mathcal{O}_{X'}$-modules

$$(\pi\circ\iota)_*\left(P\otimes_{\mathcal{O}_{Z_\chi}} M\right)\simeq(\pi\circ\iota)_*P\otimes_{(\pi\circ\iota)_*\mathcal{O}_{Z_\chi}}(\pi\circ\iota)_*M$$

implies that Zariski locally over X', $(\pi\circ\iota)_*\left(P\otimes_{\mathcal{O}_{Z_\chi}} M\right)$ is isomorphic to a direct sum of $p^d=\mathrm{rk}_{\mathcal{O}_{Z_\chi}} P$ copies of $(\pi\circ\iota)_*M$. Therefore, $(\pi\circ\iota)_*M$ has characteristic polynomial χ if and only if $(\pi\circ\iota)_*\left(P\otimes_{\mathcal{O}_{Z_\chi}} M\right)$ has characteristic polynomial χ^{p^d}.

The above arguments are clearly functorial and work for any $\chi\in B'(T)$. So these arguments together with the Morita equivalence Proposition 2.2.1, and the two equivalences we proved in the previous sections, fit into depicted equivalences

$$\begin{tikzcd}
\boxed{\text{(HB)}} \arrow[r, leftrightarrow, "\S 2.2"] \arrow[d, leftrightarrow, "\S 4.3.2"'] & \boxed{\text{(LB)}} \arrow[d, leftrightarrow, "\S 5.5.1"] \\
\boxed{\text{(HA)}} & \boxed{\text{(LA)}}
\end{tikzcd}$$

So our goal is to find splittings of the Azumaya algebra.

7 Existence of Splittings, for abelian varieties

From now on, assume that $X/S := X/\operatorname{Spec} k := X/k$ is an abelian variety over an *algebraically closed* field k of characteristic $p > 0$. Denote by $e : \operatorname{Spec} k \longrightarrow X$ the zero section. Then X'/k is again an abelian variety. Recall Example 4.2.4 that, the Hitchin base $\mathbf{B}' := \mathbf{B}'_r := \mathbf{B}_{X'/k,r}$ is representable by a k-scheme $B' := B'_r := B_{X'/k,r}$. Moreover, recall Example 4.2.6 that, we have a universal spectral cover $Z/X'_{B'}$, and a larger scheme $\widetilde{Z} \supseteq Z$. In case of $r = 1$, $\widetilde{Z} = Z$.

7.1 Rank one case

This result is due to Roman Bezrukavnikov, see [OV07, Thm. 4.14] and [CZ17, Appendices. B and C]. We reproduce it as follows.

Noting that when $r = 1$, we know that the Hitchin base $B' := B'_1 = \mathbf{V}(\Gamma(X', \Omega^1_{X'/k}))$ is the k-vector space of global sections of $\Omega^1_{X'/k}$. Hence a T-point of B' is the same as a one-form ω on X'_T. Any such ω determines a closed subscheme Z_ω of the cotangent bundle of X'_T/T. In fact, recall Example 4.2.1 that, the inclusion of Z_ω into the cotangent bundle is the same as a section of the projection from the cotangent bundle to X'_T. In particular, we may identify the inclusion of the universal spectral cover Z into $\mathrm{T}^*(X'_{B'}/B')$ as the section $X'_{B'} \longrightarrow \mathrm{T}^*(X'_{B'}/B')$ determined by the 1-form $\chi_{\mathrm{univ}} \in \Gamma(X'_{B'}, \Omega^1_{X'\times B'/B'})$. Meanwhile, in this case, $\mathbf{LocSys}_{X/k,1}$ is usually denoted by $\mathbf{Pic}^\natural_{X/k}$. That is, for any k-scheme T, $\mathbf{Pic}^\natural_{X/k}(T)$ is the groupoïd of invertible sheaves with flat T-connections on X_T, i.e, of $D_{X_T/T}$-modules that are invertible as $\mathscr{O}_{X_T}$-modules. Let $\mathbf{Pic}_{X/k}$ be the (relative) *Picard stack*[12] of invertible sheaves, i.e., $\mathbf{Pic}_{X/k}(T)$ is the groupoïd of invertible sheaves on X_T for any T/k. In other words, $\mathbf{Pic}_{X/k} = f_*(\mathrm{B}\mathbb{G}_{\mathrm{m},X}) = \mathrm{Res}_{X/k}(\mathrm{B}\mathbb{G}_{\mathrm{m},X})$ (see [SGA 4_{III}, Exposé XVIII, §§1.4.21 and 1.5.1]).

For any scheme T/k, a *rigidified invertible sheaf* (L, α) consists of an invertible sheaf L on X_T together with an isomorphism $\alpha : e_T^*L \simeq \mathscr{O}_T$, where e_T is the pullback of the unit section e. A flat connection on (L, α) is just a flat T-connection on L. Then there are *k-group*

[12] Picard stack is sometimes a confusing name: it can refer to a *(strictly) commutative group stack*, see [SGA 4_{III}, Exposé XVIII, §1.4].

schemes $\mathrm{Pic}_{X/k,e}$ and $\mathrm{Pic}^{\natural}_{X/k,e}$ over k satisfying that for any T/k,

$$\begin{aligned}\mathrm{Pic}_{X/k,e}(T) &= \{\text{isom. classes of rigidified invertible sheaves } (L,\alpha) \text{ over } X_T\},\\ \mathrm{Pic}^{\natural}_{X/k,e}(T) &= \{\text{isom. classes of rigidified invertible sheaves}\\ &\qquad\qquad \text{with a flat connection } (L,\alpha,\nabla) \text{ over } X_T\}.\end{aligned}$$

The existence of the scheme $\mathrm{Pic}_{X/k,e}$ (i.e., the representability of the associated fppf-sheaf of the functor as above) is the classical theory on Picard functors, see [Kle05] for a detailed exposition; while the existence of the scheme $\mathrm{Pic}^{\natural}_{X/k,e}$ is discussed in [MM74] and see also [OV07, Proposition 4.11] and [BK09, Appendix B]. Moreover, $\mathbf{Pic}_{X/k} = \mathrm{Pic}_{X/k} \times_k \mathbf{B}\mathbb{G}_{\mathrm{m}}$, and $\mathbf{Pic}^{\natural}_{X/k} = \mathrm{Pic}^{\natural}_{X/k} \times_k \mathbf{B}\mathbb{G}_{\mathrm{m}}$. Clearly there are natural maps $\mathbf{Pic}^{\natural}_{X/k} \to \mathbf{Pic}_{X/k}$ and $\mathrm{Pic}^{\natural}_{X/S,e} \to \mathrm{Pic}_{X/k,e}$, compatible with the projections.

7.1.1 Proposition The map

$$c_{\mathrm{dR}} : \mathbf{Pic}^{\natural}_{X/k} \longrightarrow B'_1$$

assigning to each invertible sheaf with a flat connection its p-curvature as defined in Proposition 5.4.1 induces a k-group scheme homomorphism,

$$c_{\mathrm{dR}} : \mathrm{Pic}^{\natural}_{X/k,e} \longrightarrow B'_1.$$

Moreover, $\mathscr{D}_{X/k}$ splits canonically over $Z_{c_{\mathrm{dR}}} \subseteq \mathrm{T}^*(X' \times \mathrm{Pic}^{\natural}/\mathrm{Pic}^{\natural})$. In other words, once pulled back along the composition of natural maps (with the identification in Example 4.2.1)

$$Z_{c_{\mathrm{dR}}} \simeq X' \times_k \mathrm{Pic}^{\natural}_{X/k,e} \xrightarrow{\mathrm{id}\times c_{\mathrm{dR}}} X' \times_k B' \simeq Z \longleftrightarrow \mathrm{T}^*(X' \times B'/B') \longrightarrow \mathrm{T}^*(X'/k),$$

the Azumaya algebra $\mathscr{D}_{X/k}$ splits.

Proof This scheme version is given [OV07, §4.3] and a stack version is given in [CZ17, Proposition B.3.4]. Actually, this follows directly from the fact that if A s an Azumaya algebra of rank r^2 as an $\mathscr{O}_X$-module, and if M is an A-module locally free of rank r as an $\mathscr{O}_X$-module, then M is a splitting module of A.

7.1.2 Corollary There is an equivalence $\mathbf{Pic}^{\natural}_{X/k} \simeq \mathrm{Res}_{Z/B'}(\mathbf{S}_{\mathscr{D}}|_Z)$ of stacks over B'. And in particular, $\mathrm{Res}_{Z/B'}(\mathbf{S}_{\mathscr{D}}|_Z)$ is algebraic.

Proof This is just a stack version [OV07, Proposition 4.13, 1)], which proved a *rigiedfied* version of this proposition. Recall that, for any $\omega \in B'(T)$, $\mathbf{Pic}^{\natural}_{X/k}(T)$ is the category of invertible sheaves on X_T with a flat T-connection, and that $\mathrm{Res}_{Z/B'}(\mathbf{S}_{\mathscr{D}}|_Z)(T)$ is the category of splittings of the pull back of $\mathscr{D}_{X/k}$ to $Z_\omega \subseteq \mathrm{T}^*(X' \times T/T)$. Write $\iota : Z_\omega \hookrightarrow \mathrm{T}^*(X' \times T/T)$ and $\pi_T : \mathrm{T}^*(X' \times T/T) \to X'_T$ for the inclusion and the projection.

In this case, we have (Example 4.2.1) that $\pi_T \circ \iota$ is an isomorphism. Let M be $\mathscr{O}_{Z_\omega}$, which defines a rank 1 Higgs bundle on X'. Then according to the arguments in the splitting principle in §6, we know that

$$\begin{aligned} \mathrm{Res}_{Z/B'}(\mathbf{S}_{\mathscr{D}}|_Z)(T) &\longrightarrow \mathbf{Pic}^{\natural}_{X/k}(T) = \mathbf{LocSys}_{X/k,1}(T) \\ P &\longmapsto \widetilde{(\pi_T \circ \iota)_*(P)} \end{aligned}$$

is an equivalence.

Algebraicity of $\mathrm{Res}_{Z/B'}(\mathbf{S}_{\mathscr{D}}|Z)$ follows from that $\mathbf{Pic}^{\natural}_{X/k}$ is algebraic. This also follows from [Ols06, Thm. 1.5], because of the fact that Z/B' is proper and flat.

7.1.3 Remark In [CZ17], It is further proved that $X^\natural \simeq \mathscr{T}_{\mathscr{D}}$, where $X^\natural = \mathrm{Pic}^{\natural}_{X/k} \times_{\mathrm{Pic}_{X/k}} X^\vee$ is the *universal (vector) extension* of X [MM74], and $\mathscr{T}_{\mathscr{D}}$ is the stack of *tensor splittings*, or *multiplicative splittings* in the terminology of [OV07].

7.1.4 Proposition For abelian varieties, the morphism tangent map $c_{\mathrm{dR}} : \mathbf{Pic}^{\natural}_{X/k} \to B'_1$ is *smooth* and *surjective*. In particular, it is formally smooth.

Proof Smoothness follows from the fact that $\mathrm{d}c_{\mathrm{dR}} : \mathbf{V}(\mathrm{H}^1_{\mathrm{dR}}(X/k)) \to \mathbf{V}(\Gamma(X', \Omega^1_{X'/k}))$ is surjective (see [OV07, Thm. 4.14]). Surjectivity is a consequence of Example 5.2.6.

7.1.5 Corollary (Bezrukavnikov) The Azumaya algebra $\mathscr{D}_{X/S}$ splits over the *formal neighbourhood* of each 1-form (cf. [OV07, Thm 4.14]).

Proof Recall Example 4.2.1 that $\mathscr{D}_{X/S}$ splits over the graph of 1-forms. Using the formal smoothness of c_{dR} in Proposition 7.1.4, as well as the equivalence in Corollary 7.1.2, we can conclude that the splitting lifts to the formal neighbourhood.

7.2 Higher rank case

Now we deal with the higher rank case. To this aim, we will mainly use the computations that we have done in Examples 4.2.3, 4.2.4 and 4.2.7. In particular, recall (4.7), for any $\chi \in B'_r(k)$, we know that Z_χ is the closed subscheme of the cotangent bundle of X'_T/T, cut out by the $S(d,r)$ equations $g_1, \dots, g_d, \dots$ appeared as coefficients of $(\pi^*\underline{\omega}^{\underline{i}})$ in (4.5). And we defined a larger closed subscheme (4.8) that is cut out by d polynomials $g_1, \dots, g_d$, each of which is a polynomial in only one variable, that has coefficients in $\Gamma(X', \mathscr{O}'_X) = k$. Let $Z/X'_{B'}$ and $\widetilde{Z}/X'_{B'}$ be the universal families as defined in §4.2.6.

7.2.1 Corollary For any $\chi \in B'_r(k)$, $\mathscr{D}_{X/S}$ splits over the formal neighbourhood of $\widetilde{Z}_\chi$ (hence over that of Z_χ if $Z_\chi \neq \emptyset$).

Proof Recall that $\widetilde{Z}_\chi$ is cut out by d polynomials $g_i \in k[\partial_i]$, $1 \leq i \leq d$. Since k is *algebraically closed* by assumption, each g_i factors as a product $\prod_{m=1}^r (\partial_i - c_{i,m})$, $c_{i,m} \in k$. So we can conclude that $\widetilde{Z}_\chi$ is a union of (possibly non-reduced) closed subschemes of the j-th infinitesimal neighbourhood of graphs of some 1-forms, $j \leq r$. Hence, according to Corollary 7.1.5, the Azumaya algebra $\mathscr{D}_{X/k}$ splits over the formal neighbourhood of $\widetilde{Z}_\chi$, *a fortiori*, over that of Z_χ.

7.2.2 Proposition Let $\mathbf{S} := \mathrm{Res}_{\widetilde{Z}/B'}(\mathbf{S}_{\mathscr{D}}|_{\widetilde{Z}}) \to B'$ be the stack of splittings $\mathscr{D}_{X/k}$ relative to $\widetilde{Z}/B'$. Then the stack $\mathbf{S}/B'$ is algebraic, and it is smooth and surjective over B'. Moreover, $\mathbf{S}$ is a $\mathbf{Pic}_{\widetilde{Z}/B'}$-torsor.

Proof Recall Remark 4.2.9 that $\widetilde{Z}$ is *proper* and *flat* over B'. So according to [Ols06, Thm. 1.5], the Weil restriction $\mathbf{S} := \mathrm{Res}_{\widetilde{Z}/B'}(\mathbf{S}_{\mathscr{D}}|_{\widetilde{Z}})$ is *algebraic* and *locally of finite presentation*. Hence, according to [SP, Tag 0DP0], to show that $\mathbf{S}/B'$ is smooth it suffices to show that $\mathbf{S}/B'$ is formally smooth. Note that B' is (locally) noetherian, $\mathbf{S}/B'$ is locally of finite type, and k is algebraically closed, then according to [SP, Tag 02HY], it suffices to show that the Azumaya algebra $\mathscr{D}_{X/k}$ splits over the formal neighbourhood Z_χ for all $\chi \in B'(k)$, which is exactly Corollary 7.2.1. The surjectivity also follows. Therefore, étale locally on B', $\mathbf{S}$ admits a section, or more precisely, there is an étale surjective morphism $U \to B'$, such that $\mathbf{S}(U)$ is non-empty. Since $\mathbf{S}_{\mathscr{D}}|_{\widetilde{Z}}$ is a $\mathbb{G}_{\mathrm{m},\widetilde{Z}}$-gerbe, i.e., a $\mathrm{B}\mathbb{G}_{\mathrm{m},\widetilde{Z}}$-torsor[13], so $\mathbf{S} := \mathrm{Res}_{\widetilde{Z}/B'}(\mathbf{S}_{\mathscr{D}}|_{\widetilde{Z}})$ is a pseudo $\mathrm{Res}_{\widetilde{Z}/B'}(\mathrm{B}\mathbb{G}_{\mathrm{m},\widetilde{Z}})$-torsor, i.e., a pseudo $\mathbf{Pic}_{\widetilde{Z}/B'}$-torsor. The existence of an étale local section implies that it is actually a torsor. Hence $\mathbf{S}$ is an $\mathbf{Pic}_{\widetilde{Z}/B'}$-torsor.

7.3 Main result

Recall Proposition 7.2.2 we have an $\mathbf{Pic}_{\widetilde{Z}/B'}$-torsor $\mathbf{S} := \mathrm{Res}_{\widetilde{Z}/B'}(\mathbf{S}_{\mathscr{D}}|_{\widetilde{Z}})$. Note moreover that, via the identifications in Proposition 4.3.2 and Proposition 5.5.1, tensor products define actions

$$\mathbf{Pic}_{\widetilde{Z}/B'} \times_{B'} \mathbf{Higgs}_{X'/k,r} \longrightarrow \mathbf{Higgs}_{X'/k,r}, \quad \text{and} \quad \mathbf{Pic}_{\widetilde{Z}/B'} \times_{B'} \mathbf{LocSys}_{X/k,r} \longrightarrow \mathbf{LocSys}_{X/k,r}$$

of $\mathbf{Pic}_{\widetilde{Z}/B'}$ on $\mathbf{Higgs}_{X'/k,r}$ and $\mathbf{LocSys}_{X/k,r}$ respectively over B'. Verifications of such actions are well defined, in particular on $\mathbf{LocSys}_{X/k,r}$, are similar to the arguments in §6 (cf. [CZ15, Proposition 3.5]). The formulation of the following theorem is inspired by that of [CZ15, Theorem 1.2, Remark 3.13].

[13] For the equivalence of BG-torsors and G-gerbes in case that G is a sheaf of *commutative* groups, see [DP08, Remark 2.2].

7.3.1 Theorem There is a $\mathbf{Pic}_{\widetilde{Z}/B'}$-equivariant isomorphism of stacks

$$C_{X/k}^{-1} : \mathbf{S} \times^{\mathbf{Pic}_{\widetilde{Z}/B'}} \mathbf{Higgs}_{X'/k,r} \longrightarrow \mathbf{LocSys}_{X/k,r}$$

over B'. In particular, there is an étale surjective morphism $U \to \mathbf{B}'$, such that

$$\mathbf{Higgs}_{X'/k,r} \times_{B'} U \simeq \mathbf{LocSys}_{X/k,r} \times_{B'} U.$$

Proof The first statement follows from the splitting principle described in §6 and Corollary 7.2.1. In fact the map is given as follows. For any $\chi \in B'(T)$, denote by

$$\iota : Z_\chi \overset{\gamma}{\hookrightarrow} \widetilde{Z}_\chi \overset{\tilde{\iota}}{\hookrightarrow} \mathrm{T}^*(X'_T/T),$$

the inclusions. For any object (E,θ) in $\mathbf{Higgs}_{X'/k}(T)$, consider via Proposition 4.3.2 the quasi-coherent sheaf $\tilde{E}$ on Z_χ. Any object

$$(\chi : T \to B', P, \alpha : \tilde{\iota}^* \mathscr{D}_{X_T/T} \simeq \mathscr{E}nd_{\mathscr{O}_{\widetilde{Z}_\chi}}(P))$$

in $\mathbf{S}(T)$ defines a splitting module $\gamma^* P$ of $\iota^* \mathscr{D}_{X_T/T}$ on Z_χ. Then the $\mathscr{O}_{X_T}$-module

$$C_{X/k}^{-1}(E) := (\pi_T \circ \iota)_* \widetilde{(\widetilde{E} \otimes_{\mathscr{O}_{Z_\chi}} \gamma^* P)}.$$

with the notation as in §6, is an $D_{X_T/T}$-module, i.e., an object in $\mathbf{LocSys}_{X/S}(T)$. Then clearly the assignment $((\chi, P, \alpha), (E, \theta)) \mapsto C_{X/k}^{-1}(E)$ defines a $\mathbf{Pic}_{\widetilde{Z}/B'}$-equivariant map $\mathbf{S} \times^{\mathbf{Pic}_{\widetilde{Z}/B'}} \mathbf{Higgs}_{X'/k,r} \to \mathbf{LocSys}_{X/k,r}$. This is an isomorphism follows directly from the discussion in §6. The second part follows from Proposition 7.2.2 that there is an étale cover $U \to B'$, such that $\mathbf{S} \times_{B'} U \simeq \mathbf{Pic}_{\widetilde{Z}_\chi/B'} \times_{B'} U$.

Bibliography

[Ber02] Pierre Berthelot. “Introduction à la théorie arithmétique des $\mathscr{D}$-modules.” In: *Cohomologies p-adiques et applications arithmétiques (II)*. Paris: Société Mathématique de France, 2002, pp. 1–80. Zbl: 1098.14010 (cit. on p. 11).

[Ber74] Pierre Berthelot. *Cohomologie cristalline des schemas de caractéristique $p > 0$.* Lecture Notes in Mathematics. 407. Berlin-Heidelberg-New York: Springer-Verlag. 604 p. 1974. Zbl: 0298.14012 (cit. on pp. 11, 26, 27).

[Ber96] Pierre Berthelot. “$\mathscr{D}$-modules arithmétiques. I: Opérateurs différentiels de niveau fini.” In: *Ann. Sci. Éc. Norm. Supér. (4)* 29.2 (1996), pp. 185–272. Zbl: 0886.14004 (cit. on pp. 1, 11).

[BK09] Jean-Benoît Bost and Klaus Künnemann. “Hermitian vector bundles and extension groups on arithmetic schemes. II: The arithmetic Atiyah extension.” In: *From probability to geometry I. Volume in honor of the 60th birthday of Jean-Michel Bismut.* Paris: Société Mathématique de France (SMF), 2009, pp. 361–424. Zbl: 1200.14049 (cit. on p. 36).

[BMR08] Roman Bezrukavnikov, Ivan Mirkovi, and Dmitriy Rumynin. “Localization of modules for a semisimple Lie algebra in prime characteristic (with an appendix by R. Bezrukavnikov and S. Riche, Computation for $\mathfrak{sl}(3)$).” In: *Ann. Math. (2)* 167.3 (2008), pp. 945–991. Zbl: 1220.17009 (cit. on pp. 1, 2, 11–13).

[BNR89] Arnaud Beauville, M. S. Narasimhan, and S. Ramanan. “Spectral curves and the generalised theta divisor”. In: *J. Reine Angew. Math.* 398 (1989), pp. 169–179. MR: 998478 (cit. on p. 23).

[BO78] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Mathematical Notes. Princeton, New Jersey: Princeton University Press. Tokyo: University of Tokyo Press. VI, not consecutively paged. 1978. Zbl: 0383.14010 (cit. on p. 27).

[Bou07] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Reprint of the 1970 original. Berlin: Springer, 2007, pp. xiii + 630. Zbl: 1111.00001 (cit. on p. 5).

[Car58] Pierre Cartier. “Questions de rationalité des diviseurs en géométrie algébrique.” In: *Bull. Soc. Math. Fr.* 86 (1958), pp. 177–251. Zbl: 0091.33501 (cit. on p. 29).

[Cha98] Antoine Chambert-Loir. “Cohomologie cristalline: Un survol.” In: *Expo. Math.* 16.4 (1998), pp. 333–382. Zbl: 0944.14008 (cit. on p. 29).

[CN19] Tsao-Hsien Chen and Ngô Bao Châu. *On the Hitchin morphism for higher dimensional varieties*. 2019. arXiv: 1905.04741 (cit. on p. 22).

[CZ15] Tsao-Hsien Chen and Xinwen Zhu. "Non-abelian Hodge theory for algebraic curves in characteristic p." In: *Geom. Funct. Anal.* 25.6 (2015), pp. 1706–1733. Zbl: 1330.14015 (cit. on pp. 1, 29–31, 38).

[CZ17] Tsao-Hsien Chen and Xinwen Zhu. "Geometric Langlands in prime characteristic." In: *Compos. Math.* 153.2 (2017), pp. 395–452. Zbl: 06782031 (cit. on pp. 14, 35–37).

[DP08] Ron Donagi and Tony Pantev. "Torus fibrations, gerbes, and duality. With an appendix by Dmitry Arinkin." In: *Mem. Am. Math. Soc.* 901 (2008), p. 90. Zbl: 1140.14001 (cit. on p. 38).

[EG17] Hélène Esnault and Michael Groechenig. *Rigid connections, F-isocrystals and integrality*. 2017. arXiv: 1707.00752 (cit. on p. 32).

[Gir71] Jean Giraud. *Cohomologie non abelienne.* 1971. Zbl: 0226.14011 (cit. on p. 6).

[GLQ10] Michel Gros, Bernard Le Stum, and Adolfo Quirós. "A Simpson correspondence in positive characteristic." In: *Publ. Res. Inst. Math. Sci.* 46.1 (2010), pp. 1–35. Zbl: 1200.14036 (cit. on pp. 1, 12–14, 27).

[Gro16] Michael Groechenig. "Moduli of flat connections in positive characteristic." In: *Math. Res. Lett.* 23.4 (2016), pp. 989–1047. Zbl: 1368.14021 (cit. on pp. i, 1, 2, 23, 28, 30, 32, 33).

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I. Schemes. With examples and exercises.* Wiesbaden: Vieweg+Teubner, 2010, pp. vii + 615. Zbl: 1213.14001 (cit. on pp. 4, 10).

[HKR06] Yoshitake Hashimoto, Masaharu Kaneda, and Dmitriy Rumynin. "On localization of $\overline{D}$-modules." In: *Representations of algebraic groups, quantum groups, and Lie algebras. AMS-IMS-SIAM joint summer research conference, Snowbird, UT, USA, July 11–15, 2004.* Providence, RI: American Mathematical Society (AMS), 2006, pp. 43–62. Zbl: 1121.14041 (cit. on p. 11).

[HKR62] G. Hochschild, Bertram Kostant, and Alex Rosenberg. "Differential forms on regular affine algebras." In: *Trans. Am. Math. Soc.* 102 (1962), pp. 383–408. Zbl: 0102.27701 (cit. on p. 11).

[Kat70] Nicholas M. Katz. "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin." In: *Publ. Math., Inst. Hautes Étud. Sci.* 39 (1970), pp. 175–232. Zbl: 0221.14007 (cit. on pp. 1, 11, 26–28).

[Kat72] Nicholas M. Katz. "Algebraic solutions of differential equations (p-curvature and the Hodge filtration)." In: *Invent. Math.* 18 (1972), pp. 1–118. Zbl: 0278.14004 (cit. on p. 29).

[Kle05] Steven L. Kleiman. "The Picard scheme". In: *Fundamental algebraic geometry*. Vol. 123. Math. Surveys Monogr. Amer. Math. Soc., Providence, RI, 2005, pp. 235–321. MR: 2223410 (cit. on p. 36).

[Lam99] T. Y. Lam. *Lectures on modules and rings.* Vol. 189. New York, NY: Springer, 1999, pp. xxiii + 557. Zbl: `0911.16001` (cit. on p. 6).

[Lan14] Adrian Langer. "Semistable modules over Lie algebroids in positive characteristic." In: *Doc. Math.* 19 (2014), pp. 509–540. Zbl: `1330.14017` (cit. on p. 28).

[LP01] Yves Laszlo and Christian Pauly. "On the Hitchin morphism in positive characteristic." In: *Int. Math. Res. Not.* 2001.3 (2001), pp. 129–143. Zbl: `0983.14004` (cit. on pp. 30, 31).

[LSZ15] Guitang Lan, Mao Sheng, and Kang Zuo. "Nonabelian Hodge theory in positive characteristic via exponential twisting." In: *Math. Res. Lett.* 22.3 (2015), pp. 859–879. Zbl: `1326.14016` (cit. on p. 1).

[Mil80] J.S. Milne. *Étale cohomology.* Princeton Mathematical Series. 33. Princeton, New Jersey: Princeton University Press. XIII, 323 p. 1980. Zbl: `0433.14012` (cit. on pp. 6, 29).

[MM74] Barry Mazur and William Messing. *Universal extensions and one dimensional crystalline cohomology.* Lecture Notes in Mathematics. 370. Berlin-Heidelberg-New York: Springer-Verlag. VII, 134 p. 1974. Zbl: `0301.14016` (cit. on pp. 36, 37).

[Ols06] Martin C. Olsson. "$\underline{\mathrm{Hom}}$-stacks and restriction of scalars." In: *Duke Math. J.* 134.1 (2006), pp. 139–164. Zbl: `1114.14002` (cit. on pp. 37, 38).

[Ols16] Martin Olsson. *Algebraic spaces and stacks.* Providence, RI: American Mathematical Society (AMS), 2016, pp. xi + 298. Zbl: `1346.14001` (cit. on p. 6).

[OV07] A. Ogus and V. Vologodsky. "Nonabelian Hodge theory in characteristic p." In: *Publ. Math., Inst. Hautes Étud. Sci.* 106 (2007), pp. 1–138. Zbl: `1140.14007` (cit. on pp. 1, 13, 27, 28, 35–37).

[Oya17] Hidetoshi Oyama. "PD Higgs crystals and Higgs cohomology in characteristic p." In: *J. Algebr. Geom.* 26.4 (2017), pp. 735–802. Zbl: `1400.14059` (cit. on p. 1).

[Rin63] G. S. Rinehart. "Differential forms on general commutative algebras." In: *Trans. Am. Math. Soc.* 108 (1963), pp. 195–222. Zbl: `0113.26204` (cit. on p. 28).

[Shi15] Atsushi Shiho. "Notes on generalizations of local Ogus-Vologodsky correspondence." In: *J. Math. Sci., Tokyo* 22.3 (2015), pp. 793–875. Zbl: `1332.14032` (cit. on p. 1).

[Sim92] Carlos T. Simpson. "Higgs bundles and local systems." In: *Publ. Math., Inst. Hautes Étud. Sci.* 75 (1992), pp. 5–95. Zbl: `0814.32003` (cit. on p. 1).

[Sim94] Carlos T. Simpson. "Moduli of representations of the fundamental group of a smooth projective variety. I." In: *Publ. Math., Inst. Hautes Étud. Sci.* 79 (1994), pp. 47–129. Zbl: `0891.14005` (cit. on p. 28).

[Sim95] Carlos T. Simpson. "Moduli of representations of the fundamental group of a smooth projective variety. II." In: *Publ. Math., Inst. Hautes Étud. Sci.* 80 (1995), pp. 5–79. Zbl: 0891.14006 (cit. on p. 17).

[Sim97] Carlos Simpson. "The Hodge filtration on nonabelian cohomology." In: *Algebraic geometry. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, July 9–29, 1995*. Providence, RI: American Mathematical Society, 1997, pp. 217–281. Zbl: 0914.14003 (cit. on p. 28).

[Xu17] Daxin Xu. *Lifting the Cartier transform of Ogus-Vologodsky modulo p^n*. 2017. arXiv: 1705.06241 (cit. on p. 1).

Abbreviation

[EGA $\mathrm{I_n}$] Alexander Grothendieck and Jean A. Dieudonné. *Éléments de géométrie algébrique. I.* Vol. 166. Springer, Berlin, 1971. Zbl: 0203.23301 (cit. on p. 33).

[EGA II] A. Grothendieck. "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. Rédigé avec la colloboration de Jean Dieudonné." In: *Publ. Math., Inst. Hautes Étud. Sci.* 8 (1961), pp. 5–222. Zbl: 0118.36206 (cit. on pp. 4, 6, 9, 10).

[EGA $\mathrm{III_1}$] A Grothendieck. "Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I". In: *Inst. Hautes Études Sci. Publ. Math.* 11 (1961), p. 167. MR: 217085 (cit. on p. 9).

[EGA $\mathrm{IV_4}$] A. Grothendieck. "Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas (Quatrième partie). Rédigé avec la colloboration de Jean Dieudonné." In: *Publ. Math., Inst. Hautes Étud. Sci.* 32 (1967), pp. 1–361. Zbl: 0153.22301 (cit. on pp. 11, 27).

[SGA 4_{III}] *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de P. Deligne, B. Saint-Donat. Tome 3. Exposés IX à XIX.* Lecture Notes in Mathematics. 305. Berlin-Heidelberg-New York: Springer-Verlag. vi, 640 pp. 1973. MR: 0354654. Zbl: 0245.00002 (cit. on p. 35).

[SGA 5] *Seminaire de géométrie algébrique du Bois-Marie 1965-66 SGA 5 dirige par A. Grothendieck avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou et J. -P. Serre. Cohomologie ℓ-adique et fonctions L.* Lecture Notes in Mathematics. 589. Berlin-Heidelberg-New York: Springer-Verlag. XII, 484 p. 1977. Zbl: 0345.00011 (cit. on p. 3).

[SGA 7_{II}] *Séminaire de géométrie algébrique du Bois-Marie 1967–1969. Groupes de monodromie en géométrie algébrique (SGA 7 II) par P. Deligne et N. Katz. Exposés X à XXII.* Lecture Notes in Mathematics. 340. Berlin-Heidelberg-New York: Springer-Verlag. x, 438 p. 1973. Zbl: `0258.00005` (cit. on p. 29).

[SP] Johan de Jong et al. *Stacks Project*. 2018. URL: `http://stacks.math.columbia.edu` (cit. on pp. 3, 4, 6, 10, 31, 38).