Fachbereich Mathematik und Informatik
der Freien Universität Berlin

Dissertation

zur Erlangung des Doktorgrades (Dr. rer. nat.)

Monodromy groups in positive characteristic

vorgelegt von

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Berlin 2019

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Tag der Disputation: 07.02.2019

Zusammenfassung

In dieser Doktorarbeit studieren wir die Monodromiegruppen von lisse Garben und Isokristallen in positiver Charakteristik. Das erste Ziel ist es, die Unabhängigkeit für Objekte mit derselben L-Funktion zu zeigen. Im letzten Abschnitt zeigen wir die Endlichkeit perfekter Torsionspunkte einer abelschen Varietät. Dies erweitert den Satz von Lang–Néron und beantwortet positiv eine Frage von Esnault.

Abstract

In this thesis we study the monodromy groups of lisse sheaves and isocrystals in positive characteristic. The first aim is to prove independence results for objects with the same L-function. In the last section we show the finiteness of perfect torsion points of an abelian variety. This extends a theorem of Lang–Néron and answers positively a question of Esnault.

Monodromy groups in positive characteristic

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June 24, 2019

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Introduction

In the first half of the twentieth century, Weil started his visionary work aimed to extend the known results in algebraic geometry to varieties over fields or rings arising from arithmetic. His main contribution in this direction was the proof of the Riemann Hypothesis for smooth projective curves over finite fields. The missing piece for a treatment of the problem in higher dimensions was the lack of a suitable cohomology theory, with characteristic 0 coefficients, for varieties over finite fields. For a smooth projective curve, a direct construction is available by means of the Jacobian.

Grothendieck, influenced by the work of Serre, had the bright idea that étale covers was the key tool to define a "nice" $\mathbb{Q}_\ell$-linear cohomology theory, in positive characteristic p, when ℓ is a prime different from p. This cohomology is called *ℓ-adic étale cohomology*. Some years later, he also suggested a way to define a cohomology with p-adic coefficients, which is called *crystalline cohomology*.

To a cohomology theory one can usually associate *coefficient objects* which represent variations of cohomology groups of constant rank. In the case of ℓ-adic étale cohomology, these objects are the *ℓ-adic local systems* or *lisse sheaves*. For crystalline cohomology, the initial notion of Grothendieck was the one of *crystal*, or its rational version, called *isocrystal*. The category of isocrystals is very large. One important feature of the isocrystals "coming from geometry" is that they are naturally endowed with a *Frobenius structure*, namely an isomorphism of the object with the Frobenius pullback of itself. Dwork discovered that the F-structure forces a certain local convergence property which is not verified in general. The isocrystals endowed with a Frobenius structure are called *F-isocrystals*.

When Berthelot introduced rigid cohomology, which is a certain variation of crystalline cohomology, he also defined the category of *overconvergent F-isocrystals*. This category is constructed Zariski-locally using Raynaud's generic fibre of formal lifts. The characterizing property of these objects is a certain convergence condition "at infinity".

The common theme during my PhD has been the study of the category of lisse sheaves, as well as the categories of convergent and overconvergent isocrystals. Under suitable assumptions these categories are all *Tannakian*. Therefore, after possibly extending the field of constants, they are equivalent to the category of linear representations of some pro-algebraic group scheme. The image of the representation associated to an object is what we call the *monodromy group* of the object. In the case of ℓ-adic local systems, the monodromy groups have been already extensively studied in the past. For the categories of F-isocrystals much less is known. In the latter situation, an interesting feature, which does not admit an ℓ-adic analogue, is the interplay between the monodromy groups of convergent and overconvergent isocrystals.

Overconvergent F-isocrystals have many properties in common with ℓ-adic local systems. Crucial examples of this analogy are the *theory of weights*, developed by Kedlaya, and the *Lang-*

lands correspondence, proven by Abe (on a smooth curve over a finite fields). This allows one to prove that the monodromy groups of overconvergent F-isocrystals "behave like" the ones of lisse shaves. The monodromy groups of convergent F-isocrystals remain instead quite mysterious.

Our thesis is divided in three sections. The last section is written in collaboration with Emiliano Ambrosi. Let us briefly summarize the main results of each section.

Section 1: The monodromy groups of lisse sheaves and overconvergent F-isocrystals

We extend previously known results of Serre, Larsen-Pink, Chin on the structure of the monodromy groups of ℓ-adic local systems and their independence of ℓ to overconvergent F-isocrystals on smooth varieties over a finite field. We extend, for example, Chin's result on the independence of the neutral component of the monodromy groups. For this purpose, we introduce and study the Frobenius tori of overconvergent F-isocrystals. These were firstly introduced by Serre for ℓ-adic Galois representations. We also show that the slope polygons of an F-isocrystal defined on an abelian variety over a finite field are constant. This recovers a result of Tsuzuki. To do this we prove that in this case the monodromy groups are commutative via an Eckmann-Hilton argument.

Section 2: Remarks on the companions conjecture for normal varieties

We study the *companions conjecture* for lisse sheaves on normal varieties over a finite field. The conjecture has been proven for smooth varieties by Drinfeld. We analyze the obstruction to extending it to normal singular varieties. We formulate and study a related conjecture which we verify in some particular cases.

Section 3: Maximal tori of monodromy groups of F-isocrystals and applications (joint with Emiliano Ambrosi)

We use the work done in Section 1 to study the monodromy groups of convergent F-isocrystals which have an overconvergent extension. Thanks to the theory of Frobenius tori we show that these groups are "big". This fact has many consequences. On the one hand, we use it to prove a special case of a conjecture proposed by Kedlaya on F-isocrystals. On the other hand, we prove a finiteness result for the perfect torsion points of an abelian variety, giving a positive answer to a question of Esnault. As an additional outcome of our work, we prove a weak (weak) semi-simplicity statement for p-adic representations coming from pure overconvegent F-isocrystals.

Acknowledgments

During my PhD I have benefited from the help of many people. I would like to thank all of them. In particular, I would like to thank the following ones.

My advisor Hélène Esnault for her guidance through the years of my PhD. Many of the problems and ideas of this thesis have been suggested by her.

My second advisor Vasudevan Srinivas for all the helpful discussions.

Tomoyuki Abe for his help understanding various subtle points about the categories of F-isocrystals.

Nobuo Tsuzuki for agreeing to be a referee for this thesis. Also his article [Tsu17] played an important role in §1.5.

Emiliano Ambrosi for his collaboration in our work on maximal tori of monodromy groups of F-isocrystals.

Gregorio Baldi for his support and all the conversations about math.

Anna Cadoret who has organized a workshop in Paris on Deligne's conjectures.

Damian Rössler for the discussions about the problem of perfect torsion points on abelian varieties.

My colleagues at the Freie Universität for all their efforts to create at the department a very nice work environment. I am in debt to them for all the things I have learned from countless conversations, seminars, etc.

Further thanks go to Piotr Achinger, Michel Brion, Brian Conrad, Chun Yin Hui, Kiran Kedlaya, Daniel Litt and Jacob Stix for their comments on my work.

1 The monodromy groups of lisse sheaves and overconvergent F-isocrystals

1.1 Introduction

1.1.1 Background

Deligne in [Del80] generalized the Riemann Hypothesis over finite fields, previously proven by himself, to a result on the behaviour of the *weights* of lisse sheaves under higher direct image. He also formulated a conjecture on other expected properties of lisse sheaves [*ibid.*, Conjecture 1.2.10]. This conjecture was inspired by the Langlands reciprocity conjecture for GL_r.

Conjecture 1.1.1.1. [1] *Let X_0 be a normal scheme of finite type over a finite field $\mathbb{F}_q$ of characteristic p and let $\mathcal{V}_0$ be an irreducible Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant has finite order.*

(i) *$\mathcal{V}_0$ is pure of weight 0.*

(ii) *There exists a number field $E \subseteq \overline{\mathbb{Q}}_\ell$ such that for every closed point x_0 in X_0, the characteristic polynomial $\det(1 - F_{x_0}t, \mathcal{V}_x)$ has coefficients in E, where F_{x_0} is the geometric Frobenius at x_0.*

(iii) *For every prime $\ell \neq p$, the eigenvalues of the Frobenii at closed points of $\mathcal{V}_0$ are ℓ-adic units.*

(iv) *For a suitable field E (maybe larger than in (ii)) and for every finite place λ not dividing p, there exists a lisse E_λ-sheaf compatible with $\mathcal{V}_0$, namely a lisse E_λ-sheaf with the same characteristic polynomials of the Frobenii at closed points as $\mathcal{V}_0$.*

(v) *When λ divides p, there exists some compatible crystalline object ("des petits camarades cristallins").*

Objects with the same characteristic polynomials at closed points are also called *companions* and the conjunction of (iv) and (v) is also known as the *companions conjecture*. The companions conjecture for Weil lisse sheaves, namely part (iv), admits the following weaker form.

(iv') *If E is a number field as in (ii), for every finite place λ not dividing p, there exists a lisse $\overline{E}_\lambda$-sheaf compatible with $\mathcal{V}_0$.*

In [Laf02] L. Lafforgue proved the Langlands reciprocity conjecture for GL_r over function fields. As a consequence, he obtained (i), (ii), (iii) and (iv'), when X_0 is a smooth curve. Chin

[1] We will omit the part of the conjecture on the p-adic valuations of the Frobenius eigenvalues at closed points.

then showed that in arbitrary dimension, if (ii) and (iv') are true for every finite étale cover of X_0, then (iv) is also true [Chi03]. As a consequence, one gets part (iv) of the conjecture when X_0 is a curve.

The lack of a Langlands correspondence for higher dimensional varieties (even at the level of the formulation) forced one to generalize Deligne's conjectures, reducing geometrically to the case of curves. One of the difficulties is that one cannot rely on a Lefschetz theorem for the étale fundamental group in positive characteristic (see for example [Esn17, Lemma 5.4]). This means that one cannot, in general, find a curve C_0 in X_0 such that for every irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{V}_0$ of X_0, the inverse image of $\mathcal{V}_0$ on C_0 remains irreducible.

Luckily, one can replace the Lefschetz theorem with a weaker result. Rather than considering all the lisse sheaves at the same time, one can fix the lisse sheaf and find a suitable curve where the lisse sheaf remains irreducible (Theorem 1.3.7.5). In this way one can prove (i) and (iii) for arbitrary varieties, using Lafforgue's result. Parts (ii) and (iv') require some more effort. The former was obtained by Deligne in [Del12], the latter by Drinfeld for smooth varieties in [Dri12] and it is still open in general. Following the ideas of Wiesend in [Wie06] Drinfeld used a gluing theorem for lisse sheaves [Drinfeld, *op. cit.*, Theorem 2.5].

Passing to (v), Crew conjectured in [Cre92a, Conjecture 4.13] that the correct p-adic analogue of lisse sheaves might be *overconvergent F-isocrystals*, introduced by Berthelot [Ber96a]. To endorse his conjecture, he proved the *global monodromy theorem* for these isocrystals, over a smooth curve [Crew, *op. cit.*, Theorem 4.9]. Many people have then worked in the direction suggested by Crew (see for example [Ked04a] and [Ked06] for references).

Finally, Abe proves the Langlands reciprocity conjecture for overconvergent F-isocrystals, over a smooth curve [Abe18]. In his work he used the theory of *arithmetic $\mathscr{D}$-modules* introduced by Berthelot in [Ber96b] and mainly developed by Abe, Berthelot, Caro, and Tsuzuki. Abe's result, combined with Lafforgue's theorem, shows that on smooth curves there is a correspondence between (certain) lisse sheaves and (certain) overconvergent F-isocrystals. Abe–Esnault, and later Kedlaya, generalized one direction of the correspondence by constructing on smooth varieties of arbitrary dimension, lisse sheaves that are compatible with overconvergent F-isocrystals (see [AE16] and [Ked18]). Even in this case, they construct lisse sheaves via a reduction to the case of curves. They both prove and use some Lefschetz type theorem, in combination with Drinfeld's gluing theorem for lisse sheaves.

1.1.2 Main results

Following Kedlaya [Ked18], we refer to lisse sheaves and overconvergent F-isocrystals as *coefficient objects*. Let X_0 be a smooth connected variety over $\mathbb{F}_q$. Suppose that $\mathcal{E}_0$ is a coefficient object on X_0 with all the eigenvalues of the Frobenii at closed points algebraic over $\mathbb{Q}$. Thanks to the known cases of the companions conjecture, $\mathcal{E}_0$ sits in an *E-compatible system* $\{\mathcal{E}_{\lambda,0}\}_{\lambda\in\Sigma}$, where E is a number field, Σ is a set of finite places of E, containing all the places which do not

divide p, and $\{\mathcal{E}_{\lambda,0}\}_{\lambda\in\Sigma}$ is a family of pairwise E-compatible E_λ-coefficient objects (as in (iv)), one for each $\lambda \in \Sigma$ (Theorem 1.3.8.2).

We use the new tools, presented above, to extend the results of λ-independence of the *monodromy groups*. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$ and x an $\mathbb{F}$-point of X_0. For every $\lambda \in \Sigma$, let $G(\mathcal{E}_{\lambda,0}, x)$ be the *arithmetic monodromy group* of $\mathcal{E}_{\lambda,0}$ and $G(\mathcal{E}_\lambda, x)$ its *geometric monodromy group* (see Definition 1.3.2.3). We generalize the result of Serre and Larsen–Pink on the λ-independence of the π_0 of the monodromy groups (see [Ser00] and [LP95, Proposition 2.2]).

Theorem 1.1.2.1 (Theorem 1.4.1.1)**.** *The groups of connected components of $G(\mathcal{E}_{\lambda,0}, x)$ and $G(\mathcal{E}_\lambda, x)$ are independent of λ.*

To prove such a theorem for overconvergent F-isocyrstals we have to relate their monodromy groups with the étale fundamental group of X_0. This is done in §1.3.3 and relies on some previous work done by Crew in [Cre92a]. Then the proof follows [LP95, Proposition 2.2].

We assume now, in addition, that for every $\lambda \in \Sigma$, the coefficient object $\mathcal{E}_{\lambda,0}$ is semi-simple. Denote by $\rho_{\lambda,0}$ the tautological representation of $G(\mathcal{E}_{\lambda,0}, x)$. We obtain the following generalization of [Chi04, Theorem 1.4].

Theorem 1.1.2.2 (Theorem 1.4.3.2)**.** *After possibly replacing E with a finite extension, there exists a connected split reductive group G_0 over E such that, for every $\lambda \in \Sigma$, the extension of scalars $G_0 \otimes_E E_\lambda$ is isomorphic to the neutral component of $G(\mathcal{E}_{\lambda,0}, x)$. Moreover, there exists a faithful E-linear representation ρ_0 of G_0 and isomorphisms $\varphi_{\lambda,0} : G_0 \otimes_E E_\lambda \xrightarrow{\sim} G(\mathcal{E}_{\lambda,0}, x)^\circ$ for every $\lambda \in \Sigma$ such that $\rho_0 \otimes_E E_\lambda$ is isomorphic to $\rho_{\lambda,0} \circ \varphi_{\lambda,0}$.*

Notice that in Theorem 1.1.2.2 we have removed from [*ibid.*, Theorem 1.4] the purity and p-plain assumptions (cf. §1.3.1.15). Chin proves his result exploiting a reconstruction theorem for connected split reductive groups (Theorem 1.4.3.4). To apply his theorem, he extends the result of Serre in [Ser00] on *Frobenius tori* of étale lisse sheaves in [Chi04, Lemma 6.4]. We further generalize Chin's result on Frobenius tori.

Theorem 1.1.2.3 (Theorem 1.4.2.10)**.** *Let $\mathcal{E}_0$ be an algebraic coefficient object over X_0. There exists a Zariski-dense subset $\Delta \subseteq X(\mathbb{F})$ such that for every $\mathbb{F}$-point $x \in \Delta$ and every object $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle$, the torus $T(\mathcal{F}_0, x)$ is a maximal torus of $G(\mathcal{F}_0, x)$. Moreover, if $\mathcal{G}_0$ is a coefficient object compatible with $\mathcal{E}_0$, the subset Δ satisfies the same property for the objects in $\langle \mathcal{G}_0 \rangle$.*

We first prove Theorem 1.1.2.3 for algebraic étale lisse sheaves, by improving Serre's finiteness result in Corollary 1.4.2.7. This is done using Deligne's conjectures. Then we deduce the general case using a *dimension data* argument due to Larsen and Pink (see Proposition 1.4.2.8). Thanks to Theorem 1.1.2.3, we are able to prove Theorem 1.1.2.2 following Chin's method. Theorem 1.1.2.3 is also used in §3 as a starting point to prove some rigidity results for the *convergent F-isocrystals* which admit an overconvergent extension. Thanks to Theorem 1.1.2.2 we are able to prove a semi-simplicity statement for the Frobenii at closed points.

Corollary 1.1.2.4 (Corollary 1.4.3.9)**.** *Let $\mathcal{E}_0$ be a semi-simple $\overline{\mathbb{Q}}_\ell$-coefficient object. The set of closed points where the Frobenius is semi-simple is Zariski-dense in X_0.*

This result is proven in [LP92, Proposition 7.2] for étale lisse sheaves. For overconvergent F-isocrystals, the statement is new and it is obtained using Theorem 1.1.2.2. We will need the full strength of Theorem 1.1.2.2, as we will also use the independence of the tautological representation of compatible coefficient objects. Another outcome of the previous techniques, is an independence result for the Lefschetz theorem for coefficient objects.

Theorem 1.1.2.5 (Theorem 1.4.4.2)**.** *Let $f_0 : (Y_0, y) \to (X_0, x)$ be a morphism of geometrically connected smooth pointed varieties. Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be compatible geometrically semi-simple coefficient objects over X_0. Denote by $\varphi_0 : G(f_0^*\mathcal{E}_0, y) \to G(\mathcal{E}_0, x)$ and $\psi_0 : G(f_0^*\mathcal{F}_0, y) \to G(\mathcal{F}_0, x)$ the morphisms induced by f_0^* and by φ and ψ their restriction to the geometric monodromy groups.*

(i) *If φ is an isomorphism, the same is true for ψ.*

(ii) *If φ_0 is an isomorphism, the same is true for ψ_0.*

We give a "cheap" proof of Theorem 1.1.2.5 which for ι-mixed coefficient objects avoids Deligne's conjecture. The proof relies on the Tannakian lemma [AE16, Lemma 1.6]. Finally, we prove the following theorem for coefficient objects defined on abelian varieties.

Theorem 1.1.2.6 (Theorem 1.5.1.1)**.** *Let X_0 be an abelian variety. Every absolutely irreducible coefficient object with finite order determinant is finite. In particular, every ι-pure coefficient object on X_0 becomes constant after passing to a finite étale cover.*

We propose two proofs. The first one uses the Künneth formula (Proposition 1.3.4.4), an Eckmann–Hilton argument and the global monodromy theorem (Theorem 1.3.5.4). This proof does not rely on the Langlands programme. The second proof is a consequence of the known cases of the companions conjecture and the global monodromy theorem for lisse sheaves. Using this method one could actually prove a more general statement (see Remark 1.5.1.5).

As a consequence, we give a proof of Deligne's conjectures for coefficient objects that does not use automorphic representations (Corollary 1.5.1.3). We also obtain in Corollary 1.5.2.2 an alternative proof of a theorem of Tsuzuki on the constancy of the Newton polygons of F-isocrystals on abelian varieties [Tsu17, Theorem 3.7].

1.1.3 Comparison with previous work

In [Pal15, Theorem 8.23], Pál gives a proof of a special case of Theorem 1.1.2.2 for curves. It relies on a strong Čebotarev density theorem for overconvergent F-isocrystals [*ibid.*, Theorem 4.13], which is now proven in [HP18]. Using the result on Frobenius tori, we do not use Hartl–Pál's

theorem. It is also worth mentioning that in [Dri18], Drinfeld proves the independence of the entire arithmetic monodromy groups (not only the neutral component), over $\overline{\mathbb{Q}}_\ell$. He uses a stronger representation-theoretic reconstruction theorem (see Remark 1.4.3.11).

1.1.4 The structure of §1

We define in §1.3.1 the categories of coefficient objects and geometric coefficient objects, and we prove some basic results. We also recall some definitions related to the characteristic polynomials of the Frobenii at closed points, and we show that p-plain (cf. §1.3.1.15) lisse sheaves are étale (Proposition 1.3.1.17).

In §1.3.2, we define the arithmetic and the geometric monodromy groups of coefficient objects, using the Tannakian formalism. We also introduce the Tannakian fundamental groups classifying coefficient objects and geometric coefficient objects. We present a *fundamental exact sequence* relating these groups (Proposition 1.3.2.6). The result is essentially all proven in the appendix for general *neutral Tannakian categories with Frobenius*. Then in §1.3.3 we show that the groups of connected components of these fundamental groups are isomorphic to the arithmetic and the geometric étale fundamental group (Proposition 1.3.3.3). We also prove a complementary result, namely Proposition 1.3.3.4.

In §1.3.4 we prove the Künneth formula for the fundamental group classifying geometric coefficient objects for projective connected varieties with a rational point. In §1.3.5 we recall the main result on rank 1 coefficient objects (Theorem 1.3.5.1). We introduce in §1.3.5.3 the notion of *type* and we prove some structural properties for them. In §1.3.6 we recollect some theorems from Weil II that are now known for coefficient objects of both kinds. For example, the main theorem on weights (Theorem 1.3.6.1). We present in §1.3.7 the state of Deligne's conjectures. In §1.3.8 we give the definition of compatible systems of lisse sheaves and overconvergent F-isocrystals and we state a stronger form of the companions conjecture in Theorem 1.3.8.2, due to the work of Chin.

In §1.4, we investigate the properties of λ-independence of the monodromy groups varying in a compatible system of coefficient objects. We start by proving in §1.4.1 the λ-independence of the groups of connected components, generalizing the theorem of Serre and Larsen–Pink (Theorem 1.1.2.1). In §1.4.2 we extend the theory of Frobenius tori to algebraic coefficient objects and we prove Theorem 1.1.2.3. In §1.4.3 we prove Theorem 1.1.2.2 and Corollary 1.1.2.4 and in §1.4.4 we prove Theorem 1.1.2.5.

In §1.5 we focus on coefficient objects on abelian varieties. We give the two proofs of Theorem 1.5.1.1, we prove Deligne's conjectures for abelian varieties and we recover Tsuzuki's theorem in Corollary 1.5.2.2.

1.2 Notation and conventions

1.2.0.1. We fix a prime number p and a positive power q. Let $\mathbb{F}_q$ be a field with q elements and $\mathbb{F}$ an algebraic closure of $\mathbb{F}_q$. For every positive integer s we denote by $\mathbb{F}_{q^s}$ the subfield of $\mathbb{F}$ with q^s elements. If k is a field we will say that a separated scheme of finite type over k is a *variety* over k. A *curve* will be a one dimensional variety. We denote by X_0 a smooth variety over some finite field k. If k is not specified, then it is assumed to be $\mathbb{F}_q$. In this case, we denote by X the extension of scalars $X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$ over $\mathbb{F}$. In general, we denote with a subscript $_0$ objects and morphisms defined over $\mathbb{F}_q$, and the suppression of the subscript will mean the extension to $\mathbb{F}$. We write k_{X_0} for the algebraic closure of $\mathbb{F}_q$ in $\Gamma(X_0, \mathcal{O}_{X_0})$. Sometimes it will be useful to consider X_0 as a variety over k_{X_0}, just changing the structural morphism.

We denote by $|X_0|$ the set of closed points of X_0. If x_0 is a closed point of X_0, the degree of x_0 will be $\deg(x_0) := [\kappa(x_0) : \mathbb{F}_q]$. A variety is said ($\mathbb{F}$-)*pointed* if it is endowed with the choice of an $\mathbb{F}$-point. A *morphism of pointed varieties* $(Y_0, y) \to (X_0, x)$ is a morphism of varieties $Y_0 \to X_0$ which sends y to x. An $\mathbb{F}$-point x of X_0 determines a unique closed point of the variety that we denote by x_0. Moreover, x determines an identification $k_{X_0} = \mathbb{F}_{q^s}$, for some $s \in \mathbb{Z}_{>0}$.

1.2.0.2. The letter ℓ will denote a prime number. In general we allow ℓ to be equal to p. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For every number field E, we denote by $|E|_\ell$ the set of finite places of E dividing ℓ. We define $|E|_{\neq p} := \bigcup_{\ell \neq p} |E|_\ell$ and $|E| := \bigcup_\ell |E|_\ell$. We choose in a compatible way, for every number field $E \subseteq \overline{\mathbb{Q}}$ and every $\lambda \in |E|$, a completion of E by λ, denoted by E_λ. For every prime ℓ, we denote by $\overline{\mathbb{Q}}_\ell$ the union of all the E_λ, when E varies among the number fields $E \subseteq \overline{\mathbb{Q}}$ and λ is any place in $|E|_\ell$. If $\mathbb{K}$ is a field of characteristic 0, an element $a \in \mathbb{K}$ is said to be an *algebraic number* if it is algebraic over $\mathbb{Q}$. If a is an algebraic number we will say that it is *p-plain*[2] if it is an ℓ-adic unit for every $\ell \neq p$.

1.2.1 Tannakian categories and affine group schemes

1.2.1.1. Let $\mathbb{K}$ be a field. We denote by $\mathbf{Vec}_\mathbb{K}$ the category of finite dimensional $\mathbb{K}$-vector spaces. A *Tannakian category* over $\mathbb{K}$ will be a rigid abelian symmetric $\otimes$-category $\mathbf{C}$ together with an isomorphism $\mathrm{End}(\mathbb{1}) \simeq \mathbb{K}$, that admits a faithful exact $\mathbb{K}$-linear $\otimes$-functor $\omega : \mathbf{C} \to \mathbf{Vec}_\mathbb{L}$, for some field extension $\mathbb{K} \subseteq \mathbb{L}$. We will call such a functor a *fibre functor* of $\mathbf{C}$ over $\mathbb{L}$. If in addition $\mathbf{C}$ admits a fibre functor over $\mathbb{K}$ itself, we say that $\mathbf{C}$ is a *neutral Tannakian category*.

For every Tannakian category $\mathbf{C}$ over $\mathbb{K}$, we say that an object in $\mathbf{C}$ is a *trivial object* if it is isomorphic to $\mathbb{1}^{\oplus n}$ for some $n \in \mathbb{N}$. We say that an object $V \in \mathbf{C}$ is *irreducible* if the only subobjects of V are 0 and V itself. We say that $V \in \mathbf{C}$ is *absolutely irreducible* if for ever finite extension $\mathbb{L}/\mathbb{K}$, the extension of scalars $V \otimes_\mathbb{K} \mathbb{L}$ is irreducible. A *Tannakian subcategory* of $\mathbf{C}$

[2]This is an abbreviation for the expression *plain of characteristic p* in [Chi04].

is a strictly full abelian subcategory, closed under $\otimes$, duals, subobjects (and thus quotients). If V is an object of $\mathbf{C}$, we denote by $\langle V \rangle$ the smallest Tannakian subcategory of $\mathbf{C}$ containing V.

1.2.1.2. If ω is a fibre functor of $\mathbf{C}$, over an extension $\mathbb{L}$, the affine group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ over $\mathbb{L}$ will be the *Tannakian group* of $\mathbf{C}$ with respect to ω. For every object $V \in \mathbf{C}$, a fibre functor ω of $\mathbf{C}$ induces, by restriction, a fibre functor for the Tannakian category $\langle V \rangle$, that we will denote again by ω. We also say that the Tannakian group of $\langle V \rangle$ with respect to ω is the *monodromy group* of V (with respect to ω). If the monodromy group of V is finite, we say that V is a *finite object*.

1.2.1.3. For every affine group scheme G, we denote by $\pi_0(G)$ the *group of connected components* of G and G° will be the connected component of G containing the neutral element, called the *neutral component* of G. When G is an algebraic group, the *reductive rank* of G will be the dimension of any maximal torus of G.

1.2.1.4. Let $\varphi : G \to H$ be a morphism of affine group schemes over $\mathbb{K}$ and let $f : \mathbf{Rep}_{\mathbb{K}}(H) \to \mathbf{Rep}_{\mathbb{K}}(G)$ be the induced restriction functor. By [DM82, Proposition 2.21], the morphism φ is faithfully flat if and only if the functor f is fully faithful and it is closed under the operation of taking subobjects. Moreover, φ is a closed immersion if and only if every object of $\mathbf{Rep}_{\mathbb{K}}(G)$ is a subquotient of an object in the essential image of f. In particular, if for a tensor generator V of $\mathbf{Rep}_{\mathbb{K}}(H)$ (cf. *ibid.*), the object $f(V)$ is a tensor generator of $\mathbf{Rep}_{\mathbb{K}}(G)$, then φ is a closed immersion. We will repeatedly use these facts in §1 without further comments. For simplicity, when $\mathbb{K}$ is a characteristic 0 field, we will say that a morphism between affine group schemes $\varphi : G \to H$ is *surjective* if it is faithfully flat and we will say that φ is *injective* if it is a closed immersion.

1.2.2 Weil lisse sheaves

We mainly use the notations and conventions for lisse sheaves as in [Del80].

1.2.2.1. If x is a geometric point of X_0, we denote by $\pi_1^{\text{ét}}(X_0, x)$ and $\pi_1^{\text{ét}}(X, x)$ the étale fundamental groups of X_0 and X respectively. If k is a finite extension of $\mathbb{F}_q$ and $\overline{k}$ is an algebraic closure of k, the inverse of the $q^{[k:\mathbb{F}_q]}$-power Frobenius will be the *geometric Frobenius of* k (with respect to $\overline{k}$). We denote by F the geometric Frobenius of $\mathbb{F}_q$ with respect to $\mathbb{F}$. For every $n \in \mathbb{Z}_{>0}$ we denote by $W(\mathbb{F}/\mathbb{F}_{q^n})$ the Weil group of $\mathbb{F}_{q^n}$ (it is generated by F^n). We also denote by $W(X_0, x)$ the Weil group of X_0.

Let x_0' be a closed point of X_0 in the same connected component of x. For any choice of a geometric point x' over x_0', the geometric Frobenius of x_0' with respect to x' determines by functoriality an element $\gamma \in W(X_0, x')$. If we choose an étale path from x' to x, it induces an

isomorphism $W(X_0, x') \xrightarrow{\sim} W(X_0, x)$. The conjugacy class of the image of γ in $W(X_0, x)$, that we will denote by $F_{x'_0} \subseteq W(X_0, x)$, depends only on x'_0. The elements in $F_{x'_0}$ will be *the Frobenii at* x'_0.

1.2.2.2. For every $\ell \neq p$ we have a category $\mathbf{LS}(X, \overline{\mathbb{Q}}_\ell)$ of *lisse* $\overline{\mathbb{Q}}_\ell$*-sheaves* over X, that is the 2-colimit of the categories $\mathbf{LS}(X, E_\lambda)$ of *lisse* E_λ*-sheaves*, where E_λ varies among the finite extensions of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$. If X_0 is not geometrically connected over $\mathbb{F}_q$ then these categories are not Tannakian (the unit objects have too many endomorphisms). If x is a geometric point of X_0, we denote by $X^{(x)}$ the connected component of X containing x. The categories $\mathbf{LS}(X^{(x)}, E_\lambda)$ and $\mathbf{LS}(X^{(x)}, \overline{\mathbb{Q}}_\ell)$ are then always neutral Tannakian categories.

If $\mathcal{V}$ is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X, an *n-th Frobenius structure* on $\mathcal{V}$ is an action of $W(\mathbb{F}/\mathbb{F}_{q^n})$ on the pair $(X, \mathcal{V})$ such that $W(\mathbb{F}/\mathbb{F}_{q^n})$ acts on $X = X_0 \otimes \mathbb{F}$ via the natural action on $\mathbb{F}$. An n-th Frobenius structure is equivalent to the datum of an isomorphism $(F^n)^*\mathcal{V} \xrightarrow{\sim} \mathcal{V}$. The category of lisse E_λ-sheaves equipped with a 1-st Frobenius structure will be the category of *Weil lisse* E_λ*-sheaves* of X_0, denoted by $\mathbf{Weil}(X_0, E_\lambda)$. The categories $\mathbf{Weil}(X_0, E_\lambda)$ are Tannakian. We will often refer to Weil lisse sheaves simply as *lisse sheaves of* X_0.

For every geometric point x of X_0 and every E_λ we define a functor

$$\Psi_{x,E_\lambda} : \mathbf{Weil}(X_0, E_\lambda) \to \mathbf{LS}(X, E_\lambda) \to \mathbf{LS}(X^{(x)}, E_\lambda)$$

where the first functor forgets the Frobenius structure and the second one is the inverse image functor with respect to the open immersion $X^{(x)} \hookrightarrow X$. If $\mathcal{V}_0$ is a Weil lisse sheaf, we remove the subscript $_0$ to indicate the lisse sheaf $\Psi_{x,E_\lambda}(\mathcal{V}_0)$.

1.2.2.3. If we fix a geometric point x of X_0, there exists an equivalence between the category of Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaves over X_0 and the finite-dimensional continuous $\overline{\mathbb{Q}}_\ell$-representations of the Weil group $W(X_0, x)$. The equivalence sends a Weil lisse sheaf $\mathcal{V}_0$ to the representation of $W(X_0, x)$ on the stalk $\mathcal{V}_x$. A Weil lisse sheaf such that the associated representation of the Weil group factors through the étale fundamental group will be an *étale lisse sheaf*.

1.2.2.4. Notation as in §1.2.2.3. If $\mathcal{V}_0$ is a Weil lisse E_λ-sheaf, for every closed point $x'_0 \in |X_0|$ the elements in $F_{x'_0}$ act on $\mathcal{V}_x$. Even if these automorphisms are a priori different, their characteristic polynomials do not change. We define (with a small abuse of notation) $P_{x'_0}(\mathcal{V}_0, t) := \det(1 - tF_{x'_0}|\mathcal{V}_x) \in E_\lambda[t]$. This will be the *(Frobenius) characteristic polynomial* of $\mathcal{V}_0$ at x'_0.

For every natural number n, a lisse $\overline{\mathbb{Q}}_\ell$-sheaf is said to be *pure* of *weight* n, if for every closed point x'_0 of X_0, the eigenvalues of any element in $F_{x'_0}$ are algebraic numbers and all the conjugates have complex absolute value $(\#\kappa(x'_0))^{n/2}$. If $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ and w is a real number, we say that a lisse sheaf is ι*-pure* of ι*-weight* w if for every closed point x'_0 of X_0 the eigenvalues of $F_{x'_0}$, after applying ι, have complex absolute value $(\#\kappa(x_0))^{w/2}$. Moreover, we say that a

lisse $\overline{\mathbb{Q}}_\ell$-sheaf is *mixed* (resp. *ι-mixed*) if it admits a filtration of lisse $\overline{\mathbb{Q}}_\ell$-sheaf with pure (resp. ι-pure) successive quotients.

1.2.3 Overconvergent F-isocrystals

1.2.3.1. Let k be a perfect field. We denote by $W(k)$ the ring of p-typical Witt vectors over k and by $K(k)$ its fraction field. For every $s \in \mathbb{Z}_{>0}$, we denote by $\mathbb{Z}_{q^s}$ the ring of Witt vectors over $\mathbb{F}_{q^s}$ and by $\mathbb{Q}_{q^s}$ its fraction field. We suppose chosen compatible morphisms $\mathbb{Q}_{q^s} \to \overline{\mathbb{Q}}_p$.

Let X_0 be a smooth variety over k, we denote by $\mathbf{Isoc}^\dagger(X_0/K(k))$ the category of Berthelot's *overconvergent isocrystals* of X_0 over $K(k)$. See [Ber96a] for a precise definition and [Cre87] or [Ked16] for a shorter presentation. The category $\mathbf{Isoc}^\dagger(X_0/K(k))$ is a $K(k)$-linear rigid abelian $\otimes$-category, with unit object $\mathcal{O}^\dagger_{X_0}$, that we will denote by $K(k)_{X_0}$. The endomorphism ring of $K(k)_{X_0}$ is isomorphic to $K(k)^s$, where s is the number of connected components of X.

We will recall now the notation for the extension of scalars and the Frobenius structure of overconvergent isocrystals. We mainly refer to [Abe18, §1.4].

1.2.3.2. For every finite extension $K(k) \hookrightarrow \mathbb{K}$ we denote by $\mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{K}$ the category of $\mathbb{K}$-*linear overconvergent isocrystals* of X_0 over $K(k)$, namely the category of pairs $(\mathcal{M}, \gamma)$, where $\mathcal{M} \in \mathbf{Isoc}^\dagger(X_0/K(k))$ and $\gamma : \mathbb{K} \to \mathrm{End}(\mathcal{M})$ is a morphism of (noncommutative) $K(k)$-algebras, called the $\mathbb{K}$-*structure*. The morphisms in $\mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{K}$ are morphisms of overconvergent isocrystals over $K(k)$ which commute with the $\mathbb{K}$-structure. We will often omit γ in the notation.

For $(\mathcal{M}, \gamma), (\mathcal{M}', \gamma') \in \mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{K}$, their tensor product in $\mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{K}$ is defined in the following way. We start by considering the tensor product of the two isocrystals $\mathcal{M} \otimes \mathcal{M}'$ in $\mathbf{Isoc}^\dagger(X_0/K(k))$. On this object $\mathbb{K}$ acts via $\gamma \otimes \mathrm{id}$ and $\mathrm{id} \otimes \gamma'$ at the same time. We define $\mathcal{N}$ as the greatest quotient of $\mathcal{M} \otimes \mathcal{M}'$ such that the two $\mathbb{K}$-structures agree. Then we define δ as the unique $\mathbb{K}$-structure induced on $\mathcal{N}$. Finally, we define $(\mathcal{M}, \gamma) \otimes (\mathcal{M}', \gamma') := (\mathcal{N}, \delta)$.

1.2.3.3. If $\mathbb{K} \subseteq \mathbb{L}$ are finite extensions of $K(k)$ and $\{\alpha_1, \dots, \alpha_d\}$ is a basis of $\mathbb{L}$ over $\mathbb{K}$ we define a functor of *extension of scalars*

$$(-) \otimes_\mathbb{K} \mathbb{L} : \ \mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{K} \to \mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{L}.$$

An object $(\mathcal{M}, \gamma) \in \mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{K}$ is sent to $\left(\bigoplus_{i=1}^d \mathcal{M}_i, \delta\right)$, where $\{\mathcal{M}_i\}_{1 \le i \le d}$ are copies of $\mathcal{M}$ and δ is defined as follows. We denote by ι_i the inclusion of $\mathcal{M}_i$ in the direct sum. For every $\alpha \in \mathbb{L}$ and $1 \le i \le d$, we write $\alpha \cdot \alpha_i = \sum_{j=1}^d a_{ij}\alpha_j$, where $a_{ij} \in \mathbb{K}$. The restriction of $\delta(\alpha)$ to $\mathcal{M}_i$ is defined as $\sum_{j=1}^d \iota_j \circ \gamma(a_{ij})$. Different choices of a basis of $\mathbb{L}$ over $\mathbb{K}$ induce functors that are canonically isomorphic.

1.2.3.4. For every finite field k, and for every finite extension $K(k) \subseteq \mathbb{K}$ we choose as a unit object of $\mathbf{Isoc}^\dagger(X_0/K(k))_\mathbb{K}$, the object $\mathbb{K}_{X_0} := K(k)_{X_0} \otimes \mathbb{K}$. We have endowed $\mathbf{Isoc}^\dagger(X_0/K(k_{X_0}))_\mathbb{K}$ with a structure of a $\mathbb{K}$-linear rigid abelian $\otimes$-category. When X_0 is geometrically connected over k, the endomorphism ring of $\mathbb{K}_{X_0}$ is isomorphic to $\mathbb{K}$.

1.2.3.5. We want to define now the inverse image functor for overconvergent isocrystals with $\mathbb{K}$-structure. Suppose given a commutative diagram

$$\begin{array}{ccc} Y_0 & \xrightarrow{f_0} & X_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{F}_{q^{s'}}) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_{q^s}) \end{array}$$

that we will denote by $f_0 : Y_0/\mathbb{F}_{q^{s'}} \to X_0/\mathbb{F}_{q^s}$ with $1 \leq s \leq t$.

We have a *naïve* inverse image functor $f_0^+ : \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_{\mathbb{Q}_{q^{s'}}} \to \mathbf{Isoc}^\dagger(Y_0/\mathbb{Q}_{q^{s'}})$ sending $(\mathcal{M}, \gamma)$ to $(f_0^+\mathcal{M}, f_0^+\gamma)$, where $f_0^+\mathcal{M}$ is the inverse image of $\mathcal{M}$ to Y_0 as an overconvergent isocrystal over $\mathbb{Q}_{q^s}$ and $f_0^+\gamma$ is the $\mathbb{Q}_{q^{s'}}$-structure on $f_0^+\mathcal{M}$, given by the composition of γ with the morphism $\mathrm{End}(\mathcal{M}) \to \mathrm{End}(f_0^+\mathcal{M})$, induced by f_0^+. The functor f_0^+ does not commute in general with the tensor structure, thus one needs to "normalize" it.

The isocrystal $f_0^+\mathcal{M}$ is endowed with two $\mathbb{Q}_{q^{s'}}$-structures. One is $f_0^+\gamma$, the other is the structural $\mathbb{Q}_{q^{s'}}$-structure as an object in $\mathbf{Isoc}^\dagger(Y_0/\mathbb{Q}_{q^{s'}})$. We define $f_0^*(\mathcal{M}, \gamma)$ as the greatest quotient of $f_0^+(\mathcal{M}, \gamma)$ such that the two $\mathbb{Q}_{q^{s'}}$-structures agree. We equip it with the unique induced $\mathbb{Q}_{q^{s'}}$-structure. For every finite extension $\mathbb{Q}_{q^{s'}} \subseteq \mathbb{K}$, this construction extends to a functor $f_0^* : \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_\mathbb{K} \to \mathbf{Isoc}^\dagger(Y_0/\mathbb{Q}_{q^{s'}})_\mathbb{K}$. This will be the inverse image functor we will mainly use.

1.2.3.6. We denote by $F : X_0 \to X_0$ the q-power Frobenius[3]. Let $\mathbb{K}$ be a finite extension for $\mathbb{Q}_q$. For every $\mathcal{M} \in \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ and every $n \in \mathbb{Z}_{>0}$, an isomorphism between $(F^n)^*\mathcal{M}$ and $\mathcal{M}$ will be an *n-th Frobenius structure* of $\mathcal{M}$. We denote by $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ the category of *overconvergent F-isocrystals with $\mathbb{K}$-structure*, namely the category of pairs $(\mathcal{M}, \Phi)$ where $\mathcal{M} \in \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ and Φ is a *1-st Frobenius structure* of $\mathcal{M}$, called the *Frobenius structure* of the F-isocrystal. The morphisms in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ are the morphisms in $\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ that commute with the Frobenius structure. For every positive integer n, the isomorphism

$$\Phi_n := \Phi \circ F^*\Phi \circ \cdots \circ (F^{n-1})^*\Phi$$

[3]The letter F will denote two different types of Frobenius endomorphisms, depending if we are working with lisse sheaves or isocrystals.

will be the *n-th Frobenius structure* of $(\mathcal{M}, \Phi)$. The category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ is a $\mathbb{K}$-linear rigid abelian $\otimes$-category. In this case, if X_0 is connected, but not necessarily geometrically connected, the ring of endomorphisms of the unit object is isomorphic to $\mathbb{K}$.

When X_0 is a smooth variety over $\mathbb{F}_{q^s}$, for every finite extension $\mathbb{Q}_{q^s} \subseteq \mathbb{K}$, the category of $\mathbb{K}$-linear isocrystals over $\mathbb{Q}_{q^s}$ with s-th Frobenius structure is equivalent to the category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ (see [Abe18, Corollary 1.4.11]). We will use this equivalence without further comments.

1.2.3.7. We extend the functor of the extension of scalars to F-isocrystals, imposing $(\mathcal{M}, \Phi) \otimes_\mathbb{K} \mathbb{L} := (\mathcal{M} \otimes_\mathbb{K} \mathbb{L}, \Phi \otimes_\mathbb{K} \mathrm{id}_\mathbb{L})$, where $\Phi \otimes_\mathbb{K} \mathrm{id}_\mathbb{L}$ is a map from $F^*\mathcal{M} \otimes_\mathbb{K} \mathbb{L} = F^*(\mathcal{M} \otimes_\mathbb{K} \mathbb{L})$ to $\mathcal{M} \otimes_\mathbb{K} \mathbb{L}$.

Let $f_0 : Y_0 \to X_0$ be a morphism, for every extension $\mathbb{Q}_q \subseteq \mathbb{K}$, the functor f_0^* defined in §1.2.3.5 for $\mathbb{K}$-linear overconvergent isocrystals extends to a functor

$$f_0^* : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K} \to \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(Y_0/\mathbb{Q}_q)_\mathbb{K}$$

which sends $(\mathcal{M}, \Phi)$ to $(f_0^*\mathcal{M}, f_0^*(\Phi))$. If (X_0, x) is a smooth pointed variety, geometrically connected over $\mathbb{F}_{q^s}$ and $\mathbb{K}$ is a finite extension of $\mathbb{Q}_{q^s}$, the natural morphism $f_0 : X_0/\mathbb{F}_{q^s} \to X_0/\mathbb{F}_q$, induces a functor

$$\Psi_{x,\mathbb{K}} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K} \to \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K} \to \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_\mathbb{K}$$

which sends $(\mathcal{M}, \Psi)$ to $f_0^*\mathcal{M}$. We denote the objects in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K}$ with a subscript $_0$ and we will remove it when we consider the image by $\Psi_{x,\mathbb{K}}$ in $\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_\mathbb{K}$.

1.2.3.8. For every finite extension $K(k) \subseteq \mathbb{K}$, the category $\mathbf{Isoc}^\dagger(\mathrm{Spec}(k)/K(k))_\mathbb{K}$ is equivalent to $\mathbf{Vec}_\mathbb{K}$ as a rigid abelian $\otimes$-category. Moreover, if $k \subseteq k'$ is an extension of finite fields, and $K(k') \subseteq \mathbb{K}$, the Tannakian category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathrm{Spec}(k')/K(k))_\mathbb{K}$ is equivalent to the category of (finite-dimensional) $\mathbb{K}$-vector spaces endowed with an automorphism.

1.2.3.9. Let (X_0, x) be a smooth pointed variety, geometrically connected over $\mathbb{F}_{q^s}$. Let E_λ be a finite extension of $\mathbb{Q}_{q^s}$ in $\overline{\mathbb{Q}}_p$. The category $\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_{E_\lambda}$ admits a fibre functor over some finite extension of E_λ. Assume that $\deg(x_0) = n$. Let $i_0 : x_0/\mathbb{F}_{q^n} \hookrightarrow X_0/\mathbb{F}_{q^s}$ the immersion of the closed point x_0 in X_0 (notation as in §1.2.3.5). Let $E_\lambda^{(x_0)}$ be the compositum of E_λ and $\mathbb{Q}_{q^n}$ in $\overline{\mathbb{Q}}_p$. Then the functor

$$\omega_{x,E_\lambda} : \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_{E_\lambda} \xrightarrow{\otimes_{E_\lambda} E_\lambda^{(x_0)}} \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_{E_\lambda^{(x_0)}} \xrightarrow{i_0^*} \mathbf{Isoc}^\dagger(x_0/\mathbb{Q}_{q^n})_{E_\lambda^{(x_0)}} \simeq \mathbf{Vec}_{E_\lambda^{(x_0)}}$$

is a fibre functor, as proven in [Cre92a, Lemma 1.8]. This means that for every finite extension E_λ of $\mathbb{Q}_{q^s}$, the category $\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_{E_\lambda}$ is *Tannakian*. Moreover, the composition of Ψ_{x,E_λ} with

ω_{x,E_λ} is a fibre functor for $\mathbf{F\text{-}Isoc}^\dagger(X_0/\mathbb{Q}_q)_{E_\lambda}$ over $E_\lambda^{(x_0)}$, that we will denote by the same symbol. Thus, for every finite extension $\mathbb{Q}_{q^s} \subseteq E_\lambda$, the category $\mathbf{F\text{-}Isoc}^\dagger(X_0/\mathbb{Q}_q)_{E_\lambda}$ is *Tannakian.*[4]

1.2.3.10. Let $i_0 : x_0' \hookrightarrow X_0$ be the immersion of a closed point of degree n. Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_q$ and $\mathbb{L}$ a finite extension of $\mathbb{K}$ which contains $\mathbb{Q}_{q^n}$. For every $\mathcal{M}_0 \in \mathbf{F\text{-}Isoc}^\dagger(X_0/\mathbb{Q}_q)_{\mathbb{K}}$, we denote by $F_{x_0'}$ the n-th Frobenius structure of $i_0^*(\mathcal{M}_0) \otimes_{\mathbb{K}} \mathbb{L}$. This will be the *(linearized geometric) Frobenius* of $\mathcal{M}_0$ at x_0'. By §1.2.3.8, it corresponds to a linear automorphism of an $\mathbb{L}$-vector space. The characteristic polynomial

$$P_{x_0'}(\mathcal{M}_0, t) := \det(1 - tF_{x_0'} | i_0^*(\mathcal{M}_0) \otimes_{\mathbb{K}} \mathbb{L}) \in \mathbb{K}[t]$$

will be the *(Frobenius) characteristic polynomial* of $\mathcal{M}_0$ at x_0'. It is independent of the choice of i_0 and $\mathbb{L}$.

In analogy with lisse sheaves, we say that overconvergent F-isocrystals are pure, ι-pure, mixed or ι-mixed, if they satisfy the similar conditions on the eigenvalues of the Frobenii at closed points.

1.3 Generalities

1.3.1 Coefficient objects

Let X_0 be a smooth variety over $\mathbb{F}_q$. Following [Ked18], we use a notation to work with lisse sheaves and overconvergent F-isocrystals at the same time.

Definition 1.3.1.1 (Coefficient objects)**.** For every prime $\ell \neq p$ and every finite field extension $\mathbb{K}/\mathbb{Q}_\ell$, a $\mathbb{K}$-*coefficient object* will be a Weil lisse $\mathbb{K}$-sheaf. If $\mathbb{K}$ is a finite field extension of $\mathbb{Q}_q$, a $\mathbb{K}$-*coefficient object* will be an object in $\mathbf{F\text{-}Isoc}^\dagger(X_0/\mathbb{Q}_q)_{\mathbb{K}}$. For a field $\mathbb{K}$ of one of the two kinds, we denote by $\mathbf{Coef}(X_0, \mathbb{K})$ the category of $\mathbb{K}$-coefficient objects. For every object in $\mathbf{Coef}(X_0, \mathbb{K})$, the field $\mathbb{K}$ will be its *field of scalars.* A *coefficient object* will be a $\mathbb{K}$-coefficient object for some unspecified field of scalars $\mathbb{K}$. For every prime ℓ, the 2-colimit of the categories $\mathbf{Coef}(X_0, E_\lambda)$ with $E_\lambda \subseteq \overline{\mathbb{Q}}_\ell$ will be the category of $\overline{\mathbb{Q}}_\ell$-*coefficient objects* and it will be denoted by $\mathbf{Coef}(X_0, \overline{\mathbb{Q}}_\ell)$.

We will also work with a category of *geometric coefficient objects.* This is built from the category of coefficient objects by forgetting the Frobenius structure. To get Tannakian categories, in this case, we will put an additional assumption on the fields of scalars.

[4]The category $\mathbf{F\text{-}Isoc}^\dagger(X_0/\mathbb{Q}_q)_{E_\lambda}$ is actually Tannakian even when E_λ is just a finite extension of $\mathbb{Q}_q$. For simplicity, in what follows, we will mainly work with finite extensions of $\mathbb{Q}_{q^s}$, in order to make $\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_{E_\lambda}$ a Tannakian category.

Notation 1.3.1.2. From now on in §1, except when explicitly stated otherwise, (X_0, x) will be a smooth pointed variety over $\mathbb{F}_q$, geometrically connected over $\mathbb{F}_{q^s}$ for some $s \in \mathbb{Z}_{>0}$.

Definition 1.3.1.3 (Admissible fields)**.** We say that a finite extension of $\mathbb{Q}_{q^s}$ is a *p-adic admissible field (for X_0).* To uniformize the notation, when ℓ is a prime different from p, we also say that every finite extension of $\mathbb{Q}_\ell$ is an *ℓ-adic admissible field.* We will refer to this second kind of fields as *étale admissible fields.* When E_λ is an admissible field, we will say that the place λ is *admissible.*

Definition 1.3.1.4 (Geometric coefficient objects)**.** For every p-adic admissible field $\mathbb{K}$, we have a functor of Tannakian categories $\Psi_{x,\mathbb{K}} : \mathbf{F\text{-}Isoc}^\dagger(X_0/\mathbb{Q}_q)_\mathbb{K} \to \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_\mathbb{K}$ which forgets the Frobenius structure (see §1.2.3.7). We denote by $\mathbf{Coef}(X^{(x)}, \mathbb{K})$ the smallest Tannakian subcategory of $\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_\mathbb{K}$ containing the essential image of $\Psi_{x,\mathbb{K}}$. We will say that the category $\mathbf{Coef}(X^{(x)}, \mathbb{K})$ is the category of *geometric $\mathbb{K}$-coefficient objects (with respect to x).*

When $\mathbb{K}$ is an étale admissible field, we have again a functor $\Psi_{x,\mathbb{K}} : \mathbf{Weil}(X_0, \mathbb{K}) \to \mathbf{LS}(X^{(x)}, \mathbb{K})$ which forgets the Frobenius structure (see §1.2.2.2). The category of *geometric $\mathbb{K}$-coefficient objects (with respect to x)* will be the smallest Tannakian subcategory of $\mathbf{LS}(X^{(x)}, \mathbb{K})$ containing the essential image of $\Psi_{x,\mathbb{K}}$ and it will be denoted by $\mathbf{Coef}(X^{(x)}, \mathbb{K})$.

For every ℓ, the category of geometric $\overline{\mathbb{Q}}_\ell$-coefficient objects will be the 2-colimit of the categories of geometric E_λ-coefficient objects when E_λ varies among the admissible fields for X_0 in $\overline{\mathbb{Q}}_\ell$. It will be denoted by $\mathbf{Coef}(X^{(x)}, \overline{\mathbb{Q}}_\ell)$ and $\Psi_{x,\overline{\mathbb{Q}}_\ell}$ will be the functor induced by the functors $\Psi_{x,\mathbb{K}}$. If $\mathcal{E}_0$ is a $\overline{\mathbb{Q}}_\ell$-coefficient object, we drop the subscript $_0$ to indicate $\Psi_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{E}_0)$, thus we write $\mathcal{E}$ for $\Psi_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{E}_0)$. When X_0 is geometrically connected over $\mathbb{F}_q$ we drop the superscript $^{(x)}$ in the notation for the categories of coefficient objects, as they do not depend on x.

Definition 1.3.1.5 (Geometric properties)**.** Let $\mathcal{E}_0$ a $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}_0$. We say that $\mathcal{E}_0$ is *geometrically semi-simple*, *geometrically trivial* or *geometrically finite* if the associated geometric coefficient object $\mathcal{E}$ is semi-simple, trivial or finite in $\mathbf{Coef}(X^{(x)}, \overline{\mathbb{Q}}_\ell)$. Notice that although the object $\mathcal{E}$ depends on x, these properties for $\mathcal{E}$ depends only on $\mathcal{E}_0$.

Definition 1.3.1.6 (Cohomology of coefficient objects)**.** Let $\mathcal{E}_0$ be an E_λ-coefficient over X_0. If $\mathcal{E}_0$ is a lisse sheaf, we denote by $H^i(X^{(x)}, \mathcal{E})$ (resp. $H^i_c(X^{(x)}, \mathcal{E})$) the λ-adic étale cohomology (resp. the λ-adic étale cohomology with compact support) of $X^{(x)}$ with coefficients in $\mathcal{E}$ and by $H^i(X_0, \mathcal{E}_0)$ (resp. $H^i_c(X_0, \mathcal{E}_0)$) the fixed points by the action of F^s on $H^i(X^{(x)}, \mathcal{E})$ (resp. $H^i_c(X^{(x)}, \mathcal{E})$). When E_λ is p-adic, we denote by $H^i(X^{(x)}, \mathcal{E})$ (resp. $H^i_c(X^{(x)}, \mathcal{E})$) the rigid cohomology (resp. the rigid cohomology with compact support) of X_0 with coefficients in $\mathcal{E}$. We also denote by $H^i(X_0, \mathcal{E}_0)$ and $H^i_c(X_0, \mathcal{E}_0)$ the respective E_λ-vector spaces of fixed points by the action of the q^s-power absolute Frobenius F^s of X_0.

Remark 1.3.1.7. For both kinds of coefficient objects, if $E_{\lambda,X}$ is the unit object of $\mathbf{Coef}(X^{(x)}, E_\lambda)$, the E_λ-vector space $\mathrm{Hom}(E_{\lambda,X}, \mathcal{E})$ is canonically isomorphic to $H^0(X^{(x)}, \mathcal{E})$.

We also have a canonical isomorphism between $\mathrm{Hom}(E_{\lambda,X_0}, \mathcal{E}_0)$ and $H^0(X_0, \mathcal{E}_0)$, where E_{λ,X_0} is the unit object in $\mathbf{Coef}(X_0, E_\lambda)$.

Proposition 1.3.1.8. *Let (X_0, x) be a smooth connected pointed variety, geometrically connected over $\mathbb{F}_{q^s}$. The functor $(F^s)^*$ is a $\otimes$-autoequivalence of $\mathbf{Coef}(X^{(x)}, E_\lambda)$. In particular, when $E_\lambda^{(x_0)} = E_\lambda$, the pair $(\mathbf{Coef}(X^{(x)}, E_\lambda), (F^s)^*)$ is a neutral Tannakian category with Frobenius, as defined in A.1.1.*

Proof. For lisse sheaves the result is well-known. In the p-adic case see [Abe18, Remark in §1.1.3] or [Laz17, Corollary 6.2] for a proof which does not use arithmetic $\mathscr{D}$-modules. □

Corollary 1.3.1.9. *Any irreducible object in $\mathbf{Coef}(X^{(x)}, E_\lambda)$ admits an n-th Frobenius structure for some $n \in \mathbb{Z}_{>0}$.*

Proof. By definition, an irreducible object $\mathcal{F}$ in $\mathbf{Coef}(X^{(x)}, E_\lambda)$ is a subquotient of some geometric coefficient object $\mathcal{E}$ that admits a Frobenius structure. By Proposition 1.3.1.8, the functor $(F^s)^*$ is an autoequivalence, thus it permutes the isomorphism classes of the irreducible subquotients of $\mathcal{E}$. This implies that there exists $n > 0$ such that $(F^{ns})^*\mathcal{F} \simeq \mathcal{F}$, as we wanted. □

Remark 1.3.1.10. When X_0 is geometrically connected over $\mathbb{F}_q$, the category $\mathbf{Coef}(X, \mathbb{Q}_q)$ is the same category as the one considered by Crew to define the fundamental group at the end of §2.5 in [Cre92a]. A priori, this category is not equivalent to the one considered by Abe to define, for example, the fundamental group in [Abe18, §2.4.17]. By Corollary 1.3.1.9, the category $\mathbf{Coef}(X, \mathbb{Q}_q)$ is a Tannakian subcategory of the one defined by Abe.

Definition 1.3.1.11. A $\mathbb{K}$-coefficient object is said *constant* if it is geometrically trivial, i.e. if after applying $\Psi_{x,\mathbb{K}}$ it becomes isomorphic to a direct sum of unit objects. We denote by $\mathbf{Coef}_{cst}(X_0, E_\lambda)$ the (strictly) full subcategory of $\mathbf{Coef}(X_0, E_\lambda)$ of constant E_λ-coefficient objects. It is a Tannakian subcategory of $\mathbf{Coef}(X_0, E_\lambda)$ which does not depend on x. We define the category of constant $\overline{\mathbb{Q}}_\ell$-coefficient objects, as the 2-colimit of the categories of constant E_λ-coefficient objects.

1.3.1.12. For every prime ℓ, the category $\mathbf{Coef}(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ is canonically equivalent to the category of $\overline{\mathbb{Q}}_\ell$-vector spaces endowed with an automorphism. For every $a \in \overline{\mathbb{Q}}_\ell^\times$ we define $\overline{\mathbb{Q}}_\ell^{(a)}$ as the rank 1 coefficient object over $\mathrm{Spec}(\mathbb{F}_q)$ associated to the vector space $\overline{\mathbb{Q}}_\ell$ endowed with the multiplication by a.

Definition. Let $p_{X_0} : X_0 \to \mathrm{Spec}(\mathbb{F}_q)$ be the structural morphism. For every $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}_0$ and every $a \in \overline{\mathbb{Q}}_\ell^\times$, we define

$$\mathcal{E}_0^{(a)} := \mathcal{E}_0 \otimes p_{X_0}^* \left(\overline{\mathbb{Q}}_\ell^{(a)}\right)$$

as the *twist* of $\mathcal{E}_0$ by a. A twist is *algebraic* if a is algebraic.

Remark 1.3.1.13. The operation of twisting coefficient objects by an element $a \in \overline{\mathbb{Q}}_\ell^\times$ gives an exact autoequivalence of the category $\mathbf{Coef}(X_0, \overline{\mathbb{Q}}_\ell)$. In particular, for every coefficient object, the property of being absolutely irreducible is preserved by any twist.

1.3.1.14. For every $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}_0$ of rank r, we can associate at every closed point x_0 of X_0 the *(Frobenius) characteristic polynomial* of $\mathcal{E}_0$ at x_0, denoted $P_{x_0}(\mathcal{E}_0, t) = 1 + a_1 t + \dots a_r t^r$, where $(a_1, \dots, a_r) \in \overline{\mathbb{Q}}_\ell^{r-1} \times \overline{\mathbb{Q}}_\ell^\times$.

Definition. For every coefficient object $\mathcal{E}_0$, the *Frobenius characteristic polynomial function* associated to $\mathcal{E}_0$ is the function of sets $P_{\mathcal{E}_0} : |X_0| \to \overline{\mathbb{Q}}_\ell^{r-1} \times \overline{\mathbb{Q}}_\ell^\times$ that sends x_0 to the coefficients of $P_{x_0}(\mathcal{E}_0, t)$.

Definition 1.3.1.15. Let ℓ be a prime number, $\mathbb{K}$ a field endowed with an inclusion $\tau : \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_\ell$. We will say that a $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}_0$ is $\mathbb{K}$-*rational with respect to* τ if the characteristic polynomials at closed points have coefficients in the image of τ. A $\mathbb{K}$-*rational coefficient object* will be the datum of $\tau : \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_\ell$ and a $\overline{\mathbb{Q}}_\ell$-coefficient object that is $\mathbb{K}$-rational with respect to τ. We will also say that an E_λ-coefficient object is E-rational if it is E-rational with respect to the natural embedding $E \hookrightarrow E_\lambda \subseteq \overline{\mathbb{Q}}_\ell$. We say that a coefficient object is *algebraic* if it is $\overline{\mathbb{Q}}$-rational for one (or equivalently any) map $\tau : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. A coefficient object is said *p-plain* if it is algebraic and all the eigenvalues at closed points are p-plain (see 1.2.0.2 for the notation).

We can compare two $\mathbb{K}$-rational coefficient objects with different fields of scalars looking at their characteristic polynomial functions.

Definition 1.3.1.16. Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be two coefficient objects that are $\mathbb{K}$-rational with respect to τ and τ' respectively. We say that $\mathcal{E}_0$ and $\mathcal{F}_0$ are $\mathbb{K}$-*compatible* if their characteristic polynomials at closed points are the same as polynomials in $\mathbb{K}[t]$, after the identifications given by τ and τ'.

Our general aim in §1 will be to convert the numerical data provided by the Frobenius characteristic polynomials at closed points to structural properties of the coefficient objects. As an example, we prove the following general statement on Weil lisse sheaves.

Proposition 1.3.1.17. *Let ℓ be a prime different from p and let $\mathcal{V}_0$ be a Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0. If all the eigenvalues of the Frobenius at x_0 are ℓ-adic units, then $\mathcal{V}_0$ is an étale lisse sheaf. In particular, p-plain lisse sheaves are étale.*

Proof. The property on the eigenvalues is preserved after an extension of the base field. Thus, we can assume that x_0 is a rational point, because étale lisse sheaves satisfy étale descent. Let ρ_0 be the ℓ-adic representation of $W(X_0, x)$ associated to $\mathcal{V}_0$ and denote by $\Pi_0 \subseteq \mathrm{GL}(\mathcal{V}_x)$ the image of ρ_0. Write Π for the image of $\pi_1^{\text{ét}}(X, x)$ via ρ_0. The group Π_0 is generated by Π and $\rho_0(\gamma)$, where γ is some element in F_{x_0}.

Let Γ be the closure in $\mathrm{GL}(\mathcal{V}_x)$ of the group generated by $\rho_0(\gamma)$. By the assumption on the eigenvalues of $\rho_0(\gamma)$, the topological group Γ is compact, hence profinite. Moreover, the group Γ normalizes Π, so that $\Pi \cdot \Gamma \subseteq \mathrm{GL}(\mathcal{V}_x)$ is a profinite group. By construction, Π_0 is contained in the profinite group $\Pi \cdot \Gamma$. Therefore, the ℓ-adic representation ρ_0 factors through the profinite completion of $W(X_0, x)$, which is $\pi_1^{\text{ét}}(X_0, x)$. This concludes the proof. $\square$

1.3.2 Monodromy groups

We introduce now the main characters of §1: the fundamental groups and the monodromy groups of coefficient objects. They will sit in a *fundamental exact sequence*, which is the analogue of the sequence relating the geometric étale fundamental group and the arithmetic étale fundamental group. We have presented this exact sequence for general *neutral Tannakian categories with Frobenius* in §A.2.

1.3.2.1. For every étale admissible field E_λ we take the fibre functor

$$\omega_{x,E_\lambda} : \mathbf{Weil}(X_0, E_\lambda) \to \mathbf{Vec}_{E_\lambda}$$

attached to x, which sends a lisse sheaf $\mathcal{V}_0$ to the stalk $\mathcal{V}_x$. When E_λ is a p-adic admissible field, we have defined in §1.2.3.9 a fibre functor for $\mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_{q^s})_{E_\lambda}$ over $E_\lambda^{(x_0)}$, denoted by ω_{x,E_λ}. For symmetry reasons, when E_λ is an étale field, we set $E_\lambda^{(x_0)} := E_\lambda$. Thus for every admissible field E_λ, we have a fibre functor ω_{x,E_λ} of $\mathbf{Coef}(X^{(x)}, E_\lambda)$ over $E_\lambda^{(x_0)}$. We will denote with the same symbol the fibre functor induced on $\mathbf{Coef}(X_0, E_\lambda)$. As the fibre functors commute with the extension of scalars, for every ℓ we also have a fibre functor over $\overline{\mathbb{Q}}_\ell$ for $\overline{\mathbb{Q}}_\ell$-coefficient objects. We will denote it by $\omega_{x,\overline{\mathbb{Q}}_\ell}$.

Definition (Fundamental groups)**.** For every admissible field E_λ, we denote by $\pi_1^\lambda(X_0, x)$ the Tannakian group over $E_\lambda^{(x_0)}$ of $\mathbf{Coef}(X_0, E_\lambda)$ with respect to ω_{x,E_λ}. We write $\pi_1^\lambda(X, x)$ for the Tannakian group of $\mathbf{Coef}(X^{(x)}, E_\lambda)$ with respect to the restriction of ω_{x,E_λ}. The functor

$$\Psi_{x,E_\lambda} : \mathbf{Coef}(X_0, E_\lambda) \to \mathbf{Coef}(X^{(x)}, E_\lambda)$$

induces a closed immersion $\pi_1^\lambda(X, x) \hookrightarrow \pi_1^\lambda(X_0, x)$. We also denote by $\pi_1^\lambda(X_0, x)^{cst}$ the quotient of $\pi_1^\lambda(X_0, x)$, corresponding to the inclusion of $\mathbf{Coef}_{cst}(X_0, E_\lambda)$ in $\mathbf{Coef}(X_0, E_\lambda)$.

Remark 1.3.2.2. Suppose that $E_\lambda^{(x_0)} = E_\lambda$, then there exists an isomorphism of functors $\eta : \omega_{x,E_\lambda} \Rightarrow \omega_{x,E_\lambda} \circ (F^s)^*$. For lisse sheaves, this is induced by the choice of an étale path between x and the $\mathbb{F}$-point over x with respect to $F^s : X \to X$. In the case of overconvergent F-isocrystals, this is constructed in [Abe18, §2.4.18]. Let us briefly recall it.

Let $i_0 : x_0 \hookrightarrow X_0$ be the inclusion of x_0, the closed point underlying x. Let σ be the lift to $\mathbb{Q}_{q^n}$ of the q^s-power Frobenius of $\mathbb{F}_{q^n}$, where n is the degree of x_0. For every overconvergent F-isocrystal $\mathcal{M}$ on X_0 we define

$$\eta_{\mathcal{M}} : i_0^+(\mathcal{M}) \otimes_{\mathbb{Q}_{q^n} \otimes E_\lambda} E_\lambda \xrightarrow{\sim} \left(\mathbb{Q}_{q^n} \otimes_{\sigma \nwarrow \mathbb{Q}_{q^n}} i_0^+(\mathcal{M}_0)\right) \otimes_{\mathbb{Q}_{q^n} \otimes E_\lambda} E_\lambda,$$

as the isomorphism which maps $m \otimes e$ to $1 \otimes m \otimes e$. The functor i_0^+ is the *naïve* pullback defined in §1.2.3.5. The isomorphisms $\eta_{\mathcal{M}}$ induce an isomorphism of fibre functors $\eta : \omega_{x,E_\lambda} \Rightarrow \omega_{x,E_\lambda} \circ (F^s)^*$. Thanks to this and Proposition 1.3.1.8, one can define a *Weil group* for coefficient objects over the field E_λ (see §A.1.4).

1.3.2.3. Every E_λ-coefficient object $\mathcal{E}_0$ generates three E_λ-linear Tannakian categories, the *arithmetic* one $\langle \mathcal{E}_0 \rangle \subseteq \mathbf{Coef}(X_0, E_\lambda)$, the *geometric* one $\langle \mathcal{E} \rangle \subseteq \mathbf{Coef}(X^{(x)}, E_\lambda)$ and the Tannakian category of constant objects $\langle \mathcal{E}_0 \rangle_{cst} \subseteq \langle \mathcal{E}_0 \rangle$. We will consider these categories endowed with the fibre functors obtained by restricting ω_{x,E_λ}.

Definition. (Monodromy groups) We denote by $G(\mathcal{E}_0, x)$ the *(arithmetic) monodromy group* of $\mathcal{E}_0$, namely the Tannakian group of $\langle \mathcal{E}_0 \rangle$. The *geometric monodromy group* of $\mathcal{E}_0$ will be instead the Tannakian group of $\langle \mathcal{E} \rangle$ and it will denoted by $G(\mathcal{E}, x)$. We will also consider the quotient $G(\mathcal{E}_0, x) \twoheadrightarrow G(\mathcal{E}_0, x)^{cst}$, which corresponds to the inclusion $\langle \mathcal{E}_0 \rangle_{cst} \subseteq \langle \mathcal{E}_0 \rangle$. These three groups are quotients of the fundamental groups defined in §1.3.2.1.

Remark 1.3.2.4. When $\mathcal{V}_0$ is a lisse sheaf and $\rho_0 : W(X_0, x) \to \mathrm{GL}(\mathcal{V}_x)$ is the associated ℓ-adic representation, then $G(\mathcal{V}_0, x)$ is the Zariski-closure of the image of ρ_0 and $G(\mathcal{V}, x)$ is the Zariski-closure of $\rho_0(\pi_1^{\text{ét}}(X, x))$. When $\mathcal{M}_0$ is an overconvergent F-isocrystal, $G(\mathcal{M}, x)$ is the same group defined by Crew in [Cre92a] and denoted by $\mathrm{DGal}(\mathcal{M}, x)$. This group is even isomorphic to the group $\mathrm{DGal}(\mathcal{M}, x)$ which appears in [AE16]. This agrees with our previous Remark 1.3.1.10.

Remark 1.3.2.5. As X_0 is connected, the étale fundamental groups associated to two different $\mathbb{F}$-points of X_0 are (non-canonically) isomorphic. Hence, in the case of lisse sheaves, the isomorphism class of the monodromy groups does not depend on the choice of x. For overconvergent F-isocrystals, by the result of Deligne in [Del90], the monodromy groups associated to two different $\mathbb{F}$-points become isomorphic after passing to a finite extension of E_λ. We do not know any better result in this case.

Let us present now the fundamental exact sequence of X_0 attached to some admissible place λ. The sequence is a generalization of the one proven in [Pal15, Proposition 4.7].

Proposition 1.3.2.6. *Let (X_0, x) be a smooth pointed variety, geometrically connected over $\mathbb{F}_{q^s}$ and let λ be an admissible place for X_0 such that $E_\lambda^{(x_0)} = E_\lambda$.*

(i) *The natural morphisms previously presented give an exact sequence*

$$1 \to \pi_1^\lambda(X,x) \to \pi_1^\lambda(X_0,x) \to \pi_1^\lambda(X_0,x)^{cst} \to 1.$$

(ii) *For every E_λ-coefficient object $\mathcal{E}_0$ and every $\mathcal{F} \in \langle \mathcal{E} \rangle$, there exists $\mathcal{G}_0 \in \langle \mathcal{E}_0 \rangle$ such that $\mathcal{F} \subseteq \mathcal{G}$.*

(iii) *For every E_λ-coefficient object $\mathcal{E}_0$, the exact sequence of (i) sits into the following commutative diagram with exact rows and surjective vertical arrows*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^\lambda(X,x) \arrow[r] \arrow[d, two heads] & \pi_1^\lambda(X_0,x) \arrow[r] \arrow[d, two heads] & \pi_1^\lambda(X_0,x)^{cst} \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & G(\mathcal{E},x) \arrow[r] & G(\mathcal{E}_0,x) \arrow[r] & G(\mathcal{E}_0,x)^{cst} \arrow[r] & 1.
\end{tikzcd}$$

(iv) *The affine group scheme $\pi_1(\mathbf{C}_0,\omega_0)^{cst}$ is isomorphic to the pro-algebraic completion of $\mathbb{Z}$ over $\mathbb{K}$ and $G(\mathcal{E}_0,x)^{cst}$ is a commutative algebraic group.*

(v) *The affine group scheme $\pi_1^\lambda(X_0,x)^{cst}$ is canonically isomorphic to $\pi_1^\lambda(\mathrm{Spec}(\mathbb{F}_{q^s}),x)$. In particular, the profinite group $\pi_0(\pi_1^\lambda(X_0,x)^{cst})$ is canonically isomorphic to $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^s})$.*

Proof. By Proposition 1.3.1.8, the datum of $(\mathbf{Coef}(X^{(x)},E_\lambda),(F^s)^*)$ is a neutral Tannakian category with Frobenius, in the sense of Definition A.1.1. Thus by Proposition A.2.3 we get all the parts from (i) to (iv).

For (v), let $q_{X_0} : X_0 \to \mathrm{Spec}(\mathbb{F}_{q^s})$ be the morphism induced by the $\mathbb{F}$-point x. We have a inverse image functor

$$q_{X_0}^* : \mathbf{Coef}(\mathrm{Spec}(\mathbb{F}_{q^s}),E_\lambda) \to \mathbf{Coef}_{cst}(X_0,E_\lambda).$$

We want to construct a quasi-inverse q_{X_0*}. For every $\mathcal{E}_0 \in \mathbf{Coef}_{cst}(X_0,E_\lambda)$, we have a canonical identification $H^0(X^{(x)},F^*\mathcal{E}) = H^0(X^{(x)},\mathcal{E})$, thus the s-th Frobenius structure Φ_s of $\mathcal{E}_0$ induces an automorphism of $H^0(X^{(x)},\mathcal{E})$ that we denote by $q_{X_0*}(\Phi_s)$. We define $q_{X_0*}(\mathcal{E}_0)$ as the pair

$$(H^0(X^{(x)},\mathcal{E}),q_{X_0*}(\Phi_s)) \in \mathbf{Coef}(\mathrm{Spec}(\mathbb{F}_{q^s}),E_\lambda).$$

The functor q_{X_0*} is a quasi-inverse of $q_{X_0}^*$, thus $q_{X_0}^*$ induces an isomorphism

$$\pi_1^\lambda(X_0,x)^{cst} \xrightarrow{\sim} \pi_1^\lambda(\mathrm{Spec}(\mathbb{F}_{q^s}),x).$$

Since $\mathbf{Coef}(\mathrm{Spec}(\mathbb{F}_{q^s}),E_\lambda)$ is canonically equivalent to $\mathbf{Rep}_{E_\lambda}(W(\mathbb{F}/\mathbb{F}_{q^s}))$, the profinite group $\pi_0(\pi_1^\lambda(X_0,x)^{cst})$ is canonically isomorphic to $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^s})$. $\square$

1.3.3 Comparison with the étale fundamental group

1.3.3.1. We continue our analysis of the fundamental groups of coefficient objects focusing on their groups of connected components. The statements of this section are fairly easy for lisse sheaves and difficult for overconvergent F-isocrystals. In the latter case, Crew had already studied the problem when X_0 is a smooth curve [Cre92a]. Later in [Ete02], Étesse proved that overconvergent isocrystals with and without Frobenius structure over smooth varieties of arbitrary dimension satisfy étale descent[5]. This allows a generalization of Crew's work.

Drinfeld and Kedlaya have presented in [DK17, Appendix B] how to perform such a generalization for the arithmetic fundamental group of overconvergent F-isocrystals. We will be mainly interested in the extension of their result to the geometric fundamental group.

1.3.3.2. Let E_λ be an admissible field for X_0 such that $E_\lambda^{(x_0)} = E_\lambda$. Following [DK17, Remark B.2.5], we define

$$\mathbf{Rep}_{E_\lambda}^{\mathrm{smooth}}(\pi_1^{\text{ét}}(X_0,x)) := 2\text{-}\varinjlim_{H} \mathbf{Rep}_{E_\lambda}(\pi_1^{\text{ét}}(X_0,x)/H)$$

where H varies among the normal open subgroups of $\pi_1^{\text{ét}}(X_0,x)$. This category is endowed with a fully faithful embedding

$$\mathbf{Rep}_{E_\lambda}^{\mathrm{smooth}}(\pi_1^{\text{ét}}(X_0,x)) \hookrightarrow \mathbf{Coef}(X_0,E_\lambda).$$

The essential image is closed under subobjects. This functor induces a surjective morphism $\pi_1^\lambda(X_0,x) \twoheadrightarrow \pi_1^{\text{ét}}(X_0,x)$, where $\pi_1^{\text{ét}}(X_0,x)$ denotes here the (pro-constant) profinite group scheme over E_λ associated to the profinite group $\pi_1^{\text{ét}}(X_0,x)$. The subcategory

$$\mathbf{Rep}_{E_\lambda}^{\mathrm{smooth}}(\mathrm{Gal}(\mathbb{F}/k_{X_0})) \subseteq \mathbf{Rep}_{E_\lambda}^{\mathrm{smooth}}(\pi_1^{\text{ét}}(X_0,x)),$$

of representations which factor through $\mathrm{Gal}(\mathbb{F}/k_{X_0})$ is sent by the functor to $\mathbf{Coef}(X_0,E_\lambda)^{cst}$. Therefore, the composition of the morphisms

$$\pi_1^\lambda(X_0,x) \twoheadrightarrow \pi_1^{\text{ét}}(X_0,x) \twoheadrightarrow \mathrm{Gal}(\mathbb{F}/k_{X_0})$$

factors through $\pi_1^\lambda(X_0,x)^{cst}$. By Proposition 1.3.2.6.(v), the induced morphism $\pi_1^\lambda(X_0,x)^{cst} \twoheadrightarrow \mathrm{Gal}(\mathbb{F}/k_{X_0})$ is surjective with connected Kernel. Finally, the homotopy exact sequence for the étale fundamental group and the fundamental exact sequence of Proposition 1.3.2.6.(i) fit in a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^\lambda(X,x) \arrow[r] \arrow[d] & \pi_1^\lambda(X_0,x) \arrow[r] \arrow[d, two heads] & \pi_1^\lambda(X_0,x)^{cst} \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(X,x) \arrow[r] & \pi_1^{\text{ét}}(X_0,x) \arrow[r] & \mathrm{Gal}(\mathbb{F}/k_{X_0}) \arrow[r] & 1.
\end{tikzcd} \tag{1.3.3.1}$$

[5]In the article he states the result for overconvergent F-isocrystals, but the same proof works without Frobenius structure.

The central and the right vertical arrows are the morphisms previously constructed. The left one is the unique morphism making the diagram commutative.

Proposition 1.3.3.3. *Let (X_0, x) be a smooth connected pointed variety. For every admissible place λ, we have a commutative diagram*

$$\begin{tikzcd} 1 \arrow[r] & \pi_0(\pi_1^\lambda(X,x)) \arrow[r] \arrow[d, "\wr"', "\varphi"] & \pi_0(\pi_1^\lambda(X_0,x)) \arrow[r] \arrow[d, "\wr"', "\varphi_0"] & \pi_0(\pi_1^\lambda(X_0,x)^{cst}) \arrow[r] \arrow[d, "\wr"', "\varphi_0^{cst}"] & 1 \\ 1 \arrow[r] & \pi_1^{ét}(X,x) \arrow[r] & \pi_1^{ét}(X_0,x) \arrow[r] & \mathrm{Gal}(\mathbb{F}/k_{X_0}) \arrow[r] & 1, \end{tikzcd} \tag{1.3.3.2}$$

where the vertical arrows are isomorphisms and the rows are exact. The diagram is functorial in (X_0, x), when it varies among the smooth connected pointed varieties.

Proof. The diagram is constructed applying the functor π_0 to (1.3.3.1), hence it is functorial. We start by showing that the upper row is exact. As the functor π_0 is right exact, it is enough to prove the injectivity of the morphism $\pi_0(\pi_1^\lambda(X,x)) \to \pi_0(\pi_1^\lambda(X_0,x))$. We first extend the field of scalars to $\overline{\mathbb{Q}}_\ell$. The π_0 of the Tannakian group of a Tannakian category is the Tannakian group of the subcategory of finite objects. Thus, we have to prove that for every absolutely irreducible finite geometric $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}$, there exists a finite object $\mathcal{F}_0 \in \mathbf{Coef}(X_0, \overline{\mathbb{Q}}_\ell)$, such that $\mathcal{E}$ is a subquotient of $\mathcal{F}$.

By Lemma A.2.2, there exists $\mathcal{F}_0' \in \mathbf{Coef}(X_0, \overline{\mathbb{Q}}_\ell)$ such that $\mathcal{E}$ is a subobject of $\mathcal{F}'$. As $\mathcal{E}$ is absolutely irreducible, we can even assume $\mathcal{F}_0'$ to be absolutely irreducible. In particular, there exist $g_1, \dots, g_n \in G(\mathcal{F}_0', x)(\overline{\mathbb{Q}}_\ell)$ such that $\omega_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{F}_0') = \sum_{i=1}^n g_i(\omega_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{E}))$. The algebraic group $G(\mathcal{F}', x)$ is normal in $G(\mathcal{F}_0', x)$, thus the vector spaces $g_i(\omega_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{E}))$ are $G(\mathcal{F}', x)$-stable for every i. In addition, their monodromy groups as representations of $G(\mathcal{F}', x)$ are all finite, as they are conjugated to the monodromy group of $\mathcal{E}$. Therefore $\mathcal{F}'$, being a sum of finite objects, is a finite object.

Let $W(\mathcal{F}_0', x)$ be the Weil group of $\langle \mathcal{F}' \rangle$, as defined in §A.1.4. Since $G(\mathcal{F}', x)$ is finite, there exists $n \in \mathbb{Z}_{>0}$ such that $(F^n)^*$ acts trivially on it. If ρ' is the representation of $G(\mathcal{F}', x)$ associated to $\mathcal{F}'$, then $(F^n)^*\rho' = \rho'$. Thus $\rho := \bigoplus_{i=0}^{n-1} (F^i)^*\rho'$ can be endowed with a Frobenius structure

$$\Phi : F^* \left(\bigoplus_{i=0}^{n-1} (F^i)^*\rho' \right) \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} (F^i)^*\rho'$$

such that, for every $1 \le i \le n-1$, the restriction of Φ to $F^*\left((F^i)^*\rho'\right)$ is the canonical isomorphism $F^*\left((F^i)^*\rho'\right) = (F^{i+1})^*\rho'$. The pair (ρ, Φ) induces a representation of $W(\mathcal{F}_0', x)$ with finite image and thus a finite coefficient object $\mathcal{F}_0$. The original geometric coefficient object $\mathcal{E}$ is a subobject of $\mathcal{F}$, therefore $\mathcal{F}_0$ satisfies the properties we wanted.

Finally, we prove that the vertical arrows of (1.3.3.2) are isomorphisms. The morphism φ_0^{cst} is an isomorphism by Proposition 1.3.2.6.(v). By diagram chasing, it remains to prove that φ_0 is

an isomorphism. For lisse sheaves, this is quite immediate. If a lisse sheaf has finite arithmetic monodromy group, its associate ℓ-adic representation factors through a finite quotient of the Weil group of X_0. In the p-adic case one can prove that φ_0 is an isomorphism using [Ked11, Theorem 2.3.7], as it is explained in [DK17, Proposition B.7.6.(i)]. □

Proposition 1.3.3.4. *Let $\mathcal{E}_0$ be an E_λ-coefficient object on (X_0, x).*

(i) *For every finite étale morphism $f_0 : (Y_0, y) \to (X_0, x)$ of pointed varieties, after extending E_λ to an admissible field for Y_0, the natural maps $G(f_0^*\mathcal{E}_0, y) \to G(\mathcal{E}_0, x)$ and $G(f^*\mathcal{E}, y) \to G(\mathcal{E}, x)$ are open immersions.*

(ii) *There exists a choice of $f_0 : (Y_0, y) \to (X_0, x)$ such that, after extending E_λ to an admissible field for Y_0, the natural maps of the previous point induce isomorphisms $G(f_0^*\mathcal{E}_0, y) \xrightarrow{\sim} G(\mathcal{E}_0, x)^\circ$ and $G(f^*\mathcal{E}, y) \xrightarrow{\sim} G(\mathcal{E}, x)^\circ$.*

Proof. We notice that by Proposition 1.3.3.3 the group of connected components of the arithmetic monodromy group (resp. geometric monodromy group) are quotients of the arithmetic étale fundamental group (resp. geometric étale fundamental group), thus (i) implies (ii).

When $\mathcal{E}_0$ is a lisse sheaf, (i) is well-known. If $\mathcal{E}_0$ is an overconvergent F-isocrystal, the result on the arithmetic monodromy groups is a consequence of [DK17, Proposition B.7.6.(ii)]. It remains to prove (i) for the geometric monodromy groups of overconvergent F-isocrystals. It is enough to treat the case when $Y_0 \to X_0$ is a Galois cover with Galois group H and Y_0 is geometrically connected over $\mathbb{F}_q$. As Y_0 is geometrically connected over $\mathbb{F}_q$, the group H acts on $\langle f^*\mathcal{E}\rangle$ via E_λ-linear autoequivalences. Let $\langle f^*\mathcal{E}\rangle^H$ be the category of H-equivariant objects in $\langle f^*\mathcal{E}\rangle$. After possibly extending E_λ, we can find isomorphisms of fiber functors between ω_{y,E_λ} and $\omega_{h(y),E_\lambda}$ for every $h \in H$. A choice of these isomorphisms induces an action of H on $G(f^*\mathcal{E}, y)$.

By [Ete02], overconvergent isocrystals with and without F-structure satisfy étale descent. Therefore, there exist fully faithful embeddings $\langle\mathcal{E}\rangle \hookrightarrow \langle f^*\mathcal{E}\rangle^H$ and $\langle f^*\mathcal{E}\rangle^H \hookrightarrow \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_{E_\lambda}$. The former embedding induces a morphism on the Tannakian groups $\varphi : G(f^*\mathcal{E}, y) \rtimes H \to G(\mathcal{E}, x)$. By definition, the subcategory $\langle\mathcal{E}\rangle \subseteq \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_{E_\lambda}$ is closed under the operation of taking subquotients. Thus, the same is true for $\langle\mathcal{E}\rangle \subseteq \langle f^*\mathcal{E}\rangle^H$. This proves that φ is surjective, which in turn implies that $G(f^*\mathcal{E}, y)$ has finite index in $G(\mathcal{E}, x)$. □

Corollary 1.3.3.5. *Let $\mathcal{E}_0$ be a coefficient object on (X_0, x). For every finite étale morphism $f_0 : (Y_0, y) \to (X_0, x)$ of pointed varieties, $\mathcal{E}_0$ is semi-simple (resp. geometrically semi-simple) if and only if $f_0^*\mathcal{E}_0$ is semi-simple (resp. geometrically semi-simple).*

Remark 1.3.3.6. We will see later a variant of Corollary 1.3.3.5 in Corollary 1.3.7.10. In that case the result is obtained as a consequence of the theory of weights and the Langlands correspondence.

1.3.4 The Künneth formula

1.3.4.1. To prove Theorem 1.5.1.1 without using the existence of companions we will need the Künneth formula for the fundamental group parameterizing geometric coefficient objects. For simplicity we will prove it just for smooth connected projective varieties admitting a rational point.

The main ingredient for the Künneth formula is the existence of a direct image functor for smooth and proper morphisms of coefficient objects, that is a right adjoint of the inverse image functor and that satisfies the proper base change. For lisse sheaves the classical direct image has the desired properties. In the p-adic case the construction is more problematic. We will use the direct image functor for *arithmetic $\mathscr{D}$-modules*.

1.3.4.2. For every smooth projective variety X_0 we take the triangulated category with t-structure of *holonomic complexes* $D^\flat_{\mathrm{hol}}(X/\mathbb{Q}_q)$, see [Abe18, Definition 1.1.1]. We have chosen X_0 to be projective in order to make X_0 *realizable* (cf. *loc. cit.*). We also consider the category of holonomic complexes with F-structure, denoted $D^\flat_{\mathrm{hol}}(X_0/\mathbb{Q}_q)$, and for every p-adic admissible field E_λ the categories enriched with E_λ-structure, denoted $D^\flat_{\mathrm{hol}}(X/\mathbb{Q}_q)_{E_\lambda}$ and $D^\flat_{\mathrm{hol}}(X_0/\mathbb{Q}_q)_{E_\lambda}$ [*ibid.* §1.4].

For every proper smooth morphism $f_0 : Y_0 \to X_0$ between smooth geometrically connected projective varieties we dispose of adjuctions (f^+, f_+) and (f_0^+, f_{0+}) of inverse and direct image for holonomic complexes and holonomic complexes with F-structure respectively. They satisfy the proper base change (see [*ibid.* §1.1.3] and [AC13, §1.3.14]).

We also consider the *specialization functors*

$$\widetilde{\mathrm{sp}}_+ : \mathbf{Isoc}^\dagger(X_0/\mathbb{Q}_q)_{E_\lambda} \to D^\flat_{\mathrm{hol}}(X/\mathbb{Q}_q)_{E_\lambda},\ \widetilde{\mathrm{sp}}_{0+} : \mathbf{F\text{-}Isoc}^\dagger(X_0/\mathbb{Q}_q)_{E_\lambda} \to D^\flat_{\mathrm{hol}}(X_0/\mathbb{Q}_q)_{E_\lambda}$$

defined in §1.1.3.11 and §2.4.15 of [Abe, *op. cit.*]. They are fully faithful functors commuting with the inverse image functors.

In light of [Car15, Théorème 3.3.1], for every object $\mathcal{M} \in \mathbf{Isoc}^\dagger(Y_0/\mathbb{Q}_q)_{E_\lambda}$ and $\mathcal{M}_0 \in \mathbf{F\text{-}Isoc}^\dagger(Y_0/\mathbb{Q}_q)_{E_\lambda}$, the complexes $f_+\widetilde{\mathrm{sp}}_+(\mathcal{M})$ and $f_{0+}\widetilde{\mathrm{sp}}_{0+}(\mathcal{M}_0)$ are in the essential image of the specialization functors. Thus f_+ and f_{0+} induce functors $f_* : \mathbf{Coef}(Y, E_\lambda) \to \mathbf{Coef}(X, E_\lambda)$ and $f_{0*} : \mathbf{Coef}(Y_0, E_\lambda) \to \mathbf{Coef}(X_0, E_\lambda)$ that are right adjoints to f^* and f_0^* respectively.

Remark 1.3.4.3. We notice that when $X_0 = \mathrm{Spec}(\mathbb{F}_q)$ and f_0 is the structural morphism, by the adjuction property, for every $\mathcal{M} \in \mathbf{Coef}(Y, E_\lambda)$, the push $f_*\mathcal{M} \in \mathbf{Coef}(\mathrm{Spec}(\mathbb{F}), E_\lambda)$ is the vector space of global sections of $\mathcal{M}$.

Proposition 1.3.4.4. *Let (X_0, x) and (Y_0, y) be two smooth projective connected pointed varieties such that x_0 and y_0 are rational points. For every admissible field E_λ, the projections of $X_0 \times Y_0$ to its factors induce an isomorphism*

$$\pi_1^\lambda(X \times_{\mathbb{F}} Y, x \times y) \xrightarrow{\sim} \pi_1^\lambda(X, x) \times \pi_1^\lambda(Y, y).$$

Proof. Let's denote $f_0 : X_0 \times Y_0 \to Y_0$ the projection to the second factor and $g_0 : X_0 \times y_0 \hookrightarrow X_0 \times Y_0$ the natural inclusion. The morphism g_0 induces a closed immersion $X_0 \hookrightarrow X_0 \times Y_0$ that we denote by the same letter. If φ and ψ are the morphisms induced by g and f on the fundamental groups classifying geometric coefficient objects, we obtain a sequence

$$1 \to \pi_1^\lambda(X, x) \xrightarrow{\varphi} \pi_1^\lambda(X \times_{\mathbb{F}} Y, x \times y) \xrightarrow{\psi} \pi_1^\lambda(Y, y) \to 1$$

with $\psi \circ \varphi$ trivial.

We want to use Theorem A.2.1 to show that it is an exact sequence. This will imply the original statement. Let's consider the sequence of functors

$$\mathbf{Coef}(Y, E_\lambda) \xrightarrow{f^*} \mathbf{Coef}(X \times_{\mathbb{F}} Y, E_\lambda) \xrightarrow{g^*} \mathbf{Coef}(X, E_\lambda).$$

The point (i) of Theorem A.1 (*loc. cit.*) follows from the existence of a section of f_0, namely the closed immersion of $x_0 \times Y_0 \hookrightarrow X_0 \times Y_0$. The point (ii) and (c) are consequence of the existence of a retraction for g_0, given by the first projection $X_0 \times Y_0 \to X_0$.

We want to show now that (a) and (b) are satisfied. Let's consider the commutative square

$$\begin{tikzcd} X_0 \times Y_0 \arrow[r, "f_0"] & Y_0 \\ X_0 \arrow[u, "g_0"] \arrow[r, "f_0'"] & y_0 \arrow[u, "g_0'"'] \end{tikzcd}$$

where f_0' is the restriction of f_0 to $X_0 \times y_0 = X_0$ and g_0' is the closed immersion of y_0 in Y_0.

Lemma 1.3.4.5. *For every geometric E_λ-coefficient object $\mathcal{E}$, the adjuction morphism $f^* f_* \mathcal{E} \to \mathcal{E}$ is injective. Moreover, after applying g^*, the morphism $g^* f^* f_* \mathcal{E} \to g^* \mathcal{E}$ makes $g^* f^* f_* \mathcal{E}$ the maximal trivial subobject of $g^* \mathcal{E}$.*

Proof. To show the injectivity of the adjuction morphism, we use the fiber functors of the Tannakian categories, associated to the rational points we are considering. Let G and H be the affine group schemes $\pi_1^\lambda(X \times_{\mathbb{F}} Y, x \times y)$ and $\pi_1^\lambda(Y, y)$. We know that the functor f^* is equivalent to the functor $\mathrm{Res}_G^H : \mathbf{Rep}_{E_\lambda}(H) \to \mathbf{Rep}_{E_\lambda}(G)$, induced by $\psi : G \to H$.

As we have already proven that $\psi : G \to H$ is surjective, the induction functor $\mathrm{Ind}_H^G : \mathbf{Rep}_{E_\lambda}(G) \to \mathbf{Rep}_{E_\lambda}(H)$ is well defined at the level of finite-dimensional representations and it is the right adjoint of Res_G^H. If we take $N := \mathrm{Ker}(\psi)$, the functor Ind_H^G sends $V \in \mathbf{Rep}_{E_\lambda}(G)$ to V^N, the induced representation of H on the subspace of V fixed by N.

By the uniqueness of the right adjoint of f^*, the counit of (f^*, f_*) is isomorphic to the counit of the adjunction $(\mathrm{Res}_G^H, \mathrm{Ind}_H^G)$, induced by $\psi : G \to H$. If we apply the counit of $(\mathrm{Res}_G^H, \mathrm{Ind}_H^G)$ to a representation $V \in \mathbf{Rep}_{E_\lambda}(G)$ we obtain the natural inclusion $V^N \hookrightarrow V$, in particular an injective map. As a consequence, the maps induced by the counit of the adjunction (f^*, f_*) on geometric coefficient objects are always injective.

We show now the second part of the statement. As g^* commutes with the fiber functors, the morphism $g^*f^*f_*\mathcal{E} \to g^*\mathcal{E}$ is injective. We have natural isomorphisms,

$$g^*f^*f_*\mathcal{E} \simeq f'^*g'^*f_*\mathcal{E} \simeq f'^*f'_*g^*\mathcal{E},$$

the second one given by the proper base change. At the same time we know that $f'_*g^*\mathcal{E} \simeq H^0(X, g^*\mathcal{E})$, in the p-adic case thanks to the Remark 1.3.4.3. Thus $g^*f^*f_*\mathcal{E} \simeq f'^*H^0(X, g^*\mathcal{E})$ is the maximal trivial subobject of $g^*\mathcal{E}$, as we wanted. □

We now verify (a). It is enough to show that if $\mathcal{E}$ is an E_λ-geometric coefficient object of $X \times_{\mathbb{F}} Y$ such that $g^*\mathcal{E}$ is trivial, then $f^*f_*\mathcal{E} \simeq \mathcal{E}$. As $g^*\mathcal{E}$ is trivial, by Lemma 1.3.4.5, we know that $g^*f^*f_*\mathcal{E}$ and $g^*\mathcal{E}$ are isomorphic, thus $f^*f_*\mathcal{E}$ and $\mathcal{E}$ have the same rank. Therefore we know that the adjuction map $f^*f_*\mathcal{E} \to \mathcal{E}$ is an injective map between two objects of the same rank. This means that it is an isomorphism.

To check (b) we have to show that for every geometric coefficient object $\mathcal{E}$, there exists $\mathcal{F} \subseteq \mathcal{E}$, such that $g^*\mathcal{F}$ is the maximal trivial subobject of $g^*\mathcal{E}$. We know by Lemma 1.3.4.5 that $f^*f_*\mathcal{E}$, equipped with the adjunction morphism $f^*f_*\mathcal{E} \to \mathcal{E}$, is a subobject of $\mathcal{E}$. We also know by the lemma that after applying g^*, the pullback $g^*f^*f_*\mathcal{E}$ becomes the maximal trivial subobject of $g^*\mathcal{E}$. Thus $\mathcal{F} := f^*f_*\mathcal{E}$ fulfills the required property. □

1.3.5 Rank 1 coefficient objects

This section is an interlude on rank 1 coefficient objects. One of the starting points of Weil II is a finiteness result for rank 1 lisse sheaves, which is a consequence of class field theory. Thanks to a reduction to unramified p-adic representations of the étale fundamental group, the same statement is now known for overconvergent F-isocrystals of rank 1.

Theorem 1.3.5.1 ([Del80, Proposition 1.3.4], [Abe15, Lemma 6.1])**.** *Let X_0 be a smooth variety over $\mathbb{F}_q$. Every E_λ-coefficient object of rank 1 is a twist of a finite E_λ-coefficient object.*

Corollary 1.3.5.2. *For every $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}_0$ over X_0, there exist a positive integer n and elements $a_1, \dots, a_n \in \overline{\mathbb{Q}}_\ell^\times$ such that*

$$\mathcal{E}_0^{\mathrm{ss}} \simeq \bigoplus_{i=1}^{n} \mathcal{F}_{i,0}^{(a_i)},$$

where for each i the coefficient object $\mathcal{F}_{i,0}$ is irreducible with finite order determinant. If $\mathcal{E}_0$ is E-rational, the elements $a_1, \dots, a_n$ can be chosen so that $a_i^{r_i} \in E$ for every i, where r_i is the rank of $\mathcal{F}_{i,0}$.

Corollary 1.3.5.2 is important as it allows to reduce many statements on coefficient objects to the case of absolutely irreducible coefficient objects with finite order determinant. It is convenient to introduce the following definitions.

Definition 1.3.5.3 (Types)**.** We denote by Θ_ℓ the torsion-free abelian group $\overline{\mathbb{Q}}_\ell^\times/\mu_\infty(\overline{\mathbb{Q}}_\ell)$. The elements of Θ_ℓ will be called the *(ℓ-adic) types*. We will refer to the class of 1 in Θ_ℓ as the *trivial type*. Let $\mathcal{E}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient object and let $a_1, \ldots, a_n \in \overline{\mathbb{Q}}_\ell^\times$ be as in Corollary 1.3.5.2. We denote by $\Theta(\mathcal{E}_0)$ the set of classes of $a_1, \ldots, a_n$ in Θ_ℓ. They will be the types of $\mathcal{E}_0$. Notice that $\Theta(\mathcal{E}_0)$ is a set which depends only on $\mathcal{E}_0$. We also denote by $\mathbb{X}(\mathcal{E}_0)$ the group generated by $\Theta(\mathcal{E}_0)$ in Θ_ℓ and by $\mathbb{X}(\mathcal{E}_0)_\mathbb{Q}$ the $\mathbb{Q}$-linear subspace $\mathbb{X}(\mathcal{E}_0) \otimes_\mathbb{Z} \mathbb{Q} \subseteq \Theta_\ell \otimes_\mathbb{Z} \mathbb{Q}$.

Theorem 1.3.5.1 is used to prove a global version of Grothendieck's *local monodromy theorem*, usually known as the *global monodromy theorem*. Here the extension to F-isocrystals is due to Crew.

Theorem 1.3.5.4 (Grothendieck, Crew)**.** *For every coefficient object $\mathcal{E}_0$, the radical subgroup*[6] *of $G(\mathcal{E}, x)$ is unipotent.*

Proof. In the case of lisse sheaves, this is a theorem of Grothendieck, and it is proven in [Del80, Théorème 1.3.8]. In the p-adic case, Crew has proven the result when X_0 is a smooth curve [Cre92a, Theorem 4.9]. One obtains the result in higher dimensions replacing [*ibid.*, Proposition 4.6] by Proposition 1.3.3.4 and [*ibid.*, Corollary 1.5] by Theorem 1.3.5.1.

□

Corollary 1.3.5.5. *Let $\mathcal{E}_0$ be a geometrically semi-simple coefficient object. The neutral component $G(\mathcal{E}, x)^\circ$ is a semi-simple algebraic group which coincides with the derived subgroup of $G(\mathcal{E}_0, x)^\circ$.*

Proof. By Cororollary 1.3.6.4, the geometric coefficient object $\mathcal{E}$ is semi-simple, thus the algebraic group $G(\mathcal{E}, x)^\circ$ is reductive. Thanks to Theorem 1.3.5.4, this implies that $G(\mathcal{E}, x)^\circ$ is semi-simple, therefore

$$G(\mathcal{E}, x)^\circ = [G(\mathcal{E}, x)^\circ, G(\mathcal{E}, x)^\circ] \subseteq [G(\mathcal{E}_0, x)^\circ, G(\mathcal{E}_0, x)^\circ] \,.$$

By Proposition 1.3.2.6.(iv), the quotient $G(\mathcal{E}_0, x)^\circ/G(\mathcal{E}, x)^\circ$ is commutative, hence

$$[G(\mathcal{E}_0, x)^\circ, G(\mathcal{E}_0, x)^\circ] \subseteq G(\mathcal{E}, x)^\circ.$$

This concludes the proof. □

Thanks to Theorem 1.3.5.1, one can even prove that, under certain assumptions, the neutral component of the arithmetic monodromy group is semi-simple and it is equal to the neutral component of the geometric monodromy group. These results can be found for lisse sheaves in [Dri18, §3.6].

[6] For us, the radical subgroup of an algebraic group will always be connected by definition.

Proposition 1.3.5.6. *For every coefficient object $\mathcal{E}_0$ and every connected finite étale cover $f_0 : (Y_0, y) \to (X_0, x)$, we have $\Theta(\mathcal{E}_0) = \Theta(f_0^*\mathcal{E}_0)$.*

Proof. After taking semi-simplification and twists, we may assume that $\mathcal{E}_0$ is absolutely irreducible with finite order determinant. In this case, the result is proven in [Dri18, Proposition 3.6.1] for lisse sheaves. The proof is the same for overconvergent F-isocrystals, as they satisfy étale descent by [Ete02]. □

Proposition 1.3.5.7. *Let $\mathcal{E}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient object. The following properties are equivalent.*

(i) *The algebraic group $G(\mathcal{E}_0, x)^\circ$ is semi-simple and it is equal to $G(\mathcal{E}, x)^\circ$.*

(ii) *The coefficient object $\mathcal{E}_0$ is semi-simple and has trivial types.*

In particular, semi-simple coefficient objects with trivial types form a Tannakian subcategory of $\mathbf{Coef}(X_0, \overline{\mathbb{Q}}_\ell)$.

Proof. If $\mathcal{E}_0$ is a coefficient object which satisfies (i), all the rank 1 coefficient objects in $\langle \mathcal{E}_0 \rangle$ have finite order under tensor. Thus, if $\overline{\mathbb{Q}}_\ell^{(a)} \in \langle \mathcal{E}_0 \rangle$ with $a \in \overline{\mathbb{Q}}_\ell$, then a is a root of unity. This implies that every type of $\mathcal{E}_0$ is trivial. Conversely, let us assume now that $\mathcal{E}_0$ satisfies (ii). Thanks to Proposition 1.3.3.4, there exists a connected finite étale cover $f_0 : (Y_0, y) \to (X_0, x)$ such that $G(f_0^*\mathcal{E}_0, y) \xrightarrow{\sim} G(\mathcal{E}_0, x)^\circ$ and $G(f^*\mathcal{E}, y) \xrightarrow{\sim} G(\mathcal{E}, x)^\circ$. By Proposition 1.3.5.6, the inverse image $f_0^*\mathcal{E}_0$ satisfies the same assumptions as $\mathcal{E}_0$. We have reduced the problem to the case when $G(\mathcal{E}_0, x)$ is connected. Thus it is enough to show that the center Z of $G(\mathcal{E}_0, x)$ is finite.

We notice that we may also assume $\mathcal{E}_0$ irreducible. Indeed, if $\mathcal{E}_0 = \mathcal{F}_0 \oplus \mathcal{G}_0$ and Z_1 and Z_2 are the centers of $G(\mathcal{F}_0, x)$ and $G(\mathcal{G}_0, x)$ respectively, then $Z \subseteq Z_1 \times Z_2$. Therefore, if Z_1 and Z_2 are finite the same holds for Z. Since Z is a group of multiplicative type, its representation on $\omega_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{E}_0)$ decomposes as a direct sum of characters $\bigoplus_{i=1}^r \chi_i$, where r is the rank of $\mathcal{E}_0$. By construction, the representation is faithful, thus $\chi_1, \ldots, \chi_r$ generate the group of all the characters of Z. On the other hand, as $\mathcal{E}_0$ is irreducible, the characters $\chi_1, \ldots, \chi_r$ are all isomorphic. Hence, by the assumption on the determinant, they are also finite. This implies that the group of characters of Z is finite, hence Z is finite. This proves that $G(\mathcal{E}_0, x)$ is semi-simple. By virtue of Proposition 1.3.2.6, the quotient $G(\mathcal{E}_0, x)/G(\mathcal{E}, x)$ is a commutative group. Since $G(\mathcal{E}_0, x)$ is semi-simple, this implies that $G(\mathcal{E}_0, x)/G(\mathcal{E}, x)$ is finite, as we wanted. □

Corollary 1.3.5.8. *Let $\mathcal{E}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient object.*

(i) *For every $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle$, we have $\Theta(\mathcal{F}_0) \subseteq \mathbb{X}(\mathcal{E}_0)$.*

(ii) *There exists a canonical map $X^*(G(\mathcal{E}_0, x)) \to \mathbb{X}(\mathcal{E}_0)$ which becomes an isomorphism when we tensor by $\mathbb{Q}$.*

Proof. It is enough to prove (i) for the objects of the form $\mathcal{E}_0^{\otimes m} \otimes (\mathcal{E}_0^\vee)^{\otimes n}$ with $m, n \in \mathbb{N}$. In addition, we may assume that $\mathcal{E}_0$ has only one type. Write $\mathcal{E}_0$ as $\mathcal{F}_0^{(a)}$ with $\mathcal{F}_0$ with trivial types. By 1.3.5.7, the coefficient object $\mathcal{F}_0^{\otimes m} \otimes (\mathcal{F}_0^\vee)^{\otimes n}$ has trivial types. Therefore, $\mathcal{E}_0^{\otimes m} \otimes (\mathcal{E}_0^\vee)^{\otimes n}$ has type $(m-n)[a] \in \mathbb{X}(\mathcal{E}_0)$. This proves part (i). Let $X^*(G(\mathcal{E}_0, x)) \to \mathbb{X}(\mathcal{E}_0)$ be the map which associates to a rank 1 coefficient object its type. After tensoring by $\mathbb{Q}$, the map becomes injective, as a rank 1 coefficient object of trivial type has finite order under tensor. To prove the surjectivity we have to prove that for every type $[a]$ of $\mathcal{E}_0$, there exists $n \geq 1$ and a rank 1 coefficient object $\mathcal{L}_0 \in \langle \mathcal{E}_0 \rangle$ of type $[a^n]$. Since, by definition, there exists an irreducible object $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle$ of type $[a]$, we can pick $\mathcal{L}_0 := \det(\mathcal{F}_0)$. □

1.3.6 Weights

In Weil II Deligne introduced the *theory of weights* for lisse sheaves. The same theory is now available for overconvergent F-isocrystals, thanks to the work of Kedlaya in [Ked06]. Here the main theorem.

Theorem 1.3.6.1 (Deligne, Kedlaya)**.** *Let X_0 be a smooth geometrically connected variety over $\mathbb{F}_q$ and $\mathcal{E}_0$ a ι-mixed coefficient object over X_0 of ι-weights $\leq w$. If α is an eigenvalue of F acting on $H_c^n(X, \mathcal{E})$, then $|\iota(\alpha)| \leq q^{(w+n)/2}$.*

Proof. For lisse sheaves this is the main result in [Del80]. For overconvergent F-isocrystals it is proven by Kedlaya in [Ked06]. □

Corollary 1.3.6.2. *Let*

$$0 \to \mathcal{F}_0 \to \mathcal{E}_0 \to \mathcal{G}_0 \to 0$$

be an exact sequence of coefficient objects such that $\mathcal{F}_0$ and $\mathcal{G}_0$ are ι-pure of weights w_1 and w_2 respectively.

(i) *If $w_1 > w_2 - 1$ the sequence splits geometrically.*

(ii) *If $w_1 \neq w_2$ and the sequence splits geometrically, then it splits.*

Proof. We have an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{G}, \mathcal{F})_F \to \mathrm{Ext}^1(\mathcal{G}_0, \mathcal{F}_0) \to \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})^F.$$

The group $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$ is equal to $H^1(X, \mathcal{F} \otimes \mathcal{G}^\vee)$ and the coefficient object $\mathcal{F}_0 \otimes \mathcal{G}_0^\vee$ is ι-pure of weight $w_1 - w_2$. Thus, by Theorem 1.3.6.1, the weights of $H^1(X, \mathcal{F} \otimes \mathcal{G}^\vee)$ are at least equal to $w_1 - w_2 + 1$. When $w_1 > w_2 - 1$, then $H^1(X, \mathcal{F} \otimes \mathcal{G}^\vee)$ has positive weights, which implies that F does not admit fixed points. Therefore in this case $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F})^F$ vanishes, which implies (i). For (ii), suppose that the class associated to our extension is zero in $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$, then it comes from $\mathrm{Hom}(\mathcal{G}, \mathcal{F})_F$. As $\mathcal{F}_0 \otimes \mathcal{G}_0^\vee$ does not admit ι-weight 0 by assumption, $\mathrm{Hom}(\mathcal{G}, \mathcal{F})_F = 0$, hence the result. □

Corollary 1.3.6.3. *For every ι-mixed coefficient object there exists an increasing filtration*

$$0 = W_{-1}(\mathcal{E}_0) \subsetneq W_0(\mathcal{E}_0) \subsetneq \cdots \subsetneq W_n(\mathcal{E}_0) = \mathcal{E}_0$$

where for every $0 \leq i \leq n$, the quotient $W_i(\mathcal{E}_0)/W_{i-1}(\mathcal{E}_0)$ is ι-pure of weight w_i and $w_0 < w_1 < \cdots < w_n$.

Corollary 1.3.6.4. *Every ι-pure coefficient object is geometrically semi-simple. Conversely, every ι-mixed geometrically semi-simple coefficient object is a direct sum of ι-pure coefficient objects.*

1.3.6.5. For every $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}_0$ on X_0, we can put together all the characteristic polynomials at closed points and form a formal series

$$L_{X_0}(\mathcal{E}_0, t) := \prod_{x_0 \in |X_0|} P_{x_0}(\mathcal{E}_0, t^{\deg(x_0)})^{-1} \in \overline{\mathbb{Q}}_\ell[[t]].$$

This is called the *L-function* of $\mathcal{E}_0$.

Theorem (Trace formula)**.** *If X_0 is geometrically connected over $\mathbb{F}_q$, for every coefficient object $\mathcal{E}_0$ we have*

$$L_{X_0}(\mathcal{E}_0, t) = \prod_{i=1}^{2d} \det(1 - Ft, H_c^i(X, \mathcal{E}))^{(-1)^{i+1}}.$$

Proof. For lisse sheaves, this is the classical Grothendieck's formula, in the p-adic case see [ES93, Théorème 6.3]. □

Thanks to the theory of weight, this formula can be used to compare the global sections of compatible coefficient objects. The theory of weights is needed to control the possible cancellations between the factors of the numerator and the denominator.

Proposition 1.3.6.6 ([Laf02, Cor. VI.3], [Abe18, Prop. 4.3.3])**.** *Let X_0 be a smooth geometrically connected variety over $\mathbb{F}_q$ of dimension d. For every ι-pure coefficient object $\mathcal{E}_0$ of ι-weight w, the dimension of $H^0(X, \mathcal{E})$ is equal to the number of poles of $\iota(L(X_0, \mathcal{E}_0^\vee(d)))$, counted with multiplicity, with absolute value $q^{w/2}$. If we also assume $\mathcal{E}_0$ to be semi-simple, the dimension of $H^0(X_0, \mathcal{E}_0)$ is equal to the order of the pole of $L(X_0, \mathcal{E}_0^\vee(d))$ at 1.*

Proof. By Poincaré duality, the dimension of $H^0(X, \mathcal{E})$ is equal to the dimension of $H_c^{2d}(X, \mathcal{E}^\vee(d))$ and the eigenvalues of F acting on $H_c^{2d}(X, \mathcal{E}^\vee(d))$ have ι-weight $-\alpha$. At the same time, by Theorem 1.3.6.1, for every $0 \leq i \leq 2d$, the groups $H_c^i(X, \mathcal{E}^\vee(d))$ have ι-weights less or equal than $-\alpha - 2d + i$. The first part of the statement is then a consequence of the trace formula (Theorem 1.3.6.5) applied to $L(X_0, \mathcal{E}_0^\vee(d))$. Indeed, by the observations on the

weights, the polynomial $\det(1 - Ft, H_c^{2d}(X, \mathcal{E}^\vee(d)))$ is relatively prime to the numerator of the L-function. Hence, the number of poles of $\iota(L(X_0, \mathcal{E}_0^\vee(d)))$ with absolute value $q^{w/2}$ is equal to $\deg(\det(1 - Ft, H_c^{2d}(X, \mathcal{E}^\vee(d)))) = \dim(H^0(X, \mathcal{E}))$.

For the second part, we also use that by assumption the endomorphism F acts semi-simply on $H^0(X, \mathcal{E})$. In particular, the geometric and the algebraic multiplicities of the eigenvalue 1 are the same. Therefore, thanks to Poincaré duality, the dimension of $H^0(X_0, \mathcal{E}_0)$ is equal to the multiplicity of 1 of $\det(1 - Ft, H_c^{2d}(X, \mathcal{E}^\vee(d)))$. By the previous reasoning, this is the same as the order of the pole of $L(X_0, \mathcal{E}_0^\vee(d))$ at 1. □

Corollary 1.3.6.7. *Let (X_0, x) be a smooth connected pointed variety over $\mathbb{F}_q$, let $\mathcal{E}_0$ and $\mathcal{F}_0$ be E-compatible coefficients objects and suppose that $\mathcal{E}_0$ is ι-mixed. The following statements are true.*

(i) *If $\mathcal{E}_0$ and $\mathcal{F}_0$ are geometrically semi-simple, then* $\dim(H^0(X^{(x)}, \mathcal{E})) = \dim(H^0(X^{(x)}, \mathcal{F}))$.

(ii) *If $\mathcal{E}_0$ and $\mathcal{F}_0$ are semi-simple, then* $\dim(H^0(X_0, \mathcal{E}_0)) = \dim(H^0(X_0, \mathcal{F}_0))$.

Proof. We may assume X_0 to be geometrically connected over $\mathbb{F}_q$ by extending the base field. Let $\overline{\mathbb{Q}}_{\ell'}$ be the algebraic closure of the field of scalars of $\mathcal{F}_0$. Let $\iota' : \overline{\mathbb{Q}}_{\ell'} \xrightarrow{\sim} \mathbb{C}$ be an isomorphism which agrees with ι on E. Then $\mathcal{F}_0$ is ι'-mixed and its weights are equal to the ones of $\mathcal{E}_0$. In addition, if $W_*(\mathcal{E}_0)$ and $W_*(\mathcal{F}_0)$ are the weight filtrations of Corollary 1.3.6.3, for each i the quotients $W_i(\mathcal{E}_0)/W_{i-1}(\mathcal{E}_0)$ and $W_i(\mathcal{F}_0)/W_{i-1}(\mathcal{F}_0)$ are E-compatible. By the geometric semi-simplicity assumption, it is enough to check (i) on these subquotients. But in this case, this follows by Proposition 1.3.6.6. For (ii) we argue similarly. □

Proposition 1.3.6.8. *Two ι-mixed $\overline{\mathbb{Q}}_\ell$-coefficient objects with the same characteristic polynomial functions have isomorphic semi-simplifications.*

Proof. For étale lisse sheaf, this is the classical Čebotarev's density theorem, explained in [Ser66, Theorem 7]. For general lisse sheaf, one can take the semi-simplification and then take suitable twists of the absolutely irreducible components, to reduce to the étale case, by [Del80, Proposition 1.3.14]. In this case, the assumption that the coefficient object is ι-mixed is not needed. For the p-adic case the statement was proven by Tsuzuki using Proposition 1.3.6.6, see [Abe18, A.3]. His proof works for ι-mixed lisse sheaves as well. □

Remark 1.3.6.9. We will see later that thanks to the Langlands correspondence, it is possible to show that every coefficient object is ι-mixed (Theorem 1.3.7.6). Therefore, Proposition 1.3.6.8 applies to every coefficient object.

1.3.7 Deligne's conjecture

We are ready to present Conjecture 1.1.1.1 for arbitrary coefficient objects. The extension of the statement to overconvergent F-isocrystals was firstly proposed by Crew in [Cre92a, Conjecture 4.13]. This corresponds to the choice of the category of overconvergent F-isocrystals as a possible candidate for Deligne's "petits camarades cristallins".

Conjecture 1.3.7.1. *Let X_0 be a smooth variety over $\mathbb{F}_q$, let ℓ be a prime number and let $\mathcal{E}_0$ be an absolutely irreducible $\overline{\mathbb{Q}}_\ell$-coefficient object whose determinant has finite order. The following statements hold.*

(i) *$\mathcal{E}_0$ is pure of weight 0.*

(ii) *There exists a number field $E \subseteq \overline{\mathbb{Q}}_\ell$ such that $\mathcal{E}_0$ is E-rational.*

(iii) *$\mathcal{E}_0$ is p-plain.*

(iv') *If E is a number field as in (ii), then for every prime ℓ' (even $\ell' = \ell$ or $\ell' = p$) and for every inclusion $\tau : E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$, there exists an absolutely irreducible $\overline{\mathbb{Q}}_{\ell'}$-coefficient object, E-rational with respect to τ, which is E-compatible with $\mathcal{E}_0$.*

We shall see that the conjecture, except part (iv'), is now known to be true. The missing case of (iv') is when $\ell' = p$ and X_0 has dimension at least 2. We will postpone the discussions on the analogue of Conjecture 1.1.1.1.(iv) in §1.3.8. We first recall an equivalent form of (i), which for lisse sheaves is [Del80, Conjecture 1.2.9].

Conjecture 1.3.7.2. *Every $\overline{\mathbb{Q}}_\ell$-coefficient object on X_0 is ι-mixed.*

Proposition 1.3.7.3. *Conjecture 1.3.7.1.(i) is equivalent to Conjecture 1.3.7.2.*

Proof. We first prove that Conjecture 1.3.7.1.(i) implies Conjecture 1.3.7.2. By taking Jordan–Hölder filtrations, Conjecture 1.3.7.2 reduces to the case of irreducible $\overline{\mathbb{Q}}_\ell$-coefficient objects. Let $\mathcal{E}_0$ be an irreducible $\overline{\mathbb{Q}}_\ell$-coefficient objects. Thanks to Corollary 1.3.5.2, it is isomorphic to $\mathcal{F}_0^{(a)}$, where $\mathcal{F}_0$ is an irreducible $\overline{\mathbb{Q}}_\ell$-coefficient object with finite order determinant and $a \in \overline{\mathbb{Q}}_\ell^\times$. By Conjecture 1.3.7.1.(i), the coefficient object $\mathcal{F}_0$ is ι-pure of weight 0, thus $\mathcal{E}_0$ is ι-pure of ι-weight $2\log_q(|\iota(a)|)$. This gives the desired result.

Conversely, assume Conjecture 1.3.7.2 and let $\mathcal{E}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient object with finite order determinant. Then, $\mathcal{E}_0$ is ι-pure for every isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Since its determinant has finite order, all the ι-weights of $\mathcal{E}_0$ are 0. This shows that $\mathcal{E}_0$ is pure of weight 0 (in particular, $\mathcal{E}_0$ is algebraic). □

When X_0 is a smooth curve, Conjecture 1.3.7.1 is a consequence of the Langlands correspondence for GL_r, over function fields, and the Ramanujan–Petersson conjecture. The Langlands correspondence for lisse sheaves and the Ramanujan–Petersson conjecture was proven by L. Lafforgue. Abe proved later the Langlands correspondence for overconvergent F-isocrystals.

Theorem 1.3.7.4 ([Laf02, Théorème VII.6], [Abe18, §4.4])**.** *If X_0 is a smooth curve, Conjecture 1.3.7.1 and Conjecture 1.3.7.2 are true.*

The extension of the results to higher dimensional varieties is performed via a reduction to curves. One of the key ingredients is a Lefschetz theorem for coefficient objects.

Theorem 1.3.7.5 (Katz, Abe–Esnault)**.** *Let X_0 be a smooth geometrically connected variety over $\mathbb{F}_q$. For every lisse sheaf $\mathcal{E}_0$ over X_0 and every reduced finite closed subscheme $S_0 \subseteq X_0$, there exists a geometrically connected smooth curve C_0 and a morphism $f_0 : C_0 \to X_0$ with a section $S_0 \to C_0$, such that the inverse image functor $\langle \mathcal{E}_0 \rangle \to \langle f_0^* \mathcal{E}_0 \rangle$ is an equivalence of categories. The same is true when $\mathcal{E}_0$ is a ι-pure overconvergent F-isocrystal.*

Proof. For lisse sheaves see [Kat99, Lemma 6 and Theorem 8] as well as [Kat01]. In the p-adic case see the (proof of) [AE16, Theorem 3.10]. □

Thanks to Theorem 1.3.7.5 and the work of Deligne in [Del12], the first three parts of the conjecture follow from the curves' case.

Theorem 1.3.7.6 (L. Lafforgue, Abe, Deligne, Abe–Esnault, Kedlaya)**.** *Parts (i), (ii) and (iii) of Conjecture 1.3.7.1 and Conjecture 1.3.7.2 are true for every smooth variety over $\mathbb{F}_q$.*

Proof. For lisse sheaves, parts (i) and (iii) follow directly from Theorem 1.3.7.4, thanks to Theorem 1.3.7.5. Switching to overconvergent F-isocrystals, Conjecture 1.3.7.2 is proven in [AE16, Theorem 2.7] and independently in [Ked18, Theorem 3.1.9]. This implies Conjecture 1.3.7.1.(i) for overconvergent F-isocrystals. Part (iii) then follows from Theorem 1.3.7.4 thanks to (i) and Theorem 1.3.7.5. Part (ii) is proven in [Del12, Theorem 3.1] for lisse sheaves and in [AE16, Lemma 4.1] and [Ked18, Theorem 3.4.2] for overconvergent F-isocrystals. □

The generalization of part (iv') to higher dimensional varieties is yet incomplete. For the moment we know how to construct from a coefficient object of both kinds, compatible lisse sheaves. In dimension greater than one, we do not know how to construct, in general, compatible overconvergent F-isocrystals.

Theorem 1.3.7.7 (L. Lafforgue, Abe, Drinfeld, Abe–Esnault, Kedlaya)**.** *Let X_0 be a smooth variety over $\mathbb{F}_q$ and E a number field. Let $\mathcal{E}_0$ be an absolutely irreducible E-rational coefficient object with finite order determinant on X_0. For every prime ℓ different from p and every embedding $\tau : E \hookrightarrow \overline{\mathbb{Q}}_\ell$, there exists a $\overline{\mathbb{Q}}_\ell$-coefficient object which is E-rational with respect to τ and E-compatible with $\mathcal{E}_0$.*

Proof. Drinfeld has proven the theorem when $\mathcal{E}_0$ is a lisse sheaf [Dri12]. The proof uses L. Lafforgue's result and a certain gluing theorem for lisse sheaves [*ibid.*, Theorem 2.5]. The gluing theorem is inspired by the seminal work of Wiesend in [Wie06]. When $\mathcal{E}_0$ is an overconvergent F-isocrystal the result was proven in [AE16] and later in [Ked18]. They both use Drinfeld's gluing

theorem for lisse sheaves. In [AE16] they prove and use Theorem 1.3.7.5 for overconvergent F-isocrystals. In [Ked18] it is proven a weaker form, namely [*ibid.*, Lemma 3.2.1], which is enough to conclude. □

The known parts of Deligne's conjecture have many important consequences. First, as every coefficient object is ι-mixed, one may apply Proposition 1.3.6.8 to every coefficient object. We list here other corollaries, which we will use later.

Corollary 1.3.7.8. *Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be two compatible coefficient objects. If $\mathcal{E}_0$ is absolutely irreducible, the same is true for $\mathcal{F}_0$. Moreover, if $\mathcal{E}$ is absolutely irreducible, even $\mathcal{F}$ is absolutely irreducible.*

Proof. Suppose that the coefficient object $\mathcal{E}_0$ is absolutely irreducible. Then by Theorem 1.3.7.6, $\mathcal{E}_0$ is ι-pure. Thus the coefficient object $\mathcal{E}nd(\mathcal{E}_0)$ is semi-simple and pure of weight 0. Since $\mathcal{E}_0$ is absolutely irreducible, the vector space $\mathrm{End}(\mathcal{E}_0) = H^0(X_0, \mathcal{E}nd(\mathcal{E}_0))$ is one dimensional. After replacing $\mathcal{F}_0$ by its semi-simplification, we may assume that it is a semi-simple coefficient object. Thus, the coefficient object $\mathcal{E}nd(\mathcal{F}_0)$ is a semi-simple coefficient object which is compatible with $\mathcal{E}nd(\mathcal{E}_0)$. By Corollary 1.3.6.7, the vector space $\mathrm{End}(\mathcal{F}_0)$ is one dimensional as well. This implies that $\mathcal{F}_0$ is absolutely irreducible, as we wanted. For the second part of the statement we proceed in the same way, applying Corollary 1.3.6.7 to $\mathcal{E}nd(\mathcal{E})$ and $\mathcal{E}nd(\mathcal{F})$. □

Corollary 1.3.7.9. *For every algebraic $\overline{\mathbb{Q}}_\ell$-coefficient object $\mathcal{E}_0$, there exists a number field $E \subseteq \overline{\mathbb{Q}}_\ell$, such that $\mathcal{E}_0$ is E-rational.*

Proof. It is enough to prove the result when $\mathcal{E}_0$ is absolutely irreducible. Thanks to Corollary 1.3.5.2, the coefficient object $\mathcal{E}_0$ is isomorphic to $\mathcal{F}_0^{(a)}$, where $\mathcal{F}_0$ is a coefficient object with finite order determinant and $a \in \overline{\mathbb{Q}}_\ell^\times$. As the determinant characters of $\mathcal{E}_0$ and $\mathcal{F}_0$ are algebraic, even the number a is algebraic. Theorem 1.3.7.6 implies that $\mathcal{F}_0$ is E-rational for some number field $E \subseteq \overline{\mathbb{Q}}_\ell$, thus $\mathcal{E}_0$ is $E(a)$-rational. □

Finally, thanks to Theorem 1.3.7.6, we also have following analogue of [Moc04, Theorem 7.1] for smooth varieties over finite fields.

Corollary 1.3.7.10. *Over a smooth variety over $\mathbb{F}_q$, a $\overline{\mathbb{Q}}_\ell$-coefficient object is geometrically semi-simple if and only if it is a direct sum of ι-pure $\overline{\mathbb{Q}}_\ell$-coefficient objects. In particular, for every morphism $f_0 : Y_0 \to X_0$ of smooth varieties, if $\mathcal{E}_0$ is a geometrically semi-simple coefficient object over X_0, then $f_0^*\mathcal{E}_0$ is a geometrically semi-simple coefficient object over Y_0.*

Proof. Thanks to Theorem 1.3.7.6, every coefficient object is ι-mixed. Therefore, by Corollary 1.3.6.4 we get the first part of the result. The property of a coefficient object to be a direct sum of ι-pure coefficient objects is manifestly preserved by the inverse image functor f_0^*. This concludes the proof. □

1.3.8 Compatible systems

Thanks to Theorem 1.3.7.7, from a coefficient object which satisfies suitable properties, we can construct many compatible coefficient objects with different fields of scalars. Looking at Conjecture 1.1.1.1.(iv), one notices that Deligne also predicted that these fields should be the completions at different finite places of a given number field. It is possible to upgrade Theorem 1.3.7.7 to a stronger form thanks to the work of Chin in [Chi03]. To state the result, we use Serre's notion of *compatible systems*, which we extend to arbitrary coefficient objects. As we cannot construct, at the moment, p-adic companions in general, we do not ask that the compatible system contains a p-adic coefficient object for every p-adic place. On the other hand, we do ask that it includes lisse sheaves for every finite place which do not divide p.

Definition 1.3.8.1 (Compatible systems)**.** If E is a number field, an *E-compatible system* over X_0, denoted $\underline{\mathcal{E}_0}$, is the datum of a set Σ of finite places of E containing $|E|_{\neq p}$ and a family $\{\mathcal{E}_{\lambda,0}\}_{\lambda\in\Sigma}$, where each $\mathcal{E}_{\lambda,0}$ is an E-rational E_λ-coefficient object and every pair of coefficient objects is E-compatible. For every λ, the coefficient object $\mathcal{E}_{\lambda,0}$ will be the *λ-component* of the compatible system.

If $E \subseteq E'$ is a finite extension of number fields and $\underline{\mathcal{E}_0}$ is an E-compatible system, the compatible system obtained from $\underline{\mathcal{E}_0}$ by *extending the scalars to E'* will be the E'-compatible system $\{\mathcal{E}'_{\lambda',0}\}_{\lambda'\in\Sigma'}$, where Σ' is the set of places of E' over the places in Σ and for every $\lambda' \in \Sigma'$ over $\lambda \in \Sigma$, the λ' component is $\mathcal{E}'_{\lambda',0} := \mathcal{E}_{\lambda,0} \otimes_{E_\lambda} E'_{\lambda'}$. We say that a compatible system is *trivial, geometrically trivial, pure, irreducible, absolutely irreducible or semi-simple* if each λ-component has the respective property.

Theorem 1.3.8.2 (after Chin)**.** *Let X_0 be a smooth variety over $\mathbb{F}_q$ and $\mathcal{E}_0$ an algebraic $\overline{\mathbb{Q}}_\ell$-coefficient object of X_0. There exists a number field E, a finite place $\nu \in |E|$ and an E-compatible system $\underline{\mathcal{E}_0}$, such that $\mathcal{E}_0$ is a ν-component of $\underline{\mathcal{E}_0}$. When X_0 is a curve, we can further find such an E-compatible system $\underline{\mathcal{E}_0}$ with $\Sigma = |E|$.*

Proof. By extending the field of scalars of $\mathcal{E}_0$ and taking semi-simplification, we reduce to the case when $\mathcal{E}_0$ is absolutely irreducible. By Corollary 1.3.7.9, the coefficient object $\mathcal{E}_0$ is E-rational. Thus, thanks to Corollary 1.3.5.2, after possibly enlarging E there exists $a \in E^\times$ such that $\mathcal{E}_0$ is isomorphic to $\mathcal{F}_0^{(a)}$, where $\mathcal{F}_0$ is an E-rational coefficient object with finite order determinant. When $\ell \neq p$, thanks to Theorem 1.3.7.7 and [Chi03, Main Theorem, page 3], after possibly enlarging E again, the lisse sheaf $\mathcal{F}_0$ sits in an E-compatible system. By twisting all the components by a, the same holds true for $\mathcal{E}_0$. When $\ell = p$, thanks to Theorem 1.3.7.7, $\mathcal{E}_0$ is E-compatible with some lisse sheaf $\mathcal{V}_0$. The result then follows from the previous case.

When X_0 is a curve, we obtain the stronger result thanks to the existence of p-adic companions provided by Theorem 1.3.7.4. After possibly replacing E with a finite extension, we may add to the compatible system previously constructed λ-components, for every place λ which

divides p. Here we do not need a new finiteness result for overconvergent F-isocrystals, namely a p-adic analogue of Chin's theorem, because the set of places we are adding is finite.

□

Remark 1.3.8.3. Even if a coefficient object $\mathcal{E}_0$ is E-rational for some number field E, it could be still necessary to enlarge E to obtain the E-compatible system $\underline{\mathcal{E}_0}$. For example, let Q_8 be the quaternion group and let X_0 be a connected smooth variety that admits a Galois cover with Galois group Q_8. Let $\mathbb{H}$ be the natural four-dimensional $\mathbb{Q}$-linear representation of Q_8 on the algebra of Hamilton's quaternions.

The representation $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is irreducible over $\mathbb{Q}_\ell$ if and only if $\ell = 2$. If we take $\ell \neq 2$, then $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ decomposes as a direct sum of two copies of an absolutely irreducible two dimensional $\mathbb{Q}_\ell$-representation V_ℓ with traces in $\mathbb{Q}$. The representation V_ℓ corresponds to an absolutely irreducible $\mathbb{Q}$-rational $\mathbb{Q}_\ell$-coefficient object which does not admit any $\mathbb{Q}$-compatible $\mathbb{Q}_2$-coefficient object. Indeed, suppose that there exists a semi-simple $\mathbb{Q}_2$-coefficient object V_2, that is $\mathbb{Q}$-compatible with V_ℓ. Then $V_2^{\oplus 2}$ would be $\mathbb{Q}$-compatible with $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_2$. By Proposition 1.3.6.8, the coefficient object $V_2^{\oplus 2}$ would be isomorphic to $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_2$. However, this is impossible, as $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is irreducible.

1.4 Independence of monodromy groups

1.4.1 The group of connected components

In [Ser00] and [LP95, Proposition 2.2] Serre and Larsen–Pink have proven some results of ℓ-independence for the groups of connected components of the monodromy groups of lisse sheaves. In this section, we shall generalize their results to general coefficient objects. We will adapt Larsen–Pink's proof. The main issue for p-adic coefficient objects is to relate the monodromy groups with the étale fundamental group of X_0. We have already treated this problem in §1.3.3. By Proposition 1.3.3.3, for every coefficient object $\mathcal{E}_0$ we have functorial surjective morphisms $\psi_{\mathcal{E}_0} : \pi_1^{\text{ét}}(X_0, x) \to \pi_0(G(\mathcal{E}_0, x))$ and $\psi_{\mathcal{E}} : \pi_1^{\text{ét}}(X, x) \to \pi_0(G(\mathcal{E}, x))$ of profinite groups.

Theorem 1.4.1.1. *Let (X_0, x) be a smooth geometrically connected pointed variety over $\mathbb{F}_q$. Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be two compatible E_λ-coefficient objects over X_0.*

(i) *There exists an isomorphism $\varphi_0 : \pi_0(G(\mathcal{E}_0, x)) \xrightarrow{\sim} \pi_0(G(\mathcal{F}_0, x))$ as abstract finite groups, such that $\psi_{\mathcal{F}_0} = \varphi_0 \circ \psi_{\mathcal{E}_0}$.*

(ii) *The isomorphism φ_0 restricts to an isomorphism $\varphi : \pi_0(G(\mathcal{E}, x)) \xrightarrow{\sim} \pi_0(G(\mathcal{F}, x))$.*

Following [LP95, Proposition 2.2], we need two lemma to prove Theorem 1.4.1.1.

Construction 1.4.1.2. Let $\mathcal{E}_0$ be a E_λ-coefficient object of rank r. We fix a basis of $\omega_{x,E_\lambda}(\mathcal{E}_0)$ and we take the representation $\rho_{\mathcal{E}_0} : G(\mathcal{E}_0, x) \to \mathrm{GL}_{r,E_\lambda}$ associated to $\mathcal{E}_0$. For every $\mathbb{Q}$-linear

representation $\theta : \mathrm{GL}_{r,\mathbb{Q}} \to \mathrm{GL}(V)$ we denote by $\mathcal{E}_0(\theta)$ the coefficient object associated to $(\theta \otimes_{\mathbb{Q}} E_\lambda) \circ \rho_{\mathcal{E}_0}$. Even if $\mathcal{E}_0(\theta)$ depends on the choice of a basis, its isomorphism class is uniquely determined.

Lemma 1.4.1.3. *Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be compatible semi-simple objects. For every representation θ of $\mathrm{GL}_{r,\mathbb{Q}}$, we have that $\dim(H^0(X, \mathcal{E}(\theta))) = \dim(H^0(X, \mathcal{F}(\theta)))$ and $\dim(H^0(X_0, \mathcal{E}_0(\theta))) = \dim(H^0(X_0, \mathcal{F}_0(\theta)))$*

Proof. Each pair $(\mathcal{E}_0(\theta), \mathcal{F}_0(\theta))$ is again compatible and semi-simple. Moreover, by Theorem 1.3.7.6, for every representation θ, the coefficient object $\mathcal{E}_0(\theta)$ is ι-mixed. Therefore, for every θ, we may apply Corollary 1.3.6.7 to $(\mathcal{E}_0(\theta), \mathcal{F}_0(\theta))$. □

Remark 1.4.1.4. Using the terminology of [LP90], Lemma 1.4.1.3 proves that $G(\mathcal{E}, x)$ and $G(\mathcal{F}, x)$ (resp. $G(\mathcal{E}_0, x)$ and $G(\mathcal{F}_0, x)$) have the same *dimension data*.

Lemma 1.4.1.5 ([LP95, Lemma 2.3])**.** *Let $\mathbb{K}$ be a field and G a reductive algebraic subgroup of $\mathrm{GL}_{r,\mathbb{K}}$. If for every finite-dimensional representation V of $\mathrm{GL}_{r,\mathbb{K}}$ the dimension of V^{G° is equal to the dimension of V^G, then the group G is connected.*

1.4.1.6Proof of Theorem 1.4.1.1. We explain the proof for the arithmetic monodromy groups. For the geometric monodromy groups the proof is the same *mutatis mutandis*.

We notice that taking semi-simplification we do not change the group of connected components of the arithmetic monodromy group. Thus we reduce to the case when $\mathcal{F}_0$ and $\mathcal{G}_0$ are semi-simple. We firstly prove a weaker statement.

(i') $G(\mathcal{E}_0, x)$ is connected if and only if $G(\mathcal{F}_0, x)$ is connected.

For every finite étale connected cover $f_0 : Y_0 \to X_0$, we denote by a_{Y_0} and b_{Y_0} the functions from the set of isomorphism classes of representations of $\mathrm{GL}_{r,\mathbb{Q}}$ to the natural numbers, defined by

$$a_{Y_0}(\theta) := \dim(H^0(Y_0, (f_0^*\mathcal{E}_0)(\theta))) \text{ and } b_{Y_0}(\theta) := \dim(H^0(Y_0, (f_0^*\mathcal{F}_0)(\theta))).$$

By Lemma 1.4.1.3, for every finite étale connected cover $Y_0 \to X_0$, we have $a_{Y_0} = b_{Y_0}$. Suppose that $G(\mathcal{E}_0, x)$ connected. By Proposition 1.3.3.4, for every étale connected cover $f_0 : (Y_0, y) \to (X_0, x)$, the groups $G(f_0^*\mathcal{E}_0, y)$ and $G(\mathcal{E}_0, x)$ are isomorphic via the natural morphisms, thus the functions $a_{Y_0}(\theta)$ and $a_{X_0}(\theta)$ are equal. Thanks to Proposition 1.3.3.3, we also know that there exists an étale Galois cover $f_0 : (Y_0, y) \to (X_0, x)$ such that $G(f_0^*\mathcal{F}_0, y)$ is isomorphic to $G(\mathcal{F}_0, x)^\circ$. The functions $b_{Y_0}(\theta)$ and $b_{X_0}(\theta)$ are equal because of the comparison with $a_{Y_0}(\theta)$ and $a_{X_0}(\theta)$. Therefore, by Lemma 1.4.1.5, the group $G(\mathcal{F}_0, x)$ is connected. This concludes the proof of (i').

To prove (i) we show that $\mathrm{Ker}(\psi_{\mathcal{E}_0})$ and $\mathrm{Ker}(\psi_{\mathcal{F}_0})$ are the same subgroups of $\pi_1^{\text{ét}}(X_0, x)$. By symmetry it is enough to show that $\mathrm{Ker}(\psi_{\mathcal{E}_0}) \subseteq \mathrm{Ker}(\psi_{\mathcal{F}_0})$. This is equivalent to proving

that if $f_0 : (Y_0, y) \to (X_0, x)$ is the Galois cover associated to $\mathrm{Ker}(\psi_{\mathcal{E}_0})$, then the natural map $\mathrm{Ker}(\psi_{\mathcal{E}_0}) \to \pi_1^{\text{ét}}(X_0, x)/\mathrm{Ker}(\psi_{\mathcal{F}_0})$ is the trivial map. In other words, it is enough to show that $G(f_0^*\mathcal{F}_0, y)$ is connected. As $G(f_0^*\mathcal{E}_0, y)$ is connected by construction, this is a consequence of (i').

□

1.4.2 Frobenius tori

We extend here the theory of *Frobenius tori* developed by Serre and Chin in [Ser00] and [Chi04, §5.1] to algebraic coefficient objects over varieties of arbitrary dimension. The result for overconvergent F-isocrystal is completely new. In §3, it is used to get information on the monodromy groups of *convergent F-isocrystals*.

Construction 1.4.2.1 (Frobenius tori)**.** For every E_λ-coefficient object $\mathcal{E}_0$ and every closed point $i_0 : x_0 \hookrightarrow X_0$ we have a functor $\langle \mathcal{E}_0 \rangle \to \langle i_0^*\mathcal{E}_0 \rangle$ of inverse image. For every $\mathbb{F}$-point x over x_0, this functor induces a closed immersion $G(i_0^*\mathcal{E}_0, x) \hookrightarrow G(\mathcal{E}_0, x)$. Let F_{x_0} be the $E_\lambda^{(x_0)}$-point of $G(i_0^*\mathcal{E}_0, x)$ corresponding to the Frobenius automorphism and let $F_{x_0}^{\mathrm{ss}}$ be its semi-simple part. The Zariski closure of the group generated by $F_{x_0}^{\mathrm{ss}}$ is the maximal subgroup of multiplicative type of $G(i_0^*\mathcal{E}_0, x)$. This will be called the *Frobenius group* attached to x_0 and it will be denoted by $M(\mathcal{E}_0, x)$. Its connected component will be the *Frobenius torus* attached to x_0, denoted by $T(\mathcal{E}_0, x)$. If $\mathcal{E}_0$ is E-rational, the torus $T(\mathcal{E}_0, x)$ descends to a torus $\widetilde{T}(\mathcal{E}_0, x)$ over E, such that $T(\mathcal{E}_0, x) \simeq \widetilde{T}(\mathcal{E}_0, x) \otimes_E E_\lambda^{(x_0)}$.

To prove our main theorem on Frobenius tori we first need another outcome of Deligne's conjecture. This is a finiteness result for the set of all the possible valuations of the eigenvalues of the Frobenii at closed points.

Notation 1.4.2.2. Let us fix a prime ℓ. For every prime ℓ' (even $\ell' = \ell$ or $\ell' = p$), we denote by $I_{\ell'}(\overline{\mathbb{Q}}_\ell)$ the set of field isomorphisms $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \overline{\mathbb{Q}}_{\ell'}$ and by $I_\infty(\overline{\mathbb{Q}}_\ell)$ the set of field isomorphisms $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. For every $\ell' \neq p$ we endow $\overline{\mathbb{Q}}_{\ell'}$ with the ℓ'-adic valuation $v : (\overline{\mathbb{Q}}_{\ell'}^\times, \times) \to (\mathbb{R}, +)$, normalized such that $v(\ell') = 1$. On $\overline{\mathbb{Q}}_p$ we consider the p-adic valuation v, normalized so that $v(q) = 1$. Finally, we endow $\mathbb{C}$ with the morphism $v : (\mathbb{C}^\times, \times) \to (\mathbb{R}, +)$ defined by $a \mapsto \log_q(|a|)$.

Definition 1.4.2.3. Let $\mathcal{E}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient object. For every closed point $x_0 \in |X_0|$, let $A_{x_0}(\mathcal{E}_0)$ be the set of Frobenius eigenvalues at x_0. For $* = \ell', \infty$ we define

$$V_*(\mathcal{E}_0) := \left\{ v(\iota(a))/\deg(x_0) \mid \ x_0 \in |X_0|,\ a \in A_{x_0}(\mathcal{E}_0),\ \iota \in I_*(\overline{\mathbb{Q}}_\ell) \right\}.$$

We denote by $V_{\neq p}(\mathcal{E}_0)$ the union of all the subsets $V_{\ell'}(\mathcal{E}_0) \subseteq \mathbb{R}$ with $\ell' \neq p$ and $V_\infty(\mathcal{E}_0) \subseteq \mathbb{R}$ and by $V(\mathcal{E}_0)$ the subset $V_p(\mathcal{E}_0) \cup V_{\neq p}(\mathcal{E}_0) \subseteq \mathbb{R}$.

We also define for $* = \ell', \infty$, the set

$$V_*^{\Theta}(\mathcal{E}_0) := \left\{ v(\iota(a)) \mid [a] \in \Theta(\mathcal{E}_0),\ \iota \in I_*(\overline{\mathbb{Q}}_\ell) \right\}$$

and $V_{\neq p}^{\Theta}(\mathcal{E}_0)$ and $V^{\Theta}(\mathcal{E}_0)$ as before (cf. §1.3.5.3).

Proposition 1.4.2.4. *Let $\mathcal{E}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient object.*

(i) $V_{\neq p}(\mathcal{E}_0) = V_{\neq p}^{\Theta}(\mathcal{E}_0)$.

(ii) *If $\mathcal{E}_0$ is algebraic, the set $V(\mathcal{E}_0)$ is finite.*

Proof. By the definition of $\Theta(\mathcal{E}_0)$, it is enough to prove part (i) when $\mathcal{E}_0$ is irreducible with finite order determinant. In this case, by Theorem 1.3.7.6, the coefficient object $\mathcal{E}_0$ is pure of weight 0 and p-plain. Therefore, we have $V_{\neq p}(\mathcal{E}_0) = V_{\neq p}^{\Theta}(\mathcal{E}_0) = \{0\}$. When $\mathcal{E}_0$ is algebraic, by [Ked18, Lemma-Definition 4.3.2], the set $V_p(\mathcal{E}_0)$ is finite. Moreover, as the types of $\mathcal{E}_0$ are algebraic, the set $V^{\Theta}(\mathcal{E}_0)$ is finite as well. Thanks to part (i), this implies that $V(\mathcal{E}_0)$ is finite. □

1.4.2.5. Let $(X_0, \tilde{x})$ be a smooth connected pointed variety over $\mathbb{F}_q$ and $\mathcal{E}_0$ an algebraic $\overline{\mathbb{Q}}_\ell$-coefficient object of rank r. Let GL_r be the algebraic group $\mathrm{GL}_{r,\overline{\mathbb{Q}}_\ell}$. For every $x \in X_0(\mathbb{F})$, we choose an isomorphism between $\omega_{\tilde{x},\overline{\mathbb{Q}}_\ell}$ and $\omega_{x,\overline{\mathbb{Q}}_\ell}$ (which exists by [Del90]) and a basis of $\omega_{\tilde{x},\overline{\mathbb{Q}}_\ell}(\mathcal{E}_0)$. This determines in turn an embedding $G(\mathcal{E}_0, x) \hookrightarrow \mathrm{GL}_r$ for every x. Let $\mathbb{G}_m^r \subseteq \mathrm{GL}_r$ be the standard maximal torus. We denote by $\chi_1, \ldots, \chi_r$ the standard basis of $X^*(\mathbb{G}_m^r)$. The Frobenius torus $T(\mathcal{E}_0, x) \subseteq G(\mathcal{E}_0, x) \hookrightarrow \mathrm{GL}_r$ is conjugated, by a $\overline{\mathbb{Q}}_\ell$-points of GL_r, to some subtorus $T_x \subseteq \mathbb{G}_m^r$. The torus T_x is uniquely determined up to the action of the permutation group S_r on $\mathbb{G}_m^r$.

Definition. Let $C(\mathcal{E}_0)$ be the set of GL_r-conjugacy classes of Frobenius tori $T(\mathcal{E}_0, x)$ where x varies among the $\mathbb{F}$-points of X_0. Let $D(\mathcal{E}_0)$ be the set of $\mathbb{R}$-linear subspaces of $X^*(\mathbb{G}_m^r)_{\mathbb{R}}$ which admit a set of generators in $V(\mathcal{E}_0)^r \subseteq X_*(\mathbb{G}_m^r)_{\mathbb{R}}$. We have a natural action of S_r on $D(\mathcal{E}_0)$.

Proposition 1.4.2.6. *Let $\mathcal{E}_0$ be an algebraic coefficient object. There exists an injective map of sets $\delta : C(\mathcal{E}_0) \hookrightarrow D(\mathcal{E}_0)/S_r$.*

Proof. Let x be an $\mathbb{F}$-point of X_0 and $\alpha_{x,1}, \ldots, \alpha_{x,r}$ the eigenvalues of the Frobenius at x. We define $Y_x \subseteq \mathbb{R}^r = X_*(\mathbb{G}_m^r)_{\mathbb{R}}$ as the $\mathbb{R}$-linear subspace generated by the elements $y_x^\iota := \left(y_{x,1}^\iota, \ldots, y_{x,1}^\iota\right)$, where ι is an element in $I(\overline{\mathbb{Q}}_\ell)$ and $y_{x,i}^\iota := v(\iota(\alpha_{x,i}))$. By definition, $Y_x \in D(\mathcal{E}_0)$ and its class $[Y_x]$ in $D(\mathcal{E}_0)/S_r$ does not depend on the order of the Frobenius eigenvalues $\alpha_{x,1}, \ldots, \alpha_{x,r}$. We set $\delta([T_x]) := [Y_x]$. We want to show that δ is injective.

The natural pairing $(\cdot,\cdot) : X^*(\mathbb{G}_m^r) \times X_*(\mathbb{G}_m^r) \to \mathbb{Z}$ induces a map $f_x : X^*(\mathbb{G}_m^r)_{\mathbb{Q}} \to \mathrm{Hom}(Y_x, \mathbb{R})$. The group $X^*(T_x)_{\mathbb{Q}}$ is the quotient $X^*(\mathbb{G}_m^r)_{\mathbb{Q}}/K_x$, where K_x is the $\mathbb{Q}$-linear subspace of elements $\chi \otimes a/b$ in $X^*(\mathbb{G}_m^r)_{\mathbb{Q}}$ with $\chi|_{T_x}$ of finite order under tensor. We want to prove

that the Kernel of f_x is K_x. In particular, that Y_x uniquely determines K_x, hence the subtorus $T_x \subseteq \mathbb{G}_m^r$. We first prove that $f_x(K_x) = 0$. Let $\chi = \chi_1^{\otimes a_1} \otimes \cdots \otimes \chi_r^{\otimes a_r} \in X^*(\mathbb{G}_m^r)$ be a character which is finite on T_x. Since $(\alpha_{x,1}, \ldots, \alpha_{x,r}) \in T_x(\overline{\mathbb{Q}}_\ell)$, we have that $\beta_x := \alpha_{x,1}^{a_1} \cdot \ldots \cdot \alpha_{x,r}^{a_r}$ is a root of unity. Therefore, for every $\iota \in I(\overline{\mathbb{Q}}_\ell)$,

$$f_x(\chi)(y_x^\iota) = a_1 y_{x,1}^\iota + \cdots + a_r y_{x,r}^\iota = v(\iota(\beta_x)) = 0.$$

This implies that $f_x(\chi) = 0$.

On the other hand, let $\chi = \chi_1^{\otimes a_1} \otimes \ldots \otimes \chi_r^{\otimes a_r} \in X^*(\mathbb{G}_m^r)$ be a character which is sent to 0 by f_x. We want to show that the restriction of χ to T_x is finite. Since the subgroup generated by the point $(\alpha_{x,1}, \ldots, \alpha_{x,r}) \in T_x(\overline{\mathbb{Q}}_\ell)$ is Zariski dense in T_x, it is enough to show that $\beta_x := \alpha_{x,1}^{a_1} \ldots \alpha_{x,r}^{a_r}$ is a root of unity. By the assumption $\mathcal{E}_0$ algebraic, we know that β_x is an algebraic number. Moreover, for every $\iota \in I(\overline{\mathbb{Q}}_\ell)$, we have that $v(\iota(\beta_x)) = a_1 y_{x,1}^\iota + \cdots + a_r y_{x,r}^\iota = f_x(\chi)(y_x^\iota) = 0$. Thus, by Kronecker's theorem, β_x is a root of unity. □

Corollary 1.4.2.7. *Let $\mathcal{E}_0$ be an algebraic $\overline{\mathbb{Q}}_\ell$-coefficient object. The set $C(\mathcal{E}_0)$ is finite.*

Proof. By Proposition 1.4.2.4.(ii), the set $V(\mathcal{E}_0)$ is finite. Therefore, by definition, $D(\mathcal{E}_0)$ is finite as well. Thanks to Proposition 1.4.2.6, this proves that $C(\mathcal{E}_0)$ is finite. □

From here we could prove directly Theorem 1.4.2.10 for étale lisse sheaves exploiting Čebotarev's density theorem as in the proof of [Ser00, Théorème at page 12]. We need instead two other results to deal with non-étale lisse sheaves and p-adic coefficient objects.

Proposition 1.4.2.8 (after Larsen–Pink)**.** *Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be two compatible coefficient objects on X_0. The reductive ranks of $G(\mathcal{E}_0, x)$ and $G(\mathcal{F}_0, x)$ are equal.*

Proof. We may assume that $\mathcal{E}_0$ and $\mathcal{F}_0$ are semi-simple. Then the result follows from [LP90, Proposition 1] applied to $G(\mathcal{E}_0, x)$ and $G(\mathcal{F}_0, x)$, thanks to Lemma 1.4.1.3. □

Lemma 1.4.2.9. *Let $\underline{\mathcal{E}_0}$ be an E-compatible system. For all but finitely many $\lambda \in |E|_{\neq p}$, the λ-component of $\underline{\mathcal{E}_0}$ is an étale lisse sheaves.*

Proof. Let $\mathcal{E}_0$ be a component of $\underline{\mathcal{E}_0}$ and x_0 a closed point of X_0. As $\mathcal{E}_0$ is algebraic, for all but finitely many primes $\ell \neq p$, the Frobenius eigenvalues at x_0 are ℓ-adic units. Therefore, by Proposition 1.3.1.17, for every $\lambda \in |E|$ which divides such an ℓ, the λ-component of $\underline{\mathcal{E}_0}$ is an étale lisse sheaf. □

Theorem 1.4.2.10. *Let X_0 be a smooth connected variety over $\mathbb{F}_q$ and $\mathcal{E}_0$ an algebraic coefficient object. There exists a Zariski-dense subset $\Delta \subseteq X(\mathbb{F})$ such that for every $\mathbb{F}$-point $x \in \Delta$ and every object $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle$, the torus $T(\mathcal{F}_0, x)$ is a maximal torus of $G(\mathcal{F}_0, x)$. Moreover, if $\mathcal{G}_0$ is a coefficient object compatible with $\mathcal{E}_0$, the subset Δ satisfies the same property for the objects in $\langle \mathcal{G}_0 \rangle$.*

Proof. Let x be a geometric point and $i_0 : x_0 \hookrightarrow X_0$ the embedding of the underlying closed point. For every object $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle$, we have a commutative square of functors

$$\begin{array}{ccc} \langle \mathcal{F}_0 \rangle & \hookrightarrow & \langle \mathcal{E}_0 \rangle \\ \downarrow{\scriptstyle i_0^*} & & \downarrow{\scriptstyle i_0^*} \\ \langle i_0^* \mathcal{F}_0 \rangle & \hookrightarrow & \langle i_0^* \mathcal{E}_0 \rangle. \end{array}$$

It induces a square on monodromy groups

$$\begin{array}{ccc} G(\mathcal{F}_0, x) & \twoheadleftarrow & G(\mathcal{E}_0, x) \\ {\scriptstyle i_{0*}}\uparrow & & {\scriptstyle i_{0*}}\uparrow \\ M(\mathcal{F}_0, x) & \twoheadleftarrow & M(\mathcal{E}_0, x). \end{array}$$

If $T(\mathcal{E}_0, x)$ is a maximal torus in $G(\mathcal{E}_0, x)$, then the same is true for $T(\mathcal{F}_0, x)$ in $G(\mathcal{F}_0, x)$ (see for example [Hum75, Corollary C, page 136]). This shows that it is enough to prove the result when $\mathcal{F}_0 = \mathcal{E}_0$. Moreover, we may assume that $\mathcal{E}_0$ is semi-simple, because semi-simplification does not change the reductive rank of the monodromy group.

We notice that by Proposition 1.4.2.8, if $\mathcal{G}_0$ is a coefficient object compatible with $\mathcal{E}_0$, the torus $T(\mathcal{E}_0, x)$ is maximal in $G(\mathcal{E}_0, x)$ if and only if $T(\mathcal{G}_0, x)$ is maximal in $G(\mathcal{G}_0, x)$. Therefore, it is enough to prove the result for some coefficient object compatible with $\mathcal{E}_0$. By Theorem 1.3.8.2, $\mathcal{E}_0$ sits in a semi-simple compatible system $\underline{\mathcal{E}_0}$. By Lemma 1.4.2.9, there exists a component of $\underline{\mathcal{E}_0}$ which is an étale lisse sheaf. Let us denote it by $\mathcal{V}_0$. After replacing X_0 by a connected finite étale cover we may assume by Proposition 1.3.3.4 that $G(\mathcal{V}_0, x)$ is connected for any choice of x. We choose an $\mathbb{F}$-point $\tilde{x}$ of X. By Corollary 1.4.2.7, the set of conjugacy classes of Frobenius tori $T(\mathcal{V}_0, x)$ in $GL(\omega_{\tilde{x}, \overline{\mathbb{Q}}_\ell}(\mathcal{V}_0))$, where x varies among the $\mathbb{F}$-points of X_0, is finite. Arguing as in [Ser00, theorem at page 12], as a consequence of Čebotarev's density theorem for the étale fundamental group of X_0, there exists a Zariski-dense subset $\Delta \subseteq X(\mathbb{F})$ such that for every $\mathbb{F}$-point $x \in \Delta$, the torus $T(\mathcal{V}_0, x)$ is maximal inside $G(\mathcal{V}_0, x)$ (see also [Chi04, Theorem 5.7] for more details). This concludes the proof. □

Remark 1.4.2.11. In the proof of Theorem 1.4.2.10, we need the results of §1.3.7 to prove the following properties of the coefficient object $\mathcal{E}_0$.

(i) $\mathcal{E}_0$ is ι-mixed.

(ii) The set $V(\mathcal{E}_0)$ is finite.

(iii) $\mathcal{E}_0$ admits a compatible étale lisse sheaf.

For many coefficient objects "coming from geometry", it is possible to prove these properties directly, without using Theorem 1.3.7.4.

1.4.3 The neutral component

We start with a first result on the independence of the neutral components of the monodromy groups of coefficient objects. As in Theorem 1.4.1.1, the independence result we need is Corollary 1.3.6.7.

Proposition 1.4.3.1. *Let X_0 be a smooth geometrically connected variety over $\mathbb{F}_q$. Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be two compatible coefficient objects over X_0.*

(i) *If $\mathcal{E}_0$ and $\mathcal{F}_0$ are semi-simple, $\mathcal{E}_0$ is finite if and only if $\mathcal{F}_0$ is finite.*

(ii) *If $\mathcal{E}_0$ and $\mathcal{F}_0$ are geometrically semi-simple, $\mathcal{E}$ is finite if and only if $\mathcal{F}$ is finite.*

Proof. Thanks to Proposition 1.3.3.4, we may assume that the arithmetic and the geometric monodromy groups of $\mathcal{E}_0$ and $\mathcal{F}_0$ are connected. By Theorem 1.3.7.6, we know that $\mathcal{E}_0$ and $\mathcal{F}_0$ are ι-mixed. Thus, thanks to Corollary 1.3.6.7, the coefficient object $\mathcal{E}_0$ is trivial (resp. geometrically trivial) if and only if the same is true for $\mathcal{F}_0$. □

1.4.3.2. The next result we want to prove is a generalization of [Chi04, Theorem 1.4]. Let (X_0, x) be a smooth connected pointed variety over $\mathbb{F}_q$. Let E be a number field and $\underline{\mathcal{E}_0}$ a semi-simple E-compatible system over X_0. For every $\lambda \in \Sigma$, we denote by $\rho_{\lambda,0}$ the associated representation on $\omega_{x,E_\lambda}(\mathcal{E}_{\lambda,0})$.

Theorem. *After possibly replacing E by a finite extension, there exists a connected split reductive group G_0 over E such that, for every $\lambda \in \Sigma$, the extension of scalars $G_0 \otimes_E E_\lambda$ is isomorphic to $G(\mathcal{E}_{\lambda,0}, x)^\circ$. Moreover, there exists a faithful E-linear representation ρ_0 of G_0 and isomorphisms $\varphi_{\lambda,0} : G_0 \otimes_E E_\lambda \xrightarrow{\sim} G(\mathcal{E}_{\lambda,0}, x)^\circ$ for every $\lambda \in \Sigma$ such that $\rho_0 \otimes_E E_\lambda$ is isomorphic to $\rho_{\lambda,0} \circ \varphi_{\lambda,0}$.*

Following Chin, we use a reconstruction theorem of a reductive group from the Grothendieck semiring of its category of finite-dimensional representations.

Notation 1.4.3.3. If $\mathbf{C}$ is a Tannakian category, we denote by $K^+(\mathbf{C})$ its *Grothendieck semiring*. If $\mathcal{E}_0$ is a coefficient object and $\mathbf{C} = \langle \mathcal{E}_0 \rangle$, we denote it by $K^+(\mathcal{E}_0)$. Finally, when $\mathbf{C} = \mathbf{Rep}(G)$ with G an algebraic group, we will write $K^+(G)$.

Theorem 1.4.3.4 ([Chi08, Theorem 1.4])**.** *Let G and G' be two connected split reductive groups, defined over a field $\mathbb{K}$ of characteristic 0. Let T and T' be maximal tori of G and G' respectively. For every pair of isomorphisms $\varphi_{T'} : T' \xrightarrow{\sim} T$ and $f : K^+(G) \xrightarrow{\sim} K^+(G')$ making the following diagram commuting*

$$\begin{array}{ccc} K^+(G) & \xrightarrow{f} & K^+(G') \\ \downarrow & & \downarrow \\ K^+(T) & \xrightarrow{\varphi^*_{T'}} & K^+(T'), \end{array}$$

there exists an isomorphism $\varphi : G' \xrightarrow{\sim} G$ of algebraic groups such that the induced homomorphism φ^* on the Grothendieck semirings is equal to f and the restriction of φ to T' is equal to $\varphi_{T'}$.

Remark 1.4.3.5. The maximal tori that we will use to apply Theorem 1.4.3.4 will be the Frobenius tori provided by Theorem 1.4.2.10. Suppose that $\mathcal{E}_0$ is a coefficient object and for some $\mathbb{F}$-point x, the group $M(\mathcal{E}_0, x)$ is connected and $\widetilde{T}(\mathcal{E}_0, x)$ is a split torus over E. Then, the group of characters of $\widetilde{T}(\mathcal{E}_0, x)$ is canonically isomorphic to the subgroup of $E^\times$ generated by the eigenvalues of F_{x_0}. The isomorphism is given by the evaluation of a character at the point $F^{\mathrm{ss}}_{x_0}$. In particular, if $\mathcal{E}_0$ sits in an E-compatible system $\underline{\mathcal{E}_0}$ and $\widetilde{T}(\mathcal{E}_{\lambda,0}, x)$ is split over E for one $\lambda \in \Sigma$ (or equivalently every $\lambda \in \Sigma$), the semirings $K^+(\widetilde{T}(\mathcal{E}_{\lambda,0}, x))$ are all canonically isomorphic when λ varies in Σ. Moreover, notice that for every $\lambda \in \Sigma$, the semiring $K^+(\widetilde{T}(\mathcal{E}_{\lambda,0}, x))$ is canonically isomorphic to $K^+(T(\mathcal{E}_{\lambda,0}, x))$.

The known cases of the companions conjecture provide isomorphisms of the Grothendieck semiring of compatible objects. A bit surprisingly, we have these isomorphisms even if we do not dispose, at the moment, of a general way to construct compatible overconvergent F-isocrystals in dimension greater than 1.

Proposition 1.4.3.6. *Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be two compatible coefficient objects such that all the irreducible objects in $\langle \mathcal{E}_0 \rangle$ and $\langle \mathcal{F}_0 \rangle$ are absolutely irreducible. There exists a unique isomorphism of semirings $K^+(\mathcal{E}_0) \xrightarrow{\sim} K^+(\mathcal{F}_0)$ preserving the characteristic polynomial functions.*

Proof. The uniqueness and the injectivity of the map are consequences of Theorem 1.3.6.8 that we can apply thanks to Theorem 1.3.7.6. By Theorem 1.3.8.2, when $\mathcal{F}_0$ is a lisse sheaf, there exists a morphism of semirings $f : K^+(\mathcal{E}_0) \to K^+(\mathcal{F}_0)$ preserving the characteristic polynomial functions. We notice that to prove the final statement, it is enough to show that f is an isomorphism in this case. Indeed if $\mathcal{E}_0$ and $\mathcal{F}_0$ are two compatible overconvergent F-isocrystals, we can always find, by Theorem 1.3.8.2, a compatible lisse sheaf $\mathcal{G}_0$. Then, the isomorphism $K^+(\mathcal{E}_0) \xrightarrow{\sim} K^+(\mathcal{F}_0)$, is obtained via the composition

$$K^+(\mathcal{E}_0) \xrightarrow{\sim} K^+(\mathcal{G}_0) \xrightarrow{\sim} K^+(\mathcal{F}_0).$$

By Corollary 1.3.7.8 and the hypothesis, f sends irreducible objects to irreducible objects. Hence, if $[\mathcal{H}_0] \in K^+(\mathcal{E}_0)$ and $\sum_{i=0}^n m_i[\mathcal{H}_0^i]$ is the isotypic decomposition of $[\mathcal{H}_0]$, then $\sum_{i=0}^n m_i f([\mathcal{H}_0^i])$ is the isotypic decomposition of $f([\mathcal{H}_0])$. In particular, a summand of an element in the image of f is again in the image of f. By construction, we know that for every $n, m \in \mathbb{N}$, the classes $\left[\mathcal{F}_0^{\otimes n} \otimes (\mathcal{F}_0^\vee)^{\otimes m}\right]$ are in the image of f. This shows that f is surjective. $\square$

Remark 1.4.3.7. The assumption that the irreducible objects in $\langle\mathcal{E}_0\rangle$ and $\langle\mathcal{F}_0\rangle$ are absolutely irreducible is verified, for example, when $G(\mathcal{E}_0, x)$ and $G(\mathcal{F}_0, x)$ are split reductive groups. In particular, it is always possible to obtain this condition after a finite extension of the fields of scalars of the coefficient objects.

1.4.3.8Proof of Theorem 1.4.3.2. Thanks to Theorem 1.4.1.1, there exists a Galois cover of X_0 such that all the arithmetic monodromy groups of the compatible system are connected. By Proposition 1.3.3.4, the neutral components of the monodromy groups remain unchanged when we pass to the cover. By Remark 1.3.2.5, after extending E, we may change the $\mathbb{F}$-point x without changing the isomorphism class of the monodromy groups. Thanks to Theorem 1.4.2.10, we may choose x, so that $T(\mathcal{E}_{\lambda,0}, x)$ is a maximal torus in $G(\mathcal{E}_{\lambda,0}, x)$ for every $\lambda \in \Sigma$. Moreover, after enlarging the base field, we may assume that x_0 is an $\mathbb{F}_q$-point. We fix a place $\mu \in \Sigma$. As $G(\mathcal{E}_{\mu,0}, x)$ is connected and reductive, up to replacing E with a finite extension, there exists a connected split reductive group G_0, defined over E, which contains a maximal torus T_0, such that $G_0 \otimes_E E_\mu \simeq G(\mathcal{E}_{\mu,0}, x)$ and $T_0 \simeq \widetilde{T}(\mathcal{E}_{\lambda,0}, x)$. We choose $\rho_0 : G_0 \hookrightarrow \mathrm{GL}_{r,E}$ and $\varphi_{\mu,0} : G_0 \otimes_E E_\mu \xrightarrow{\sim} G(\mathcal{E}_{\mu,0}, x)$ such that $\varphi_{\mu,0}(T_0 \otimes_E E_\mu) = T(\mathcal{E}_{\mu,0}, x)$ and $\rho_0 \otimes_E E_\mu \simeq \rho_{\mu,0} \circ \varphi_{\mu,0}$. The isomorphism $\varphi_{\mu,0}$ induces an isomorphism

$$\varphi^*_{\mu,0} : K^+(\mathcal{E}_{\mu,0}) \xrightarrow{\sim} K^+(G_0 \otimes_E E_\mu)$$

which sends $[\mathcal{E}_{\mu,0}]$ to $[\rho_0 \otimes_E E_\mu]$. As T_0 is split over E, for every $\lambda \in \Sigma$ the reductive group $G(\mathcal{E}_{\lambda,0}, x)$ is split. By Proposition 1.4.3.6, for every $\lambda \in \Sigma$, there exists a unique isomorphism $g_{\lambda,\mu} : K^+(\mathcal{E}_{\lambda,0}) \simeq K^+(\mathcal{E}_{\mu,0})$ preserving the characteristic polynomial functions, hence it sends $[\mathcal{E}_{\lambda,0}]$ to $[\mathcal{E}_{\mu,0}]$. As G_0 is split and connected, there exists a canonical isomorphism $h_{\mu,\lambda} : K^+(G_0 \otimes_E E_\mu) \xrightarrow{\sim} K^+(G_0 \otimes_E E_\lambda)$. We take

$$f_{\lambda,0} := h_{\mu,\lambda} \circ \varphi^*_{\mu,0} \circ g_{\lambda,\mu} : K^+(\mathcal{E}_{\lambda,0}) \to K^+(G_0 \otimes_E E_\lambda).$$

By construction it commutes with the natural isomorphism $K^+(T(\mathcal{E}_{\lambda,0}, x)) \simeq K^+(T_0 \otimes_E E_\lambda)$. Thanks to Theorem 1.4.3.4, the isomorphism $f_{\lambda,0}$ induces an isomorphism $\varphi_{\lambda,0} : G_0 \otimes_E E_\lambda \xrightarrow{\sim} G(\mathcal{E}_{\lambda,0}, x)$ such that $f_{\lambda,0} = \varphi^*_{\lambda,0}$. As $f_{\lambda,0}([\mathcal{E}_{\lambda,0}]) = [\rho_0 \otimes_E E_\lambda]$, the representations $\rho_0 \otimes_E E_\lambda$ and $\rho_{\lambda,0} \circ \varphi_{\lambda,0}$ are isomorphic. □

Corollary 1.4.3.9. *Let $\mathcal{E}_0$ be a semi-simple $\overline{\mathbb{Q}}_\ell$-coefficient object. The set of closed points where the Frobenius is semi-simple is Zariski-dense in X_0.*

Proof. Twisting the irreducible summands of $\mathcal{E}_0$, we may assume that $\mathcal{E}_0$ has trivial types. Hence by Theorem 1.3.7.6, the coefficient object $\mathcal{E}_0$ is algebraic. Let $\tilde{x}$ be an $\mathbb{F}$-point of X_0 and $\rho_{\mathcal{E}_0} : G(\mathcal{E}_0, \tilde{x}) \hookrightarrow \mathrm{GL}(\omega_{\tilde{x},\overline{\mathbb{Q}}_\ell}(\mathcal{E}_0))$ be the tautological representation. If $\mathcal{E}_0$ is an étale lisse sheaf, by [LP92, Proposition 7.2], there exists a Zariski-dense set of closed points x_0 such that the Frobenius at x_0 is Γ-*regular* with respect to $\rho_{\mathcal{E}_0} : G(\mathcal{E}_0, \tilde{x}) \hookrightarrow \mathrm{GL}(\omega_{\tilde{x},\overline{\mathbb{Q}}_\ell}(\mathcal{E}_0))$ (cf. *ibid.*). In

particular, at these points the Frobenius is (regular) semi-simple by [LP92, Proposition 4.6]. If $\mathcal{E}_0$ is not an étale lisse sheaf, by Theorem 1.3.8.2 and Lemma 1.4.2.9, there exists a semi-simple compatible étale lisse sheaf $\mathcal{V}_0$. By the previous discussion, there exists a Zariski-dense set of closed points of X_0 which are Γ-regular with respect to the tautological representation $\rho_{\mathcal{V}_0} : G(\mathcal{V}_0, x) \hookrightarrow \mathrm{GL}(\omega_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{V}_0))$. By Theorem 1.4.3.2, for all these closed points, the Frobenius is Γ-regular even with respect to the tautological representation $\rho_{\mathcal{E}_0} : G(\mathcal{E}_0, x) \hookrightarrow \mathrm{GL}(\omega_{x,\overline{\mathbb{Q}}_\ell}(\mathcal{E}_0))$. Thus we conclude again thanks to [LP92, Proposition 4.6]. □

Remark 1.4.3.10. As a consequence of Theorem 1.4.3.2, we obtain the same result of independence for the geometric monodromy groups. Indeed, by Corollary 1.3.5.5 if $\mathcal{E}_0$ is a geometrically semi-simple coefficient object, $G(\mathcal{E}, x)^\circ$ is the derived subgroup of $G(\mathcal{E}_0, x)^\circ$.

Remark 1.4.3.11. If we weaken the statement of Theorem 1.4.3.2, asking that all the isomorphisms between G_0 and the monodromy groups are defined over $\overline{\mathbb{Q}}_\ell$, rather than E_λ, one can prove it differently. One can use [KLV14, Theorem 1.2], a stronger version of Theorem 1.4.3.4, in combination with Proposition 1.4.3.6. This proof does not use Frobenius tori.

The author became aware of the theorem of Kazhdan–Larsen–Varshavsky reading [Dri18]. In his paper, Drinfeld uses this result as a starting point to prove the independence of the entire monodromy groups over $\overline{\mathbb{Q}}_\ell$ (not only the neutral components).

1.4.4 Lefschetz theorem

In this section, we prove an independence result for Theorem 1.3.7.5. This could also be obtained as a consequence of Theorem 1.4.3.2. Here we have preferred to give a proof which exploits the full strength of the Tannakian lemma [AE16, Lemma 1.6]. A similar argument is used in [*ibid.*, Corollary 3.7]. In our proof, when the coefficient objects are ι-mixed, we do not need the results in §1.3.7. We first prove a lemma which relates the arithmetic and the geometric situation.

Lemma 1.4.4.1. *Let $(Y_0, y) \to (X_0, x)$ be a morphism of geometrically connected pointed varieties over $\mathbb{F}_q$. Let $\mathcal{E}_0$ be a coefficient object on X_0.*

(i) *If the natural morphism $f_* : G(f^*\mathcal{E}, y) \to G(\mathcal{E}, x)$ is an isomorphism, the same is true for $f_{0*} : G(f_0^*\mathcal{E}_0, y) \to G(\mathcal{E}_0, x)$.*

(ii) *If $\mathcal{E}_0$ is geometrically semi-simple and $f_{0*} : G(f_0^*\mathcal{E}_0, y)^\circ \to G(\mathcal{E}_0, x)^\circ$ is an isomorphism, even $f_* : G(f^*\mathcal{E}, y)^\circ \to G(\mathcal{E}, x)^\circ$ is an isomorphism.*

Proof. We want to use the functorial diagram of Proposition 1.3.2.6.(iii) to show that the morphism f_{0*} in (i) is surjective. As Y_0 and X_0 are geometrically connected, $f_{0*} : \pi_1^\lambda(Y_0, y)^{cst} \to \pi_1^\lambda(X_0, x)^{cst}$ is an isomorphism, thus $f_{0*} : G(f^*\mathcal{E}_0, y)^{cst} \to G(\mathcal{E}_0, x)^{cst}$ is surjective. On the other hand, at the level of geometric monodromy groups, $f_* : G(f^*\mathcal{E}, y) \to G(\mathcal{E}, x)$ is surjective by assumption. The surjectivity of $f_{0*} : G(f_0^*\mathcal{E}_0, y) \to G(\mathcal{E}_0, x)$ is then a consequence of the other

two. For (ii) we notice that by Corollary 1.3.5.5, the algebraic groups $G(f^*\mathcal{E}, y)^\circ$ and $G(\mathcal{E}, x)^\circ$ are the derived subgroups of $G(f_0^*\mathcal{E}_0, y)^\circ$ and $G(\mathcal{E}_0, x)^\circ$ respectively. Thus we get the result. $\square$

Theorem 1.4.4.2. *Let $f_0 : (Y_0, y) \to (X_0, x)$ be a morphism of geometrically connected smooth pointed varieties. Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be compatible geometrically semi-simple coefficient objects over X_0. Let $\varphi_0 : G(f_0^*\mathcal{E}_0, y) \to G(\mathcal{E}_0, x)$ and $\psi_0 : G(f_0^*\mathcal{F}_0, y) \to G(\mathcal{F}_0, x)$ be the morphisms induced by f_0^* and let φ and ψ be their restriction to the geometric monodromy groups.*

(i) *If φ is an isomorphism, the same is true for ψ.*

(ii) *If φ_0 is an isomorphism, the same is true for ψ_0.*

Proof. By Lemma 1.4.4.1, part (i) implies part (ii). Notice that φ and ψ are always injective, thus to prove part (i) it is enough to prove that if φ is surjective, even ψ is so. Suppose that φ is surjective, we want to apply [AE16, Lemma 1.6] to prove that ψ is surjective. Indeed, the morphism ψ satisfies the hypothesis $(\star)$ of the lemma by Theorem 1.3.5.1. Thus we are reduced to show that $f^* : \langle \mathcal{F} \rangle \to \langle f^*\mathcal{F} \rangle$ is fully faithful.

A functor of Tannakian categories commuting with fibre functors is always faithful. Hence it is enough to prove that f^* preserves the dimensions of the Hom-sets, or equivalently that for every $\mathcal{G} \in \langle \mathcal{F} \rangle$ we have

$$h^0(\mathcal{G}) = h^0(f^*\mathcal{G}), \tag{1.4.4.1}$$

where we denote by h^0 the dimension of the space of global sections of geometric coefficient objects.

We proceed by steps. First we prove that for every pair of coefficient objects $\mathcal{G}', \mathcal{G}'' \in \langle \mathcal{F} \rangle$, they satisfy the equality (1.4.4.1) if and only if the same is true for $\mathcal{G}' \oplus \mathcal{G}''$. By the additivity of h^0, it is clear that if the geometric coefficient objects satisfy the equality individually, then the same is true for their direct sum. Conversely, if $h^0(\mathcal{G}' \oplus \mathcal{G}'') = h^0(f^*(\mathcal{G}' \oplus \mathcal{G}''))$, then

$$h^0(\mathcal{G}') - h^0(f^*\mathcal{G}') + h^0(\mathcal{G}'') - h^0(f^*\mathcal{G}'') = 0.$$

Since f^* is faithful, then $h^0(\mathcal{G}') - h^0(f^*\mathcal{G}') \leq 0$ and $h^0(\mathcal{G}'') - h^0(f^*\mathcal{G}'') \leq 0$. Thus $h^0(\mathcal{G}') = h^0(f^*\mathcal{G}')$ and $h^0(\mathcal{G}'') = h^0(f^*\mathcal{G}'')$, as we wanted. In particular, as $\langle \mathcal{F} \rangle$ is a semi-simple category, we have proven that it is enough to show (1.4.4.1) for the objects of the form $\mathcal{F}^{\otimes m} \otimes (\mathcal{F}^\vee)^{\otimes n}$ with $m, n \in \mathbb{N}$.

We fix $m, n \in \mathbb{N}$. By the hypothesis, the $\otimes$-functor $f^* : \langle \mathcal{E} \rangle \to \langle f^*\mathcal{E} \rangle$ is fully faithful, therefore the equality (1.4.4.1) holds for $\mathcal{E}^{\otimes m} \otimes (\mathcal{E}^\vee)^{\otimes n}$. By Corollary 1.3.6.7.(i), as we know by Theorem 1.3.7.6 that every coefficient object is ι-mixed, we have that $h^0(\mathcal{E}^{\otimes m} \otimes (\mathcal{E}^\vee)^{\otimes n}) = h^0(\mathcal{F}^{\otimes m} \otimes (\mathcal{F}^\vee)^{\otimes n})$ and $h^0(f^*(\mathcal{E}^{\otimes m} \otimes (\mathcal{E}^\vee)^{\otimes n})) = h^0(f^*(\mathcal{F}^{\otimes m} \otimes (\mathcal{F}^\vee)^{\otimes n}))$. Hence, we get $h^0(\mathcal{F}^{\otimes m} \otimes (\mathcal{F}^\vee)^{\otimes n}) = h^0(f^*(\mathcal{F}^{\otimes m} \otimes (\mathcal{F}^\vee)^{\otimes n}))$. This concludes the proof. $\square$

1.5 Coefficient objects on abelian varieties

1.5.1 A finiteness result

Let X_0 be an abelian variety over $\mathbb{F}_q$ with identity x_0 and let x be a geometric point over x_0. We prove a finiteness statement for coefficient objects defined on X_0.

Theorem 1.5.1.1. *Let X_0 be an abelian variety. Every absolutely irreducible E_λ-coefficient object with finite order determinant is finite. In particular, every ι-pure E_λ-coefficient object on X_0 becomes constant after passing to a finite étale cover.*

1.5.1.2 *Proof without companions.* After extending E_λ we can assume that λ is an admissible place for X_0. We want to prove that the fundamental group $\pi_1^\lambda(X,x)$ is commutative via an Eckmann–Hilton argument [EH62, Theorem 5.4.2]. As X_0 is projective, by Proposition 1.3.4.4, the two projections of $X_0 \times X_0$ to its factors induce an isomorphism

$$\pi_1^\lambda(X \times_{\mathbb{F}} X, x \times x) \xrightarrow{\sim} \pi_1^\lambda(X,x) \times \pi_1^\lambda(X,x).$$

If $m : X \times_{\mathbb{F}} X \to X$ is the multiplication map of X, the morphism

$$\widetilde{m_*} : \pi_1^\lambda(X,x) \times \pi_1^\lambda(X,x) \xrightarrow{\sim} \pi_1^\lambda(X \times_{\mathbb{F}} X, x \times x) \xrightarrow{m_*} \pi_1^\lambda(X,x)$$

endows $\pi_1^\lambda(X,x)$ with the structure of a group object in the category of affine group schemes. By Eckmann–Hilton, this implies that $\pi_1^\lambda(X,x)$ is commutative.

Let $\mathcal{E}_0$ be an absolutely irreducible coefficient object with finite order determinant. We notice first that the geometric monodromy group $G(\mathcal{E},x)$, being a quotient of $\pi_1^\lambda(X,x)$ is also commutative. On the other hand, as $\mathcal{E}_0$ is semi-simple, $G(\mathcal{E},x)$ is reductive. Thus we have proven that $G(\mathcal{E},x)$ is a group of multiplicative type. By Theorem 1.3.5.4, we deduce that $G(\mathcal{E},x)$ is finite. As the coefficient object $\mathcal{E}_0$ is absolutely irreducible with finite order determinant, by Proposition 1.3.5.7 $G(\mathcal{E},x)$ has finite index in $G(\mathcal{E}_0,x)$. Thus $\mathcal{E}_0$ is a finite coefficient object.

To prove the second part, by Proposition 1.3.3.3, it is enough to show that a ι-pure coefficient object $\mathcal{E}_0$ is geometrically finite. By Corollary 1.3.6.4, we already know that $\mathcal{E}_0$ is geometrically semi-simple. Thus, after extending the field of scalars and taking the semi-simplification, we can assume that $\mathcal{E}_0$ is absolutely irreducible. By Corollary 1.3.5.2, there exists a twist $\mathcal{F}_0$ of $\mathcal{E}_0$ with finite order determinant. By the first part of the statement, we know that $\mathcal{F}_0$ is finite, thus $\mathcal{E}_0$ is geometrically finite, as we wanted. □

The previous proof does not use Theorem 1.3.7.4. It relies mainly on Theorem 1.3.5.4 and Proposition 1.3.4.4. As a consequence, we obtain a proof of Deligne's conjectures for abelian varieties that does not use the Langlands correspondence.

Corollary 1.5.1.3. *If X_0 is an abelian variety, then Conjecture 1.3.7.1 holds.*

Proof. Let $\mathcal{E}_0$ be an absolutely irreducible $\overline{\mathbb{Q}}_\ell$-coefficient object whose determinant has finite order. By the previous proposition $\mathcal{E}_0$ is finite, thus the eigenvalues of the Frobenii at closed points are roots of unity. In particular, $\mathcal{E}_0$ is pure of weight 0 and p-plain. Moreover, $\mathcal{E}_0$ corresponds to an absolutely irreducible $\overline{\mathbb{Q}}_\ell$-linear representation ρ of a finite quotient G of $\pi_1^{\text{ét}}(X_0, x)$. As G is a finite group, there exists a number field $E \subseteq \overline{\mathbb{Q}}_\ell$ and an absolutely irreducible E-linear representation $\widetilde{\rho}$ of G such that $\widetilde{\rho} \otimes_E \overline{\mathbb{Q}}_\ell \simeq \rho$.

We deduce first that $\mathcal{E}_0$ is E-rational. In addition, if ℓ' is a prime, for every embedding $\tau : E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$ the representation $\widetilde{\rho} \otimes_E \overline{\mathbb{Q}}_{\ell'}$ corresponds to an absolutely irreducible $\overline{\mathbb{Q}}_{\ell'}$-coefficient object, E-rational with respect to τ and E-compatible with $\mathcal{E}_0$. This concludes the proof. □

1.5.1.4 *Proof of Theorem 1.5.1.1 with companions.* By Corollary 1.3.5.2 there exists a twist $\mathcal{F}_0$ of $\mathcal{E}_0$ with finite order determinant. In light of Theorem 1.3.7.6, there exists a number field E such that $\mathcal{F}_0$ is E-rational and ι-pure. By Theorem 1.3.7.7 there exists an E-rational lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}_0$, E-compatible with $\mathcal{F}_0$. By Corollary 1.3.7.8, the lisse sheaf $\mathcal{G}_0$ is absolutely irreducible. As $\pi^{\text{ét}}(X, x)$ is commutative, thanks to Theorem 1.3.5.4 for lisse sheaves, we know that $\mathcal{G}$ is finite. By Proposition 1.3.5.7, also $\mathcal{G}_0$ is finite. By virtue of Proposition 1.4.3.2, the coefficient object $\mathcal{F}_0$, being compatible with $\mathcal{G}_0$ and absolutely irreducible, is finite as well. This proves the first part of the statement. For the second part we can proceed as in the proof in §1.5.1.2. □

Remark 1.5.1.5. Even if the second proof uses a deep result as Theorem 1.3.7.7, it has the advantage that can be adapted to a wider class of varieties. Indeed, to apply the reasoning, it is enough that $\pi_1^{\text{ét}}(X, x)$ contains an open subgroup that is a *solvable profinite group*, namely an open subgroup which contains a finite normal series of closed subgroups with abelian successive quotients.

1.5.2 Newton polygons

Tsuzuki has proven that the *Newton polygons* of F-isocrystals on an abelian variety are constant [Tsu17, Theorem 3.7]. We use Theorem 1.5.1.1 to recover his result. We recall first the notion of Newton polygon at closed points of a coefficient object.

Definition 1.5.2.1. Let X_0 be a smooth variety, ℓ a prime number and ι_p an isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_p$. Let v_p the p-adic valuation of $\overline{\mathbb{Q}}_p$, normalized such that $v_p(q) = 1$. We denote by the same symbol the valuation induced by ι_p on $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{E}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient object on X_0 and x_0' a closed point of X_0. We consider the characteristic polynomial $P_{x_0'}(\mathcal{E}_0, t) = a_0 + a_1 t + \cdots + a_r t^r$ of $\mathcal{E}_0$ at x_0', where $a_0 = 1$ and $(a_1, \ldots, a_r) \in \overline{\mathbb{Q}}_\ell^{r-1} \times \overline{\mathbb{Q}}_\ell^\times$. Let $\Lambda_{x_0', \iota_p}(\mathcal{E}_0)$ be the polygonal chain in $\mathbb{R}^2$ with vertexes

$$\Big(\big(i, v_p(a_i)/\deg(x_0') \big) \Big)_{0 \le i \le r}.$$

We define the *ι_p-Newton polygon* of $\mathcal{E}_0$ at x_0' as the boundary of the upper convex hull of $\Lambda_{x_0',\iota_p}(\mathcal{E}_0)$. We say that $\mathcal{E}_0$ is *ι_p-isoclinic of ι_p-slope v* if for every closed point x_0', the ι_p-Newton polygon of $\mathcal{E}_0$ at y_0 consists of one segment of slope v. We say that $\mathcal{E}_0$ is *ι_p-unit-root* if it is ι_p-isoclinic of ι_p-slope 0.

Corollary 1.5.2.2. *Let X_0 be an abelian variety. The ι_p-Newton polygon at closed points of a coefficient object $\mathcal{E}_0$ defined on X_0 is independent of the point. Moreover, if $\mathcal{E}_0$ is absolutely irreducible, then it is ι_p-isoclinic.*

Proof. After extending the field of scalars and taking the semi-simplification, $\mathcal{E}_0$ can be expressed as a direct sum of absolutely irreducible coefficient objects. This operation does not change the ι_p-Newton polygons at closed points. As a consequence, it is enough to show that if $\mathcal{E}_0$ is absolutely irreducible, then it is ι_p-isoclinic.

We know by Corollary 1.3.5.2 that such a coefficient object is the twist of a coefficient object with finite order determinant. Hence, in light of Theorem 1.5.1.1, it is a twist of a finite coefficient object. A finite coefficient object is ι_p-unit-root, because the eigenvalues of the characteristic polynomials at closed points are all roots of unity. At the same time, the twist of a ι_p-unit-root coefficient object by $a \in \overline{\mathbb{Q}}_\ell^\times$ is ι_p-isoclinic of ι_p-slope $v_p(a)$. This yields the desired result. □

Remark 1.5.2.3. One could even show that Tsuzuki's result (Corollary 1.5.2.2) implies Theorem 1.5.1.1. First, taking twists, one reduces the theorem to the case of ι_p-unit-root coefficient objects. By [Ked11, Theorem 2.3.7], the coefficient object is then determined by a representation of the étale fundamental group. As the geometric étale fundamental group of an abelian variety is commutative, the finiteness of the geometric monodromy group becomes a consequence of class field theory.

2 Some remarks on the companions conjecture for normal varieties

2.1 Introduction

2.1.1 The companions conjecture

Let $\mathbb{F}_q$ be a finite field of characteristic p and X_0 a normal variety over $\mathbb{F}_q$. Let ℓ be a prime different from p and $\mathcal{V}_0$ an absolutely irreducible Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 with finite order determinant. For every closed point of X_0, we have a well-defined characteristic polynomial associated to the action of the Frobenius at that point. Let $E \subseteq \overline{\mathbb{Q}}_\ell$ be the field generated by the coefficients of all these polynomials. Deligne has proven in [Del12], that E is a finite extension of $\mathbb{Q}$. He shows this finiteness exploiting the known case when X_0 is a curve, proven by L. Lafforgue in [Laf02] as a consequence of the Langlands correspondence. This property of the field E was conjectured by Deligne in [Del80, Conjecture 1.2.10] together with other properties for $\mathcal{V}_0$. One of them is the following one.

Conjecture 2.1.1.1 (Companion conjecture)**.** *After possibly replacing E with a finite extension, for every finite place λ not dividing p, there exists a Weil lisse E_λ-sheaf compatible with $\mathcal{V}_0$ (same characteristic polynomials of the Frobenii at closed points).*

When X_0 has dimension 1, the conjecture is again a consequence of the Langlands correspondence. In higher dimension, Drinfeld in [Dri12] proved Conjecture 2.1.1.1 when X_0 is smooth. He uses some ideas of Wiesend developed in [Wie06] to deduce the result from the case of curves. Unluckily his method can not be applied directly to prove the full conjecture [Drinfeld, *op. cit.*, §6].

2.1.2 The obstruction

Suppose for simplicity that the singular locus of X_0 consists of one closed point and that we can solve the singularity. In other words, suppose that there exists a smooth variety Y_0 and a proper morphism $f_0 : Y_0 \to X_0$ which sends $Z_0 \subseteq Y_0$ to a closed point $x_0 \in |X_0|$ and which is an isomorphism outside Z_0. Assume also that $Z := Z_0 \otimes_{\mathbb{F}_q} \mathbb{F}$ is connected. We have an exact sequence

$$\pi_1^{\text{ét}}(Z, z) \xrightarrow{i_*} \pi_1^{\text{ét}}(Y_0, z) \xrightarrow{f_{0*}} \pi_1^{\text{ét}}(X_0, x) \to 1$$

in the sense that the smallest normal closed subgroup containing the image of i_* is the kernel of f_{0*} (see Lemma 2.3.1.2).

This means that every Weil lisse sheaf $\mathcal{V}_0$ on Y_0, such that $\mathcal{V}_0|_Z \simeq \overline{\mathbb{Q}}_\ell^{\oplus r}$ for some r, is the inverse image of a Weil lisse sheaf defined over X_0. As we know the companions conjecture on Y_0, in order to deduce it for X_0, we have to verify the following.

Property 2.1.2.1. For every pair $(\mathcal{V}_0, \mathcal{W}_0)$ of compatible absolutely irreducible Weil lisse sheaves with finite order determinant on Y_0, the sheaf $\mathcal{V}_0$ is trivial on $Z := Z_0 \otimes \mathbb{F}$ if and only if the same is true for $\mathcal{W}_0$.

In general, if this property holds we say that the morphism $Z_0 \to X_0$ is *balanced* (Definition 2.3.2.4).

Conjecture 2.1.2.2 (Conjecture 2.3.2.5)**.** *Let X_0 and Z_0 be varieties over $\mathbb{F}_q$. If X_0 is normal, every morphism $g_0 : Z_0 \to X_0$ is balanced.*

We show that the property of a morphism to be balanced is "invariant under deformations" (Proposition 2.3.2.8). We also prove the conjecture in some cases.

Theorem 2.1.2.3 (Theorem 2.4.2.3)**.** *Let $g_0 : Z_0 \to X_0$ be a morphism between two varieties over $\mathbb{F}_q$. Suppose X_0 normal, then g_0 is balanced in the following cases.*

(i) *When Z_0 is a normal variety.*

(ii) *When Z_0 is a semi-stable curve with simply connected dual graph.*

(iii) *If the smallest closed normal subgroup of $\pi_1^{\acute{e}t}(X_0, x)$ containing the image of $\pi_1^{\acute{e}t}(Z, z)$ is open inside $\pi_1^{\acute{e}t}(X, x)$.*

(iv) *If $\pi_1^{\acute{e}t}(X, x)$ contains an open solvable profinite subgroup.*

2.2 Notation

We fix a prime number p and we denote by q a certain power of p. Let $\mathbb{F}_q$ be a field with q elements and $\mathbb{F}$ an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. A variety over a field k will be a separated scheme of finite type over k. We denote by $X_0, Y_0, Z_0, ...$ varieties over $\mathbb{F}_q$ and by $X, Y, Z, ...$ the base change of them to $\mathbb{F}$. In general we always denote with a subscript $_0$ objects and morphisms defined over $\mathbb{F}_q$ and the suppression of it will mean the extension to $\mathbb{F}$.

If X_0 is connected and x is a geometric point of X_0 then we denote by $\pi_1^{\text{ét}}(X_0, x)$ and $\pi_1^{\text{ét}}(X, x)$ the étale fundamental groups of X_0 and X, by $W(\mathbb{F}/\mathbb{F}_q)$ the Weil group of $\mathbb{F}_q$ and by $W(X_0, x)$ the Weil group of X_0. If y_0 is a closed point of X_0 then we write $F_{y_0} \subseteq W(X_0, x)$ for the conjugacy class of geometric Frobenii of y_0 and we call F_{y_0} the *Frobenii at* y_0.

The letter ℓ will denote a prime number different from p. For every ℓ we fix an algebraic closure of $\mathbb{Q}_\ell$, denoted by $\overline{\mathbb{Q}}_\ell$. We have a category $\mathbf{LS}(X, \overline{\mathbb{Q}}_\ell)$ of *lisse sheaves* defined over X. Adding an action of $W(\mathbb{F}/\mathbb{F}_q)$ we obtain the category of *Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaves* defined over X_0, denoted by $\mathbf{Weil}(X_0, \overline{\mathbb{Q}}_\ell)$. We often call them simply lisse sheaves. We denote Weil lisse sheaves with a calligraphic letter, as for example $\mathcal{V}_0$, and their restriction to X removing $_0$.

For every natural number n, a lisse sheaf is said *pure* of *weight* n if for every closed point x_0 of X_0 and every isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, the eigenvalues of any *Frobenius automorphism* induced by F_{x_0} have complex absolute value $(\#\kappa(x_0))^{n/2}$. In the rest of the exposition the representations of the Weil group and the étale fundamental group will always be continuous and finite dimensional even if not said explicitly.

If E is a number field, an *E-compatible system* over X_0, denoted $\underline{\mathcal{V}_0}$, is a family $\{\mathcal{V}_{0,\lambda}\}_{\lambda \nmid p}$ where:

(i) For every $\lambda \nmid p$, $\mathcal{V}_{0,\lambda}$ is an E-rational lisse E_λ-sheaf with respect the natural inclusion $E \hookrightarrow E_\lambda$.

(ii) The lisse sheaves $\mathcal{V}_{0,\lambda}$ are pairwise E-compatible.

For every λ, the lisse sheaf $\mathcal{V}_{0,\lambda}$ will be the *λ-component* of the compatible system.

2.3 Balanced morphisms

2.3.1 Finite étale covers of a contraction

Definition 2.3.1.1. Let $f_0 : Y_0 \to X_0$ be a morphism between two varieties and Z_0 a closed subvariety of Y_0. We say that f_0 is a *contraction* of Z_0 if f_0 is a surjective morphism, which is an isomorphism outside Z_0 and such that the image of Z_0 is a subvariety W_0 of X_0 of dimension 0.

Let Y_0 be a geometrically connected normal variety over $\mathbb{F}_q$ and $i_0 : Z_0 \hookrightarrow Y_0$ a closed immersion, where Z_0 is proper and geometrically integral over $\mathbb{F}_q$. Suppose in addition that there exists a normal variety X_0 over $\mathbb{F}_q$ and $f_0 : Y_0 \to X_0$ a contraction of Z_0.

Lemma 2.3.1.2 (Corollaire 6.11, SGA I, Exposé XI)**.** *For every geometric point x of X_0 over x_0 and every geometric point z of Z_0 over x. We have an exact sequence*

$$\pi_1^{ét}(Z, z) \xrightarrow{i_*} \pi_1^{ét}(Y_0, z) \xrightarrow{f_{0*}} \pi_1^{ét}(X_0, x) \to 1$$

in the sense that the smallest normal closed subgroup containing the image of i_* is the Kernel of f_{0*}.

Remark 2.3.1.3. Under the hypothesis of the lemma, the datum of a lisse sheaf on X_0 is equivalent to the datum of a lisse sheaf on Y_0, geometrically trivial on Z_0. Hence if the companions conjecture holds for Y_0, the conjecture on X_0 is equivalent to the following conjecture.

Conjecture 2.3.1.4. *Let $\mathcal{V}_0$ be an E-rational, lisse E_λ-sheaf on Y_0. If $\mathcal{V}_0$ is geometrically trivial on Z_0, for every other finite place $\lambda' \nmid p$, every semi-simple compatible lisse $\overline{E}_{\lambda'}$-sheaf is geometrically trivial on Z_0.*

2.3.2 The conjecture

We study Conjecture 2.3.1.4 in a slightly more general setting.

Definition 2.3.2.1. Let Z_0 be a connected variety. A compatible system $\underline{\mathcal{V}_0}$ on Z_0 is *balanced* if one of the following conditions is verified.

(i) For every λ, the lisse sheaf $\mathcal{V}_{0,\lambda}$ is geometrically trivial.

(ii) It does not exist any λ such that the lisse sheaf $\mathcal{V}_{0,\lambda}$ is geometrically trivial.

A compatible system $\underline{\mathcal{V}_0}$ on Z_0 is *strongly balanced* if the dimension of $H^0(Z, \mathcal{V}_\lambda)$ does not depend on λ. If Z_0 is not connected we say that a compatible system is *balanced* (resp. *strongly balanced*) if it is balanced (resp. strongly balanced) for every connected component.

Remark 2.3.2.2. To prove that a compatible system is balanced, we can always make some dévissage. Indeed, the property to be balanced remains invariant when we take a finite extension of the base field. Secondly, thanks to Corollary 1.3.6.4, every pure lisse sheaf on X_0 is geometrically semi-simple, hence we can always assume that the compatible system is semi-simple. Using [Del80, Proposition 1.3.4], we can also reduce to the case when $\underline{\mathcal{V}_0}$ has finite order determinant. In particular, we can work with $\underline{\mathcal{V}_0}$ étale.

Proposition 2.3.2.3. *Let Z_0 be a normal variety over $\mathbb{F}_q$. Every pure compatible system on Z_0 is strongly balanced.*

Proof. We can suppose Z_0 geometrically connected. As Z_0 is normal, if U_0 is a dense open of Z_0, the étale fundamental group of U maps surjectively onto the étale fundamental group of Z. Thus a lisse sheaf over Z_0 is geometrically trivial if and only if the restriction to U_0 is geometrically trivial. This means we can also assume Z_0 to be smooth. In this case we may apply Corollary 1.3.6.7. □

Definition 2.3.2.4. Let $g_0 : Z_0 \to X_0$ be a morphism between two varieties over $\mathbb{F}_q$. We say that g_0 is a *balanced morphism* if for every pure compatible system $\underline{\mathcal{V}_0}$ of X_0, the pullback $g_0^*\underline{\mathcal{V}_0}$ is balanced.

We reformulate Conjecture 2.3.1.4 in a more general setting.

Conjecture 2.3.2.5. *Let X_0 and Z_0 be varieties over $\mathbb{F}_q$. If X_0 is normal, every morphism $g_0 : Z_0 \to X_0$ is balanced.*

Notation 2.3.2.6. For simplicity, let us a assume from now on Z_0 geometrically connected. We have a morphism $\pi_1^{\text{ét}}(Z, z) \xrightarrow{g_*} \pi_1^{\text{ét}}(X_0, x)$. For every étale compatible system $\underline{\mathcal{V}_0}$ on X_0 we denote by $\{\rho_{0,\lambda}\}_{\lambda \nmid p}$ the associated family of ℓ-adic representations of $\pi_1^{\text{ét}}(X_0, x)$. Let $\overline{\text{Im}(g_*)}$ be the smallest normal closed subgroup of $\pi_1^{\text{ét}}(X_0, x)$ containing the image of g_*.

Remark 2.3.2.7. A morphism g_0 is balanced if and only if for every étale pure compatible system on X_0, the inclusion $\overline{\mathrm{Im}(g_*)} \subseteq \mathrm{Ker}(\rho_{0,\lambda})$ for one λ implies the same inclusion for every other place λ. In particular, the property of a morphism to be balanced depends only on the inclusion $\overline{\mathrm{Im}(g_*)} \subseteq \pi_1^{\text{ét}}(X_0, x)$ as topological groups together with the assignment of the conjugacy classes of the Frobenii at closed points of $\pi_1^{\text{ét}}(X_0, x)$ and their degrees.

As a consequence of the remark, we prove an "homotopy invariance" of balanced morphisms. Let Y_0 and S_0 be two varieties over $\mathbb{F}_q$ and $h_0 : Y_0 \to S_0$ a proper and flat morphism with connected and reduced geometric fibers. Let s_0 and s_0' be closed points of S_0 and write Z_0 and Z_0' for the fibers of h_0 above these points.

Proposition 2.3.2.8. *Let $f_0 : Y_0 \to X_0$ be any morphism. The restriction $g_0 := f_0|_{Z_0}$ is balanced if and only if $g_0' := f_0|_{Z_0'}$ is balanced.*

Proof. Let z and z' be two geometric points of Z_0 and Z_0' respectively and x and x' their images via f_0. By the homotopy exact sequence, the pairs of topological groups $(\pi_1^{\text{ét}}(X_0, x), \overline{\mathrm{Im}(g_*)})$ and $(\pi_1^{\text{ét}}(X_0, x'), \overline{\mathrm{Im}(g_*')})$ are isomorphic, with isomorphism given by an isomorphism between $\pi_1^{\text{ét}}(X_0, x)$ and $\pi_1^{\text{ét}}(X_0, x')$ induced by a path from x to x'. This isomorphism respects the conjugacy classes of Frobenii at closed points and their degrees. We conclude by Remark 2.3.2.7. □

2.4 Some examples

We can verify now Conjecture 2.3.2.5 in some cases. Notice that thanks to Proposition 2.3.2.3 we already know the conjecture when Z_0 is normal.

2.4.1 Semi-stable curves

Let Z_0 be a connected semi-stable curve over $\mathbb{F}_q$. Denote by $\{Z^{(i)}\}_{1 \le i \le n}$ the set of the irreducible components of Z. Assume that for every i, the component $Z^{(i)}$ is smooth. Let z be a geometric point of Z and for every $1 \le i \le n$, let $z^{(i)}$ be a generic geometric point of $Z^{(i)}$. We denote by Γ the dual graph of Z and by P the point of Γ associated to the connected component where z lies.

Proposition 2.4.1.1 ([Sti06])**.** *The choice of étale paths $\{\gamma^{(i)}\}_{1 \le i \le n}$ joining z to $z^{(i)}$ for every i determines an isomorphism*

$$\pi_1^{\acute{e}t}(Z, z) \simeq \pi_1^{\acute{e}t}(Z^{(1)}, z^{(1)}) * \cdots * \pi_1^{\acute{e}t}(Z^{(n)}, z^{(n)}) * \pi_1(\Gamma, P)^{\wedge},$$

where $\pi_1(\Gamma, P)^{\wedge}$ is the profinite completion of the topological fundamental group of Γ.

Corollary 2.4.1.2. *If Γ is a tree, every pure compatible system on Z_0 is balanced.*

Proof. Proposition 2.4.1.1 shows that when Γ is a tree, the geometric étale fundamental group of Z_0 is generated by the étale fundamental groups of the components $Z^{(i)}$, which are smooth by assumption. Hence, by Proposition 2.3.2.3, we get the result. □

2.4.2 Finite monodromy

The invariance of the L-function of compatible lisse sheaves can be used, as in Proposition 2.3.2.3, to prove that some morphisms are balanced. This kind of method works only under strong finiteness conditions.

Proposition 2.4.2.1. *If Z_0 is geometrically connected, a morphism $g_0 : Z_0 \to X_0$ is balanced in the following cases.*

(i) *If the smallest closed normal subgroup of $\pi_1^{ét}(X_0, x)$ containing the image of $\pi_1^{ét}(Z, z)$ is open inside $\pi_1^{ét}(X, x)$.*

(ii) *If $\pi_1^{ét}(X, x)$ contains an open solvable profinite subgroup.*

Proof. Let $\underline{\mathcal{V}_0}$ be a pure compatible system on X_0. For every $\lambda \nmid p$ let G_λ be the geometric monodromy group of $\mathcal{V}_{0,\lambda}$.

(i) If $g_0^*(\mathcal{V}_{0,\lambda})$ is geometrically trivial on Z_0, then by assumption G_λ is finite. By [LP95, Proposition 2.2] the same is true for every λ and the groups $\mathrm{Ker}(\rho_\lambda) \cap \pi_1^{\text{ét}}(X, x)$ are all equal when λ varies. This implies that for every λ, the group $\pi_1^{\text{ét}}(Z, z)$ is contained in $\mathrm{Ker}(\rho_\lambda)$. Therefore, for every λ, the lisse sheaf $g_0^*(\mathcal{V}_{0,\lambda})$ is geometrically trivial.

(ii) By Corollary 1.3.6.4, the groups G_λ are all semi-simple as $\underline{\mathcal{V}_0}$ is pure. Thanks to the assumption that $\pi_1^{\text{ét}}(X, x)$ is solvable, we also know that all the groups G_λ are solvable. Hence they are finite and we can proceed as in the previous case.

□

Corollary 2.4.2.2. *A dominant morphism $g_0 : Z_0 \to X_0$ is balanced. In particular, if X_0 is a smooth curve, every morphism $g_0 : Z_0 \to X_0$ is balanced.*

Proof. As g_0 is dominant, we can find a smooth connected variety $Z_0' \subseteq Z_0$ such that $g_0|_{Z_0'}$ is again dominant. In particular, the morphism $g_0|_{Z_0'}$ is generically finite, hence the image of the geometric étale fundamental group of Z_0 has finite index in $\pi_1^{\text{ét}}(X, x)$. Thus we can apply Theorem 2.4.2.1 to conclude. □

To summarize our results we state them as a unique theorem.

Theorem 2.4.2.3 (Proposition 2.3.2.3, Corollary 2.4.1.2 and Proposition 2.4.2.1)**.** *Let $g_0 : Z_0 \to X_0$ be a morphism between two varieties over $\mathbb{F}_q$. Suppose X_0 normal, then g_0 is balanced in the following cases.*

(i) *When Z_0 is a normal variety.*

(ii) *When Z_0 is a semi-stable curve with simply connected dual graph.*

(iii) *If the smallest closed normal subgroup of $\pi_1^{ét}(X_0, x)$ containing the image of $\pi_1^{ét}(Z, z)$ is open inside $\pi_1^{ét}(X, x)$.*

(iv) *If $\pi_1^{ét}(X, x)$ contains an open solvable profinite subgroup.*

3 Maximal tori of monodromy groups of F-isocrystals and applications (joint with Emiliano Ambrosi)

3.1 Introduction

3.1.1 Convergent and overconvergent isocrystals

Let $\mathbb{F}_q$ be a finite field of characteristic p and let X_0 be a smooth geometrically connected variety over $\mathbb{F}_q$. The first Weil cohomology introduced to study X_0 is the *ℓ-adic étale cohomology*, where ℓ is a prime different from p. Its associated category of local systems is the category of Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaves, denoted by $\mathbf{Weil}(X_0, \overline{\mathbb{Q}}_\ell)$. While p-adic étale cohomology is not a Weil cohomology theory, moving from ℓ to p one encounters two main p-adic cohomology theories: *crystalline cohomology* and *rigid cohomology*. These two give rise to different categories of "local systems". We have the category $\mathbf{F\text{-}Isoc}(X_0)$ of $\overline{\mathbb{Q}}_p$-linear *convergent* F-isocrystals over X_0 and the category $\mathbf{F\text{-}Isoc}^\dagger(X_0)$ of $\overline{\mathbb{Q}}_p$-linear *overconvergent* F-isocrystals over X_0.

By [Ked04b], these two categories are related by a natural fully faithful functor

$$\epsilon : \mathbf{F\text{-}Isoc}^\dagger(X_0) \to \mathbf{F\text{-}Isoc}(X_0).$$

When X_0 is proper, the functor ϵ is an equivalence of categories. In general, the two categories have different behaviours. While $\mathbf{F\text{-}Isoc}^\dagger(X_0)$ shares many properties with $\mathbf{Weil}(X_0, \overline{\mathbb{Q}}_\ell)$ as we have seen in §1, the category $\mathbf{F\text{-}Isoc}(X_0)$ has some exceptional p-adic features.

For example, thanks to [Ked18, Prop. 1.2.7 and Prop. 1.2.8], for every $\mathcal{E}_0 \in \mathbf{F\text{-}Isoc}(X_0)$, after possibly shrinking X_0, there exists a filtration

$$0 = \mathcal{E}_0^0 \subseteq \mathcal{E}_0^1 \subseteq ... \subseteq \mathcal{E}_0^n = \mathcal{E}_0$$

where for each i the quotient $\mathcal{E}_0^{i+1}/\mathcal{E}_0^i$ has a unique slope s_i at closed points and the sequence $s_1, \ldots, s_n$ is increasing. When $\mathcal{E}_0 = \epsilon(\mathcal{E}_0^\dagger)$ for some $\mathcal{E}_0^\dagger \in \mathbf{F\text{-}Isoc}^\dagger(X_0)$, the subobjects $\mathcal{E}_0^i$ in general are not in the essential image of ϵ as well (see for example [Ked16, Remark 5.12]). Our main result highlights a new relationship between the subquotients of $\mathcal{E}_0^\dagger$ in $\mathbf{F\text{-}Isoc}^\dagger(X_0)$ and the ones of $\mathcal{E}_0^1$ in $\mathbf{F\text{-}Isoc}(X_0)$.

Theorem 3.1.1.1 (Theorem 3.3.1.2)**.** *Let $\mathcal{E}_0^\dagger$ be an irreducible $\overline{\mathbb{Q}}_p$-linear overconvergent F-isocrystal. If $\mathcal{E}_0$ admits a subobject of minimal slope $\mathcal{F}_0 \subseteq \mathcal{E}_0$ with a non-zero morphism $\mathcal{F}_0 \to \mathcal{O}_{X_0}$ of convergent isocrystals, then $\mathcal{E}_0^\dagger$ has rank* 1.

Remark 3.1.1.2. Theorem 3.1.1.1 proves a particular case of the conjecture in [Ked16, Remark 5.14]. Even if the conjecture turned out to be false in general, Theorem 3.1.1.1 corresponds to the case when $\mathcal{F}_1 \subseteq \mathcal{E}_1$ has minimal slope and $\mathcal{E}_2$ is the convergent isocrystal $\mathcal{O}_{X_0}$, endowed with some Frobenius structure (notation as in [*ibid.*]).

3.1.2 Torsion points of abelian varieties

Before explaining the main ingredients of the proof of Theorem 3.1.1.1, let us describe an application to torsion points of abelian varieties. This was our main motivation to prove Theorem 3.1.1.1. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$ and $\mathbb{F} \subseteq k$ be a finitely generated field extension. Recall the classical Lang–Néron theorem (see [LN59] or [Con06] for a more modern presentation).

Theorem 3.1.2.1 (Lang–Néron)**.** *If A be an abelian variety over k such that $\mathrm{Tr}_{k/\mathbb{F}}(A) = 0$, then the group $A(k)$ is finitely generated.*

By Theorem 3.1.2.1, denoting by $A^{(n)}$ the Frobenius twist of A by the p^n-power Frobenius, we have a tower of finite groups $A(k)_{\mathrm{tors}} \subseteq A^{(1)}(k)_{\mathrm{tors}} \subseteq A^{(2)}(k)_{\mathrm{tors}} \subseteq \dots$. In June 2011, in a correspondence with Langer and Rössler, Esnault asked whether this chain is *eventually stationary*. An equivalent way to formulate the question is to ask whether the group of k^{perf}-rational torsion points $A(k^{\mathrm{perf}})_{\mathrm{tors}}$ is a finite group, where k^{perf} is a perfect closure of k. As an application of Theorem 3.1.1.1, we give a positive answer to her question.

Theorem 3.1.2.2 (Theorem 3.4.1.1)**.** *If A be an abelian variety over k such that $\mathrm{Tr}_{k/\mathbb{F}}(A) = 0$, then the group $A(k^{\mathrm{perf}})_{\mathrm{tors}}$ is finite.*

Remark 3.1.2.3. Theorem 3.1.2.2 was already known for elliptic curves, by the work of Levin in [Lev68], and for ordinary abelian varieties, by [Rös17, Theorem 1.4]. When ℓ is a prime different from p, the group $A[\ell^\infty]$ is étale, hence $A[\ell^\infty](k^{\mathrm{perf}}) = A[\ell^\infty](k)$. Therefore, in Theorem 3.1.2.2, the finiteness of torsion points of prime-to-p order is guaranteed by Theorem 3.1.2.1.

To relate Theorem 3.1.2.2 to Theorem 3.1.1.1 we use the *crystalline Dieudonné theory*, as developed in [BBM82]. The proof of Theorem 3.1.2.2 is by contradiction. If $|A[p^\infty](k^{\mathrm{perf}})| = \infty$, then there exists a monomorphism $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty]^{\text{ét}}$ from the trivial p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ over k and the étale part of the p-divisible group of A. Spreading out to a "nice" model $\mathfrak{A}/X$ of A/k and applying the contravariant crystalline Dieudonné functor $\mathbb{D}$, one gets an epimorphism of F-isocrystals $\mathbb{D}(\mathfrak{A}[p^\infty]^{\text{ét}}) \twoheadrightarrow \mathbb{D}((\mathbb{Q}_p/\mathbb{Z}_p)_X) \simeq \mathcal{O}_X$ over X. By a descent argument and Theorem 3.1.1.1, the quotient extends to a quotient $\mathbb{D}(\mathfrak{A}[p^\infty]) \twoheadrightarrow \mathcal{O}_X$ over X. Going back to p-divisible groups, this gives an injective map $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty]$ over k. Therefore, $A[p^\infty](k)$ would be an infinite group, contradicting Theorem 3.1.2.1.

3.1.3 Monodromy groups

The categories $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ and $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ and their versions without Frobenius structures $\mathbf{Isoc}(X_0)$ and $\mathbf{Isoc}^\dagger(X_0)$ are neutral Tannakian categories. The choice of an $\mathbb{F}$-point x of X_0 induces fibre functors for all these categories. To prove Theorem 3.1.1.1, we study the *monodromy groups* associated to the objects involved. For every $\mathcal{E}_0^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$, we have already

seen that we can associate an object $\mathcal{E}_0 := \epsilon(\mathcal{E}_0^\dagger) \in \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$. We denote by $\mathcal{E}^\dagger \in \mathbf{Isoc}^\dagger(X_0)$ (resp. $\mathcal{E} \in \mathbf{Isoc}(X_0)$) the isocrystals obtained from $\mathcal{E}_0^\dagger$ (resp. $\mathcal{E}_0$) by forgetting their Frobenius structure. Using the Tannakian formalism, we associate to each of these objects an algebraic group $G(-)$. They all sit naturally in a commutative diagram of closed immersions

$$\begin{array}{ccc} G(\mathcal{E}) & \hookrightarrow & G(\mathcal{E}_0) \\ \downarrow & & \downarrow \\ G(\mathcal{E}^\dagger) & \hookrightarrow & G(\mathcal{E}_0^\dagger). \end{array}$$

If $\mathcal{E}_0^\dagger$ is irreducible and its determinant has finite order, by Proposition 1.3.5.7, the group $G(\mathcal{E}_0^\dagger)/G(\mathcal{E}^\dagger)$ is finite. We prove that the same is true for $G(\mathcal{E}_0)/G(\mathcal{E})$.

Proposition 3.1.3.1 (Proposition 3.3.1.1)**.** *Let $\mathcal{E}_0^\dagger$ be an irreducible overconvergent F-isocrystal with finite order determinant. The quotient $G(\mathcal{E}_0)/G(\mathcal{E})$ is finite.*

To prove Proposition 3.1.3.1, we have to show that $G(\mathcal{E})$ is "big". We study $G(\mathcal{E})$ as a subgroup of $G(\mathcal{E}^\dagger)$ and we prove our fundamental result.

Theorem 3.1.3.2 (Theorem 3.2.3.9)**.** *If $\mathcal{E}_0^\dagger$ is an overconvergent F-isocrystal, then $G(\mathcal{E})$ contains a maximal torus of $G(\mathcal{E}^\dagger)$.*

To prove Theorem 3.1.3.2, we use the existence of *Frobenius tori* which are maximal tori of $G(\mathcal{E}_0^\dagger)$ (Theorem 3.2.3.4). First we reduce to the case when $\mathcal{E}_0^\dagger$ is semi-simple and *algebraic* (cf. Definition 3.2.3.3). By Theorem 3.2.3.4, there exists a closed point $i_0 : x_0 \hookrightarrow X_0$ such that the subgroup $G(i_0^*\mathcal{E}_0^\dagger) \subseteq G(\mathcal{E}_0^\dagger)$ contains a maximal torus of $G(\mathcal{E}_0^\dagger)$. Since over a closed point every F-isocrystal admits an overconvergent extension, one has $G(i_0^*\mathcal{E}_0^\dagger) = G(i_0^*\mathcal{E}_0)$. Hence, $G(\mathcal{E}_0)$ contains a maximal torus of $G(\mathcal{E}_0^\dagger)$. To pass from $G(\mathcal{E}_0) \subseteq G(\mathcal{E}_0^\dagger)$ to $G(\mathcal{E}) \subseteq G(\mathcal{E}^\dagger)$, we will apply Theorem 3.2.3.4 to an auxiliary overconvergent F-isocrystal $\widetilde{\mathcal{E}}_0^\dagger$ over X_0, such that $G(\widetilde{\mathcal{E}}^\dagger) = G(\mathcal{E}^\dagger)$, $G(\widetilde{\mathcal{E}}) = G(\mathcal{E})$ and with the additional property that the natural map $G(\widetilde{\mathcal{E}}_0)/G(\widetilde{\mathcal{E}}) \to G(\widetilde{\mathcal{E}}_0^\dagger)/G(\widetilde{\mathcal{E}}^\dagger)$ is an isomorphism.

Remark 3.1.3.3. In [Cre92a, page 460] Crew asks whether, under the assumptions of Theorem 3.1.3.2, the group $G(\mathcal{E})$ is a parabolic subgroup of $G(\mathcal{E}^\dagger)$. In two subsequent articles [Cre92b] and [Cre94], he gives a positive answer to his question in some particular cases. Since parabolic subgroups of reductive groups always contain a maximal torus, Theorem 3.1.3.2 is an evidence for Crew's expectation.

To deduce Theorem 3.1.1.1 from Proposition 3.1.3.1, we first reduce ourself to the situation where the determinant of $\mathcal{E}_0^\dagger$ has finite order. To simplify, let us assume that $\mathcal{F}_0 = \mathcal{E}_0^1$ and $G(\mathcal{E}_0)$ is connected. Proposition 3.1.3.1 implies that $G(\mathcal{E}) = G(\mathcal{E}_0)$ hence that the morphism $\mathcal{E}_0^1 \to \mathcal{O}_{X_0}$

commutes with the trivial Frobenius structure on $\mathcal{O}_{X_0}$. In particular, $\mathcal{E}_0^1$ has slope 0, so that the minimal slope of $\mathcal{E}_0$ is 0. Since the determinant of $\mathcal{E}_0$ has finite order, this implies that $\mathcal{E}_0^1 = \mathcal{E}_0$ hence that $\mathcal{E}_0$ admits a quotient $\mathcal{E}_0 \twoheadrightarrow \mathcal{O}_{X_0}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$. As $\epsilon : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0) \to \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ is fully faithful, $\mathcal{E}_0^\dagger$ admits a quotient $\mathcal{E}_0^\dagger \twoheadrightarrow \mathcal{O}_{X_0}^\dagger$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$. On the other hand, $\mathcal{E}_0^\dagger$ is irreducible, so that the quotient gives actually an isomorphism $\mathcal{E}_0^\dagger \simeq \mathcal{O}_{X_0}^\dagger$.

3.1.4 Weak (weak) semi-simplicity

As an additional outcome of Theorem 3.1.3.2, we get a semi-simplicity result for extensions of *constant* convergent F-isocrystals. For us, a constant F-isocrystal will be an object $\mathcal{E}_0 \in \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ such that its image in $\mathbf{Isoc}(X_0)$ is isomorphic to $\mathcal{O}_{X_0}^{\oplus n}$ for some $n \in \mathbb{Z}_{>0}$.

Let $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(X_0)$ denote the Tannakian subcategory of $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ generated by the essential image via $\epsilon : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0) \to \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ of pure objects in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$. This category is large enough to contain all the F-isocrystals "coming from geometry". More precisely, for every smooth and proper morphism $f_0 : Y_0 \to X_0$ and every $i \in \mathbb{N}$, the subquotients of the higher direct image $R^i f_{0,\mathrm{crys}*}\mathcal{O}_{Y_0}$ are in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(X_0)$ by [KM74] and [Shi08] (see [Amb18, Fact 3.1.1.2 and Fact 3.2.1.1]). Thanks to a group-theoretic argument (Lemma 3.2.3.8), Theorem 3.1.3.2 implies the following.

Corollary 3.1.4.1 (Corollary 3.2.3.12)**.** *A convergent F-isocrystal in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(X_0)$ which is an extension of constant F-isocrystals is constant.*

Remark 3.1.4.2. One can construct on $\mathbb{A}^1_{\mathbb{F}_q}$ non-constant extensions of constant convergent F-isocrystals. Corollary 3.1.4.1 shows that these extensions are outside $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(\mathbb{A}^1_{\mathbb{F}_q})$. One can further construct these extensions in such a way that the resulting convergent F-isocrystal has *log-decay*, in the sense of [KM16]. Therefore, as a consequence of Corollary 3.1.4.1, we get new examples of convergent F-isocrystals with log-decay which do not "come from geometry".

Remark 3.1.4.3. Let $\mathcal{E}_0$ be a convergent F-isocrystal with constant Newton polygons. Corollary 3.1.4.1 implies that $G(\mathcal{E})$ has no unipotent quotients. Let $\mathcal{E}^1$ be the convergent isocrystal which underlies the subobject of $\mathcal{E}_0$ of minimal slope. Since $G(\mathcal{E}^1)$ is a quotient of $G(\mathcal{E})$, it does not have unipotent quotients as well. In [Cha13, Conjecture 7.4 and Remark 7.4.1], Chai conjectures that if $\mathcal{E}_0^\dagger$ is the higher direct image of a family of ordinary abelian varieties, then $G(\mathcal{E}^1)$ is reductive. Corollary 3.1.4.1 may be thought as a first step towards his conjecture.

3.1.5 Organization of §3

In Section 3.2 we introduce the monodromy groups of the various categories of isocrystals and we prove Theorem 3.1.3.2. In Section 3.3 we prove Theorem 3.1.1.1 and some of its consequences. Finally, in Section 3.4 we prove Theorem 3.1.2.2.

3.1.6 Notation

3.1.6.1. Let $\mathbb{K}$ be a characteristic zero field and $\mathbf{C}$ a $\mathbb{K}$-linear Tannakian category. A *Tannakian subcategory* of $\mathbf{C}$ is a strictly full subcategory of $\mathbf{C}$ closed under direct sums, tensor products, duals and subobjects. For $\mathcal{E} \in \mathbf{C}$, we denote by $\langle \mathcal{E} \rangle$ the smallest Tannakian subcategory of $\mathbf{C}$ containing $\mathcal{E}$. Let $\omega : \mathbf{C} \to \mathbb{K}$ a fibre functor. For every $\mathcal{E} \in \mathbf{C}$, the restriction of ω to $\langle \mathcal{E} \rangle$ defines a fibre functor of $\langle \mathcal{E} \rangle$. We denote by $G(\mathcal{E})$ the Tannakian group of $\langle \mathcal{E} \rangle$ with respect to this fibre functor. In general, the fibre functor will be clear from the context, so that we do not keep ω in the notation. The group $G(\mathcal{E})$ will be called the *monodromy group* of $\mathcal{E}$.

3.1.6.2. When G is an algebraic group, we denote by $\mathrm{rk}(G)$ the dimension of a maximal torus of G and we will call it the *reductive rank* of G. We say that a subgroup H of G is of maximal rank if $\mathrm{rk}(H) = \mathrm{rk}(G)$. Let $\mathbb{K}$ be a characteristic 0 field, G and H two affine groups over $\mathbb{K}$ and $f : G \to H$ a morphism of affine group schemes. We will say that f is *injective* if it is a closed immersion and that f is *surjective* if it is faithfully flat. Since over a characteristic 0 field every affine group scheme is reduced, this does not generate any confusion.

3.2 Monodromy of convergent isocrystals

3.2.1 Review of isocrystals

We recall in this section some basic facts about isocrystals. See [Ked16, §2] for more details. Throughout §3.2.1, let k be a subfield of $\mathbb{F}$. We denote by $W(k)$ the ring of Witt vectors of k and by K its field of fractions. We write $\overline{\mathbb{Q}}_p$ for a fixed algebraic closure of $\mathbb{Q}_p$, and we choose an embedding of $W(\mathbb{F})$ in $\overline{\mathbb{Q}}_p$. Let X be a smooth variety over k.

Definition 3.2.1.1. We denote by $\mathbf{Isoc}(X)$ the category of $\overline{\mathbb{Q}}_p$*-linear convergent isocrystals*. Let $\mathcal{O}_X$ be the convergent isocrystal associated to the structural sheaf. We also denote by $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ the category of $\overline{\mathbb{Q}}_p$*-linear convergent F-isocrystals*. This category consists of pairs $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is a convergent isocrystal and Φ is a *Frobenius structure* on $\mathcal{E}$, namely an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. Let $\mathbf{Crys}(X/W(k))$ be the category of crystals of finite $\mathcal{O}_{X,\mathrm{crys}}$-modules and let $\mathbf{Crys}(X/W(k))_{\overline{\mathbb{Q}}_p}$ be the extension of scalars of $\mathbf{Crys}(X/W(k))$ to $\overline{\mathbb{Q}}_p$. Let $\mathbf{F}\text{-}\mathbf{Crys}(X/W(k))_{\overline{\mathbb{Q}}_p}$ be the category of objects in $\mathbf{Crys}(X/W(k))_{\overline{\mathbb{Q}}_p}$ endowed with a Frobenius structure.

Theorem 3.2.1.2 (Ogus, Berthelot)**.** *There exists a canonical equivalence of categories*

$$\mathbf{F}\text{-}\mathbf{Crys}(X/W(k))_{\overline{\mathbb{Q}}_p} \xrightarrow{\sim} \mathbf{F}\text{-}\mathbf{Isoc}(X).$$

Proof. The result follows from [Ked16, Theorem 2.2] after extending the field of scalars from K to $\overline{\mathbb{Q}}_p$. □

Remark 3.2.1.3. In light of Theorem 3.2.1.2, we will feel free to refer to convergent F-isocrystals simply as F-isocrystals.

Definition 3.2.1.4. Let $\mathbf{Isoc}^\dagger(X)$ be the category of $\overline{\mathbb{Q}}_p$*-linear overconvergent isocrystals* and $\mathbf{F\text{-}Isoc}^\dagger(X)$ the category of $\overline{\mathbb{Q}}_p$*-linear overconvergent F-isocrystals.* The overconvergent isocrystal associated to the structural sheaf will be denoted by $\mathcal{O}_X^\dagger$.

By definition, there is a natural functor $\epsilon : \mathbf{F\text{-}Isoc}^\dagger(X) \to \mathbf{F\text{-}Isoc}(X)$.

Theorem 3.2.1.5 (Kedlaya)**.** *The functor $\epsilon : \mathbf{F\text{-}Isoc}^\dagger(X) \to \mathbf{F\text{-}Isoc}(X)$ is fully faithful.*

Remark 3.2.1.6. Even if this functor is fully faithful, the essential image is not closed under subquotients. Thus, the essential image is not a Tannakian subcategory in the sense of §3.1.6.1, so that the induced morphism on Tannakian groups is not surjective. Nevertheless, the morphism is an *epimorphism* in the category of affine group schemes (see [BB92]). See Remark 3.2.3.11 for further comments.

Definition 3.2.1.7. Suppose that X is connected and let $\mathcal{E}$ be an F-isocrystal of rank r. We denote by $\{a_i^\eta(\mathcal{E})\}_{1\leq i\leq r}$ the set of generic slopes of $\mathcal{E}$. We use the convention that $a_1^\eta(\mathcal{E}) \leq \cdots \leq a_r^\eta(\mathcal{E})$, thus the choice of the ordering does not agree with [DK17]. We say that $\mathcal{E}$ is *isoclinic* if $a_1^\eta(\mathcal{E}) = a_r^\eta(\mathcal{E})$. A subobject $\mathcal{F}$ of $\mathcal{E}$ is *of minimal slope* if it is isoclinic of slope $a_1^\eta(\mathcal{E})$. See [Ked16, §3 and §4] for more details on the theory of slopes.

3.2.2 The fundamental exact sequence

We shall briefly review the theory of *monodromy groups* of F-isocrystals. These monodromy groups have been firstly studied by Crew in [Cre92a]. In Proposition 3.2.2.4, we introduce a fundamental diagram of monodromy groups that we will extensively use in the next sections.

Notation 3.2.2.1. Let X_0 be a smooth geometrically connected variety over $\mathbb{F}_q$. We choose once and for all an $\mathbb{F}$-point y of X_0. This defines fibre functors for all the Tannakian categories of isocrystals previously defined. We write $\mathbb{1}_0^\dagger$ for the overconvergent F-isocrystal $\mathcal{O}_{X_0}^\dagger$ endowed with its canonical Frobenius structure. For every $\mathcal{E}_0^\dagger \in \mathbf{F\text{-}Isoc}^\dagger(X_0)$ we consider three associated objects. We denote by $\mathcal{E}^\dagger \in \mathbf{Isoc}^\dagger(X_0)$ the overconvergent isocrystal obtained from $\mathcal{E}_0^\dagger$ by forgetting the Frobenius structure. The image of $\mathcal{E}_0^\dagger$ in $\mathbf{F\text{-}Isoc}(X_0)$ will be denoted by removing the superscript †. At the same time, $\mathcal{E}$ will be the convergent isocrystal in $\mathbf{Isoc}(X_0)$, obtained from $\mathcal{E}_0 \in \mathbf{F\text{-}Isoc}(X_0)$ by forgetting its Frobenius structure. Here a summary table.

	Isocrystal	F-Isocrystal
Convergent	$\mathcal{E}$	$\mathcal{E}_0$
Overconvergent	$\mathcal{E}^\dagger$	$\mathcal{E}_0^\dagger$

For each of these objects we have a monodromy group $G(-)$ (see §3.1.6.1) with respect to the fibre functor associated to our $\mathbb{F}$ point y.

Definition 3.2.2.2. We say that a convergent isocrystal is *trivial* if it is isomorphic to $\mathbb{1}^{\oplus r}$ for some $r \in \mathbb{N}$. An F-isocrystal $\mathcal{E}_0$ is said *constant* if the convergent isocrystal $\mathcal{E}$ is trivial. We denote by $\mathbf{F\text{-}Isoc}_{cst}(X_0)$ the strictly full subcategory of $\mathbf{F\text{-}Isoc}(X_0)$ of constant objects. For $\mathcal{E}_0 \in \mathbf{F\text{-}Isoc}(X_0)$, we denote by $\langle \mathcal{E}_0 \rangle_{cst} \subseteq \langle \mathcal{E}_0 \rangle$ the Tannakian subcategory of constant objects and by $G(\mathcal{E}_0)^{cst}$ the Tannakian group of $\langle \mathcal{E}_0 \rangle_{cst}$. Finally, for $\alpha \in \overline{\mathbb{Q}}_p$ and $\mathcal{E}_0 \in \mathbf{F\text{-}Isoc}(X_0)$, we denote by $\mathcal{E}_0^{(\alpha)}$ the F-isocrystal obtained from $\mathcal{E}_0$ multiplying its Frobenius structure by α. We will call $\mathcal{E}_0^{(\alpha)}$ the *twist* of $\mathcal{E}_0$ by α. We give analogues definitions for overconvergent isocrystals.

Remark 3.2.2.3. The category $\mathbf{F\text{-}Isoc}_{cst}(X_0)$ is a Tannakian subcategory of $\mathbf{F\text{-}Isoc}(X_0)$ in the sense of §3.1.6.1. Let $p_{X_0} : X_0 \to \mathrm{Spec}(\mathbb{F}_q)$ be the structural morphism of X_0. Every constant F-isocrystal is the inverse image via p_{X_0} of an F-isocrystal defined over $\mathrm{Spec}(\mathbb{F}_q)$. The category $\mathbf{F\text{-}Isoc}(\mathrm{Spec}(\mathbb{F}_q))$ is equivalent to the category of $\overline{\mathbb{Q}}_p$-vector spaces endowed with a linear automorphism. The automorphism is induced by the Frobenius structure. Since the functor $p_{X_0}^*$ is fully faithful, the same is true for $\mathbf{F\text{-}Isoc}_{cst}(X_0)$. As a consequence, the monodromy group of any constant object is commutative. Finally, the natural functor $\epsilon : \mathbf{F\text{-}Isoc}^\dagger(X_0) \to \mathbf{F\text{-}Isoc}(X_0)$ induces an equivalence of categories between $\mathbf{F\text{-}Isoc}^\dagger_{cst}(X_0)$ and $\mathbf{F\text{-}Isoc}_{cst}(X_0)$.

Proposition 3.2.2.4. *The natural morphisms induce a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & G(\mathcal{E}) \arrow[r] \arrow[d, hook] & G(\mathcal{E}_0) \arrow[r] \arrow[d, hook] & G(\mathcal{E}_0)^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^\dagger) \arrow[r] & G(\mathcal{E}_0^\dagger) \arrow[r] & G(\mathcal{E}_0^\dagger)^{cst} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The left and the central vertical arrows are injective and the right one is surjective.

Proof. The inverse image functor with respect to the q-power Frobenius of X_0, is equivalence of categories both for the convergent and overconvergent isocrystals over X_0 (see [Ogu84, Corollary 4.10] and [Laz17]). The exactness of the rows then follows from Proposition A.2.3. In addition, the right vertical arrow is surjective because, by the discussion in Remark 3.2.2.3, the functor $\langle \mathcal{E}_0^\dagger \rangle_{cst} \to \langle \mathcal{E}_0 \rangle_{cst}$ is fully faithful and the essential image is closed under subquotients. □

Remark 3.2.2.5. We do not know whether the natural quotient $\varphi : G(\mathcal{E}_0)^{cst} \twoheadrightarrow G(\mathcal{E}_0^\dagger)^{cst}$ is an isomorphism in general. Via the Tannakian formalism, to prove the injectivity of φ, one has to show that the embedding $\langle \mathcal{E}_0^\dagger \rangle_{cst} \hookrightarrow \langle \mathcal{E}_0 \rangle_{cst}$ is essentially surjective. While every $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle_{cst}$ comes from an object $\mathcal{F}_0^\dagger$ in $\mathbf{F\text{-}Isoc}^\dagger(X_0)$, we do not know whether such an $\mathcal{F}_0^\dagger$ lies in $\langle \mathcal{E}_0^\dagger \rangle$. This

will be the main issue in the proof of Theorem 3.2.3.9. We bypass the problem by embedding $\mathcal{E}_0^\dagger$ in an auxiliary overconvergent F-isocrystal $\widetilde{\mathcal{E}}_0^\dagger$ with $G(\widetilde{\mathcal{E}}_0)^{cst} \simeq G(\widetilde{\mathcal{E}}_0^\dagger)^{cst}$. One can use Theorem 3.2.3.9 to show that if $\mathcal{E}_0^\dagger$ is algebraic (cf. §3.2.3.3) and semi-simple, then φ is an isogeny.

3.2.3 Maximal tori

In this section, we briefly recall the main theorem on *Frobenius tori* of overconvergent F-isocrystals in §1.4.2 and we use it to prove Theorem 3.2.3.9. For this task, the main issue is to pass from the arithmetic situation (Corollary 3.2.3.6) to the geometric one (Theorem 3.2.3.9). We keep the notation as in §3.2.2.1

Definition 3.2.3.1. Let $i_0 : x_0 \hookrightarrow X_0$ be the closed immersion of a closed point of X_0. For every overconvergent F-isocrystal $\mathcal{E}_0^\dagger$ we have an inclusion $G(i_0^*\mathcal{E}_0^\dagger) \hookrightarrow G(\mathcal{E}_0^\dagger)$, with $G(i_0^*\mathcal{E}_0^\dagger)$ commutative. The image of the maximal torus of $G(i_0^*\mathcal{E}_0^\dagger)$ in $G(\mathcal{E}_0^\dagger)$ is the *Frobenius torus* of $\mathcal{E}_0^\dagger$ at x_0, denoted by $T_{x_0}(\mathcal{E}_0^\dagger)$.

3.2.3.2. Thanks to Deligne's conjecture for lisse sheaves and overconvergent F-isocrystals (cf. §1.3.7), for a certain class of overconvergent F-isocrystals it is possible to construct *ℓ-adic companions* (cf. [*ibid.*]) where ℓ is a prime different from p. From this construction one can translate some results known for lisse sheaves to overconvergent F-isocrystals. Theorem 3.2.3.4 is an example of such a technique (see also §3.4.3.2). For the existence of companions one needs some mild assumptions on the eigenvalues of the Frobenii at closed points.

Definition 3.2.3.3. An overconvergent F-isocrystal $\mathcal{E}_0^\dagger$ is *algebraic* if the eigenvalues of the Frobenii at closed points are algebraic numbers.

Theorem 3.2.3.4 (Theorem 1.4.2.10)**.** *Let $\mathcal{E}_0^\dagger$ be an algebraic overconvergent F-isocrystal. There exists a Zariski-dense set of closed points x_0 of X_0 such that the torus $T_{x_0}(\mathcal{E}_0^\dagger)$ is a maximal torus of $G(\mathcal{E}_0^\dagger)$.*

Remark 3.2.3.5. It is worth mentioning that when $\mathcal{E}_0^\dagger$ is pure Theorem 3.2.3.4 is also a consequence of the new crystalline Čebotarev density theorem proven by Hartl and Pál [HP18, Theorem 12.2].

Corollary 3.2.3.6. *Let $\mathcal{E}_0^\dagger$ be an algebraic overconvergent F-isocrystal. The closed subgroup $G(\mathcal{E}_0) \subseteq G(\mathcal{E}_0^\dagger)$ is a subgroup of maximal rank.*

Proof. Thanks to Theorem 3.2.3.4, we can find a closed embedding of a closed point $i_0 : x_0 \hookrightarrow X_0$ such that $T_{x_0}(\mathcal{E}_0^\dagger)$ is a maximal torus of $G(\mathcal{E}_0^\dagger)$. We have a commutative diagram

$$\begin{tikzcd} G(i_0^*\mathcal{E}_0) \arrow[r, hook] \arrow[d, "\wr"'] & G(\mathcal{E}_0) \arrow[d, hook] \\ G(i_0^*\mathcal{E}_0^\dagger) \arrow[r, hook] & G(\mathcal{E}_0^\dagger), \end{tikzcd}$$

where the morphism $G(i_0^*\mathcal{E}_0) \to G(i_0^*\mathcal{E}_0^\dagger)$ is an isomorphism by Remark 3.2.2.3. Since $G(i_0^*\mathcal{E}_0^\dagger)$ is a subgroup of $G(\mathcal{E}_0^\dagger)$ of maximal rank, the same is true for the subgroup $G(\mathcal{E}_0) \subseteq G(\mathcal{E}_0^\dagger)$. □

Corollary 3.2.3.7. *If $\mathcal{E}_0^\dagger$ be an algebraic semi-simple overconvergent F-isocrystal, then $G(\mathcal{E}_0)^{cst}$ and $G(\mathcal{E}_0^\dagger)^{cst}$ are groups of multiplicative type.*

Proof. As discussed in Remark 3.2.2.3, the groups $G(\mathcal{E}_0^\dagger)^{cst}$ and $G(\mathcal{E}_0)^{cst}$ are commutative. It suffices to verify that they are also reductive. The former is a quotient of $G(\mathcal{E}_0^\dagger)$, which is reductive because $\mathcal{E}_0^\dagger$ is semi-simple. The latter is a quotient of $G(\mathcal{E}_0)$, which by Corollary 3.2.3.6 is a subgroup of $G(\mathcal{E}_0^\dagger)$ of maximal rank. Since $G(\mathcal{E}_0)^{cst}$ is commutative, $R_u(G(\mathcal{E}_0)^{cst})$ is a quotient of $G(\mathcal{E}_0)$. Thus $R_u(G(\mathcal{E}_0)^{cst})$ is trivial by the group-theoretic Lemma 3.2.3.8 below. This concludes the proof. □

Lemma 3.2.3.8. *Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, let G be a reductive group over $\mathbb{K}$ and let H be a subgroup of G of maximal rank. Every morphism from H to a unipotent group is trivial. Equivalently, the group $\mathrm{Ext}^1_H(\mathbb{K}, \mathbb{K})$ vanishes.*

Proof. Every unipotent group is an iterated extension of copies of $\mathbb{G}_a$. Therefore, it is enough to show that every morphism from H to $\mathbb{G}_a$ is trivial. Suppose there exists a non-trivial morphism $\varphi : H \to \mathbb{G}_a$. As $\mathrm{char}(\mathbb{K}) = 0$, the image of φ is $\mathbb{G}_a$ itself. We write K for the kernel of φ. Every map from a torus to $\mathbb{G}_a$ is trivial, thus the subgroup $K \subseteq G$ has maximal rank as well. This implies by [Mil15, Lemma 18.52] that $N_G(K^\circ)^\circ = K^\circ$. By construction, K is normal in H, thus H is contained in $N_G(K)$, which in turn is contained in $N_G(K^\circ)$. This implies that $K^\circ = H^\circ$, thus that H/K is a finite group scheme, against the fact that $H/K \simeq \mathbb{G}_a$. □

Theorem 3.2.3.9. *Let $\mathcal{E}_0^\dagger$ be an overconvergent F-isocrystal. The subgroup $G(\mathcal{E}) \subseteq G(\mathcal{E}^\dagger)$ has maximal rank.*

Proof. If we replace $\mathcal{E}_0^\dagger$ with its semi-simplification with respect to a Jordan–Hölder filtration, we do not change the reductive rank of $G(\mathcal{E}^\dagger)$ and $G(\mathcal{E})$. Thus we may and do assume that $\mathcal{E}_0^\dagger$ is semi-simple. This implies that $\mathcal{E}^\dagger$ is semi-simple as well. We also notice it is harmless to twist the irreducible summands of $\mathcal{E}_0^\dagger$. Thus, we may assume that all the irreducible subobjects of $\mathcal{E}_0^\dagger$ have finite order determinant, hence that $\mathcal{E}_0^\dagger$ is algebraic and pure of weight 0 (Theorem 1.3.7.6). Choose a set of generators $\chi_{1,0}, \ldots, \chi_{n,0}$ of $X^*(G(\mathcal{E}_0)^{cst})$. Let $V_0^\dagger$ (resp. V_0) be the representation of $G(\mathcal{E}_0^\dagger)$ (resp. $G(\mathcal{E}_0)$) associated to $\mathcal{E}_0^\dagger$ (resp. $\mathcal{E}_0$). As every constant F-isocrystal comes from

an overconvergent F-isocrystal, for every i, the character $\chi_{i,0}$ extends to a character $\chi_{i,0}^\dagger$ of $\pi_1^{\mathbf{F\text{-}Isoc}^\dagger}(X_0)$, the Tannakian group of $\mathbf{F\text{-}Isoc}^\dagger(X_0)$. Take

$$\widetilde{V}_0^\dagger := V_0^\dagger \oplus \bigoplus_{i=1}^{n} \chi_{i,0}^\dagger$$

and write $\widetilde{V}_0$ for the induced representation of $\pi_1^{\mathbf{F\text{-}Isoc}}(X_0)$, the Tannakian group of $\mathbf{F\text{-}Isoc}(X_0)$. By construction, the groups of constant characters $X^*(G(\widetilde{V}_0)^{cst})$ and $X^*(G(V_0)^{cst})$ are canonically isomorphic. Moreover, since $\widetilde{V}^\dagger \simeq V^\dagger \oplus \overline{\mathbb{Q}}_p^{\oplus n}$ and $\widetilde{V} \simeq V \oplus \overline{\mathbb{Q}}_p^{\oplus n}$, we get isomorphisms $G(\widetilde{V}^\dagger) \simeq G(\mathcal{E}^\dagger)$ and $G(\widetilde{V}) \simeq G(\mathcal{E})$. Thus it is enough to show that $\mathrm{rk}(G(\widetilde{V}^\dagger)) = \mathrm{rk}(G(\widetilde{V}))$.

By Proposition 3.2.2.4, there exists a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G(\widetilde{V}) \arrow[r] \arrow[d, hook] & G(\widetilde{V}_0) \arrow[r] \arrow[d, hook] & G(\widetilde{V}_0)^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\widetilde{V}^\dagger) \arrow[r] & G(\widetilde{V}_0^\dagger) \arrow[r] & G(\widetilde{V}_0^\dagger)^{cst} \arrow[r] & 0
\end{tikzcd}$$

where the first two vertical arrows are injective and the last one is surjective. As $\widetilde{V}_0^\dagger$ is still algebraic and pure of weight 0, by Corollary 3.2.3.6, $\mathrm{rk}(G(\widetilde{V}_0)) = \mathrm{rk}(G(\widetilde{V}_0^\dagger))$. Since the reductive rank is additive in exact sequences, it is enough to show that $G(\widetilde{V}_0)^{cst}$ and $G(\widetilde{V}_0^\dagger)^{cst}$ have the same reductive rank. We will show that the morphism $\varphi : G(\widetilde{V}_0)^{cst} \to G(\widetilde{V}_0^\dagger)^{cst}$ of the previous diagram is actually an isomorphism. We already know that φ is surjective. As $G(\widetilde{V}_0)^{cst}$ and $G(\widetilde{V}_0^\dagger)^{cst}$ are groups of multiplicative type by Corollary 3.2.3.7, it remains to show that the map $\varphi^* : X^*(G(\widetilde{V}_0^\dagger)^{cst}) \to X^*(G(\widetilde{V}_0)^{cst})$ is surjective. This is a consequence of the construction of $\widetilde{V}_0^\dagger$. Indeed, $X^*(G(\widetilde{V}_0)^{cst}) = X^*(G(V_0)^{cst})$ is generated by $\chi_{1,0}, \ldots, \chi_{n,0}$ and for every i, the character $\chi_{i,0}^\dagger \in X^*(G(\widetilde{V}_0^\dagger)^{cst})$ is sent by φ^* to $\chi_{i,0}$. □

Corollary 3.2.3.10. *Let $\mathcal{E}_0^\dagger$ be an algebraic overconvergent F-isocrystal. The reductive rank of $G(\mathcal{E}_0^\dagger)^{cst}$ is the same as the one of $G(\mathcal{E}_0)^{cst}$.*

Proof. The result follows from Corollary 3.2.3.6 and Theorem 3.2.3.9, thanks to Proposition 3.2.2.4 and the additivity of the reductive ranks with respect to exact sequences. □

Remark 3.2.3.11. Using Theorem 3.2.1.5, one can show that when $\mathcal{E}^\dagger$ is semi-simple, the functor $\langle \mathcal{E}^\dagger \rangle \to \langle \mathcal{E} \rangle$ is fully faithful. Therefore, in this case, $G(\mathcal{E}) \subseteq G(\mathcal{E}^\dagger)$ is an epimorphic subgroup (cf. Remark 3.2.1.6). Nevertheless, Theorem 3.2.3.9 does not follow directly from this, because epimorphic subgroups can have, in general, lower reductive rank. For example, let $\mathbb{K}$ be any field and let G be the algebraic group $\mathrm{SL}_{3,\mathbb{K}}$. The subgroup H of G defined by the matrices of the form

$$\begin{pmatrix} a & 0 & * \\ 0 & a & * \\ 0 & 0 & a^{-2} \end{pmatrix},$$

with $a \in \mathbb{K}^\times$, is the radical of a maximal parabolic subgroup of G. Therefore, by [BB92, §2.(a) and §2.(d)], H is an epimorphic subgroup of G. On the other hand, the reductive rank of H is 1. More surprisingly, in characteristic 0, *every* almost simple group contains an epimorphic subgroup of dimension 3 [*ibid.*, §5.(b)].

Another consequence of Theorem 3.2.3.9 is given by the following result that we will not use, but which has its own interest. We have already discussed it in §3.1.4.

Corollary 3.2.3.12. *Let $\mathcal{E}_0^\dagger$ be a overconvergent F-isocrystal and assume that $\mathcal{E}^\dagger$ is semi-simple. Every $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle$ which is an extension of constant F-isocrystals is constant.*

Proof. The statement is equivalent to the fact that the group $\mathrm{Ext}^1_{G(\mathcal{E})}(\overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_p)$ vanishes. The result then follows from Theorem 3.2.3.9 thanks to Lemma 3.2.3.8. □

3.3 A special case of a conjecture of Kedlaya

3.3.1 Proof of the main theorem

As a consequence of the results of §3.2.3, we obtain a special case of the conjecture in [Ked16, Remark 5.14]. We shall start with a finiteness result. We retain the notation as in §3.2.2.1.

Proposition 3.3.1.1. *If $\mathcal{E}_0^\dagger$ is an irreducible overconvergent F-isocrystal with finite order determinant, then $G(\mathcal{E}_0)^{cst}$ is finite. In particular, every constant subquotient of the F-isocrystal $\mathcal{E}_0$ is finite.*

Proof. We first notice that $\mathcal{E}_0^\dagger$ is algebraic thanks to Deligne's conjecture (Theorem 1.3.7.6). By Corollary 3.2.3.7, we deduce that the algebraic groups $G(\mathcal{E}_0)^{cst}$ and $G(\mathcal{E}_0^\dagger)^{cst}$ are of multiplicative type and by Corollary 3.2.3.10 that they have the same dimensions. The algebraic group $G(\mathcal{E}_0^\dagger)^{cst}$ is finite thanks to Proposition 1.3.5.7. Therefore, the same is true for $G(\mathcal{E}_0)^{cst}$. □

Theorem 3.3.1.2. *Let $\mathcal{E}_0^\dagger$ be an irreducible overconvergent F-isocrystal. If $\mathcal{E}_0$ admits a subobject of minimal slope $\mathcal{F}_0 \subseteq \mathcal{E}_0$ with a non-zero morphism $\mathcal{F} \to \mathbb{1}$, then $\mathcal{F} = \mathcal{E}$ and $\mathcal{E} \simeq \mathbb{1}$.*

Proof. Observe that both the hypothesis and the conclusion are invariant under twist. Thus, by [Abe15, Lemma 6.1], we may assume that the determinant of $\mathcal{E}_0^\dagger$ is of finite order, hence unit-root. We first prove that $\mathcal{E}_0^\dagger$ is unit-root as well. If r is the rank of $\mathcal{E}_0^\dagger$, since

$$\sum_{i=1}^{r} a_i^\eta(\mathcal{E}_0^\dagger) = a_1^\eta(\det(\mathcal{E}_0^\dagger)) = 0 \quad \text{and} \quad a_1^\eta(\mathcal{E}_0^\dagger) \leq \cdots \leq a_r^\eta(\mathcal{E}_0^\dagger),$$

it suffices to show that $a_1^\eta(\mathcal{E}_0^\dagger) = 0$. Let $\mathcal{F} \twoheadrightarrow \mathcal{T}$ be the maximal trivial quotient of $\mathcal{F}$. By maximality, it lifts to a quotient $\mathcal{F}_0 \twoheadrightarrow \mathcal{T}_0$, where $\mathcal{T}_0$ is a constant F-isocrystal. The overconvergent F-isocrystal $\mathcal{E}_0^\dagger$ satisfies the assumptions of Proposition 3.3.1.1, hence $\mathcal{T}_0$ is finite. As the F-isocrystal $\mathcal{F}_0$ is isoclinic and it admits a non-zero quotient which is finite, it is unit-root. This implies that $a_1^\eta(\mathcal{E}_0^\dagger) = 0$, as we wanted.

We now prove that $\mathcal{E}_0^\dagger$ has rank 1. Since $\mathcal{E}_0^\dagger$ is unit-root, by [Ked16, Theorem 3.9], the functor $\langle \mathcal{E}_0^\dagger \rangle \to \langle \mathcal{E}_0 \rangle$ is an equivalence of categories. Therefore, if $\mathcal{E}_0$ has a constant subquotient, the same is true for $\mathcal{E}_0^\dagger$. But $\mathcal{E}_0^\dagger$ is irreducible by assumption, thus it has to be itself a constant F-isocrystal. Since irreducible constant ($\overline{\mathbb{Q}}_p$-linear) F-isocrystals have rank 1, this ends the proof. □

Remark 3.3.1.3. The statement of Theorem 3.3.1.2 is false in general if we do not assume that $\mathcal{F}_0 \subseteq \mathcal{E}_0$ is of minimal slope. A counterexample is provided in [Ked16, Example 5.15].

3.3.2 Some consequences

Corollary 3.3.2.1. *Let $\mathcal{E}_0^\dagger$ be an overconvergent F-isocrystals and $\mathcal{F}_0$ a subobject of $\mathcal{E}_0$ of minimal generic slope. If $\mathcal{E}^\dagger$ is semi-simple, then the restriction morphism* $\mathrm{Hom}(\mathcal{E}, \mathbb{1}) \to \mathrm{Hom}(\mathcal{F}, \mathbb{1})$ *is surjective.*

Proof. As $\mathcal{E}^\dagger$ is semi-simple if we replace $\mathcal{E}_0^\dagger$ with its semi-simplification with respect to a Jordan–Hölder filtration, we do not change the isomorphism class of $\mathcal{E}^\dagger$. Thus we may and do assume that $\mathcal{E}_0^\dagger$ is semi-simple. The proof is then an induction on the number n of summands of some decomposition of $\mathcal{E}_0^\dagger$ in irreducible overconvergent F-isocrystals. If $n = 1$ this is an immediate consequence of Theorem 3.3.1.2. Suppose now that the result is known for every positive integer $m < n$. Take an irreducible subobject $\mathcal{G}_0^\dagger$ of $\mathcal{E}_0^\dagger$, write $\mathcal{H}_0 := \mathcal{G}_0 \times_{\mathcal{E}_0} \mathcal{F}_0$ and consider the following commutative diagram with exact rows and injective vertical arrows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d, hook] & \mathcal{F} \arrow[r] \arrow[d, hook] & \mathcal{F}/\mathcal{H} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}/\mathcal{G} \arrow[r] & 0.
\end{tikzcd}$$

As $\mathcal{E}_0^\dagger$ is semi-simple, the quotient $\mathcal{E}_0^\dagger \twoheadrightarrow \mathcal{E}_0^\dagger/\mathcal{G}_0^\dagger$ admits a splitting. This implies that the lower exact sequence splits. We apply the functor $\mathrm{Hom}(-, \mathbb{1})$ and we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathcal{E}/\mathcal{G}, \mathbb{1}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{E}, \mathbb{1}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{G}, \mathbb{1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{F}/\mathcal{H}, \mathbb{1}) \arrow[r] & \mathrm{Hom}(\mathcal{F}, \mathbb{1}) \arrow[r] & \mathrm{Hom}(\mathcal{H}, \mathbb{1}). &
\end{tikzcd}$$

Since $\mathcal{H}_0$ and $\mathcal{F}_0/\mathcal{H}_0$ are subobjects of minimal slope of $\mathcal{G}_0$ and $\mathcal{E}_0/\mathcal{G}_0$ respectively, by the inductive hypothesis, the left and the right vertical arrows are surjective. By diagram chasing, this implies that the central vertical arrow is also surjective, as we wanted. $\square$

Remark 3.3.2.2. By the theory of weights, if $\mathcal{E}_0^\dagger$ is pure then $\mathcal{E}^\dagger$ is semi-simple, hence one can apply Theorem 3.3.2.1 in this situation. The theorem is instead false without the assumption that $\mathcal{E}^\dagger$ is semi-simple. For example, when $X_0 = \mathbb{G}_{m,\mathbb{F}_q}$, there exits a non-trivial extension

$$0 \to \mathbb{1}_0^\dagger \to \mathcal{E}_0^\dagger \to (\mathbb{1}_0^\dagger)^{(q)} \to 0,$$

which does not split in $\mathbf{Isoc}^\dagger(X_0)$. If $\mathcal{F}_0 \subseteq \mathcal{E}_0$ is the rank 1 trivial subobject of $\mathcal{E}_0$, then the map $\mathrm{Hom}(\mathcal{E}, \mathbb{1}) \to \mathrm{Hom}(\mathcal{F}, \mathbb{1})$ is the zero map, even if $\mathrm{Hom}(\mathcal{F}, \mathbb{1}) = \overline{\mathbb{Q}}_p$.

We end the section presenting a variant of Corollary 3.3.2.1, where we consider morphisms in $\mathbf{F\text{-}Isoc}^\dagger(X_0)$.

Corollary 3.3.2.3. *Let $\mathcal{E}_0^\dagger$ be an algebraic semi-simple overconvergent F-isocrystals with constant Newton polygons and of minimal slope equal to* 0*. The restriction morphism* $\mathrm{Hom}(\mathcal{E}_0, \mathbb{1}_0) \to \mathrm{Hom}(\mathcal{E}_0^1, \mathbb{1}_0)$ *is an isomorphism.*

Proof. Since $\mathcal{E}_0^\dagger$ is semi-simple, $\mathcal{E}^\dagger$ is semi-simple as well. By Theorem 3.3.2.1, the restriction morphism $\mathrm{Hom}(\mathcal{E}, \mathbb{1}) \to \mathrm{Hom}(\mathcal{E}^1, \mathbb{1})$ is surjective. As the group $G(\mathcal{E}_0)^{cst}$ is reductive (Corollary 3.2.3.7), the action of the absolute Frobenius F on $\mathrm{Hom}(\mathcal{E}, \mathbb{1})$ and $\mathrm{Hom}(\mathcal{E}^1, \mathbb{1})$ is semi-simple, so that the restriction morphism

$$\mathrm{Hom}(\mathcal{E}_0, \mathbb{1}_0) = \mathrm{Hom}(\mathcal{E}, \mathbb{1})^F \to \mathrm{Hom}(\mathcal{E}^1, \mathbb{1})^F = \mathrm{Hom}(\mathcal{E}_0^1, \mathbb{1}_0)$$

is still surjective. The injectivity follows from the fact that $\mathcal{E}_0/\mathcal{E}_0^1$ has positive slopes. Indeed, this implies that there are no non-zero morphisms from $\mathcal{E}_0$ to $\mathbb{1}_0$ which factor through $\mathcal{E}_0/\mathcal{E}_0^1$. $\square$

3.4 An extension of the theorem of Lang–Néron

3.4.1 p-torsion and p-divisible groups

We exploit here Corollary 3.3.2.1 to prove the following result on the torsion points of abelian varieties. Let $\mathbb{F} \subseteq k$ be a finitely generated field extension and let k^{perf} be a perfect closure of k.

Theorem 3.4.1.1. *If A be an abelian variety over k such that $\mathrm{Tr}_{k/\mathbb{F}}(A) = 0$, then the group $A(k^{\mathrm{perf}})_{\mathrm{tors}}$ is finite.*

As we have already discussed in Remark 3.1.2, thanks to Theorem 3.1.2.1, it is enough to show that $A[p^\infty](k^{\mathrm{perf}})$ is finite.

Notation 3.4.1.2. Throughout §3.4, let $k_0 \subseteq k$ be a finitely generated field such that $k = \mathbb{F}k_0$ and such that there exists an abelian variety A_0/k_0 with $A \simeq A_0 \otimes_{k_0} k$. Let $\mathbb{F}_q$ be the algebraic closure of $\mathbb{F}_p$ in k_0. We choose a smooth geometrically connected variety X_0 over $\mathbb{F}_q$, with $\mathbb{F}_q(X_0) \simeq k_0$ and such that there exists an abelian scheme $f_0 : \mathfrak{A}_0 \to X_0$ with constant Newton polygons and generic fiber A_0/k_0. We denote by X and $\mathfrak{A}$ the extension of scalars of X_0 and $\mathfrak{A}_0$ to $\mathbb{F}$.

Lemma 3.4.1.3. *If $|A[p^\infty](k^{\mathrm{perf}})| = \infty$, then there exists an injective morphism $(\mathbb{Q}_p/\mathbb{Z}_p)_X \hookrightarrow \mathfrak{A}[p^\infty]^{\text{ét}}$.*

Proof. We first prove that the group $A[p^\infty](k^{\mathrm{perf}})$ is isomorphic to $\mathfrak{A}[p^\infty]^{\text{ét}}(X)$, showing thereby that $\mathfrak{A}[p^\infty]^{\text{ét}}(X)$ is infinite as well. As k^{perf} is a perfect field, the map

$$A[p^\infty](k^{\mathrm{perf}}) \to A[p^\infty]^{\text{ét}}(k^{\mathrm{perf}}) = A[p^\infty]^{\text{ét}}(k),$$

induced by the quotient $A[p^\infty] \twoheadrightarrow A[p^\infty]^{\text{ét}}$, is an isomorphism. In addition, as the Newton polygons of $f : \mathfrak{A} \to X$ are constant, the étale sheaves $\mathfrak{A}[p^i]^{\text{ét}}$ are locally constant on X. Therefore, since X is smooth, the restriction morphism $\mathfrak{A}[p^\infty]^{\text{ét}}(X) \to A[p^\infty]^{\text{ét}}(k)$ induced by the inclusion of the generic point $\mathrm{Spec}(k) \hookrightarrow X$ is an isomorphism. These two observations show that $A[p^\infty](k^{\mathrm{perf}}) \simeq \mathfrak{A}[p^\infty]^{\text{ét}}(X)$, as we wanted.

Since $|\mathfrak{A}[p^\infty]^{\text{ét}}(X)| = |A[p^\infty](k^{\mathrm{perf}})| = \infty$, a standard compactness argument shows that there exists a morphism $(\mathbb{Q}_p/\mathbb{Z}_p)_X \hookrightarrow \mathfrak{A}[p^\infty]^{\text{ét}}$. For the reader convenience, we quickly recall it. We define a partition of $\mathfrak{A}[p^\infty]^{\text{ét}}(X)$ in subsets $\{\Delta_i\}_{i\in\mathbb{N}}$ in the following way. Let $\Delta_0 := \{0\}$ and for $i > 0$, let $\Delta_i := \mathfrak{A}[p^i]^{\text{ét}}(X) \setminus \mathfrak{A}[p^{i-1}]^{\text{ét}}(X)$. When $j \geq i$, the multiplication by p^{j-i} induces a map $\Delta_j \to \Delta_i$. These maps make $\{\Delta_i\}_{i\in\mathbb{N}}$ an inverse system. We claim that every Δ_i is non-empty. Suppose by contradiction that for some $N \in \mathbb{N}$, the set Δ_N is empty. By construction, for every $i \geq N$ the sets Δ_i are empty as well. Since every Δ_i is finite, this implies that $\mathfrak{A}[p^\infty]^{\text{ét}}(X)$ is also finite, which is a contradiction. As every Δ_i is non-empty, by Tychonoff's theorem, the projective limit $\varprojlim \Delta_i$ is non-empty. The choice of an element $(P_i)_{i\in\mathbb{N}} \in \varprojlim \Delta_i$ induces an injective map $(\mathbb{Q}_p/\mathbb{Z}_p)_X \hookrightarrow \mathfrak{A}[p^\infty]^{\text{ét}}$, given by the assignment $[1/p^i] \mapsto P_i$. This yields the desired result. □

3.4.2 Reformulation with the crystalline Dieudonné theory

We restate the classical crystalline Dieudonné theory in our setup.

3.4.2.1. Let

$$\mathbb{D} : \{p\text{-divisible groups } /X\} \to \mathbf{F\text{-}Crys}(X/W(\mathbb{F}))$$

be the crystalline Dieudonné module (contravariant) functor, where $\mathbf{F\text{-}Crys}(X/W(\mathbb{F}))$ is the category of F-crystals (cf. [BBM82]). In [*ibid.*], it is proven that this functor is fully faithful

and $\mathbb{D}(\mathfrak{A}[p^\infty]) \simeq R^1 f_{\mathrm{crys}*}\mathcal{O}_{\mathfrak{A}}$. Extending the scalars to $\overline{\mathbb{Q}}_p$ and post-composing with the functor of Theorem 3.2.1.2, we define a $\overline{\mathbb{Q}}_p$-linear fully faithful contravariant functor

$$\mathbb{D}_{\overline{\mathbb{Q}}_p} : \{p\text{-divisible groups } /X\}_{\overline{\mathbb{Q}}_p} \to \mathbf{F}\text{-}\mathbf{Isoc}(X).$$

The functor sends the trivial p-divisible group $(\mathbb{Q}_p/\mathbb{Z}_p)_{X,\overline{\mathbb{Q}}_p}$ to the F-isocrystal $(\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$ on X. When $X = \mathrm{Spec}(\mathbb{F})$, the functor induces an equivalence

$$\mathbb{D}_{\overline{\mathbb{Q}}_p} : \{p\text{-divisible groups } / \mathrm{Spec}(\mathbb{F})\}_{\overline{\mathbb{Q}}_p} \xrightarrow{\sim} \mathbf{F}\text{-}\mathbf{Isoc}_{[0,1]}(\mathrm{Spec}(\mathbb{F})),$$

where $\mathbf{F}\text{-}\mathbf{Isoc}_{[0,1]}(\mathrm{Spec}(\mathbb{F}))$ is the category of F-isocrystals with slopes between 0 and 1. Since $\mathbb{D}_{\overline{\mathbb{Q}}_p}$ is compatible with base change, this implies that for every X, the functor $\mathbb{D}_{\overline{\mathbb{Q}}_p}$ is exact, it preserves the heights/ranks and it sends étale p-divisible groups to unit-root F-isocrystals.

3.4.2.2. By [Ete02, Théorème 7], the F-isocrystal $R^1 f_{0,\mathrm{crys}*}\mathcal{O}_{\mathfrak{A}_0}$ over X_0 comes from an over-convergent F-isocrystal, which we denote by $\mathcal{E}_0^\dagger$. Let $\mathcal{F}_0$ be the maximal unit-root subobject of $\mathcal{E}_0$ and let $(\mathcal{F}_0)_X$ be the inverse image of $\mathcal{F}_0$ to X, as an F-isocrystal. By the discussion in §3.4.2.1, we have the following result.

Lemma 3.4.2.3. *The quotient $\mathfrak{A}[p^\infty] \twoheadrightarrow \mathfrak{A}[p^\infty]^{\acute{e}t}$ is sent by $\mathbb{D}_{\overline{\mathbb{Q}}_p}$ to the natural inclusion $(\mathcal{F}_0)_X \hookrightarrow (\mathcal{E}_0)_X$.*

Thanks to Lemma 3.4.2.3, we can reformulate Lemma 3.4.1.3 in the language of F-isocrystals.

Corollary 3.4.2.4. *If $|A[p^\infty](k^{\mathrm{perf}})| = \infty$, then there exists a quotient $(\mathcal{F}_0)_X \twoheadrightarrow (\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$.*

Proof. Thanks to Lemma 3.4.1.3, if $|A[p^\infty](k^{\mathrm{perf}})| = \infty$, then there exists an injective morphism $(\mathbb{Q}_p/\mathbb{Z}_p)_X \hookrightarrow \mathfrak{A}[p^\infty]^{\text{ét}}$. By Lemma 3.4.2.3, after we extend the scalars to $\overline{\mathbb{Q}}_p$, this morphism is sent by $\mathbb{D}_{\overline{\mathbb{Q}}_p}$ to a quotient $(\mathcal{F}_0)_X \twoheadrightarrow (\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$. □

3.4.3 End of the proof

We need to rephrase the finiteness of torsion points given by the theorem of Lang–Néron in terms of morphisms of isocrystals on X_0. This will lead to the proof of Theorem 3.4.1.1. Retain notation as in §3.4.2.2.

Proposition 3.4.3.1. *If there exists a morphism $\mathcal{E} \to \mathcal{O}_{X_0}$ which is non-zero on $\mathcal{F}$, then $\mathrm{Tr}_{k/\mathbb{F}}(A) \neq 0$.*

Proof. The maximal trivial quotient $\mathcal{E} \twoheadrightarrow \mathcal{T}$, descends to a quotient $\mathcal{E}_0 \twoheadrightarrow \mathcal{T}_0$, where $\mathcal{T}_0$ is a constant F-isocrystal. We base change this quotient from X_0 to X, as a morphism of F-isocrystals, obtaining a quotient $(\mathcal{E}_0)_X \twoheadrightarrow (\mathcal{T}_0)_X$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$. Since $\mathcal{T}_0$ is an F-isocrystal

coming from $\mathrm{Spec}(\mathbb{F}_q)$, the F-isocrystal $(\mathcal{T}_0)_X$ comes from $\mathrm{Spec}(\mathbb{F})$. Thanks to [Ked16, Theorem 3.5], $(\mathcal{T}_0)_X$ decomposes in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ as

$$(\mathcal{T}_0)_X = (\mathcal{T}_0')_X \oplus (\mathcal{O}_X^{\oplus n}, \mathrm{id}_{\mathcal{O}_X^{\oplus n}})$$

where $(\mathcal{O}_X^{\oplus n}, \mathrm{id}_{\mathcal{O}_X^{\oplus n}})$ is the maximal unit-root subobject of $(\mathcal{T}_0)_X$ and $n \geq 0$. As $\mathcal{F}_0$ is unit-root, it is sent via the quotient $(\mathcal{E}_0)_X \twoheadrightarrow (\mathcal{T}_0)_X$ to a non-zero unit-root F-isocrystal, so that $n > 0$. Thus, $(\mathcal{E}_0)_X$ admits a quotient to $(\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$. Since $\mathbb{D}_{\overline{\mathbb{Q}}_p}$ is fully faithful, such a quotient comes from a monomorphism $(\mathbb{Q}_p/\mathbb{Z}_p)_{X,\overline{\mathbb{Q}}_p} \hookrightarrow \mathfrak{A}[p^\infty]_{\overline{\mathbb{Q}}_p}$ in the category of p-divisible groups with coefficients in $\overline{\mathbb{Q}}_p$. The map, after possibly multiplying it by some power of p, comes from an injection $(\mathbb{Q}_p/\mathbb{Z}_p)_X \hookrightarrow \mathfrak{A}[p^\infty]$ of p-divisible groups over X. By Theorem 3.1.2.1, this implies that $\mathrm{Tr}_{k/\mathbb{F}}(A) \neq 0$. □

Proof of Theorem 3.4.1.1. Assume by contradiction that $|A[p^\infty](k^{\mathrm{perf}})| = \infty$. By Corollary 3.4.2.4, we have a quotient $(\mathcal{F}_0)_X \twoheadrightarrow (\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$. Forgetting the Frobenius structure, we get a quotient $\mathcal{F}_X \twoheadrightarrow \mathcal{O}_X$ in $\mathbf{Isoc}(X)$. By a descent argument (see for example the proof of [Kat99, Proposition 1.3.2]), the morphism $\mathcal{F}_X \twoheadrightarrow \mathcal{O}_X$ descends to a quotient $\mathcal{F} \twoheadrightarrow \mathcal{O}_{X_0}$ in $\mathbf{Isoc}(X_0)$. By Theorem 3.3.2.1, the map extends to a quotient $\mathcal{E} \twoheadrightarrow \mathcal{O}_{X_0}$ in $\mathbf{Isoc}(X_0)$. We obtain then a contradiction by Proposition 3.4.3.1. □

Remark 3.4.3.2. The proofs of Theorem 3.2.3.4 and Proposition 3.3.1.1 rely on the known cases of Deligne's conjecture. In particular, they rely on the Langlands correspondence for lisse sheaves proven in [Laf02] and the Langlands correspondence for overconvergent F-isocrystals proven in [Abe18]. We want to point out that to prove Theorem 3.4.1.1 we do not need to use this theory. More precisely, when $\mathcal{E}_0^\dagger$ is an overconvergent F-isocrystal which "comes from geometry", for example any overconvergent F-isocrystals appearing in §3.4, Theorem 3.2.3.4 can be proven more directly, as explained in Remark 1.4.2.11. Even in the proof of Proposition 3.3.1.1, if $\mathcal{E}_0^\dagger$ "comes from geometry" we do not need Theorem 1.3.7.6.

A Neutral Tannakian categories with Frobenius

We introduce in this appendix the notion of *neutral Tannakian categories with Frobenius*, and we present a *fundamental exact sequence* for these categories. This formalism applies to the categories of coefficient objects, as explained in Proposition 1.3.1.8. We have preferred to work here in a more general setting in order to include some other categories, such as the category of *convergent isocrystals*.

A.1 Definition and Weil group

Definition A.1.1. A *neutral Tannakian category with Frobenius* is a neutral Tannakian category over some field $\mathbb{K}$, endowed with a $\mathbb{K}$-linear $\otimes$-autoequivalence $F^* : \widetilde{\mathbf{C}} \to \widetilde{\mathbf{C}}$.

Construction A.1.2. We fix a neutral Tannakian category with Frobenius $(\widetilde{\mathbf{C}}, F^*)$ over some field $\mathbb{K}$. We denote by $\mathbf{C}_0$ the category of pairs $(\mathcal{E}, \Phi)$, where $\mathcal{E} \in \widetilde{\mathbf{C}}$ and Φ is an isomorphism between $F^*\mathcal{E}$ and $\mathcal{E}$. A morphism between two objects $(\mathcal{E}, \Phi)$ and $(\mathcal{E}', \Phi')$ is a morphism $f : \mathcal{E} \to \mathcal{E}'$ such the following diagram commutes

$$\begin{tikzcd} F^*\mathcal{E} \arrow[r, "\Phi"] \arrow[d, "F^*f"] & \mathcal{E} \arrow[d, "f"] \\ F^*\mathcal{E}' \arrow[r, "\Phi'"] & \mathcal{E}'. \end{tikzcd}$$

Let $\Psi : \mathbf{C}_0 \to \widetilde{\mathbf{C}}$ be the forgetful functor sending $(\mathcal{E}, \Phi)$ to $\mathcal{E}$. Write $\mathbf{C}$ for the smallest Tannakian subcategory of $\widetilde{\mathbf{C}}$ which contains the essential image of Ψ.

Choose a fiber functor ω of $\widetilde{\mathbf{C}}$ over $\mathbb{K}$. It restricts to a fibre functor of $\mathbf{C}$ which we will denote by the same symbol. We write ω_0 for the fibre functor of $\mathbf{C}_0$ given by the composition $\omega \circ \Psi$. We define $\pi_1(\mathbf{C}, \omega)$ and $\pi_1(\mathbf{C}_0, \omega_0)$ as the Tannakian groups of $\mathbf{C}$ and $\mathbf{C}_0$ with respect to ω and ω_0 respectively. The functor Ψ induces a closed immersion $\pi_1(\mathbf{C}, \omega) \hookrightarrow \pi_1(\mathbf{C}_0, \omega_0)$ and for every $\mathcal{E}_0 = (\mathcal{E}, \Phi) \in \mathbf{C}_0$ a closed immersion $G(\mathcal{E}, \omega) \hookrightarrow G(\mathcal{E}_0, \omega_0)$.

Definition A.1.3. We say that an object in $\mathbf{C}_0$ is *constant* if its image in $\mathbf{C}$ is trivial, i.e. isomorphic to $\mathbb{1}^{\oplus n}$ for some $n \in \mathbb{N}$. The constant objects of $\mathbf{C}_0$ form a Tannakian subcategory $\mathbf{C}_{cst} \subseteq \mathbf{C}_0$. Let $\pi_1(\mathbf{C}_0, \omega_0)^{cst}$ be the Tannakian group of $\mathbf{C}_{cst}$ with respect to ω_0. The inclusion $\mathbf{C}_{cst} \subseteq \mathbf{C}_0$ induces a faithfully flat morphism $\pi_1(\mathbf{C}_0, \omega_0) \twoheadrightarrow \pi_1(\mathbf{C}_0, \omega_0)^{cst}$. For every object $\mathcal{E}_0 \in \mathbf{C}_0$, we denote by $G(\mathcal{E}_0, \omega_0)^{cst}$ the Tannakian group of the full subcategory $\langle \mathcal{E}_0 \rangle_{cst} \subseteq \langle \mathcal{E}_0 \rangle$ of constant objects. This induces a faithfully flat morphism $G(\mathcal{E}_0, \omega_0) \twoheadrightarrow G(\mathcal{E}_0, \omega_0)^{cst}$.

A.1.4. Suppose that $(\widetilde{\mathbf{C}}, F^*)$ admits an isomorphism of fibre functors $\eta : \omega \Rightarrow \omega \circ F^*$. The group $\pi_1(\mathbf{C}, \omega)$ is then endowed with an automorphism φ which is constructed in the following

way. For every $\mathbb{K}$-algebra R, the automorphism φ sends $\alpha \in \pi_1(\mathbf{C}, \omega)(R)$ to $\eta_R^{-1} \circ \alpha \circ \eta_R$, where η_R is the extension of scalars of η from $\mathbb{K}$ to R.

Definition (Weil group)**.** Let $W(\mathbf{C}_0, \omega_0)$ be the group scheme over $\mathbb{K}$ which is the semi-direct product $\pi_1(\mathbf{C}, \omega) \rtimes \mathbb{Z}$, where $1 \in \mathbb{Z}$ acts on $\pi_1(\mathbf{C}, \omega)$ as φ acts on $\pi_1(\mathbf{C}, \omega)$. We will call $W(\mathbf{C}_0, \omega_0)$ the *Weil group* of $\mathbf{C}_0$.

Remark A.1.5. Thanks to [Del90], we know that if $\mathbb{K}$ is algebraically closed, an isomorphism η as in §A.1.4 always exists. This is not the case in general when $\mathbb{K}$ is not algebraically closed. This is due to the existence of non-isomorphic algebraic groups with equivalent categories of linear representations. For coefficient objects, an isomorphism η can be constructed each time that a fibre functor exists, as it is explained in Remark 1.3.2.2.

Lemma A.1.6. *Let $(\widetilde{\mathbf{C}}, F^*)$ be a neutral Tannakian category with Frobenius which admits a fibre functor ω isomorphic to $\omega \circ F^*$. There exists a natural equivalence of categories $\mathbf{C}_0 \xrightarrow{\sim} \mathbf{Rep}_{\mathbb{K}}(W(\mathbf{C}_0, \omega_0))$ and a natural morphism $\iota : W(\mathbf{C}_0, \omega_0) \to \pi_1(\mathbf{C}_0, \omega_0)$ such that the following diagram commutes*

$$\begin{tikzcd}
 & \mathbf{Rep}_{\mathbb{K}}(\pi_1(\mathbf{C}_0, \omega_0)) \arrow[d, "\iota^*"] \\
\mathbf{C}_0 \arrow[ur, "\sim"] \arrow[r, "\sim"'] & \mathbf{Rep}_{\mathbb{K}}(W(\mathbf{C}_0, \omega_0)),
\end{tikzcd}$$

where the equivalence $\mathbf{C}_0 \xrightarrow{\sim} \mathbf{Rep}_{\mathbb{K}}(\pi_1(\mathbf{C}_0, \omega_0))$ is the one induced by the fibre functor ω_0. In addition, the image of ι is Zariski-dense in $\pi_1(\mathbf{C}_0, \omega_0)$.

Proof. For every $(\mathcal{E}, \Phi) \in \mathbf{C}_0$, we extend the natural representation of $\pi_1(\mathbf{C}, \omega)$ on the vector space $\omega(\mathcal{E})$ to a representation of $W(\mathbf{C}_0, \omega_0)$. Write e for the identity point in $\pi_1(\mathbf{C}, \omega)(\mathbb{K})$. We impose that $(e, 1) \in W(\mathbf{C}_0, \omega_0)(\mathbb{K})$ acts on $\omega(\mathcal{E})$ via $\omega(\Phi) \circ \eta_{\mathcal{E}}$, where $\eta_{\mathcal{E}}$ is the isomorphism induced by η between $\omega(\mathcal{E})$ and $\omega(F^*\mathcal{E})$. This defines an equivalence $\mathbf{C}_0 \xrightarrow{\sim} \mathbf{Rep}_{\mathbb{K}}(W(\mathbf{C}_0, \omega_0))$ and a morphism $\iota : W(\mathbf{C}_0, \omega_0) \to \pi_1(\mathbf{C}_0, \omega_0)$ satisfying the required properties. By the Tannaka reconstruction theorem, the affine group $\pi_1(\mathbf{C}_0, \omega)$ is the pro-algebraic completion of $W(\mathbf{C}_0, \omega_0)$, thus the image of ι is Zariski-dense in $\pi_1(\mathbf{C}_0, \omega)$. $\square$

A.2 The fundamental exact sequence

A.2.1. We briefly recall the general criterion for the exactness of sequences of Tannakian groups. Let $L \xrightarrow{q} G \xrightarrow{p} A$ be a sequence of affine group schemes over a field $\mathbb{K}$. Write

$$\mathbf{Rep}_{\mathbb{K}}(A) \xrightarrow{p^*} \mathbf{Rep}_{\mathbb{K}}(G) \xrightarrow{q^*} \mathbf{Rep}_{\mathbb{K}}(L)$$

for the induced sequence of functors.

Theorem ([EHS07, Theorem A.1])**.** *Suppose that p is faithfully flat and q is a closed immersion. Then the sequence $L \xrightarrow{q} G \xrightarrow{p} A$ is exact if and only if the following conditions are fulfilled.*

(a) *For every $V \in \mathbf{Rep}_{\mathbb{K}}(G)$, the image $q^*(V)$ in $\mathbf{Rep}_{\mathbb{K}}(L)$ is trivial if and only if $V \simeq p^*U$ for some $U \in \mathbf{Rep}_{\mathbb{K}}(A)$.*

(b) *For every $V \in \mathbf{Rep}_{\mathbb{K}}(G)$, if we write W for the maximal trivial subobject of $q^*(V)$ in $\mathbf{Rep}_{\mathbb{K}}(L)$, there exists $V' \subseteq V$ in $\mathbf{Rep}_{\mathbb{K}}(G)$ such that $q^*(V') = W$.*

(c) *For every $W \in \mathbf{Rep}_{\mathbb{K}}(L)$, there exists $V \in \mathbf{Rep}_{\mathbb{K}}(G)$ such that W is a subobject of $q^*(V)$.*

Lemma A.2.2. *Let $(\widetilde{\mathbf{C}}, F^*)$ be a neutral Tannakian category with Frobenius and let ω be a fibre functor of $\widetilde{\mathbf{C}}$. The subgroup $\pi_1(\mathbf{C}, \omega) \subseteq \pi_1(\mathbf{C}_0, \omega_0)$ is a normal subgroup. In particular, for every $\mathcal{F} \in \mathbf{C}$ there exists $\mathcal{G}_0 \in \mathbf{C}_0$ such that $\mathcal{F} \subseteq \Psi(\mathcal{G}_0)$.*

Proof. Thanks to Theorem A.2.1, the second part of the statement follows from the first one. We may verify that the subgroup is normal after extending the field $\mathbb{K}$ to its algebraic closure. Under the additional assumption that $\mathbb{K}$ is algebraically closed, by Remark A.1.5 there exists an isomorphism between ω and $\omega \circ F^*$, so that we can construct the Weil group $W(\mathbf{C}_0, \omega_0)$ as defined in §A.1.4. By Lemma A.1.6, the group scheme $W(\mathbf{C}_0, \omega_0)$ is endowed with a natural morphism $\iota : W(\mathbf{C}_0, \omega_0) \to \pi_1(\mathbf{C}_0, \omega_0)$ with Zariski-dense image. Let H be the normalizer of $\pi_1(\mathbf{C}, \omega)$ in $\pi_1(\mathbf{C}_0, \omega_0)$. The group $\pi_1(\mathbf{C}, \omega)$ is normal in $W(\mathbf{C}_0, \omega_0)$, hence the $\mathbb{K}$-point $(e, 1) \in W(\mathbf{C}_0, \omega_0)(\mathbb{K})$ normalizes $\pi_1(\mathbf{C}, \omega)$. As a consequence, $\iota(e, 1) \in \pi_1(\mathbf{C}_0, \omega_0)(\mathbb{K})$ is contained in $H(\mathbb{K})$. The group $W(\mathbf{C}_0, \omega_0)$ is generated by $\pi_1(\mathbf{C}, \omega)$ and $(e, 1)$, thus the image of ι is contained in H. This implies that $H = \pi_1(\mathbf{C}_0, \omega_0)$, which shows that $\pi_1(\mathbf{C}, \omega)$ is normal in $\pi_1(\mathbf{C}_0, \omega_0)$, as we wanted.

□

Proposition A.2.3. *Let $(\widetilde{\mathbf{C}}, F^*)$ be a neutral Tannakian category over $\mathbb{K}$ with Frobenius and let ω be a fibre functor of $\widetilde{\mathbf{C}}$. The following statements hold.*

(i) *The morphisms constructed in §A.1.2 and §A.1.3 form an exact sequence*

$$1 \to \pi_1(\mathbf{C}, \omega) \to \pi_1(\mathbf{C}_0, \omega_0) \to \pi_1(\mathbf{C}_0, \omega_0)^{cst} \to 1.$$

(ii) *For every $\mathcal{E}_0 = (\mathcal{E}, \Phi) \in \mathbf{C}_0$ and every $\mathcal{F} \in \langle \mathcal{E} \rangle$, there exists $\mathcal{G}_0 \in \langle \mathcal{E}_0 \rangle$ such that $\mathcal{F} \subseteq \Psi(\mathcal{G}_0)$.*

(iii) *For every object $\mathcal{E}_0 = (\mathcal{E}, \Phi) \in \mathbf{C}_0$, the exact sequence of (i) sits in a commutative diagram with exact rows*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbf{C}, \omega) \arrow[r] \arrow[d, two heads] & \pi_1(\mathbf{C}_0, \omega_0) \arrow[r] \arrow[d, two heads] & \pi_1(\mathbf{C}_0, \omega_0)^{cst} \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & G(\mathcal{E}, x) \arrow[r] & G(\mathcal{E}_0, x) \arrow[r] & G(\mathcal{E}_0, x)^{cst} \arrow[r] & 1,
\end{tikzcd}$$

where the vertical arrows are the natural quotients.

(iv) *The affine group scheme* $\pi_1(\mathbf{C}_0, \omega_0)^{cst}$ *is isomorphic to the pro-algebraic completion of* $\mathbb{Z}$ *over* $\mathbb{K}$ *and* $G(\mathcal{E}_0, x)^{cst}$ *is a commutative algebraic group.*

Proof. We already know that the sequence of part (i) is exact on the left and on the right. It remains to show the exactness in the middle using Theorem A.2.1. Condition (a) is satisfied by construction. For condition (b) we notice that a $\otimes$-functor sends trivial objects to trivial objects. Therefore, for every $(\mathcal{E}, \Phi) \in \mathbf{C}_0$, the maximal trivial subobject $\mathcal{F} \subseteq \mathcal{E}$ is sent by F^* to the maximal trivial subobject of $F^*(\mathcal{E})$. This means that the restriction of Φ to $F^*(\mathcal{F})$ defines an isomorphism between $F^*(\mathcal{F})$ and $\mathcal{F}$ that we denote by $\Phi|_{\mathcal{F}}$. The pair $(\mathcal{F}, \Phi|_{\mathcal{F}})$ is the subobject of $(\mathcal{E}, \Phi)$ with the desired property. Condition (c) is proven in Lemma A.2.2.

For part (ii) we notice that the subgroup $G(\mathcal{E}, \omega) \subseteq G(\mathcal{E}_0, \omega_0)$ is a quotient of $\pi_1(\mathbf{C}, \omega) \subseteq \pi_1(\mathbf{C}_0, \omega_0)$, thus it is normal. By Theorem A.2.1, this implies the desired result. The diagram of part (iii) is obtained by taking the natural morphisms of the Tannakian groups. To prove that the lower sequence is exact we proceed as in part (i), replacing Lemma A.2.2 with part (ii). Finally, the category $\mathbf{C}_{cst}$ is equivalent to $\mathbf{Rep}_{\mathbb{K}}(\mathbb{Z})$, thus $\pi_1(\mathbf{C}_0, \omega_0)^{cst}$ is isomorphic to the pro-algebraic completion of $\mathbb{Z}$ over $\mathbb{K}$. In particular, for every $\mathcal{E}_0 \in \mathbf{C}_0$, the algebraic group $G(\mathcal{E}_0, \omega_0)^{cst}$, being a quotient of $\pi_1(\mathbf{C}_0, \omega_0)^{cst}$, is commutative. This concludes the proof. $\square$

References

[Abe15] T. Abe, Langlands program for p-adic coefficients and the petits camarades conjecture, *J. reine angew. Math* (published online 2015).

[Abe18] T. Abe, Langlands correspondence for isocrystals and the existence of crystalline companions for curves, *J. Amer. Math. Soc.* **31** (2018), 921–1057.

[AC13] T. Abe and D. Caro, *Theory of weights in p-adic cohomology*, arXiv:1303.0662 (2013).

[AC18] T. Abe and D. Caro, On Beilinson's equivalence for p-adic cohomology, *Selecta Mathematica* **24** (2018), 591–608.

[AE16] T. Abe and H. Esnault, A Lefschetz theorem for overconvergent isocrystals with Frobenius structure, preprint, available at http://www.mi.fu-berlin.de/users/esnault/preprints/helene (2016), to appear in *Ann. Sci. École Norm. Sup.* .

[Amb18] E. Ambrosi, *Specialization of Néron–Severi groups in positive characteristic*, arXiv:1810.06481 (2018).

[BB92] F. Bien and A. Borel, Sous-groupe épimorphiques de groupe algebriques linéaires I, *C.R. Acad. Sci. Paris* **315** (1992), 694–653.

[Bri16] M. Brion, *Epimorphic subgroups of algebraic groups*, arXiv:1605.07769 (2016).

[Ber96a] P. Berthelot, Cohomologie rigide et cohomologie rigide à supports propres, première partie, *Prépublication IRMAR*, available at https://perso.univ-rennes1.fr/pierre.berthelot (1996).

[Ber96b] P. Berthelot, $\mathcal{D}$-modules arithmétiques. I. Opérateurs différentiels de niveau fini, *Ann. Sci. École Norm. Sup.* **29** (1996), 185–272.

[BBM82] P. Berthelot, L. Breen, W. Messing, *Théorie de Dieudonné cristalline II*, Lecture Notes in Mathematics 930, Springer-Verlag, Berlin, 1982.

[Car15] D. Caro, Sur la préservation de la surconvergence par l'image directe d'un morphisme propre et lisse, *Ann. Sci. École Norm. Sup.* **48** (2015), 131–169.

[Cha13] C. Chai, *Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy,Tate-linear subvarieties and Hecke orbits*, preprint available at https://www.math.upenn.edu/ chai/papers.html (2013).

[Chi03] C. Chin, Independence of ℓ in Lafforgue's Theorem, *Adv. Math.* **180** (2003), 64–86.

[Chi04] C. Chin, Independence of ℓ of monodromy groups, *J. Amer. Math. Soc.* **17** (2004), 723–747.

[Chi08] C. Chin, Determining a split connected reductive group from its irreducible representations, *Journal of Group Theory* **11** (2008), 799–812.

[Con06] B. Conrad, Chow's K/k-image and K/k-trace and the Lang–Néron theorem, *Enseign. Math.* **52** (2006), 37–108.

[Cre87] R. Crew, F-isocrystals and p-adic representations, *Proc. Symp. Pure Math* **46** (1987), 111–138.

[Cre92a] R. Crew, F-isocrystals and their monodromy groups, *Ann. Sci. École Norm. Sup.* **25** (1992), 429–464.

[Cre92b] R. Crew, The p-adic monodromy of a generic abelian scheme in characteristic p, *p-adic methods in number theory and algebraic geometry*, Contemp. Math. 133, Amer. Math. Soc., 1992, 59–72.

[Cre94] R. Crew, Kloosterman sums and monodromy of a p-adic hypergeometric equation, *Compos. Math.* **91** (1994), 1–36.

[Del80] P. Deligne, La Conjecture de Weil, II, *Publ. Math. IHES* **52** (1980), 137–252.

[Del90] P. Deligne, Catégories tannakiennes, *The Grothendieck Festschrift, Vol. II*, Progr. Math. Birkhauser Boston, 1990, 111–195

[Del12] P. Deligne, Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius, en caractéristique finie, *Mosc. Math. J.* **12** (2012), 497–514.

[DM82] P. Deligne, J. S. Milne, Tannakian Categories, *Lecture Notes in Math.* **900** (1982), Springer–Verlag, Berlin–Heidelberg, 101–228.

[Dri12] V. Drinfeld, On a conjecture of Deligne, *Moscow Math. J.* **12** (2012), 515–542.

[Dri18] V. Drinfeld, On the pro-semisimple completion of the fundamental group of a smooth variety over a finite field, *Adv. Math.* **327** (2018), 708–788.

[DK17] V. Drinfeld and K. S. Kedlaya, Slopes of indecomposable F-isocrystals, *Pure Appl. Math. Quart.* **13** (2017), 131–192.

[EH62] B. Eckmann and P. J. Hilton, Group-like structures in general categories. I. Multiplications and comultiplications, *Math. Ann.*, **145** (1962), 227–255.

[Esn17] H. Esnault, Survey on some aspects of Lefschetz theorems in algebraic geometry, *Revista Matemática Complutense* **30** (2017), 217–232.

[EHS07] H. Esnault, P. H. Hai and X. Sun, On Nori's Fundamental Group Scheme, *Progress in Mathematics* **265** (2007), 377–398.

[EK11] H. Esnault, M. Kerz, Notes on Deligne's letter to Drinfeld dated March 5, 2007, preprint available at https://www.uni-due.de/ mat903/preprints/helene/103-110617.pdf (2011).

[Ete02] J.-Y. Étesse, Descente étale des F-isocristaux surconvergents et rationalité des fonctions L de schémas abéliens, *Ann. Sci. École Norm. Sup.* **35** (2002), 575–603.

[ES93] J.-Y. Étesse, B. Le Stum, Fonctions L associèes aux F-isocristaux surconvergent I. Interprétation cohomologique, *Math. Ann.* **296** (1993), 557–576.

[HP18] U. Hartl and A. Pál, *Crystalline Chebotarëv density theorems*, arXiv:1811.07084 (2018).

[Hum75] J. E. Humphreys, *Linear algebraic groups*, GTM 21, Springer–Verlag, New York–Heidelberg, 1975.

[Kat99] N. M. Katz, Space filling curves over finite fields, *Math. Res. Lett.* **6** (1999), 613–624.

[Kat01] N. M. Katz, Corrections to: Space filling curves over finite fields, *Math. Res. Lett.* **8** (2001), 689–691.

[KM74] N. Katz and W. Messing, Some Consequences of the Riemann Hypothesis for Varieties over Finite Fields, *Invent. Math.* **23** (1974), 73–78.

[KLV14] D. Kazhdan, M. Larsen and Y. Varshavsky, The Tannakian formalism and the Langlands conjectures, *Algebra Number Theory* **8** (2014), 243–256.

[Ked04a] K. S. Kedlaya, A p-adic local monodromy theorem, *Ann. Math.* **160** (2004), 93–184.

[Ked04b] K. S. Kedlaya, Full faithfulness for overconvergent F-isocrystals, *Geometric Aspects of Dwork Theory*, de Gruyter, 2004, 819–835.

[Ked06] K. S. Kedlaya, Fourier transforms and p-adic "Weil II", *Compos. Math.* **142** (2006), 1426–1450.

[Ked11] K. S. Kedlaya, Swan conductors for p-adic differential modules. II: global variation, *J. Inst. Math. Jussieu* **10** (2011), 191–224.

[Ked16] K. S. Kedlaya, *Notes on isocrystals*, arXiv:1606.01321 (2016).

[Ked18] K. S. Kedlaya, *Étale and crystalline companions*, arXiv:1811.00204 (2018).

[KM16] J. Kramer-Miller, *The monodromy of F-isocrystals with log-decay*, arXiv:1612.01164 (2016).

[Laf02] L. Lafforgue, Chtoucas de Drinfeld et correspondance de Langlands, *Invent. Math.* **147** (2002), 1–241.

[LN59] S. Lang and A. Néron, Rational points of abelian varieties over function fields, *Amer. J. Math.* **81** (1959), 95–118.

[LP90] M. Larsen and R. Pink, Determining representations from invariant dimensions, *Invent. Math.* **102** (1990), 377–398.

[LP92] M. J. Larsen and R. Pink, On ℓ-independence of algebraic monodromy groups in compatible systems of representations, *Invent. Math.* **107** (1992), 603–636.

[LP95] M. Larsen and R. Pink, Abelian varieties, ℓ-adic representations, and ℓ-independence, *Math. Ann.* **302** (1995), 561–580.

[Laz17] C. Lazda, *A note on effective descent for overconvergent isocrystals*, arXiv:1706.05300 (2017).

[Lev68] M. Levine, On the Group of Rational Points on Elliptic Curves Over Function Fields, *Amer. J. Math.* **90** (1968), 456–462.

[Mil15] J. S. Milne, *Algebraic Groups* (v2.00), available at www.jmilne.org/math (2015).

[Moc04] T. Mochizuki, *A characterization of semisimple local system by tame pure imaginary pluri-harmonic metric*, arXiv:0402122 (2004).

[Ogu84] A. Ogus, F-isocrystals and de Rham cohomology II–Convergent isocrystals, *Duke Math. J.* **51** (1984), 765–850.

[Pal15] A. Pál, *The p-adic monodromy group of abelian varieties over global function fields of characteristic p*, arXiv:1512.03587 (2015).

[Rös11] D. Rössler, *Torsion points of abelian varieties in the perfect closure of a function field*, question on the site Mathoverflow.net, available at https://mathoverflow.net/questions/73867.

[Rös17] D. Rössler, *On the group of purely inseparable points of an abelian variety defined over a function field of positive characteristic II*, arXiv:1702.07142 (2017).

[Ser66] J.-P. Serre, Zeta and L-functions, *Arithmetical Algebraic Geometry* (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, 1966, 82–92.

[Ser68] J.-P. Serre, Groupes de Grothendieck des schémas en groupes réductifs déployés, *Inst. Hautes Etudes Sci. Publ. Math.* **34** (1968), 37–52.

[Ser00] J.-P. Serre, Lettres à Ken Ribet du 1/1/1981 et du 29/1/1981, *Œuvres - Collected Papers IV*, Springer–Verlag, Heidelberg, 2000, 1–12.

[Shi08] A. Shiho, *Relative log convergent cohomology and relative rigid cohomology II*, arXiv:0707.1743 (2008).

[Sti06] J. Stix, A general Seifert-Van Kampen theorem for algebraic fundamental groups, in *RIMS* **42** (2006), 763–786.

[Tsu02] N. Tsuzuki, Morphisms of F-isocrystals and the finite monodromy theorem for unit-root F-isocrystals, *Duke Math. J.* **111** (2002), 385–418.

[Tsu17] N. Tsuzuki, *Constancy of Newton polygons of F-isocrystals on abelian varieties and isotriviality of families of curves*, arXiv:1704.00856 (2017).

[Wie06] G. Wiesend, A construction of covers of arithmetic schemes, *J. Number Theory* **121** (2006), 118–131.

Selbständigkeitserklärung

Hiermit versichere ich, Marco D'Addezio,

- dass ich alle Hilfsmittel und Hilfen angegeben habe,
- dass ich auf dieser Grundlage die Arbeit selbständig verfasst habe,
- dass diese Arbeit nicht in einem früheren Promotionsverfahren eingereicht worden ist.

Berlin, den 03.12.2018
Marco D'Addezio