The Hysteretic Limit of a Reaction-Diffusion System with a Small Parameter

eingereichte Dissertation am Fachbereich Mathematik und Informatik der Freien Universität Berlin.

vorgelegt von

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August 2018
(revidiert März 2019)

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Disputations Datum: 18. Oktober 2018

English Summary

This thesis is concerned with the study of a reaction-diffusion system with a nonlinearity that obeys a hysteresis law. This law is realized as an ensemble of scalar hysteresis operators, one defined at each spatial point and operating independently of one another. This independent ensemble approach is inspired by initial biological applications where combinations of diffusing and non-diffusing substances interact according to a hysteresis law.

The individual operators are either non-ideal relays or solutions to an ordinary differential equation with a small parameter. Under a very general condition called *spatial transversality* we prove the well-posedness of the system with non-ideal relays and that it is approximated by the system with ordinary differential equations as the parameter goes to zero. For the first time in the partial differential equation setting, we prove explicit asymptotics with respect to this parameter.

This work was supported by the *Project A9: Reaction-diffusion systems: hysteresis and nonlocal interactions* as part of Collaborative Research Center 910 *Control of self-organizing nonlinear systems: Theoretical methods and concepts of application.*

Acknowledgements

Whether I shall turn out to be the hero of my own life, or whether that station will be held by anybody else, these pages must show.

Opening Sentence of *David Copperfield* by *Charles Dickens.*

As with any undertaking of this magnitude, the name of the author is merely a proxy for the many, many people who helped along the way.

I am above all indebted to my supervisor Pavel Gurevich, who continued to provide scientific insight, even when my incomprehension tested his patience. Thank you for giving me the opportunity to complete a PhD.

To the rest of my colleagues in the Nonlinear Dynamics Group, my deepest thanks. In particular my fellow junior researchers Isabelle Schneider, Alejandro López Nieto, Yuya Tokuta, Nicola Vassena, Jia-Yuan Dai, Adem Güngör, Hannes Stuke, Nikita Begun, Ignacio Gonzalez Betegon and Phillipo Lappicy. Your feedback and good company was much appreciated.

I would like to thank Christian Kuehn for introducing me to Slow-Fast Systems, Professor Bernold Fiedler for his exhausting but fruitful seminars, Dmitry Rachinskiy for his offer to referee this work, and Sergey Tikhomirov for his interest in my research and the many deep insights he offered.

To those who kept me sane the whole time. My family who constantly sent me photos of my nephews and who were willing to Skype in the wee hours. The Forever Alone Club and their noble steeds, the Charité Chor Berlin for a weekly break from mathematics, and the countless other friends who shared the last four years with me.

I am most grateful.

Contents

Selected Notation Index

Basic Terminology:

$B_\delta(X)$, $\overline{B}_\delta(X)$	Open/closed ball respectively of width δ centered at X.
π_u, π_v	Projection onto the u and v axes respectively.

Constants:

q	Chosen in accordance with the regularity of the initial data φ; Condition 1.2.14.
γ	Chosen in accordance with the regularity of the initial data φ; Condition 1.2.14.
u_∞, v_∞	Global L_∞ bounds on solutions to (1.2.6)–(1.2.8); Theorem 1.2.18.
C_{dx}	Lower bound on a spatial derivative; Definition 1.2.20 and Lemma 3.2.2.
C_{space}	Width of region in x where simple transversality applies; Definition 1.2.20.
C_{strong}	The constant in inequality 2.1.5.
C_{Lip}	Lipschitz constant of $H^-(\cdot)$ in A^-_{hyp}; Lemma 5.2.1.
C_γ, $c(\gamma)$	Upper bounds on the on the regularity of the forcing term the forced fast-ODE; Condition 6.1.2.

Hysteresis, Nonlinearities, Apriori Estimates:

$\mathcal{H}(\cdot,\cdot)$	The non-ideal relay; Definition 1.2.2.
$f(u,v)$	$f:\mathbb{R}^2\to\mathbb{R}$. The nonlinearity of the fast variable (1.2.6); Condition 1.2.6.
f_0	The nonlinearity of the semilinear auxiliary problem (2.2.1)–(2.2.3); see Theorem 2.2.3 for the general result and Theorem 3.3.1 for an important specific choice of f_0.
$P^\lambda(Q_T)$	The set on which many apriori estimates are valid; Definition 3.1.1.

Parts of the ODE Phase Space that do not scale with ε:

$A^{\pm}_{\text{quad}}$	A neighborhood of the fold points where $f(u,v)$ is exactly equal to a quadratic function; Condition 1.2.6.
$A^{\pm}_{\text{crit}}$	Critical Region. A neighborhood of the fold point whose width is determined by the paramaters C_{dx} and C_{space}; Condition **??**.
$A^{\pm}_{\text{hyp}}$	Normally Hyperbolic Region. A set containing a normally hyperbolic piece of $H^{\pm}$, i.e., it does not contain a fold point; Lemma 5.2.1.
$\mathcal{S}_{\text{fast}}$	A section at the edge of A^{-}_{crit}. $f(u,v) > 0$ for all (u,v) in a neighborhood of this section; Definition 5.3.1.
$\mathcal{S}_{\text{drop}}$	A section that is used as initial data for trajectories of (6.1.1) that are in A^{+}_{hyp} but not yet in $\chi^{+}_{\text{hyp}}(\varepsilon)$; Definition 5.3.1.

Parts of the ODE Phase Space that Scale with ε:

$\chi^{-}_{\text{hyp}}(\varepsilon)$, $\chi^{+}_{\text{hyp}}(\varepsilon)$	The normally hyperbolic sleeve. An area contained inside of $A^{\pm}_{\text{hyp}}$; Definition 5.2.4.
$\chi^{-}_{\text{crit}}(\varepsilon)$	The critical sleeve. Contained inside of A^{-}_{crit}. Adjoined to $A^{-}_{\text{hyp}}(\varepsilon)$; Definition 5.1.8.
D_j, δ_j	Determine the height of $\chi^{-}_{\text{crit}}(\varepsilon)$; Definition 5.1.6.
d, δ_{nh}, δ_J	Determines the height of $\chi^{-}_{\text{hyp}}(\varepsilon)$ and $\chi^{+}_{\text{hyp}}(\varepsilon)$.

1. Introduction

This thesis is concerned with the study of a reaction-diffusion equation with a nonlinearity that obeys a hysteresis law. This law is realized as an ensemble of scalar hysteresis operators, one defined at each spatial point and operating independently of one another. This independent ensemble approach is inspired by initial biological applications where combinations of diffusing and non-diffusing substances interact according to a hysteresis law.

The two constituent operators we study are a non-ideal relay which switches states instantaneously (cf. Section 1.2.1), and a Fast-ODE system that switches on a "fast" time scale compared to the evolution of its input (cf. Section 1.2.3). We encode this time scale difference in a small parameter ε where $0 < \varepsilon \ll 1$. The question of how the ODE system approximates the system with non-ideal relays as $\varepsilon \to 0$ will be a central scientific question of this thesis.

We answer this question in the affirmative by demonstrating that the system with non-ideal relays is well-posed under very general assumptions, and that it is approximated by the system of Fast-ODEs in the L_∞-norm. We also show that this convergence takes place at a rate arbitrarily close to $\varepsilon^{\frac{2}{3}}$ depending on the regularity of the initial data.

1.1. Problem Overview

These two systems straddle multiple research areas, which we have collated into several loose categories:

(i) *Biological Applications.* Applications of our mathematical problem in biology provides crucial context in so much as it provides helpful working examples for the technical exposition, as well as highlighting the relevant phenomena we should be on the look out for.

(ii) *Limiting PDE Hysteresis Problem.* This refers to a system with instantaneous jumps, the well posedness of which has thus far only been partially addressed in the literature.

(iii) *General Theory of Slow-Fast ODE Systems.* There is already substantial literature on systems of coupled "Fast" and "Slow" ODEs. Such systems also contain parameters that encode different time scales, and we will describe some of the general machinery for analysing these systems for small parameter values.

(iv) *Hysteresis Phenomena in ODEs.* Beyond the general machinery, we will highlight how hysteresis plays a role in several classical ODE settings.

1.1.1. Biological Applications

The first appearance of equations of this type came in [HJ80] and [HJP84] in an attempt to model spatio-temporal patterns observed in bacterial growth. More specifically, the emergence of a concentric ring pattern (see Figure 1.2) when several drops of a requisite amino acid are added to the center of a colony of Salmonella Typhimurium fixed to a petri dish. The top layer of agar in the dish is stiff which stops the bacteria from migrating. The authors considered a system of two diffusing and one non-diffusing substance, which in their notation reads

$$\begin{aligned} B_t &= \alpha V B, \\ H_t &= D_H \Delta H - \beta V B, \\ G_t &= D_G \Delta G - \gamma V B. \end{aligned} \tag{1.1.1}$$

In (1.1.1) B is the concentration of bacteria, H is the concentration of the amino acid (Histidine) and G is the concentration of an acidic by-product which inhibits bacterial growth, and $V(G,H)$ is a hysteresis operator describing a bacterium's internal metabollic processes.

Because the bacteria are not migrating, one only needs to know the values of G and H at a given spatial point in order to evaluate V at the point in question. If a spatial point is fixed then we should think of G and H as two positive, time varying scalar inputs that parametrize a curve in the upper left quadrant of the plane as in Figure 1.1. To define V, take two non-intersecting curves Γ_{on} and Γ_{off} in the first quadrant of the G-H plane. If $V(G,H)$ is initially equal to one, then V remains equal to one until the time parametrized curve reaches the curve Γ_{off} at which point it switches to zero. Switching from zero to one can only occur upon reaching Γ_{on}.

The authors of [HJP84] implemented a functional V with an instantaneous jump, and though their numerics agreed well with experiments, the existence and uniqueness of solutions to (1.1.1) was not addressed.

Moreover, the authors noted that it may have been more natural to treat the V as a solution to an ordinary differential equation with a small parameter ε encoding the problem's two time scales, i.e., the difference between diffusion rates (slow) and the bacteria's reaction to the changing environment (fast). The ordinary differential equation would take the form

$$\varepsilon v_t = f(u,v), \tag{1.1.2}$$

where u is a time varying input representing a bacterium's immediate environment, ε is a small parameter and the curve $f(u,v) = 0$ is a curve with two disjoint branches of stable equilibria. Such an f is often called *S-shaped*. In model (1.1.1), u would be determined by the value of the diffusing variables at the bacterium's location.

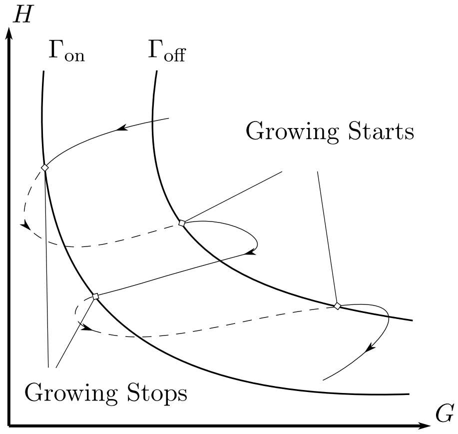

Figure 1.1.: The functional $V(G,H)$ follows the law of mass-action, i.e., the switch curves are nullclines of $G \cdot H = a_{\text{on/off}} + b_{\text{on/off}}$ for appropriate choices of $a_{\text{on/off}}$ and $b_{\text{on/off}}$.

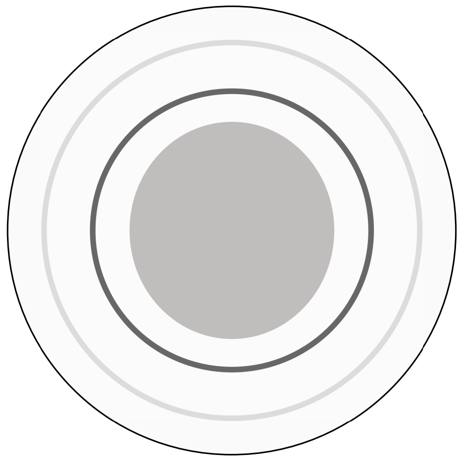

Figure 1.2.: Typical concentric ring pattern observed over a seven day incubation period. The central circle grows first, and the rings further out grow in succession with the first band from the center being particularly high density.

A familiar working example with a scalar input $u = u(t)$ is the Van der Pol oscillator $f(u,v) = v - \frac{v^3}{3} - u$ (see Figure 1.3). Moreover, the two disjoint branches indicates that the system can exhibit hysteresis (see Section 1.1.4).

The other hysteresis operator we consider is the solution v to (1.1.2), which we call the *Fast-ODE*. Of particular concern is how the solution v converges to the operator V as $\varepsilon \to 0$. If (1.1.2) takes the place of V in systems of the form (1.1.1), then we call the resulting set of equations the corresponding *Slow-Fast System*.

Such systems appeared in the study of reaction-diffusion systems where all of the species diffuse but at wildly different rates. In [Fif76] and [MTH80], the authors replaced species with slow diffusion by equations of the form (1.1.2). Because the solution v to (1.1.5) does not necessarily have spatial regularity, the authors of [Fif76] and [MTH80] were able to find solutions with jump discontinuities. Using the implicit function theorem, one can show that these solutions perturb to functions with steep smooth interfaces when a slowly diffusing term is introduced to (1.1.2).

A related question which falls beyond the scope of this thesis is to study the limit where the diffusion rates of species' with fast diffusion tends to infinity. This leads to the so called *shadow system*, the dynamics of which is treated in detail in [Kok+97]. A detailed description of the bifurcations that occur in these systems can be found in [Nis82].

More recent research on slow-fast systems can be found in [Mar03], [Mar06] and [Köt13]. Their model organism was a fresh water Polyp (or Hydra) which can regenerate into

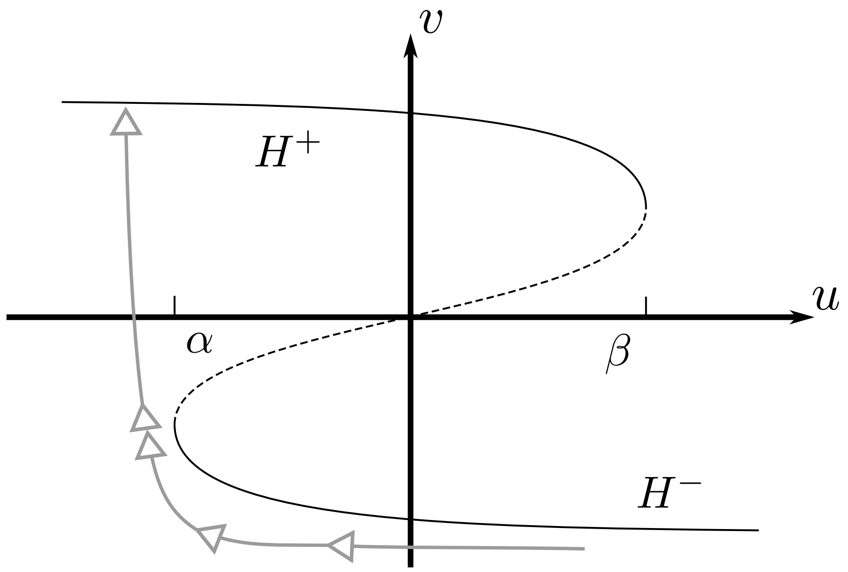

Figure 1.3.: A cubic, with input $u(t)$ whose distance from the stable branch increases as $u(t)$ approaches the critical point $u = \alpha$. After this, a fast motion occurs and the trajectory is close to the stable branch H^+. Note that the variable t parametrizes $u(t)$ and $v(t)$, where the curve $(u(t), v(t))$ is light grey. In particular, t does not correspond to one of the axes.

two separate Hydras after being cut in two. Moreover, upon having cells anterior to its central axis (the head) grafted onto cells at its posterior (the foot), a second head can grow at the site of the transplantation. A diffusing substance present in all models under consideration was a ligand which would bind to non-diffusing receptors on the cell's surface. In [Mar03] and [Mar06] the concentration of bound receptors then informs the extent to which the cell should differentiate into anterior or posterior parts of the Hydra's body.

If the bounded-receptor concentration has a monotone pattern, this represents an adult hydra without a transplant. Steady state solutions that divide the spatial domain into distinct regions of high bound-receptor concentration indicates the presence of multiple heads.

None of the models studied in [Mar03] contained hysteresis, however one the main mechanisms that was addressed was whether cells could autocatalyse additional free receptors on their surface. In a model with 1 diffusing substance, this autocatalysis was a necessary condition for Turing instabilty. Furthermore, multiple heads could be found numerically but it was necessary to increase the size of the domain. In a model with two diffusing substances without autocatalysis of free receptors, numerics indicated that the outcomes of some cutting experiments could be correctly modelled, and other simulations of random perturbations of the constant steady state evolved into a monotone pattern.

Hysteresis entered into the models of [Mar06] as the production rate of the diffusing ligands was modelled as the solution to an ODE with an S-shaped nonlinearity, i.e., one that exhibits hysteretic behavior. For a model with one diffusing substance, solutions with monotone bound-receptor concentration could be found analytically. Moreover, a model with two diffusing substances contained several numerical phenomena. Solutions with monotone bound-receptors and solutions with multiple peaks of bound-receptors could be found. However, in contrast to a model without hysteresis, the only cutting experiments that could be successfully simulated would only allow one half of the hydra to regenerate.

The last results we will discuss in this class of models was [Köt13]. The author considered one diffusing substance (the ligands) and one ODE for ligand production. When this ODE had a nonlinearity with monotone nullcline that is not S-shaped, all non-homogenous stationary solutions are unstable, but for S-shaped nullclines it is possible to construct infinitely many (for any diffusion rate) stable non-homogenous solutions with spatial discontinuities in the non-diffusing variable.

Bear in mind that the term slow-fast system is widely used when the "input variable" u is a solution to an ODE, not a PDE (see [Kue15]). This thesis contains new results for both the ODE (cf. Chapter 7) and PDE (cf. Chapters 3 and 4) case, but the distinction will always be made clear.

1.1.2. Limiting PDE Hysteresis Problem

First attempts to prove the well-posedness of (1.1.1) can be attributed to Alt [Alt85] and Visintin [Vis86]. More recent work can be found in [GTS13], [GT12] and [Cur14]. All these papers deal with the scalar reaction diffusion equation (see [AK08], [Kop06] and [GT14] for systems of equations)

$$u_t = \Delta u + v, \tag{1.1.3}$$

where $(x, t) \in Q_T := Q \times (0, T)$, Q is a bounded domain with smooth boundary, u satisfies Neumann boundary conditions and v is an ensemble of hysteresis operators defined at every spatial point. By delinieating regions in Q_T where hysteresis is is in one state or the other, (1.1.3) becomes a free boundary problem, and the regularity of u near this free boundary was studied in [AU15]. A summary of all these works can be found in [CGT16], but in the present discussion we will ask two questions: How is v defined and is the solution to (1.1.3) in such a setting unique?

Each invidual hysteresis operator responds to a continuous scalar input, i.e., for fixed $x \in Q$, $u(x, \cdot)$ is a scalar input into an operator with output $v(x, \cdot)$. For the purposes of explanation let us write $u(t)$ and $v(t)$, and let us denote the constituent scalar operators of [Alt85], [Vis86] and [GTS13] by $v = \mathcal{H}_{\text{Alt}}(\xi_0, u)$, $v = \mathcal{H}_{\text{Vis.}}(\xi_0, u)$ and $v = \mathcal{H}(\xi_0, u)$ respectively. A schematic of these scalar operators is shown in Figure 1.4. Arrows indicate how the $v(t)$ is permitted to vary for varying values of $u(t)$.

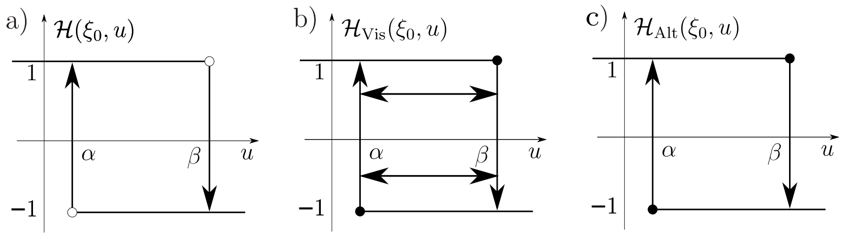

Figure 1.4.: A comparison of three hysteresis operators as displayed in [CGT16]. In this simplified diagram the counterpart to the branches of stable equilibria from Figure 1.3 are just straight lines where the hysteresis output is ± 1.

All three operators are defined in reference to two threshold values for u which are denoted α and β. If $u(t) > \beta$ then $v(t) = -1$ for all three operators, and if $u(t) < \alpha$, then $v(t) = 1$ for all three operators. Moreover, if $u(t) \in (\alpha, \beta)$, then $v(t)$ is constant in a smal neighborhood of t. If $u(0) \in [\alpha, \beta]$ then $v(0)$ is not uniquely defined, so we let $\xi_0 = v(0)$.

The key differences occurs when $u(t) \in \{\alpha, \beta\}$. Suppose for instance that $u(t^*) = \beta$, and $u(t^*) = \beta$ is a local maximum.

(i) $\mathcal{H}(\xi_0, u)(t^*) = -1$ and $\mathcal{H}(\xi_0, u)(t) = -1$ for all $t > t^*$ provided $u(t) > \alpha$.

(ii) $\mathcal{H}_{\mathrm{Vis.}}(\xi_0, u)(t^*) \in [-1, 1]$ and $\mathcal{H}_{\mathrm{Vis.}}(\xi_0, u)(t) = \mathcal{H}_{\mathrm{Vis.}}(\xi_0, u)(t^*)$ for $t > t^*$ provided $u(t) \in (\alpha, \beta)$.

(iii) $\mathcal{H}_{\mathrm{Alt}}(\xi_0, u)(t^*) \in [-1, 1]$ and $\mathcal{H}_{\mathrm{Alt}}(\xi_0, u)(t)$ is non-decreasing for t in a neighborhood of t^*.

Note that $\mathcal{H}_{\mathrm{Vis.}}(\xi_0, u)$ and $\mathcal{H}_{\mathrm{Alt}}(\xi_0, u)$ are not uniquely defined.

The operator $\mathcal{H}(\cdot, \cdot)$ is called the non-ideal relay. The precise Definition used in this thesis is given in Section 1.2.1.

Returning to the PDE setting with u and v taking values in Q_T and $Q \subset \mathbb{R}^n$, the author of [Vis86] proves that there exists a solution to a suitable weak formulation of (1.1.3) with $v = \mathcal{H}_{\mathrm{Vis.}}(\xi_0, u)$ and $n \geq 1$. In [Alt85], the author proves that there exists a solution to a suitable weak formulation of (1.1.3) with $v = \mathcal{H}_{\mathrm{Alt}}(\xi_0, u)$ and $n = 1$. We will omit the technical details and refer the reader to [CGT16] for more information.

The case of $v = \mathcal{H}(\xi_0, u)$ depends crucially on the initial data for u, which we denote by $\varphi : Q \to \mathbb{R}$. In [GTS13; GT12], the authors proved the existence and uniqueness of a solution to (1.1.3) for a class of initial data referred to as *transverse initial data*, which give rise to *transverse solutions*. Loosely speaking this means that if $u(x,t) \in \{\alpha, \beta\}$ and $v(x,t)$ has a discontinuity at (x,t), i.e., there is no neighborhood of (x,t) where v is constant, then $\nabla_x u(x,t) \neq 0$. These solutions are "strong" in the sense that u has two weak derivatives in x one in t that belong $L_q(Q_T)$ (the space of q integrable functions).

The key result is that transverse solutions are in fact unique. This is also the case for systems of diffusing and non-diffusing substances [GT14], and there are partial results for $n \geq 2$ [Cur14]. For the $n \geq 2$ case, the point of discontinuity of v could possibly be an $n-1$ dimensional manifold, and in [Cur14] an additional regularity assumption on this manifold was required. In the current work, we will prove well-posedness for $n \geq 2$ requiring only that $\nabla_x u(x,t) \neq 0$ where v is discontinuous.

1.1.3. General Theory of Slow-Fast ODE Systems

Though we will only study Fast-ODEs coupled with a PDE, there is an expansive literature on Fast-ODEs couple with a second ODE, appropriately called the Slow-ODE. We will discuss three sub topics that most resemble properties of the PDE system. Let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ satisfy the equations

$$\dot{u} = g(u,v), \tag{1.1.4}$$

$$\varepsilon \dot{v} = f(u,v). \tag{1.1.5}$$

If we let $\tau = \frac{t}{\varepsilon}$ and $'$ denote differentiation with respect to τ we also have

$$u' = \varepsilon g(u,v), \tag{1.1.6}$$

$$v' = f(u,v). \tag{1.1.7}$$

For $\varepsilon > 0$, (1.1.4)–(1.1.5) is equivalent to (1.1.6)–(1.1.7). Let $\overline{u} \in \mathbb{R}^m$ and $\overline{v} \in \mathbb{R}^n$ with $(\overline{u}, \overline{v}) := \overline{m} \in \mathbb{R}^{m+n}$ denoting initial data. Moreover, let $\psi_{[t_1,t_2]}(\overline{m})$ be the piece of the trajectory between t_1 and t_2 of the initial data $\overline{m} \in \mathbb{R}^{m+n}$ under the flow of (1.1.6)–(1.1.7).

Fenichel Theorems

For $\varepsilon = 0$, (1.1.6)–(1.1.7) is called the fast subsystem. Of particular importance is the set

$$\mathcal{M}_{\text{crit}} := \{(\overline{u}, \overline{v}) \in \mathbb{R}^{n+m} \mid f(\overline{u}, \overline{v}) = 0\}. \tag{1.1.8}$$

This is just the set of equilibria of the fast subsystem (1.1.7) with the variable u treated as a parameter. We call a connected compact subset of $M_0 \subset \mathcal{M}_{\text{crit}}$ *normally hyperbolic* if for every $(\overline{u}, \overline{v}) \in M_0$ the linearization $D_v f(\overline{u}, \overline{v})$ is hyperbolic (no purely imaginary eigenvalues). Note that by the implicit function theorem $\mathcal{M}_{\text{crit}}$ is locally the graph of some function, e.g., $\overline{u} \mapsto H^-(\overline{u})$ and the $\overline{v}$ coordinate is the normal direction to this graph.

Note that the term *normally hyperbolic* has a more specific meaning that compares the expansion/contraction of (1.1.4)–(1.1.5) in both u and v directions [Fen71], however, for ε sufficiently small slow-fast systems always fulfil these criteria.

Given any $M_0 \subset \mathbb{R}^{n+m}$, we say that M_0 is locally invariant if there exists a neighborhood $\mathcal{O}$ of M_0 such that for all $\overline{m} \in M_0$, $\psi_{[0,t]}(\overline{m}) \subset \mathcal{O}$ implies that $\psi_{[0,t]}(\overline{m}) \subset M_0$, and $\psi_{[-t,0]}(\overline{m}) \subset \mathcal{O}$ implies that $\psi_{[-t,0]}(\overline{m}) \subset M_0$. In other words a trajectory cannot leave M_0 without also leaving $\mathcal{O}$.

The Fenichel Theorems relate M_0 to a nearby invariant manifold M_ε of (1.1.6)–(1.1.7) when $\varepsilon > 0$ is sufficiently small. The Theorems are valid for $\mathbb{R}^n$ with $n \geq 2$. An overview of these results can be found in [Jon95].

Theorem 1.1.1 ([Jon95, Thm. 1],[Kue15, Thm. 3.1.4]). *Suppose that M_0 is compact, C^r, and normally hyperbolic. Then there is an ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, there is a manifold M_ε that is:*

(i) Diffeomorphic to M_0.

(ii) Is C^r for any $r < \infty$, including in ε.

(iii) Hausdorff distance $O(\varepsilon)$ from M_0.

(iv) Locally invariant.

Let $W^s(M_0)$ and $W^u(M_0)$ be the union of the stable (resp. unstable) manifolds of the set equilibria M_0 of (1.1.7). Suppose that the negative eigenvalues of $D_v f(\overline{u}, \overline{v})$ are bounded above by $\alpha_s < 0$, uniformly for every $(\overline{u}, \overline{v}) \in M_0$. Because M_0 is connected and normally hyperbolic, it follows that the number of negative eigenvalues, denoted n_s, is independent of $(\overline{u}, \overline{v}) \in M_0$.

Theorem 1.1.2 ([Jon95, Thm. 3]). *Under the assumptions of Theorem 1.1.1, there is an ε_0 such that if $0 < \varepsilon < \varepsilon_0$, there exists manifolds $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ that are diffeomorphic to and lie within $O(\varepsilon)$ of $W^s(M_0)$ and $W^u(M_0)$ respectively. They are both locally invariant and C^r, including C^r in the paramater ε.*

Theorem 1.1.3 ([Jon95, Thm. 5]). *For every $\overline{m} \in W^s(M_\varepsilon)$, there is a constant C_s*

$$\operatorname{dist}(\psi_{[0,t]}(\overline{m}), M_\varepsilon) \leq C_s \exp(\alpha_s t),$$

as long as $\psi_{[0,t]}(\overline{m}) \subset \mathcal{O}$. A similar statement for $t < 0$ holds for $W^u(M_0)$.

Due to the implicit function theorem (cf. Appendix C), we can locally consider M_0 as being a graph over some subset $\mathcal{O}_u \subset \mathbb{R}^m$. For sufficiently small ε_0, so too is M_ε. Let H^0 and H^ε be two functions such that

$$\begin{aligned} M_0 &= \{(\overline{u}, H^0(\overline{u})) \mid \overline{u} \in \mathcal{O}_u\}, \\ M_\varepsilon &= \{(\overline{u}, H^\varepsilon(\overline{u})) \mid \overline{u} \in \mathcal{O}_u\}. \end{aligned}$$

Theorem 1.1.4 ([Jon95, Thm. 6]). *For every $(\overline{u}, H^\varepsilon(\overline{u})) \in M_\varepsilon$, there is an n_s-dimensional manifold $W^s((\overline{u}, H^\varepsilon(\overline{u}))) \subset W^s(M_\varepsilon)$, that is Hausdorff distance $O(\varepsilon)$ from $W^s(H^0(\overline{u}))$. This manifold is locally invariant in the sense that*

$$\psi_{[0,t]}\Big(W^s((\overline{u}, H^\varepsilon(\overline{u})))\Big) \subset W^s(\psi_{[0,t]}\Big((\overline{u}, H^\varepsilon(\overline{u}))\Big),$$

provided that $\psi_{[0,t^*]}((\overline{u}, H^\varepsilon(\overline{u}))) \subset \mathcal{O}$ for every $t^* \in [0,t]$.

Asymptotic Expansions near Fold Points

This is the first of two methods for studying (1.1.6)–(1.1.7) at points where $\mathcal{M}_{\text{crit}}$ is not normally hyperbolic. These techniques are not well studied in arbitrarily high dimensions [Kue15], so we will only discuss the planar case $u \in \mathbb{R}$, $v \in \mathbb{R}$. Let $(0,0) \in \mathcal{M}_{\text{crit}}$ satisfy $f'(0,0) = 0$, $f'(0,0) > 0$ and $g'(0,0) < 0$. Such a critical point is called a generic fold. As an example consider Figure 1.11 where $A^{\pm}_{\text{quad}}$ are neighborhoods of generic folds.

For generic folds it is always possible to change coordinates such that in a neighborhood of $(0,0)$, $f(u,v) = v^2 - u$ and

$$g = -1 + cv + O(v^2, u), \tag{1.1.9}$$

so let's assume from the outset that this is the case (see [Kue15, Chap.4] and the references therein). Note that for any initial condition where the Fast-ODE is normally hyperbolic ($u > 0$), (1.1.9) implies that such an initial condition will drift below the fold point, inevitably resulting in a bifurcation. It is also clear that any trajectory of (1.1.6)–(1.1.7) can be written in the form $u = u(v)$. [MR80] attempts to find an asymptotic expansion for u in terms of ε raised to some power. The exact form of the expansion depends crucially on the proximity to the fold, also expressed as ε raised to some power.

The authors refer to the region $-\varepsilon^{\mu} \leq v \leq -C$ with $\mu \in (0, \frac{1}{3})$ as the *initial part* of the trajectory in a neighborhood of the fold (see Figure 1.5). For such values of u, the authors prove that $u(v)$ has an asymptotic expansion

$$u(v) = v^2 + \sum_{j=1}^{J} v_j \varepsilon^j + O(\varepsilon^{J+1-\mu(3J+1)}). \tag{1.1.10}$$

Since the hysteresis branch is just the 0-th order approximation, we are interested in the case

$$u(v) = v^2 + O(\varepsilon^{1-\mu}). \tag{1.1.11}$$

More specifically we want to know how well v approximates the hysteresis branch, so if $u > \varepsilon^{2\mu}$ then the displacement of v from the hysteresis branch is $\varepsilon^{1-2\mu}$.

In the neighborhood $(0,0)$ where $|v| \leq \varepsilon^{\frac{1}{3}}$ the expansion (1.1.10) is no longer valid, and an asymptotic expansion is developed in powers $\varepsilon^{-\frac{1}{3}}$. In fact such an expansion needs to be split into smaller expansions depending on if v is positive or negative, but the more pertinent question is what to take as the 0-th order approximation.

Under the rescaling

$$\begin{aligned} u &= \varepsilon^{\frac{2}{3}} U, \\ v &= \varepsilon^{\frac{1}{3}} V, \\ t &= \varepsilon^{\frac{2}{3}} \tau. \end{aligned} \tag{1.1.12}$$

One obtains an equation for V of the form

$$V_\tau = -U + V^2. \tag{1.1.13}$$

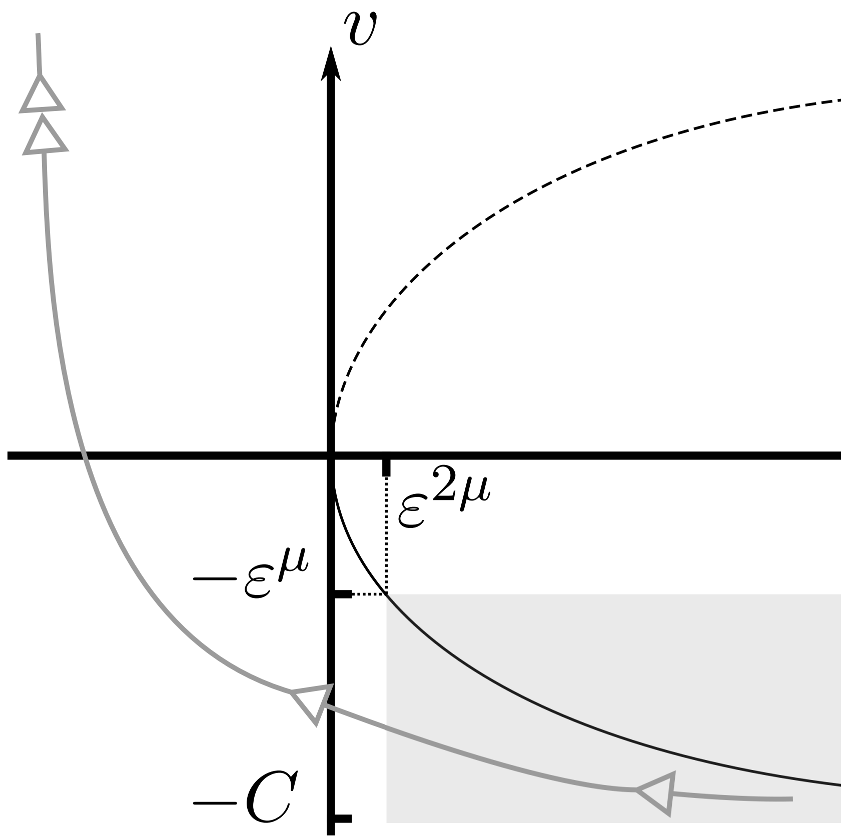

Figure 1.5.: The generic fold with a trajectory that transitions from slow to fast motion (grey line). The piece of the trajectory in the shaded region is the *initial part* of the trajectory (in the nomenclature of [MR80]).

The equation for U_τ depends on the higher order terms of (1.1.9). One can divide U_τ by V_τ, and substituting into the author's proposed asymptotic expansion [MR80, Chap. 2,(10.5)] which we omit here, the equation for the 0th-order approximation reads

$$\frac{dU}{dV} = \frac{1}{V^2 - U}. \tag{1.1.14}$$

Note that this equation is defined with $V \in \mathbb{R}$, i.e., when $\varepsilon = 0$ the domain of the 0th-order approximation becomes unbounded. (1.1.14) is a Riccati equation that can be solved in terms of Bessel functions, and there is a unique solution that is asymptotic to the parabola $V^2 = U$ for negative values of V.

Blow-up Techniques

An alternative method for studying (1.1.6)–(1.1.7), first used by [DRR96] and later by [KS01a], was based on making a coordinate change in a neighborhood of the fold point that is singular at $(0, 0)$. In a sense, we already know the behavior of the system at $(0, 0)$; it is an equilibrium. As such, we are not compelled to make a coordinate change that is defined there. A simple example would be changing from Cartesian to polar coordinates, though the variant that is often applied actually blows up $(0, 0)$ to a circle S^1.

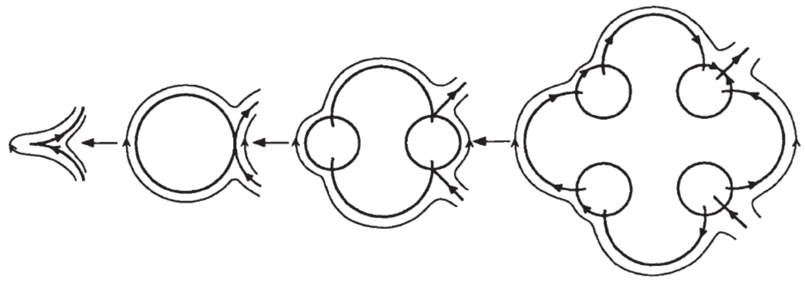

Figure 1.6.: Unfolding a cusp singularity as shown in [Dum93][Fig.2]. A series of polar coordinate changes creates new equilibria which eventually desingularize the vector field.

Assume that $(0, 0)$ is mapped under the blow-up map to a set A. The central idea of the blow-up technique is that after applying the blow-up map to the non-hyperbolic fixed point $(0, 0)$, one can always extend the vector field to A but might have introduced additional equilibria on A itself. One can hope that these equilibria will be hyperbolic.

Fortunately, Dumotier [Dum77] proved that if one keeps applying blow-up maps to any newly created non-hyperbolic equilibria, a large class of planar vector fields eventually become desinguralized vector field on $\mathbb{R}^2 \backslash A$, where A is homeomorphic to A. See Figure 1.6 for a schematic of this technique applied to a cusp singularity. In general, all critical poins on ∂A are either

(i) Isolated, and if the critical point is not hyperbolic, the vector field is not identically zero on its center manifold.

(ii) Smooth closed curves along which the vector field is normally hyperbolic.

In [DRR96], the authors designularized the generic fold as it appears in the Van der Pol equation

$$\dot{u} = \varepsilon(v - a), \tag{1.1.15}$$

$$\dot{v} = u - \frac{v^2}{2} - \frac{v^3}{3}, \tag{1.1.16}$$

$$\dot{\varepsilon} = 0. \tag{1.1.17}$$

In the desingularized coordinates, they proved the already well known [Eck83] existence of Canard orbits, i.e., orbits that follow the unstable branch of $\mathcal{M}_{\text{crit}}$ for an appropriately defined "long" time. Note that in (1.1.15)–(1.1.17) the singularity $(u, v, \varepsilon) = (0, 0, 0)$ blows up to S^2, but there is no general theorem guaranteeing this is possible in $\mathbb{R}^3$.

Because these orbits are close to the critical manifold $\mathcal{M}_{\text{crit}}$ of (1.1.16), with the exception of jumps between attracting branches, the approach of [KS01b] was to use the desingularized vector field only near the fold point, and to use Fenichel Theorems to track the orbits near the normally hyperbolic parts of $\mathcal{M}_{\text{crit}}$. This required linking the manifolds M_ε to orbits on the local center manifold of $(0, 0, 0)$, which was the work of [KS01a].

Tracking orbits across S^2 meant using three different coordinate charts. In the chart normal to the ε-axis, the desingularized vector field takes the form of (1.1.14), and the analysis mimics that of [MR80]. A schematic of the blown-up phase space of (1.1.15)–(1.1.17), including M_ε is shown in Figure 1.7, with an overline indicating one is in the blow-up coordinates.

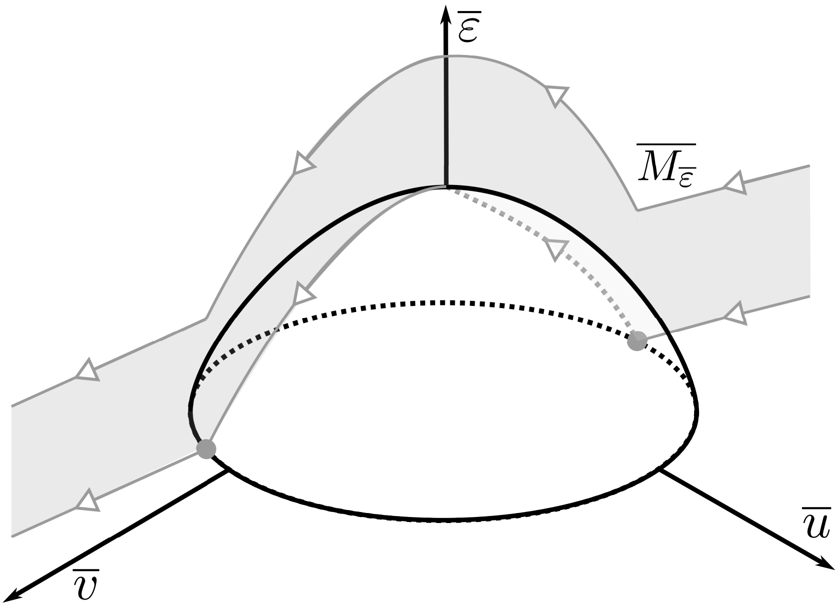

Figure 1.7.: The blown-up phase space of (1.1.15)–(1.1.17). On the set $\{\overline{\varepsilon} = 0\}$ there are four hyperbolic equilibrium points on the circumfrence of the circle. One of these connects to the blown up representation of M_ε (shaded grey), which by an abuse of notation we have also used to label the trajectories that traverse the sphere and exit via the second equilibrium point. This diagram is a simplified reproduction of [KS01a, Fig. 2.7] with our notation.

1.1.4. Hysteresis Phenomena in ODEs

ODE Approximation of Hysteresis Operators

The question of whether hysteresis operators could be approximated using Fast-ODEs was first attributed to Nuteschil (in the Russian literature), however the results are

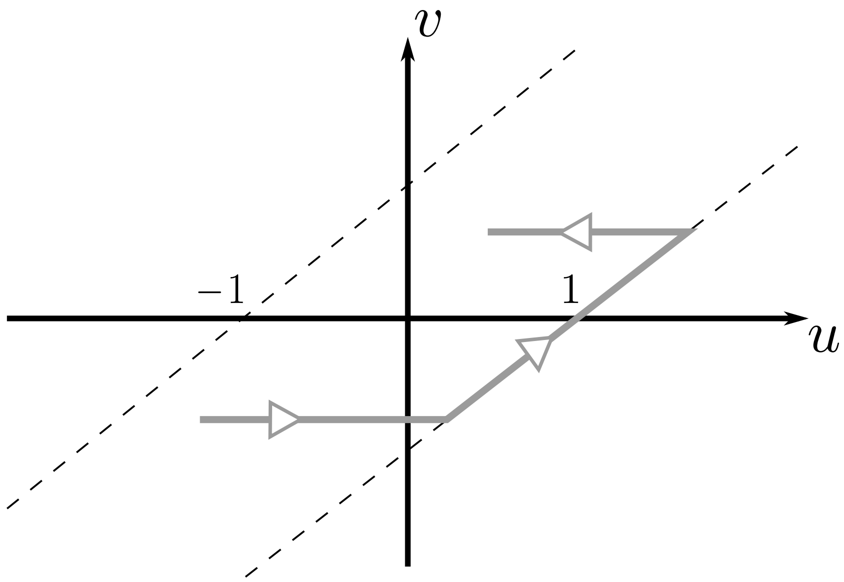

Figure 1.8.: A comparison of three hysteresis operators as displayed in [CGT16]. The counterpart to the branches of stable equilibria from Figure 1.3 are just straight lines where the hysteresis output is ± 1

formulated in English in [KNP12]. These results in fact concern the Play operator, a schematic of which is shown in Figure 1.8.

The value of $\mathcal{H}_{\text{Play}}(\xi_0, u)$ remains constant unless it reaches one of the two lines Γ_+ or Γ_-. To formulate Nuteschil's result, let ψ be the function such that

$$\begin{aligned} &\psi(v) = v + 1, \text{ if } v \leq -1, \\ &\psi(v) = 0, \text{ if } v \in (-1, 1), \\ &\psi(v) = v - 1, \text{ if } v \geq 1, \end{aligned} \tag{1.1.18}$$

and let v be the solution to

$$\varepsilon v_t = \psi(u - v). \tag{1.1.19}$$

Nuteschil's Theorem as stated in [KNP12] is that

$$\lim_{\varepsilon \to 0} \|\mathcal{H}_{\text{Play}}(\xi_0, u) - v\|_{L_\infty(0,T)} = 0. \tag{1.1.20}$$

If u is the solution to an additional ODE coupled to v, then an appropriate reformulation of (1.1.20) was recently obtained in [KM17].

Similar results for the non-ideal relay can be found [LSK93] and [Kre05]. Here the authors study a Fast-ODE with a forcing of the form

$$\varepsilon \dot{v} = h(v) - u. \tag{1.1.21}$$

where h is a negative cubic function such that in the language of Section 1.1.3, $\mathcal{M}_{\text{crit}}$ has two branches of stable equilibria, H^+ and H^-, and two folds at $u = \alpha$, $u = \beta$ (cf. Figure 1.3). Moreover, u is a given regulated function, i.e., for every $t \in (0, T)$, both one-sideded limits $u(s) \to u(t)_+$ and $u(s) \to u(t)_-$ exist. The result we state here also assumes that u is left continuous.

Theorem 1.1.5 ([Kre05]). *Let $[t_1, t_2] \subset (0, T)$. If $u(t_1^+) > \beta$ and $u(t) \geq \alpha$ for every $t \in [t_1, t_2]$, then for every $t^* \in (t_1, t_2]$ there exists an $\varepsilon_0 > 0$ such that*

$$v^\varepsilon(t) > H^-(\alpha), \text{ for all } t \in [t^*, t_2], \;\; \varepsilon \in (0, \varepsilon_0].$$

Moroever, one has

$$\lim_{\varepsilon \to 0^+} v^\varepsilon(t) = H^-(u(t)).$$

Note that under this hypothesis the non-ideal relay never switches back to H^+ because $u(t) \geq \alpha$ for $t \in [t_1, t_2]$. This Theorem does not make conclusions about the rate of convergence with respect to ε.

Implicit Hysteresis Phenomena in Slow-Fast ODEs

We have already observed in Section 1.1.1, that in the applications we would like to study the stable equilibria of the fast subsystem form the branches of the non-ideal relay $\mathcal{H}(\cdot, \cdot)$ we defined in Section 1.1.2. The slow subsystem is the differential algebraic equation obtained by taking $\varepsilon = 0$ in (1.1.4)–(1.1.5). One would expect the slow subsystem to approximate the output of the operator $\mathcal{H}(\xi_0, u)$.

Several classical slow-fast ODE systems have solutions where several mutually disjoint components of the trajectory spend an $O(1)$ amount of time near one of the two branches of $\mathcal{H}(\xi_0, u)$. In certain cases this corresponds to $\mathcal{H}(\xi_0, u)$ switching from one branch to the other, and then switching back.

The Van der Pol oscillator is one such example, the full form of which is

$$\dot{u} = v, \tag{1.1.22}$$

$$\varepsilon \dot{v} = v - \frac{v^3}{3} - u. \tag{1.1.23}$$

Periodic orbits of (1.1.22)–(1.1.23) that switch between fast and slow motion are called as *relaxation oscillations*. In [MR80], the authors use the stable branches of 1.1.23 as the 0th-order approximation in their asymptotic expansions of the solution segments away from the fold points. When 1.1.23 has a large amplitude sinusodal forcing term, there are solutions whose period is a multiple of the driving frequency [GRA80], [GNV84]. These authors also use the stable branches as the 0th-order approximation in asymptotic expansions of solution segments away from the fold point.

In Section 1.1.3, we alluded to [KS01b] which combined the blow up technique with the Fenichel theorems to construct the Canard cycle. The same technique of linking slow manifolds with their extension's near the fold points can be used to find relaxation oscillations. It was imperative to know the rates at which orbits are attracted to the slow manifolds (and their extensions near the fold point) to allow one to construct a contraction mapping on a section perpendicular to one the stable branches.

The FitzHugh-Nagumo equations model electrical pulses propogating along a nerve fibre [Jon84]. They are a set of reaction-diffusion equations with a diffusing fast variable,

$$u_\tau = \varepsilon(v - \delta u), \tag{1.1.24}$$

$$v_\tau = v_{xx} + h(v) - u, \tag{1.1.25}$$

where h is a cubic function, $x \in \mathbb{R}$ and $0 < \epsilon, \delta \ll 1$. Even though the general model falls outside the scope of the thesis (for our purposes the slow variable diffuses), a travelling wave solution turns (1.1.24)–(1.1.25) into a 3 dimensional ODE. If we define the variable $y = x - ct$ for some $c > 0$, then a travelling wave solution is a solution of the form $u(x,t) = u(y)$ and $v(x,t) = v(y)$. The goal is to find solutions of the system

$$u_y = -\frac{\varepsilon}{c}(v - \delta u), \tag{1.1.26}$$

$$v_y = w, \tag{1.1.27}$$

$$w_y = -cw - h(v) + u. \tag{1.1.28}$$

Solutions that satisfy $(u, v, w) \to (0, 0, 0)$ as $y \to \pm\infty$ are called travelling pulses, in other words, they are homoclinic solutions of (1.1.26)–(1.1.28).

Homoclinic solutions to (1.1.26)–(1.1.28) have been found using a number of different methods. In [Has76] the author defines a plane in $\mathbb{R}^3$ which cuts through the middle of the set

$$\{(h(v) + cw, v, w) \in \mathbb{R}^3 \mid (v, w) \in \mathbb{R}^2\},$$

If c is small this is approximately slow subsystem of (1.1.23)–(1.1.23). From here one can show that the set of parameters (ε, c) where there is an orbit that cuts the plane twice, has two connected components. One can prove the simultaneous existence of pulses of different speeds.

[Car77] constructs isolating blocks around the stable branches of 1.1.27 and 1.1.28, and uses results attributed to [CE71] to infer the existence of k-pulses, i.e., pulses that make several loops before converging to $(0, 0, 0)$.

[Jon84] proves stability of pulse solutions in the context of perturbations to (1.1.24)–(1.1.24). The key step is augmenting (1.1.26)–(1.1.28) with the equation

$$c_y = 0,$$

and showing that the center unstable and center stable manifolds of $(0, 0, 0, c^*)$ intersect (where c^* is the wave speed of the pulse solution). This necessitates tracking the orientation of these manifolds globally, even as the trajectory changes direction during the transitions between fast and slow motion. The technical tool used here was the *exchange lemma*, which first appeared in the PhD thesis of Robert Langer, but was not published until [JKL91] (private communication with CKRT Jones).

1.2. Technical Setting

We now introduce our technical setting. This will include fixing notation for the remainder of the thesis beginning with the declaration $\alpha = 0$.
Condition 1.2.1. Set the left most threshold $\alpha = 0$.

Condition 1.2.1 is purely for notational convenience and preventing clutter in diagrams. Conditions of mathematical importance will be declared shortly.

1.2.1. Hysteresis Operators

Definition 1.2.2 ([CGT16])**.** *Let $H^- : [0, \infty) \to \mathbb{R}$, $H^+ : (-\infty, \beta] \to \mathbb{R}$ and $\xi_0 \in \{+, -\}$. Let $u, v : [0, T] \to \mathbb{R}$, where u is a continuous function. We say that $v = \mathcal{H}(\xi_0, u)$ if the following hold:*

(i) $(u(t), v(t)) \in \{(u, H^-(u)) \mid u > 0\} \cup \{(u, H^+(u)) \mid u < \beta\}$ *for every* $t \in [0, T]$.

(ii) If $u(0) \in (\alpha, \beta)$, *then* $v(0) = H^{\xi_0}(u(0))$.

(iii) If $u(t_0) \in (\alpha, \beta)$, *then* $v(t)$ *is continuous in a neighorhood of* t_0.

If $\mathcal{H}(\xi_0, u)(t) = H^{\pm}(u(t))$, then we call $\xi(t) = \pm$ the configuration of $\mathcal{H}(\xi_0, u)$ at the moment t, and we call ξ_0 the initial configuration.

Note that we define $H^-(0)$ and $H^+(\beta)$ for later convenience, even though $\mathcal{H}(\xi_0, u)$ cannot attain these values.
Condition 1.2.3. If $u(0) < 0$ then $\xi_0 = -$, if $u(0) > \beta$ then $\xi_0 = +$.

Let $Q \subset \mathbb{R}^n$ with $n \geq 1$ be a bounded domain with smooth boundary, and for $T > 0$ let

$$Q_T = Q \times (0, T),$$

be the corresponding cylinder of height T. We will denote points in Q_T with the letters $(x, t) \in Q_T$.
Definition 1.2.4. *Let $u : Q_T \to \mathbb{R}$ and $\xi_0 : Q \to \{+, -\}$. Then $\mathcal{H}(\xi_0, u)(x, t)$ is defined as per Definition 1.2.2 by treating x as a parameter. In this case, we call $\mathcal{H}(\xi_0, u)$ spatially distributed hysteresis and $\xi : Q_T \to \{+, -\}$ the configuration function.*

1.2.2. Reaction-Diffusion Equations with Hysteresis

The functions u and v without superscript satisfy the system

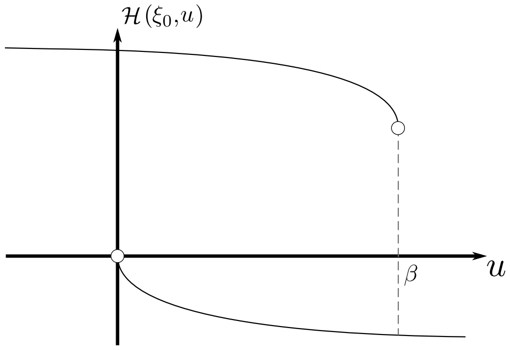

Figure 1.9.: A schematic of the non-ideal relay with the left most threshold at $u = 0$.

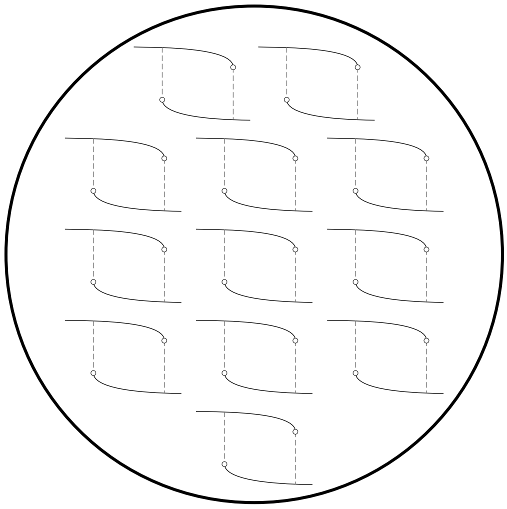

Figure 1.10.: A schematic of the spatially distributed hysteresis.

$$\begin{cases} u_t = \Delta u + v, & (1.2.1) \\ v = \mathcal{H}(\xi_0, u), & (1.2.2) \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial Q} = 0, \qquad u|_{t=0} = \varphi(x), & (1.2.3) \\ v|_{t=0} = H^{\xi_0}(\varphi). & (1.2.4) \end{cases}$$

We will also consider functions u^ε and v^ε, defined on the same spatial domain and with the same the initial data as u and v. We will only study the functions u^ε and v^ε for $n = 1$, and they will satisfy the system

$$\begin{cases} u^\varepsilon_t = u^\varepsilon_{xx} + v^\varepsilon, & (1.2.5) \\ \varepsilon v^\varepsilon_t = f(u^\varepsilon, v^\varepsilon), & (1.2.6) \\ \left.\dfrac{\partial u^\varepsilon}{\partial \nu}\right|_{\partial \Omega} = 0, \qquad u^\varepsilon|_{t=0} = \varphi, & (1.2.7) \\ v^\varepsilon|_{t=0} = H^{\xi_0}(\varphi). & (1.2.8) \end{cases}$$

It is possible to formulate the results of this thesis for the case where the initial data of the non-diffusing variable $v^\varepsilon|_{t=0}$ for $\varepsilon > 0$ also depends on ε. However, for technical clarity we have also assumed that for for every $\varepsilon > 0$, $v^\varepsilon|_{t=0}$ only takes values on the hysteresis branches.

1.2.3. Assumptions on the Fast-ODE

Definition 1.2.5 (Critical Manifold)**.** *We call the nullcline of f the critical manifold and denote it*

$$\mathcal{M}_{\text{crit}} := \{(u,v) \in \mathbb{R}^2 \mid f(u,v) = 0\}.$$

We now list our assumptions on f. Our goal is to encompass general "S-shaped" nullclines and it may be helpful to remember that the Van der Pol oscillator $f(u,v) = v - \frac{v^3}{3} - u$ satisfies the following Condition if you ignore the higher order terms near the fold point (cf. Figure 1.11).

Condition 1.2.6 (Assumptions on the Nonlinearity)**.**

(i) f is a C^2 function.

(ii) There is a C^1 function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{M}_{\text{crit}} = \{(g(v), v) \mid v \in \mathbb{R}\}.$$

(iii) There exists a real number $0 < \beta$ and two functions $H^- : [0,\infty) \to \mathbb{R}$ and $H^+ : (-\infty, \beta] \to \mathbb{R}$ such that

$$\begin{aligned}
\{(u,v) \in \mathcal{M}_{\text{crit}} \mid g'(v) < 0\} &= \{(u, H^-(u)) \mid u > 0\} \cup \{(u, H^+(u)) \mid u < \beta\}, \\
\{(u,v) \in \mathcal{M}_{\text{crit}} \mid g'(v) = 0\} &= \{(0,0), (\beta, H^+(\beta))\}, \\
\{(u,v) \in \mathcal{M}_{\text{crit}} \mid g'(v) > 0\} &= \{(g(v), v) \mid v \in (0, H^+(\beta))\}.
\end{aligned} \tag{1.2.9}$$

(iv) For every $u \in (0,\infty)$

$$\frac{\partial f}{\partial v}(u, H^-(u)) < 0, \quad \frac{\partial f}{\partial u}(u, H^-(u)) < 0.$$

Similarly, for every $u \in (-\infty, \beta)$

$$\frac{\partial f}{\partial v}(u, H^+(u)) < 0, \quad \frac{\partial f}{\partial u}(u, H^+(u)) < 0.$$

(v) There exists a neighborhood A^-_{quad} (resp. A^+_{quad}) of $(0,0)$ (resp. $(\beta, H^+(\beta))$) such that

$$\begin{aligned}
f(u,v) &= v^2 - u, \text{ for all } (u,v) \in \mathcal{M}_{\text{crit}} \cap A^-_{\text{quad}}, \\
f(u,v) &= \beta - u - (v - H^-(\beta))^2, \text{ for all } (u,v) \in \mathcal{M}_{\text{crit}} \cap A^+_{\text{quad}}.
\end{aligned} \tag{1.2.10}$$

As a simple consequence of item (iii) in Condition 1.2.6, there is a $u_\infty > 0$ such that $H^-(u_\infty) < 0$, $H^+(-u_\infty) > 0$ and $\|\varphi\|_{L_\infty(Q)} \leq u_\infty$. We fix such a $u_\infty > 0$ and then fix

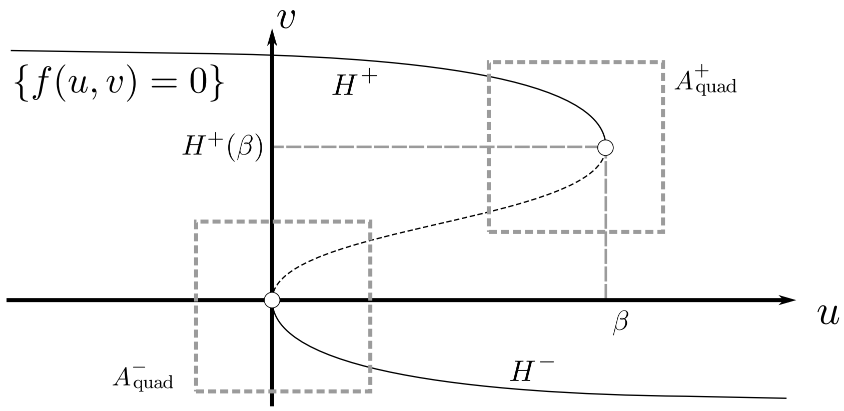

Figure 1.11.: The nullcline of f with neighborhoods indicating where the function is locally quadratic.

an additional constant $c_\infty > 0$ such that the following inequalities are satisfied:

$$
\begin{aligned}
&H^-(u_\infty) + c_\infty u_\infty < 0,\\
&H^+(-u_\infty) - c_\infty u_\infty > 0,\\
&H^-(u_\infty) < 0,\ H^+(-u_\infty) > 0 \text{ and } \|\varphi\|_{L_\infty(Q)} \le u_\infty,
\end{aligned}
\tag{1.2.11}
$$

where the redundant third inequality in (1.2.11) is included to simplify referencing (1.2.11) in future Chapters.

1.2.4. Function Spaces and Embedding Theorems

We now introduce some standard function spaces:

(i) Let $L_q(\cdot)$ be the standard Lebesgue space, $q > 1$.

(ii) Let $C^\gamma(\cdot)$ denote the standard Hölder space, where $\gamma \in (0,1)$.

(iii) For an integer l, let $W_q^l(\cdot)$ denote the Sobolev space of functions with weak derivatives up to and including order l, all of which are in L_q.

Definition 1.2.7 ([LSU68, Chap. 2,Sec. 3.3])**.** *For $l > 0$, noninteger (with integer part $[l]$) we denote by $W_q^l(Q)$ the space consisting of all functions $u \in W_q^l(Q)$) with the norm*

$$
\|u\|_{W_q^l(Q)} = \|u\|_{W_q^{[l]}(Q)} + \sum_{j=[l]} \left(\int_Q dx \int_Q \frac{|D_x^j u(x) - D_y^j u(y)|^q}{|x-y|^{n+q(l-[l])}}\, dy \right)^{\frac{1}{q}}, \tag{1.2.12}
$$

where D_x^j and D_y^j denotes the weak derivative with respect to the multi-index j.

Definition 1.2.8. *Denote by $W_\infty^{0,1}(Q_T)$ the space of functions such that $u \in L_\infty(Q_T)$ and the weak derivative with respect to the variable t exists with $u_t \in L_\infty(Q_T)$.*

Definition 1.2.9. *Denote by* $W_q^{2,1}$ *the anisotropic Sobolev space, i.e., the space of functions with weak derivatives* $D_t^r D_x^s u \in L_q(Q_T)$ *where* $2r + s \leq 2$. *The space* $W_q^{2,1}(Q_T)$ *is endowed with the norm*

$$\|u\|_{W_q^{2,1}(Q_T)} = \sum_{2r+s\leq 2} \|D_t^r D_x^s u\|_{L_q(Q_T)}.$$

Lemma 1.2.10 ([LSU68, Chap. 2,Lemma 3.3])**.** *For* $q > n+2$, $0 \leq \gamma < 1 - \frac{n+2}{q}$ *and* $T_0 > 0$, *the inclusion* $W_q^{2,1}(Q_{T_0}) \subset C^\gamma(\overline{Q_{T_0}})$ *is a compact embedding, and there is a constant* $C > 0$ *such that*

$$\|u\|_{C^\gamma(\overline{Q_{T_0}})} + \sum_{i=1}^{n} \|u_{x_i}\|_{C^\gamma(\overline{Q_{T_0}})} < C\left(\|u\|_{W_q^{2,1}(Q_{T_0})}\right), \tag{1.2.13}$$

where $C = C(T_0, q, n, \gamma)$, *but* C *is independent* u.
Lemma 1.2.11 ([LSU68, chap. 2, Lemma 3.4])**.** *If* $u \in W_q^{2,1}(Q_T)$ *with* $q > 2$, *then for every* $t \in [0,T]$ *the trace* $u(\cdot,t)$ *is well defined and* $u(\cdot,t) \in W_q^{2-\frac{2}{q}}(Q)$.
Lemma 1.2.12 ([Tri78, Section 4.6.1])**.** *If* $q > n+2$, $1 < \gamma \leq 1 - \frac{n+2}{q}$, *then there is a constant* $C > 0$ *such for every* $\varphi \in W_q^{2-\frac{2}{q}}(Q)$

$$\sum_{i=1}^{n} \|\varphi_{x_i}\|_{C^\gamma(\overline{Q})} + \|\varphi\|_{C^\gamma(\overline{Q})} \leq C\|\varphi\|_{W_q^{2-\frac{2}{q}}(Q)}, \tag{1.2.14}$$

where $C = C(n, q, \gamma)$, *but* C *is independent of* u.
Definition 1.2.13. *The functions from* $W_q^{2-\frac{2}{q}}(Q)$ *and* $q > 2$ *with homogeneous Neumann boundary conditions form a well defined subspace [Tri78, section 4.3.3], and we will denote it by* $\mathcal{W}$.

We now fix some constants for the remainder of the thesis. There will be more constants to fix as we delve into the techincalities of the coming Chapters, but to begin with we need a functional setting.
Condition 1.2.14. If we specify n and φ, then we shall assume we have also specified a q, u_∞, c_∞ and γ such that:

- $q > n+2$.
- $0 < \gamma < 1 - \frac{n+2}{q}$.
- u_∞ and c_∞ satisfy inequalities (1.2.11).
- $T_0 > 0$, where we always assume that $T \in (0, T_0)$.
- $\varphi \in \mathcal{W}$ (cf. Definition 1.2.13).

In particular, we shall assume that such a choice of q, u_∞, c_∞ and γ exists.

We will also need uniform bounds on the nonlinearities of (1.2.2) and (1.2.6). In order to define these we specify two further constants.

Condition 1.2.15. Given u_∞ and c_∞ satisfying inequalities (1.2.11), choose v_∞ such that $v_\infty < \max_{|u|\le u_\infty}\{|H^+(u)| + |H^-(u)| + 2c_\infty\}$ where we mean $H^-(u) = 0$ for $u < 0$ and $H^+(u) = 0$ for $u > \beta$. Moreover, let

$$f_\infty = \max_{|u|+|v|\le v_\infty + u_\infty} |f(u,v)|.$$

Finally, defnine the constant

$$f_{0,q} := \left(\int_0^{T_0}\int_Q (f_\infty + v_\infty + 2c_\infty u_\infty)^q\right)^{\frac{1}{q}}. \tag{1.2.15}$$

1.2.5. Definition of Solution

Definition 1.2.16. *We say u is a solution of* (1.2.1)–(1.2.4) *with* $\varphi \in \mathcal{W}$ *if the following hold:*

(i) $u \in W_q^{2,1}(Q_T)$ *and* $\mathcal{H}(u,\xi_0) \in L_q(Q_T)$.

(ii) (1.2.1), (1.2.2) *and* (1.2.4) *are satisfied almost everywhere.*

(iii) (1.2.3) *is satisfied in terms of traces.*

Definition 1.2.17. *We say* $(u^\varepsilon, v^\varepsilon)$ *is a solution of* (1.2.5)–(1.2.8) *with* $\varphi \in \mathcal{W}$ *if the following hold:*

(i) $u^\varepsilon \in W_q^{2,1}(Q_T)$ *and* $v^\varepsilon \in W_\infty^{0,1}(Q_T)$.

(ii) (1.2.5), (1.2.6) *and* (1.2.8) *are satisfied almost everywhere.*

(iii) (1.2.7) *is satisfied in terms of traces.*

We will now formulate a well-posedness Theorem for (1.2.5)–(1.2.8) but relegate an outline of its proof to Appendix D. The reason being is that the proof borrows on the functional setting used in the proof of the well-posedness of (1.2.1)–(1.2.4) as well as some additional standard results on parabolic equations that we introduce in Chapter 2. Deriving the necessary apriori estimates for the well-posedness (1.2.1)–(1.2.4) is the novel part of its proof, while the corresponding estimates for (1.2.5)–(1.2.8) simply boil down to the nonlinearity f in (1.2.6) being locally Lipshitz continuous.

Theorem 1.2.18. *The system* (1.2.5)–(1.2.8) *has a unique solution* $(u^\varepsilon, v^\varepsilon)$ *in the sense of Definition 1.2.17, and for every* $\varepsilon > 0$ *both* u^ε *and* v^ε *are uniformly bounded by the constants*

$$\|u^\varepsilon\|_{L_\infty(Q_T)} \le u_\infty, \quad \|v^\varepsilon\|_{L_\infty(Q_T)} \le v_\infty.$$

1.2.6. Spatial Transversality

Definition 1.2.19. *Let* $Q^\pm := \{x \in Q \mid \xi_0(x) = \pm\}$. *We say that* φ *is transverse with respect to* ξ_0 *if the following hold:*

(i) Q^- *and* Q^+ *are measurable.*

(ii) If $\varphi(x) = 0$ *and* $\nabla\varphi(x) = 0$*, then there exists a neighborhood* A *of* x *such that* $\xi_0(x) = +$ *for a.e.* $x \in A$.

(iii) If $\varphi(x) = \beta$ *and* $\nabla\varphi(x) = 0$*, then there exists a neighborhood* A *of* x *such that* $\xi_0(x) = -$ *for a.e.* $x \in A$.

We say a function $u : Q_T \to \mathbb{R}$ *is transverse with respect to* $\mathcal{H}(\xi_0, u)$ *if for every* $t \in [0, T]$, $u(\cdot, t) : Q \to \mathbb{R}$ *is transverse with respect to* $\xi(\cdot, t) : Q \to \{-, +\}$*, i.e., transverse with respect to the configuration of* $\mathcal{H}(\xi_0, u)(\cdot, t)$.

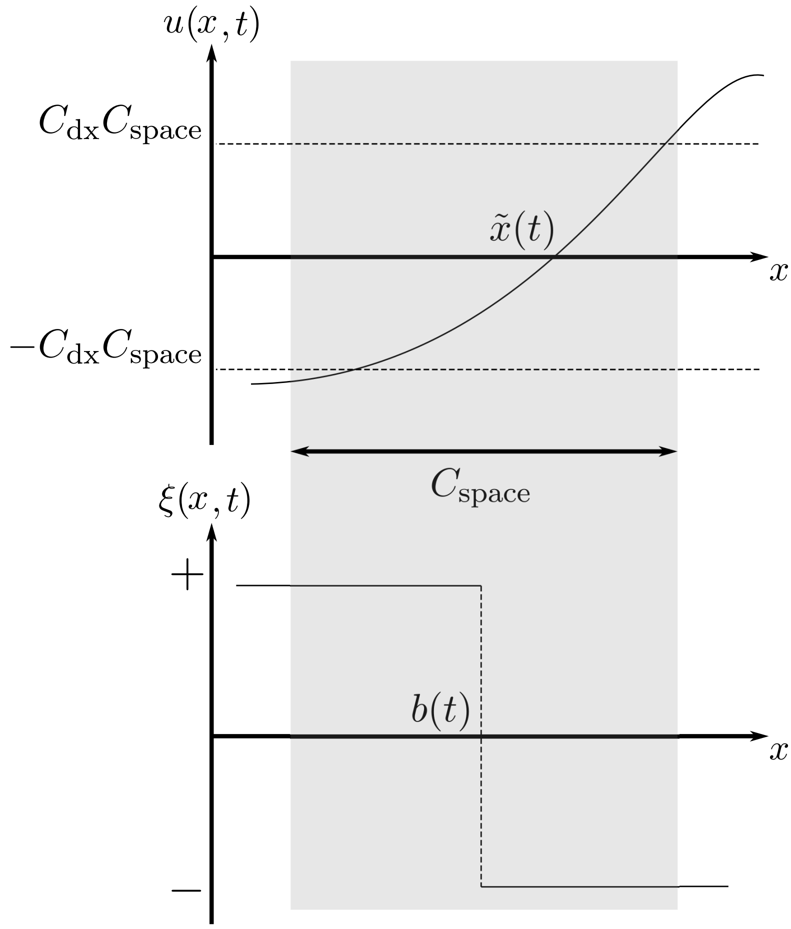

Figure 1.12.: An example of simply transverse initial data for case $b(t) \neq \tilde{x}(t)$.

In [GTS13] and [GT12], the authors also address the possibility of $u(\cdot, T)$ failing to be transverse. More specifically, they find the maximal time $T_{\max} > 0$ such that (1.2.1)–(1.2.4) has a unique solution on the time interval $[0, T_{\max}]$ where $u(\cdot, t)$ is transverse for every $t \in [0, T_{\max})$ and $u(\cdot, T_{\max})$ is not transverse. In the current exposition we note that finding such a $T_{\max}$ is possible using similar arguments, however we choose to focus

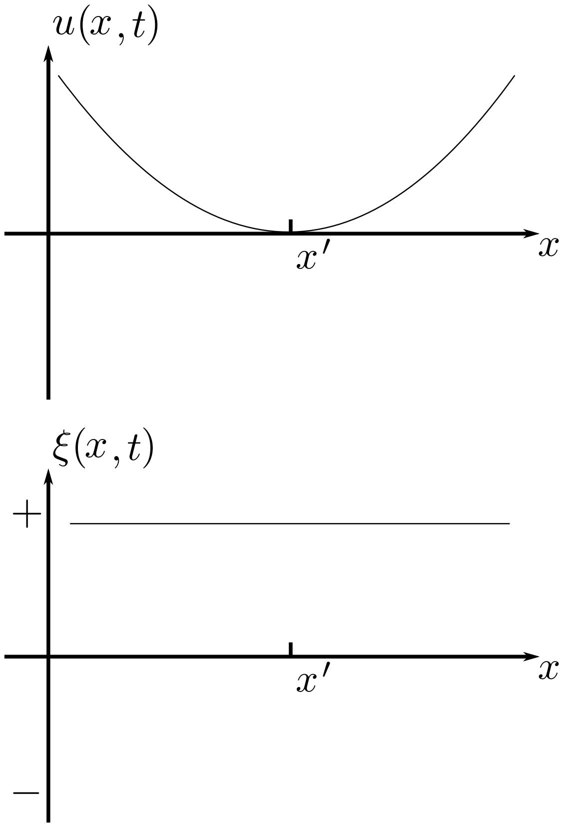

Figure 1.13.: An example of transverse initial data where $n = 1$ and $\varphi_x(x') = 0$. Note that ξ_0 must be constant in in neighborhood of x'.

on the essential issues of existence and uniqueness of solutions for $n \geq 2$.

We will in fact only prove that (1.2.6)–(1.2.8) converges to (1.2.1)–(1.2.4) for a simplified transversality assumption, and only in the case $n = 1$. Nonetheless, this model case still addresses the novel technical difficulties that need to be overcome to prove a corresponding result using Definition 1.2.19

Definition 1.2.20. *We say φ is simply transverse with respect to ξ_0 if $n = 1$, and there are constants $C_{\text{dx}} > 0$ and $C_{\text{space}} > 0$ such that the following holds:*

(i) ξ_0 has only one discontinuity point denoted $b(0)$.

(ii) If $\tilde{x}(0) \in \overline{B}_{C_{\text{space}}}(b(0))$ (the closed ball of radius C_{space}) and $\varphi(\tilde{x}(0)) = 0$, then for all $x \in \overline{B}_{C_{\text{space}}}(b(0))$ one has $\varphi_x(x) > C_{\text{dx}}$. In particular, such an $\tilde{x}(0)$ is unique.

(iii) $\varphi(x) < -C_{\text{dx}}C_{\text{space}}$ for $x \leq b(0) - C_{\text{space}}$.

(iv) $\varphi(x) > C_{\text{dx}}C_{\text{space}}$ for $x > b(0) + C_{\text{space}}$.

We say a function $u : Q_T \to \mathbb{R}$ is simply transverse if there are constants C_{space} and

C_{dx} independent of $t \in [0,T]$ such that $u(\cdot,t) : Q \to \mathbb{R}$ is simply transverse with respect to $\xi(\cdot,t) : Q \to \{-,+\}$, i.e., simply transverse with respect to the configuration of $\mathcal{H}(\xi_0,u)(\cdot,t)$. In this case let $\tilde{x}(t)$ and $b(t)$ denote the equivalent quantities to $b(0)$ and $\tilde{x}(0)$ respectively.

In Definition 1.2.20 we defined simple transversality as a spatially local phenomena in the sense that $\xi(x,t)$ always has one discontinuity point where $\mathcal{H}(\xi_0,u)$ is transitioning at the threshold $u(x,t) = 0$. To describe how $u(x,t)$ could behave more generally let $\{T_i\}_{i=1}^{\infty}$, $\{C_{\text{dx;i}}\}_{i=1}^{\infty}$ and $\{C_{\text{space;i}}\}_{i=1}^{\infty}$ be sequences such that $u(x,t)$ is simply transverse on the interval $[T_i, T_{i+1}]$ with the associated constants $C_{\text{dx;i}}$ and $C_{\text{space;i}}$.

(i) Suppose that $T_i \to T_{\max}$, and either $C_{\text{dx;i}} \to 0$ or $C_{\text{space;i}} \to 0$. Then there could exist an $x_0 \in \overline{Q}$ such that $x_0 \neq b(T_{\max})$ and $u(x_0,T_{\max}) = u_x(x_0,T_{\max}) = 0$, i.e., the function has become non-transverse away from the discontinuity point $b(T_{\max})$. Observe that (iii) and (iv) in Definition 1.2.20 no longer exclude this outcome.

(ii) Suppose that $T_i \to T_{\max}$ and $b(T_i) \to 1$. Then due to (1.2.3), one necessarily has $C_{\text{dx;i}} \to 0$. If $u(\cdot,T_{\max})$ doesn't lose transversaliy like in the previous item, then $u(x,T_{\max})$ is transverse and $\xi(T_{\max})$ has no discontinuity.

(iii) If $\xi(t)$ has more than one discontinuity, denoted b_1 and b_2, then we could have $\lim_{i\to\infty} b_1(T_i) = \lim_{i\to\infty} b_2(T_i)$, i.e., the discontinuities merge together. In this case the constants $C_{\text{dx;i}}$ for b_1 (or b_2) approach zero, but $u(x,T_{\max})$ could still be transverse, albeit $\xi(T_{\max})$ has less discontinuities (cf. Figure 1.14).

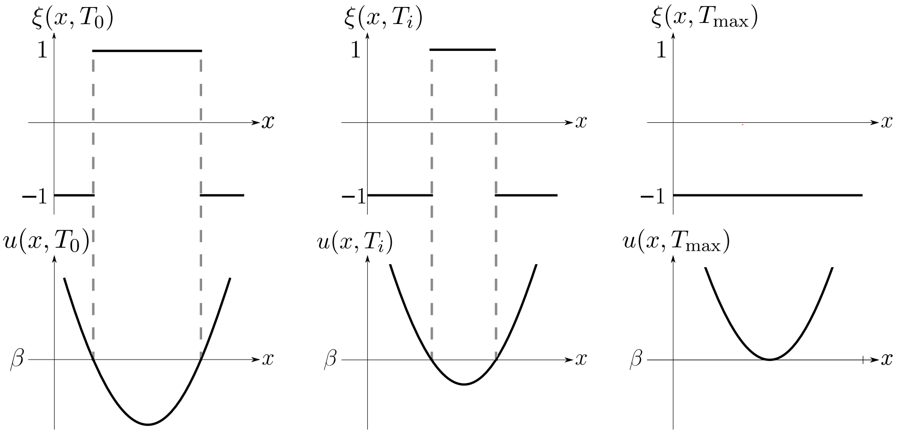

Figure 1.14.: [CGT16, Figure 6.]. Discontinuities merging at the threshold β. As $T_i \to T_{\max}$, $u(x,T_{\max})$ is still transverse with respect to $\xi(x,T_{\max})$ even though transversality holds in ever shrinking neighborhoods of the two discontinuity points.

1.3. Statement of Main Results

This thesis contains three Theorems divided into two categories. The first category contains two Theorems concerning the well-posedness of the hysteresis problem (1.2.1)–(1.2.4). Their proof is the content of Chapter 3.

Theorem 1.3.1 (Existence). *Assume that $n \geq 1$ and $\varphi \in \mathcal{W}$ is transverse with respect to ξ_0. Then there is a $T_{\mathrm{dx}} > 0$ such that*

(i) There is at least one transverse solution $u \in W_q^{2,1}(Q_{T_{\mathrm{dx}}})$ to (1.2.1)–(1.2.4).

(ii) Any solution $u \in W_q^{2,1}(Q_{T_{\mathrm{dx}}})$ must be transverse.

Theorem 1.3.2 (Uniqueness). *Let $n \geq 1$. Given any $T_{\max} > 0$, if $u_1, u_2 \in W_q^{2,1}(Q_{T_{\max}})$ are two transverse solutions to (1.2.1)–(1.2.4), then $u_1 = u_2$ almost everywhere.*

The second category concerns the behavior of (1.2.5)–(1.2.8).

Definition 1.3.3. *Let $q > 3$ and $\frac{2}{3}(\frac{2}{q-1}) < \lambda_0 < \frac{2}{3}$. Define*

$$\frac{2}{3_-} := \frac{2}{3} - \lambda_0.$$

Theorem 1.3.4. *Let $n = 1$, $\varphi \in \mathcal{W}$ and assume that there are three positive constants $T, C_{\mathrm{space}}, C_{\mathrm{dx}} > 0$ such that the solution u to (1.2.1)–(1.2.4) is simply transverse on Q_T with said constants.*

Then there exists four constants $C_i, \varepsilon_0 > 0$ ($i = 1, 2, 3$) such that for every $\varepsilon \in (0, \varepsilon_0)$ the following holds:

(i) The solution u^ε to (1.2.6) is simply transverse on Q_T.

(ii)

$$\|u^\varepsilon - u\|_{L_\infty(Q_T)} \leq \frac{C_1\sqrt{T}}{C_{\mathrm{dx}}(1-\gamma)} \exp\left(\left(\frac{C_2}{C_{\mathrm{dx}}}\right)^2 T\right) \varepsilon^{\frac{2}{3}_-}. \tag{1.3.1}$$

(iii) For $p \in \{1, q\}$ and $t \in [0, T]$

$$\|v^\varepsilon(\cdot, t) - \mathcal{H}(u^\varepsilon, \xi_0)(\cdot, t)\|_{L_p(Q)} \leq \frac{C_3}{C_{\mathrm{dx}}(1-\gamma)} \varepsilon^{\frac{2}{3}_- \cdot \frac{1}{p}} \tag{1.3.2}$$

The constants C_i and ε_0 do not depend on ε or p, however, $\varepsilon_0 \to 0$ as $\lambda_0 \to \frac{2}{3}(\frac{2}{q-1})$.

Definition 1.3.5. *We define the constant ε_0 explicitly*

$$\varepsilon_0 = \min_{1 \leq i \leq 6} \varepsilon_i,$$

where the quantities ε_i are, in increasing order, defined in:

1) Lemma 4.2.1.

2) Definition 5.1.9.

3) Definition 5.2.6.

4) Definition 5.3.2.

5) Definition 7.3.5.

6) The end of Section 7.3, in the proof of Lemma 6.1.5.

The most important feature to highlight is that the exponent of $\varepsilon^{\frac{2}{3}-}$ is strictly less than $\frac{2}{3}$. In other words we do not prove that

$$\|(u^\varepsilon, v^\varepsilon) - (u, v)\|_{L_\infty(Q_T)} = O(\varepsilon^{\frac{2}{3}}),$$

(recall that the solution to (1.2.1)–(1.2.4) is always written without superscript). The reason this is significant is related to the question of whether there exists a $\varphi \in \mathcal{W}$ and a $T > 0$ such that $(u^\varepsilon, v^\varepsilon)$ converges to (u, v) on Q_T with a rate faster than $O(\varepsilon^{\frac{2}{3}-})$. We cannot construct such an example, however we conjecture that $O(\varepsilon^{\frac{2}{3}})$ should be the best possible rate, i.e., the convergence rate of the PDE should not be faster that of the ODE (1.1.6)–(1.1.7) in the planar case (see (1.1.10) and the accompanying discussion).

What limits our methods is the time regularity of $u^\varepsilon \in C^\gamma(\overline{Q_T})$, where Hölder continuity in place of Lipshitz continuity severly complicates matters. This will become clear in Chapter 7. Observe that γ is in turn informed by the regularity of $\varphi \in \mathcal{W}$ (cf. Lemma 1.2.12 and Condition 1.2.14). If φ is smooth, then we can take the q in Definition 1.2.13 arbitrarily large and thus we can make our convergence rate in Definition 1.3.3 as close to $\varepsilon^{\frac{2}{3}}$ as possible.

1.4. Structure of the Thesis

In Chapter 2 we present more auxiliary results that will be needed to prove Theorems 1.3.1 and 1.3.2. The proofs themselves appear in Chapter 3, with Section 3.2 devoted to establishing the necessary apriori estimates.

The remaining Chapters are devoted to proving Theorem 1.3.4, however the order is nonlinear. In Chapter 4 we prove items (i) and (ii), however, assuming that (iii) already holds.

Chapters 5—7 are devoted to proving item (iii) of Theorem 1.3.4. In Chapter 5 we describe the phase space of 1.2.6. This description is indispensible to understanding the proofs contained in 6 and 7.

Chapter 6 proves item (iii) of Theorem 1.3.4, modulo three technical Lemmas pertaining to the behavior of (1.2.6) with u treated as a non-autonomous forcing. These Lemmas are ODE results.

Chapter 7 is where the necessary study of the Fast-ODE take place, and the three aforementions Lemmas are proved. Those who are particularly interested in the Forced Fast-

ODE can read Chapter 5 to familliarize oneself with the terminology, and then read Chapter 7 independent of the rest of the thesis.

The linear order of the exposition would be Chapters; 2, 3, 5, 7, 6 then 4.

At the end of the thesis we make some concluding remarks and discuss directions for future research.

2. Auxilary Results

2.1. Strong Solutions in $W_q^{2,1}(Q_T)$

Theorem 2.1.1 ([LSU68, Chap. 4, Sec. 9])**.** *Let $T_0 > 0$, $F \in L_q(Q_{T_0})$ and $\varphi \in \mathcal{W}$. Consider the equation*

$$\begin{cases} u_t = \Delta u + F(x,t), & (2.1.1) \\ u|_{t=0} = \varphi, & (2.1.2) \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial Q} = 0. & (2.1.3) \end{cases}$$

Then (2.1.1)–(2.1.3) *has a unique solution $u \in W_q^{2,1}(Q_{T_0})$ that satisfies*

$$\|u\|_{W_q^{2,1}(Q_{T_0})} \le C\left(\|F\|_{L_q(Q_{T_0})} + \|\varphi\|_{\mathcal{W}}\right), \tag{2.1.4}$$

where $C = C(T_0, n)$ but C does not depend on u, φ or F. By combining with (1.2.13) *one obtains*

$$\|u\|_{C^\gamma(\overline{Q_{T_0}})} + \sum_{i=1}^{n} \|u_{x_i}\|_{C^\gamma(\overline{Q_{T_0}})} \le C_{strong}\left(\|F\|_{L_q(Q_{T_0})} + \|\varphi\|_{\mathcal{W}}\right), \tag{2.1.5}$$

where $C_{strong} = C_{strong}(T_0, n, q, \gamma,)$ but C_{strong} does not depend on u, φ or F.

Note that for any $T \le T_0$ and $F \in L_q(Q_T)$, one can extend F by zero, to an $F \in L_q(Q_{T_0})$ and use the same constant C_{strong} as (2.1.5).

2.2. Mild Solutions to Semilinear Parabolic Problems

Let $f_0 : \mathbb{R} \times Q \times [0, \infty) \to \mathbb{R}$ and consider the semilinear problem.

$$\begin{cases} u_t = \Delta u + f_0(u, x, t), & (2.2.1) \\ u|_{t=0} = \varphi, \qquad \dfrac{\partial u}{\partial \nu}\Big|_{\partial Q_T} = 0, & (2.2.2) \\ f_0|_{t=0} = f_0(\varphi(x), x, 0). & (2.2.3) \end{cases}$$

Condition 2.2.1. For every bounded set $\Omega \subset \mathbb{R} \times Q \times [0, \infty)$:

(i) $|f_0(u,x,t)| \leq L(\Omega)$ for all $(u,x,t) \in \Omega$.

(ii) $|f_0(u_1,x,t) - f_0(u_2,x,t)| \leq L(\Omega)|u_1 - u_2|$ for all $(u_1,x,t),(u_2,x,t) \in \Omega$.

(iii) For every fixed $u \in \mathbb{R}$, the function $f_0(u,x,t)$ is measurable in Q_T.

Definition 2.2.2. *We say that u is an $E_{\infty,T}$-mild solution of (2.2.1)–(2.2.3) for initial data $\varphi \in L_\infty(Q)$ on the interval $[0,T)$, if u is a measureable function on Q_T and satisfies the following:*

(i) $u(\cdot,t) \in L_\infty(Q)$ *for a.e.* $t \in [0,T)$.

(ii) $sup_{s\in(0,t)}\|u(\cdot,s)\|_{L_\infty(Q)} < \infty$ *for all* $t \in (0,T)$.

(iii) $u(\cdot,t) = \mathcal{P}(t)\varphi + \int_0^t \mathcal{P}(t-s)(f_0(u(\cdot,s),\cdot,s)\,ds$ *for all* $t \in (0,T)$.

Here $\mathcal{P}$ is a semigroup on $L_\infty(Q)$ defined in [Rot84, pg.111] and the integral is an absolutely convergent Bochner integral in $L_\infty(Q)$.

Theorem 2.2.3. *If f_0 satisfies Condition 2.2.1, then for each initial function $\varphi \in L_\infty(Q)$, there exists a $T \in (0,\infty]$ such that (2.2.1)–(2.2.3) has a unique $E_{\infty,T}$ mild solution on the interval $[0,T)$.*

Proof. The proof of Theorem 2.2.3 is formulated in [Rot84, Theorem 1, pg.111]. □

Theorem 2.2.4. *Let f_0 satisfy Condition 2.2.1. Then the $E_{\infty,T}$ solution u coincides with the strong solution $u \in W_q^{2,1}(Q_T)$.*

Proof. See [Rot84, pg. 120,Thm. 2]. □

2.3. Uniformly Bounded Mild/Strong Solutions

Lemma 2.3.1. *Assume that $u \in W_q^{2,1}(Q_T)$ satisfies (2.2.1)–(2.2.3), $\varphi \in \mathcal{W}$ and $F(x,t) = f_0(u,x,t) \in L_q(Q_T)$. Further assume that there is some $u_\infty > 0$ such that*

(i) $f_0(\cdot,x,t) : \mathbb{R} \to \mathbb{R}$ *is continuous at the points $\pm u_\infty$ uniformly with respect to $(x,t) \in Q_T$, i.e., for all $\delta > 0$ there is a $\delta' > 0$ such that $|u_\infty - \overline{u}^\infty| < \delta'$ implies that $|f_0(u_\infty,x,t) - f_0(\overline{u}^\infty,x,t)| < \delta$ where δ' can be chosen independently of (x,t).*

(ii) $f_0(u_\infty,x,t) < 0$, $f_0(-u_\infty,x,t) > 0$ *for all* $(x,t) \in Q_T$.

(iii) $\|\varphi\|_{C(\overline{Q})} < u_\infty$.

Then $\|u\|_{C(\overline{Q_T})} < u_\infty$.

Proof. We refer the reader to Appendix B. □

Definition 2.3.2. *We say that $T_1 \in (0,\infty)$ is a maximal existence time for the initial data $\varphi \in L_\infty(Q)$ if problem (2.2.1)–(2.2.3) has an $E_{\infty,T}$-mild solution on the interval*

$[0, T_1)$ *but for any* $T' > T_1$*, there does not exist an* E_{∞,T_1}*-mild solution on the interval* $[0, T')$.

Lemma 2.3.3. *Assume that* $\varphi \in L_\infty(Q)$ *and that Condition 2.2.1 is satisfied. Then there is a maximal existence time* $T_1 \in (0, \infty]$ *and problem* (2.2.1)–(2.2.3) *has a unique* $E_{\infty,T}$*-mild solution on the interval* $[0, T_1)$*. If* T_1 *is finite then*

$$\lim_{t \to T_1} \|u(\cdot, t)\|_{L_\infty(Q)} = \infty.$$

Proof. The proofs of Theorem 2.2.3 and 2.3.3 are formulated in [Rot84, Theorem 1, pg.111]. □

2.4. Green Functions

Theorem 2.4.1 ([CK13, Thm. 3.21])**.** *Let* $n \geq 1$ *and let* $Q \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary. Then for* $0 < t - s \leq T$ *and* $x, y \in Q$*, the Green function for the heat equation satisfies*

$$|\mathcal{G}(x, y, t, s)| \leq \frac{C_1}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{C_2\|x-y\|^2}{t-s}\right),$$

where C_1 *and* C_2 *depend on* n*,* u_∞ *and* Q *but not on* x*,* y*,* t *or* s*.*

3. Well-Posedness of the Limiting Hysteresis Problem

3.1. Specific Functional Setting for Solutions to Reaction-Diffusion Equations

For the remainder of this Chapter, let $n \geq 1$ and $\varphi \in \mathcal{W}$ unless specifically stated otherwise (cf. Condition 1.2.14).

For $\lambda \in (0, \gamma)$, let $c_{\lambda,\gamma} > 0$ be the embedding constant

$$\|u\|_{C^\lambda(\overline{Q_T})} \leq c_{\lambda,\gamma} \|u\|_{C^\gamma(\overline{Q_T})}.$$

Moreover, let $f_{0,q}$ be the constant defined in Condition 1.2.15 and C_{strong} the embedding constant defined in Theorem 2.1.1. Now define the constant

$$C_P := c_{\lambda,\gamma} C_{\text{strong}} (f_{0,q} + \|\varphi\|_{\mathcal{W}}). \tag{3.1.1}$$

Definition 3.1.1. *Let $P^\lambda(Q_T)$ be the set of all functions $u \in C^\lambda(\overline{Q_T})$ such that $u(x,0) = \varphi(x)$, there exists $u_{x_j} \in C^\lambda(\overline{Q_T})$ $(j = 1, \cdots, n)$, and*

$$\|u\|_{P^\lambda(\overline{Q_T})} \leq C_P, \tag{3.1.2}$$

where

$$\|u\|_{P^\lambda(\overline{Q_T})} := \|u\|_{C^\lambda(\overline{Q_T})} + \sum_{j=1}^{n} \|u_{x_j}\|_{C^\lambda(\overline{Q_T})}. \tag{3.1.3}$$

Note that the set of functions $u \in C^\lambda(\overline{Q_T})$ such that (3.1.3) *is finite forms a Banach space and $P^\lambda(Q_T)$ is a closed covex set. For brevity, we will write $\|\cdot\|_{P^\lambda(Q_T)}$ as the restriction of* (3.1.3) *to the set $P^\lambda(Q_T)$. Using the same C_P, let $P^\gamma(Q_T)$ be defined similarly.*

3.2. Apriori Estimates

3.2.1. Temporal Input

Lemma 3.2.1. *Suppose $a_j \in C[0,T]$ and let $b_j(t) = \max_{0 \leq s \leq t} a_j(s)$. Then*

$$\|b_1 - b_2\|_{C[0,T]} \leq \max_{0 \leq s \leq t} |a_1(s) - a_2(s)|.$$

Proof. Define the time t^* via the formula

$$t^* = \min_{t\in[0,T]} \{|b_1(t) - b_2(t)| = \|b_1 - b_2\|_{C[0,T]}\},$$

and suppose without loss of generality that $b_1(t^*) - b_2(t^*) = \|b_1 - b_2\|_{C[0,T]}$. Note that if $t^* = 0$ then we are done.

For the case $t^* > 0$, we claim that $b_1(t^*) = a_1(t^*)$. If not, then there is a $t \in [0, t^*]$ such that $b_1(t^*) = a_1(t)$, and since b_1 is non-decreasing this means that $b_1(t^*) = b_1(t)$. However, b_2 is also non-decreasing which leads to

$$b_1(t) - b_2(t) \geq b_1(t^*) - b_2(t^*) = \|b_1 - b_2\|_{C[0,T]},$$

a clear contradiction of the definition of t^*. We are now left with the inequality

$$\|b_1 - b_2\|_{C[0,T]} = b_1(t^*) - b_2(t^*) = a_1(t^*) - b_2(t^*) \leq a_1(t^*) - a_2(t^*),$$

from which the two results follow. □

3.2.2. Local Apriori Estimates in Q_T

Let us write the spatial variable x as $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$. If $n = 1$ then x' as a singleton. An alternative derivation of some of these estimates for $n = 1$ can be found in [GT12].

We recall Theorem C.0.9 from Appendix C. This Theorem relates to a mapping between Banach spaces, so first let us first address some simple technicalities.

Suppose there is an $x_0 \in Q$ such that $\varphi(x_0', x_{0n}) = 0$ and $\varphi_{x_n}(x_0', x_{0n}) > 0$. Let us choose a neighborhood of (x_0', x_{0n}) of a particular form

$$(x_0', x_{0n}) \in Q' \times B_\delta(x_{n0}) := Q^{\text{loc}}, \qquad Q_T^{\text{loc}} := Q^{\text{loc}} \times [0, T], \tag{3.2.1}$$

where $Q' \subset \mathbb{R}^{n-1}$ is an open set. Since φ_{x_n} is continous, let us further suppose that $\varphi_{x_n}(x', x_n) > 0$ for every $(x', x_n) \in Q^{\text{loc}}$. When we write functions $u(x', x_n, t)$, with $(x', x_n) \in Q^{\text{loc}}$, and $t \in [0, T]$ we tacitly mean an extension of u to $\mathbb{R}^{n+1}$ that is zero outside of a neighborhood of Q_T^{loc}. We will also extend φ into Q_T^{loc} using the shorthand $\varphi(x', x_n, t) := \varphi(x', x_n)$. Note that this means that if $\|u - \varphi\|_{P^\lambda(Q_T)} \leq C$, then for every $t \in [0, T]$

$$\|u(\cdot, \cdot, t) - \varphi\|_{C^\lambda(\overline{Q^{\text{loc}}})} + \sum_{j=1}^{n} \|u(\cdot, \cdot, t)_{x_j} - \varphi_{x_j}\|_{C^\lambda(\overline{Q^{\text{loc}}})} \leq C. \tag{3.2.2}$$

If $(x', x_n) \in \partial Q$, then it a straightforward procedure to straighten the boundary and reflect φ over the plane tangent to the boundary of Q. Note that by using the Neumann boundary conditions this reflection gives an extension that is also in P^λ, albeit $P^\lambda(\mathbb{R}^n_T)$.

Finally note that we are applying Theorem C.0.9 on the Banach space of functions such that (3.1.3) is finite, however we restrict to functions belonging to $P^\lambda(Q_T)$ (cf. Definition 3.1.1).

Lemma 3.2.2. *Suppose $\varphi(x_0', x_{0n}) = 0$, $\varphi_{x_n}(x_0', x_{0n}) > 0$ and consider $\varphi \in P^\lambda(Q_T^{\text{loc}})$. Then there are constants $C_{\text{dx}}, \delta, p' > 0$, and a neighborhood $Q' \subset \mathbb{R}^{n-1}$ such that for all $x' \in Q'$, $t \in [0,T]$ and $u_1, u_2 \in B_{p'}(\varphi) \subset P^\lambda(Q_T^{\text{loc}})$*

$$\int_{-\delta}^{\delta} |\mathcal{H}(\xi_0, u_1) - \mathcal{H}(\xi_0, u_2)|(x', x_n, t)dx_n \leq \frac{C}{C_{\text{dx}}} \|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})}, \tag{3.2.3}$$

where C_{dx} is a lower bound on $\|\nabla u_1\|$ and $\|\nabla u_2\|$ in $B_{p'}(\varphi)$.

Proof.
(Step 1/5) Apply implicit function theorem:

Recall that Q^{loc} can be chosen such that for every $(x', x_n) \in Q^{\text{loc}}$ the spatial derivative $\varphi_{x_n}(x_0', x_{0n}) > 0$. Define the function

$$\mathcal{F} : \{P^\lambda(Q_T) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}\} \to \mathbb{R}; \qquad \mathcal{F}(u, x', x_n, t) = u(x', x_n, t),$$

where $\{P^\lambda(Q_T) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}\}$ has the L_1-norm.

Possibly by reducing the size of Q', δ, p', we see that

$$\mathcal{F} : \{B_{p'}(\varphi) \times Q' \times B_\delta(x_{n0}) \times (0,T)\} \to \mathbb{R}$$

satisfies the following:

(i) Because $\varphi(x_0', x_{0n}) = 0$, we can shrink the domain of $\mathcal{F}$ such that for every (u, x', x_n, t) in its domain, $u(x', x_n, t) \in \pi_u A^-_{\text{quad}}$ (cf. Condition 1.2.6).

(ii) There is a constant C such that $u(x', x_n, t) < C < \beta$ for every (u, x', x_n, t) in the domain of $\mathcal{F}$.

(iii) $\mathcal{F}$ is Lipshitz continuous in u, x' and x_n.

(iv) $\mathcal{F}(\varphi, x_0', x_{0n}, 0) = 0$.

(v) The Fréchet derivative $D_{x_n}\mathcal{F}(u, x', x_n, t) = u_{x_n}(x', x_n, t)$ exists and moreover $D_{x_n}\mathcal{F}(\varphi, x_0', x_{0n}, 0)$ is invertible.

(vi) $D_{x_n}\mathcal{F}(u, x', x_n, t) \in \mathcal{L}(\mathbb{R}, \mathbb{R})$ is Lipshitz continuous in the variables u and x'.

Observe that once we have chosen $B_{p'}(\varphi)$, (3.2.2) proves that the above properties are true irrespective of the value of $t \in (0,T)$. By Theorem C.0.9 there exists a unique function such that

$$a : B_{p'}(\varphi) \times Q' \times (0,T) \to B_\delta(x_n^0), \qquad \mathcal{F}(u, x', a(u, x', t), t) = 0.$$

Note that Theorem C.0.9 does not facilitate a non-Lipschitz parameter, however, we apply the Theorem anyway because we will not need any conclusions about t regularity in this Lemma. Now fix an $x' \in Q'$, and for any $u_1, u_2 \in B_{p'}(\varphi)$ define

$$a_1(x', t) = a(u_1, x', t), \qquad b_1(x', t) = \max_{0 \leq s \leq t} a_1(x', t),$$

with a_2 and b_2 defined similarly. Assume without loss of generality that $b_2(x', t) < b_1(x', t)$. To finish the proof we will need three crude estimates.

(Step 2/5) Estimate distance between b_1 and b_2:

First consider

$$\int_{b_2(x',t)}^{b_1(x',t)} |\mathcal{H}(\xi_0, u_1) - \mathcal{H}(\xi_0, u_2)|(x', x_n, t) dx_n. \tag{3.2.4}$$

Now using the Mean-Value Theorem (cf. Figure 3.1) one obtains

$$\begin{aligned} |a_1(x',t) - a_2(x',t)| \leq & \frac{1}{C_{\text{dx}}} |u_1(x', a_1(x',t), t) - u_1(x', a_2(x',t), t)| \\ \leq & \frac{C}{C_{\text{dx}}} |u_2(x', a_2(x',t), t) - u_1(x', a_2(x',t), t)|, \end{aligned} \tag{3.2.5}$$

where C_{dx} is a lower bound of $\|\nabla u_1\|$ and $\|\nabla u_2\|$ on the neighborhood $B_{p'}(\varphi)$.

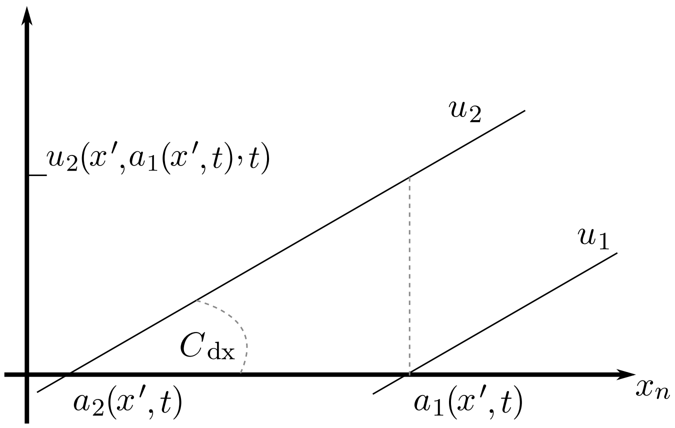

Figure 3.1.: The Mean-Value Theorem for the functions u_1 and u_2.

Now applying Lemma 3.2.1 to (3.2.5) we get the first estimate

$$\begin{aligned} \|b_1(x', \cdot) - b_2(x', \cdot)\|_{C[0,t]} &\leq \max_{0 \leq s \leq t} \|u_1(\cdot, \cdot, s) - u_2(\cdot, \cdot, s)\|_{L_\infty(Q^{\text{loc}})} \\ &\leq \|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})}. \end{aligned} \tag{3.2.6}$$

(Step 3/5) Estimate on the H^+ branch:

The upper hysteresis branch H^+ is locally Lipshitz away from $u = \beta$ (cf. item (iii) of Condition 1.2.6) so we have our second estimate

$$\int_{-\delta}^{\delta} |H^+(u_1) - H^+(u_2)|(x', x_n, t) dx_n \leq C\|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})}. \tag{3.2.7}$$

(Step 4/5) Estimate on the H^- branch:

$\xi(x', x_n, t) = -1$ is possible only if $u_1(x', x_n, s) > 0$ for all $s \in [0, t]$, which in turn means $x_n > b_1(x', t) > b_2(x', t)$. Moreover, because $u_1(x', x_n, t) \in \pi_u A^-_{\text{quad}}$ we have

$$\mathcal{H}(\xi_0, u_1)(x', x_n, t) = H^-(u_1(x', x_n, t)) = -\sqrt{u_1(x', x_n, t)},$$

where the same conclusions hold for u_2. Notice that for $x_n > b_1(x', t)$

$$\begin{aligned} |u_1(x', x_n, t)| &= |u_1(x', x_n, t) - u_1(x', a_1(x', t), t)| \\ &\geq C_{\text{dx}} |x_n - a_1(x', t)| \\ &\geq C_{\text{dx}} |x_n - b_1(x', t)|, \end{aligned} \tag{3.2.8}$$

where C_{dx} is a lower bound of $\|\nabla u_1\|$ and $\|\nabla u_2\|$ on the neighborhood Q^{loc}. Let's omit the variable (x', x_n, t) to write the next inequality, namely

$$|H^-(u^1) - H^-(u^2)| \leq \frac{|u_1 - u_2|}{|\sqrt{u_1} + \sqrt{u_2}|} \leq \frac{|u_1 - u_2|}{\sqrt{|x_n - b_1(x', t)|}}. \tag{3.2.9}$$

With this algebraic manipulation in hand we can now write our third estimate

$$\begin{aligned} \int_{b_1(x',t)}^{\delta} |H^-(u_1) - H^-(u_2)|(x', x_n, t) dx_n &\leq \int_{b_1(x',t)}^{\delta} \frac{|u_1 - u_2|(x', x_n, t)}{\sqrt{C_{\text{dx}} |x_n - b_1(x', t)|}} dx_n \\ &\leq \frac{C}{\sqrt{C_{\text{dx}}}} \|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})}. \end{aligned} \tag{3.2.10}$$

(Step 5/5) Final estimate:

By combining (3.2.6), (3.2.7) and (3.2.10) we obtain

$$\int_{-\delta}^{\delta} |\mathcal{H}(\xi_0, u_1) - \mathcal{H}(\xi_0, u_2)|(x', x_n, t) dx_n \leq \frac{C}{C_{\text{dx}}} \|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})}.$$

□

Lemma 3.2.3. *Suppose $\varphi(x'_0, x_{0n}) = 0$ and $\varphi_{x_n}(x'_0, x_{0n}) > 0$. Then there are constants $\delta, C_{\text{dx}} > 0$, a neighborhood $B_{p'}(\varphi)$ of $\varphi \in B_{p'}(\varphi) \subset P^\lambda(Q_T^{\text{loc}})$, and a neighborhood $Q' \subset \mathbb{R}^{n-1}$ such that for every $t \in [0, T]$*

$$\begin{aligned} &\|\mathcal{H}(\xi_0, u_1)(\cdot, t) - \mathcal{H}(\xi_0, u_2)(\cdot, t)\|_{L_q(Q^{\text{loc}})} \\ &\leq \frac{C}{C_{\text{dx}}} \left(\|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})} + \|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})}^{\frac{1}{2}} + \|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc}})}^{\frac{1}{q}} \right). \end{aligned} \tag{3.2.11}$$

Moreover, $u(x', x_n, t) \in \pi_u A^-_{\text{quad}}$ for every $(u, x', x_n, t) \in B_{p'}(\varphi) \times Q' \times B_\delta(x_n^0) \times (0, T)$, and C_{dx} is a lower bound of $\|\nabla u\|$ on $B_{p'}(\varphi)$.

Proof. One repeats the steps of the proof of Lemma 3.2.2, however one calculates the three key inequalities (3.2.6), (3.2.7) and (3.2.10) in the L_q norm. (3.2.6) becomes

$$\begin{aligned}&\left(\int_{b_2(x',t)}^{b_1(x',t)} |\mathcal{H}(\xi_0,u_1)-\mathcal{H}(\xi_0,u_2)|^q(x',x_n,t)dx_n\right)^{\frac{1}{q}}\\ &\le C\|b_1(x',\cdot)-b_2(x',\cdot)\|_{C[0,t]}^{\frac{1}{q}}\\ &\le \frac{C}{C_{\mathrm{dx}}}\|u_1-u_2\|_{L_\infty(Q_t^{\mathrm{loc}})}^{\frac{1}{q}}.\end{aligned} \tag{3.2.12}$$

(3.2.7) becomes

$$\begin{aligned}&\left(\int_{-\delta}^{\delta} |H^+(u_1)-H^+(u_2)|^q(x',x_n,t)dx_n\right)^{\frac{1}{q}}\\ \le& C\left(\int_{-\delta}^{\delta}|u_1-u_2|^q(x',x_n,t)dx_n\right)^{\frac{1}{q}}\\ \le& C\|u_1-u_2\|_{L_\infty(Q_t^{\mathrm{loc}})}.\end{aligned} \tag{3.2.13}$$

To finish the proof one uses the simple algebraic trick (written here for $u_1 > u_2 > 0$)

$$|\sqrt{u_1}-\sqrt{u_2}| = \frac{|u_1-u_2|}{\sqrt{u_1}+\sqrt{u_2}} = \sqrt{|u_1-u_2|}\frac{\sqrt{u_1}}{\sqrt{u_1}+\sqrt{u_2}} \le \sqrt{|u_1-u_2|},$$

and calculates (3.2.10) in the L_q norm which yields

$$\begin{aligned}&\left(\int_{b_1(x',t)}^{\delta} |H^-(u_1)-H^-(u_2)|^q(x',x_n,t)dx_n\right)^{\frac{1}{q}}\\ \le&\left(\int_{b_1(x',t)}^{\delta}|u_1-u_2|^{\frac{q}{2}}(x',x_n,t)dx_n\right)^{\frac{1}{q}}\\ \le&\|u_1-u_2\|_{L_\infty(Q_t^{\mathrm{loc}})}^{\frac{1}{2}}.\end{aligned} \tag{3.2.14}$$

□

3.2.3. Estimates on $\mathcal{H}(\xi_0, u)$ for Simply Transverse Functions

We finish this subsection by rephrasing the previous two Lemmas for functions that are simply transverse. These results do not directly relate to the proofs of Theorem 1.3.1 and Theorem 1.3.2, but interrupting the flow of the present Chapter will be offset by streamlining the exposition later on.

Lemma 3.2.4. *Let $n = 1$ and $\varphi \in \mathcal{W}$ be given. Suppose that $u_1 \in P^\gamma(Q_T)$ is simply transverse. Then there is a constant $p_{\mathrm{dx}} > 0$ such that any $u_2 \in \overline{B}_{p_{\mathrm{dx}}}(u) \subset P^\gamma(Q_T)$ is simply transverse with the same defining constants $C_{\mathrm{dx}}, C_{\mathrm{space}} > 0$. If $\tilde{x}_j(t)$ are the values where $u_j(\tilde{x}_j(t),t) = 0$, then $\tilde{x}_2(t) \in B_{C_{\mathrm{space}}}(\tilde{x}_1(t))$.*

Proof. The statement is more or less clear in light of the definition of the norm for P^γ (cf. Definition 3.1.1). Indeed, if $\|u_1 - u_2\|_{C^\gamma(\overline{Q_T})}$ and $\|u_1 - u_2\|_{C^\gamma(\overline{Q_T})}$ are small enough, then u_2 also satisfies Definition 1.2.20. $\square$

Lemma 3.2.5. *Let $n = 1$ and φ be given. Suppose that $u_1 \in P^\gamma(Q_T)$ is simply transverse. Then there is a constant p_{dx} such that for every $u_2 \in \overline{B}_{p_{\mathrm{dx}}}(u_1) \subset P^\gamma(Q_T)$ and every $t \in [0,T]$ satisfies*

$$\|\mathcal{H}(\xi_0, u_1)(\cdot,t) - \mathcal{H}(\xi_0, u_2)(\cdot,t)\|_{L_1(Q_T)} \leq \frac{C}{C_{\mathrm{dx}}}\|u_1 - u_2\|_{L_\infty(Q_t)}. \tag{3.2.15}$$

One also has

$$\begin{aligned}&\|\mathcal{H}(\xi_0, u_1)(\cdot,t) - \mathcal{H}(\xi_0, u_2)(\cdot,t)\|_{L_q(Q)} \\ \leq &\frac{C}{C_{\mathrm{dx}}}\left(\|u_1 - u_2\|_{L_\infty(Q_t)} + \|u_1 - u_2\|^{\frac{1}{2}}_{L_\infty(Q_t)} + \|u_1 - u_2\|^{\frac{1}{q}}_{L_\infty(Q_t)}\right).\end{aligned} \tag{3.2.16}$$

Proof. For (3.2.15) one follows the procedure laid out in the proof of Lemma 3.2.2 and notes that for $j = 1,2$ we considered $u_j \in \overline{B}_{p'}(\varphi)$, where p' was chosen so that we could find $(x_j, t_j) \in Q_T^{\mathrm{loc}}$ with $u_j(x_j, t_j) = 0$. If we have a $u_1 \in P^\gamma(Q_T)$ that is apriori simply transverse, then using Lemma 3.2.4 there is a $p_{\mathrm{dx}} > 0$ such that every $u_2 \in B_{p_{\mathrm{dx}}}(u_1)$ is also simply transverse. In particular, we have the points $\tilde{x}_j(t)$ where $u_j(\tilde{x}_j(t), t) = 0$ and a lower bound $|u_x(x,t)| > C_{\mathrm{dx}}$ which is all one needs to obtain (3.2.15).

(3.2.16) follows by making the necessary adjustments found in the proof of Lemma 3.2.3. $\square$

3.2.4. Global Apriori Estimates

Lemma 3.2.6. *Suppose that φ is transverse with respect to ξ_0. Then there exists two constants $p_{\mathrm{dx}} > 0$ and $C_{\mathrm{dx}} > 0$ with the following properties:*

(i) Every $u \in \overline{B}_{p_{\mathrm{dx}}}(\varphi) \subset P^\lambda(Q_T)$ is transverse with respect to $\mathcal{H}(\xi_0, u)$.

(ii) For every $u_1, u_2 \in \overline{B}_{p_{\mathrm{dx}}}(\varphi) \subset P^\lambda(Q_T)$ one has

$$\|\mathcal{H}(\xi_0, u_1) - \mathcal{H}(\xi_0, u_2)\|_{L_1(Q_T)} \leq \frac{C}{\sqrt{C_{\mathrm{dx}}}}\|u_1 - u_2\|_{L_\infty(Q_T)}.$$

Proof. We begin by proving item *(ii)*. Consider the set $Q^0 := \{x \in Q \mid \varphi(x) = 0\}$ and the two subsets

$$\begin{aligned}\{x \in Q \mid \varphi(x) = 0, \nabla\varphi(x) = 0\} &:= Q^{0,\nabla=0}, \\ \{x \in Q \mid \varphi(x) = 0, \nabla\varphi(x) \neq 0\} &:= Q^{0,\nabla\neq0}.\end{aligned} \tag{3.2.17}$$

Define $Q^{\beta,\nabla=0}$ and $Q^{\beta,\nabla\neq0}$ similarly, however for simplicity assume these sets are empty. The set Q^0 can be covered by a combination of two types of open sets:

(i) The spatial transversality assumption (Definition 1.2.19) prescribes that for all $x \in Q^{0,\nabla=0}$, there exists a neighborhood $Q^{\mathrm{loc},k} \subset Q$ of x such that $\xi_0(x) = +$ for all $x \in Q^{\mathrm{loc},k}$. As such, there is a $B_{p'}(\varphi) \subset P^{\lambda}(Q_T)$ for some $p' > 0$, such that $u(x,t) < p' < \beta$ for every $u \in B_{p'}(\varphi)$ and $(x,t) \in Q_T^{\mathrm{loc},k}$.

(ii) For all $x \in Q^{0,\nabla\neq 0}$, there is an integer $1 \le i \le n$ such that $\varphi(x)_{x_i} \neq 0$ and there exists a neighborhood $Q^{\mathrm{loc}} \subset Q$ and a neighborhood $B_{p'}(\varphi) \subset P^{\lambda}(Q_T^{\mathrm{loc}})$ such that Lemma 3.2.2 applies with i in place of n. To keep our notation concise assume that $i = n$ for every such Q^{loc}, and that $\varphi_{x_n}(x) > 0$.

Since the set Q^0 is compact, consider a finite subcover of Q^0

$$\mathcal{O} = \bigcup_{k=1}^{K} \bigcup_{j=K+1}^{J} \{Q^{\mathrm{loc},k}, Q^{\mathrm{loc},j}\},$$

where $Q^{\mathrm{loc},k}$ are sets of the type in item (i) and $Q^{\mathrm{loc},j}$ are sets of the type in item (ii), more specifically

$$(x', x_n) \in Q^{\mathrm{loc},j} = Q' \times B_{\delta_j}(0).$$

Note that for $x \in Q^{\mathrm{loc},0} := Q \backslash \mathcal{O}$, there is a $p_0 > 0$ such that both $|\varphi(x)| > 2p_0$ and $|\varphi(x) - \beta| > 2p_0$.

Let p_k and p_j be the radii of the sets $B_{p'}(\varphi)$ prescribed in items (i) and (ii) above, and define

$$p_{\mathrm{dx}} := \frac{1}{2} \min_{1 \le k \le K, K+1 \le j \le J} \{p_0, p_k, p_j\}.$$

For all $x \in Q^{\mathrm{loc},k}$ with $0 \le k \le K$, $|u(x)| > p_{\mathrm{dx}}$ if $\mathcal{H}(\xi_0, u) = H^+(u)$. As such, for $0 \le k \le K$, $t \in (0,T)$ and $u_1, u_2 \in \overline{B}_{p_{\mathrm{dx}}}(\varphi)$ one has

$$\begin{aligned} \|\mathcal{H}(\xi_0, u_1)(\cdot,t) - \mathcal{H}(\xi_0, u_2)(\cdot,t)\|_{L_\infty(Q^{\mathrm{loc},k})} &\le \|H^+(u_1) - H^+(u_2)\|_{L_\infty(Q_t^{\mathrm{loc},k})} \\ &\le C\|u_1 - u_2\|_{L^\infty(Q_t^{\mathrm{loc},k})}. \end{aligned} \tag{3.2.18}$$

Note that (3.2.18) also applies for $k = 0$.

Next, observe that for any $u_1, u_2 \in \overline{B}_{p_{\mathrm{dx}}}(\varphi)$ and $K+1 \le j \le J$, Lemma 3.2.2 applies. In particular, one can write $Q^{\mathrm{loc},j} = Q' \times B_{\delta_j}(0)$ and

$$\int_{Q'} \int_{-\delta_j}^{\delta_j} |\mathcal{H}(\xi_0, u_1) - \mathcal{H}(\xi_0, u_2)|(x', x_n, t) dx_n dx' \le \frac{C}{\sqrt{C_{\mathrm{dx}}^j}} \|u_1 - u_2\|_{L_\infty(Q_t^{\mathrm{loc}})}, \tag{3.2.19}$$

holds for any $t \in (0,T)$, where C_{dx}^j is a lower bound on $\|\nabla u_1\|$ and $\|\nabla u_2\|$ on the neighborhood $Q^{\mathrm{loc},j}$. The current Lemma now follows from (3.2.18) and (3.2.19), where in the later we replace C_{dx}^j with

$$C_{\mathrm{dx}} = \min_{K+1 \le j \le J} C_{\mathrm{dx}}^j.$$

To prove item *(i)*, note that if $u(x,t)=0$, then $x \in Q^{\mathrm{loc},j}$ for some $K+1 \le j \le J$. On such a set $\|\nabla u\| \neq 0$ by construction. □

Lemma 3.2.7. *Suppose φ is transverse with respect to ξ_0, then there is a p_{dx} such that for all $u_1, u_2 \in \overline{B}_{p_{\mathrm{dx}}}(\varphi) \subset P^\lambda(Q_T)$*

$$\begin{aligned} &\|\mathcal{H}(\xi_0,u_1)(\cdot,t) - \mathcal{H}(\xi_0,u_2)(\cdot,t)\|_{L_q(Q)} \\ \le& \frac{C}{C_{\mathrm{dx}}}\left(\|u_1-u_2\|_{L_\infty(Q_t)} + \|u_1-u_2\|_{L_\infty(Q_t)}^{\frac{1}{2}} + \|u_1-u_2\|_{L_\infty(Q_t)}^{\frac{1}{q}} \right). \end{aligned} \tag{3.2.20}$$

Proof. Repeat the steps of the proof of Lemma 3.2.6, however applying the local L_q estimates of Lemma 3.2.3 instead of the local L_1 estimates of Lemma 3.2.2. □

3.3. Proof of Theorem 1.3.1

Theorem 3.3.1. *Suppose that φ is transverse with respect to ξ_0. Then there is a $T_{\mathrm{dx}} > 0$ such that for any $u_0 \in P^\lambda(Q_{T_{\mathrm{dx}}})$, there exists a unique solution $u \in W_q^{2,1}(Q_{T_{\mathrm{dx}}})$ to* (2.2.1)–(2.2.3) *with nonlinearity*

$$f_0(u,x,t) = \mathcal{H}(\xi_0,u_0) + c_\infty(u_0 - u), \tag{3.3.1}$$

that satisfies the following:

(i) $u \in P^\lambda(Q_T)$ *and* $u \in P^\gamma(Q_{T_{\mathrm{dx}}})$.

(ii) *For every* $t \in [0, T_{\mathrm{dx}}]$

$$\|u(\cdot,t) - \varphi\|_{P^\lambda(Q_{T_{\mathrm{dx}}})} \le p_{\mathrm{dx}},$$

where p_{dx} is the constant from Lemmas 3.2.6 and 3.2.7.

(iii) *Let $u_{0n} \in P^\lambda(Q_{T_{\mathrm{dx}}})$ be a sequence of functions such that $u_{0n} \to u_0$ in $P^\lambda(Q_{T_{\mathrm{dx}}})$. If u_n denotes the solution to* (2.2.1)–(2.2.3) *with nonlinearity*

$$f_{0n}(u_n,x,t) = \mathcal{H}(\xi_0,u_{0n}) + c_\infty(u_{0n} - u_n),$$

then $u_n \to u$ in $P^\gamma(Q_{T_{\mathrm{dx}}})$.

Proof. We first establish that for any $T > 0$ and any $u_0 \in P^\lambda(Q_T)$, there is a unique solution to (2.2.1)–(2.2.3) with nonlinearity (3.3.1) such that both $u \in W_q^{2,1}(Q_T)$ and $\|u\|_{L_\infty(Q_T)} \le u_\infty$.

Clearly f_0 satisfies Condition 2.2.1. By Theorem 2.2.3, there is an $E_{\infty,T}$-mild solution u, and a maximal existence time $T_1 > 0$ (cf. Definition 2.3.2 and Lemma 2.3.3).

Since f_0 is not defined beyond time T clearly $T_1 \le T$. We claim in fact that $T_1 = T$. Indeed, we see from (1.2.11), that our nonlinearity satisfies the dissipativity conditions of Lemma 2.3.1. In particular, $\|u(\cdot,T_1)\|_{L_\infty(Q)} < u_\infty < \infty$ which would not be the case if $T_1 < T$. Finally observe that by Theorem 2.2.4, u is also a strong solution $u \in W_q^{2,1}(Q_T)$.

Now let's address item (i) of the present Theorem. Applying inequality (2.1.5) we see that

$$\begin{aligned} \|u\|_{P^\gamma(Q_T)} \le \|u\|_{C^\gamma(\overline{Q_T})} + \sum_{i=1}^{n} \|u_{x_i}\|_{C^\gamma(\overline{Q_T})} \quad &\le C_{\text{strong}}\left(\|f_{0n}\|_{L_q(Q_T)} + \|\varphi\|_{\mathcal{W}}\right), \\ &\le C_{\text{strong}}(f_{0,q} + \|\varphi\|_{\mathcal{W}}). \end{aligned} \tag{3.3.2}$$

Because $\|u\|_{P^\lambda(Q_T)} \le c_{\lambda,\gamma}\|u\|_{P^\gamma(Q_T)}$, (3.3.2) implies that $u \in P^\lambda(Q_T)$ (cf. Definition 3.1.1). Note that $\|u\|_{P^\gamma(Q_T)} \le C_P$ is already clear from (3.3.2) and thus $u \in P^\gamma(Q_T)$.

We can now determine T_{dx} and prove item (ii). Indeed because $\|u\|_{P^\gamma(Q_T)} \le C_P$ one obtains

$$\|u(\cdot,t) - \varphi\|_{C^\gamma(\overline{Q_T})} + \sum_{j=1}^{n} \|u_{x_j}(\cdot,t) - \varphi_{x_j}\|_{C^\gamma(\overline{Q_T})} \le C_P T_{\text{dx}} \le p_{\text{dx}}, \tag{3.3.3}$$

for T_{dx} sufficiently small.

For item (iii) take the said sequence $\{u_{0n}\}$ and note that it is bounded in $P^\lambda(Q_T)$. Using inequality (2.1.5) the sequence of solutions $\{u_n\}$ is also bounded in $W^{2,1}(Q_T)$, so the compactness of the embedding $W^{2,1}(Q_T) \subset C(\overline{Q_T})$ (cf. Lemma 1.2.10), yields a convergent subsequence (not relabelled)

$$\|u_n - u'\|_{L_\infty(Q_{T_{\text{dx}}})} \to 0. \tag{3.3.4}$$

Consider the inequality

$$\begin{aligned} \|f_{0n}(u_n,x,t) - f_0(u',x,t)\|_{L_q(Q_{T_{\text{dx}}})} \le & C\|\mathcal{H}(\xi_0,u_{0n}) - \mathcal{H}(\xi_0,u_0)\|_{L_q(Q_{T_{\text{dx}}})} \\ &+ c_\infty(\|u_{0n} - u_0\|_{L_q(Q_{T_{\text{dx}}})} \\ &+ \|u_n - u'\|_{L_q(Q_{T_{\text{dx}}})}). \end{aligned} \tag{3.3.5}$$

Using the assumptions of the present Theorem, (3.3.4) and Lemma 3.2.7, we see that the right-hand side of (3.3.5) goes to zero. This means that $\|u_n - u'\|_{W_q^{2,1}(Q_{T_{\text{dx}}})}$ goes to zero and hence u' solves

$$\begin{cases} u'_t - \Delta u' = f_0(u',x,t), & (3.3.6) \\ u'|_{t=0} = \varphi, & (3.3.7) \\ \left.\dfrac{\partial u'}{\partial \nu}\right|_{\partial Q_T} = 0. & (3.3.8) \end{cases}$$

This problem has a unique $E_{\infty,T}$-mild solution (cf. Theorem 2.2.3), and hence for every subsequence $u_{0n} \to u_0$, one has $u_n \to u'$. □

Proof of Theorem 1.3.1. Consider $u_0 \in P^\lambda(Q_{T_{\mathrm{dx}}})$ and let u solve the auxiliary problem (2.2.1)–(2.2.3) with nonlinearity

$$f_0(u,x,t) = \mathcal{H}(\xi_0, u_0) + c_\infty(u_0 - u).$$

By Theorem 3.3.1 we have $u \in P^\gamma(Q_{T_{\mathrm{dx}}})$ and $u \in P^\lambda(Q_{T_{\mathrm{dx}}})$, therefore the map $u_0 \mapsto u$ maps $P^\lambda(Q_{T_{\mathrm{dx}}})$ into itself. This map is continuous by item (iii) of Theorem 3.3.1, and compact since $C^\gamma(\overline{Q_{T_{\mathrm{dx}}}}) \subset C^\lambda(\overline{Q_{T_{\mathrm{dx}}}})$ is a compact embedding. More specifically

$$C^\lambda(Q_{T_{\mathrm{dx}}}) \underset{u_0 \to u}{\longrightarrow} C^\gamma(Q_{T_{\mathrm{dx}}}) \underset{\text{Identity}}{\longrightarrow} C^\lambda(Q_{T_{\mathrm{dx}}}),$$

where the first map is continuous and the second is compact. By the Schauder Fixed Point Theorem [GT01, Corollary 10.2, pg.222] $u_0 \mapsto u$ has a fixed point which is a solution to (1.2.1)–(1.2.4).

□

3.4. Proof of Theorem 1.3.2

Lemma 3.4.1. *There is a $T \le T_{\max}$ such that if $u_1, u_2 \in W_q^{2,1}(Q_T)$ are two solutions to (1.2.1)–(1.2.4), then $u_1 = u_2$ a.e. on Q_T.*

Proof. Let u_1, u_2 be two solutions to (1.2.1)–(1.2.4) on Q_T. If we define

$$h = \mathcal{H}(\xi_0, u_1) - \mathcal{H}(\xi_0, u_2), \quad w = u_1 - u_2,$$

then w satisfies the differential equation,

$$w_t = \Delta w + h, \tag{3.4.1}$$

$$w|_{t=0} = 0, \qquad \frac{\partial w}{\partial \nu}\Big|_{\partial Q} = 0, \tag{3.4.2}$$

and w can be represented as a convolution with the Green function

$$w(x,t) = \int_0^t \int_Q \mathcal{G}(x,y,t,s) h(y,s) dy ds. \tag{3.4.3}$$

Let $T \le T_{\mathrm{dx}}$. By item (ii) of Theorem 3.3.1, Lemma 3.2.6 is applicable, and so let us again consider the open cover

$$\{Q^{\mathrm{loc},k}, Q^{\mathrm{loc},j}\}_{0 \le k \le K, K+1 \le j \le J} \quad Q^{\mathrm{loc},j} = Q' \times B_{\delta_j}(0),$$

from the proof of the Lemma.

For a fixed $(x,t) \in Q_T$ consider the integral (3.4.3), where the variable of integration y is restricted to $Q^{\text{loc},k}$, with $0 \le k \le K$. Applying inequality (3.2.18) (which also holds for $k = 0$) and Theorem 2.4.1 one obtains

$$\begin{aligned}
&\int_0^t \int_{Q^{\text{loc},k}} |\mathcal{G}(x,y,t,s)| h(y,s) dy ds, \\
\le& C\|u_1 - u_2\|_{L_\infty(Q_T^{\text{loc},k})} \int_0^t \int_{Q^{\text{loc},k}} |\mathcal{G}(x,y,t,s)| dy ds, \\
\le& C\|u_1 - u_2\|_{L_\infty(Q_T^{\text{loc},k})} \int_0^t ds \int_{\mathbb{R}^n} \frac{C_1}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{\|x-y\|^2}{t-s}\right) dy \\
\le& Ct\|u_1 - u_2\|_{L_\infty(Q_T^{\text{loc},k})} \\
\le& CT\|u_1 - u_2\|_{L_\infty(Q_T^{\text{loc},k})}.
\end{aligned} \tag{3.4.4}$$

Note that the constant C in the last line of (3.4.4) does note depend on $(x,t) \in Q_T$.

For the case $Q^{\text{loc},j}$ we let $y = (y', y_n)$ with $y' \in \mathbb{R}^{n-1}$, and focus on the integral with respect to the variable $y_n \in \mathbb{R}$. To this end, let's factorize the Green Function as follows:

$$\begin{aligned}
|\mathcal{G}(x,y,t,s)| &\le \frac{C_1}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{C_2\|x-y\|^2}{t-s}\right) \\
&\le \left(\frac{1}{(t-s)^{\frac{n-1}{2}}} \exp\left(-\frac{C_2(x'-y')^2}{t-s}\right)\right) \frac{C_1}{\sqrt{t-s}}.
\end{aligned} \tag{3.4.5}$$

We can now invoke Lemma 3.2.2 and integrate over y_n and y' separately, more specifically

$$\begin{aligned}
&\int_0^t \int_{Q^{\text{loc},j}} |\mathcal{G}(x,y,t,s)| h(y,s) dy ds, \\
\le& \int_0^t \int_{Q'} \frac{C_1}{(t-s)^{\frac{n-1}{2}}} \exp\left(-\frac{C_2(x'-y')^2}{t-s}\right) \int_{-\delta}^{\delta} \frac{C_1}{\sqrt{t-s}} |h(y', y_n, s)| dy_n dy' ds \\
\le& \int_0^t \frac{C\|u_1 - u_2\|_{L_\infty(Q_t^{\text{loc},j})}}{\sqrt{t-s}} \int_{Q'} \frac{1}{(t-s)^{\frac{n-1}{2}}} \exp\left(-\frac{C_2(x'-y')^2}{t-s}\right) dy' ds \\
\le& C\sqrt{T}\|u_1 - u_2\|_{L_\infty(Q_T^{\text{loc},j})}.
\end{aligned} \tag{3.4.6}$$

Note that the constant C in the last line of (3.4.6) does not depend on $(x,t) \in Q_T$.

Combining (3.4.4) and (3.4.6) one obtains

$$h(x,t) \le C\sqrt{T}\|u_1 - u_2\|_{L_\infty(Q_T)}.$$

Taking the supremum over all $(x,t) \in Q_T$ and $T > 0$ sufficiently small one obtains

$$\|w\|_{L_\infty(Q_T)} \le \frac{1}{2}\|w\|_{L_\infty(Q_T)},$$

which is only possible if $\|w\|_{L_\infty(Q_T)} = 0$. □

Proof of Theorem 1.3.2. Suppose that $u_1 \neq u_2$ in $Q_{T_{\max}}$ on a set of positive measure. Let

$$T' = \sup\{T \in [0, T_{\max}] \mid u_1 \neq u_2 \text{ a.e. in } Q_T\}.$$

Note that T' is well defined because $u_1(\cdot, 0) = u_2(\cdot, 0)$. Because both u_1 and u_2 are continuous, $u_1(\cdot, T') = u_2(\cdot, T')$. Moreover, because $u_1(\cdot, T')$ is transverse by assumption, Lemma 3.4.1 implies that there exists a $T > T'$ such that $u_1 = u_2$ in Q_T, a contradiction. □

4. Convergence of the Diffusing Variable

The standing assumptions for this Chapter are that $n = 1$, $\varphi \in \mathcal{W}$ is simply transverse with constants C_{dx} and C_{space}, and that Condition 1.2.14 holds.

4.1. Convergence Results when u^ε is apriori Simply Transverse

Lemma 4.1.1. *Suppose that the solution $u^\varepsilon \in W_q^{2,1}$ to* (1.2.6)–(1.2.8) *is simply transverse and let $t \in [0, T]$. Then the inequality*

$$\|v^\varepsilon(\cdot, t) - \mathcal{H}(u^\varepsilon, \xi_0)(\cdot, t)\|_{L_p(Q)} \leq \frac{C}{C_{\text{dx}}(1-\gamma)} \varepsilon^{\frac{2}{3} - \cdot \frac{1}{p}}, \tag{4.1.1}$$

holds when $p = 1$ or $p = q$.

Lemma 4.1.2. *Suppose that the solution $u \in W_q^{2,1}(Q_T)$ to* (1.2.1)–(1.2.4) *is simply transverse and that $u^\varepsilon \in B_{p_{\text{dx}}}(u) \subset P^\gamma(Q_T)$ where p_{dx} is the constant from Lemma* (3.2.4) *and* (3.2.5)*. Moreover, assume that Lemma 4.1.1 holds. Then for every $t \in [0, T]$*

$$\|u^\varepsilon - u\|_{L_\infty(Q_t)} \leq \frac{C_1 \sqrt{T}}{C_{\text{dx}}(1-\gamma)} \exp\left(\left(\frac{C_2}{C_{\text{dx}}}\right)^2 t\right) \varepsilon^{\frac{2}{3}-}, \tag{4.1.2}$$

where $C_1, C_2 > 0$.

Proof. Consider $u^\varepsilon - u$ and the diffusion equation

$$(u^\varepsilon - u)_t = (u^\varepsilon - u)_{xx} + (v^\varepsilon - \mathcal{H}(\xi_0, u)), \tag{4.1.3}$$

with zero Neumann boundary conditions and zero initial conditions. If we fix $(x, t) \in Q_T$, then we can represent $u^\varepsilon - u$ as a convolution with the Green function on Q_T to obtain

$$\begin{aligned} |(u^\varepsilon - u)|(x, t) \leq & \int_0^t \int_Q |v^\varepsilon - \mathcal{H}(\xi_0, u^\varepsilon)|(y, s)|\mathcal{G}(x, y, t, s)| dy ds \\ & + \int_0^t \int_Q |\mathcal{H}(\xi_0, u^\varepsilon) - \mathcal{H}(\xi_0, u)|(y, s)|\mathcal{G}(x, y, t, s)| dy ds. \end{aligned} \tag{4.1.4}$$

Consider the second term in (4.1.4),

$$\int_0^t \int_Q |\mathcal{H}(\xi_0, u^\varepsilon) - \mathcal{H}(\xi_0, u)|(y, s)|\mathcal{G}(x, y, t, s)| dy ds. \tag{4.1.5}$$

If one applies Lemma 3.2.5 and Theorem 2.4.1 one obtains,

$$
\begin{aligned}
(4.1.5) &\le \int_0^t \frac{C}{\sqrt{t-s}} \|(\mathcal{H}(\xi_0, u^\varepsilon) - \mathcal{H}(\xi_0, u))(\cdot, t)\|_{L_1(Q)} ds \\
&\le \int_0^t \frac{C}{C_{\mathrm{dx}}\sqrt{t-s}} \max_{\sigma\in[0,s]} \|(u^\varepsilon - u)(\cdot, \sigma)\|_{L_\infty(Q)} ds.
\end{aligned}
$$

For convenience define the following notation:

$$
\begin{aligned}
\mathrm{Max}^\varepsilon(s) &:= \max_{\sigma\in[0,s]} \|(u^\varepsilon - u)(\cdot, \sigma)\|_{L_\infty(Q)}, \\
h^\varepsilon(t) &:= \sup_{x\in Q} \int_0^t \int_Q |v^\varepsilon - \mathcal{H}(u^\varepsilon, \xi_0)|(y,s)|\mathcal{G}(x,y,t,s)| dy ds.
\end{aligned}
$$

Taking the supremum over all $x \in Q$ in inequality (4.1.4), one has the inequality

$$
\|(u^\varepsilon - u)(\cdot, t)\|_{L_\infty(Q)} \le h^\varepsilon(t) + \int_0^t \frac{C}{C_{\mathrm{dx}}\sqrt{t-s}} \mathrm{Max}^\varepsilon(s). \tag{4.1.6}
$$

Consider $t^* = \inf_{s\in[0,t]}\{\mathrm{Max}^\varepsilon(s) = \mathrm{Max}^\varepsilon(t)\}$. Then because $\mathrm{Max}^\varepsilon(t^*) \ge \mathrm{Max}^\varepsilon(s)$ for every $s \in [0, t^*]$,

$$
\begin{aligned}
&\int_0^t \frac{1}{\sqrt{t-s}} \mathrm{Max}^\varepsilon(s) ds - \int_0^{t^*} \frac{1}{\sqrt{t^*-s}} \mathrm{Max}^\varepsilon(s) ds \\
=&\int_{t^*}^t \frac{1}{\sqrt{t-s}} \mathrm{Max}^\varepsilon(t^*) ds - \int_0^{t^*} \left(\frac{1}{\sqrt{t^*-s}} - \frac{1}{\sqrt{t-s}}\right) \mathrm{Max}^\varepsilon(s) ds \\
\ge&\mathrm{Max}^\varepsilon(t^*) \int_{t^*}^t \frac{1}{\sqrt{t-s}} ds - \int_0^{t^*} \frac{1}{\sqrt{t^*-s}} - \frac{1}{\sqrt{t-s}} ds \\
\ge&2\mathrm{Max}^\varepsilon(t^*)(\sqrt{t} - \sqrt{t^*}) \\
\ge&0.
\end{aligned}
$$

Now rewrite (4.1.6) to obtain

$$
\begin{aligned}
\mathrm{Max}^\varepsilon(t) = \mathrm{Max}^\varepsilon(t^*) &\le \sup_{\tau\in[0,t^*]} h^\varepsilon(\tau) + \int_0^{t^*} \frac{\tilde{C}_2}{C_{\mathrm{dx}}\sqrt{t^*-s}} \mathrm{Max}^\varepsilon(s) ds \\
&\le \sup_{\tau\in[0,t^*]} h^\varepsilon(\tau) + \int_0^t \frac{\tilde{C}_2}{C_{\mathrm{dx}}\sqrt{t-s}} \mathrm{Max}^\varepsilon(s) ds,
\end{aligned}
$$

where we have labelled the constant $\tilde{C}_2$ to help keep track of its role later on. Using Lemma 4.1.1 and Theorem 2.4.1 we can bound $h^\varepsilon(\tau)$ by a constant independent of τ, namely

$$
h^\varepsilon(\tau) \le \frac{\tilde{C}_1\sqrt{T}}{C_{\mathrm{dx}}(1-\gamma)} \varepsilon^{\frac{2}{3}-}, \tag{4.1.7}
$$

where $\tilde{C}_1$ is a positive constant. We can now apply Grönwall's Lemma (cf. Lemma A.0.2 in Appendix A) to obtain

$$\|u^\varepsilon - u\|_{L_\infty(Q_t)} \le \mathrm{Max}^\varepsilon(t) \le \frac{2\tilde{C}_1\sqrt{T}}{C_{\mathrm{dx}}(1-\gamma)} \exp\left(\left(\frac{\tilde{C}_2}{C_{\mathrm{dx}}}\right)^2 \pi t\right) \varepsilon^{\frac{2}{3}-}. \tag{4.1.8}$$

□

4.2. Proof of Theorem 1.3.4

Lemma 4.2.1. *Let $T_{\max} > 0$ and suppose that the solution u to (1.2.1)–(1.2.4) is simply transverse on $Q_{T_{\max}}$. Then there exists an $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$,*

$$\|u - u^\varepsilon\|_{P^\gamma(Q_{T_{\max}})} \le \frac{p_{\mathrm{dx}}}{2},$$

where p_{dx} is the constant from Lemma (3.2.4) and (3.2.5), and ε_1 does not depend on $T_{\max}$.

Proof. Fix $\varepsilon > 0$ and consider

$$T' = \inf_{T \in [0, T_{\max}]} \{\|u^\varepsilon - u\|_{P^\gamma(Q_T)} \le \frac{p_{\mathrm{dx}}}{2}\}.$$

We claim that $T' > 0$. Indeed, by combining inequality (2.1.5), Theorem 1.2.18, and Condition 1.2.15 we conclude that

$$\|u\|_{P^\gamma(Q_T)} \le C_P T, \qquad \|u^\varepsilon\|_{P^\gamma(Q_T)} \le C_P T. \tag{4.2.1}$$

In particular, for T sufficiently small

$$\|u - \varphi\|_{P^\gamma(Q_T)} \le \frac{p_{\mathrm{dx}}}{4}, \qquad \|u^\varepsilon - \varphi\|_{P^\gamma(Q_T)} \le \frac{p_{\mathrm{dx}}}{4},$$

which proves that $T' > 0$. If $T' = T_{\max}$, then we are done, so suppose that $T' < T_{\max}$.

Let $T' < T' + \delta < T_{\max}$ where $\delta > 0$ will be specified shortly. Using (4.2.1) again one obtains

$$\begin{aligned} \|u - u^\varepsilon\|_{P^\gamma(Q_{T'+\delta})} \le& \|u - u^\varepsilon\|_{P^\gamma(Q_{T'})} + \|u - u^\varepsilon\|_{P^\gamma(Q\times(T', T'+\delta))}, \\ \le& \|u - u^\varepsilon\|_{P^\gamma(Q_{T'})} + C_P\delta. \end{aligned} \tag{4.2.2}$$

□

Since $u^\varepsilon \in B_{p_{\text{dx}}}(u) \subset P^\gamma(Q_{T'})$ we can apply Lemmas 3.2.5, 4.1.1 and 4.1.2, as well as (2.1.5) which yields

$$\begin{aligned} \|u - u^\varepsilon\|_{P^\gamma(Q_{T'})} &\leq C_{\text{strong}} \|v - v^\varepsilon\|_{L_q(Q_{T'})} \\ &\leq \|\mathcal{H}(\xi_0, u) - \mathcal{H}(\xi_0, u^\varepsilon)\|_{L_q(Q_{T'})} + \|\mathcal{H}(\xi_0, u^\varepsilon) - v^\varepsilon\|_{L_q(Q_{T'})} \\ &\leq \frac{C_3\sqrt{T_{\max}}}{C_{\text{dx}}(1-\gamma)} \exp\left(\left(\frac{C_2}{C_{\text{dx}}}\right)^2 T'\right) \varepsilon^{\frac{2}{3} - \cdot \frac{1}{q}}, \end{aligned} \tag{4.2.3}$$

where C_3 is a positive constant, denoted as such because it is not the same as C_1 in Lemma 4.1.2. Choose ε_1 such that

$$\frac{C_3\sqrt{T_{\max}}}{C_{\text{dx}}(1-\gamma)} \exp\left(\left(\frac{C_2}{C_{\text{dx}}}\right)^2 T'\right) \varepsilon_1^{\frac{2}{3} - \cdot \frac{1}{q}} \varepsilon_1^{\frac{2}{3} - \cdot \frac{1}{q}} = \frac{p_{\text{dx}}}{4}, \tag{4.2.4}$$

and $\delta \in (0, T_{\max} - T')$ such that

$$C_P \delta \leq \frac{p_{\text{dx}}}{4}.$$

Returning to (4.2.2), we see that if we define ε_1 as in (4.2.4), which only depends on C_{dx}, $T_{\max}$ and γ, then we can find a δ that contradicts the assumption that $T' < T$. The Lemma follows because this argument is valid whenever $\varepsilon < \varepsilon_1$.

Proof of Theorem 1.3.4. By Lemma 4.2.1 there is an $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$ $\|u - u^\varepsilon\|_{P^\gamma(Q_T)} \leq \frac{p_{\text{dx}}}{2}$. In particular, Lemmas 3.2.4 and 4.1.2 are applicable which finishes the proof . □

5. Fast-ODE Phase Space

In this Chapter we divide the phase space of (1.2.6) into a number of smaller regions. One must be familiar with these Definitions in order to follow the proofs in Chapters 6 and 7. The division of the phase space will facilitate the analysis of simply transverse functions (cf. Definition 1.2.20). In particular, the subdivision will depend on the constants C_{space} and C_{dx}, as well as ε. Let us begin with a common piece of notation.

Definition 5.0.2. *For any set $A \subset \mathbb{R}^2$, let $\pi_u A$ be the projection onto the u-axis and π_v the projection onto the v-axis.*

5.1. Critical Region

Definition 5.1.1. *We define a set A^-_{crit} of the form $A^-_{\text{crit}} = (-M, M) \times (-2M^2, 2M^2)$ (cf. Figure 5.1), where A^-_{crit} satisfies the following:*

- $A^-_{\text{crit}} \subset A^-_{\text{quad}}$ *(cf. Condition 1.2.6).*
- $\pi_u A^-_{\text{crit}} \subset B_{2C_{\text{dx}}C_{\text{space}}}(0)$.

It's vital to emphasize that $M \to 0$ if $C_{\text{dx}} \to 0$ (cf. Section 1.2.6). However, for the purposes of simplifying our calculations we let the following hold.

Condition 5.1.2. Take $M = 1$, in particular, $(u, v) = (1, -1) \in A^-_{\text{crit}}$.

This Condition will not distract from the central idea of the proof of Theorem 1.3.4.

Inside A^-_{crit} we will denote quantities using capital letters wherever possible, with the exception of the coordinate v.

5.1.1. Critical Sleeve

Recall the constants λ_0 and γ from Definition 1.3.3. In addition, define the constant

$$\mu_0 := \frac{1}{3_-} = \frac{1}{3} - \frac{\lambda_0}{2}. \tag{5.1.1}$$

If λ_0 and γ satisfy Definition 1.3.3 and μ_0 is as defined as in (5.1.3), it is possible to choose two additional constants λ_1, λ_2 satisfying the following Condition.

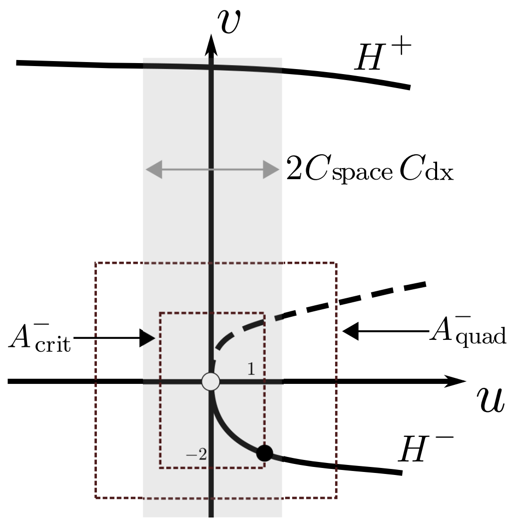

Figure 5.1.: A^-_{crit} lies in A^-_{quad} with the point $(u, v) = (1, -2)$ indicated with a black dot. The quadratic scaling near $(u, v) = 0$ will be very important in tracking trajectories as they approach the fold. That A^-_{crit} is contained in the shaded region will help us choose good coordinates for calculating the L_q-norm in Theorem (iii).

Definition 5.1.3. *Choose two constants* $\frac{1}{3} > \lambda_1, \lambda_1 > 0$ *such that*

$$\gamma(1 - \lambda_1) - (2 + \gamma)\mu_0 > \lambda_2 > 0. \tag{5.1.2}$$

Note that if $\mu_j < \mu_0$*, then* (5.1.2) *holds with* μ_j *in place of* μ_0*.*

These convoluted definitions can be understood (in principle) in the following way. If the initial data φ is smooth, then q can be taken arbitrarily large, or formally speaking $q = \infty$. In this case one can formally take

- $\gamma = 1$.
- $\lambda_0 = 0$, i.e. $\frac{2}{3}_- = \frac{2}{3}$.
- $\mu_0 = \frac{1}{3}_- = \frac{1}{3}$.
- $\lambda_1 = \lambda_2 = 0$.

In other words, we have matched the asymptotics of the 0th-order approximation of the Slow-Fast ODE (cf. Section 1.1.3).

Definition 5.1.3 will give a precise description of how close we can get to the Slow-Fast ODE asymptotics. The role of λ_1 and λ_2 will become clear shortly.

Definition 5.1.4. *As a matter of notation, if we are given any* $u, U \in \mathbb{R}$*, then let* $w = \sqrt{|u|}$ *and* $W = \sqrt{|U|}$*. Moreover, let*

$$u_+^\varepsilon := \varepsilon^{\frac{2}{3}-}, \qquad w_+^\varepsilon := \varepsilon^{\frac{1}{3}-}. \tag{5.1.3}$$

Definition 5.1.5. *Given a sequence* $\{\mu_j\}_{j\geq 0} \subset \mathbb{R}$ *define*

$$\begin{aligned} W_j &= \varepsilon^{\mu_j}, \\ U_j &= \varepsilon^{2\mu_j}, \\ \delta_j &= \gamma(1-\lambda_1) - \mu_j(1+\gamma) - \lambda_2, \\ D_j &= \varepsilon^{\delta_j}. \end{aligned} \tag{5.1.4}$$

Definition 5.1.6. *Let* $W_0 = \varepsilon^{\frac{1}{3}-}$ *(cf.* (5.1.3) *and Definition 5.1.4), and let* δ_0 *be defined as in Definition* (5.1.5). *Define* W_{j+1} *by the recursive formula*

$$W_{j+1} = W_j + \varepsilon^{\delta_j+\lambda_2}, \tag{5.1.5}$$

and define δ_{j+1} *via* (5.1.4).

Lemma 5.1.7. *There exists a positive integer* J *such that* $U_J < 1 \leq U_{J+1}$.

Proof. If there is no such J, then the step size ε^{δ_j} must go to zero ($\delta_j \to \infty$), which by (5.1.4) implies that $W_j = \varepsilon^{\mu_j}$ and $U_j = \varepsilon^{2\mu_j}$ must go to infinity, i.e., $\mu_j \to -\infty$, a clear contradiction. □

Definition 5.1.8. *We define the critical sleeve (denoted by* $\chi^-_{\text{crit}}(\varepsilon)$ *and schematized in Figures 5.2 and 5.3) via the formula*

$$\chi^-_{\text{crit}}(\varepsilon) := \bigcup_{j=0}^{J-1} \{(u,v) \mid (u,v) \in [U_j, U_{j+1}] \times [-U_{j+1} - D_{j+1}, -U_j + D_j]\}.$$

Due to Definition 5.1.3,

$$0 \leq \varepsilon^{\mu_0} - \varepsilon^{\delta_0} = \varepsilon^{\mu_0}(1 - \varepsilon^{\gamma(1-\lambda_1)-\mu_0(2+\gamma)-\lambda_2}), \tag{5.1.6}$$

for ε sufficiently small. Note that (5.1.6) holds for all $j \in \{0, \cdots, J\}$. Moreover, because $\lambda_1, \lambda_2 > 0$ we can formalize the following Condition.

Definition 5.1.9. *Let* ε_2 *be the supremum of all* ε *satisfying the inequalities*

$$\begin{aligned} &2\left(\exp\left(-2c(\gamma)\varepsilon^{-\lambda_1}\right) + \varepsilon^{\lambda_2}\right) \leq 1, \\ &0 \leq \varepsilon^{\mu_0} - \varepsilon^{\delta_0} \leq \varepsilon^{\mu_0}(1 - \varepsilon^{\gamma(1-\lambda_1)-\mu_0(2+\gamma)-\lambda_2}). \end{aligned} \tag{5.1.7}$$

5.2. Normally Hyperbolic Region

Lemma 5.2.1 (Normally hyperbolic region)**.** *There are two bounded open sets* $A^{\pm}_{\text{hyp}} \subset \mathbb{R}^2$ *which we call normally hyperbolic regions (cf. Figure 5.4), where* A^-_{hyp} *satisfies the following conditions (with analogous conditions for* A^+_{hyp}*):*

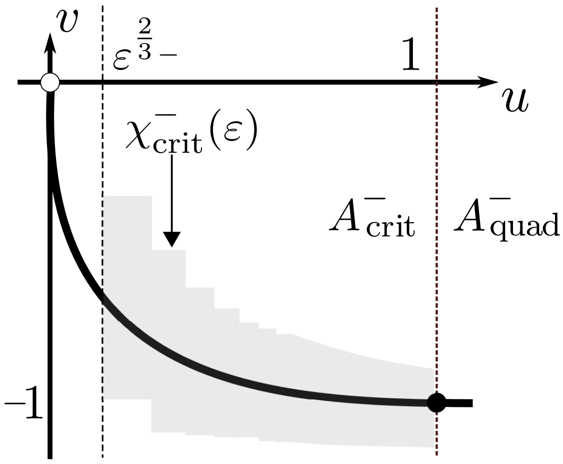

Figure 5.2.: The critical sleeve in u coordinates between U_0 and U_J. The point $(u, v) = (1, -1)$ on $\partial A^{-}_{\text{crit}}$ is indicated by a black dot.

(i) $\pi_u A^{-}_{\text{hyp}} \subset (0, u_\infty)$ *and* $(1, -1) \in A^{-}_{\text{hyp}}$. *In particular,* $A^{-}_{\text{crit}} \bigcap A^{-}_{\text{hyp}} \neq \emptyset$.

(ii) *For every* $(u, v) \in \pi_u A^{-}_{\text{hyp}}$, $f(u, v) = 0$ *if and only if* $u = H^{-}(u)$.

(iii) *If* (u, v) *is in the closure of* A^{-}_{hyp}, *then*

$$\frac{\partial f}{\partial v}(u, v) < 0, \frac{\partial f}{\partial u}(u, v) < 0.$$

In particular $f(u, v) < 0$ *if* $v > H^{-}(v)$ *and* $f(u, v) > 0$ *if* $v < H^{-}(v)$.

(iv) *If we define the constant*

$$\frac{1}{C_{\text{Lip}}} = \min_{(u,v) \in A^{-}_{\text{hyp}}} \{1, \left|\frac{\partial f}{\partial v}(u, v)\right| \left|\frac{\partial f}{\partial u}(u, v)\right|^{-1}\}, \tag{5.2.1}$$

then C_{Lip} *satisfies the inequality*

$$|H^{-}(u_1) - H^{-}(u_2)| \leq C_{\text{Lip}} |u_1 - u_2|. \tag{5.2.2}$$

Proof. For every $u > 0$ there exists a $\delta_u > 0$ and a neighborhood $B_{\delta_u}(u, H^{-}(u))$, such that for every $(\overline{u}, \overline{v}) \in \overline{B}_{\delta_u}(u, H^{-}(u))$

$$\begin{gathered} \frac{\partial f}{\partial v}(\overline{u}, \overline{v}) < 0, \\ \frac{\partial f}{\partial u}(\overline{u}, \overline{v}) < 0, \\ f(\overline{u}, \overline{v}) = 0 \text{ iff } \overline{v} = H^{-}(\overline{u}). \end{gathered} \tag{5.2.3}$$

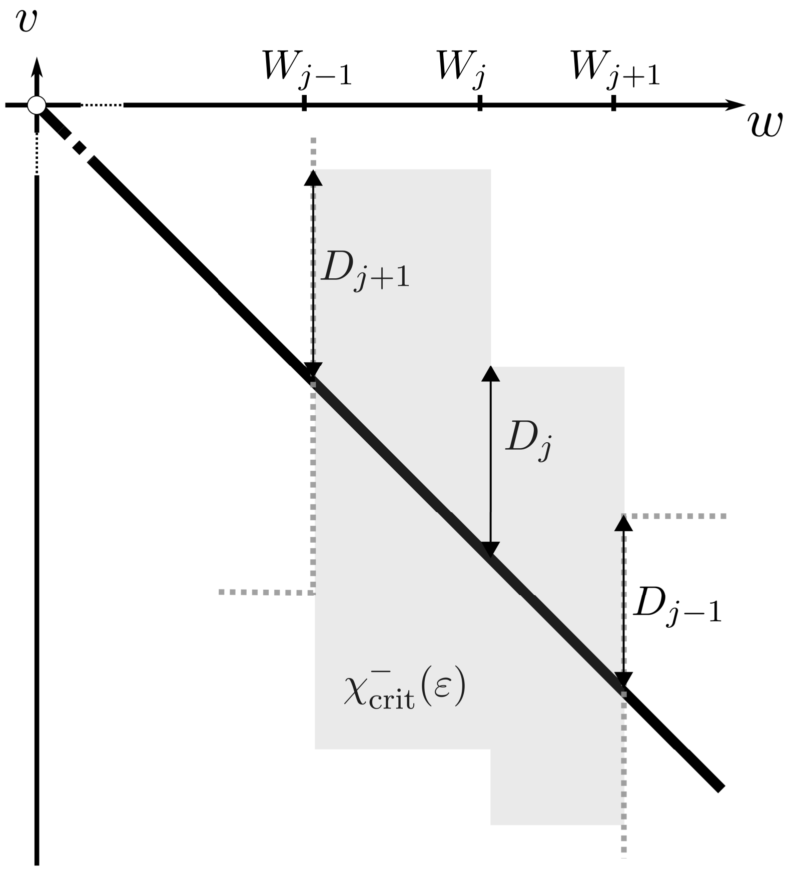

Figure 5.3.: A diagram of the internals of $\chi^-_{\text{crit}}(\varepsilon)$ where only two "boxes" are highlighted grey. Parts of the adjacent boxes are shown with dotted lines. Note that this is a plot of w and v so the quadratic scaling of f in a neighborhood of the fold point means that H^- is a straight line in these coordinates.

This follows from items (i) and (iv) of Condition 1.2.6. Let $C \in (\frac{1}{2}, 1)$ and define

$$C_{\text{height}} := \frac{1}{2} \min\{\delta_u, |H^+(-u_\infty)|, |H^-(u_\infty)| \quad \text{where } u \in [\frac{1}{2}, u_\infty]\}.$$

Next, define A^-_{hyp} via the formula

$$A^-_{\text{hyp}} := \{(u,v) \mid u \in (\frac{1}{2}, u^\infty) \text{ and } |v - H^-(u)| \le 2C_{\text{height}}\}.$$

All that remains is to find the constant C_{Lip}. To this end, note that H^- is the local invserse of the nullcline $v \mapsto g(v)$ when $v < 0$. Moreover, g' can be found by calculating

$$0 = \frac{\partial}{\partial v} f(g(v), v) = g'(v) \frac{\partial f}{\partial u}(g(v), v) + \frac{\partial f}{\partial v}(g(v), v).$$

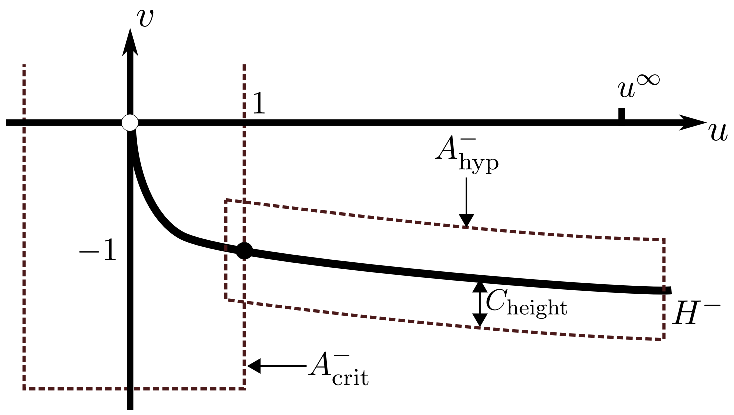

Figure 5.4.: The normally hyperbolic region connects the edge of the critical region A^-_{crit} to the edge of the invariant region for u^ε at the value u_∞ (cf. Theorem 1.2.18). A^-_{hyp} is constructed to have constant height C_{height}.

If we define

$$\frac{1}{C_{\text{Lip}}} = \min\{(u,v) \in A^-_{\text{hyp}} \mid |\frac{\partial f}{\partial v}(u,v)| \cdot \left(|\frac{\partial f}{\partial u}(u,v)|\right)^{-1}\},$$

Then C_{Lip} satisfies (5.2.2). □

Note that C_{height}, A^-_{hyp} and C_{Lip} do not depend on ε.

5.2.1. Normally Hyperbolic Sleeve

Definition 5.2.2. *Define $\delta_{\text{nh}} = \delta_J$ (cf. Lemma 5.1.7) and $d = \varepsilon^{\delta_{\text{nh}}}$, in particular*

$$\varepsilon^{\delta_{\text{nh}}} = d = D_J. \tag{5.2.4}$$

Definition 5.2.3. *Choose $\mu > 0$ satisfying $1 > \gamma > \mu > \delta_{nh}$.*

Definition 5.2.4 (Normally Hyperbolic Sleeve)**.** *Let $u_0 = U_J \in \{\pi_u A^-_{\text{hyp}}\}$ and define $u_k = u_0 + k\varepsilon^\mu$, up to some integer K such that $u_{K-1} \le u_\infty < u_K$. We define the normally hyperbolic sleeve $\chi^-_{\text{hyp}}(\varepsilon) \subset \mathbb{R}^2$ (see Figure 5.5) as the set of pairs*

$$\chi^-_{\text{hyp}}(\varepsilon) := \bigcup_{k=0}^{K-1} \{(u,v) \mid (u,v) \in [u_k, u_{k+1}] \times [H^{-1}(u_k) - d, H^{-1}(u_{k+1}) + d]\}, \tag{5.2.5}$$

with a corresponding Definition of $\chi^+_{\text{hyp}}(\varepsilon)$ for H^+.

Remark 5.2.5. We make a couple of observations about Definition 5.2.4. First, note that for all $(u,v) \in \chi^-_{\text{hyp}}(\varepsilon)$

$$|v - H^-(u)| \le 2\varepsilon^{\text{nh}}.$$

Second, because of the strict inequalities specified in Condition 1.2.15, $\chi^-_{\text{hyp}}(\varepsilon)$ should always be contained in the invariant region of the solution $(u^\varepsilon, v^\varepsilon)$ of (1.2.6)–(1.2.8) (cf. Definition 1.2.17), provided the height of $\chi^-_{\text{hyp}}(\varepsilon)$ in the v-direction is sufficiently small.

Definition 5.2.6. *Let $\varepsilon_3 > 0$ is the supremum over all $\varepsilon > 0$ satisfying:*

$$\exp\left(-c(\gamma)C_{\text{Lip}}^{-1-\frac{1}{\gamma}}\varepsilon^{-1+\frac{\mu}{\gamma}}\right) + C_{\text{Lip}}\varepsilon^{-\delta_{\text{nh}}+\mu} \leq 1,$$

$$\{U_J\} \times (-1 - \varepsilon^{\delta_{\delta_{\text{nh}}}}, -1 + \varepsilon^{\delta_{\text{nh}}}) \subset A^-_{\text{hyp}} \cap A^-_{\text{crit}},$$

$$\chi^-_{\text{hyp}}(\varepsilon) \subset (-u_\infty, u_\infty) \times (-v_\infty, v_\infty).$$

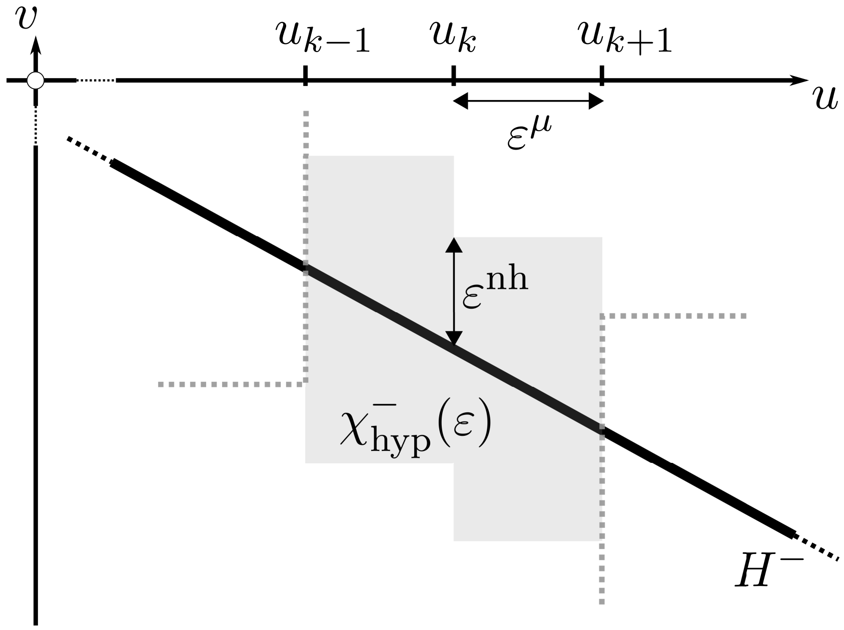

Figure 5.5.: A segment of $\chi^-_{\text{hyp}}(\varepsilon)$ is where $H^-(u)$ is drawn as a straight line for simplicity. Note that unlike Figure 5.3, here u is plotted against v.

5.3. Drop and Fast-Motion Sections

Definition 5.3.1. *There is a positive constant $C_1 > 0$ and two constants $0 < v_{\text{fast}} < v_{\text{drop}}$ that taken together define two sections $\mathcal{S}_{\text{drop}}$ and $\mathcal{S}_{\text{fast}}$ (cf. Figure 5.6) via the formulas*

$$\mathcal{S}_{\text{drop}} = (-C_1, C_1) \times \{v_{\text{drop}}\},$$

$$\mathcal{S}_{\text{fast}} = (-C_1, C_1) \times \{v_{\text{fast}}\}.$$

$\mathcal{S}_{\text{drop}}$ and $\mathcal{S}_{\text{fast}}$ have the following properties:

(i) *There is a neighborhood of the rectangle* $(-C_1, C_1) \times (v_{\text{fast}}, v_{\text{drop}})$ *where* $f(\cdot,\cdot)$ *is strictly positive.*

(ii) $\mathcal{S}_{\text{fast}} \subset A^-_{\text{crit}}$ *and* $\mathcal{S}_{\text{drop}} \subset A^+_{\text{hyp}}$.

Definition 5.3.2. *Let* ε_4 *be the supremum over all* ε *such that* $-\varepsilon^{\frac{2}{3}-} \in \pi_u \mathcal{S}_{\text{fast}}$

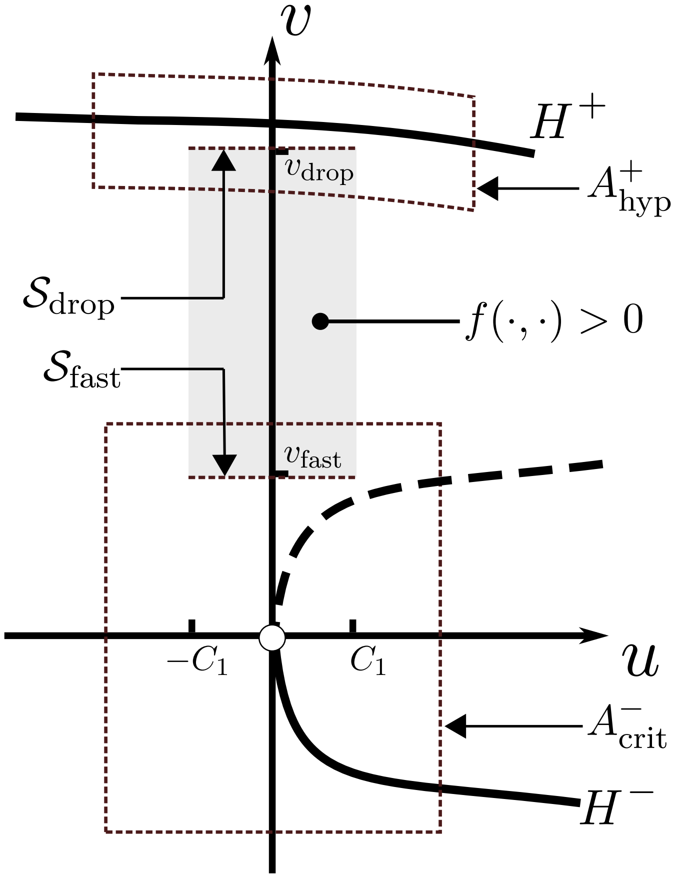

Figure 5.6.: The two sections $\mathcal{S}_{\text{drop}}$ and $\mathcal{S}_{\text{fast}}$. The highlighted region between the two indicates where the fast motion will occur.

5.4. Critical Sleeve Scaling Lemmas

In this section we present Lemmas describing how the quantities D_j and W_j scale as j increases. We will need them in Chapters 6 and 7, but because they are just contortions of the Definitions encountered so far in the present Chapter, we will include them here.

Lemma 5.4.1. *If* $\varepsilon < \min_{1 \le i \le 4}\{\varepsilon_i\}$ *and* $(U, v) \in \chi^-_{\text{crit}}(\varepsilon)$*, then* $v < 0$.

Proof. We must check that for every $j \in \{0, \cdots, J\}$, $\varepsilon^{\mu_j} - \varepsilon^{\delta_j} \ge 0$. It sufficies to check

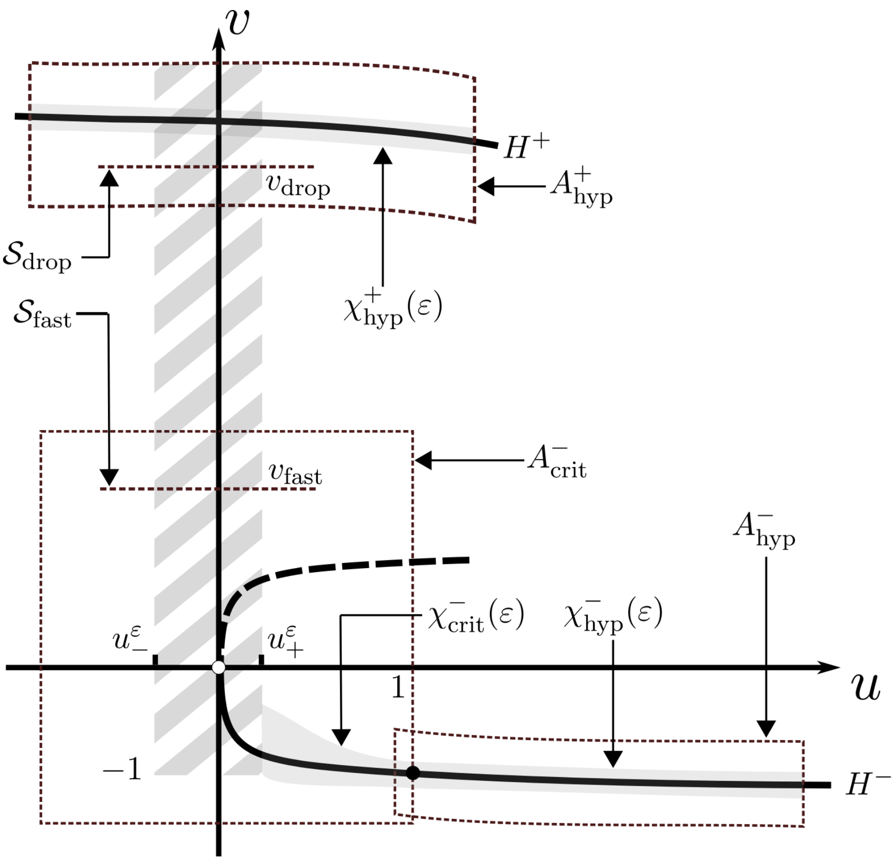

Figure 5.7.: A complete picture of the phase space for the forced Fast-ODE. The quantity u^ε_- will be rigorously defined in Chapter 7.

for $j = 0$, which is just Definition 5.1.9. □

Lemma 5.4.2. *If* $\varepsilon < \min_{1 \leq i \leq 4}\{\varepsilon_i\}$, *then for all* $j > 0$

$$\varepsilon^{\delta_j} - \varepsilon^{\delta_{j+1}} < |W_{j+1} - W_j| < \varepsilon^{\delta_{j+1}}.$$

Proof. We treat each side of the inequality separately.

Inequality $|W_{j+1} - W_j| < \varepsilon^{\delta_{j+1}}$:

From the Definition of $|W_{j+1} - W_{j-1}|$ in (5.1.5) we must show that

$$\varepsilon^{\delta_j - \delta_{j+1}} < \varepsilon^{-\lambda_2},$$

so it is sufficient to show that $\varepsilon^{\delta_j - \delta_{j+1}}$ is bounded. Calculating directly yields

$$\begin{aligned}
\varepsilon^{\delta_j - \delta_{j+1}} &= \varepsilon^{(1+\gamma)(\mu_{j+1} - \mu_j)} \\
&= \left(\frac{W_{j+1} + (W_j - W_j)}{W_j} \right)^{1+\gamma} \\
&= \left(\varepsilon^{\delta_j + \lambda_2 - \mu_j} + 1 \right)^{1+\gamma}. \qquad (5.4.1)
\end{aligned}$$

A sufficient condition for the inequality to hold is that $\delta_j + \lambda_2 - \mu_j$ is positive, which is indeed the case since Definition 5.1.3 states that

$$\delta_j + \lambda_2 - \mu_j = \gamma(1 - \lambda_1) - (2 + \gamma)\mu_j > \lambda_2 > 0. \qquad (5.4.2)$$

Inequality $\varepsilon^{\delta_j} - \varepsilon^{\delta_{j+1}} < \varepsilon^{\delta_j + \lambda_2}$:

First observe that the inequality

$$\varepsilon^{\delta_j} - \varepsilon^{\delta_{j+1}} < \varepsilon^{\delta_j + \lambda_2}, \qquad (5.4.3)$$

is equivalent to

$$\varepsilon^{\delta_j - \delta_{j+1}} < (1 - \varepsilon^{\lambda_2})^{-1}. \qquad (5.4.4)$$

Moreover, we already know the RHS of (5.4.4) is equal to (5.4.1). Using Definition 5.1.3 we see that (5.4.1) satisfes the inequality

$$\left(\varepsilon^{\delta_j + \lambda_2 - \mu_j} + 1 \right)^{1+\gamma} \leq (\varepsilon^{2\lambda_2} + 1)^2 \leq 1 + (2\varepsilon^{\lambda_2})\varepsilon^{\lambda_2} + \varepsilon^{4\lambda_2} \leq \sum_{m=0}^{\infty} \varepsilon^{m\lambda_2}, \qquad (5.4.5)$$

where $2\varepsilon^{\lambda_2} \leq 1$ due to Definition 5.1.9. Now note that the right-most term in (5.4.5) is the geometric series representation of the RHS of (5.4.3). □

Lemma 5.4.3. *If* $\varepsilon < \min_{1 \leq i \leq 4}\{\varepsilon_i\}$, *then*

$$[W_{j-1}, W_{j+1}] \times [-W_j - D_j, -W_j + D_j] \subset \chi^-_{\text{crit}}(\varepsilon).$$

Proof. The Lemma can be divided into two smaller statements. We already know from Definition 5.1.8 that

$$[W_{j-1}, W_j] \times [-W_j - D_j, -W_{j-1} + D_{j-1}] \subset \chi^-_{\text{crit}}(\varepsilon),$$

so the first thing to check is if

$$-W_j + D_j \leq -W_{j-1} - D_{j-1}.$$

However, this is trivial because $D_j - D_{j-1}$ is negative and $W_j - W_{j-1}$ is positive. The second thing to check, again in light of Definition 5.1.8, is if

$$-W_{j+1} - D_{j+1} \leq -W_j - D_j.$$

But this inequality is a restatement of Lemma 5.4.2. □

6. Convergence of the non-Diffusing Variable

6.1. Standing Assumptions about the Forced Fast-ODE

To begin this section we formulate an ODE with its associated notation convention. It will be analogous to the case where $x \in Q$ is fixed and $v^\varepsilon(x, \cdot)$ is interpreted as a scalar valued function satisfying the non-autonomous ODE (1.2.6) with forcing term $u^\varepsilon(x, \cdot)$.

Definition 6.1.1 (Notation Convention for the Forced Fast-ODE). $v^\varepsilon : [0, T] \to \mathbb{R}$ *is a solution to*

$$\varepsilon \dot{v}^\varepsilon = f(u, v^\varepsilon), \tag{6.1.1}$$

where $u : [0, T] \to \mathbb{R}$ *is a non-autonomous input. This notation convention will be in place in Section 6.1 and the entirety of Chapter 7.*

Condition 6.1.2. $u \in C^\gamma[0, T]$ with

$$\|u\|_{C^\gamma[0,T]} \le C_\gamma, \quad c(\gamma) := C_\gamma^{-\frac{1}{\gamma}}.$$

This Condition will be in place in Section 6.1 and the entirety of Chapter 7.

In order to proceed directly to a proof of Theorem 1.3.4 we need three Lemmas pertaining to (6.1.1). We delay the proofs until the subsequent Chapter so that we can prove our main result as quickly as possible. If a reader is so inclined, they may read Chapter 7 first and return to the current Chapter afterwards.

Lemma 6.1.3. *Suppose that* u *satisfies Condition 6.1.2,* $\varepsilon < \varepsilon_3$ *(cf. Definition 5.2.6), and that for every* $t \in [0, T]$*,* $u(t) \in (u_0, u_K)$ *(cf. Definition 5.2.4). Also assume that one of the following holds (see Figure 6.1):*

(i) $u(0) = u_k$ *and* $|v^\varepsilon - H^-(u_k)| \le \varepsilon^{\delta_{\text{nh}}}$.

(ii) $v^\varepsilon(0) = H^-(u(0))$.

Then for all $t \in [0, T]$*,*

$$(u(t), v^\varepsilon(t)) \in \chi^-_{\text{hyp}}(\varepsilon).$$

Lemma 6.1.4. *Suppose that* (u, v^ε) *satisfy Condition 6.1.2,* $\varepsilon < \varepsilon_2$*, and that* $w(t) \in (W_0, W_J)$ *for every* $t \in [0, T]$ *(cf. Definitions 5.1.4–5.1.6 and Definition 5.1.8). Also assume that one of the following holds (see Figure 6.2):*

(i) $w(0) = W_j$ *and* $|v^\varepsilon(0) + W_j| \le \varepsilon^{\mu_j}$.

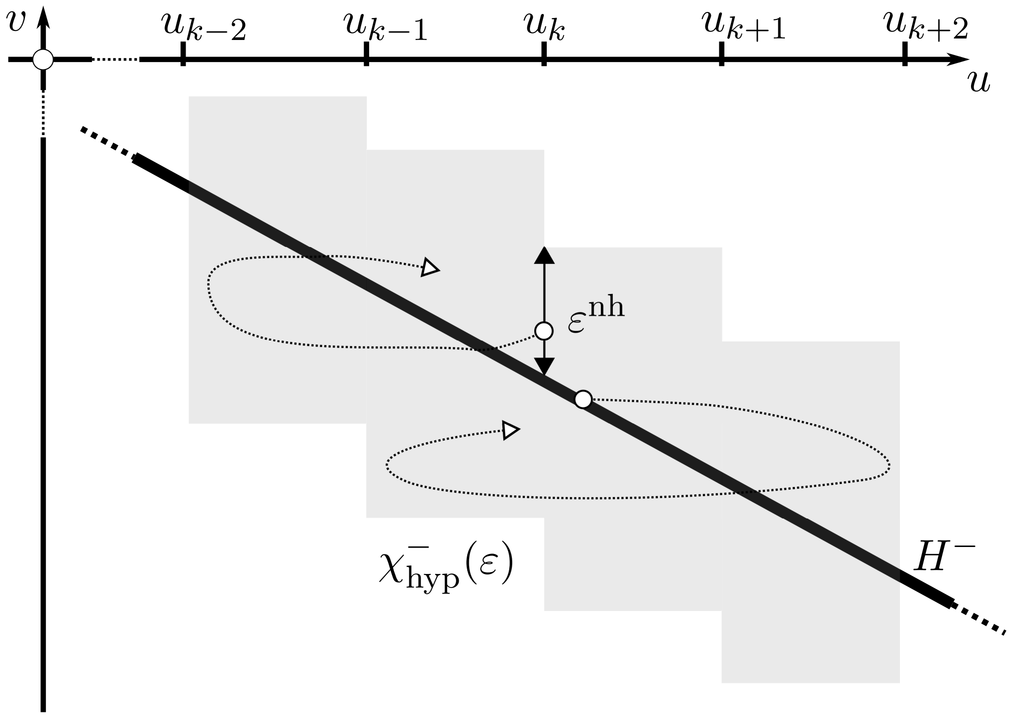

Figure 6.1.: An illustration of Lemma 6.1.3 with two sample trajectories, one of which has an initial condition satisfying item (i), the other has an initial condition satisfying item (ii).

(ii) $v^\varepsilon(0) = w(0)$.

Then for all $t \in [0, T]$,

$$(u(t), v^\varepsilon(t)) \in \chi^-_{\mathrm{crit}}(\varepsilon).$$

Lemma 6.1.5. *Suppose that* (u, v^ε) *satisfy Condition 6.1.2 . Then there exists an* ε_6 *and a* $C_0 > 1$ *that defines a value* $u^\varepsilon_- := -C_0\varepsilon^{\frac{2}{3}-}$ *with the following properties: If* $\varepsilon < \varepsilon_6$ *and if there is a* $t^* > 0$ *such that (see Figure 6.3):*

(i) $u(0) = 0$.

(ii) $u(t) \in (u^\varepsilon_-, 0)$ *for* $t \in (0, t^*)$.

(iii) $u(t^*) = u^\varepsilon_-$.

Then $(u(t^*), v^\varepsilon(t^*)) \in \chi^+_{\mathrm{hyp}}(\varepsilon)$. *Moreover,* C_0 *is independent of* ε.

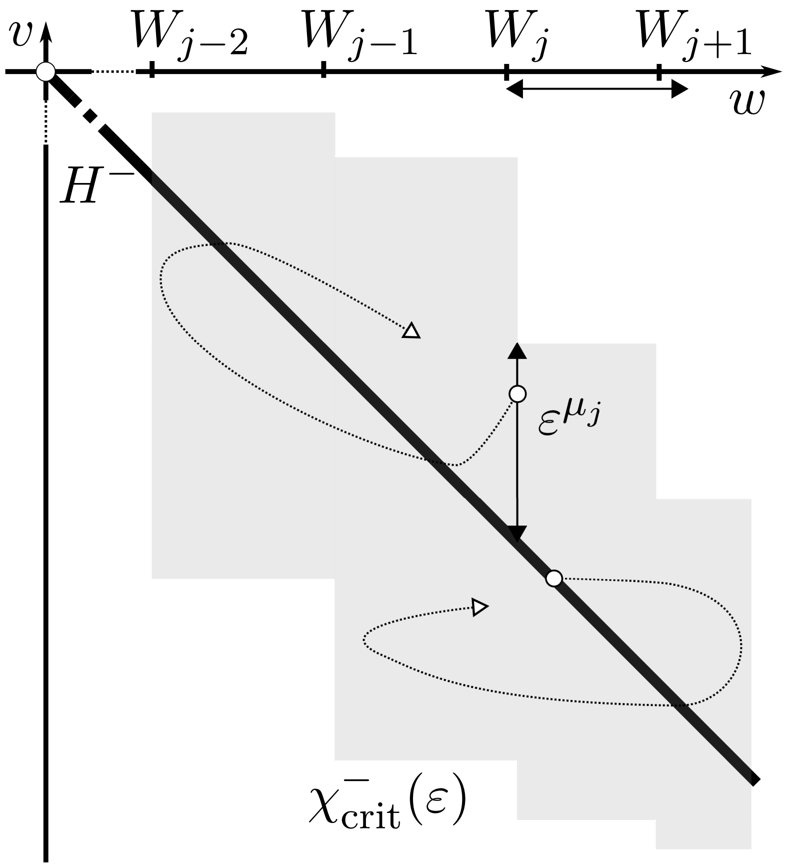

Figure 6.2.: An illustration of Lemma 6.1.4 with two sample trajectories, one of which has an initial condition satisfying item (i), the other has an initial condition satisfying item (ii).

6.2. Unstable Interfaces

Definition 6.2.1. *If $u^\varepsilon : Q_T \to \mathbb{R}$ is simply transverse (cf. Definition 1.2.20) we mean the following:*

(i) For $u^\varepsilon(\cdot, t) : Q \to \mathbb{R}$, we let $\tilde{x}^\varepsilon(t)$ denote the equivalent of $\tilde{x}(0)$ and $b^\varepsilon(t)$ denote the equivalent of $b(0)$.

(ii) The constants C_{dx} and C_{space} are independent of ε.

We now state a straightforward consequence of the Mean Value Theorem that is true of any simply transverse function.

Lemma 6.2.2. *If there exists an $\tilde{x}(t) \in B_{C_{\mathrm{space}}}(b^\varepsilon(t))$ (cf. Definition 6.2.1), then the following hold:*

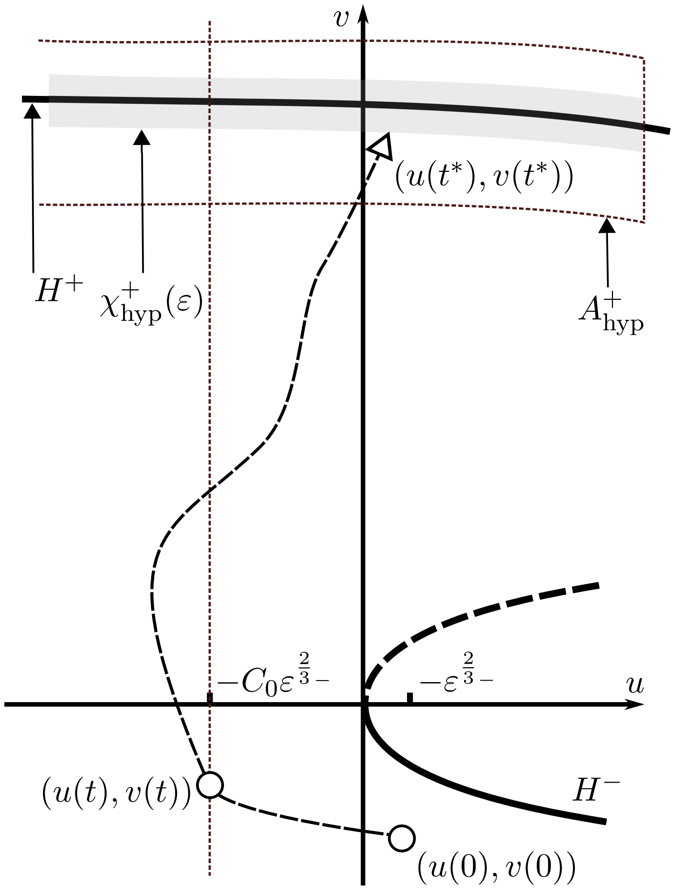

Figure 6.3.: An illustration of Lemma 6.1.5 with a sample trajectory (dashed line with an arrow head). After the trajectory crosses the line $\{u = -C_0\varepsilon^{\frac{2}{3}-}\}$, then it will eventually reach $\chi^+_{\text{hyp}}(\varepsilon)$, even if $u(\cdot)$ subsequently increases.

(i) For every $x, y \in B_{C_{\text{space}}}(b^\varepsilon(t))$*, one has the inequality*

$$|x - y| < \frac{1}{C_{\text{dx}}} |u^\varepsilon(x,t) - u^\varepsilon(y,t)|.$$

(ii) For $x \in B_{C_{\text{space}}}(b^\varepsilon(t))$ *the map* $x \mapsto u^\varepsilon(x,t)$ *is injective and*

$$|u^\varepsilon(x,t)| \le 2C_{\text{dx}} \cdot C_{\text{space}}.$$

We are now ready to define two fundamental objects that act as the equivalent to the free boundary in the study of (1.2.1)–(1.2.4). In the language of [MR80] and [KS01a], we are describing the analogue of two sections they place on either side of the fold point to separate slow and fast motion. This is where the stable branch of $\mathcal{M}_{\text{crit}}$ dissapears in a saddle node bifurcation hence our descision to use the following nomenclature.

Definition 6.2.3 (Unstable Interfaces)**.** *If u^ε is simply transverse then we define the unstable interfaces as the quantities*

$$b_-^\varepsilon(t) = \max_{s\in[0,t]}\{\tilde{x}_-^\varepsilon(s), b(0)\},$$

$$b_+^\varepsilon(t) = \max_{s\in[0,t]}\{\tilde{x}_+^\varepsilon(s), b(0)\},$$

where $\tilde{x}_\pm^\varepsilon(t)$ are the unique values such that $u^\varepsilon(\tilde{x}_\pm^\varepsilon(t),t) = u_\pm^\varepsilon$ and $|\tilde{x}_\pm^\varepsilon(t) - b^\varepsilon(t)| \leq C_{\text{space}}$. Note that $\tilde{x}_\pm^\varepsilon(t)$ need not necessarily exist.

The most important property of the *unstable interfaces* is that they separate the *critical sleeve* $\chi_{\text{crit}}^-(\varepsilon)$ (cf. Definition 5.1.8) and the *normally hyperbolic sleeve* on the opposite branch $\chi_{\text{hyp}}^+(\varepsilon)$ (cf. Defnition 5.2.4). We know the height (in the v-direction) of both of these regions and that a solution $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ is in a certain sense contained inside of them (cf. Lemmas 6.1.3 and 6.1.4). However, between $b_-^\varepsilon(t)$ and $b_+^\varepsilon(t)$ our estimate of the integrand of (4.1.1) is $O(1)$. For the remainder of this section we will formalize how to transition between these regions.

Lemma 6.2.4. *If $x < b_-^\varepsilon(t)$ and u^ε is simply transverse (cf. Definition 6.2.1) then $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi_{\text{hyp}}^+(\varepsilon)$.*

Proof. In Definition 1.2.20 we stipulated that $u^\varepsilon(x,s) < -C_{\text{dx}}C_{\text{space}}$ for every $s \in [0,t]$, and as such if

$$(u^\varepsilon(x,0), v^\varepsilon(x,0)) = \big(u^\varepsilon(x,0), H^+(u^\varepsilon(x,0))\big) \in \chi_{\text{hyp}}^+(\varepsilon),$$

then there does not exist a time $t > 0$ such that $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi_{\text{hyp}}^-(\varepsilon)$. In other words $|u^\varepsilon(x,t) - \beta| = O(1)$ so we may suppose that $u^\varepsilon(x,t) \notin \pi_u\chi_{\text{crit}}^+(\varepsilon)$. The other possibility is that we started on the branch H^-, namely

$$(u^\varepsilon(x,0), v^\varepsilon(x,0)) = (u^\varepsilon(x,0), H_-(u^\varepsilon(x,0))) \in \chi_{\text{hyp}}^-(\varepsilon) \cup \chi_{\text{crit}}^-(\varepsilon).$$

Then because $x < b_-^\varepsilon(t)$, this means that there is a $t^* \in [0,t]$ such that $x = \tilde{x}_-^\varepsilon(t^*)$. Now Lemma 6.1.5 is applicable and hence $u^\varepsilon(x,t^*) \in \chi_{\text{hyp}}^+(\varepsilon)$. □

Lemma 6.2.5. *If $x > b_+^\varepsilon(t)$ then one the following holds:*

(i) $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi_{\text{hyp}}^-(\varepsilon)$.

(ii) $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi_{\text{crit}}^-(\varepsilon)$.

Proof. Before starting the proof, recall that (1.2.8) reads $v^\varepsilon(x,0) = H_-(\varphi(x))$, i.e., we start on the hysteresis branch.

Let's begin with treating the cases where $u^\varepsilon(x,s) \neq U_J$ (cf. Definition 5.1.8 and Definition 5.2.6) for every $s \in [0,t]$.

- Suppose that $(\varphi(x), H_-(\varphi(x)) \in \chi^-_{\text{hyp}}(\varepsilon)$ and $u^\varepsilon(x,s) \geq U_J$ for every $s \in [0,t]$. In this case one can apply Lemma 6.1.3 directly to conclude that $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi^-_{\text{hyp}}(\varepsilon)$.
- Suppose that $(\varphi(x), H_-(\varphi(x)) \in \chi^-_{\text{crit}}(\varepsilon)$ and $u^\varepsilon(x,s) \leq U_J$ for every $s \in [0,t]$. Now note that the assumption of the current Lemma states that $u^\varepsilon(x,s) > u^\varepsilon_+$ for every $s \in [0,T]$, or equivalently $w^\varepsilon(x,s) > w^\varepsilon_+$ (cf. Definition 5.1.4) . In this case Lemma 6.1.4 implies that $(w^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi^-_{\text{crit}}(\varepsilon)$.

To treat the case where u^ε crosses U_J suppose that $(u^\varepsilon(x,0), v^\varepsilon(x,0)) \in \chi^-_{\text{hyp}}(\varepsilon)$ and there exists a $t_1 \in [0,t]$ such that $u^\varepsilon(x,t_1) = U_J$. Note that this is equivalent to $w^\varepsilon(x,t_1) = W_J$. However, it is also true that $|v^\varepsilon(x,t_1) - W_J| \leq \varepsilon^{\delta_{\text{nh}}} \leq D_J$, and as such Lemma 6.1.4 holds until there exists a time $t_2 > t_1$ where $w^\varepsilon(x,t_2) = W_J$. But indeed this is just $u^\varepsilon(x,t_2) = U_J$ and one repeats a similar argument under the auspices of Lemma 6.1.3.

□

Lemma 6.2.6. *There is a constant C, such that for every $t \in [0,T]$,*

$$|b^\varepsilon_+(t) - b^\varepsilon_-(t)| \leq C\varepsilon^{\frac{2}{3}-}.$$

Proof. Combining Lemmas 3.2.1 and 6.2.2 yields

$$|b^\varepsilon_+(t) - b^\varepsilon_-(t)| \leq |\tilde{x}^\varepsilon_-(t) - \tilde{x}^\varepsilon_+(t)| = \frac{1}{C_{\text{dx}}}|u^\varepsilon_- - u^\varepsilon_+| \leq \frac{1}{C_{\text{dx}}}\varepsilon^{\frac{2}{3}-}.$$

□

6.3. Convergence of the non-Diffusing Variable when u^ε is apriori Simply Transverse

Remark 6.3.1. Lemma 1.2.10 implies that for every fixed $x \in Q$, $u^\varepsilon(x,\cdot) \in C^\gamma[0,T]$. Moreover, Theorem 1.2.18 and inequality (2.1.5) imply that there is a C_γ such that for every $x \in Q$ and $\varepsilon > 0$, one has $\|u^\varepsilon(x,\cdot)\|_{C^\gamma[0,T]} \leq C_\gamma$, i.e., every $u^\varepsilon(x,\cdot)$ satisfies Condition 6.1.2.

Definition 6.3.2.

$$Q^\varepsilon_{\text{crit}}(t) := \left\{(x,t) \in Q_T \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi^-_{\text{crit}}(\varepsilon)\right\},$$

$$Q^\varepsilon_{\text{hyp}}(t) := \left\{(x,t) \in Q_T \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \chi^-_{\text{hyp}}(\varepsilon) \cup \chi^+_{\text{hyp}}(\varepsilon)\right\}.$$

Proof of Lemma 4.1.1. We will provide a proof for $q = 1$ and highlight where one needs to modify for the case $q > 1$.

If one recalls Lemmas 6.2.4 and 6.2.5, any $y \in Q$ must belong to one of three sets:

(i) $y \in Q^{\varepsilon}_{\text{crit}}(t)$.

(ii) $y \in Q^{\varepsilon}_{\text{hyp}}(t)$.

(iii) $y \in (b^-(t), b^+(t))$.

Step-One: Integrate over $Q^{\varepsilon}_{\text{crit}}(t)$

Assume that $Q^{\varepsilon}_{\text{crit}}(t) \neq \emptyset$. Consider the integral

$$\int_{Q^{\varepsilon}_{\text{crit}}(t)} |v^{\varepsilon} - \mathcal{H}(\xi_0, u^{\varepsilon})|(y,t)dy.$$

Note that the map $y \mapsto u^{\varepsilon}(y,t)$ with domain $y \in Q^{\varepsilon}_{\text{crit}}(t)$ is invertible on $Q^{\varepsilon}_{\text{crit}}(t)$ (cf. Lemma 6.2.2). As such, we can make a coordinate change that treats $U := u^{\varepsilon}(y,t)$ as paramaterizing a curve in the ODE phase space.

$$\int_{Q^{\varepsilon}_{\text{crit}}(t)} |v^{\varepsilon} - \mathcal{H}(u^{\varepsilon}, \xi_0)|(y,t)dy = \frac{1}{C_{\text{dx}}} \int_{u^{\varepsilon}(Q^{\varepsilon}_{\text{crit}}(t),t)} |v^{\varepsilon}(U,t) - H^-(U)|dU. \tag{6.3.1}$$

If $U \in [U_0, U_J]$ but $U \notin [U_0, U_J]$ define $|v^{\varepsilon}(U,t) - \sqrt{U}| = 0$. In addition, replace $H^-(U)$ with $-\sqrt{U}$ (cf. item (v) of Condition 1.2.6). With these substitions, let us continue the calculation in (6.3.1).

$$\begin{aligned}
&\frac{1}{C_{\text{dx}}} \int_{u^{\varepsilon}(Q^{\varepsilon}_{\text{crit}}(t),t)} |v^{\varepsilon}(U,t) - H^-(U)|dU \\
=&\frac{1}{C_{\text{dx}}} \int_{U_0}^{U_J} |v^{\varepsilon}(U,t) + \sqrt{U}|dU \\
=&\frac{1}{C_{\text{dx}}} \int_{W_0}^{W_J} |v(W^2,t) + W| \cdot wdw.
\end{aligned} \tag{6.3.2}$$

The width of the j-th component of $\chi^-_{\text{crit}}(\varepsilon)$ is $(W_{j+1} - W_j) + D_{j+1} + D_j$. This allows us to write (6.3.2) as

$$\begin{aligned}
&\frac{1}{C_{\text{dx}}} \int_{W_0}^{W_J} |v^{\varepsilon}(t,W) + W| \cdot wdw \\
&\leq \frac{1}{C_{\text{dx}}} \sum_{j=0}^{J-1} |W_{j+1} - W_j| \cdot |W_{j+1} - W_j + D_{j+1} + D_j| \cdot W_{j+1} \\
&\leq \frac{C}{C_{\text{dx}}} \sum_{j=0}^{J-1} |W_{j+1} - W_j| \cdot \varepsilon^{\mu_{j+1}} \cdot W_{j+1},
\end{aligned} \tag{6.3.3}$$

where in (6.3.3) we have used $D_j \leq 2D_{j+1}$ (cf. Definition 5.1.6 and Lemma 5.4.2).

This sum will in fact be bounded using a second integral (see Figure 6.4). To this end note that $D_j = \varepsilon^{\gamma(1-\lambda_1)-\lambda_2} W_j^{-(1+\gamma)}$ (cf. Definition 5.1.5).

We now encounter a step where we use $q \geq 1$. One would need to replace ε^{μ_j} in (6.3.3) by $\varepsilon^{q\mu_j}$, but since $\varepsilon^{\mu_j} < 1$ one could argue that $\varepsilon^{q\mu_j} < \varepsilon^{\mu_j}$ and proceed. We opt for the more transparent option of introducing the factor q into the computation. In the next series of inequalities we will need to bound the quantities $W_{j+1}^{-q(1+\gamma)+1}$ by an integral as indicated in Figure 6.4.

$$\begin{aligned}
(6.3.3) &\leq \frac{C}{C_{\mathrm{dx}}} \varepsilon^{q(\gamma(1-\lambda_1)-\lambda_2)} \sum_{j=0}^{J-1} W_{j+1}^{-q(1+\gamma)+1} |W_{j+1} - W_j| \\
&\leq \frac{C}{C_{\mathrm{dx}}} \varepsilon^{q(\gamma(1-\lambda_1)-\lambda_2)} \int_{W_0}^{W_J} W^{-q(1+\gamma)+1} dW \\
&= \frac{C}{C_{\mathrm{dx}}} \varepsilon^{q(\gamma(1-\lambda_1)-\lambda_2)} W_0^{-q(1+\gamma)+2} \\
&\leq \frac{C\varepsilon^{2\mu_0+q}}{C_{\mathrm{dx}}(1-\gamma)}.
\end{aligned} \tag{6.3.4}$$

Note that in (6.3.4) we have used Definition 5.1.3. If one recalls (5.1.3) then it's immediately clear that

$$\int_{Q_{\mathrm{crit}}^{\varepsilon}(t)} |v^{\varepsilon} - \mathcal{H}(u^{\varepsilon}, \xi_0)|(y,t) dy \leq \frac{C\varepsilon^{\frac{2}{3}-}}{C_{\mathrm{dx}}(1-\gamma)}. \tag{6.3.5}$$

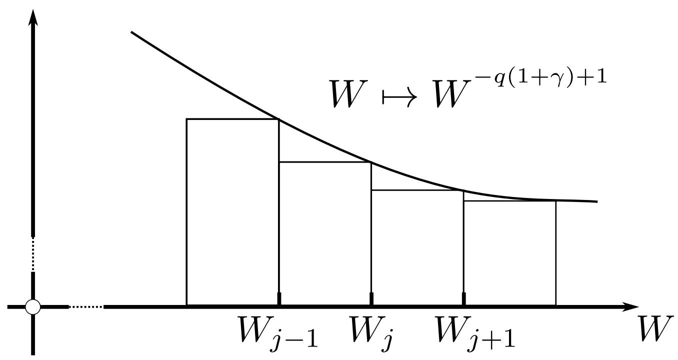

Figure 6.4.: The area enclosed by $\chi_{\mathrm{crit}}^{-}(\varepsilon)$ is bounded above by the integral of $W \mapsto W^{-q(1+\gamma)-1}$.

Step Two: Integrate over $Q_{\mathrm{hyp}}^{\varepsilon}(t)$

Assume that $Q_{\mathrm{hyp}}^{\varepsilon}(t) \neq \emptyset$ and let us take $y \in Q_{\mathrm{hyp}}^{\varepsilon}(t)$ such that $(u^{\varepsilon}(y,t), v^{\varepsilon}(y,t)) \in \chi_{\mathrm{hyp}}^{-}(\varepsilon)$ (the case $y \in \chi_{\mathrm{hyp}}^{+}(\varepsilon)$ is treated similarly). In particular, there is a $k \in \{0, \cdots, K\}$ such that $u^{\varepsilon}(x,t) \in [u_k, u_{k+1}]$, and the maximum distance of $v^{\varepsilon}(x,t)$ from $H^{-}(u^{\varepsilon}(x,t))$ is $\varepsilon^{nh} + \varepsilon^{\mu} \leq C\varepsilon^{nh}$. If we recall Definitions 5.1.6 and 5.2.2, then

$$|v^{\varepsilon}(x,t) - H^{-}(u^{\varepsilon}(x,t))| \leq C\varepsilon^{\delta_{\mathrm{nh}}} \leq C\varepsilon^{\delta_J} \leq C\varepsilon^{\delta_0}.$$

One can now consult Definition 5.1.3 to see that $\varepsilon^{\delta_0} \leq \varepsilon^{\mu_0}$, and as such

$$\int_{Q^{\varepsilon}_{\text{hyp}}(t)} |v^{\varepsilon} - \mathcal{H}(\xi_0, u^{\varepsilon})|(y,t)dy \leq C\varepsilon^{\delta_{\text{nh}}} \leq C\varepsilon^{\frac{2}{3}-}. \tag{6.3.6}$$

Step Three: Integrate between the unstable interfaces

It remains to calculate the integral over $(b^{\varepsilon}_{-}(t), b^{\varepsilon}_{+}(t))$. Using Lemma 6.2.6 and that $(u^{\varepsilon}, v^{\varepsilon})$ is uniformly bounded (cf. Theorem 1.2.18), one has that

$$\left(\int_{b^{\varepsilon}_{-}(t)}^{b_{+}(\varepsilon)} |v^{\varepsilon} - \mathcal{H}(u^{\varepsilon}, \xi_0)|^q(y,t)dy\right)^{\frac{1}{q}} \leq \frac{C}{C_{\text{dx}}}|U_{-} - U_{+}|^{\frac{1}{q}} \leq \frac{C}{C_{\text{dx}}}\varepsilon^{\frac{2}{3}-\cdot\frac{1}{q}}. \tag{6.3.7}$$

Combining (6.3.5), (6.3.6) and (6.3.7) one obtains inequality (4.1.1). □

7. Forced Fast-ODE

The goal of this chapter is to prove Lemmas 6.1.3–6.1.5 under Condition 6.1.2. This means we will be treating (1.2.6) where u^ε is replaced with a scalar function acting as a non-autonomous forcing term. We already discussed this briefly in the beginning of the previous Chapter where we decided to write u without superscript for the forcing term. We also emphasized that the regularity of u should be the same as that of $u^\varepsilon(x, \cdot)$.

7.0.1. Some Preliminaries

Lemma 7.0.3. *Let v_j^ε and u_j for $j = 1, 2$ satisfy Condition 6.1.2. Suppose that there is a set A such that $(u_j(t), v_j^\varepsilon(t)) \in A$ for every $t \in [0, T]$, and that for every $(u, v) \in A$*

$$\frac{\partial f}{\partial v}(u, v) < 0, \quad \frac{\partial f}{\partial u}(u, v) < 0.$$

Suppose that for every $t \in [0, T]$, $u_1(t) > u_2(t)$, and $v_1(0) = v_2(0)$. Then for all $t \in (0, T]$, $v_1(t) < v_2(t)$.

Proof. Observe that u_j, v_j and f are continuous and that

$$f(u_1(0), v_1(0)) - f(u_2(0), v_2(0)) < 0.$$

In particular, there exists a $t^* \in (0, T]$ such that for every $s \in (0, t^*]$

$$f(u_1(s), v_1(s)) - f(u_2(s), v_2(s)) < 0. \tag{7.0.1}$$

Now taking $t \in (0, t^*]$ one sees that $v_1(t) > v_2(t)$ since

$$v_1(t) - v_2(t) = \int_0^t (f(u_1(s), v_1(s)) - f(u_2(s), v_2(s)))\, ds < 0. \tag{7.0.2}$$

Define $t_{\min} = \inf\{s \in (0, T] \mid v_1(s) = v_2(s)\}$ and note that if $t_{\min}$ is well defined, then due to (7.0.2) one has $t_{\min} > 0$ and $v_1(t) - v_2(t) < 0$ for every $t \in (0, t_{\min}]$.

Since $u_1(t_{\min}) - u_2(t_{\min}) > 0$, (7.0.1) holds with $s = t_{\min}$, and because the functions u_j, v_j and f are continuous, there exists a $\delta > 0$ such that (7.0.1) indeed holds for $s \in (t_{\min} - \delta, t_{\min})$. As a consequence

$$\begin{aligned} v_1(t_{\min}) - v_2(t_{\min}) &= (v_1(t_{\min} - \delta) - v_2(t_{\min} - \delta)) \\ &\quad + \int_{t_{\min}-\delta}^{t_{\min}} (f(u_1(s), v_1(s)) - f(u_2(s), v_2(s)))\, ds < 0, \end{aligned}$$

contradicting the definition $t_{\min}$. Therefore, $t_{\min}$ is not well defined and the Lemma follows.

□

Observe that we have so far only encountered regions where f fits the framework of Lemma 7.0.3, namely $f(u, v)$ for $(u, v) \in A^{\pm}_{\text{hyp}}$ and $f(u, v) = v^2 - u$ for $(u, v) \in A^{-}_{\text{crit}}$ (cf. Condition 1.2.6).

7.1. Invariance Lemma for the Normally Hyperbolic Sleeve

Definition 7.1.1. *Consider a fixed $u_k \in \pi_u \chi^{-}_{\text{hyp}}(\varepsilon)$ with $k \in \{0, 1, \cdots, K\}$ (cf. Definition 5.2.4). The anchored solution v^{ε}_k is the solution to*

$$\varepsilon \dot{v}^{\varepsilon}_k = f(u_k, v^{\varepsilon}_k). \tag{7.1.1}$$

Lemma 7.1.2. *Anchored solutions satisfy the bound*

$$|v^{\varepsilon}_k(t) - H^{-}(u_k)| \leq |v^{\varepsilon}_k(0) - H^{-}(u_k)| \exp\left(-\frac{t}{C_{\text{Lip}}\varepsilon}\right).$$

Proof. Suppose $v^{\varepsilon}_k(0) > H^{-}(u_k)$, in particular $f(u_k, v^{\varepsilon}_k(t)) < 0$ for all $t \geq 0$ (cf. item (iii) of Lemma 5.2.1). Expressing $v^{\varepsilon}_k(t) - H^{-}(u_k)$ with the variation of constants formula yields

$$v^{\varepsilon}_k(t) - H^{-}(u_k) = v^{\varepsilon}_k(0) - H^{-}(u_k) + \frac{1}{\varepsilon}\int_0^t \big(f(u_k, v^{\varepsilon}_k(s)) - f(u_k, H^{-}(u_k))\big)\, ds. \tag{7.1.2}$$

Note that $f(u_k, H^{-}(u_k)) = 0$. If we recall the definition of C_{Lip} (cf. item (iv) of Lemma 5.2.1), then a simple application of the mean value theorem yields

$$f(u_k, v^{\varepsilon}_k(s)) - f(u_k, H^{-}(u_k)) \leq -\frac{1}{C_{\text{Lip}}}(v^{\varepsilon}_k(s) - H^{-}(u_k)). \tag{7.1.3}$$

Substituting (7.1.2) into (7.1.3) yields

$$v^{\varepsilon}_k(t) - H^{-}(u_k) \leq v^{\varepsilon}_k(0) - H^{-}(u_k) - \frac{1}{\varepsilon}\int_0^t \frac{1}{C_{\text{Lip}}}(v^{\varepsilon}_k(s) - H^{-}(u_k))ds.$$

The result now follows from Grönwall's Lemma (cf. Appendix A). □

Proof of Lemma 6.1.3.

- **Case:** $u(0) = u_k$ and $|v^{\varepsilon} - H^{-}(u_k)| \leq d$.

Define the time t_1 as

$$t_1 = \inf\{t \in [0,T] \mid u(t) = u_{k-1} \text{ or } u(t) = u_{k+1}\},$$

otherwise let $t_1 = T$. We now treat three subcases.

Subcase $t_1 = T$:

We claim that the set

$$(u,v) \in [u_{k-1}, u_{k+1}] \times [H^-(u_k) - d, H^-(u_k) + d], \tag{7.1.4}$$

is invariant as long as $u(t) \in (u_{k-1}, u_{k+1})$. Indeed the vector field for v^ε always points towards $\{H^-(u) = u\}$ (cf. item (iii) of Lemma 5.2.1). Moreover, (7.1.4) is a subset of $\chi^-_{\text{hyp}}(\varepsilon)$.

Subcase $u(t_1) = u_{k+1}$:

Note that since $u(t_0) = u_k$ and u is continuous, there exists a $t_0 \geq 0$ such that $u(t) \in [u_k, u_{k+1}]$ for $t \in [t_0, t_1]$. Also note that because the RHS of (7.1.4) is invariant we need only concern ourselves with the possibility that $|v^\varepsilon(0) - H^-(u_{k+1})| > d$, or more specifically, that $v^\varepsilon(0) > H^-(u_{k+1}) + d$ (see Figure 7.1).

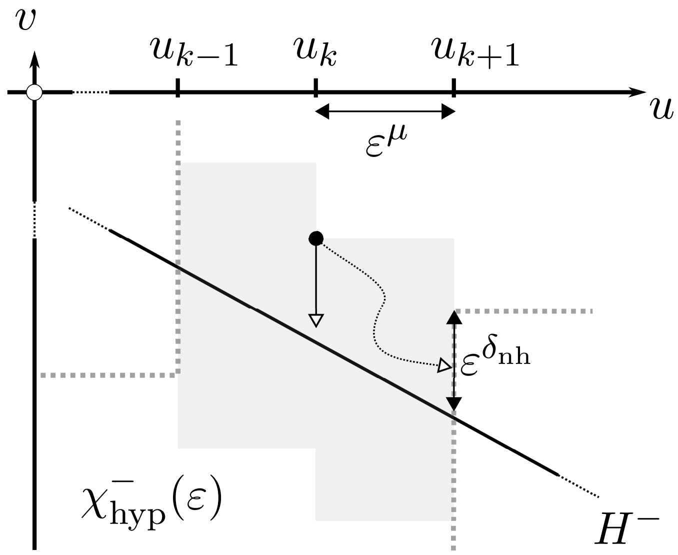

Figure 7.1.: Illustration of the proof of Lemma 6.1.3. $v^\varepsilon_k(\cdot)$ (solid line) is an upper bound for $v^\varepsilon(\cdot)$ (curved dashed line).

Let v^ε_k be the anchored solution satisfying (7.1.1) with $v^\varepsilon_k(0) = v^\varepsilon(0)$. Then because $(u(t), v^\varepsilon(t))$ and $(u_k, v^\varepsilon_k(t))$ satisfy Lemma 7.0.3, we can conclude that $v^\varepsilon(t_1) < v^\varepsilon_k(t_1)$ (cf. Figure 7.1).

Moreover, the direction of the vector field in $\chi^-_{\text{hyp}}(\varepsilon)$ implies that $v^\varepsilon_k(t_1)$ is also bounded below, namely

$$H^-(u_{k+1}) < v^\varepsilon_k(t_1).$$

It now suffices to check if $v_k^\varepsilon(t_1) - H^-(u_{k+1}) \leq d$.

Recall that $d = \varepsilon^{\delta_{\text{nh}}}$ by assumption and that $v^\varepsilon(0) - H^-(u_k) \leq \varepsilon^{\delta_{\text{nh}}}$, so applying Lemma 7.1.2 yields

$$\begin{aligned}(v_k^\varepsilon(t_1) - H^-(u_{k+1})) &\leq (v_k^\varepsilon(t_1) - H^-(u_k)) + (H^-(u_k) - H^-(u_{k+1})), \\ &\leq \varepsilon^{\delta_{\text{nh}}} \exp\left(-\frac{(t_1 - t_0)}{\varepsilon C_{\text{Lip}}}\right) \\ &\quad + C_{\text{Lip}}|u_k - u_{k+1}|.\end{aligned} \tag{7.1.5}$$

One can factor $\varepsilon^{\delta_{\text{nh}}}$ from the RHS of the last line of (7.1.5) to obtain

$$\varepsilon^{\delta_{nh}} \left(\exp\left(-\frac{(t_1 - t_0)}{\varepsilon C_{\text{Lip}}}\right) + \varepsilon^{-\delta_{nh}} C_{\text{Lip}}|u_k - u_{k+1}|\right). \tag{7.1.6}$$

Because u is Hölder continuous and H^- is Lipschitz continuous (cf. Conditions 6.1.2 and 1.2.6 respectively), we can substitute the inequality

$$|t_0 - t_1| \geq \left(\frac{1}{C_\gamma C_{\text{Lip}}}\right)^{\frac{1}{\gamma}} |u_k - u_{k+1}|^{\frac{1}{\gamma}} = c(\gamma) C_{\text{Lip}}^{-\frac{1}{\gamma}} |u_k - u_{k+1}|^{\frac{1}{\gamma}},$$

into (7.1.6), which when taken together with (7.1.5) means that

$$(v_k^\varepsilon(t_1) - H^-(u_{k+1})) \leq \varepsilon^{\delta_{nh}} \left(\exp\left(-c(\gamma) C_{\text{Lip}}^{-1-\frac{1}{\gamma}} \varepsilon^{-1+\frac{\mu}{\gamma}}\right) + C_{\text{Lip}} \varepsilon^{-\delta_{nh}+\mu}\right).$$

The term in the brackets is strictly less than one by Definition of ε_3 (cf. Definition 5.2.6), which completes the proof of the subcase when $u(t_1) = u_{k+1}$.

Subcase $u(t_1) = u_{k-1}$:

This is proved in an almost identical manner to the subcase $u(t_1) = u_{k+1}$ (cf. Figure 7.2).

- **Case:** $v(0) = H^-(u(0))$

If $u(t) \in (u_k, u_{k+1})$ for $t \in [0, t_1)$, then the maximum distance $v^\varepsilon(t)$ can move from the critical manifold is $|H^-(u_k) - H^-(u_{k+1})|$ (see Figure 7.3). Therefore, because $\varepsilon_0 < \varepsilon_3$ (cf. Definitions 5.2.6 and 1.3.5) we have

$$\begin{aligned}|v^\varepsilon(t) - H^-(u(t))| &\leq |H^-(u_k) - H^-(u_{k+1})|, \\ &\leq C_{\text{Lip}} \varepsilon^\mu, \\ &\leq \varepsilon^{\delta_{\text{nh}}}.\end{aligned} \tag{7.1.7}$$

If $u(t_1) = u_k$, then we are back in the case where $u(0) = u_k$ and $|v^\varepsilon(t_1) - H^-(u_k)| \leq \varepsilon^{\delta_{\text{nh}}}$, albeit with t_1 in place of 0. The same argument applies for $u(t_1) = u_{k+1}$. □

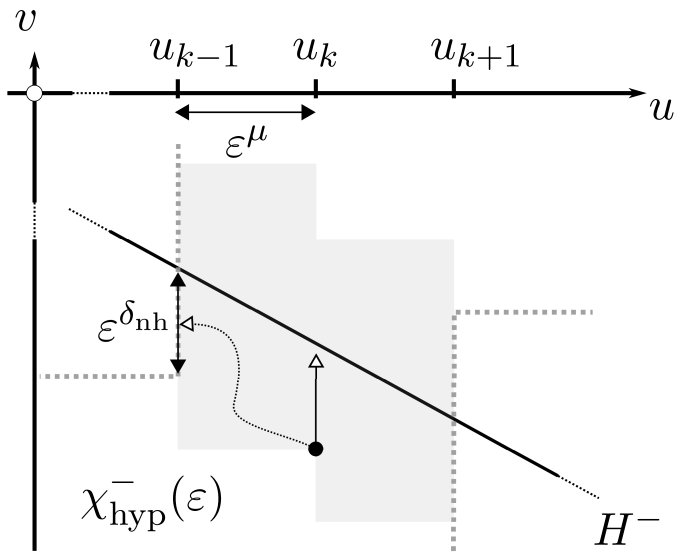

Figure 7.2.: Illustration of the proof of Lemma 6.1.3. $v_k^\varepsilon(\cdot)$ (solid line) is a lower bound for $v^\varepsilon(\cdot)$ (curved dashed line).

7.2. Invariance Lemma for the Quadratic Sleeve

Lemma 7.2.1. *For every $t \in [0,T]$ assume $u(t) \geq 0$ and define $w(t) = \sqrt{u(t)}$ (cf. Definition 5.1.4). Then for every $t_0, t_1 \in [0,T]$*

$$\min\{w(t_0), w(t_1)\}|w(t_0) - w(t_1)| \leq C_\gamma |t_0 - t_1|^\gamma. \tag{7.2.1}$$

Proof. Write $w(t)$ explicitly as $\sqrt{u(t)} = w(t)$, apply the mean value theorem to $u(t) \mapsto \sqrt{u(t)}$, and note that $u(t)$ is Hölder continuous in t (cf. Condition 6.1.2). □

Definition 7.2.2. *Consider a fixed $W_j \in (\varepsilon^{\frac{1}{3}-}, 1)$ with $j \in \{1, \cdots, J\}$ (cf. Definition 5.1.6). The anchored solution v_j^ε is the solution to*

$$\varepsilon \dot{v}_j^\varepsilon = (v_j^\varepsilon)^2 - (W_j)^2. \tag{7.2.2}$$

Lemma 7.2.3. *There is a C independent of j and ε such that if $v_j^\varepsilon(0) < 0$, then the anchored solution v_j^ε satisfies the bound*

$$|v_j^\varepsilon(0) + W_j| \leq C|v_j^\varepsilon(0) + W_j| \exp\left(-\frac{2W_j t}{\varepsilon}\right), \tag{7.2.3}$$

In particular, Definition 5.1.9 means that (7.2.3) *if $(W_j, v_j^\varepsilon(0)) \in \chi^-_{\mathrm{crit}}(\varepsilon)$.*

Proof. Start by writing

$$(v_j^\varepsilon)^2 - W_j^2 = (v_j^\varepsilon - W_j)(v_j^\varepsilon + W_j).$$

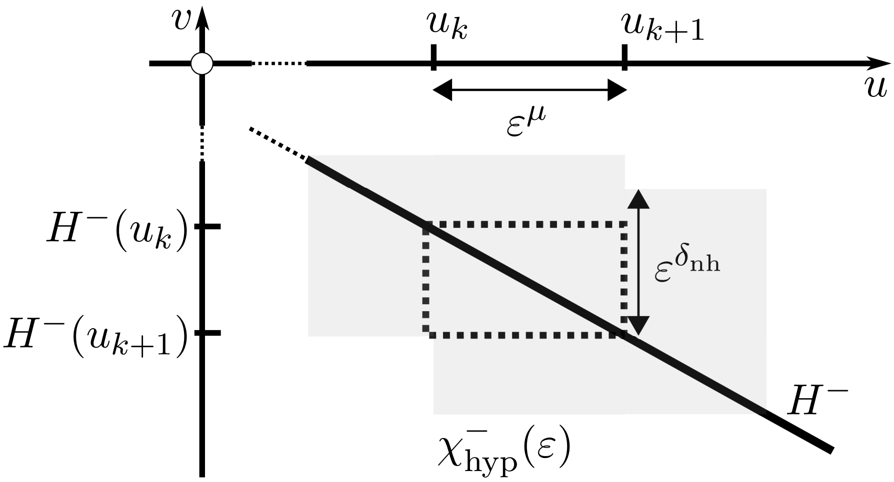

Figure 7.3.: Illustration of the case $v(0) = H^-(u(0))$ in the proof of Lemma 6.1.3. Because the vector field point towards H^-, all trajectories that start in the dotted box will remain there until there is a time t_1 such that $u(t_1) = u_k$ or $u(t_1) = u_{k+1}$.

As such, the reciprocal of $(v_j^\varepsilon)^2 - (W_j)^2$ can be expressed in the following way

$$\dot{v}_j^\varepsilon \left(\frac{1}{v_j^\varepsilon - W_j} - \frac{1}{v_j^\varepsilon + W_j} \right) = \frac{2W_j}{\varepsilon}.$$

We can now integrate from 0 to t to obtain

$$|v_j^\varepsilon(t) - W_j| = K(t)|v_j^\varepsilon(0) - W_j| \exp(-\frac{2W_k}{\varepsilon}), \qquad K(t) = \frac{|v_j^\varepsilon(t) - W_j|}{|v_j^\varepsilon(0) - W_j|}.$$

Observe that $|v_j^\varepsilon(t) - W_j| \leq 2W_j$, and that $|v_j^\varepsilon(0) - W_j| \geq W_j$. Therefore, $K(t)$ is bounded independent of j and ε. □

Proof of Lemma 6.1.4.

- **Case:** $w(0) = W_j$ and $|v^\varepsilon(0) + W_j| \leq D_j$:

Define the the time t_1 as

$$t_1 = \inf\{t \in [0, T] \mid u(t) = W_{j-1} \text{ or } u(t) = W_{j+1}\},$$

otherwise define $t_1 = T$. We now treat three sub-cases.

Subcase $t_1 = T$:

First note that as a consequence of Lemma 5.4.3 we have the inclusion

$$[W_{j-1}, W_{j+1}] \times [-W_j - D_j, -W_j + D_j] \subset \chi^-_{\text{crit}}(\varepsilon). \tag{7.2.4}$$

Moreover, it's clear that $w(t) \in (W_{j-1}, W_{j+1})$ and as such the LHS of (7.2.4) is invariant due to the direction of the vector field in A^-_{crit} (see Figure 7.4).

Subcase $w(t_1) = W_{j-1}$:

Because $w(\cdot)$ is continuous and $w(0) = W_j$, there is a $t_0 \in [0, t_1]$, such that $w(t_0) = W_j$ and

$$w(t) \in (W_{j-1}, W_j) \text{ for } t \in (t_0, t_1).$$

We know from (7.2.4) (see also Lemma 5.4.3) that we only need to to concern ourselves with the case where

$$v^\varepsilon(t_0) \notin [-W_j - D_j, -W_j + D_j].$$

The only remaining possibility in this scenario is that $v^\varepsilon(t_0) < -W_{j-1} - D_{j-1}$ (cf. Figure 7.4).

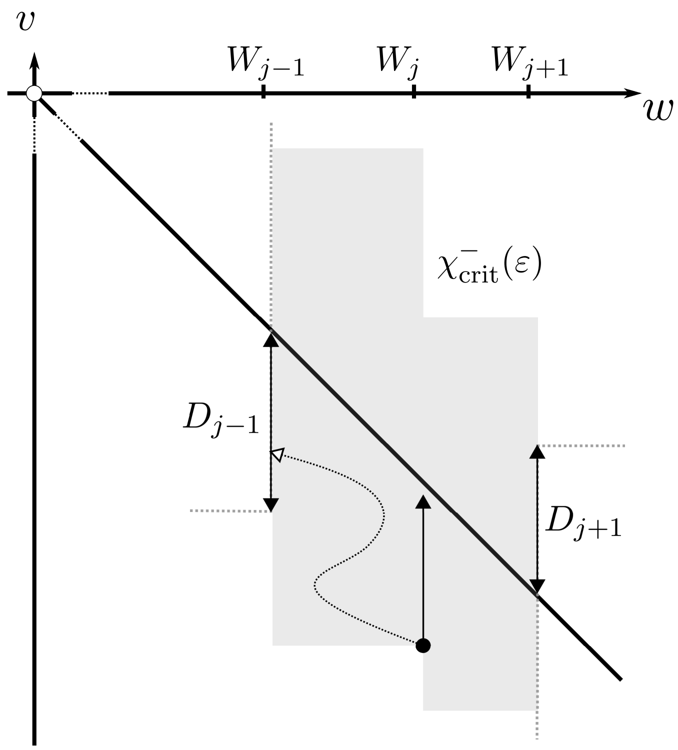

Figure 7.4.: Illustration of proof of Lemma 6.1.4 for the case $w(t_1) = W_{j-1}$. The vector field in the grey region points toward the line $\{v + w = 0\}$. $v^\varepsilon_j(\cdot)$ (solid line) is a lower bound for $v^\varepsilon(\cdot)$ (cuved dashed line).

Consider the anchored solution v^ε_j with $v^\varepsilon_j(0) = v^\varepsilon(0)$. Because of Lemma 7.0.3, we know that $v^\varepsilon(t_1) > v^\varepsilon_j(t_1)$. Moreover, $v^\varepsilon_j(t_1) < W_{j-1}$ because of the direction of the vector field in A^-_{crit}. We therefore obtain

$$|v^\varepsilon(t_1) + W_{j-1}| \leq |v^\varepsilon_j(t_1) + W_{j-1}|.$$

Applying Lemma 7.2.3 and taking out $\varepsilon^{\delta_{j-1}}$ as a pre-factor leads to the series of inequalities

$$\begin{aligned} |v_j^\varepsilon(t_1) + W_{j-1}| &\le |v_j^\varepsilon(t_1) + W_j| + |W_j - W_{j-1}|, \\ &\le \varepsilon^{\delta_{j-1}} \Big[C\varepsilon^{\delta_j - \delta_{j-1}} \exp\left(-2\varepsilon^{-1} W_j (t_1 - t_0)\right) \\ &\quad + \varepsilon^{-\delta_{j-1}} |W_j - W_{j-1}| \Big]. \end{aligned} \tag{7.2.5}$$

Let's now treat the second line of (7.2.5). By the definition of $|W_{j-1} - W_j|$ in (5.1.5), we have

$$|W_{j-1} - W_j| \varepsilon^{-\delta_{j-1}} = \varepsilon^{\lambda_2}. \tag{7.2.6}$$

Moreover, using Lemma 7.2.1 and replacing W_j with W_{j-1} (note that $W_{j-1} < W_j$) we obtain

$$\begin{aligned} \exp\left(-2\varepsilon^{-1} W_j (t_1 - t_0)\right) &\le \exp\left(-c(\gamma)\varepsilon^{-1+\mu_{j-1}+\frac{1}{\gamma}(\delta_{j-1}+\lambda_2+\mu_{j-1})}\right) \\ &\le \exp\left(-c(\gamma)\varepsilon^{-\lambda_1}\right), \end{aligned} \tag{7.2.7}$$

where in the second line of (7.2.7) we have used the definition of δ_j (cf. Definition 5.1.5). By recalling the Definition of ε_2 (Definition 5.1.9) we see that the sum of the RHS of (7.2.6) and the RHS of (7.2.7) is strictly less than one. This in turn implies that the term in the square brackets on the RHS of (7.2.5) is strictly less than one.

Subcase $w(t_1) = W_{j+1}$:

As a consequence of Lemma 5.4.3 we have the inclusion

$$[W_j, W_{j+1}] \times [-W_j - D_j, -W_j + D_j] \subset \chi^-_{\text{crit}}(\varepsilon),$$

and the direction of the vector field in A^-_{crit} (see Figure 7.5) tells us we only need to consider the initial data

$$v^\varepsilon(t_0) \in [-W_{j+1} + D_{j+1}, -W_j + D_j].$$

We define t_0 as in the previous case with W_{j+1} in place of W_{j-1}, and again consider the anchored solution v_j^ε with $v_j^\varepsilon(0) = v^\varepsilon(0)$.

Because of Lemma 7.0.3, we know that $v^\varepsilon(t_1) < v_j^\varepsilon(t_1)$. Moreover, $v_j^\varepsilon(t_1) > W_{j+1}$ because of the direction of the vector field in A^-_{crit}. Using Lemmas 7.2.3 and 7.2.1 we get a series of inequalities:

$$\begin{aligned} |v^\varepsilon(t_1) + W_{j+1}| &\le |v_j^\varepsilon(t_1) + W_{j+1}| \\ &\le |v_j^\varepsilon(t_1) + W_j| + |W_{j+1} - W_j| \\ &\le \varepsilon^{\delta_j} \exp\left(-2\varepsilon^{-1} W_j (t_1 - t_0)\right) + |W_{j+1} - W_j| \\ &\le \varepsilon^{\delta_{j+1}} \left(\frac{\varepsilon^{\delta_j}}{\varepsilon^{\delta_{j+1}}}\right) \left(\exp\left(-2c(\gamma)\varepsilon^{-1+\frac{\mu_j}{\gamma}(1+\gamma)+\frac{1}{\gamma}(\delta_j+\lambda_2)}\right) + \varepsilon^{\lambda_2}\right) \\ &\le 2\varepsilon^{\delta_{j+1}} \left(\exp\left(-2c(\gamma)\varepsilon^{-\lambda_1}\right) + \varepsilon^{\lambda_2}\right), \end{aligned}$$

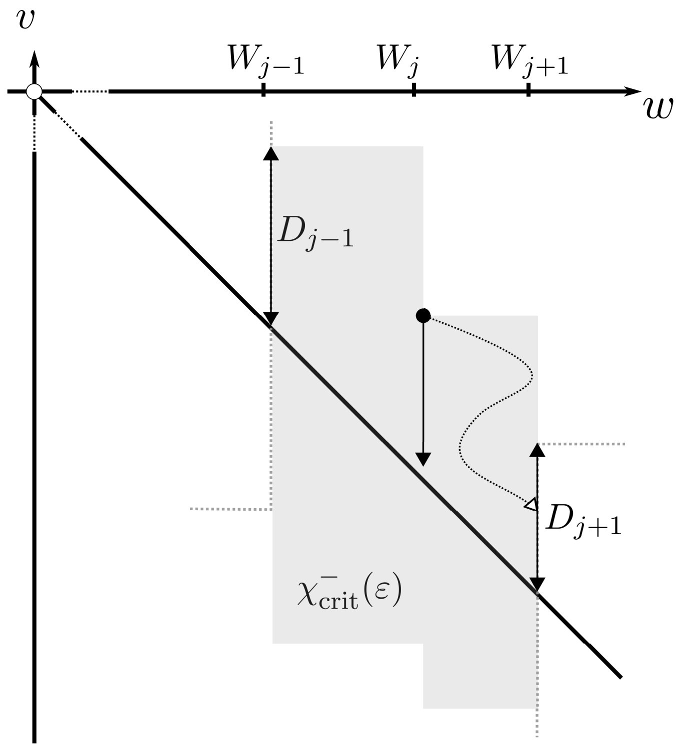

Figure 7.5.: Illustration of proof of Lemma 6.1.4 for the case $w(t_1) = W_{j+1}$. The vector field in the grey region points towards $\{U + w = 0\}$. $v_j^\varepsilon(\cdot)$ (solid line) is an upper bound for $v^\varepsilon(\cdot)$ (curved dashed line).

where to make the f to the last inequality we used $D_j < 2D_{j+1}$ (cf. Lemma 5.4.2). Now invoke the Definition of ε_2 (cf. Definition 5.1.9) to obtain

$$|v^\varepsilon(t_1) + W_{j+1}| \leq |v_j^\varepsilon(t_1) + W_{j+1}| \leq \varepsilon^{\delta_{j+1}}.$$

Case $v^\varepsilon(0) = w(0)$:

If $w(t) \in (W_{j-1}, W_j)$ for $t \in [0, t_0]$, then $v^\varepsilon(t) \in (W_{j-1}, W_j)$ (the analagous argument for the normally hyperbolic sleeve is shown in Figure 7.3) and hence by Lemma 5.4.3 $(w(t), v^\varepsilon(t)) \in \chi^-_{\text{crit}}(\varepsilon)$. If $w(t_0) = W_{j-1}$, then

$$|v^\varepsilon(t_0) - W_{j-1}| \leq |W_j - W_{j-1}| \leq \varepsilon^{\delta_j} \leq D_{j-1}, \tag{7.2.8}$$

where the second inequality is just Lemma 5.4.2. This puts you back in the first case of the current Lemma, in other words, the case which this proof has already addressed. If $w(0) = W_j$, then replace $|v^\varepsilon(t_0) - W_{j-1}|$ by $|v^\varepsilon(t_0) - W_j|$ in (7.2.8). □

7.3. Fast Motion and Drop Part of a Trajectory

Definition 7.3.1. *Given $U_{\text{exit}} < 0$, let $v^{\varepsilon}_{\text{exit}}$ be the solution to the ODE*

$$\varepsilon \dot{v}^{\varepsilon}_{\text{exit}} = (v^{\varepsilon}_{\text{exit}})^2 - U_{\text{exit}}. \tag{7.3.1}$$

Lemma 7.3.2. *The exact solution to* (7.3.1) *on the time interval $t \in (t_0, t_1)$ is*

$$\arctan(\frac{v^{\varepsilon}_{\text{exit}}(t_1)}{U_{\text{exit}}}) - \arctan(\frac{v^{\varepsilon}_{\text{exit}}(t_0)}{U_{\text{exit}}}) = \frac{U_{\text{exit}}(t_1 - t_0)}{\varepsilon}.$$

Proof. Recall the well known formula.

$$\frac{d}{d\theta}\arctan(\theta) = \frac{1}{\theta^2 + 1}. \tag{7.3.2}$$

We also have

$$\varepsilon \dot{v}^{\varepsilon}_{\text{exit}} = (v^{\varepsilon}_{\text{exit}})^2 + w^2_{\text{exit}},$$

where $W_{\text{exit}} = \sqrt{|U_{\text{exit}}|}$. The result know follows by making the substitution $\theta = \frac{v^{\varepsilon}_{\text{exit}}}{W_{\text{exit}}}$ into (7.3.2). □

Lemma 7.3.3 (Exiting the Critical Region)**.** *Let $\varepsilon < \varepsilon_4$ (cf. Definition 5.3.2) and suppose that $t_0 \geq 0$ and that $(-\varepsilon^{\frac{2}{3}-}, v^{\varepsilon}(t_0)) \in A^{-}_{\text{crit}}$. Then there exists a $t_1 > t_0$ such that the solution $v^{\varepsilon}_{\text{exit}}$ to* (7.3.1) *with fixed $U_{\text{exit}} = -\varepsilon^{\frac{2}{3}-}$ satisfies $(-\varepsilon^{\frac{2}{3}-}, v^{\varepsilon}(t_1)) \in \mathcal{S}_{\text{fast}}$ and $|t_1 - t_0|^{\gamma} \leq C\varepsilon^{\frac{2}{3}-}$, where C is independent of ε.*

Proof. From Lemma 7.3.2 we have an explicit formula for $v^{\varepsilon}_{\text{exit}}$ (cf. Definition 5.3.1) which we can solve for $v^{\varepsilon}_{\text{exit}}(t_1) = v_{\text{fast}}$. Since $\arctan(\cdot)$ is bounded we obtain

$$|t_1 - t_0| \leq C\varepsilon^{(1-\frac{1}{3}-)}, \quad |t_1 - t_0|^{\gamma} \leq C\varepsilon^{\frac{2}{3}-}, \tag{7.3.3}$$

where the last inequality follows from $\gamma(1-\mu_0) - 2\mu_0 > \gamma\lambda_1 + \lambda_2 > 0$ (cf. Definition 5.1.3). □

Lemma 7.3.4 (Fast Motion)**.** *Suppose that $t_1 \geq 0$ and $(u_{\text{fast}}, v^{\varepsilon}(t_1)) \in \mathcal{S}_{\text{fast}}$ where $v^{\varepsilon}(\cdot)$ is a solution to* (6.1.1) *with constant forcing term $u \equiv u_{\text{fast}} \in \pi_u\mathcal{S}_{\text{fast}}$. Then there exists a $t_2 > t_1$ such that $(u_{\text{fast}}, v^{\varepsilon}(t_2)) \in \mathcal{S}_{\text{drop}}$, and $|t_1 - t_2|^{\gamma} \leq C\varepsilon^{\frac{2}{3}-}$ where C is independent of ε.*

Proof. Because $u_{\text{fast}} \in \pi_u\mathcal{S}_{\text{fast}} = \pi_u\mathcal{S}_{\text{drop}}$, there is a constant $C > 0$ such that $|f(u_{\text{fast}}, v)| > C > 0$ for $v \in (v_{\text{fast}}, v_{\text{drop}})$ (cf. Definition 5.3.1). As such, there must exists a t_2 such that $(u_{\text{fast}}, v^{\varepsilon}(t_2)) \in \mathcal{S}_{\text{fast}}$. In particular, one has

$$v^{\varepsilon}(t_2) > v^{\varepsilon}(t_1) + \frac{C(t - t_1)}{\varepsilon}.$$

Note that $|v_{\text{drop}} - v_{\text{fast}}|$ does not depend on ε and hence

$$|t_1 - t_2| \leq C\varepsilon, \quad |t_1 - t_2|^\gamma \leq C\varepsilon^{\frac{2}{3}-},$$

where the last inequality follows from $\gamma - \mu_0 > \gamma\lambda_1 + \lambda_2 + \mu_0 + \gamma\mu_0 > 0$ (cf. Definition 5.1.3). $\square$

Definition 7.3.5. *Let ε_5 be the supremum over all ε such that*

$$C_{\text{Lip}} C_\gamma \left(\varepsilon \log \left(\frac{6v_\infty}{\varepsilon^{\text{nh}}} \right) C_{\text{Lip}} \right)^\gamma \leq \frac{\varepsilon^{\delta_{\text{nh}}}}{3}.$$

Note that ε_5 is well defined because $\gamma > \delta_{\text{nh}}$ (cf. Definition 5.2.3).

Lemma 7.3.6 (Dropping to χ^+_{hyp})**.** *Suppose that $t_2 \geq 0$, $u \in C^\gamma[t_2, \infty)$, and $(u(t_2), v^\varepsilon(t_2)) \in \mathcal{S}_{\text{drop}}$. Then there exists a $t_3 \geq t_2$ such that for all $\varepsilon < \varepsilon_5$, $(u(t_3), v^\varepsilon(t_3)) \in \chi^+_{\text{hyp}}(\varepsilon)$, and $|t_3 - t_2|^\gamma \leq C\varepsilon^{\frac{2}{3}-}$ where C is independent of ε.*

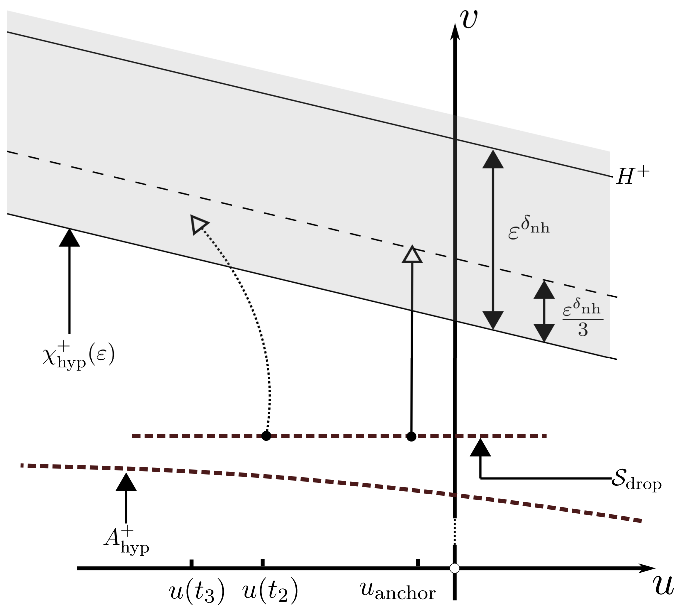

Figure 7.6.: Illustration of the proof of Lemma 7.3.6. $v^\varepsilon_{\text{anchor}}(\cdot)$ with $u = u_{\text{anchor}}$ (solid line) is a lower bound for $v^\varepsilon(\cdot)$ (curved dashed line).

Proof. The maximum height of $\chi^+_{\text{hyp}}(\varepsilon)$ is $2\varepsilon^{\delta_{\text{nh}}}$ so the Lemma is certainly true if $|v^\varepsilon(t_3) - H^+(u^\varepsilon(t_3))| \leq \varepsilon^{\delta_{\text{nh}}}$ (cf. Remark 5.2.5). Note that if there exists a $t > t_2$ such that $H^+(u(t)) = v^\varepsilon(t)$, then we are done. Therefore, we need to find a $t_3 > t_2$ such that

$$|H^+(u(t_3)) - v^\varepsilon(t_3)| = H^+(u(t_3)) - v^\varepsilon(t_3) \leq \varepsilon^{\delta_{\text{nh}}}.$$

For all fixed $u_{\text{anchor}} \in \pi_u A^+_{\text{crit}}$, let $v^\varepsilon_{\text{anchor}}$ be the solution of (7.1.1) with u_{anchor} in place of u_k. Lemma 7.1.2 and Theorem 1.2.18 still hold, in particular, the terminal and initial conditions

$$|v^\varepsilon_{\text{anchor}}(t_3) - H^+(u_{\text{anchor}})| = \frac{\varepsilon^{\delta_{\text{nh}}}}{3}, \qquad |v^\varepsilon_{\text{anchor}}(t_2) - H^+(u_{\text{anchor}})| \leq 2v_\infty, \tag{7.3.4}$$

hold if

$$|t_3 - t_2| \leq C_{\text{Lip}}\varepsilon \log(\frac{6v_\infty}{\varepsilon^{\text{nh}}}). \tag{7.3.5}$$

Moreover, if $\varepsilon < \varepsilon_5$ (cf. Definition 7.3.5), then

$$|H^+(u(t_2)) - H^+(u(t_3))| \leq C_{\text{Lip}}|u(t_2) - u(t_3)| \leq C_{\text{Lip}}C_\gamma|t_2 - t_3|^\gamma \leq \frac{\varepsilon^{\delta_{\text{nh}}}}{3}. \tag{7.3.6}$$

Note that because of (7.3.6) we also have

$$|u(t_2) - u(t_3)| \leq \frac{\varepsilon^{\delta_{\text{nh}}}}{3C_{\text{Lip}}}. \tag{7.3.7}$$

Now let's make an informed choice of u_{anchor}, namely

$$u_{\text{anchor}} = u(t_2) + \frac{\varepsilon^{\delta_{\text{nh}}}}{3C_{\text{Lip}}}, \tag{7.3.8}$$

which by (7.3.7) means that Lemma 7.0.3 applies with $u_{\text{anchor}} > u(t)$ for $t \in (t_2, t_3)$ (cf. Figure 7.6). From here we conclude that $v^\varepsilon_{\text{anchor}}(t_3) < v^\varepsilon(t_3)$ and more generally

$$|H^+(u(t_3)) - v^\varepsilon(t_3)| = H^+(u(t_3)) - v^\varepsilon(t_3) \leq H^+(u(t_3)) - v^\varepsilon_{\text{anchor}}(t_3).$$

Note that $|H^+(u(t_2)) - H^+(u_{\text{anchor}})| \leq \frac{\varepsilon^{\delta_{\text{nh}}}}{3}$, and that we can combine this observation (7.3.4), (7.3.6) to obtain

$$H^+(u(t_3)) - v^\varepsilon(t_3) \leq \varepsilon^{\delta_{\text{nh}}}.$$

To conclude the proof note that by (7.3.5) and because $\gamma > 2\mu_0$ (cf. Definition 5.1.3)

$$|t_3 - t_2|^\gamma \leq C\varepsilon^{\frac{2}{3}-}.$$

□

Proof of Lemma 6.1.5.

To begin with let $C_0 = 1$, though we will increase C_0 if necessary. Choose ε_6 sufficiently small so that $-\varepsilon_6^{\frac{2}{3}-} \in \pi_u A^-_{\text{crit}}$, in other words $\varepsilon_6 \leq \varepsilon_4$ (cf. Definition 5.3.2). Because $u(0) = 0$ we have a lower bound on t^*, namely

$$|u(0) - u(t^*)| = |u^\varepsilon_-| = C_0\varepsilon^{\frac{2}{3}-} \leq (t^*)^\gamma. \tag{7.3.9}$$

The goal is to show that we can choose C_0 in a way that forces t^* to satisfy both $(u^\varepsilon(t^*), v^\varepsilon(t^*)) \in \chi^+_{\text{hyp}}(\varepsilon)$ and that $(t^*)^\gamma \leq C\varepsilon^{\frac{2}{3}-}$ (cf. Figure 7.7). Because u is continuous, there is a $t_0 \in (0, t^*)$ such that $u(t) \in [u^\varepsilon_-, -\varepsilon^{\frac{2}{3}-}]$ for every $t \in [t_0, t^*]$. By Lemma 7.3.3 there is a $\tau_1 > t_0$ such that

$$(-\varepsilon^{\frac{2}{3}-}, v^\varepsilon_{\text{exit}}(\tau_1)) \in \mathcal{S}_{\text{fast}}.$$

Increase C_0 so that $C_0\varepsilon^{\frac{2}{3}-} < \tau_1^\gamma$ and decrease ε_6 so that $-C_0\varepsilon_6^{\frac{2}{3}-} \in \pi_u A^-_{\text{crit}}$. By Lemma 7.0.3 one has $v^\varepsilon(\tau_1) > v^\varepsilon_{\text{exit}}(\tau_1)$. In particular, there is a $t_1 \in (t_0, \tau_1)$ such that $(u(t_1), v^\varepsilon(t_1)) \in \mathcal{S}_{\text{fast}}$.

Next, by Lemma 7.3.4 there is a $t_2 > t_1$ such that

$$(u(t_2), v^\varepsilon(t_2)) \in \mathcal{S}_{\text{drop}}.$$

Increase C_0 so that $C_0\varepsilon_6^{\frac{2}{3}-} < t_2^\gamma$ and decrease ε_6 so that $-C_0\varepsilon_6^{\frac{2}{3}-} \in \pi_u A^-_{\text{crit}}$.

If is not already the case, let $\varepsilon_6 < \varepsilon_5$ and using Lemma 7.3.6, let t_3 by a time such that $(u^\varepsilon(t_3), v^\varepsilon(t_3)) \in \chi^+_{\text{hyp}}(\varepsilon)$. Increase C_0 so that $C_0\varepsilon_6^{\frac{2}{3}-} < t_3^\gamma$ and decrease ε_6 so that $-C_0\varepsilon_6^{\frac{2}{3}-} \in \pi_u A^-_{\text{crit}}$.

Lemmas 7.3.3–7.3.6 also establish that $|t_3 - t_0|^\gamma \leq C\varepsilon^{\frac{2}{3}-}$, more specifically

$$|t_3 - t_0|^\gamma \leq 3 \max_{i=1,2,3} |t_i - t_{i-1}|^\gamma \leq C\varepsilon^{\frac{2}{3}-}.$$

□

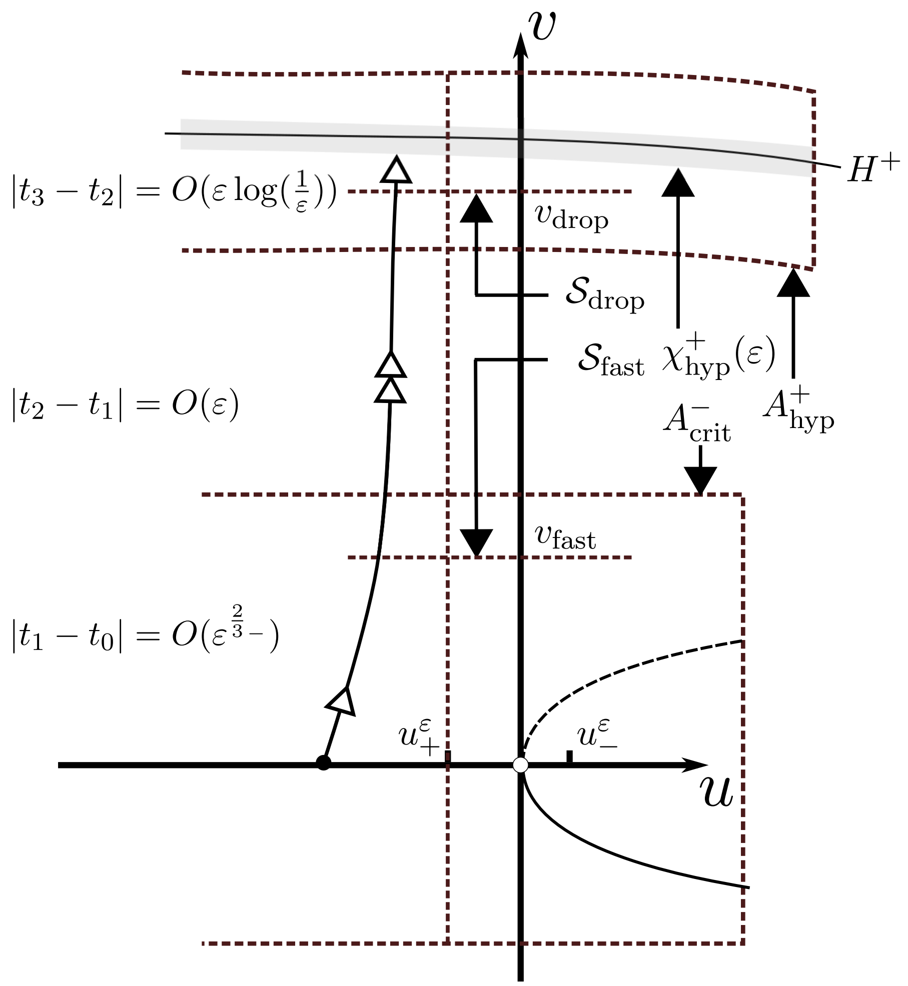

Figure 7.7.: The combined motion of exiting the critical region, making a fast transition and dropping to A^+_{hyp}. Note that this picture does not indicate the estimates in the statements of Lemma 7.3.3–7.3.6, but in their respective proof's.

8. Conclusion

In this thesis, our main object of study was a reaction-diffusion system with a nonlinearity that obeys a hysteresis law. This law is realized as an ensemble of scalar hysteresis operators, one defined at each spatial point and operating independently of one another.

The two scalar hysteresis operators we studied were the non-ideal relay (the *limiting hysteresis problem* (1.2.1)–(1.2.4)) and an ODE with a small parameter ε (the Slow-Fast System (1.2.6)–(1.2.8)) where the section of its nullcline consisting of stable equilibria corresponds to the branches of the non-ideal relay (cf. Condition 1.2.6).

The two issues we addressed were:

(i) Well-posedness of the limiting hysteresis problem; Chapter 3

(ii) In what way does the Slow-Fast System approximate the limiting hysteresis problem as $\varepsilon \to 0$, in particular, with what asymptotics; Chapters 4–7.

We were able to address item (i) in any spatial dimension when the initial data is transverse (cf. Definition 1.2.19) and item (ii) in one spatial dimension when the data is simply transverse (cf. Definition 1.2.20). The later is restricted to only one spatial dimension, however, the novel technical difficulties of a general result would aleady appear in the one-dimensional case. Though item (ii) had been addressed in other PDE settings (cf. Sections 1.1.1 and 1.1.4), this is the first result to include asymptotics in the variable ε. Moreover, we established that the order of convergence can be made arbitrarily close to $\varepsilon^{\frac{2}{3}}$ depending on the spatial regularity of the initial data. Note that this means one can be arbitrarily close to the 0th-order approximation of a system of a Fast-ODE coupled with a Slow-ODE (cf. Section 1.1.3).

8.1. Directions for Future Research

8.1.1. Slow-Fast Approximations for General Transverse Initial Data

In Theorem 1.3.4 we proved that system (1.2.6)–(1.2.8) approximates (1.2.1)–(1.2.4) in the L_∞-norm and that a spatially distributed Forced Fast-ODE (1.2.6) approximates the corresponding spatially distributed non-ideal relays under the same forcing (1.2.2). Both had the asymptotics described above.

The main assumption was simple transversality, which can fail if C_{dx} approaches zero. Moreover, this can occur if the discontinuities $b^\varepsilon(t) \to \{0, 1\}$, in which case the function

$u^\varepsilon(x,t)$ is still transverse in the general sense, and the limiting problem $u(x,t)$ is still well defined. The question of how to proceed was not addressed in this thesis.

A possible way forward is to adapt the method of "merging discontinuities" described in [GTS13], where the authors described how the limiting problem behaves as the discontinuity in $\mathcal{H}(\xi_0, u)$ approaches the boundary of the domain. In this thesis, we circumvented this complication by proving uniqueness of the limiting problem with what was effectively an application of Zorn's Lemma. (cf. Theorem 1.3.2)

8.1.2. Nonlinearity with Generic Folds

In this thesis we always assumed that the nonlinearity f in the Fast-ODE (1.2.6) was identical to a quadratic function in a neighborhood of the fold point. However, any generic fold can be transformed into a quadratic in a neighborhood of a the fold [MR80]. The other crucial Lemma we needed to study the forced Fast-ODE was Lemma 7.2.3. This Lemma concerned the convergence rate of the fast-subsystem to the equilibria on the parabola $u = v^2$, in particular, the eigenvalues had to approach zero linearly as $v \to 0$. This is still true after the coordinate transform described in [MR80] and [Kue15] that transforms a generic fold into a parabola of equilibria. As such, treating the generic case should be a technicality.

8.1.3. Blow-up Technique for Non-transverse Initial Data

We have alluded to the similarity of the $\varepsilon^{\frac{2}{3}}-$ asymptotics of Theorem 1.3.4 and its resemblance to the $\varepsilon^{\frac{2}{3}}$ asymptotic expansions of [MR80] and [KS01a]. The authors of both results tackle the region

$$\{(u,v) \in \mathbb{R}^2 \mid |u| \leq \varepsilon^{\frac{2}{3}}, |v| \leq \varepsilon^{\frac{1}{3}}\},$$

by applying a rescaling in u, v and t.

$$\begin{aligned} u &= \varepsilon^{\frac{2}{3}} U, \\ v &= \varepsilon^{\frac{1}{3}} V, \\ t &= \varepsilon^{\frac{2}{3}} \tau. \end{aligned} \tag{8.1.1}$$

We amend this rescaling with an additional rescaling in space, namely

$$x = y\varepsilon^{\frac{1}{3}}. \tag{8.1.2}$$

Under the rescalings (8.1.1) and (8.1.2), the system (1.2.5)–(1.2.8) becomes

$$U_\tau = U_{yy} + \varepsilon^{\frac{1}{3}} V, \tag{8.1.3}$$

$$V_\tau = -U + V^2. \tag{8.1.4}$$

Of particular interest is how the initial data φ scales in a neighborhood of zero. Let $\varphi \in C^\infty(Q)$ with a Taylor expansion

$$\varphi(x) = \varphi(0) + \varphi'(0)x + \varphi''(0)x^2 + \varphi'''(0)x^3 + O(x^4).$$

If φ is not transverse, then necessarily $\varphi(0) = \varphi'(0) = 0$, and under the rescaling (8.1.2) one has

$$\varphi(y) = \varphi''(0)y^2 + \varphi'''(0)y^3\varepsilon^{\frac{1}{3}} + O(\varepsilon^{\frac{2}{3}}y^4).$$

So for $\varepsilon = 0$ the 0th-order approximation is $\varphi(y) = \varphi''(0)y^2$ with $y \in \mathbb{R}$.

We know the Riccati ODE is solvable in terms of Bessel functions, and there is one solution which is asymptotic to the stable part of the parabola for large values of U. In (8.1.3)–(8.1.4), these special solutions are unfortunately not available. However, recent work in [GT17] and [GT18] proved the existence of oscillating patterns (which the authors called *rattling*) in a spatially discrete version of (1.2.1)–(1.2.4). These oscillations were independent of the size of the discretization, and as such they may provide some insight into what sort of asymptotics one should expect in (8.1.4)–(8.1.3).

A. Grönwall's Lemma

Lemma A.0.1 ([Hen93, Lemma 7.1.1]). *Suppose u is nonnegative and locally integrable on $0 \le t \le T < +\infty$, a is an arbitrary real constant, $\beta > 0$, and $b > 0$. If*

$$u(t) \le a + b\int_0^t (t-s)^{\beta-1}u(s)ds, \tag{A.0.1}$$

then

$$u(t) \le aE_\beta(\theta t), \tag{A.0.2}$$

where $\theta = (b\Gamma(\beta))^{\frac{1}{\beta}}$ and $E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta+1)}$.

We will use Lemma A.0.1 in the following way.

Lemma A.0.2. *Suppose u is nonnegative and locally integrable on $0 \le t \le T < +\infty$, $a \in \mathbb{R}$, and $b > 0$. If*

$$u(t) \le a + b\int_0^t (t-s)^{\frac{1}{2}}u(s)ds, \tag{A.0.3}$$

then

$$u(t) \le 2a\exp\left(\pi b^2 t\right), \tag{A.0.4}$$

Proof. The results will follow from Lemma A.0.1 if we can prove the estimate (cf. [Hen93, Chap. 7,Exercise 1])

$$\frac{d}{dz}E_{\frac{1}{2}}(z) = \frac{1}{(\pi z)^{\frac{1}{2}}} + E_{\frac{1}{2}}(z). \tag{A.0.5}$$

Let's begin by writing out the first few terms in $E_{\frac{1}{2}}(z)$, more specifically

$$E_{\frac{1}{2}}(z) = 1 + \frac{z^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \sum_{n=2}^{\infty}\frac{z^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{A.0.6}$$

Since $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, we get the first term on the RHS of (A.0.5)

$$\frac{d}{dz}E_{\frac{1}{2}}(z) = \frac{1}{(\pi z)^{\frac{1}{2}}} + \sum_{n=2}^{\infty}\lambda(n)z^{\frac{n-2}{2}}, \text{ where } \lambda(n) = \frac{n}{2\Gamma(\frac{n}{2}+1)}. \tag{A.0.7}$$

It now suffices to show that

$$\lambda(n) = \frac{1}{\Gamma(\frac{n-2}{2}+1)}.$$

We divide this last step into two cases.

- **Case:** $n = 2j$

In this case a simple calculation yields

$$\lambda(n) = \lambda(2j) = \frac{j}{\Gamma(j+1)} = \frac{1}{(j-1)!} = \frac{1}{\Gamma(\frac{2j-2}{2}+1)} = \frac{1}{\Gamma(\frac{n-2}{2}+1)}. \qquad \text{(A.0.8)}$$

- **Case:** $n = 2j - 1$

We resort first to the well known formula

$$\Gamma(j + \frac{1}{2}) = \frac{(2j)!}{2^{2j} j!},$$

which allows us to rewrite $\lambda(2j-1)$ as

$$\lambda(2j-1) = \frac{2j-1}{2\Gamma(j+\frac{1}{2})} = \frac{2^{2j-1}(2j-1)j!}{(2j)!}.$$

Writing $(2j)! = 2j(2j-1)!$ yields

$$\lambda(2j-1) = \frac{2^{2j-2}(j-1)!}{(2j-1)!} = \frac{1}{\Gamma((j-1)+\frac{1}{2})} = \frac{1}{\Gamma(\frac{2j-3}{2}+1)} = \frac{1}{\Gamma(\frac{n-2}{2}+1)}.$$

□

B. Uniformly Bounded Solutions to Scalar Reaction-Diffusion Equations

The proofs of these statements are largely based on [Cur14] but for convenience are reproduced here. Let $Q \subset \mathbb{R}^n$ be a bounded domain.

Definition B.0.3. *Let $u \in C(\overline{Q_T})$ and suppose that $u(x,0) < U_\infty$ for all $x \in \overline{Q}$. We say that $t \in (0,T]$ is a U_∞-*attainability moment *of u if there exists an $x \in \overline{Q}$ such that $u(x,t) = U_\infty$. We call the set of all U_∞-attainability moments the U_∞-*attainability set*, and denote it τ.*

If (x_j, t_j) is a sequence of points such that $t_j \to t'$ and $u(x_j, t_j) = U_\infty$, then by considering a convergent subsequence on the compact set $\overline{Q_T}$ and noting that u is continuous, it becomes clear that τ is a closed set.

Definition B.0.4. *For any $t \in \tau$ let $X(u,t) = \{x \in \overline{Q} \mid u(x,t) = U_\infty\}$.*

Definition B.0.5. *We call the minimal element of τ the* first U_∞-attainability moment.

Lemma B.0.6. *Let $u \in C(\overline{Q_T})$ and $\{u_j\}_{j>0} \subset C(\overline{Q_T})$ have U_∞-attainability sets $\tau \neq \emptyset$ (resptively $\tau_j \neq \emptyset$) and first U_∞-attainability moments t (resptively t_j). Moreover, assume that the following hold:*

(i) $\|u_j - u\|_{C(\overline{Q_T})} \to 0$.

(ii) For all $x \in \overline{Q}$, both $u(x,0) < U_\infty$ and $u_j(x,0) < U_\infty$.

(iii) $|t_j - t'| \to 0$.

Then $t' \in \tau$ and for any $\delta > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, the set $X(u_j, t_j)$ lies in the δ neighbourhood of $X(u, t')$.

Proof. We will first show that $t' \in \tau$. Take $x_j \in X(u_j, t_j)$ and form a sequence $(x_j, t_j) \in \overline{Q_T}$. Choose a convergent subsequence (not relabelled) such that $(x_j, t_j) \to (x', t')$ in $\overline{Q_T}$. Then

$$\begin{aligned} |u(x',t') - U_\infty| \leq & |u(x',t') - u(x_j,t_j)| + |u(x_j,t_j) - u_j(x_j,t_j)| \\ & + |u_j(x_j,t_j) - U_\infty|. \end{aligned} \tag{B.0.1}$$

Consider the RHS of inequality (B.0.1). Reading from left to right, the first term goes to zero because u is continuous, the second because $u_j \to u$ in $C(\overline{Q_T})$ and the term on the second line is equal to zero because $(x_j, t_j) \in X(u_j, t_j)$. Thus $u(x', t') = U_\infty$.

It remains to show that for all $\delta > 0$ there exists an $N \in \mathbb{N}$ such that if $x_j \in X(u_j, t_j)$ and $n \geq N$, then there is an $x' \in X(u, t')$ such that $\|x' - x_j\| < \delta$. Consider a sequence of points (x_j, t_j) with $x_j \in X(u_j, t_j)$. Take a convergent subsequence $(x_j, t_j) \to (x', t')$, then reasoning as above we conclude that $u(x', t') = U_\infty$. If there was a $\delta > 0$ and a sequence $x_j \in X(u_j, t_j)$ such that

$$\inf_{x \in X(u,t')} \{\|x_j - x\|\} > \delta,$$

then converse to this assumption we have just shown that there is necessarily a subsequence converging to something in $X(u, t')$. □

Proof of Lemma 2.3.1. Choose $U_\infty < u_\infty$ sufficiently close to u_∞ such that

$$\|\varphi\|_{C(\overline{Q_T})} < U_\infty,$$

and using the continuity of f_0 in the first argument further assume that there is a $C > 0$ such that for all $(x, t) \in \overline{Q_T}$ and U sufficiently close to U_∞

$$f_0(U, x, t) < C \text{ and } f_0(-U, x, t) > -C. \tag{B.0.2}$$

We will show that $\max_{(x,t)\in\overline{Q_T}} u(x, t) \leq U_\infty$ (the statement $\min_{(x,t)\in\overline{Q_T}} u(x, t) \geq -U_\infty$ is proved similarly).

Let $\varphi_j \in C^\infty(\overline{Q})$ and $F_j \in C^\infty(\overline{Q_T})$ be sequences of functions such that $\varphi_j \to \varphi$ in $\mathcal{W}$ and $F_j \to F$ in $L_q(Q_T)$. The functions F_j and φ_j can be constructed by mollifiers (c.f. [Eva10, App. C,Thm. 6]).

By [LSU68, Chap. 4,Thm. 5.3], for each j the problem

$$\begin{cases} u_{j;t} = \Delta u_j + F_j(x, t), & \text{(B.0.3)} \\ u_j|_{t=0} = \varphi_j, & \text{(B.0.4)} \\ \left.\dfrac{\partial u_j}{\partial \nu}\right|_{\partial Q_T} = 0, & \text{(B.0.5)} \end{cases}$$

has a unique classical solution where $u_{j;t}$ means the partial derivative of u_j with respect to t. By Theorem 2.1.1 $u_j \to u$ in $W_q^{2,1}(Q_T)$ and thus by Lemma 1.2.10 also in $C(\overline{Q_T})$. Therefore, it suffices to show that $\max_{(x,t)\in\overline{Q_T}} u_j(x, t) < U_\infty$.

To this end, we begin by taking j sufficiently large and then relabel the index in such a way that $\varphi_j(x) < U_\infty$ for all $x \in \overline{Q}$ and $j \in \mathbb{N}$. Let τ_j, τ' be the U_∞-attainability sets of u_j, u respectively.

If $\tau' = \emptyset$ or if there is a subsequence $\tau_j = \emptyset$ (i.e. $u_j(x, t) < U_\infty$ for all $(x, t) \in \overline{Q_T}$), then the Lemma is proved. Therefore, assume that $\tau', \tau_j \neq \emptyset$. Let t_j be the first U_∞-attainability moment of u_j. Choose a converging subsequence with limit $t_j \to t'$ (and relabelled index j) and note by Lemma B.0.6 that $t' \in \tau'$.

Because of (B.0.2)

$$f_0(u(x', t'), x, t) = f_0(U_\infty, x, t) < -c,$$

for all $(x', t') \in X(u, t')$ and $(x, t) \in \overline{Q_T}$. Let Y_δ be the intersection of the δ-neighbourhood of the set $X(u, t') \times \{t'\}$ with $\overline{Q_T}$. Since u is continuous and f_0 is continuous in the first argument, (B.0.2) implies we can choose δ small enough so that for all $(x, t) \in Y_\delta$, $u(x, t)$ is sufficiently close to U_∞ to imply that

$$F(x, t) = f_0(u(x, t), x, t) < -C.$$

If the F_j were constructed via mollifiers, then for sufficiently large j

$$F_j(x, t) < C - c, \tag{B.0.6}$$

for all $(x, t) \in Y_{\frac{\delta}{2}}$. By Lemma B.0.6 $X(u_j, t_j) \times \{t_j\}$ can be chosen within distance $\frac{\delta}{2}$ of $X(u, t') \times \{t'\}$ for sufficiently large j, i.e., $X(u_j, t_j) \times \{t_j\} \subset Y_{\frac{\delta}{2}}$. Hence

$$F_j(x, t_j) \leq -C, \tag{B.0.7}$$

for all $x \in X(u_j, t_j)$. However since $\varphi_j(x) < U_\infty$ and t_j was was the first U_∞-attainability moment we must have for $k = 1, \cdots, n$

$$\frac{\partial^2 u_j}{\partial x_k^2}(x, t_j) < 0, \tag{B.0.8}$$

for every $x \in X(u_j, t_j)$. Moreover, by (B.0.6) and (B.0.8) every $x \in X(u_j, t_j)$ satisfies

$$\frac{\partial u_j}{\partial t}(x, t_j) < 0.$$

Therefore, t_j is not the first U_∞-attainability moment of u_j. This contradiction proves that $\tau_j = \emptyset$, which in turn proves the Lemma. □

C. The Implicit Function Theorem with Lipshitz Parameters

In this Appendix let X, Y and Z be Banach spaces, $X \times Y$ has the L_1 -metric

$$\mathcal{F} : X \times Y \to Z,$$

and for every $x \in X$, let $\mathcal{F}^x(y) := \mathcal{F}(x, y)$. This Appendix is loosely based on [CH12, Chapt. 2].

Condition C.0.7 (Differentiable with Lipshitz Parameter)**.** We assume throughout that $(x_0, y_0) \in X \times Y$ and that there exists a convex neighborhood $\mathcal{U}$ of (x_0, y_0) with the following properties:

(i) $\mathcal{F}$ is Lipschitz continuous in $\mathcal{U}$.

(ii) $\mathcal{F}(x_0, y_0) = 0$

(iii) For every $(x, y) \in \mathcal{U}$, the Fréchet derivative $D_y\mathcal{F}^x(y) \in \mathcal{L}(Y, Z)$ exists, where $\mathcal{L}(Y, Z)$ is the space of bounded linear operators from Y to Z.

(iv) The map from $X \times Y$ to $\mathcal{L}(Y, Z)$ defined by $(x, y) \mapsto D_y\mathcal{F}^x(y)$ is Lipschitz continuous.

Lemma C.0.8. *If $\mathcal{F}$ satisfies Condition C.0.7, then for every $(x, y) \in \mathcal{U}$ and $h \in Y$ such that $y + h \in \mathcal{U}$, the function $\mathcal{F}$ satisfies*

$$\|\mathcal{F}(x, y + h) - \mathcal{F}(x, y)\|_Z \leq \max_{0 \leq t \leq 1} \|D_y\mathcal{F}(x, y + th)\|_{\mathcal{L}(Y,Z)} \|h\|_Y. \tag{C.0.1}$$

Proof. This is a simple consequence of the formula found in [CH12, Chap. 2,Thm. 1.3] which holds for any function with a Fréchet derivative, namely

$$\mathcal{F}(x, y + h) - \mathcal{F}(x, y) = \int_0^1 D_y\mathcal{F}^x(y + sh)ds.$$

□

Theorem C.0.9 (Implicit function theorem with a Lipschitz parameter)**.** *If Condition C.0.7 holds and $D_y\mathcal{F}^{x_0}(y_0)$ is invertible, then there exists a $\delta > 0$, a neighborhood $\mathcal{V} \subset B_\delta(x_0)$ and a function $a : \mathcal{V} \to B_\delta(y_0)$ such that for every $(x, y) \in \mathcal{V} \times B_\delta(y_0)$, $\mathcal{F}(x, y) = 0$ if and only if $y = a(x)$.*

Proof. Consider the map $\mathcal{A}^x : Y \to Y$ defined by

$$\mathcal{A}^x(y) = y - [D_y\mathcal{F}^{x_0}(y_0)]^{-1}\mathcal{F}^x(y). \tag{C.0.2}$$

We first claim that for $\delta > 0$ sufficiently small, $\mathcal{A}^x$ maps $B_\delta(y_0)$ to itself. More specifically, first choose $\delta > 0$ so that $B_{2\delta}(x_0, y_0) \subset \mathcal{U}$. Now for $\|h\|_Y \le \delta$ consider the series of inequalities

$$\begin{aligned}
&\|\mathcal{A}^x(y_0 + h) - y_0\|_Y \\
\le&\|\mathcal{A}^x(y_0 + h) - \mathcal{A}^x(y_0)\|_Y + \|\mathcal{A}^x(y_0) - y_0\|_Y, \\
\le& \max_{0\le t\le 1} \|D_y\mathcal{A}^x(y_0 + th)\|_{\mathcal{L}(Z,Z)}\|h\|_Y + \|(D_y\mathcal{F}^{x_0}(y_0))^{-1}\|_{\mathcal{L}(Z,Y)}\|\mathcal{F}^x(y_0)\|_Z,
\end{aligned} \tag{C.0.3}$$

where in the second inequality we have used Lemma C.0.8. We now treat each term on the RHS of (C.0.3). Firstly, note that for every $(x, y) \in \mathcal{U}$,

$$\begin{aligned}
D_y\mathcal{A}^x(y) &= I - [D_y\mathcal{F}^{x_0}(y_0)]^{-1}D_y\mathcal{F}^x(y), \\
&= [D_y\mathcal{F}^{x_0}(y_0)]^{-1}[D_y\mathcal{F}^{x_0}(y) - D_y\mathcal{F}^x(y_0)].
\end{aligned} \tag{C.0.4}$$

Now use that $D_y\mathcal{F}^x(y)$ is Lipshitz in both parameters to obtain

$$\max_{0\le t\le 1} \|D_y\mathcal{A}^x(y_0 + th)\|_{\mathcal{L}(Y,Z)} \le C\|D_y\mathcal{F}^{x_0}(y_0)\|^{-1}_{\mathcal{L}(Y,Z)}(\delta + \|x - x_0\|_X). \tag{C.0.5}$$

We also know that $\mathcal{F}^x(y_0)$ is Lipshitz in the the parameter x and that $\mathcal{F}^{x_0}(y_0) = 0$, hence

$$\|\mathcal{F}^x(y_0)\|_Z \le C\|x - x_0\|_X. \tag{C.0.6}$$

By choosing δ and $\|x - x_0\|_X$ small enough we see that both terms on the RHS of (C.0.3) are less than $\frac{\delta}{2}$. In particular, one obtains

$$\|\mathcal{A}^x(y_0 + h) - y_0\|_Y \le \delta.$$

We now show that for a suitable choice of δ, $\mathcal{A}^x$ is a contraction. To this end, choose two $h_1, h_2 \in B_\delta(y_0)$ and consider the difference

$$\begin{aligned}
&\|\mathcal{A}^x(y_0 + h_1) - \mathcal{A}^x(y_0 + h_2)\|_Y, \\
&\|\mathcal{A}^x(y_0 + h_1) - \mathcal{A}^x(y_0 + h_1 + (h_2 - h_1))\|_Y.
\end{aligned} \tag{C.0.7}$$

We now recall that $D_y\mathcal{A}^x(y_0 + h_1)$ exists on $B_\delta(y_0 + h_1)$ because at the outset we chose δ such that $B_{2\delta}(x_0, y_0) \subset \mathcal{U}$. Using lemma C.0.8 on the map $D_y\mathcal{A}^y(y_0 + h_1)$ yields

$$\|\mathcal{A}^x(y_0 + h_1 + (h_2 - h_1))\|_Y \le \max_{0\le t\le 1} \|D_y\mathcal{A}^x(y_0 + h_1 + t(h_2 - h_1))\|_{\mathcal{L}(Y,Y)}\|h_1 - h_2\|_Y.$$

Because $D_y\mathcal{A}^{x_0}(y_0) = 0$ we have the estimate

$$\begin{aligned}
&\max_{0\le t\le 1} \|D_y\mathcal{A}^x(y_0 + h_1 + t(h_2 - h_1))\|_{\mathcal{L}(Y,Y)}. \\
\le&C \max_{0\le t\le 1} \|(x_0, y_0) - (x, y_0 + t(h_2 - h_1))\|_{X\times Y}, \\
\le&C(\|x_0 - x\|_X + 2\delta).
\end{aligned} \tag{C.0.8}$$

Now for $\|x-x_0\|_X < C^{-1}(1-2\delta)$ we get a contraction. By [CH12, Chap. 2,Thm. 2.1] there is a unique fixed point of (C.0.2) which we denote $y = a(x)$, such that $a(x) = \mathcal{A}^x(a(x))$, i.e., $\mathcal{F}^x(a(x)) = 0$. □

D. Well-Posedness of the Slow-Fast System

In this Appendix we outline a proof that there is a $u^\varepsilon \in W_q^{2,1}(Q_T)$ and $v^\varepsilon \in W_\infty^{0,1}(Q_T)$ that is a solution (1.2.5)–(1.2.8) in the sense of Definition 1.2.17, and that both u^ε and v^ε are uniformly bounded independent of ε and T. The proof will borrow on the general framework from Chapter 3 and use some results from Chapter 7.

Firstly consider $u_0 \in R^\lambda(Q_T)$. We will determine a precise value of T shortly. According to [Hen93, Theorem 3.3.3], for every $x \in Q$ there exists a solution $v_0(x, \cdot) \in C^1(0, T)$ to the ordinary differential equation

$$\varepsilon v_{0;t}(x,t) = f(u_0, v_0). \tag{D.0.1}$$

where the subscript $0;t$ means differentiation with respect to the variable t. If we assume that u_∞ is taken sufficiently removed from the fold point at zero, then whenever $u_0(x,t)$ is in a neighborhood of u_∞, we must have $v_0(x,t) \in A^-_{\text{hyp}}$. If A^-_{hyp} is narrow enough (in other words c_∞ is small enough, cf. (1.2.11)), then $v_0(x,t) < 0$. A similar conclusion holds for A^+_{hyp} and $-u_\infty$. Note that $\|v_{0j}\|_{L_\infty(Q_T)} < v_\infty$ where v_∞ is defined as in Condition 1.2.15. Now let

$$f_0(u,x,t) = v_0 + c_\infty(u_0 - u), \tag{D.0.2}$$

and consider the semi-linear equation (2.2.1)–(2.2.3). We would like to prove a result corresponding to Theorem 3.3.1 by following the steps of that proof. The only main point of difference is showing continuous dependence of the solution on u_0.

Let $u_{0j} \to u_0$ in $R^\lambda(Q_T)$ and let v_{0j} be the solution of (D.0.1) with nonlinearity $f(u_{0j}, v_{0j})$. We want to know if $u_j \to u$ where u_j solves (2.2.1)—(2.2.3) with nonlinearity

$$f_{0n}(u_j,x,t) = v_{0j} + c_\infty(u_{0j} - u_j).$$

Note that because of Lemma 2.3.1 one has $\|u_j\|_{L_\infty(Q_T)} < u_\infty$. The main thing one needs to check is

$$\|v_0 - v_{0j}\|_{L_q(Q_T)} \to 0.$$

We can in fact prove an inequality for $L_\infty(Q_T)$. To this end let $u_0, u_0' \in R^\lambda(Q_T)$ and temporarily dropping the variable x to make the equations compact, consider

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^t |f(u_0(s), v_0(s)) - f(u_0'(s), v_0'(s))|ds, \\ &\leq \frac{Ct}{\varepsilon}\|u_0 - u_0'\|_{L_\infty(Q_T)} + \frac{C}{\varepsilon}\int_0^t |v_0(s) - v_0'(s)|ds. \end{aligned} \tag{D.0.3}$$

Then according to Appendix A one has

$$\|v_0(\cdot,t) - v_0'(\cdot,t)\|_{L_\infty(Q)} \leq \frac{Ct}{\varepsilon}\|u_0 - u_0'\|_{L_\infty(Q_T)} \exp\left(\frac{Ct}{\sqrt{\varepsilon}}\right). \tag{D.0.4}$$

Thus as $\|u_0 - u_0'\|_{L_\infty(Q_T)}$ goes to zero, so too does $\|v_0 - v_0'\|_{L_\infty(Q_T)}$. This proves existence of solutions to (1.2.5)–(1.2.8).

For uniqueness, one can use (D.0.4) for T sufficiently small to conclude that for two pairs of solutions (u, v) and (u', v'), one has

$$\|v - v'\|_{L_1(Q_T)} < \|u - u'\|_{L_\infty(Q_T)}. \tag{D.0.5}$$

This gives an analogue of Lemma 3.4.1 from which the uniqueness of solutions follows.

E. Formalities (Deutsche Zusammenfassung, Selbstständigkeitserklärung, Curriculum Vitae)

Deutsche Zusammenfassung

In dieser Arbeit untersuchen wir Reaktions-Diffusionsgleichungen, deren Nichtlinearität einem Hysterese-Gesetz folgt. Das Hysterese-Gesetz wird als eine Sammlung skalarer Operatoren umgesetzt, je einer an jedem räumlichen Punkt, die alle unabhängig voneinander reagieren. Diese Herangehensweise wird durch biologische Anwendungen gerechtfertigt, bei denen Kombinationen diffundierender und nicht-diffundierender Substanzen interagieren und dabei einem Hysterese-Gesetz folgen.

Die einzelnen Operatoren sind entweder nichtideale Schalter („Relais") oder Lösungen einer gewöhnlichen Differentialgleichung mit einem kleinen Parameter. Unter einer sehr allgemeinen Bedingung, der sogenannten räumlichen Transversalität, beweisen wir die Existenz und Eindeutigkeit der Lösungen des Systems mit den nichtidealen Schaltern. Außerdem beweisen wir, dass das System mit den gewöhnlichen Differentialgleichungen das System mit den nichtidealen Schaltern approximiert, wenn der kleine Parameter gegen Null geht. Im Zusammenhang mit partiellen Differentialgleichungen beweisen wir zum ersten Mal die explizite Asymptotik in Bezug auf diesen Parameter.

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Selbstständigkeitserklärung

Hiermit bestätige ich, Mark Curran, dass ich die vorliegende Dissertation mit dem Thema

The Hysteretic Limit of a Reaction-Diffusion System with a Small Parameter

selbstständig angefertigt und nur die genannten Quellen und Hilfen verwendet habe.

Die erste Version dieser Arbeit wurde erstmalig am 14.8.2018 ausschließlich an der Freien Universität Berlin eingereicht.

Die revidierende Version ist am untergeschriebenen Datum erstmalig und nur an der Freien Universität eingereicht worden.

Berlin, den

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